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# Coloured topological operads and moduli spaces of surfaces with multiple boundary curves

Dissertation
zur Erlangung des Doktorgrades (Dr. rer. nat.)

Mathematisch-Naturwissenschaftliche Fakultät der
Rheinischen Friedrich-Wilhelms-Universität Bonn

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# Abstract

This thesis is based on the following observation: while it is a classical result that the collection $\coprod_g \mathfrak{M}_{g,1}$ of moduli spaces of surfaces with a single boundary curve is an $E_2$-algebra (more precisely: it admits an action of the little 2-cubes operad $\mathscr{C}_2$), we need a coloured version of $\mathscr{C}_2$ which understands a *cluster* of squares as a single input with a certain multiplicity, if we want to establish an action on the collection of moduli spaces $\mathfrak{M}_{g,n}$ of surfaces with *multiple* boundary curves in a similar way.

Moreover, Bödigheimer introduced a finite multisimplicial model $\mathfrak{P}_{g,n}$ for $\mathfrak{M}_{g,n}$, which is useful for explicit homological calculations. In order to construct an operadic action on this specific model, we have to additionally require a certain coupling behaviour among squares belonging to the same input. This gives rise to a family of suboperads, called *vertical operads*.

We analyse these operads from several perspectives: on the one hand, their operation spaces and free algebras are modelled by clustered and vertical configuration spaces, whose homology, homological stability, and iterated bar constructions we investigate in the first chapters. On the other hand, we study the homotopy theory and the homology of their algebras and use the arising operations to describe the unstable homology of moduli spaces.

Finally, it turns out that the developed methods are also useful to solve a problem of a seemingly different flavour: for a space $A$, the collection of *parametrised* moduli spaces $\coprod_g \mathfrak{M}_{g,1}[A]$ is itself an $E_2$-algebra, and its group completion is an infinite loop space. We identify the underlying spectrum in the spirit of Madsen and Weiss.

Keywords: coloured operad, configuration space, moduli space, homological stability, bar construction, homology operation, group completion, infinite loop space.

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# Conventions

The following conventions will be used without any further explanation:

- We denote the *natural numbers* (including 0) by $\mathbb{N} := \{0, 1, 2, \dots\}$, the *integers* by $\mathbb{Z}$, the *rational numbers* by $\mathbb{Q}$, the *real numbers* by $\mathbb{R}$, and the *complex numbers* by $\mathbb{C}$. The set of strictly positive integers will be denoted by $\overline{\mathbb{N}} := \{1, 2, \dots\}$.
- For an integer $n > 0$, we denote the cyclic group with $n$ elements by $\mathbb{Z}_n$, and for a prime $p$ we denote the field with $p$ elements by $\mathbb{F}_p$.
- Closed intervals are denoted by $[a; b]$, while open intervals are denoted by $(a; b)$, and half-open intervals by $[a; b)$ and $(a; b]$, respectively.
- We denote the *disc*, the *sphere*, and the *torus* by

$$\begin{aligned} \mathbb{D}^d &:= \{(x_1, \dots, x_d) \in \mathbb{R}^d;\ x_1^2 + \cdots + x_d^2 \leqslant 1\}, \\ \mathbb{S}^{d-1} &:= \{(x_1, \dots, x_d) \in \mathbb{R}^d;\ x_1^2 + \cdots + x_d^2 = 1\}, \\ \mathbb{T}^d &:= (\mathbb{S}^1)^d. \end{aligned}$$

- We abbreviate $\underline{r} := \{1, \dots, r\}$ for each $r \geqslant 0$, and we write $\mathfrak{S}_r$ for the $r^{\text{th}}$ *symmetric group*, which contains all bijections on the set $\underline{r}$.
- By a *space*, we mean a topological space which is compactly generated and has the weak Hausdorff property. Let **Top** be the category of them. Moreover, we let $\mathbf{Top}_*$ be the category of (possibly degenerately) pointed spaces, with pointed maps as morphisms. The space with a single point is denoted by $*$.

# Overview

> *... nos esse quasi nanos*
> *gigantum umeris insidentes.*[1]
>
> Bernard de Chartres

This thesis continues a classical approach which has proved very useful: it exploits the interplay between configuration spaces, operads, and moduli spaces of Riemann surfaces.

Configuration spaces of smooth manifolds have been playing a central rôle in algebraic topology since the influential works of [FN62a; FN62b]. There are plenty of perspectives from which configurations can be considered; let me mention only those which appear in this thesis: firstly, configuration spaces give geometric models for classifying spaces of Artin's braid groups [Art25] and of the symmetric groups. Secondly, their homology has been extensively studied in [Arn69; Fuc70; CLM76], and the sequence of unordered configuration spaces which are related by adding a new particle 'far away' takes shape in many classical sequences for which homological stability has been shown [Nak61; Arn69; Seg79; Ran13]. Thirdly, for a based space $X$, one can consider configuration spaces of particles which come together with a *label*, and a particle vanishes if its label reaches the basepoint. These labelled configuration spaces have been compared to iterated loop spaces [Seg73; MS76], or, more generally, to section and mapping spaces [Böd87; BCT89], and they admit a stable splitting [Sna74; Böd87].

The concept of an operad has its origins in the study of iterated loop spaces by Boardman, Vogt, and May [BV68; May72; BV73], and has become an indispensable part of modern homotopy theory since then. The master-example of a topological operad, the operad $\mathscr{C}_d$ of little $d$-cubes introduced in [May72], models the homotopy theory of $E_d$-algebras. On the other hand,

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[1] *... that we are like dwarves perched on the shoulders of giants.*

it is connected to the aforementioned configuration spaces: the operation spaces of $\mathscr{C}_d$ are equivalent to configuration spaces of *ordered* points in $\mathbb{R}^d$, while their free algebras are equivalent to configuration spaces of *unordered* and *labelled* points. Therefore, studying the homotopy theory of $E_d$-algebras is closely related to studying the homotopy type of configuration spaces. The groundbreaking work of [CLM76] classifies all homology operations for $E_d$-algebras: they are known as the *Pontrjagin product* [Pon39], the *Browder bracket* [Bro60], and, in the case of finite characteristic, a collection of divided power operations called *Araki–Kudo operations* or *Dyer–Lashof operations* [AK56; DL62]. These operations can be used in order to give a very concise description of the rational and the $\mathbb{F}_p$-homology of unordered configuration spaces [CLM76; Coh95].

There is a profound connection between $E_d$-algebras and iterated loop spaces: each $E_d$-algebra $M$ admits an iterated bar construction $B^d M$, introduced in [May72], and comes together with a map $M \to \Omega BM \simeq \Omega^d B^d M$ of H-spaces. It has been shown in [BP72; May74; MS76] that this map is a group completion: in particular, if $\pi_0(M) \cong \mathbb{N}$, then the *stable space* $M_\infty$ is homology equivalent to $\Omega^d_0 B^d M$, the path component of the constant loop. Finally, I ought to mention that operads have also been considered in purely algebraic contexts, starting with the works of Ginzburg and Kapranov [GK94]. However, their work is going to play a minor rôle in this thesis, since all algebraic operads we consider are induced from topological ones.

Moduli spaces are much older: already Riemann noticed that for $g \geqslant 2$, there are $3g - 3$ complex parameters describing the possible complex structures that a Riemann surface of genus $g$ can have [Rie57]. A rigorous way to study these parameter spaces has been established in [Tei40]. In this thesis, we consider moduli spaces $\mathfrak{M}^m_{g,n}$ of surfaces of genus $g \geqslant 0$ with $n \geqslant 1$ parametrised boundary curves and $m \geqslant 0$ permutable punctures. Since we have at least one boundary curve, the moduli space $\mathfrak{M}^m_{g,n}$ is a classifying space for the topological group of orientation-preserving diffeomorphisms which, by a result of [ES70], is equivalent to the mapping class group $\Gamma^m_{g,n}$. Hence, studying the (co-)homology of moduli spaces is the same as studying characteristic classes for oriented surface bundles. Here, significant progress has been achieved over the past fourty years: the stability theorem of Harer [Har84; Iva90; Bol12; Ran16] states that the map $H_h(\mathfrak{M}_{g,1}) \to H_h(\mathfrak{M}_{g+1,1})$, which is induced by forming the boundary-connected sum with a bounded torus, is an isomorphism for $h \leqslant \frac{2}{3} \cdot (g-1)$. The rational stable cohomology $H^\bullet(\mathfrak{M}_{\infty,1};\mathbb{Q})$ in turn contains a polynomial algebra $\mathbb{Q}[\kappa_1,\kappa_2,\dots]$ generated by the Mumford–Miller–Morita classes $\kappa_i$ in degree $2i$, see [Mum83; Mil86; Mor87]. The famous Mumford conjecture states that this inclusion is even an equality, see below.

Secondly, the collection $\coprod_{g\geqslant 0} \mathfrak{M}_{g,1}$ of moduli spaces with a single boundary curve forms an $E_2$-algebra [Mil86; Böd90b] by sewing of surfaces. Its group completion $\Omega B \coprod_g \mathfrak{M}_{g,1}$ is an *infinite* loop space [Til97], and the celebrated Madsen–Weiss theorem [MW07] tells us which one: it is the infinite loop space associated with the affine oriented Thom spectrum $\mathbf{MTSO}(2)$. This shows that the stable moduli space $\mathfrak{M}_{\infty,1}$ is homology equivalent to $\Omega_0^\infty \mathbf{MTSO}(2)$, which was the key ingredient for the proof of the aforementioned Mumford conjecture. The homotopical methods of Madsen and Weiss have been refined in the study of topological cobordism categories [Gal+09] and admit higher-dimensional analogues [GR14; GR17; GR18]. Finally, the work of [Til97] has been generalised in [Til00; Bas+17], where infinite loop spaces are recognised by operads which do not necessarily have an underlying $E_\infty$-operad, but instead enjoy certain homological stability properties with respect to their operation spaces. One example of such an operad with homological stability is the *surface operad*, whose initial algebra is $\coprod_g \mathfrak{M}_{g,1}$.

While the above results give a comprehensive description of the *stable* homology of moduli spaces, their *unstable* homology is notoriously complicated. One approach for explicit low-genus calculations uses a finite multisimplicial model $\mathfrak{P}^m_{g,n}$ for $\mathfrak{M}^m_{g,n}$ that is based on an old work of Hilbert [Hil09] and has been established in the context of moduli spaces by Bödigheimer [Böd90a]. In recent decades, many explicit unstable homology groups and their generators have been identified in works of [Ehr98; Abh05; God07; ABE08; Vis11; Meh11; Wan11; BH14; Boe18]. Here, the aforementioned homology operations from [CLM76] for $E_2$-algebras can be used to describe explicit generators.

This thesis starts with the following idea: for the above operadic description of the collection of moduli spaces as an $E_2$-algebra, it was crucial that we considered surfaces with a *single* boundary curve. However, even though the operation of 'capping off' additional boundary curves becomes stably irrelevant, the unstable homology of moduli spaces with *multiple* boundary curves requires some further investigation. In order to apply operadic techniques to this problem, we need a coloured version of the little 2-cubes operad $\mathscr{C}_2$ which understands a cluster of little squares as a single input with a certain multiplicity. Secondly, Bödigheimer's model $\mathfrak{P}^m_{g,n}$ of parallel slit domains requires a certain coupling behaviour among squares which belong to the same input. Several special cases of this approach have already been studied in [Böd90b], and a unified operadic approach has been suggested in [Böd13]. This is the situation from which our journey starts.

**Chapter 1. Vertical configuration spaces** Following the classical path, we start by looking at configuration spaces that model our desired operads: we consider configuration spaces of points inside $\mathbb{R}^d$ as in [Seg73; CLM76], together with the information that some of them form a *cluster*: for each tuple $K = (k_1, \dots, k_r)$ of integers $k_i \geqslant 1$, we obtain an ordered and an unordered *clustered* configuration space: $\tilde{C}_K(\mathbb{R}^d)$ and $C_K(\mathbb{R}^d)$. Both have been studied for their own sake [TP14; Pal21] and in relation to Hurwitz spaces [Tie16].

The aforementioned coupling constraint, which we would like to use later in order to establish an action on $\mathfrak{P}^m_{g,n}$, gives rise to a family of subspaces of $\tilde{C}_K(\mathbb{R}^d)$ and $C_K(\mathbb{R}^d)$: given a decomposition $d = p + q$, we define the *vertical* configuration spaces $\tilde{V}_K(\mathbb{R}^{p,q})$ and $V_K(\mathbb{R}^{p,q})$ as subspaces containing configurations of clusters of points in $\mathbb{R}^d$ where points from the same cluster share their first $p$ coordinates. The adjective 'vertical' is motivated by the low-dimensional example $(p, q) = (1, 1)$, where the coupling asks all points from the same cluster to lie on a common vertical line in the plane. These vertical configuration spaces go back to a construction by Bödigheimer [Böd90b, § 5] and have been studied in [Her14; Rös14; Lat17].

In joint work with Andrea Bianchi, I calculated the cohomology of the ordered vertical configuration spaces, see Section 1.2. For $K = (1, \dots, 1)$, this, of course, recovers the classical result of [Arn69]. Moreover, we proved a homological stability result for the unordered spaces $V^k_r(\mathbb{R}^{p,q}) = V_{k,\dots,k}(\mathbb{R}^{p,q})$, which generalises the results of [TP14; Lat17; Pal21]:

> Theorem 1.3.3 (Bianchi–K.). *If $(p, q) \neq (0, 1)$, then the stabilisation map $V^k_r(\mathbb{R}^{p,q}) \to V^k_{r+1}(\mathbb{R}^{p,q})$ induces an isomorphism in homology in degrees $\leqslant \frac{r}{2}$.*

Apart from these homological results, a few more things can be said about the topology of vertical configuration spaces: for the case $(p, q) = (1, 1)$, we establish a cellular decomposition inspired by [FN62b; Fuc70], which, by a geometric version of Poincaré–Lefschetz duality similar to [Mun84], yields the following complexity result:

> Theorem 1.4.13. *For each* $K = (k_1, \dots, k_r)$*, the spaces* $\tilde{V}_K(\mathbb{R}^{1,1})$ *and* $V_K(\mathbb{R}^{1,1})$ *are equivalent to* $(r-1)$*-dimensional cell complexes.*

In the last section, we study the relationship between $\pi_1(\tilde{V}_K(\mathbb{R}^{1,1}))$ and the classical pure braid groups [Art25], which we recall in Appendix A; here we use methods from combinatorial group theory. Their higher homotopy groups remain, except for a few special cases which have been studied in [Lat17], unknown.

**Chapter 2. Clustered configuration spaces as $E_d$-algebras** In a second step, we consider those spaces which are going to model free algebras over the desired operads, viz. *labelled* clustered configuration spaces $C(\mathbb{R}^d; \boldsymbol{X})$ and vertical configuration spaces $V(\mathbb{R}^{p,q}; \boldsymbol{X})$ for a sequence $\boldsymbol{X} = (X_k)_{k \geqslant 1}$ of based spaces, where each cluster of size $k \geqslant 1$ carries a label in $X_k$. These spaces admit the structure of an $E_{p+q}$-algebra, and I showed that the $p$-fold bar construction resolves the verticality constraint, similar to [Seg73]:

> Theorem 2.2.2. *There is an equivalence* $B^p V(\mathbb{R}^{p,q}; \boldsymbol{X}) \simeq C(\mathbb{R}^q; \Sigma^p \boldsymbol{X})$ *of* $E_q$*-algebras, where* $(\Sigma \boldsymbol{X})_k := \Sigma X_k$*.*

It remains to understand the iterated bar construction of clustered configuration spaces $C(\mathbb{R}^d; \boldsymbol{X})$ without any verticality constraint. While this question turns out to be hard in general, the case $d = 1$ is feasible: using an $E_1$-cellular decomposition in the sense of [GKR18; GKR19], we will see:

> Theorem 2.3.4. *There is a weak equivalence* $BC(\mathbb{R}; \boldsymbol{X}) \simeq \Sigma \bigvee_e \boldsymbol{X}^{\wedge K(e)}$*, where* $e$ *ranges over a countable set and determines* $K(e) = (k_1, \dots, k_r)$*, which defines* $\boldsymbol{X}^{\wedge K(e)} := X_{k_1} \wedge \cdots \wedge X_{k_r}$*. In particular, if each* $X_k$ *is path connected, then* $C(\mathbb{R}; \boldsymbol{X})$ *is equivalent to a free* $E_1$*-algebra.*

These results can be used to study the *stable* homology of the unordered and unlabelled vertical configuration spaces: to this aim, we show a stable splitting result similar to [Sna74; Böd87] and employ the Thom isomorphism for vertical configuration spaces with labels in spheres, as in [BCT89].

**Chapter 3. Coloured and dyed operads** The main purpose of this chapter is to set up the operadic language that we are going to use in the remainder of the thesis: we recall the notion of a *coloured* operad from [BM07; Yau16] and extend many classical constructions for monochromatic operads to the coloured case.

Secondly, we explicate the idea of 'clustering inputs' by combining two well-known concepts: the PROP associated to a monochromatic operad and the $\overline{\mathbb{N}}$-coloured operad represented by a PROP. The result will be called the *dyeing construction* $\overline{\mathbb{N}}(\mathcal{C})$ of a monochromatic operad $\mathcal{C}$. We study various properties of the dyeing construction and make some first general observations regarding the structure of their algebras.

Finally, we consider a suboperad $\overline{\mathbb{N}}^c(\mathcal{C}) \subseteq \overline{\mathbb{N}}(\mathcal{C})$, called the *connective part*, which, roughly speaking, contains only those operations which yield a connected result when all its arguments are connected.

**Chapter 4. Vertical operads and their algebras** This chapter contains the already announced central construction of the thesis: as suggested by [Böd13], we introduce, for each decomposition $d = p + q$, the *vertical* operad $\mathcal{V}_{p,q}$ as a suboperad of $\overline{\mathbb{N}}(\mathcal{C}_d)$. These operads are closely related to the extended Swiss cheese operad [Vor99; Wil17].

The connection to the vertical *configuration spaces* from the first chapters is immediate: the operation space $\mathcal{V}_{p,q}\binom{K}{n}$ is equivalent to $\tilde{V}_K(\mathbb{R}^{p,q} \times \underline{n})$, and the levels of the free $\mathcal{V}_{p,q}$-algebra over $\boldsymbol{X}$ are equivalent to $V(\mathbb{R}^{p,q} \times \underline{n}; \boldsymbol{X})$, as we carry out in Proposition 4.1.3. Here $\mathbb{R}^{p,q} \times \underline{n} = \coprod^n \mathbb{R}^{p,q}$, i.e. we consider configurations of vertical clusters that are distributed on $n$ layers.

It is easy to see that we have a pairing of operads $(\mathcal{C}_p, \overline{\mathbb{N}}(\mathcal{C}_q)) \to \mathcal{V}_{p,q}$ by forming products of boxes, and, similar to [Dun86], we show by methods of [Bri00] the following additivity result, where '$\odot$' denotes the Boardman–Vogt tensor product:

Theorem 4.2.2. *The induced map $\mathscr{C}_p \odot \overline{\mathbb{N}}(\mathscr{C}_q) \to \mathscr{V}_{p,q}$ is an equivalence of $\mathfrak{S}$-cofibrant operads.*

This shows that the homotopy theory of $\mathscr{V}_{p,q}$-algebras is equivalent to the homotopy theory of $\overline{\mathbb{N}}(\mathscr{C}_q)$-algebras, which come equipped with levelwise and interchangeable $\mathscr{C}_p$-actions.

In a second step, we restrict ourselves to the case $(p,q) = (1,1)$, the one for which we want to consider the action on Bödigheimer's model for moduli spaces, and study the system of homology operations which a $\mathscr{V}_{1,1}$- or a $\mathscr{V}^c_{1,1}$-action on a sequence of spaces imposes on their respective homology groups. Here is a rough summary of these investigations:

Section 4.4. *If $\boldsymbol{X} = (X_n)_{n\geqslant 1}$ is a $\mathscr{V}^c_{1,1}$-algebra, then its homology groups $A_{n,h} := H_h(X_n)$ can be endowed with:*

- *homomorphisms $f\colon A_{k,h} \to A_{n,h}$ for each surjection $f\colon \underline{k} \to \underline{n}$ that comes with total orders of its fibres $f^{-1}(\ell)$,*
- *a unit $1 \in A_{1,0}$,*
- *a vertical Pontrjagin product $- \cdot -\colon A_{n,h} \otimes A_{n',h'} \to A_{n+n'-1,h+h'}$,*
- *a vertical Browder bracket $[-,-]\colon A_{n,h} \otimes A_{n',h'} \to A_{n+n'-1,h+h'+1}$.*

*If we work over $\mathbb{F}_2$ or if $h$ is even, then we additionally have:*

- *a vertical Dyer–Lashof square $Q\colon A_{n,h} \to A_{n,1+2h}$.*

*These operations satisfy various relations (e.g. a coloured Jacobi identity).*

This result is very similar to Cohen's work [CLM76, § III] which treats the case of the little $d$-cubes, but its proof is more geometrical: it uses the basic concepts of discrete Morse theory [For98], which we recall in Appendix B, as well as intersection theory.

**Chapter 5. Homology operations on moduli spaces of surfaces** At the beginning of this chapter, we recall explicit models for moduli spaces $\mathfrak{M}^m_{g,n}$. We then observe that the connective part of the dyed operad $\overline{\mathbb{N}}^c(\mathscr{C}_2)$ acts on the collection $\left(\coprod_{g,m} \mathfrak{M}^m_{g,n}\right)_{n\geqslant 1}$ of moduli spaces with multiple boundary curves, the action given by sewing of surfaces as in [Mil86; Seg88; Böd90b].

The vertical suboperad $\mathcal{V}_{1,1} \subseteq \overline{\mathbb{N}}(\mathcal{C}_2)$ is tailor-made to act on Bödigheimer's simplicial model $\mathfrak{P}^m_{g,n}$ of parallel slit domains [Böd90a] for $\mathfrak{M}^m_{g,n}$:

> Construction 5.2.14. *There is a model for $\mathcal{V}^c_{1,1}$ acting on the collection $\big(\coprod_{g,m} \mathfrak{P}^m_{g,n}\big)_{n \geqslant 1}$ of spaces of parallel slit domains, and this action is a restriction of the $\overline{\mathbb{N}}^c(\mathcal{C}_2)$-action on $\big(\coprod_{g,m} \mathfrak{M}^m_{g,n}\big)_{n \geqslant 1}$.*

This restriction, and its resulting homology operations, are useful for unstable calculations as in [Ehr98; Abh05; God07; ABE08; Vis11; Meh11; Wan11; BH14; Boe18], to which Section 5.3 contributes. Their work comprises, among many other results, six toric classes $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$, and $\mathbf{f}$ in $H_\bullet(\mathfrak{M}^m_{g,1})$, as well as two ad-hoc constructions: a so-called *T-operation* and an *E-operation*.

First of all, we introduce classes $\mathbf{c}_2$, $\mathbf{d}_2$, and $\mathbf{e}_2$ in the homology of moduli spaces of surfaces with *two* boundary curves, which differ from the old ones by gluing a pair of pants. Using these classes, we can describe the 3-torus $\mathbf{f}$ as a composition $s^1[\mathbf{a}, \mathbf{e}_2]$, see Remark 5.3.6, using the bracket which comes from the $\mathcal{V}^c_{1,1}$-action, as well as the codegeneracy $s^1 \colon \underline{2} \to \underline{1}$.

Similarly, we lift the $T$-operation to a map $\hat{T} \colon H_h(\mathfrak{M}^m_{g,n}) \to H_{h+1}(\mathfrak{M}^{m-1}_{g,n+1})$, and in Construction 5.3.10, we introduce a new generator $\mathbf{g}_2 \in H_2(\mathfrak{M}_{1,2})$ which is supported on a 2-torus. Using these constructions, we can fully describe the generators of $H_\bullet(\mathfrak{M}_{1,2})$, here, for example, over $\mathbb{F}_2$:

> Proposition 5.3.12. *The $\mathbb{F}_2$-homology of $\mathfrak{M}_{1,2}$ is generated as follows:*

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\mathfrak{M}_{1,2}$ | $\mathbf{c}_2\mathbf{c}$ | $\mathbf{c}_2\mathbf{d}, \mathbf{d}_2\mathbf{c}$ | $\mathbf{d}_2\mathbf{d}, \mathbf{g}_2$ | $\hat{T}\mathbf{e}, Q\mathbf{d}_2$ | $\hat{T}E\mathbf{b}$ |

**Chapter 6. Parametrised moduli spaces of surfaces as infinite loop spaces** This last chapter contains a side project which, even though it starts with a different question, relies on the coloured operadic approach to moduli spaces of surfaces with multiple boundary curves that has been established in the previous chapters of this thesis.

Recall that the Madsen–Weiss theorem [MW07] identifies the group completion $\Omega B \coprod_g \mathfrak{M}_{g,1}$ of the $E_2$-algebra $\coprod_g \mathfrak{M}_{g,1}$ with the infinite loop space associated with the affine oriented Thom spectrum $\mathbf{MTSO}(2)$.

In joint work with Andrea Bianchi and Jens Reinhold, we considered a generalisation of the problem: for a space $A$, the moduli space $\coprod_g \mathfrak{M}_{g,1}[A]$ of orientable surface bundles over $A$ is itself an $E_2$-algebra, and we aim to understand its group completion $\Omega B \coprod_g \mathfrak{M}_{g,1}[A]$. This question can be seen as a first step towards the homotopy type of *parametrised cobordism categories* studied in [RS17]. Our main result is the following:

> Theorem 6.4.1 (Bianchi–K.–Reinhold). *There is a space $C[A]$ with*
>
> $$\Omega B \coprod_g \mathfrak{M}_{g,1}[A] \simeq \Omega^\infty \mathbf{MTSO}(2) \times \Omega^\infty \Sigma^\infty_+ C[A].$$

More precisely, we prove that the space $C[A]$ is a union of classifying spaces of centralisers of certain *∂-irreducible* mapping classes in mapping class groups of surfaces with multiple boundary curves.

To this aim, we showed that $\coprod_g \mathfrak{M}_{g,1}[A]$ is the first component of a relatively free algebra $F^{\mathscr{M}}_{\mathbb{T}}(\mathfrak{C}[A])$ over a coloured version of the surface operad $\mathscr{M}$ [Til00], with base a sequence $\mathfrak{C}[A] := (\mathfrak{C}_k[A])_{k\geqslant 1}$ of spaces which carries an action by the sequence $\mathbb{T} := (\mathbb{T}^k \rtimes \mathfrak{S}_k)_{k\geqslant 1}$ of twisted tori.

The second ingredient generalises one of the main statements from [Bas+17] on operads with homological stability to the coloured and relative case:

> Theorem 6.3.11. *Let $\mathscr{O}$ be an $N$-coloured operad with homological stability and let $\mathbf{I}$ be a topological category with a map $\mathscr{B} \odot \mathbf{I} \to \mathscr{O}$. Furthermore, let $\boldsymbol{X} := X_\bullet \colon \mathbf{I} \to \mathbf{Top}_*$ be an enriched functor, i.e. a $(\mathscr{B} \odot \mathbf{I})$-algebra. Then, under the mild point-set assumptions of 6.3.10, we have, for each colour $n \in N$, an equivalence of loop spaces*
>
> $$\Omega B \tilde{F}^{\mathscr{O}}_{\mathscr{B}\odot\mathbf{I}}(\boldsymbol{X})_n \simeq \Omega B\, \mathscr{O}\binom{}{n} \times \Omega^\infty \Sigma^\infty \mathrm{hocolim}_{\mathbf{I}}(X_\bullet).$$

Here $\mathscr{B} \odot \mathbf{I}$ is precisely the operad whose algebras are enriched functors $\mathbf{I} \to \mathbf{Top}_*$ to *based* spaces, and $\tilde{F}^{\mathscr{O}}_{\mathscr{B}\odot\mathbf{I}}$ is the derived free algebra.

For $\mathscr{O} = \mathscr{M}$ and $n = 1$, the first factor can be identified with $\Omega^\infty \mathbf{MTSO}(2)$, and for $\mathbf{I} = \mathbb{T}$ and the specific base $\boldsymbol{X} = \mathfrak{C}[A]_+$, the derived free algebra is equivalent to the strict one. This finally tells us which infinite loop space the group completion of $\coprod_g \mathfrak{M}_{g,1}[A]$ is. We end by discussing similar results for further well-known $E_d$-algebras.

Parts of this thesis have been published in the following articles, as I will additionally indicate at the beginning of the corresponding sections:

[BK21] A. Bianchi and F. Kranhold. 'Vertical configuration spaces and their homology'. To appear 2022 in *Q. J. Math.* 2021. DOI: `10.1093/qmath/haab061`. arXiv: `2103.12137 [math.AT]`

[Kra21] F. Kranhold. 'Configuration spaces of clusters as $E_d$-algebras'. 2021. arXiv: `2104.02729 [math.AT]`

[BKR21] A. Bianchi, F. Kranhold, and J. Reinhold. 'Parametrised moduli spaces of surfaces as infinite loop spaces'. To appear in *Forum Math. Sigma*. 2021. arXiv: `2105.05772 [math.AT]`

In these sections, I will leave out those technicalities which have been spelled out mainly by my coauthors.

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## Chapter 1

# Vertical configuration spaces

> Ἀρχὰς εἶναι τῶν ὅλων ἀτόμους καὶ κενὸν,
> τὰ δ′ ἄλλα πάντα νενομίσθαι.[1]
>
> DEMOCRITUS

The purpose of this chapter is to introduce families of *clustered* and *vertical configuration spaces* and to study their algebraic invariants, such as homology, homological stability, and homotopy groups.

Our interest in clustered configuration spaces comes from an operadic perspective on moduli spaces of surfaces with multiple boundary curves; however, they have also been considered for their own sake [TP14] and in relation to Hurwitz spaces [Tie16], and they are special cases of spaces of (disconnected) submanifolds, which have been studied in [Pal21].

The notion of vertical configuration spaces is based on work of Bödigheimer, who used small examples of them in order to describe symplectic operations on moduli spaces of surfaces [Böd90b], and who suggested a unified operadic approach [Böd13], which we carry out in the later chapters.

Vertical configuration spaces are closely related to *fibrewise* configuration spaces, which have been studied in [Cno19] in order to formulate an approximation theorem for configurations with twisted labels and labels in partial abelian monoids. From this viewpoint, vertical configuration spaces can be regarded as *multi*-fibrewise configuration spaces. Additionally, they describe certain subcomplexes of the Fox–Neuwirth–Fuchs complex [FN62b; Fuc70] for classical configuration spaces.

Several bachelors' and masters' theses studied vertical configuration spaces [Her14; Rös14; Lat17], in particular their higher homotopy groups and their homological stability.

---

[1] *The beginning of everything is atoms and void, everything else is perception.*

## 1.1. Preliminaries

Let us start with a short reminder on classical configuration spaces: if $E$ is a space (usually, a smooth manifold) and $r \geqslant 0$ is a non-negative integer, then we define the *ordered configuration space of r points in E* as

$$\tilde{C}_r(E) := \{(z_1, \dots, z_r) \in E^r; z_i \neq z_j \text{ for } i \neq j\},$$

topologised as a subspace of $E^r$. The $r^{\text{th}}$ symmetric group $\mathfrak{S}_r$ acts freely on $\tilde{C}_r(E)$ by permuting coordinates, and we call the quotient $C_r(E) := \tilde{C}_r(E)/\mathfrak{S}_r$ the *unordered configuration space of r points in E*. Intuitively, the space $C_r(E)$ parametrises subsets of $E$ of cardinality $r$.

We would like to declare that some of the points within a configuration 'belong together'. More precisely, given a tuple $K = (k_1, \dots, k_r)$ of integers $k_i \geqslant 1$, we want to consider configurations of $|K| := k_1 + \cdots + k_r$ points, clustered in $r$ blocks, of sizes $k_1, \dots, k_r$. To this end, we start with a simple reindexing: we let $\tilde{C}_K(E) := \tilde{C}_{|K|}(E)$, but we denote its elements by tuples $\boldsymbol{z} = (\vec{z}_1, \dots, \vec{z}_r)$ where $\vec{z}_i = (z_{i,1}, \dots, z_{i,k_i})$, and we call $\vec{z}_i$ a *cluster of size $k_i$*.

The idea of a *vertical configuration space* is to consider subspaces of $\tilde{C}_K(E)$ where points from the same cluster behave similarly in a certain sense. This is made precise by the following definition.

**Definition 1.1.1.** Fix a decomposition $d = p + q$ and assume that $E$ is the total space of a manifold bundle $\mathrm{pr}\colon E \to B$ with $\dim(B) = p$, i.e. the fibre $F$ has dimension $q$. Then a cluster $\vec{z} = (z_1, \dots, z_k)$ is called *vertical* if all of its $k$ points have the same projection to $B$, that is $\mathrm{pr}(z_1) = \cdots = \mathrm{pr}(z_k)$.

We introduce the *(ordered) vertical configuration space* $\tilde{V}_K(E) \subseteq \tilde{C}_K(E)$ as the subspace of all configurations $(\vec{z}_1, \dots, \vec{z}_r)$ where each cluster $\vec{z}_i$ is vertical.

Note that we adopt the usual abuse of notation and denote the entire bundle datum by the total space $E$.

**Example 1.1.2.** For $p \geqslant 0$ and $q \geqslant 1$, the map $\mathrm{pr}_{p,q}\colon \mathbb{R}^{p,q} := \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ leads to vertical configuration spaces which have been considered in [Her14; Rös14; Lat17]. For $p = 1$ and $q = 1$, we are requiring all points of a cluster to lie on the same vertical line of $\mathbb{R}^2$, whence the terminology.

Slightly more generally, we will consider the trivial bundle $\mathbb{R}^{p,q} \times \underline{n} \to \mathbb{R}^p$, whose total space is given by $n$ copies of $\mathbb{R}^{p,q}$.

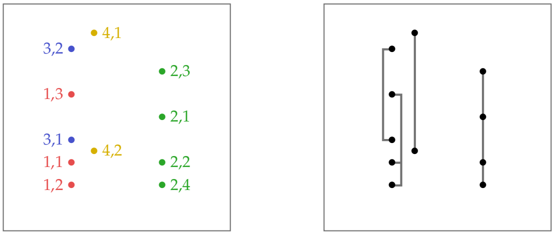

**Figure 1.1.** A configuration in the ordered configuration space $\tilde{V}_{3,4,2,2}(\mathbb{R}^{1,1})$ and its image under the projection to the unordered vertical configuration space $V_{3,4,2,2}(\mathbb{R}^{1,1})$.

There is an unordered counterpart of the above construction: we denote by $\mathfrak{S}_K \subseteq \mathfrak{S}_{|K|}$ the subgroup containing all permutations which preserve the *unordered* partition of $\{1, \dots, |K|\}$ into $r$ blocks of sizes $k_i$. If we denote by $r(k) \geqslant 0$ the number of occurrences of $k$ in $K$, then $\mathfrak{S}_K$ is isomorphic to

$$\mathfrak{S}_K \cong \prod_{k \geqslant 1} \mathfrak{S}_k \wr \mathfrak{S}_{r(k)} = \prod_{k \geqslant 1} (\mathfrak{S}_k)^{r(k)} \rtimes \mathfrak{S}_{r(k)}.$$

**Definition 1.1.3.** The group $\mathfrak{S}_K$ acts freely on $\tilde{V}_K(E)$ by permuting the labels $1 \leqslant i \leqslant r$ of clusters of the same size and permuting the labels $1 \leqslant j \leqslant k_i$ of the points of each cluster. We denote the quotient space by

$$V_K(E) := \tilde{V}_K(E) / \mathfrak{S}_K.$$

Roughly speaking, and using the above notation, a point in $V_K(E)$ consists of a collection of $r$ clusters, of which $r(k)$ have size $k$; clusters of the same size are unordered, and points inside a cluster are also unordered.

**Notation 1.1.4.** We denote elements of $\tilde{V}_K(E)$ and $V_K(E)$ as follows:

- An element of $\tilde{V}_K(E)$ is an ordered collection $\boldsymbol{z} = (\vec{z}_1, \dots, \vec{z}_r)$ of clusters $\vec{z}_i = (z_{i,1}, \dots, z_{i,k_i})$ with $z_{i,j} \in E$. We also write $\boldsymbol{z} = (z_{1,1}, \dots, z_{r,k_r})$.
- For the unordered version, we make use of the suggestive sum notation: a generic element in $V_K(E)$ is an unordered collection $[\boldsymbol{z}] := \sum_{i=1}^{r} [\vec{z}_i]$ of unordered (vertical) clusters $[\vec{z}_i] := [z_{i,1}, \dots, z_{i,k_i}] = \{z_{i,1}, \dots, z_{i,k_i}\}$.

Note that we use a calligraphic letter $\mathscr{j}$ for the index corresponding to points *inside* a cluster. We stick to this distinction throughout the entire thesis: this will be particularly useful in order to differentiate between *inputs* and *layers*.

Moreover, we will occasionally restrict to situations where all clusters have the same size, i.e. $K = (k, \dots, k)$ for some $k \geqslant 1$. If we let $r$ be the length of this tuple, we may simplify our notation and write $V_r^k(E) := V_{k,\dots,k}(E)$.

Finally, if the base space $B$ is just a point, then the verticality condition is empty. While in this case the *ordered* configuration space $\tilde{V}_K(E) \cong \tilde{C}_{|K|}(E)$ is well-known, its *unordered* version $V_K(E)$ is a covering of $C_{|K|}(E)$, since we keep the information which of the points belong together: these spaces have been studied in [TP14; Tie16; Pal21]. In this case, we also write[2] $C_K(E) := V_K(E)$, and in the case where all clusters have size $k$, we write $C_r^k(E)$.

**Remark 1.1.5.** The space $\tilde{V}_K(E)$ is locally an open subspace of $B^r \times F^{|K|}$ and hence an orientable smooth manifold of dimension $p \cdot r + q \cdot |K|$. The action of $\mathfrak{S}_K$ is free, so $V_K(E)$ is again a manifold of the same dimension.

For our main example $E = \mathbb{R}^{p,q}$, the path components of $V_K(\mathbb{R}^{p,q})$ are readily classified and the question of orientability can easily be answered:

**Remark 1.1.6** (Path components)**.** If we fix numbers $p \geqslant 0$ and $q \geqslant 1$, then the following holds for the ordered vertical configuration spaces $\tilde{V}_K(\mathbb{R}^{p,q})$:

- For $q \geqslant 2$, the space $\tilde{V}_K(\mathbb{R}^{p,q})$ is path connected.
- For $q = 1$ and $p \geqslant 1$, the space $\tilde{V}_K(\mathbb{R}^{p,1})$ has one component $\tilde{V}_K(\mathbb{R}^{p,1})_{\boldsymbol{\sigma}}$ for each tuple $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_r) \in \prod_i \mathfrak{S}_{k_i}$ of permutations. This component contains all configurations $(z_{1,1}, \dots, z_{r,k_r})$ with

  $$z^{p+1}_{i,\sigma_i(1)} < \cdots < z^{p+1}_{i,\sigma_i(k_i)}$$

  for all $1 \leqslant i \leqslant r$, where $z = (z^1, \dots, z^{p+1}) \in \mathbb{R}^{p+1}$.
- For $q = 1$ and $p = 0$, we recall that $\tilde{V}_K(\mathbb{R}^{0,1}) = \tilde{C}_{|K|}(\mathbb{R})$, whence each permutation $\sigma \in \mathfrak{S}_{|K|}$ corresponds to a connected component which contains all configurations $(z_{1,1}, \dots, z_{r,k_r}) = (z_1, \dots, z_{|K|})$ with $z_i < z_{\sigma(i)}$.

---

[2] Of course, as *clustered* configuration spaces are special cases of *vertical* configuration spaces, it is enough to state the upcoming results for $V$; however, it feels more natural to call clustered configuration spaces, where no verticality constraint is involved, $C$ instead of $V$.

We have inclusions $\prod_i \mathfrak{S}_{k_i} \subseteq \mathfrak{S}_K \subseteq \mathfrak{S}_{|K|}$ and the group $\mathfrak{S}_K = \prod_k \mathfrak{S}_k \wr \mathfrak{S}_{r(k)}$ acts on $\pi_0\big(\tilde{V}_K(\mathbb{R}^{p,q})\big)$ with quotient equal to $\pi_0(V_K(\mathbb{R}^{p,q}))$. Since the action is transitive in the first two cases, the space $V_K(\mathbb{R}^{p,q})$ is path connected for $(p,q) \neq (0,1)$, whereas for $(p,q) = (0,1)$ we have $\pi_0(V_K(\mathbb{R}^{0,1})) \cong \mathfrak{S}_{|K|}/\mathfrak{S}_K$.

The latter set can also be identified with the set of unordered partitions of $\{1,\ldots,|K|\}$ into subsets of sizes $k_1,\ldots,k_r$.

**Remark 1.1.7** (Orientability). The manifold $\tilde{V}_K(\mathbb{R}^{p,q})$ is clearly orientable as an open subset of $(\mathbb{R}^p)^r \times (\mathbb{R}^q)^{|K|}$. However, $V_K(\mathbb{R}^{p,q})$ is non-orientable if and only if at least one of the following holds:

- $q \geqslant 3$ is odd and there is at least one cluster of some size $k \geqslant 2$: then a path in $V_K(\mathbb{R}^{p,q})$ interchanging two points of this cluster, while fixing all other points, reverses the local orientation.

- $p + q \geqslant 2$ and there is some $k \geqslant 1$ such that $p + q \cdot k$ is odd and $r(k) \geqslant 2$: then interchanging two clusters of size $k$ while preserving their internal ordering and fixing all other points reverses the local orientation.

In the upcoming parts of this chapter, we study the spaces $\tilde{V}$ and $V$ from different topological perspectives. Here is a short overview:

§ 1.2 THE COHOMOLOGY IN THE ORDERED CASE
We give a a complete description of $H^\bullet(\tilde{V}_K(\mathbb{R}^{p,q}))$ for all $p$, $q$, and $K$. This section also appeared as part of [BK21].

§ 1.3 HOMOLOGICAL STABILITY FOR THE UNORDERED CASE
Using methods from [Pal18], we show that for a fixed $k$, the spaces $(V_r^k(\mathbb{R}^{p,q}))_{r \geqslant 0}$ satisfy homological stability if $(p,q) \neq (0,1)$. This generalises results from [TP14; Lat17; Pal21] and appeared as part of [BK21].

§ 1.4 VERTICAL CONFIGURATION SPACES AS RELATIVE CELL COMPLEXES
We give a cellular decomposition of $\tilde{V}_K(\mathbb{R}^{1,1})$ and $V_K(\mathbb{R}^{1,1})$ in the spirit of [FN62b] and draw some first conclusions from this description. For example, we show that both $\tilde{V}_K(\mathbb{R}^{1,1})$ and $V_K(\mathbb{R}^{1,1})$ are homotopy equivalent to finite cellular complexes of dimension $r - 1$ if $K = (k_1,\ldots,k_r)$.

§ 1.5 HOMOTOPY GROUPS
This section mainly deals with the fundamental groups of $\tilde{V}_K(\mathbb{R}^{p,q})$ and their relation to Artin's braid groups. We then close the section by mentioning results from [Her14; Rös14; Lat17] on asphericity.

## 1.2. The cohomology in the ordered case

In this section we calculate the integral cohomology of the spaces $\tilde{V}_K(\mathbb{R}^{p,q})$ for all dimensions $p \geqslant 0$ and $q \geqslant 1$. In the case $p = 0$ we recover the calculations of [Arn69; CLM76] for the classical ordered configuration spaces $\tilde{C}_{|K|}(\mathbb{R}^d)$. This section appeared as a part of the article [BK21].

For simplicity, let us exclude the case $(p, q) = (0, 1)$, where all components are contractible anyway.

### 1.2.1. *Ray partitions*

We fix a partition $K = (k_1, \ldots, k_r)$ for the entire section. Before we can state our main result, we need to introduce a few more combinatorial notions.

**Definition 1.2.1.** The *tableau*[3] associated with the partition $K$ is the set

$$\mathbb{Y}_K := \{(i, \jmath);\ 1 \leqslant i \leqslant r \text{ and } 1 \leqslant \jmath \leqslant k_i\}.$$

We order $\mathbb{Y}_K$ lexicographically, which means that we write $(i, \jmath) < (i', \jmath')$ if either $i < i'$, or $i = i'$ and $\jmath < \jmath'$ holds.

**Notation 1.2.2.** For each partition $\mathcal{Q}$ of $\mathbb{Y}_K$ into non-empty subsets $\mathcal{Q}_1, \ldots, \mathcal{Q}_l$ we consider two positive integers: firstly, the number $l(\mathcal{Q}) := l$ is called the *length* of the partition, and in general we have $1 \leqslant l(\mathcal{Q}) \leqslant |K|$.

Secondly, consider on $\{1, \ldots, l\}$ the equivalence relation spanned by $\beta \sim \beta'$ if there are $1 \leqslant i \leqslant r$ and $1 \leqslant \jmath, \jmath' \leqslant k_i$ with $(i, \jmath) \in \mathcal{Q}_\beta$ and $(i, \jmath') \in \mathcal{Q}_{\beta'}$ (i.e. the $i^{\text{th}}$ cluster intersects both $\mathcal{Q}_\beta$ and $\mathcal{Q}_{\beta'}$). The number of equivalence classes $s(\mathcal{Q})$ is called the *agility* of the partition, see Figure 1.2.

---

[3] The letter $\mathbb{Y}$ should remind us a of *Young tableau*, although our notion is slightly more general, as the sizes of the columns do not have to be non-increasing.

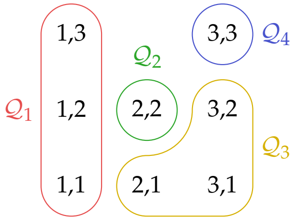

**Figure 1.2.** A partition $\mathcal{Q}$ of $\mathbb{Y}_{3,2,3}$ into four components, i.e. $l(\mathcal{Q}) = 4$. Note that $2 \sim 3 \sim 4$, whence the agility $s(\mathcal{Q})$ is 2.

**Definition 1.2.3.** A *ray partition* $\mathcal{Q}$ *of type* $K$ is a partition $\mathcal{Q}_1, \dots, \mathcal{Q}_l$ of $\mathbb{Y}_K$, with a total order $\prec_\beta$ on each $\mathcal{Q}_\beta$ (called *ray*), such that the following hold:

R1. the components are labelled from 1 to $l$ according to their minimum with respect to the global order $<$;

R2. for each $\beta$, the minima of $\mathcal{Q}_\beta$ with respect to $<$ and $\prec_\beta$ coincide.

If $q = 1$, then $p \geqslant 1$, since we excluded $(p, q) = (0, 1)$, and $\tilde{V}_K(\mathbb{R}^{p,1})$ is disconnected, with components indexed by tuples $\boldsymbol{\sigma} \in \prod_{i=1}^{r} \mathfrak{S}_{k_i}$. We would like to calculate the homology of a single path component. In order to do so, we assign to each ray partition $\mathcal{Q}$ of type $K$ such a tuple $\boldsymbol{\sigma}$ as follows:

**Definition 1.2.4.** Given a ray partition $\mathcal{Q}$, the *stacked total order* $\prec$ on

$$\mathbb{Y}_K = (\mathcal{Q}_l, \prec_l) \sqcup \cdots \sqcup (\mathcal{Q}_1, \prec_1)$$

is determined by the property that it restricts on each $\mathcal{Q}_\beta$ to $\prec_\beta$, and that all elements from $\mathcal{Q}_{\beta+1}$ are $\prec$-smaller than all elements from $\mathcal{Q}_\beta$.

For each $i$, there is a unique $\sigma_i \in \mathfrak{S}_{k_i}$ such that $(i, \sigma_i(\jmath)) \prec (i, \sigma_i(\jmath + 1))$ for all $1 \leqslant \jmath < k_i$, and we define $\boldsymbol{\sigma}(\mathcal{Q}) := (\sigma_1, \dots, \sigma_r) \in \prod_i \mathfrak{S}_{k_i}$.

**Theorem 1.2.5.** *Let* $K = (k_1, \dots, k_r)$ *with* $k_i \geqslant 1$.

1. *The integral cohomology* $H^\bullet(\tilde{V}_K(\mathbb{R}^{p,q}))$ *is freely generated by classes* $u_\mathcal{Q}$ *for each ray partition* $\mathcal{Q}$, *and the cohomological degree of* $u_\mathcal{Q}$ *is*

$$|u_\mathcal{Q}| = p \cdot (r - s(\mathcal{Q})) + (q - 1) \cdot (|K| - l(\mathcal{Q})).$$

2. *For* $q = 1$, *the class* $u_\mathcal{Q}$ *is supported on the component* $\tilde{V}_K(\mathbb{R}^{p,1})_{\boldsymbol{\sigma}(\mathcal{Q})}$.

### 1.2.2. *The weight filtration and the proof of Theorem 1.2.5*

Throughout this subsection, we fix $K$, $p$, and $q$ as before, and we treat the cases $q \geqslant 2$ and $q = 1$ simultaneously, putting in parentheses the differences needed in the case $q = 1$. For $q \geqslant 2$ we abbreviate $\tilde{V} := \tilde{V}_K(\mathbb{R}^{p,q})$, while for $q = 1$ we fix a path component $\boldsymbol{\sigma} \in \prod_i \mathfrak{S}_{k_i}$ and abbreviate $\tilde{V} := \tilde{V}_K(\mathbb{R}^{p,q})_{\boldsymbol{\sigma}}$.

**Notation 1.2.6.** We denote by $\mathrm{pr}\colon \mathbb{R}^d \cong \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ the projection to the first $p$ coordinates. However, in some situations, we make use of the decomposition $\mathbb{R}^d \cong \mathbb{R}^{p+q-1} \times \mathbb{R}$ and write $(\zeta, t)$ for a generic point in $\mathbb{R}^d$.

Hence, we have two further projections, namely $\mathrm{pr}_\zeta\colon \mathbb{R}^d \to \mathbb{R}^{p+q-1}$ and $\mathrm{pr}_t\colon \mathbb{R}^d \to \mathbb{R}$. Clearly, if $q = 1$, then $\mathrm{pr}_\zeta$ and pr coincide.

**Definition 1.2.7.** Let $\boldsymbol{z} = (z_{1,1}, \ldots, z_{r,k_r}) \in \tilde{V}_K(\mathbb{R}^{p,q})$. Then we say that a ray partition $\mathcal{Q}$ is *witnessed by* $\boldsymbol{z}$ if the following conditions hold:

- w1. all $z_{i,\jmath}$ with $(i,\jmath) \in \mathcal{Q}_\beta$ project along $\mathrm{pr}_\zeta$ to the same point in $\mathbb{R}^{d-1}$;
- w2. if $(i,\jmath) \prec_\beta (i',\jmath')$ holds in $\mathcal{Q}_\beta$, then $\mathrm{pr}_t(z_{i,\jmath}) < \mathrm{pr}_t(z_{i',\jmath'})$ holds in $\mathbb{R}$.

Condition w1 says that the points $z_{i,\jmath}$ with $(i,\jmath) \in \mathcal{Q}_\beta$ lie on a line in $\mathbb{R}^d$ parallel to the $t$-axis; condition w2 ensures that the same points are assembled on this line according to the order $\prec_\beta$ of their indices. In particular, the points $z_{i,\jmath}$ with $(i,\jmath) \in \mathcal{Q}_\beta$ lie on a *ray*, namely the half-line starting at $z_{\min(\mathcal{Q}_\beta, \prec_\beta)}$ and running in the positive $t$-direction; see Figure 1.3 for an example.

**Remark 1.2.8** (Poincaré–Lefschetz duality)**.** For a space $X$, we denote by $X^\infty$ its one-point compactification and denote the point at infinity by $\infty$. Since $\tilde{V}_K(\mathbb{R}^{p,q})$ is an open and orientable manifold of dimension $p \cdot r + q \cdot |K|$, we can apply Poincaré–Lefschetz duality and obtain the isomorphism

$$H^\bullet\big(\tilde{V}_K(\mathbb{R}^{p,q})\big) \cong H_{p\cdot r+q\cdot|K|-\bullet}\big(\tilde{V}_K(\mathbb{R}^{p,q})^\infty, \infty\big).$$

**Notation 1.2.9.** For a positive integer $\Lambda \geqslant 0$, we denote by $\mathbb{P}(\Lambda)$ the set of all ordered partitions of $\Lambda$, i.e. sequences $\lambda = (\lambda_1, \ldots, \lambda_l)$ of integers $\lambda_i \geqslant 1$, for some $1 \leqslant l \leqslant \Lambda$, with $\lambda_1 + \cdots + \lambda_l = \Lambda$. The number $l$ is called the *length* of the sequence. We have an injection $\mathbb{P}(\Lambda) \hookrightarrow \{0, \ldots, \Lambda\}^\Lambda$ by adding a suitable number of zeros at the end, and consider on $\mathbb{P}(\Lambda)$ the inherited lexicographic order. We denote by $\mathbb{P}(K)$ the set $\mathbb{P}(|K|)$, and by $N$ its cardinality.

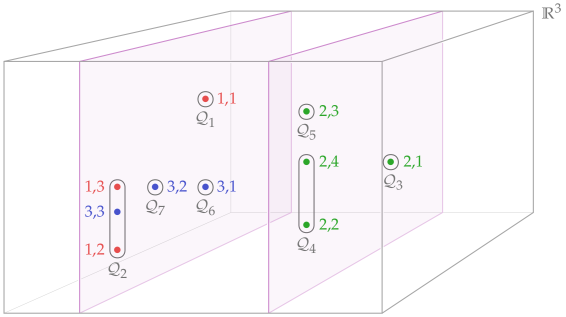

**Figure 1.3.** A configuration inside the space $\tilde{V}_{3,4,3}(\mathbb{R}^{1,2})$ which witnesses the ray partition $(\mathcal{Q}_1, \dots, \mathcal{Q}_7)$, where e.g. $\mathcal{Q}_2 = \{(1,2) \prec (3,3) \prec (1,3)\}$. The components $\mathcal{Q}_\beta$ are numbered according to their smallest label (R1), and the point carrying the minimal label lies at the bottom of each ray (R2). The verticality condition demands that all points belonging to the same cluster have to lie in the same purple plane.

**Definition 1.2.10.** The *weight* of a ray partition $\mathcal{Q}$ is defined as

$$\omega(\mathcal{Q}) := \big(\#\mathcal{Q}_1, \dots, \#\mathcal{Q}_l\big) \in \mathbb{P}(K).$$

**Lemma 1.2.11.** *Let $\boldsymbol{z} \in \tilde{V}$. There is a unique ray partition, called $\mathcal{Q}^{\boldsymbol{z}}$, which is witnessed by $\boldsymbol{z}$ and has maximal weight among all ray partitions witnessed by $\boldsymbol{z}$. (If $q = 1$, we have moreover that $\boldsymbol{\sigma}(\mathcal{Q}^{\boldsymbol{z}})$ agrees with the $\boldsymbol{\sigma}$ to start with.)*

We prove Lemma 1.2.11 in [BK21, Lem. 3.10] by constructing $\mathcal{Q}^{\boldsymbol{z}}$ inductively: we let $\mathcal{Q}^{\boldsymbol{z}}_\beta$ contain the minimum $(i, \jmath)$ of $\mathbb{Y}_K \setminus \bigcup_{\beta'<\beta} \mathcal{Q}_{\beta'}$ and all $(i', \jmath')$ such that $z_{i',\jmath'}$ lies on the *ray* starting at $z_{i,\jmath}$ and running in the *positive* $t$-direction, with $\prec^{\boldsymbol{z}}_\beta$ determined by the condition W2, see Figure 1.4.

**Definition 1.2.12.** Given any ray partition $\mathcal{Q}$, we denote by $W_\mathcal{Q} \subset \tilde{V}$ the subspace containing all points $\boldsymbol{z}$ with $\mathcal{Q}^{\boldsymbol{z}} = \mathcal{Q}$.

We define a filtration $F_\bullet$ on $\tilde{V}^\infty$ indexed by the linearly ordered set $\mathbb{P}(K)$: for each $\lambda \in \mathbb{P}(K)$, we define the filtration level $F_\lambda := F_\lambda \tilde{V}^\infty$ as the subspace containing $\infty$ and all $\boldsymbol{z} \in \tilde{V}$ with $\omega(\mathcal{Q}^{\boldsymbol{z}}) \geqslant \lambda$. Note that for $\lambda < \lambda'$ in $\mathbb{P}(K)$ we have an inclusion $F_{\lambda'} \subset F_\lambda$. It is straightforward [BK21, Lem. 3.12] to check that the inclusion $F_\lambda \subseteq \tilde{V}^\infty$ is a closed cofibration.

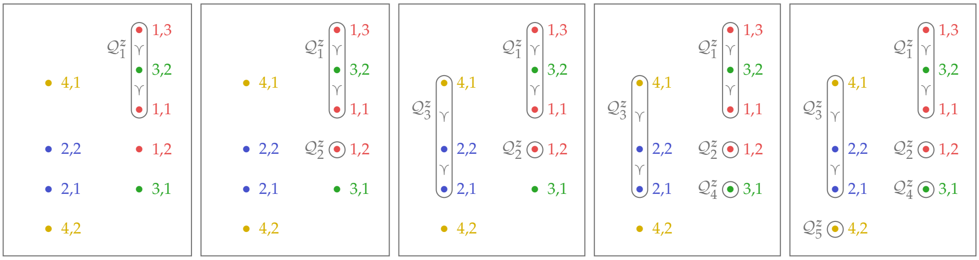

**Figure 1.4.** This is how our algorithm proceeds to cover all points.

**Notation 1.2.13.** We switch our indexing set of the filtration $F_\bullet$ from $\mathbb{P}(K)$ to the natural numbers $1 \leqslant \nu \leqslant N$ as follows: let $\chi\colon \{1, \dots, N\} \to \mathbb{P}(K)$ be the unique order-*reversing* bijection; then, for $1 \leqslant \nu \leqslant N$, we define $F_\nu := F_{\chi(\nu)}$. Moreover, we set $F_0 := \{\infty\} \subset \tilde{V}^\infty$ and obtain an *ascending* filtration

$$\{\infty\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_N = \tilde{V}^\infty.$$

We also denote $F_{-1} := \varnothing$, and for $0 \leqslant \nu \leqslant N$ we denote by $\mathfrak{F}_\nu$ the $\nu^{\text{th}}$ filtration stratum of the filtration $F_\bullet$, i.e. the difference $\mathfrak{F}_\nu := F_\nu \setminus F_{\nu-1}$.

**Lemma 1.2.14.** *The strata satisfy the following properties:*

1. *For each ray partition $\mathcal{Q}$ (with $\boldsymbol{\sigma}(\mathcal{Q}) = \boldsymbol{\sigma}$), the subspace $W_\mathcal{Q}$ is a contractible open manifold of dimension $|K| + p \cdot s(\mathcal{Q}) + (q-1) \cdot l(\mathcal{Q})$ and a path component of the stratum $\mathfrak{F}_\nu$, where $1 \leqslant \nu \leqslant N$ satisfies $\chi(\nu) = \omega(\mathcal{Q})$.*

2. *All connected components of a stratum $\mathfrak{F}_\nu$ with $\nu \geqslant 1$ arise in this way.*

3. *The closure $\overline{W_\mathcal{Q}}$ of $W_\mathcal{Q}$ inside $\tilde{V}$ is also a smooth, orientable submanifold of $\tilde{V}$, and of dimension $|K| + p \cdot s(\mathcal{Q}) + (q-1) \cdot l(\mathcal{Q})$ as well.*

This lemma is proven in [BK21, Lem. 3.14]; let us here only emphasise two instructive steps: firstly, in order to calculate the dimension of $W_\mathcal{Q}$, note that the following parameters describe a point inside $W_\mathcal{Q}$: on the one hand, we have, for each $1 \leqslant \beta \leqslant l$, a parameter $\zeta_\beta = (\zeta_\beta^1, \zeta_\beta^2) \in \mathbb{R}^p \times \mathbb{R}^{q-1}$ which corresponds to the value attained by $\mathrm{pr}_\zeta(z_{i,\jmath})$ for all $(i, \jmath) \in \mathcal{Q}_\beta$. However, if two rays $\mathcal{Q}_\beta$ and $\mathcal{Q}_{\beta'}$ share a cluster, then their further projections $\zeta_\beta^1$ and $\zeta_{\beta'}^1$ coincide. Hence, we get for each equivalence class of rays a choice in $\mathbb{R}^p$, and

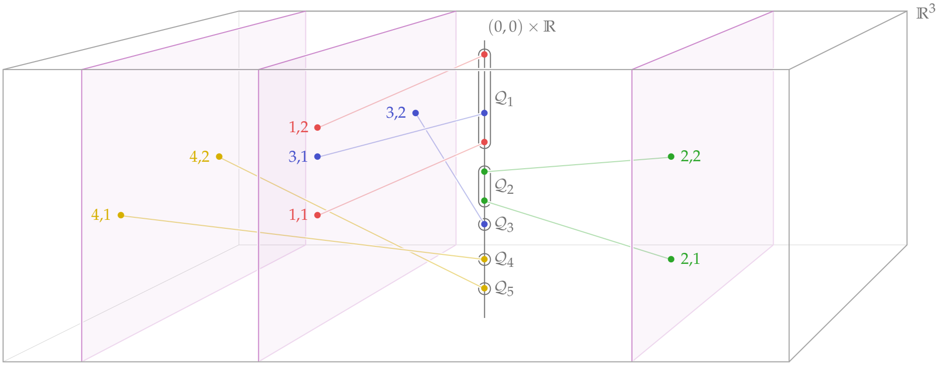

**Figure 1.5.** The linear interpolation from $z$ to the configuration $\mathring{z}$

for each ray a choice in $\mathbb{R}^{q-1}$, yielding $p \cdot s(\mathcal{Q}) + (q-1) \cdot l(\mathcal{Q})$ parameters. On the other hand, we have for each $(i,\jmath) \in \mathbb{Y}_K$ a parameter $\mathrm{pr}_t(z_{i,\jmath}) \in \mathbb{R}$.

Secondly, in order to see that $W_{\mathcal{Q}}$ is contractible, we choose distinct numbers $\mathring{t}_{i,\jmath} \in \mathbb{R}$ such that $\mathring{t}_{i',\jmath'} < \mathring{t}_{i,\jmath}$ if and only if either $\beta' > \beta$, or $\beta' = \beta$ and $(i',\jmath') \prec_\beta (i,\jmath)$, and define $\mathring{z}_{i,\jmath} := (0, \mathring{t}_{i,\jmath}) \in \mathbb{R}^d$. Then $\mathring{z} = (\mathring{z}_{1,1}, \ldots, \mathring{z}_{r,k_r})$ lies in $W_{\mathcal{Q}}$, and an explicit contraction can be given by linear interpolation $H(z,s) := s \cdot z + (1-s) \cdot \mathring{z}$, see Figure 1.5.

Now we are ready to prove Theorem 1.2.5.

*Proof of Theorem 1.2.5.* We consider the Leray spectral sequence associated with the filtered space $\tilde{V}^\infty$ and compute its homology: the $E^1$-page reads

$$E^1_{\nu,\mu} = H_{\nu+\mu}(F_\nu, F_{\nu-1}) = \tilde{H}_{\nu+\mu}(F_\nu / F_{\nu-1}).$$

By Lemma 1.2.14, each $\mathfrak{F}_\nu$ is the disjoint union of the open manifolds $W_{\mathcal{Q}}$ for $\mathcal{Q}$ varying in the finite set of ray partitions with $\omega(\mathcal{Q}) = \chi(\nu)$, and we have

$$F_\nu / F_{\nu-1} \cong \mathfrak{F}_\nu^\infty \cong \bigvee_{\omega(\mathcal{Q}) = \chi(\nu)} W_{\mathcal{Q}}^\infty.$$

Even for $\nu = 0$ we have that $F_0 = F_0 / F_{-1} = \{\infty\}$ is formally homeomorphic to the *empty bouquet*. By Lemma 1.2.14, $W_{\mathcal{Q}}$ is an open manifold of dimension

$d(\mathcal{Q}) := |K| + p \cdot s(\mathcal{Q}) + (q-1) \cdot l(\mathcal{Q})$ for all ray partitions; hence we can apply Poincaré–Lefschetz duality and obtain for all $\nu, \mu \geqslant 0$ an isomorphism

$$E^1_{\nu,\mu} \cong \bigoplus_{\omega(\mathcal{Q})=\chi(\nu)} H_{\nu+\mu}(W^{\infty}_{\mathcal{Q}}, \infty) \cong \bigoplus_{\omega(\mathcal{Q})=\chi(\nu)} H^{d(\mathcal{Q})-\nu-\mu}(W_{\mathcal{Q}}).$$

Again by Lemma 1.2.14, $W_{\mathcal{Q}}$ is contractible for all ray partitions $\mathcal{Q}$; hence $H^{d(\mathcal{Q})-\nu-\mu}(W_{\mathcal{Q}})$ contributes to the first page of the spectral sequence only in the case $\mu + \nu = d(\mathcal{Q})$. We can rewrite, for each $\nu \geqslant 0$ and by considering all degrees $\mu$ at the same time,

$$\bigoplus_{\mu \geqslant 0} E^1_{\nu,\mu} \cong \bigoplus_{\omega(\mathcal{Q})=\chi(\nu)} H_{d(\mathcal{Q})}(W^{\infty}_{\mathcal{Q}}, \infty).$$

Since $F_\bullet$ is a closed filtration and $W_{\mathcal{Q}}$ is a path component of $\mathfrak{F}_\nu$, we can now, for all ray partitions $\mathcal{Q}$, replace the relative homology of the pair $(W^{\infty}_{\mathcal{Q}}, \infty)$ with the relative homology of the pair $(F_\nu, F_\nu \setminus W_{\mathcal{Q}})$ or, using excision, with the relative homology of the pair $\big(\overline{W}^{\infty}_{\mathcal{Q}}, \overline{W}^{\infty}_{\mathcal{Q}} \setminus W_{\mathcal{Q}}\big)$, where we denote by $\overline{W}^{\infty}_{\mathcal{Q}}$ the one-point compactification of $\overline{W}_{\mathcal{Q}}$ (which coincides with the closure of $W_{\mathcal{Q}}$ in $\tilde{V}^{\infty}$ for $\nu \geqslant 1$); then we obtain

$$\bigoplus_{\mu \geqslant 0} E^1_{\nu,\mu} \cong \bigoplus_{\omega(\mathcal{Q})=\chi(\nu)} H_{d(\mathcal{Q})}\big(\overline{W}^{\infty}_{\mathcal{Q}}, \overline{W}^{\infty}_{\mathcal{Q}} \setminus W_{\mathcal{Q}}\big).$$

Each direct summand in the previous decomposition is isomorphic to $\mathbb{Z}$, generated by the fundamental class of the relative manifold $\big(\overline{W}^{\infty}_{\mathcal{Q}}, \overline{W}^{\infty}_{\mathcal{Q}} \setminus W_{\mathcal{Q}}\big)$.

By Lemma 1.2.14, $\big(\overline{W}^{\infty}_{\mathcal{Q}}, \infty\big)$ is also a relative manifold, and its fundamental class projects to that of $\big(\overline{W}^{\infty}_{\mathcal{Q}}, \overline{W}^{\infty}_{\mathcal{Q}} \setminus W_{\mathcal{Q}}\big)$ under the natural map

$$H_{d(\mathcal{Q})}\big(\overline{W}^{\infty}_{\mathcal{Q}}, \infty\big) \to H_{d(\mathcal{Q})}\big(\overline{W}^{\infty}_{\mathcal{Q}}, \overline{W}^{\infty}_{\mathcal{Q}} \setminus W_{\mathcal{Q}}\big).$$

The previous analysis shows in particular that for all integers $\nu \geqslant 0$ the natural map $H_\bullet(F_\nu, \infty) \to H_\bullet(F_\nu, F_{\nu-1})$ is surjective. This suffices to prove that the spectral sequence collapses on its first page: any element on the first page is represented by a genuine relative cycle of a pair $(F_\nu, \infty)$, so it must survive to the limit. This shows that $H_\bullet(\tilde{V}^{\infty}, \infty)$ is freely generated by the fundamental classes of the relative submanifolds $(\overline{W}^{\infty}_{\mathcal{Q}}, \infty)$, so, via

Poincaré–Lefschetz duality, $H^\bullet(\tilde{V})$ is generated by their duals, which we call $u_{\mathcal{Q}}$. For each ray partition $\mathcal{Q}$, we finally see

$$\begin{aligned} |u_{\mathcal{Q}}| &= p \cdot r + q \cdot |K| - d(\mathcal{Q}) \\ &= p \cdot (r - s(\mathcal{Q})) + (q-1) \cdot (|K| - l(\mathcal{Q})). \end{aligned}$$

For $q = 1$ and a fixed component $\sigma \in \prod_i \mathfrak{S}_{k_i}$, the entire argument takes place inside $\tilde{V} = \tilde{V}_K(\mathbb{R}^{p,q})_\sigma$; more precisely: for each ray partition $\mathcal{Q}$ with $\sigma(\mathcal{Q}) = \sigma$, we have $\overline{W_{\mathcal{Q}}} \subset \tilde{V}_K(\mathbb{R}^{p,q})_\sigma$. □

## 1.3. Homological stability for the unordered case

In this section we prove homological stability for the family $(V^k_r(\mathbb{R}^{p,q}))_{r\geqslant 0}$ of unordered configuration spaces of vertical clusters of size $k$ for all values of $p$ and $q$ except for the one pair $(p,q)$ where it is obviously false. By doing so, we extend results by [TP14; Lat17; Pal21]. This section also appeared as part of the article [BK21].

### 1.3.1. *Setting and results*

Throughout the section, we fix a cluster size $k \geqslant 1$ and we save one index by abbreviating $V_r(\mathbb{R}^{p,q}) := V^k_r(\mathbb{R}^{p,q})$. If $p$ and $q$ are fixed and clear from the context, we may also just write $V_r$.

**Construction 1.3.1.** For each number $r \geqslant 0$ of clusters and dimensions $p \geqslant 0$ and $q \geqslant 1$, we have *stabilisation maps*

$$\mathrm{stab}\colon V_r(\mathbb{R}^{p,q}) \to V_{r+1}(\mathbb{R}^{p,q})$$

by adding an extra cluster on the far right with respect to the first coordinate of $\mathbb{R}^{p+q}$, as depicted in Figure 1.6.

If the clusters were ordered, a natural candidate for a topological retraction of stab would be given by just forgetting the *last* cluster. In our situation, the clusters are unordered, but still, we can define a 'multiretraction' by summing over all possibilities of forgetting clusters. Such a multiretraction lands in the symmetric product, so it can, via the Dold–Thom theorem [DT58, Satz 6.10], be used to construct a map in homology.

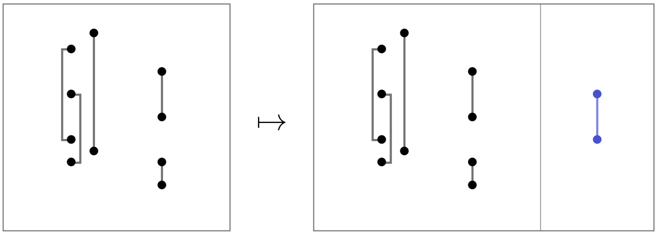

**Figure 1.6.** The stabilisation map stab: $V_5(\mathbb{R}^{1,1}) \to V_6(\mathbb{R}^{1,1})$, which adds a new cluster (blue) on the far right; here $k = 2$.

**Proposition 1.3.2.** *The induced maps in homology*

$$\mathrm{stab}_*\colon H_h(V_r(\mathbb{R}^{p,q})) \to H_h(V_{r+1}(\mathbb{R}^{p,q})).$$

*are split injective for each* $h, r, p \geqslant 0$ *and* $q \geqslant 1$.

*Proof.* We generalise the method of proof from [Pal21, Lem. 5.1], which is based on the following lemma from [Dol62]:

> Lemma 2. *Suppose we are given a sequence* $(0 = A_0 \xrightarrow{s_0} A_1 \xrightarrow{s_1} \cdots)$ *of abelian groups, and assume that there are* $\tau_{j,r}\colon A_r \to A_j$ *for* $1 \leqslant j \leqslant r$ *such that* $\tau_{r,r} = \mathrm{id}$ *and* $\tau_{j,r} - \tau_{j,r+1} \circ s_r\colon A_r \to A_j$ *lies in the image of* $s_{j-1}$. *Then every* $s_r$ *is split monic.*

To do so, we firstly note that $V_0 = *$, whence all spaces $V_r = V_r(\mathbb{R}^{p,q})$ are canonically based by $\mathrm{stab}^r(*) \in V_r$, and the stabilisation maps are basepoint-preserving by definition. For a fixed $h \geqslant 0$, let $A_r := \tilde{H}_h(V_r)$; then we have maps $s_r\colon A_r \to A_{r+1}$ induced by the stabilisation.

Now recall for $l \geqslant 0$ the $l$-fold symmetric product $\mathrm{SP}^l V_r := (V_r)^l/\mathfrak{S}_l$ where $\mathfrak{S}_l$ acts by coordinate permutation. We denote elements of $\mathrm{SP}^l V_r$ as formal sums of elements of $V_r$, but we use the symbol $\oplus$ in order to avoid confusion with Notation 1.1.4. Using this convention, we consider the maps

$$\gamma_{j,r}\colon V_r \to \mathrm{SP}^{\binom{r}{j}} V_j, \quad \sum_{i=1}^{r}[\vec{z}_i] \mapsto \bigoplus_{\substack{S\subseteq\{1,\dots,r\}\\ \#S=j}} \sum_{i\in S}[\vec{z}_i].$$

A priori, $\gamma_{j,r}$ is not based, but it can be homotoped to a based map since $V_r$ is well-based. Then $\gamma_{r,r} = \mathrm{id}$ and we have a homotopy

$$\gamma_{j,r+1} \circ \mathrm{stab}_r \simeq \gamma_{j,r} \oplus \mathrm{SP}^{\binom{r}{j-1}}\big(\mathrm{stab}_{j-1}\big) \circ \gamma_{j-1,r}.$$

of maps $V_r \to \mathrm{SP}^{\binom{r+1}{j}} V_j$. Applying the functor $\pi_h \circ \mathrm{SP}^\infty$ and using the 'flattening' map $\varphi_l \colon \mathrm{SP}^\infty \mathrm{SP}^l V_r \to \mathrm{SP}^\infty V_r$, we obtain the desired system $(\tau_{j,r})_{1 \leqslant j \leqslant r}$ of morphisms for Dold's lemma by the dashed arrow in the following diagram, where the vertical identifications are due to the Dold–Thom theorem:

$$\begin{tikzcd}
A_r \arrow[rr, dashed, "\tau_{j,r}"] \arrow[d, equal] & & A_j \arrow[d, equal] \\
\pi_h(\mathrm{SP}^\infty V_r) \arrow[r, "\pi_h\left(\mathrm{SP}^\infty \gamma_{j,r}\right)"'] & \pi_h\big(\mathrm{SP}^\infty \mathrm{SP}^{\binom{r}{j}} V_j\big) \arrow[r, "\pi_h\left(\varphi_{\binom{r}{j}}\right)"'] & \pi_h\big(\mathrm{SP}^\infty V_j\big).
\end{tikzcd}$$ □

In contrast to this, surjectivity of $\mathrm{stab}_* \colon H_h(V_r(\mathbb{R}^{p,q})) \to H_h(V_{r+1}(\mathbb{R}^{p,q}))$ holds only in a certain range. The rest of this section is devoted to the proof of the following stability theorem:

**Theorem 1.3.3.** *For all $p \geqslant 0$ and $q \geqslant 1$ with $(p,q) \neq (0,1)$, the induced maps*

$$H_h(V_r(\mathbb{R}^{p,q})) \to H_h(V_{r+1}(\mathbb{R}^{p,q}))$$

*are isomorphisms for $h \leqslant \frac{r}{2}$.*

Many cases of Theorem 1.3.3 have already been solved:

- We know that $\pi_0\big(V_r(\mathbb{R}^{0,1})\big) \cong \mathfrak{S}_{r \cdot k}/(\mathfrak{S}_k \wr \mathfrak{S}_r)$, so there is no stability result to be expected in the case $p = 0$ and $q = 1$.

- For $p = 0$, we are in the case without any vertical coupling condition. This can alternatively be described by embeddings of (disconnected) 0-dimensional manifolds into $\mathbb{R}^q$. For these cases, the theorem was proven for $q = 2$ in [TP14] and for $q \geqslant 3$ in [Pal21].

- In [Lat17], the case $p + q \geqslant 3$ was considered and proven. To be precise, Latifi writes down the proof only for $p = 2$ and $q = 1$, but her strategy works whenever $p + q \geqslant 3$.

Hence we only have to prove the single remaining case $(p,q) = (1,1)$. However, since the method is the same, we will provide an argument for arbitrary pairs $(p,1)$ with $p \geqslant 1$, which uses different techniques than Latifi's proof.

### 1.3.2. *The dexterity filtration*

In the remainder of the section, we assume $q = 1$, so if we use Notation 1.2.6, then pr and $\mathrm{pr}_\zeta$ agree. Moreover, each vertical cluster $[\vec{z}] = \{z_1, \dots, z_k\}$ is canonically ordered by the last coordinate $t_j := \mathrm{pr}_t(z_j) \in \mathbb{R}$ of each point, and $[\vec{z}]$ is determined by their common projection $\zeta := \mathrm{pr}_\zeta(z) \in \mathbb{R}^p$ and by the real numbers $t_1, \dots, t_k$. Hence we can write $\{z_1, \dots, z_k\} = (\zeta; t_1 < \cdots < t_k)$.

We introduce an invariant which measures, for a given configuration in $V_r^k(\mathbb{R}^{p,1})$, how 'entangled' the clusters are, and we want to use this invariant in order to establish a filtration of $V_r^k(\mathbb{R}^{p,1})$.

**Definition 1.3.4.** Let $\boldsymbol{z} := (\vec{z}_1, \dots, \vec{z}_r) \in \tilde{V}_r$ be an *ordered* configuration, where $\vec{z}_i = (z_{i,1}, \dots, z_{i,k})$. We define an equivalence relation $\sim_{\boldsymbol{z}}$ on the set $\{1, \dots, r\}$. Firstly, we set $i \sim_{\boldsymbol{z}} i'$ whenever the two following conditions hold:

- the clusters $\vec{z}_i$ and $\vec{z}_{i'}$ are *aligned*, i.e. they are contained in the same $t$-line, or, equivalently, $\mathrm{pr}_\zeta(\vec{z}_i) = \mathrm{pr}_\zeta(\vec{z}_{i'})$ in $\mathbb{R}^p$;
- the clusters $\vec{z}_i$ and $\vec{z}_{i'}$ are *entangled*, i.e. their convex hulls (contained in the vertical line) intersect each other, see Figure 1.7.

Let $\sim_{\boldsymbol{z}}$ be the equivalence relation generated by the above relations. We define the *dexterity* of $\boldsymbol{z}$, denoted $\delta(\boldsymbol{z})$, to be the number of equivalence classes of $\sim_{\boldsymbol{z}}$. Since the notion of dexterity is invariant under the permutation action of the group $\mathfrak{S}_k \wr \mathfrak{S}_r$, it descends to unordered configurations in $V_r$.

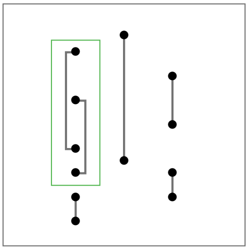

**Figure 1.7.** The leftmost two upper clusters are entangled and hence form an equivalence class. Therefore, the dexterity is 5, while the number of clusters is 6.

**Definition 1.3.5** (Dexterity filtration)**.** For $s \geqslant -1$ we let $F_s V_r \subseteq V_r$ be the subspace of all configurations with dexterity *at least* $r - s$. Then we have a chain of inclusions

$$\varnothing = F_{-1} V_r \subseteq F_0 V_r \subseteq \cdots \subseteq F_{r-1} V_r = V_r,$$

and we denote by $\mathfrak{F}_s V_r := F_s V_r \setminus F_{s-1} V_r$ the $s^{\text{th}}$ stratum of the filtration.

Note that each filtration level $F_s V_r$ is an open subspace of $V_r$; in particular, it is a manifold of the same dimension $r \cdot (p + k)$. The stratum $\mathfrak{F}_s V_r$ is a closed subset of $F_s V_r$, and one easily checks [BK21, Lem. 4.7] that it is even a closed submanifold of codimension $p \cdot s$, i.e. of dimension $p \cdot (r - s) + r \cdot k$.

Additionally, the stabilisation map $\mathrm{stab}\colon V_r \to V_{r+1}$ is filtration-preserving, i.e. it restricts to maps $F_s V_r \to F_s V_{r+1}$ of filtration components, and even to maps of strata $\mathfrak{F}_s V_r \to \mathfrak{F}_s V_{r+1}$.

### 1.3.3. *Coloured configuration spaces*

We want to use the dexterity filtration in order to replace *vertical* configurations by *coloured* configurations, starting with the following idea: we describe each point inside the stratum $\mathfrak{F}_s V_r$ by a configuration of $r - s$ points in $\mathbb{R}^{p+1}$, one point for each equivalence class of entangled clusters, carrying as a colour a combinatorial type which describes the entanglement. This leads us to the notion of coloured configuration spaces, which we assign to a each map $\alpha$ telling us how often which combinatorial type appears.

**Notation 1.3.6.** Let $E$ be an index set. A *distribution* is a map $\alpha\colon E \to \mathbb{N}$ with finite support. We write $\alpha_e := \alpha(e)$ and $\alpha = \sum_e \alpha_e \cdot e$.

In particular, for a fixed $e_0 \in E$, we denote by $\alpha + e_0$ the distribution which coincides with $\alpha$, except for the fact that $\alpha_{e_0}$ is increased by 1.

**Definition 1.3.7.** Let $E$ be a set and $\alpha\colon E \to \mathbb{N}$ be a distribution. Then we define $|\alpha| := \sum_{e \in E} \alpha_e$ and $\mathfrak{S}(\alpha) := \prod_{e \in E} \mathfrak{S}_{\alpha_e} \subseteq \mathfrak{S}_{|\alpha|}$. Moreover, for a family $\boldsymbol{X} := (X_e)_{e \in E}$ of spaces, we define the *coloured configuration space*

$$C_\alpha(\mathbb{R}^{p+1}; \boldsymbol{X}) := \tilde{C}_{|\alpha|}(\mathbb{R}^{p+1}) \times_{\mathfrak{S}(\alpha)} \prod_{e \in E} X_e^{\alpha_e}.$$

In the case $X_e = *$ for all $e$, we just write $C_\alpha(\mathbb{R}^{p+1}) := \tilde{C}_{|\alpha|}(\mathbb{R}^{p+1}) / \mathfrak{S}(\alpha)$.

Informally, $C_\alpha(\mathbb{R}^{p+1};\boldsymbol{X})$ is the space of unordered configurations of $|\alpha|$ unordered points, each equipped with a colour $e \in E$ and a parameter in $X_e$, such that for all $e \in E$, there are precisely $\alpha_e$ points with colour $e$.

For unordered labelled configurations as before, we use again the 'sum notation': for distinct points $y_1, \dots, y_{|\alpha|} \in \mathbb{R}^{p+1}$, colours $e_1, \dots, e_{|\alpha|} \in E$ and labels $x_l \in X_{e_l}$, we denote by $\sum_l y_l \otimes (e_l, x_l)$ the unordered labelled configuration in which the point $y_l$ carries the colour $e_l$ and the parameter $x_l$.

**Definition 1.3.8.** For $w \geqslant 1$, a partition of $\{1, \dots, w \cdot k\}$ into subsets $S_1, \dots, S_w$ of size $k$ is called *irreducible* if there is no $1 \leqslant i \leqslant w-1$ for which the subset $\{1, \dots, i \cdot k\}$ is a union of some pieces $S_b$ of the partition.

As usual, we consider a partition $e = (S_1, \dots, S_r)$ as *unordered* by demanding that $\min(S_1) < \cdots < \min(S_r)$ holds, and we denote by $\mathbb{E}_w$ the set of all irreducible, unordered partitions of $\{1, \dots, w \cdot k\}$ into subsets of size $k$.

We let $\mathbb{E} := \coprod_{w \geqslant 1} \mathbb{E}_w$, and for $e \in \mathbb{E}_w \subseteq \mathbb{E}$, we write $w(e) := w$ for the *weight* of $e$. Note that there is precisely one partition $e_0 \in \mathbb{E}$ with $w(e_0) = 1$.

**Figure 1.8.** We can depict partitions of $\{1, \dots, w \cdot k\}$ as in this figure. Here we see a reducible and an irreducible partition, with $k = 2$ and $w = 3$.

**Remark 1.3.9.** The notion of irreducibility is related to the dexterity filtration: given an irreducible partition $e = (S_1, \dots, S_w)$ with $S_b = \{m_{b,1} < \cdots < m_{b,k}\}$ and $\zeta_1, \dots, \zeta_w \in \mathbb{R}^p$, consider the configuration $[z] = \sum_b [z_{b,1}, \dots, z_{b,k}] \in V_w$ with $z_{b,j} = (\zeta_b, m_{b,j})$. Then $[z]$ has dexterity 1 if and only if $\zeta_1 = \cdots = \zeta_b$ holds. This is used, in the following, for the special case $\zeta_1 = 0$, i.e. we vary the $w-1$ coordinates $\zeta_2, \dots, \zeta_w$ inside $p$-dimensional discs $\mathbb{D}^p \subseteq \mathbb{R}^p$.

**Construction 1.3.10.** For each $e \in \mathbb{E}$ we denote by $\mathbb{D}_e$ the product $(\mathbb{D}^p)^{w(e)-1}$, and thus obtain a family of discs $\mathbb{D} := (\mathbb{D}_e)_{e \in \mathbb{E}}$. We denote by $\xi_2, \dots, \xi_{w(e)}$ the $w(e) - 1$ parameters of $\mathbb{D}_e$, each taking values in $\mathbb{D}^p$; each parameter $\xi_b$ consists of $p$ coordinates $\xi_b^1, \dots, \xi_b^p \in \mathbb{R}$.

For each distribution $\alpha\colon \mathbb{E} \to \mathbb{N}$, we obtain a coloured configuration space $C_\alpha(\mathbb{R}^{p+1};\mathbb{D})$. By allowing only coloured configurations with labels in the centres of the discs, we obtain an inclusion $C_\alpha(\mathbb{R}^{p+1}) \subseteq C_\alpha(\mathbb{R}^{p+1};\mathbb{D})$, which is a closed embedding of a submanifold of codimension $p \cdot s(\alpha)$. We denote the complement by $C^*_\alpha(\mathbb{R}^{p+1};\mathbb{D}) := C_\alpha(\mathbb{R}^{p+1};\mathbb{D}) \smallsetminus C_\alpha(\mathbb{R}^{p+1})$.

**Definition 1.3.11.** For a distribution $\alpha\colon \mathbb{E} \to \mathbb{N}$, we define the *degree* $\deg(\alpha)$ as the pair of non-negative numbers

$$\deg(\alpha) := (r(\alpha), s(\alpha)) := \Big(\textstyle\sum_e \alpha_e \cdot w(e),\, \sum_e \alpha_e \cdot (w(e)-1)\Big).$$

We let $C_{r,s} := C_{r,s}(\mathbb{R}^{p+1};\mathbb{D})$ be the union of all spaces $C_\alpha(\mathbb{R}^{p+1};\mathbb{D})$, where $\alpha$ ranges among distributions $\alpha\colon \mathbb{E} \to \mathbb{N}$ with $\deg(\alpha) = (r,s)$. Moreover, we define $C^0_{r,s} \subseteq C_{r,s}$ to be the union of all $C_\alpha(\mathbb{R}^{p+1})$ which satisfy $\deg(\alpha) = (r,s)$, and define its complement $C^*_{r,s} := C_{r,s} \smallsetminus C^0_{r,s}$.

A generic point in $C_{r,s}$ is of the form $\sum_{l=1}^{r-s} y_l \otimes (e_l, \boldsymbol{\zeta}_l)$, where the points $y_1, \ldots, y_{r-s} \in \mathbb{R}^{p+1}$ are all distinct, $e_l \in \mathbb{E}$, and the parameter $\boldsymbol{\zeta}_l \in \mathbb{D}_{e_l}$ is expanded as $\boldsymbol{\zeta}_l = (\zeta_{l,2}, \ldots, \zeta_{l,w(e_l)})$ with $\zeta_{l,b} \in \mathbb{D}^p$.

There are stabilisation maps $C_{r,s} \to C_{r+1,s}$ given by placing a new point with label in $\mathbb{D}_{e_0} = *$ on the far right with respect to the first coordinate of $\mathbb{R}^{p+1}$. The stabilisation increases the parameter $r$ by 1, but leaves $s$ constant. Along this map, $C^0_{r,s}$ gets sent to $C^0_{r+1,s}$ and $C^*_{r,s}$ gets sent to $C^*_{r+1,s}$.

### 1.3.4. *The insertion map*

As we have already indicated earlier, we want to connect the filtration pairs $(F_sV_r, F_{s-1}V_r)$ to the pairs $(C_{r,s}, C^*_{r,s})$ of coloured configurations via a so-called *insertion map*.

**Idea 1.3.12.** For each $1 \leqslant s \leqslant r$, we construct a map of pairs

$$\varphi_{r,s}\colon (C_{r,s}, C^*_{r,s}) \to (F_sV_r, F_{s-1}V_r),$$

which pictorially does the following, see Figure 1.9: given a coloured configuration in $C_{r,s}$, we draw pairwise disjoint cylinders around each point in $\mathbb{R}^{p+1}$ and place inside each cylinder a small 'standard configuration' which corresponds to the given indecomposable partition and is 'perturbed' by the $w(e)-1$ disc parameters from the label, where in each cylinder, one cluster stays in the centre.

By Remark 1.3.9, the dexterity of the resulting vertical configuration is $r - s$ if and only if *all* clusters inside each cylinder stay in the centre, i.e. if all disc parameters are 0. Thus, if the labelled configuration to start with lies in $C^*_{r,s}$, then the dexterity is at least $r - (s - 1)$, whence we land in the filtration component $F_{s-1}V_r$.

**Construction 1.3.13.** The insertion map $\varphi_{r,s}$ is formally constructed as follows:

- For each $w \geqslant 1$, for each subset $S \subseteq \{1, \dots, w \cdot k\}$ of cardinality $k$, and for each $\xi \in \mathbb{D}^p$, we define the unordered standard cluster

  $$T_S(\xi) \coloneqq \big(\xi, \big(-1 + \tfrac{2}{k \cdot w + 1} \cdot m\big)_{m \in S}\big).$$

  Pictorially, $T_S(\xi)$ is the unordered vertical cluster of $k$ points which projects to $\xi \in \mathbb{D}^p$ and whose $t$-coordinate takes the values corresponding to $S$, among all values arising from a uniform distribution of $w \cdot k$ points in the interior of the interval $[-1; 1]$.

- For a partition $e \in \mathbb{E}$, we write $S_1, \dots, S_{w(e)} \subseteq \{1, \dots, w \cdot k\}$ for the partition components; recall that $\min(S_1) < \cdots < \min(S_{w(e)})$. For all $\xi_2, \dots, \xi_{w(e)} \in \mathbb{D}^p$, we set $\xi_1 \coloneqq 0 \in \mathbb{D}^p$ and define

  $$T_e(\xi_2, \dots, \xi_{w(e)}) \coloneqq \sum_{b=1}^{w(e)} T_{S_b}(\xi_b) \in V_{w(e)}.$$

  Note that $S_b$ and $S_{b'}$ are disjoint for $b \neq b'$; hence the clusters $T_{S_b}(\xi_b)$ and $T_{S_{b'}}(\xi_{b'})$ are also disjoint, and the sum which defines $T_e(\xi_2, \dots, \xi_{w(e)})$ is admissible.

- Consider on $\mathbb{R}^p \times \mathbb{R}$ the *product distance*

  $$d((\zeta, t), (\zeta', t')) \coloneqq \max(d(\zeta, \zeta'), d(t, t'))$$

  with respect to the Euclidean distances in $\mathbb{R}^p$ and $\mathbb{R}$, respectively. This means that for a radius $\rho > 0$, the closed $\rho$-ball around $(\zeta, t)$ is given by the cylinder $B_\rho(\zeta, t) = (\zeta + \rho \cdot \mathbb{D}^p) \times [t - \rho; t + \rho]$.

Now we have everything together to define the desired insertion map: given an element $\Theta = \sum_l y_l \otimes (e_l, \xi_l)$ in $C_{r,s}$, we define[4]

$$\rho(\Theta) := \begin{cases} \frac{1}{5} \cdot \min_{l \neq l'} d(y_l, y_{l'}) & \text{for } r - s \geqslant 2, \\ 1 & \text{for } r - s = 1. \end{cases}$$

and, accordingly, the map $\varphi_{r,s} \colon C_{r,s} \to F_s V_r$ by

$$\varphi_{r,s}\left( \Theta = \sum_{l=1}^{r-s} y_l \otimes (e_l, \xi_l) \right) := \sum_{l=1}^{r-s} (y_l + \rho(\Theta) \cdot T_{e_l}(\xi_l)).$$

Note that the signs '+' and '·' in the expression '$y_l + \rho(\Theta) \cdot T_{e_l}(\xi_l)$' denote a translation and a dilation in $\mathbb{R}^{p+1}$, while the sum sign always describes an unordered collection (of points, or of vertical clusters). The second sum is well-defined as the configurations $y_l + \rho(\Theta) \cdot T_{e_l}(\xi_l)$ lie inside the cylinders $B_\rho(y_l)$ and are therefore disjoint. Finally, we have $\delta(\varphi_{r,s}(\Theta)) \geqslant r - s$, so the image of $\varphi_{r,s}$ is actually contained in the filtration level $F_s V_r \subseteq V_r$.

As indicated before, Remark 1.3.9 ensures that the point $\varphi_{r,s}(\Theta)$ lies in the stratum $\mathfrak{F}_s V_r$ if and only if $\xi_l = 0_{e_l}$ for all $1 \leqslant l \leqslant r - s$. In particular, $\varphi_{r,s}$ restricts to maps $C^*_{r,s} \to F_{s-1} V_r$ and $C^0_{r,s} \to \mathfrak{F}_s V_r$.

**Remark 1.3.14.** In more abstract language, and up to homotopy, the insertion map can be described as follows: we define a map $\varphi \colon \coprod_e \mathbb{D}_e \to \coprod_r V_r$ which restricts for all $e \in \mathbb{E}$ to a map $\mathbb{D}_e \to V_{w(e)}$ similar to the map $T_e$ above. Then we use that $\coprod_r V_r$ is an algebra over the little $(p+1)$-cubes operad $\mathscr{C}_{p+1}$, see Example 3.1.7, and consider the adjoint map of $\mathscr{C}_{p+1}$-algebras

$$\bar{\varphi} \colon C(\mathbb{R}^{p+1}; \mathbb{D}) \simeq F^{\mathscr{C}_{p+1}}(\textstyle\coprod_e \mathbb{D}_e) \to \textstyle\coprod_r V_r.$$

The left hand side decomposes as a disjoint union of the spaces $C_{r,s}$ as before, while the right hand side decomposes into the spaces $V_r$, which are filtered by $F_s V_r$, and the map $\varphi$ is compatible with this decomposition and filtration.

---

[4] In order to ensure that $\varphi_{r,s}$ is well-defined, it would have been enough to choose $\rho$ slightly smaller than $\frac{1}{2} \cdot \min_{l \neq l'} d(y_l, y_{l'})$. However, we want to ensure that $\varphi_{r,s} \colon C_{r,s} \to F_s V_r$ is even an embedding, so we need that $\varphi_{r,s}(\Theta)$ still 'knows' which clusters come from the same label: these have to be closer to each other than to foreign ones.

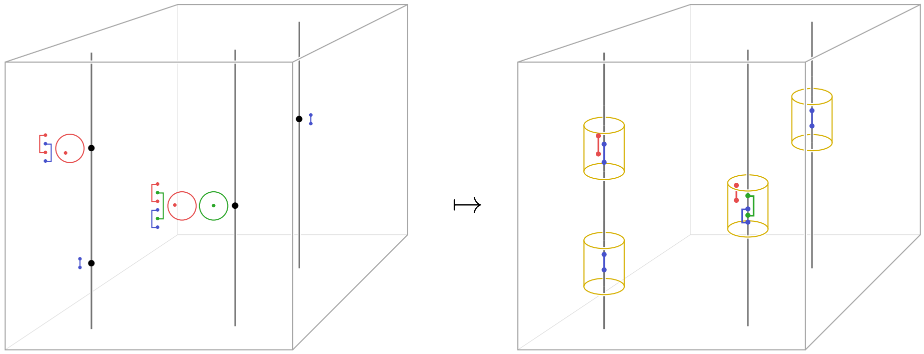

**Figure 1.9.** An instance of the insertion map $\varphi_{7,3}\colon (C_{7,3}, C^*_{7,3}) \to (F_3V_7, F_2V_7)$. The result even lies in the deeper filtration component $F_1V_7$.

Note that the insertion maps respect the stabilisation maps on both sides up to homotopy: the stabilisation maps can easily be chosen such that the diagram

$$\begin{tikzcd}
(C_{r,s}, C^*_{r,s}) \arrow[r, "\varphi_{r,s}"] \arrow[d, "\mathrm{stab}"'] & (F_sV_r, F_{s-1}V_r) \arrow[d, "\mathrm{stab}"] \\
(C_{r+1,s}, C^*_{r+1,s}) \arrow[r, "\varphi_{r+1,s}"'] & (F_sV_{r+1}, F_{s-1}V_{r+1})
\end{tikzcd}$$

commutes on the nose. In order to compare the two stabilisation morphisms on both sides, we need the following statement:

**Proposition 1.3.15.** *For each $1 \leqslant s \leqslant r$, the map $\varphi_{r,s}\colon (C_{r,s}, C^*_{r,s}) \to (F_sV_r, F_{s-1}V_r)$ induces an isomorphism in relative homology.*

This Proposition is shown in [BK21, Prop. 4.19], and follows essentially from the excision property of singular homology, since $\varphi_{r,s}\colon C_{r,s} \hookrightarrow F_sV_r$ is an inclusion of a complement of a subspace $W$ such that $\varphi_{r,s}(C^*_{r,s}) = F_{s-1}V_r \smallsetminus W$, and the closure of $W$ is contained in the open subspace $F_{s-1}V_r$.

### 1.3.5. *The stability proof*

After having translated the filtration pairs $(F_sV_r, F_{s-1}V_r)$ to the pairs $(C_{r,s}, C^*_{r,s})$ of coloured configuration spaces, and after having compared their respective stabilisations as well, we have to show homological stability for the sequence of relative coloured configuration spaces with disc parameters. To do so, we relate our situation to the case of coloured configuration spaces without disc parameters from [Pal18, Ex. 4.6] via the Thom isomorphism. This leads us to the following statement:

**Lemma 1.3.16.** *The stabilisation map $C_{r,s} \to C_{r+1,s}$ induces isomorphisms*

$$H_h(C_{r,s}, C^*_{r,s}) \to H_h(C_{r+1,s}, C^*_{r+1,s})$$

*in homology for $h \leqslant \frac{r}{2}$ and all $s \leqslant r$.*

*Proof.* For each distribution $\alpha$ of degree $(r, s)$, we have a (not always orientable) disc bundle $C_\alpha(\mathbb{R}^{p+1}; \mathbb{D}) \to C_\alpha(\mathbb{R}^{p+1})$ of dimension $p \cdot s$ and structure group $\mathfrak{S}(\alpha)$; this gives rise to a Thom isomorphism

$$H_h\big(C_\alpha(\mathbb{R}^{p+1}; \mathbb{D}), C^*_\alpha(\mathbb{R}^{p+1}; \mathbb{D})\big) \cong H_{h-p\cdot s}\big(C_\alpha(\mathbb{R}^{p+1}); \mathrm{pr}^*_\alpha \mathcal{O}_\alpha\big) =: M_{h,\alpha},$$

where $\mathrm{pr}_\alpha \colon \pi_1(C_\alpha(\mathbb{R}^{p+1})) \to \mathfrak{S}(\alpha)$ is the projection and $\mathcal{O}_\alpha$ is of the form

$$\mathcal{O}_\alpha \colon \mathfrak{S}(\alpha) \to \{\pm 1\}, \quad (\sigma_e)_{e \in \mathbb{E}} \mapsto \prod_e \mathrm{sg}(\sigma_e)^{p\cdot(w(e)-1)}.$$

In particular, we have a natural isomorphism $\mathrm{pr}^*_\alpha \mathcal{O}_\alpha \cong \mathrm{stab}^* \mathrm{pr}^*_{\alpha+e_0} \mathcal{O}_{\alpha+e_0}$ for the stabilisation $C_\alpha(\mathbb{R}^{p+1}) \to C_{\alpha+e_0}(\mathbb{R}^{p+1})$, which gives us induced stabilisation morphisms $M_{h,\alpha} \to M_{h,\alpha+e_0}$. We have a commutative square

$$\begin{tikzcd}
H_h(C_{r,s}, C^*_{r,s}) \arrow[r, "\cong"] \arrow[d] & \bigoplus\limits_{\deg(\alpha)=(r,s)} M_{h,\alpha} \arrow[d] \\
H_h(C_{r+1,s}, C^*_{r+1,s}) \arrow[r, "\cong"'] & \bigoplus\limits_{\deg(\alpha)=(r,s)} M_{h,\alpha+e_0} \oplus \bigoplus\limits_{\substack{\deg(\alpha)=(r+1,s) \\ \alpha_{e_0}=0}} M_{h,\alpha},
\end{tikzcd}$$

where the left vertical arrow is the desired stabilising map and the right side is a sum of maps $M_{h,\alpha} \to M_{h,\alpha+e_0}$. Therefore, we can prove the statement by showing that, for $h \leqslant \frac{r}{2}$, we firstly have $M_{h,\alpha} = 0$ for each distribution $\alpha$ of degree $(r+1, s)$ with $\alpha_{e_0} = 0$, and, secondly, the stabilising map $M_{h,\alpha} \to M_{h,\alpha+e_0}$

is an isomorphism for each distribution $\alpha$ of degree $(r,s)$. For the first part, we use that $w(e) \geqslant 2$ for all $e \in \mathbb{E}$ with $\alpha_e \neq 0$ to obtain

$$p \cdot s \geqslant p \cdot \sum_e \alpha_e \cdot \big(w(e) - \tfrac{1}{2} \cdot w(e)\big) = p \cdot \tfrac{r+1}{2} \geqslant \tfrac{r+1}{2},$$

so $h - p \cdot s < 0$, whence $M_{h,\alpha} = 0$. For the second part, we are left to check that $H_{h-p\cdot s}(C_\alpha(\mathbb{R}^{p+1}); \mathrm{pr}_\alpha^* \mathcal{O}_\alpha) \to H_{h-p\cdot s}(C_{\alpha+e_0}(\mathbb{R}^{p+1}); \mathrm{pr}_{\alpha+e_0}^* \mathcal{O}_{\alpha+e_0})$ is an isomorphism for $h \leqslant \frac{r}{2}$. To do so, we first observe that $\frac{r}{2} \geqslant \frac{1}{2} \cdot \alpha_{e_0} + \sum_{e \neq e_0} \alpha_e$ since $w(e) \geqslant 2$ for $e \neq e_0$, and, as $p \geqslant 1$, we obtain

$$h - p \cdot s \leqslant \tfrac{r}{2} - p \cdot \sum_e \alpha_e \cdot (w(e) - 1) \leqslant -\tfrac{r}{2} + \sum_e \alpha_e \leqslant \tfrac{1}{2} \cdot \alpha_{e_0}. \tag{$\star$}$$

As already indicated, we want to use a technique from [Pal18], so we adapt his notation by writing $\lambda := (\alpha_e)_{e \neq e_0}$, i.e. $|\lambda| = \sum_{e \neq e_0} \alpha_e$, as well as $\lambda[n]$ for the distribution with $\lambda[n]_{e_0} := n - |\lambda|$ and $\lambda[n]_e := \alpha_e$ for $e \neq e_0$. Then we have $|\lambda[n]| = n$, and in particular, $\lambda[r-s] = \alpha$ and $\lambda[r-s+1] = \alpha + e_0$. We have a stabilisation map $C_{\lambda[n]}(\mathbb{R}^{p+1}) \to C_{\lambda[n+1]}(\mathbb{R}^{p+1})$ by placing an additional point with label $e_0$, which for $n = r - s$ is our map from before.

Next, we construct a signed version of [Pal18, Ex. 4.6]: let **PInj** be the category whose objects are non-negative integers and whose morphisms $n \to n'$ are partially defined injections $\eta\colon \{1,\dots,n\} \dashrightarrow \{1,\dots,n'\}$. Now we define a functor $\mathcal{P}_\lambda\colon \mathbf{PInj} \to \mathbf{Ab}$ to the category of abelian groups: we set

$$\mathcal{P}_\lambda(n) := \mathbb{Z}\big\langle (P_e)_{e \neq e_0};\ P_e \subseteq \{1,\dots,n\},\ P_e \cap P_{e'} = \varnothing, \text{ and } \#P_e = \lambda_e \big\rangle,$$

and for each partially-defined injection $\eta\colon n \to n'$, we define the homomorphism $\eta_*\colon \mathcal{P}_\lambda(n) \to \mathcal{P}_\lambda(n')$ on additive generators $P := (P_e)_{e \neq e_0}$ by

$$\eta_*(P) := \begin{cases} \prod_{e \neq e_0} \mathrm{sg}(\eta|_{P_e})^{p\cdot(w(e)-1)} \cdot (\eta(P_e))_{e \neq e_0} & \text{if } \eta \text{ is defined on } \bigcup_e P_e, \\ 0 & \text{else,} \end{cases}$$

where in the first case, the restriction $\eta|_{P_e}\colon P_e \to \eta(P_e)$ is canonically identified with a permutation in $\mathfrak{S}_{\alpha_e}$ by using that $P_e$ and $\eta(P_e)$ are totally ordered as subsets of $\{1 < \dots < n\}$ and $\{1 < \dots < n'\}$, respectively. By the inductive argument from [Pal18, Lem. 4.7], we obtain that $\mathcal{P}_\lambda$ is a polynomial coefficient system of degree $|\lambda| = r - s - \alpha_{e_0}$ and since $\mathbb{Z}[\mathfrak{S}_n] \otimes_{\mathfrak{S}(\lambda[n])} \mathcal{O}_{\lambda[n]}$ and $\mathcal{P}_\lambda(n)$ are isomorphic as $\mathfrak{S}_n$-representations, we have natural identifications

$$\begin{tikzcd}
H_\bullet\big(C_{\lambda[n]}(\mathbb{R}^{p+1}); \mathrm{pr}^*_{\lambda[n]}\mathcal{O}_{\lambda[n]}\big) \arrow[r] \arrow[d, equal] & H_\bullet\big(C_{\lambda[n+1]}(\mathbb{R}^{p+1}); \mathrm{pr}^*_{\lambda[n+1]}\mathcal{O}_{\lambda[n+1]}\big) \arrow[d, equal] \\
H_\bullet\big(C_n(\mathbb{R}^{p+1}); \mathrm{pr}^*_n\mathcal{P}_\lambda(n)\big) \arrow[r] & H_\bullet\big(C_{n+1}(\mathbb{R}^{p+1}); \mathrm{pr}^*_{n+1}\mathcal{P}_\lambda(n+1)\big),
\end{tikzcd}$$

where $\mathrm{pr}_n \colon \pi_1\big(C_n(\mathbb{R}^{p+1})\big) \to \mathfrak{S}_n$ is the projection. As $p+1 \geqslant 2$, the bottom map is an isomorphism for $\bullet \leqslant \frac{1}{2} \cdot (n - r + s + \alpha_{e_0})$ by [Pal18, Thm. A]. For us, $n = r - s$ and $\bullet = h - p \cdot s$, so we are in the stable range by ($\star$). $\square$

Now we have all tools to prove Theorem 1.3.3.

*Proof of Theorem 1.3.3.* Let $E(r)$ denote the Leray spectral sequence associated with the filtered space $F_\bullet V_r$. Then the filtration-preserving stabilisation map $V_r \to V_{r+1}$ induces a morphism $f \colon E(r) \to E(r+1)$ of spectral sequences, which on the first page is of the form

$$\begin{tikzcd}
E(r)^1_{s,t} \arrow[r, "f^1_{s,t}"] \arrow[d, equal] & E(r+1)^1_{s,t} \arrow[d, equal] \\
H_{s+t}(F_s V_r, F_{s-1} V_r) \arrow[r, "\mathrm{stab}_*"'] & H_{s+t}(F_s V_{r+1}, F_{s-1} V_{r+1})
\end{tikzcd}$$

By Proposition 1.3.15 and Lemma 1.3.16, the bottom map is an isomorphism for $s + t \leqslant \frac{r}{2}$, so by a standard comparison argument [Zee57] for spectral sequences, $H_h(V_r) \to H_h(V_{r+1})$ is an isomorphism for $h \leqslant \frac{r}{2}$. $\square$

The strategy of proof of Theorem 1.3.3 generalises to the following case: let $K = (k_1, \ldots, k_r)$, and for $k \geqslant 1$ let $r(k) \geqslant 0$ be the number of indices $1 \leqslant i \leqslant r$ with $k_i = k$. If we construct a stabilisation map $V_K(\mathbb{R}^{p,1}) \to V_{(K,k)}(\mathbb{R}^{p,1})$ by placing a new vertical cluster of size $k$, then the induced map in homology

$$\mathrm{stab}_* \colon H_h(V_K(\mathbb{R}^{p,1})) \to H_h(V_{(K,k)}(\mathbb{R}^{p,1}))$$

is an isomorphism for $h \leqslant \frac{r(k)}{2}$. We leave to the reader the details of the straightforward generalisation of the proof.

**Outlook 1.3.17.** We believe that the Leray spectral sequence for the filtration $F_\bullet V_r$ collapses on its first page and that the extension problem is trivial. This would then imply that, using the notation from the proof of Lemma 1.3.16,

$$H_h(V_r(\mathbb{R}^{p,1})) \cong \bigoplus_{s=0}^{r-1} H_h(C_{r,s}, C^*_{r,s}) \cong \bigoplus_{s=0}^{r-1} \bigoplus_{\deg(\alpha)=(r,s)} M_{h,\alpha}.$$

Our motivation is the description of the stable homology as

$$H_h(V_\infty(\mathbb{R}^{p,1})) \cong \bigoplus_{\alpha} M^\infty_{h,\alpha},$$

given in Application 2.4.1, where the last direct sum is extended over all distributions $\alpha\colon \mathbb{E} \to \mathbb{N}$ with $\alpha_{e_0} = 0$ and where $M^\infty_{h,\alpha} := \varinjlim_t M_{h,\alpha+te_0}$.

## 1.4. Vertical configuration spaces as relative cell complexes

In the spirit of [FN62b; Fuc70], we want to construct (relative, dual) cellular decompositions of $\tilde{V}_K(\mathbb{R}^{p,q})$ and $V_K(\mathbb{R}^{p,q})$, where cells correspond to allocations of clusters into 'columns'. For simplicity, we restrict ourselves to the 2-dimensional case, i.e. $p + q = 2$, although these methods easily generalise to arbitrary $p$ and $q$. The case $(p, q) = (0, 2)$ coincides, up to some colouring, with the classical situation, so we are left with $(p, q) = (1, 1)$. The upshot of this construction is not very surprising: we end up with a *subcomplex* of the classical Fox–Neuwirth–Fuchs complex.

Although we have already fully understood the homology $H_\bullet(\tilde{V}_K(\mathbb{R}^{1,1}))$ additively in Section 1.2, the cellular decomposition will still be useful for us in order to understand $H_\bullet(\tilde{V}_K(\mathbb{R}^{1,1}))$, or, more precisely, the collection $H_\bullet(\tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n}))_{K,n}$, as an *operad* in Section 4.3. In the current chapter, we will just draw some qualitative conclusions from the decomposition; most importantly, the statement of Theorem 1.4.13: for $K = (k_1, \dots, k_r)$, the spaces $\tilde{V}_K(\mathbb{R}^{1,1})$ and $V_K(\mathbb{R}^{1,1})$ are equivalent to $(r-1)$-dimensional cell complexes.

Before coming to the desired cellular description, we have to introduce a bit of combinatorial notation.

**Definition 1.4.1.** Among tuples of positive integers, we have a pushforward construction: if $K = (k_1, \dots, k_r)$ is a tuple of positive integers and $c\colon \underline{r} \to \underline{s}$ is surjective, then we obtain a tuple $c_* K$ of length $s$ by

$$c_* K := \Big( \textstyle\sum_{i \in c^{-1}(1)} k_i, \dots, \sum_{i \in c^{-1}(s)} k_i \Big).$$

**Definition 1.4.2.** Let $K = (k_1, \dots, k_r)$ be a tuple and let $1 \leqslant s \leqslant r$. Recall from Definition 1.2.1 the *tableau* $\mathbb{Y}_K = \{(i, \jmath);\ 1 \leqslant i \leqslant r \text{ and } 1 \leqslant \jmath \leqslant k_i\}$. An *allocation of K into s columns* is encoded by the following datum:

1. another tuple $M$ of length $s$;
2. a map $\pi\colon \mathbb{Y}_M \to \underline{r}$ such that each preimage $\pi^{-1}(i) \subseteq \mathbb{Y}_M$ is of cardinality $k_i$ and contained in a single column $\big\{(a_\pi(i), \ell);\ 1 \leqslant \ell \leqslant m_{a_\pi(i)}\big\}$.

We denote such an allocation by $(M, \pi)$; however, we will sometimes suppress $M$ and just write $\pi$. Let $\Pi_{K,s}$ be the set of all allocations of $K$ into $s$ columns. Note that each such allocation gives rise to two maps:

- the surjection $a_\pi\colon \underline{r} \to \underline{s}$ that assigns to each index $1 \leqslant i \leqslant r$ the column inside $\mathbb{Y}_M$ where its preimage lies; note that $M = (a_\pi)_* K$;
- the bijection $\mathbb{Y}_\pi\colon \mathbb{Y}_K \to \mathbb{Y}_M$ which identifies, for each $1 \leqslant i \leqslant r$, the $i^{\text{th}}$ column $\{(i, \jmath);\ 1 \leqslant \jmath \leqslant k_i\}$ of $\mathbb{Y}_K$ with $\pi^{-1}(i)$ in a monotone way.

**Notation 1.4.3.** To save notation, we abbreviate allocations by the tuple

$$(\pi(1,1) \cdots \pi(1, m_1), \dots, \pi(s,1) \cdots \pi(s, m_s)),$$

where the entries are written as formal products of elements from $\underline{r}$. For example, if $K = (2, 1, 3, 2)$, then one possible allocation of $K$ into two columns is given by the tuple $(1\,4\,1\,4, 3\,2\,3\,3)$: for each $1 \leqslant i \leqslant 4$, there are precisely $k_i$ many occurrences of $i$, and they are all contained in a single entry.

Using this notation, we can describe the cellular structure on the ordered vertical configuration spaces: for simplicity, let us restrict to the path component $\tilde{V}_K^{<}(\mathbb{R}^{1,1})$ which contains all configurations $(\zeta_1, \dots, \zeta_r; t_{1,1}, \dots, t_{r,k_r})$ with $t_{i,\jmath} < t_{i,\jmath+1}$, and call its one-point compactification $\tilde{V}^\infty := \tilde{V}_K^{<}(\mathbb{R}^{1,1}) \cup \{\infty\}$.

**Construction 1.4.4.** We have a filtration

$$\{\infty\} = \tilde{V}_0^\infty = \cdots = \tilde{V}_{|K|}^\infty \subseteq \tilde{V}_{|K|+1}^\infty \subseteq \cdots \subseteq \tilde{V}_{|K|+r}^\infty = \tilde{V}^\infty,$$

where $\tilde{V}_{|K|+s}^\infty$ contains all configurations on at most $s$ columns. More precisely, it contains all configurations $(\zeta_1, \dots, \zeta_r; t_{1,1}, \dots, t_{r,k_r})$ with $\#\{\zeta_1, \dots, \zeta_r\} \leqslant s$. Then $\tilde{V}_{|K|+s}^\infty$ arises from $\tilde{V}_{|K|+(s-1)}^\infty$ by attaching, for each allocation $\pi \in \Pi_{K,s}$ into $s$ columns, a $(|K|+s)$-cell. If we identify the standard $n$-simplex $\Delta^n$ with $\{-\infty \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant +\infty\}$, then the characteristic map of the cell corresponding to $\pi$ is of the form $F_\pi \colon \Delta^s \times \Delta^{m_1} \times \cdots \times \Delta^{m_s} \to \tilde{V}_{|K|+s}^\infty$ with

$$F_\pi(\boldsymbol{\zeta}, \boldsymbol{t}) = \begin{cases} \infty & \text{if 1. } \vec{t}_a \in \partial\Delta^{m_a} \text{ for some } a, \text{ or} \\ & \quad\text{2. } (\zeta_a, t_{a,\mathscr{b}}) = (\zeta_{a+1}, t_{a+1,\mathscr{b}'}) \text{ for some } a, \mathscr{b}, \mathscr{b}', \text{ or} \\ & \quad\text{3. } \zeta_1 = -\infty \text{ or } \zeta_s = +\infty, \\ (a_\pi^* \boldsymbol{\zeta}, \mathbb{Y}_\pi^* \boldsymbol{t}) & \text{else,} \end{cases}$$

where $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_s) \in \Delta^s$ and $\boldsymbol{t} = (\vec{t}_1, \dots, \vec{t}_s)$ with $\vec{t}_a = (t_{a,1}, \dots, t_{a,m_a})$.

**Remark 1.4.5.** The Short notation 1.4.3 already looks like a configuration of points in $\tilde{V}_K(\mathbb{R}^{1,1})$, up to rotating each entry of the tuple anticlockwise by 90°. However, we intend to save space by denoting the entries horizontally.

Pictorially, the differentials in the corresponding cellular chain complex capture all possibilities of merging two consecutive columns. In order to formalise this, we need to introduce the notion of a shuffle.

**Definition 1.4.6.** For two integers $r, r' \geqslant 0$, a $(r, r')$-*shuffle* is a monotone injective map $\imath \colon \underline{r} \hookrightarrow \underline{r + r'}$. Each shuffle has a counterpart $\imath' \colon \underline{r'} \hookrightarrow \underline{r + r'}$ which is the unique monotone inclusion of the complement.

The *standard shuffle* is defined to be the map $\imath_0 \colon \underline{r} \hookrightarrow \underline{r + r'}$ with $\imath_0(a) = a$. For each $(r, r')$-shuffle, there is a unique permutation $\tau_\imath \in \mathfrak{S}_{r+r'}$ such that $\imath = \tau_\imath \circ \imath_0$ and $\imath' = \tau_\imath \circ \imath_0'$, and we put $\mathrm{sg}(\imath) := \mathrm{sg}(\tau_\imath)$.

We will occasionally write $\imath \colon (r, r')$ to express that $\imath$ is an $(r, r')$-shuffle.

Note that, in principle, an $(r, r')$-shuffle contains the same information as an allocation of $(r, r')$ on a single column, but we are going to use these two notions in different contexts; roughly speaking, shuffles 'act' on the set of allocations in the following way:

**Definition 1.4.7.** Let $(M,\pi)$ be an allocation of $K$ into $s$ columns, pick an index $1 \leqslant \alpha \leqslant s-1$, and let $\imath$ be an $(m_\alpha, m_{\alpha+1})$-shuffle. Then we have a new allocation $d_{\alpha,\imath}(M,\pi) := (M',\pi')$ with $M' := (m_1,\dots,m_\alpha + m_{\alpha+1},\dots,m_s)$ and

$$\pi'(a,b) := \begin{cases} \pi(a,b) & \text{for } a < \alpha, \\ \pi(a,\bar{b}) & \text{for } a = \alpha \text{ and } b = \imath(\bar{b}), \\ \pi(a+1,\bar{b}) & \text{for } a = \alpha \text{ and } b = \imath'(\bar{b}), \\ \pi(a+1,b) & \text{for } a > \alpha. \end{cases}$$

**Construction 1.4.8.** Consider the multisimplicial complex

$$\tilde{V}^\blacktriangle := \coprod_{s=1}^{r} \coprod_{(M,\pi)\in\Pi_{K,S}} \Delta^s \times (\Delta^{m_1} \times \dots \times \Delta^{m_s}) \Big/ (\pi; d^\alpha \boldsymbol{\zeta}, d^{\alpha,\imath}\boldsymbol{t}) \sim (d_{\alpha,\imath}\pi; \boldsymbol{\zeta}, \boldsymbol{t})$$

with $d^{\alpha,\imath}\boldsymbol{t} := (\vec{t}_1,\dots,\vec{t}_{\alpha-1},\imath^*\vec{t}_\alpha,\imath'^*\vec{t}_\alpha,\vec{t}_{\alpha+1},\dots,\vec{t}_{s-1})$ for $\boldsymbol{t} = (\vec{t}_1,\dots,\vec{t}_{s-1})$. Then we obtain a subcomplex $\tilde{V}^\vartriangle \subseteq \tilde{V}^\blacktriangle$ which consists of all points that can be represented by $(\pi;\boldsymbol{\zeta},\boldsymbol{t})$ with $\zeta_a = \pm\infty$ or $\vec{t}_a \in \partial\Delta^{m_a}$ for some $1 \leqslant a \leqslant s$, called the *degenerate subcomplex*.

Note that $\tilde{V}^\infty = \tilde{V}^\blacktriangle / \tilde{V}^\vartriangle$ and $\tilde{V}_K(\mathbb{R}^{1,1}) = \tilde{V}^\blacktriangle \setminus \tilde{V}^\vartriangle$. The differentials in the relative cellular chain complex of $(\tilde{V}^\blacktriangle, \tilde{V}^\vartriangle)$ can now be calculated simplicially: for each commutative ring $R$, the relative cellular chain complex of $(\tilde{V}^\blacktriangle, \tilde{V}^\vartriangle)$ with coefficients in $R$ is given by $C^{\mathrm{cell}}_{|K|+s}(\tilde{V}^\blacktriangle, \tilde{V}^\vartriangle; R) = R\langle \Pi_{K,s}\rangle$ for $1 \leqslant s \leqslant r$, no further cells, and with differentials

$$\partial\pi = \sum_{\alpha=1}^{s-1} (-1)^\alpha \cdot \sum_{\imath\colon (m_\alpha, m_{\alpha+1})} \mathrm{sg}(\imath) \cdot d_{\alpha,\imath}\pi.$$

**Remark 1.4.9.** By Poincaré–Lefschetz duality, we get isomorphisms

$$H_\bullet(\tilde{V}^<_K(\mathbb{R}^{1,1})) \cong H^{|K|+r-\bullet}(\tilde{V}^\infty, \infty) \cong H^{|K|+r-\bullet}(\tilde{V}^\blacktriangle, \tilde{V}^\vartriangle).$$

This has some immediate consequences:

1. The homology $H_\bullet(\tilde{V}_K(\mathbb{R}^{1,1}))$ is of finite type and $\tilde{V}_K(\mathbb{R}^{1,1})$ has no non-trivial homology groups above degree $r-1$. This coincides with Theorem 1.2.5 for the case $(p,q) = (1,1)$.

2. The Euler characteristic of $\tilde{V}_K(\mathbb{R}^{1,1})$ can be calculated by

$$\begin{aligned}\chi(\tilde{V}_K(\mathbb{R}^{1,1})) &= \sum_{s=1}^{r}(-1)^{r-s}\cdot \#\Pi_{K,s} \\ &= \sum_{s=1}^{r}(-1)^{r-s}\cdot \sum_{\substack{c\colon \underline{r}\to\underline{s} \\ \text{surjective}}} \frac{(c_*K)_1!\cdots(c_*K)_s!}{k_1!\cdots k_r!}.\end{aligned}$$

Since Theorem 1.2.5 already gives a full description for the homology of $\tilde{V}_K(\mathbb{R}^{1,1})$, there is no need for a more explicit inspection of the cellular chain complex *at this point*. However, the efficient ray filtration from Section 1.2 behaves poorly with respect to the operadic structure of Chapter 4, in particular with respect to a permutation of clusters. We will therefore come back to this cellular chain complex later and simplify it by using methods from discrete Morse theory.

A similar story can be told for the unordered vertical configuration spaces: if $r(k) \geqslant 0$ denotes, for each $k \geqslant 1$, the number of occurrences of $k$ in the tuple $K = (k_1, \dots, k_r)$, then the action of $\mathfrak{S}_K$ on $\tilde{V}_K(\mathbb{R}^{1,1})$ from Definition 1.1.3 restricts to an action of $\mathfrak{S}^K := \prod_{k\geqslant 1}\mathfrak{S}_{r(k)}$ on $\tilde{V}_K^{<}(\mathbb{R}^{1,1})$ and in both cases, the quotient is given by the unordered vertical configuration space $V_K(\mathbb{R}^{1,1})$.

**Remark 1.4.10.** The action of $\mathfrak{S}^K$ on $\tilde{V}^{<}$ which exchanges clusters of the same size extends to a cellular action on $\tilde{V}^{\blacktriangle}$ which preserves the subcomplex $\tilde{V}^{\vartriangle}$ of degenerates. On cells, the action is given by $\tau\cdot(M,\pi) = (M, \tau\circ\pi)$. Thus, we get a relative multisimplicial complex $(V^{\blacktriangle}, V^{\vartriangle})$ with $V^{\blacktriangle}\smallsetminus V^{\vartriangle} = V_K(\mathbb{R}^{1,1})$, and the multisimplices are indexed in $\Pi_{K,s}/\mathfrak{S}^K$. Likewise, we get for the cellular chain complex $C_\bullet^{\mathrm{cell}}(V^{\blacktriangle}, V^{\vartriangle}) = C_\bullet^{\mathrm{cell}}(\tilde{V}^{\blacktriangle}, \tilde{V}^{\vartriangle})\otimes_{\mathfrak{S}^K} R$.

However, explicit homology calculations by means of this cellular complex are complicated: firstly, recall that $V_K(\mathbb{R}^{1,1})$ is not always orientable, compare Remark 1.1.7, and secondly, even if we work over $\mathbb{F}_2$, a parity trick similar to the one in [Fuc70] is not quite to be expected, as the combinatorics of the differentials are much more involved. However, the Euler characteristic is readily calculated as

$$\chi(V_K(\mathbb{R}^{1,1})) = \left(\prod_{k\geqslant 1}\frac{1}{r(k)!}\right)\cdot\sum_{s=1}^{r}(-1)^{r-s}\cdot\sum_{\substack{c\colon \underline{r}\to\underline{s} \\ \text{surjective}}}\frac{(c_*K)_1!\cdots(c_*K)_s!}{k_1!\cdots k_r!}.$$

Additionally, it follows immediately that $H_h(V_K(\mathbb{R}^{1,1});\mathbb{Z})$ vanishes for $h \geqslant r$ and is torsion-free for $h = r-1$, and we can calculate the first stable homology of $V_r^k(\mathbb{R}^{1,1})$ for each $k \geqslant 1$ as follows:

**Corollary 1.4.11.** *For each number $r \geqslant 2$ of clusters, each cluster size $k \geqslant 1$, and each a commutative ring $R$, we have*

$$H_1(V_r^k(\mathbb{R}^{1,1});R) \cong R^{\frac{1}{2}\cdot\binom{2k}{k}}.$$

*Proof.* By the Stability theorem 1.3.3, it is sufficient to show the statement for $r = 2$. Here we note that the cell complex of $(V^\infty, \infty)$ has only cells in dimension $2k+1$ and $2k+2$. As there is only a single $(2k+2)$-cell and since $H^{2k+2}(V^\infty,\infty;\mathcal{O}) \cong H_0(V_2^k(\mathbb{R}^{1,1})) \cong R$ holds for the orientation system $\mathcal{O}$, the twisted relative cellular cochain complex $C^\bullet_{\mathrm{cell}}(V^\infty,\infty;\mathcal{O})$ has to be formal. Hence $H_1(V_2^k(\mathbb{R}^{1,1}))$ is freely generated by all cells of codimension 1, and there are precisely $\#\Pi_{(k,k),1}/\mathfrak{S}_2 = \frac{1}{2}\cdot\binom{2k}{k}$ many of them. ☐

**Example 1.4.12.** We can entirely calculate the integral homology of $V_r^k(\mathbb{R}^{1,1})$ for arbitrary $k \geqslant 1$ and $r \in \{1,2,3\}$: in all three cases, it is free abelian, and the Betti numbers are as in Table 1.1. The last entry in this table comes from the fact that we know all other Betti numbers and the Euler characteristic.

| | 0 | 1 | 2 |
|---|---|---|---|
| $V_1^k$ | 1 | | |
| $V_2^k$ | 1 | $\frac{1}{2}\cdot\binom{2k}{k}$ | |
| $V_3^k$ | 1 | $\frac{1}{2}\cdot\binom{2k}{k}$ | $\frac{1}{2}\cdot\binom{2k}{k}\cdot\left(\frac{1}{3}\cdot\binom{3k}{k}-1\right)$ |

**Table 1.1.** The integral/rational Betti numbers of $V_r^k(\mathbb{R}^{1,1})$ for $r \in \{1,2,3\}$

**Theorem 1.4.13.** *For $K = (k_1,\dots,k_r)$, the spaces $\tilde{V}_K(\mathbb{R}^{1,1})$ and $V_K(\mathbb{R}^{1,1})$ are homotopy equivalent to $(r-1)$-dimensional cell complexes.*

*Proof.* We want to use from [Mun84, Lem. 70.1] the following geometric version of Poincaré–Lefschetz duality: let $(X^\blacktriangle, X^\vartriangle)$ be a finite relative simplicial complex such that $X := X^\blacktriangle \smallsetminus X^\vartriangle$ is an open $m$-dimensional manifold;

consider the barycentric subdivison of $X^{\blacktriangle}$ and let $X^{\#}$ be the union of all flags $\Phi = |\sigma_0 < \cdots < \sigma_\kappa| \subseteq X^{\blacktriangle}$ satisfying $\Phi \cap X^{\vartriangle} = \varnothing$. Then the inclusion $X^{\#} \hookrightarrow X$ is a deformation retract, and, on the other hand, $X^{\#}$ itself carries the structure of a finite cell complex: for each $\mu$-dimensional cell $\sigma$ which does not belong to $X^{\vartriangle}$, we define the *dual cell* $D(\sigma)$ to be the union of all flags which start with $\sigma$. If we define $X^{\#}_{\mu} \subseteq X^{\#}$ to be the union of all dual cells $D(\sigma)$ for $\sigma$ of dimension at least $m - \mu$, then the chain of inclusions $\varnothing = X^{\#}_{-1} \subseteq X^{\#}_{0} \subseteq \cdots \subseteq X^{\#}_{m} = X^{\#}$ is a cellular filtration, i.e. $X^{\#}_{\mu}$ arises from $X^{\#}_{\mu-1}$ by attaching (dual) cells of dimension $\mu$, one for each simplex $\sigma$ of dimension $m - \mu$ which does not belong to $X^{\vartriangle}$.

Exactly the same can be done for the multisimplicial complex $(\tilde{V}^{\blacktriangle}, \tilde{V}^{\vartriangle})$ after we have subdivided the multisimplices into simplices: note that in $\tilde{V}^{\blacktriangle}$, the attaching maps for each multisimplex are injective on vertices, so after cutting the multisimplices into simplices, we obtain an honest finite simplicial complex. Even after the subdivision, there are no cells outside $\tilde{V}^{\vartriangle}$ which are of dimension smaller than $|K| + 1$, and hence, there are no dual cells inside $\tilde{V}^{\#}$ of dimension larger than $r - 1$.

Finally, the $\mathfrak{S}^K$-action on $\tilde{V}^{<}_{K}(\mathbb{R}^{1,1})$ restricts to a cellular action on the dual complex $\tilde{V}^{\#}$, which implies that the inclusion $\tilde{V}^{\#} \subseteq \tilde{V}^{<}_{K}(\mathbb{R}^{1,1})$ is $\mathfrak{S}^K$-equivariant and we obtain an induced map $\tilde{V}^{\#}/\mathfrak{S}^K \hookrightarrow V_K(\mathbb{R}^{1,1})$. Since the action is free on the component $\tilde{V}^{<}_{K}(\mathbb{R}^{1,1})$, its restriction to $\tilde{V}^{\#}$ is also free, so the quotient actually calculates the homotopy quotient on both sides. Hence, the induced map is still a homotopy equivalence. As the action on $\tilde{V}^{\#}$ was cellular, we get an induced cellular structure on $\tilde{V}^{\#}/\mathfrak{S}^K$ which lifts to the previous one. This proves the claim. $\square$

## 1.5. Homotopy groups

In this last section, we study the homotopy groups of vertical configuration spaces, extending the works of [Her14; Rös14; Lat17]. Since $\tilde{V}_K(\mathbb{R}^{p,q})$ is a covering of $V_K(\mathbb{R}^{p,q})$, we have $\pi_\bullet(V_K(\mathbb{R}^{p,q})) \cong \pi_\bullet(\tilde{V}_K(\mathbb{R}^{p,q}))$ for $\bullet \geqslant 2$.

We also would like to relate the fundamental groups of $\tilde{V}_K(\mathbb{R}^{p,q})$ to Artin's *pure braid groups* $\mathrm{PBr}_r$, see Appendix A for a short reminder on them. Clearly, $\pi_1(V_K(\mathbb{R}^{p,q}))$ is an extension of $\mathfrak{S}_K$ (or of the stabilising subgroup of the corresponding path component) by $\pi_1\big(\tilde{V}_K(\mathbb{R}^{p,q})\big)$, so we will focus on the fundamental groups of $\tilde{V}_K(\mathbb{R}^{p,q})$. In all dimensions apart from $(p,q)=(1,1)$, these are easy to understand, as the following statement shows:

**Proposition 1.5.1.** *For a tuple $K$ as before and $(p,q)\neq(1,1)$, the fundamental groups of $\tilde{V}_K(\mathbb{R}^{p,q})$ are as follows:*

$$\pi_1\big(\tilde{V}_K(\mathbb{R}^{p,q})\big)\cong\begin{cases}0 & \text{for } q\geqslant 3 \text{ or } (q=1 \text{ and } p\geqslant 2),\\ \prod_i \mathrm{PBr}_{k_i} & \text{for } q=2 \text{ and } p\geqslant 1,\\ \mathrm{PBr}_{|K|} & \text{for } q=2 \text{ and } p=0.\end{cases}$$

*Proof.* Recall that $\tilde{V}_K(\mathbb{R}^{p,q})$ can be regarded as a subspace of $\mathbb{R}^{p\cdot r}\times\mathbb{R}^{q\cdot|K|}$ which is the complement of a polyhedral subcomplex, which we call the *collision subcomplex*. Now we can use in the fact that if we remove from a smooth manifold without boundary a subcomplex of codimension at least 3, we do not change the fundamental group, compare [God71, Thm. 2.3].

Suppose first $q\geqslant 3$: in this case, the collision subcomplex is contained in the product of $\mathbb{R}^{p\cdot r}$ and the 'fat diagonal' of $(\mathbb{R}^q)^{|K|}$; hence it has codimension at least $q$ and we are done. If instead $q=1$, then we can restrict to the open, convex subspace of $\mathbb{R}^{p\cdot r}\times\mathbb{R}^{|K|}$ containing all points $(\zeta_1,\dots,\zeta_r;t_{1,1},\dots,t_{r,k_r})$ with $t_{i,\jmath}<t_{i,\jmath+1}$, as this subspace contains one of the mutually homeomorphic path components of $\tilde{V}_K(\mathbb{R}^{p,q})$. This already excludes collisions of points from the same cluster, and the subspace is of the form $\mathbb{R}^{p\cdot r}\times\Delta$, for an open, convex subset $\Delta$ of $\mathbb{R}^{|K|}$. We obtain a path component of $\tilde{V}_K(\mathbb{R}^{p,1})$ if we additionally exclude collisions of points from *different* clusters. However, each condition that two points from different clusters collide ties together $p+1\geqslant 3$ coordinates, whence the collision subcomplex is again of codimension $\geqslant 3$.

For $q=2$, we consider the subspace of $\mathbb{R}^{p\cdot r}\times\mathbb{C}^{|K|}$ which contains points $(\zeta_1,\dots,\zeta_r;w_{1,1},\dots,w_{r,k_r})$ with $w_{i,\jmath}\neq w_{i,\jmath'}$ for all $1\leqslant i\leqslant r$ and $\jmath\neq\jmath'$. This space is homeomorphic to $\mathbb{R}^{p\cdot r}\times\prod_i\tilde{C}_{k_i}(\mathbb{C})$, so its fundamental group is identified with $\prod_i\mathrm{PBr}_{k_i}$. In order to describe $\tilde{V}_K(\mathbb{R}^{p,2})$, we additionally have to exclude collisions of points from *different* clusters as before. Here we again tie together $p+2$ coordinates, so the collision subcomplex is again of codimension $\geqslant 3$ for $p\geqslant 1$. If instead $p=0$, then we have no verticality condition and get $\tilde{V}_K(\mathbb{R}^{0,2})\cong\tilde{C}_{|K|}(\mathbb{R}^2)$. □

In contrast to this, the fundamental groups of $\tilde{V}_K(\mathbb{R}^{1,1})$ are more complicated: first of all, let us restrict ourselves again to the path component $\tilde{V}_K^{<}(\mathbb{R}^{1,1})$ which consists of configurations $(\zeta_1, \dots, \zeta_r; t_{1,1}, \dots, t_{r,k_r})$ with $t_{i,\jmath} < t_{i,\jmath+1}$. We fix a standard configuration in this path component as a basepoint and still write $\pi_1\big(\tilde{V}_K(\mathbb{R}^{1,1})\big)$ for the fundamental group (which only sees the component $\tilde{V}_K^{<}(\mathbb{R}^{1,1})$). For the case $K = (2, \dots, 2)$, these groups have been given the name *double braid groups* in [Böd90b, § 5.5].

Consider the *relaxation map* $\eta\colon \tilde{V}_K^{<}(\mathbb{R}^{1,1}) \hookrightarrow \tilde{V}_K(\mathbb{R}^{0,2}) = \tilde{C}_{|K|}(\mathbb{R}^2)$ which forgets the verticality constraint and induces a map among fundamental groups $\eta_*\colon \pi_1\big(\tilde{V}_K(\mathbb{R}^{1,1})\big) \to \mathrm{PBr}_{|K|}$. Our first aim is to understand the image of this map. Intuitively, points from the same cluster cannot spin around each other, so we hit only pure braids whose restrictions to a single block become trivial. The next proposition claims that we hit all of them.

More precisely, recall the tableau $\mathbb{Y}_K := \{(i,\jmath);\, 1 \leqslant i \leqslant r \text{ and } 1 \leqslant \jmath \leqslant k_i\}$ from Definition 1.2.1 and consider the flattening bijection

$$\Phi\colon \mathbb{Y}_K \to \{1, \dots, |K|\}, \quad (i,\jmath) \mapsto k_1 + \dots + k_{i-1} + \jmath.$$

We denote its inverse as a pair of maps $(i_K, \jmath_K)$: in particular, the constituent $i_K\colon \{1, \dots, |K|\} \to \{1, \dots, r\}$ assigns to $1 \leqslant u \leqslant |K|$ the unique $1 \leqslant i \leqslant r$ such that $u = k_1 + \dots + k_{i-1} + \jmath$ holds for some $1 \leqslant \jmath \leqslant k_i$.

**Proposition 1.5.2.** *For each $1 \leqslant i \leqslant r$ consider the projection $p_i\colon \mathrm{PBr}_{|K|} \to \mathrm{PBr}_{k_i}$ which remembers only the strands $\Phi(i,1), \dots, \Phi(i,k_i)$. Then*

$$\eta_*\Big(\pi_1\big(\tilde{V}_K(\mathbb{R}^{1,1})\big)\Big) = \bigcap_{i=1}^{r} \ker(p_i).$$

The proof of Proposition 1.5.2 relies on a structure result for pure braid groups which, to the best of my knowledge, does not appear in the literature: recall that the pure braid group $\mathrm{PBr}_{|K|}$ is generated by pure braids $\alpha_{u,v}$ as shown in Figure 1.10, for each $1 \leqslant u < v \leqslant |K|$. An explicit presentation with these braids $\alpha_{u,v}$ as generators can be found in Appendix A. Note that if $i_K(u) \neq i_K(v)$, then its $i^{\text{th}}$ projection $p_i(\alpha_{u,v})$ is trivial for each $1 \leqslant i \leqslant r$.

**Proposition 1.5.3.** *The intersection of kernels $\bigcap_{i=1}^{r} \ker(p_i) \subseteq \mathrm{PBr}_{|K|}$ is generated by all elementary pure braids $\alpha_{u,v}$ with $i_K(u) \neq i_K(v)$.*

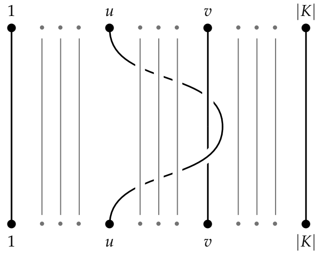

**Figure 1.10.** The generator $\alpha_{u,v}$ inside $\mathrm{PBr}_{|K|}$.

Before proving Proposition 1.5.3, let us prove Proposition 1.5.2 under the assumption of Proposition 1.5.3:

*Proof of Proposition 1.5.2.* The inclusion '$\subseteq$' is easy: the map $p_i$ of groups is induced by the topological map $\mathrm{pr}_i^C \colon \tilde{C}_{|K|}(\mathbb{R}^2) \to \tilde{C}_{k_i}(\mathbb{R}^2)$ which forgets all points apart from the ones which belong to the $i^{\text{th}}$ block. On the other hand, we have a map $\mathrm{pr}_i^V \colon \tilde{V}_K^{<}(\mathbb{R}^{1,1}) \to \tilde{V}_{k_i}^{<}(\mathbb{R}^{1,1})$ which forgets all clusters apart from the $i^{\text{th}}$ one, and we obtain diagram

$$\begin{tikzcd}
\tilde{V}_K^{<}(\mathbb{R}^{1,1}) \arrow[r, "\eta^K"] \arrow[d, "\mathrm{pr}_i^V"'] & \tilde{C}_{|K|}(\mathbb{R}^2) \arrow[d, "\mathrm{pr}_i^C"] \\
\tilde{V}_{k_i}^{<}(\mathbb{R}^{1,1}) \arrow[r, "\eta^{k_i}"'] & \tilde{C}_{k_i}(\mathbb{R}^2).
\end{tikzcd}$$

However, the bottom left corner $\tilde{V}_{k_i}^{<}(\mathbb{R}^{1,1})$ is contractible.

By using Proposition 1.5.3, the inclusion '$\supseteq$' can be shown by constructing, for each $1 \leqslant u < v \leqslant |K|$ with $i_K(u) \neq i_K(v)$, a loop $[0;1] \to \tilde{V}_K^{<}(\mathbb{R}^{1,1})$ which gets sent to $\alpha_{u,v}$ along $\eta$. Here is one possibility to do so: choose as basepoint the tuple $* = (\zeta_1, \dots, \zeta_r; t_{1,1}, \dots, t_{r,k_r}) = (0, \dots, 0; 1, 2, \dots, k_1 + \cdots + k_r)$, i.e. all clusters have the same first coordinate, namely 0, and the $i^{\text{th}}$ cluster stays completely below the $(i+1)^{\text{st}}$ one. Then we take $\eta(*) \in \tilde{C}_{|K|}(\mathbb{R}^2)$ to be the basepoint on the right side, and the braids arise by looking 'from the right'.

For $1 \leqslant u < v \leqslant |K|$, let $(i, \jmath) := (i_K, \jmath_K)(u)$ and $(i', \jmath') := (i_K, \jmath_K)(v)$. Since $i_K(u) \neq i_K(v)$, we have $i < i'$. Next, we construct a loop $\gamma$ which consists of seven paths $\gamma_1, \dots, \gamma_7$ and is depicted in Figure 1.11: intuitively, it moves the $i^{\text{th}}$ cluster to the background, then moves 'upper part' of the $i^{\text{th}}$ cluster, starting with the $\jmath^{\text{th}}$ point, upwards, lets the $\Phi(i,\jmath)^{\text{th}}$ strand spin around the $\Phi(i',\jmath')^{\text{th}}$ strand, and goes back to the basepoint. More formally, the seven paths can be described as follows:

1. $\gamma_1$ moves $\zeta_i$ from $0$ to $-1$;
2. $\gamma_2$ moves $(t_{i,\ell})_{\jmath+1\leqslant\ell\leqslant k_i}$ to $(|K|-\jmath+\ell)_{\jmath+1\leqslant\ell\leqslant k_i}$ and $t_{i,\jmath}$ to $\Phi(i',\jmath')+\frac{1}{2}$;
3. $\gamma_3$ moves $\zeta_i$ from $-1$ to $1$;
4. $\gamma_4$ moves $t_{i,\jmath}$ from $\Phi(i',\jmath')+\frac{1}{2}$ to $\Phi(i',\jmath')-\frac{1}{2}$;
5. $\gamma_5$ moves $\zeta_i$ from $1$ to $-1$;
6. $\gamma_6$ moves $(t_{i,\jmath})_{\jmath\leqslant\ell\leqslant k_i}$ back to $(\Phi(i,\ell))_{\jmath\leqslant\ell\leqslant k_i}$;
7. $\gamma_7$ moves $\zeta_i$ from $-1$ to $0$.

One can easily see that $\eta_*[\gamma]$ is not quite $\alpha_{u,v}$ if $\jmath\neq k_i$, but its conjugate by another element in this image is: in a similar fashion we can construct a loop $\delta\colon[0;1]\to\tilde{V}^{<}_{K}(\mathbb{R}^{1,1})$ which is depicted in Figure 1.12: intuitively, it lets the 'remaining part' of the $i^{\text{th}}$ cluster, starting with the $(\jmath+1)^{\text{st}}$ point, spin around everything which lies above it. More formally, it consists of five paths which can be described as follows:

1. $\delta_1$ moves $\zeta_i$ from $0$ to $1$;
2. $\delta_2$ moves $(t_{i,\ell})_{\jmath+1\leqslant\ell\leqslant k_i}$ to $(|K|-\jmath+\ell)_{\jmath+1\leqslant\ell\leqslant k_i}$;
3. $\delta_3$ moves $\zeta_i$ from $1$ to $-1$;
4. $\delta_4$ moves $(t_{i,\ell})_{\jmath+1\leqslant\ell\leqslant k_i}$ back to $(\Phi(i,\ell))_{\jmath+1\leqslant\ell\leqslant k_i}$;
5. $\delta_5$ moves $\zeta_i$ from $-1$ to $0$.

Then $\eta_*\big([\delta]\cdot[\gamma]\cdot[\delta]^{-1}\big)=\alpha_{u,v}$, as Figure 1.13 shows. □

In order to prove Proposition 1.5.3, we need the following lemma:

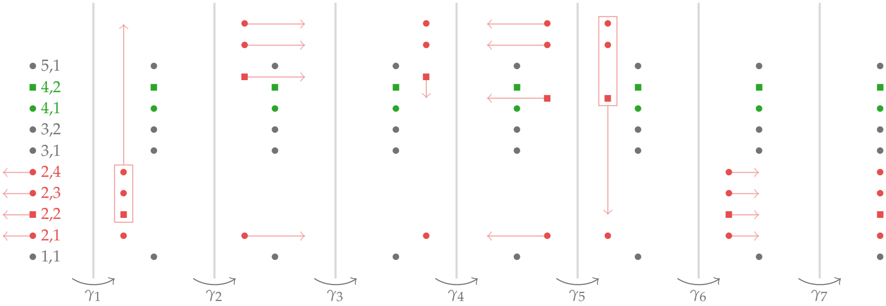

**Figure 1.11.** The different stages of the loop $\gamma$. Here we have $K = (1, 4, 2, 2, 1)$ and $(i, j) = (2, 2)$ while $(i', j') = (4, 2)$, so $u = 3$ and $v = 9$.

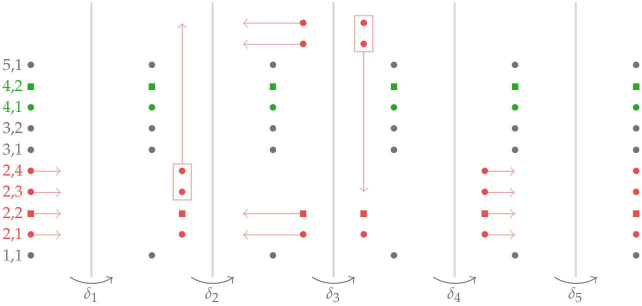

**Figure 1.12.** The different stages of the loop $\delta$ for the same situation.

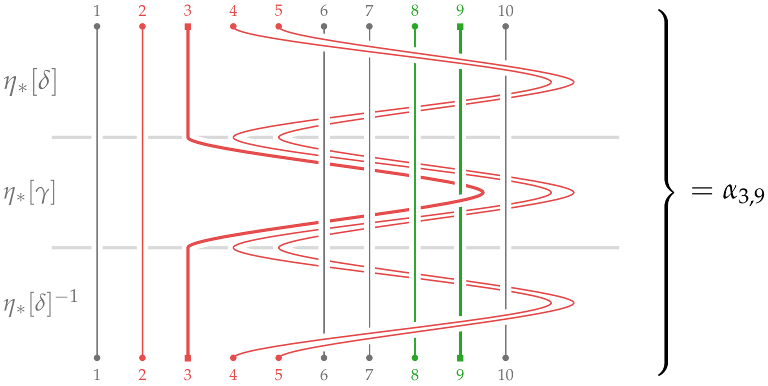

**Figure 1.13.** The conjugate $\eta_*[\delta]^{-1} \cdot \eta_*[\gamma] \cdot \eta_*[\delta]$ is the desired generator $\alpha_{u,v}$, here for $u = 3$ and $v = 9$.

**Lemma 1.5.4.** *Let $G$ be a group, $N \subseteq G$ be a normal subgroup, and let $H \subseteq G$ be another subgroup. Moreover, assume that:*

1. *the projection $p\colon G \to G/N$ admits a section $s\colon G/N \to G$ of groups;*
2. *there is a generating set $X \subseteq G$ such that $x \cdot s(p(x))^{-1} \in H$ for all $x \in X$;*
3. *there is a subset $Y \subseteq X$ such that:*
    - *$Y \subseteq s(G/N)$ is a generating set for $s(G/N)$;*
    - *for each $y \in Y$ and each $h \in H$, we have $yhy^{-1}, y^{-1}hy \in H$.*

*Then the normal subgroup $N$ is contained in $H$.*

The main advantage of Lemma 1.5.4 is that for condition 2, we only have to check $x \cdot s(p(x))^{-1} \in H$ for a generating set. As the proof of Lemma 1.5.4 requires some combinatorial group theory, let us first prove Proposition 1.5.3 and then develop the setting in which we can prove Lemma 1.5.4.

*Proof of Proposition 1.5.3.* The problem is already very close to the situation of the lemma: if we define $G := \mathrm{PBr}_{|K|}$ and let $p := \prod_i p_i\colon \mathrm{PBr}_{|K|} \to \prod_i \mathrm{PBr}_{k_i}$ as well as $N := \ker(p)$ and $H := \langle \alpha_{u,v};\, i_K(u) \neq i_K(v) \rangle \subseteq G$, then we know already that $H \subseteq N$ holds, and we want to show that $N \subseteq H$ holds.

We have a section $s\colon \prod_i \mathrm{PBr}_{k_i} \to \mathrm{PBr}_{|K|}$ by forming block sums of pure braids. As generating set, we take $X := \{\alpha_{u,v}\}_{1 \leqslant u < v \leqslant |K|}$, and, accordingly,

$$Y := \big\{\alpha_{u,v};\, \Phi(i,1) \leqslant u < v \leqslant \Phi(i,k_i) \text{ for some } i\big\} \subseteq X.$$

Then $Y$ generates the image $s\big(\prod_i \mathrm{PBr}_{k_i}\big) \subseteq \mathrm{PBr}_{|K|}$ of the $s$; we only have to show that, additionally, $\alpha_{u,v} \cdot s(p(\alpha_{u,v}))^{-1} \in H$ holds for all $1 \leqslant u < v \leqslant |K|$, and that $\alpha_{u,v} \cdot h \cdot \alpha_{u,v}^{-1}, \alpha_{u,v}^{-1} \cdot h \cdot \alpha_{u,v} \in H$ holds for all $h \in H$ and $i_K(u) = i_K(v)$.

The first subclaim is easy: if $i_K(u) = i_K(v)$, then $s(p(\alpha_{u,v})) = \alpha_{u,v}$, so we obtain that $\alpha_{u,v} \cdot s(p(\alpha_{u,v}))^{-1} = 1$. If, however, $i_K(u) \neq i_K(v)$, then we have $p(\alpha_{u,v}) = 1$ and obtain $\alpha_{u,v} \cdot s(p(\alpha_{u,v}))^{-1} = \alpha_{u,v} \in H$.

For the second part, we have to show that the conjugate $\alpha_{u',v'} \cdot \alpha_{u,v} \cdot \alpha_{u',v'}^{-1}$ lies in $H$ for each $i_K(u) \neq i_K(v)$ and $i := i_K(u') = i_K(v')$: then the mirrored statement for the second way of conjugating follows analogously, and for the general case, recall that each $h \in H$ is a product of elements of the form

$\alpha^{\pm 1}_{u,v}$. In order to understand the above conjugate, we first have to understand which values $u, v, u', v'$ can attain. Note that among these four indices, there can be at most one equality: either $u = u'$ or $v = v'$ or $u = v'$ or $u' = v$, but as soon as two of them hold, the cluster condition is violated. This leaves us with the following nine possible cases:

1. $u < v < u' < v'$,
2. $u < u' < v' < v$,
3. $u' < v' < u < v$,
4. $u = u' < v' < v$,
5. $u < u' < v' = v$,
6. $u' < v' = u < v$,
7. $u < v = u' < v'$,
8. $u' < u < v' < v$,
9. $u < u' < v < v'$.

Now one verifies in all these cases the following elementary relations among generators by using the presentation from Appendix A. Here '$\star$' abbreviates the inverted first factor of the term, i.e. $y^{-1} \cdot h \cdot (\star) = y^{-1} \cdot h \cdot y$:

$$\alpha_{u',v'} \cdot \alpha_{u,v} \cdot \alpha^{-1}_{u',v'} = \begin{cases} \alpha_{u,v} & \text{for 1, 2, and 3,} \\ \alpha^{-1}_{v',v} \cdot \alpha_{u,v} \cdot (\star) & \text{for 4,} \\ \alpha^{-1}_{u,u'} \cdot \alpha_{u,v} \cdot (\star) & \text{for 5,} \\ \alpha^{-1}_{u',v} \cdot \alpha_{u,v} \cdot (\star) & \text{for 6,} \\ \left(\alpha_{u,v'} \cdot \alpha_{u,v}\right)^{-1} \cdot \alpha_{u,v} \cdot (\star) & \text{for 7,} \\ \left(\alpha^{-1}_{v',v} \cdot \alpha^{-1}_{u',v} \cdot \alpha_{v',v} \cdot \alpha_{u',v}\right)^{-1} \cdot \alpha_{u,v} \cdot (\star) & \text{for 8,} \\ \left(\alpha^{-1}_{u,v'} \cdot \alpha^{-1}_{u,u'} \cdot \alpha_{u,v'} \cdot \alpha_{u,u'}\right)^{-1} \cdot \alpha_{u,v} \cdot (\star) & \text{for 9.} \end{cases}$$

The important point of these identities is that in each case, the right side contains only products of $\alpha_{\bar{u},\bar{v}}$ where $i_K(\bar{u}) \neq i_K(\bar{v})$; for example **8**: Since $u' < u < v'$, we have $i_K(u') = i_K(u) = i_K(u') < i_K(u)$, and thus, all three generators $\alpha_{u,v}, \alpha_{u',v}, \alpha_{v',v}$ are contained in $H$. This concludes the proof. □

To prove Lemma 1.5.4, we need the notion of a Schreier transversal:

**Definition 1.5.5.** Let $F$ be a free group and $M \subseteq F$ be a subgroup. A subset $T \subseteq F$ is called a *Schreier transversal of* $M$ if the following conditions hold:

S1. for each $t, t' \in T$ with $t \neq t'$ we have $Mt \neq Mt'$,

S2. we have $\bigcup_{t \in T} Mt = F$, and

S3. for each $f, f' \in F$ with $f \cdot f' \in T$, we have $f \in T$.

Note that for a given Schreier transversal and for each $f \in F$, there is a unique element $\bar{f} \in F$ such that $\bar{f} \in T$ and $Mf = M\bar{f}$, and we have $f \cdot \bar{f}^{-1} \in M$.

Using the notation of a Schreier transversal, we can formulate a special case of the Reidemeister–Schreier method, which is proven in [LS01, Prop. II.4.1] in much higher generality; we are only interested in free groups and only care about generators:

**Theorem 1.5.6** (Reidemeister–Schreier method, special case)**.** *Let $F := F(X)$ be the free group over a set $X$, let $M \subseteq F$ be a subgroup, and $T \subseteq F$ be a Schreier transversal for $M$. Then $M$ is generated by elements of the form $(tx) \cdot (\overline{tx})^{-1}$.*

Now we can finally come to the proof of Lemma 1.5.4:

*Proof of Lemma 1.5.4.* We denote by $F(X)$ the free group generated by $X$ and, correspondingly, by $F(Y) \subseteq F(X)$ the free group generated by $Y$, and denote the projections by $q_X \colon F(X) \to G$ and $q_Y \colon F(Y) \to s(G/N)$, respectively.

Consider the Cayley graph for the subgroup $s(G/N)$ with generating set $Y^{\pm} := \{y^{\pm 1}; y \in Y\}$ and choose a spanning tree. Then we obtain a map *of sets* $a \colon s(G/N) \to F(Y)$ which assigns to each $g \in s(G/N)$ the product along the unique $Y^{\pm}$-labelled path inside the spanning tree from 1 to $g$, and we clearly have $q_Y \circ a = \mathrm{id}_{s(G/N)}$.

We let $\tilde{N} := q_X^{-1}(N) \subseteq F(X)$ and claim that $T := a(s(G/N)) \subseteq F(Y)$ is a Schreier transversal for $\tilde{N}$: clearly, S3 is satisfied by construction, as we took products along a spanning tree. For S1, let $t, t' \in T$ be distinct and let $z := q_Y(t)$ as well as $z' := q_Y(t')$. Then $z \neq z'$, and since $N \cap s(G/N) = \{1\}$, we get $Nz \neq Nz'$. This shows that $\tilde{N}t = q_X^{-1}(Nz) \neq q_X^{-1}(Nz') = \tilde{N}t'$. Finally, we verify S2 by

$$\bigcup_{t \in T} \tilde{N}t = \bigcup_{z \in s(G/N)} q_X^{-1}(Nz) = q_X^{-1}\Big(\bigcup_{z \in s(G/N)} Nz\Big) = q_X^{-1}(G) = F.$$

The Reidemeister–Schreier method tells us that $\tilde{N}$ is generated by elements of the form $(tx) \cdot (\overline{tx})^{-1}$ with $t \in T$ and $x \in X$, and thus, $N$ is generated by elements of the form $q_Y(t) \cdot x \cdot q_Y(\overline{tx})^{-1}$. To understand $q_Y(\overline{tx})$, note that $Nq_Y(\overline{tx}) = Nq_Y(t)x$ holds by definition, so since $q_Y(\overline{tx}) \in s(G/N)$, we get $q_Y(\overline{tx}) = q_Y(t) \cdot s(p(x))$. Hence, if we put $z := q_Y(t)$, then $N$ is generated by products $z \cdot x \cdot s(p(x))^{-1} \cdot z^{-1}$ for $z \in s(G/N)$ and $x \in X$.

Now we can apply our assumptions to conclude that these generators lie in $H$: each $z \in s(G/N)$ is of the form $y_1^{\pm 1} \cdots y_m^{\pm 1}$ for some minimal $m$, and by induction on this $m$, we show that $z \cdot x \cdot s(p(x))^{-1} \cdot z^{-1} \in H$: if $m = 0$, then $z = 1$ and assumption 2 can directly be applied. For the induction step '$m - 1 \to m$', we can write $z = y^{\pm 1} \cdot z_0$ where $z_0$ can be built out of $m - 1$ generators from $Y$, and

$$z \cdot x \cdot s(p(x))^{-1} \cdot z^{-1} = y^{\pm 1} \cdot \left( z_0 \cdot x \cdot s(p(x))^{-1} \cdot z_0^{-1} \right) \cdot y^{\mp 1}.$$

By the induction hypothesis, the inner term lies in $H$, and by assumption 3, its conjugate by $y$ enjoys this property as well. □

To summarise, we understood the image of $\eta_* \colon \pi_1(\tilde{V}_K(\mathbb{R}^{1,1})) \to \mathrm{PBr}_{|K|}$. However, the kernel of $\eta_*$ can be quite large; here is an example which shows non-triviality:

**Example 1.5.7.** Consider the loop $\gamma \colon [0;1] \to \tilde{V}_{2,2}(\mathbb{R}^{1,1})$ which is depicted at the left side of Figure 1.14: it consists of four paths, which are painted in red, green, blue, and yellow. Its image $\eta \circ \gamma$, which is drawn at the right side of Figure 1.14 from the perspective 'bottom right', is a trivial pure braid in $\mathrm{PBr}_4$.

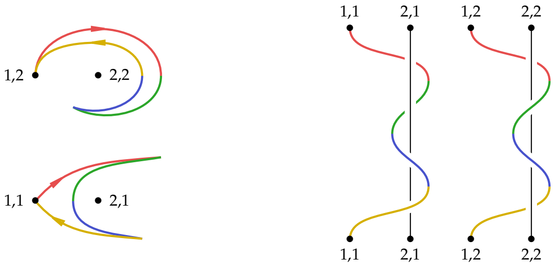

**Figure 1.14.** The loop $\gamma$ in $\tilde{V}_{2,2}(\mathbb{R}^{1,1})$ and its image $\eta \circ \gamma$: it is trivial in $\mathrm{PBr}_4$.

However, $[\gamma]$ is non-trivial in $\tilde{V}_{2,2}(\mathbb{R}^{1,1})$: intuitively, each contraction of one of the two curves would force the other one to touch a point from the second cluster. A rigorous argument can be given by considering the first homology of $\tilde{V}_{2,2}(\mathbb{R}^{1,1})$: if we use the cellular decomposition of $\tilde{V}^\infty$ from the previous section, then each intersection of $\gamma$ with a cell of codimension 1 is transverse, and $\gamma$ touches no cells of lower dimension (as there are none, apart from $\infty$). Thus, under Poincaré–Lefschetz duality, the 5-dimensional cohomology class dual to the 1-dimensional homology class $[\gamma]$ is represented by a (signed) sum of all cells which $\gamma$ crosses, i.e. $(1\,2\,1\,2) - (2\,1\,1\,2) + (2\,1\,2\,1) - (1\,2\,2\,1)$. This cocycle is not a coboundary in the cellular chain complex $C^\bullet_{\mathrm{cell}}(\tilde{V}^\infty, \infty)$ as there are no cells of dimension 4, and so $[\gamma] \neq 0$.

Let us close this section with a short discussion on the higher homotopy groups of $\tilde{V}_K(\mathbb{R}^{1,1})$, or, equivalently, of $V_K(\mathbb{R}^{1,1})$.

**Remark 1.5.8.** The ordered configuration spaces $\tilde{C}_r(\mathbb{R}^2)$ of the plane are aspherical by an application of the Fadell–Neuwirth fibrations: we have a fibre bundle $\mathbb{R}^2 \smallsetminus \{x_1, \ldots, x_{r-1}\} \to \tilde{C}_r(\mathbb{R}^2) \to \tilde{C}_{r-1}(\mathbb{R}^2)$ by forgetting the last point, and now we use that $\tilde{C}_0(\mathbb{R}^2)$ is just a point and hence aspherical, and the fibres are homotopy equivalent to a bouquet of circles, in order to prove asphericity inductively.

However, the natural candidates for Fadell–Neuwirth maps for the vertical configuration spaces of type $\mathbb{R}^{1,1}$, namely

$$\tilde{V}_{k_1,\ldots,k_r}(\mathbb{R}^{1,1}) \to \tilde{V}_{k_1,\ldots,k_{r-1}}(\mathbb{R}^{1,1}),$$
$$\tilde{V}_{k_1,\ldots,k_r}(\mathbb{R}^{1,1}) \to \tilde{V}_{k_1,\ldots,k_r-1}(\mathbb{R}^{1,1}),$$

which either forget the last cluster or forget a single point from the last cluster, fail to be fibrations if $k_r \geqslant 2$, as the bachelors' theses by Rösner [Rös14] and Herberz [Her14] show. Therefore, the problem of asphericity for the spaces $\tilde{V}_K(\mathbb{R}^{1,1})$, which was claimed for clusters of size 2 in [Böd90b, Prop. 5.1.6], remains an intricate question.

So far, we can only repeat what has already been observed in the master's thesis of Latifi [Lat17]: one can easily see that forgetting a cluster of size 1 is indeed a fibration, with fibre a perforated plane. In combination with the fact that, according to Theorem 1.4.13, the space $V_{k,k'}(\mathbb{R}^{1,1})$ is homotopy equivalent to a graph, we can conclude inductively that all vertical configuration spaces of the form $V_{k,k',1,\ldots,1}(\mathbb{R}^{1,1})$ are aspherical.

Therefore, the smallest example for which we do not yet know its higher homotopy groups is the vertical configuration space $V_{2,2,2}(\mathbb{R}^{1,1})$.

I would like to close this chapter with a list of open questions which we encountered, and which the motivated reader may see as a challenge:

1. Calculate the homology of the unordered vertical configuration spaces $V_K(\mathbb{R}^{1,1})$ with coefficients in $\mathbb{F}_2$ and compare the result to the calculations of [Fuc70].

2. Show that the spectral sequence associated with the dexterity filtration $F_\bullet V_r$ from Subsection 1.3.2 collapses on the first page and that the extension problems are trivial. Conclude that the homology of $V_r(\mathbb{R}^{p,1})$ splits into the modules $M_{h,\alpha}$ from Outlook 1.3.17.

3. Give a description of the kernel $N$ of $\pi_1\big(\tilde{V}_K(\mathbb{R}^{1,1})\big) \to \mathrm{PBr}_{|K|}$ as a group and understand the extension problem
$$1 \to N \to \pi_1\big(\tilde{V}_K(\mathbb{R}^{1,1})\big) \to \textstyle\bigcap_i \ker(p_i) \to 1.$$

4. Prove or disprove that all spaces $\tilde{V}_K(\mathbb{R}^{1,1})$ are aspherical. As they are finite-dimensional, this would imply that $\pi_1\big(\tilde{V}_K(\mathbb{R}^{1,1})\big)$ is torsion-free.

Chapter 2

# Clustered configuration spaces as $E_d$-algebras

> *Ubi materia, ibi geometria.*[1]
>
> Johannes Kepler

After having considered ordered and unordered vertical configuration spaces in the previous chapter, we now want to turn our attention to *labelled* vertical configuration spaces $V(\mathbb{R}^{p,q};\boldsymbol{X})$ for some decomposition $d = p + q$. These spaces naturally generalise the classical notation of labelled configuration spaces [Seg73; McD75; Sna74] to the clustered and vertical case.

Our interest in labelled vertical configuration spaces has two reasons: firstly, they model free algebras over the vertical operads which we study in the later chapters, and secondly, the spaces $V(\mathbb{R}^{p,q};\boldsymbol{X})$ are instances of $E_d$-algebras which, roughly speaking, are 'close enough' to being free, whence we can study their iterated bar construction [May72] by classical scanning techniques [Seg73] and by recent $E_d$-cellular methods from [GKR18; GKR19].

These results, in turn, can be used to describe the stable homology of unordered vertical configuration spaces, as we carry out in the last section. This chapter appeared as the preprint [Kra21].

## 2.1. Labelled clusters and a stable splitting result

In this section, we introduce a labelled version $V(E;\boldsymbol{X})$ of the vertical configuration spaces for each manifold bundle $E \to B$, and prove a stable splitting result in the spirit of [Sna74; Böd87] for them.

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[1] *Where there is matter, there is geometry.*

### 2.1.1. *Labelled vertical configuration spaces*

In a first step, we want to define the labelled vertical configuration spaces in analogy to the configuration spaces $C(M; X)$ of points in a manifold $M$ with labels in a based space $X$, as considered in [Seg73; McD75; Böd87].

One natural generalisation is to assign to each cluster a label and to balance the internal ordering of each cluster with a given symmetric action on the labelling space. Therefore, instead of a single based space, we start with a *sequence* $\boldsymbol{X} = (X_n)_{n\geqslant 1}$ of based spaces together with basepoint-preserving left actions of $\mathfrak{S}_n$ on $X_n$, called a *based symmetric sequence*.

To make the definition precise, we introduce an indexing category, which is a special case of the Grothendieck construction, generalises the notion of a wreath product $G \wr \mathfrak{S}_r = G^r \rtimes \mathfrak{S}_r$, and which helps us in several situations to encode the bulk of equivariancies and balancing relations.

Since we will later use a version which involves topological groups as well, let us give a definition entirely in a topologically enriched setting.

**Notation 2.1.1.** Let **Inj** be the category whose objects are non-negative integers $r \geqslant 0$ and whose morphisms are all injections $\underline{r} \hookrightarrow \underline{r}'$, and let $\boldsymbol{\Sigma} \subseteq \mathbf{Inj}$ be the subgroupoid of all bijections.

Then **Inj** is generated[2] by all maps from $\boldsymbol{\Sigma}$, i.e. all permutations $\sigma\colon \underline{r} \to \underline{r}$, and *cofaces* $d^i\colon \underline{r-1} \to \underline{r}$ for each $1 \leqslant i \leqslant r$, given by the unique monotone injective map omitting the $i^{\text{th}}$ element from $\underline{r}$.

**Definition 2.1.2.** Let $N$ be a indexing set and let $\boldsymbol{G} := (G_n)_{n\in N}$ be a family of topological groups. We define the *wreath product* $\boldsymbol{G} \wr \mathbf{Inj}$ as the following small and topologically enriched category:

1. the objects of $\boldsymbol{G} \wr \mathbf{Inj}$ are tuples $K = (k_1, \ldots, k_r)$ with $r \geqslant 0$ and $k_i \in N$;
2. for two tuples $K$ and $K'$, we define $(\boldsymbol{G} \wr \mathbf{Inj})\binom{K}{K'} \subseteq \boldsymbol{G}^K \times \mathbf{Inj}\binom{r}{r'}$ as the subspace of pairs $w = (\gamma, u)$ consisting of a tuple $\gamma \in \boldsymbol{G}^K$, and of an injective map $u\colon \underline{r} \hookrightarrow \underline{r}'$ satisfying $K = u^*K' := (k'_{u(1)}, \ldots, k'_{u(r)})$;
3. we let $(\gamma', u') \circ (\gamma, u) := (u^*\gamma' \cdot \gamma, u' \circ u)$, where $u^*\gamma' = (\gamma'_{u(1)}, \ldots, \gamma'_{u(r)})$, and '$\cdot$' denotes component-wise multiplication.

---

[2] It is enough to consider permutations and the *top* cofaces $d^r\colon \underline{r-1} \to \underline{r}$ to generate **Inj**.

For each tuple $K$, we define $\boldsymbol{G}[K] \subseteq \boldsymbol{G} \wr \mathbf{Inj}$ as the full subcategory spanned by objects of the form $\tau^* K$ for $\tau \in \mathfrak{S}_r$. Moreover we let $\boldsymbol{G} \wr \boldsymbol{\Sigma}$ be the subgroupoid with morphism spaces given by $\boldsymbol{G}(K) \times \boldsymbol{\Sigma}\binom{K}{K'}$.

If $(G_n)_{n \in N}$ is the trivial sequence $G_n = 1$ of groups, then we write $N \wr \mathbf{Inj}$ for the wreath product, and we similarly write $N \wr \boldsymbol{\Sigma}$ and $N[K]$.

**Construction 2.1.3.** If $\boldsymbol{X} = (X_n)_{n \in N}$ is a family of spaces, then we obtain a functor $\boldsymbol{X}^- \colon N \wr \boldsymbol{\Sigma} \to \mathbf{V}$ with $\boldsymbol{X}^K := X_{k_1} \times \cdots \times X_{k_r}$. If each $X_n$ is based, then we get an extension $N \wr \mathbf{Inj} \to \mathbf{Top}$ by $u_*(x_1, \ldots, x_r) := (x_{u^{-1}(1)}, \ldots, x_{u^{-1}(r')})$, where we define $x_\varnothing$ to be the basepoint.

Finally, if $\boldsymbol{G} = (G_n)_{n \in N}$ is a family of topological groups and if, in addition, each $X_n$ carries a basepoint-preserving left action of $G_n$, then $\boldsymbol{X}$ can be extended to a functor $\boldsymbol{G} \wr \mathbf{Inj} \to \mathbf{Top}$ as follows: since $\boldsymbol{G}^K$ acts on $\boldsymbol{X}^K$ component-wise, we can define $(\gamma, u)_*(\boldsymbol{x}) := u_*(\gamma \cdot \boldsymbol{x})$.

**Definition 2.1.4.** Consider the sequence $\mathfrak{S} := (\mathfrak{S}_n)_{n \geqslant 1}$ of symmetric groups. For each manifold bundle $E \to B$, the family of spaces $\tilde{V}_K(E)$ constitutes a functor $(\mathfrak{S} \wr \mathbf{Inj})^{\mathrm{op}} \to \mathbf{Top}$ by declaring that for each $1 \leqslant i \leqslant r$ the $i^{\text{th}}$ face map $d_i \colon \tilde{V}_K(E) \to \tilde{V}_{d_i K}(E)$ forgets the $i^{\text{th}}$ cluster. If we are additionally given a based symmetric sequence $\boldsymbol{X} = (X_n)_{n \geqslant 1}$, we define $V(E; \boldsymbol{X})$ to be the coend

$$\begin{aligned} V(E; \boldsymbol{X}) &:= \int^{K \in \mathfrak{S} \wr \mathbf{Inj}} \tilde{V}_K(E) \times \boldsymbol{X}^K \\ &= \operatorname{coeq}\left( \coprod_{K, K'} \tilde{V}_{K'}(E) \times (\mathfrak{S} \wr \mathbf{Inj})\tbinom{K}{K'} \times \boldsymbol{X}^K \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_K V_K(E) \times \boldsymbol{X}^K \right), \end{aligned}$$

where $\alpha(\boldsymbol{z}, w, \boldsymbol{x}) = (w^* \boldsymbol{z}, \boldsymbol{x})$ and $\beta(\boldsymbol{z}, w, \boldsymbol{x}) = (\boldsymbol{z}, w_* \boldsymbol{x})$.

We denote elements in $\tilde{V}_K(E) \times \boldsymbol{X}^K$ as tuples $\Theta = (\vec{z}_1, \ldots, \vec{z}_r, x_1, \ldots, x_r)$ and use the sum notation to denote $[\Theta] = \sum_i \vec{z}_i \otimes x_i$, where $\vec{z}_i = (z_{i,1}, \ldots, z_{i,k_i})$ is a cluster, $x_i \in X_{k_i}$, and $(\sigma^* \vec{z}_i) \otimes x_i = \vec{z}_i \otimes (\sigma \cdot x_i)$ for $\sigma \in \mathfrak{S}_{k_i}$.

**Remark 2.1.5.** For each indexing set $N$ and each integer $r \geqslant 0$, we have a right action of the symmetric group $\mathfrak{S}_r$ on $N^r$ by coordinate permutation, and we denote the orbit of a tuple $K \in N^r$ by $[K]$.

For a tuple $K = (k_1, \ldots, k_r)$ of positive integers $k_i \geqslant 1$ and a permutation $\tau \in \mathfrak{S}_r$, we have a canonical isomorphism $V_K(E) \cong V_{\tau^* K}(E)$, whence $V_K(E)$ depends, up to *canonical* isomorphism, only on $[K]$.

**Example 2.1.6.** It is perhaps surprising how many different variations of these spaces can be produced by a suitable choice of the labelling sequence:

1. If all $X_k$ carry a trivial $\mathfrak{S}_k$-action, then $V(E;\boldsymbol{X})$ contains unordered collections of labelled and internally unordered clusters. If $\boldsymbol{X} = \underline{\mathbb{S}}^0$, i.e. $X_k = \mathbb{S}^0$, endowed with the trivial $\mathfrak{S}_k$-action for each $k \geqslant 1$, then
$$V(E;\underline{\mathbb{S}}^0) \cong \coprod_{[K]} V_K(E).$$

2. More generally, if $\mathfrak{G} := (\mathfrak{G}_k)_{k\geqslant 1}$ is a sequence of subgroups $\mathfrak{G}_k \subseteq \mathfrak{S}_k$ and if $\boldsymbol{X} = (X_k)_{k\geqslant 1}$ is a based $\mathfrak{G}$-sequence, then we can consider the *induction* $(\mathfrak{S}_+ \wedge_{\mathfrak{G}} \boldsymbol{X})_k := (\mathfrak{S}_k)_+ \wedge_{\mathfrak{G}_k} X_k$, where $(\mathfrak{S}_k)_+ := \mathfrak{S}_k \sqcup \{*\}$. For $X = \underline{\mathbb{S}}^0$, endowed with trivial $\mathfrak{G}$-actions, we obtain
$$V\big(E;\mathfrak{S} \wedge_{\mathfrak{G}} \underline{\mathbb{S}}^0\big) \cong \coprod_{[K]} \tilde{V}_K(E)/\mathrm{Aut}_{\mathfrak{G}\wr\mathbf{Inj}}(K).$$
In particular, the space $V(E;\mathfrak{S}_+ \wedge \underline{\mathbb{S}}^0)$ contains unordered collections of unlabelled, but internally *ordered* vertical clusters.

3. For a family $\boldsymbol{X} = (X_k)_{k\geqslant 1}$ of based spaces, we let $(\boldsymbol{X}^{\wedge})_k = X_k^{\wedge k}$, with $\mathfrak{S}_k$ acting by coordinate permutation. Then $V(E;\boldsymbol{X}^{\wedge})$ contains configurations of clusters where *each* point inside a $k$-cluster carries a label in $X_k$; and if one of these labels reaches the basepoint, the cluster vanishes.

4. For $k \geqslant 1$ and a based space $X$ with based $\mathfrak{S}_k$-action let $X[k] := (X_l)_{l\geqslant 1}$ be the sequence concentrated in degree $k$, i.e. $X_k := X$ and $X_l := *$ else. Then the space $V(E;X[k])$ contains only configurations where all clusters have size $k$. In particular,
$$V(E;\mathbb{S}^0[k]) \cong \coprod_{r\geqslant 0} V_r^k(E).$$
We will occasionally use the short notation $V^k(E;X) := V(E;X[k])$.

**Notation 2.1.7.** If the base space $B$ is just a point, then the verticality condition is again empty and $V(E;\boldsymbol{X})$ is the space of labelled clusters in $E$.

In this case, we also write $C(E;\boldsymbol{X})$, and for a based space $X$, we abbreviate $C^k(E;X) := C(E;X[k])$ as we did for $V$. Note that $C^1(E;X)$ is the same as $C(E;X)$, the classical configuration spaces of points in $E$, with labels in $X$.

**Definition 2.1.8** (Well-basedness)**.** We will sometimes assume that the labelling sequence $\boldsymbol{X} = (X_k)_{k \geqslant 1}$ is *equivariantly well-based*, i.e. the basepoint inclusion $* \hookrightarrow X_k$ is a $\mathfrak{S}_k$-cofibration for each $k \geqslant 1$ as in [Die79, § 8].

### 2.1.2. *The stable splitting*

Having constructed the spaces $V(E; \boldsymbol{X})$ for a manifold bundle $E$ and a based symmetric sequence $\boldsymbol{X}$, we want to filter these spaces by the number of clusters and prove a stable splitting result as in [Sna74; Böd87].

**Definition 2.1.9.** For two tuples $K'$ and $K$, we say that $K' \leqslant K$ holds if there is an injective map $u \colon \underline{r'} \hookrightarrow \underline{r}$ such that $K' = u^*K$. For a fixed tuple $K$, we let $V_K(E; \boldsymbol{X}) \subseteq V(E; \boldsymbol{X})$ be the subspace of all configurations of at most $K$ clusters. More precisely, we have a quotient map

$$\coprod_{\text{tuples } K'} \tilde{V}_{K'}(E) \times \boldsymbol{X}^{K'} \to V(E; \boldsymbol{X}),$$

and we let $V_K(E; \boldsymbol{X})$ be the union of all images of $\tilde{V}_{K'}(E) \times \boldsymbol{X}^{K'}$ with $K' \leqslant K$. For each $K' \leqslant K$, we have inclusions $V_{K'}(E; \boldsymbol{X}) \hookrightarrow V_K(E; \boldsymbol{X})$, which are cofibrations if $X$ is equivariantly well-based. We define the *filtration quotient*

$$D_K(E; \boldsymbol{X}) := \frac{V_K(E; \boldsymbol{X})}{\bigcup_{K' < K} V_{K'}(E; \boldsymbol{X})} = \frac{V_K(E; \boldsymbol{X})}{\bigcup_{i=1}^{r} V_{d_i K}(E; \boldsymbol{X})}.$$

Clearly, the filtration component $V_K(E; \boldsymbol{X}) \subseteq V(E; \boldsymbol{X})$ depends only on the unordered collection $[K]$, and hence also $D_K(E; \boldsymbol{X})$ only depends on $[K]$.

If $X_{k_i} = *$ for some $1 \leqslant i \leqslant r$, then $V_{d_i K}(E; \boldsymbol{X}) = V_K(E; \boldsymbol{X})$ and $D_K(E; \boldsymbol{X}) = *$. As in [BCT89, § 2.6], the homology of these filtration quotients can be related to unlabelled vertical configuration spaces with fixed cluster sizes:

**Proposition 2.1.10.** *Let $K = (k_1, \dots, k_r)$ and, for each $1 \leqslant i \leqslant r$, let $X_{k_i}$ be a sphere $\mathbb{S}^{d_{k_i}}$ for some $d_{k_i} \geqslant 0$, endowed with the trivial $\mathfrak{S}_{k_i}$-action. Then we have*

$$\tilde{H}_\bullet(D_K(E; \boldsymbol{X})) \cong H_{\bullet - (d_{k_1} + \cdots + d_{k_r})}(V_K(E); \mathcal{O}_K),$$

*where $\mathcal{O}_K$ is an orientation system of the following form*

$$\begin{tikzcd}
\pi_1(V_K(E)) \arrow[r] & \mathfrak{S}_K \arrow[r] & \{\pm 1\}. \\
& \prod_k(\sigma_{k,1}, \dots, \sigma_{k,r(k)}; \tau_k) \arrow[r, mapsto] & \prod_k \mathrm{sg}(\tau_k)^{d_k}.
\end{tikzcd}$$

*Proof.* We have a vector bundle over $V_K(E)$ given by

$$R := \left( \tilde{V}_K(E) \Big/ \mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_r} \right) \times_{\prod_k \mathfrak{S}_{r(k)}} \prod_{k \geqslant 1} (\mathbb{R}^{d_k})^{r(k)},$$

and its Thom space $\mathrm{Th}(R)$ is isomorphic to the quotient $D_K(E; \boldsymbol{X})$ by the same argument as in [Mil72, Thm. 1.3.2]. Moreover, $R$ has rank $\sum_k r(k) \cdot d_k = \sum_i d_{k_i}$ and the associated orientation bundle is exactly the bundle assigned to $\mathcal{O}_K$. Hence the result is an instance of the (twisted) Thom isomorphism. □

In two special cases, we know that the orientation system is trivial: if all spheres are of even dimension, or if the monodromy $\pi_1(V_K(E)) \to \mathfrak{S}_K$ attains only even permutations $\tau_k$. The latter is e.g. the case if $E = \mathbb{R}^{0,1}$, as each component of $V_K(\mathbb{R}^{0,1})$ is contractible.

The method of proof shows that it is possible to additionally incorporate orthogonal representations $\mathfrak{S}_k \to \mathrm{O}(d_k) \hookrightarrow \mathrm{Aut}(\mathbb{S}^{d_k}, *)$ which would then yield a further factor $\prod_k \prod_i \det((\sigma_{k,i})_*)$ for the orientation system.

As in [Sna74, Thm. 1.1] and [Böd87, § 3], we can show that the aforementioned filtration stably splits as a bouquet of filtration quotients:

**Theorem 2.1.11** (Splitting theorem). *Let $\boldsymbol{X} := (X_k)_{k \geqslant 1}$ be a sequence of based spaces with a left action of $\mathfrak{S}_k$ on $X_k$, such that each basepoint is equivariantly non-degenerate. Then there is a stable equivalence of suspension spectra*

$$\Sigma^\infty V(E; \boldsymbol{X}) \to \Sigma^\infty \textstyle\bigvee_{[K] \neq \varnothing} D_K(E; \boldsymbol{X}).$$

*Proof.* We start by fixing some notation: let us abbreviate $D_K := D_K(E; \boldsymbol{X})$ and $W := \bigvee_K D_K$. Recall moreover the classical spaces $C(M; Y)$ of configurations of points in a space $M$ with labels in a based space $Y$; we use also here the suggestive sum notation $\sum_i \zeta_i \otimes y_i$ for $\zeta_i \in M$ and $y_i \in Y$.

Now we adapt the strategy of [Böd87, § 3] and define a 'power set map' $P \colon V(E; \boldsymbol{X}) \to C(\mathbb{R}^\infty; W)$: to do so, we note that all $V_K(E)$ are smooth manifolds and pick an embedding $\iota \colon \coprod_K V_K(E) \hookrightarrow \mathbb{R}^\infty$ by Whitney's embedding theorem. Next, we construct maps $\tilde{P}_K \colon \tilde{V}_K(E) \times \boldsymbol{X}^K \to C(\mathbb{R}^\infty; W)$ as follows: for each tuple $\Theta = (\vec{z}_1, \ldots, \vec{z}_r, x_1, \ldots, x_r) \in \tilde{V}_K(E) \times \boldsymbol{X}^K$ and each non-empty indexing subset $T \subseteq \{1, \ldots, r\}$, we define

$$[\Theta^T] := \sum_{i \in T} [\vec{z}_i] \in V_{K|_T}(E) \quad \text{and} \quad [\Theta_T] := \sum_{i \in T} \vec{z}_i \otimes x_i \in V_{K|_T}(E; \boldsymbol{X}),$$

where $K|_T$ arises from $K$ by removing all entries indexed outside $T$. Recall that $[\vec{z}_i]$ does not know its internal ordering any more, whereas in $\vec{z}_i \otimes x_i$, the ordering is balanced with the label. Secondly, since $[\Theta^T] \neq [\Theta^{T'}]$ for $T \neq T'$, the tuple of subconfigurations $(\imath[\Theta^T])_T$ lives in $\tilde{C}_{2^r-1}(\mathbb{R}^\infty)$. Now consider the map $\jmath\colon V_{K|_T}(E;\boldsymbol{X}) \to D_{K|_T} \subseteq W$ and set

$$\tilde{P}_K(\Theta) := \sum_{T \neq \varnothing} \imath[\Theta^T] \otimes \jmath[\Theta_T] \in C(\mathbb{R}^\infty; W).$$

Finally, the maps $\tilde{P}_K$ are continuous and $\coprod_K \tilde{P}_K$ factors over $V(E;\boldsymbol{X})$ since we sum over all non-empty subsets $T$, and since $\jmath[\Theta_T]$ is the basepoint if there is an $i \in T$ such that $x_i = *$. Thus, we get the desired map $P$.

We have a map $\omega\colon C(\mathbb{R}^\infty; W) \to \Omega^\infty\Sigma^\infty W$ from [Seg73], and by passing to the homotopy adjoint, we obtain a stable map $\bar{\omega}\colon \Sigma^\infty C(\mathbb{R}^\infty; W) \to \Sigma^\infty W$. By precomposing with the power set map $P$, we reach a stable morphism $\bar{\omega} \circ \Sigma^\infty P\colon \Sigma^\infty V(E;\boldsymbol{X}) \to \Sigma^\infty W$, and we claim that it is a stable equivalence. To this aim, note that the map $P$ is filtered in the following *coarse* way: for each $r \geqslant 0$, we let $V_r(E;\boldsymbol{X}) := \bigcup_{\#[K] \leqslant r} V_K(E;\boldsymbol{X})$ and $W_r := \bigvee_{\#[K] \leqslant r} D_K$. Then $P$ sends $V_r(E;\boldsymbol{X})$ to $C(\mathbb{R}^\infty; W_r)$, so we get $\bar{\omega} \circ \Sigma^\infty P_r\colon \Sigma^\infty V_r(E;\boldsymbol{X}) \to \Sigma^\infty W_r$.

We show by induction on $r$ that each $\bar{\omega} \circ \Sigma^\infty P_r$ is a stable equivalence, which yields the statement since stable homotopy groups commute with filtered colimits. For $r = 0$, the statement is clear, and for the induction step '$r-1 \to r$', we note that $V_{r-1} \hookrightarrow V_r$ and $W_{r-1} \hookrightarrow W_r$ are cofibrations, and that we have a diagram of cofibre sequences

$$\begin{tikzcd}
\Sigma^\infty V_{r-1}(E;\boldsymbol{X}) \arrow[r] \arrow[d, "\bar{\omega}\circ\Sigma^\infty P_{r-1}"'] & \Sigma^\infty V_r(E;\boldsymbol{X}) \arrow[r] \arrow[d, "\bar{\omega}\circ\Sigma^\infty P_r"'] & \Sigma^\infty \bigvee_{\#[K]=r} D_K(E;\boldsymbol{X}) \arrow[d, equal] \\
\Sigma^\infty W_{r-1} \arrow[r] & \Sigma^\infty W_r \arrow[r] & \Sigma^\infty(W_r/W_{r-1}),
\end{tikzcd}$$

in the homotopy category of spectra. Thus, the induction step follows by the five lemma applied to the long exact sequence of stable homotopy groups. □

**Example 2.1.12.** In the case $\boldsymbol{X} = X[k]$, the splitting theorem has an easier form: first of all, note that $D_K(E; X[k]) = *$ whenever $K \neq (k, \ldots, k)$, so we actually have only the filtration quotients $D_r^k(E;X) := D_{k,\ldots,k}(E; X[k])$ and obtain the more common form, namely a stable equivalence

$$\Sigma^\infty V^k(E;X) \to \Sigma^\infty \bigvee_{r\geqslant 1} D^k_r(E;X).$$

**Remark 2.1.13.** One can without any further effort consider a *relative* version $V(E,E';\boldsymbol{X})$ where $E' := \mathrm{pr}^{-1}(B')$ for a cofibration $B' \hookrightarrow B$. In the same way, we can define filtration components $V_K(E,E';\boldsymbol{X})$ and filtration quotients $D_K(E,E';\boldsymbol{X})$ to prove a corresponding splitting result which claims the existence of a stable equivalence

$$\Sigma^\infty V(E,E';\boldsymbol{X}) \to \Sigma^\infty \bigvee_{[K]} D_K(E,E';\boldsymbol{X}).$$

However, since we are not going to use it in the remainder of this thesis, we will not go into more detail.

## 2.2. Bar constructions of vertical configuration spaces

In this section, we study the spaces $V(\mathbb{R}^{p,q};\boldsymbol{X})$ as algebras over the operad $\mathscr{C}_d$ of little $d$-cubes for $d = p+q$, and we give a geometric model for its $p$-fold bar construction.

**Construction 2.2.1.** If $\boldsymbol{X} = (X_k)_{k\geqslant 1}$ is a sequence as before, then $V(\mathbb{R}^{p,q};\boldsymbol{X})$ admits the structure of a $\mathscr{C}_d$-algebra: recall that elements in $\mathscr{C}_d(r)$ are tuples $(c_1,\dots,c_r)$ of rectilinear embeddings $c_i\colon [0;1]^d \hookrightarrow [0;1]^d$ with pairwise disjoint image, see Example 3.1.7 for more details. If we additionally identify $\mathbb{R} \cong (0;1)$, and hence $\mathbb{R}^d \cong (0;1)^d$, then we can construct the desired structure maps $\lambda_r\colon \mathscr{C}_d(r) \times V(\mathbb{R}^{p,q};\boldsymbol{X})^r \to V(\mathbb{R}^{p,q};\boldsymbol{X})$ by

$$\lambda_r\left((c_1,\dots,c_r), \left(\sum_{a=1}^{s_i} \vec{z}_{i,a} \otimes x_{i,a}\right)_{1\leqslant i\leqslant r}\right) := \sum_{i=1}^{r}\sum_{a=1}^{s_i} c_i(\vec{z}_{i,a}) \otimes x_{i,a},$$

where for a cluster $\vec{z} = (z_1,\dots,z_k)$ and a rectilinear embedding $c\colon \mathbb{R}^d \hookrightarrow \mathbb{R}^d$, we write $c(\vec{z}) := (c(z_1),\dots,c(z_k))$. Here we see that $\mathrm{pr}(c(z_{\jmath})) = \mathrm{pr}(c(z_{\jmath'}))$ if $\mathrm{pr}(z_{\jmath}) = \mathrm{pr}(z_{\jmath'})$ holds, since $c$ is rectilinear and the identification $\mathbb{R}^d \cong (0;1)^d$ is defined coordinate-wise, whence the verticality constraint is preserved by the structure maps.

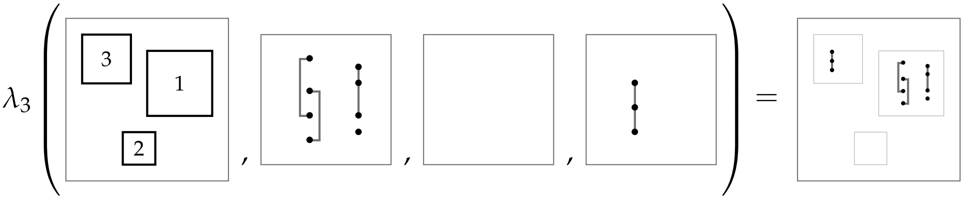

**Figure 2.1.** An instance of $\lambda_3\colon \mathscr{C}_2(3) \times V(\mathbb{R}^{1,1}; \underline{\mathbb{S}}^0)^3 \to V(\mathbb{R}^{1,1}; \underline{\mathbb{S}}^0)$.

The first $p$ steps of the iterated bar construction can be calculated by the following theorem, and they gradually 'resolve' the verticality constraint, whence we are left with a *clustered* configuration space without any verticality condition:

**Theorem 2.2.2.** *If $\boldsymbol{X}$ is levelwise equivariantly well-based, then there is a homotopy equivalence of $\mathscr{C}_q$-algebras*

$$B^p V(\mathbb{R}^{p,q}; \boldsymbol{X}) \simeq C(\mathbb{R}^q; \Sigma^p \boldsymbol{X}),$$

*where we write $(\Sigma^p \boldsymbol{X})_k := \Sigma^p X_k$, endowed with the induced $\mathfrak{S}_k$-actions.*

If $\boldsymbol{X}$ is levelwise path connected, then also $V(\mathbb{R}^{p,q}; \boldsymbol{X})$ is path connected and the theorem implies that we have an equivalence of $\mathscr{C}_d$-algebras,

$$V(\mathbb{R}^{p,q}; \boldsymbol{X}) \simeq \Omega^p C(\mathbb{R}^q; \Sigma^p \boldsymbol{X}).$$

The rest of this subsection aims to prove Theorem 2.2.2. As we follow the strategy of [Seg73], this proof is quite similarly organised: we emphasise the passages where the usage of clusters makes a difference, but we are short on those technicalities which go through without any modifications.

We proceed inductively: since there is nothing to show for $p = 0$, we can assume that $p \geqslant 1$. Then $V := V(\mathbb{R}^{p,q}; \boldsymbol{X})$ is in particular a $\mathscr{C}_1$-algebra and we will construct a weak equivalence

$$BV \simeq V(\mathbb{R}^{p-1,q}; \Sigma \boldsymbol{X})$$

of $\mathscr{C}_{p+q-1}$-algebras, which clearly implies the statement by induction on $p$.

**Definition 2.2.3.** Let $M$ be a partial topological monoid in the sense of [Seg73, Def. 2.2]. Then $M$ gives rise to a simplicial space $NM$, the *nerve*, with

$$(NM)_n := \{(m_1, \dots, m_n) \in M^n;\ m_1 \cdots m_n \text{ defined}\},$$

and face and degeneracy maps

$$\begin{aligned} d_l(m_1, \dots, m_n) &= \begin{cases} (m_2, \dots, m_n) & \text{for } l = 0, \\ (m_1, \dots, m_l \cdot m_{l+1}, \dots, m_n) & \text{for } 1 \leqslant l \leqslant n-1, \\ (m_1, \dots, m_{n-1}) & \text{for } l = n, \end{cases} \\ s_l(m_1, \dots, m_n) &= (m_1, \dots, m_l, 1, m_{l+1}, \dots, m_n). \end{aligned}$$

The *bar construction* $BM := |NM|$ is the geometric realisation of $NM$.

The following is the analogue of [Seg73, Prop. 2.3]:

**Lemma 2.2.4.** *Consider the (abelian) partial monoid $M := V(\mathbb{R}^{p-1,q}; \boldsymbol{X})$ where two labelled vertical configurations are summable if the clusters are disjoint, and in that case, their sum is the union. Then $BM \cong V(\mathbb{R}^{p-1,q}; \Sigma\boldsymbol{X})$ as $\mathscr{C}_{p+q-1}$-algebras.*

*Proof.* We define maps $\varphi_n \colon (NM)_n \times \Delta^n \to V(\mathbb{R}^{p-1,q}; \Sigma\boldsymbol{X})$ as follows: we use the simplex coordinates $\Delta^n = \{0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1\}$, so a generic point in $(NM)_n \times \Delta^n$ is of the form $(m_1, \dots, m_n, t_1, \dots, t_n)$ with $m_l = \sum_{h=1}^{r_l} \vec{z}_{l,h} \otimes x_{l,h}$, each $\vec{z}_{l,h}$ being a vertical cluster in $\mathbb{R}^{p-1,q}$. Then we set, using $\Sigma X_k = \mathbb{S}^1 \wedge X_k$ with the symmetric action $\sigma \cdot (t \wedge x) = t \wedge (\sigma x)$,

$$\varphi_n(m_1, \dots, m_n, t_1, \dots, t_n) := \sum_{l=1}^{n} \sum_{h=1}^{r_l} \vec{z}_{l,h} \otimes (t_l \wedge x_{l,h}).$$

Note that $\varphi_n$ is well-defined since $(m_1, \dots, m_n)$ is summable and the balancing relation is preserved. Moreover, one readily checks the identities

$$\begin{aligned} \varphi_{n-1}(d_l(m_1, \dots, m_n), t_1, \dots, t_{n-1}) &= \varphi_n(m_1, \dots, m_n, d^l(t_1, \dots, t_{n-1})), \\ \varphi_{n+1}(s_l(m_1, \dots, m_n), t_1, \dots, t_{n+1}) &= \varphi_n(m_1, \dots, m_n, s^l(t_1, \dots, t_{n+1})), \end{aligned}$$

so the maps $(\varphi_n)_{n \geqslant 0}$ can be glued together to a map $\varphi \colon BM \to V(\mathbb{R}^{p-1,q}; \Sigma\boldsymbol{X})$. Then $\varphi$ is a homeomorphism as it has a continuous inverse which can be constructed as

$$\begin{tikzcd}
\coprod_K \tilde{V}_K(\mathbb{R}^{p-1,q}) \times \boldsymbol{X}^K \times [0;1]^r \arrow[r] \arrow[d, two heads] & \coprod_r (NM)_r \times [0;1]^r \arrow[d, two heads] \\
V(\mathbb{R}^{p-1,q};\Sigma \boldsymbol{X}) \arrow[r, dashed] & BM,
\end{tikzcd}$$

where on the right side, the cube $[0;1]^r$ is subdivided into $r!$ many simplices $\Delta^r_\tau = \{0 \leqslant t_{\tau(1)} \leqslant \cdots \leqslant t_{\tau(r)} \leqslant 1\}$ for each $\tau \in \mathfrak{S}_r$, and the top horizontal map combines the map $(\vec{z}_1, \ldots, \vec{z}_r, x_1, \ldots, x_r) \mapsto (\vec{z}_1 \otimes x_1, \ldots, \vec{z}_r \otimes x_r)$ with the identity on $[0;1]^r$. Finally, it is a straightforward task to check that for each $\mu = (c_1, \ldots, c_r) \in \mathscr{C}_{p+q}(r)$, $(m_{1,i}, \ldots, m_{n,i}) \in (NM)_n$ for $1 \leqslant i \leqslant r$, and $0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1$, we have

$$\begin{aligned}
&\varphi_n(\lambda_r(\mu, m_{1,1}, \ldots, m_{1,r}), \ldots, \lambda_r(\mu, m_{n,1}, \ldots, m_{n,r}), t_1, \ldots, t_n) \\
&\qquad = \lambda_r(\mu, \varphi_n(m_{1,1}, \ldots, m_{n,1}, t_1, \ldots, t_n), \ldots, \varphi_n(m_{1,r}, \ldots, m_{n,r}, t_1, \ldots, t_n)),
\end{aligned}$$

so we indeed have an isomorphism of $\mathscr{C}_{p+q+1}$-algebras. □

**Construction 2.2.5.** An advantage of our inductive procedure is that we can describe the bar construction elementary: we first 'thicken' $V = V(\mathbb{R}^{p,q}; \boldsymbol{X})$ to a true topological monoid $\boldsymbol{V}$, and then take its classical bar construction.

This true monoid $\boldsymbol{V}$ is constructed as follows: each $\sum_i \vec{z}_i \otimes x_i \in V$ has a *support* $\bigcup_i [z_i] \subseteq \mathbb{R}^{p,q}$. Now consider the projection $\mathrm{pr}_1 \colon \mathbb{R}^{p,q} \to \mathbb{R}$ to the very first coordinate and define

$$\boldsymbol{V} := \Big\{ \big(t, \textstyle\sum_i \vec{z}_i \otimes x_i\big) \in \mathbb{R}_{\geqslant 0} \times V;\ \mathrm{pr}_1\big(\bigcup_i [z_i]\big) \subseteq (0;t) \Big\},$$

where $(0;t)$ is the open interval. On $\boldsymbol{V}$, we define the *Moore concatenation*

$$\Big(t, \textstyle\sum_i \vec{z}_i \otimes x_i\Big) \cdot \Big(t', \sum_i \vec{z}'_i \otimes x'_i\Big) := \Big(t + t', \sum_i \vec{z}_i \otimes x_i + \sum_i \mathrm{sh}_t(\vec{z}'_i) \otimes x'_i\Big),$$

where $\mathrm{sh}_t \colon \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ translates the first coordinate by $t$, here applied to an entire cluster. This construction coincides with the thickening from [Dun86, § 1] for general $\mathscr{C}_1$-algebras, whence $B\boldsymbol{V} = |N\boldsymbol{V}|$ serves as a model for $BV$.

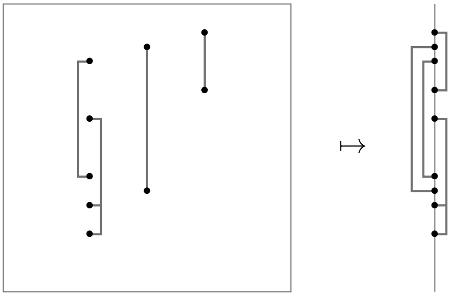

**Figure 2.2.** An instance of the map $\psi\colon \boldsymbol{V}^{\mathrm{proj}} \to M$

**Definition 2.2.6.** We have a second projection $\mathrm{pr}_2\colon \mathbb{R}^{p,q} \to \mathbb{R}^{p-1,q}$ and we call $[\Theta] = \sum_i \vec{z}_i \otimes x_i \in V(\mathbb{R}^{p,q}; \boldsymbol{X})$ *projectable* if the restriction

$$\mathrm{pr}_2|_{\bigcup_i [\vec{z}_i]}\colon \bigcup_i [\vec{z}_i] \to \mathbb{R}^{p-1,q}$$

is injective. Let $\boldsymbol{V}^{\mathrm{proj}} \subseteq \boldsymbol{V}$ be the subspace of pairs $(t, [\Theta])$ with projectable $[\Theta]$. Then $\boldsymbol{V}^{\mathrm{proj}}$ is a *partial* submonoid with respect to the concatenation, where two configurations can be multiplied if their product is again projectable.

Moreover, we have a morphism $\psi\colon \boldsymbol{V}^{\mathrm{proj}} \to M$ of partial monoids which is induced by the projection $\mathrm{pr}_2$, applied to entire clusters, see Figure 2.2,

$$\psi\Big(t, \textstyle\sum_i \vec{z}_i \otimes x_i\Big) := \textstyle\sum_i \mathrm{pr}_2(\vec{z}_i) \otimes x_i.$$

The projectability condition is exactly the one we need to ensure that all clusters $\mathrm{pr}_2(\vec{z}_i)$ are pairwise disjoint, and hence $\psi$ is well-defined. Moreover, $\boldsymbol{V}^{\mathrm{proj}}$ is also a $\mathscr{C}_{p+q-1}$-subalgebra and $\psi$ is a morphism of $\mathscr{C}_{p+q-1}$-algebras.

As in [Seg73], the maps $(N\psi)_n\colon (N\boldsymbol{V}^{\mathrm{proj}})_n \to (NM)_n$ between the spaces of summable tuples are homotopy equivalences and, since $\boldsymbol{X}$ was assumed to be equivariantly well-based, our simplicial spaces are proper, so we have a homotopy equivalence among the classifying spaces $B\psi\colon B\boldsymbol{V}^{\mathrm{proj}} \to BM$.

The following is the analogue of [Seg73, Prop. 2.4]:

**Lemma 2.2.7.** *The inclusion $\gamma\colon \boldsymbol{V}^{\mathrm{proj}} \hookrightarrow \boldsymbol{V}$ induces a homotopy equivalence of $\mathscr{C}_{p+q-1}$-algebras $B\gamma\colon B\boldsymbol{V}^{\mathrm{proj}} \to B\boldsymbol{V}$ among classifying spaces.*

**Remark 2.2.8.** Everything we did so far made no particular use of the fact that points from the same cluster project to the same value along $\mathrm{pr}_1$, and we might as well define $\boldsymbol{C}^{\mathrm{proj}}(\mathbb{R}^q;\boldsymbol{X})$ and a projection morphism as above, which now is of the form $\psi\colon \boldsymbol{C}^{\mathrm{proj}}(\mathbb{R}^q;\boldsymbol{X}) \to C(\mathbb{R}^{q-1};\boldsymbol{X})$ and still induces a homotopy equivalence of classifying spaces.

However, Lemma 2.2.7 fails to be true if we drop the condition '$p \geqslant 1$': we will highlight the stages of the proof where we need verticality, and we will give a precise counterexample later in Example 2.4.3.

Before proving Lemma 2.2.7, let us combine all arguments and prove the main result, Theorem 2.2.2, under the assumption of Lemma 2.2.7.

*Proof of Theorem 2.2.2.* We obtain a zig-zag of morphisms of $\mathscr{C}_{p+q-1}$-algebras whose underlying maps are homotopy equivalences, given by

$$\begin{tikzcd}
& B\boldsymbol{V}^{\mathrm{proj}} \arrow[dl, "B\gamma"', "\simeq"] \arrow[dr, "B\psi", "\simeq"'] & \\
BV(\mathbb{R}^{p,q};\boldsymbol{X}) = B\boldsymbol{V} & & BM \cong V(\mathbb{R}^{p-1,q};\Sigma\boldsymbol{X}).
\end{tikzcd}$$

□

*Proof of Lemma 2.2.7.* There is an equivalent description of $BM$ for a (partial) topological monoid $M$, see [Seg73, A1]: consider the category $M_{\#}$ with object space $M$, morphism space $(NM)_3$, domain assignment $d_1(m_1,m_2,m_3) = m_2$, and codomain assignment $d_0(m_1,m_2,m_3) = m_1 \cdot m_2 \cdot m_3$, i.e. arrows $m \to m'$ are pairs $(\underline{m},\overline{m}) \in M \times M$ with $\underline{m} \cdot m \cdot \overline{m} = m'$; then $BM \cong |M_{\#}|$.

We define the space $\mathcal{N}$ whose elements are triples $\nu = (a,b,[\Theta])$ in which $a \leqslant 0 \leqslant b$ and $[\Theta] = \sum_i \vec{z}_i \otimes x_i \in V(\mathbb{R}^{p,q};\boldsymbol{X})$ with $\mathrm{pr}_1(\bigcup_i [\vec{z}_i]) \subseteq (a;b)$. In order to give $\mathcal{N}$ a partial order, we introduce the following construction: for $L \subseteq \mathbb{R}$ and $[\Theta] = \sum_i \vec{z}_i \otimes x_i$ with $\mathrm{pr}_1[\vec{z}_i] \notin \partial L$ for all $1 \leqslant i \leqslant r$, we define

$$[\Theta] \cap \left(L \times \mathbb{R}^{p-1,q}\right) := \sum_{\mathrm{pr}_1[\vec{z}_i] \in L} \vec{z}_i \otimes x_i.$$

Then $\mathrm{pr}_1[\vec{z}_i] \notin \partial L$ ensures that the result projects to the interior of $L$. Here we use that $\mathrm{pr}_1[\vec{z}_i]$ is a single value in $\mathbb{R}$ since all points in $[\vec{z}_i]$ project to the same real number. Now we let $(a,b,[\Theta]) \leqslant (a',b',[\Theta'])$ in $\mathcal{N}$ whenever

- $[a;b] \subseteq [a';b']$, and
- $a,b \notin \mathrm{pr}_1(\bigcup_i [\vec{z}'_i])$ and $[\Theta] = [\Theta'] \cap \left([a;b] \times \mathbb{R}^{p-1,q}\right)$,

compare Figure 2.3. Then we can regard $\mathcal{N}$ as a topological category and obtain a functor $\rho\colon \mathcal{N} \to \boldsymbol{V}_\#$ by $\rho(a, b, [\Theta]) = (b - a, -a + [\Theta])$, where the term '$-a + [\Theta]$' denotes a translation by $a$ to the left in the first coordinate.

We can copy [Seg73, Lem. 2.6] verbatim to show that $|\rho|\colon |\mathcal{N}| \to |\boldsymbol{V}_\#|$ is shrinkable, i.e. it admits a section $s$ such that $s \circ |\rho| \simeq \mathrm{id}_{|\mathcal{N}|}$ by a homotopy which respects $|\rho|$. Let $\mathcal{M} \subseteq \mathcal{N}$ be the ordered subspace consisting of all triples $(a, b, [\Theta])$ with projectable $[\Theta]$. Then $\rho(\mathcal{M}) = \boldsymbol{V}_\#^{\mathrm{proj}}$ holds, whence it is enough to show that $|\mathcal{M}| \to |\mathcal{N}|$ is a homotopy equivalence. In order to do so, we use from [Seg73] the following tool:

Proposition 2.7. *Let $\mathcal{N}$ be a good*[3] *ordered space such that:*

- G1. *for each $v_1, v_2, v \in \mathcal{N}$ with $v_1, v_2 \leqslant v$ there exists* $\inf(v_1, v_2)$*;*
- G2. *wherever defined,* $(v_1, v_2) \mapsto \inf(v_1, v_2)$ *is continuous.*

*Moreover, let $\mathcal{N}' \subseteq \mathcal{N}$ be open such that:*

- G3. *for $v' \in \mathcal{N}'$ and $v \leqslant v'$, we have $v \in \mathcal{N}'$,*
- G4. *there is a numerable cover $(\mathcal{W}_i)_{i \in I}$ and maps $w_i\colon \mathcal{W}_i \to \mathcal{N}'$ such that $w_i(v) \leqslant v$ for all $v \in \mathcal{W}_i$.*

*Then the induced map $|\mathcal{N}'| \to |\mathcal{N}|$ is a homotopy equivalence.*

Since $\boldsymbol{X}$ is equivariantly well-based, there are contractible and $\mathfrak{S}_k$-invariant neighbourhoods $U_k \subseteq X_k$ around the respective basepoints $*_k$ and equivariant homotopies $h^k_\bullet\colon X_k \to X_k$ which move $U_k$ into $*_k$. If we set $\boldsymbol{U} := (U_k)_{k \geqslant 1}$, then $\boldsymbol{V}(\mathbb{R}^{p,q}; \boldsymbol{U})$ is a contractible neighbourhood around the empty configuration $[\varnothing]$, which witnesses that the topological monoid is good. Moreover, $\mathcal{N} \to \boldsymbol{V}_\#$ is shrinkable, whence $\mathcal{N}$ is good as well.

Additionally, the assumptions G1 and G2 involving the infimum are clearly satisfied by the explicit construction of our order. Now we use $\boldsymbol{U}$ to thicken $\mathcal{M}$ to an open subset $\mathcal{N}' \subseteq \mathcal{N}$ containing all configurations which are projectable if we ignore clusters labelled in $\boldsymbol{U}$; we call these configurations *almost projectable*. Then $|\mathcal{M}| \to |\mathcal{N}'|$ is a homotopy equivalence, with inverse induced by $(h^k_\bullet)_{k \geqslant 1}$ from above. Moreover, $\mathcal{N}'$ satisfies assumption G3 since

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[3] A *good ordered space* is a space $\mathcal{N}$ such that its nerve is a good simplicial space. A topological monoid $(M, 1)$ is good if 1 has a contractible neighbourhood, see [Seg73, A2].

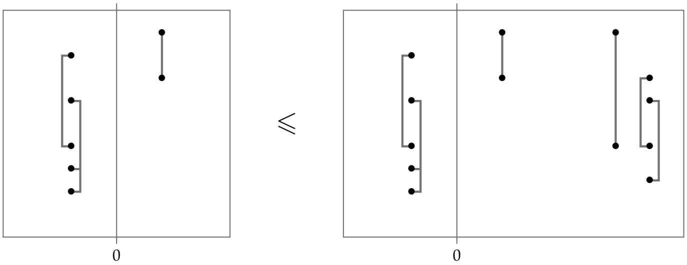


**Figure 2.3.** Inside $\mathcal{N}$, the left configuration is smaller than the right one since it is a restriction of the latter.

restrictions of projectables are still projectable. As a cover for G4, we define

$$\mathcal{W}_\delta := \left\{ (a, b, [\Theta]) \in \mathcal{N};\ \substack{\pm\delta \notin \mathrm{pr}_1[\Theta], \text{ and} \\ [\Theta] \cap ([-\delta;\delta] \times \mathbb{R}^{p-1,q}) \text{ is almost projectable}} \right\}$$

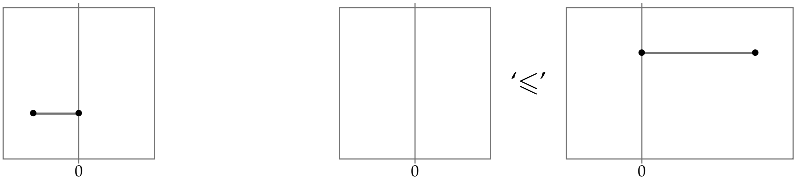
for each $\delta > 0$. Then $\mathcal{W}_\delta \subseteq \mathcal{N}$ is open, the cover $(\mathcal{W}_\delta)_{\delta>0}$ is numerable, and since each $[\Theta]$ has only finitely many clusters, each $[\Theta]$ admits a $\delta > 0$ such that $[\Theta] \cap \big([-\delta;\delta] \times \mathbb{R}^{p-1,q}\big)$ projects to a single point in $(-\delta;\delta)$, hence the restriction has to be projectable, and so $(\mathcal{W}_\delta)_{\delta>0}$ is exhaustive. Now we define the maps $w_\delta \colon \mathcal{W}_\delta \to \mathcal{N}'$ by

$$w_\delta(a, b, [\Theta]) := \Big( \max(a, -\delta), \min(b, \delta), [\Theta] \cap \big([-\delta;\delta] \times \mathbb{R}^{p-1,q}\big) \Big),$$

which is continuous since $\pm\delta \notin \mathrm{pr}_1(\bigcup_i [z_i])$ for each $[\Theta] = \sum_i \vec{z}_i \otimes x_i$ and $a, b$ with $(a, b, [\Theta]) \in \mathcal{W}_\delta$. Hence, all four assumptions of [Seg73, Prop. 2.7] are satisfied and $|\mathcal{N}'| \to |\mathcal{N}|$ is a homotopy equivalence. □

Let us close this subsection by emphasising stages of the proof which break down if we drop the condition that a cluster projects along $\mathrm{pr}_1$ to a single point: we needed that for each $(a, b, [\Theta]) \in \mathcal{N}$, there is an $a \leqslant a' \leqslant 0$ and a $0 \leqslant b' \leqslant b$ such that $[\Theta] \cap \big([a';b'] \times \mathbb{R}^{p-1,q}\big)$ is projectable. For example, in $V(\mathbb{R}^{0,2}; \underline{\mathbb{S}}^0)$, this is not always possible, as Figure 2.4a shows.

One may try to remedy this issue by allowing restrictions which exclude *parts* of a cluster, and in that case, we remove the entire cluster. However, the so-defined order on $\mathcal{N}$ clearly does not reflect the monoid structure of $\boldsymbol{V}$ correctly, which formally means that the functor $\rho \colon \mathcal{N} \to \boldsymbol{V}_\#$ cannot be defined on morphisms, as Figure 2.4b shows.



**(a)** A non-projectable configuration of a single 2-cluster in $V(\mathbb{R}^{0,2};\underline{\mathbb{S}}^0)$, where each restriction which is projectable would 'break' the cluster.

**(b)** If one also removes clusters which are only partially excluded, $\rho$ would have to deal with the above 'morphism'. However, the right side is not a concatenation $\underline{m} \cdot m \cdot \overline{m}$, where $m$ is the left side.

**Figure 2.4.** Two examples in $\mathbb{R}^{0,2}$ where the proof strategy is not applicable.

## 2.3. An $E_1$-cellular decomposition

We have already seen that $B^p V(\mathbb{R}^{p,q};\boldsymbol{X}) \simeq C(\mathbb{R}^q;\Sigma^p \boldsymbol{X})$ holds for each based symmetric sequence $\boldsymbol{X}$. However, the right side is still a $\mathscr{C}_q$-algebra which we want to deloop further. In general, finding a geometric model for the bar construction of $C(\mathbb{R}^d;\boldsymbol{X})$ appears to be quite complicated, but we can give an answer to this question for the case $d = 1$: here we provide a (strictified) $E_1$-cellular decomposition in the sense of [GKR18; GKR19].

Let us start with a few combinatorial preliminaries, and then discuss the easy and instructive example of $C(\mathbb{R};\underline{\mathbb{S}}^0)$.

**Notation 2.3.1.** We extend Definition 1.3.8: for each $\Lambda \geqslant 0$, an *unordered partition* of $\{1,\dots,\Lambda\}$ is a tuple $e = (S_1,\dots,S_r)$ such that:

1. the collection $\{S_1,\dots,S_r\}$ is a partition of $\{1,\dots,\Lambda\}$;
2. the entries are ordered by their minimum, i.e. $\min(S_i) < \min(S_{i+1})$.

We write $|e| := \Lambda$ and $K(e) = (\#S_1,\dots,\#S_r)$, and we call $\#e = r$ the *weight*. Moreover, let $\Xi$ be the set of all unordered partitions for varying $\Lambda \geqslant 0$.

**Construction 2.3.2.** We have a product $\Xi \times \Xi \to \Xi$ by *stacking* partitions: for two partitions $e = (S_1,\dots,S_r)$ and $e' = (S'_1,\dots,S'_{r'})$, we let

$$e \sqcup e' := \big(S_1,\dots,S_r, |e| + S'_1,\dots,|e| + S'_{r'}\big),$$

where the rear $r'$ components are shifted by $|e|$, as the notation indicates. Thus, $\Xi$ becomes a monoid with neutral element the empty partition $\varnothing$.

This monoid is freely generated: we call a partition $e$ *irreducible* if it is neither $\varnothing$ nor the product of two non-empty partitions, and we denote the subset of them by $\mathbb{E} \subseteq \Xi$. Then $\Xi$ is, as a monoid, freely generated by $\mathbb{E}$.

As a notational mnemonic, the letters $\Xi$ and $\mathbb{E}$ were mainly chosen because $\Xi$ looks like a 'decomposable' version of $\mathbb{E}$.

**Example 2.3.3.** We have a morphism $\chi\colon C(\mathbb{R};\underline{\mathbb{S}}^0) \to \Xi$ of $\mathscr{C}_1$-algebras given by identifying, for each clustered configuration $\sum_i [\vec{z}_i]$ inside $C(\mathbb{R};\underline{\mathbb{S}}^0)$, the set $\bigcup_i [\vec{z}_i] \subseteq \mathbb{R}$ with $\{1,\dots,\sum_i k_i\}$ in a monotone way, see Figure 2.5.

$= (\{1,5,7\},\{2,4\},\{3,8\},\{6\})$

**Figure 2.5.** An instance of $\chi\colon V(\mathbb{R}^{0,1};\underline{\mathbb{S}}^0) \to \Xi$

This map admits a section $s\colon \Xi \to C(\mathbb{R};\underline{\mathbb{S}}^0)$ by including $\{1,\dots,\sum_i k_i\}$ into $\mathbb{R}$, and the composition $s \circ \chi$ is clearly homotopic to the identity by linear interpolation. Thus, $\chi$ is an equivalence of $\mathscr{C}_1$-algebras.

Since $\Xi$ is a freely generated by $\mathbb{E}$, or, in other words, the (reduced) James product of $\mathbb{E}_+$, we get $BC(\mathbb{R};\underline{\mathbb{S}}^0) \simeq B\Xi \cong BJ\mathbb{E}_+ \simeq \Sigma\mathbb{E}_+ \cong \bigvee_{e\in\mathbb{E}} \mathbb{S}^1$.

If we replace $\underline{\mathbb{S}}^0$ by a general sequence $\boldsymbol{X}$ of labelling spaces, then the bar construction is calculated as follows:

**Theorem 2.3.4.** *For a sequence $\boldsymbol{X} = (X_k)_{k\geqslant 1}$ of well-based spaces (with arbitrary based $\mathfrak{S}_k$-actions on $X_k$), we have a weak homotopy equivalence*

$$BC(\mathbb{R};\boldsymbol{X}) \simeq \Sigma \bigvee_{e\in\mathbb{E}} \boldsymbol{X}^{\wedge K(e)},$$

*where we write $\boldsymbol{X}^{\wedge K} := X_{k_1} \wedge \cdots \wedge X_{k_r}$ for $K = (k_1,\dots,k_r)$.*

Before proving Theorem 2.3.4, let us discuss some remarks and examples.

**Remark 2.3.5.** The reader should not be surprised by the fact that the symmetric actions on $\boldsymbol{X}$ do not occur on the right side: they are also irrelevant for the left side since we have $q = 1$. Note that the action of $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_r}$

on $\tilde{C}_K(\mathbb{R})$ induces a free action on $\pi_0$, so we can alternatively restrict to the subspace $\tilde{C}_K^<(\mathbb{R})$ containing configurations $(\vec{z}_1, \ldots, \vec{z}_r)$ of clusters where each cluster $\vec{z}_i$ is of the form $\vec{z}_i = (z_{i,1}, \ldots, z_{i,k_i})$ with $z_{i,\jmath} < z_{i,\jmath+1}$ in $\mathbb{R}$: then we get the coend description, writing $\overline{\mathbb{N}} := \{1, 2, \ldots\}$,

$$C(\mathbb{R}; \boldsymbol{X}) \cong \int^{K \in \overline{\mathbb{N}} \wr \mathbf{Inj}} \tilde{C}_K^<(\mathbb{R}) \times \boldsymbol{X}^K,$$

where $\boldsymbol{X}$ is considered only as a sequence of based spaces. This is also the reason why we do not have to assume that $\boldsymbol{X}$ is equivariantly well-based.

**Example 2.3.6.** In some cases, Theorem 2.3.4 recovers old results: for $\boldsymbol{X} = \underline{\mathbb{S}}^0$, we see that $(\underline{\mathbb{S}}^0)^{\wedge K(e)} = \mathbb{S}^0$ and we get precisely Example 2.3.3.

If $\boldsymbol{X} = X[1]$ for a well-based space $X$, i.e. we only see clusters of size 1, then $C(\mathbb{R}; X[1])$ is the classical labelled configuration space. On the other hand, we have $X[1]^{\wedge K(e)} = *$ whenever $K(e)$ is not of the form $(1, \ldots, 1)$. However, there is only one irreducible partition $e = (S_1, \ldots, S_r)$ with $\#S_i = 1$ for all $i$, namely $e = (\{1\})$, so we recover $BC(\mathbb{R}; X) \simeq \Sigma X$ from [Seg73].

The idea of the proof of Theorem 2.3.4 is the following: we construct a filtration $F_\bullet C(\mathbb{R}; \boldsymbol{X})$ of $C(\mathbb{R}; \boldsymbol{X})$ by the maximal weight of irreducible partitions. Then each $F_r C(\mathbb{R}; \boldsymbol{X})$ is a sub-$E_1$-algebra, and, up to homotopy, $F_r C(\mathbb{R}; \boldsymbol{X})$ arises from $F_{r-1} C(\mathbb{R}; \boldsymbol{X})$ by attaching a *free* $E_1$-algebra: pictorially, we have to add one letter for each irreducible partition $e$ of weight $r$, and each such letter comes with a parameter domain $\boldsymbol{X}^{K(e)}$, corresponding to the labels of the involved clusters. If at least one coordinate of the parameter attains the basepoint, then the letter breaks apart and we land in $F_{r-1} C(\mathbb{R}; \boldsymbol{X})$. Finally, we have to understand bar constructions of such attachments.

In order to make this idea precise, it is convenient to discard the contractible information which tells us how far apart the particles are, and to switch from the category of $E_1$-algebras to the category of topological monoids. To do so, we first consider again the thickened version $\boldsymbol{C}$ of $C(\mathbb{R}; \boldsymbol{X})$ as in Construction 2.2.5, which is now of the form

$$\boldsymbol{C} := \left\{ (t, \textstyle\sum_i \vec{z}_i \otimes x_i) \in \mathbb{R}_{\geqslant 0} \times C(\mathbb{R}; \boldsymbol{X});\ \bigcup_i [\vec{z}_i] \subseteq (0; t) \right\},$$

again endowed with the Moore concatenation as product. Then $BC(\mathbb{R}; \boldsymbol{X})$ and $B\boldsymbol{C}$ are equivalent. Subsequently, we construct a cellular monoid $M$, which arises from the trivial monoid by attaching free topological monoids (or, in other words, reduced James products), and give an explicit monoid homomorphism $\varphi\colon \boldsymbol{C} \to M$ whose underlying map is a homotopy equivalence. Finally, we calculate $BM$ cellularly.

**Definition 2.3.7.** Let $\boldsymbol{X} = (X_k)_{k\geqslant 1}$ be a sequence of well-based spaces and let $K = (k_1, \dots, k_r)$ be a tuple of positive integers. We denote the basepoint of $X_k$ by $*_k$ and define the *thick bouquet*

$$\boldsymbol{X}_0^K := \Big\{(x_1, \dots, x_r) \in \boldsymbol{X}^K;\ x_i = *_{k_i} \text{ for some } 1 \leqslant i \leqslant r\Big\} \subseteq \boldsymbol{X}^K$$

as the subspace of *degenerated tuples*. For $T \subseteq \{1, \dots, r\}$, we let $\boldsymbol{X}^{K,T} \subseteq \boldsymbol{X}^K$ be the subspace of all tuples $(x_1, \dots, x_r)$ with $x_i = *_{k_i}$ for $i \notin T$. Then we have

$$\boldsymbol{X}_0^K = \operatorname*{colim}_{T \neq \{1,\dots,r\}} \boldsymbol{X}^{K,T} \simeq \operatorname*{hocolim}_{T \neq \{1,\dots,r\}} \boldsymbol{X}^{K,T},$$

where the last equivalence comes from the fact that in the diagram $\boldsymbol{X}^{K,-}$, all arrows are cofibrations since $\boldsymbol{X}$ is assumed to be levelwise well-based.

Given a topological monoid $M$, we write $UM$ for its underlying based space, and, conversely, given a based space $Y$, we write $JY$ for the James product, i.e. the free topological monoid over the based space $Y$.

**Construction 2.3.8.** We construct the topological monoid $M$ inductively:

- we give a sequence $(\iota_r\colon A_r \hookrightarrow Y_r)_{r\geqslant 1}$ of based cofibrations,
- we start with the trivial monoid $M_0 := 1$,
- given $M_{r-1}$, we give a based map $f_r\colon A_r \to UM_{r-1}$, consider its adjoint $\bar{f}_r\colon JA_r \to M_{r-1}$, and define $M_r$ to be the pushout of monoids

$$\begin{tikzcd} JA_r \arrow[r, "\bar{f}_r"] \arrow[d, "J\iota_r"'] & M_{r-1} \arrow[d, "j_r"] \\ JY_r \arrow[r] \arrow[ur, phantom, "\ulcorner", near end] & M_r. \end{tikzcd}$$

- finally, we let $M := \varinjlim_r M_r$ be the colimit of topological monoids.

As cofibrations $A_r \hookrightarrow Y_r$, we choose $\bigvee_{\#e=r} \boldsymbol{X}_0^{K(e)} \hookrightarrow \bigvee_{\#e=r} \boldsymbol{X}^{K(e)}$, and we write points in $Y_r$ as $e \wedge (x_1, \ldots, x_r)$ where $e \in \mathbb{E}$ and $x_i \in X_{k_i}$.

In order to construct $f_r$, we provide maps $f_e^T \colon \boldsymbol{X}^{K(e),T} \to UM_{r-1}$ for each $e = (S_1, \ldots, S_r) \in \mathbb{E}$ and each $T \subsetneq \{1, \ldots, r\}$: consider the partition $e^T$ where we have removed all components $S_i$ for which $i \notin T$. Then there are unique irreducibles $e_1, \ldots, e_s \in \mathbb{E}$ such that $e^T = e_1 \sqcup \cdots \sqcup e_s$, and by construction, we have $\#e_l \leqslant r-1$ for each $1 \leqslant l \leqslant s$. If we write $e_l = (S_{l,1}, \ldots, S_{l,r_l})$, then there is a unique bijection $\Phi \colon \mathbb{Y}_{r_1,\ldots,r_s} \to T$ such that $S_{\Phi(l,h)}$ inside $e$ corresponds to $S_{l,h}$ inside $e_l$, where $\mathbb{Y}_{r_1,\ldots,r_s}$ denotes the tableau from Definition 1.2.1. We write $x_{l,h} := x_{\Phi(l,h)}$ and define the map $f_e^T$ by

$$f_e^T(x_1, \ldots, x_r) := \prod_{l=1}^{s} e_l \wedge (x_{l,1}, \ldots, x_{l,r_l}) \in UM_{r-1}.$$

Then we have $f_e^{T'}|_{\boldsymbol{X}^{K(e),T}} = f_e^T$ for $T \subseteq T'$ by the already existing attaching maps $f_1, \ldots, f_{r-1}$, so we get an amalgamated map $f_e \colon \boldsymbol{X}_0^{K(e)} \to UM_{r-1}$.

Similar to the classical proof that the thickening of the free $E_1$-algebra over a well-based space $X$ is equivalent to the reduced James product over $X$, we show that the monoids $\boldsymbol{C}$ and $M$ are equivalent.

**Lemma 2.3.9.** *There is a homomorphism $\varphi \colon \boldsymbol{C} \to M$ of topological monoids, whose underlying map is a homotopy equivalence.*

*Proof.* We construct for each tuple $K$ a map $\tilde{\varphi}_K \colon \tilde{C}_K^{<}(\mathbb{R}) \times \boldsymbol{X}^K \to M$ as follows: given a configuration $(\vec{z}_1, \ldots, \vec{z}_r) \in \tilde{C}_K^{<}(\mathbb{R})$, we define:

- the partition $e = (S_1, \ldots, S_r) = \chi(\sum_i [z_i])$ as in Example 2.3.3,
- a permutation $\tau \in \mathfrak{S}_r$ with $z_{\tau(1),1} < \cdots < z_{\tau(r),1}$, which means that the clusters $\vec{z}_{\tau(1)}, \ldots, \vec{z}_{\tau(r)}$ are ordered by their minimum.

The pair $(e, \tau)$ only depends on the path component of $\tilde{C}_K^{<}(\mathbb{R})$, so the assignment is continuous. We proceed as above and decompose $e = e_1 \sqcup \cdots \sqcup e_s$ into irreducibles. Again we write $e_l := (S_{l,1}, \ldots, S_{l,r_l})$ and consider the bijection $\Phi \colon \mathbb{Y}_{r_1,\ldots,r_s} \to \underline{r}$ from above. For $(x_1, \ldots, x_r) \in \boldsymbol{X}^K$, we define

$$\tilde{\varphi}_K(\vec{z}_1, \ldots, \vec{z}_r, x_1, \ldots, x_r) := \prod_{l=1}^{s} e_l \wedge (x_{(\tau\circ\Phi)(l,1)}, \ldots, x_{(\tau\circ\Phi)(l,r_l)}).$$

Then one readily checks that $\coprod_K \tilde{\varphi}_K$ factors through $\tilde{\varphi}\colon C(\mathbb{R};\boldsymbol{X}) \to M$. Finally, we define $\varphi\colon \boldsymbol{C} \to M$ by $\varphi(t, \sum_i \vec{z}_i \otimes x_i) := \tilde{\varphi}(\sum_i \vec{z}_i \otimes x_i)$, which clearly is a homomorphism of monoids.

In order to see that $\varphi$ is a homotopy equivalence, we use that all $X_k$ are well-based, so they admit Urysohn functions $u_k\colon X_k \to [0;1]$ with $u_k^{-1}(0) = \{*_k\}$. Now for each $e \in \mathbb{E}$, we write $K := K(e)$ and define maps $\bar{\alpha}_e\colon \boldsymbol{X}^K \to U\boldsymbol{C}$ as follows: write $e = (S_1, \dots, S_r)$ and $K = (k_1, \dots, k_r)$, which means $k_i = \#S_i$ and $|e| = \sum_i k_i$. There are tuples $\vec{z}_i = (z_{i,1}, \dots, z_{i,k_i})$ of positive integers which are uniquely determined by the properties $z_{i,\jmath} < z_{i,\jmath+1}$ and $\{z_{i,1}, \dots, z_{i,k_i}\} = S_i$.

Moreover, the partition determines a map $\pi\colon \{1, \dots, |e|\} \to \underline{r}$ such that $S_i = \pi^{-1}(i)$, and for each tuple $(x_1, \dots, x_r) \in \boldsymbol{X}^K$, we stretch $\{1, \dots, |e|\} \subseteq \mathbb{R}$ such that each $j$ is the mid-point of a segment of length $u_{k_{\pi(j)}}(x_{\pi(j)})$, see Figure 2.6: formally, we define $\mathrm{str}\colon \boldsymbol{X}^K \times \{1, \dots, |e|\} \to \mathbb{R}$ by

$$\mathrm{str}(x_1, \dots, x_r, \jmath) := \tfrac{1}{2} \cdot u_{k_{\pi(j)}}(x_{\pi(j)}) + \sum_{j'=1}^{j-1} u_{k_{\pi(j')}}(x_{\pi(j')}).$$

and, applying str to an entire cluster, we put

$$\bar{\alpha}_e(x_1, \dots, x_r) := \left( \sum_{i=1}^{r} k_i \cdot u_{k_i}(x_i), \sum_{i=1}^{r} \mathrm{str}(x_1, \dots, x_r, \vec{z}_i) \otimes x_i \right) \in \boldsymbol{C}.$$

Let $\alpha_r\colon JY_r \to \boldsymbol{C}$ be the adjoint of $\bigvee_{\#e=r} \bar{\alpha}_e\colon Y_r \to U\boldsymbol{C}$. These maps can be used to inductively define a system $(\psi_r\colon M_r \to \boldsymbol{C})_{r \geqslant 0}$ of homomorphisms satisfying $\psi_r \circ \jmath_r = \psi_{r-1}$ and $\alpha_{r+1} \circ (J\iota_{r+1}) = \psi_r \circ \bar{f}_{r+1}$ as follows: we define $\psi_0$ to be the trivial map, and for the induction step '$r-1 \to r$', we define $\psi_r$ to be the pushout map of the diagram

$$\begin{tikzcd}
JA_r \arrow[r, "\bar{f}_r"] \arrow[d, "J\iota_r"'] & M_{r-1} \arrow[d, "\jmath_r"] \arrow[ddr, bend left, "\psi_{r-1}"] & \\
JY_r \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] \arrow[drr, bend right, "\alpha_r"'] & M_r \arrow[dr, dashed, "\psi_r"] & \\
& & \boldsymbol{C},
\end{tikzcd}$$

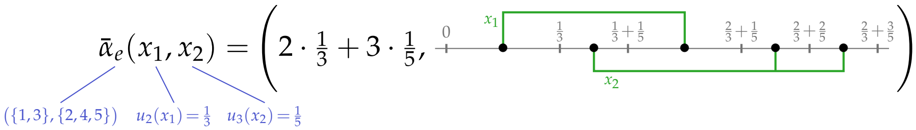


**Figure 2.6.** An instance of $\bar{\alpha}_e \colon \boldsymbol{X}^K \to \boldsymbol{C}$ for $e = (\{1,3\},\{2,4,5\})$.

using that we already know $\alpha_r \circ (J\iota_r) = \psi_{r-1} \circ \bar{f}_r$. To complete the induction step, assume that $\#e = r+1$ and let $K := K(e)$ and $(x_1, \ldots, x_{r+1}) \in \boldsymbol{X}^K$. By construction, if $x_i = *$ for some $1 \leqslant i \leqslant r+1$, its corresponding Urysohn map attains 0 and the $k_i$ segments are entirely removed, which means that the maps $\bar{\alpha}_e \circ \iota_e$ and $U\psi_r \circ f_e$ coincide for each $e$, and so do their bouquets $(\bigvee_e \bar{\alpha}_e) \circ \iota_{r+1}$ and $U\psi_r \circ f_{r+1}$, and their adjoints $\alpha_{r+1} \circ (J\iota_{r+1})$ and $\psi_r \circ \bar{f}_{r+1}$.

Finally, we let $\psi := \varinjlim_r \psi_r \colon M \to \boldsymbol{C}$. Then $\varphi \circ \psi$ is the identity on $M$, and $\psi \circ \varphi$ is homotopic to the identity on $\boldsymbol{C}$, again by linear interpolation. □

Since both topological monoids, $\boldsymbol{C}$ and $M$, are well-based, the homomorphism $\varphi$ induces a weak equivalence $B\varphi \colon B\boldsymbol{C} \to BM$, so we can equivalently study the homotopy type of $BM$.

To this aim, we use that the bar construction respects such attachments and hence have to understand how for each tuple $K$, the suspension $\Sigma\boldsymbol{X}^K$ can be attached with respect to the subspace $\Sigma\boldsymbol{X}_0^K$. Here we use that the cofibre sequence associated with $\boldsymbol{X}_0^K \hookrightarrow \boldsymbol{X}^K$ splits after a single suspension step in order to reach the following statement:

**Lemma 2.3.10.** *Let $\boldsymbol{X}$ be as before. For each tuple $K$, there are homotopy equivalences $\mu_{K,0} \colon \Sigma\boldsymbol{X}_0^K \to \bigvee_{Q \neq \{1,\ldots,r\}} \Sigma\boldsymbol{X}^{\wedge K|_Q}$ and $\mu_K \colon \Sigma\boldsymbol{X}^K \to \bigvee_Q \Sigma\boldsymbol{X}^{\wedge K|_Q}$, where $Q$ ranges over non-empty subsets of $\{1, \ldots, r\}$, such that*

$$\begin{tikzcd}
\Sigma\boldsymbol{X}_0^K \arrow[r, hook] \arrow[d, "\mu_{K,0}"'] & \Sigma\boldsymbol{X}^K \arrow[d, "\mu_K"] \\
\bigvee_{Q \neq \{1,\ldots,r\}} \Sigma\boldsymbol{X}^{\wedge K|_Q} \arrow[r, hook] & \bigvee_Q \Sigma\boldsymbol{X}^{\wedge K|_Q}.
\end{tikzcd} \tag{$\diamond$}$$

*commutes up to homotopy.*

*Proof.* Since each $X_k$ is well-based, the inclusion $\boldsymbol{X}_0^K \hookrightarrow \boldsymbol{X}^K$ is a cofibration, and we have a cofibre sequence $\boldsymbol{X}_0^K \to \boldsymbol{X}^K \to \boldsymbol{X}^{\wedge K}$. Recall for each subset $T \subseteq \{1, \dots, r\}$ the subspaces $\boldsymbol{X}^{K,T} \subseteq \boldsymbol{X}^K$ from Definition 2.3.7, and note that we have an isomorphism $\boldsymbol{X}^{K,T} \cong \boldsymbol{X}^{K|_T}$. For each non-empty subset $Q \subseteq T$, we have a map $\mathrm{pr}_Q^T \colon \boldsymbol{X}^{K,T} \to \boldsymbol{X}^{\wedge K|_Q}$, and the coproduct of them,

$$\mu_K^T := \left(\bigvee\nolimits_Q \Sigma\mathrm{pr}_Q^T\right) \circ \nabla_T^{2^{\#T}-1} \colon \Sigma\boldsymbol{X}^{K,T} \to \bigvee\nolimits_{\varnothing \neq Q \subseteq T} \Sigma\boldsymbol{X}^{\wedge K|_Q},$$

is a homotopy equivalence by an inspection of the corresponding cofibre sequence. For each subset $T \subseteq T'$, the inclusion map $\Sigma\boldsymbol{X}^{K,T} \hookrightarrow \Sigma\boldsymbol{X}^{K,T'}$ is a morphism of co-H-spaces, so by counitality, the diagram

$$\begin{tikzcd}
\Sigma\boldsymbol{X}^{K,T} \arrow[r, hook] \arrow[d, "\mu_K^T"'] & \Sigma\boldsymbol{X}^{K,T'} \arrow[d, "\mu_K^{T'}"] \\
\bigvee_{\varnothing \neq Q \subseteq T} \Sigma\boldsymbol{X}^{\wedge K|_Q} \arrow[r, hook] & \bigvee_{\varnothing \neq Q \subseteq T'} \Sigma\boldsymbol{X}^{\wedge K|_Q}.
\end{tikzcd} \qquad (\diamond\diamond)$$

commutes up to homotopy. We obtain a map between homotopy colimits

$$\mu_{K,0} \colon \underset{T \neq \{1,\dots,r\}}{\mathrm{hocolim}}\ \Sigma\boldsymbol{X}^{K,T} \to \underset{T \neq \{1,\dots,r\}}{\mathrm{hocolim}}\ \bigvee\nolimits_{Q \subseteq T} \Sigma\boldsymbol{X}^{\wedge K|_Q},$$

which is an equivalence since the morphism of diagrams is a levelwise equivalence. On the other hand, since $\boldsymbol{X}$ is well-based, both diagrams are cofibrant, whence the two homotopy colimits are equivalent to the honest amalgamations $\Sigma\boldsymbol{X}_0^K$ and $\bigvee_{Q \neq \{1,\dots,r\}} \Sigma\boldsymbol{X}^{\wedge K|_Q}$. If we finally define $\mu_K := \mu_K^{\{1,\dots,r\}}$, then the H-commutativity of $(\diamond)$ follows from the H-commutativity of $(\diamond\diamond)$. □

Now we have everything together to prove Theorem 2.3.4.

*Proof of Theorem 2.3.4.* Recall that it is enough to see that $BM$ has the weak homotopy type of $\bigvee_e \Sigma\boldsymbol{X}^{\wedge K(e)}$. First of all, since $B$ commutes with filtered colimits, we obtain $BM \cong \varinjlim_r BM_r$. Moreover, we have $BM_0 = *$ and, by a variation of [KM18, Prop. 98], $BM_r$ arises from $BM_{r-1}$ as the pushout

$$\begin{tikzcd}
\bigvee_{\#e=r} \Sigma\boldsymbol{X}_0^{K(e)} \arrow[r, "\kappa_{M_{r-1}} \circ (\Sigma f_r)"] \arrow[d, hook] & BM_{r-1} \arrow[d, hook] \\
\bigvee_{\#e=r} \Sigma\boldsymbol{X}^{K(e)} \arrow[r] & BM_r, \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where $\kappa\colon \Sigma U \Rightarrow B$ is the natural inclusion. We aim to show that the inclusion $BM_{r-1} \hookrightarrow BM_r$ is equivalent to $BM_{r-1} \hookrightarrow BM_{r-1} \vee \bigvee_{\#e=r} \Sigma \boldsymbol{X}^{\wedge K(e)}$, which implies the statement. To do so, we pick for each tuple $K$ a homotopy inverse $\nu_{K,0}\colon \bigvee_{Q\neq\{1,\dots,r\}} \Sigma \boldsymbol{X}^{\wedge K|_Q} \to \Sigma \boldsymbol{X}_0^K$ and define

$$f'_r := \kappa_{M_{r-1}} \circ (\Sigma f_r) \circ \big(\bigvee_{\#e=r} \nu_{K(e),0}\big)\colon \bigvee_{\#e=r} \bigvee_{Q\neq\{1,\dots,r\}} \Sigma \boldsymbol{X}^{\wedge K(e)|_Q} \to BM_{r-1}.$$

Then we obtain a morphism of spans

$$\begin{tikzcd}
BM_{r-1} \arrow[d, equal] & \bigvee_{\#e=r} \Sigma \boldsymbol{X}_0^{K(e)} \arrow[l, "\kappa_{M_{r-1}}\circ(\Sigma f_r)"'] \arrow[r, hook] \arrow[d, "\bigvee_{\#e=r} \mu_{K(e),0}"'] & \bigvee_{\#e=r} \Sigma \boldsymbol{X}^{K(e)} \arrow[d, "\bigvee_{\#e=r} \mu_{K(e)}"'] \\
BM_{r-1} & \bigvee_{\#e=r} \bigvee_{Q\neq\{1,\dots,r\}} \Sigma \boldsymbol{X}^{\wedge K(e)|_Q} \arrow[l, "f'_r"] \arrow[r, hook] & \bigvee_{\#e=r} \bigvee_{Q} \Sigma \boldsymbol{X}^{\wedge K(e)|_Q},
\end{tikzcd}$$

where the right square commutes by Lemma 2.3.10, while the left square commutes by definition, using that $\nu_{K,0} \circ \mu_{K,0}$ is homotopic to the identity on $\Sigma \boldsymbol{X}_0^K$. Moreover, all three morphisms are homotopy equivalences, so we obtain a homotopy equivalence between their induced homotopy pushouts. Finally, the homotopy pushout of the bottom span is equal to the bouquet of $BM_{r-1}$ with those summands $\Sigma \boldsymbol{X}^{\wedge K(e)|_Q}$ where $Q = \{1, \dots, r\}$. ☐

**Corollary 2.3.11.** *Let $\boldsymbol{X}$ be a sequence of well-based spaces with based $\mathfrak{S}_k$-actions.*

1. *If $\boldsymbol{X}$ is equivariantly well-based, then we have a weak equivalence*

$$B^{p+1} V(\mathbb{R}^{p,1}; \boldsymbol{X}) \simeq \bigvee_e \Sigma^{1+p\cdot\#e} \boldsymbol{X}^{\wedge K(e)}.$$

2. *If $\boldsymbol{X}$ is levelwise connected, then $C(\mathbb{R}; \boldsymbol{X})$ is equivalent to a classical labelled configuration space*

$$C(\mathbb{R}; \boldsymbol{X}) \simeq \Omega\Sigma \bigvee_e \boldsymbol{X}^{\wedge K(e)} \simeq C\big(\mathbb{R}; \bigvee_e \boldsymbol{X}^{\wedge K(e)}\big).$$

**Outlook 2.3.12.** The proof of Theorem 2.3.4 is obviously a peculiarity of dimension 1: in order to switch from one combinatorial type to another one, at least one label has to meet the basepoint.

The easiest case for which we do not have a geometric model of its (iterated) bar construction is the $E_2$-algebra $C^2(\mathbb{R}^2; \mathbb{S}^0)$ of unordered configurations of unordered pairs in the plane.

## 2.4. Applications and calculations

In this short section, we discuss a few applications of the results from the previous sections, which mainly show how understanding the homotopy theory of labelled vertical configuration spaces helps us to understand the homology of the unlabelled ones.

**Application 2.4.1** (Stable homology)**.** For each $p, q, k \geqslant 1$, we see

$$V^k(\mathbb{R}^{p,q}; \mathbb{S}^0) \cong \coprod_{r \geqslant 0} V^k_r(\mathbb{R}^{p,q}).$$

Since all spaces $V^k_r(\mathbb{R}^{p,q})$ are connected, the monoid $\pi_0(V^k(\mathbb{R}^{p,q}; \mathbb{S}^0))$ with the addition induced by the $\mathscr{C}_{p+q}$-action is isomorphic to $\mathbb{N}$.

Moreover, we saw in Theorem 1.3.3 that the sequence $(V^k_r(\mathbb{R}^{p,q}))_{r \geqslant 0}$ is homologically stable, so since $V^k(\mathbb{R}^{p,q}; \mathbb{S}^0)$ is at least a $\mathscr{C}_2$-algebra, we calculate the stable homology by the group completion theorem [MS76, Prop. 1],

$$H_h\big(V^k_\infty(\mathbb{R}^{p,q})\big) \cong H_h\big(\Omega^p_0 B^p V^k(\mathbb{R}^{p,q}; \mathbb{S}^0)\big) \cong H_h\big(\Omega^p_0 C^k(\mathbb{R}^q; \mathbb{S}^p)\big),$$

where $\Omega_0$ denotes the path component of the constant loop. In the case $q = 1$, we obtain, summing over all irreducibles of the form $K(e) = (k, \ldots, k)$,

$$\begin{aligned} H_h\big(V^k_\infty(\mathbb{R}^{p,1})\big) &\cong H_h\big(\Omega^{p+1}_0 \Sigma^{p+1} \textstyle\bigvee_e \mathbb{S}^{p\cdot(w(e)-1)}\big) \\ &\cong H_h\big(C_\infty\big(\mathbb{R}^{p+1}; \textstyle\bigvee_e \mathbb{S}^{p\cdot(w(e)-1)}\big)\big), \end{aligned}$$

where $w(e)$ denotes the weight as in Definition 1.3.8. For the last identification, we use that $\bigvee_e \mathbb{S}^{p\cdot(w(e)-1)} = \bigvee_{w(e)\geqslant 2} \mathbb{S}^{p\cdot(w(e)-1)} \sqcup \{e_0\}$ has exactly two path components, and we stabilise by adding a point with label $e_0$.

This is very similar to our method of proof of the Stability theorem 1.3.3, and indeed, we can say a bit more: if we let $C_t \subseteq C(\mathbb{R}^{p+1}; \bigvee_e \mathbb{S}^{p\cdot(w(e)-1)})$ be the component which has exactly $t$ points with label $e_0$, then by [Sna74, Thm. 1.1], we obtain a stable splitting $\Sigma^\infty(C_t)_+ \simeq \Sigma^\infty \bigvee_\alpha D_\alpha$, where $\alpha$ ranges over all distributions $\alpha\colon \mathbb{E} \to \mathbb{N}$ with $\alpha_{e_0} = t$, and $D_\alpha$ is the quotient of the subspace of $C(\mathbb{R}^{p+1}; \bigvee_e \mathbb{S}^{p\cdot(w(e)-1)})$ having for each $e$ at most $\alpha_e$ points with labels in the sphere corresponding to $e$, quotiented by the subspace where at least one of these labels is the basepoint. Note that if $\alpha = t \cdot e_0$, then $D_\alpha = C_t(\mathbb{R}^{p+1})_+$.

As in [BCT89, § 2.6] and Proposition 2.1.10, $D_\alpha$ is a Thom space, whence we get $\tilde{H}_h(D_\alpha) \cong H_{h-p\cdot s(\alpha)}(C_\alpha; \mathrm{pr}_\alpha^* \mathcal{O}_\alpha) =: M_{h,\alpha}$, where $s(\alpha) := \sum_e \alpha_e \cdot (w(e) - 1)$ as in Definition 1.3.11, $C_\alpha := C_\alpha(\mathbb{R}^{p+1})$ is the coloured configuration space from Definition 1.3.7, and $\mathrm{pr}_\alpha^* \mathcal{O}_\alpha$ is the sign system from Lemma 1.3.16. In other words, we have $H_h(C_t) \cong \tilde{H}_h((C_t)_+) \cong \bigoplus_\alpha M_{h,\alpha}$, where $\alpha$ ranges over all distributions with $\alpha_{e_0} = t$.

Under this identification, the stabilisation maps $H_h(C_t) \to H_h(C_{t+1})$ split into maps $M_{h,\alpha} \to M_{h,\alpha+e_0}$, which eventually become isomorphisms, as we saw in the proof of Lemma 1.3.16. For a given distribution $\alpha \colon \mathbb{E} \to \mathbb{N}$, which is *normalised*, i.e. $\alpha_{e_0} = 0$, we define the stable summand $M_{h,\alpha}^\infty := \varinjlim_t M_{h,\alpha+te_0}$. Then we finally can calculate the stable homology of $V_r^k(\mathbb{R}^{p,1})$ by

$$H_h(V_\infty^k(\mathbb{R}^{p,1})) \cong \bigoplus_{\alpha \text{ normalised}} M_{h,\alpha}^\infty.$$

**Application 2.4.2.** Similarly, we can take as labelling space the 2-sphere $\mathbb{S}^2$, whose main merit is to be connected and of even dimension. By the Splitting theorem 2.1.11, we have a stable equivalence

$$\Sigma^\infty V^k\big(\mathbb{R}^{p,q}; \mathbb{S}^2\big) \simeq \Sigma^\infty \textstyle\bigvee_{r\geqslant 1} D_r^k(\mathbb{R}^{p,q}; \mathbb{S}^2).$$

In combination with the Thom isomorphism in the form of Proposition 2.1.10, and by noticing that the orientation system is trivial as the labelling sphere is even-dimensional, we get

$$\bigoplus_{r\geqslant 1} H_{h-2r}(V_r^k(\mathbb{R}^{p,q})) \cong \bigoplus_{r\geqslant 1} \tilde{H}_h(D_r^k(\mathbb{R}^{p,q}; \mathbb{S}^2)) \cong \tilde{H}_h(V^k(\mathbb{R}^{p,q}; \mathbb{S}^2)). \qquad (\diamond)$$

Since $\mathbb{S}^2$ is connected, we have $V^k(\mathbb{R}^{p,q}; \mathbb{S}^2) \simeq \Omega^p C^k(\mathbb{R}^q; \mathbb{S}^{p+2})$. For $q = 1$, the homology of $C^k(\mathbb{R}; \mathbb{S}^{p+2})$ is easy to understand: we can apply the splitting theorem and the Thom isomorphism again (now the orientation system is trivial since each component of $C_r^k(\mathbb{R})$ is simply connected) to get

$$\tilde{H}_h\big(C^k(\mathbb{R}; \mathbb{S}^{p+2})\big) \cong \bigoplus_{r\geqslant 1} H_{h-(p+2)\cdot r}\big(C_r^k(\mathbb{R})\big).$$

The space $C_r^k(\mathbb{R})$ is homotopy discrete and the path components are indexed by partitions $e$ of type $(k, \ldots, k)$. If we let $\Xi_r^k \subseteq \Xi$ be the set of them, then

$H_h\big(C_r^k(\mathbb{R})\big) \cong \mathbb{Z}\langle \Xi_r^k \rangle$ if $h = 0$ and is trivial else. Therefore, we get, including the empty partition in degree 0,

$$H_h\big(C^k(\mathbb{R}; \mathbb{S}^{p+2})\big) \cong \begin{cases} \mathbb{Z}\langle \Xi_r^k \rangle & \text{if } h = (p+2) \cdot r \text{ for some } r \geqslant 0, \\ 0 & \text{else.} \end{cases}$$

If also $p = 1$, we can say a bit more: firstly, we know from Theorem 1.4.13 that $V_r^k(\mathbb{R}^{1,1})$ is equivalent to an $(r-1)$-dimensional cell complex, which means that in the decomposition ($\diamond$), the direct summand $H_{h-2r}(V_r^k(\mathbb{R}^{1,1}))$ is non-trivial only if $2r \leqslant h \leqslant 3r - 1$. This is visualised in Table 2.1.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $V_1^k$ | | | $\mathbb{Z}$ | | | | | | | | | | | | |
| $V_2^k$ | | | | | $\mathbb{Z}$ | ? | | | | | | | | | |
| $V_3^k$ | | | | | | | $\mathbb{Z}$ | ? | ? | | | | | | |
| $V_4^k$ | | | | | | | | | $\mathbb{Z}$ | ? | ? | ? | | | |
| $V_5^k$ | | | | | | | | | | | $\mathbb{Z}$ | ? | ? | ? | ? |

**Table 2.1.** The shifted homology groups $H_{\bullet-2r}(V_r^k(\mathbb{R}^{1,1}))$, where '?' denotes an unknown summand. The columns sum up to $\tilde{H}_\bullet(V^k(\mathbb{R}^{1,1}; \mathbb{S}^2))$.

Moreover, we have a path space fibration $V^k(\mathbb{R}^{1,1}; \mathbb{S}^2) \to * \to C^k(\mathbb{R}; \mathbb{S}^3)$, and the homology of the base of this sequence is as easy as described above. One could now invoke the Serre spectral sequence assigned to it, in order to get information about the homology of the fibre, which decomposes into the homology of the spaces $V_r^k(\mathbb{R}^{1,1})$.

Finally, we provide the counterexample that we promised in Remark 2.2.8 and which shows that, in general, for $p = 0$, the inclusion $\boldsymbol{C}^{\mathrm{proj}} \hookrightarrow \boldsymbol{C}$ from Lemma 2.2.7 does not induce an equivalence among classifying spaces.

**Example 2.4.3.** Consider the space $X := \{-1, 0, +1\}$ with basepoint $0$ and with $\mathfrak{S}_2$ acting on $X$ by sign. Then $C^2(\mathbb{R}^2; X) = \coprod_r Y_r$ with $Y_r = \tilde{C}_{2r}(\mathbb{R}^2)/\mathfrak{S}_r$, where $\mathfrak{S}_r \subseteq \mathfrak{S}_{2r}$ permutes blocks of size 2. In other words, $Y_r$ is the space of unordered configurations of ordered pairs in the plane.

We want to show that the inclusion $\boldsymbol{C}^{2,\mathrm{proj}}(\mathbb{R}^2;X) \hookrightarrow \boldsymbol{C}^2(\mathbb{R}^2;X)$ does not induce an equivalence among classifying spaces, and we do so by showing that the induced map in $H_1(\Omega_0 B(-))$ cannot be injective.

Exactly as in Section 1.3, one sees that the stabilisation $Y_r \to Y_{r+1}$ induces split monic maps in homology. Moreover, $Y_r$ is aspherical (as a covering of the configuration space of $2r$ points in the plane) and if we put $G_r := \pi_1(Y_r)$, then we obtain a short exact sequence

$$1 \to \mathrm{PBr}_{2r} \to G_r \to \mathfrak{S}_r \to 1,$$

where $\mathrm{PBr}_{2r}$ denotes the pure braid group on $2r$ strands. By a careful inspection of the Lyndon–Hochschild–Serre spectral sequence assigned to this short exact sequence, we see that $H_1(Y_r) \cong H_1(\mathrm{PBr}_{2r})_{\mathfrak{S}_r} \cong \mathbb{Z}^4$ for $r \geqslant 2$, the four generators being orbits of elementary pure braids $\alpha_{u,v}$ in $\mathrm{PBr}_{2r}$, compare Appendix A. In particular, the stabilisation maps induce isomorphisms in $H_1$ for $r \geqslant 2$, so we see by the group completion theorem [MS76, Thm. 1.1] that $H_1(\Omega_0 B\boldsymbol{C}^2(\mathbb{R}^2;X)) \cong H_1(Y_\infty) \cong \mathbb{Z}^4$ holds.

On the other hand, the projection $\boldsymbol{C}^{2,\mathrm{proj}}(\mathbb{R}^2;X) \to C^2(\mathbb{R};X)$ induces an equivalence of classifying spaces $B\boldsymbol{C}^{2,\mathrm{proj}}(\mathbb{R}^2;X) \simeq BC^2(\mathbb{R};X) \cong C^2(\mathbb{R};\Sigma X)$, where $C^2(\mathbb{R};X)$ is treated as a partial monoid. Now $\Sigma X = \mathbb{S}^1 \vee \mathbb{S}^1$, with $\mathfrak{S}_2$ acting by permuting the two summands of the bouquets. Hence we get

$$BC^2(\mathbb{R};\Sigma X) \simeq \bigvee_{r\geqslant 1} \#\mathbb{E}^2_r \cdot \Sigma(\mathbb{S}^1 \vee \mathbb{S}^1)^{\wedge r} = \bigvee_{r\geqslant 1} (2^r \cdot \#\mathbb{E}^2_r) \cdot \mathbb{S}^{r+1},$$

where we write $a \cdot Z := \bigvee_{i=1}^{a} Z$ for each based space $Z$ and each non-negative integer $a \geqslant 0$. The right side contains $8 \cdot \mathbb{S}^3$ as a summand, since $\#\mathbb{E}^2_2 = 2$; in particular, $\pi_3$ of the right hand side has $\mathbb{Z}^8$ as a direct summand. Since $\Sigma X$ is connected, we have $C^2(\mathbb{R};\Sigma X) \simeq \Omega BC^2(\mathbb{R};\Sigma X)$, and thus,

$$H_1(\Omega_0 B\boldsymbol{C}^{2,\mathrm{proj}}(\mathbb{R}^2;X)) \cong H_1(\Omega_0\Omega BC^2(\mathbb{R};\Sigma X)).$$

Since $\pi_1\big(\Omega_0\Omega BC^2(\mathbb{R};\Sigma X)\big)$ is abelian, we have

$$H_1(\Omega_0 B\boldsymbol{C}^{2,\mathrm{proj}}(\mathbb{R}^2;X)) \cong \pi_1(\Omega_0\Omega BC^2(\mathbb{R};\Sigma X)) \cong \pi_3(BC^2(\mathbb{R};\Sigma X)),$$

and we saw that the right side has $\mathbb{Z}^8$ as a direct summand. We conclude that the inclusion $\boldsymbol{C}^{2,\mathrm{proj}}(\mathbb{R}^2;X) \hookrightarrow \boldsymbol{C}^2(\mathbb{R}^2;X)$ from Lemma 2.2.7 does not induce a homotopy equivalence among classifying spaces.

Again, we close this chapter with a short list of open problems:

1. Find a geometric model for the $d$-fold delooping of $C(\mathbb{R}^d; \boldsymbol{X})$ for $d \geqslant 2$. Prove or disprove that $C(\mathbb{R}^d; \boldsymbol{X})$ is free if $\boldsymbol{X}$ is levelwise path connected.
2. Approximate $C(\mathbb{R}^d; \boldsymbol{X})$ by an $E_d$-cellular algebra, as we did for the case $d = 1$. A possible higher-dimensional analogue of irreducibility might have something to do with non-trivial intersections of the convex hulls of the clusters.
3. Understand the Serre spectral sequence of $V^k(\mathbb{R}^{1,1}; \mathbb{S}^2) \to * \to C^k(\mathbb{R}; \mathbb{S}^3)$ and how it can help us understanding the homology of $V_r^k(\mathbb{R}^{1,1})$. The cup product structure on the cohomology of the base can easily be described and might be useful for these calculations.

Chapter 3

# Coloured and dyed operads

> *The inputs are numbered red, yellow, and green;*
> *and their colours are three, two, and four!*
>
> Forschungsseminar Bonn

The purpose of this chapter is mostly foundational: it aims to establish the operadic setting we want to use, and claims no ingenuity; however, I am not aware of any textbook or treatise which presents these concepts in sufficient generality and exhaustiveness. Wherever possible, references to the standard literature are given.

Roughly speaking, an operad $\mathbb{O}$ contains *operations* $\mu$, which have several inputs and a single output. However, there are situations in which we need a type distinction for the inputs and the output: one way to formalise this is the concept of a coloured operad. From this viewpoint, a classical operad is an operad with a single colour, and hence is called *monochromatic*.

Given a monochromatic operad $\mathscr{C}$, we want to construct a coloured operad $\overline{\mathbb{N}}(\mathscr{C})$, in which several inputs of $\mathscr{C}$ are allowed to form a common input of higher multiplicity: we call this the *dyeing* („Färbung“) of $\mathscr{C}$. This construction is straightforward, and it combines several ideas from [LV12].

## 3.1. The notion of a coloured operad

### 3.1.1. *Prolegomena on monoidal categories*

We have to fix some basic axioms and conventions for working with monoidal and enriched categories. To do so, we assume that the reader is familiar with the basic concepts of enriched category theory, as it is for example presented in the standard textbook [Kel82].

For us, the following closed symmetric monoidal categories are relevant:

1. the category $(\mathbf{Set}, \times, *)$ of sets, together with the cartesian product and the singleton as monoidal unit;
2. the category $(\mathbf{Top}, \times, *)$ of compactly generated and weak Hausdorff spaces, together with the $k$-fied product and the singleton space as monoidal unit, the internal hom given by mapping spaces;
3. for a commutative ring $R$, the category $(R\text{-}\mathbf{Mod}, \otimes, R)$ of $R$-modules, together with the tensor product of $R$-modules, and $R$, regarded as a module over itself, as monoidal unit;
4. Similarly, the category $(\mathbf{Ch}_R, \otimes, R^{[0]})$ of chain complexes of $R$-modules, together with the graded tensor product of chain complexes, and $R$, concentrated in degree 0, as monoidal unit. This contains the monoidal subcategory $R\text{-}\mathbf{Mod}^{\mathbb{Z}}$ of $\mathbb{Z}$-graded $R$-modules, regarded as chain complexes with trivial differentials.

**Setting 3.1.1.** Let us collect various conventions for a general treatment:

- A *nice monoidal category* $(\mathbf{V}, \otimes, \mathbf{1})$ is a bicomplete closed symmetric monoidal category, and we will only consider nice ones. The isomorphisms $X \otimes Y \to Y \otimes X$ coming from the symmetric structure are called *twists*. We denote the isomorphism type of the initial object by $\varnothing$.
- For a $\mathbf{V}$-enriched category $\mathbf{I}$, we denote by $\mathbf{I}\binom{k}{n}$ the $\mathbf{V}$-object of morphisms from $k$ to $n$. A single morphism $k \to n$ in $\mathbf{I}$ is a map $\mathbf{1} \to \mathbf{I}\binom{k}{n}$ and the identity of an object $n$ is denoted by $\mathbb{1}_n$. As $\mathbf{V}$ is closed, $\mathbf{V}$ is enriched over itself and we denote the evaluation by $\mathrm{ev}_{X,Y} \colon \mathbf{V}\binom{X}{Y} \otimes X \to Y$.
- Since $\mathbf{V}$ is closed, the bifunctor $\otimes$ preserves colimits in both arguments, and since $\mathbf{V}$ is cocomplete, we have a copowering $[-] \colon \mathbf{Set} \to \mathbf{V}$ by $[S] = \coprod_{s \in S} \mathbf{1}$, which is strong monoidal: $[*] \cong \mathbf{1}$ and $[S \times T] \cong [S] \otimes [T]$. If $\mathbf{I}$ is a locally small category, then it can be enhanced to a $\mathbf{V}$-enriched category $\mathbf{I}^{\mathbf{V}}$ by $\mathbf{I}^{\mathbf{V}}\binom{k}{n} := [\mathbf{I}\binom{k}{n}]$.
- For a $\mathbf{V}$-enriched functor $F \colon \mathbf{I} \to \mathbf{J}$, we write $F\binom{k}{n} \colon \mathbf{I}\binom{k}{n} \to \mathbf{J}\binom{Fk}{Fn}$ for the induced $\mathbf{V}$-morphism among morphism objects.

We continue with a minimalistic reminder on the *enriched* coend construction, which generalises the classical one from Definition 2.1.4.

**Construction 3.1.2.** Let $(\mathbf{V}, \otimes, \mathbf{1})$ be a nice monoidal category and let $\mathbf{I}$ be $\mathbf{V}$-enriched and small. Moreover, let $H\colon \mathbf{I}^{\mathrm{op}} \times \mathbf{I} \to \mathbf{V}$ be an enriched functor. Then we define the *enriched coend* to be the coequaliser

$$\int^{k \in \mathbf{I}} H(k,k) := \operatorname{coeq}\Big( \coprod_{k,n} \mathbf{I}\tbinom{k}{n} \otimes H(n,k) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_{k} H(k,k) \Big),$$

where the two parallel morphisms are given by

$$\alpha := \mathrm{ev}_{H(n,k),H(k,k)} \circ \big(H\tbinom{k,k}{n,k} \otimes H(n,k)\big) \circ \Big(\mathbf{I}\tbinom{k}{n} \otimes \mathbb{1}_k \otimes H(n,k)\Big),$$
$$\beta := \mathrm{ev}_{H(n,k),H(n,n)} \circ \big(H\tbinom{n,k}{n,n} \otimes H(n,k)\big) \circ \Big(\mathbb{1}_n \otimes \mathbf{I}\tbinom{k}{n} \otimes H(n,k)\Big).$$

### 3.1.2. *Examples of classical operads*

We assume that the reader is familiar with the basic notion of a (symmetric, monochromatic) operad $\mathscr{C}$ in a nice monoidal category and the notion of an algebra over $\mathscr{C}$, as it is for example presented in [MSS02]. Before coming to the formal definition of a coloured operad in full generality in the next subsection, let us start with a few examples of well-known operads in sets.

If $\mathbf{V}$ is nice in the sense of Setting 3.1.1, then each operad $\mathscr{C}$ in sets gives rise to an operad $\mathscr{C}^{\mathbf{V}}$ in $\mathbf{V}$ with $\mathscr{C}^{\mathbf{V}}(r) := [\mathscr{C}(r)]$.

**Example 3.1.3** (The trivial operad $\mathscr{I}$)**.** We define an operad $\mathscr{I}$ with operations

$$\mathscr{I}(r) := \begin{cases} \{\mathbb{1}\} & \text{for } r = 1, \\ \varnothing & \text{else,} \end{cases}$$

which means that $\mathscr{I}$ has only a single operation, namely the identity $\mathbb{1}$. Then $\mathscr{I}^{\mathbf{V}}$-algebras are objects in $\mathbf{V}$ without any extra structure.

**Example 3.1.4** (The operad of based sets $\mathscr{B}$)**.** We define an operad $\mathscr{B}$ by

$$\mathscr{B}(r) := \begin{cases} \{\mathfrak{v}\} & \text{for } r = 0, \\ \{\mathbb{1}\} & \text{for } r = 1, \\ \varnothing & \text{else,} \end{cases}$$

where we additionally have a single nullary operation $\mathfrak{v}$. The fraktur letter $\mathfrak{v}$ stands for 'void' („Leere") for reasons which will become apparent soon. A $\mathcal{B}^{\mathbf{V}}$-algebra is an object $X$ which comes together with a map $\mathbf{1} \to X$. For example, if $\mathbf{V} = \mathbf{Top}$, then $\mathcal{B}$-algebras are the same as based spaces.

**Example 3.1.5** (The commutative operad $\mathcal{C}om$)**.** We define an operad $\mathcal{C}om$ with operation sets

$$\mathcal{C}om(r) := \{\mathfrak{p}_r\} \quad \text{for all } r \geqslant 0,$$

again with the only possible structure maps, and $\mathbb{1} := \mathfrak{p}_1$. The fraktur letter $\mathfrak{p}$ stands for 'product', and we often write $\mathfrak{p} := \mathfrak{p}_2$ for the binary one. Then $\mathcal{C}om^{\mathbf{V}}$-algebras are the same as commutative monoid objects in $\mathbf{V}$. Note that the commutativity comes from the fact that $\mathfrak{S}_2 \cong \langle \tau \mid \tau^2 \rangle$ acts trivially on $\mathcal{C}om(2)$, and hence $\mathfrak{p}(x, x') = (\tau^*\mathfrak{p})(x, x') = \mathfrak{p}(x', x)$.

**Example 3.1.6** (The associative operad $\mathcal{A}ss$)**.** We define an operad $\mathcal{A}ss$ by

$$\mathcal{A}ss(r) := \mathfrak{S}_r \quad \text{for all } r \geqslant 0,$$

with $\mathbb{1} := \mathrm{id}_1$, input permutation $\tau^*\sigma := \sigma \circ \tau$ for $\sigma \in \mathcal{A}ss(r)$ and $\tau \in \mathfrak{S}_r$, and with composition given by *cabling*: let $r \geqslant 0$ and $s_1, \ldots, s_r \geqslant 0$:

- for $\sigma \in \mathfrak{S}_r$ we let $\sigma_{s_1,\ldots,s_r} \in \mathfrak{S}_{s_1+\cdots+s_r}$ be the *block permutation*

  $$(\sigma_{s_1,\ldots,s_r})(s_1 + \cdots + s_{i-1} + j) := j + \sum_{\sigma(i')<\sigma(i)} s_i.$$

- for $\sigma_i \in \mathfrak{S}_{s_i}$, we let $\sigma_1 \sqcup \cdots \sqcup \sigma_r \in \mathfrak{S}_{s_1+\cdots+s_r}$ be the *block sum*

  $$(\sigma_1 \sqcup \cdots \sqcup \sigma_r)(s_1 + \cdots + s_{i-1} + j) := s_1 + \cdots + s_{i-1} + \sigma_i(j).$$

Then we put $\sigma \circ (\sigma_1, \ldots, \sigma_r) := \sigma_{s_1,\ldots,s_r} \circ (\sigma_1 \sqcup \cdots \sqcup \sigma_r) \in \mathcal{A}ss(s_1 + \cdots + s_r)$. Then $\mathcal{A}ss^{\mathbf{V}}$-algebras are precisely monoid objects in $\mathbf{V}$, with multiplication given by $x \cdot x' := \mathrm{id}_2(x \otimes x')$.

We will now turn to the perhaps most important example of a *topological* operad, which has already appeared in Construction 2.2.1, namely the *little d-cubes operad*, which go back to [BV68; May72].

**Example 3.1.7.** For $1 \leqslant d < \infty$ we consider the *little d-cubes*[1] *operad* $\mathscr{C}_d$: we let $\mathscr{C}_d(r)$ be the space of tuples $(c_1, \dots, c_r)$ of rectilinear embeddings

$$c_i \colon [0;1]^d \hookrightarrow [0;1]^d, \qquad \begin{pmatrix} z^1 \\ \vdots \\ z^d \end{pmatrix} \mapsto \begin{pmatrix} a_i^1 + (b_i^1 - a_i^1) \cdot z^1 \\ \vdots \\ a_i^d + (b_i^d - a_i^d) \cdot z^d \end{pmatrix},$$

such that the interiors of their images $c_i((0;1)^d)$ are disjoint. We topologise $\mathscr{C}_d(r)$ as a subspace of $[0;1]^{2dr}$ containing the parameters $a_1^1, \dots, a_r^d, b_1^1, \dots, b_r^d$. The symmetric action is given by renumbering the boxes, and the composition is given by placing boxes inside each other and shift the numbers of the boxes accordingly, as in Figure 3.1, formally

$$\begin{aligned} &(c_1, \dots, c_r) \circ ((c_{1,1}, \dots, c_{1,s_1}), \dots, (c_{r1}, \dots, c_{r,s_r})) \\ &\quad := (c_1 \circ c_{1,1}, \dots, c_1 \circ c_{1,s_1}, \dots, c_r \circ c_{r,1}, \dots, c_r \circ c_{r,s_r}) \end{aligned}$$

Note that $\mathscr{C}_d(0)$ contains exactly one configuration of boxes, namely the empty one. This configuration deserves the name *void* $\mathfrak{v}$.

A variation of $\mathscr{C}_d$, called the *little d-discs operad* $\mathscr{D}_d$, uses discs instead of cuboids: we define $\mathscr{D}_d(r)$ to be the space of tuples $(c_1, \dots, c_r)$ of embeddings $c_i \colon \mathbb{D}^d \hookrightarrow \mathbb{D}^d$, which are of the form $c_i(z) = \hat{z}_i + \varepsilon_i \cdot z$, such that the interiors of their images $c_i(\mathring{\mathbb{D}}^d)$ are disjoint. Pictorially, $\hat{z}_i \in \mathbb{D}^d$ is the midpoint and $\varepsilon_i > 0$ is the radius of the $i^{\text{th}}$ small disc, and we topologise $\mathscr{D}_d(r)$ as a subspace of $(\mathbb{D}^d)^r \times \mathbb{R}^r_{>0}$ containing these parameters. The operadic structure is defined in exactly the same way as for $\mathscr{C}_d$.

It is an expectable and classical result, see for example [MSS02, § 1.4.1], that $\mathscr{C}_d$ and $\mathscr{D}_d$ are equivalent as topological operads, i.e. there is a zig-zag of operad morphisms which consist of weak equivalences among operation spaces for each arity. Thus, we may freely swap between both models.

There are embeddings $\mathscr{C}_d \hookrightarrow \mathscr{C}_{d+1}$ and we obtain a limit operad $\mathscr{C}_\infty$. For $1 \leqslant d \leqslant \infty$, algebras over $\mathscr{C}_d$ (or, equivalently, $\mathscr{D}_d$) are called *$E_d$-algebras*. It is a classical result [May72] that $d$-fold loop spaces are $E_d$-algebras, and conversely, each $E_d$-algebra $X$ admits a $d$-fold bar construction $B^d X$ which comes with a map $\Sigma^d X \to B^d X$ such that the adjoint $X \to \Omega^d B^d X$ is a group completion in the sense of [MS76].

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[1] The name 'little *d-cuboids* operad' would be more adequate, as all images $\operatorname{im}(c_i) \subseteq [0;1]^d$ are cuboids, rather than honest cubes.

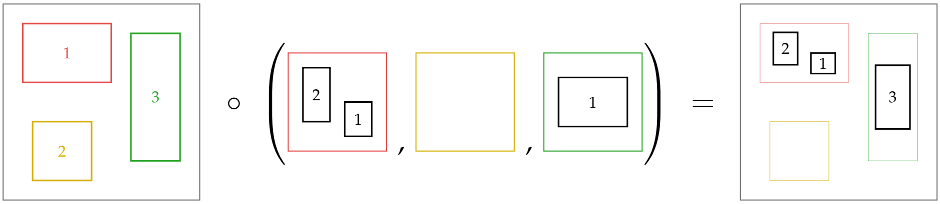


**Figure 3.1.** The composition $\mathscr{C}_2(3) \times (\mathscr{C}_2(2) \times \mathscr{C}_2(0) \times \mathscr{C}_2(1)) \to \mathscr{C}_2(3)$

### 3.1.3. *Coloured operads in symmetric monoidal categories*

Here we give the formal definition of a coloured operad in a general symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$. We stick to the notation of [Yau16; BM07], but we try to be less technical.

**Notation 3.1.8.** If $L_i = (l_{i,1}, \ldots, l_{i,s_i}) \in N^{s_i}$ is a tuple of elements in $N$ for each $1 \leqslant i \leqslant r$, then define the *concatenation*

$$L_1 \cdots L_r := (l_{1,1}, \ldots, l_{1,s_1}, \ldots, l_{r,1}, \ldots, l_{r,s_r}) \in N^{s_1+\cdots+s_r}.$$

**Definition 3.1.9.** Let $(\mathbf{V}, \otimes, \mathbf{1})$ be a nice monoidal category as in Setting 3.1.1 and let $N$ be a set. Let $N \wr \mathbf{\Sigma}$ be as in Definition 2.1.2; then an *$N$-coloured operad* $\mathscr{O}$ is a family $\left(\mathscr{O}\binom{-}{n} \colon (N \wr \mathbf{\Sigma})^{\mathrm{op}} \to \mathbf{V}\right)_{n \in N}$ of functors, equipped with:

1. identities $\mathbb{1}_n \colon \mathbf{1} \to \mathscr{O}\binom{n}{n}$ for each $n \in N$;
2. composition maps for each $n \in N$ and tuples $K, L_1, \ldots, L_r$ in $N$, which are of the form
$$\kappa \colon \mathscr{O}\binom{K}{n} \otimes \bigotimes_{i=1}^{r} \mathscr{O}\binom{L_i}{k_i} \to \mathscr{O}\binom{L_1 \cdots L_r}{n},$$

such that the expectable axioms of unitality, associativity, and two-sided equivariancy from [Yau16, § 11.2] are satisfied. When considering $\mathscr{O}\binom{K}{n}$, we call $K$ the *input profile*, $n$ the *output colour*, and $r$ the *arity*. The induced maps $\tau^* \colon \mathscr{O}\binom{K}{n} \to \mathscr{O}\binom{\tau^* K}{n}$ are called *input permutations*.

We call $\mathscr{O}$ *monochromatic* if $N = *$ is just a singleton. In this case, we also write $\mathscr{O}(r) := \mathscr{O}\binom{*, \ldots, *}{*}$ for the object of $r$-ary operations. For an $N$-coloured operad $\mathscr{O}$ and a subset $N' \subseteq N$, we consider the restriction $\mathscr{O}|_{N'}$ to those operation spaces where all input colours and the output colour live in $N'$. In particular, for $n \in N$, there is a monochromatic operad $\mathbb{O}|_n := \mathbb{O}|_{\{n\}}$ with operation spaces $(\mathbb{O}|_n)(r) := \mathbb{O}\binom{n,\dots,n}{n}$.

For a fixed colour set $N$, a morphism of $N$-coloured operads $\rho\colon \mathbb{O} \to \mathcal{P}$ is a collection of natural transformations $(\rho_n^- : \mathbb{O}\binom{-}{n} \Rightarrow \mathcal{P}\binom{-}{n})_{n\in N}$ which commutes with units and compositions in the obvious way. This gives rise to a category $\mathbf{Op}_N(\mathbf{V})$ of $N$-coloured operads in $\mathbf{V}$.

In principle, as functors between categories do not have to fix the object sets, there is a more general notion of morphisms between differently coloured operads, which we will not use in this thesis.

**Example 3.1.10.** A small enriched category $\mathbf{I}$ is the same as an $\mathrm{ob}(\mathbf{I})$-coloured operad with $\mathbf{I}\binom{k_1,\dots,k_r}{n} = \varnothing$ for $r \neq 1$, where $\varnothing$ is the initial object. Conversely, coloured operads can be seen as categories in which morphisms have multiple inputs and hence are sometimes called *(symmetric) multicategories*.

**Remark 3.1.11.** An *operation* $\mu \in \mathbb{O}\binom{K}{n}$ is a map $\mu\colon \mathbf{1} \to \mathbb{O}\binom{K}{n}$, and we call $\mu$ a *nullary*, *unary*, or *binary* operation if $r$ is 0, 1, or 2, respectively. For a collection of operations $\mu_i \in \mathbb{O}\binom{L_i}{k_i}$, we consider $\mu_1 \otimes \cdots \otimes \mu_r \in \bigotimes_i \mathbb{O}\binom{L_i}{k_i}$ and denote the composition by $\mu \circ (\mu_1 \otimes \cdots \otimes \mu_r)$.

We also use a short notation for *partial composition*: for each $\mu \in \mathbb{O}\binom{K}{n}$ and $\mu' \in \mathbb{O}\binom{L}{k_i}$, we write $\mu \circ_i \mu' := \mu \circ (\mathbb{1}_{k_1} \otimes \cdots \otimes \mathbb{1}_{k_{i-1}} \otimes \mu' \otimes \mathbb{1}_{k_{i+1}} \otimes \cdots \otimes \mathbb{1}_{k_r})$.

In some sense, operads are 'designed' to act on other objects. This can be formalised in the definition of an *algebra* over a given operad.

**Definition 3.1.12.** Let $\mathbb{O}$ be an $N$-coloured operad in $(\mathbf{V}, \otimes, \mathbf{1})$. An $\mathbb{O}$-*algebra* is an $N$-indexed family $\boldsymbol{X} := (X_n)_{n\in N}$ of objects in $\mathbf{V}$, together with maps

$$\lambda\colon \mathbb{O}\binom{k_1,\dots,k_r}{n} \otimes (X_{k_1} \otimes \cdots \otimes X_{k_r}) \to X_n,$$

such that the expectable axioms of unitality, associativity, and equivariancy from [Yau16, § 13] hold. We call $X_n$ the $n^{th}$ *level* of $\boldsymbol{X}$.

A map $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ of $\mathbb{O}$-algebras is a compatible family $(f_n\colon X_n \to Y_n)_{n\in N}$ of maps: this gives rise to the category $\mathbb{O}$-$\mathbf{Alg}$ of $\mathbb{O}$-algebras.

**Example 3.1.13.** For each $N$-coloured operad $\mathbb{O}$, the family $(\mathbb{O}\binom{}{n})_{n\in N}$ of nullaries constitutes an $\mathbb{O}$-algebra, which is initial in the category $\mathbb{O}$-$\mathbf{Alg}$ by construction, and hence is called the *initial* $\mathbb{O}$-*algebra*.

**Construction 3.1.14.** Each $\mathcal{O}$-algebra has an underlying $N$-indexed family of objects. If we denote by $\mathbf{V}^N$ the category of them, then the forgetful functor $U^{\mathcal{O}}\colon \mathcal{O}\text{-}\mathbf{Alg}\to\mathbf{V}^N$ admits a left adjoint $F^{\mathcal{O}}$, called the *free $\mathcal{O}$-algebra functor*: each family $\boldsymbol{X}=(X_n)_{n\in N}$ in $\mathbf{V}$ gives rise to a functor $\boldsymbol{X}^{\otimes -}\colon N\wr\boldsymbol{\Sigma}\to\mathbf{V}$ as in Construction 2.1.3, and we define $F^{\mathcal{O}}(\boldsymbol{X})$ levelwise for each $n\in N$ by

$$F^{\mathcal{O}}(\boldsymbol{X})_n := \int^{K\in N\wr\boldsymbol{\Sigma}} \mathcal{O}\tbinom{K}{n}\otimes\boldsymbol{X}^{\otimes K}.$$

The $\mathcal{O}$-action is given by the top horizontal arrow in

$$\begin{tikzcd}
\mathcal{O}\tbinom{K}{n}\otimes\big(F^{\mathcal{O}}(\boldsymbol{X})_{k_1}\otimes\cdots\otimes F^{\mathcal{O}}(\boldsymbol{X})_{k_r}\big) \arrow[r, dashed] \arrow[d, equal] & F^{\mathcal{O}}(\boldsymbol{X})_n \arrow[dd, equal] \\
\mathcal{O}\tbinom{K}{n}\otimes\bigotimes_i\int^{L_i}\mathcal{O}\tbinom{L_i}{k_i}\otimes\boldsymbol{X}^{\otimes L_i} \arrow[d, equal] & \\
\int^{L_1,\dots,L_r}\mathcal{O}\tbinom{K}{n}\otimes\bigotimes_i\mathcal{O}\tbinom{L_i}{k_i}\otimes\boldsymbol{X}^{\otimes L_1\cdots L_r} \arrow[r] & \int^{L}\mathcal{O}\tbinom{L}{n}\otimes\boldsymbol{X}^{\otimes L}.
\end{tikzcd}$$

**Construction 3.1.15.** Each map $\rho\colon\mathcal{P}\to\mathcal{O}$ of $N$-coloured operads gives rise to a *base-change adjunction* $\rho_!\colon\mathcal{P}\text{-}\mathbf{Alg}\rightleftarrows\mathcal{O}\text{-}\mathbf{Alg}:\rho^*$ as follows: each $\mathcal{O}$-algebra is an $\mathcal{P}$-algebra by restriction; conversely, note that each $\mathcal{P}$-algebra $\boldsymbol{X}$ is the reflexive coequaliser of $F^{\mathcal{P}}U^{\mathcal{P}}F^{\mathcal{P}}U^{\mathcal{P}}\boldsymbol{X}\rightrightarrows F^{\mathcal{P}}U^{\mathcal{P}}\boldsymbol{X}$, so $\rho_!\boldsymbol{X}$ is the reflexive coequaliser of $F^{\mathcal{O}}U^{\mathcal{P}}F^{\mathcal{P}}U^{\mathcal{P}}\boldsymbol{X}\rightrightarrows F^{\mathcal{O}}U^{\mathcal{P}}\boldsymbol{X}$: see [BM07, § 4] for details.

When $\rho$ is clear from the context, we also write $F^{\mathcal{O}}_{\mathcal{P}}\colon\mathcal{P}\text{-}\mathbf{Alg}\rightleftarrows\mathcal{O}\text{-}\mathbf{Alg}:U^{\mathcal{O}}_{\mathcal{P}}$ for the base-change adjunction.

Let me point out that the adjunction $\rho_!\vdash\rho^*$ is *monadic*, i.e. it is equivalent to the free–forgetful adjunction for algebras over the monad $\rho^*\rho_!$ on the category $\mathcal{P}$-**Alg**. To see this, note that $\rho^*$ reflects isomorphisms and preserves reflexive coequalisers (since both monads $U^{\mathcal{O}}F^{\mathcal{O}}$ and $U^{\mathcal{P}}F^{\mathcal{P}}$ are sifted [GKR18, Cor. 4.12]), and invoke the 'crude' monadicity theorem [BW85, § 3.5].

**Construction 3.1.16.** Let $H\colon(\mathbf{V},\otimes,\mathbf{1})\to(\mathbf{V}',\otimes',\mathbf{1}')$ be a lax monoidal functor between nice monoidal categories.

If we are given an $N$-coloured operad $\mathcal{O}$ in $\mathbf{V}$, then we obtain an induced an $N$-coloured operad $H\mathcal{O}$ inside $\mathbf{V}'$ by transferring the structure along $H$. Similarly, for each $\mathcal{O}$-algebra $\boldsymbol{X}=(X_n)_{n\in N}$, we get an $H\mathcal{O}$-algebra $H\boldsymbol{X}$.

**Example 3.1.17.** The functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ is even strong monoidal. Thus, if $\mathscr{C}$ is a topological operad, then $\pi_0(\mathscr{C})$ is an operad in sets. For example, it is an easy exercise to see that for the little $d$-cubes operad, we have

$$\pi_0(\mathscr{C}_d) = \begin{cases} \mathscr{Ass} & \text{for } d = 1, \\ \mathscr{Com} & \text{for } d \geqslant 2. \end{cases}$$

**Example 3.1.18.** The singular chain complex functor

$$C_\bullet^{\text{sing}}(-;R)\colon (\mathbf{Top}, \times, *) \to (\mathbf{Ch}_R, \otimes, R^{[0]})$$

from the category of topological spaces to the category of chain complexes in $R$-modules is lax monoidal by the Eilenberg–Zilber transformation. Thus, if $\mathbb{O}$ is a topological operad, then $C_\bullet^{\text{sing}}(\mathbb{O}; R)$ is an operad in $\mathbf{Ch}_R$, and if $\boldsymbol{X}$ is an $\mathbb{O}$-algebra, then $C_\bullet^{\text{sing}}(\mathbb{O}; R)$ acts on $C_\bullet^{\text{sing}}(\boldsymbol{X}; R)$. This action gives rise to a collection of homology operations on $\mathbb{O}$-algebras, see Subsection 3.3.1.

By the Künneth theorem, taking homology is a lax monoidal functor from $\mathbf{Ch}_R$ to the category $R$-$\mathbf{Mod}^{\mathbb{Z}}$ of graded modules. Thus, $H_\bullet(\mathbb{O}; R)$ becomes an operad in $R$-$\mathbf{Mod}^{\mathbb{Z}}$ acting on $H_\bullet(\boldsymbol{X}; R)$ for each $\mathbb{O}$-algebra $\boldsymbol{X}$.

## 3.2. Presentation of operads

In this section, we describe how to *present* a coloured operad $\mathbb{O}$. By this, we roughly mean a description of $\mathbb{O}$ via generators and relations, which helps us to understand what extra structure an $\mathbb{O}$-algebra carries.

Before coming to these notions, we give a short introduction to the language of trees. For details we refer to [LV12, Apx. C] and [MSS02, § II.1.5].

**Definition 3.2.1.** A *graph* is a pair $\Gamma = (V, E)$ where $V$ is a finite set and $E$ is a subset of the power set of $V$, such that each $e \in E$ has exactly two elements.

We call each $v \in V$ a *vertex* and each $e \in E$ an *edge*, and we say that $e$ is an *adjacent edge* for $v$ if $v \in e$. For each $v \in V$, we define the *degree* of $v$ to be the number of adjacent edges $\#v := \#\{e \in E;\ v \in e\}$.

Note that $\Gamma$ can be regarded as a 1-dimensional simplicial complex and we can consider its geometric realisation $|\Gamma|$. We call $\Gamma$ a *tree* if $|\Gamma|$ is contractible, or, in other words, if the graph is connected and has no cycles.

**Definition 3.2.2.** An *admissible tree* is a tuple $Y = (V, E, \bar{v}, V_\circ)$, where:

1. $(V, E)$ is a tree with at least one edge;
2. $\bar{v} \in V$ is a distinguished vertex of degree 1, called the *root*;
3. $V_\circ$ is a subset of the vertex set whose elements all have to be of degree 1 and have to be different from the root. A vertex in $V_\circ$ is called a *leaf*.

All vertices which are neither the root nor a leaf are called *internal vertices*, and we let $V_\bullet$ be the set of them, i.e. $V = V_\bullet \,\dot{\cup}\, V_\circ \,\dot{\cup}\, \{\bar{v}\}$.

Standing at a single vertex, an adjacent edge is called *incoming* if we increase the distance towards the root when following it, and *outgoing* if we come closer to the root when following it. By definition, the root vertex has only one adjacent edge, which is incoming and which deserves the name $e_Y$, and each other vertex $v$ has a unique outgoing edge $e_v$. While a leaf has no incoming edges, an internal vertex may have several. For each internal vertex $v \in V_i$, let $r_v := \#v - 1 \geqslant 0$ be the number of incoming edges, called the *arity* of $v$. Moreover, we let $E_v \subseteq E$ be the subset of incoming edges at $v$.

While admissibility only describes the *shape* of the trees we are interested in, we want to systematically label parts of a tree in order to endow it with all information we need in order to describe operadic structures. This is captured by the following definition:

**Definition 3.2.3.** An *N-tree* is an admissible tree $Y = (V, E, \bar{v}, V_\circ)$, with:

1. for $r_Y := \#V_\circ$, a bijection $\lambda\colon \{1, \dots, r_Y\} \to V_\circ$, called the *leaf numbering*;
2. for each internal vertex $v \in V_\bullet$, a bijection $\lambda_v\colon \{1, \dots, r_v\} \to E_v$, called the *input ordering at v*;
3. a labelling of the edges $\varepsilon\colon E \to N$, called the *typification*.

The *input profile* of $Y$ is given by $K_Y := (\varepsilon(e_{\lambda(1)}), \dots, \varepsilon(e_{\lambda(r)}))$ and the *output colour* is given by $n_Y := \varepsilon(e_Y)$. Likewise, for each internal vertex $v$, we define its input profile $K_v := (\varepsilon\lambda_v(1), \dots, \varepsilon\lambda_v(r_v))$ and its output colour $n_v := \varepsilon(e_v)$.

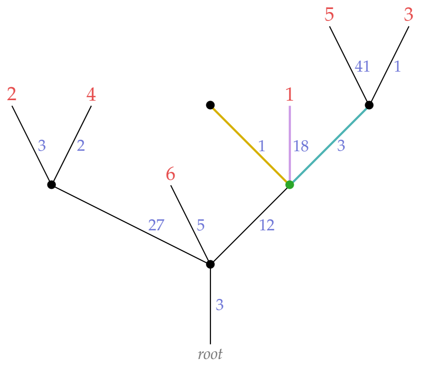


**Figure 3.2.** An $\mathbb{N}$-tree: the numbering $\lambda$ of the outer leaves is written in red, the labels $\varepsilon$ of the edges are written in blue. Accordingly, we have $K = (18, 3, 1, 2, 41, 5)$ as well as $n = 3$. There are five internal vertices, depicted by a bullet. If we call the green vertex $v$, then $K_v = (1, 18, 3)$ and $n_v = 12$. Moreover, $\lambda_v(1)$ is the yellow, $\lambda_v(2)$ the violet, and $\lambda_v(3)$ the turquoise edge. Finally, the other end of the yellow edge is 1-valent, but no outer leaf: it is an internal vertex of arity 0.

**Remark 3.2.4.** One typically replaces the input orderings at each internal vertex by an isotopy class of an embedding $|Y| \hookrightarrow \mathbb{R}^2$. These definitions can be translated into each other: given an $N$-tree $Y$, we embed $|Y|$ inductively: at each internal vertex, we place the $r(v)$ incoming edges on top, ordered as it is prescribed by $\lambda_v$, from left to right. Then we continue with the branches starting at $\lambda_v(1), \dots, \lambda_v(r)$, see Figure 3.2 for an example.

Conversely, given an isotopy type of an embedding $|Y| \hookrightarrow \mathbb{R}^2$, we can read off the input ordering at each internal vertex $v$ from the image of $|Y|$.

The final combinatorial ingredient we need before coming back to operads is the notion of *grafting*. I ought to point out that this English word may only be common among arborists and mathematicians (a German translation would be „Pfropfung“): in horticulture, it desribes a technique to place parts of a tree—or, more generally, a plant—onto another one, such that they continue their growth together. This is precisely what we are going to do.

**Definition 3.2.5** (Grafting)**.** If $Y, Y_1, \dots, Y_r$ are $N$-trees with $K_Y = (n_{Y_1}, \dots, n_{Y_r})$, then we can form a new $N$-tree $Y \circ (Y_1, \dots, Y_r)$ as follows: we remove for

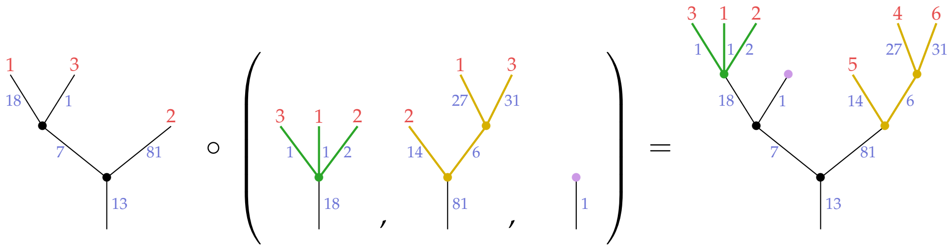


**Figure 3.3.** A grafting $Y \circ (Y_1, Y_2, Y_3)$ of $\mathbb{N}$-trees

each $Y_i$ the root and the edge adjacent to the root, and place the remaining tree which starts at the first internal vertex of $Y_i$ at the $i^{\text{th}}$ outer leaf of $Y$, as we see in Figure 3.3. Then the outer leaves of $Y \circ (Y_1, \ldots, Y_r)$ are exactly the joint outer leaves of $Y_1, \ldots, Y_r$, and we number them by cabling the old numberings: $\lambda(r_{Y_1} + \cdots + r_{Y_{i-1}} + j) = \lambda_{Y_i}(j)$.

The internal vertices of $Y \circ (Y_1, \ldots, Y_r)$ are exactly the joint internal vertices of $Y, Y_1, \ldots, Y_r$, having the same input profile and output colour.

In order to present an operad, we need a reasonable notion of freeness, and we choose our bases to be collections of operation spaces which, heuristically, already carry the information how to permute inputs, but lack neutral elements and compositions. The following definition makes this precise:

**Definition 3.2.6.** Let $N$ be a set and $(\mathbf{V}, \otimes, \mathbf{1})$ a symmetric monoidal category. An *$N$-coloured quiver* $\mathscr{A}$ is a family of functors $(\mathscr{A}\binom{-}{n} \colon (N \wr \mathbf{\Sigma})^{\mathrm{op}} \to \mathbf{V})_{n \in N}$. A morphism $\vartheta \colon \mathscr{A} \to \mathscr{A}'$ of quivers is a family $(\vartheta_n \colon \mathscr{A}\binom{-}{n} \Rightarrow \mathscr{A}'\binom{-}{n})_{n \in N}$ of natural transformations; this gives rise to the category $\mathbf{Qu}_N(\mathbf{V})$ of quivers.

Each $N$-coloured operad $\mathscr{O}$ has an underlying quiver by forgetting compositions and identities, and we get a forgetful functor $U \colon \mathbf{Op}_N(\mathbf{V}) \to \mathbf{Qu}_N(\mathbf{V})$.

This forgetful functor admits a left adjoint $\Psi \colon \mathbf{Qu}_N(\mathbf{V}) \to \mathbf{Op}_N(\mathbf{V})$, which generalises the monochromatic construction from [MSS02, § 1.9]:

**Construction 3.2.7.** Let $\mathscr{A}$ be an $N$-coloured quiver in $\mathbf{V}$. Then we define an operad $\Psi\mathscr{A}$ as follows: for each $N$-tree $Y$, we write $\mathscr{A}(Y) := \bigotimes_v \mathscr{A}\binom{K_v}{n_v}$, where $v$ ranges over all internal vertices of $Y$. If $\tau$ is a family of permutations

$\tau_v \in \mathfrak{S}_{r_v}$, then we let $\tau_* Y$ be the $N$-tree which arises from $Y$ by replacing the input ordering $\lambda_v$ at each internal vertex by $\lambda_v \circ \tau_v^{-1}$. We define

$$(\Psi\mathscr{A})\tbinom{K}{n} := \operatorname{coeq}\Big( \textstyle\coprod_Y \coprod_\tau \mathscr{A}(\tau_* Y) \rightrightarrows \coprod_Y \mathscr{A}(Y) \Big),$$

where $Y$ ranges over all $N$-trees with $K_Y = K$ and $n_Y = n$, where $\tau$ ranges over all families of permutations $\tau_v \in \mathfrak{S}_{r_v}$, and where the summand $\mathscr{A}(\tau_* Y)$ indexed by $(Y, \tau)$ gets sent along the upper arrow via $\bigotimes_v \tau_v^*$ to $\mathscr{A}(Y)$ and along the lower arrow via the identity to $\mathscr{A}(\tau_* Y)$. The operadic structure on $\Psi\mathscr{A}$ is declared as follows:

1. $\Psi\mathscr{A}$ is again an $N$-coloured quiver: given an $N$-tree $Y$ and a permutation $\tau \in \mathfrak{S}_{r_Y}$, then we let $\tau^* Y$ be the $N$-tree which arises from $Y$ by replacing the *global* input ordering $\lambda$ by $\lambda \circ \tau$. Since permuting the outer leaves does not affect the internal vertices, there are canonical identifications $\mathscr{A}(Y) \cong \mathscr{A}(\tau^* Y)$ that assemble into a map $\tau^* \colon (\Psi A)\binom{K}{n} \to (\Psi A)\binom{\tau^* K}{n}$.

2. For each colour $n \in N$, there is a distinguished $N$-tree $I_n$ which has a single outer leaf and no internal vertex, and the single edge between the root and the outer leaf carries the label $n$. Then $K_{I_n} = (n)$ and $n_{I_n} = n$; the product $\mathscr{A}(I_n)$ is empty and hence equals the monoidal unit $\mathbf{1}$. We put $\mathbb{1}_n \colon \mathbf{1} \cong \mathscr{A}(I_n) \hookrightarrow (\Psi\mathscr{A})\binom{n}{n}$.

3. For the composition, we use that $\otimes$ commutes with colimits in each argument, so it is enough to describe the composition consistently on the union $(\coprod_Y \mathscr{A}(Y)) \otimes \bigotimes_i \big(\coprod_{Y_i} \mathscr{A}(Y_i)\big) \cong \coprod_{Y, Y_1, \ldots, Y_r} (\mathscr{A}(Y) \otimes \bigotimes_i \mathscr{A}(Y_i))$ where $Y$ ranges over all $N$-trees as before and $Y_i$ ranges over all $N$-trees with $K_{Y_i} = L_i$ and $n_{Y_i} = k_i$. This is the point where we graft trees: as $Y \circ (Y_1, \ldots, Y_r)$ has exactly the joint internal vertices of $Y, Y_1, \ldots, Y_r$, we have a map $\mathscr{A}(Y) \otimes \bigotimes_i \mathscr{A}(Y_i) \cong \mathscr{A}(Y \circ (Y_1, \ldots, Y_r)) \hookrightarrow (\Psi\mathscr{A})\binom{L_1 \cdots L_r}{n}$.

**Remark 3.2.8.** We give some intuition what $\Psi\mathscr{A}$ looks like for $\mathbf{V} = \mathbf{Top}$: elements of $(\Psi\mathscr{A})\binom{K}{n}$ are $N$-trees $Y$, together with an element of $\mathscr{A}\binom{K}{n}$ for each internal vertex $v$ with $K_v = K$ and $n_v = n$, called an *$\mathscr{A}$-labelled $N$-tree.*

Note that we have an inclusion of $N$-coloured quivers $\mathscr{A} \hookrightarrow U\Psi\mathscr{A}$ by identifying each $\mathfrak{a} \in \mathscr{A}\binom{k_1, \ldots, k_r}{n}$ with the $\mathscr{A}$-labelled $N$-tree having only a single internal vertex labelled by $\mathfrak{a}$. Clearly, this is the unit of the adjunction.

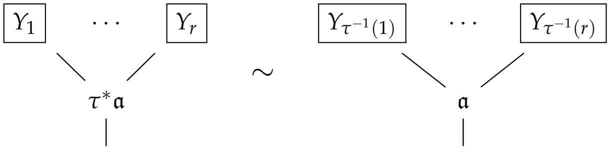


**Figure 3.4.** The balancing relation pictorially: here $Y_1, \ldots, Y_r$ are the incoming branches at $\tau^*\mathfrak{a}$.

The balancing relation which is coequalised can be described as follows: if one internal vertex is labelled with $\tau^*\mathfrak{a}$ for some $\mathfrak{a} \in \mathscr{A}$, then we might as well change its label to $\mathfrak{a}$ and instead alter the order in which the incoming branches enter the vertex, see Figure 3.4.

In order to present an operad, we need the correct notion of an operadic 'kernel'. Here it makes sense to give two separate definitions; one for the category of topological spaces and one for algebraic categories:

**Definition 3.2.9.** Let $\mathfrak{O}$ be an $N$-coloured topological operad. An *operadic relation* $I$ on $\mathfrak{O}$ is a system of equivalence relations $I\binom{K}{n}$ on each operation space $\mathfrak{O}\binom{K}{n}$, which we just denote by '$\sim$', such that:

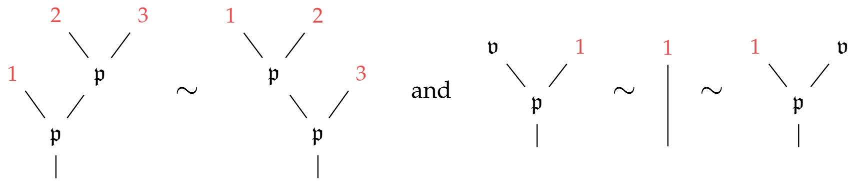
1. if $\mu \sim \mu'$ for $\mu, \mu' \in \mathfrak{O}\binom{K}{n}$, then $\tau^*\mu \sim \tau^*\mu'$ for each $\tau \in \mathfrak{S}_r$;
2. if $\mu \sim \mu'$ and $\mu_i \sim \mu_i'$, then $\mu \circ (\mu_1, \ldots, \mu_r) \sim \mu' \circ (\mu_1', \ldots, \mu_r')$.

If $\rho\colon \mathfrak{O} \to \mathscr{P}$ is a morphism of operads, then the system of relations, which declares $\mu$ to be related to $\mu'$ if and only if $\rho(\mu) = \rho(\mu')$ holds, is an operadic relation. Conversely, given an operadic relation $I$, we can form the *quotient* $\mathfrak{O}/I$ by $(\mathfrak{O}/I)\binom{K}{n} := \mathfrak{O}\binom{K}{n}/I\binom{K}{n}$, together with the induced input permutations, neutral elements, and composition maps. This quotient operad enjoys the obvious universal property: each morphism $\rho\colon \mathfrak{O} \to \mathscr{P}$ of operads satisfying $\rho(\mu) = \rho(\mu')$ for all $\mu \sim \mu'$ extends uniquely to a map $\bar{\rho}\colon \mathfrak{O}/I \to \mathscr{P}$.

Clearly, the intersection of a family of operadic relations is again an operadic relation; hence, given a system $J$ of relations on $\mathfrak{O}$, there is a minimal operadic relation $\langle J \rangle$ which contains $J$. One way of describing an operad is to *present* it; that is: providing an $N$-coloured quiver $\mathscr{A}$ and a family of relations $J$ on $\Psi\mathscr{A}$. Then we define $\langle \mathscr{A} \mid J \rangle := (\Psi\mathscr{A})/\langle J \rangle$.

**Example 3.2.10.** The first four examples from Subsection 3.1 admit an easy presentation: note that a monochromatic quiver is just a symmetric sequence $\mathscr{A}\colon \mathbf{\Sigma}^{\mathrm{op}} \to \mathbf{V}$, and we write its constituents as $\mathscr{A}(r)$.

1. The trivial operad $\mathscr{I}$ is freely generated by the empty quiver.
2. The operad $\mathscr{B}$ of based spaces is freely generated by $\mathscr{A}(0) = \{\mathfrak{v}\}$ and $\mathscr{A}(r) = \varnothing$ for $r \geqslant 1$.
3. The commutative operad $\mathscr{C}om$ is generated by the symmetric sequence with $\mathscr{A}(0) = \{\mathfrak{v}\}$ and $\mathscr{A}(2) = \{\mathfrak{p}\}$, and related by the identifications $\mathfrak{p} \circ_1 \mathfrak{p} \sim \mathfrak{p} \circ_2 \mathfrak{p}$ and $\mathfrak{p} \circ_1 \mathfrak{v} \sim \mathbb{1} \sim \mathfrak{p} \circ_2 \mathfrak{v}$, where we use the $\circ_i$-notation from Remark 3.1.11. The relations can be expressed in trees as follows:



4. The associative operad $\mathscr{A}ss$ is generated by a quiver with $\mathscr{A}(0) = \{\mathfrak{v}\}$ and $\mathscr{A}(2) = \{\mathfrak{p}, \mathfrak{p}'\}$, where $\mathfrak{S}_2$ permutes the two points in $\mathscr{A}(2)$. The relations are the same as for $\mathscr{C}om$; it is enough to state them for $\mathfrak{p}$, then we can conclude e.g. $\mathfrak{p}' \circ_1 \mathfrak{p}' = (1\,3)^*(\mathfrak{p} \circ_2 \mathfrak{p}) \sim (1\,3)^*(\mathfrak{p} \circ_1 \mathfrak{p}) = \mathfrak{p}' \circ_2 \mathfrak{p}'$.

We will see further advantages of presenting topological operads in Subsection 3.3.2. Let us now come to presentations of algebraic operads. We restrict ourselves to the category of graded $R$-modules $R$-$\mathbf{Mod}^{\mathbb{Z}}$ for a commutative ring $R$ and we follow [LV12, § 5.2], who discussed the monochromatic case.

**Definition 3.2.11.** Let $\mathscr{O}$ be an $N$-coloured operad in $R$-$\mathbf{Mod}^{\mathbb{Z}}$. An *operadic ideal* $I \subseteq \mathscr{O}$ is a system of submodules $I\binom{K}{n} \subseteq \mathscr{O}\binom{K}{n}$, such that:

1. if $\mu \in I\binom{K}{n}$, then $\tau^*\mu \in I\binom{\tau^*K}{n}$ for each $\tau \in \mathfrak{S}_r$;
2. if out of $\mu, \mu_1, \ldots, \mu_r$, at least one operation lies in $I$, then so does the composition $\mu \circ (\mu_1 \otimes \cdots \otimes \mu_r)$.

If $\rho\colon \mathcal{O} \to \mathcal{P}$ is a morphism of operads, then its kernel $\ker(\rho)$, given by the family of submodules $\{\mu \in \mathcal{O}\binom{K}{n};\ \rho(\mu) = 0\}$, is an operadic ideal. Conversely, given an operadic ideal $I \subseteq \mathcal{O}$, we can form the *quotient operad* $\mathcal{O}/I$ with operation modules $(\mathcal{O}/I)\binom{K}{n} := \mathcal{O}\binom{K}{n}/I\binom{K}{n}$, together with the induced input permutations, neutral elements, and composition maps. This quotient has the obvious universal property: each morphism $\rho\colon \mathcal{O} \to \mathcal{P}$ of operads satisfying $I \subseteq \ker(\rho)$ extends uniquely to an operad map $\bar{\rho}\colon \mathcal{O}/I \to \mathcal{P}$.

Clearly, the intersection of a family of operadic ideals is again an operadic ideal; hence, given a system $J$ of submodules $J\binom{K}{n} \subseteq \mathcal{O}\binom{K}{n}$, there is a minimal operadic ideal $\langle J \rangle \subseteq \mathcal{O}$ which contains $J$. One way of describing an algebraic operad is to *present* it, that is: providing an $N$-coloured quiver $\mathcal{A}$ and a family of submodules $J$ of $\Psi\mathcal{A}$. Then we define $\langle \mathcal{A} \mid J \rangle := (\Psi\mathcal{A})/\langle J \rangle$.

**Remark 3.2.12.** The main purpose of presenting an algebraic operad is to gain a better description of its algebras: if a presented operad $\mathcal{O} = \langle \mathcal{A} \mid J \rangle$ acts on an algebra $\boldsymbol{X} = (X_n)_{n\in N}$, then the action can equivalently be given as follows: each $\mathcal{A}\binom{K}{n}$ is a graded $R$-module, with the grading components $\mathcal{A}\binom{K}{n}_s$. If we denote the grading components of $X_n$ by $(X_{n,h})_{h\in\mathbb{Z}}$, then each generator $\mathfrak{a} \in \mathcal{A}\binom{K}{n}_s$ can be regarded as a map $\mathfrak{a}\colon X_{k_1,h_1} \otimes \cdots \otimes X_{k_r,h_r} \to X_{n,s+h_1+\cdots+h_r}$ and we have the relation

$$(\tau^*\mathfrak{a})(x_1 \otimes \cdots \otimes x_r) = \mathrm{sg}_{x_1,\ldots,x_r}(\tau) \cdot \mathfrak{a}(x_{\tau^{-1}(1)} \otimes \cdots \otimes x_{\tau^{-1}(r)}),$$

where $\mathrm{sg}_{x_1,\ldots,x_r}(\tau)$ is the sign weighted by the degrees of $x_1, \ldots, x_r$. Moreover, for each relation in $J$, we get relations among these multilinear maps, and morphisms of $\mathcal{O}$-algebras are equivariant with respect to them.

We close this section with an example which relates an interesting topological operad to an interesting algebraic operad.

**Example 3.2.13** (The Poisson $d$-operad)**.** The *Poisson $d$-operad $\mathcal{P}ois_d^R$* is a monochromatic operad in $R$-$\mathbf{Mod}^{\mathbb{Z}}$ whose algebras are Poisson $d$-algebras. A full presentation starts with the following three generators:

1. a *void* $\mathfrak{v} \in \mathcal{P}ois_d^R(0)_0$ in arity 0 and degree 0,
2. a *product* $\mathfrak{p} \in \mathcal{P}ois_d^R(2)_0$ in arity 2 and degree 0, and
3. a *bracket* $\mathfrak{b} \in \mathcal{P}ois_d^R(2)_{d-1}$ in arity 2 and degree $d-1$,

where $\mathfrak{S}_2 = \langle \tau \mid \tau^2 \rangle$ acts on the two binary generators by $\tau^*\mathfrak{p} = \mathfrak{p}$ and $\tau^*\mathfrak{b} = (-1)^d \cdot \mathfrak{b}$. The Poisson $d$-operad is related as follows:

1. unitality of the product: $\mathfrak{p} \circ_1 \mathfrak{v} \sim \mathbb{1} \sim \mathfrak{p} \circ_2 \mathfrak{v}$,
2. associativity of the product: $\mathfrak{p} \circ_1 \mathfrak{p} \sim \mathfrak{p} \circ_2 \mathfrak{p}$,
3. annihilation of the bracket by the void: $\mathfrak{b} \circ_1 \mathfrak{v} \sim 0$,
4. the Jacobi identity: $(1 + (1\,2\,3)^* + (1\,3\,2)^*)(\mathfrak{b} \circ_2 \mathfrak{b}) \sim 0$,
5. the Leibniz rule: $\mathfrak{b} \circ_2 \mathfrak{p} \sim \mathfrak{p} \circ_1 \mathfrak{b} + (1\,2)^*(\mathfrak{p} \circ_2 \mathfrak{b})$.

This fully explains what a Poisson $d$-algebra is: in order to align with the literature, we put $n := d - 1$, and in order to save space, we put $a' := n + |a|$ for a homogeneous element. For each $\mathscr{Pois}_d^R$-algebra $A$, we write $1 := \mathfrak{v}()$, $a \cdot b := \mathfrak{p}(a \otimes b)$ and[2] $[a, b] := (-1)^{na} \cdot \mathfrak{b}(a \otimes b)$.

Then the symmetric action on the generators yields $a \cdot b = (-1)^{ab} \cdot (b \cdot a)$, i.e. the product is graded commutative, and $[a, b] = -(-1)^{a'b'} \cdot [b, a]$. The five relations look as follows (relation **5** is the usual Leibniz rule for $[a, -]$):

1. unitality of the product: $a \cdot 1 = a = 1 \cdot a$,
2. associativity of the product: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
3. annihilation of the bracket by the unit: $[a, 1] = 0$,
4. the Jacobi identity:

$$(-1)^{a'c'} \cdot [a, [b, c]] + (-1)^{b'a'} \cdot [b, [c, a]] + (-1)^{c'b'} \cdot [c, [a, b]] = 0,$$

5. the Leibniz rule: $[a, b \cdot c] = [a, b] \cdot c + (-1)^{ba'} \cdot b \cdot [a, c]$.

It is well-known (by old work of [Arn69] and [CLM76, § III.6], reformulated in the language of operads in [Sin06, Thm. 6.3]) that the homology of the little $d$-cubes operad $H_\bullet(\mathscr{C}_d; R)$ is isomorphic to the Poisson $d$-operad $\mathscr{Pois}_d^R$: the void is given by the ground class of $\mathscr{C}_d(0) \simeq *$, and product and bracket are given by the ground class and a choice of fundamental class of $\mathscr{C}_d(2) \simeq \mathbb{S}^{d-1}$, respectively (for $d = 1$, the product is a choice of path component of $\mathscr{C}_1(2)$

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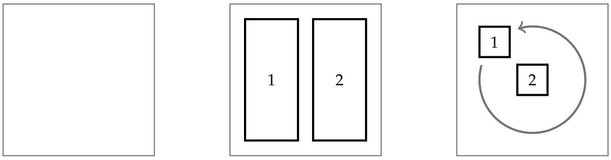
[2] This additional sign has only cosmetic reasons and aims to produce nicer formulæ. Our convention coincides with the one in [Coh95] and differs from the one in [CLM76] by a constant single $(-1)$-factor, which does not effect the relations.



$\mathfrak{v} \in H_0(\mathscr{C}_2(0))$ $\mathfrak{p} \in H_0(\mathscr{C}_2(2))$ $\mathfrak{b} \in H_1(\mathscr{C}_2(2))$

**Figure 3.5.** The three generators of $H_\bullet(\mathscr{C}_2)$.

and the bracket is a choice of difference between the path components), see Figure 3.5. Thus, if $X$ is a $\mathscr{C}_d$-algebra, then $H_\bullet(X;R)$ is a Poisson $d$-algebra. We call 1 the *unit*, the induced product *Pontrjagin product* after [Pon39], and the induced bracket *Browder bracket* after [Bro60].

## 3.3. Coloured topological operads

In this section, we turn our attention to (coloured) topological operads and the homotopy theory and homology of their algebras.

### 3.3.1. *Homology operations*

One important feature of topological operads is the fact that they endow the homology of their algebras with a variety of extra structure: for example, the homology of each $\mathscr{C}_1$-algebra carries the structure of a graded, unital algebra.

Sometimes, it even happens that this structure fully describes the homology of *free* $\mathbb{O}$-algebras; for example, if $\mathbb{F}$ is a field, then the homology of the free $\mathscr{C}_1$-algebra $F^{\mathscr{C}_1}(X) \simeq \coprod_r X^r$ with coefficients in $\mathbb{F}$ is the free graded, unital algebra over $H_\bullet(X;\mathbb{F})$ by the Künneth theorem.

**Definition 3.3.1.** Let $\mathbb{O}$ be an $N$-coloured topological operad and $R$ a commutative ring. Then we have, for each $k \in N$ and each $h \geqslant 0$, a functor

$$H_{k,h}\colon \mathbb{O}\text{-}\mathbf{Alg} \to \mathbf{Set}, \quad (X_n)_{n\in N} \mapsto H_h(X_k;R).$$

An *$\mathbb{O}$-homology operation of degree $s \in \mathbb{Z}$ and of profile $\binom{k_1,\dots,k_r}{n}$* is just a natural transformation of functors $H_{k_1,h_1} \times \cdots \times H_{k_r,h_r} \Rightarrow H_{n,s+h_1+\cdots+h_r}$.

Note that, as usual, we do not require (multi-)linearity, and we also require no dependency among the colours $k_1,\dots,k_r,n \in N$.

**Example 3.3.2.** One straightforward way to construct $\mathcal{O}$-homology operations has been described in Example 3.1.18: if an $N$-coloured operad acts on a sequence $\boldsymbol{X} = (X_n)_{n \in N}$, then for each commutative ring $R$, the algebraic operad $H_\bullet(\mathcal{O}; R)$ acts on $H_\bullet(\boldsymbol{X}; R)$; this means that each class $m \in H_s(\mathcal{O}; R)\binom{k_1,\dots,k_r}{n}$ can, for a given collection $h_1, \dots, h_r \geqslant 0$, be regarded as a map

$$m\colon H_{h_1}(X_{k_1}) \otimes \cdots \otimes H_{h_r}(X_{k_r}) \to H_{s+h_1+\dots+h_r}(X_n),$$

and this construction is clearly natural with respect to maps of $\mathcal{O}$-algebras. Note that all operations which arise in this way are by construction multilinear. We call these $\mathcal{O}$-homology operations *Künneth operations*, as the lax monoidality of $H_\bullet(-; R)$ uses the Künneth map.

Recall from Example 3.2.13 that in the case of the little $d$-cubes operad $\mathscr{C}_d$, the operad $H_\bullet(\mathscr{C}_d; R)$ is, for each commutative ring $R$, generated by only three classes, namely the *void* $\mathfrak{v} \in H_0(\mathscr{C}_d; R)(0)$, the *product* $\mathfrak{p} \in H_0(\mathscr{C}_d; R)(2)$ and the *bracket* $\mathfrak{b} \in H_{d-1}(\mathscr{C}_d; R)(2)$; hence the Künneth operations on the homology of a $\mathscr{C}_d$-algebra $X$ are entirely captured by the Poisson $d$-algebra structure on $H_\bullet(X; R)$.

For special coefficient rings, these are *all* operations in the following sense, compare with [MK95, Thm. 1.5.2] for the monochromatic case:

**Proposition 3.3.3.** *Let $\mathbb{F}$ be a field of characteristic 0 and let $\boldsymbol{X} = (X_n)_{n \in N}$ be a family of spaces, as well as $\mathcal{O}$ an $N$-coloured topological operad. Then $H_\bullet(F^{\mathcal{O}}(\boldsymbol{X}); \mathbb{F})$ is isomorphic to the free $H_\bullet(\mathcal{O}; \mathbb{F})$-algebra over the graded family $H_\bullet(\boldsymbol{X}; \mathbb{F})$.*

However, the story is more complicated over arbitrary coefficient rings: just as cup powers give rise to interesting cohomology operations, we can evaluate an operation $m \in H_s(\mathcal{O}; R)\binom{k_1,\dots,k_r}{n}$ multiple times at the same class, if the input colours coincide; for example, a class $m \in H_s(\mathcal{O}; R)\binom{k,k}{n}$ gives rise to $m_2\colon H_h(X_k; R) \to H_{s+2h}(X_n; R)$ with $m_2(x) = m(x \otimes x)$. Of course, each $H_\bullet(\mathcal{O}; R)$-algebra $\boldsymbol{A}$ contains these formal powers, but if $\boldsymbol{A}$ is the homology of some (topological) $\mathcal{O}$-algebra, then it happens that these powers are *divisible*; for example, in the case of $\mathscr{C}_d$-algebras, $\mathfrak{b}(x \otimes x)$ is divisible by 2 if $x$ is a class of even dimension.

To systematically construct their indivisible variant, called *divided power operations*, we have to switch to the symmetric monoidal category $(\mathbf{Ch}_R, \otimes, R^{[0]})$ of chain complexes of $R$-modules. Let us start with a sign correction.

**Definition 3.3.4.** Let $\mathfrak{G} \subseteq \mathfrak{S}_t$ be a subgroup and let $h \in \mathbb{Z}$. Then, for each left $R[\mathfrak{G}]$-chain complex $A$, we define $A(h)$ to be the $R[\mathfrak{G}]$-chain complex whose underlying $R$-chain complex is $A$ itself, but on which we consider the 'disturbed' action that identifies $\tau_*\alpha$ in $A(h)$ with $\mathrm{sg}(\tau)^h \cdot \tau_*\alpha$ in $A$ for $\tau \in \mathfrak{G}$.

Note that $A(h) = A(h+2)$. Moreover, if $\mathrm{char}(R) = 2$, or if $h$ is even, or if $\mathfrak{G}$ contains only even permutations, then $A(h) \cong A$. In this case we call the combination $(R, \mathfrak{G}, h)$ *admissible*.

**Definition 3.3.5.** If $P$ is a right $R[\mathfrak{G}]$-chain complex, we write $P_{\mathfrak{G}} := P \otimes_{\mathfrak{G}} R$ for the $R$-chain complex of coinvariants. Moreover, we write $\bar{\mu} := \mu \otimes_{\mathfrak{G}} 1$ for each $\mu \in P$, and we write $d$ for the differential in both $P$ and $P_{\mathfrak{G}}$. Note that if $P = C_\bullet^{\mathrm{sing}}(Y; R)$ for some right $\mathfrak{G}$-space $Y$, then $P_{\mathfrak{G}} \cong C_\bullet^{\mathrm{sing}}(Y/\mathfrak{G}; R)$.

**Reminder 3.3.6.** In the classical, monochromatic case, the divided power operations are concisely constructed by using from [May70] the following:

> LEMMA 1.1 (4). *Let $\mathfrak{G} \subseteq \mathfrak{S}_t$ and let $P$ be a non-negative, right $R[\mathfrak{G}]$-free chain complex and $X$ a chain complex of $R$-modules. If $\mu \in P$ such that $d\bar{\mu} = 0$ and if $\xi_0, \xi_1 \in X_h$ are homologous cycles, then $\mu \otimes_{\mathfrak{G}} \xi_0^{\otimes t}$ and $\mu \otimes_{\mathfrak{G}} \xi_1^{\otimes t}$ are two homologous cycles in $P \otimes_{\mathfrak{G}} X^{\otimes t}(h)$.*
>
> *Moreover, if $\mu' \in P$ such that $\bar{\mu}'$ is homologous to $\bar{\mu}$, and $\xi \in X_h$ is a cycle, then $\mu' \otimes_{\mathfrak{G}} \xi^{\otimes t}$ is homologous to $\mu \otimes_{\mathfrak{G}} \xi^{\otimes t}$.*

If $P$ is a non-negative, right $R[\mathfrak{G}]$-free chain complex and $X$ is a chain complex of $R$-modules, then each $c = [\bar{\mu}] \in H_s(P_{\mathfrak{G}})$ gives rise to a map

$$\tilde{Q}_c \colon H_h(X) \to H_{s+th}\big(P \otimes_{\mathfrak{G}} X^{\otimes t}(h)\big), \quad [\xi] \mapsto [\mu \otimes_{\mathfrak{G}} \xi^{\otimes t}].$$

If $P = \mathcal{O}(t)$ holds for some operad $\mathcal{O}$ and if $X$ is an $\mathcal{O}$-algebra, then the action of $\mathcal{O}$ on $X$ gives rise to a chain map $\lambda \colon P \otimes X^{\otimes t} \to X$, and since this map is $\mathfrak{S}_t$-equivariant, it factors through a chain map $\bar{\lambda} \colon P \otimes_{\mathfrak{G}} X^{\otimes t} \to X$. Thus, if $\mathfrak{S}_t$ acts freely on $\mathcal{O}(t)$ and the triple $(R, \mathfrak{G}, h)$ is admissible, then we obtain maps in homology $Q_c \colon H_h(X) \to H_{s+th}(X)$, called *divided power operations*, which are natural with respect to maps of $\mathcal{O}$-algebras.

The first example of such divided power operations are the *Dyer–Lashof squares* which are operations on the $\mathbb{F}_p$-homology of $E_d$-algebras. The case

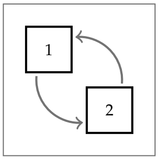


**Figure 3.6.** A path $\mu\colon [0;1] \to \mathscr{C}_2(2)$ with $[\bar{\mu}] = c_1 \in H_1(\mathscr{C}_2(2)/\mathfrak{S}_2)$

$p = 2$ is due to Araki and Kudo [AK56], while the construction for odd primes is due to Dyer and Lashof [DL62]. We will use the name 'Dyer–Lashof' in all cases, and restrict for the moment to the case $p = 2$.

**Example 3.3.7** (Dyer–Lashof squares)**.** Let $R = \mathbb{F}_2$ and let $\mathbb{O} := C_\bullet^{\mathrm{sing}}(\mathscr{C}_d; \mathbb{F}_2)$ be the singular chain complex of the little $d$-cubes operad with $\mathbb{F}_2$-coefficients.

Note that $\mathscr{C}_d(2)/\mathfrak{S}_2$ is equivalent to the real projective space $\mathbb{RP}^{d-1}$, so $H_s(\mathbb{O}(2)_{\mathfrak{S}_2}) = \mathbb{F}_2\langle c_s\rangle$ for each $0 \leqslant s \leqslant d-1$. If $\mathscr{C}_d$ acts on a space $X$, then each of these generators gives rise to an operation $Q_{c_s}\colon H_h(X; \mathbb{F}_2) \to H_{s+2h}(X; \mathbb{F}_2)$.

If $\mu \in \mathbb{O}(2)_{d-1}$ represents the top generator $c_{d-1}$, then $\mu + \tau^*\mu \in \mathbb{O}(2)_{d-1}$ is a cycle representing the fundamental class $\mathfrak{b}$ of $\mathscr{C}_d(2) \simeq \mathbb{S}^{d-1}$. Pictorially, $\mu$ parametrises how the first box moves in a single hemisphere around the second one. For $d = 2$, the top generator $c_1$ can be represented by a path $\mu\colon [0;1] \to \mathscr{C}_2(2)$ parametrising the two boxes spinning around each other, performing a half-rotation and, thus, changing places, see Figure 3.6.

Usually, one defines $Q^s := Q_{c_{s-h}}\colon H_h(X) \to H_{h+s}(X)$ for $h \leqslant s \leqslant h+d-1$, which is an operation of degree $s$. In [CLM76, § III.1], many formulæ among these operations $Q^s$ have been proven, which are very similar to the ones for Steenrod squares in $\mathbb{F}_2$-cohomology. If one defines a *$W_d$-algebra* to be a Poisson $d$-algebra $A$ over $\mathbb{F}_2$ together with operations $Q^s\colon A_h \to A_{h+s}$ for $h \leqslant s \leqslant h+d-1$ satisfying all relations from [CLM76, § III.1], then a statement similar to Proposition 3.3.3 holds: the $\mathbb{F}_2$-homology $H_\bullet(F^{\mathscr{C}_d}(X); \mathbb{F}_2)$ of the free $E_d$-algebra over a space $X$ is isomorphic to the free $W_d$-algebra over the graded $\mathbb{F}_2$-vector space $H_\bullet(X; \mathbb{F}_2)$. This entirely describes the homology of unordered configuration spaces over $\mathbb{F}_2$, see [CLM76, § III.4].

One should point out that in some situations, these Dyer–Lashof squares can equally well be defined over arbitrary commutative rings $R$: for even $d$, we have $H_{d-1}(\mathscr{C}_d(2); R) \cong R\langle c_{d-1}\rangle$, and if also $h$ is even, then $(R, \mathfrak{S}_2, h)$ is admissible and we consider the top operation $Q_{c_{d-1}}\colon H_h(X;R) \to H_{d-1+2h}(X;R)$. Again, if $c_{d-1} = [\bar{\mu}]$ for some $\mu \in C^{\mathrm{sing}}_{d-1}(\mathscr{C}_d(2);R)$, then $\mu + \tau^*\mu$ represents the bracket $\mathfrak{b} \in H_{d-1}(\mathscr{C}_d(2);R)$. Hence we see for $x = [\xi] \in H_h(X;R)$

$$\begin{aligned}[x,x] &= (-1)^{(d-1)\cdot h} \cdot [\lambda((\mu + \tau^*\mu) \otimes_{\mathfrak{S}} (\xi \otimes \xi))] \\ &= 2 \cdot (-1)^{(d-1)\cdot h} \cdot Q_{c_{d-1}}(x),\end{aligned}$$

so the Browder bracket $[x,x]$ is indeed divisible by 2.

**Remark 3.3.8.** In the case $d = 2$, one can give a bit more intuition what $Qx$ looks like geometrically: if $x \in H_n(X)$ is supported on an $h$-dimensional closed manifold $M$, then $Qx$ is supported on the mapping torus $\mathbb{S}^1 \ltimes M^2$ of the *twist* $\mathrm{tw}\colon M^2 \to M^2$ with $\mathrm{tw}(p,p') = (p',p)$ as follows:

Consider the map $\gamma\colon [0;1] \times X^2 \to X$ with $\gamma(t,x,x') = \mu(t)(x,x')$, where $\mu$ is the path from Figure 3.6. Then we have $\gamma(0,x,x') = \gamma(1,x',x)$, whence $\gamma$ factors over $\bar{\gamma}\colon \mathbb{S}^1 \ltimes X^2 \to X$, and for each map $\alpha\colon M \to X$, we obtain the description $Q(\alpha_*[M]) = \bar{\gamma}_*(\mathrm{id}_{\mathbb{S}^1} \ltimes \alpha^2)_*[\mathbb{S}^1 \ltimes M^2]$, where $[M]$ and $[\mathbb{S}^1 \ltimes M^2]$ are the fundamental classes mod 2.

If we work integrally and $M$ is oriented, then $\mathbb{S}^1 \ltimes M^2$ is orientable if and only if $h = \dim(M)$ is even: this explains our earlier restrictions.

We want to generalise this construction to the case of coloured operads. Note that in the coloured setting we can only permute inputs of the same type if we want to stay in a single operation complex. This forces us to work with a slightly more complicated version of May's lemma: let $t_1, \dots, t_r \geqslant 0$ and let $\mathfrak{G}_i \subseteq \mathfrak{S}_{t_i}$ be subgroups. We write $\mathfrak{G} := \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_r \subseteq \mathfrak{S}_{t_1+\cdots+t_r}$.

**Lemma 3.3.9.** *Let $P$ be a non-negative, right $R[\mathfrak{G}]$-free chain complex and let $X_1, \dots, X_r$ be arbitrary chain complexes of $R$-modules. If $\mu \in P$ such that $d\bar{\mu} = 0$ and if $\xi_{i,0}$ and $\xi_{i,1}$ are are homologous cycles in $X_{i,h_i}$ for each $i$, then $\mu \otimes_{\mathfrak{G}} \bigotimes_i \xi_{i,0}^{\otimes t_i}$ and $\mu \otimes_{\mathfrak{G}} \bigotimes_i \xi_{i,1}^{\otimes t_i}$ are homologous cycles in $P \otimes_{\mathfrak{G}} \bigotimes_i X_i^{\otimes t_i}(h_i)$.*

*Moreover, if $\mu' \in P$ such that $\bar{\mu}'$ is homologous to $\bar{\mu}$, and if $\xi_i \in X_{i,h_i}$ are cycles, then $\mu' \otimes_{\mathfrak{G}} \bigotimes_i \xi_i^{\otimes t_i}$ is homologous to $\mu \otimes_{\mathfrak{G}} \bigotimes_i \xi_i^{\otimes t_i}$.*

The proof is a straightforward generalisation of [May70, Lem. 1.1] and left to the reader. Now we can define homology operations for algebras over coloured operads in the same way as for the monochromatic case.

**Construction 3.3.10.** Again we fix numbers $t_1, \dots, t_r \geqslant 0$ and subgroups $\mathfrak{G}_i \subseteq \mathfrak{S}_{t_i}$, and we write $\mathfrak{G} := \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_r$. Given a set $N$ and $k_1, \dots, k_r \in N$, then we write $(t_1 \times k_r, \dots, t_r \times k_r) \in N^{t_1 + \cdots + t_r}$ for the tuple where the entry $k_i$ is repeated $t_i$ times.

If $\mathcal{O}$ is an $N$-coloured operad in $R$-chain complexes, then $\mathfrak{G}$ acts $R$-linearly on the operation complex $\mathcal{O}\binom{t_1 \times k_1, \dots, t_r \times k_r}{n}$, and if this action is $R[\mathfrak{G}]$-free, then Lemma 3.3.9 enables us to construct for $c = [\bar{\mu}] \in H_s(\mathcal{O}\binom{t_1 \times k_1, \dots, t_r \times k_r}{n}_{\mathfrak{G}})$ a map

$$\tilde{Q}_c \colon \prod_{i=1}^{r} H_{h_i}(X_{k_i}) \to H_{s+t_1h_1+\cdots+t_rh_r}\left(\mathcal{O}\binom{t_1 \times k_1, \dots, t_r \times k_r}{n} \otimes_{\mathfrak{G}} \bigotimes_{i=1}^{r} X_{k_i}^{\otimes t_i}(h_i)\right).$$

by $\tilde{Q}_c([\xi_1], \dots, [\xi_r]) := [\mu \otimes_{\mathfrak{G}} (\xi_1^{\otimes t_1} \otimes \cdots \otimes \xi_r^{\otimes t_r})]$. The (equivariant) operadic action gives rise to a chain map $\bar{\lambda} \colon \mathcal{O}\binom{t_1 \times k_1, \dots, t_r \times k_r}{n} \otimes_{\mathfrak{G}} \bigotimes_i X_{k_i}^{\otimes t_i} \to X_n$, and if each $(R, \mathfrak{G}_i, h_i)$ is admissible, then $X_{k_i}^{\otimes t_i}(h_i) = X_{k_i}^{\otimes t_i}$ and postcomposing $\tilde{Q}_c$ with $\bar{\lambda}$ defines the desired *divided power operation*

$$Q_c \colon \prod_{i=1}^{r} H_{h_i}(X_{k_i}) \to H_{s+t_1h_1+\cdots+t_rh_r}(X_n).$$

**Remark 3.3.11.** If $\mathcal{O}$ is an $N$-coloured topological operad which is $\mathfrak{S}$*-free*, i.e. for each tuple $K$ and each $n \in N$, the action of $\mathfrak{S}^K := \mathrm{Aut}_{N \wr \Sigma}(K)$ is free on $\mathcal{O}\binom{K}{n}$, then the operad $C_\bullet^{\mathrm{sing}}(\mathcal{O}; R)$ in the category $R$-chain complexes has $R[\mathfrak{G}]$-free operation complexes for each choice of subgroup $\mathfrak{G}$.

**Definition 3.3.12** (Transfer). Let $\mathfrak{G}$ be a finite group and $P$ be a right $R[\mathfrak{G}]$-free chain complex. We write $\mathrm{pr}_* \colon H_\bullet(P) \to H_\bullet(P_{\mathfrak{G}})$ for the map induced by the projection to the coinvariants. There is also a map $\mathrm{pr}^! \colon H_\bullet(P_{\mathfrak{G}}) \to H_\bullet(P)$ in the other direction, called the *transfer*, and it is induced by the chain map

$$P_{\mathfrak{G}} \to P, \quad \bar{\mu} \to \sum_{\tau \in \mathfrak{G}} \tau^* \mu.$$

Note that, by construction, we have $\mathrm{pr}_* \circ \mathrm{pr}^! = \#\mathfrak{G} \cdot \mathrm{id}_{H_\bullet(P_{\mathfrak{G}})}$.

The following observations are not particularly deep; they follow directly from the construction. However, I have not found them in the literature, and they turn out to be quite useful for concrete calculations.

**Remark 3.3.13.** For each $N$-coloured operad $\mathcal{O}$ in $R$-chain complexes with sufficiently free operation complexes, and for each $\mathcal{O}$-algebra $\boldsymbol{X}$, the divided power operations satisfy the following relations:

$$\begin{aligned} Q_{ac+a'c'}(x_1,\dots,x_r) &= a \cdot Q_c(x_1,\dots,x_r) + a' \cdot Q_{c'}(x_1,\dots,x_r), \\ \#\mathfrak{G} \cdot Q_c(x_1,\dots,x_r) &= (\mathrm{pr}^! c)(x_1^{\otimes t_1} \otimes \cdots \otimes x_r^{\otimes t_r}), \\ Q_{\mathrm{pr}_* m}(x_1,\dots,x_r) &= m(x_1^{\otimes t_1} \otimes \cdots \otimes x_r^{\otimes t_r}), \\ Q_c(x_1,\dots,a \cdot x_i,\dots,x_r) &= a^{t_i} \cdot Q_c(x_1,\dots,x_r). \end{aligned}$$

**Remark 3.3.14.** Considering sums in one of the entries is a bit more intricate: one easily sees that for $c = [\bar{\mu}]$, $x_i = [\zeta_i]$, and $x_i' = [\zeta_i']$, if $\mathfrak{G}_j = \mathfrak{S}_{t_j}$ for some $j$, then $\tilde{Q}_c(x_1,\dots,x_j + x_j',\dots,x_r)$ is represented by the cycle

$$\sum_{l=1}^{t_j} \sum_{\tau \in \mathfrak{S}_{t_j}/\mathfrak{S}_l \times \mathfrak{S}_{t_j - l}} \tau^* \mu \otimes_{\mathfrak{G}} (\xi_1^{\otimes t_1} \otimes \cdots \otimes \xi_j^{\otimes l} \otimes \xi_j'^{\otimes t_j - l} \otimes \cdots \otimes \xi_r^{\otimes t_r}).$$

In the easiest case $r = 1$ and $t_1 = 2$, this expression boils down to

$$Q_c(x+y) = Q_c(x) + (\mathrm{pr}^! c)(x \otimes y) + Q_c(y).$$

There is another class of 'universal' formulæ which hold for all sufficiently free operads in chain complexes: let us restrict to the case where the groups $\mathfrak{G}$ are the entire respective symmetric group, so either $\mathrm{char}(R) = 2$ or the degrees of all arguments are even. Then the coinvariants $\mathcal{O}\binom{t_1 \times l_1,\dots,t_s \times l_s}{k}_{\mathfrak{S}_{t_1} \times \cdots \times \mathfrak{S}_{t_s}}$ assemble into a specific $\mathcal{O}$-algebra on their own: if we endow the colour set $N$ with a total order and if we let $\underline{R} := (R^{[0]})_{n \in N}$, then

$$F^{\mathcal{O}}(\underline{R})_k \cong \bigoplus_{s \geqslant 0} \bigoplus_{t_1,\dots,t_s \geqslant 0} \bigoplus_{l_1 < \cdots < l_s} \mathcal{O}\binom{t_1 \times l_1,\dots,t_s \times l_s}{k}_{\mathfrak{S}},$$

where we abbreviate $\mathfrak{S} := \prod_j \mathfrak{S}_{t_j}$. The following lemma explains how the $H_\bullet(\mathcal{O})$-action on $H_\bullet(F^{\mathcal{O}}(\underline{R}))$ interacts with the divided power operations:

**Lemma 3.3.15.** *Let $l_1 < \cdots < l_s$ be colours in $N$ and let $t_{i,j} \geqslant 0$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$. Moreover, let $m \in H_\bullet(\mathcal{O}\binom{k_1,\dots,k_r}{n})$ and $c_i \in H_\bullet(\mathcal{O}\binom{t_{i,1} \times l_1,\dots,t_{i,s} \times l_s}{k_i}_{\mathfrak{S}})$. Note that $m(c_1 \otimes \cdots \otimes c_r) \in H_\bullet(\mathcal{O}\binom{t_1 \times l_1,\dots,t_s \times l_s}{n}_{\mathfrak{S}})$ for $t_j := \sum_i t_{i,j}$. Then, for each class $x_j \in H_\bullet(X_{l_j})$, we have the equality*

$$Q_{m(c_1 \otimes \cdots \otimes c_r)}(x_1,\dots,x_s) = m\big(Q_{c_1}(x_1,\dots,x_s) \otimes \cdots \otimes Q_{c_r}(x_1,\dots,x_s)\big).$$

*Proof.* We find $\mu \in \mathfrak{O}\binom{k_1,\dots,k_r}{n}$ and $\nu_i \in \mathfrak{O}\binom{t_{i,1}\times l_1,\dots,t_{i,s}\times l_s}{k_i}$ such that $m := [\mu]$ and $c_i := [\bar{\nu}_i]$. If we let $\tau \in \mathfrak{S}_{t_1+\dots+t_s}$ be the permutation that identifies $\coprod_i \coprod_j \underline{t}_{i,j}$ with $\coprod_j \coprod_i \underline{t}_{i,j}$, then $\tau^*(\mu \circ \bigotimes_i \nu_i)$ has input profile $(t_1 \times l_1, \dots, t_s \times l_s)$, and if $x_j$ is represented by the cycle $\xi_j \in X_{l_j}$, then we get

$$\begin{aligned} Q_{m(c_1\otimes\cdots\otimes c_r)}(x_1,\dots,x_s) &= \Big[\tau^*(\mu \circ \textstyle\bigotimes_i \nu_i)(\bigotimes_j \xi_j^{\otimes t_j})\Big] \\ &= \Big[(\mu \circ \textstyle\bigotimes_i \nu_i)(\bigotimes_i \bigotimes_j \xi_j^{\otimes t_{i,j}})\Big] \\ &= \Big[\mu(\textstyle\bigotimes_i \nu_i(\bigotimes_j \xi_j^{\otimes t_{i,j}}))\Big] \\ &= m\big(Q_{c_1}(x_1,\dots,x_s) \otimes \cdots \otimes Q_{c_r}(x_1,\dots,x_s)\big). \end{aligned}$$

Note that there is no Koszul sign appearing since we only exchange instances of $\xi_j$ with something else, and either $\operatorname{char}(R) = 2$ or $|\xi_j|$ is even. □

Let me point out that, although it looks as if $c_1,\dots,c_r$ all have the same input colours $l_1,\dots,l_s$, the multiplicities $t_{i,j}$ can be very different; in particular, we allowed $t_{i,j} = 0$ as well, whence the profiles can be entirely disjoint.

### 3.3.2. *The Boardman–Vogt tensor product*

In this subsection, we repeat the contruction of the Boardman–Vogt tensor product of two topological operads, which has originally been introduced by [BV73, § II.3] in a different language.

As we need this notion in higher generality at several occasions, let us give a short survey which mainly follows [Wei11, § 2.2].

**Notation 3.3.16.** We form exterior products of tuples and permutations:

- Let $K = (k_1,\dots,k_r)$ be a tuple in $N$ and $K = (k'_1,\dots,k'_{r'})$ a tuple in $N'$. Then we define the *tuple product* $K \times K' \in (N \times N')^{r\cdot r'}$ by

$$K \times K' := \big((k_1,k'_1),\dots,(k_1,k'_{r'}),\dots,(k_r,k'_1),\dots,(k_r,k'_{r'})\big).$$

- Let $\tau \in \mathfrak{S}_r$ and $\tau' \in \mathfrak{S}_{r'}$ be two permutations. Then we put

$$\tau \times \tau' := \tau \circ_{\mathcal{A}ss} (\tau',\dots,\tau') \in \mathfrak{S}_{r\cdot r'}.$$

  If $\tau\colon \tau^*K \to K$ is a map in $N \wr \mathbf{\Sigma}$ and $\tau'\colon \tau'^*K' \to K'$ is a map in $N' \wr \mathbf{\Sigma}$, then we get a map $\tau \times \tau'\colon (\tau \times \tau')^*(K \times K') \to K \times K'$ in $(N \times N') \wr \mathbf{\Sigma}$.

**Definition 3.3.17.** Let $\mathcal{O}$ be an $N$-coloured, $\mathcal{O}'$ be an $N'$-coloured, and $\mathcal{P}$ be an $(N \times N')$-coloured topological operad. A *pairing* $(\mathcal{O}, \mathcal{O}') \to \mathcal{P}$ is declared by a family of maps

$$\zeta \colon \mathcal{O}\binom{K}{n} \times \mathcal{O}'\binom{K'}{n'} \to \mathcal{P}\binom{K \times K'}{(n,n')},$$

such that the following properties hold:

1. $\zeta(\mathbb{1}_n^{\mathcal{O}}, \mathbb{1}_{n'}^{\mathcal{O}'}) = \mathbb{1}_{(n,n')}^{\mathcal{P}}$,
2. $\zeta(\tau^* \mu, \tau'^* \mu') = (\tau \times \tau')^* \zeta(\mu, \mu')$,
3. $\zeta(\mu \circ (\mu_1, \ldots, \mu_r), \mu' \circ (\mu'_1, \ldots, \mu'_{r'})) = \zeta(\mu, \mu')(\zeta(\mu_1, \mu'_1), \ldots, \zeta(\mu_r, \mu'_{r'}))$.

**Remark 3.3.18.** Let $\zeta$ be a pairing as above.

- For each $n' \in N'$ we have a map $\zeta(-, \mathbb{1}_{n'}) \colon \mathcal{O} \to \mathcal{P}|_{N \times \{n'\}}$.
- For each $n \in N$ we have a map $\zeta(\mathbb{1}_n, -) \colon \mathcal{O}' \to \mathcal{P}|_{\{n\} \times N'}$.

Thus, if $\boldsymbol{X} = (X_{n,n'})_{n \in N, n' \in N'}$ is a $\mathcal{P}$-algebra, then each $\boldsymbol{X}_{\bullet,n'} := (X_{n,n'})_{n \in N}$ is an $\mathcal{O}$-algebra and each $\boldsymbol{X}_{n,\bullet} := (X_{n,n'})_{n' \in N'}$ is an $\mathcal{O}'$-algebra.

**Construction 3.3.19.** If $\mathcal{O}$ is an $N$-coloured operad and $\mathcal{O}'$ is an $N'$-coloured operad, then we define its *Boardman–Vogt tensor product* $\mathcal{O} \odot \mathcal{O}'$ as an $(N \times N')$-coloured operad that comes with a pairing $\chi \colon (\mathcal{O}, \mathcal{O}') \to \mathcal{O} \odot \mathcal{O}'$ which is initial in the following sense: for each pairing $\zeta \colon (\mathcal{O}, \mathcal{O}') \to \mathcal{P}$, there is a unique morphism $\zeta^{\odot} \colon \mathcal{O} \odot \mathcal{O}' \to \mathcal{P}$ of $(N \times N')$-coloured operads such that $\zeta^{\odot}(\chi(\mu, \mu')) = \zeta(\mu, \mu')$ holds, i.e. for all $K$, $K'$, $n$, and $n'$, the triangle

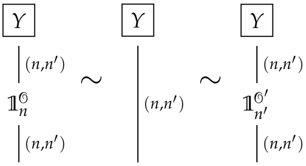
$$\begin{tikzcd}
\mathcal{O}\binom{K}{n} \times \mathcal{O}'\binom{K'}{n'} \arrow[r, "\zeta"] \arrow[d, "\chi"'] & \mathcal{P}\binom{K \times K'}{(n,n')} \\
(\mathcal{O} \odot \mathcal{O}')\binom{K \times K'}{(n,n')} \arrow[ur, dashed, "\zeta^{\odot}"']
\end{tikzcd}$$

commutes: this clearly determines $\mathcal{O} \odot \mathcal{O}'$ uniquely up to unique isomorphism. An explicit construction can be given in form of a presentation: we start with the $(N \times N')$-coloured quiver $\mathcal{O} * \mathcal{O}' := (\mathcal{O} \times N') \sqcup (N \times \mathcal{O}')$, where

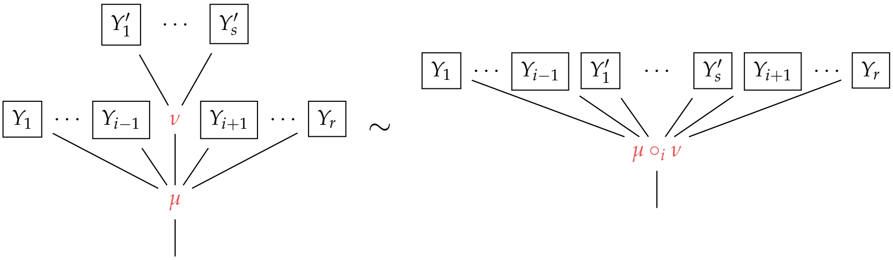
$$(\mathcal{O} \times N')\binom{(k_1,n'),\ldots,(k_r,n')}{(n,n')} := \mathcal{O}\binom{k_1,\ldots,k_r}{n},$$
$$(N \times \mathcal{O}')\binom{(n,k'_1),\ldots,(n,k'_{r'})}{(n,n')} := \mathcal{O}'\binom{k'_1,\ldots,k'_{r'}}{n'},$$

together with the induced structure maps. This means that elements in the free operad $\Psi(\mathcal{O} * \mathcal{O}')$ are trees with two types of internal vertices: those labelled in $\mathcal{O}$ and those labelled in $\mathcal{O}'$. For those of the first type, all incoming edges and the outgoing edge have the same second colour component, and for those of the second type, all incoming edges and the outgoing edge have the same first colour component. Now we divide out the following relations:

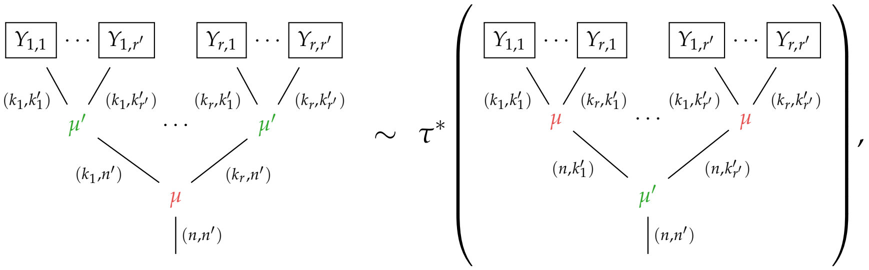
1. the already existing identities $\mathbb{1}_n^{\mathcal{O}}$ and $\mathbb{1}_{n'}^{\mathcal{O}'}$ can be skipped:



2. two neighboured internal vertices of the same type can be composed: if $\mu$ and $\nu$ both come from $\mathcal{O}$ or both come from $\mathcal{O}'$, then



3. the *interchange law*: if $\mu$ comes from $\mathcal{O}$ and $\mu'$ from $\mathcal{O}'$, then we relate



where $\tau \in \mathfrak{S}_{\sum_{i,j} r_{Y_{i,j}}}$ is the $(r_{Y_{1,1}}, \ldots, r_{Y_{r,r'}})$-block version of $\tau_0 \in \mathfrak{S}_{r \cdot r'}$ that changes the reading direction from 'row-wise' to 'column-wise'.

Given a pairing $\zeta\colon (\mathcal{O},\mathcal{O}') \to \mathscr{P}$, we obtain a map $\mathcal{O} \odot \mathcal{O}' \to \mathscr{P}$ of $(N \times N')$-coloured operads in a canonical way: the pairing $\zeta$ induces morphisms of quivers $\coprod_{n'} \zeta(-,\mathbb{1}^{\mathcal{O}'}_{n'})\colon \mathcal{O} \times N' \to \mathscr{P}$ and likewise $\coprod_{n} \zeta(\mathbb{1}^{\mathcal{O}}_{n},-)\colon N \times \mathcal{O}' \to \mathscr{P}$, and their union $\mathcal{O} * \mathcal{O}' \to \mathscr{P}$ has an adjoint map $\bar{\zeta}\colon \Psi(\mathcal{O} * \mathcal{O}') \to \mathscr{P}$. Now the properties of a pairing ensure that $\bar{\zeta}$ factors over the operadic relation above.

This construction is bifunctorial: if $\rho\colon \mathcal{O} \to \mathscr{P}$ and $\rho'\colon \mathcal{O}' \to \mathscr{P}'$ are morphisms of operads, then we get a morphism $\rho \odot \rho'\colon \mathcal{O} \odot \mathcal{O}' \to \mathscr{P} \odot \mathscr{P}'$. It is even possible to use the Boardman–Vogt tensor product in order to construct a monoidal structure on the category of coloured operads with varying colour set, and the trivial operad $\mathscr{J}$ serves as a monoidal unit, see [Wei11, Thm. 2.22].

**Remark 3.3.20** (Nullaries)**.** Let $\zeta\colon (\mathcal{O},\mathcal{O}') \to \mathscr{P}$ be a pairing. If $\mathfrak{v} \in \mathcal{O}\binom{}{n}$ and $\mathfrak{v}' \in \mathcal{O}'\binom{}{n'}$ are nullary operations, then

$$\zeta(\mathfrak{v},\mathfrak{v}') = \zeta(\mathbb{1}_n \circ \mathfrak{v}, \mathfrak{v}' \circ ()) = \zeta(\mathbb{1}_n, \mathfrak{v}')() = \zeta(\mathbb{1}_n, \mathfrak{v}'),$$

and for symmetry reasons, we get $\zeta(\mathfrak{v},\mathbb{1}_{n'}) = \zeta(\mathfrak{v},\mathfrak{v}') = \zeta(\mathbb{1}_n,\mathfrak{v}')$. Employing this equality multiple times, we can identify *all* nullaries of the form $\zeta(\mathfrak{v},\mathbb{1}_{n'})$ with *all* nullaries of the form $\zeta(\mathbb{1}_n,\mathfrak{v}')$. In particular, if $\mathcal{O}\binom{}{n} \neq \varnothing$, then all elements of the form $\zeta(\mathbb{1}_n,\mathfrak{v}')$ with $\mathfrak{v}' \in \mathcal{O}'\binom{}{n'}$ coincide, and the same holds mutatis mutandis if $\mathcal{O}'\binom{}{n'} \neq \varnothing$.

This behaviour is reflected by the explicit construction of the tensor product: if $\mathcal{O}\binom{}{n}$ and $\mathcal{O}'\binom{}{n'}$ are both non-empty, then $(\mathcal{O} \odot \mathcal{O}')\binom{}{(n,n')}$ is a singleton.

In many cases, this construction has an easier form: one situation which is important for us is the case where one factor is monochromatic and the second factor is a small, enriched category, i.e. it has only unaries:

**Remark 3.3.21.** Let $\mathscr{C}$ be a monochromatic operad and $\mathbf{I}$ be a topologically enriched category with object set $N$. Then $\mathscr{C} \odot \mathbf{I}$ has operation spaces

$$(\mathscr{C} \odot \mathbf{I})\binom{k_1,\dots,k_r}{n} = \mathscr{C}(r) \times \prod_{i=1}^{r} \mathbf{I}\binom{k_i}{n},$$

together with the obvious input permutations, identities, and compositions. For each $n \in N$, we also abbreviate $\mu \odot n := \mu \odot (\mathbb{1}_n, \dots, \mathbb{1}_n) \in (\mathscr{C} \odot \mathbf{I})\binom{n,\dots,n}{n}$. Note that $(\mathscr{C} \odot \mathbf{I})$-algebras are the same as enriched functors $\mathbf{I} \to \mathscr{C}$-**Alg**.

**Example 3.3.22.** If $N$ is the discrete category with object set $N$, then we get

$$(\mathscr{C} \odot N)\binom{k_1,\dots,k_r}{n} = \begin{cases} \mathscr{C}(r) & \text{for } k_1 = \cdots = k_r = n, \\ \varnothing & \text{else,} \end{cases}$$

and $(\mathscr{C} \odot N)$-algebras are just $N$-indexed families of $\mathscr{C}$-algebras.

We close this subsection with a reminder on Dunn's additivity theorem which studies the Boardman–Vogt tensor product of two little cubes operads.

**Example 3.3.23.** There is a pairing $\zeta\colon (\mathscr{C}_p, \mathscr{C}_q) \to \mathscr{C}_{p+q}$ of monochromatic operads given by forming products of cuboids

$$\zeta\big((c_1,\dots,c_r),(c'_1,\dots,c'_{r'})\big) := (c_1 \times c'_1,\dots,c_1 \times c'_{r'},\dots,c_r \times c'_1,\dots,c_r \times c'_{r'})$$

as depicted in Figure 3.7. It was shown by [Dun86; Bri00] that the induced map $\mathscr{C}_p \odot \mathscr{C}_q \to \mathscr{C}_{p+q}$ is a weak equivalence.

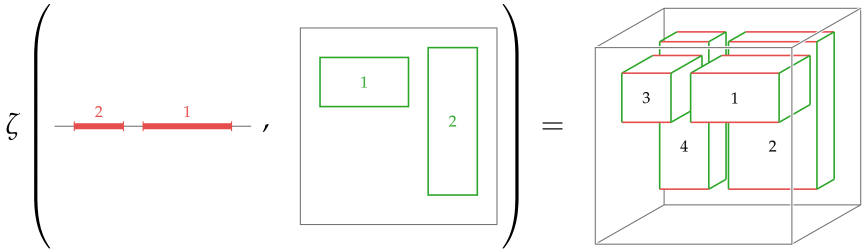


**Figure 3.7.** An instance of $\zeta\colon \mathscr{C}_1(2) \times \mathscr{C}_2(2) \to \mathscr{C}_3(4)$

### 3.3.3. *Base-change for topological operads*

In this last subsection, I would like to discuss the base-change adjunction from Construction 3.1.15 for a given operad morphism $\rho\colon \mathscr{P} \to \mathscr{O}$ in the special case of topological operads.

Without spending too much time on model-categorical subtleties, let me point out that one important property of the base-change adjunction is the fact that it helps us comparing the homotopy theory of algebras over different operads. This is made precise in the following way:

**Remark 3.3.24.** As done in detail in [BM03; BM07], one can transfer the Quillen model structure on $\mathbf{Top}^N$ along the adjunction $F^{\mathcal{O}}\colon \mathbf{Top}^N \rightleftarrows \mathcal{O}\text{-}\mathbf{Alg} : U^{\mathcal{O}}$ to the category of $\mathcal{O}$-algebras: then the equivalences are exactly the morphisms of $\mathcal{O}$-algebras whose underlying maps are (levelwise) weak equivalences.

An $N$-coloured operad $\mathcal{O}$ is called $\mathfrak{S}$*-cofibrant* if all operation spaces $\mathcal{O}\binom{K}{n}$ are cofibrant as $\mathfrak{S}^K$-spaces, where $\mathfrak{S}^K := \mathrm{Aut}_{N\wr\Sigma}(K)$. This is for example the case if $\mathfrak{S}^K$ acts freely on $\mathcal{O}\binom{K}{n}$, i.e. if $\mathcal{O}$ is $\mathfrak{S}$-free as in Remark 3.3.11, and if, additionally, $\mathcal{O}\binom{K}{n}/\mathfrak{S}^K$ is a retract of a cell complex. Moreover, $\mathcal{O}$ is called *well-based* if the inclusions $\{\mathbb{1}_n\} \hookrightarrow \mathcal{O}\binom{n}{n}$ are cofibrations. We call a morphism $\rho\colon \mathcal{P} \to \mathcal{O}$ of $N$-coloured operads a *weak equivalence* if all $\rho\binom{K}{n}\colon \mathcal{P}\binom{K}{n} \to \mathcal{O}\binom{K}{n}$ are weak equivalences.

Then [BM07, Thm. 4.1] tells us that if $\rho\colon \mathcal{P} \to \mathcal{O}$ is a weak equivalence between $\mathfrak{S}$-cofibrant and well-based operads, then the corresponding base-change adjunction $\rho_!\colon \mathcal{P}\text{-}\mathbf{Alg} \rightleftarrows \mathcal{O}\text{-}\mathbf{Alg} : \rho^*$ is a Quillen equivalence. In other words, the homotopy theory of $\mathcal{O}$-algebras coincides with the homotopy theory of $\mathcal{P}$-algebras.

If the operad $\mathcal{P}$ is sufficiently small, then the base-change adjunction is easy to understand, as we see in the following examples.

**Example 3.3.25** (Free algebras over based spaces)**.** Recall the monochromatic operad $\mathcal{B}$ of based spaces. Then a map $\mathcal{B} \to \mathcal{O}$ is the same as a monochromatic operad $\mathcal{O}$ together with a choice of void $\mathfrak{v} \in \mathcal{O}(0)$. Since $\mathcal{B}$-algebras are the same as based spaces, the adjunction is of the form

$$F^{\mathcal{O}}_{\mathcal{B}}\colon \mathbf{Top}_* \rightleftarrows \mathcal{O}\text{-}\mathbf{Alg} : U^{\mathcal{O}}_{\mathcal{B}},$$

and for a based space $(X, *)$, the relatively free $\mathcal{O}$-algebra $F^{\mathcal{O}}_{\mathcal{B}}(X)$ is given by quotienting $\coprod_{r\geqslant 0} \mathcal{O}(r) \times_{\mathfrak{S}_r} X^r$ by the usual basepoint relations.

In the case of the little $d$-cubes operad $\mathcal{C}_d$, we have a unique void $\mathfrak{v} \in \mathcal{C}_d(0)$ given by the empty configuration of boxes, and for a based space $X$, the free $\mathcal{C}_d$-algebra $F^{\mathcal{C}_d}_{\mathcal{B}}(X)$ is equivalent to the space $C(\mathbb{R}^d; X)$ of labelled configurations as considered in [Seg73; Sna74; McD75; Böd87].

A similar story can be told in the coloured setting: note that $(\mathcal{B} \odot N)$-algebras are the same as $N$-indexed families of based spaces; in particular, each $(\mathcal{B} \odot N)$-algebra $\boldsymbol{X} = (X_k)_{k\in N}$ gives rise to a functor $\boldsymbol{X}^-\colon N \wr \mathbf{Inj} \to \mathbf{Top}$ with $\boldsymbol{X}^K = X_{k_1} \times \cdots \times X_{k_r}$ as in Construction 2.1.3.

On the other hand, a morphism $\mathcal{B} \odot N \to \mathcal{O}$ is the same as a choice of void $\mathfrak{v}_n \in \mathcal{O}\binom{}{n}$ for each colour $n$. This can be used in order to 'block' inputs by precomposing with them: thus, we obtain cofaces

$$d_i \colon \mathcal{O}\tbinom{K}{n} \to \mathcal{O}\tbinom{d_i K}{n}, \quad \mu \mapsto \mu \circ_i \mathfrak{v}_{k_i},$$

each $\mathcal{O}\binom{-}{n}$ can be enhanced to a functor $(N \wr \mathbf{Inj})^{\mathrm{op}} \to \mathbf{Top}$, and we have

$$F^{\mathcal{O}}_{\mathcal{B}\odot N}(\boldsymbol{X})_n \cong \int^{K \in N \wr \mathbf{Inj}} \mathcal{O}\tbinom{K}{n} \times \boldsymbol{X}^K.$$

**Example 3.3.26** (Free algebras over $G$-spaces)**.** If $\boldsymbol{G} = (G_n)_{n \in N}$ is a family of topological groups, regarded as a topologically enriched groupoid, then a $\boldsymbol{G}$-algebra is the same as a family $\boldsymbol{X} = (X_n)_{n \in N}$ of spaces, with an action of $G_n$ on $X_n$. As in Construction 2.1.3, we obtain a functor $\boldsymbol{X}^- \colon \boldsymbol{G} \wr \boldsymbol{\Sigma} \to \mathbf{Top}$.

On the other hand, a morphism $\rho \colon \boldsymbol{G} \to \mathcal{O}$ gives rise to a right action of $\boldsymbol{G}^K$ on each operation space $\mathcal{O}\binom{K}{n}$ by precomposition: this defines a collection of functors $\mathcal{O}\binom{-}{n} \colon (\boldsymbol{G} \wr \boldsymbol{\Sigma})^{\mathrm{op}} \to \mathbf{Top}$, and the functor $F^{\mathcal{O}}_{\boldsymbol{G}}$ has the description

$$F^{\mathcal{O}}_{\boldsymbol{G}}(\boldsymbol{X})_n \cong \int^{K \in \boldsymbol{G} \wr \boldsymbol{\Sigma}} \mathcal{O}\tbinom{K}{n} \times \boldsymbol{X}^K \cong \coprod_{[K]} \mathcal{O}\tbinom{K}{n} \times_{\boldsymbol{G}^K} \boldsymbol{X}^K.$$

In combination with Example 3.3.25, we note that $(\mathcal{B} \odot \boldsymbol{G})$-algebras are the same as families $\boldsymbol{X} = (X_n)_{n \in N}$ of based spaces, together with a basepoint-preserving action of $G_n$ on $X_n$ for each $n \in N$. As in Construction 2.1.3, we obtain a functor $\boldsymbol{X}^- \colon \boldsymbol{G} \wr \mathbf{Inj} \to \mathbf{Top}$. Similarly, a morphism $\mathcal{B} \odot \boldsymbol{G} \to \mathcal{O}$ gives rise to $\mathcal{O}\binom{-}{n} \colon (\boldsymbol{G} \wr \mathbf{Inj})^{\mathrm{op}} \to \mathbf{Top}$ and the functor $F^{\mathcal{O}}_{\mathcal{B}\odot\boldsymbol{G}}$ is of the form

$$F^{\mathcal{O}}_{\mathcal{B}\odot\boldsymbol{G}}(\boldsymbol{X})_n \cong \int^{K \in \boldsymbol{G} \wr \mathbf{Inj}} \mathcal{O}\tbinom{K}{n} \times \boldsymbol{X}^K.$$

## 3.4. Dyeing of monochromatic operads

In this section, we construct operads with colour set $\overline{\mathbb{N}} = \{1, 2, \dots\}$ out of monochromatic ones, in a way that, heuristically, allows us to 'bunch' several inputs together to a single one, whose colour then agrees with the number of inputs it comprises. Moreover, we give a description of their algebras.

### 3.4.1. *A motivating example*

We still owe the reader a convincing example of an honest coloured operad. This subsection aims to give at least some visual impression of which operads we are going to need in Chapter 5 when dealing with moduli spaces of Riemann surfaces. By doing so, we anticipate some notions, which will be made precise later, for dramaturgical reasons. The reader who prefers a concise and formal treatise may skip this subsection without compunction.

Our example starts with the following classical result [Mil86; Böd90b]: if $\mathfrak{M}_{g,n}$ denotes the moduli space of Riemann surfaces of genus $g \geqslant 0$ with $n \geqslant 1$ parametrised boundary curves, then the collection $\coprod_{g \geqslant 0} \mathfrak{M}_{g,1}$ carries, up to equivalence, the structure of an $E_2$-algebra. We will encounter several comparable ways of making this action precise; for the moment, the following description shall suffice: let $\mathfrak{M}_{g,n\cdot\circ}$ be the moduli space of Riemann surfaces with a parametrisation of a *collared neighbourhood* of each boundary curve; it is easy to see that $\mathfrak{M}_{g,n\cdot\circ} \simeq \mathfrak{M}_{g,n}$. Given collared surfaces $\mathcal{C}_i \in \mathfrak{M}_{g_i,1\cdot\circ}$ and a configuration $\mu \in \mathscr{D}_2(r)$ of discs, we can form a new compound surface $\mu(\mathcal{C}_1, \dots, \mathcal{C}_r) \in \mathfrak{M}_{g_1+\dots+g_r,1\cdot\circ}$ out of it by removing the $r$ discs from the large disc and by gluing in the given surfaces at the emerging holes, using the parametrisations of the collars.

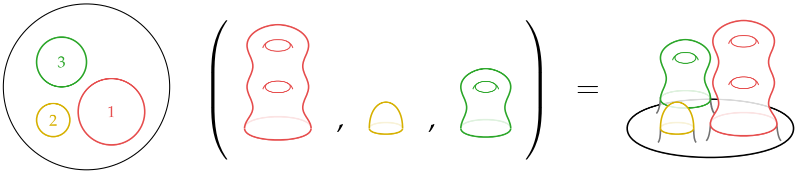


**Figure 3.8.** An instance of $\mathscr{D}_2(3) \times (\mathfrak{M}_{2,1\cdot\circ} \times \mathfrak{M}_{0,1\cdot\circ} \times \mathfrak{M}_{1,1\cdot\circ}) \to \mathfrak{M}_{3,1\cdot\circ}$

It is conspicious that for this construction, we have to restrict to the case of precisely one boundary curve. While it is not surprising that closed surfaces have to be treated differently, we would like to incorporate surfaces with multiple boundary curves into the picture. To do so, we need a version of $\mathscr{D}_2$, which is coloured in $\overline{\mathbb{N}} = \{1, 2, \dots\}$, which we call $\overline{\mathbb{N}}(\mathscr{D}_2)$, the *dyeing* („Färbung“) of $\mathscr{D}_2$, and whose operation spaces look as follows: the space $\overline{\mathbb{N}}(\mathscr{D}_2)\binom{k_1,\dots,k_r}{n}$ contains configurations of $k_1 + \dots + k_r$ small discs on $n$ large discs: the first $k_1$ discs form a common input, the next $k_2$ discs form a common input, and so on. If we are given such an operation $\mu$, we can, in the same fashion, glue in Riemann surfaces $\mathcal{C}_i \in \mathfrak{M}_{g_i,k_i\cdot\circ}$ and obtain a Riemann surface $\mu(\mathcal{C}_1,\dots,\mathcal{C}_r)$ with $n$ boundary curves. The result may be disconnected, but in fact there is a suboperad $\overline{\mathbb{N}}^c(\mathscr{D}_2)$, called the *connective suboperad*, which enjoys the property that if the inputs are connected, the output is connected as well. This gives us an action of $\overline{\mathbb{N}}^c(\mathscr{D}_2)$ on the sequence of moduli spaces $\mathfrak{M} := \big(\coprod_{g\geqslant 0}\mathfrak{M}_{g,n\cdot\circ}\big)_{n\geqslant 1}$, which is depicted in Figure 3.9. The genus grading is a bit more complicated: the genera do not add up, but for the Euler characteristic, we have $\chi(\mu(\mathcal{C}_1,\dots,\mathcal{C}_r)) = \sum_i(\chi(\mathcal{C}_i) - k_i) + n$.

Here we follow a different approach than [GK98], who encapsulated the combinatorical difficulties arising from the genus grading in a concept called a *modular operad*. Roughly speaking, we follow a more classical approach in order to attack more classical questions with it.

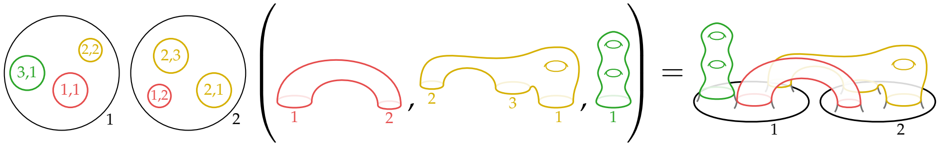


**Figure 3.9.** An instance of $\overline{\mathbb{N}}^c(\mathscr{D}_2)\binom{2,3,1}{2} \times (\mathfrak{M}_{0,2\cdot\circ} \times \mathfrak{M}_{1,3\cdot\circ} \times \mathfrak{M}_{2,1\cdot\circ}) \to \mathfrak{M}_{4,2\cdot\circ}$.

The remainder of this section aims to formally construct $\overline{\mathbb{N}}(\mathscr{C})$ out of a monochromatic operad $\mathscr{C}$ and to understand its properties.

### 3.4.2. *PROPs associated to operads*

Before coming to the dyeing construction $\overline{\mathbb{N}}(-)$, we recall a classical construction, which can be found in standard textbooks [Lei04; LV12], namely the PROP ('**pro**ducts and **p**ermutations category') assigned to a monochromatic operad. The notion of a PROP is comparably old and goes back to [Mac65].

**Definition 3.4.1.** Let $(\mathbf{V},\otimes,\mathbf{1})$ be a nice monoidal category as in Setting 3.1.1. A *PROP* is a strict symmetric monoidal $\mathbf{V}$-enriched category $(\mathbf{P},\sqcup,0)$ with object set $\mathbb{N} = \{0,1,\dots\}$, where the monoidal sum on objects is given by the usual addition of natural numbers. For a morphism $a\colon k \to n$, we call $k$ the *arity* („Stelligkeit") and $n$ the *valency* („Wertigkeit") of $a$.

Morphisms of PROPs $F\colon \mathbf{P} \to \mathbf{Q}$ are strict monoidal functors, i.e. functors which satisfy $\mathrm{ob}(F) = \mathrm{id}_{\mathbb{N}}$ and $F(a \sqcup a') = Fa \sqcup Fa'$.

**Remark 3.4.2** (Twists)**.** Let $(\mathbf{P}, \sqcup, 0)$ be a PROP. The twist maps in $\mathbf{P}$ which come from the *symmetric* structure can be expressed as a collection of right $\mathfrak{S}_k$-actions and left $\mathfrak{S}_n$-actions on the morphism objects $\mathbf{P}\binom{k}{n}$, which are interchangeable and satisfy the following compatibility properties, compare [Mac65, § 24], for which we abbreviate $\tau a := \tau_* a$ and $a\tau := \tau^* a$, and denote by $(1\,2)_{n,n'} \in \mathfrak{S}_{n+n'}$ the block transposition assigned to $n, n' \geqslant 0$:

$$\begin{aligned} b \circ (a\tau) &= (b\tau) \circ a, & (\tau a) \sqcup (\tau' a') &= (\tau \sqcup \tau')(a \sqcup a'), \\ (b \circ a)\tau &= b \circ (a\tau), & (a\tau) \sqcup (a'\tau') &= (a \sqcup a')(\tau \sqcup \tau'), \\ \tau(b \circ a) &= (\tau b) \circ a, & (1\,2)_{n,n'}(a \sqcup a') &= (a' \sqcup a)(1\,2)_{k,k'}. \end{aligned}$$

**Example 3.4.3.** Many well-known categories with object set $\mathbb{N}$ carry the structure of a PROP in the symmetric monoidal category of sets:

1. The discrete category $\mathbb{N}$, which has as objects the natural numbers and has no morphisms apart from the identities, carries a unique structure of a PROP. Note that $\mathbb{N}$ is initial among PROPs in sets.

2. Consider the category **Fin** whose morphisms $k \to n$ are all maps of the form $\underline{k} \to \underline{n}$, where we abbreviate $\underline{n} := \{1, \ldots, n\}$ as before. Then **Fin** carries the structure of a PROP by forming the disjoint sum of maps. The same applies to the subcategories **Inj** and $\mathbf{\Sigma}$.

3. An example of a slightly different flavour is given by the 2-dimensional (homotopy) cobordism category $\mathbf{hCob}_2$, where morphisms $k \to n$ are given by isomorphism classes of (possibly disconnected) orientable surfaces with $k$ incoming and $n$ outgoing numbered boundary curves. The composition is given by sewing surfaces and the monoidal sum is given by the disjoint union of surfaces. There is a topological analogue, which is called **M** and which we encounter in Construction 5.1.11.

4. The terminal PROP is given by the category $E\mathbb{N}$, with morphism sets $(E\mathbb{N})\binom{k}{n} = *$ for all natural numbers $k$ and $n$. This category is sometimes called the *chaotic category*.

Again we can use the canonical copowering $[-]\colon \mathbf{Set} \to \mathbf{V}$ to construct, for each of the above examples $\mathbf{P}$, a $\mathbf{V}$-enriched version $\mathbf{P}^{\mathbf{V}}$ by $\mathbf{P}^{\mathbf{V}}\binom{k}{n} = [\mathbf{P}\binom{k}{n}]$. If it is clear which category we consider, then we skip the decoration $(-)^{\mathbf{V}}$.

The notion of a PROP is closely related to the notion of a monochromatic operad, as the following construction from [LV12, Def. 5.3.11] shows:

**Construction 3.4.4.** Each PROP $(\mathbf{P}, \sqcup, 0)$ gives rise to a monochromatic operad $\mathbf{P}\binom{-}{1} = (\mathbf{P}\binom{r}{1})_{r \geqslant 0}$, with input permutations given by the twists in $(\mathbf{P}, \sqcup, 0)$, identity given by the identity on the object 1, and compositions induced by the monoidal sum and the composition in $\mathbf{P}$ as follows:

$$\mathbf{P}\tbinom{r}{1} \otimes \big(\mathbf{P}\tbinom{s_1}{1} \otimes \cdots \otimes \mathbf{P}\tbinom{s_r}{1}\big) \xrightarrow{\ \sqcup\ } \mathbf{P}\tbinom{r}{1} \otimes \mathbf{P}\tbinom{s_1+\cdots+s_r}{r} \xrightarrow{\ \circ\ } \mathbf{P}\tbinom{s_1+\cdots+s_r}{1}.$$

If $\mathbf{V}$ is cocomplete and $\otimes$ commutes with colimits in both arguments, then the functor $\mathbf{P} \mapsto \mathbf{P}\binom{-}{1}$ from the category of PROPs to the category of monochromatic operads admits a left adjoint, which is called the *associated PROP*: given an operad $\mathscr{C}$, we construct the associated PROP $\operatorname{cat}(\mathscr{C})$ by setting

$$\operatorname{cat}(\mathscr{C})\tbinom{k}{n} := \coprod_{f\colon \underline{k} \to \underline{n}} \mathscr{C}(\#f^{-1}(1)) \otimes \cdots \otimes \mathscr{C}(\#f^{-1}(n)),$$

endowed with the following extra structure: the monoidal sum is given on summands indexed by $f\colon \underline{k} \to \underline{n}$ and $f'\colon \underline{k'} \to \underline{n'}$ via the identification

$$\bigotimes_{\ell=1}^{n} \mathscr{C}(\#f^{-1}(\ell)) \otimes \bigotimes_{\ell'=1}^{n'} \mathscr{C}(\#f'^{-1}(\ell')) \cong \bigotimes_{\ell=1}^{n+n'} \mathscr{C}(\#(f \sqcup f')^{-1}(\ell)).$$

For the description of the twists, note that $\mathfrak{S}_k$ right acts on $\operatorname{cat}(\mathscr{C})\binom{k}{n}$ by mapping the summand indexed by $f$ to the (canonically isomorphic) summand indexed by $f \circ \tau$ for each $\tau \in \mathfrak{S}_k$, and similarly, $\mathfrak{S}_n$ left acts on $\operatorname{cat}(\mathscr{C})\binom{k}{n}$ by mapping the summand indexed by $f$ to the summand indexed by $\tau \circ f$ for each $\tau \in \mathfrak{S}_n$.

The composition is defined on each summand: for $g\colon \underline{n} \to \underline{m}$, we write $g^{-1}(\hbar) = \{\ell_{\hbar,1} < \cdots < \ell_{\hbar,v_\hbar}\} \subseteq \{1 < \cdots < n\}$ for each $1 \leqslant \hbar \leqslant m$; then, given $f\colon \underline{k} \to \underline{n}$, we send the summand of $\operatorname{cat}(\mathscr{C})\binom{n}{m} \otimes \operatorname{cat}(\mathscr{C})\binom{k}{n}$ indexed by the pair $(g, f)$ to the summand of $\mathrm{cat}(\mathscr{C})\binom{k}{m}$ indexed by $g \circ f$ via

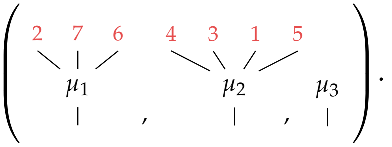
$$\begin{aligned}\bigotimes_{\hbar=1}^{m} \mathscr{C}(v_\hbar) \otimes \bigotimes_{\ell=1}^{n} \mathscr{C}(\#f^{-1}(\ell)) &\to \bigotimes_{\hbar=1}^{m}\left(\mathscr{C}(v_\hbar) \otimes \bigotimes_{i=1}^{v_\hbar} \mathscr{C}(\#f^{-1}(\ell_{\hbar,i}))\right)\\ &\to \bigotimes_{\hbar=1}^{m} \mathscr{C}(\#(g \circ f)^{-1}(\hbar)),\end{aligned}$$

where we use the operadic composition for the last step. By construction, $\mathrm{cat}(\mathscr{C})\binom{-}{1}$ is the same as $\mathscr{C}$ itself. The running index $\ell$ is called the *layer parameter*, and the upcoming examples will justify this name.

**Remark 3.4.5.** The morphism objects in $\mathrm{cat}(\mathscr{C})$ can also be written as

$$\mathrm{cat}(\mathscr{C})\binom{k}{n} \cong \coprod_{u_1+\cdots+u_n=k} (\mathscr{C}(u_1) \otimes \cdots \otimes \mathscr{C}(u_n)) \otimes_{\mathfrak{S}_{u_1} \times \cdots \times \mathfrak{S}_{u_n}} \mathbf{\Sigma}^{\mathbf{V}}\binom{k}{u_1+\cdots+u_n},$$

and this description gives a little bit more intuition: for $\mu_\ell \in \mathscr{C}(u_\ell)$ and a bijection $\varphi\colon \underline{k} \to \underline{u_1} \sqcup \cdots \sqcup \underline{u_n}$, we get an element $(\bigotimes_\ell \mu_\ell) \otimes_{\mathfrak{S}} \varphi \in \mathrm{cat}(\mathscr{C})\binom{k}{n}$, and we depict this element as a tuple $(Y_1, \ldots, Y_n)$ of admissible trees, each $Y_\ell$ with a single internal vertex of arity $u_\ell$, which is labelled by $\mu_\ell$, together with a bijection between the outer leaves of *all* trees and the set $\{1, \ldots, k\}$. For example, if we have $\mu_1 \in \mathscr{C}(3)$, $\mu_2 \in \mathscr{C}(4)$, and $\mu_3 \in \mathscr{C}(0)$, then we declare an element of $\mathrm{cat}(\mathscr{C})\binom{7}{3}$ by



**Example 3.4.6.** It is an easy exercise to check the following identifications:

$$\begin{aligned}\mathrm{cat}(\mathscr{I}) &\cong \mathbf{\Sigma},\\ \mathrm{cat}(\mathscr{B}) &\cong \mathbf{Inj},\\ \mathrm{cat}(\mathscr{C}om) &\cong \mathbf{Fin}.\end{aligned}$$

In particular, as $\mathscr{I}$ is the initial operad, each operad $\mathscr{C}$ gives rise to a map $\mathbf{\Sigma} \to \mathrm{cat}(\mathscr{C})$ of PROPs, and if $\mathscr{C}$ comes with a preferred void $\mathfrak{v} \in \mathscr{C}(0)$, i.e. an operad morphism $\mathscr{B} \to \mathscr{C}$, then we obtain a preferred map $\mathbf{Inj} \to \mathrm{cat}(\mathscr{C})$.

**Example 3.4.7** (Non-commutative finite sets)**.** The PROP $\mathrm{cat}(\mathscr{A}ss)$ has a similar description: it is the PROP whose morphisms $k \to n$ are maps $f\colon \underline{k} \to \underline{n}$ that come together with a total order $\prec$ on each fibre. The composition is given as follows: for $f\colon \underline{k} \to \underline{n}$, $g\colon \underline{n} \to \underline{m}$, and $\hbar \in \underline{m}$, we have an identification

$$(g \circ f)^{-1}(\hbar) \cong \coprod_{\ell \in g^{-1}(\hbar)} \{\ell\} \times f^{-1}(\ell),$$

and we endow the right side with the lexicographic order with respect to the existing orders on $g^{-1}(\hbar)$ and $f^{-1}(\ell)$. This category was introduced in [FL91] and is called the *category of non-commutative finite sets* $\mathbf{\Delta\Sigma}$, see [RP02].

The symbol '$\mathbf{\Delta\Sigma}$' comes from the fact that this category is the bicrossed product of the simplex category[3] $\mathbf{\Delta}$ with $\mathbf{\Sigma}$, see [Kas95, § IX]. In particular, $\mathbf{\Delta\Sigma}$ is generated by $\mathbf{\Delta}$ and $\mathbf{\Sigma}$, i.e. by three classes of morphisms:

1. permutations $\sigma\colon \underline{n} \to \underline{n}$,
2. cofaces $d^\ell : \underline{n-1} \to \underline{n}$ for $1 \leqslant \ell \leqslant n$, where $d^\ell$ is the unique monotone injective map omitting the $\ell^{\text{th}}$ element, and
3. codegeneracies $s^\ell : \underline{n} \to \underline{n-1}$ with $(s^\ell)^{-1}(\ell) = \{\ell \prec \ell{+}1\}$ for each $1 \leqslant \ell \leqslant n-1$,

**Example 3.4.8.** For $1 \leqslant d \leqslant \infty$, we let $\mathbf{Fin}_d := \mathrm{cat}(\mathscr{C}_d)$ be the PROP associated to the little $d$-cubes operad (or, equivalently, $\mathrm{cat}(\mathscr{D}_d)$, the PROP associated to the little $d$-discs operad).

Then $\mathbf{Fin}_d\binom{k}{n}$ contains configurations of $k$ numbered, small boxes on $n$ large cubes $[0;1]^d \times \underline{n}$, and the composition is given by placing boxes inside each other. Formally, an arrow in $\mathbf{Fin}_d\binom{k}{n}$ is a map $\vec{c}\colon [0;1]^d \times \underline{k} \hookrightarrow [0;1]^d \times \underline{n}$ of the following form: if $d < \infty$ and if we write $\vec{c} = (c_1, \dots, c_k)$, each $c_\jmath$ being a map $[0;1]^d \hookrightarrow [0;1]^d \times \underline{n}$, then for each $1 \leqslant \jmath \leqslant k$, there is, as a *layer parameter*, an integer $1 \leqslant \ell_\jmath \leqslant n$ and there are coordinates $0 \leqslant a_\jmath^\omega < b_\jmath^\omega \leqslant 1$ for each $1 \leqslant \omega \leqslant d$, such that for each $z = (z^1, \dots, z^d) \in [0;1]^d$, we have

$$c_\jmath(z) = \left( \begin{pmatrix} a_\jmath^1 + (b_\jmath^1 - a_\jmath^1)\cdot z^1 \\ \vdots \\ a_\jmath^d + (b_\jmath^d - a_\jmath^d)\cdot z^d \end{pmatrix}, \ell_\jmath \right).$$

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[3] More precisely, the augmented and shifted simplex category.

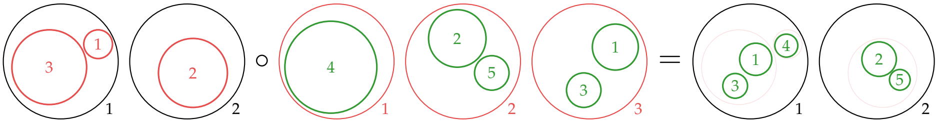


**Figure 3.10.** An instance of the composition $\mathbf{Fin}_2\binom{3}{2} \times \mathbf{Fin}_2\binom{5}{3} \to \mathbf{Fin}_2\binom{5}{2}$ if we use the equivalent little 2-discs description. Intuitively, each large disc is interpreted as a 'layer' on which content can be placed.

If $d = \infty$, then a similar description holds true: again we have for each index $1 \leqslant \jmath \leqslant k$ a layer parameter $1 \leqslant \ell_\jmath \leqslant n$, and for each $\omega \geqslant 1$, we require that there are $0 \leqslant a_\jmath^\omega < b_\jmath^\omega \leqslant 1$, only finitely many $a_\jmath^\omega$ different from 0 and only finitely many $b_\jmath^\omega$ different from 1, such that $c_\jmath(z)^\omega = a_\jmath^\omega + (b_\jmath^\omega - a_\jmath^\omega) \cdot z^\omega$ holds for each coordinate $\omega$.

The composition in $\mathbf{Fin}_d$ is given by composing the rectilinear embeddings $\vec{c}$ as maps. Finally, the terminal map $\mathscr{C}_d \to \mathscr{C}\!om$ of operads induces a map $\mathbf{Fin}_d \to \mathbf{Fin}$ of PROPs, which is given by remembering only on which of the $n$ discs each of the $k$ discs is placed.

### 3.4.3. *Representable operads*

In this subsection, we describe how each PROP $\mathbf{P}$ gives rise to an $\overline{\mathbb{N}}$-coloured operad $\operatorname{rep}(\mathbf{P})$. This straightforward construction is well-known for general symmetric monoidal categories.

**Construction 3.4.9.** Let $(\mathbf{P}, \sqcup, 0)$ be a PROP in $\mathbf{V}$. Then we construct an $\overline{\mathbb{N}}$-coloured operad $\operatorname{rep}(\mathbf{P})$, called the *representable operad underlying* $\mathbf{P}$, with

$$\operatorname{rep}(\mathbf{P})\binom{k_1,\dots,k_r}{n} := \mathbf{P}\binom{k_1+\dots+k_r}{n},$$

with input permutation given by permuting blocks of sizes $k_1, \dots, k_r$, with identities $\mathbb{1}_n \colon \mathbf{1} \to \mathbf{P}\binom{n}{n} = \operatorname{rep}(\mathbf{P})\binom{n}{n}$, and with composition induced by the monoidal sum and the composition in $\mathbf{P}$, where $|K| := k_1 + \dots + k_r$:

$$\begin{tikzcd}
\operatorname{rep}(\mathbf{P})\binom{K}{n} \otimes \bigotimes_i \operatorname{rep}(\mathbf{P})\binom{L_i}{k_i} \arrow[rr, dashed] \arrow[d, equal] & & \operatorname{rep}(\mathbf{P})\binom{L_1\cdots L_r}{n} \arrow[d, equal] \\
\mathbf{P}\binom{|K|}{n} \otimes \bigotimes_i \mathbf{P}\binom{|L_i|}{k_i} \arrow[r, "\sqcup"'] & \mathbf{P}\binom{|K|}{n} \otimes \mathbf{P}\binom{|L_1|+\dots+|L_r|}{|K|} \arrow[r, "\circ"'] & \mathbf{P}\binom{|L_1\cdots L_r|}{n}.
\end{tikzcd}$$

**Remark 3.4.10.** The term 'representable operad' is due to [Her00, § 1] and has the following reason: note that the operad $\mathrm{rep}(\mathbf{P})$ contains a distinguished class of morphisms

$$U_{k_1,\dots,k_r} := \mathbb{1}_{k_1} \sqcup \cdots \sqcup \mathbb{1}_{k_r} \in \mathrm{rep}(\mathbf{P})\binom{k_1,\dots,k_r}{k_1+\cdots+k_r},$$

called *universal morphisms*, and every operation $f \in \mathrm{rep}(\mathbf{P})\binom{k_1,\dots,k_r}{n}$ can be written uniquely as $\hat{f} \circ U_{k_1,\dots,k_r}$ where $\hat{f} \in \mathbf{P}\binom{k_1+\cdots+k_r}{n}$ is a unary. In this way, as Hermida puts it, the multiary structure is 'universally represented by multi[ary] classifying tensors'.

**Remark 3.4.11.** It may be surprising that we decided to skip the monoidal unit 0 as a colour. This has two reasons: firstly, we want to avoid redundancy, and an extra colour 0 would cause many coinciding operation spaces, e.g.

$$\mathrm{rep}(\mathbf{P})\binom{k_1,k_1,k_3}{n} = \mathrm{rep}(\mathbf{P})\binom{k_1,0,k_2,k_3}{n} = \mathrm{rep}(\mathbf{P})\binom{k_1,0,k_2,0,0,k_3}{n} = \dots,$$

and secondly, we have as a desired application the collection $(\coprod_{g\geqslant 0} \mathfrak{M}_{g,n})_{n\geqslant 1}$ of moduli spaces in mind, and we want to avoid closed Riemann surfaces.

**Remark 3.4.12.** Here are some immediate observations:

1. If we denote by $\overline{\mathbf{P}}$ the full subcategory of $\mathbf{P}$ with object set $\overline{\mathbb{N}}$, regarded as an operad with only unaries, then we obtain a morphism of $\overline{\mathbb{N}}$-coloured operads $\overline{\mathbf{P}} \to \mathrm{rep}(\mathbf{P})$.
2. By construction, if we restrict $\mathrm{rep}(\mathbf{P})$ to the colour 1, then we obtain the monochromatic operad $\mathbf{P}\binom{-}{1}$.

### 3.4.4. *Dyeing of operads*

Now we have everything at hand to dye an operad, as we wanted to do in Subsection 3.4.1. Even though this construction is just a composition of two well-known ones, I am not aware of any appearance in the literature.

**Definition 3.4.13** (Dyeing of operads)**.** Let $\mathscr{C}$ be a monochromatic operad inside $\mathbf{V}$. Then we define its *dyeing* to be the $\overline{\mathbb{N}}$-coloured operad

$$\overline{\mathbb{N}}(\mathscr{C}) := \mathrm{rep}(\mathrm{cat}(\mathscr{C})).$$

Since both constructions, cat and rep, are functorial, this gives rise to a functor $\overline{\mathbb{N}}\colon \mathbf{Op}(\mathbf{V}) \to \mathbf{Op}_{\overline{\mathbb{N}}}(\mathbf{V})$ from the category of monochromatic operads in $\mathbf{V}$ to the category of $\overline{\mathbb{N}}$-coloured operads in $\mathbf{V}$.

Before coming to the properties of the dyeing construction, let us convince ourselves that for $\mathscr{C} = \mathscr{D}_d$ this is exactly the construction we motivated in Subsection 3.4.1.

**Example 3.4.14** (Dyeing of $\mathscr{C}_d$)**.** Let $1 \leqslant d \leqslant \infty$. Then $\overline{\mathbb{N}}(\mathscr{C}_d) = \operatorname{rep}(\mathbf{Fin}_d)$, and we saw in Example 3.4.8 what operations in $\mathbf{Fin}_d$ look like. Consequently, operations in $\overline{\mathbb{N}}(\mathscr{C}_d)\binom{k_1,\dots,k_r}{n}$ are tuples $(\vec{c}_1, \dots \vec{c}_r)$ of rectilinear embeddings $\vec{c}_i\colon [0;1]^d \times \underline{k_i} \hookrightarrow [0;1]^d \times \underline{n}$ with mutually disjoint interiors, and the operadic composition is given by

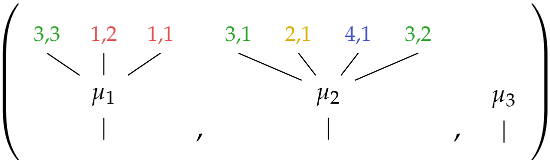
$$\begin{aligned}(\vec{c}_1, \dots, \vec{c}_r) \circ \big((\vec{c}_{1,1}, \dots, \vec{c}_{1,s_1}), \dots, (\vec{c}_{r,1}, \dots, \vec{c}_{r,s_r})\big)\\ = (\vec{c}_1 \circ \vec{c}_{1,1}, \dots, \vec{c}_1 \circ \vec{c}_{1,s_1}, \dots, \vec{c}_r \circ \vec{c}_{r,1}, \dots, \vec{c}_r \circ \vec{c}_{r,s_r}).\end{aligned}$$

The same holds for $\mathscr{D}_d$: we only have to replace rectilinear embeddings by embeddings of discs, see Figure 3.11.

If $\mathscr{C}$ is a general monochromatic operad, then we can visualise operations in $\overline{\mathbb{N}}(\mathscr{C})$ in a similar way by using trees as in Remark 3.4.5; we only have to refine our input assignment:

**Remark 3.4.15.** Operations in $\overline{\mathbb{N}}(\mathscr{C})\binom{k_1,\dots,k_r}{n}$ can be given by tuples $(Y_1, \dots, Y_n)$ of admissible trees, each $Y_\ell$ with a single internal vertex of arity $u_\ell$ which is labelled in $\mathscr{C}(u_\ell)$, together with a bijection between the outer leaves of *all* trees and the tableau $\mathbb{Y}_{k_1,\dots,k_r}$ from Definition 1.2.1.

For example, if $\mu_1 \in \mathscr{C}(3)$, $\mu_2 \in \mathscr{C}(4)$, and $\mu_3 \in \mathscr{C}(0)$, then we obtain an operation in $\overline{\mathbb{N}}(\mathscr{C})\binom{2,1,3,1}{3}$ by



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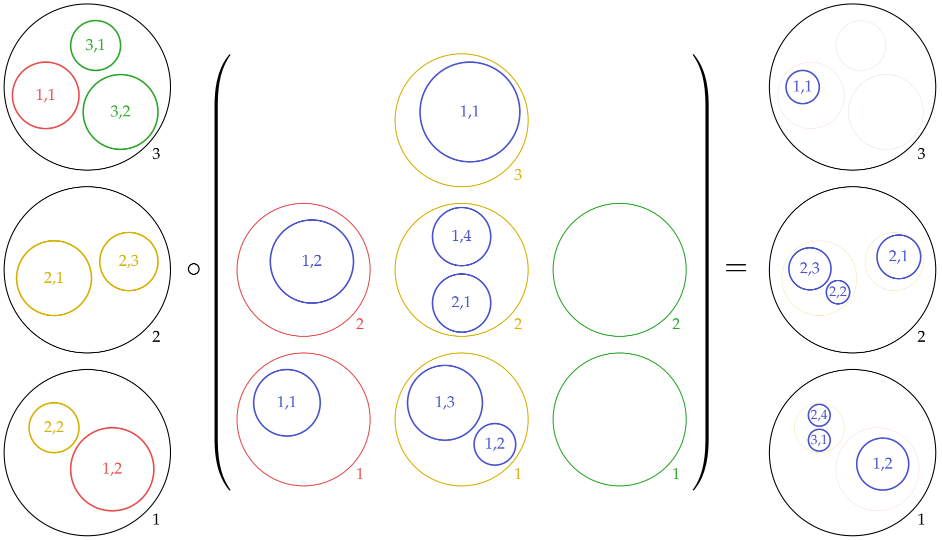

**Figure 3.11.** $\overline{\mathbb{N}}(\mathscr{D}_2)\binom{2,3,2}{3} \times \big(\overline{\mathbb{N}}(\mathscr{D}_2)\binom{2}{2} \times \mathscr{O}\binom{4,1}{3} \times \overline{\mathbb{N}}(\mathscr{D}_2)\binom{}{2}\big) \to \overline{\mathbb{N}}(\mathscr{D}_2)\binom{2,4,1}{3}$. Note that, in contrast to Figure 3.9 and Figure 3.10, we draw the multiple large discs of a given operation from bottom to top.

**Remark 3.4.16.** Note that $\overline{\Sigma}$ is the same as the sequence of symmetric groups $\mathfrak{S} = (\mathfrak{S}_n)_{n\geqslant 1}$, and hence, we will use the abbreviation $\mathfrak{S}$ instead.

For each monochromatic operad $\mathscr{C}$, we get a pairing $\zeta\colon (\mathscr{C},\mathfrak{S}) \to \overline{\mathbb{N}}(\mathscr{C})$ as follows: if $\mu \in \mathscr{C}(r)$ and $\sigma \in \mathfrak{S}_n$, then we put

$$\zeta(\mu,\sigma) := \left( \begin{array}{c} 1,\sigma^{-1}(1)\ \cdots\ r,\sigma^{-1}(1) \\ \diagdown \quad \diagup \\ \mu \\ | \end{array} ,\ \dots,\ \begin{array}{c} 1,\sigma^{-1}(n)\ \cdots\ r,\sigma^{-1}(n) \\ \diagdown \quad \diagup \\ \mu \\ | \end{array} \right) \in \overline{\mathbb{N}}(\mathscr{C})\binom{n,\dots,n}{n}.$$

This gives rise to a map of $\overline{\mathbb{N}}$-coloured operads $\iota\colon \mathscr{C} \odot \mathfrak{S} \to \overline{\mathbb{N}}(\mathscr{C})$, whence each $\overline{\mathbb{N}}(\mathscr{C})$-algebra has an underlying functor $\mathfrak{S} \to \mathscr{C}$-**Alg**. In other words, if $\boldsymbol{X} = (X_n)_{n\geqslant 1}$ is a $\overline{\mathbb{N}}(\mathscr{C})$-algebra, then each $X_n$ is a $\mathscr{C}$-algebra, together with a left $\mathfrak{S}_n$-action by $\mathscr{C}$-automorphisms.

If $\mathscr{C}$ has a preferred nullary operation $\mathfrak{v} \in \mathscr{C}(0)$, then we can extend the above pairing $\zeta$ to $(\mathscr{C},\overline{\mathbf{Inj}}) \to \overline{\mathbb{N}}(\mathscr{C})$ by filling, for each $f\colon \underline{k} \hookrightarrow \underline{n}$, the $\ell^{\text{th}}$ entry of the tuple $\zeta(\mu,f) \in \overline{\mathbb{N}}(\mathscr{C})\binom{k,\dots,k}{n}$ with $\mathfrak{v}$ whenever $f^{-1}(\ell) = \varnothing$.

The following paragraph shows how the dyeing construction in different symmetric monoidal categories can be compared via lax monoidal functors.

**Construction 3.4.17.** Let $H\colon (\mathbf{V}, \otimes, \mathbf{1}) \to (\mathbf{V}', \otimes', \mathbf{1}')$ be a lax monoidal functor and let $\mathscr{C}$ be a monochromatic operad in $\mathbf{V}$. Then there is a natural morphism $\eta_{\mathscr{C}}\colon \overline{\mathbb{N}}(H\mathscr{C}) \to H\overline{\mathbb{N}}(\mathscr{C})$ of $\overline{\mathbb{N}}$-coloured operads in $\mathbf{V}'$, which is comprised of maps of the form

$$\coprod_{f\colon \underline{k}\to\underline{n}} \bigotimes_{\ell=1}^{n}{}' H\mathscr{C}(\#f^{-1}(\ell)) \to H\left(\coprod_{f\colon \underline{k}\to\underline{n}} \bigotimes_{\ell=1}^{n} \mathscr{C}(\#f^{-1}(\ell))\right)$$

Clearly, if $H$ commutes with finite coproducts and if, for all $u_1, \ldots, u_n \geqslant 0$, the transformation $\bigotimes'_\ell H\mathscr{C}(u_\ell) \to H(\bigotimes_\ell \mathscr{C}(u_\ell))$ is an isomorphism, then $\eta_{\mathscr{C}}$ is an isomorphism of $\overline{\mathbb{N}}$-coloured operads. Note that the second condition is a fortiori satisfied if $H$ is strong monoidal.

**Example 3.4.18.** The functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ is strong monoidal and preserves coproducts. Thus, $\pi_0(\overline{\mathbb{N}}(\mathscr{C})) \cong \overline{\mathbb{N}}(\pi_0(\mathscr{C}))$; for example, we get

$$\pi_0(\overline{\mathbb{N}}(\mathscr{C}_d)) = \begin{cases} \overline{\mathbb{N}}(\mathscr{A}\!ss) & \text{for } d = 1, \\ \overline{\mathbb{N}}(\mathscr{C}\!om) & \text{for } d \geqslant 2. \end{cases}$$

For a principal ideal domain $R$, the functor $H_\bullet(-;R)\colon \mathbf{Top} \to R\text{-}\mathbf{Mod}^{\mathbb{Z}}$ is lax monoidal and also preserves coproducts. If $\mathbb{F}$ is a field, then $H_\bullet(-;\mathbb{F})$ is even strong monoidal, whence we get $H_\bullet(\overline{\mathbb{N}}(\mathscr{C});\mathbb{F}) \cong \overline{\mathbb{N}}(H_\bullet(\mathscr{C};\mathbb{F}))$ for each monochromatic operad $\mathscr{C}$.

For a general principal ideal domain $R$, this is not the case, but if $H_\bullet(\mathscr{C};R)$ is degreewise and aritywise free, then the torsion terms in the Künneth theorem vanish and we obtain again $H_\bullet(\overline{\mathbb{N}}(\mathscr{C});R) \cong \overline{\mathbb{N}}(H_\bullet(\mathscr{C};R))$. This is for example the case for $\mathscr{C} = \mathscr{C}_d$, as [Arn69] shows. Therefore, we obtain

$$H_\bullet(\overline{\mathbb{N}}(\mathscr{C}_d);R) \cong \overline{\mathbb{N}}(H_\bullet(\mathscr{C}_d;R)) \cong \overline{\mathbb{N}}(\mathscr{P}\!ois_d^R).$$

Let us close this subsection with a collection of examples for which we aim to understand algebras over their dyed operads.

**Example 3.4.19** (Dyeing of $\mathscr{C}om$)**.** Using that $\mathrm{cat}(\mathscr{C}om) = \mathbf{Fin}$, the operation spaces of $\overline{\mathbb{N}}(\mathscr{C}om)$ are readily identified as

$$\overline{\mathbb{N}}(\mathscr{C}om)\binom{k_1,\dots,k_r}{n} = \mathbf{Fin}\binom{k_1+\dots+k_r}{n} = \prod_{i=1}^{r} \mathbf{Fin}\binom{k_i}{n},$$

and the composition is given entrywise by

$$\begin{aligned}(g_1,\dots,g_r) \circ \big((f_{1,1},\dots,f_{1,s_1}),\dots,(f_{r,1},\dots,f_{r,s_r})\big)\\ = (g_1 \circ f_{1,1},\dots,g_1 \circ f_{1,s_1},\dots,g_r \circ f_{r,1},\dots,g_r \circ f_{r,s_r}).\end{aligned}$$

One readily checks that the pairing $\zeta\colon (\mathscr{C}om, \mathfrak{S}) \to \overline{\mathbb{N}}(\mathscr{C}om)$ from Remark 3.4.16 extends to an isomorphism $\mathscr{C}om \odot \overline{\mathbf{Fin}} \to \overline{\mathbb{N}}(\mathscr{C}om)$ of operads, whence $\overline{\mathbb{N}}(\mathscr{C}om)$-algebras are the same as functors $\overline{\mathbf{Fin}} \to \mathscr{C}om\text{-}\mathbf{Alg}$ into the category of (topological) commutative monoids.

### 3.4.5. *Connective operations*

The last general construction, which we promised in Subsection 3.4.1, is the one of the *connective suboperad* $\overline{\mathbb{N}}^c(\mathscr{C})$: for $\mathscr{C} = \mathscr{D}_2$, the suboperad $\overline{\mathbb{N}}^c(\mathscr{D}_2)$ should contain only those configurations of discs which yield connected surfaces if we start with connected surfaces as arguments. Pictorially, an operation in $\overline{\mathbb{N}}(\mathscr{D}_2)\binom{K}{n}$ is connective if we can reach each of the $n$ large discs by jumping along discs which belong to the same input. This is a purely combinatorial property, which can be formalised as follows:

**Construction 3.4.20.** Recall from Example 3.4.19 that $\overline{\mathbb{N}}(\mathscr{C}om)\binom{k_1,\dots,k_r}{n}$ consists of tuples $f = (f_1,\dots,f_r)$ of maps $f_i\colon \underline{k_i} \to \underline{n}$. We call such a tuple *connective* („verbindend“) if the equivalence relation on $\underline{n}$, which is spanned by $\ell \sim \ell'$ if there is a $1 \leqslant i \leqslant r$ such that $\ell, \ell' \in \mathrm{im}(f_i)$, is full. This property is clearly invariant under input permutation and we denote the $\overline{\mathbb{N}}$-coloured quiver of these operations by $\overline{\mathbb{N}}^c(\mathscr{C}om) \subseteq \overline{\mathbb{N}}(\mathscr{C}om)$.

Moreover, $\mathbb{1}_n = (\mathrm{id}_n) \in \overline{\mathbb{N}}(\mathscr{C}om)\binom{n}{n}$ is clearly connective for each $n$, and it is straightforward to check that the composition of two connective operations is again connective. This shows that $\overline{\mathbb{N}}^c(\mathscr{C}om) \subseteq \overline{\mathbb{N}}(\mathscr{C}om)$ is a suboperad.

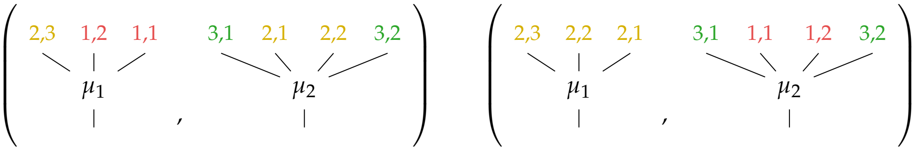

**Figure 3.12.** Two operations in $\overline{\mathbb{N}}(\mathscr{C})\binom{2,3,2}{2}$, with $\mu_1 \in \mathscr{C}(3)$ and $\mu_2 \in \mathscr{C}(4)$. The left one is connective, while the right one is not.

**Construction 3.4.21.** For each monochromatic operad $\mathscr{C}$ there is a unique (terminal) morphism of operads $\rho\colon \mathscr{C} \to \mathscr{C}om^{\mathbf{V}}$, and we define the *connective suboperad* $\overline{\mathbb{N}}^c(\mathscr{C})$ as the pullback of $\overline{\mathbb{N}}$-coloured operads (i.e. for each colour profile, the pullback in $\mathbf{V}$, together with the induced structure maps)

$$\begin{tikzcd} \overline{\mathbb{N}}^c(\mathscr{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathbb{N}}^c(\mathscr{C}om^{\mathbf{V}}) \arrow[d, hook] \\ \overline{\mathbb{N}}(\mathscr{C}) \arrow[r, "\overline{\mathbb{N}}(\rho)"'] & \overline{\mathbb{N}}(\mathscr{C}om^{\mathbf{V}}). \end{tikzcd}$$

If $\mathbf{V} = \mathbf{Top}$, then $\overline{\mathbb{N}}(\mathscr{C}om^{\mathbf{V}})$ consists of discrete operation spaces, so $\overline{\mathbb{N}}^c(\mathscr{C})$ is, topologically, just a restriction to certain connected components.

**Remark 3.4.22.** We have $\overline{\mathbb{N}}^c\binom{K}{1} = \overline{\mathbb{N}}\binom{K}{1}$, i.e. each operation with output colour 1 is connective. On the other hand, note that $\overline{\mathbb{N}}^c\binom{}{n} = \varnothing$ if $n \geqslant 2$, as the equivalence relation imposed on $\underline{n}$ is discrete.

Moreover, there is a replacement for Remark 3.4.16, where we skip the nullaries of $\mathscr{C}$: if we denote by $\mathscr{C}^c$ the monochromatic operad with operation spaces $\mathscr{C}^c(r) = \mathscr{C}(r)$ for $r \geqslant 1$ and $\mathscr{C}(r) = \varnothing$, then the aforementioned pairing corestricts to $(\mathscr{C}^c, \mathfrak{S}) \to \overline{\mathbb{N}}^c(\mathscr{C})$ and we obtain a map of $\overline{\mathbb{N}}$-coloured operads $\mathscr{C}^c \odot \mathfrak{S} \to \overline{\mathbb{N}}^c(\mathscr{C})$.

Put differently, each $\overline{\mathbb{N}}^c(\mathscr{C})$-algebra has an underlying sequence $(X_n)_{n\geqslant 1}$ of $\mathscr{C}^c$-algebras, and each $X_n$ carries an $\mathfrak{S}_n$-action by $\mathscr{C}^c$-automorphisms.

Chapter 4

# Vertical operads and their algebras

*First recall what trees themselves are.*

Tom Leinster

For $d = p + q$, we define the vertical operad $\mathcal{V}_{p,q}$ as a suboperad of $\overline{\mathbb{N}}(\mathcal{C}_d)$. As mentioned earlier, this operad is related to the extended Swiss cheese operad from [Wil17] and generalises ideas from [Böd90b; Böd13].

Using methods from [Dun86; Bri00], we show that $\mathcal{V}_{p,q}$ splits, up to equivalence of operads, as the Boardman–Vogt tensor product of $\mathcal{C}_p$ and $\overline{\mathbb{N}}(\mathcal{C}_q)$, which implies that the homotopy theory of $\mathcal{V}_{p,q}$-algebras is equivalent to the one of $\overline{\mathbb{N}}(\mathcal{C}_q)$-algebras with compatible $\mathcal{C}_p$-structures on each level.

In a consequent section, we extend our cellular calculations of Section 1.4 and construct a Morse flow similar to the one in [Bia21] in order to calculate the homology of $\mathcal{V}_{1,1}$ cellularly: on the one hand, this recovers Theorem 1.2.5 for the case $(p, q) = (1, 1)$, but in addition, the cellular decomposition behaves well with respect to the operadic structure. This allows us to give a presentation of the algebraic operad $H_\bullet(\mathcal{V}_{1,1})$ in the spirit of [Sin06] and to classify all (binary) divided power operations as in [CLM76, § III].

## 4.1. Definition and first observations

**Definition 4.1.1.** Let $1 \leqslant d < \infty$ and recall that operations in $\overline{\mathbb{N}}(\mathcal{C}_d)\binom{k_1,\dots,k_r}{n}$ are tuples $(c_{1,1}, \dots, c_{r,k_r})$ where $c_{i,\jmath} \colon [0;1]^d \hookrightarrow [0;1]^d \times \underline{n}$ is a rectilinear embedding for each $1 \leqslant i \leqslant r$ and each $1 \leqslant \jmath \leqslant k_i$, such that the interiors of $\operatorname{im}(c_{i,\jmath})$ are mutually disjoint.

Now fix a decomposition $d = p + q$ with $p \geqslant 0$ and $q \geqslant 1$ and consider for each $n \geqslant 1$ the projection $\mathrm{pr}_p^n \colon [0;1]^d \times \underline{n} = [0;1]^p \times [0;1]^q \times \underline{n} \to [0;1]^p$

which forgets the rear $q$ coordinates and layer parameter. The *vertical operad* $\mathcal{V}_{p,q}$ is then defined to be the suboperad of $\overline{\mathbb{N}}(\mathcal{C}_d)$ where $\mathcal{V}_{p,q}\binom{k_1,\dots,k_r}{n}$ contains only those configurations $(c_{1,1},\dots,c_{r,k_r})$ of boxes where $\mathrm{pr}^n_p \circ c_{i,\jmath} = \mathrm{pr}^n_p \circ c_{i,\jmath'}$ holds for all $1 \leqslant i \leqslant r$ and each $1 \leqslant \jmath, \jmath' \leqslant k_i$.

Pictorially, this condition requires that boxes which belong to the same input share their first $p$ coordinates. Clearly, this condition is invariant under input permutation, $\mathcal{V}_{p,q}$ contains the identities $\mathbb{1}_n$, and it is an easy task to check that $\mathcal{V}_{p,q}$ is closed under composition in $\overline{\mathbb{N}}(\mathcal{C}_d)$, see Figure 4.1. Hence $\mathcal{V}_{p,q}$ is a suboperad of $\overline{\mathbb{N}}(\mathcal{C}_d)$.

This operad $\mathcal{V}_{p,q}$ will play a central rôle throughout the rest of this chapter and the next one. Let us start by highlighting the connection to a variation of the so-called 'Swiss cheese operad'.

**Remark 4.1.2** (Extended Swiss cheese operad)**.** The operad $\mathcal{V}_{p,q}$ is related to a version of the Swiss cheese operad [Vor99], namely the *extended Swiss cheese operad* $\mathcal{S}w_{q,d}$ which is due to [Wil17] and can be described as follows:

The operad $\mathcal{S}w_{q,d}$ has two colours: $\boldsymbol{a}$ (for '**a**rbitrary') and $\boldsymbol{m}$ (for '**m**iddle'), and we define, using the notation from Construction 3.3.10, $\mathcal{S}w_{q,d}\binom{r\times\boldsymbol{a},s\times\boldsymbol{m}}{\boldsymbol{a}}$ to be $\mathcal{D}_d(r)$ for $s=0$, and empty for $s \geqslant 1$, while we let $\mathcal{S}w_{q,d}\binom{r\times\boldsymbol{a},s\times\boldsymbol{m}}{\boldsymbol{m}}$ be the subspace of $\mathcal{D}_d(r+s)$ containing configurations $(c_1,\dots,c_r,c'_1,\dots,c'_s)$ of discs such that all discs $c'_i\colon \mathbb{D}^d \hookrightarrow \mathbb{D}^d$ are centred inside $\mathbb{D}^q$, i.e. if we write $c'_i(z) = \hat{z} + \varepsilon\cdot z$, then we require that $\hat{z} \in \{0\}^p \times \mathbb{D}^q$. Composition is given by placing discs inside each other as usual, and one easily sees that the colour distinction ensures that the new centring constraint is preserved. There are several variations of these operads, see [PT21, § 4] for an overview.

Now we see that, up to equivalence, these operation spaces are special cases of our vertical operad $\mathcal{V}_{p,q}$. More precisely, we construct an equivalence

$$\mathcal{S}w_{q,d}\binom{r\times\boldsymbol{a},s\times\boldsymbol{m}}{\boldsymbol{m}} \simeq \mathcal{V}_{p,q}\binom{r\times 1,s}{1}$$

as follows: there is a deformation retract $\mathcal{S}w'_{q,d}\binom{r\times\boldsymbol{a},s\times\boldsymbol{m}}{\boldsymbol{m}} \hookrightarrow \mathcal{S}w_{q,d}\binom{r\times\boldsymbol{a},s\times\boldsymbol{m}}{\boldsymbol{m}}$ containing those configurations where all discs $c'_i$ have the same radius. For each embedded disc $c\colon \mathbb{D}^d \hookrightarrow \mathbb{D}^d$, there is a maximal rectilinear embedding $\tilde{c}\colon [0;1]^d \hookrightarrow [0;1]^d$ such that $\mathrm{im}(\tilde{c}) \subseteq \mathrm{im}(c)$, and for each $(c,c') \in \mathcal{S}w'_{q,d}$, the corresponding tuple $(\tilde{c},\tilde{c}')$ can be regarded as an element in $\overline{\mathbb{N}}(\mathcal{C}_d)\binom{r\times 1,s}{1}$.

Moreover, all boxes $\tilde{c}'_1, \dots, \tilde{c}'_s$ project to the same first $p$ coordinates, as they are maximal inside equally sized discs centred at $\{0\}^p \times \mathbb{D}^q \subseteq \mathbb{D}^d$. This gives rise to a map $\mathcal{S}w'_{q,d}\binom{r\times \boldsymbol{a}, s\times \boldsymbol{m}}{\boldsymbol{m}} \hookrightarrow \mathcal{V}_{p,q}\binom{r\times 1, s}{1}$, which is again a deformation retract, the retraction given by shrinking boxes to cubes which are contained in mutually disjoint discs, and by recentring the configuration such that the last cluster projects down to a mid-centred cube in $[0;1]^p$.

It is worth pointing out that, although *some* operation spaces of $\mathcal{V}_{p,q}$ coincide with the ones from $\mathcal{S}w_{q,d}$, a crucial difference between the two operads lies in the combinatorics of their respective compositions: roughly speaking, in $\mathcal{V}_{p,q}$, clusters from different inputs can never be merged, while in $\mathcal{S}w_{q,d}$, different inputs can contribute central discs.

We continue by establishing a connection between vertical operads and vertical configuration spaces from the previous chapters. To do so, note that the pairing $\zeta\colon (\mathcal{C}_d, \mathfrak{S}) \to \overline{\mathbb{N}}(\mathcal{C}_d)$ from Remark 3.4.16 is of the following form: if $\mu = (c_1, \dots, c_r) \in \mathcal{C}_d(r)$ and $\sigma \in \mathfrak{S}_n$, then we have

$$\zeta(\mu, \sigma) = (c_1 \times \sigma, \dots, c_r \times \sigma).$$

In particular, $\zeta(\mu, \sigma)$ is contained in the suboperad $\mathcal{V}_{p,q}$ for all $p + q = d$, so if $\boldsymbol{X} = (X_n)_{n \geqslant 1}$ is an $\mathcal{V}_{p,q}$-algebra, then each $X_n$ is a $\mathcal{C}_d$-algebra, together with a left $\mathfrak{S}_n$-action on $X_n$ by $\mathcal{C}_d$-automorphisms.

We want to focus on the suboperad $\mathcal{B} \odot \mathfrak{S} \hookrightarrow \mathcal{C}_d \odot \mathfrak{S}$. A fortiori, each $\mathcal{V}_{p,q}$-algebra is a $(\mathcal{B} \odot \mathfrak{S})$-algebra by restriction; that is, it has an underlying based symmetric sequence.

**Proposition 4.1.3.** *If we denote by* $\mathbb{R}^{p,q} \times \underline{n} := \mathbb{R}^p \times (\mathbb{R}^q \times \underline{n}) \to \mathbb{R}^p$ *the trivial fibre bundle, then there are two sorts of homotopy equivalences:*

1. *For each colour profile* $K = (k_1, \dots, k_r)$ *and each* $n \geqslant 1$*, we have a homotopy equivalence*
$$\mathcal{V}_{p,q}\tbinom{K}{n} \to \tilde{V}_K(\mathbb{R}^{p,q} \times \underline{n}).$$

2. *For an equivariantly well-based symmetric sequence* $\boldsymbol{X} = (X_n)_{n \geqslant 1}$ *and each colour* $n \geqslant 1$*, we have a based homotopy equivalence*
$$F^{\mathcal{V}_{p,q}}_{\mathcal{B}\odot\mathfrak{S}}(\boldsymbol{X})_n \to V(\mathbb{R}^{p,q} \times \underline{n}; \boldsymbol{X}).$$

*Proof.* A choice of homeomorphism $\mathbb{R} \cong (0;1)$ gives rise to an isomorphism of bundles $\mathbb{R}^{p,q} \times \underline{n} \cong (0;1)^{p,q} \times \underline{n}$, so we can equally well work with the interior of a cube instead.

Similar to the classical monochromatic case [May72, Thm. 4.8], we have a map $\Phi := \Phi^K_n \colon \mathcal{V}_{p,q}\binom{K}{n} \to \tilde{V}_K((0;1)^{p,q} \times \underline{n})$ by taking mid-points, i.e.

$$\Phi(c_{1,1}, \dots, c_{r,k_r}) := \big(c_{1,1}\big(\tfrac{1}{2}, \dots, \tfrac{1}{2}\big), \dots, c_{r,k_r}\big(\tfrac{1}{2}, \dots, \tfrac{1}{2}\big)\big).$$

It admits a homotopy inverse $\Psi \colon \tilde{V}_K((0;1)^{p,q} \times \underline{n}) \to \mathcal{V}_{p,q}\binom{K}{n}$ that determines, for each vertical configuration $(z_{1,1}, \dots, z_{r,k_r})$ on $(0;1)^{p,q} \times \underline{n}$, the largest box diameter such that boxes which are centered in $z_{1,1}, \dots, z_{r,k_r}$ are still contained in $[0;1]^d \times \underline{n}$ and have disjoint interior. Then $\Phi \circ \Psi = \mathrm{id}_{\tilde{V}}$ and a homotopy $\Psi \circ \Phi \Rightarrow \mathrm{id}_{\mathcal{V}}$ is given by rescaling boxes.

The second equivalence uses the same maps $\Phi^K_n$: if we invoke the coend description for $F^{\mathcal{V}_{p,q}}_{\mathcal{B}\odot\mathfrak{S}}$ from Example 3.3.26, then we obtain an induced map

$$\begin{tikzcd}
F^{\mathcal{V}_{p,q}}_{\mathcal{B}\odot\mathfrak{S}}(\boldsymbol{X})_n \arrow[r, dashed] \arrow[d, equal] & V(\mathbb{R}^{p,q} \times \underline{n}; \boldsymbol{X}) \arrow[d, equal] \\
\int^{K \in \mathfrak{S}\wr\mathbf{Inj}} \mathcal{V}_{p,q}\binom{K}{n} \times \boldsymbol{X}^K \arrow[r, "\int^K \Phi^K_n \times \mathrm{id}_{\boldsymbol{X}^K}"'] & \int^{K \in \mathfrak{S}\wr\mathbf{Inj}} \tilde{V}_K(\mathbb{R}^{p,q} \times \underline{n}) \times \boldsymbol{X}^K.
\end{tikzcd}$$

Since each $\mathrm{Aut}_{\mathfrak{S}\wr\mathbf{Inj}}(K)$ acts freely on both $\mathcal{V}_{p,q}\binom{K}{n}$ and $\tilde{V}_K(\mathbb{R}^{p,q} \times \underline{n})$ and each coface $d^i \colon \boldsymbol{X}^{d_i K} \to \boldsymbol{X}^K$ is a cofibration, the bottom horizontal map agrees, up to equivalence with the map induced among the respective homotopy coends, which is an equivalence as each constituent $\Phi^K_n \times \mathrm{id}_{\boldsymbol{X}^K}$ is. □

In light of the previous Proposition, the results from Chapter 2 can be interpreted as an attempt to study the $E_d$-algebra structure which underlies the free $\mathcal{V}_{p,q}$-algebra $F^{\mathcal{V}_{p,q}}_{\mathcal{B}\odot\mathfrak{S}}(\boldsymbol{X})$ levelwise: the statement of Theorem 2.2.2 can without any difficulties be pushed a bit further, stating that the $p$-fold bar construction $B^p V(\mathbb{R}^{p,q} \times \underline{n}; \boldsymbol{X})$ is $E_q$-equivalent to $C(\mathbb{R}^q \times \underline{n}; \Sigma\boldsymbol{X})$, one just has to additionally entrain the projection $\underline{n} \to \underline{1}$ throughout the entire proof. While this completely describes the $\mathcal{C}_p$-algebra structure in the case where $\boldsymbol{X}$ is levelwise path connected, the next delooping steps are still hard to understand. Our second delooping result Theorem 2.3.4 gives a partial answer, by describing the *first* level of the $\mathcal{C}_{p+1}$-algebra $F^{\mathcal{V}_{p,1}}_{\mathcal{B}\odot\mathfrak{S}}(\boldsymbol{X})_1$ entirely.

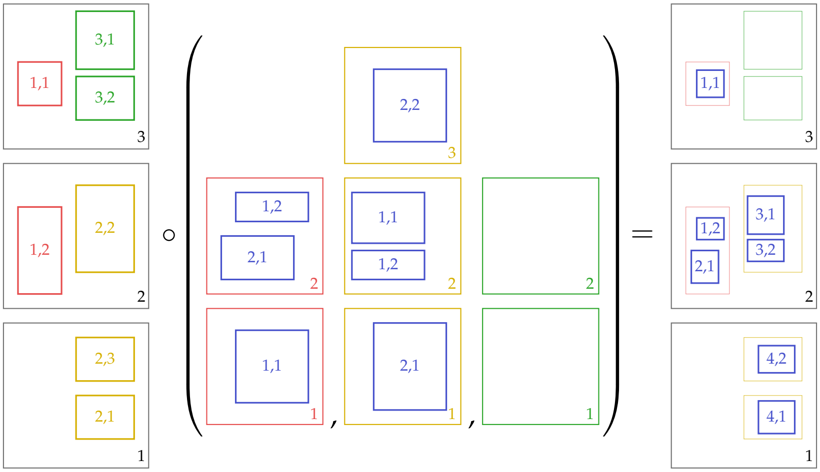

**Figure 4.1.** $\mathcal{V}_{1,1}\binom{2,3,2}{3} \times \left(\mathcal{V}_{1,1}\binom{2,1}{2} \times \mathcal{V}_{1,1}\binom{2,2}{3} \times \mathcal{V}_{1,1}\binom{}{2}\right) \to \mathcal{V}_{1,1}\binom{2,1,2,2}{3}$

## 4.2. Dunn additivity for vertical operads

In this section, we show that $\mathcal{V}_{p,q}$ splits, up to equivalence, into easier operads: the pairing among the little cubes operad which gave rise to Dunn's additivity theorem, see Example 3.3.23, has a coloured analogue as follows:

**Construction 4.2.1.** If we denote elements in $\mathscr{C}_p(r)$ as tuples $(c_1, \dots, c_r)$ with $c_i \colon [0;1]^p \hookrightarrow [0;1]^p$ a rectilinear embedding, and elements of $\overline{\mathbb{N}}(\mathscr{C}_q)\binom{k_1,\dots,k_s}{n}$ by $(\vec{c}_1, \dots, \vec{c}_s)$ with $\vec{c}_i \colon [0;1]^q \times \underline{k_i} \hookrightarrow [0;1]^q \times \underline{n}$ a rectilinear embedding, then we have a pairing $\zeta \colon (\mathscr{C}_p, \overline{\mathbb{N}}(\mathscr{C}_q)) \to \mathcal{V}_{p,q}$ with

$$\zeta\big((c_1, \dots, c_r), (\vec{c}_1, \dots, \vec{c}_s)\big) = (c_1 \times \vec{c}_1, \dots, c_1 \times \vec{c}_s, \dots, c_r \times \vec{c}_1, \dots, c_r \times \vec{c}_s).$$

**Theorem 4.2.2.** *The induced map of $\overline{\mathbb{N}}$-coloured operads*

$$\mathscr{C}_p \odot \overline{\mathbb{N}}(\mathscr{C}_q) \to \mathcal{V}_{p,q}$$

*is an equivalence of $\mathfrak{S}$-cofibrant operads.*

The proof of Theorem 4.2.2 is inspired by the method from [Bri00] and will occupy the rest of this section.

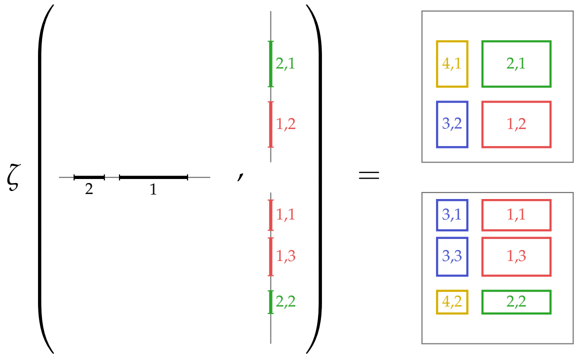

**Figure 4.2.** An instance of $\zeta\colon \mathscr{C}_1(2) \times \overline{\mathbb{N}}(\mathscr{C}_1)\binom{3,2}{2} \to \mathscr{V}_{1,1}\binom{3,2,3,2}{2}$

**Notation 4.2.3.** During the proof of Theorem 4.2.2, we will occasionally argue for a general pairing $\zeta\colon (\mathscr{C}, \mathscr{O}) \to \mathscr{P}$ of operads, where $\mathscr{C}$ is monochromatic and both $\mathscr{O}$ and $\mathscr{P}$ are $N$-coloured. In all these cases, we assume that these operads are *reduced*, i.e. $\mathscr{C}(0) = \mathscr{O}\binom{}{n} = \mathscr{P}\binom{}{n} = *$.

### 4.2.1. *Decomposable operations*

In order to prove Theorem 4.2.2, we show two substatements: firstly, the induced map $\zeta^{\odot}\colon \mathscr{C}_p \odot \overline{\mathbb{N}}(\mathscr{C}_q) \to \mathscr{V}_{p,q}$ is an isomorphism onto its image, and secondly, the inclusion of the image into $\mathscr{V}_{p,q}$ is an equivalence of operads.

While the first statement needs some further preparation, we prove the second statement in this subsection.

**Notation 4.2.4.** Let $\zeta\colon (\mathscr{C}, \mathscr{O}) \to \mathscr{P}$ be a pairing. For each pair of suboperads $\mathscr{C}' \subseteq \mathscr{C}$ and $\mathscr{O}' \subseteq \mathscr{O}$, we denote by $\mathscr{C}' \circledast \mathscr{O}' \subseteq \mathscr{P} := \zeta^{\odot}(\mathscr{C}' \odot \mathscr{O}')$ the image of $\mathscr{C}' \odot \mathscr{O}'$ under the induced operad map $\zeta^{\odot}$; it is a suboperad of $\mathscr{P}$.

Using this notation, the precise statement we want to prove in this subsection is the following:

**Proposition 4.2.5.** *The inclusion* $\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q) \hookrightarrow \mathscr{V}_{p,q}$ *is an equivalence.*

The proof of Proposition 4.2.5 is rather similar to the proof of [Dun86, Prop. 2.3]; however, we need a slightly more involved notion of decomposability, with respect to a system of binary operations.

**Definition 4.2.6.** Let $\mathcal{O}$ be an $N$-coloured topological operad. A family of subspaces $\mathscr{A}\binom{k_1,k_2}{n} \subseteq \mathcal{O}\binom{k_1,k_2}{n}$ of binary operations is called a *decomposer* if, for each unary $\mu \in \mathcal{O}\binom{n}{n'}$ and $\alpha \in \mathscr{A}\binom{k_1,k_2}{n}$, there is an operation $\alpha' \in \mathscr{A}\binom{k'_1,k'_2}{n'}$ and unaries $\mu_i \in \mathcal{O}\binom{k_i}{k'_i}$ such that $\mu \circ \alpha = \alpha' \circ (\mu_1, \mu_2)$ holds.

Let $\mathscr{A} \subseteq \mathcal{O}$ be a decomposer. We define what it means for an operation $\mu$ to be *decomposable* by induction on the arity $r := r(\mu)$: if $r \leqslant 1$, $\mu$ is called decomposable without any requirement; and if $r \geqslant 2$, we call $\mu$ decomposable if there is a permutation $\tau \in \mathfrak{S}_r$, a binary operation $\alpha \in \mathscr{A}$ and decomposable operations $\mu_i$ with $r(\mu_i) < r$, such that $\mu = \tau^*(\alpha \circ (\mu_1, \mu_2))$ holds. We denote by $\mathrm{d}\mathcal{O}\binom{K}{n} \subseteq \mathcal{O}\binom{K}{n}$ the subspace of all decomposable operations.

**Lemma 4.2.7.** *The collection* $\mathrm{d}\mathcal{O}$ *of operation spaces forms a suboperad of* $\mathcal{O}$*, and if* $\rho\colon \mathcal{O} \to \mathcal{O}'$ *is an operad morphism and* $\mathscr{A} \subseteq \mathcal{O}$ *and* $\mathscr{A}' \subseteq \mathcal{O}'$ *are decomposers with* $\rho(\mathscr{A}) \subseteq \mathscr{A}'$*, then* $\rho(\mathrm{d}\mathcal{O}) \subseteq \mathrm{d}\mathcal{O}'$ *and we obtain an induced map* $\mathrm{d}\rho\colon \mathrm{d}\mathcal{O} \to \mathrm{d}\mathcal{O}'$*.*

The proof of Lemma 4.2.7 is straightforward and left to the reader.

**Example 4.2.8.** For the little $p$-cubes operad $\mathscr{C}_p$, we let the decomposer be the system of *all* binary operations. Then $\mathrm{d}\mathscr{C}_p \subseteq \mathscr{C}_p$ is exactly the suboperad of decomposables which was considered in [Dun86, Def. 2.1].

For the dyeing $\overline{\mathbb{N}}(\mathscr{C}_q)$, we again let the decomposer be the system of *all* binary operations. Then we get $\mathrm{d}\overline{\mathbb{N}}(\mathscr{C}_q) = \overline{\mathbb{N}}(\mathscr{C}_q)$, since each $\nu \in \overline{\mathbb{N}}(\mathscr{C}_q)\binom{k_1,\dots,k_r}{n}$ can be written as $\nu' \circ U_{k_1,\dots,k_r}$, where $\nu'$ is a unary and $U_{k_1,\dots,k_r}$ are the universal morphisms from Remark 3.4.10; these universal morphisms, in turn, decompose into a binary tree, as $U_{k_1,\dots,k_r} = U_{k_1,k_2+\dots+k_r} \circ (\mathbb{1}_{k_1}, U_{k_2,\dots,k_r})$.

**Example 4.2.9.** For the vertical operad, the pairing $\zeta\colon (\mathscr{C}_p, \overline{\mathbb{N}}(\mathscr{C}_q)) \to \mathscr{V}_{p,q}$ gives rise to operad maps $\zeta(\mathbb{1}, -)\colon \overline{\mathbb{N}}(\mathscr{C}_q) \to \mathscr{V}_{p,q}$ and $\zeta(-, \mathbb{1}_n)\colon \mathscr{C}_p \to \mathscr{V}_{p,q}|_n$ for each $n \geqslant 1$. We define $\mathscr{A}\binom{k_1,k_2}{n}$ to be

$$\begin{cases} \{\zeta(\mu, \mathbb{1}_n);\, \mu \in \mathscr{C}_p(2)\} \cup \{\zeta(\mathbb{1}, \nu);\, \nu \in \overline{\mathbb{N}}(\mathscr{C}_q)\binom{n,n}{n}\} & \text{for } k_1 = k_2 = n, \\ \{\zeta(\mathbb{1}, \nu);\, \nu \in \overline{\mathbb{N}}(\mathscr{C}_q)\binom{k_1,k_2}{n}\} & \text{else.} \end{cases}$$

One easily checks that $\mathscr{A}$ indeed forms a decomposer.

The pairing $\zeta\colon (\mathscr{C}_p, \overline{\mathbb{N}}(\mathscr{C}_q)) \to \mathscr{V}_{p,q}$ behaves nicely with respect to the suboperads of decomposable operations, as the following Lemma shows.

**Lemma 4.2.10.** *We have* $\mathrm{d}\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q) = \mathrm{d}\mathscr{V}_{p,q}$.

*Proof.* In order to show '$\subseteq$', it is sufficient to show that $\zeta$ sends pairs of decomposable operations to decomposable ones: recall that $\mathrm{d}\overline{\mathbb{N}}(\mathscr{C}_q) = \overline{\mathbb{N}}(\mathscr{C}_q)$. By construction, the operad morphism $\zeta(\mathbb{1}, -)\colon \overline{\mathbb{N}}(\mathscr{C}_q) \to \mathscr{V}_{p,q}$ maps the decomposer of $\overline{\mathbb{N}}(\mathscr{C}_q)$ into the decomposer of $\mathscr{V}_{p,q}$; and thus, by Lemma 4.2.7, it corestricts to an operad map $\overline{\mathbb{N}}(\mathscr{C}_q) \to \mathrm{d}\mathscr{V}_{p,q}$. Likewise, for each $n \geqslant 1$, the operad map $\zeta(-, \mathbb{1}_n)\colon \mathscr{C}_p \to \mathscr{V}_{p,q}|_n$ corestricts to $\mathrm{d}\mathscr{C}_p \to (\mathrm{d}\mathscr{V}_{p,q})|_n$.

Now let $\mu \in \mathrm{d}\mathscr{C}_p$ and $\nu \in \overline{\mathbb{N}}(\mathscr{C}_q)$ be two operations. Then the previous paragraph shows that $\zeta(\mu, \mathbb{1}_n) \in \mathrm{d}\mathscr{V}_{p,q}$ and $\zeta(\mathbb{1}, \nu) \in \mathrm{d}\mathscr{V}_{p,q}$, and we obtain the equality $\zeta(\mu, \nu) = \zeta(\mu, \mathbb{1}_n) \circ (\zeta(\mathbb{1}, \nu), \ldots, \zeta(\mathbb{1}, \nu)) \in \mathrm{d}\mathscr{V}_{p,q}$.

The inclusion '$\supseteq$' is shown by induction on the arity $r$: clearly, the map $\zeta^{\odot}\binom{}{n}\colon (\mathscr{C}_p \odot \overline{\mathbb{N}}(\mathscr{C}_q))\binom{}{n} \to \mathscr{V}_{p,q}\binom{}{n} = *$ is surjective, and for each $\xi \in \mathscr{V}_{p,q}\binom{k}{n}$, there is a $\mu \in \mathscr{C}_p(1) = (\mathrm{d}\mathscr{C}_p)(1)$ and a $\nu \in \overline{\mathbb{N}}(\mathscr{C}_q)\binom{k}{n}$ such that $\xi = \zeta(\mu, \nu)$. If $r \geqslant 2$ and $\xi \in \mathrm{d}\mathscr{V}_{p,q}\binom{k_1, \ldots, k_r}{n}$ is an $r$-ary operation, then, by construction, there is a decomposition $\xi = \tau^*(\alpha \circ (\xi_1, \xi_2))$ with $\alpha$ inside the decomposer and $\xi_i$ of smaller arity and decomposable. By the induction hypothesis, $\xi_1$ and $\xi_2$ lie in the suboperad $\mathrm{d}\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q)$, and $\alpha$ is either of the form $\zeta(\mu, \mathbb{1}_n)$ with $\mu \in \mathscr{C}_p(2) = (\mathrm{d}\mathscr{C}_p)(2)$ or $\zeta(\mathbb{1}, \nu)$ with $\nu \in \overline{\mathbb{N}}(\mathscr{C}_q)\binom{l_1, l_2}{n}$, and in both cases, $\alpha$ lies in $\mathrm{d}\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q)$. Since $\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q)$ forms a suboperad, the composition $\xi = \tau^*(\alpha \circ (\xi_1, \xi_2))$ lies in the image as well. ☐

In [Dun86, Prop. 2.3], it is shown that the inclusion map $\mathrm{d}\mathscr{C}_p \hookrightarrow \mathscr{C}_p$ is an equivalence of operads. We can adjust Dunn's proof to show the analogue for the case of the vertical operad.

**Lemma 4.2.11.** *The inclusion* $\mathrm{d}\mathscr{V}_{p,q} \hookrightarrow \mathscr{V}_{p,q}$ *is an equivalence of operads.*

*Proof.* For each $0 \leqslant t \leqslant 1$, we let $u_{d,t}\colon [0;1]^d \hookrightarrow [0;1]^d$ be the map which shrinks the square by the factor $t$ 'towards the mid-point': formally, we let

$$u_{d,t}\Big(\tfrac{1}{2} + z^1, \ldots, \tfrac{1}{2} + z^d\Big) := \Big(\tfrac{1}{2} + t \cdot z^1, \ldots, \tfrac{1}{2} + t \cdot z^d\Big).$$

Now consider, for each input profile $K = (k_1, \ldots, k_r)$ and each output colour $n \geqslant 1$, the deformation $H\colon \mathscr{V}_{p,q}\binom{K}{n} \times (0;1] \to \mathscr{V}_{p,q}\binom{K}{n}$ given by

$$H(c_{1,1}, \ldots, c_{r,s_r}; t) := (c_{1,1} \circ u_{d,t}, \ldots, c_{r,s_r} \circ u_{d,t}),$$

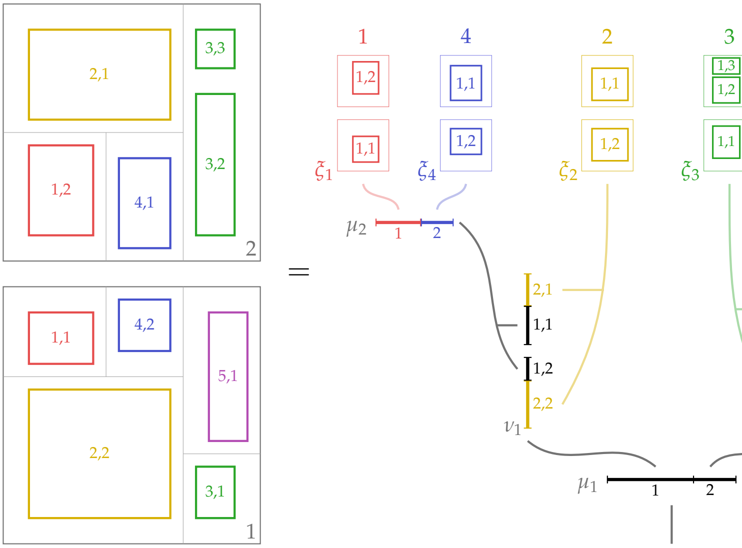

**Figure 4.3.** An operation in $\mathscr{V}_{1,1}\binom{2,2,3,2,1}{2}$, as well as a possible decomposition as $(2\,4\,3)^*(\zeta(\mu_1,\mathbb{1}_2)\circ(\zeta(\mathbb{1},\nu_1)\circ(\zeta(\mu_2,\mathbb{1}_2)\circ(\xi_1,\xi_4),\xi_2),\zeta(\mathbb{1},\nu_2)\circ(\xi_3,\xi_5)))$ with $\mu_1,\mu_2 \in \mathscr{C}_1(2)$ and $\nu_1 \in \overline{\mathbb{N}}(\mathscr{C}_1)\binom{2,2}{2}$ and $\nu_2 \in \overline{\mathbb{N}}(\mathscr{C}_1)\binom{2,1}{2}$, and $\xi_1,\dots,\xi_5 \in \mathscr{V}_{1,1}$ are unaries.

which leaves all boxes at their place, but makes them smaller for $t < 1$.

The following argument shows that for each $\xi \in \mathscr{V}_{p,q}\binom{K}{n}$, there is an $\varepsilon > 0$ such that $H(\xi,\varepsilon) \in \mathrm{d}\mathscr{V}_{p,q}\binom{K}{n}$: we write $\xi = (\vec{c}_1,\dots,\vec{c}_r)$ with $\vec{c}_i = (c_{i,1},\dots,c_{i,k_i})$. Let $\mathrm{pr}_p^n \colon [0;1]^d \times \underline{n} \to [0;1]^p$ be again the projection to the first $p$ coordinates and let $\eta_p \colon [0;1]^p \cong [0;1]^p \times \{0\}^q \hookrightarrow [0;1]^d$ be the obvious inclusion. For each $i$, we define the $p$-cube $c_i := \mathrm{pr}_n^p \circ c_{i,1} \circ \eta_p \colon [0;1]^p \to [0;1]^p$ and we denote its mid-point by $z_i := c_i(\frac{1}{2},\dots,\frac{1}{2}) \in [0;1]^p$. We say that two indices $1 \leqslant i, i' \leqslant r$ are *equivalent* if $z_i = z_{i'}$, and we write $S_1,\dots,S_h$ for the equivalence classes, ordered by their minimum. For each $1 \leqslant l \leqslant h$ we write $s_l := \#S_l$, and we let $\bar{c}_l \colon [0;1]^p \hookrightarrow [0;1]^p$ be the smallest little $p$-cube which contains all boxes $c_i$ with $i \in S_l$. Then there clearly is an $\varepsilon' > 0$ such that $(\bar{c}_1 \circ u_{\varepsilon'},\dots,\bar{c}_h \circ u_{\varepsilon'})$ is a tuple of little $p$-cubes with mutually disjoint interior, i.e. an element of $\mathscr{C}_p(h)$, and by the argument in [Dun86, Lem. 2.2], there is an $0 < \varepsilon < \varepsilon'$ such that $\mu := (\bar{c}_1 \circ u_\varepsilon,\dots,\bar{c}_h \circ u_\varepsilon)$ is even a decomposable operation in $\mathscr{C}_p(h)$.

On the other hand, let $\mathrm{pr}_q \colon [0;1]^d \to [0;1]^q$ be the projection to the last $q$ co-

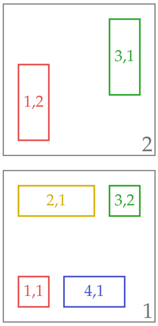

**Figure 4.4.** An indecomposable configuration in $\mathscr{V}_{1,1}\binom{2,1,2,1}{2}$: clearly, a decomposition cannot start with a vertical cut, and the only possible horizontal cut would separate the first input from the third one; however, they are 'entangled' on the second layer.

ordinates and let $\eta_q \colon [0;1]^q \cong \{0\}^p \times [0;1]^q \hookrightarrow [0;1]^d$ be the obvious inclusion. For each $1 \leqslant l \leqslant h$, we write $S_l = \{i_{l,1} < \cdots < i_{l,s_l}\}$, and for each $1 \leqslant m \leqslant s_l$, we let $\vec{c}_{l,m} := (\mathrm{pr}_q \times \underline{n}) \circ \vec{c}_{i_{l,m}} \circ (\eta_q \times \underline{k}_{i_{l;m}\cdot})$. Since these boxes share their midpoint with respect to the first $p$ coordinates, $\nu'_l := (\vec{c}_{l,1}, \dots, \vec{c}_{l,s_l})$ is a tuple of clusters of little $q$-cubes in $[0;1]^q \times \underline{n}$ with disjoint interior, i.e. an element of $\overline{\mathbb{N}}(\mathscr{C}_q)$, and the same applies to $\nu_l := (\vec{c}_{l,1} \circ u_{q,\varepsilon}, \dots, \vec{c}_{l,s_l} \circ u_{q,\varepsilon})$. Finally, for each $1 \leqslant l \leqslant h$ and each $1 \leqslant m \leqslant s_l$, there is a unary $\mu_{l,m} = (c_{l,m}) \in \mathscr{C}_p(1)$ with $c_{l,m} \colon [0;1]^p \hookrightarrow [0;1]^p$ measuring the difference between the maximal box $\bar{c}_l$ and the individual one $c_{i_{l,m}}$. Now it is an easy task to see that

$$H(\xi,\varepsilon) = \tau^*\big(\zeta(\mu,\mathbb{1}) \circ (\zeta(\mathbb{1},\nu_1),\dots,\zeta(\mathbb{1},\nu_h)) \circ (\zeta(\mu_{1,1},\mathbb{1}),\dots,\zeta(\mu_{h,s_h},\mathbb{1}))\big),$$

where $\tau \in \mathfrak{S}_r$ compares the total order on $\underline{r}$ with the lexicographic one on $\coprod_l \{l\} \times S_l$, and all operations in the above expression are decomposable.

For each $\xi \in \mathscr{V}_{p,q}\binom{K}{n}$, we define $\varepsilon_\xi := \sup(\varepsilon;\, H(\xi,\varepsilon) \in \mathrm{d}\mathscr{V}_{p,q})$. The previous paragraphs ensure that $\varepsilon_\xi > 0$. Moreover, $\mathrm{d}\mathscr{V}_{p,q}\binom{K}{n} \subseteq \mathscr{V}_{p,q}\binom{K}{n}$ is closed, whence we have $H(\xi,\varepsilon_\xi) \in \mathrm{d}\mathscr{V}_{p,q}\binom{K}{n}$. Finally, it is a tedious, but straightforward exercise to check that the assignment $\xi \mapsto \varepsilon_\xi$ is continuous. Since $\varepsilon_\xi = 1$ for all $\xi \in \mathrm{d}\mathscr{V}_{p,q}\binom{K}{n}$, this gives rise to a retraction of $\iota \colon \mathrm{d}\mathscr{V}_{p,q}\binom{K}{n} \hookrightarrow \mathscr{V}_{p,q}\binom{K}{n}$ by

$$\rho \colon \mathscr{V}_{p,q}\binom{K}{n} \to \mathrm{d}\mathscr{V}_{p,q}\binom{K}{n}, \quad \xi \mapsto H(\xi,\varepsilon_\xi).$$

and we have a homotopy $\hat{H} \colon \mathrm{id}_{\mathscr{V}} \Rightarrow \iota \circ \rho$ by $\hat{H}(\xi,t) := H(\xi, 1 - t + t \cdot \varepsilon_\xi)$. $\square$

**Corollary 4.2.12.** *The inclusion* $\mathrm{d}\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q) \hookrightarrow \mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q)$ *inside the surrounding operad* $\mathscr{V}_{p,q}$ *is an equivalence of operads.*

*Proof.* Recall that we saw in Lemma 4.2.10 that $\mathrm{d}\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q) = \mathrm{d}\mathscr{V}_{p,q}$. Now for each $0 < t \leqslant 1$, the map $H(-,t)\colon \mathscr{V}_{p,q}\binom{K}{n} \to \mathscr{V}_{p,q}\binom{K}{n}$ is given by precomposing with the unaries $u_{d,t} \times \underline{k}_i$, and $\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q)$ contains *all* unaries, so we obtain that $H(\zeta, t) \in (\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q))\binom{K}{n}$ holds for all $\zeta \in (\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q))\binom{K}{n}$, whence both the retraction $\rho$ and the homotopy $\hat{H}$ restrict to $(\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q))\binom{K}{n}$, leaving the subspace $\mathrm{d}\mathscr{V}_{p,q}\binom{K}{n}$ fixed. □

*Proof of Proposition 4.2.5.* Lemma 4.2.10 identifies $\mathrm{d}\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q)$ and $\mathrm{d}\mathscr{V}_{p,q}$ as suboperads of $\mathscr{V}_{p,q}$. Altogether, we have a square of suboperad inclusions

$$\begin{tikzcd}
\mathrm{d}\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q) \arrow[r, equal] \arrow[d, hook, "\simeq"'] & \mathrm{d}\mathscr{V}_{p,q} \arrow[d, hook, "\simeq"] \\
\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q) \arrow[r, hook] & \mathscr{V}_{p,q},
\end{tikzcd}$$

where the two vertical maps are equivalences of operads by Lemma 4.2.11 and Corollary 4.2.12. By the 2-out-of-3 property, the bottom inclusion is an equivalence as well. □

### 4.2.2. *A combinatorial interlude*

In order to proceed with the proof of Theorem 4.2.2 in the way we indicated before, we need a few combinatorial tools which apply to general operads.

Let us quickly recall from Construction 3.2.7 and Construction 3.3.19 how the Boardman–Vogt tensor product of two operads is constructed, here in a slightly simpler way, as we can assume one factor to be monochromatic:

**Reminder 4.2.13.** Let $\mathscr{C}$ be a monochromatic operad and $\mathscr{O}$ be an $N$-coloured operad. We define the $N$-coloured quiver $\mathscr{C} * \mathscr{O} := (\mathscr{C} \times N) \sqcup (* \times \mathscr{O})$ and consider the free operad $\Psi(\mathscr{C} * \mathscr{O})$ over $\mathscr{C} * \mathscr{O}$: as done in Construction 3.2.7, we define $(\mathscr{C} * \mathscr{O})(Y) := \prod_v (\mathscr{C} * \mathscr{O})\binom{K_v}{n_v}$, using the notation of Definition 3.2.3. Then the operation space $\Psi(\mathscr{C} * \mathscr{O})\binom{K}{n}$ is given by quotienting $\coprod_Y (\mathscr{C} * \mathscr{O})(Y)$ by the relation which equalises the formal and the given input permutation at each internal vertex, where $Y$ ranges over all $N$-trees with input profile $K$ and output colour $n$.

In this section, it is convenient to use a more visual language: a *two-parted tree* is an $N$-tree $Y$, together with a decomposition $V_\bullet = {\color{red}V_\bullet} \,\dot\cup\, {\color{green}V_\bullet}$ of the set of internal vertices, such that for each $v \in {\color{red}V_\bullet}$, we have $K_v = (n_v, \ldots, n_v)$; then we can rewrite $(\mathscr{C} * \mathscr{D})(Y) = \prod_{v \in {\color{red}V_\bullet}} \mathscr{C}(r_v) \times \prod_{w \in {\color{green}V_\bullet}} \mathscr{D}\binom{K_w}{n_w}$. Accordingly, we call an internal vertex *red* if it belongs to ${\color{red}V_\bullet}$, and *green* if it belongs to ${\color{green}V_\bullet}$, and we call this distinction the *domain* of $v$. Moreover, we call the vertex which is next to the root the *crotch*.

In a second step, we quotient the free operad $\Psi(\mathscr{C} * \mathscr{D})$ by the operadic relation from Construction 3.3.19: we abridge internal vertices labelled by ${\color{red}\mathbb{1}^{\mathscr{C}}}$ or ${\color{green}\mathbb{1}_n^{\mathscr{D}}}$, we compose adjacent internal vertices of the same domain, and we employ the interchange law.

For the remainder of the proof of Theorem 4.2.2, it is necessary to bring these representatives into certain normal forms, and this subsection aims to introduce them properly. Let us start with the notion of a *reduced* two-parted tree, which is similar to [Bri00, Def. 5.2]:

**Definition 4.2.14.** We call a two-parted tree *reduced* if it is either the stump, i.e. the tree with a single internal vertex of arity 0, or its arity is at least 1 and the following holds:

R1. there are no adjacent internal vertices of the same domain;

R2. there is no vertex of arity 0;

R3. there are no two adjacent internal vertices of arity 1, which are both not adjacent to an outer leaf.

The following elementary Lemma is proven in [Bri00, Lem. 5.3]:

**Lemma 4.2.15.** *Let $\mathscr{C}$ and $\mathscr{D}$ be operads and let $\vartheta \in \mathscr{C} \odot \mathscr{D}$ be an operation.*

1. *For each domain (red or green), $\vartheta$ can be represented by a reduced two-parted tree whose crotch has the chosen domain.*
2. *If the arity of $\vartheta$ is at least 2, then $\vartheta$ can be represented by a reduced two-parted tree whose crotch has at least arity 2.*

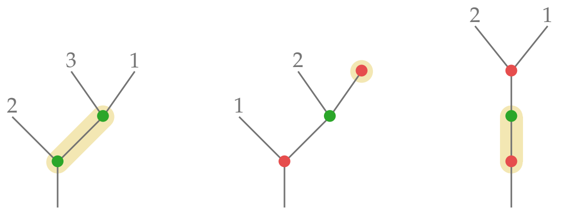

**Figure 4.5.** Here are three two-parted trees: the $i^{\text{th}}$ example violates the condition R$i$ (coloured in yellow) and none of the others.

In the special case where the underlying colour set is $\overline{\mathbb{N}}$, we are additionally interested in a property which affects the labels of the edges.

**Definition 4.2.16.** We call a reduced two-parted $\overline{\mathbb{N}}$-tree *bounded* if the following holds: let $v$ be a red internal vertex which is neither adjacent to the root nor to an outer leaf. By R1, all incoming edges start at a green internal vertex, and we let $w_1, \ldots, w_s$ be the collection of them. In this case, we require that

$$|K_{w_1}| + \cdots + |K_{w_s}| \geqslant n_v,$$

where for $K = (k_1, \ldots, k_r)$, we write $|K| := k_1 + \cdots + k_r$ as in Section 1.1.

The main merit of this property is captured by the following Lemma.

**Lemma 4.2.17.** *For each input profile $K$ and each output colour $n \geqslant 1$, there are only finitely many bounded two-parted $\overline{\mathbb{N}}$-trees of type $\binom{K}{n}$.*

*Proof.* If we ignore the labels for a moment, then [Bri00, § 5] gives an elementary argument why there are only finitely many reduced trees.

We are left to label the edges by positive integers. Note that the output colour $n$ and the input profile $K$, together with the numbering of the outer leaves, already determines the label of the edge ending at the root and the labels of the edges which start at an outer leaf. Now we proceed 'from top to bottom': it is sufficient to show that at each internal vertex, given all labels of the incoming edges (there is at least one, as we do not have nullaries), we have only finitely many possibilities to label the outgoing edge.

If the respective internal vertex is red, this statement is clear, as red internal vertices come from the monochromatic factor. Else, if the respective internal vertex $w$ is green, then the outgoing edge is either the root or it has to end in a red internal vertex $v$. If $v$ is adjacent to the root, then the global output colour $n$ determines the label $n_w = n_v = n$. If $v$ has an adjacent outer leaf with input number $i$, then we already know the label of their incoming edges $n_w = n_v = k_i$. If not, we make use of the bounding property: we let $w = w_1, \ldots, w_s$ be the neighbours of $w$, i.e. the green internal vertices whose outgoing edges land in $v$. Since we proceed from top to bottom, we have already chosen the labels of all incoming edges of $w_1, \ldots, w_s$; in particular, we have already determined $|K_{w_1}|, \ldots, |K_{w_s}|$. Now the bounding property tells us that $n_w = n_v$ can be at most $|K_{w_1}| + \cdots + |K_{w_s}|$. ☐

**Lemma 4.2.18.** *Let $\mathscr{C}$ and $\mathscr{D}$ be two monochromatic operads. Then each operation $\vartheta \in \mathscr{C} \odot \overline{\mathbb{N}}(\mathscr{D})$ can be represented by a bounded $\overline{\mathbb{N}}$-tree.*

*Proof.* By Lemma 4.2.15, we can find a representative for $\vartheta$ which is a reduced two-parted tree $Y$. We call a red vertex $v$ of $Y$ *bad* if it violates the bounding condition, and let $l$ be the sum of all distances from a bad red vertex to the crotch. Clearly, $Y$ is bounded if and only if $l = 0$ holds. We give an inductive procedure how to make $l$ smaller as long as $l \geqslant 1$, see Figure 4.6:

We pick a bad vertex $v$ and let $w_1, \ldots, w_s$ be the green vertices above $v$, with labels $\nu_i \in \overline{\mathbb{N}}(\mathscr{D})\binom{K_{w_i}}{n_v}$. If we define $k_i := |K_{w_i}|$, then each $\nu_i$ encodes a map $f_i \colon \underline{k_i} \to \underline{n_v}$. Let $k$ be the cardinality of $\bigcup_i \operatorname{im}(f_i)$. Then clearly $k \leqslant \sum_i k_i$. Moreover, let $f \colon \underline{k} \hookrightarrow \underline{n_v}$ be the monotone inclusion of the image into $\underline{n_v}$, which can be regarded as a unary operation in $\overline{\mathbb{N}}(\mathscr{D})\binom{k}{n_v}$ by using the pairing $(\mathscr{D}, \overline{\mathbf{Inj}}) \to \overline{\mathbb{N}}(\mathscr{D})$ from Remark 3.4.16.

Then we can write $\nu_i = f \circ \nu_i'$ with $\nu_i' \in \overline{\mathbb{N}}(\mathscr{D})\binom{K_{w_i}}{k}$ for each $1 \leqslant i \leqslant s$. Thus, we can split each green vertex $w_j$ into two green vertices, the lower one $\bar{w}_j$ being a unary labelled by $f$ and the upper one labelled by $\nu_i$. Applying the interchange law, we can exchange $\bar{w}_1, \ldots, \bar{w}_s$, which are all labelled by $f$, and $v$, resulting in a single green vertex $\bar{w}$ below $v$. Finally, since $v$ was not the root, $\bar{w}$ is adjacent to a green vertex and we can compose them. The modified tree has the same shape as the old one; in particular, it is again reduced, and moreover, the vertex $v$ does not violate the bounding property any more. It may happen that we have caused a single new bad red vertex below by moving $f$ downwards, but this one is now strictly nearer to the root; hence we succeeded in making $l$ smaller. ☐

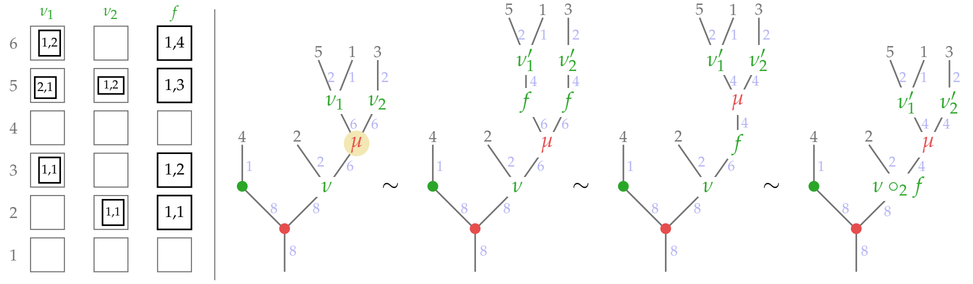

**Figure 4.6.** An inductive step in order to become bounded: here we are given explicit operations $\nu_1 \in \overline{\mathbb{N}}(\mathscr{C}_2)\binom{2,1}{6}$ and $\nu_2 \in \overline{\mathbb{N}}(\mathscr{C}_2)\binom{2}{6}$. This gives rise to $f \in \overline{\mathbb{N}}(\mathscr{C}_2)\binom{4}{6}$ covering all layers attained by $\nu_1$ or $\nu_2$.

### 4.2.3. *Brinkmeier's compactification*

In order to prove Theorem 4.2.2, we are left to show that the induced map $\zeta^{\odot} \colon \mathscr{C}_p \odot \overline{\mathbb{N}}(\mathscr{C}_q) \to \mathscr{V}_{p,q}$ is a homeomorphism onto its image. Although we in principle could show injectivity 'by hand', we have to deal with the fact that the operation spaces are not compact.

To remedy that, we invoke a compactification $\bar{\mathscr{C}}_p$ of $\mathscr{C}_p$, which is due to [Bri00, § 4]: the subtlety which prevents $\mathscr{C}_p(r)$ from being compact is the fact that all boxes must have positive size. A possible compactification would allow 'degeneracies' in some directions.

**Construction 4.2.19.** For a fixed $r \geqslant 0$, let $\square^p$ be the space of pairs $(a, b)$ with $a, b \in [0;1]^{pr}$, with coordinates by $a_i^\omega$ and $b_i^\omega$ for $1 \leqslant i \leqslant r$ and $1 \leqslant \omega \leqslant p$, satisfying the condition $a_i^\omega \leqslant b_i^\omega$. Note that $\square^p$ is compact.

Recall that $\mathscr{C}_p(r)$ is topologised as a subspace of $\square^p$: each tuple $(a, b) \in \square^p$ gives rise to a collection $(c_1, \dots, c_r)$ of maps $c_i \colon [0;1]^p \to [0;1]^p$ which are defined coordinate-wise by $c_i(z)^\omega = a_i^\omega + (b_i^\omega - a_i^\omega) \cdot z^\omega$. We call $(a, b)$ *non-degenerate* if $a_i^\omega < b_i^\omega$ holds strictly for each $i$ and $\omega$, or, equivalently, if each $c_i$ is an embedding. An element $(a, b)$ belongs to $\mathscr{C}_p(r)$ if and only if it is non-degenerate and the interiors $(a_i; b_i) = \prod_\omega (a_i^\omega; b_i^\omega)$ are mutually disjoint.

Each rectilinear embedding $c_i$ determines $2p$ *exterior sectors*, namely

$$\begin{aligned} A_i^\omega &:= [0;1]^{\omega-1} \times [0; a_i^\omega] \times [0;1]^{p-\omega}, \\ B_i^\omega &:= [0;1]^{\omega-1} \times [b_i^\omega; 1] \times [0;1]^{p-\omega}, \end{aligned}$$

and $[0;1]^p \smallsetminus \bigcup_\omega (A_i^\omega \cup B_i^\omega)$ is the interior of $c_i$. Note that for $p = 2$, these exterior sectors are exactly 'above', 'below', 'left', and 'right'.

For a pair of input indices $1 \leqslant i < i' \leqslant r$, we define $D_{i,i'} \subseteq \square^p$ to contain all non-degenerate pairs $(a, b)$ such that the corresponding boxes $c_i$ and $c_{i'}$ have disjoint interior. Then clearly $\mathscr{C}_p(r) = \bigcap_{i<i'} D_{i,i'}$. The disjointness condition can be dismantled further: if $c_i$ and $c_{i'}$ have disjoint interior, then $c_{i'}$ has to lie in one of the exterior halves of $c_i$: if we put

$$\begin{aligned} A_{i,i'}^\omega &:= \{(c_1, \dots, c_r) \in D_{i,i'};\, \mathrm{im}(c_{i'}) \subseteq A_i^\omega\}, \\ B_{i,i'}^\omega &:= \{(c_1, \dots, c_r) \in D_{i,i'};\, \mathrm{im}(c_{i'}) \subseteq B_i^\omega\}, \end{aligned}$$

then we get $D_{i,i'} = \bigcup_\omega (A_{i,i'}^\omega \cup B_{i,i'}^\omega)$. Note that the closure $\bar{A}_{i,i'}^\omega$ inside $\square^p$ is compact and contains boxes $(a, b)$ which are possibly degenerate, but still $c_{i'}$ lies inside one of the exterior sectors of $c_i$; and similarly for $\bar{B}_{i,i'}^\omega$. We let

$$\bar{\mathscr{C}}_p(r) := \bigcap_{i<i'} \Big( \bigcup_\omega (\bar{A}_{i,i'}^\omega \cup \bar{B}_{i,i'}^\omega) \Big)$$

and write elements in $\bar{\mathscr{C}}_p(r)$ as tuples $(c_1, \dots, c_r)$ of *maps* $c_i \colon [0;1]^p \to [0;1]^p$, which are rectilinear, but possibly degenerate, and for each pair of boxes, each one lies in one of the exterior sectors of the other.

Then $\bar{\mathscr{C}}_p(r)$ is compact, and the operadic structure given by composing maps from $\mathscr{C}_p$ extends to $\bar{\mathscr{C}}_p$; hence we obtain an operad $\bar{\mathscr{C}}_p$ with compact operation spaces containing the little $p$-cubes operad $\mathscr{C}_p$ as a suboperad.

The same can be done with the dyeing construction and the tensor product of the two operads:

**Construction 4.2.20.** Consider the dyeing $\overline{\mathbb{N}}(\bar{\mathscr{C}}_q)$: operations in $\overline{\mathbb{N}}(\bar{\mathscr{C}}_q)$ are given by tuples $(\vec{c}_1, \dots, \vec{c}_r)$ where each $\vec{c}_i$ is a rectilinear, but possibly degenerate map $[0;1]^q \times \underline{k}_i \to [0;1]^q \times \underline{n}$, and if we write $\vec{c}_i = (c_{i,1}, \dots, c_{i,k_i})$, then for each two pairs $(i, \jmath), (i', \jmath')$ with $c_{i,\jmath}$ and $c_{i',\jmath'}$ landing on the same layer, the box $c_{i',\jmath'}$ lies in one of the exterior sectors of $c_{i,\jmath}$.

Again we obtain a pairing $\bar{\zeta} \colon (\bar{\mathscr{C}}_p, \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)) \to \overline{\mathbb{N}}(\bar{\mathscr{C}}_d)$ by concatenating the coordinates, i.e. $\bar{\zeta}((c_1, \dots, c_r), (\vec{c}_1, \dots, \vec{c}_s)) = (c_1 \times \vec{c}_1, \dots, c_r \times \vec{c}_s)$. Then $\bar{\zeta}$ extends the old pairing, which we write as $\zeta \colon (\mathscr{C}_p, \overline{\mathbb{N}}(\mathscr{C}_q)) \to \mathscr{V}_{p,q} \subseteq \overline{\mathbb{N}}(\bar{\mathscr{C}}_d)$, and if we corestrict these maps to their respective images, we obtain a square of operad maps

$$\begin{tikzcd}
\mathscr{C}_p \odot \overline{\mathbb{N}}(\mathscr{C}_q) \arrow[r, "\zeta^{\odot}"] \arrow[d] & \mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q) \arrow[d, hook] \\
\bar{\mathscr{C}}_p \odot \overline{\mathbb{N}}(\bar{\mathscr{C}}_q) \arrow[r, "\bar{\zeta}^{\odot}"'] & \bar{\mathscr{C}}_p \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_q),
\end{tikzcd}$$

where the horizontal maps are by construction levelwise surjective and the right vertical map is levelwise an inclusion of subspaces. Moreover, the left vertical map is levelwise injective, with image the preimage of $\mathscr{C}_p \circledast \overline{\mathbb{N}}(\mathscr{C}_q)$ along $\bar{\zeta}^{\odot}$. It is even levelwise an inclusion of subspaces; in particular, if we can show that $\bar{\zeta}^{\odot}$ is an isomorphism onto its image, the same is true for $\zeta^{\odot}$.

*Proof.* The suboperads $\mathscr{C}_p \subseteq \bar{\mathscr{C}}_p$ and $\overline{\mathbb{N}}(\mathscr{C}_q) \subseteq \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)$ are levelwise open. Therefore, $\coprod_Y(\mathscr{C}_p * \overline{\mathbb{N}}(\mathscr{C}_q))(Y) \subseteq \coprod_Y(\bar{\mathscr{C}}_p * \overline{\mathbb{N}}(\bar{\mathscr{C}}_q))(Y)$ is the inclusion of an open subspace, and one easily checks that it is also saturated with respect to the relation to be quotiented out: the requirement that all internal vertices are labelled by non-degenerate configurations of boxes is preserved by input permutation, adjacent composition, inserting units, and the interchange law. This shows that the canonical map from the quotient of the subspace to the subspace of the quotient is a homeomorphism. □

The rest of this section aims to show that the map $\bar{\zeta}^{\odot}$ from the above square is levelwise a homeomorphism. This is achieved in two steps: firstly, we show that the domain $\bar{\mathscr{C}}_p \odot \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)$ is levelwise compact. Since the right side is clearly Hausdorff, it then only remains to show that $\bar{\zeta}^{\odot}$ is levelwise bijective. Let us start by showing the compactness of $\bar{\mathscr{C}}_p \odot \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)$:

**Proposition 4.2.21.** *For each input profile $K$ and each output colour $n$, the operation space $(\bar{\mathscr{C}}_p \odot \overline{\mathbb{N}}(\bar{\mathscr{C}}_q))\binom{K}{n}$ is compact.*

*Proof.* By Lemma 4.2.18, the composition

$$\coprod_{\substack{Y \text{ of type } \binom{K}{n} \\ \text{and bounded}}} (\bar{\mathscr{C}}_p * \overline{\mathbb{N}}(\bar{\mathscr{C}}_q))(Y) \hookrightarrow \coprod_{Y \text{ of type } \binom{K}{n}} (\bar{\mathscr{C}}_p * \overline{\mathbb{N}}(\bar{\mathscr{C}}_q))(Y)$$
$$\to (\bar{\mathscr{C}}_p \odot \overline{\mathbb{N}}(\bar{\mathscr{C}}_q))\binom{K}{n}$$

is surjective, and by Lemma 4.2.17, there are only finitely many bounded two-parted $\overline{\mathbb{N}}$-trees $Y$ of type $\binom{K}{n}$. Finally, the space

$$(\bar{\mathscr{C}}_p * \overline{\mathbb{N}}(\bar{\mathscr{C}}_q))(Y) = \prod_{v \in \textcolor{red}{V_\bullet}} \bar{\mathscr{C}}_p(r_v) \times \prod_{w \in \textcolor{green}{V_\bullet}} \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)\binom{K_w}{n_w}$$

is compact: we have already seen that $\bar{\mathscr{C}}_p(r_v)$ is compact, and for the second factor, recall from Construction 3.4.4 that for $k := |K_w|$, we have

$$\overline{\mathbb{N}}(\bar{\mathscr{C}}_q)\binom{K_w}{n_w} = \mathrm{cat}(\bar{\mathscr{C}}_q)\binom{k}{n_w} = \coprod_{f\colon \underline{k} \to \underline{n}_w} \bar{\mathscr{C}}_q(f^{-1}(1)) \times \cdots \times \bar{\mathscr{C}}_q(f^{-1}(n_w)).$$

For the right side, we see that the indexing set is finite and each summand is compact, as a product of compact spaces. ☐

### 4.2.4. *An injectivity criterion for pairings*

In order to prove Theorem 4.2.2, we still have to show that the induced map $\zeta^{\odot}\colon \bar{\mathscr{C}}_p \odot \overline{\mathbb{N}}(\bar{\mathscr{C}}_q) \to \overline{\mathbb{N}}(\bar{\mathscr{C}}_{p+q})$ is levelwise injective. To this aim, we establish an injectivity criterion for pairings.

This follows essentially [Bri00, Def. 6.1]; however, we phrase it in terms of a general pairing $\zeta\colon (\mathscr{C}, \mathscr{O}) \to \mathscr{P}$ of reduced operads, where $\mathscr{C}$ is monochromatic, and we close a small gap in Brinkmeier's argument, see below. Since all our operads are reduced, $\zeta^{\odot}\binom{}{n}$ is trivially bijective and we can exclude nullaries at this stage of the proof, as there is nothing to show for them.

The idea the following: given an element $\xi \in \mathscr{C} \circledast \mathscr{O}$, we want to inductively build a labelled tree $(Y, \mathfrak{a}) \in \coprod_Y (\mathscr{C} * \mathscr{O})(Y)$ such that each subtree starts with a 'minimal' crotch in the sense that it factors over all other potential crotches of the same domain, and that all subtrees represent elements in $\mathscr{C} \circledast \mathscr{O}$ which are maximal among all potential arguments which, after composition, yield the original element $\xi$. We call a pairing *trackable*, if such a procedure is possible, and we will show that, in this case, the element $\xi^\sharp \in \mathscr{C} \odot \mathscr{O}$ which is represented by such a tree is uniquely determined by these properties. Finally, we show that, under mild further assumptions, the assignment $\xi \mapsto \xi^\sharp$ is a left inverse for $\zeta^{\odot}$.

**Definition 4.2.22.** The *core* $\mathrm{core}(\mathscr{C})$ of an operad $\mathscr{C}$ is the subcategory that contains all invertible unaries. E.g. $\mathrm{core}(\mathscr{C}_p) = \{\mathbb{1}\}$, while $\mathrm{core}(\overline{\mathbb{N}}(\mathscr{C}_q)) = \mathfrak{S}$.

**Definition 4.2.23.** A pairing is called *trackable* if the following holds for both domains, red and green, here formulated only for red:

T1. **Minimal crotches**
For each $\xi \in (\mathscr{C} \circledast \mathbb{O})\binom{K}{n}$, there is a $\mu \in \mathscr{C}(r)$ such that:

- A. there are $\xi_1, \dots, \xi_r \in \mathscr{C} \circledast \mathbb{O}$, each one of arity at least 1, such that $\xi = \zeta(\mu, \mathbb{1}_n) \circ (\xi_1, \dots, \xi_r)$ holds up to input permutation;
- B. for each other $\bar{\mu} \in \mathscr{C}(s)$ satisfying A, there are $\mu_1, \dots, \mu_s \in \mathscr{C}$ such that $\mu = \bar{\mu} \circ (\mu_1, \dots, \mu_s)$ holds up to input permutation.

We call such a $\mu$ a *minimal red crotch*. Moreover, we require the following:

- T1$^{\mathrm{a}}$. for each other minimal red crotch $\mu'$, there are $\alpha_1, \dots, \alpha_r \in \operatorname{core}(\mathscr{C})$ such that $\mu' = \mu \circ (\alpha_1, \dots, \alpha_r)$ holds up to input permutation.
- T1$^{\mathrm{b}}$. if $\mu_1, \dots, \mu_r$ are minimal red crotches for $\xi_1, \dots, \xi_r$ and $\mu \in \mathscr{C}$, then $\mu \circ (\mu_1, \dots, \mu_r)$ is a minimal red crotch for $\zeta(\mu, \mathbb{1}_n) \circ (\xi_1, \dots, \xi_r)$.
- T1$^{\mathrm{c}}$. let $s_i$ be the arity of $\xi_i$, define $S := (s_1, \dots, s_r)$, and for $\tau \in \mathfrak{S}_r$, let $\tau_S \in \mathfrak{S}_{\#K}$ be the corresponding block permutation. Then we require that $\tau_S^* \xi = \xi$ in $\mathscr{C} \circledast \mathbb{O}$ implies $\tau^* \mu = \mu$ in $\mathscr{C}$.

T2. **Maximal arguments**
If $\mu$ is a minimal red crotch for $\xi$, then there are $\xi_1, \dots, \xi_r \in \mathscr{C} \circledast \mathbb{O}$ with:

- A. $\xi = \zeta(\mu, \mathbb{1}_n) \circ (\xi_1, \dots, \xi_r)$ holds up to input permutation;
- B. for each other collection $\bar{\bar{\xi}}_1, \dots, \bar{\bar{\xi}}_r$ satisfying A, there are $\xi_{i,j}$ and $\tau \in \mathfrak{S}_r$ with $\tau^* \mu = \mu$ and $\bar{\bar{\xi}}_{\tau(i)} = \xi_i \circ (\xi_{i,1}, \dots, \xi_{i,s_i})$.

We call such a collection $\xi_1, \dots, \xi_r$ *maximal arguments for $\mu$ to get $\xi$*. In addition, we require the following:

- T2$^{\mathrm{a}}$. for each other maximal collection $\xi'_1, \dots, \xi'_r$, there are $\alpha_i \in \operatorname{core}(\mathscr{C})$ and $\tau \in \mathfrak{S}_r$ with $\tau^* \mu = \mu \circ (\alpha_1, \dots, \alpha_r)$ and $\xi'_{\tau(i)} = \zeta(\alpha_i, \mathbb{1}_n) \circ \xi_i$.
- T2$^{\mathrm{b}}$. if $\xi_1, \dots, \xi_r$ are maximal arguments for $\mu$ to get $\xi$ and if, moreover, $\gamma_i \in \operatorname{core}(\mathscr{C} \circledast \mathbb{O})$, then $\gamma_1 \circ \xi_1, \dots, \gamma_r \circ \xi_r$ are maximal arguments for $\mu$ to get $\zeta(\mu, \mathbb{1}_n) \circ (\gamma_1 \circ \xi_1, \dots, \gamma_r \circ \xi_r)$.

T3. **Uniqueness for unaries**
If $\xi$ has arity 1, then there is a minimal red crotch $\mu$ which has a maximal argument $\zeta(\mathbb{1}, \nu)$; this property determines both $\mu$ and $\nu$ uniquely.

**Remark 4.2.24.** Let $\zeta\colon (\mathscr{C},\mathscr{O}) \to \mathscr{P}$ be a trackable pairing and $\xi \in \mathscr{C} \circledast \mathscr{O}$.

1. If $\alpha_i \in \mathrm{core}(\mathscr{C})$ and $\xi_1,\dots,\xi_r$ are maximal for $\mu \circ (\alpha_1,\dots,\alpha_r)$ to get $\xi$, then $\zeta(\alpha_1,\mathbb{1}_n) \circ \xi_1,\dots,\zeta(\alpha_r,\mathbb{1}_n) \circ \xi_r$ are maximal for $\mu$ to get $\xi$.
2. If $\nu$ is a minimal green crotch for $\xi$ and if $\alpha \in \mathrm{core}(\mathscr{C})$, then $\nu$ is also a minimal green crotch for $\zeta(\alpha,\mathbb{1}_n) \circ \xi$.

Given an operation $\xi \in \mathscr{C} \circledast \mathscr{O}$, we would like to construct a two-parted tree representing it. This is formalised as follows:

**Definition 4.2.25.** Recall that for each profile $\binom{K}{n}$, we have a chain of maps

$$\coprod\nolimits_{Y \text{ of type } \binom{K}{n}} (\mathscr{C} * \mathscr{O})(Y) \overset{\mathrm{pr}}{\twoheadrightarrow} (\mathscr{C} \odot \mathscr{O})\tbinom{K}{n} \xrightarrow{\zeta^{\odot}} (\mathscr{C} \circledast \mathscr{O})\tbinom{K}{n} \subseteq \mathscr{P}\tbinom{K}{n}.$$

If $\zeta$ is trackable and $\xi \in \mathscr{C} \circledast \mathscr{O}$, then we call a labelled two-parted tree $(Y,\mathfrak{a})$ with $(\zeta^{\odot} \circ \mathrm{pr})(Y,\mathfrak{a}) = \xi$ an *optimal representative* if the following holds:

- M1. $Y$ is a reduced two-parted $N$-tree with a red crotch;
- M2. each internal vertex is a minimal crotch of the element in $\mathscr{C} \circledast \mathscr{O}$ which is represented by the subtree starting at that vertex;
- M3. at each vertex, the incoming subtrees represent maximal arguments.

**Lemma 4.2.26.** *If $\zeta\colon (\mathscr{C},\mathscr{O}) \to \mathscr{P}$ is a trackable pairing, then each $\xi \in \mathscr{C} \circledast \mathscr{O}$ has such a minimal representative $(Y,\mathfrak{a})$ and the element $\xi^{\sharp} := \mathrm{pr}(Y,\mathfrak{a}) \in \mathscr{C} \odot \mathscr{O}$ is uniquely determined by $\xi$.*

*Proof.* We build $(Y,\mathfrak{a})$ inductively: we choose a minimal red crotch, and then consider its maximal arguments. For each of them, we find minimal green crotches. If a subtree has arity 1, then we choose the two unary vertices by property T3. In order to show that this process terminates, we have to see that if an element $\xi'$ represented by a subtree has arity at least 2 and if a minimal crotch of $\xi'$ is of arity 1, then the next crotch is of higher arity: by Lemma 4.2.15, $\xi'$ has either a green or a red crotch of arity at least 2. Now note that a *minimal* crotch is always of *maximal* arity among all crotches of the same domain. It may happen that we have to choose the 'wrong' domain, say 'green', i.e. there are only red crotches of arity 1. In this case, we have $\xi' = \zeta(\mathbb{1}, \nu) \circ \xi''$ for a unary $\nu \in \mathcal{O}$. If $\xi''$ had a green crotch of arity at least 2, then this would also apply to $\xi'$, which is not possible. Therefore, $\xi''$ has a red crotch of arity at least 2, and hence, the minimal red crotch of $\xi'$ is of higher arity. We end up with a tree having the properties M2 and M3. Its crotch is red and the tree is reduced, where R1 and R2 are immediate, while R3 follows from the previous paragraph, hence M1 holds as well.

We are left to show that $\mathrm{pr}(\Upsilon, \mathfrak{a}) \in \mathcal{C} \odot \mathcal{O}$ is uniquely determined by $\xi$, and we do so by induction on the arity $r \geqslant 1$ of $\xi$: the case $r = 1$ is clear by T3. For $r \geqslant 2$, let $\mu$ and $\mu'$ be two choices of red minimal crotches of $\xi$ and let $\xi_1, \dots, \xi_r$ and $\xi'_1, \dots, \xi'_r$ be the chosen maximal arguments for $\mu$ and $\mu'$, respectively, to get $\xi$. Moreover, we pick minimal green crotches $\nu_i$ for $\xi_i$ and $\xi'_i$ for $\nu'_i$, and let $\xi_{i,j}$ and $\xi'_{i,j}$ be their maximal arguments to get $\xi_i$ and $\xi'_i$, respectively. Then the arity of each $\xi_{i,j}$ has to be strictly smaller than $r$, and, by the induction hypothesis, all optimal representatives for $\xi_{i,j}$ and $\xi'_{i,j}$ encode the same elements $\xi^\sharp_{i,j}$ and $\xi'^\sharp_{i,j}$, respectively, inside $\mathcal{C} \odot \mathcal{O}$. Since within the two optimal trees, the subtrees are optimal as well, we only have to show that inside $\mathcal{C} \odot \mathcal{O}$, we have, writing $\mu \odot \nu := \chi(\mu, \nu)$ for the pairing $\chi \colon (\mathcal{C}, \mathcal{O}) \to \mathcal{C} \odot \mathcal{O}$,

$$(\mu' \odot \mathbb{1}) \circ (\mathbb{1} \odot \nu'_\bullet) \circ \xi'^\sharp_{\bullet,\bullet} = (\mu \odot \mathbb{1}) \circ (\mathbb{1} \odot \nu_\bullet) \circ \xi^\sharp_{\bullet,\bullet},$$

where we shorten $\nu_\bullet := (\nu_1, \dots, \nu_r)$ and similarly $\xi^\sharp_{\bullet,\bullet}$.

To this aim, we note that by property T1$^{\mathrm{a}}$, there are $\alpha_i \in \mathrm{core}(\mathcal{C})$ and $\tau \in \mathfrak{S}_r$ such that $\tau^*\mu = \mu' \circ (\alpha_1, \dots, \alpha_r)$. Since $\tau$ is uniquely determined up to those permutations whose respective block permutations fix $\xi$, and since those, in turn, fix $\mu$ by T1$^{\mathrm{c}}$, we can without loss of generality assume that $\tau = \mathrm{id}$. Then $\zeta(\alpha_1, \mathbb{1}) \circ \xi_1, \dots, \zeta(\alpha_r, \mathbb{1}) \circ \xi_r$ are maximal arguments for $\mu'$ to get $\xi$. By axiom T2$^{\mathrm{a}}$, this implies that there are $\alpha'_i \in \mathrm{core}(\mathcal{C})$ and $\tau \in \mathfrak{S}_r$ such that $\mu' \circ (\alpha'_1, \dots, \alpha'_r) = \mu'$ and $\xi'_{\tau(i)} = \zeta(\bar{\alpha}_i, \mathbb{1}) \circ \xi_i$ for $\bar{\alpha}_i := \alpha'_i \circ \alpha_i$, and again, we can assume $\tau$ to be trivial. Moreover, we see that $\mu' \circ (\bar{\alpha}_1, \dots, \bar{\alpha}_r) = \mu$.

Since $\xi'_i$ and $\xi_i$ differ only by postcomposition with $\zeta(\bar{\alpha}_i, \mathbb{1}_n)$, we see by Remark 4.2.24 that the minimal green crotches of $\xi_i$ and $\xi'_i$ coincide, i.e. $\nu_i = \nu'_i$ and $s_i = s'_i$. Hence we obtain by the interchange law

$$\begin{aligned}
\xi'_i &= \zeta(\bar{\alpha}_i, \mathbb{1}) \circ \zeta(\mathbb{1}, \nu_i) \circ (\xi_{i,1}, \dots, \xi_{i,s_i}) \\
&= \zeta(\mathbb{1}, \nu_i) \circ (\zeta(\bar{\alpha}_i, \mathbb{1}) \circ \xi_{i,1}, \dots, \zeta(\bar{\alpha}_i, \mathbb{1}) \circ \xi_{i,s'_i}).
\end{aligned}$$

As $\zeta(\bar{\alpha}_i,\mathbb{1})$ is invertible, $\zeta(\bar{\alpha}_i,\mathbb{1})\circ\xi_{i,1},\dots,\zeta(\bar{\alpha}_i,\mathbb{1})\circ\xi_{i,s_i}$ are maximal arguments for $\nu_i$ to obtain $\xi'_i$. By T2$^{\mathrm{a}}$, there are $\beta_{i,j}\in\operatorname{core}(\mathbb{O})$ with $\nu_i\circ(\beta_{i,1},\dots,\beta_{i,s_i})=\nu_i$ and $\zeta(\bar{\alpha}_i,\mathbb{1})\circ\xi_{i,j}=\zeta(\mathbb{1},\beta_{i,j})\circ\xi'_{i,j}$. A tedious, but straightforward check shows that there is an optimal representative for $\zeta(\mathbb{1},\beta_{i,j})\circ\xi'_{i,j}$ which is equivalent to an optimal representative for $\xi'_{i,j}$, postcomposed with a single green vertex labelled by $\beta_{i,j}$, and thus, $(\mathbb{1}\odot\beta_{i,j})\circ\xi'^{\sharp}_{i,j}=(\zeta(\mathbb{1},\beta_{i,j})\circ\xi'_{i,j})^{\sharp}$.

Moreover, we have $(\zeta(\bar{\alpha}_i,\mathbb{1})\circ\xi_{i,j})'^{\sharp}=(\bar{\alpha}_i\odot\mathbb{1})\circ\xi'^{\sharp}_{i,j}$, which, altogether, implies the equality $(\mathbb{1}\odot\beta_{i,j})\circ\xi'^{\sharp}_{i,j}=(\bar{\alpha}_i\odot\mathbb{1})\circ\xi^{\sharp}_{i,j}$. As desired, we get

$$\begin{aligned}
(\mu'\odot\mathbb{1})\circ(\mathbb{1}\odot\nu'_{\bullet})\circ\xi'^{\sharp}_{\bullet,\bullet} &= (\mu'\odot\mathbb{1})\circ(\mathbb{1}\odot\nu_{\bullet})\circ\xi'^{\sharp}_{\bullet,\bullet}\\
&= (\mu'\odot\mathbb{1})\circ(\mathbb{1}\odot(\nu_{\bullet}\circ\beta_{\bullet,\bullet}))\circ\xi'^{\sharp}_{\bullet,\bullet}\\
&= (\mu'\odot\mathbb{1})\circ(\mathbb{1}\odot\nu_{\bullet})\circ\big((\mathbb{1}\odot\beta_{\bullet,\bullet})\circ\xi'^{\sharp}_{\bullet,\bullet}\big)\\
&= (\mu'\odot\mathbb{1})\circ(\mathbb{1}\odot\nu_{\bullet})\circ\big((\bar{\alpha}_{\bullet}\odot\mathbb{1})\circ\xi^{\sharp}_{\bullet,\bullet}\big)\\
&= \big((\mu'\circ\bar{\alpha}_{\bullet})\odot\mathbb{1}\big)\circ(\mathbb{1}\odot\nu_{\bullet})\circ\xi^{\sharp}_{\bullet,\bullet}\\
&= (\mu\odot\mathbb{1})\circ(\mathbb{1}\odot\nu_{\bullet})\circ\xi^{\sharp}_{\bullet,\bullet}. \qquad \square
\end{aligned}$$

**Remark 4.2.27.** I believe that the analogous statement in [Bri00, Prop. 6.10] for the special case of the (monochromatic) compactified little cubes operad misses the maximality assumption in order to guarantee the uniqueness of the optimal representative: if $\mu$ is degenerate, then the operations $\mu_1,\dots,\mu_r$ are *not* uniquely determined by $\mu\circ(\mu_1,\dots,\mu_r)$ and $\mu$; however, the weaker, and—in the end—sufficient, property T2 still holds.

Next, we want to show that, under mild extra assumptions, the assignment $(-)^{\sharp}$ is indeed a left inverse for $\zeta^{\odot}$:

**Proposition 4.2.28.** *Let $\zeta\colon(\mathscr{C},\mathbb{O})\to\mathscr{P}$ be a trackable pairing, and assume that:*

- P1. *for all colours $k,n\in N$, the map $\zeta\colon\mathscr{C}(1)\times\mathbb{O}\binom{k}{n}\to\mathscr{P}\binom{k}{n}$ is injective;*
- P2. *if $\vartheta\in\mathscr{C}\odot\mathbb{O}$ such that $\zeta^{\odot}(\vartheta)$ has a minimal red crotch of arity $\geqslant 2$, then $\vartheta$ itself has a reduced representation with a red crotch of arity $\geqslant 2$.*

*Then for each $\vartheta\in\mathscr{C}\odot\mathbb{O}$, we have $(\zeta^{\odot}(\vartheta))^{\sharp}=\vartheta$; in particular, the induced map $\zeta^{\odot}$ is levelwise injective.*

*Proof.* This proof essentially follows the one of [Bri00, Thm. 6.12]; however, I think that in the general setting, we need to additionally assume P2.

Let $\xi := \zeta^{\odot}(\vartheta)$, and we aim to show that $\xi^{\sharp} = \vartheta$. To do so, we proceed by induction over the arity of $\vartheta$: if $\vartheta$ is a unary, then $\vartheta$ is of the form $\mu \odot \nu$ with $\mu \in \mathscr{C}(1)$ and $\nu \in \mathfrak{O}\binom{k}{n}$, and similarly, $\xi^{\sharp} = \mu' \odot \nu'$ holds. Since $\zeta^{\odot}(\xi^{\sharp}) = \xi$, we get $\zeta(\mu, \nu) = \zeta(\mu', \nu')$, whence, by P1, we get both $\mu = \mu'$ and $\nu = \nu'$; in particular, $\xi^{\sharp} = \vartheta$.

For the induction step, we have so switch between red and green crotches. Clearly, we can define optimal representatives with a *green* crotch in the same way, and we define $\xi^{\flat} := \mathrm{pr}(Y, \mathfrak{a})$ for $(Y, \mathfrak{a})$ being an optimal representative with a green crotch, and show that $\xi^{\flat}$ is uniquely determined by $\xi$. The start of the induction works equally well, showing that $\vartheta = \xi^{\flat}$ if there is an optimal representative with a green crotch having only one internal vertex.

Now assume that $\vartheta$ has arity at least 2. By Lemma 4.2.15, there is a reduced representative for $\vartheta$, which has either a green or a red crotch of arity at least 2. If we find a red crotch of arity at least 2 and with label $\mu \in \mathscr{C}$, then we write $\vartheta = (\mu \odot \mathbb{1}) \circ (\vartheta_1, \dots, \vartheta_r)$. If we put $\xi_i := \zeta^{\odot}(\vartheta_i)$ and let $\mu_i$ be a minimal red crotch of $\xi_i$, then, by T1$^{\mathrm{b}}$, $\mu \circ (\mu_1, \dots, \mu_r)$ is a minimal crotch of $\xi$. Moreover, let $\xi_{i,1}, \dots, \xi_{i,s_i}$ be maximal arguments for $\mu_i$ to get $\xi_i$, and let $\vartheta_{i,j} := \xi^{\flat}_{i,j}$. Since each $\vartheta_i$ has strictly less inputs than $\vartheta$, we see that $\mu_i \circ (\vartheta_{i,1}, \dots, \vartheta_{i,s_i}) = \xi^{\sharp}_i = \vartheta_i$, and we therefore get, using the same notation as before,

$$\begin{aligned} \xi^{\sharp} &= ((\mu \circ \mu_{\bullet}) \odot \mathbb{1}) \circ \vartheta_{\bullet,\bullet} \\ &= (\mu \circ \mathbb{1}) \circ (\mu_{\bullet} \odot \mathbb{1}) \circ \vartheta_{\bullet,\bullet} \\ &= (\mu \circ \mathbb{1}) \circ \vartheta_{\bullet} \\ &= \vartheta. \end{aligned}$$

However, if $\vartheta$ only admits a reduced representative which has a green crotch of arity 2, then we have to argue differently: as in the previous paragraph, we can inductively show that $\xi^{\flat} = \vartheta$. Moreover, by P2, the minimal *red* crotch $\mu$ of $\xi$ has arity 1 as well. Hence, if $(Y, \mathfrak{a})$ is an optimal red representative for $\xi$, we can employ the interchange law and exchange the unary crotch and its adjacent green internal vertex (and reduce the tree again), which has to be of arity at least 2. Hence we reached a representative for $\xi^{\sharp}$ which has a green crotch of arity 2, and we see, as above, that $\xi^{\sharp} = (\zeta^{\odot}(\xi^{\sharp}))^{\flat} = \xi^{\flat} = \vartheta$. □

### 4.2.5. *Injectivity of the compactified pairing*

In order to finish the proof of Theorem 4.2.2, we are left to prove that the compactified pairing $\bar{\zeta}\colon (\bar{\mathscr{C}}_p, \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)) \to \overline{\mathbb{N}}(\bar{\mathscr{C}}_{p+q})$ meets the requirements of the injectivity criterion from Proposition 4.2.28. It is clear that $\bar{\zeta}$ satisfies P1, as the two inclusions affect disjoint coordinates of $[0;1]^{p+q} \times \underline{n}$, but we still have to show trackability and P2:

**Proposition 4.2.29.** *The pairing* $\bar{\zeta}\colon (\bar{\mathscr{C}}_p, \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)) \to \overline{\mathbb{N}}(\bar{\mathscr{C}}_{p+q})$ *is trackable and satisfies property* P2.

In order to prove Proposition 4.2.29, we adapt the framing argument from [Bri00, § 4] to the coloured situation.

**Preparation 4.2.30.** Let $F_p$ be the set of tuples $\mu = (c_1, \dots, c_r)$ of (possibly degenerate, possibly overlapping) rectilinear maps $[0;1]^p \to [0;1]^p$. Then we have a preorder on $F_p$ by declaring $(c_1, \dots, c_r) \leqslant (c'_1, \dots, c'_{r'})$ whenever there is a surjection $\pi\colon \underline{r} \to \underline{r}'$ with $c_i \subseteq c'_{\pi(i)}$ for each $i$. Likewise, we let $F_{q,n}$ be the set of tuples $\nu = (c_{1,1}, \dots, c_{r,k_r})$ of rectilinear maps $c_{i,\jmath}\colon [0;1]^q \to [0;1]^q \times \underline{n}$, and we have a preorder on $F_{q,n}$ by letting $(c_{i,\jmath})_{(i,\jmath)\in K} \leqslant (c'_{i,\jmath})_{(i,\jmath)\in K'}$ if there is a column-preserving surjection $\pi\colon \mathbb{Y}_K \to \mathbb{Y}_{K'}$ of tableaux with $c_{i,\jmath} \subseteq c'_{\pi(i,\jmath)}$.

Although we do not need to give $F_p$ and $(F_{q,n})_{n\geqslant 1}$ the formal structure of an operad, we can compose elements by composing rectilinear maps, and permute their inputs by permuting the boxes. Clearly, if $\mu = \mu' \circ (\mu_1, \dots, \mu_r)$ holds up to input permutation, then $\mu \leqslant \mu'$, and similarly for $\nu$ and $\nu'$. The converse also holds, see [Bri00, Cor. 4.8].

We have inclusions $F'_p := \coprod_r \bar{\mathscr{C}}_p(r) \subseteq F_p$ and $F'_{q,n} := \coprod_K \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)\binom{K}{n} \subseteq F_{q,n}$, and for $\mu, \mu' \in F'_p$, we have $\mu \leqslant \mu' \leqslant \mu$ if and only if $\mu$ and $\mu'$ differ by an input permutation, compare [Bri00, Lem. 4.5]. Similarly, for $\nu, \nu' \in F'_{q,n}$, we have $\nu \leqslant \nu' \leqslant \nu$ if and only if $\nu$ and $\nu'$ differ by an input permutation and a layer permutation, i.e. precomposition with elements from the core.

**Construction 4.2.31.** For each $\mu \in F_p$, there is a smallest $\bar{\mu} \in F'_p$ with $\mu \leqslant \bar{\mu}$, and similarly, for each $\nu \in F_{q,n}$, there is a smallest $\bar{\nu} \in F'_{q,n}$ with $\nu \leqslant \bar{\nu}$. We give details for the construction of $\bar{\mu}$; for $\bar{\nu}$ see Figure 4.7:

- We define an equivalence relation on $\underline{r}$ that relates $i$ and $i'$ if $\mu$ does not lie in $\bigcup_\omega \bar{A}^\omega_{i,i'} \cup \bar{B}^\omega_{i,i'}$ from Construction 4.2.19.

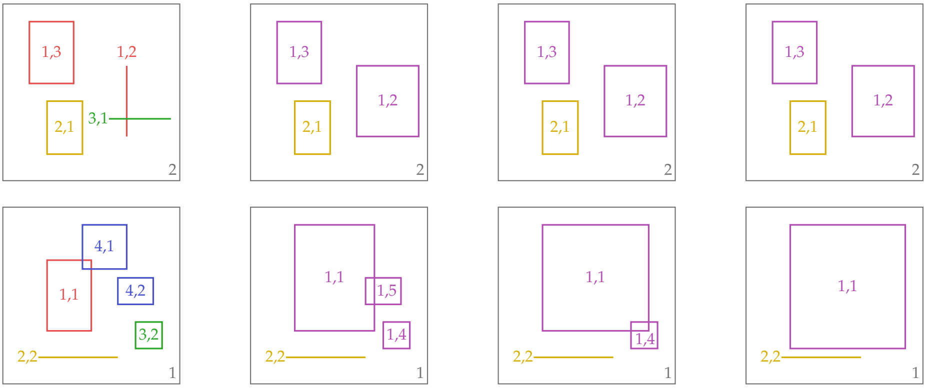

**Figure 4.7.** The inductive procedure which starts at an arbitrary configuration $(c_{1,1},\dots,c_{4,2})$ of squares and ends up with a configuration that lies in $\overline{\mathbb{N}}(\mathscr{C}_2)\binom{3,2}{2}$, where the colours indicate the equivalence relation.

- For each equivalence class, we find the smallest rectangle that contains all $c_i$ from this class: it is given by the intersection of all rectangles with this property, and we order these two new rectangles by the minimum of their respective equivalence classes.

- If the new collection of boxes does not lie in $F'_p$, then we repeat the previous two steps.

In each step, the number of boxes decreases, so this procedure terminates after finitely many steps and gives the desired collection $\bar{\mu}$. By the previous paragraph, $\bar{\mu}$ is determined, up to input permutation, by the property of being a smallest upper bound.

While we use the previous construction to produce minimal crotches, the following construction is necessary to construct maximal arguments:

**Construction 4.2.32.** Let $c = \prod_\omega [a^\omega; b^\omega]$ be a box and $c' = \prod_\omega [a'^\omega; b'^\omega] \subseteq c$. Then we define the *inverse rescaling* $c^{-1} \cdot c' := \prod_\omega [\bar{a}^\omega; \bar{b}^\omega]$ as follows: for all $1 \leqslant \omega \leqslant p$ with $a^\omega \neq b^\omega$, we let

$$\bar{a}^\omega := \frac{a'^\omega - a^\omega}{b^\omega - a^\omega} \quad \text{and} \quad \bar{b}^\omega := \frac{b'^\omega - a^\omega}{b^\omega - a^\omega},$$

and for each other coordinate $\omega$, we set $\bar{a}^\omega_i := 0$ and $\bar{b}^\omega_i := 1$.

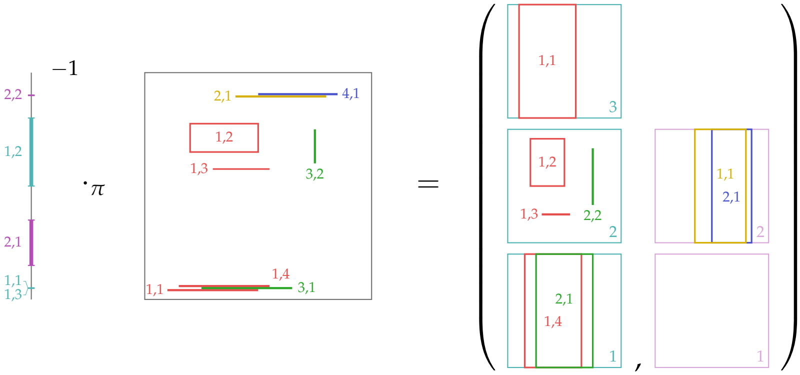

**Figure 4.8.** Calculating $\nu^{-1} \cdot_\pi \xi$ for $\nu \in \overline{\mathbb{N}}(\bar{\mathscr{C}}_1)\binom{3,2}{1}$ and $\xi \in \overline{\mathbb{N}}(\bar{\mathscr{C}}_2)\binom{4,1,2,1}{1}$, resulting in $\xi_1 \in F_{2,3}$ and $\xi_2 \in F_{2,2}$. Both $\xi_1$ and $\xi_2$ have overlaps, but still $\xi = \zeta(\mathbb{1}, \nu) \circ (\xi_1, \xi_2)$ holds up to input permutation. The map $\pi \colon \mathbb{Y}_{4,1,2,1} \to \mathbb{Y}_{3,2}$ lets $\pi(1,1) = (1,3)$ and $\pi(1,4) = \pi(3,1) = (1,1)$.

For $\xi = (c_{1,1}, \dots, c_{s,k_s}) \in \overline{\mathbb{N}}(\bar{\mathscr{C}}_{p+q})\binom{K}{n}$, if $\mu = (c_1, \dots, c_r) \in \bar{\mathscr{C}}_p(r)$ is a red crotch for $\xi$, then we have $\xi \leqslant \zeta(\mu, \mathbb{1}_n)$, so there is a surjection $\pi \colon \underline{\#K} \to \underline{r}$ with $c_{i,\jmath} \subseteq c_{\pi(i)} \times [0;1]^q \times \underline{n}$ for all $(i,\jmath)$. For each explicit choice of $\pi$, we define $\mu^{-1} \cdot_\pi \xi := (\xi_1, \dots, \xi_r)$, where $\xi_l := ((c_l \times [0;1]^q)^{-1} \cdot c_{i,\jmath})_{\pi(i)=l}$. Similarly, we define $\nu^{-1} \cdot_\pi \xi$ for $\nu \in \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)\binom{K'}{n}$ and $\pi \colon \mathbb{Y}_K \to \mathbb{Y}_{K'}$, see Figure 4.8.

Then we clearly have $\xi = \zeta(\mu, \mathbb{1}_n) \circ (\mu^{-1} \cdot_\pi \xi)$ and $\xi = \zeta(\mathbb{1}, \nu) \circ (\nu^{-1} \cdot_\pi \xi)$ up to input permutation.

The collection $\mu^{-1} \circ_\pi \xi$ should play the rôle of a collection of maximal arguments; however, in general, $\xi_l$ does not lie inside $\overline{\mathbb{N}}(\bar{\mathscr{C}}_{p+q})$: as we insisted that if the $\omega^{\text{th}}$ coordinate of $c_l$ is degenerate, the respective coordinates of $c_l^{-1} \cdot c_{i,\jmath}$ are scaled to full size, there might be overlaps. Furthermore, the operations $\xi_1, \dots, \xi_r$ depend on $\pi$, see Figure 4.8. The following criterion impedes this:

**Construction 4.2.33.** Let $\mu \in \bar{\mathscr{C}}_p(r)$ and let $1 \leqslant i \leqslant r$ such that the $i^{\text{th}}$ box of $\mu$ has a degenerate coordinate, and let $\xi_i \in (\bar{\mathscr{C}}_p \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_q))\binom{k_1, \dots, k_s}{n}$.

- Let $\mu' \in \bar{\mathscr{C}}_p(r + s - 1)$ be the operation where we have $s$ boxes at the *same* place as the original $i^{\text{th}}$ box. They are numbered $i, \dots, i + s - 1$, and all successors get shifted by $s - 1$.

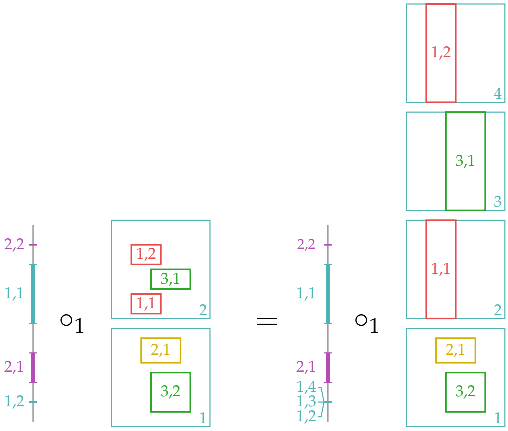

**Figure 4.9.** Two operations $\nu \in \overline{\mathbb{N}}(\bar{\mathscr{C}}_1)\binom{2,2}{1}$ and $\zeta_1 \in (\bar{\mathscr{C}}_1 \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_1))\binom{2,1,2}{2}$, and their refinements $\nu' \in \overline{\mathbb{N}}(\bar{\mathscr{C}}_1)\binom{4,2}{1}$ and $\zeta_1' \in (\bar{\mathscr{C}}_1 \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_1))\binom{2,1,2}{4}$.

- For $1 \leqslant h \leqslant s$, let $\xi_{i,h} \in \bar{\mathscr{C}}_p \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_q))\binom{k_h}{n}$ be the operation that arises from $\xi_i$ by keeping only the $h^{\text{th}}$ cluster, and in all directions $1 \leqslant \omega \leqslant p$ in which the $i^{\text{th}}$ box of $\mu$ is flat, each of the $k_h$ boxes are scaled to full size (this does not produce any overlap, since all boxes in $\xi_{i,h}$ share their first $p$ coordinates, and if two boxes in $\xi_{i,h}$ are at the same place, then they are degenerate in a direction $p+1 \leqslant \omega \leqslant p+q$ as well).

Then $\zeta(\mu, \mathbb{1}_n) \circ_i \xi_i = \zeta(\mu', \mathbb{1}_n) \circ_{\{i,\dots,i+s-1\}} (\xi_{i,1}, \dots, \xi_{i,s})$. For $\xi \in \bar{\mathscr{C}}_p \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)$, we call $\mu$ *optimal* for $\xi$ if there are $\xi_1, \dots, \xi_r$ with $\xi = \zeta(\mu, \mathbb{1}_n) \circ (\xi_1, \dots, \xi_r)$ up to input permutation, and for each $1 \leqslant i \leqslant r$, the operation $\mu$ cannot be refined any more as above, i.e. if $c_i$ is degenerate, then $\xi_i$ is a unary.

A similar construction can be done for $\nu \in \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)$, by putting all boxes that are to be placed into a degenerate box on separate *layers*, see Figure 4.9.

**Remark 4.2.34.** If $\mu$ is optimal for $\xi$, then $\mu^{-1} \cdot_\pi \xi = (\xi_1, \dots, \xi_r)$ does not have the aforementioned problems any more: each $\xi_i$ lives in $\bar{\mathscr{C}}_p \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)$, since whenever we want to rescale a box to full size, there is enough space; and secondly, all surjections $\pi \colon \underline{s} \to \underline{r}$ which satisfy $c_{i,\jmath} \subseteq c_{\pi(i)}$ differ only by precomposition with a permutation. Hence we write $\mu^{-1} \cdot \xi$ without the index $\pi$: the collection $(\xi_1, \dots, \xi_r) = \mu^{-1} \cdot \xi$ is uniquely determined up to input and layer permutation.

Now we have everything together to prove Proposition 4.2.29.

*Proof of Proposition 4.2.29.* We start by showing the trackability of $\bar{\zeta}$: given $\xi = (c_{1,1}, \dots, c_{r,k_r}) \in (\bar{\mathscr{C}}_p \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_q))\binom{K}{n}$, with $c_{i,\jmath}\colon [0;1]^{p+q} \to [0;1]^{p+q} \times \underline{n}$, we want to construct minimal crotches for $\xi$, and maximal arguments for them.

To do so, we consider again the projections $\mathrm{pr}^n_p\colon [0;1]^{p+q} \times \underline{n} \to [0;1]^p$ and $\mathrm{pr}_q\colon [0;1]^{p+q} \to [0;1]^{p+q}$. Then $\mathrm{pr}^n_p \circ c_{i,\jmath} = \mathrm{pr}^n_p \circ c_{i,\jmath'}$ holds for all $1 \leqslant i \leqslant r$ and $1 \leqslant \jmath, \jmath' \leqslant k_i$, as the pairing lands in the suboperad of $\overline{\mathbb{N}}(\mathscr{C}_{p+q})$ where clusters satisfy this condition. Now we consider the two configurations

$$\begin{aligned}\tilde{\mu}(\xi) &:= (\mathrm{pr}^n_p \circ c_{1,1}, \dots, \mathrm{pr}^n_p \circ c_{r,1}) \in F_p,\\ \tilde{\nu}(\xi) &:= \big((\mathrm{pr}_q \times \underline{n}) \circ c_{1,1}, \dots, (\mathrm{pr}_q \times \underline{n}) \circ c_{r,k_r}\big) \in F_{q,n}.\end{aligned}$$

The operation $\tilde{\mu}(\xi)$ contains $r$ boxes, while $\tilde{\nu}(\xi)$ consists of $|K|$ boxes, tied together into $r$ inputs. Clearly, $\tilde{\mu}(\xi)$ and $\tilde{\nu}(\xi)$ do not have to lie in $F'_p$ and $F'_{q,n}$, respectively, but we can consider their smallest upper bounds, $\mu$ and $\nu$, as in Construction 4.2.31: see Figure 4.10. Then $\mu$ and $\nu$ are by construction optimal for $\xi$, and we put $(\xi_1, \dots, \xi_r) := \mu^{-1} \cdot \xi$, and similarly for $\nu$.

We are left to show that $\mu$ is a minimal red crotch and $(\xi_1, \dots, \xi_r)$ is a collection of maximal arguments: if there is $\bar{\mu} \in \bar{\mathscr{C}}_p(s)$ and $\bar{\xi}_1, \dots, \bar{\xi}_s$ with $\zeta(\bar{\mu}, \mathbb{1}_n) \circ (\bar{\xi}_1, \dots, \bar{\xi}_s) = \xi$ up to input permutation, then clearly $\tilde{\mu}(\xi) \leqslant \bar{\mu}$. Then we have $\mu \leqslant \bar{\mu}$, whence there are $\mu_1, \dots, \mu_s$ with $\mu = \bar{\mu} \circ (\mu_1, \dots, \mu_s)$ up to input permutation. The maximality of $(\xi_1, \dots, \xi_r)$ can be checked in the same way. Moreover, both $\mu$ and $(\xi_1, \dots, \xi_r)$ are uniquely determined up to (input or layer) permutation, showing T1$^{\mathrm{a}}$ and T2$^{\mathrm{a}}$, and similarly, T2$^{\mathrm{c}}$ holds, using that $\mathrm{core}(\bar{\mathscr{C}}_p \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)) \subseteq \mathrm{core}(\overline{\mathbb{N}}(\mathscr{C}_{p+q})) = \mathfrak{S}$, so we only have to deal with layer permutations.

For T1$^{\mathrm{b}}$ we use that, for $\mu \in \bar{\mathscr{C}}_p$ and $\xi_1, \dots, \xi_r \in \bar{\mathscr{C}}_p \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)$, we have $\tilde{\mu}(\zeta(\mu, \mathbb{1}_n) \circ (\xi_1, \dots, \xi_r)) = \mu \circ (\tilde{\mu}(\xi_1), \dots, \tilde{\mu}(\xi_r))$. The property T1$^{\mathrm{c}}$ is immediate from the construction, see the two degenerate black intervals 3 and 4 in Figure 4.10, which can be permuted since swapping the above clusters 5 and 6 fixes the operation. If $\xi$ is a unary, then it is of the form $\zeta(\mu, \nu)$, and this decomposition is unique, showing T3.

To show P2, let $\vartheta \in \bar{\mathscr{C}}_p \odot \overline{\mathbb{N}}(\bar{\mathscr{C}}_q)$ be an operation such that $\xi := \bar{\zeta}^{\odot}(\vartheta)$ has a red crotch $\mu$ with arity at least 2, and we choose a reduced representative

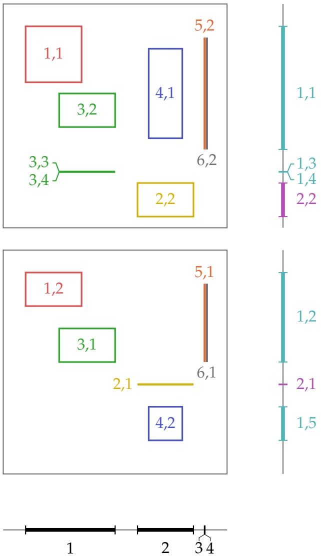

**Figure 4.10.** An element $\xi \in (\bar{\mathscr{C}}_1 \circledast \overline{\mathbb{N}}(\bar{\mathscr{C}}_1))\binom{2,2,4,2,2,2}{2}$ and its minimal red crotch $\mu \in \bar{\mathscr{C}}_1(4)$ and minimal green crotch $\nu \in \overline{\mathbb{N}}(\bar{\mathscr{C}}_1)\binom{5,2}{2}$.

for $\vartheta$ with a red crotch. Standing at a green vertex without adjacent leaves, we denote the labels of the red vertices directly above it by $\mu_1, \ldots, \mu_s$, and let $\bar{\mu}$ be their smallest common upper bound inside $F'_p$, as in Construction 4.2.31. Then we can write $\mu_i = \bar{\mu} \circ (\mu_{i,1}, \ldots, \mu_{i,s_i})$, and by the interchange law, we can move $\bar{\mu}$ down and compose it with the next red ones. In this way, we inductively reach a reduced representative for $\vartheta$ where each red vertex is maximal with respect to its neighbours. For this representative, the (modified) red crotch $\mu'$ can be at most $\mu$, as for each input $1 \leqslant i \leqslant r$ of $\vartheta$, the box inside $[0;1]^p$ which is given by composing the respective entries inside the red vertices along the unique path from the leaf numbered by $i$ down to the root, has to lie in one of the boxes of $\mu$, and for each input $1 \leqslant h \leqslant s$ of $\mu'$, the collection of boxes arising in the same way, for the inputs of the subtree starting with the $h^{\text{th}}$ incoming edge at $\mu'$, fill the entire cube $[0;1]^p$ by the maximality property. Therefore, the arity of $\mu'$ is at least as large as the arity of $\mu$, and we have constructed a reduced representative for $\vartheta$ with a red crotch $\mu'$ of arity at least 2. ☐

This concludes the long proof of Theorem 4.2.2, and it additionally shows the $\mathfrak{S}$-cofibrancy of $\mathscr{C}_p \odot \overline{\mathbb{N}}(\mathscr{C}_q)$ as follows: since $\zeta^{\odot} \colon \mathscr{C}_p \odot \overline{\mathbb{N}}(\mathscr{C}_q) \to \mathscr{V}_{p,q}$ is levelwise injective, the input permutation acts freely on $\mathscr{C}_p \odot \overline{\mathbb{N}}(\mathscr{C}_q)$, and moreover, we saw that the subspaces of decomposable operations are retracts of the larger ones, and these, in turn, are open manifolds.

Therefore, the category of $\mathscr{V}_{p,q}$-algebras is Quillen equivalent to the category of $\overline{\mathbb{N}}(\mathscr{C}_q)$-algebras which are levelwise $\mathscr{C}_p$-algebras in a compatible way.

Let me finally point out that, by restricting to appropriate connected components, we can show in the same way that the connective suboperad $\mathscr{V}^c_{p,q} \subseteq \mathscr{V}_{p,q}$ is equivalent to the Boardman–Vogt tensor product $\mathscr{C}^c_p \odot \overline{\mathbb{N}}^c(\mathscr{C}_q)$.

## 4.3. A cellular description of $\mathscr{V}_{1,1}$

From now on, we restrict our attention to the vertical operad $\mathscr{V} := \mathscr{V}_{1,1}$. In this section, we want to see how the operadic structure of $\mathscr{V}$ is related to the cellular decomposition from Section 1.4. These results will be exploited further in the next section in order to describe the algebraic operad $H_\bullet(\mathscr{V}_{1,1})$.

### 4.3.1. *Path components of $\mathscr{V}_{1,1}$ and their decompositions*

Recall that, by Proposition 4.1.3, the space $\mathscr{V}_{1,1}\binom{K}{n}$ is homotopy equivalent to the ordered vertical configuration space $\tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n})$. Let us start by understanding its path components, extending Remark 1.1.6:

**Remark 4.3.1.** Recall from Example 3.4.7 the category $\mathbf{\Delta\Sigma}$ of non-commutative finite sets, whose arrows $k \to n$ are maps $f \colon \underline{k} \to \underline{n}$, together with a total order $\prec$ on each fibre. Let $K = (k_1, \dots, k_r)$ be a tuple of positive integers $k_i \geqslant 1$ and let $n \geqslant 1$. Then we have

$$\pi_0(\mathscr{V}_{1,1}\tbinom{K}{n}) \cong \pi_0(\tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n})) \cong \mathbf{\Delta\Sigma}\tbinom{k_1}{n} \times \cdots \times \mathbf{\Delta\Sigma}\tbinom{k_r}{n}.$$

The identification is done in the following way: given a tuple $\boldsymbol{f} = (f_1, \dots, f_r)$ of arrows $f_i \in \mathbf{\Delta\Sigma}\binom{k_i}{n}$, we let $\tilde{V}_{\boldsymbol{f}} \subseteq \tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n})$ be the subspace of all tuples $(z_{1,1}, \dots, z_{r,k_r})$ such that, if we write $f_i^{-1}(\ell) = \{\jmath_1 \prec \cdots \prec \jmath_m\}$, then the $\ell^{\text{th}}$ layer contains the points $z_{i,\jmath_1}, \dots, z_{i,\jmath_m}$, and, if we write $z_{i,\jmath} = (\zeta_i, t_{i,\jmath})$, then

$t_{i,\jmath_1} < \cdots < t_{i,\jmath_m}$. One readily checks that $\tilde{V}_f$ is path connected and that $\tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n})$ decomposes as the disjoint union $\coprod_f \tilde{V}_f$.

This identification is operadic in the following sense: we let $\mathbf{\Delta\Sigma}^\times$ be the $\overline{\mathbb{N}}$-coloured operad with $\mathbf{\Delta\Sigma}^\times\binom{K}{n} := \prod_i \mathbf{\Delta\Sigma}\binom{k_i}{n}$, with $\tau^* \boldsymbol{f} = (f_{\tau(1)}, \ldots, f_{\tau(r)})$ and with entrywise composition as usual. Then the operad $\pi_0(\mathcal{V}_{1,1})$ of sets is isomorphic to the operad $\mathbf{\Delta\Sigma}^\times$.

The map $\pi_0(\mathcal{V}_{1,1} \hookrightarrow \overline{\mathbb{N}}(\mathcal{C}_2))$ of operads is of the form $\mathbf{\Delta\Sigma}^\times \to \overline{\mathbb{N}}(\mathcal{C}om)$ and forgets, for each arrow $f_i \colon k_i \to n$ in $\mathbf{\Delta\Sigma}$, the order of the fibres.

**Notation 4.3.2.** An explicit arrow $f \in \mathbf{\Delta\Sigma}\binom{k}{n}$ can be denoted as follows: if $f^{-1}(\ell) = \{\jmath_{\ell,1} \prec \ldots \prec \jmath_{\ell,m_\ell}\}$, then we represent $f$ as the tuple

$$(\jmath_{1,1} \cdots \jmath_{1,m_1} \mid \cdots \mid \jmath_{n,1} \cdots \jmath_{1,m_n}).$$

For each element $\boldsymbol{f} \in \mathbf{\Delta\Sigma}^\times\binom{K}{n}$, we want to calculate the homology of the corresponding component $\tilde{V}_{\boldsymbol{f}}$. Note that for $n = 1$, all these components are isomorphic and we calculated their integral homology in Theorem 1.2.5. The case of arbitrary $n$ would be just an easy generalisation; however, this description does not behave nicely with respect to the operadic structure. Instead we employ a multi-layered version of the cellular description from Section 1.4. We want to emphasise the similarity between these constructions and advice the reader to compare them.

**Definition 4.3.3.** Fix $K$, $n$, and $\boldsymbol{f}$, and let $1 \leqslant s \leqslant r$. An *allocation of* $\boldsymbol{f}$ *into* $s$ *colums* is given by a pair $(M, \pi)$ where:

1. $M = (M_1, \ldots, M_n)$ with $M_\ell = (m_{1,\ell}, \ldots, m_{s,\ell})$, where $m_{a,\ell} \geqslant 0$;
2. $\pi = (\pi_1, \ldots, \pi_n)$ with $\pi_\ell \colon \mathbb{Y}_{M_\ell} \to \underline{r}$,

such that the following holds:

- for each $i \in \underline{r}$, there is an $a_\pi(i) \in \underline{s}$ such that for each $\ell \in \underline{n}$, the fibre $\pi_\ell^{-1}(i)$ is contained in the *column* $\mathbb{Y}_{M_\ell}^{a_\pi(i)} := \{(a_\pi(i), b);\ 1 \leqslant b \leqslant m_{a,\ell}\}$;
- we have $\#\pi_\ell^{-1}(i) = \# f_i^{-1}(\ell)$.

We usually suppress $M$ and just write $\pi$ for a given allocation. Let $\Pi_{\boldsymbol{f},s}$ be the set of all allocations of $\boldsymbol{f}$ into $s$ columns. Note that $\Pi_{\boldsymbol{f},s}$ only depends on the tuple of *maps* which underlies $\boldsymbol{f}$, not on the orders on the fibre.

**Notation 4.3.4.** Given an explicit allocation $\pi$ of $\boldsymbol{f}$ into $s$ columns, it will turn out to be convenient to denote $\pi$ by the matrix

$$\begin{pmatrix} \pi_n(1{,}1)\cdots\pi_n(1{,}m_{1,n}) & \cdots & \pi_n(s{,}1)\cdots\pi_n(s{,}m_{s,n}) \\ \vdots & & \vdots \\ \pi_1(1{,}1)\cdots\pi_1(1{,}m_{1,1}) & \cdots & \pi_1(s{,}1)\cdots\pi_1(s{,}m_{s,1}) \end{pmatrix}.$$

**Example 4.3.5.** Let $\boldsymbol{f} = (4\,||\,2\,1\,3,1\,|\,3\,|\,2,\,|\,1\,2\,|\,) \in \boldsymbol{\Delta\Sigma}^{\times}\binom{4,3,2}{3}$. Then a possible allocation of $\boldsymbol{f}$ into two columns is given by the matrix

$$\pi = \left(\begin{smallmatrix} 2 & & 1\,1\,1 \\ 3 & 3\,2 & \\ 2 & & 1 \end{smallmatrix}\right).$$

Note that, with regards to the type of an allocation, $\boldsymbol{f}$ only prescribes how many instances of 1, 2, and 3 are on which layer, and pictorially, each entry of $\boldsymbol{f}$ has to be rotated anticlockwise by 90° in order to 'fit' into the matrix.

**Construction 4.3.6.** Let $\pi \in \Pi_{\boldsymbol{f},s}$ and $1 \leqslant \alpha \leqslant s-1$, and let $\imath = (\imath_1, \dots, \imath_n)$ with $\imath_\ell$ an $(m_{\alpha,\ell}, m_{\alpha+1,\ell})$-shuffle. Then we define $d_{\alpha,\imath}(\pi) := (\pi'_1, \dots, \pi'_n)$ with $\pi'_\ell := d_{\alpha,\imath_\ell}(\pi_\ell)$ as in Definition 1.4.7. Consider the simplicial complex

$$\tilde{V}^{\blacktriangle} := \coprod_{i=1}^{r} \coprod_{\pi\in\Pi_{\boldsymbol{f},s}} \Delta^s \times \prod_{\ell=1}^{n} \Delta^{M_\ell} \Big/ (\pi; d^{\alpha}\boldsymbol{\zeta}, (d^{\alpha,\imath_\ell}\boldsymbol{t}_\ell)_{\ell\in\underline{n}}) \sim (d_{\alpha,\imath}\pi; \boldsymbol{\zeta}, (\boldsymbol{t}_\ell)_{\ell\in\underline{n}}),$$

where $\Delta^{M_\ell} = \prod_a \Delta^{m_{a,\ell}}$ and $\boldsymbol{t}_\ell \in \Delta^{M_\ell}$, and with $d^{\alpha,\imath_\ell}\boldsymbol{t}_\ell$ as in Construction 4.3.6. Again, we have a degenerate subcomplex $\tilde{V}^{\vartriangle} \subseteq \tilde{V}^{\blacktriangle}$ and, for each $\pi$, a map $\boldsymbol{f}_\pi \colon \Delta^s \times \prod_\ell \Delta^{M_\ell} \to \tilde{V}_{\boldsymbol{f}}^{\infty}$ to the one-point compactification of $\tilde{V}_{\boldsymbol{f}}$:

- let $\mathbb{Y}_{\boldsymbol{f},\pi} \colon \mathbb{Y}_K \to \coprod_\ell \{\ell\} \times \mathbb{Y}_{M_\ell} =: \mathbb{Y}_M$ be the unique bijection that identifies, for $f_i^{-1}(\ell) = \{\jmath_1 \prec \cdots \prec \jmath_m\}$, the subset $\{(i, \jmath_1) \prec \cdots \prec (i, \jmath_m)\}$ with the fibre $\pi_\ell^{-1}(i) \subseteq \mathbb{Y}_{M_\ell}$ in a monotone way.

- elements in $\tilde{V}_{\boldsymbol{f}}$ can be written as $(\zeta_1, \dots, \zeta_r; t_{1,1}, \dots, t_{r,k_r})$ with $\zeta_i \in \mathbb{R}$ and $t_{i,\jmath} \in \mathbb{R}$; they are identified with tuples $(z_{1,1}, \dots, z_{r,k_r})$ of points in $\mathbb{R}^2 \times \underline{n}$ via $z_{i,\jmath} := (\zeta_i, t_{i,\jmath}, f_i(\jmath)) \in \mathbb{R}^2 \times \underline{n}$.

Now we put again $\Delta^s = \{-\infty \leqslant t_1 \leqslant \cdots \leqslant t_s \leqslant +\infty\}$, write elements in the multisimplex $\prod_\ell \Delta^{M_\ell}$ as $\boldsymbol{t} = (t_{\ell,a,\mathscr{b}})$, and define

$$\boldsymbol{f}_\pi(\boldsymbol{\zeta}, \boldsymbol{t}) := \begin{cases} (a_\pi^*\boldsymbol{\zeta}, \mathbb{Y}_{\boldsymbol{f},\pi}^*\boldsymbol{t}) & \text{if } (\boldsymbol{\zeta}, \boldsymbol{t}) \notin \tilde{V}^{\vartriangle}, \\ \infty & \text{if } (\boldsymbol{\zeta}, \boldsymbol{t}) \in \tilde{V}^{\vartriangle}. \end{cases}$$

Then the union $\coprod_i \coprod_\pi f_\pi$ factors through $f_*\colon \tilde{V}^\blacktriangle \to \tilde{V}_f^\infty$. Moreover, we get $f_*^{-1}(\infty) = \tilde{V}^\vartriangle$, and the corestriction $\tilde{V}^\blacktriangle \setminus \tilde{V}^\vartriangle \to \tilde{V}_f$ is an isomorphism. By Poincaré–Lefschetz duality, we get

$$H_\bullet(V_f(\mathbb{R}^{1,1} \times \underline{n})) \cong H^{|K|+r-\bullet}(\tilde{V}^\blacktriangle, \tilde{V}^\vartriangle).$$

**Remark 4.3.7.** The relative cellular chain complex $C_\bullet^{\mathrm{cell}}(\tilde{V}^\blacktriangle, \tilde{V}^\vartriangle)$ is generated by $\Pi_{f,s}$ in degree $|K| + s$, and the differential is of the form

$$\partial\pi = \sum_{\alpha=1}^{s-1} (-1)^\alpha \cdot \sum_{\substack{\imath=(\imath_1,\dots,\imath_n) \\ \imath_\ell:(m_{\alpha,\ell},m_{\alpha+1,\ell})}} \underbrace{\prod_{\ell=1}^{n} \mathrm{sg}(\imath_\ell)}_{=:\mathrm{sg}(\imath)} \cdot d_{\alpha,\imath}(\pi).$$

**Example 4.3.8.** If we let $f = (2\,3\,|\,1,1\,|\,2) \in \mathbf{\Delta\Sigma}^\times\binom{3,2}{2}$, then $\left(\begin{smallmatrix} 2 & 1 \\ 2 & 11 \end{smallmatrix}\right) \in \Pi_{f,2}$ is a 7-dimensional multisimplex in $(\tilde{V}^\blacktriangle, \tilde{V}^\vartriangle)$, and we get

$$\partial\left(\begin{smallmatrix} 2 & 1 \\ 2 & 11 \end{smallmatrix}\right) = -\left(\begin{smallmatrix} 21 \\ 211 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 21 \\ 121 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 21 \\ 112 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 12 \\ 211 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 12 \\ 121 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 12 \\ 112 \end{smallmatrix}\right).$$

Similarly, if $f = (1\,|\,3\,2, 2\,|\,1, \,|\,1,1\,|\,) \in \mathbf{\Delta\Sigma}^\times\binom{3,2,1,1}{2}$, then $\left(\begin{smallmatrix} 32 & & 11 \\ 2 & 4 & 1 \end{smallmatrix}\right) \in \Pi_{f,3}$ is a 10-dimensional multisimplex and we get

$$\partial\left(\begin{smallmatrix} 32 & & 11 \\ 2 & 4 & 1 \end{smallmatrix}\right) = -\left(\begin{smallmatrix} 32 & 11 \\ 24 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 32 & 11 \\ 42 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 32 & 11 \\ 2 & 41 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 32 & 11 \\ 2 & 14 \end{smallmatrix}\right).$$

As in Remark 1.4.5, these matrices for $\pi$ already look like a configuration $f_\pi(\boldsymbol{\zeta}, \boldsymbol{t}) \in \tilde{V}_f$, one just has to rotate all entries anticlockwise by 90° and add the internal order for each cluster as prescribed by the order of the fibres.

### 4.3.2. *The operadic structure through simplices*

Using the equivalences $\Phi\colon \mathcal{V}_{1,1}\binom{K}{n} \rightleftarrows \tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n}) :\!\Psi$ from Proposition 4.1.3, the operadic structure on $\mathcal{V}$, more specifically the input permutation and the composition law, gives rise to several maps of the form

$$\varphi\colon \prod_i \tilde{V}_{f_i} \xrightarrow[\simeq]{\Psi} \prod_i (\mathcal{V}_{1,1})_{f_i} \longrightarrow (\mathcal{V}_{1,1})_g \xrightarrow[\simeq]{\Phi} \tilde{V}_g$$

for some $f_i \in \mathbf{\Delta\Sigma}^\times\binom{K_i}{n}$ and $g \in \mathbf{\Delta\Sigma}^\times\binom{L}{m}$. If we choose a commutative ring $R$ and consider (co-)homology over $R$, then our aim is to find a combinatorial construction $\varphi\colon \bigotimes_i R\langle\Pi_{f_i,s_i}\rangle \to R\langle\Pi_{g,t}\rangle$ that *reflects* the map $\varphi$ in homology in the following sense: after identifying $R\langle\Pi_{f_i,s_i}\rangle \cong C^{|K_i|+s_i}_{\mathrm{cell}}(\tilde V^{\blacktriangle}_{f_i}, \tilde V^{\vartriangle}_{f_i})$ as well as $R\langle\Pi_{g,t}\rangle \cong C^{|L|+t}_{\mathrm{cell}}(\tilde V^{\blacktriangle}_{g}, \tilde V^{\vartriangle}_{g})$, the following square commutes:

$$\begin{tikzcd}
\bigotimes_i H^{|K_i|+s_i}(\tilde V^{\blacktriangle}_{f_i}, \tilde V^{\vartriangle}_{f_i}) \arrow[r, "\otimes_i[\sum_u \pi_{i,u}]\mapsto[\varphi(\otimes_i \sum_u \pi_{i,u})]"] \arrow[d, "\text{Poincaré–Lefschetz}"', "\cong"] & H^{|L|+t}(\tilde V^{\blacktriangle}_{g}, \tilde V^{\vartriangle}_{g}) \arrow[d, "\cong"', "\text{P.–L.}"] \\
\bigotimes_i H_{\#K_i-s_i}(\tilde V_{f_i}) \arrow[r, "\varphi_*"'] & H_{\#L-t}(\tilde V_g).
\end{tikzcd}$$

Our approach is differential-topologically flavoured: if we denote by $\pi^\circ$ the interior of a cell $\pi$, then we will show, in several cases, that for each collection of $(|K_i|+s_i)$-cells $\pi_i$, there is a specific $(|L|+t)$-cell $\pi'$ such that for each transversal intersection $S \subseteq \prod_i \tilde V_{f_i}$ with $\prod_i \pi_i^\circ \subseteq \prod_i \tilde V_{f_i}$, the image $\varphi(S)$ is a single transversal intersection with $\pi'^\circ$.

Calculating the correct signs of these intersections, however, turns out to be a rather lengthy, yet completely straightforward task. Therefore, we decided to skip a general sign discussion and to calculate the sign by hand in all situations where it is necessary: in this subsection, we just write '±', which is justified by a single transversal intersection.

If the reader gets wearied by the heaviness of the upcoming combinatorial details, the author, who felt the same way, invites them to start with the Examples 4.3.13 in order to get a feeling for what the lemmata aim at.

**Lemma 4.3.9** (Input permutation). *Let $f \in \mathbf{\Delta\Sigma}^\times\binom{K}{n}$ and $\tau \in \mathfrak{S}_r$. Then the input permutation $\tau^*\colon H_{r-s}(\tilde V_f) \to H_{r-s}(\tilde V_{\tau^* f})$ is reflected by the map*

$$\tau^*\colon R\langle\Pi_{f,s}\rangle \to R\langle\Pi_{\tau^* f,s}\rangle, \quad \pi \mapsto \pm(\tau^{-1}\circ\pi).$$

*Proof.* The map $\tau^*\colon \tilde V_f \to \tilde V_{\tau^* f}$ is an isomorphism, with inverse given by $(\tau^{-1})^*$, in particular an open embedding.

Let $\pi \in \Pi_{f,s}$ be a simplex. Then for each point $\boldsymbol{z}' \in (\tau^{-1}\circ\pi)^\circ$, the fibre $(\tau^*)^{-1}(\boldsymbol{z}') \subseteq \pi^\circ$ contains a single point. Additionally, for each transversal intersection $S \subseteq \tilde V_f$ through $\boldsymbol{z} \in \pi^\circ$, the image $\tau^*(S)$ is a single transversal intersection with the interior $(\tau^{-1}\circ\pi)^\circ$.

Thus, if $\sum_l \pi_l$ is a representing cocycle for the Poincaré–Lefschetz dual of a class $\xi \in H_{r-s}(\tilde V_f)$, then $\sum_l \omega_l \cdot (\tau^{-1}\circ\pi)$ is a representing cocycle for the dual of $\tau^*\xi$, where $\omega_l$ is the sign comparing the orientation of the tangent space $T_z\pi^\circ \oplus T_zS$ with the one of $T_{\tau^*z}(\tau^{-1}\circ\pi)^\circ \oplus T_{\tau^*z}(\tau^*S)$ for any transversal intersection $S$ and a choice of orientation of $T_zS$. $\square$

The next two lemmata are shown in exactly the same way, using again that the maps we consider are open embeddings; hence we omit their proofs.

**Lemma 4.3.10** (Postcomposing with unaries). *Let $\boldsymbol{f} \in \mathbf{\Delta\Sigma}^\times\binom{K}{n}$ and $g \in \mathbf{\Delta\Sigma}\binom{n}{n'}$. Then the map $g_*\colon H_{r-s}(\tilde{V}_{\boldsymbol{f}}) \to H_{r-s}(\tilde{V}_{g\circ \boldsymbol{f}})$ induced by postcomposing with an arbitrary operation $\tilde{g}$ inside the contractible component $\mathcal{V}_{1,1}\binom{n}{n'}_g$ is reflected by*

$$g_*\colon R\langle\Pi_{\boldsymbol{f},s}\rangle \to R\langle\Pi_{g\circ \boldsymbol{f},s}\rangle,\quad \pi \mapsto \pm(\pi\circ g^*),$$

*where $(g_*M)_{a,\ell} = \sum_{\mathscr{j}\in g^{-1}(\ell)} m_{a,\mathscr{j}}$ and $(g^*)^{-1}\colon \mathbb{Y}_M \to \mathbb{Y}_{g_*M}$ sends $(\mathscr{j},a,\mathscr{b})$ with $g(\mathscr{j}) = \ell$ to $(\ell,a,m+\mathscr{b})$, where $m = \sum_{\mathscr{j}'\prec\mathscr{j}} m_{\mathscr{j}',a}$.*

**Lemma 4.3.11** (Universal morphisms). *Let $\boldsymbol{f} \in \mathbf{\Delta\Sigma}^\times\binom{K}{n}$ and $\boldsymbol{f}' \in \mathbf{\Delta\Sigma}^\times\binom{K'}{n'}$. Consider the universal morphisms $U_{n,n'} \in \mathcal{V}_{1,1}\binom{n,n'}{n+n'}$ from Remark 3.4.10. Then the postcomposition $U_{n,n'}\colon H_{r-s}(\tilde{V}_{\boldsymbol{f}}) \otimes H_{r'-s'}(\tilde{V}_{\boldsymbol{f}'}) \to H_{(r-s)+(r'-s')}(\tilde{V}_{U_{n,n'}\circ(\boldsymbol{f},\boldsymbol{f}')})$ is reflected by the map*

$$\begin{aligned} U_{n,n'}\colon R\langle\Pi_{\boldsymbol{f},s}\rangle \otimes R\langle\Pi_{\boldsymbol{f}',s'}\rangle &\to R\langle\Pi_{U_{n,n'}\circ(\boldsymbol{f},\boldsymbol{f}'),s+s'}\rangle,\\ \pi\otimes\pi' &\mapsto \pm(\pi\sqcup\pi'), \end{aligned}$$

*where $\pi\sqcup\pi'\colon \mathbb{Y}_{M\sqcup M'} \cong \mathbb{Y}_M \sqcup \mathbb{Y}_{M'} \to \underline{r}\sqcup\underline{r'} \cong \underline{r+r'}$, with*

$$(M\sqcup M')_{a,\ell} = \begin{cases} m_{a,\ell} & \text{for } a \leqslant s \text{ and } \ell \leqslant n,\\ m'_{a-s,\ell-n} & \text{for } a > s \text{ and } \ell > n,\\ 0 & \text{else.} \end{cases}$$

**Lemma 4.3.12** (Single columns). *Let $\boldsymbol{g} \in \mathbf{\Delta\Sigma}^\times\binom{K}{n}$ and $\boldsymbol{f}_i \in \mathbf{\Delta\Sigma}^\times\binom{L_i}{k_i}$ with $\#K = r$ and $\#L_i = s_i$. Then the map $H_{r-1}(\tilde{V}_{\boldsymbol{g}}) \otimes \bigotimes_i H_{s_i-1}(\tilde{V}_{\boldsymbol{f}_i}) \to H_{s_1+\cdots+s_r-1}(\tilde{V}_{\boldsymbol{g}\circ(\boldsymbol{f}_1,\ldots,\boldsymbol{f}_r)})$ induced in the top-dimensional homology is reflected by the map of cells*

$$\begin{aligned} (-)\circ_{\boldsymbol{g}}(-)\colon R\langle\Pi_{\boldsymbol{g},1}\rangle \otimes \textstyle\bigotimes_i R\langle\Pi_{\boldsymbol{f}_i,1}\rangle &\to R\langle\Pi_{\boldsymbol{g}\circ(\boldsymbol{f}_1,\ldots,\boldsymbol{f}_r),1}\rangle,\\ \pi\otimes(\pi_1\otimes\cdots\otimes\pi_r) &\mapsto \pm\pi', \end{aligned}$$

*where $\pi'$ is defined as follows: if $M = (m_1, \dots, m_n)$ is the partition that underlies $\pi$ and $Q_i = (q_{i,1}, \dots, q_{i,k_i})$ is the partition that underlies $\pi_i$, then we use the bijection $\varphi := \mathbb{Y}_{\pi,g} \colon \mathbb{Y}_K \to \mathbb{Y}_M$ from Construction 4.3.6 and put $m'_\ell := \sum_{b=1}^{m_\ell} q_{\varphi^{-1}(\ell,b)}$. We obtain $\Phi \colon \mathbb{Y}_{M'} \to \coprod_i \mathbb{Y}_{Q_i}$ and let $\pi' := \big(\coprod_i \tilde{\pi}_i\big) \circ \Phi$ with $\tilde{\pi}_i = \pi_i + \sum_{i'<i} s_{i'}$.*

*Proof.* The main difference to the previous three lemmata is the obstacle that the map $\tilde{V}_g \times \prod_i \tilde{V}_{f_i} \to \tilde{V}_{g\circ(f_1,\dots,f_r)}$ is not an embedding. To remedy this, we replace both sides by slightly different relative multisimplicial complexes: for each $f \in \mathbf{\Delta\Sigma}^\times\binom{K}{n}$, we define $\tilde{V}_{f,0} \subseteq \tilde{V}_f$ to be the subspace containing all configurations of clusters in $\mathbb{R}^{1,1} \times \underline{n}$ where the horizontal coordinate $\zeta_1$ of the *first* cluster is 0. Then $\jmath_f \colon \tilde{V}_{f,0} \hookrightarrow \tilde{V}_f$ is a deformation retract, the retraction $\rho_f$ given by shifting the entire configuration horizontally by $-\zeta_1$. Moreover, $\tilde{V}_{f,0}$ is the interior of a relative simplicial complex $(\tilde{V}^{\blacktriangle}_{f,0}, \tilde{V}^{\vartriangle}_{f,0})$ where we have a multisimplex $\Delta^{s-1} \times \prod_\ell \Delta^{M_\ell}$ for each $\pi \in \Pi_{f,s}$; here we omit the parameter for the horizontal coordinate of the column which contains the first cluster.

Via Poincaré–Lefschetz duality, the maps in homology which are induced by $\jmath_f \colon \tilde{V}_{f,0} \rightleftarrows \tilde{V}_f : \rho_f$ correspond to the maps in cohomology induced by the canonical identification

$$C^{|K|+s-1}_{\text{cell}}(\tilde{V}^{\blacktriangle}_{f,0}, \tilde{V}^{\vartriangle}_{f,0}) \cong R\langle \Pi_{f,s} \rangle \cong C^{|K|+s}_{\text{cell}}(\tilde{V}^{\blacktriangle}_{f}, \tilde{V}^{\vartriangle}_{f})$$

up to sign. In the same fashion, the product $\tilde{V}_{f,0} \times \mathbb{R}^d$ is the interior of the relative simplicial complex $(\tilde{V}^{\blacktriangle}_{f,0}, \tilde{V}^{\vartriangle}_{f,0}) \times ([-\infty;\infty]^d, \partial[-\infty;\infty]^d)$, where we have a multisimplex $\Delta^{s-1} \times \prod_\ell \Delta^{M_\ell} \times [-\infty;\infty]^d$ for each $\pi \in \Pi_{f,s}$. The maps $\jmath'_f \colon \tilde{V}_{f,0} \times \mathbb{R}^d \rightleftarrows \tilde{V}_f : \rho'_f$ enlarging the aforementioned ones by $\mathbb{R}^d \rightleftarrows \{0\}$, are, via Poincaré–Lefschetz duality, again induced by the canonical identifications of relative cellular cochain complexes.

Now we consider the map $\kappa \colon \tilde{V}_{g,0} \times \prod_i \tilde{V}_{f_i,0} \to \tilde{V}_{g\circ(f_1,\dots,f_r),0} \times \mathbb{R}^{|K|}$, which consists of the composition map and the map which remembers the vertical coordinates from the first factor. Then $\jmath'_{g\circ(f_1,\dots,f_r)} \circ \kappa \circ (\rho_g \times \prod_i \rho_{f_i})$ is the composition map, and we can alternatively give a cellular description for $\kappa$.

It is a straightforward task to check that $\kappa$ is indeed an open embedding. Moreover, if $S \subseteq \tilde{V}_{g,0} \times \prod_i \tilde{V}_{f_i,0}$ intersects the $(|K| + \sum_i |L_i|)$-dimensional multisimplex $\pi^\circ \times \prod_i \pi_i^\circ$ transversally, then $\kappa(S)$ intersects the multisimplex $\pi'$ from the statement transversally. Now the strategy of proof for Lemma 4.3.9 applies. $\square$

**Example 4.3.13.** Here are some examples for the four previous lemmata:

1. *Input permutation*

$$(1\,4\,2)^*\left(\begin{smallmatrix} 242 & 31 \\ 42 & 1 \end{smallmatrix}\right) = \pm\left(\begin{smallmatrix} 414 & 32 \\ 14 & 2 \end{smallmatrix}\right)$$

2. *Postcomposition with unaries*

$$\begin{aligned} (1\,2)_*\left(\begin{smallmatrix} 242 & 31 \\ 42 & 1 \end{smallmatrix}\right) &= \pm\left(\begin{smallmatrix} 42 & 1 \\ 242 & 31 \end{smallmatrix}\right) \\ s^1_*\left(\begin{smallmatrix} 242 & 31 \\ 42 & 1 \end{smallmatrix}\right) &= \pm\left(\begin{smallmatrix} 42242 & 131 \end{smallmatrix}\right) \\ d^2_*\left(\begin{smallmatrix} 242 & 31 \\ 42 & 1 \end{smallmatrix}\right) &= \pm\left(\begin{smallmatrix} 242 & 31 \\ & \\ 42 & 1 \end{smallmatrix}\right) \end{aligned}$$

3. *Universal morphisms*

$$U_{2,1}\left(\left(\begin{smallmatrix} 242 & 35 & 11 \\ 42 & 5 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 23 & 11 \end{smallmatrix}\right)\right) = \pm\left(\begin{smallmatrix} & & & 78 & 66 \\ 242 & 35 & 11 & & \\ 42 & 5 & 1 & & \end{smallmatrix}\right)$$

4. *Cells with a single column*
   Let $\boldsymbol{g} = (\,|\,1\,|\,,\,|\,2\,|\,1\,,\,2\,|\,|\,1\,3\,) \in \mathbf{\Delta\Sigma}^{\times}\binom{1,2,3}{3}$. Then

$$\left(\begin{smallmatrix} 233 \\ 12 \\ 3 \end{smallmatrix}\right) \circ_{\boldsymbol{g}} \left(\left(\begin{smallmatrix} 21 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 31 \\ 21 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 111 \\ 21 \\ 22 \end{smallmatrix}\right)\right) = \pm\left(\begin{smallmatrix} 4377666 \\ 2153 \\ 76 \end{smallmatrix}\right)$$

### 4.3.3. *A Morse flow for $\tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n})$*

In this subsection, we fix a commutative ring $R$, a profile $K = (k_1, \ldots, k_r)$, an output color $n \geqslant 1$, and a path component $\boldsymbol{f} = (f_1, \ldots, f_r) \in \mathbf{\Delta\Sigma}^{\times}\binom{K}{n}$, let $(\tilde{V}^{\blacktriangle}, \tilde{V}^{\vartriangle})$ be the relative simplicial complex with $\tilde{V}_{\boldsymbol{f}} = \tilde{V}^{\blacktriangle} \setminus \tilde{V}^{\vartriangle}$, and declare a discrete Morse flow on the relative cellular chain complex $C^{\mathrm{cell}}_{\bullet}(\tilde{V}^{\blacktriangle}, \tilde{V}^{\vartriangle}; R)$, or, equivalently, the reversed cochain complex $C^{|K|+r-\bullet}_{\mathrm{cell}}(\tilde{V}^{\blacktriangle}, \tilde{V}^{\vartriangle}; R)$.

We assume that the reader is familiar with the techniques of discrete Morse theory; we give a short summary of them in Appendix B.

**Remark 4.3.14.** We have a preferred basis $C^{\mathrm{cell}}_{|K|+s}(\tilde{V}^{\blacktriangle}, \tilde{V}^{\vartriangle}; R) = R\langle \Pi_{\boldsymbol{f},s} \rangle$, so we can write the chain complex as a based complex $R(\Omega, \varepsilon)$, with $\Omega_{|K|+s} = \Pi_{\boldsymbol{f},s}$ and incidence numbers $\varepsilon_{\pi,\pi'} = (-1)^{\alpha} \cdot \mathrm{sg}(\imath)$ if $\pi' = d_{\alpha,\imath}\pi$, and 0 else.

The pair $(\pi, d_{\alpha,\imath}\pi)$ determines the parameters $\alpha$ and $\imath$ uniquely, whence there is no ambiguity in the above expression.

**Notation 4.3.15.** Let $(M, \pi)$ be a cell. Let us abbreviate $\mathbb{Y}_M := (\mathbb{Y}_{M_1}, \dots, \mathbb{Y}_{M_n})$ and $\mathbb{Y}^a_M := (\mathbb{Y}^a_{M_1}, \dots, \mathbb{Y}^a_{M_n})$ for the $a^{\text{th}}$ column, and write $C_\pi(i) := \mathbb{Y}^{a_\pi(i)}_M$ for the one that contains the $i^{\text{th}}$ cluster, and $P_\pi(i) := (\pi_1^{-1}(i), \dots, \pi_n^{-1}(i)) \subseteq C_\pi(i)$ for the actual points of the $i^{\text{th}}$ cluster inside the $a_\pi(i)^{\text{th}}$ column.

There should be no confusion with the prior definition $\mathbb{Y}_M := \coprod_\ell \{\ell\} \times \mathbb{Y}_{M_\ell}$, which contains exactly the same information as the tuple $(\mathbb{Y}_{M_1}, \dots, \mathbb{Y}_{M_n})$.

**Definition 4.3.16.** Let $1 \leqslant s \leqslant r$ as well as $\pi \in \Pi_{f,s}$ and $1 \leqslant a \leqslant s$. A *ray (in the $a^{th}$ column)* is a system of subsets $Q = (Q_1, \dots, Q_n)$ with $Q_\ell \subseteq \mathbb{Y}^a_{M_\ell}$, such that the following holds:

1. there are $1 \leqslant i_1, \dots, i_t \leqslant r$ such that $Q = P_\pi(i_1) \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} P_\pi(i_t)$ holds, i.e. $Q$ is a union of clusters;
2. each $Q_\ell$ is an *upper half ray*: if $(a, \mathscr{b}) \in Q_\ell$ and $\mathscr{b}' \geqslant \mathscr{b}$, then $(a, \mathscr{b}') \in Q_\ell$.

**Example 4.3.17.** Here are four examples of families $Q = (Q_1, Q_2, Q_3)$ of subsets for the same cell $\pi$, depicted by dyeing a number in red if and only if it lies in $Q$. The first two examples violate the first condition of being a ray (in the first column), as the red collection is not a union of clusters. The third example violates the second condition, as in the bottom line, there is a black number on the right side of a red one. However, the last example is an honest ray (in the second column):

$$\begin{pmatrix} {\color{red}1} & 2 \\ 33{\color{red}1} & 244 \\ 3{\color{red}113} & 24 \end{pmatrix} \quad \begin{pmatrix} {\color{red}1} & 2 \\ 33{\color{red}1} & 244 \\ 3113 & 24 \end{pmatrix} \quad \begin{pmatrix} {\color{red}1} & 2 \\ 33{\color{red}1} & 244 \\ 3{\color{red}11}3 & 24 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 331 & 2{\color{red}44} \\ 3113 & 2{\color{red}4} \end{pmatrix}$$

For each $i$, we want to consider the smallest ray containing all clusters of number $i' \leqslant i$ inside the same column. This is formalised as follows:

**Construction 4.3.18.** For two rays $Q$ and $Q'$ inside the $a^{\text{th}}$ column, the intersection $Q \cap Q' := (Q_1 \cap Q'_1, \dots, Q_n \cap Q'_n)$ is again a ray, and moreover, the entire column $\mathbb{Y}^a_M$ itself is a ray. Hence, for each collection $S = (S_1, \dots, S_n) \subseteq \mathbb{Y}_M$ of subsets, there is a smallest ray containing $S$.

For each $1 \leqslant i \leqslant r$, we let $Q_\pi(i)$ be the smallest ray that contains the (levelwise) union $\bigcup_{i' \leqslant i} P_\pi(i') \cap C_\pi(i)$, i.e. that contains all clusters in the same column as $i$, which have a number at most $i$.

Note that $P_\pi(i) \subseteq Q_\pi(i) \subseteq C_\pi(i)$ and for $i' \leqslant i$, we have $Q_\pi(i') \subseteq Q_\pi(i)$.

**Example 4.3.19.** If $\pi \in \Pi_{f,3}$ is as below, then we have for example

$$\begin{aligned} Q_\pi(1) &= \begin{pmatrix} 1 & 6 & 4 \\ 331 & 662 & 45 \\ 3131 & 22 & 5 \end{pmatrix}, \\ Q_\pi(2) &= \begin{pmatrix} 1 & 6 & 4 \\ 331 & 662 & 45 \\ 3131 & 22 & 5 \end{pmatrix}, \\ Q_\pi(5) &= \begin{pmatrix} 1 & 6 & 4 \\ 331 & 662 & 45 \\ 3131 & 22 & 5 \end{pmatrix}. \end{aligned}$$

The idea for the Morse flow on $(\Omega, \varepsilon)$ is the following: if $Q_\pi(1) \subsetneq C_\pi(1)$, then we declare $\pi$ to be redundant, and we construct its collapsible partner $\pi^\sharp$ by putting the ray $Q_\pi(1)$ in a new column on the right side of $C_\pi(1) \setminus Q_\pi(1)$. If not, and if $\pi$ is not exhibited as a collapsible cell by the first step, we look at the second ray $Q_\pi(2)$. If $Q_\pi(2) \subsetneq C_\pi(2)$, then we declare $\pi$ to be redundant, and we proceed as above. This is done until we reach the highest cluster number $r$. This idea is formalised as follows:

**Definition 4.3.20.** For a cell $\pi$, we define the two numbers, which attain values in $\{1, \ldots, r, \infty\}$:

$$\begin{aligned} i_{\mathrm{red}}(\pi) &:= \min\big(1 \leqslant i \leqslant r;\ Q_\pi(i) \subsetneq C_\pi(i)\big), \\ i_{\mathrm{coll}}(\pi) &:= \min\left(1 \leqslant i \leqslant r;\ \begin{array}{l} Q_\pi(i) = C_\pi(i) \\ a_\pi(i) \neq 1 \\ a_\pi(i') \neq a_\pi(i) - 1 \text{ for } i' < i \end{array}\right). \end{aligned}$$

We call $\pi$ *redundant* if $i_{\mathrm{red}}(\pi) < i_{\mathrm{coll}}(\pi)$, and *collapsible* if $i_{\mathrm{coll}}(\pi) < i_{\mathrm{red}}(\pi)$. The bijection $(-)^\sharp \colon \Omega_h^{\mathrm{red}} \rightleftarrows \Omega_{h+1}^{\mathrm{coll}} : (-)^\flat$ is defined as follows:

- If $\pi$ is redundant, then we let $i := i_{\mathrm{red}}(\pi) < \infty$ and $\alpha := a_\pi(i)$, as well as $Q := Q_\pi(i) \subsetneq C_\pi(i)$, and we define
$$M_\ell^\sharp := (m_{1,\ell}, \ldots, m_{\alpha,\ell} - \#Q_\ell, \#Q_\ell, \ldots, m_{s,\ell}).$$
Then there is a unique (levelwise) map $\varphi \colon \mathbb{Y}_{M^\sharp} \to \mathbb{Y}_M$ that identifies $\mathbb{Y}_{M^\sharp}^\alpha$ with $\mathbb{Y}_M^\alpha \setminus Q_\pi(i)$, $\mathbb{Y}_{M^\sharp}^{\alpha+1}$ with $Q_\pi(i)$, and the other columns as they are in a monotone way, and we put $\pi^\sharp := \pi \circ \varphi$.

- If $\pi$ is collapsible, let $i := i_{\mathrm{coll}}(\pi) < \infty$ and $\alpha := a_\pi(i) - 1$. We define the shuffle $\imath := (\imath_1, \ldots, \imath_n)$ to be the tuple of standard $(m_{\alpha,\ell}, m_{\alpha+1,\ell})$-shuffles that put the respective left components entirely under the right ones, and we set $\pi^\flat := d_{\alpha,\imath}\pi$.

**Proposition 4.3.21.** *The previous definition declares a Morse flow $\Lambda$ and the derived chain complex $R(\Omega, \varepsilon)^{\Lambda}$ is formal, i.e. $\partial^{\Lambda} = 0$.*

*Proof.* Huge parts of the proof are straightforward combinatorial arguments, which we leave to the reader: let me say a word on acyclicity of the modified graph and on formality of the derived chain complex.

In order to show acyclicity, we consider, for each cell $\pi \in \Omega_h$ and each $1 \leqslant i \leqslant r$, the integer $\delta_i(\pi) := \# \bigcup_{i' \leqslant i} Q_{\pi}(i')$ and $\delta(\pi) := (\delta_1(\pi) \ldots, \delta_r(\pi))$. Then one can show that $\delta(\pi) \leqslant \delta(\pi')$ holds if $\varepsilon_{\pi,\pi'} \neq 0$, and $\delta(\pi^{\sharp}) = \delta(\pi)$ for all redundant cells $\pi$. Therefore, it is enough to exclude cycles inside the subgraph where all cells attain the same value along $\delta$.

To do so, we consider the assignment $\nu(\pi) = (a_{\pi}(1), \ldots, a_{\pi}(r)) \in \mathbb{Z}^r$, and we show that for each two-step path $\pi' \searrow \pi \nearrow \pi^{\sharp}$ inside the subgraph with fixed $\delta$, the difference $\nu(\pi^{\sharp}) - \nu(\pi')$ lies in the subset of tuples which are of the form $(0, \ldots, 0, 1, a_{r-s}, \ldots, a_r) \in \mathbb{Z}^r$. As this subset is positively linearly independent, no cycles can appear.

In order to show formality, we fix an essential cell $\pi$ and consider all *column-permuted* cells $\pi^{\tau}$: we consider the partition $M^{\tau}$ with $m^{\tau}_{\alpha,\ell} := m_{\tau^{-1}(\alpha),\ell}$, and if $\mathbb{Y}^{\tau} \colon \mathbb{Y}_{M^{\tau}} \to \mathbb{Y}_M$ describes the permutation of columns, then we set $\pi^{\tau} := \pi \circ \mathbb{Y}^{\tau}$. One easily shows that $\pi^{\tau}$ is collapsible if $\tau \neq \mathrm{id}$. Now for each $1 \leqslant \ell \leqslant n$, let $\tau_{M_{\ell}} \in \mathfrak{S}_{m_{1,\ell}+\cdots+m_{s,\ell}}$ be the block permutation with respect to $M_{\ell}$, and consider the chain

$$\tilde{\pi} := \sum_{\tau} \mathrm{sg}(\tau) \cdot \prod_{\ell=1}^{n} \mathrm{sg}(\tau_{M_{\ell}}) \cdot \pi^{\tau} \in R(\Omega, \varepsilon)_{|K|+s}.$$

Under the chain map $\Phi^{\Lambda} \colon R(\Omega, \varepsilon) \to R(\Omega, \varepsilon)^{\Lambda}$, the chain $\tilde{\pi}$ gets sent to the single essential cell $\pi \in R(\Omega, \varepsilon)^{\Lambda}_{|K|+s}$, as all other summands are collapsible. We claim that $\tilde{\pi}$ is a cycle in $R(\Omega, \varepsilon)$, which implies that also $\pi = \Phi^{\Lambda}(\tilde{\pi})$ is a cycle, and, thus, concludes the proof.

To this aim, note that each boundary $\pi'$ of some $\pi^{\tau}$ occurs exactly twice as a summand in $\partial \tilde{\pi}$, since $d_{\alpha,\imath} \pi^{\tau} = d_{\alpha',\imath'} \pi^{\tau'}$ holds if and only if $\alpha = \alpha'$ and $\tau' = (\alpha \;\; \alpha{+}1) \cdot \tau$, and the shuffle $\imath'$ is complementary to $\imath$ as in Definition 1.4.6. In this case, the signs cancel out: if we put $\mu := M_{\tau^{-1}(\alpha)}$ and $\nu := M_{\tau^{-1}(\alpha+1)}$, then we see $\mathrm{sg}((\alpha \;\; \alpha{+}1)_{M^{\tau}_{\ell}}) = (-1)^{\mu_{\ell} \nu_{\ell}}$ and $\mathrm{sg}(\imath') = (-1)^{\mu_{\ell} \nu_{\ell}} \cdot \mathrm{sg}(\imath)$. Hence we get the calculation

$$
\begin{aligned}
&(-1)^{\alpha} \cdot \mathrm{sg}(\tau') \cdot \prod_{\ell=1}^{n} \mathrm{sg}(\tau'_{M_\ell}) \cdot \mathrm{sg}(\imath'_\ell) \\
&\quad = (-1)^{\alpha} \cdot \mathrm{sg}(\tau) \cdot \prod_{\ell=1}^{n} \mathrm{sg}((\alpha\ \ \alpha{+}1)_{M_\ell^\tau}) \cdot \mathrm{sg}(\tau_{M_\ell}) \cdot (-1)^{\mu_\ell \nu_\ell} \cdot \mathrm{sg}(\imath_\ell) \\
&\quad = -(-1)^{\alpha} \cdot \mathrm{sg}(\tau) \cdot \prod_{\ell=1}^{n} \mathrm{sg}(\tau_{M_\ell}) \cdot \mathrm{sg}(\imath_\ell). \qquad \square
\end{aligned}
$$

By Remark B.8, the dual discrete Morse flow on the dualised and mirrored chain complex $C_{\mathrm{cell}}^{|K|+r-\bullet}(\tilde{V}^{\blacktriangle}, \tilde{V}^{\vartriangle}; R)$ yields again a formal derived complex, generated by the (dual) essential cells. Using once again Poincaré–Lefschetz duality, we obtain the following statement:

**Corollary 4.3.22.** $H_{r-s}(\tilde{V}_f; R) \cong H_{r-s}(C_{\mathrm{cell}}^{|K|+r-\bullet}(\tilde{V}^{\blacktriangle}, \tilde{V}^{\vartriangle}; R))$ *is freely generated by the essential cells in* $\Pi_{f,s}$*.*

This result can be compared with the homology calculations from Theorem 1.2.5 in the following way:

**Remark 4.3.23.** A cell $\pi \in \Pi_{f,s}$ is essential if and only if $Q_\pi(i) = C_\pi(i)$ holds for all $i$, and for all $i$ with $a_\pi(i) \neq 1$, there is an $i' < i$ with $a_\pi(i') = a_\pi(i) - 1$.

For $n = 1$, this recovers Theorem 1.2.5: if $(\mathcal{Q}_1, \dots, \mathcal{Q}_l)$ is a ray partition of agility $1 \leqslant s \leqslant r$ with $\boldsymbol{\sigma}(\mathcal{Q}) = \mathbf{id}$, then we can construct an essential cell out of it: we can write $\{1, \dots, l\} = \{\beta_{1,1}, \dots, \beta_{s,h_s}\}$ such that $\beta_{1,1} < \cdots < \beta_{s,1}$ and $\beta_{a,\mathscr{b}} < \beta_{a,\mathscr{b}'}$ and that $(\{\beta_{a,1}, \dots, \beta_{a,h_a}\})_{1 \leqslant a \leqslant s}$ are exactly the equivalence classes with respect to the agility relation in Notation 1.2.2.

Then we let $M := (m_1, \dots, m_s)$ with $m_a := \sum_{\mathscr{b}} \#\mathcal{Q}_{\beta_{a,\mathscr{b}}}$ and for each $a$, we let $\varphi_a \colon \coprod_{\mathscr{b}} \mathcal{Q}_{\beta_{a,\mathscr{b}}} \to \mathbb{Y}_M^a$ be the unique bijection that stacks $\mathcal{Q}_{\beta_{a,\mathscr{b}}}$ below $\mathcal{Q}_{\beta_{a,\mathscr{b}+1}}$. Moreover, consider $\pi_\beta \colon \mathcal{Q}_\beta \to \underline{r}$ with $\pi_\beta(i, \jmath) = i$ for each $\beta$. Then we let

$$
\pi := \left(\textstyle\coprod_a \coprod_{\mathscr{b}} \pi_{\beta_{a,\mathscr{b}}}\right) \circ \left(\textstyle\coprod_a \varphi_a^{-1}\right) \colon \mathbb{Y}_M \to \underline{r}.
$$

It is easy to see that $\pi$ is essential, and that this construction gives a bijection between ray partitions $\mathcal{Q}$ of agility $s$ with $\boldsymbol{\sigma}(\mathcal{Q}) = \mathbf{id}$ and essential cells with $s$ columns, yielding a generator in homology of degree $r - s$.

## 4.4. The homology of $\mathcal{V}_{1,1}$-algebras

In this section, we describe the algebraic operad $H_\bullet(\mathcal{V};R)$ for $\mathcal{V} := \mathcal{V}_{1,1}$ in terms of generators and relations in the sense of Section 3.2. By doing so, we classify all Künneth operations (with coefficients in $R$) that $\mathcal{V}$ imposes on all its algebras. The result will be quite similar to the classical calculations [CLM76, § III] for the little $d$-cubes operad.

### 4.4.1. *Generators*

The aim of this subsection is to give an explicit system of generators for the algebraic operad $H_\bullet(\mathcal{V})$. While nearly all generators are ground classes of certain path components, there is a single family of generators in degree 1, which generalises the idea of a Browder bracket from Example 3.2.13.

**Definition 4.4.1** (Vertical Browder bracket). For each $n, n' \geqslant 1$, we consider the circle of vertical configurations

$$\begin{aligned}\gamma_{n,n'} \colon \mathbb{S}^1 &\to \tilde{V}_{n,n'}(\mathbb{R}^{1,1} \times \underline{n+n'-1}),\\ (x,y) &\mapsto \big((x,y,1),\dots,(x,y,n);(0,0,1),(0,0,n+1)\dots,(0,0,n+n'-1)\big).\end{aligned}$$

If we choose the standard orientation on $\mathbb{S}^1$ and let $[\mathbb{S}^1] \in H_1(\mathbb{S}^1)$ be its fundamental class, then $(\gamma_{n,n'})_*[\mathbb{S}^1]$ generates the first homology of the respective path component of $\tilde{V}_{n,n'}(\mathbb{R}^{1,1} \times \underline{n+n'-1})$.

Using the equivalence $\Psi \colon \tilde{V}_{n,n'}(\mathbb{R}^{1,1} \times \underline{n+n'-1}) \to \mathcal{V}\binom{n,n'}{n+n'-1}$, we define the *vertical Browder bracket* $\mathfrak{b}_{n,n'} := (\Psi \circ \gamma_{n,n'})_*[\mathbb{S}^1]$, see Figure 4.11.

Now we can already formulate the main theorem of this subsection:

**Theorem 4.4.2.** *The operad $H_\bullet(\mathcal{V})$ is generated by the following classes:*

1. *the ground class $\mathfrak{v}_1 \in H_0(\mathcal{V}\binom{}{1})$, called the 'void';*
2. *for each $f \colon k \to n$ in $\mathbf{\Delta\Sigma}$, the ground class $f \in H_0(\mathcal{V}\binom{k}{n}_f)$;*
3. *for each $n, n' \geqslant 1$, the ground class $\mathfrak{u}_{n,n'} \in H_0(\mathcal{V}\binom{n,n'}{n+n'})$ of the component containing the universal morphism;*
4. *for each $n, n' \geqslant 1$, the vertical Browder bracket $\mathfrak{b}_{n,n'} \in H_1(\mathcal{V}\binom{n,n'}{n+n'-1})$.*

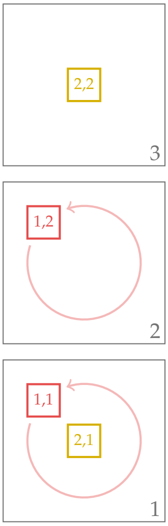

**Figure 4.11.** The loop $\Upsilon \circ \gamma_{2,2} \colon \mathbb{S}^1 \to \mathscr{V}\binom{2,2}{3}$

This formally means the following: we consider the free symmetric quiver $\mathscr{A}$ over the set of these classes, e.g. $\mathscr{A}\binom{n,n'}{n+n'-1} = R\langle \mathfrak{b}_{n,n'}, (1\,2)^*\mathfrak{b}_{n,n'} \rangle$. Then we get a map of quivers $\mathscr{A} \to H_\bullet(\mathscr{V})$, and the theorem claims that its adjoint $\Upsilon(\mathscr{A}) \to H_\bullet(\mathscr{V})$ is surjective for each colour profile.

In order to prove the theorem, we make use of the Morse flow from the previous subsection. Therefore, we have to relate the cellular description of the operadic structure from Subsection 4.3.2 to the derived complex coming from the Morse flow from Subsection 4.3.3.

**Notation 4.4.3.** Throughout the subsection, we write $\bar{C}_\bullet := C_{\text{cell}}^{|K|+r-\bullet}(\tilde{V}^\blacktriangle, \tilde{V}^\vartriangle)$ whenever it is clear which input profile, which output colour, and which path component we consider.

Moreover, we let $\bar{\Omega}_h$ be the set of all (dual) cells of codimension $h$, which we shall denote by $\pi$ as well. By Remark B.8, the Morse flow from Subsection 4.3.3 gives rise to a dual Morse flow on $\bar{\Omega}_\bullet$, which we denote by $\Lambda$ as well, and where the rôles of collapsible and redundant cells are switched.

The derived complex $\bar{C}_\bullet^\Lambda$ is formal and we have a pair of chain homotopy equivalences $\varphi^\Lambda \colon \bar{C}_\bullet \rightleftarrows \bar{C}_\bullet^\Lambda : \psi^\Lambda$. Since $\bar{C}_\bullet^\Lambda$ is formal, the chain map $\psi^\Lambda$ attains values in the cycles of $\bar{C}_\bullet$.

For a cycle $c \in \bar{C}_h$, we denote by $[c] \in H_h(\mathcal{V})$ the induced homology class, and for an essential cell $\eta \in \bar{\Omega}_h^{\mathrm{ess}}$, we denote by $[\![\eta]\!] := [\psi^\Lambda \eta]$ the homology class represented by $\psi^\Lambda \eta \in \bar{C}_h$. If $\pi$ is collapsible, then we set $[\![\pi]\!] := 0$.

Let us start with a general observation which actually holds for each based chain complex, together with a discrete Morse flow on it:

**Remark 4.4.4.** Let $\eta \in \bar{\Omega}_h^{\mathrm{ess}}$ be an essential cell and let $c \in \bar{C}_h$ be a linear combination of collapsible cells such that $\eta + c \in C_h$ is a cycle. Then we see

$$[\eta + c] = \psi_*^\Lambda \varphi_*^\Lambda [\eta + c] = [\psi^\Lambda \eta] = [\![\eta]\!].$$

Now we can start with the first ingredient of the proof of Theorem 4.4.2, which decomposes homology classes which are not of the highest dimension into classes of smaller arity:

**Proposition 4.4.5.** *Let $r \geqslant 2$ and $h \leqslant r - 2$ and let $x \in H_h(\mathcal{V}\binom{K}{n})$ be an additive generator. Then there are:*

1. *a decomposition $\{1, \dots, r\} = A \,\dot{\cup}\, A'$ into two non-empty subsets,*
2. *integers $k, k' \geqslant 1$ and $y \in H_{\#A-1}(\mathcal{V}\binom{K|_A}{k})$ and $y' \in H_{h-\#A+1}(\mathcal{V}\binom{K|_{A'}}{k'})$,*
3. *a map $f \colon \underline{k + k'} \to \underline{n}$ in $\mathbf{\Delta\Sigma}$, and*
4. *a permutation $\tau \in \mathfrak{S}_r$,*

*such that $\tau^* x = \pm f \circ \mathrm{u}_{k,k'} \circ (y \otimes y')$ holds.*

The proof of Proposition 4.4.5 requires some preparing lemmata, for which it may be useful to repeat the constructions from Subsection 4.3.2.

**Lemma 4.4.6.** *Let $f \colon \underline{k + k'} \to \underline{n}$ be a map in $\mathbf{\Delta\Sigma}$ such that the two restrictions $f|_{\underline{k}}$ and $f|_{\underline{k'}}$ are injective. Let $\eta$ be an essential cell of $\mathcal{V}\binom{K}{k}$ and $\eta'$ be an essential cell of $\mathcal{V}\binom{K'}{k'}$. Then $(\eta \sqcup \eta') \circ f^*$ is an essential cell in $\mathcal{V}\binom{K \cdot K'}{n}$ and we have*

$$f \circ \mathrm{u}_{k,k'} \circ ([\![\eta]\!] \otimes [\![\eta']\!]) = \pm [\![(\eta \sqcup \eta') \circ f^*]\!].$$

*Proof.* Since $f|_{\underline{k}}$ and $f|_{\underline{k'}}$ are both injective, the columns of $\bar{\eta} := (\eta \sqcup \eta') \circ f^*$ agree, up to layer permutation, empty layers, and shifts of all cluster numbers inside the rear columns, with the ones of $\eta$ and $\eta'$. Therefore, there is no $1 \leqslant i \leqslant r + r'$ with $Q_{\bar{\eta}}(i) \subsetneq C_{\bar{\eta}}(i)$. Secondly, if $a_{\bar{\eta}}(i) \neq 1$, then there is an $i' < i$ with $a_{\bar{\eta}}(i') = a_{\bar{\eta}}(i) - 1$: the only interesting situation is the case where the $i^{\text{th}}$ cluster lies in the first column that comes from $\eta'$; however, in this case, $a_{\bar{\eta}}(i) - 1$ is the last column that comes from $\eta$ and, thus, contains at least one cluster with number $i' \leqslant r < i$. This shows that $\bar{\eta}$ is essential.

Moreover, we can write $\psi^\Lambda \eta = \eta + \sum_b \mu_b \cdot \pi_b$ and $\psi^\Lambda \eta' = \eta' + \sum_{b'} \mu'_{b'} \cdot \pi'_{b'}$, where $\pi_b$ and $\pi'_{b'}$ are collapsible, and $\mu_b$ and $\mu'_{b'}$ are non-trivial coefficients. It is now easy to see that cells of the form $(\eta \sqcup \pi'_{b'}) \circ f^*$ or $(\pi_b \sqcup \eta') \circ f^*$ or $(\pi_b \sqcup \pi_{b'}) \circ f^*$ are again collapsible, as in all these cases, there is an index $1 \leqslant i \leqslant r + r'$ such that the corresponding ray is not the entire column.

By Lemma 4.3.10 and Lemma 4.3.11, we obtain that $f \circ \mathrm{u}_{k,k'} \circ (\llbracket \eta \rrbracket \otimes \llbracket \eta' \rrbracket)$ is represented by the cocycle $f_* U_{k,k'}(\psi^\Lambda \eta \otimes \psi^\Lambda \eta')$, which is of the form

$$\pm\bar{\eta} + \underbrace{\sum_b \pm\mu_b \cdot (\pi_b \sqcup \eta')f^* + \sum_{b'} \pm\mu'_{b'} \cdot (\eta \sqcup \pi'_{b'})f^* + \sum_{b,b'} \pm(\mu_b \mu'_{b'}) \cdot (\pi_b \sqcup \pi'_{b'})f^*}_{=:c}.$$

Now note that $c$ is a linear combination of collapsible cells. By Remark 4.4.4, we get $[\pm\bar{\eta} + c] = \pm\llbracket \bar{\eta} \rrbracket$, as desired. □

**Lemma 4.4.7.** *Let $\eta$ be an essential cell and let $a_\eta^{-1}(1) = \{1 = i_1 < \cdots < i_t\}$. Now we consider the permutation $\tau := (i_2\ i_2{-}1 \cdots 2) \cdots (i_t\ i_t{-}1 \cdots t) \in \mathfrak{S}_r$. Then $\bar{\eta} := \tau^{-1} \circ \eta$ is essential with $a_{\bar{\eta}}^{-1}(1) = \{1, \ldots, t\}$, and we have $\tau^* \llbracket \eta \rrbracket = \pm \llbracket \bar{\eta} \rrbracket$.*

Before proving Lemma 4.4.7, let us prove Proposition 4.4.5 with the help of Lemma 4.4.7.

*Proof of Proposition 4.4.5.* There is an essential dual cell $\eta$ of codimension $h$ such that $x = \pm\llbracket \eta \rrbracket$. Since we have $h \leqslant r - 2$, the cell $\eta$ must have at least two columns. Let $A := a_\eta^{-1}(1)$ and $A' := \{1, \ldots, r\} \smallsetminus A$ and define $\tau \in \mathfrak{S}_r$ as in the previous Lemma 4.4.7, as well as $\bar{\eta} := \tau^{-1} \circ \eta$. By Lemma 4.4.7, $\bar{\eta}$ is again essential and $\tau^* x = \pm\tau^* \llbracket \eta \rrbracket = \pm \llbracket \bar{\eta} \rrbracket$.

By looking only at the first column of $\bar{\eta}$ and removing empty layers, we obtain an essential cell $\vartheta$ of $\mathcal{V}\binom{K|_A}{k}$, as well as an injective map $g \colon \underline{k} \hookrightarrow \underline{n}$.

Similarly, we look at the remaining columns of $\bar{\eta}$ and shift all cluster numbers down. By removing empty layers, we obtain an essential cell $\vartheta'$ of $\mathscr{V}\binom{K|_{A'}}{k'}$, as well as an injective map $g' \colon \underline{k'} \hookrightarrow \underline{n}$.

If we let $f := g \sqcup g' \colon \underline{k+k'} \cong \underline{k} \sqcup \underline{k'} \to \underline{n}$, then both restrictions $f|_{\underline{k}}$ and $f|_{\underline{k'}}$ are injective and we get $\bar{\eta} = (\vartheta \sqcup \vartheta') \circ f^*$. If we finally set $y := [\![\vartheta]\!]$ and $y' := [\![\vartheta']\!]$, then by Lemma 4.4.6, we get $x = \pm[\![\bar{\eta}]\!] = \pm f \circ \mathfrak{u}_{k,k'} \circ (y \otimes y')$. $\square$

In order to prove Lemma 4.4.7, we need a further auxiliary statement, which describes what $\psi^\Lambda \eta$ looks like for an essential cell $\eta$:

**Lemma 4.4.8.** *Let $\eta$ be an essential cell and let $\pi$ be a summand of the cycle $\psi^\Lambda \eta$. Then we have $a_\pi = a_\eta$.*

*Proof.* Let $(\Gamma^\Lambda, \lambda^\Lambda)$ be the modified graph from Definition B.4. Using the notation from Construction B.6, we show $\mu^\Lambda_{\eta,\pi} = \sum_{\gamma \colon \eta \rightsquigarrow \pi} \lambda^\Lambda(\gamma) = 0$ for $a_\pi \neq a_\eta$.

To do so, let $T$ be the set of all collapsible cells $\rho$ with $a_\rho \neq a_\eta$ such that there is a path $\gamma_1 \colon \eta = \pi_0 \searrow \cdots \nearrow \pi_{2m} = \rho$ with $a_{\pi_{2l}} = a_\eta$ for $l < m$. For each $\rho \in M$, we let $P(\rho)$ be the set of all such paths. If $\pi$ is any cell with $a_\pi \neq a_\eta$, then there is a bijection between the set of all paths $\gamma \colon \eta \rightsquigarrow \pi$ and the set of all triples $(\rho, \gamma_1, \gamma_2)$ with $\rho \in T$, $\gamma_1 \in P(\rho)$, and $\gamma_2 \colon \rho \rightsquigarrow \pi$, by setting $\rho$ to be the first collapsible cell along $\gamma$ with $a_\rho \neq a_\eta$. Then we see

$$\mu^\Lambda_{\eta,\pi} = \sum_{\rho \in T} \underbrace{\Bigg( \sum_{\gamma_1 \in P(\rho)} \lambda^\Lambda(\gamma_1) \Bigg)}_{=:\lambda(\rho)} \cdot \Bigg( \sum_{\gamma_2 \colon \rho \rightsquigarrow \pi} \lambda^\Lambda(\gamma_2) \Bigg).$$

It hence suffices to show that $\lambda(\rho) = 0$ holds for each $\rho \in T$. To this aim, we construct a pairing on $P(\rho)$ that matches two paths with a different sign: for each path $\gamma_1 \in P(\rho)$, consider its final two segments $\pi_{2m-2} \searrow \pi_{2m-1} \nearrow \rho$. We let $(M, \pi) := \pi_{2m-1}$ and let $i := i_{\mathrm{red}}(\pi)$, as well as $\alpha = a_\pi(i) - 1 \geqslant 1$. Be aware that the notions of being redundant and being collapsible have been toggled when passing to the dual complex, i.e. we have $Q_\pi(i) = C_\pi(i)$. Then $\rho = d_{\alpha,\imath}\pi$, where $\imath$ is the tuple of standard shuffles, see Definition 1.4.6.

Moreover, there is an index $1 \leqslant \beta \leqslant s$ and an $(M_\beta, M_{\beta+1})$-shuffle $\jmath$ with $\pi_{2m-2} = d_{\beta,\jmath}\pi$. We argue that $\beta = \alpha - 1$: as $a_{\pi_{2m-2}} = a_\eta \neq a_\rho$, we have $\beta \neq \alpha$. If $\beta = \alpha + 1$, then $\min(a_\eta^{-1}(\beta)) = i$, while $\min(a_\eta^{-1}(\beta - 1)) > i$. Moreover,

$Q_\eta(i) = C_\eta(i)$, since $\eta$ is not collapsible, and $a_\eta(i) = \alpha + 1 \geqslant 2$, whence $\eta$ is redundant: this is not possible. The same issue occurs if two completely different neighboured columns are merged, which excludes the remaining cases in which $\beta \neq \alpha - 1$.

In order to simplify the notation a bit, let us denote the columns of the cell $\pi$ by $A\colon \mathbb{Y}_M^{\alpha-1} \to \underline{r}$, $B\colon \mathbb{Y}_M^{\alpha} \to \underline{r}$ and $C\colon \mathbb{Y}_M^{\alpha+1} \to \underline{r}$. Then, schematically, the path $\gamma_1$ looks as follows:

$$\begin{tikzcd}
\eta \arrow[dr] & & (*,\jmath(A,B),C,*) \arrow[dr] & & (*,A,\genfrac{}{}{0pt}{}{C}{B},*) \\
& \cdots \arrow[ur] & & (*,A,B,C,*) \arrow[ur] &
\end{tikzcd}$$

We call $\gamma_1$ of *type 1* if $\jmath$ is the *complementary* standard shuffle, i.e. it puts $A$ on top of $B$, and of *type 2* otherwise, compare Definition 1.4.6.

Consider the cell $\bar\pi = (*,B,A,C,*)$ which arises from $\pi$ by switching the $(\alpha-1)^{\text{st}}$ and the $\alpha^{\text{th}}$ column. We claim that $\pi'$ is again redundant: let $i'$ be the minimum of $\operatorname{im}(A)\,\dot\cup\,\operatorname{im}(B) \subseteq \underline{r}$. Since $a_\eta = a_{\pi_{2m-2}}$ holds and $\eta$ is essential, we have $i' < i$. However, $i'$ cannot lie in $B$, as otherwise, $i_{\text{red}}(\pi) \leqslant i' < i$. Therefore, $i'$ is the minimum of $\operatorname{im}(A)$. Moreover, $Q_{\bar\pi}(i') = C_{\bar\pi}(i')$ holds, as otherwise, $Q_\pi(i') \subsetneq C_\pi(i')$ and $i_{\text{coll}}(\pi) \leqslant i' < i$. As no further columns are affected, $\bar\pi$ is indeed redundant and its collapsible partner can be written as $\bar\pi^\sharp = (*,\genfrac{}{}{0pt}{}{A}{B},C,*)$.

If $\gamma_1$ is of type 1, then $\pi_{2m-2} = \bar\pi^\sharp$ holds; in particular, $\pi_{2m-3} = \bar\pi$. Since $a_{\pi_{2m-4}} = a_{\pi_{2m-2}}$, there is a shuffle $\bar\jmath$ with $\pi_{2m-4} = d_{\beta,\bar\jmath}\bar\pi = (*,\bar\jmath(B,A),C,*)$. In this case, we can shorten the path $\gamma_1$ as follows:

$$\begin{tikzcd}
\eta \arrow[dr] & & (*,\bar\jmath(B,A),C,*) \arrow[dr] \arrow[drrr, crossing over] & & (*,\genfrac{}{}{0pt}{}{A}{B},C,*) \arrow[dr] & & (*,A,\genfrac{}{}{0pt}{}{C}{B},*) \\
& \cdots \arrow[ur] & & (*,B,A,C,*) \arrow[ur] & & (*,A,B,C,*) \arrow[ur] &
\end{tikzcd}$$

One checks that the green arrow carries the sign $(-1)^{\alpha-1} \cdot \operatorname{sg}(\bar\jmath) \cdot \prod_\ell (-1)^{\#A_\ell \cdot \#B_\ell}$ while the alternative concatenation of three black arrows has exactly the opposite sign, using that the signs of the arrows which go upwards are toggled in the Morse graph $\Gamma^\Lambda$. Conversely, if $\gamma_1$ is of type 2, then $\pi_{2m-2} \neq \bar\pi^\sharp$ and we can prolongate the path $\gamma_1$ as follows:

$$\begin{tikzcd}
\eta \arrow[dr] & & (\ast, \jmath(A,B), C, \ast) \arrow[dr, red] \arrow[drrr] & & {\color{red}(\ast, \genfrac{}{}{0pt}{}{A}{B}, C, \ast)} \arrow[dr, red] & & (\ast, A, \genfrac{}{}{0pt}{}{C}{B}, \ast) \\
& \cdots \arrow[ur] & & {\color{red}(\ast, B, A, C, \ast)} \arrow[ur, red] & & (\ast, A, B, C, \ast) \arrow[ur] &
\end{tikzcd}$$

These constructions are clearly inverses of each other, yielding the desired pairing on $P(\rho)$. ☐

Now we can give the outstanding proof of Lemma 4.4.7, which then also concludes the proof of Proposition 4.4.5.

*Proof of Lemma* 4.4.7. Recall that we wrote $\bar{\eta} := \tau^{-1} \circ \eta$; hence $a_{\bar{\eta}} = a_\eta \circ \tau$ and one readily checks that $\tau(l) = i_l$ for $1 \leqslant l \leqslant t$, and so $a_{\bar{\eta}}^{-1}(1) = \{1, \dots, t\}$. Now let $\pi$ be any cell with $a_\pi^{-1} = \{i_1 < \cdots < i_t\}$. Then, in $\tau^{-1} \circ \pi$, every cluster number outside the first column stays as it is or increases by one. In particular, the rays stay as they are, and the ordering of the columns by their minima remains. This shows that if $\eta$ is essential, then also $\tau^{-1} \circ \eta = \bar{\eta}$ is essential, and if $\pi$ is collapsible, then $\tau^{-1} \circ \pi$ is collapsible as well.

We have $\psi^\Lambda \eta = \eta + c$, where $c = \sum_b \mu_b \cdot \pi_b$ is a linear combination of collapsible cells. By Lemma 4.4.8, we have $a_{\pi_b} = a_\eta$ for each summand $\pi_b$, in particular, $a_{\pi_b}^{-1} = \{i_1 < \cdots < i_t\}$ holds. Moreover, $\tau^*[\![\eta]\!] = \tau^*[\psi^\Lambda \eta]$ is via Lemma 4.3.9 represented by $\pm\bar{\eta} + \sum_b (\pm\mu_b) \cdot (\tau^{-1} \circ \pi)$. By the previous paragraph, $\tau^{-1} \circ \pi$ is again essential. Finally, by Remark 4.4.4, the homology class represented by the above cycle coincides with $\pm[\![\bar{\eta}]\!]$. ☐

We are left to decompose top-dimensional cells into our desired generators. Here we have to deal only with a single column, which facilitates some of our combinatorial considerations.

**Remark 4.4.9.** If we have a single column, then the multi-layered $\mathbb{Y}_M$ can be written as a usual tableau

$$\mathbb{Y}_M = \{(\ell, \jmath); 1 \leqslant \ell \leqslant n \text{ and } 1 \leqslant \jmath \leqslant m_\ell\},$$

and for each cell $\pi$ of $\mathscr{V}\binom{K}{n}_f$ and each $S = \{i_1 < \cdots < i_s\}$, we have a map $f_{S,\pi} \in \mathbf{\Delta\Sigma}\binom{k_{i_1} + \cdots + k_{i_s}}{n}$, where the ordering of the fibres reflects the ordering of the points inside the single column.

In order to run inductive arguments, we have to partially order the set of cells, and therefore introduce a measure for them:

**Construction 4.4.10.** For each path component $f \in \mathbf{\Delta\Sigma}^{\times}\binom{K}{n}$, there is a specific top-dimensional cell $\pi_f \colon \mathbb{Y}_M \to \underline{r}$ such that, for each $1 \leqslant \mathscr{b} < \mathscr{b}' \leqslant m_\ell$, we have $\pi_f(\ell, \mathscr{b}) \leqslant \pi_f(\ell, \mathscr{b}')$. For an arbitrary cell $\pi$ in $\mathcal{V}\binom{K}{n}_f$, we define its *height*

$$\mathrm{ht}(\pi) := \#\big\{(\ell, \mathscr{b}, \mathscr{b}');\ \mathscr{b} < \mathscr{b}' \text{ and } \pi(\ell, \mathscr{b}) > \pi(\ell, \mathscr{b}')\big\}.$$

Intuitively, the height of $\pi$ measures the distance from $\pi$ to $\pi_f$, and we have $\pi = \pi_f$ if and only if $\mathrm{ht}(\pi) = 0$. We note the following:

1. It is easy to see that $\pi_f$ is essential if and only if, for each $1 \leqslant i \leqslant r$, there is an $i' < i$ and a layer $\ell$ containing both $i$ and $i'$.
2. Let $\pi \neq \pi_f$ be a cell. Then there is a smallest pair $(\ell, \mathscr{b}')$, with respect to the lexicographic order on $\mathbb{Y}_M$, such that there is a $\mathscr{b} < \mathscr{b}'$ satisfying $\pi(\ell, \mathscr{b}) > \pi(\ell, \mathscr{b}')$, and in this case, $\mathscr{b}$ can be chosen to be $\mathscr{b}' - 1$, since if $\pi(\ell, \mathscr{b}' - 1) \leqslant \pi(\ell, \mathscr{b}')$, then $\mathscr{b}'$ would not have been minimal.

Our strategy is as follows: given a top-dimensional essential cell $\eta$ of height $t$, we want to write $[\![\eta]\!]$ as a sum of $[\![\bar{\eta}]\!]$ and an element in the operadic span, such that $\bar{\eta}$ has height $t - 1$.

For the induction start, we have to write $[\![\pi_f]\!]$ itself as an element in the operadic span. To do so, we construct a family of toric classes, which can be decomposed into the vertical Browder brackets from Definition 4.4.1.

**Construction 4.4.11.** For each $n \geqslant 1$ we consider the map

$$\begin{aligned} \gamma_n \colon \mathbb{S}^1 &\to \tilde{V}_{n,n}(\mathbb{R}^{1,1} \times \underline{n}), \\ (x,y) &\mapsto ((0,0,1), \dots, (0,0,n); (x,y,1), \dots, (x,y,n)), \end{aligned}$$

and under the homotopy equivalence $\Psi \colon \tilde{V}_{n,n}(\mathbb{R}^{1,1} \times \underline{n}) \to \mathcal{V}\binom{n,n}{n}$, we let $\mathfrak{b}_n := (\Psi \circ \gamma_n)_*[\mathbb{S}^1] \in H_1(\mathcal{V}\binom{n,n}{n})$. Then $\mathfrak{b}_n$ can be decomposed into vertical Browder brackets as follows: if we let $\mathfrak{b}_n^\ell := g_\ell \circ \mathfrak{b}_{n,n} \circ ((1 \cdots \ell) \otimes (1 \cdots \ell))$ with $g_\ell := (n{+}1\ \ 2 \,|\, \cdots \,|\, n{+}\ell{-}1\ \ \ell \,|\, 1 \,|\, \ell{+}1\ \ n{+}\ell \,|\, \cdots \,|\, n\ \ 2n{-}1) \in \mathbf{\Delta\Sigma}\binom{2n-1}{n}$, then we have $\mathfrak{b}_n = \mathfrak{b}_n^1 + \cdots + \mathfrak{b}_n^n$, as one can check 'by hand', by tracking which edges of the dual cell complex are crossed.

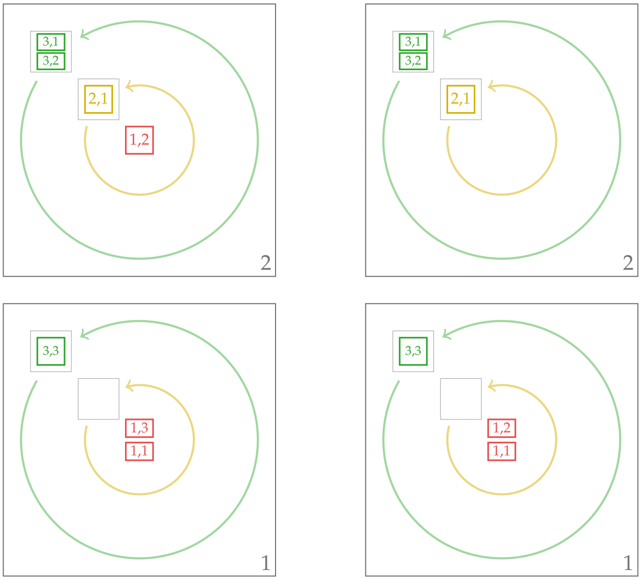

**Figure 4.12.** The two-dimensional toric class $\mathfrak{b}_f$, once for $f = (1\,3\,|\,2\,,\,|\,1\,,\,3\,|\,2\,1)$ and once for $f = (1\,2\,|\,,\,|\,1\,,\,3\,|\,2\,1)$. The cell $\pi_f$ is crossed when all boxes are aligned at the bottom of their respective circles. In the first case, the cell $\pi_f$ is essential, while in the second case, the ray $Q_{\pi_f}(1)$ does not include the box 2,1.

For each $f \in \boldsymbol{\Delta\Sigma}^{\times}\binom{K}{n}$ we define a class $\mathfrak{b}_f \in H_{r-1}(\mathscr{V}\binom{K}{n}_f)$ recursively by

$$\begin{aligned}\mathfrak{b}_{f_1} &:= f_1,\\ \mathfrak{b}_{f_1,\dots,f_{s+1}} &:= \mathfrak{b}_n \circ (\mathfrak{b}_{f_1,\dots,f_s} \otimes f_{s+1}).\end{aligned}$$

Then $\mathfrak{b}_f$ lies by construction in the operadic span and can geometrically be realised by an embedding $\gamma_f \colon (\mathbb{S}^1)^{r-1} \hookrightarrow \tilde{V}_f$ which has $2^{r-1}$ transversal intersections with cells of codimension $r-1$, namely at $\gamma_f((0,\pm 1),\dots,(0,\pm 1))$.

It is an easy (and actually quite satisfying) exercise left to the reader to check that there is only one intersection with a cell which has a chance to be essential, namely $\pi_f$, which is hit in $\gamma_f((0,1),\dots,(0,1))$, see Figure 4.12. We therefore obtain $[\![\pi_f]\!] = \pm\mathfrak{b}_f$.

Now we can carry out the advertised inductive argument:

**Proposition 4.4.12.** *Let $\#K = r$ and $x \in H_{r-1}(\mathcal{V}\binom{K}{n})$. Then $x$ lies in the operadic span of the generators from Theorem 4.4.2.*

*Proof.* We proceed by induction on the arity $r$. For $r = 1$, the statement is clear since $H_0(\mathcal{V}\binom{k}{n}) = R\langle \mathbf{\Delta\Sigma}\binom{k}{n}\rangle$.

For the induction step '$r-1 \to r$' we do the following: we can assume that $x = [\![\eta]\!]$ for an essential cell in $\mathcal{V}\binom{K}{n}_f$. Now we start a second induction on the height $t := \mathrm{ht}(\eta)$. For $t = 0$, we have $\eta = \pi_f$, and we have already seen that $[\![\pi_f]\!] = \pm\mathfrak{b}_f$. For the induction step '$t-1 \to t$', let $(\ell_0, \mathscr{b}'_0)$ be the minimal pair from Construction 4.4.10, and let $\bar\eta$ be the cell where $\eta(\ell_0, \mathscr{b}'_0 - 1)$ and $\pi(\ell_0, \mathscr{b}'_0)$ are toggled. Then $\mathrm{ht}(\bar\eta) = \mathrm{ht}(\eta) - 1$, so either $\bar\eta$ is collapsible and $[\![\bar\eta]\!] = 0$, or $\bar\eta$ is essential, and by the (inner) induction hypothesis, $[\![\bar\eta]\!]$ lies in the operadic span.

Now let $i' := \eta(\ell_0, \mathscr{b}'_0)$ and $i := \eta(\ell_0, \mathscr{b}'_0 - 1)$, i.e. $i' < i$, and consider the permutation $\tau := (i'{+}1 \cdots i) \in \mathfrak{S}_r$. Then one readily checks that $\tau^{-1} \circ \eta$ is again essential and $\tau^*[\![\eta]\!] = [\![\tau^{-1} \circ \eta]\!]$ holds, and moreover, $\tau^{-1} \circ \bar\eta$ is essential if and only if $\bar\eta$ is, and in this case we have $[\![\tau^{-1} \circ \bar\eta]\!] = \tau^*[\![\bar\eta]\!]$.

There are $1 \leqslant \jmath \leqslant k_i$ and $1 \leqslant \jmath' \leqslant k_{i'}$ such that $(\ell_0, \mathscr{b}'_0)$ corresponds to $(i', \jmath')$ and $(\ell_0, \mathscr{b}'_0 - 1)$ corresponds to $(i, \jmath)$. We let $\sigma' := (1 \cdots \jmath') \in \mathfrak{S}_{k_{i'}}$ and $\sigma := (1 \cdots \jmath) \in \mathfrak{S}_{k_i}$, and we let $g := (g_1, \dots, g_{r-1})$, where $g_j = f_{d^i j}$ for $j \neq i'$, and $g_{i'} \in \mathbf{\Delta\Sigma}\binom{k_{i'}+k_i-1}{n}$ is determined by $g_{i'} \circ (1 \cdots k_{i'}) \circ s^{k_{i'}} \circ (\sigma' \sqcup \sigma) = f_{\bar\eta,\{i,i'\}}$ as in Remark 4.4.9.

Now we let $K' := (k_1, \dots, k_{i'-1}, k_i + k_{i'} - 1, k_{i'-1}, \dots, \hat{k}_i, \dots, k_r)$ and define a top-dimensional cell $\vartheta$ of $\mathcal{V}\binom{K'}{n}_g$: we decrease $m_{\ell_0}$ by 1 and let

$$\vartheta(\ell, \mathscr{b}) := \begin{cases} (s^{i'} \circ \tau \circ \eta)(\ell_0, d^{\mathscr{b}'_0}\mathscr{b}) & \text{if } \ell = \ell_0, \\ (s^{i'} \circ \tau \circ \eta)(\ell, \mathscr{b}) & \text{else.} \end{cases}$$

Pictorially, the two problematic points are fused into a single one, all points with cluster number $i'$ or $i$ now carry the number $i'$, and all cluster numbers above $i$ are shifted down by 1.

One readily checks that $\vartheta$ is again essential, mainly because we merged two clusters. Since the arity of $\vartheta$ is $r-1$, the outer induction hypothesis tells us that $[\![\vartheta]\!]$ is contained in the operadic span.

Now we employ Lemma 4.3.12: the circle $\mathfrak{b} := \mathfrak{b}_{k_{i'},k_i} \circ (\sigma' \otimes \sigma)$ intersects two cells, called $\delta$ and $\bar{\delta}$, of $\mathscr{V}\binom{k_{i'},k_i}{k_{i'}+k_i-1}$ transversally. Furthermore, the cellular composition $\vartheta \circ_g (\mathbb{1}^{\otimes i'-1} \otimes \delta \otimes \mathbb{1}^{\otimes r-1-i'})$ agrees, up to sign, with $\tau^{-1} \circ \eta$, and the same holds for $\bar{\delta}$ and $\tau^{-1} \circ \bar{\eta}$. If we write $\psi^{\Lambda}\vartheta = \vartheta + \sum_b \mu_b \cdot \pi_b$, then one readily checks that all cells of the form $\pi_b \circ_g (\mathbb{1}^{\otimes i'-1} \otimes \delta \otimes \mathbb{1}^{\otimes r-1-i'})$ are again collapsible, and the same holds for $\bar{\delta}$. This shows by Lemma 4.3.12 that, abbreviating $s := r - i' - 1$, we have

$$\begin{aligned}
&[\![\vartheta]\!] \circ \left(\mathbb{1}^{\otimes i'-1} \otimes \mathfrak{b} \otimes \mathbb{1}^{\otimes s}\right) \\
&\quad = \Big[ \pm \vartheta \circ_g (\mathbb{1}^{\otimes i'-1} \otimes \delta \otimes \mathbb{1}^{\otimes s}) \pm \vartheta \circ_g (\mathbb{1}^{\otimes i'-1} \otimes \bar{\delta} \otimes \mathbb{1}^{\otimes s}) \\
&\qquad + \textstyle\sum_b \mu_b \cdot \big( \pm \pi_b \circ_g (\mathbb{1}^{\otimes i'-1} \otimes \delta \otimes \mathbb{1}^{\otimes s}) \pm \pi_b \circ_g (\mathbb{1}^{\otimes i'-1} \otimes \bar{\delta} \otimes \mathbb{1}^{\otimes s})\big)\Big] \\
&\quad = \pm[\![\tau^{-1} \circ \eta]\!] \pm [\![\tau^{-1} \circ \bar{\eta}]\!] \\
&\quad = \pm\tau^*[\![\eta]\!] \pm \tau^*[\![\bar{\eta}]\!].
\end{aligned}$$

Since both $\tau^*[\![\bar{\eta}]\!]$ and $[\![\vartheta]\!] \circ (\mathbb{1}^{\otimes i'-1} \otimes \mathfrak{b} \otimes \mathbb{1}^{\otimes s})$ lie in the operadic span, the same applies to $\tau^*[\![\eta]\!]$, and hence also to $[\![\eta]\!] = x$. □

This concludes the proof of Theorem 4.4.2 as well:

*Proof of Theorem 4.4.2.* We show that each additive generator $x \in H_h(\mathscr{V}\binom{K}{n})$, can be written as a linear combination of operadic compositions applied to our generators, and we procced by induction over the arity $r := \#K$ of $x$.

For $r = 0$, we have to deal with the ground class $\mathfrak{v}_n$ of $\mathscr{V}\binom{}{n}$. Here we see $\mathfrak{v}_n = f \circ \mathfrak{v}_1$, where $f\colon \underline{1} \to \underline{n}$ is an arbitrary map. For $r = 1$, note that $\mathscr{V}\binom{k}{n}$ has contractible components and $H_0(\mathscr{V}\binom{k}{n}) = R\langle \mathbf{\Delta\Sigma}\binom{k}{n}\rangle$, so $x$ is, up to sign, one of the maps $f\colon k \to n$.

For the induction step '$r - 1 \to r$', we use the previous results: if $h = r - 1$, then we can directly apply Proposition 4.4.12, and if $h \leqslant r - 2$, then we first apply Proposition 4.4.5 in order to decompose $\tau^*x = \pm f \circ \mathfrak{u}_{k,k'} \circ (y \otimes y')$, where both $y$ and $y'$ have strictly smaller arity. By the induction hypothesis, both $y$ and $y'$ lie in the span of our generators. □

Let us close this long subsection with a similar statement for the homology $H_\bullet(\mathscr{V}^c)$ of the connective suboperad.

**Theorem 4.4.13.** *The operad $H_\bullet(\mathcal{V}^c)$ is generated by the following classes:*

1. *the void* $\mathfrak{v}_1 \in H_0(\mathcal{V}_c(_1))$*;*
2. *for each surjective* $f\colon k \to n$ *in* $\mathbf{\Delta\Sigma}$*, the ground class* $f \in H_0(\mathcal{V}^c\binom{k}{n})$*;*
3. *the ground class* $\mathfrak{p}_{n,n'} := s^1 \circ (2 \cdots n{+}1) \circ \mathfrak{u}_{n,n'} \in H_0(\mathcal{V}^c\binom{n,n'}{n+n'-1})$*, for each* $n, n' \geqslant 1$*, called the 'vertical Pontrjagin product';*
4. *for each* $n, n' \geqslant 1$*, the vertical Browder bracket* $\mathfrak{b}_{n,n'} \in H_1(\mathcal{V}^c\binom{n,n'}{n+n'-1})$*.*

In principle, we could have put $s^n \circ \mathfrak{u}_{n,n'}$ for the vertical Pontrjagin product; however, we wanted it to be a class in the same path component as $\mathfrak{b}_{n,n'}$.

The main reason for us to call the classes $\mathfrak{p}_{n,n'}$ 'vertical Pontrjagin products' is the fact that $\mathfrak{p}_{1,1} \in H_0(\mathcal{V}^c\binom{1,1}{1}) = H_0(\mathcal{C}_2(2))$ is the ground class for the classical Pontrjagin product.

*Proof.* One can show 'by hand' that all classes $x \in H_h(\mathcal{V}^c\binom{K}{n})$ with arity $\#K \leqslant 2$ can be expressed in terms of the above generators; here all occurring spaces $\mathcal{V}^c\binom{K}{n}$ are equivalent to graphs. We are left to show that $H_\bullet(\mathcal{V}^c)$ is generated by classes of arity at most 2.

For each $x$ of arity at least 3, an iterated application of Proposition 4.4.5 gives rise to a permutation $\tau \in \mathfrak{S}_r$ which satisfies $\tau^* x = \pm \boldsymbol{f} \circ (x_1, \dots, x_{r-h})$, where $\boldsymbol{f} = (f_1, \dots, f_{r-h}) \in H_0(\mathcal{V}\binom{k'_1,\dots,k'_{r-h}}{n})$, and each $x_l$ is a top-dimensional class without empty layers. It is now easy to see that the decomposition of Proposition 4.4.12 does not create empty layers, and moreover, only path components without empty layers have non-vanishing top homology. Hence, each $x_l$ is connective and of arity strictly smaller than $r$, so by induction, $x_l$ lies in the span of the above generators.

We are left to study the 0-dimensional class $\boldsymbol{f}$. Here we can see that, by the construction of $\boldsymbol{f}$ in Proposition 4.4.5, $\boldsymbol{f}$ itself is connective, and it is an easy, purely combinatorial exercise to check that $\pi_0(\mathcal{V}^c)$ is indeed generated by unaries, i.e. surjective maps in $\mathbf{\Delta\Sigma}$, and the vertical Pontrjagin products. □

### 4.4.2. *Relations*

In the previous subsection, we gave a system of generators for $H_\bullet(\mathcal{V})$ and $H_\bullet(\mathcal{V}^c)$; now we want to find a system of relations between them. Let us start by fixing some notation.

**Notation 4.4.14.** Let $\sigma \in \mathfrak{S}_t$ be a permutation and let $n_1, \ldots, n_t \geqslant 0$ be a collection of non-negative integers. Then we denote by $\sigma_{n_1,\ldots,n_t} \in \mathfrak{S}_{n_1+\cdots+n_t}$ the corresponding *block permutation*.

Secondly, we use the short notation $\rho_{\jmath} := (1 \cdots \jmath) \in \mathfrak{S}_k$ for each $1 \leqslant \jmath \leqslant k$ whenever it is clear from the context what $k$ is.

**Remark 4.4.15** (Relations among ground classes)**.** Recall that all generators of $H_\bullet(\mathcal{V})$, apart from the vertical Browder brackets, are ground classes of a certain path component. Hence, it is a purely combinatorial task to show the following relations:

1. the composition of unaries $g \circ f$, i.e. the $\mathbf{\Delta\Sigma}$-maps in $H_\bullet(\mathcal{V})$, equals the actual composition $gf$ in the category $\mathbf{\Delta\Sigma}$. Since this is already suggested by the notation, we will not mention this any more;
2. if we write $\mathfrak{v}_n = d^1 \circ \mathfrak{v}_{n-1}$ for the empty configuration on $n$ layers, then $f \circ \mathfrak{v}_k = \mathfrak{v}_n$ for each map $f\colon k \to n$;
3. the universal morphisms are associative:
$$\mathfrak{u}_{n+n',n''} \circ (\mathfrak{u}_{n,n'} \otimes \mathbb{1}_{n''}) = \mathfrak{u}_{n,n'+n''} \circ (\mathbb{1}_n \otimes \mathfrak{u}_{n',n''});$$
4. the universal morphisms are commutative up to layer permutation:
$$(1\,2)^*\mathfrak{u}_{n,n'} = (1\,2)_{n',n} \circ \mathfrak{u}_{n',n};$$
5. using the void as one of the arguments for the universal morphism $\mathfrak{u}_{n,1}$ is the same applying a coface: $\mathfrak{u}_{n,1} \circ (\mathbb{1}_n \otimes \mathfrak{v}_1) = d^{n+1}$;
6. unaries can be 'pulled out' of a universal morphism: for $f\colon k \to n$ and $f'\colon k' \to n'$, we have
$$\mathfrak{u}_{n,n'} \circ (f \otimes f') = (f \sqcup f') \circ \mathfrak{u}_{k,k'};$$
7. precomposing with $\mathfrak{u}_{n,n'}$ equalises some of the $\mathbf{\Delta\Sigma}$-maps, as we do not remember the entire order of each fibre, but only its restrictions to $\underline{n}$ and $\underline{n'}$. Formally, if $f\colon n + n' \to m$ is a map in $\mathbf{\Delta\Sigma}$ and $\sigma \in \mathfrak{S}_{n+n'}$ is a permutation such that $\sigma|_{\underline{n}}$ and $\sigma|_{\underline{n'}}$ are monotone, and if $f\sigma$ and $f$ coincide in **Fin** (though perhaps not in $\mathbf{\Delta\Sigma}$), then $f\sigma \circ \mathfrak{u}_{n,n'} = f \circ \mathfrak{u}_{n,n'}$.

The relations involving the vertical Browder brackets are slightly more complicated, mainly because there is one layer which contains two arguments. Let us start with the relations in arity 2.

**Proposition 4.4.16** (Relations including Browder brackets in arity 2)**.** *The following relations hold inside $H_\bullet(\mathcal{V})$:*

1. *the Browder brackets are commutative up to layer permutation:*

$$(1\,2)^*\mathfrak{b}_{n,n'} = (2\,3)_{1,n'-1,n-1} \circ \mathfrak{b}_{n',n};$$

2. *the Browder brackets are cancelled by the void:*

$$\mathfrak{b}_{n,1} \circ (\mathbb{1}_n \otimes \mathfrak{v}_1) = 0;$$

3. *Browder brackets and codegeneracies are related in the following way:*

$$\mathfrak{b}_{n-1,n'} \circ (s^\ell \otimes \mathbb{1}_{n'}) = \begin{cases} s^1 \circ \mathfrak{b}_{n,n'} + s^1\rho_2 \circ \mathfrak{b}_{n,n'} \circ (\rho_2 \otimes \mathbb{1}_{n'}) & \textit{if } \ell = 1, \\ s^\ell \circ \mathfrak{b}_{n,n'} & \textit{else;} \end{cases}$$

4. *Browder brackets and cofaces are related in the following way:*

$$\mathfrak{b}_{n+1,n'} \circ (d^\ell \otimes \mathbb{1}_{n'}) = \begin{cases} 0 & \textit{if } \ell = 1, \\ d^\ell \circ \mathfrak{b}_{n,n'} & \textit{else;} \end{cases}$$

5. *Browder brackets and layer permutations are related in the following way:*

$$\mathfrak{b}_{n,n'} \circ (\sigma \otimes \mathbb{1}_{n'}) = \sigma\rho^{-1}_{\sigma^{-1}(1)} \circ \mathfrak{b}_{n,n'} \circ (\rho_{\sigma^{-1}(1)} \otimes \mathbb{1}_{n'});$$

6. *if $n, n' \geqslant 2$, then there are several ways to produce loops as in Example 1.5.7, This is captured by the 'loop interchange relation'*

$$\begin{aligned}(s^2 - s^2(2\,3))(3 \cdots n{+}1) &\circ \mathfrak{b}_{n,n'} \\ &= (1\,2)(s^2 - s^2(2\,3))(3 \cdots n{+}1) \circ \mathfrak{b}_{n,n'} \circ (\rho_2 \otimes \rho_2).\end{aligned}$$

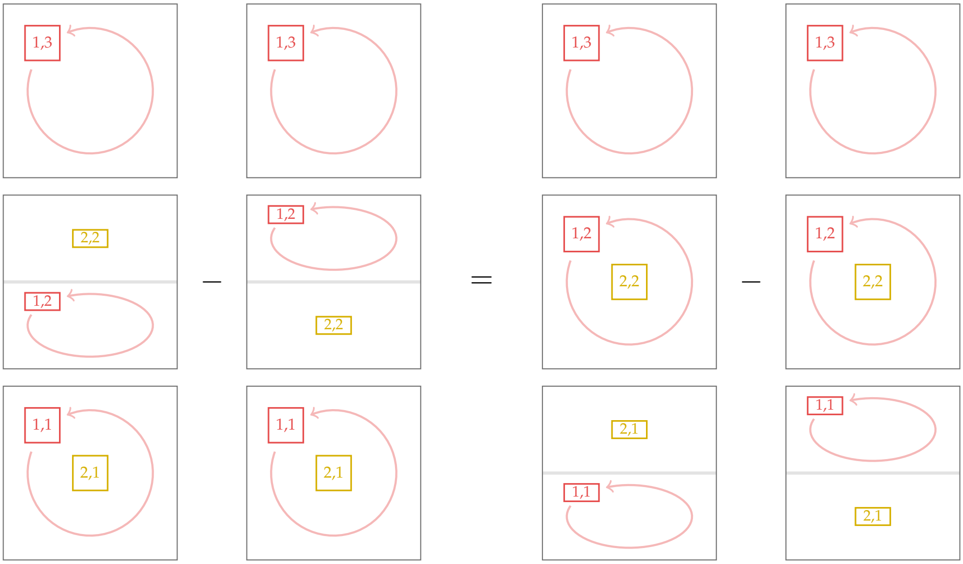

**Figure 4.13.** $(s^2 - s^2(2\,3))(3\,4) \circ \mathfrak{b}_{3,2} = (1\,2)(s^2 - s^2(2\,3))(3\,4) \circ \mathfrak{b}_{3,2} \circ (\rho_2 \otimes \rho_2)$.

*Proof.* Since all relations are proven in the same way, let us prove the loop interchange relation, as it is the most unexpected one. First of all, we see that all four summands in the formula live in the same path component of $\mathscr{V}\binom{n,n'}{n+n'-2}$, namely $f = (f_1, f_2)$ with $f_1(j) = j$, while $f_2(1) = 1$, $f_2(2) = 2$, and $f_2(j) = n - 2 + j$ else.

Now recall that $\mathscr{V}\binom{n,n'}{n+n'-2}_f$ is equivalent to a graph with two vertices $v_{1,2}$ and $v_{2,1}$, which are dual to the top-dimensional cells where the first cluster is on the left or right side of the second cluster (hence the indexing), respectively. In this easy situation, we do not have to care about the simplicial sign system and can just orient each edge such that it points from $v_{1,2}$ towards $v_{2,1}$.

We have one edge for each shuffle of the two clusters subordinate to the distribution $f$ of boxes to the layers. Since the only layers with more than one box are the two bottom ones, there are exactly four edges, which we may call $\binom{1\,2}{1\,2}$, $\binom{1\,2}{2\,1}$, $\binom{2\,1}{1\,2}$, and $\binom{2\,1}{2\,1}$, in accordance to Notation 4.3.4.

Then all four summands are represented by loops in the graph, which start at $v_{1,2}$ and cross one of the edges in positive and one of the edges in negative direction, see Figure 4.13. Hence, if we abbreviate $\sigma := (3 \cdots n{+}1)$, then the brackets are represented by the following cellular cycles:

$$\begin{aligned} s^2\sigma \circ \mathfrak{b}_{n,n'} &= \left[\binom{1\,2}{1\,2} - \binom{1\,2}{2\,1}\right], \\ s^2(2\,3)\sigma \circ \mathfrak{b}_{n,n'} &= \left[\binom{2\,1}{1\,2} - \binom{2\,1}{2\,1}\right], \\ (1\,2)s^2\sigma \circ \mathfrak{b}_{n,n'} \circ (\rho_2 \otimes \rho_2) &= \left[\binom{1\,2}{1\,2} - \binom{2\,1}{1\,2}\right], \\ (1\,2)s^2(2\,3)\sigma \circ \mathfrak{b}_{n,n'} \circ (\rho_2 \otimes \rho_2) &= \left[\binom{1\,2}{2\,1} - \binom{2\,1}{2\,1}\right]. \end{aligned}$$

Therefore, we finally get

$$\begin{aligned} (s^2 - s^2(2\,3))\sigma \circ \mathfrak{b}_{n,n'} &= \left[\binom{1\,2}{1\,2} - \binom{1\,2}{2\,1} - \binom{2\,1}{1\,2} + \binom{2\,1}{2\,1}\right] \\ &= (1\,2)(s^2 - s^2(2\,3))\sigma \circ \mathfrak{b}_{n,n'} \circ (\rho_2 \otimes \rho_2). \qquad \square \end{aligned}$$

There are clearly analogues of the formulæ 2 – 5 with nullaries and unaries in the *second* argument; here we use formula 1 in order to deduce them from the given ones. Similarly, formula 6 can be phrased for other boxes than only the first and the second one of each cluster, using formula 4.

Since all **ΔΣ**-maps can be written as $d \circ s \circ \sigma$, where $d$ is a composition of cofaces, $s$ is a composition of codegeneracies, and $\sigma$ is a permutation, we can *nearly* 'pull out' **ΔΣ**-maps as arguments of Browder brackets: it may happen that we are left with a single permutation of the form $\rho_\jmath$. This is the reason why the following proposition takes these permutations into account.

**Proposition 4.4.17** (Relations including Browder brackets in arity 3)**.** *The following relations hold inside $H_\bullet(\mathcal{V})$:*

1. *the Browder bracket and the universal morphisms are related in the following way: for each $1 \leqslant \jmath \leqslant n'$, we have*

$$\begin{aligned} &\mathfrak{b}_{n,n'+n''} \circ (\mathbb{1}_n \otimes (\rho_\jmath \circ \mathfrak{u}_{n',n''})) \\ &\qquad = \mathfrak{u}_{n+n'-1,n''} \circ ((\mathfrak{b}_{n,n'} \circ (\mathbb{1}_n \otimes \rho_\jmath)) \otimes \mathbb{1}_{n''}), \end{aligned}$$

   *while for all $1 \leqslant \jmath \leqslant n''$, we have for $\sigma := (2\,3)_{n,n''-1,n'} \in \mathfrak{S}_{n+n'+n''-1}$*

$$\begin{aligned} &\mathfrak{b}_{n,n'+n''} \circ (\mathbb{1}_n \otimes \rho_{n'+\jmath}\mathfrak{u}_{n',n''}) \\ &\qquad = (2\,3)^*(\sigma \circ \mathfrak{u}_{n+n''-1,n}((\mathfrak{b}_{n,n''} \circ (\mathbb{1}_n \otimes \rho_\jmath)) \otimes \mathbb{1}_{n'})); \end{aligned}$$

2. *if we compose two Browder brackets in a way that the two spinning layers do not meet, then we obtain an associativity property: for* $2 \leqslant \jmath \leqslant n'$*, we obtain*

$$\begin{aligned}&\mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes (\rho_\jmath \circ \mathfrak{b}_{n',n''}))\\ &\qquad = -\rho_2 \circ \mathfrak{b}_{n+n'-1,n''} \circ ((\rho_2 \circ \mathfrak{b}_{n,n'} \circ (\mathbb{1}_n \otimes \rho_\jmath)) \otimes \mathbb{1}_{n''}),\end{aligned}$$

   *while for* $2 \leqslant \jmath \leqslant n''$*, we have for* $\sigma := (2\,3)_{n,n''-1,n'-1} \in \mathfrak{S}_{n+n'+n''-2}$

$$\begin{aligned}&\mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes (\rho_{n'-1+\jmath} \circ \mathfrak{b}_{n',n''}))\\ &\qquad = (2\,3)^*(\sigma\rho_2 \circ \mathfrak{b}_{n'+n''-1,n'}((\rho_2 \circ \mathfrak{b}_{n,n''}(\mathbb{1}_n \otimes \rho_\jmath)) \otimes \mathbb{1}_{n'}));\end{aligned}$$

3. *if in a composition of two Browder brackets, two spinning layers meet, we obtain a generalised Jacobi identity. Formally, this means*

$$\begin{aligned}0 = {}&\mathfrak{b}_{n,n'+n''-1}(\mathbb{1}_n \otimes \mathfrak{b}_{n',n''})\\ &+ (1\,2\,3)^*((2\,3\,4)_{1,n'-1,n''-1,n-1} \circ \mathfrak{b}_{n',n''+n-1} \circ (\mathbb{1}_{n'} \otimes \mathfrak{b}_{n'',n}))\\ &+ (1\,3\,2)^*((2\,4\,3)_{1,n''-1,n-1,n'-1} \circ \mathfrak{b}_{n'',n+n'-1} \circ (\mathbb{1}_{n''} \otimes \mathfrak{b}_{n,n'})).\end{aligned}$$

*Proof.* Again, the proofs are very similar to each other, whence we prove, as an example, the generalised Jacobi identity: first of all, note that all three summands $x_1$, $x_2$ and $x_3$ appearing in the formula lie in the same path component of $\mathscr{V}\binom{n,n',n''}{n+n'+n''-2}$, namely $f = (f_1, f_2, f_3)$ with $f_1(\jmath) = \jmath$ being the inclusion of the first block, $f_2(1) = 1$ and $f_2(\jmath) = n - 1 + \jmath$ else, while $f_3(\jmath) = 1$ and $f_3(\jmath) = n + n' - 2 + \ell$ else.

Moreover, each summand is by construction represented by a smoothly embedded 2-torus $\mathbb{S}^1 \times \mathbb{S}^1 \hookrightarrow \mathscr{V}\binom{n,n',n''}{n+n'+n''-2}_f$, where the first factor parametrises the outer bracket and the second one the inner bracket. Each of these tori intersects exactly four cells of codimension 2 transversally. Note that since all layers apart from the first one carry only a single box, the cells are determined by the vertical alignment of the three boxes on the first layer, i.e. we have six cells of codimension 2, which we may call $\big(\sigma(1)\;\sigma(2)\;\sigma(3)\big)$, for each permutation $\sigma \in \mathfrak{S}_3$. Calculating the signs of the transversal intersections by hand, we obtain that, under Poincaré–Lefschetz duality, the three classes $x_1$,

$x_2$, and $x_3$ are represented by the dual cocycles

$$\begin{aligned} x_1 &= (-1)^{n'-1} \cdot [(123) + (132) - (321) - (231)], \\ x_2 &= (-1)^{n'-1} \cdot [(231) + (213) - (132) - (312)], \\ x_3 &= (-1)^{n'-1} \cdot [(312) + (321) - (123) - (213)]. \end{aligned}$$

This shows that $x_1 + x_2 + x_3 = 0$ holds, since each permutation occurs twice among these twelve summands, with mutually different signs. □

**Remark 4.4.18.** One can show that we have actually found *all* relations in order to give a full presentation in the sense of Section 3.2.

The proof is lengthy, but straightforward: one can introduce a 'normal form' for trees with internal vertices labelled by our generators (e.g. all unaries are gathered near the root, we only have permutations of the form $\rho_{\underline{j}}$ inside brackets, etc.), show that all trees can be written as a linear combination of trees which are in normal form, and finally give a surjection from the set of essential cells to the set of trees in normal form. However, as we will at no point need that our list of relations is exhaustive, we omit the details.

This presentation can be translated into a structure result for algebras over $H_\bullet(\mathcal{V})$: an $H_\bullet(\mathcal{V})$-algebra is the same as a functor $A_\bullet \colon \overline{\mathbf{\Delta\Sigma}} \to R\text{-}\mathbf{Mod}^{\mathbb{Z}}$, that comes together with:

- a unit $1 \in A_{1,0}$ induced by $\mathfrak{v}_1$,
- unions $- \sqcup - \colon A_{n,h} \otimes A_{n',h'} \to A_{n+n',h+h'}$ induced by $\mathfrak{u}_{n,n'}$,
- brackets $[-,-] \colon A_{n,h} \otimes A_{n',h'} \to A_{n+n'-1,h+h'+1}$ induced by $\mathfrak{b}_{n,n'}$,

such that the above relations hold with respect to the Koszul sign rule: for example, we have

$$\begin{aligned} a \sqcup b &= (-1)^{ab} \cdot (1\,2)_{n(b),n(a)}(b \sqcup a), \\ a \sqcup 1 &= d^{n(a)+1} a, \\ [a, b \sqcup c] &= [a,b] \sqcup c, \\ 0 &= (-1)^{ac} \cdot [a,[b,c]] + (-1)^{ab} \cdot \sigma_1 [b,[c,a]] + (-1)^{bc} \cdot \sigma_2 [c,[a,b]], \end{aligned}$$

where we write $n(a) := n$ and $(-1)^a := (-1)^h$ for $a \in A_{n,h}$, and where we put $\sigma_1 := (2\,3\,4)_{1,n(b)-1,n(c)-1,n(a)-1}$ and $\sigma_2 := (2\,4\,3)_{1,n(c)-1,n(a)-1,n(b)-1}$.

I would like to point out that, apart from layer permutations that enter quite often, these formulæ are very close to the ones in Example 3.2.13.

This structure result holds in particular for the homology of a $\mathcal{V}$-algebra. Furthermore, by Proposition 3.3.3, we see that if we work over a field $\mathbb{F}$ with characteristic 0, then the homology $H_\bullet(V(\mathbb{R}^{1,1} \times \underline{n}; \boldsymbol{X}); \mathbb{F})_{n\geqslant 1}$ of the free $\mathcal{V}$-algebra over a sequence $\boldsymbol{X} := (X_n)_{n\geqslant 1}$ of spaces is isomorphic to the free $H_\bullet(\mathcal{V})$-algebra over $H_\bullet(\boldsymbol{X})$.

**Remark 4.4.19.** The induced map $\varphi\colon H_\bullet(\mathcal{V}) \to H_\bullet(\overline{\mathbb{N}}(\mathcal{C}_2))$ of operads is not injective: for example, the map $H_0(\mathcal{V}\binom{k}{n}) \to H_0(\overline{\mathbb{N}}(\mathcal{C}_2)\binom{k}{n})$ is induced by the map $\boldsymbol{\Delta\Sigma}\binom{k}{n} \to \mathbf{Fin}\binom{k}{n}$ which forgets the ordering of the fibres.

One can show that the entire kernel of $\varphi$ is, as an operadic ideal, generated by the differences $f' - f$ of $\boldsymbol{\Delta\Sigma}$-maps whose underlying maps in **Fin** are equal. Along $\varphi$, the unexpected loop interchange relation becomes trivial, as already the difference $s^2 - s^2(2\,3)$ is 0 inside $R\langle\mathbf{Fin}\rangle\binom{n+n'-1}{n+n'-2}$.

A similar story can be told for the connective suboperad $H_\bullet(\mathcal{V}^c)$. Here we keep all relations that do not contain the symbols $d^\ell$ or $\mathfrak{u}_{n,n'}$, since all other generators are connective. In addition to them, we have further relations involving the vertical Pontrjagin product $\mathfrak{p}_{n,n'}$. Let us start again with relations among ground classes:

**Remark 4.4.20** (Relations among connective ground classes)**.**

1. the vertical Pontrjagin products are associative:
$$\mathfrak{p}_{n+n'-1,n''} \circ (\mathfrak{p}_{n,n'} \otimes \mathbb{1}_{n''}) = \mathfrak{p}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes \mathfrak{p}_{n',n''});$$
2. the Pontrjagin products are commutative up to layer permutation:
$$(1\,2)^*\mathfrak{p}_{n,n'} = (2\,3)_{1,n'-1,n-1} \circ \mathfrak{p}_{n',n};$$
3. using the void as one of the arguments in a vertical Pontrjagin product $\mathfrak{p}_{n,1}$ does not change anything: $\mathfrak{p}_{n,1} \circ (\mathbb{1}_n \otimes \mathfrak{v}_1) = \mathbb{1}_n$;

4. degeneracies can be 'pulled out' of one argument of a vertical Pontrjagin product: $\mathfrak{p}_{n-1,n'} \circ (s^{\ell} \otimes \mathbb{1}_{n'}) = s^{\ell} \circ \mathfrak{p}_{n,n'}$;

5. as for the Browder brackets, permutations can be 'pulled out' up to a cyclic remainder:

$$\mathfrak{p}_{n,n'} \circ (\sigma \otimes \mathbb{1}_{n'}) = \sigma\rho_{\sigma^{-1}(1)}^{-1} \circ \mathfrak{p}_{n,n'} \circ (\rho_{\sigma^{-1}(1)} \otimes \mathbb{1}_{n'})$$

6. precomposing with $\mathfrak{p}_{n,n'}$ equalises some $\mathbf{\Delta\Sigma}$-maps: if $f\colon n+n'-1 \to m$ is a map in $\mathbf{\Delta\Sigma}$ and $\sigma \in \mathfrak{S}_{n+n'-1}$ is a permutation such that $\sigma|_{\underline{n}}$ and $\sigma|_{\{1,n+1,\dots,n+n'-1\}}$ are monotone, and if $f\sigma$ and $f$ coincide as maps in **Fin** (though perhaps not in $\mathbf{\Delta\Sigma}$), then we have $f\sigma \circ \mathfrak{p}_{n,n'} = f \circ \mathfrak{p}_{n,n'}$.

Finally, there are relations between vertical Browder brackets and vertical Pontrjagin products, which are captured in the following proposition:

**Proposition 4.4.21.** *The following relations hold in $H_\bullet(\mathcal{V}^c)$:*

1. *without any cyclic permutation, the Browder bracket and the vertical product are related as follows: if we write $\sigma := (2\,3)_{n,n''-1,n'-1} \in \mathfrak{S}_{n+n'+n''-2}$, then we have a generalised Leibniz rule*

$$\begin{aligned}&\mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes \mathfrak{p}_{n',n''})\\ &\quad = \mathfrak{p}_{n+n'-1,n''} \circ (\mathfrak{b}_{n,n'} \otimes \mathbb{1}_{n''}) + (2\,3)^*(\sigma \circ \mathfrak{p}_{n+n''-1,n'} \circ (\mathfrak{b}_{n,n''} \otimes \mathbb{1}_{n'}));\end{aligned}$$

2. *if $2 \leqslant j \leqslant n'$, then we have*

$$\begin{aligned}&\mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes (\rho_j \circ \mathfrak{p}_{n',n''}))\\ &\quad = \rho_2 \circ \mathfrak{p}_{n+n'-1,n''} \circ ((\rho_2 \circ \mathfrak{b}_{n,n'} \circ (\mathbb{1}_n \otimes \rho_j)) \otimes \mathbb{1}_{n''});\end{aligned}$$

3. *if $2 \leqslant j \leqslant n''$, then we have for $\sigma = (2\,3)_{n,n''-1,n'-1} \in \mathfrak{S}_{n+n'+n''-2}$*

$$\begin{aligned}&\mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes (\rho_{n'+j-1} \circ \mathfrak{p}_{n',n''}))\\ &\quad = (2\,3)^*(\sigma\rho_2 \circ \mathfrak{p}_{n+n''-1,n'} \circ ((\rho_2 \circ \mathfrak{b}_{n,n''} \circ (\mathbb{1}_n \otimes \rho_j)) \otimes \mathbb{1}_{n'})).\end{aligned}$$

*Proof.* We show the first formula by using $\mathfrak{p}_{n',n''} = s^1(2\cdots n'{+}1) \circ \mathfrak{u}_{n',n'}$ and the relations from $H_\bullet(\mathcal{V})$. First we calculate that

$$\begin{aligned}
&\mathfrak{b}_{n,n'} \circ (\mathbb{1}_n \otimes s^1) \\
&\quad = (1\,2)^*\big((2\,3)_{1,n'-1,n-1} \circ \mathfrak{b}_{n',n} \circ (s^1 \otimes \mathbb{1}_n)\big) \\
&\quad = (2\,3)_{1,n'-1,n-1} \circ (1\,2)^*\big(s^1 \circ \mathfrak{b}_{n'+1,n} + s^1\rho_2 \circ \mathfrak{b}_{n'+1,n} \circ (\rho_2 \otimes \mathbb{1}_n)\big) \\
&\quad = s^1(2\cdots n{+}1) \circ (1\,2)^*\big((2\,3)_{1,n',n-1} \circ \mathfrak{b}_{n'+1,n}\big) \\
&\qquad\quad + s^1\rho_{n+1} \circ (1\,2)^*\big((2\,3)_{1,n',n-1} \circ \mathfrak{b}_{n'+1,n} \circ (\rho_2 \otimes \mathbb{1}_n)\big) \\
&\quad = s^1(2\cdots n{+}1) \circ \mathfrak{b}_{n,n'+1} + s^1\rho_{n+1} \circ \mathfrak{b}_{n,n'+1} \circ (\mathbb{1}_n \otimes \rho_2).
\end{aligned}$$

Similarly, we see that $\mathfrak{b}_{n,n'} \circ (\mathbb{1}_n \otimes \sigma) = \tilde{\sigma} \circ \mathfrak{b}_{n,n'}$ holds if $\sigma(1) = 1$, where $\tilde{\sigma}(n-1+\jmath) := n-1+\sigma(\jmath)$. This can be used for the remaining calculation:

$$\begin{aligned}
&\mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes \mathfrak{p}_{n',n''}) \\
&\quad = \mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes s^1(2\cdots n'{+}1) \circ \mathfrak{u}_{n',n''}) \\
&\quad = s^1(2\cdots n{+}1) \circ \mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes ((2\cdots n'{+}1) \circ \mathfrak{u}_{n',n''})) \\
&\qquad + s^1\rho_{n+1} \circ \mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes (\rho_{n+1} \circ \mathfrak{u}_{n',n''})) \\
&\quad = s^1(2\cdots n{+}n') \circ \mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes \mathfrak{u}_{n',n''}) \\
&\qquad + s^1\rho_{n+1} \circ \mathfrak{b}_{n,n'+n''-1} \circ (\mathbb{1}_n \otimes (\rho_{n+1} \circ \mathfrak{u}_{n',n''})) \\
&\quad = s^1(2\cdots n{+}n') \circ \mathfrak{u}_{n+n'-1,n''} \circ (\mathfrak{b}_{n,n'} \otimes \mathbb{1}_{n''}) \\
&\qquad + (2\,3)^*\big(s^1\rho_{n+1}(2\,3)_{n,n''-1,n'} \circ \mathfrak{u}_{n+n''-1,n'} \circ (\mathfrak{b}_{n,n''} \otimes \mathbb{1}_{n'})\big) \\
&\quad = \mathfrak{p}_{n+n'-1,n''} \circ (\mathfrak{b}_{n,n'} \otimes \mathbb{1}_{n''}) + (2\,3)^*\big(\sigma \circ \mathfrak{p}_{n+n''-1,n'} \circ (\mathfrak{b}_{n,n''} \otimes \mathbb{1}_{n'})\big). \qquad \square
\end{aligned}$$

**Remark 4.4.22.** Again, one can show that this list of relations is exhaustive, and again, this can be rephrased as a structure result for $H_\bullet(\mathcal{V}^c)$-algebras: if we denote by $\overline{\mathbf{\Delta\Sigma}}_{\mathrm{surj}} \subseteq \overline{\mathbf{\Delta\Sigma}}$ the subcategory of surjective $\mathbf{\Delta\Sigma}$-maps, then $H_\bullet(\mathcal{V}^c)$-algebras are the same as functors $A_\bullet\colon \overline{\mathbf{\Delta\Sigma}}_{\mathrm{surj}} \to R\text{-}\mathbf{Mod}^{\mathbb{Z}}$, with:

- a unit $1 \in A_{1,0}$ induced by $\mathfrak{v}_1$,
- products $-\cdot-\colon A_{n,h} \otimes A_{n',h'} \to A_{n+n'-1,h+h'}$ induced by $\mathfrak{p}_{n,n'}$,
- brackets $[-,-]\colon A_{n,h} \otimes A_{n',h'} \to A_{n+n'-1,h+h'+1}$ induced by $\mathfrak{b}_{n,n'}$,

such that the aforementioned relations hold with respect to the Koszul sign rule, for example

$$\begin{aligned}a \cdot (b \cdot c) &= (a \cdot b) \cdot c,\\ a \cdot b &= (-1)^{ab} \cdot (2\,3)_{n(b)-1,n(a)-1}(b \cdot a),\\ a \cdot 1 &= a,\\ [a, b \cdot c] &= [a,b] \cdot c + (-1)^{bc} \cdot (2\,3)_{n(a),n(c)-1,n(b)-1}([a,c] \cdot b).\end{aligned}$$

Note that the subfamily $(A_{1,h})_{h \geqslant 0}$ carries an action of $H_\bullet(\mathscr{V}^c_{1,1}|_1) = \mathscr{P}ois_2$, and in this description, the vertical Pontrjagin product $\mathfrak{p}_{1,1}$ and the classical Pontrjagin product $\mathfrak{p}$, as well as the vertical Browder bracket $\mathfrak{b}_{1,1}$ and the classical Browder bracket $\mathfrak{b}$ agree: this justifies the notation '$\cdot$' and '$[-,-]$'.

In particular, we recover the classical[1] Jacobi identity and Leibniz rule for the Pontrjagin product and the Browder bracket as the special case in which all arguments are supported on a single layer.

Finally, I would like to point out that the Cartan formula for expressions of the form $[a \cdot a', b \cdot b']$ is easily established by applying the Leibniz rule twice and by using the (graded) commutativity of the bracket.

### 4.4.3. *Squaring operations*

In this last subsection, we describe a system of divided power operations in the sense of Construction 3.3.10 that arise from the operad $\mathscr{V}$.

We start by constructing a sequence of classes $c_n \in H_1(\mathscr{V}\binom{n,n}{n})$ which give rise to operations $Q\colon H_h(X_n) \to H_{1+2h}(X_n)$ for each $\mathscr{V}$-algebra $(X_n)_{n \geqslant 1}$, and we study relations among them. Afterwards, we show that $Q$, together with the Künneth operations, exhausts *all* squaring operations, i.e. divided power operations with one input of multiplicity 2.

**Remark 4.4.23** (Unordered operation spaces)**.** For $K = (k_1, \dots, k_r)$ and $k \geqslant 1$, let $r(k)$ be the number of occurrences of $k$ inside $K$. Recall that the symmetric group $\mathfrak{S}^K = \prod_{k \geqslant 1} \mathfrak{S}_{r(k)}$ acts freely on both $\mathscr{V}\binom{K}{n}$ and $\tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n})$ by exchanging clusters of the same size.

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[1] Note that, in contrast to Cohen's work, which was summarised in Example 3.2.13, we use a different sign convention by defining $[a,b]$ to be $\mathfrak{b}_{n,n'}(a \otimes b)$ instead of $(-1)^a \cdot \mathfrak{b}_{n,n'}(a \otimes b)$.

The homotopy equivalence $\Psi\colon \tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n}) \leftrightarrows \mathscr{V}\binom{K}{n} :\Phi$ from Proposition 4.1.3 is $\mathfrak{S}^K$-equivariant, whence we get for each subgroup $\mathfrak{G} \subseteq \mathfrak{S}^K$ an induced homotopy equivalence $\bar{\Psi}\colon \tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n})/\mathfrak{G} \leftrightarrows \tilde{V}\binom{K}{n}/\mathfrak{G} :\bar{\Phi}$.

The path components of $\mathscr{V}\binom{K}{n}/\mathfrak{G}$ are given by equivalence classes of tuples $[f] := [f_1, \dots, f_r]$ with $f_i \in \mathbf{\Delta\Sigma}\binom{k_i}{n}$, and two tuples are identified if they differ only by an index permutation from $\mathfrak{G}$. We occasionally use the short notation $(\tilde{V}/\mathfrak{G})_{[f]}$ for the respective path component.

Since the action of $\mathfrak{G}$ on $\tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n})$ preserves the (relative, dual) cellular structure, we get an induced cellular structure on $\tilde{V}_K(\mathbb{R}^{1,1} \times \underline{n})/\mathfrak{G}$, with cells given by equivalence classes $[\pi]$ of maps $\mathbb{Y}_M \to \underline{r}$, where two such maps $\pi$ and $\pi'$ are identified if they differ by postcomposition with a permutation in $\mathfrak{G}$. We depict such a cell as in Notation 4.3.4 by a matrix, but we use square brackets to indicate that we speak of $[\pi]$ instead of $\pi$.

**Construction 4.4.24** (Vertical Dyer–Lashof squares)**.** For a fixed $n \geqslant 1$, we consider the path component $\mathbb{1}^2 := (\mathbb{1}, \mathbb{1}) \in \mathbf{\Delta\Sigma}^{\times}\binom{n,n}{n}$ and the loop

$$\begin{tikzcd}
{[0;1]} \arrow[rr, "\alpha"] \arrow[d] & & \tilde{V}_{\mathbb{1}^2} \arrow[d] \\
\mathbb{S}^1 \arrow[rr, dashed, "\bar{\alpha}"'] & & \left(\tilde{V}\binom{n,n}{n}/\mathfrak{S}_2\right)_{[\mathbb{1}^2]},
\end{tikzcd}$$

which is defined by

$$\alpha(t) = \big((e^{\pi i \cdot t}, 1), \dots, (e^{\pi i \cdot t}, n); (e^{\pi i \cdot (t-1)}, 1), \dots, (e^{\pi i \cdot (t-1)}, n)\big).$$

Pictorially, on the $\ell^{\text{th}}$ layer, the $\ell^{\text{th}}$ box from the first cluster and the $\ell^{\text{th}}$ box from the second cluster change places, as in Figure 4.14.

We define $c_n := (\bar{\Psi} \circ \bar{\alpha})_*[\mathbb{S}^1] \in H_1((\mathscr{V}/\mathfrak{S})_{[\mathbb{1}^2]})$. As $\mathfrak{S}_2$ acts freely on $\mathscr{V}\binom{n,n}{n}$, this gives rise to an operation, which we call *vertical Dyer–Lashof square*

$$Q := Q_{c_n}\colon H_h(X_n) \to H_{1+2h}(X_n),$$

for each $\mathscr{V}$-algebra $(X_n)_{n \geqslant 1}$, whenever $h$ is even or we work mod 2, compare Construction 3.3.10.

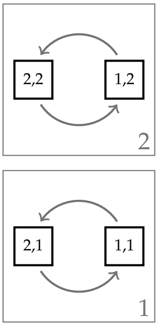


**Figure 4.14.** $\Psi \circ \alpha \colon [0;1] \to \mathcal{V}\binom{2,2}{2}$ with $(\bar{\Psi} \circ \bar{\alpha})_*[\mathbb{S}^1] = c_2 \in H_1(\mathcal{V}\binom{n,n}{n}/\mathfrak{S}_2)$.

**Proposition 4.4.25.** *For each $\mathcal{V}$-algebra $(X_n)_{n \geqslant 1}$, the following formulæ hold inside the $\mathbb{F}_2$-homology $(H_h(X_n;\mathbb{F}_2))_{n \geqslant 1, h \geqslant 0}$:*

1. *$Q$ commutes with cofaces: $Q(d^\ell x) = d^\ell Qx$;*
2. *$Q$ commutes with permutations: $Q(\sigma x) = \sigma Qx$;*
3. *$Q$ interacts with codegeneracies in the following way: define $g \in \mathbf{\Delta\Sigma}\binom{2n-1}{n}$ as $(2 \;\; n{+}1 \,|\, \cdots \,|\, \ell \;\; n{+}\ell{-}1 \,|\, \ell{+}1 \;\; 1 \;\; n{+}\ell \,|\, \ell{+}2 \;\; n{+}\ell{+}1 \,|\, \cdots \,|\, n \;\; 2n{-}1)$ and write $\rho_\ell := (1 \cdots \ell) \in \mathfrak{S}_n$ as before. Then*
$$Q(s^\ell x) = s^\ell Qx + g[\rho_{\ell+1}x, \rho_\ell x];$$
4. *$Q$ commutes with unifiers: $Q(x \sqcup x') = Qx \sqcup Qx'$;*
5. *$Q$ inside a Browder bracket can be translated into a sum of Browder brackets: if $x \in H_\bullet(X_n)$ and $x' \in H_\bullet(X_{n'})$, then we let $g_\ell \in \mathbf{\Delta\Sigma}\binom{2n-1}{n}$ as in Construction 4.4.11 and $g'_\ell := g_\ell \sqcup \mathbb{1}_{n'-1} \in \mathbf{\Delta\Sigma}\binom{2n+n'-1}{n+n'-1}$. Then we get*
$$[Qx, x'] = \sum_{\ell=1}^{n} g'_\ell[\rho_\ell x, \rho_\ell[x, x']].$$
6. *the Browder bracket measures to what extend $Q$ fails to be additive:*
$$Q(x + x') = Qx + Qx' + \sum_{\ell=1}^{n} g_\ell[\rho_\ell x, \rho_\ell x'];$$

7. *Q and the vertical Pontrjagin product satisfy a generalised Cartan formula:*

$$Q(x \cdot x') = Qx \cdot Qx' + s^2 s^4 \cdots s^{2n'+2n-4}(x \cdot [x, x'] \cdot x').$$

*Proof.* The relations 1 – 5 are all proven in the same way; thus, we exemplarily show 5. Let $\boldsymbol{E} = (E_w)_{w \geqslant 1}$ be the free $\mathscr{V}$-algebra over *two* points, which we consider as charges: a point called '+' of colour $n$, and a point called '−' of colour $n'$. If we abbreviate $K(r, r') := (r \times n, r' \times n')$ and $\boldsymbol{\mathfrak{S}}_{r,r'} := \mathfrak{S}_r \times \mathfrak{S}_{r'}$ for $r, r' \geqslant 0$, then, for each $w \geqslant 1$, the level $E_w$ decomposes, up to equivalence, as

$$E_w \simeq \coprod_{r,r' \geqslant 0} \tilde{V}_{K(r,r')}(\mathbb{R}^{1,1} \times \underline{w}) / \boldsymbol{\mathfrak{S}}_{r,r'}.$$

Each of the summands splits again into $(\tilde{V}/\boldsymbol{\mathfrak{S}})_{[f],[f']}$, with $[f] = [f_1, \ldots, f_r]$ and $[f'] = [f'_1, \ldots, f'_{r'}]$, with $f_i \in \mathbf{\Delta\Sigma}\binom{n}{w}$ and $f'_i \in \mathbf{\Delta\Sigma}\binom{n'}{w}$, and each of these components has a (relative, dual) cellular decomposition as in Remark 4.4.23. Since we work over $\mathbb{F}_2$, we do not have to care about orientations, whence each homology class of $(\tilde{V}/\boldsymbol{\mathfrak{S}})_{[f],[f']}$ has, via Poincaré–Lefschetz duality, a representation as a cellular cocycle.

With this in mind, we show relation 5 first for the special case where:

- $x$ is the ground class $e$ of $(\tilde{V}/\boldsymbol{\mathfrak{S}})_{[\mathbb{1}],\varnothing} \subseteq E_n$, the space of configurations of a single cluster of positive charge, which has one point on each layer;
- $x'$ is the ground class $e'$ of $(\tilde{V}/\boldsymbol{\mathfrak{S}})_{\varnothing,[\mathbb{1}]} \subseteq E_{n'}$, the space of configurations of a single cluster of negative charge, which has one point on each layer.

Then $[Qe, e']$ is represented by an embedded 2-torus $\mathbb{S}^1 \times \mathbb{S}^1 \hookrightarrow (\tilde{V}/\boldsymbol{\mathfrak{S}})_{[f,f],[f']}$, where $f \in \mathbf{\Delta\Sigma}\binom{n}{n+n'-1}$ is the inclusion of the first block and $f'(1) = 1$, while $f'(\ell) = n + \ell - 1$ for $2 \leqslant \ell \leqslant n' - 1$. This 2-torus intersects two dual cells of codimension 2 transversally, namely

$$\pi_1 := \begin{bmatrix} 3 \\ \vdots \\ 3 \\ 1\,2 \\ \vdots \\ 1\,2 \\ 1\,2\,3 \end{bmatrix} \quad \text{and} \quad \pi_2 := \begin{bmatrix} 3 \\ \vdots \\ 3 \\ 1\,2 \\ \vdots \\ 1\,2 \\ 3\,1\,2 \end{bmatrix}.$$

Therefore $[Qe, e']$ is represented by the cellular cocycle $\pi_1 + \pi_2$. Exactly the same holds for $y := \sum_\ell g'_\ell[\rho_\ell e, \rho_\ell[e, e']]$: there are four intersections, with

$$\pi'_1 := \pi_1, \quad \pi'_2 := \begin{bmatrix} 3 \\ \vdots \\ 3 \\ 21 \\ \vdots \\ 21 \\ 321 \end{bmatrix} = \pi_2, \quad \pi'_3 := \begin{bmatrix} 3 \\ \vdots \\ 3 \\ 12 \\ \vdots \\ 12 \\ 132 \end{bmatrix}, \quad \text{and} \quad \pi'_4 := \begin{bmatrix} 3 \\ \vdots \\ 3 \\ 21 \\ \vdots \\ 21 \\ 231 \end{bmatrix} = \pi'_3.$$

Hence $y$ is represented by $\pi'_1 + \pi'_2 + \pi'_3 + \pi'_4 = \pi_1 + \pi_2$.

The case of a general $\mathcal{V}$-algebra $\boldsymbol{X} = (X_n)_{n \geqslant 1}$ and of general homology classes $x \in H_h(X_n)$ and $x' \in H_{h'}(X_{n'})$ now follows from abstract considerations: if we write $\mathfrak{O} := C^{\mathrm{sing}}_\bullet(\mathcal{V})$ for the corresponding operad in $\mathbb{F}_2$-chain complexes and consider the formal (coloured) chain complex $B = \mathbb{F}_2\langle \beta, \beta' \rangle$, where $\beta$ is of degree $h$ and colour $n$, while $\beta'$ is of degree $h'$ and colour $n'$, then $F^{\mathfrak{O}}(B)$, the free $\mathfrak{O}$-algebra over $B$, decomposes into chain complexes $C^{\mathrm{sing}}_\bullet(\mathcal{V}_{(f,f')}) \otimes_{\mathfrak{G}} (\mathbb{F}_2\langle\beta\rangle^{\otimes r} \otimes \mathbb{F}_2\langle\beta'\rangle^{\otimes r'})$ for varying $(\boldsymbol{f}, \boldsymbol{f'}) \in \boldsymbol{\Delta\Sigma}^\times\binom{r\times n, r'\times n'}{w}$ with $r, r' \geqslant 0$ and $w \geqslant 1$, and $\mathfrak{G} \subseteq \mathfrak{G}_{r,r'}$ being the isotropy group of $(\boldsymbol{f}, \boldsymbol{f'})$. Similarly, for a fixed pair $(\boldsymbol{f}, \boldsymbol{f'})$, the singular chain complex of the component $(\mathcal{V}/\mathfrak{G})_{[f],[f']}$ is of the form $C^{\mathrm{sing}}_\bullet(\mathcal{V}_{(f,f')}) \otimes_{\mathfrak{G}} (\mathbb{F}_2\langle\varepsilon\rangle^{\otimes r} \otimes \mathbb{F}_2\langle\varepsilon'\rangle^{\otimes r'})$, where both $\varepsilon$ and $\varepsilon'$ are of degree 0. By sending $\varepsilon$ to $\beta$ and $\varepsilon'$ to $\beta'$, we obtain a chain map $\varphi_{f,f'} \colon C^{\mathrm{sing}}_\bullet((\mathcal{V}/\mathfrak{G})_{[f],[f']}) \to F^{\mathfrak{O}}(B)$ of degree $r \cdot h + r' \cdot h'$.

If we choose representing singular cycles $\xi$ for $x$ and $\xi'$ for $x'$, then the map $B \to C^{\mathrm{sing}}_\bullet(\boldsymbol{X})$ sending $\beta$ to $\xi$ and $\beta'$ to $\xi'$ has an adjoint $\psi \colon F^{\mathfrak{O}}(B) \to C^{\mathrm{sing}}_\bullet(\boldsymbol{X})$, and altogether, we obtain, for each $(\boldsymbol{f}, \boldsymbol{f'}) \in \boldsymbol{\Delta\Sigma}^\times\binom{r\times n, r'\times n'}{w}$, a map

$$\Phi_{f,f'} \colon H_\bullet((\mathcal{V}/\mathfrak{G})_{[f],[f']}) \xrightarrow{(\varphi_{f,f'})_*} H_{\bullet+rh+r'h'}(F^{\mathfrak{O}}(B)_w) \xrightarrow{(\psi_m)_*} H_{\bullet+rh+r'h'}(X_w),$$

It is straightforward to check that $\Phi_{\mathbb{1},\varnothing}(e) = x$ and $\Phi_{\varnothing,\mathbb{1}}(e') = x'$, and that $\Phi$ commutes with Künneth operations and $Q$ in the sense that

$$m(\Phi e_1 \otimes \cdots \otimes \Phi e_s) = \Phi(m(e_1 \otimes \cdots \otimes e_s)),$$
$$Q(\Phi\bar{e}) = \Phi(Q\bar{e}),$$

for each operation $m \in H_\bullet(\mathcal{V})$ and for each collection $e_1, \ldots, e_s, \bar{e}$ of fitting classes in $H_\bullet(\boldsymbol{E})$. This proves the claim.

Relation 6 follows from Remark 3.3.14 and the observation that the homological transfer $\mathrm{pr}^!(c_n)$ is given by $\sum_\ell g_\ell \circ \mathfrak{b}_{n,n} \circ (\rho_\ell \otimes \rho_\ell)$, while relation 7 follows from the previous ones and the equality $x \cdot x' = s^1(2 \cdots n{+}1)(x \sqcup x')$:

$$\begin{aligned} Q(x \cdot x') &= Q(s^1(2 \cdots n{+}1)(x \sqcup x')) \\ &= s^1(2 \cdots n{+}1)(Qx \sqcup Qx') \\ &\qquad + g[\rho_{n+1}(x \sqcup x'), (2 \cdots n{+}1)(x \sqcup x')] \\ &= Qx \cdot Qx' + g(n{+}n'{+}1 \cdots 2n{+}n')[\rho_{n+1}(x \sqcup x'), x \sqcup x'] \\ &= Qx \cdot Qx' + g(n{+}n'{+}1 \cdots 2n{+}n')([\rho_{n+1}(x \sqcup x'), x] \sqcup x') \\ &= Qx \cdot Qx' + s^2 s^4 \cdots s^{2n'+2n-4}(x \cdot [x, x'] \cdot x') \end{aligned} \qquad \square$$

**Remark 4.4.26.** If the homology classes to which we want to apply $Q$ lie in the colour-1 part of $H_\bullet(\boldsymbol{X})$, then the operation $Q$ coincides with the classical Dyer–Lashof operation coming from the suboperad $\mathscr{V}^c_{1,1}|_1 = \mathscr{C}_2$. In this case, our formulæ recover many classical relations from [CLM76, § III], e.g.

$$\begin{aligned} [Qx, x'] &= [x, [x, x']], \\ Q(x + x') &= Qx + Qx' + [x, x'], \\ Q(x \cdot x') &= Qx \cdot Qx' + x \cdot [x, x'] \cdot x'. \end{aligned}$$

The reader should not be surprised by the first formula: in some sources, e.g. [Böd90b, § 4.5] and [BH14, Prop. 4.1.12], one can find the formula '$[Qx, x'] = 0$' for $Q$ being the top operation for $E_2$-algebras. This is a misquote of [CLM76, § III, Thm. 1.2], where the 'top Dyer–Lashof operation' is given another name, whence the correct formula was included in another list.

Indeed, consider the $E_2$-algebra $F^{\mathscr{C}_2}(\{+,-\}) \simeq \coprod_{r,r'} \tilde{C}_{r+r'}(\mathbb{R}^2)/(\mathfrak{S}_r \times \mathfrak{S}_{r'})$ and let $x$ and $x'$ be the two ground classes of the components of configurations consisting of a single point. Then, by the same methods as in the previous proof, $[Qx, x']$ is represented by a relative cellular cocycle, which is not 0, and hence also not a boundary, as it is of maximal codimension.

Let me point out that since all Dyer–Lashof squares we consider are either Pontrjagin products or top-dimensional, there are no Adem relations similar to [CLM76, Thm. III.1.1] to be expected; though they might enter in the description of the homology of $\mathscr{V}_{p,q}$-algebras for $p + q \geqslant 3$.

Finally, we want to point out that this sole construction $Q$ exhausts *all* squaring operations.

**Proposition 4.4.27.** *Let $c \in H_h(\mathcal{V}\binom{k,k}{n}/\mathfrak{S}_2)$. Then $Q_c$ can be expressed in terms of $Q$ and Künneth operations.*

*Proof.* We can assume that $c$ is subordinate to a single path component $\mathcal{V}_{\boldsymbol{f}}/\mathfrak{S}_2$ with $\boldsymbol{f} = (f_1, f_2)$, as otherwise, $c$ decomposes as a sum of classes. If $f_1 \neq f_2$, then $\mathrm{pr}\colon \mathcal{V} \to \mathcal{V}/\mathfrak{S}_2$ restricts over the component $[\boldsymbol{f}]$ to a trivial covering, and there is a class $m \in H_h(\mathcal{V}\binom{k,k}{n})$ with $c = \mathrm{pr}_* m$, whence $Q_c(x) = m(x \otimes x)$ by Remark 3.3.13. Thus, we only have to consider the map $f_1 = f_2 =: f$.

We can write $f = d \circ s \circ \sigma$ with $d$ a composition of cofaces, $s$ a composition of codegeneracies and $\sigma$ a permutation. Then there is a class $c' \in H_h(\mathcal{V}\binom{k,k}{n'})$ such that $c = dc'$ and we get $Q_c = dQ_{c'}$ by Lemma 3.3.15. Thus, we can assume that $f = s \circ \sigma$ is surjective.

Note that $h \in \{0, 1\}$, and if $h = 0$, then again, there is a $m$ with $c = \mathrm{pr}_* mc$ and $Q_c(x) = m(x \otimes x)$. Hence, we only have to deal with $h = 1$. Here we see that $H_1((\mathcal{V}/\mathfrak{S})_{[\boldsymbol{f}]}) = \mathbb{F}_2\langle \Pi_{f,1}/\mathfrak{S}_2\rangle$, so we may assume that $c$ corresponds to a cell $[\pi] \in \Pi_{f,1}/\mathfrak{S}_2$.

There is maximal colour $n' \geqslant n$ such that we find a factorisation of $f$ into a surjective $f'\colon k \to n'$ and an iterated codegeneracy $s'\colon n' \to n$, and a class $c' \in H_1((\mathcal{V}/\mathfrak{S})_{[\boldsymbol{f}']})$ such that $c = s'c'$. If we write $a_\ell := \#f'^{-1}(\ell)$ for each $1 \leqslant \ell \leqslant n'$, then there are $\sigma_2, \ldots, \sigma_{n'} \in \mathfrak{S}_2$ such that $c'$ corresponds to the cell

$$c' = \begin{bmatrix} a_{n'} \times \sigma_{n'}(1) & a_{n'} \times \sigma_{n'}(2) \\ \multicolumn{2}{c}{\vdots} \\ a_2 \times \sigma_2(1) & a_2 \times \sigma_2(2) \\ a_1 \times 1 & a_2 \times 2 \end{bmatrix},$$

as otherwise, $c'$ can be decomposed further, contradicting the maximality of $n'$. (Here the term '$a \times j$' denotes an $a$-fold repetition of the symbol $j$.) This means that $c' = Q_{c''}(f')$ for some $c'' \in H_1((\mathcal{V}/\mathfrak{S}_2)_{[\mathbb{1}^2]})$. In particular, we get $Q_c(x) = sQ_{c''}(f'x)$ for each $x \in H_\bullet(X_k)$, so we only have to deal with $Q_{c''}$.

Therefore, we can assume that $c$ lies in the component $[\mathbb{1}^2] = [\mathbb{1}, \mathbb{1}]$. Again there are $\sigma_2, \ldots, \sigma_n \in \mathfrak{S}_2$ such that

$$c = \begin{bmatrix} \sigma_n(1) & \sigma_n(2) \\ \multicolumn{2}{c}{\vdots} \\ \sigma_2(1) & \sigma_2(2) \\ 1 & 2 \end{bmatrix},$$

and we prove that $Q_c$ can be expressed via $Q$ and via Künneth operations, by downwards induction on $\hbar := \mathrm{ht}(c) := \min(\ell;\, \sigma_\ell \neq \mathbb{1}) \in \{2, \dots, n+1\}$. For $h = n+1$, we have $c = c_n$, and for the induction step '$n+1, \dots, \hbar+1 \to \hbar$', we consider the $\mathbf{\Delta\Sigma}$-map

$$g := \big(2 \;\; n{+}1 \,|\, \cdots \,|\, \hbar \;\; n{+}\hbar{-}1 \,|\, 1 \,|\, \sigma^*_{\hbar+1}(\hbar{+}1 \;\; n{+}\hbar) \,|\, \cdots \,|\, \sigma^*_n(n \;\; 2n{-}1)\big),$$

where $\sigma^*(\jmath \;\; \jmath')$ interchanges the two symbols $\jmath$ and $\jmath'$ if $\sigma \neq \mathbb{1}$. Then $\bar{c} := c + g \circ \mathfrak{b}_{n,n}$ corresponds to a cell that differs from $c$ by swapping the entry in the $\hbar^{\text{th}}$ row. Hence we get $\mathrm{ht}(\bar{c}) > \mathrm{ht}(c)$ and the induction hypothesis applies. Moreover, we see that $Q_c(x) = Q_{\bar{c}}(x) - g[x, x]$ holds, as desired. ☐

Again, I would like to close this long chapter with a list of open questions:

1. Classify, up to equivalence, all levelwise path connected $\mathscr{V}_{p,q}$-algebras $\boldsymbol{X} = (X_n)_{n \geqslant 1}$. We already know that each $X_n$ is a connected $E_p$-algebra, whence it is by May's recognition principle [May72] equivalent to a $p$-fold loop space. For the remaining $q$ directions, a combinatorial enhancement of the recognition principle seems to be necessary. As a first task, one should try to find an adjunction $\tilde{\Sigma} \colon \mathbf{Top}_*^{\overline{\mathbb{N}}} \rightleftarrows \mathbf{Top}_*^{\overline{\mathbb{N}}} : \tilde{\Omega}$ such that for each based sequence $\boldsymbol{X}$, the sequence $\tilde{\Omega}(\boldsymbol{X})$ carries the structure of an $\overline{\mathbb{N}}(\mathscr{C}_1)$-algebra, and such that the arising natural transformation $U^{\overline{\mathbb{N}}(\mathscr{C}_1)} F^{\overline{\mathbb{N}}(\mathscr{C}_1)} \Rightarrow \tilde{\Omega}\tilde{\Sigma}$ is an equivalence for connected spaces.

2. Provide a presentation of the algebraic operads $H_\bullet(\mathscr{V}_{p,q})$ and $H_\bullet(\mathscr{V}^c_{p,q})$ for general $p$ and $q$. I assume that the same cellular methods can be used here; one just has to be more patient than I have been.

3. Prove or disprove that, over $\mathbb{F}_2$, all *higher* divided power operations can be expressed in terms of $Q$ and Künneth operations. After having found all such power operations and all relations between them, we should end up with a result similar to [CLM76, § III]: for each (based, symmetric) sequence $\boldsymbol{X}$, the $\mathbb{F}_2$-homology of the free $\mathscr{V}_{1,1}$-algebra $F^{\mathscr{V}_{1,1}}(\boldsymbol{X})$ is the free $W'$-algebra over $H_\bullet(\boldsymbol{X})$, where a $W'$-algebra is a $H_\bullet(\mathscr{V}_{1,1})$-algebra, together with all power operations and satisfying *all* relations: here we most likely will have to include coloured Nishida relations, describing how $Q$ interacts with the (dual) Steenrod squares. Finally, one should understand what the situation looks like mod $p$ for odd primes.

Chapter 5

# Homology operations on moduli spaces of surfaces

> *Es soll das Innere von Ω auf eine*
> *von einem geradlinigen Schlitz begrenzte*
> *Ebene konform abgebildet werden.*
>
> David Hilbert

As already anticipated in Subsection 3.4.1, we aim to apply the operadic techniques from the previous chapters in order to gain a better understanding of moduli spaces and mapping class groups of surfaces with multiple boundary components and their homology.

In order to pursue this aim, we start this chapter by making the action of dyed $E_2$-operads on models of $\mathfrak{M}$ precise in two mutually comparable ways; this extends the classical construction [Mil86; Böd90b] in a straightforward way and involves a coloured version of Tillmann's surface operad [Til00], which is related to Segal's conformalcobordism PROP [Seg88].

Secondly, we turn our attention to a specific simplicial model [Böd90a; ABE08] of $\mathfrak{M}$. The idiosyncrasy of this model makes it necessary to restrict ourselves to the vertical suboperad $\mathcal{V}_{1,1} \subseteq \overline{\mathbb{N}}(\mathcal{C}_2)$; this construction extends a description from [Böd90b, § 5]. As a consequence, the homology of $\mathfrak{M}$ can be endowed with the operations which we described in Section 4.4. Moreover, we will encounter several ad-hoc constructions of homology operations which involve multiple boundary components.

In the third and last section of this chapter, we apply these constructions to explicit low-genus calculations in the unstable range and, by doing so, contribute to the *map of the world* („Weltkarte“) which has been initiated and developed by our working group over the last decade.

## 5.1. Moduli spaces and classical operadic actions on them

This section explicates which models for moduli spaces of Riemann surfaces we want to use and what the $E_2$-actions look like in these cases.

### 5.1.1. *Surfaces, mapping class groups, and moduli spaces*

**Definition 5.1.1.** A *surface with parametrised boundary* is a pair $(\mathcal{S}, \Theta)$ where:

1. $\mathcal{S}$ is a smooth, compact, oriented, possibly disconnected surface with boundary $\partial\mathcal{S} \subseteq \mathcal{S}$, together with a decomposition $\partial\mathcal{S} = \partial^{\text{in}}\mathcal{S} \sqcup \partial^{\text{out}}\mathcal{S}$ into *incoming* and *outgoing* boundary curves, such that each component of $\mathcal{S}$ has non-empty outgoing boundary.
2. $\Theta = (\Theta^{\text{in}}, \Theta^{\text{out}})$ is a pair of diffeomorphisms $\Theta^{\text{in}} \colon \underline{k} \times \mathbb{S}^1 \to \partial^{\text{in}}\mathcal{S}$ and $\Theta^{\text{out}} \colon \underline{n} \times \mathbb{S}^1 \to \partial^{\text{out}}\mathcal{S}$, where $k := \#\pi_0(\partial^{\text{in}}\mathcal{S})$ and $n := \#\pi_0(\partial^{\text{out}}\mathcal{S})$.

To save notation, we sometimes treat $\Theta$ as a single map, which then is the disjoint sum $\underline{k+n} \to \partial\mathcal{S}$ of the parametrisations. Each path component of the boundary $\partial\mathcal{S}$ is endowed with *two* orientations: one induced from the orientation of $\mathcal{S}$, by using an *inwards*-pointing normal vector field, and one coming from the parametrisation. For a component of $\partial^{\text{out}}\mathcal{S}$ we require them to coincide, whereas for a component of $\partial^{\text{in}}\mathcal{S}$ we require them to differ.

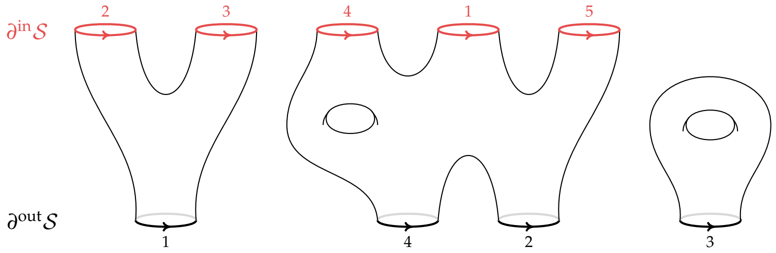


**Figure 5.1.** A surface $\mathcal{S}$ with five incoming and four outgoing boundary curves, where the arrows indicate the orientations of the parametrisations. The components of $\mathcal{S}$ receive a canonical ordering by their minimal outgoing boundary curves.

**Definition 5.1.2** (Surface types)**.** A *diffeomorphism* $\Psi\colon (\mathcal{S},\Theta) \to (\mathcal{S}',\Theta')$ of surfaces with parametrised boundary is a diffeomorphism $\Psi\colon \mathcal{S} \to \mathcal{S}'$ which preserves the parametrisations, i.e. $\Theta' = \Psi \circ \Theta$.

A *surface type* is an isomorphism class $\Sigma$ of surfaces with parametrised boundary curves. In particular, let $\Sigma_{g,n}$ be the isomorphism class of connected surfaces with genus $g \geqslant 0$ and $n \geqslant 1$ outgoing parametrised boundary curves and no incoming boundary curve.

**Remark 5.1.3.** We write $\Sigma\colon k \to n$ if $\Sigma$ is a surface type with $k$ incoming and $n$ outgoing boundary curves. Then $\Sigma$ is determined by the following data:

1. a number $1 \leqslant l \leqslant n$ of path components;
2. genera $g_1, \dots, g_l \geqslant 0$ for each path component;
3. a partition $\underline{n} = \boldsymbol{n}_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \boldsymbol{n}_l$ into non-empty subsets $\boldsymbol{n}_j$, which has no additional order, which means $\min(\boldsymbol{n}_j) < \min(\boldsymbol{n}_{j+1})$: if we number the components of $\Sigma$ by their minimal outgoing boundary curve, then the $j^{\text{th}}$ component has all outgoing boundary curves indexed in $\boldsymbol{n}_j$;
4. a partition $\underline{k} = \boldsymbol{k}_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \boldsymbol{k}_l$ into possibly empty subsets: the $j^{\text{th}}$ component has all incoming boundary curves numbered in $\boldsymbol{k}_j$.

Then these surface types assemble into a PROP $(\mathbf{hCob}_{2,\partial}, \sqcup, 0)$, which is a subcategory of $\mathbf{hCob}_2$ from Example 3.4.3, containing only isomorphism classes of cobordisms where each component has at least one outgoing boundary curve, called the *positive boundary subcategory*.

**Definition 5.1.4.** Given a surface $(\mathcal{S},\Theta)$, we denote by $\mathrm{Diff}(\mathcal{S})$ the group of automorphisms in the sense of Definition 5.1.2: it is the group of diffeomorphisms $\Phi\colon \mathcal{S} \to \mathcal{S}$ which satisfy $\Phi \circ \Theta = \Theta$. Note that this is equivalent to the condition $\Phi|_{\partial\mathcal{S}} = \mathrm{id}_{\partial\mathcal{S}}$, whence $\mathrm{Diff}(\mathcal{S})$ does not depend on $\Theta$.

We endow $\mathrm{Diff}(\mathcal{S})$ with the $C^\infty$-topology of uniform convergence of differentials of all orders and define the *mapping class group* $\Gamma(\mathcal{S}) := \pi_0(\mathrm{Diff}(\mathcal{S}))$ to be the group of isotopy classes of diffeomorphisms. We denote isotopy classes by small Greek letters and use capital Greek letters for diffeomorphisms.

The isomorphism type of $\Gamma(\mathcal{S})$ depends only on the surface type of $\mathcal{S}$: each diffeomorphism $\mathcal{S} \to \mathcal{S}'$ gives rise an isomorphism $\Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}')$ by conjugating diffeomorphisms. We denote by $\Gamma(\Sigma)$ the isomorphism type of $\Gamma(\mathcal{S})$ for some $\mathcal{S}$ of type $\Sigma$; particularly, we write $\Gamma_{g,n} := \Gamma(\Sigma_{g,n})$.

We assume that the reader is familiar with the basic properties of mapping class groups, as it is for example presented in [FM12], and hence forgo a collection of foundational examples.

**Remark 5.1.5.** Since each component of $\mathcal{S}$ has non-empty boundary, [ES70, Thm. 1D] tells us that the group $\mathrm{Diff}(\mathcal{S})$ is homotopy discrete, so the projection $\mathrm{Diff}(\mathcal{S}) \to \Gamma(\mathcal{S})$ to path components is a homotopy equivalence.

As $\mathrm{Diff}(\mathcal{S})$ is an innate structure group for surface bundles, the group cohomology of $\mathrm{Diff}(\mathcal{S})$—or, equivalently, the cohomology of $\Gamma(\mathcal{S})$—encodes characteristic classes for them.

This is the main reason why we are interested in models for the classifying spaces $B\Gamma(\mathcal{S}) \simeq B\mathrm{Diff}(\mathcal{S})$ and their (co-)homology, which leads us to the definition of a moduli space $\mathfrak{M}(\Sigma)$ of Riemann surfaces of a certain type. Here is one way of giving a precise model:

**Definition 5.1.6.** For each $\varepsilon > 0$, we define the two *$\varepsilon$-half-annuli*

$$\begin{aligned}\mathcal{A}_\varepsilon^{\mathrm{in}} &:= \{z \in \mathbb{C};\, 1 \leqslant |z| < 1+\varepsilon\},\\ \mathcal{A}_\varepsilon^{\mathrm{out}} &:= \{z \in \mathbb{C};\, 2-\varepsilon < |z| \leqslant 2\}.\end{aligned}$$

Both come with an embedding of the unit circle $\mathbb{S}^1 \subseteq \mathbb{C}$ into them as follows: we have $\alpha_\varepsilon^{\mathrm{in}} \colon \mathbb{S}^1 \hookrightarrow \mathcal{A}_\varepsilon^{\mathrm{in}}, z \mapsto z$ and $\alpha_\varepsilon^{\mathrm{out}} \colon \mathbb{S}^1 \hookrightarrow \mathcal{A}_\varepsilon^{\mathrm{out}}, z \mapsto 2 \cdot z$.

**Definition 5.1.7.** Let $\Sigma$ be a surface type. A *Riemann surface of type* $\Sigma$ is a pair $(\mathcal{F}, \Theta)$, where $\mathcal{F}$ is a Riemann surface with underlying surface $U\mathcal{F}$, such that $(U\mathcal{F}, \Theta)$ is a surface with parametrised boundary of type $\Sigma$, and the boundary parametrisations respect the complex structure as follows: there is[1] an $\varepsilon > 0$ and there are holomorphic embeddings $\tilde{\Theta}^{\mathrm{in}} \colon \underline{k} \times \mathcal{A}_\varepsilon^{\mathrm{in}} \hookrightarrow \mathcal{F}$ and $\tilde{\Theta}^{\mathrm{out}} \colon \underline{n} \times \mathcal{A}_\varepsilon^{\mathrm{out}} \hookrightarrow \mathcal{F}$ such that

$$\begin{aligned}\Theta^{\mathrm{in}} &= \tilde{\Theta}^{\mathrm{in}} \circ (\underline{k} \times \alpha_\varepsilon^{\mathrm{in}}),\\ \Theta^{\mathrm{out}} &= \tilde{\Theta}^{\mathrm{out}} \circ (\underline{n} \times \alpha_\varepsilon^{\mathrm{out}}).\end{aligned}$$

---

[1] Since there are only finitely many boundary curves, requiring a common constant $\varepsilon > 0$ for all curves does not make the condition stronger.

An isomorphism $(\mathcal{F},\Theta) \to (\mathcal{F}',\Theta')$ is a complex isomorphism $\Psi\colon \mathcal{F} \to \mathcal{F}'$ that satisfies $\Theta' = U\Psi \circ \Theta$. We denote by $\mathfrak{M}(\Sigma)$ the moduli space of Riemann surfaces of type $\Sigma$, and particularly, we abbreviate $\mathfrak{M}_{g,n} := \mathfrak{M}(\Sigma_{g,n})$. Elements in $\mathfrak{M}(\Sigma)$ are *conformal classes* $\mathcal{C} = [\mathcal{F},\Theta]$ of Riemann surfaces of type $\Sigma$.

We topologise $\mathfrak{M}(\Sigma)$ as follows: for each surface $(\mathcal{S},\Theta)$ of type $\Sigma$, we can consider the classical *Teichmüller space* $\mathfrak{T}(\mathcal{S},\Theta)$, see [Tei40], of *marked* Riemann surfaces, whose elements are equivalence classes of triples $(\mathcal{F},\Lambda,q)$, where $(\mathcal{F},\Lambda)$ is a Riemann surface and $q\colon (\mathcal{S},\Theta) \to (U\mathcal{F},\Lambda)$ is a diffeomorphism of surfaces, and where two triples $(\mathcal{F},\Lambda,q)$ and $(\mathcal{F}',\Lambda',q')$ are identified if the map $q' \circ q^{-1}\colon U\mathcal{F} \to U\mathcal{F}'$ is isotopic to a complex isomorphism $\mathcal{F} \to \mathcal{F}'$. The Teichmüller space $\mathfrak{T}(\mathcal{S},\Theta)$ can be given a metric which measures the dilatation of $q' \circ q^{-1}$, and it is a classical result that with respect to this metric, $\mathfrak{T}(\mathcal{S},\Theta)$ is contractible [Tei40].

Moreover, $\mathrm{Diff}(\mathcal{S})$ right acts on $\mathfrak{T}(\mathcal{S},\Theta)$ by $\Phi^*[\mathcal{F},\Lambda,q] = [\mathcal{F},\Lambda,q\circ\Phi]$, and this action factors through an action of $\Gamma(\mathcal{S})$. The map

$$\mathfrak{T}(\mathcal{S},\Theta) \to \mathfrak{M}(\Sigma), \quad [\mathcal{F},\Lambda,q] \mapsto [\mathcal{F},\Lambda]$$

is $\Gamma(\mathcal{S})$-invariant and the induced map $\mathfrak{T}(\mathcal{S},\Theta)/\Gamma(\mathcal{S}) \to \mathfrak{M}(\Sigma)$ is a bijection. Now we endow $\mathfrak{M}(\Sigma)$ with the quotient topology from the left side. This way of topologising $\mathfrak{M}(\Sigma)$ does not depend on the choice of $(\mathcal{S},\Theta)$ inside $\Sigma$.

Recall that every connected component of $\mathcal{S}$ has non-empty boundary. Thus, the action of $\Gamma(\mathcal{S})$ on $\mathfrak{T}(\mathcal{S},\Theta)$ is free and proper [EE69, § 5], whence $\mathfrak{M}(\Sigma)$ is indeed a classifying space for the group $\Gamma(\mathcal{S})$.

We should point out that there are many useful models for $B\mathrm{Diff}(\mathcal{S})$; for example, the space $\mathrm{Emb}_\partial(\mathcal{S},\mathbb{R}^\infty)$ of embeddings $\mathcal{S} \hookrightarrow \mathbb{R}^\infty$, which restrict to a fixed standard embedding at the boundary, is contractible, and the group $\mathrm{Diff}(\mathcal{S})$ acts freely and properly on it by precomposition. In fact, this model can equally well be used in order to describe the $E_2$-structure on $\coprod_{g\geqslant 0} B\Gamma_{g,1}$; however, the generalisation to the case with multiple boundary curves would become very technical; so we decided to stick to the other one.

**Construction 5.1.8** (Punctures)**.** The previous constructions can be repeated mutatis mutandis in combination with a finite subset $\mathcal{P} = \{P_1,\ldots,P_m\} \subseteq \mathring{\mathcal{S}}$, whose elements are called *punctures*.

A *punctured surface with parametrised boundary* is a triple $(S, \Theta, \mathcal{P})$ where $(\mathcal{S}, \Theta)$ is a surface as before, and $\mathcal{P} \subseteq \mathring{\mathcal{S}}$ is a finite subset of the interior of $\mathcal{S}$.

A *diffeomorphism* $\Psi\colon (\mathcal{S}, \Theta, \mathcal{P}) \to (\mathcal{S}', \Theta', \mathcal{P}')$ of punctured surfaces with boundary is required to additionally satisfy $\Psi(\mathcal{P}) = \mathcal{P}'$. A *punctured surface type* is an isomorphism class of punctured surfaces; in particular, let $\Sigma^m_{g,n}$ be the isomorphism class of connected surfaces with genus $g \geqslant 0$, $n \geqslant 1$ outgoing boundary curves, no incoming boundary curves, and $m \geqslant 0$ disjoint points (arbitrarily distributed over the single component of $\mathcal{S}$).

We define the group $\operatorname{Diff}(\mathcal{S}, \mathcal{P})$ to be the automorphism group of $(\mathcal{S}, \Theta, \mathcal{P})$ in the aforementioned sense; i.e. the group of diffeomorphisms $\Phi\colon \mathcal{S} \to \mathcal{S}$ which satisfy $\Phi|_{\partial\mathcal{S}} = \mathrm{id}_{\partial\mathcal{S}}$ and $\Phi(\mathcal{P}) = \mathcal{P}$; note that $\Phi$ is allowed to permute the punctures. We define the *mapping class group* $\Gamma(\mathcal{S}, \mathcal{P}) := \pi_0(\operatorname{Diff}(\mathcal{S}, \mathcal{P}))$ and write $\Gamma(\Sigma)$ for the isomorphism type of $\Gamma(\mathcal{S}, \mathcal{P})$ for some $(\mathcal{S}, \mathcal{P})$ of type $\Sigma$; particularly, we write $\Gamma^m_{g,n} := \Gamma(\Sigma^m_{g,n})$. The projection $\operatorname{Diff}(\mathcal{S}, \mathcal{P}) \to \Gamma(\mathcal{S}, \mathcal{P})$ to path components is again a homotopy equivalence.

A *punctured Riemann surface of type* $\Sigma$ is a tuple $(\mathcal{F}, \Theta, \mathcal{P})$ where $\mathcal{F}$ is a Riemann surface and $(U\mathcal{F}, \mathcal{P})$ is a punctured surface of type $\Sigma$, such that the parametrisations respect the complex structure in the above sense. An isomorphism of punctured Riemann surfaces $(\mathcal{F}, \Theta, \mathcal{P}) \to (\mathcal{F}', \Theta', \mathcal{P}')$ is a biholomorphism $\Psi\colon \mathcal{F} \to \mathcal{F}'$ such that $U\Psi$ is a diffeomorphism of punctured surfaces. We denote by $\mathfrak{M}(\Sigma)$ the moduli space of punctured Riemann surfaces of type $\Sigma$: elements are conformal classes $\mathcal{C} = [\mathcal{F}, \Theta, \mathcal{P}]$. Particularly, we write $\mathfrak{M}^m_{g,n} := \mathfrak{M}(\Sigma^m_{g,n})$. Again, $\mathfrak{M}(\Sigma)$ is a classifying space for $\Gamma(\Sigma)$.

### 5.1.2. *The coloured surface operad*

In this subsection, we rebuild the conformal cobordism category from [Seg88, § 4], with a few technical modifications which give rise to an honest (strictly unital) PROP in the sense of Subsection 3.4.2. By our general machinery from Section 3.4, we then obtain a coloured surface operad $\mathscr{M}$ which generalises the construction from [Til00].

Let us start by making the *sewing* construction precise, using collared Riemann surfaces in the following way:

**Definition 5.1.9.** Define the *standard half-annuli* $\mathcal{A}^{\mathrm{in}} := \mathcal{A}^{\mathrm{in}}_1$ and $\mathcal{A}^{\mathrm{out}} := \mathcal{A}^{\mathrm{out}}_1$, and write $\alpha^{\mathrm{in}} := \alpha^{\mathrm{in}}_1$ and $\alpha^{\mathrm{out}} := \alpha^{\mathrm{out}}_1$.

A Riemann surface $(\mathcal{F},\Theta)$ is called *regularly collared* if there are holomorphic embeddings $\bar{\Theta}^{\text{in}}\colon \underline{k}\times\mathcal{A}^{\text{in}}\hookrightarrow\mathcal{F}$ and $\bar{\Theta}^{\text{out}}\colon \underline{n}\times\mathcal{A}^{\text{out}}\hookrightarrow\mathcal{F}$ which restrict at the boundary to $\Theta^{\text{in}}$ and $\Theta^{\text{out}}$, respectively. For a punctured Riemann surface $(\mathcal{F},\Theta,\mathcal{P})$, we additionally require that $\mathcal{P}$ avoids the image of $\bar{\Theta}$. Note that if there are holomorphic extensions $\bar{\Theta}^{\text{in}}$ and $\bar{\Theta}^{\text{out}}$, then they are uniquely determined by $\Theta^{\text{in}}$ and $\Theta^{\text{out}}$ via the Cauchy–Riemann equations.

We denote by $\mathfrak{M}_\circ(\Sigma)\subseteq\mathfrak{M}(\Sigma)$ the subspace of all conformal classes of regularly collared Riemann surfaces. This inclusion is a deformation retract, a retraction given by making incoming boundary curves small enough and outgoing boundary curves large enough: formally, there is an $\varepsilon>0$ such that $\Theta^{\text{in}}$ can be extended to $\tilde{\Theta}^{\text{in}}\colon \underline{k}\times\mathcal{A}^{\text{in}}_{\varepsilon}\hookrightarrow\mathcal{F}$ and $\mathcal{P}$ avoids the image of $\tilde{\Theta}^{\text{in}}$. Let $0<\lambda\leqslant 1$ be the supremum of these $\varepsilon$, capped by 1, and glue in the collars $\underline{k}\times\{z\in\mathbb{C};\ \frac{1+\lambda}{2}\leqslant|z|<1+\lambda\}$, amalgamated over $\underline{k}\times\mathcal{A}^{\text{in}}_{\lambda}$. Then proceed similarly for the outgoing boundary curves.

This shows that $\mathfrak{M}_\circ(\Sigma)$ is another (slightly smaller) model for $B\Gamma(\Sigma)$. Particularly, we abbreviate $\mathfrak{M}^m_{g,n\cdot\circ}:=\mathfrak{M}_\circ(\Sigma^m_{g,n})$.

**Example 5.1.10.** Let $\Sigma$ be the class of connected surfaces of genus $g=0$, one incoming and one outgoing boundary curve, and no punctures. Then the moduli space $\mathfrak{M}(\Sigma)$ is homeomorphic to $\mathbb{S}^1\times(1;\infty)$ as follows: to each pair $(w,r)\in\mathbb{S}^1\times(1;\infty)$, we assign the surface $\mathcal{A}(r):=\{z\in\mathbb{C};\ 1\leqslant|z|\leqslant r\}$, together with the incoming boundary $\partial^{\text{in}}\mathcal{A}(r)=\mathbb{S}^1$, the outgoing boundary $\partial^{\text{out}}\mathcal{A}(r)=r\cdot\mathbb{S}^1$, and the parametrisations $\Theta^{\text{in}}(z)=z$ and $\Theta^{\text{out}}(z)=r\cdot w\cdot z$.

Clearly, these parametrisations can be extended to the standard half-annuli if and only if $\mathcal{A}(r)$ is 'thick enough', which means that $r\geqslant 2$. Hence, under the above identification, $\mathfrak{M}_\circ(\Sigma)$ corresponds to the subspace $[2;\infty)\times\mathbb{S}^1$.

The main merit of introducing collared versions of moduli spaces is that we gained the possibility of *sewing* two of them in a strict way. This is formalised by the construction of the following topologically enriched PROP.

**Construction 5.1.11.** We define a topological PROP $\mathbf{M}^\bullet_\partial$, called the *conformal cobordism category*, with morphism spaces

$$\mathbf{M}^\bullet_\partial\tbinom{k}{n}:=\coprod_{\Sigma\colon k\to n}\mathfrak{M}_\circ(\Sigma),$$

where $\Sigma$ ranges over all surface types of (possibly punctured) surfaces. The structure of a PROP is defined as follows:

1. **Identity.** For each $n \geqslant 0$, the identity $\mathbb{1}_n$ is given by the conformal class of $n$ disjoint standard annuli

$$\mathcal{F} := \underline{n} \times \{z \in \mathbb{C};\, 1 \leqslant |z| \leqslant 2\}.$$

2. **Composition.** If $\mathcal{C} = [\mathcal{F}, \Theta, \mathcal{P}]$ and $\mathcal{C}' = [\mathcal{F}', \Theta', \mathcal{P}']$ are two conformal classes of collared Riemann surfaces, where $\mathcal{C}\colon k \to n$ and $\mathcal{C}'\colon n \to m$, then we define $\mathcal{C}' \circ \mathcal{C}$ by *sewing* of surfaces as follows: its underlying Riemann surface is given by the pushout

$$\begin{tikzcd}
\underline{n} \times (\mathcal{A}^{\mathrm{in}} \cap \mathcal{A}^{\mathrm{out}}) \arrow[r, "\bar{\Theta}'^{\mathrm{in}}"] \arrow[d, "\bar{\Theta}^{\mathrm{out}}"'] \arrow[dr, phantom, "\ulcorner", very near end] & \mathcal{F}' \smallsetminus \partial^{\mathrm{in}} \mathcal{F}' \arrow[d] \\
\mathcal{F} \smallsetminus \partial^{\mathrm{out}} \mathcal{F} \arrow[r] & \mathcal{F}' \sqcup_{\underline{n} \times \mathcal{A}} \mathcal{F},
\end{tikzcd}$$

together with the amalgamated conformal structure: it is well-defined as the two constituents form an open cover; the new set of punctures is defined to be $\mathcal{P} \mathbin{\dot\cup} \mathcal{P}'$, using that the punctures do not touch the collars. Finally, the new boundary parametrisations of $\mathcal{C}' \circ \mathcal{C}\colon k \to m$ are given by the incoming one from $\mathcal{C}$ and the outgoing one from $\mathcal{C}'$.

3. **Monoidal sum.** For two conformal classes $\mathcal{C}$ and $\mathcal{C}'$ of collared Riemann surfaces, their monoidal sum $\mathcal{C} \sqcup \mathcal{C}'$ is defined to be the conformal class of their disjoint union, together with the disjoint union of parametrisations and puncturing subsets. The twists are given by permuting the boundary curves, that is: precomposing $\Theta^{\mathrm{in}}$ with the automorphism on $\underline{k} \times \mathbb{S}^1$ induced by the permutation on the left factor; similarly for $\Theta^{\mathrm{out}}$.

There is a subcategory $\mathbf{M}_\partial \subseteq \mathbf{M}_\partial^\bullet$ which allows only unpunctured surface types, and in fact, this is the more common one.

**Remark 5.1.12.** This construction deserves some remarks:

1. Morphisms in $\mathbf{M}_\partial$ and $\mathbf{M}_\partial^\bullet$ are called *(conformal) cobordisms*. Nowadays, topological cobordism categories are modelled in a slightly different way, see for example [Gal+09].

2. The subscript $\partial$ comes from the fact that there is also a category $\mathbf{M}$ where one drops the condition that each component of a cobordism $\mathcal{C}$ needs at least one outgoing boundary curve. Classically, $\mathbf{M}_\partial \subseteq \mathbf{M}$ is called the *positive boundary subcategory*. Note that the homotopy category $\pi_0(\mathbf{M})$ is the discrete cobordism category from Example 3.4.3.

3. The only cobordism which ends in 0 is the empty one. On the other hand, the moduli space $\mathfrak{M}^m_{g,n\cdot\circ}$ is a component of $\mathbf{M}^\bullet_\partial\binom{0}{n}$: they describe exactly the components indexed by connected surface types. Since all cobordisms in $\mathbf{M}^\bullet_\partial\binom{0}{1}$ have to be connected, we obtain
$$\mathbf{M}_\partial\tbinom{0}{1} = \coprod_{g\geqslant 0} \mathfrak{M}_{g,1\cdot\circ} \quad\text{and}\quad \mathbf{M}^\bullet_\partial\tbinom{0}{1} = \coprod_{g,m\geqslant 0} \mathfrak{M}^m_{g,1\cdot\circ}.$$

**Definition 5.1.13.** We define the *coloured surface operad* $\mathcal{M} := \mathrm{rep}(\mathbf{M}_\partial)$ to be the operad represented by $\mathbf{M}_\partial$ as defined in Construction 3.4.9, i.e. it is an $\overline{\mathbb{N}}$-coloured operad with $\mathcal{M}\binom{k_1,\dots,k_r}{n} = \mathbf{M}_\partial\binom{k_1+\dots+k_r}{n}$.

If we restrict $\mathcal{M}$ to the colour 1, then we obtain a model for the classical monochromatic surface operad which was introduced in [Til00] and which, in our language, will consistently be called $\mathcal{M}|_1$. Note that this coincides with the valency-1 part $\mathbf{M}_\partial\binom{}{1}$ of the PROP $\mathbf{M}_\partial$ as in Construction 3.4.4.

**Example 5.1.14.** In contrast to the little $d$-discs operads, the initial $\mathcal{M}$-algebra is non-trivial: for instance, its colour-1 part $\mathcal{M}\binom{}{1}$ homotopy equivalent to the familiar collection of moduli spaces
$$\mathcal{M}\tbinom{}{1} = \mathbf{M}_\partial\tbinom{0}{1} \simeq \coprod_{g\geqslant 0} \mathfrak{M}_{g,1}.$$

**Remark 5.1.15.** There is a *connective suboperad* $\mathcal{M}^c \subseteq \mathcal{M}$, similar to Subsection 3.4.5, which contains only those conformal cobordisms $\mathcal{F}$, together with input assignments, such that the equivalence relation on $\pi_0(\mathcal{F})$, which relates two components if both have an incoming boundary curve from the same input block, is full. Then the initial $\mathcal{M}^c$-algebra is given by $\mathcal{M}^c\binom{}{n} = \coprod_g \mathfrak{M}_{g,n\cdot\circ}$.

The same can be done with punctures, resulting in a coloured operad $\mathcal{M}^\bullet$ with connective part $\mathcal{M}^{\bullet,c} \subseteq \mathcal{M}^\bullet$ satisfying $\mathcal{M}^{\bullet,c}\binom{}{n} = \coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\circ}$.

**Corollary 5.1.16.** *A map $\mathfrak{O} \to \mathcal{M}^c$ of $\overline{\mathbb{N}}$-coloured operads endows $(\coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\circ})_{n\geqslant 1}$ with the structure of an $\mathfrak{O}$-algebra, which contains $(\coprod_g \mathfrak{M}_{g,n\cdot\circ})_{n\geqslant 1}$ as a subalgebra.*

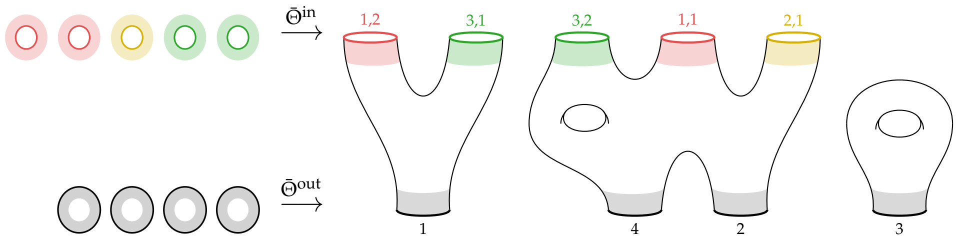


**Figure 5.2.** An operation in $\mathcal{M}\binom{2,1,2}{4}$, represented by a Riemann surface together with parametrisations of the incoming and the outgoing boundary curves, and their unique extension to the standard half-annuli.

### 5.1.3. *Coloured actions of the little 2-discs operad*

**Definition 5.1.17.** We make the little 2-discs operad $\mathcal{D}_2$ a bit smaller: recall that operations in $\mathcal{D}_2$ are tuples $(c_1, \dots, c_r)$ of embeddings $c_i \colon \mathbb{D}^2 \hookrightarrow \mathbb{D}^2$ which are of the form $c_i(z) = \hat{z}_i + \varepsilon_i \cdot z$, with $\hat{z}_i \in \mathbb{D}^2$ and $\varepsilon_i > 0$, and we require that the images $c_i(\mathring{\mathbb{D}}^2)$ of the interior of the disc are mutually disjoint.

These embeddings $c_i$ can easily be extended to $\bar{c}_i \colon \mathbb{C} \to \mathbb{C}$ by the same prescription, and we define the subspace $\mathcal{D}_2'(r) \subseteq \mathcal{D}_2(r)$ to contain all configurations $\mu = (c_1, \dots, c_r)$ of discs such that even the images $\bar{c}_i(2 \cdot \mathring{\mathbb{D}}^2)$ of the interior of the disc of radius 2 are mutually disjoint. In this case, we say the configuration $\mu$ is *2-disjoint*.

Clearly, $\mathcal{D}_2'(1) = \mathcal{D}(1)$, so the identity is contained in $\mathcal{D}_2'$, and moreover, this stronger disjointness condition is preserved by composition, whence $\mathcal{D}_2'$ is a suboperad of $\mathcal{D}_2$. The inclusion $\mathcal{D}_2' \hookrightarrow \mathcal{D}_2$ is clearly an equivalence of $\mathfrak{S}$-cofibrant operads, and in order to save notation, we might as well write $\mathcal{D}_2$ for the smaller substitute.

**Construction 5.1.18.** There is an inclusion of operads $\imath \colon \mathcal{D}_2 \hookrightarrow \mathcal{M}|_1$ as follows: given a 2-disjoint configuration $\mu = (c_1, \dots, c_r)$ with $c_i(z) = \hat{z}_i + \varepsilon_i \cdot z$, we assign to it the connected collared Riemann surface $\imath(\mu) = [\mathcal{F}, \Theta]$ which has $r$ incoming boundary curves, a single outgoing boundary curve, genus 0 and no punctures, with

$$\mathcal{F} := \{z \in \mathbb{C};\ |z| \leqslant 2\} \smallsetminus \big(c_1(\mathring{\mathbb{D}}^2) \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} c_r(\mathring{\mathbb{D}}^2)\big),$$

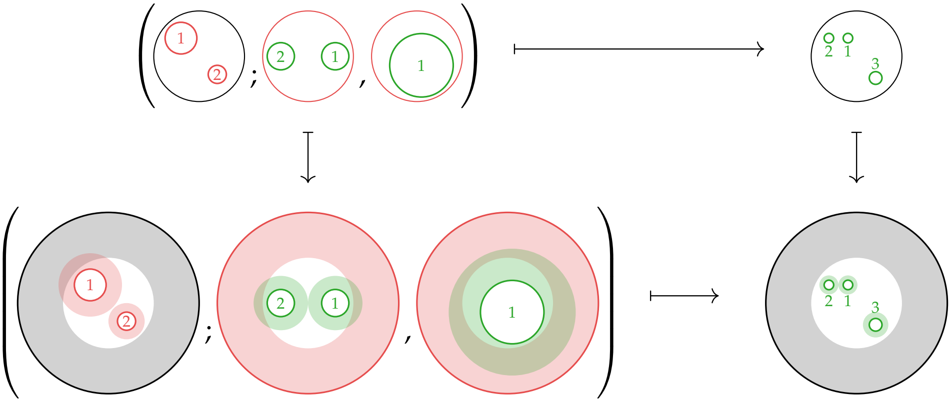


**Figure 5.3.** An instance of $\mathcal{D}_2(2) \times (\mathcal{D}_2(2) \times \mathcal{D}_2(1)) \to \mathcal{D}_2(3)$ and its transformation via $\imath$ to $\mathcal{M}|_1$.

together with the boundary parametrisations $\Theta^{\mathrm{out}}\colon \mathbb{S}^1 \hookrightarrow U\mathcal{F}, z \mapsto 2 \cdot z$ and $\Theta^{\mathrm{in}}\colon \underline{r} \times \mathbb{S}^1 \hookrightarrow U\mathcal{F}, (i,z) \mapsto c_i(z)$. Then the 2-disjointness condition ensures that $\Theta^{\mathrm{in}}$ can be extended to a holomorphic embedding $\underline{r} \times \mathcal{A}^{\mathrm{in}} \hookrightarrow \mathcal{F}$. One readily checks that $\imath$ is indeed a map of operads, i.e. it is strictly unital and commutes with the compositions, see Figure 5.3 for an example.

**Construction 5.1.19.** Since $\mathcal{M}|_1 = \mathbf{M}_\partial\binom{-}{1}$, the operad morphism $\imath\colon \mathcal{D}_2 \to \mathcal{M}|_1$ is adjoint to a map of PROPs $\bar{\imath}\colon \mathrm{cat}(\mathcal{D}_2) \to \mathbf{M}_\partial$, and by passing to representable operads, we obtain a map of $\overline{\mathbb{N}}$-coloured operads

$$\mathrm{rep}(\bar{\imath})\colon \overline{\mathbb{N}}(\mathcal{D}_2) \to \mathcal{M}.$$

This gives in particular a map $\mathcal{D}_2 \odot \mathfrak{S} \to \mathcal{M}$, so for each $\mathcal{M}$-algebra $(X_n)_{n \geqslant 1}$, the constituent $X_n$ is a $\mathcal{D}_2$-algebra together with an $\mathfrak{S}_n$-action by $\mathcal{D}_2$-automorphisms. Since the initial $\mathcal{M}$-algebra has the property $\mathcal{M}\binom{}{1} = \coprod_g \mathfrak{M}_{g,1\cdot\circ}$, this recovers the classical $E_2$-structure on moduli spaces of Riemann surfaces with a single outgoing boundary curve from [Mil86; Böd90b].

Moreover, the connective suboperad $\overline{\mathbb{N}}^c(\mathcal{D}_2)$ gets sent to the connective suboperad $\mathcal{M}^c$; in particular, by Corollary 5.1.16, the sequence $(\coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\circ})_{n \geqslant 1}$ is an $\overline{\mathbb{N}}^c(\mathcal{D}_2)$-algebra which contains $(\coprod_g \mathfrak{M}_{g,n\cdot\circ})_{n \geqslant 1}$ as a subalgebra.

**Remark 5.1.20.** The $\overline{\mathbb{N}}^c(\mathcal{D}_2)$-action respects the bigrading by the genus and the number of punctures in the following way: let $\mu \in \overline{\mathbb{N}}^c(\mathcal{D}_2)\binom{k_1,\dots,k_r}{n}$ and $\mathcal{C} = \mu(\mathcal{C}_1,\dots,\mathcal{C}_r)$. Then clearly, the punctures add up, i.e. $m(\mathcal{C}) = \sum_i m(\mathcal{C}_i)$, and a quick Euler characteristic calculation shows that for the genus, we get

$$g(\mathcal{C}) = (1-n) + \textstyle\sum_i (k_i - 1) + \sum_i g(\mathcal{C}_i).$$

**Example 5.1.21.** This construction generalises the one from [ST08] as well: they studied the space $X := \coprod_{g \geqslant -1} \mathfrak{M}_{g,2}$, where $\mathfrak{M}_{-1,2} := \mathfrak{M}(\Sigma_{0,1} \sqcup \Sigma_{0,1})$, as an algebra over $\mathcal{D}_2$, and this action is exactly the restriction of the $\mathscr{M}|_2$-action on $\coprod_{\Sigma\colon 0\to 2} \mathfrak{M}(\Sigma)$ along the aforementioned map $\mathcal{D}_2 \to \mathscr{M}|_2$, by finally noting that $\{\Sigma_{0,1} \sqcup \Sigma_{0,1}\} \cup \{\Sigma_{g,2};\, g \geqslant 0\}$ is a submonoid of $\{\Sigma\colon 0 \to 2\}$.

To be more precise, Segal and Tillmann use the *framed* little 2-discs operad $\mathcal{D}_2^{\mathrm{fr}}$, which allows, for each little disc, a rotation; clearly $\mathcal{D}_2^{\mathrm{fr}}$ can equally well be included into $\mathscr{M}|_1 = \mathbf{M}_\partial\binom{-}{1}$; we just have to additionally rotate $\Theta^{\mathrm{in}}$. Then, again, we obtain a map $\mathcal{D}_2^{\mathrm{fr}} \odot \mathfrak{S} \to \mathscr{M}$ that restricts to $\mathcal{D}_2^{\mathrm{fr}} \to \mathscr{M}|_2$.

Apart from some strictifications and combinatorial generalisations, nothing surprisingly new has happened in this chapter so far. However, to the best of my knowledge, the generalisation to the case of multiple boundary curves has not yet been exploited for either homological calculations or questions of group completions and deloopings, and this is what we will pursue in the remainder of this chapter and the next one. Let me additionally point out that the questions of Chapter 6 and their answers relie only on the material which we have already presented, and are independent of the remainder of the current chapter.

## 5.2. Operadic actions on Bödigheimer's simplicial model

From now on, we want to focus on a particular model $\mathfrak{P}^m_{g,n}$ for the moduli spaces $\mathfrak{M}^m_{g,n}$, that is based on an old work of Hilbert [Hil09], has been established by Bödigheimer [Böd90a], and admits a (relative) multisimplicial description [ABE08; BH14], which is useful for unstable calculations.

The aim of this section is to formally establish an operadic action of a variation of $\mathcal{V}^c_{1,1}$ on $(\coprod_{g,m} \mathfrak{P}^m_{g,n})_{n\geqslant 1}$ which is, up to homotopy, a restriction of the previously described action of $\overline{\mathbb{N}}^c(\mathcal{D}_2)$ on $(\coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\circ})_{n\geqslant 1}$.

### 5.2.1. *Different shapes of the collars*

For technical reasons, we first of all have to introduce two slightly different models for $\mathfrak{M}^m_{g,n}$, namely $\mathfrak{M}^m_{g,n\cdot\square}$ and $\mathfrak{M}^m_{g,n\cdot\triangleleft}$: pictorially, we replace annuli by square-shaped frames, or, more generally, by square-shaped frames where the length of the left face may be disturbed.

In order to save digits and brackets, let us abbreviate $\mathbb{I}^2 := [-1;1]^2 \subseteq \mathbb{C}$ for the centred unit square, and we write $r\mathbb{I}^2 := [-r;r]^2$ for the scaled one. Moreover, we define the *(outgoing) frame* $\mathcal{E}^{\mathrm{out}} := 2\mathbb{I}^2 \smallsetminus \mathbb{I}^2$.

**Construction 5.2.1.** Let $\mathfrak{M}^m_{g,n\cdot\square}$ be the moduli space whose elements are conformal classes $\mathcal{C} = [\mathcal{F}, \Theta, \mathcal{P}]$ of punctured Riemann surfaces (with corners[2]), together with a holomorphic parametrisation $\Theta\colon \mathcal{E}^{\mathrm{out}} \times \underline{n} \to \mathcal{F}$ which sends $\partial\mathcal{E}^{\mathrm{out}} \times \underline{n}$ to $\partial\mathcal{F}$ and avoids the punctures.

Then we clearly have a map $\mathfrak{M}^m_{g,n\cdot\circ} \to \mathfrak{M}^m_{g,n\cdot\square}$ by gluing $2\mathbb{I}^2 \smallsetminus 2\mathring{\mathbb{D}}^2$ at each boundary circle, with homotopy inverse given by gluing $(2\sqrt{2})\mathbb{D}^2 \smallsetminus 2\mathring{\mathbb{I}}^2$ at each boundary square. Hence, $\mathfrak{M}^m_{g,n\cdot\square}$ is another model for $\mathfrak{M}^m_{g,n}$.

In a similar way as $\overline{\mathbb{N}}^c(\mathscr{D}_2)$ acts on $(\coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\circ})_{n\geqslant 1}$, we can construct an action of $\overline{\mathbb{N}}^c(\mathscr{C}_2)$ on $(\coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\square})_{n\geqslant 1}$:

**Construction 5.2.2.** We modify the little 2-cubes operad by defining $\mathscr{C}_2'(r)$ to contain tuples $(c_1, \ldots, c_r)$ of conformal rectilinear embeddings

$$c_i\colon \mathbb{I}^2 \hookrightarrow \mathbb{I}^2, \quad z \mapsto \hat{z}_i + \varepsilon_i \cdot z.$$

such that the images $\bar{c}_i(2\mathring{\mathbb{I}}^2)$ of the interior of the squares with double size are mutually disjoint.

There is an inclusion $\mathscr{C}_2' \hookrightarrow \mathscr{C}_2$ by relaxing the conditions of being disjoint and conformal and by rescaling the square $\mathbb{I}^2$ to $[0;1]^2$: this is an equivalence of $\mathfrak{S}$-cofibrant operads, compare [MSS02, Prop. II.4.5]. Again, we might as well write $\mathscr{C}_2$ for the smaller and origin-centred substitute.

In the same fashion as above, $\overline{\mathbb{N}}^c(\mathscr{C}_2)$ acts on $(\coprod_{g,m} \mathfrak{M}^m_{g,\square\cdot n})_{n\geqslant 1}$: given an operation $\mu := (\vec{c}_1, \ldots, \vec{c}_r)$ with $\vec{c}_i\colon \mathbb{I}^2 \times \underline{k_i} \hookrightarrow \mathbb{I}^2 \times \underline{n}$, and $\mathcal{C}_i \in \mathfrak{M}^{m_i}_{g_i,k_i\cdot\square}$, we can construct a new surface $\mu(\mathcal{C}_1, \ldots, \mathcal{C}_r)$ by considering $(2\mathbb{I}^2 \times \underline{n}) \smallsetminus \bigcup_i \vec{c}_i(\mathring{\mathbb{I}}^2 \times \underline{k_i})$,

---

[2] One way to formalise this is to consider instead the moduli space of *closed* Riemann surfaces $\bar{\mathcal{F}}$, together with a holomorphic embedding $\bar{\Theta}\colon (\mathbb{C}\mathrm{P}^1 \smallsetminus \mathbb{I}^2) \times \underline{n} \to \bar{\mathcal{F}}$ which avoids the punctures; then $\mathcal{F} = \bar{\mathcal{F}} \smallsetminus \bar{\Theta}((\mathbb{C}\mathrm{P}^1 \smallsetminus 2\mathbb{I}^2) \times \underline{n})$ and $\Theta = \bar{\Theta}|_{\mathcal{E}^{\mathrm{out}} \times \underline{n}}$.

and by gluing in $\mathcal{C}_1, \dots, \mathcal{C}_r$ at the arising $k_1 + \dots + k_r$ square-shaped holes. The resulting surface is connected since $\mu$ was connective.

We have an equivariant family of equivalences $\mathscr{D}_2(r) \hookrightarrow \mathscr{C}_2(r)$ by taking, for each configuration of $r$ small discs, the configuration of maximal squares inside these discs. This gives rise to an equivariant family $\varphi$ of equivalences $\overline{\mathbb{N}}^c(\mathscr{D}_2)\binom{K}{n} \to \overline{\mathbb{N}}^c(\mathscr{C}_2)\binom{K}{n}$, and we easily see that the square

$$\begin{tikzcd}
\overline{\mathbb{N}}^c(\mathscr{D}_2)\binom{k_1,\dots,k_r}{n} \times \prod_i \mathfrak{M}^{m_i}_{g_i,k_i\cdot\circ} \arrow[r] \arrow[d, "\simeq"'] & \mathfrak{M}^{m_1+\dots+m_r}_{g,n\cdot\circ} \arrow[d, "\simeq"] \\
\overline{\mathbb{N}}^c(\mathscr{C}_2)\binom{k_1,\dots,k_r}{n} \times \prod_i \mathfrak{M}^{m_i}_{g_i,k_i\cdot\square} \arrow[r] & \mathfrak{M}^{m_1+\dots+m_r}_{g,n\cdot\square}
\end{tikzcd}$$

is H-commutative, where $g$ is determined as in Remark 5.1.20. Furthermore, although $\varphi$ is not a strict operad morphism, all necessary diagrams commute up to homotopy. Hence, the homology operations arising from both actions are comparable via the induced isomorphism $H_\bullet(\mathfrak{M}^m_{g,n\cdot\circ}) \to H_\bullet(\mathfrak{M}^m_{g,n\cdot\square})$.

We therefore may use squares instead of discs from now on. However, in order to use the advertised model $\mathfrak{P}^m_{g,n}$, we need a slightly more flexible boundary behaviour: for each small square, we focus on the left vertical strip of its frame; then, roughly speaking, the following happens in the context of slit pictures: firstly, each puncture 'steals' a segment of positive height from the left strips, and secondly, the boundary squares can 'steal from each other'. In this way, the size of the left vertical strip can become larger or smaller, and the sum of their differences is 0 if there are no punctures, and negative else. Formally, this is captured by the following definition:

**Definition 5.2.3.** For a real number $\omega > -2$, we define the *$\omega$-frame* $\mathcal{E}^{\mathrm{out}}(\omega)$ as the Riemann surface that arises from gluing $S := [-2;1) \times [-1;1+\omega]$ into $\mathcal{E}^{\mathrm{out}} \setminus [-2;1) \times (-1;1)$ by identifying both copies of $(x,-1)$, as well as $(x, 1+\omega)$ from $S$ with $(x,1)$ from the remainder of $\mathcal{E}^{\mathrm{out}}$; then $\mathcal{E}^{\mathrm{out}}(0) = \mathcal{E}^{\mathrm{out}}$.

For technical reasons, we also define $\mathcal{E}(\omega)$, which additionally contains the inner boundary, i.e. we start with $2\mathbb{I}^2 \setminus \mathring{\mathbb{I}}^2$, remove the left strip, and glue in $[-2;1] \times [-1;1+\omega]$. We call $\mathcal{E}(\omega) \setminus \mathcal{E}^{\mathrm{out}}(\omega)$ the *inner boundary*.

Now we are ready to define the space $\mathfrak{M}^m_{g,n\cdot\trapezium}$:

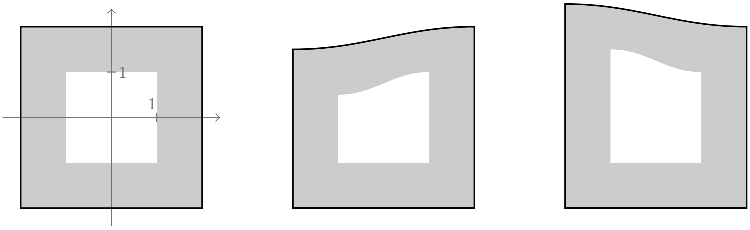


**Figure 5.4.** The three square frames $\mathcal{E}^{\mathrm{out}}$, $\mathcal{E}^{\mathrm{out}}(-1/2)$, and $\mathcal{E}^{\mathrm{out}}(1/2)$.

**Definition 5.2.4.** For $n \geqslant 1$ and $m \geqslant 0$, we define the parameter spaces

$$\Omega^m_n := \begin{cases} \{(\omega_1, \dots, \omega_n) \in (-2; \infty)^n;\ \omega_1 + \cdots + \omega_n = 0\} & \text{if } m = 0, \\ \{(\omega_1, \dots, \omega_n) \in (-2; \infty)^n;\ \omega_1 + \cdots + \omega_n \leqslant 0\} & \text{if } m > 0, \end{cases}$$

and for technical reasons, we also define $\mathring{\Omega}^0_n := \Omega^0_n$ and $\mathring{\Omega}^m_n := \Omega^m_n \setminus \Omega^0_n$. Now we let $\mathfrak{M}^m_{g,n\cdot\lhd}$ be the moduli space whose elements are conformal classes $\mathcal{C} = [\mathcal{F}, \Theta, \mathcal{P}]$ of punctured Riemann surfaces which come together with $(\omega_1, \dots, \omega_n) \in \Omega^m_n$, and holomorphic parametrisations $\Theta \colon \coprod_\ell \mathcal{E}^{\mathrm{out}}(\omega_\ell) \to \mathcal{F}$ that avoid the punctures.

Then we have a bundle map $\mathrm{pr} \colon \mathfrak{M}^m_{g,n\cdot\lhd} \to \Omega^m_n$ which assigns to each $\mathcal{C}$ the tuple $(\omega_1, \dots, \omega_n)$, and the subspace $\mathfrak{M}^m_{g,n\cdot\square} \subseteq \mathfrak{M}^m_{g,n\cdot\lhd}$ is the fibre of $(0, \dots, 0)$. Since the base space $\Omega^m_n$ is contractible, the inclusion is an equivalence. We call the parameters $\omega_1, \dots, \omega_n$ the *deviation* of the boundary curves.

### 5.2.2. *The space of slit domains*

Here we give a short survey on the relative simplicial model of *slit domains* [Böd90a; ABE08], which has been extended to the case of multiple boundary curves in [BH14]. We omit a complete description of the combinatorics of the simplicial boundary maps and refer to [BH14, § 2.3] instead.

**Definition 5.2.5.** For each parameters $g, m \geqslant 0$ and $n \geqslant 1$, there is relative $(1+n)$-semisimplicial complex $(P^m_{g,n}, P'^m_{g,n})$ whose interior $|P^m_{g,n}| \setminus |P'^m_{g,n}|$ we denote by $\mathfrak{P}^m_{g,n}$, and whose non-degenerate $(q, p_1, \dots, p_n)$-cells are given by tuples $\pi = (\sigma_q, \dots, \sigma_0)$, where $\sigma_i$ is an automorphism of the *augmented tableau*

$$\mathbb{Y}^0_{p_1,\dots,p_n} := \{(\ell, \jmath);\ 1 \leqslant \ell \leqslant n \text{ and } 0 \leqslant \jmath \leqslant p_\ell\},$$

and where $\pi$ has to satisfy the properties of [BH14, Def. 2.3.3]:

- S1. $\sigma_0 = \prod_\ell((\ell,0)\cdots(\ell,p_\ell))$, where $((\ell,0)\cdots(\ell,p_\ell))$ denotes a cycle;
- S2. $\sigma_i(\ell,p_\ell) = (\ell,0)$ for each $0 \leqslant i \leqslant q$ and $1 \leqslant \ell \leqslant n$;
- S3. $\sigma_q$ has $n+m$ cycles (including fixed points), and no cycle contains two different symbols of the form $(\ell,0)$.
- S4. if $N$ denotes the word length norm with respect to the generating set of all transpositions, then $\sum_i N(\sigma_i \cdot \sigma_{i-1}^{-1}) = 2g+m+2n-2$;
- S5. the relation on $\underline{n}$, which is spanned by $\ell \sim \ell'$ if there are $i, \jmath, \jmath'$ with $\sigma_i(\ell,\jmath) = (\ell',\jmath')$, is full;
- S6. the tuple is 'minimal' in two ways: we have $\sigma_i \neq \sigma_{i-1}$, and there is no $(\ell,\jmath)$ such that $\sigma_i(\ell,\jmath) = (\ell,\jmath+1)$ for *all* $i$.

**Construction 5.2.6.** If we use the coordinates $\Delta^r = \{-1 \leqslant t_1 \leqslant \cdots \leqslant t_r \leqslant 1\}$, then each *slit domain* $\mathcal{B} \in \mathfrak{P}^m_{g,n}$ is represented by a tuple

$$\big(\pi; -1 < x_q < \cdots < x_1 < 1; (-1 < y_{\ell,1} < \cdots < y_{\ell,p_\ell} < 1)_{1\leqslant\ell\leqslant n}\big),$$

where $\pi$ is a non-degenerate cell in the above sense. We can visualise $\mathcal{B}$ in the following way, see Figure 5.5:

- subdivide $\mathbb{I}^2 \times \underline{n}$ into rectangles $R_{i,\ell,\jmath} := [x_{i+1}; x_i] \times [y_{\ell,\jmath}; y_{\ell,\jmath+1}] \times \{\ell\}$, where $x_0 = y_{\ell,p_\ell+1} = 1$ and $x_{q+1} = y_{\ell,0} = -1$;
- glue the left face of $R_{i,\ell,\jmath}$ to the right face of $R_{i+1,\ell,\jmath}$;
- if $\sigma_i(\ell,\jmath) = (\ell,\jmath+1)$, then we glue the top face of $R_{i,\ell,\jmath}$ to the bottom face of $R_{i,\ell,\jmath+1}$;
- if not, then we mark the top face of $R_{i,\ell,\jmath}$ and the bottom face of $R_{i,\sigma_i(\ell,\jmath)}$ with the same colour. In this case, the top face of $R_{i,\ell,\jmath}$ and the bottom face of $R_{i,\ell,\jmath+1}$ form a *slit*, see Figure 5.5.

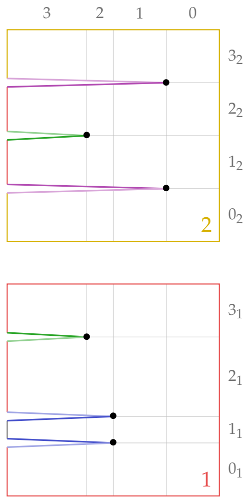


**Figure 5.5.** The visualisation of the slit domain $(\pi; x, y_1, y_2)$ with coordinates $x = (-1/4, 0, 1/2)$, $y_1 = (-1/2, -1/4, 1/2)$, and $y_2 = (-1/2, 0, 1/2)$, and the simplex $\pi = (\sigma_3, \sigma_2, \sigma_1, \sigma_0)$ with $\sigma_3 = (0_1\, 2_1\, 2_2\, 1_2\, 3_1)(1_1)(0_2\, 3_2)$, abbreviating $\jmath_\ell := (\ell, \jmath)$. We have $m = 1$ and $g = 0$, as well as the deviations $\omega_1 = 3/4$ and $\omega_2 = -1$.

**Construction 5.2.7.** Given $\mathcal{B} \in \mathfrak{P}^m_{g,n}$, we can form an element $G(\mathcal{B}) \in \mathfrak{M}^m_{g,n \cdot \triangleleft}$ by continuing the above gluing recipe as follows:

- identify faces of the same marking colour, i.e. the top face of $R_{i,\ell,\jmath}$ with the bottom face of $R_{i,\sigma_i(\ell,\jmath)}$, and call the result $\tilde{G}(\mathcal{B})$;
- for each layer $1 \leqslant \ell \leqslant n$, if $((\ell_1, \jmath_1) \cdots (\ell_s, \jmath_s))$ is the cycle of $\sigma_q$ with $(\ell_1, \jmath_1) = (\ell, 0)$, then we define the deviation

$$\omega_\ell := \sum_{a=1}^{s} y_{\ell_a, \jmath_a + 1} - y_{\ell_a, \jmath_a}.$$

- note that $\tilde{G}(\mathcal{B})$ has $n + m$ boundary curves, and for each $1 \leqslant \ell \leqslant n$, the boundary curve of $\tilde{G}(\mathcal{B})$ which contains the bottom face of $R_{q,\ell,0}$ can be identified with the inner boundary of $\mathcal{E}(\omega_\ell)$. Thus, we can attach the $\omega_\ell$-frame $\mathcal{E}(\omega_\ell)$ to it.

- to the remaining $m$ boundary curves, we attach a fixed Euclidean disc which contains a single puncture.
- the complex structure is given by noticing that each point has a neighbourhood that is canonically identified with an open subset of $\mathbb{C}$, apart from the endpoints of a slit; here we obtain a branching point, with ramification index the length of the cycle of $\sigma_i \cdot \sigma_{i-1}^{-1}$ which contains the respective point.

In this way, we have reached a Riemann surface $G(\mathcal{B})$, which is connected by s5, and a quick Euler characteristic check shows that it has genus $g$ by the norm condition s4. Now a key result of [Böd90a, § 5.2] tells us that the arising map $G\colon \mathfrak{P}_{g,n}^m \to \mathfrak{M}_{g,n\cdot\triangleleft}^m$, called the *gluing construction*, is not only continuous, but a homotopy equivalence.

**Remark 5.2.8.** Bödigheimer models the gluing construction in a slightly different manner: he uses the coordinates $\Delta^r = \{-\infty \leqslant t_1 \leqslant \cdots \leqslant t_r \leqslant +\infty\}$, constructs an open surface by discarding $\partial[-\infty;\infty]^2 \times \underline{n}$, and closes each end with a single point. Those ends which correspond to boundary curves come with a choice of tangent vector, by pointing from $+\infty$ leftwards. The result lies in $\vec{\mathfrak{M}}_{g,n}^m$, another model for $\mathfrak{M}_{g,n}^m$ where boundary curves are replaced by ordered punctures $Q_1, \ldots, Q_n$ on a *closed* surface $\bar{\mathcal{F}}$ which come together with tangential directions.

An inverse of the gluing construction is based on a uniformisation method of Hilbert [Hil09] and uses a harmonic function $u\colon \bar{\mathcal{F}} \to \mathbb{R} \cup \{\infty\}$ in order to reach a normal form for the given Riemann surface. The crucial point of the construction is the observation that this harmonic function is unique up to a contractible (even affine) choice [Koc91, Thm. 11.3], whence the inverse of the gluing construction is defined on an affine bundle over $\vec{\mathfrak{M}}_{g,n}^m$.

In this description, the parameters $\omega_1, \ldots, \omega_n$ correspond, up to rescaling $[-\infty;\infty]$ to $[-1;1]$, to the residues of $u$ at $Q_1, \ldots, Q_n$.

Recall that $\overline{\mathbb{N}}^c(\mathscr{C}_2)$ acts on $(\coprod_{g,m} \mathfrak{M}_{g,n\cdot\square}^m)_{n\geqslant 1}$. We want to construct a similar description for $(\coprod_{g,m} \mathfrak{M}_{g,n\cdot\triangleleft}^m)_{n\geqslant 1}$, and $(\coprod_{g,m} \mathfrak{P}_{g,n}^m)_{n\geqslant 1}$, by the implantation of slit domains in the spirit of [Böd90b, § 3], such that the actions are comparable via the zig-zag $\mathfrak{M}_{g,n\cdot\square}^m \hookrightarrow \mathfrak{M}_{g,n\cdot\triangleleft}^m \leftarrow \mathfrak{P}_{g,n}^m$ of homotopy equivalences.

Here, two independent problems arise:

- As one can see in Figure 5.5, the lengths of corresponding slits are coupled, even if they lie on different layers. Hence, if we want to produce a larger slit domain by placing the existing squares into new ones, the layers from a single slit domain have to stay vertically aligned. This condition is exactly the one captured by the suboperad $\mathscr{V}^c_{1,1} \subseteq \overline{\mathbb{N}}^c(\mathscr{C}_2)$. However, there is no need to assume this for $\mathfrak{M}^m_{g,n\cdot\text{◿}}$. We therefore expect an action of $\mathscr{V}^c_{1,1}$ on $(\coprod_{g,m} \mathfrak{P}^m_{g,n})_{n\geqslant 1}$ that extends, via $G\colon \mathfrak{P}^m_{g,n} \to \mathfrak{M}^m_{g,n\cdot\text{◿}}$, to an action of $\overline{\mathbb{N}}^c(\mathscr{C}_2)$ on $(\coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\text{◿}})_{n\geqslant 1}$. One special case[3] of these operations using verticality has been described in [Böd90b, § 5] where pairs of vertically aligned boxes have been considered.

- In both situations, we have to deal with the fact that the boundary of the arguments we want to glue in is not quite the boundary of a square, but has a deviated left face. To make this precise, we need an enlargement of both $\overline{\mathbb{N}}^c(\mathscr{C}_2)$ and $\mathscr{V}^c_{1,1}$, in which each operation already 'knows' the deviation of its arguments, and if the argument has $k$ boundary curves, then the deviation ranges in a contractible subspace of $(-2;\infty)^k$.

Note that the second issue does not appear in the case which is considered in [Böd90b]: if $n = 1$ and $m = 0$, i.e. we have a single boundary curve and no punctures, then all occurring deviations are trivial.

### 5.2.3. *Interlude: Operads with a colour space*

This subsection provides a setting in which the second of the aforementioned problems can be addressed: we have to enlarge the colour *set* $\overline{\mathbb{N}}$ to a colour *space* $\coprod_{n\geqslant 1}(-2;\infty)^n$, and this is formally captured by the notion of an *internal operad*, which generalises the concept of an internal category from [Gro61] in a straightforward way. We only need the following special case, which is homotopically well-behaved.

---

[3] To be precise, Bödigheimer did not consider slit domains describing surfaces with two boundary curves, but so-called 'partitioned slit domains', which, in our language, are slit domains in $\mathfrak{P}_{g-1,2}$, identified with the moduli space $\mathfrak{M}_{g,1}$ by gluing a pair of pants.

**Definition 5.2.9.** Let $N$ be a set and let $(\Omega_n)_{n\in N}$ be a family of contractible spaces. Then an $\coprod_n \Omega_n$*-coloured operad* $\mathfrak{O}$ is given by the following data:

1. for each $n \in N$, a functor $\mathfrak{O}\binom{-}{n} := (N \wr \mathbf{\Sigma})^{\mathrm{op}} \to \mathbf{Top}$,
2. for each $k_1, \dots, k_r, n \in N$, a *source map*[4] $d_1$ and a *target map* $d_0$ as in
$$\textstyle\prod_i \Omega_{k_i} \xleftarrow{\;d_1\;} \mathfrak{O}\binom{k_1,\dots,k_r}{n} \xrightarrow{\;d_0\;} \Omega_n;$$
3. for each $n \in N$, a map $\mathbb{1}_n \colon \Omega_n \to \mathfrak{O}\binom{n}{n}$, called the *unit*,
4. for $K = (k_1, \dots, k_r)$ and tuples $L_1, \dots, L_r$, a *composition map*
$$\mathfrak{O}\binom{K}{n} \times^{\prod_i \Omega_{k_i}} \prod_{i=1}^{r} \mathfrak{O}\binom{L_i}{k_i} \to \mathfrak{O}\binom{L_1 \cdots L_r}{n},$$
where '$\times^{\prod_i \Omega_{k_i}}$' denotes the fibre product of spaces,

which have to satisfy the obvious axioms. An $\mathfrak{O}$*-algebra* is given by a collection of maps $(d_0 \colon X_n \to \Omega_n)_{n\in N}$, together with maps
$$\lambda \colon \mathfrak{O}\binom{k_1,\dots,k_r}{n} \times^{\prod_i \Omega_{k_i}} \prod_{i=1}^{r} X_{k_i} \to X_n,$$
satisfying the obvious axioms. We additionally require the following two properties, which ensure a homotopically nice behaviour:

- we require that the source assignment $d_1 \colon \mathfrak{O}\binom{K}{n} \to \Omega^K$ is a fibre bundle, and that for each path component $P \subseteq \mathfrak{O}\binom{K}{n}$, the subspace $d_0(P) \subseteq \Omega_n$ is contractible and $d_0 \colon P \to d_0(P)$ is a fibre bundle;
- for an algebra, we require that for each component $P \subseteq X_n$, the subspace $d_0(P) \subseteq \Omega_n$ is contractible and $d_0 \colon P \to d_0(P)$ is a fibre bundle.

If we fix, for each $n \in N$, a basepoint $0_n \in \Omega_n$ and regard $N$ as a discrete subspace of $\coprod_n \Omega_n$ via the inclusion of basepoints, then the restriction $\mathfrak{O}|_N$ is a usual $N$-coloured operad, to which all our constructions from Chapter 3 apply. The following proposition shows that the same applies, homologically, to $\mathfrak{O}$-algebras.

---

[4] The symbols $d_0$ and $d_1$ are borrowed from the simplicial description of the corresponding topological nerve.

**Proposition 5.2.10.** *Let $\mathfrak{O}$ be an $\coprod_n \Omega_n$-coloured operad and $\boldsymbol{X}$ be an $\mathfrak{O}$-algebra.*

1. *the family $(H_\bullet(X_n))_{n\in N}$ carries the structure of an $H_\bullet(\mathfrak{O}|_N)$-algebra.*
2. *if $\mathfrak{S}_2$ acts freely on $\mathfrak{O}\binom{k,k}{n}$, and we work over $\mathbb{F}_2$ or $h$ is even, then we have, for each class $c \in H_s(\mathfrak{O}|_N\binom{k,k}{n}/\mathfrak{S}_2)$, a divided power operation*
$$Q_c \colon H_h(X_k) \to H_{s+2h}(X_n),$$
*and all relations for the homology of $\mathfrak{O}|_N$-algebras hold in $H_\bullet(\boldsymbol{X})$ as well.*

*Proof.* Let $P_i \subseteq X_{k_i}$ be a path component and write $d_0P := \prod_i d_0P_i$. Then the inclusion $d_1^{-1}(d_0P) \hookrightarrow \mathfrak{O}\binom{K}{n}$ is a restriction of the bundle $d_1 \colon \mathfrak{O}\binom{K}{n} \to \Omega^K$ to a contractible subspace of the base, and hence is itself an equivalence.

Secondly, the subspace inclusion $d_1^{-1}(d_0P) \times^{d_0P} \prod_i P_i \hookrightarrow d_1^{-1}(d_0P) \times \prod_i P_i$ is a homotopy equivalence since the fibre product is taken over a contractible space and both structure maps are fibrations. We obtain a zig-zag

$$\begin{tikzcd}
(\mathfrak{O}|_N)\binom{K}{n} \times \prod_i P_i \arrow[d, "\simeq"'] \arrow[ddd, dashed, bend left] & \\
\mathfrak{O}\binom{K}{n} \times \prod_i P_i & \\
d_1^{-1}(d_0P) \times \prod_i P_i \arrow[u, "\simeq"] & \\
d_1^{-1}(d_0P) \times^{d_0P} \prod_i P_i \arrow[u, "\simeq"] \arrow[r, "\lambda"'] & X_n,
\end{tikzcd}$$

where the dashed arrow is an arrow in the homotopy category. In particular, after applying the homotopy functor $H_\bullet$, we reach the desired morphism $H_\bullet((\mathfrak{O}|_N)\binom{K}{n}) \otimes \bigotimes_i H_\bullet(P_i) \to H_\bullet(X_n)$ for each choice of path components $P_i \subseteq X_{k_i}$. A lengthy diagram chase now shows that these maps satisfy the compatibility requirements for an operadic action, essentially because there are no choices of inverses involved.

For the divided power operations, choose a path component $P \subseteq X_k$. By Lemma 3.3.9, each choice of representative $\mu \in C_s^{\mathrm{sing}}(\mathfrak{O}|_N\binom{k,k}{n}) \subseteq C_s^{\mathrm{sing}}(\mathfrak{O}\binom{k,k}{n})$ gives rise to a map $\tilde{Q}_c \colon H_h(P) \to H_{s+2h}(\mathfrak{O}\binom{k,k}{n} \times_{\mathfrak{S}_2} P^2)$ by sending $[\xi]$ to the class $[\mu \otimes_{\mathfrak{S}_2} \xi^{\otimes 2}]$. Secondly, the inclusion $\iota \colon d_1^{-1}(d_0P^2) \times^{d_0P^2} P^2 \hookrightarrow \mathfrak{O}\binom{k,k}{n} \times P^2$

is a $\mathfrak{S}_2$-equivariant homotopy equivalence and $\mathfrak{S}_2$ acts freely on both sides. Hence the induced map $H_\bullet(d_1^{-1}(d_0P^2) \times_{\mathfrak{S}_2}^{d_0P^2} P^2) \to H_\bullet(\mathfrak{S}\binom{k,k}{n} \times_{\mathfrak{S}_2} P^2)$ is an isomorphism and we obtain the upper arrow in the square

$$\begin{tikzcd}
H_h(P) \arrow[r, dashed, "Q_c^P"] \arrow[d, "\tilde{Q}_c"'] & H_{s+2h}(X_n) \\
H_{s+2h}(\mathfrak{S}\binom{k,k}{n} \times_{\mathfrak{S}_2} P^2) & H_{s+2h}(d_1^{-1}(d_0P^2) \times_{\mathfrak{S}_2}^{d_0P^2} P^2). \arrow[l, "\cong", "H_{s+2h}(\iota_{\mathfrak{S}_2})"'] \arrow[u, "H_{s+2h}(\lambda_{\mathfrak{S}_2})"']
\end{tikzcd}$$

Finally, in order to define $Q_c$ on $H_h(X_k) = \bigoplus_P H_h(P)$, we employ the sum formula from Remark 3.3.14: let $x = x_1 + \cdots + x_t \in H_h(X_k)$ be a class with $x_l \in H_h(P_l)$, then we define $Q_c(x)$ by induction on $t$, with $Q_c(0) = 0$ and

$$Q_c(x) := Q_c^{P_1}(x - x_t) + Q_c(x_t) + (\mathrm{pr}^! c)((x - x_t) \otimes x_t),$$

where $\mathrm{pr}^!$ is the homological transfer. Again, the compatibility requirements are easily checked as there are no choices of inverses involved. □

Let me point out that the above construction for the divided power operations works in higher generality, i.e. for higher arity and not only over $\mathbb{F}_2$, compare Subsection 3.3.1. However, as we will make no use of it, we skipped this generalisation in order to save notation.

### 5.2.4. *Operadic actions on the space of slit domains*

**Construction 5.2.11.** We define the $(-2;\infty)$-*coloured little 2-cubes operad* $\mathscr{C}_2^{\trianglelefteq}$ by constructing, for each $r \geqslant 0$, the operation space $\mathscr{C}_2^{\trianglelefteq}(r)$ in several steps:

- For each integer $u \geqslant 0$, we define $\mathscr{C}_2^{\infty}(u)$ to be the space of all tuples $v := (c_1, \ldots, c_u, \omega_1, \ldots, \omega_u)$ with:
  - $\omega_l \in (-2;\infty)$ for $1 \leqslant l \leqslant u$;
  - $c_l \colon \mathbb{I}^2 \hookrightarrow [-1;1] \times [-1;\infty)$ is a rectilinear and conformal embedding, and the images of $c_1, \ldots, c_u$ have disjoint interiors;
  - we additionally require $c_l^1(1) = c_{l'}^1(1)$, where $c_l^1 \colon \mathbb{I}^1 \hookrightarrow \mathbb{I}^1$ is the restriction to the first coordinate. This means that the $u$ squares may have different size, but they are 'right-justified'.

- From each $\nu \in \mathscr{C}_2^\infty(u)$, we extract the following data:
  - its joint interior image $\operatorname{im}(\nu) := \bigcup_l c_l(\mathring{\mathbb{I}}^2)$;
  - the common right border $x_\nu := c_1^1(1)$; then $-1 < x_\nu \leqslant 1$;
  - if $\varepsilon_l > 0$ denotes the scaling factor of the embedding $c_l$, then we put $\omega_\nu := \varepsilon_1 \cdot \omega_1 + \cdots + \varepsilon_u \cdot \omega_u$;
  - the automorphism $\tilde{\varphi}_\nu$ of $[-1;\infty)$ that scales the vertical segment attained by $c_l$ by the factor $1 + \frac{\omega_l}{2}$. We write $\varphi_\nu := \mathrm{id}_{[-1;x_\nu]} \times \tilde{\varphi}_\nu$.
- For each surjection $\pi \colon \underline{r} \to \underline{s}$ with $u_a := \#\pi^{-1}(a)$, we construct a space
$$\mathscr{C}_2^{\lhd}(r)_\pi := \left\{ (\nu_a)_{a=1}^s \in \textstyle\prod_a \mathscr{C}_2^\infty(u_a);\ \begin{smallmatrix} \operatorname{im}(\nu_a) \subseteq [-1;x_{\nu_a}] \times [-1;1+\sum_{a'<a}\omega_{\nu_{a'}}], \\ \operatorname{im}(\nu_a) \cap \bigcup_{a'<a}(\varphi_{\nu_{a-1}} \cdots \varphi_{\nu_{a'}})(\operatorname{im}(\nu_{a'})) = \varnothing \end{smallmatrix} \right\}.$$
- For each surjection $\alpha \colon \underline{s} \to \underline{s'}$, we get a cofibration $\alpha^* \colon \mathscr{C}_2^{\lhd}(r)_{\alpha\pi} \hookrightarrow \mathscr{C}_2^{\lhd}(r)_\pi$ by permuting box labels and by splitting entries $\nu_a$ into several neighboured ones. We use these maps to glue $\mathscr{C}_2^{\lhd}(r)$ together as
$$\mathscr{C}_2^{\lhd}(r) := \coprod_{\pi \colon \underline{r} \twoheadrightarrow \underline{s}} \{\pi\} \times \mathscr{C}_2^{\lhd}(r)_\pi \Big/ (\alpha\pi, \boldsymbol{\nu}) \sim (\pi, \alpha^*\boldsymbol{\nu}).$$

For each surjective map $\pi \colon \underline{r} \to \underline{s}$ with $u_a := \#\pi^{-1}(a)$, we have a canonical identification $\pi_\# \colon \underline{r} \to \coprod_a \pi^{-1}(a) \cong \Upsilon_{u_1,\ldots,u_s}$ and for each $\mu := [\pi;\boldsymbol{\nu}] \in \mathscr{C}_2^{\lhd}(r)$, we can write $\boldsymbol{\nu} = (\mu_1, \ldots, \mu_s)$ with $\mu_a = (c_{a,1}, \ldots, c_{a,u_a}, \omega_{a,1}, \ldots, \omega_{a,u_a})$. We let $\omega_i := \omega_{\pi_\#(i)}$ and $c_i := c_{\pi_\#(i)}$, and define
$$d_0\mu := \omega_{\mu_1} + \cdots + \omega_{\mu_s},$$
$$d_1\mu := (\omega_1, \ldots, \omega_r).$$

Input permutation is given $\tau^*[\pi, \boldsymbol{\nu}] = [\pi\tau, \boldsymbol{\nu}]$, the unit $\mathbb{1} \colon (-2;\infty) \to \mathscr{C}_2^{\lhd}(1)$ assigns to each $\omega$ the pair $(\mathrm{id}_{\mathbb{I}^2}, \omega)$, and composition is given by composing rectilinear embeddings. For this, it is important that during this process, the deviation does not change, see Figure 5.6.

Finally, note that the maps $d_0$ and $d_1$ are indeed fibre bundles and that the monochromatic operad $\mathscr{C}_2^{\lhd}|_{\{0\}}$, in which all parameters $\omega$ are 0, is exactly the classical little 2-cubes operad $\mathscr{C}_2$.

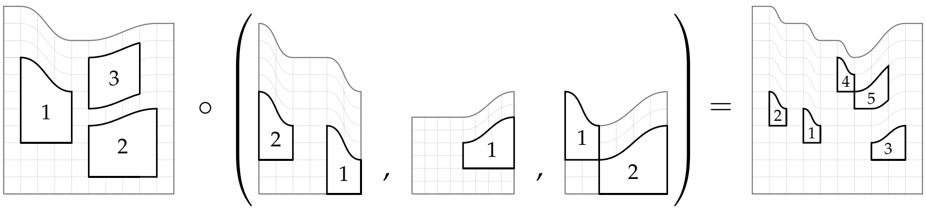


**Figure 5.6.** An instance of $\mathscr{C}_2^{\lhd}(3) \times^{\Omega^3} (\mathscr{C}_2^{\lhd}(1) \times \mathscr{C}_2^{\lhd}(2) \times \mathscr{C}_2^{\lhd}(2)) \to \mathscr{C}_2^{\lhd}(5)$. If we denote the outer operation by $\mu$, then $d_1\mu = (4/3, -1/2, 0)$, and since the scaling factors are $r_1 = r_3 = 3/10$ and $r_2 = 2/5$, we get $d_0\mu = 1/5$. The grid illustrates where area is gained or lost.

Next, we want to consider a dyed version of $\mathscr{C}_2^{\lhd}$ that extends $\overline{\mathbb{N}}^c(\mathscr{C}_2)$. We give a general recipe how to dye an operad with contractible colour space.

**Construction 5.2.12.** Let $\Omega$ be contractible and let $\mathscr{C}$ be an $\Omega$-coloured operad. Then, as in Section 3.4, we can define a dyed version $\overline{\mathbb{N}}(\mathscr{C})$ with colour space $\coprod_{n\geqslant 1}\Omega^n$ and operation spaces

$$\begin{aligned}\overline{\mathbb{N}}(\mathscr{C})\tbinom{k_1,\dots,k_r}{n} &:= \coprod_{f\colon \underline{k}\to\underline{n}} \mathscr{C}(\#f^{-1}(1)) \times \cdots \times \mathscr{C}(\#f^{-1}(n)) \\ &\cong \coprod_{u_1,\dots,u_n\geqslant 0} \big(\mathscr{C}(u_1) \times \cdots \times \mathscr{C}(u_n)\big) \times_{\prod_\ell \mathfrak{S}_{u_\ell}} \mathbf{\Sigma}\tbinom{k_1+\cdots+k_r}{u_1+\cdots+u_n}.\end{aligned}$$

where $k := k_1 + \cdots + k_r$, with source and target assignments

$$\begin{aligned}d_0[\mu_1,\dots,\mu_n,\varphi] &= (d_0\mu_1,\dots,d_0\mu_n) \in \Omega^n, \\ d_1[\mu_1,\dots,\mu_n,\varphi] &= \varphi^*(d_1\mu_1,\dots,d_1\mu_n) \in \textstyle\prod_i \Omega^{k_i}.\end{aligned}$$

Then $d_1$ is indeed a fibre bundle, as it is a product of fibre bundles, and the image of a path component of $\overline{\mathbb{N}}(\mathscr{C})\binom{k_1,\dots,k_r}{n}$ under $d_0$ is a product of images of path components of $\mathscr{C}(\#f^{-1}(\ell))$ and hence again contractible.

The rest of the construction is exactly as in Section 3.4, and in addition, each choice of basepoint $0 \in \Omega$ defines a family $0_n = (0,\dots,0) \in \Omega^n$ of basepoints and it is straightforward to check that $\overline{\mathbb{N}}(\mathscr{C})|_{\overline{\mathbb{N}}} \cong \overline{\mathbb{N}}(\mathscr{C}|_{\{0\}})$.

Finally, there is a connective suboperad $\overline{\mathbb{N}}^c(\mathscr{C}) \subseteq \overline{\mathbb{N}}(\mathscr{C})$ for which we sum only over those maps $f\colon \underline{k} \to \underline{n}$ such that if we put $f_i\colon \underline{k_i} \hookrightarrow \underline{k} \to \underline{n}$, then the tuple $(f_1,\dots,f_r)$ is connective in the sense of Subsection 3.4.5.

If we insist again that in $\mathscr{C}_2^{\lhd}$, even the double-sized images of the interiors are mutually disjoint, then the dyed operad $\overline{\mathbb{N}}^c(\mathscr{C}_2^{\lhd})$ acts on the collection $(\coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\lhd})_{n\geqslant 1}$ from Definition 5.2.4 as follows:

**Construction 5.2.13.** Abbreviating $\Omega := (-2;\infty)$, we endow $(\coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\lhd})_{n\geqslant 1}$ with the structure of an $\overline{\mathbb{N}}^c(\mathscr{C}_2^{\lhd})$-algebra:

- we have maps $d_0\colon \coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\lhd} \to \Omega^n$, the image of $\mathfrak{M}^m_{g,n\cdot\lhd}$ is the contractible space $\Omega^m_n \subseteq \Omega^n$, and $d_0\colon \mathfrak{M}^m_{g,n\cdot\lhd} \to \Omega^m_n$ is a fibre bundle.
- for $\mu \in \overline{\mathbb{N}}^c(\mathscr{C}_2^{\lhd})\binom{k_1,\dots,k_r}{n}$ and $\mathcal{C}_i \in \mathfrak{M}^{m_i}_{g_i,k_i\cdot\lhd}$ with $d_1\mu = (d_0\mathcal{C}_1,\dots,d_0\mathcal{C}_r)$, we can form a compound Riemann surface $\mu(\mathcal{C}_1,\dots,\mathcal{C}_r) \in \mathfrak{M}^{m_1+\dots+m_r}_{g,n\cdot\lhd}$ with $g$ as in Remark 5.1.20 and $d_0\mu(\mathcal{C}_1,\dots,\mathcal{C}_r) = d_0\mu$, in the same way as in Construction 5.2.2, using that a neighbourhood of the boundary of each input box can be canonically identified with the fitting $\omega$-frame $\mathcal{E}(\omega)$.

This construction clearly extends the $\overline{\mathbb{N}}^c(\mathscr{C}_2)$-action on $(\coprod_{g,m} \mathfrak{M}^m_{g,n\cdot\Box})_{n\geqslant 1}$, in the sense that the following diagram commutes:

$$\begin{tikzcd}
\overline{\mathbb{N}}^c(\mathscr{C}_2)\binom{k_1,\dots,k_r}{n} \times \prod_i \mathfrak{M}^{m_i}_{g_i,k_i\cdot\Box} \arrow[r] \arrow[d, hook, "\simeq"'] & \mathfrak{M}^{m_1+\dots+m_r}_{g,n\cdot\Box} \arrow[d, hook, "\simeq"] \\
\overline{\mathbb{N}}^c(\mathscr{C}_2^{\lhd})\binom{k_1,\dots,k_r}{n} \times \prod_i \mathfrak{M}^{m_i}_{g_i,k_i\cdot\lhd} \arrow[r] & \mathfrak{M}^{m_1+\dots+m_r}_{g,n\cdot\lhd}.
\end{tikzcd}$$

Secondly, we finally have sufficiently addressed all obstacles in order to formally define an operadic action on the collection of slit domains:

**Construction 5.2.14.** As in Chapter 4, let $\mathscr{V}^{c,\lhd}_{1,1} \subseteq \overline{\mathbb{N}}(\mathscr{C}_2^{\lhd})$ be the suboperad containing all operations where boxes from the same cluster have the same behaviour with respect to their first coordinate, i.e. they lie above each other and have the same width (though maybe different deviations).

Then we can construct an action of $\mathscr{V}^{c,\lhd}_{1,1}$ on $(\coprod_{g,m} \mathfrak{P}^m_{g,n})_{n\geqslant 1}$ by *implanting slit domains* exactly as in [Böd90b, § 3]: here we skip those technical details of the construction which are exactly the same as in Bödigheimer's work, but emphasise where we need the new apparatus.

Firstly, note that we have maps $d_0\colon \coprod_{g,m} \mathfrak{P}^m_{g,n} \to \Omega^n$, the image of each path component $\mathfrak{P}^m_{g,n}$ is the contractible subspace $\mathring{\Omega}^m_n \subseteq \Omega^n$, and the restriction $d_0\colon \mathfrak{P}^m_{g,n} \to \mathring{\Omega}^m_n$ is a fibre bundle.

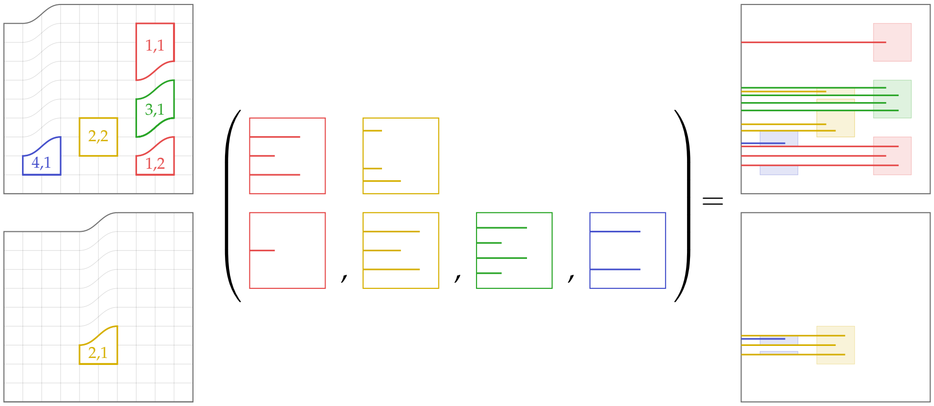


**Figure 5.7.** An instance of $\mathcal{V}_{1,1}^{c,\lhd}\binom{2,2,1,1}{2} \times^{\Omega^4} (\mathfrak{P}_{0,2} \times \mathfrak{P}^1_{0,2} \times \mathfrak{P}_{1,1} \times \mathfrak{P}^1_{0,1}) \to \mathfrak{P}^2_{2,2}$. Here we simplified the visualisation: if there are only *slit pairs*, i.e. transpositions in $\sigma_i \cdot \sigma_{i-1}^{-1}$, then it is clear how to reglue them and we can just draw a single line for each slit, and if all slit pairs have different lengths, then we do not need to distinguish them by colour.

If $\mu \in \mathcal{V}_{1,1}^{c,\lhd}\binom{k_1,\dots,k_r}{n}$ and $\mathcal{B}_i \in \mathfrak{P}^{m_i}_{g_i,k_i}$ such that $d_1\mu = (d_0\mathcal{B}_1,\dots,d_0\mathcal{B}_r)$, then we can form a compound slit domain $\mu(\mathcal{B}_1,\dots,\mathcal{B}_r) \in \mathfrak{P}^{m_1+\dots+m_r}_{g,n}$, where $g$ is again determined as in Remark 5.1.20. To do so, we implant slit pictures, as depicted in Figure 5.7: the operation $\mu$ yields a surjective map $\pi\colon \underline{r} \to \underline{s}$ as before, i.e. there are $-1 < x_1 < \dots < x_s \leqslant 1$ such that $c^1_{i,\underline{j}}(1) = x_a$ for each $i \in \pi^{-1}(a)$. Now we build up the new slit domain by proceeding from right to left: for $\pi^{-1}(s) = \{i_1,\dots,i_u\}$, we start with an empty slit domain on $\mathbb{I}^2 \times \underline{n}$ and place the slit pictures $\mathcal{B}_{i_1},\dots,\mathcal{B}_{i_u}$ on them. There is no ambiguity, since all boxes are on top of each other. Here we heavily use the verticality condition in order to get a proper slit domain, as there may be corresponding slits from the same input in different boxes.

In order to continue with the next stage and to add all slit domains $\mathcal{B}_i$ with $i \in \pi^{-1}(s-1)$, we reglue the layers according to the already implanted slits. Now we use that the operation $\mu$ anticipates the deviations that arise from this step, and thus, we have a *canonical* identification of the reglued area left of the implanted slit domains, and the area in which the remaining boxes from $\mu$ dwell. Hence we can proceed as in the first step and, finally, build the slit domain $\mu(\mathcal{B}_1,\dots,\mathcal{B}_r)$ inductively.

Finally, the gluing construction $G\colon \mathfrak{P}^m_{g,n} \to \mathfrak{M}^m_{g,n\cdot\lhd}$ translates *by definition* implanting of slit pictures into gluing in Riemann surfaces with deviated square frames, i.e. we even have an—even strictly—commuting square

$$\begin{tikzcd}
\mathcal{V}^{\lhd,c}_{1,1}\binom{k_1,\dots,k_r}{n} \times \prod_i \mathfrak{P}^{m_i}_{g_i,k_i} \arrow[r] \arrow[d] & \mathfrak{P}^{m_1+\dots+m_r}_{g,n} \arrow[d] \\
\overline{\mathbb{N}}^c(\mathcal{C}^{\lhd}_2)\binom{k_1,\dots,k_r}{n} \times \prod_i \mathfrak{M}^{m_i}_{g_i,k_i\cdot\lhd} \arrow[r] & \mathfrak{M}^{m_1+\dots+m_r}_{g,n\cdot\lhd},
\end{tikzcd}$$

where $g$ is determined as in Remark 5.1.20. Moreover, the action of $\mathcal{V}^{\lhd,c}_{1,1}$ on the sequence $(\coprod_{g,m} \mathfrak{P}^m_{g,n})_{n\geqslant 1}$ extends the (monochromatic) action of $\mathcal{C}_2$ on $\coprod_g \mathfrak{P}_{g,1}$ that has been described in [Böd90b, § 3].

Let us close this subsection with a conceptual remark: the reader may be surprised that the restriction to the vertical suboperad $\mathcal{V}^{\lhd,c}_{1,1}$ is necessary. Roughly speaking, elements in $\mathfrak{P}^m_{g,n}$ do not only encode conformal classes of Riemann surfaces $\mathcal{F}$, but come with an additional continuous map $u\colon \mathcal{F} \to [-1;1]$, that has a certain boundary behaviour, (or—as in Bödigheimer's work—a harmonic function $u\colon \bar{\mathcal{F}} \to \mathbb{R} \cup \{\infty\}$ starting from a closed surface) by projecting down the slit domain to the first coordinate. Although this additional datum ranges in a contractible domain, there is no canonical choice for a compound surface if the boundary curves from the same input are not glued to the *same* level of $u$.

The reason why we deliberately accept this 'artificial' restriction is the following: Bödigheimer's simplicial model of slit domains has been proved useful for explicit homology calculations in the unstable range [ABE08; Meh11; Wan11; BH14; Boe18], and we want to exploit our operadic description for a reformulation and an enhancement of their results: this is what we carry out in Section 5.3.

### 5.2.5. *Homology operations on the space of slit domains*

In the previous subsection, we saw that the $\coprod_{n\geqslant 1}(-2;\infty)^n$-coloured operad $\mathcal{V}^{c,\lhd}_{1,1}$ acts on the collection $(\coprod_{g,m} \mathfrak{P}^m_{g,n})_{n\geqslant 1}$. Now we want to understand the homology operations that arise from this construction.

**Construction 5.2.15.** Since $\mathscr{V}_{1,1}^{c,\triangleleft}|_{\overline{\mathbb{N}}} = \mathscr{V}_{1,1}^{c}$, we obtain, by Proposition 5.2.10, a (graded) action of the algebraic operad $H_\bullet(\mathscr{V}_{1,1}^{c})$ on the family of graded modules $H_\bullet(\mathfrak{P}_{g,n}^{m})$. Using the presentation of $H_\bullet(\mathscr{V}_{1,1}^{c})$ from Section 4.4, we obtain the following five fundamental types of operations:

1. the unit $1 \in H_0(\mathfrak{P}_{0,1})$, which is the ground class of $\mathfrak{P}_{0,1}$;
2. for each $\sigma \in \mathfrak{S}_n$ a *permutation*

   $$\sigma\colon H_h(\mathfrak{P}_{g,n}^{m}) \to H_h(\mathfrak{P}_{g,n}^{m}),$$

   which is induced by the map $\sigma\colon \mathfrak{P}_{g,n}^{m} \to \mathfrak{P}_{g,n}^{m}$ that permutes the layers of the slit domain, or, in terms of $\mathfrak{M}_{g,n}^{m}$, permutes the boundary curves. The map $\sigma$ clearly is an isomorphism.
3. for each $1 \leqslant \ell \leqslant n-1$ a *codegeneracy*

   $$s^{\ell}: H_h(\mathfrak{P}_{g,n}^{m}) \to H_h(\mathfrak{P}_{g+1,n-1}^{m}),$$

   which is induced by the map $s^{\ell}: \mathfrak{P}_{g,n}^{m} \to \mathfrak{P}_{g+1,n-1}^{m}$ that unites the $\ell^{\text{th}}$ and the $(\ell+1)^{\text{st}}$ layer of a slit picture on a single layer, or, in terms of $\mathfrak{M}_{g,n}^{m}$, glues a pair of pants joining the $\ell^{\text{th}}$ and the $(\ell+1)^{\text{st}}$ boundary component. By Harer's stability theorem [Har84; Iva90; Bol12; Ran16], $s^{\ell}$ is an isomorphism if $h \leqslant \frac{2}{3}\cdot(g-2)$ and surjective if $h \leqslant \frac{2}{3}\cdot(g+1)$.
4. the *vertical Pontrjagin product*

   $$-\cdot-\colon H_h(\mathfrak{P}_{g,n}^{m}) \otimes H_{h'}(\mathfrak{P}_{g',n'}^{m'}) \to H_{h+h'}(\mathfrak{P}_{g+g',n+n'-1}^{m+m'}),$$

   which is induced by the map $-\cdot-\colon \mathfrak{P}_{g,n}^{m} \times \mathfrak{P}_{g',n'}^{m'} \to \mathfrak{P}_{g+g',n+n'-1}^{m+m'}$ that merges, on the first layer, the respective first layers of both arguments, or, in terms of $\mathfrak{M}_{g,n}^{m}$, combining two surfaces by forming the boundary-connected sum along their respective first boundary curves.
5. the *vertical Browder bracket*

   $$[-,-]\colon H_h(\mathfrak{P}_{g,n}^{m}) \otimes H_{h'}(\mathfrak{P}_{g',n'}^{m'}) \to H_{h+h'+1}(\mathfrak{P}_{g+g',n+n'-1}^{m+m'}),$$

   which is induced by $[-,-]: \mathfrak{P}_{g,n}^{m} \times \mathfrak{P}_{g',n'}^{m'} \to \operatorname{map}(\mathbb{S}^1, \mathfrak{P}_{g+g',n+n'-1}^{m+m'})$, that lets the first layer of the first argument spin around the first layer of the second argument.

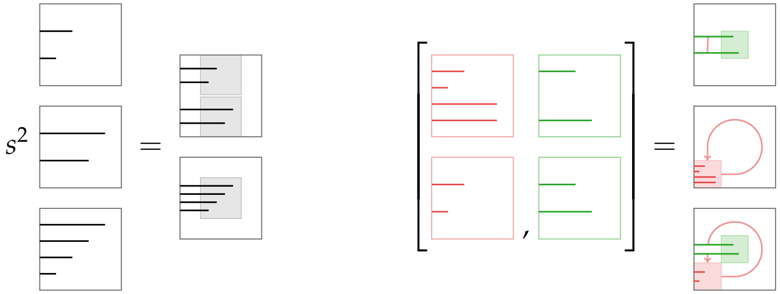


**Figure 5.8.** Instances of $s^2\colon \mathfrak{P}_{0,3} \to \mathfrak{P}_{1,2}$ and $[-,-]\colon \mathfrak{P}^1_{0,2}\times\mathfrak{P}^1_{0,2} \to \mathrm{map}(\mathbb{S}^1,\mathfrak{P}^2_{0,3})$.

**Remark 5.2.16.** We saw in Theorem 4.4.2 that $H_\bullet(\mathcal{V}^{c}_{1,1})$ is, as an algebraic operad, generated by these operations, whence these are *all* Künneth operations which arise from the action of $\mathcal{V}^{c,\lhd}_{1,1}$ on the spaces of slit domains.

Additionally, all relations we developed in Subsection 4.4.2 hold here; we can even say a little bit more: since the $\mathcal{V}^{c,\lhd}_{1,1}$-action on $(\coprod_{g,m}\mathfrak{P}^m_{g,n})_{n\geqslant 1}$ factors, up to homotopy, through the $\overline{\mathbb{N}}^c(\mathscr{C}^{\lhd}_2)$-action on $(\coprod_{g,m}\mathfrak{M}^m_{g,n\cdot\lhd})_{n\geqslant 1}$, all relations inside the operadic kernel of $H_\bullet(\mathcal{V}^c_{1,1}) \to H_\bullet(\overline{\mathbb{N}}^c(\mathscr{C}_2))$ hold: these are spelled out in Remark 4.4.19.

**Construction 5.2.17.** If we work over $\mathbb{F}_2$ or if $h$ is even, then the class $c_n$ from Construction 4.4.24 gives rise to a *vertical Dyer–Lashof square*

$$Q\colon H_h(\mathfrak{P}^m_{g,n}) \to H_{1+2h}(\mathfrak{P}^{2m}_{2g+n-1,n}),$$

and all relations from Subsection 4.4.3 hold. We give a geometric description of $Qx$ if $x$ is supported on an $h$-dimensional manifold $M$ as in Remark 3.3.8: in this case, $Qx$ is supported on the mapping torus $\mathbb{S}^1 \ltimes M^2$ of the twist:

Consider the map $\gamma\colon [0;1]\times(\mathfrak{P}^m_{g,n})^2 \to \mathfrak{P}^{2m}_{2g+n-1,n}$ which takes two slit domains on $n$ layers, puts the $\ell^{\text{th}}$ layers of the first and the second argument on the $\ell^{\text{th}}$ layer of the new one and lets them switch places on all layers simultaneously, performing a rotation around each other by 180°. Then we have $\gamma(\mathcal{B},\mathcal{B}',0) = \gamma(\mathcal{B}',\mathcal{B},1)$, whence $\gamma$ factors over $\bar{\gamma}\colon \mathbb{S}^1 \ltimes (\mathfrak{P}^m_{g,n})^2 \to \mathfrak{P}^{2m}_{2g+n-1,n}$, and for each map $\alpha\colon M \to \mathfrak{P}^m_{g,n}$, we get $Q(\alpha_*[M]) = \bar{\gamma}_*(\mathrm{id}_{\mathbb{S}^1}\ltimes\alpha^2)_*[\mathbb{S}^1\ltimes M^2]$, where $[M]$ and $[\mathbb{S}^1\ltimes M^2]$ are the fundamental classes.

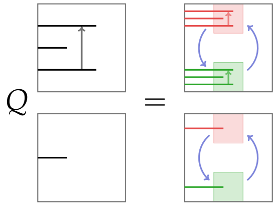


**Figure 5.9.** $Q\alpha := \bar{\gamma} \circ (\mathrm{id}_{\mathbb{S}^1} \ltimes \alpha^2) \colon \mathbb{S}^1 \ltimes \mathbb{T}^2 \to \mathfrak{P}_{1,2}$ for $\alpha \colon \mathbb{S}^1 \to \mathfrak{P}_{0,2}$

### 5.2.6. *Further ad-hoc constructions*

In this last subsection, we want to discuss two further ad-hoc constructions for homology operations on moduli spaces, which will turn out to be useful for the upcoming calculations, and make a final remark on a slightly more general model of slit domains.

The first construction lifts the $T$-operation $T \colon H_h(\mathfrak{M}^m_{g,n}) \to H_{h+1}(\mathfrak{M}^{m-1}_{g+1,n})$ from [Meh11] over the pair-of-pants gluing $s^n \colon \mathfrak{M}^{m-1}_{g,n+1} \to \mathfrak{M}^{m-1}_{g+1,n}$:

**Construction 5.2.18** ($\hat{T}$-operation)**.** Let $g \geqslant 0$ and $m, n \geqslant 1$, i.e. we have at least one puncture. Then there is an $m$-fold covering $\alpha \colon \mathfrak{M}^{m-1,1}_{g,n} \to \mathfrak{M}^m_{g,n}$ which marks one of the punctures.

We also have a projection $\mathfrak{M}^{m-1,1}_{g,n} \to \mathfrak{M}^{m-1}_{g,n}$ which forgets the chosen puncture, and this bundle is a subbundle of the universal surface bundle over $\mathfrak{M}^{m-1}_{g,n}$, where points in the fibre have to avoid the boundary curve and the given $m-1$ punctures. Over this space, we can consider the vertical unit tangent bundle $p \colon W_T := \mathbb{S}(T^{\perp}\mathfrak{M}^{m-1,1}_{g,n}) \to \mathfrak{M}^{m-1,1}_{g,n}$, which is a 1-dimensional sphere bundle. Then $W_T$ is oriented, as the structure group $\mathfrak{M}^{m-1,1}_{g,n}$ contains only orientation-preserving diffeomorphisms.

We construct a map $\vartheta \colon W_T \to \mathfrak{M}^{m-1}_{g,n+1}$, which exploits the observation that a chosen puncture together with a unit tangential direction carries the same information as a parametrised boundary curve.

Explicitly, we do the following: via the exponential map, a point in $W_T$ is given by a conformal class of a Riemann surface $\mathcal{F}$, a subset $\mathcal{P} \subseteq \mathring{\mathcal{F}}$ of cardinality $m-1$, a point $x \in \mathcal{F}$ with a small disc $D \subseteq \mathring{\mathcal{F}} \setminus \mathcal{P}$ around it, and a point $x'$ on $\partial D$. If we remove $\mathring{D}$ and parametrise $\partial D$ such that the angle 0°

is mapped to $x'$, then we have reached a new parametrised boundary curve. Finally, we use the homological transfer maps and define

$$\hat{T}\colon H_h(\mathfrak{M}^m_{g,n}) \xrightarrow{\alpha^!} H_h(\mathfrak{M}^{m-1,1}_{g,n}) \xrightarrow{p^!} H_{h+1}(W_T) \xrightarrow{\vartheta_*} H_{h+1}(\mathfrak{M}^{m-1}_{g,n+1}).$$

Under the gluing construction $\mathfrak{P}^m_{g,n} \to \mathfrak{M}^m_{g,n}$, the map $\vartheta$ can also be visualised in terms of slit domains: if $\tilde{W}_T$ denotes the pullback of the bundle $W_T$ along the equivalence $\mathfrak{P}^{m-1,1}_{g,n} \to \mathfrak{M}^{m-1,1}_{g,n}$, then an element in $\tilde{W}^T$ is given by a slit domain, together with the choice of one of the cycles of $\sigma_q$ containing no letter of the form $0_\ell$, and, on the circle given by regluing the left faces of the rectangles inside the chosen cycle, a marked point. We can turn this left face into an honest boundary curve by providing a new layer for it: we draw a small slit at the given position and pair it with a single slit on a new layer, see Figure 5.10. This gives rise to a map $\tilde{\vartheta}\colon \tilde{W}_T \to \mathfrak{P}^{m-1}_{g,n+1}$, covering $\vartheta$ up to homotopy.

We define $T := s^n \circ \hat{T}\colon H_h(\mathfrak{M}^m_{g,n}) \to H_{h+1}(\mathfrak{M}^{m-1}_{g+1,n})$, which is exactly the $T$-operation defined in [Meh11, Def. 84].

Secondly, we want to remind the reader of the so-called $E$-operation from [Meh11, Def. 86], which we do not intend to lift:

**Reminder 5.2.19** ($E$-operation)**.** Let $g \geqslant 0$, $n \geqslant 1$, and $m \geqslant 2$. Then we consider the $\binom{m}{2}$-sheeted covering $\beta\colon \mathfrak{M}^{m-2,2}_{g,n} \to \mathfrak{M}^m_{g,n}$ where two punctures are separated from the other ones, but they are not ordered.

Next, consider the torus bundle $q\colon W_E \to \mathfrak{M}^{m-2,2}_{g,n}$ given by the fibre product of the two vertical tangent bundles at the two punctures.[5]

There is a map $\eta\colon W_E \to \mathfrak{M}^{m-1}_{g+1,n}$ as follows: note that an element in $W_E$ is defined by a conformal class of a Riemann surface $\mathcal{F}$, a subset $\mathcal{P} \subseteq \mathring{\mathcal{F}}$ of cardinality $m-2$, two (unordered) punctures $x_1, x_2 \in \mathcal{F}$ with small disjoint discs $D_1, D_2 \subseteq \mathring{\mathcal{F}} \smallsetminus \mathcal{P}$ around them, and points $x'_i \in \partial D_i$. If we cut along the two straight lines from $x_1$ to $x'_1$ and from $x_2$ to $x'_2$ and reglue, then we have identified the two punctures and increased the genus by 1. Finally, we

---

[5] Formally, we consider the double covering $\mathfrak{M}^{m-2,1,1}_{g,n} \to \mathfrak{M}^{m-2,2}_{g,n}$ where the two points are ordered. Then, over $B := \mathfrak{M}^{m-2,1,1}_{g,n}$, there are two unit vertical tangent bundles $L$ and $L'$ and the fibre product $L \times^B L' \to \mathfrak{M}^{m-2,1,1}_{g,n}$ is $\mathfrak{S}_2$-equivariant. The induced map on quotients is the desired bundle.

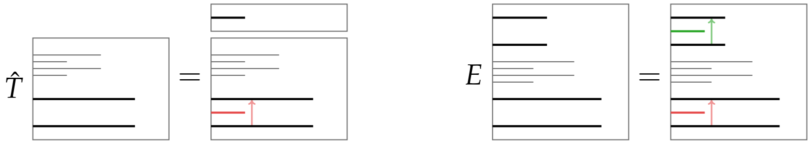


**Figure 5.10.** The two ad-hoc constructions $\hat{T}\colon H_h(\mathfrak{P}^1_{1,1}) \to H_{h+1}(\mathfrak{P}_{1,2})$ as well as $E\colon H_h(\mathfrak{P}^2_{1,1}) \to H_{h+2}(\mathfrak{P}^1_{2,1})$, here applied to ground classes. Here the first toric factor is drawn in red, while the second toric factor is drawn in green. In the second picture, a homotopy between the drawn class and the one where red and green are toggled is given by interchanging the positions of the two black slit pairs.

forget that the joint puncture is special among the other ones. Again, this construction can be visualised in terms of slit pictures, see Figure 5.10.

The torus bundle $q\colon W_E \to \mathfrak{M}^{m-2,2}_{g,n}$ is not orientable, but if we work over $\mathbb{F}_2$, then we still have a homological transfer, which can be used to define

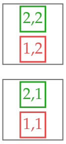
$$E\colon H_h(\mathfrak{M}^m_{g,n}) \xrightarrow{\beta^!} H_h(\mathfrak{M}^{m-2,2}_{g,n}) \xrightarrow{q^!} H_{h+2}(W_E) \xrightarrow{\eta_*} H_{h+2}(\mathfrak{M}^{m-1}_{g+1,n}).$$

Let us point out that the $E$-operation can also be applied integrally to ground classes $x \in H_0(\mathfrak{M}^m_{g,n})$: here the class $x$ is of the form $\imath[*]$, where $\imath\colon * \hookrightarrow \mathfrak{M}^m_{g,n}$ is any basepoint inclusion, and $\imath^* W_E$ is just a 2-torus. Hence we can define $Ex := \varphi_*[\mathbb{T}^2] \in H_2(\mathfrak{M}^{m-1}_{g+1,n})$, where $\varphi\colon \mathbb{T}^2 \cong \imath^* W_E \to W_E \to \mathfrak{M}^{m-1}_{g+1,n}$. The sign of $Ex$ depends *a priori* on the choice of identification $\mathbb{T}^2 \cong \imath^* W_E$; however, if $\mathrm{tw}\colon \mathbb{T}^2 \to \mathbb{T}^2$ denotes the twist, then $\varphi$ is homotopic to $\varphi \circ \mathrm{tw}$, the homotopy given by a path exchanging the two punctures; hence $2 \cdot Ex = 0$.

We close this section with a remark on a combinatorially enhanced model:

**Remark 5.2.20.** As already indicated in Definition 5.2.5, there is a more general model $\mathfrak{P}^m_g[\rho]$ for $\mathfrak{M}^m_{g,n}$, where $\rho \in \mathfrak{S}_\lambda$ is a permutation with exactly $n$ cycles (including fixed points), for some $\lambda \geqslant n$: here, the slit domains take place on $\lambda$ layers and $\sigma_q$ is allowed to contain cycles which have more than a single symbol of the form $0_\ell$, and if we remove all other symbols from $\sigma_q$, then we obtain $\rho$. We call these objects *slit domains with multiplicities*, and we clearly have $\mathfrak{P}^m_{g,n} = \mathfrak{P}^m_g[\mathrm{id}_n]$. Note that the equivalence $\mathfrak{P}^m_g[\rho] \simeq \mathfrak{M}^m_{g,n}$ is not canonical, but depends on a choice of ordering of the cycles of $\rho$.

It is not possible to extend the $\mathscr{V}^c_{1,1}$-action on $(\coprod_{g,m}\mathfrak{P}^m_{g,n})_{n\geqslant 1}$ to these more enhanced models just by implanting slit domains; this already fails on the level of path components: for the transposition $\tau = (1\,2) \in \mathfrak{S}_2$, the two operations below would induce maps between the given path components, while they themselves lie in the same component of $\mathscr{V}^c_{1,1}\binom{2,2}{2}$:



$: (\mathfrak{P}^0_0[\tau])^2 \to \mathfrak{P}^0_0[\mathrm{id}_2]$

$: (\mathfrak{P}^0_0[\tau])^2 \to \mathfrak{P}^1_0[\tau],$

In particular, the result of these operations once lies in a model for $\mathfrak{M}^1_{0,2}$ and once in a model for $\mathfrak{M}^1_{0,1}$. I think this issue has been overseen in [BH14, Prop. 4.2.2]. Here we see that the combinatorial type of the result does not only depend on the orders on the fibres of each $\mathbf{\Delta\Sigma}$-map $f_i\colon k_i \to n$, but merely on a single large map $f \in \mathbf{\Delta\Sigma}\binom{k_1+\dots+k_r}{n}$; this would also distinguish the two above operations.

More formally, one can define an action of the smaller operad $\overline{\mathbb{N}}^c(\mathscr{C}_1)$ on $(\coprod_{\rho\in\mathfrak{S}_\lambda}\coprod_{g,m}\mathfrak{P}^m_g[\rho])_{\lambda\geqslant 1}$: given a connective configuration of clusters of intervals on $\lambda$ large intervals, then we can cross $[-1;1]$ with everything and then implant slit pictures with multiplicities into the arising boxes. The three examples from [BH14, § 4.2] are indeed instances of this smaller action.

A more elaborated approach, which we have not pursued to a meaningful end, enlarges the combinatorics of $\mathscr{V}^c_{1,1}$, ending up with an operad $\mathcal{S}$ with colour set $\coprod_{\lambda\geqslant 1}\mathfrak{S}_\lambda$ and with operation spaces $\mathcal{S}\binom{\pi_1,\dots,\pi_r}{\rho}$ containing configurations of vertically aligned *slitted* boxes, together with a regluing prescription that resembles the colours $\pi_1,\dots,\pi_r$, such that the global permutation arising from the regluing of slits equals $\rho$. Such an operad deserves the name *slit operad*; it is possible, yet very technical, to define it, and the collection $(\coprod_{g,m}\mathfrak{P}^m_g[\rho])_\rho$ is an algebra over $\mathcal{S}$.

## 5.3. Low-genus calculations

In this last section, we want to employ the previously established structure results in order to describe some very concrete generators. By doing so, we contribute to a long project which involves results of [Ehr98; Abh05; God07; ABE08; Vis11; Meh11; BH14; Boe18]. We start by recalling parts of their work.

### 5.3.1. *Current 'state of the art'*

This subsection is a mere summary of low-genus calculations which have already been performed: let us start by describing six toric generators in the homology of moduli spaces with a *single* boundary curve, which have been given the names **a**, **b**, **c**, **d**, **e**, and **f**.

**Definition 5.3.1** (Toric generators)**.** We consider the following classes:

1. the ground class $\mathbf{a} \in H_0(\mathfrak{M}^1_{0,1})$, represented by a single puncture inside a disc, i.e. a single slit pair on a single layer. Note that the product

$$\mathbf{a} \cdot - \colon H_h(\mathfrak{M}^m_{g,n}) \to H_h(\mathfrak{M}^{m+1}_{g,n})$$

   is the same as forming the boundary-connected sum with a punctured disc at the first boundary curve of the given surface. In [BT01, Thm. 1.3], it was shown that this (topological) map admits a stable retraction, and hence, the induced map in homology is split monic;

2. the circle $\mathbf{b} := Q\mathbf{a} \in H_1(\mathfrak{M}^2_{0,1})$, represented by two punctures switching places, or a small slit pair inside a large one, performing a half-rotation, see Figure 5.11;

3. the ground class $\mathbf{c} \in H_0(\mathfrak{M}_{1,1})$. Note that the product

$$\mathbf{c} \cdot - \colon H_h(\mathfrak{M}^m_{g,n}) \to H_h(\mathfrak{M}^m_{g+1,n})$$

   is the classical genus stabilisation, and Harer's (improved) stability theorem [Har84; Iva90; Bol12; Ran16] tells us that it is an isomorphism for $h \leqslant \frac{2}{3} \cdot (g-1)$;

4. the circle $\mathbf{d} := T\mathbf{a} \in H_1(\mathfrak{M}_{1,1})$, which is depicted in Figure 5.11;

5. the 2-torus $\mathbf{e} := E\mathbf{a}^2 \in H_2(\mathfrak{M}^1_{1,1})$;

6. the 3-torus $\mathbf{f} \in H_3(\mathfrak{M}^2_{1,1})$ which is due to [Boe18] and is depicted in Figure 5.11 as well (we will decompose **f** into smaller classes soon).

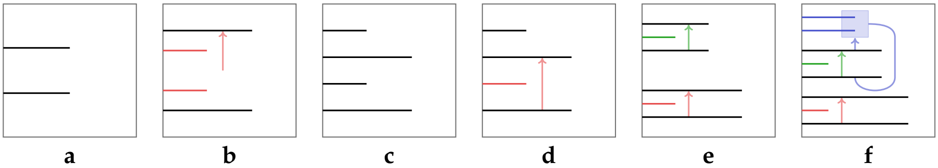


**Figure 5.11.** The six generators, depicted as maps from tori into the space of slit domains. Here the first toric factor is drawn in red, the second one in green, and the third one in blue.

There are a few relations between these toric classes, that are either immediate or have been calculated in previous work.

**Remark 5.3.2** (Relations)**.** Since $\mathbf{e} = E\mathbf{a}^2$, we already know that $\mathbf{e}$ must have 2-torsion, whence $2 \cdot \mathbf{e} = 0$. Moreover, if $x \in H_h(\mathfrak{M}_{g,n})$ is any class in some moduli space without punctures, then we obtain $T(x \cdot \mathbf{a}) = (-1)^h \cdot (x \cdot \mathbf{d})$, simply because the $T$-operation effects only the newly added puncture.

The relation $Q\mathbf{c} = 3 \cdot \mathbf{cd}$ is due to [God07, Ex. 6] and also appears in [Meh11, § 1.2]. In particular, $[\mathbf{c}, \mathbf{c}] = 2 \cdot Q\mathbf{c} = 6 \cdot \mathbf{cd} \neq 0$, showing that the $E_2$-structure on $\coprod_g \mathfrak{M}_{g,1}$ cannot be enhanced to an $E_3$-structure [FS96, Thm. 2.5], although the group completion $\Omega B \coprod_g \mathfrak{M}_{g,1}$ has the homotopy type of an *infinite* loop space [Til97, Thm. A].

Let me finally mention that [Meh11, § 1.2] also claims that $[\mathbf{c}, \mathbf{d}] = \mathbf{d}^2$ holds, a relation which we are going to disprove in the next subsection.

As we can see in the Tables 5.1 and 5.2, these toric generators describe much of the homology of low-genus moduli spaces; only a few generators are unknown. An empty field means that the respective homology group is trivial, and all homology groups that are not listed are trivial.

These tables contain two generators that have not been explained yet and which both are due to [Boe18]: on the one hand, he found a rational generator $\mathbf{s} \in H_3(\mathfrak{M}_{2,1}; \mathbb{Q})$, which means that for some rational number $\lambda \neq 0$, the class $\lambda\mathbf{s}$ is an integral generator of the free summand. On the other hand, there is a class $\mathbf{w} \in H_4(\mathfrak{M}_{2,1})$ which has 3-torsion and is obtained from the Segal–Tillmann map $C_6(\mathbb{R}^2) \to \mathfrak{M}_{2,2} \to \mathfrak{M}_{2,1}$, see [ST08], where the latter arrow is given by capping with a disc. However, both generators will play no further rôle in the upcoming discussion.

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\mathfrak{M}_{1,1}$ | $\mathbb{Z}\langle \mathbf{c} \rangle$ | $\mathbb{Z}\langle \mathbf{d} \rangle$ | | | |
| $\mathfrak{M}^1_{1,1}$ | $\mathbb{Z}\langle \mathbf{ac} \rangle$ | $\mathbb{Z}\langle \mathbf{ad} \rangle$ | $\mathbb{Z}_2\langle \mathbf{e} \rangle$ | | |
| $\mathfrak{M}^2_{1,1}$ | $\mathbb{Z}\langle \mathbf{a}^2\mathbf{c} \rangle$ | $\mathbb{Z}\langle \mathbf{a}^2\mathbf{d} \rangle \oplus \mathbb{Z}_2\langle \mathbf{bc} \rangle$ | $\mathbb{Z}_2\langle \mathbf{ae}, \mathbf{bd} \rangle$ | $\mathbb{Z}_2\langle \mathbf{f} \rangle$ | |
| $\mathfrak{M}_{2,1}$ | $\mathbb{Z}\langle \mathbf{c}^2 \rangle$ | $\mathbb{Z}_{10}\langle \mathbf{cd} \rangle$ | $\mathbb{Z}_2\langle \mathbf{d}^2 \rangle$ | $\mathbb{Z}\langle \lambda\mathbf{s} \rangle \oplus \mathbb{Z}_2\langle T\mathbf{e} \rangle$ | $\mathbb{Z}_3\langle \mathbf{w} \rangle \oplus \mathbb{Z}_2\langle ? \rangle$ |

**Table 5.1.** Generators for the integral homology of low-genus moduli spaces. Only a single generator of $H_4(\mathfrak{M}_{2,1})$ with 2-torsion is missing.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\mathfrak{M}_{1,1}$ | $\mathbf{c}$ | $\mathbf{d}$ | | | | |
| $\mathfrak{M}^1_{1,1}$ | $\mathbf{ac}$ | $\mathbf{ad}$ | $\mathbf{e}$ | $E\mathbf{b}$ | | |
| $\mathfrak{M}^2_{1,1}$ | $\mathbf{a}^2\mathbf{c}$ | $\mathbf{a}^2\mathbf{d}, \mathbf{bc}$ | $\mathbf{ae}, \mathbf{bd}, ?$ | $\mathbf{a}\,E\mathbf{b}, \mathbf{f}, ?$ | $?$ | |
| $\mathfrak{M}_{2,1}$ | $\mathbf{c}^2$ | $\mathbf{cd}$ | $\mathbf{d}^2, ?$ | $\lambda\mathbf{s}, T\mathbf{e}, Q\mathbf{d}$ | $?, TE\mathbf{b}$ | $?$ |

**Table 5.2.** Generators for the $\mathbb{F}_2$-homology of low-genus moduli spaces. Here some more gaps appear, which arise from the torsion part in the universal coefficient theorem; however, some other generators could have been described via the Dyer–Lashof square $Q$ and the $E$-operation.

In order to complete the survey, I ought to mention that calculations for genus 3 have been made in [Wan11]; however, no explicit generators have been described yet.

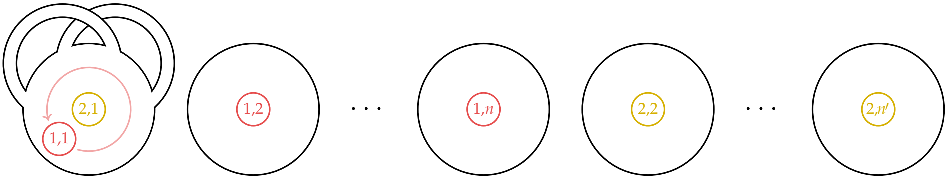
### 5.3.2. *Two folklore statements*

Before considering explicit classes in the homology of moduli spaces with multiple boundary curves, let us take the time to spell out two statements which I believe are well-known to the experts, but, to the best of my knowledge, have never been written down explicitly so far.

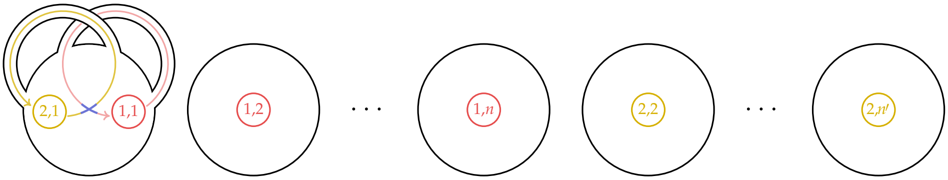
The first one claims that a single genus stabilisation step cancels the (vertical) Browder bracket. Let me point out that, in the case of a single boundary curve, it follows from abstract considerations that the Browder bracket $[x, x']$ of two classes $x \in H_\bullet(\mathfrak{M}_{g,1})$ and $x' \in H_\bullet(\mathfrak{M}_{g',1})$ vanishes after *finitely many* stabilisation steps: the group completion $\coprod_g \mathfrak{M}_{g,1} \to \mathfrak{M}_{\infty,1} \times \mathbb{Z}$ is given by iterated stabilisations and respects the $E_2$-structure on both sides. Furthermore, the right side has the homology[6] of an *infinite* loop space [Til97, Thm. A], so its $E_2$-Browder bracket vanishes. Here is a stronger statement.

**Proposition 5.3.3.** *For each pair of classes $x \in H_\bullet(\mathfrak{M}^m_{g,n})$ and $x' \in H_\bullet(\mathfrak{M}^{m'}_{g',n'})$, we have $\mathbf{c} \cdot [x, x'] = 0$.*

*Proof.* We use the (connective) coloured surface operad $\mathcal{M}^c$ from Definition 5.1.13 and consider the loop $\gamma \colon \mathbb{S}^1 \to \mathcal{M}^c\binom{n,n'}{n+n'-1}$ which is depicted as follows:



Then clearly $\mathbf{c} \cdot [x, x'] = [\gamma](x \otimes x')$, so it suffices to show that $[\gamma]$ is trivial in the first homology of $\mathcal{M}^c\binom{n,n'}{n+n'-1}$. To do so, we recycle an argument from [Bia20]: let $I \subseteq \mathbb{S}^1$ be a small closed interval on the standard circle. Then we define a map $\tilde{\gamma} \colon \mathbb{T}^2 \setminus \mathring{I}^2 \to \mathcal{M}^c\binom{n,n'}{n+n'-1}$ whose source is a bounded 2-torus and which is depicted as follows:



Here the excluded square $\mathring{I}^2$ parametrises the situation in which both discs live inside the blue region, and this is exactly the case where the disjointness condition is violated.

Now one readily checks that, after identifying $\partial I^2 \cong \mathbb{S}^1$, the boundary of $\tilde{\gamma}$ is homotopic to $\gamma$. Hence, the cycle $\gamma$ is a boundary, as desired. $\square$

---

[6] And, after applying the Quillen plus construction, also the homotopy type.

The previous proof provides a stronger result: for each $\mathscr{M}^c$-algebra $X$, we have a genus stabilising operation $\mathbf{c} \cdot -$ and a Browder bracket $[-,-]$ in homology, and their composition $\mathbf{c} \cdot [-,-]$ is trivial in $H_\bullet(X)$.

The second statement I would like to mention concerns the generator $\mathbf{b}$.

**Remark 5.3.4.** As in the work of [BT01, Thm. 1.3] on the injectivity of $\mathbf{a} \cdot -$, i.e. adding a single puncture, we can ask if the map $H_h(\mathfrak{M}^m_{g,n}) \to H_{h+1}(\mathfrak{M}^{m+2}_{g,n})$, which is given by multiplication with $\mathbf{b}$, is injective.

Integrally, this statement is wrong, as one can already see in the case of a single boundary curve and genus 0: the class $\mathbf{b}$ itself is a free generator of $H_1(\mathfrak{M}^2_{0,1}) \cong \mathbb{Z}$, while by the Koszul sign rule, we get $\mathbf{b}^2 = -\mathbf{b}^2$.

Over the field $\mathbb{F}_2$ and in the case of a single boundary curve, the situation looks different: based on work of [BCT89], Bianchi [Bia20, Thm. 6.5] recently has established an isomorphism

$$\bigoplus_{m \geqslant 0} H_\bullet(\Gamma^m_{g,1}) \cong \mathbb{F}_2[Q^i \mathbf{a}]_{i \geqslant 0} \otimes H_\bullet(\Gamma_{g,1}; \mathrm{Sym}(\mathcal{H}))$$

of modules over the Dyer–Lashof algebra $\mathbb{F}_2[Q^i\mathbf{a}]_{i \geqslant 0}$, where $\mathrm{Sym}(\mathcal{H})$ is the symmetric algebra over the symplectic representation of $\Gamma_{g,1}$. This implies that the homology of punctured moduli spaces is a *free* $\mathbb{F}_2[Q^i\mathbf{a}]_{i \geqslant 0}$-module, whence not only the multiplication with the classes $\mathbf{a}$ and $\mathbf{b}$, but with any iterated Dyer–Lashof square $Q^i\mathbf{a}$ is injective.

### 5.3.3. *Generators with multiple boundary curves*

Next, we will describe several toric classes in the low-genus homology of moduli spaces of surfaces with *multiple* boundary curves. Our approach starts with the observation that the previously described generators $\mathbf{x}_1 \in \{\mathbf{c}, \mathbf{d}, \mathbf{e}\}$ are of the form $s^1\mathbf{x}_2$ for certain lifted homology classes $\mathbf{x}_2 \in H_\bullet(\mathfrak{M}^m_{g,2})$ for suitable $g, m \geqslant 0$, and we want to understand the lifted ones.

We give these lifted classes an index corresponding to the number of boundary curves of their respective moduli space, and in all cases, we will reach the relation $\mathbf{x}_{n-1} = s^1\mathbf{x}_n$. For the sake of consistency, our old six toric generators silently carry an index 1, which we decided to skip for the sake of continuity and simplicity.

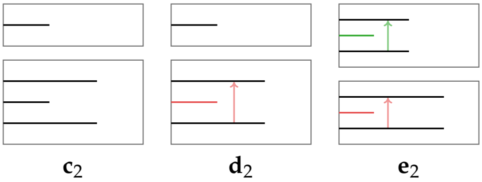


**Figure 5.12.** The generators $\mathbf{c}_2$, $\mathbf{d}_2$ and $\mathbf{e}_2$, depicted as maps from tori into the space of parallel slit domains on two layers.

**Construction 5.3.5.** We consider the following three toric classes, which have been depicted in Figure 5.12:

1. the ground class $\mathbf{c}_2 \in H_0(\mathfrak{M}_{0,2})$ of the moduli space of a cylinder. Note that the vertical product

$$\mathbf{c}_2 \cdot - \colon H_h(\mathfrak{M}^m_{g,n}) \to H_h(\mathfrak{M}^m_{g,n+1}),$$

is the same as forming the boundary-connected sum with a cylinder, or, in other words, gluing in an 'inverse' pair of pants at the first boundary curve. Again, a part of Harer's stability theorem [Har84; Iva90; Bol12; Ran16] tells us that this map is an isomorphism for $h \leqslant \frac{2}{3} \cdot (g-1)$, and an epimorphism for $h \leqslant \frac{2}{3} \cdot g$. We clearly have $\mathbf{c} = s^1\mathbf{c}_2$.

2. the circle $\mathbf{d}_2 := \hat{T}\mathbf{a} \in H_1(\mathfrak{M}_{0,2})$. By construction, $s^1\mathbf{d}_2 = s^1\hat{T}\mathbf{a} = T\mathbf{a} = \mathbf{d}$, which is a generator of $H_1(\mathfrak{M}_{1,1}) \cong \mathbb{Z}$. Therefore, $\mathbf{d}_2$ is a generator of $H_1(\mathfrak{M}_{0,2})$, that is: the standard Dehn twist on the cylinder.

3. the 2-torus $\mathbf{e}_2 \in H_2(\mathfrak{M}^1_{0,2})$ which is depicted in Figure 5.12. Note that $\mathbf{e}_2$ can be constructed by applying the $E$-operation to the ground class of $\mathfrak{M}(\Sigma^1_{0,1} \sqcup \Sigma^1_{0,1})$, the moduli space of two punctured discs[7] (but we have not introduced the $E$-operation for these cases). However, this does not imply that $\mathbf{e}_2$ has 2-torsion, as the two punctures on $\Sigma^1_{0,1} \sqcup \Sigma^1_{0,1}$ cannot be interchanged by a path. Indeed, $\mathfrak{M}^1_{0,2} \simeq \mathbb{T}^2$, as we see in the proof of Proposition 5.3.12, whence its homology is torsion-free.

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[7] In this disconnected situation, the genus does not increase. This is similar to the grading in [ST08, § 3], where the sum of two discs has been given the genus $-1$.

**Remark 5.3.6.** Here are some first observations:

1. If $x \in H_h(\mathfrak{M}_{g,n})$ is a class in the homology of a moduli space without punctures, then we clearly have $\hat{T}(x \cdot \mathbf{a}) = (-1)^h \cdot x \cdot \mathbf{d}_2$, for the same reason as in Remark 5.3.2, and similarly, $\hat{T}[x, \mathbf{a}] = (-1)^{h+1} \cdot [x, \mathbf{d}_2]$.

2. the 3-torus $\mathbf{f}$ can be lifted to a class in $\mathbf{f}_2 \in H_3(\mathfrak{M}^2_{0,2})$, but we can also express $\mathbf{f}_2$ through a vertical Browder bracket $(1\,2)[\mathbf{a}, \mathbf{e}_2]$. In particular, we can describe $\mathbf{f}$ itself as $s^1[\mathbf{a}, \mathbf{e}_2]$; here we use that $s^1(1\,2) = s^1$, as the $\mathcal{V}^c_{1,1}$-action factors, up to homotopy, through $\overline{\mathbb{N}}^c(\mathscr{C}_2)$.

Finally, we can use these enhanced generators in order to show certain divisibility properties of the classical Browder bracket.

**Proposition 5.3.7.** *For each class $x \in H_\bullet(\mathfrak{M}^m_{g,n})$, we have*

$$[\mathbf{c}, x] = 2 \cdot s^1[\mathbf{c}_2, x],$$
$$[\mathbf{d}, x] = 2 \cdot s^1[\mathbf{d}_2, x].$$

*Proof.* We prove the first relation, since the second one follows in the same way. Note that $s^1\mathbf{c}_2 = \mathbf{c}$ and $(1\,2)\mathbf{c}_2 = \mathbf{c}_2$ holds, as well as $s^1 = s^1(1\,2)$, since the action of $\mathcal{V}^{c,\triangleleft}_{1,1}$ factors through $\overline{\mathbb{N}}^c(\mathscr{C}^{\triangleleft}_2)$. Employing the relations among codegeneracies and vertical Browder brackets from Proposition 4.4.16, we get $[\mathbf{c}, x] = [s^1\mathbf{c}_2, x] = s^1[\mathbf{c}_2, x] + s^1(1\,2)[(1\,2)\mathbf{c}_2, x] = 2 \cdot s^1[\mathbf{c}_2, x]$. □

**Remark 5.3.8.** Note that the previous Proposition 5.3.7 implies that $[\mathbf{c}, \mathbf{d}]$ is divisible by 2 (even by 4), and since it lives in $H_2(\mathfrak{M}_{2,1}) \cong \mathbb{Z}_2$, it has to be trivial, and not, as [Meh11, § 1.2] claimed, the generator $\mathbf{d}^2$.

### 5.3.4. *The homology of $\mathfrak{M}_{1,2}$*

In this last subsection, we give a complete list of generators of $H_\bullet(\mathfrak{M}_{1,2})$, both integrally and over $\mathbb{F}_2$. To do so, we relate $\mathfrak{M}_{1,2}$ to $\mathfrak{M}^1_{1,1}$.

**Remark 5.3.9.** For each $g \geqslant 0$, we have a map cap: $\mathfrak{M}_{g,2} \to \mathfrak{M}^1_{g,1}$ given by capping the second boundary curve with a punctured disc. Up to homotopy, this map coincides with the vertical unit tangent bundle $\mathbb{S}(T^\perp\mathfrak{M}^1_{g,1})$ over the surface bundle $\mathfrak{M}^1_{g,1} \to \mathfrak{M}_{g,1}$, the homotopy equivalence being precisely the

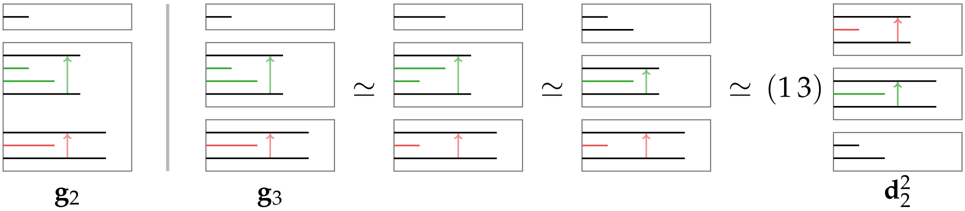

**Figure 5.13.** *Left:* the 2-torus $\mathbf{g}_2 \in H_2(\mathfrak{M}_{1,2})$: the two (unpaired) green slits move together inside the black pair, keeping a constant distance. *Right:* the class $\mathbf{g}_3 \in H_2(\mathfrak{M}_{0,3})$ and a homotopy to the 2-torus $(1\,3)\mathbf{d}_2^2$.

map $\vartheta_T \colon \mathbb{S}(T^{\perp}\mathfrak{M}_{g,1}^1) \to \mathfrak{M}_{g,2}$ from Construction 5.2.18. Under this identification, the fibre inclusion $\mathbb{S}^1 \hookrightarrow \mathfrak{M}_{g,2}$ parametrises the Dehn twist about the second boundary curve.

For $g \in \{0,1\}$, this sphere bundle is trivial: one way to see this is by noting that the corresponding second integral cohomology groups $H^2(\Gamma_{0,1}^1;\mathbb{Z})$ and $H^2(\Gamma_{1,1}^1;\mathbb{Z})$ vanish, compare Table 5.1 for the second one. Hence we may choose a section $\mathfrak{M}_{g,1}^1 \to \mathbb{S}(T^{\perp}\mathfrak{M}_{g,1}^1)$, and, postcomposing with the equivalence $\vartheta_T$, we obtain a section $\eta \colon \mathfrak{M}_{g,1}^1 \to \mathfrak{M}_{g,2}$ for $g \leqslant 2$.

**Construction 5.3.10.** We consider a new toric generator $\mathbf{g}_2 := \eta_* \mathbf{e} \in H_2(\mathfrak{M}_{1,2})$ which is depicted in Figure 5.13: the section $\mathfrak{M}_{g,1}^1 \to \mathbb{S}(T^{\perp}\mathfrak{M}_{g,1}^1)$ marks a fixed point on the circle around the puncture, and at this position, the new slit pair starts. Clearly $\mathbf{g}_2$ again has 2-torsion: we see $2 \cdot \mathbf{g}_2 = \eta_*(2 \cdot \mathbf{e}) = 0$.

The class $\mathbf{g}_2$ can be lifted to $\mathbf{g}_3 \in H_2(\mathfrak{M}_{0,3})$, which means $\mathbf{g}_2 = s^1 \mathbf{g}_3$, and this class $\mathbf{g}_3$ can, in turn, be identified with $(1\,3)\mathbf{d}_2^2$, see Figure 5.13. Note that $\mathbf{g}_3$ has no 2-torsion, as $\mathfrak{M}_{0,3} \simeq B\Gamma_{0,3} \cong \mathbb{T}^3$; indeed $\mathbf{g}_3$ is one of the free generators of $H_2(\mathfrak{M}_{0,3}) \cong \mathbb{Z}^3$: the group $\Gamma_{0,3}$ is generated by the Dehn twists about the three boundary curves and each of the tori $\mathbf{d}_2^2$, $(1\,2)\mathbf{d}_2^2$, and $(1\,3)\mathbf{d}_2^2$ parametrises two of them.

On the other hand, one may ask which class from $H_2(\mathfrak{M}_{2,1}) \cong \mathbb{Z}_2\langle \mathbf{d}^2 \rangle$ the 2-torus $\mathbf{g}_2$ itself lifts. Here we easily see that $s^1\mathbf{g}_2 = s^1 s^1 (1\,3)\mathbf{d}_2^2 = s^1 s^1 \mathbf{d}_2^2 = \mathbf{d}^2$.

Now we have all classes at hand to describe the generators of $H_\bullet(\mathfrak{M}_{1,2})$. Since we already know $\mathfrak{M}_{1,2} \simeq \mathfrak{M}_{1,1}^1 \times \mathbb{S}^1$, the previous results, together with the Künneth theorem and the universal coefficient theorem, tell us that

$$H_\bullet(\mathfrak{M}_{1,2};\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}\oplus\mathbb{Z}_2, \mathbb{Z}_2),$$
$$H_\bullet(\mathfrak{M}_{1,2};\mathbb{F}_2) = (\mathbb{F}_2, \mathbb{F}_2^2, \mathbb{F}_2^2, \mathbb{F}_2^2, \mathbb{F}_2).$$

**Proposition 5.3.11.** *The integral homology of $\mathfrak{M}_{1,2}$ is generated as follows:*

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mathfrak{M}_{1,2}$ | $\mathbb{Z}\langle \mathbf{c}_2\mathbf{c}\rangle$ | $\mathbb{Z}\langle \mathbf{c}_2\mathbf{d}, \mathbf{d}_2\mathbf{c}\rangle$ | $\mathbb{Z}\langle \mathbf{d}_2\mathbf{d}\rangle \oplus \mathbb{Z}_2\langle \mathbf{g}_2\rangle$ | $\mathbb{Z}_2\langle \hat{T}\mathbf{e}\rangle$ |

*Proof.* Clearly, the ground class $\mathbf{c}_2\mathbf{c}$ generates $H_0(\mathfrak{M}_{1,2})$. Secondly, $2\cdot\mathbf{g}_2 = 0$ and $s^1\mathbf{g}_2 = \mathbf{d}^2 \neq 0$ inside $H_2(\mathfrak{M}_{2,1})$, whence $\mathbf{g}_2$ generates the $\mathbb{Z}_2$-summand of $H_2(\mathfrak{M}_{1,2})$. Similarly, $s^1\hat{T}\mathbf{e} = T\mathbf{e} \neq 0$, whence $\hat{T}\mathbf{e}$ generates $H_3(\mathfrak{M}_{1,2}) \cong \mathbb{Z}_2$.

For the first homology, the product $\mathbf{c}_2\mathbf{d} = \eta_*(\mathbf{ad})$ generates the summand $H_1(\mathfrak{M}^1_{1,1})$ in the Künneth decomposition of $H_\bullet(\mathfrak{M}_{1,2}) \cong H_\bullet(\mathfrak{M}^1_{1,1}\times\mathbb{S}^1)$, and $\mathbf{d}_2\mathbf{c}$ is exactly the Dehn twist about the second boundary curve.

For the free part of the second homology, we note that $\mathbf{d}_2\mathbf{d}$ is supported on a 2-torus $\mathbb{S}^1\times\mathbb{S}^1 \to \mathfrak{M}_{1,2} \simeq \mathbb{S}^1\times\mathfrak{M}^1_{1,1}$, which is of the form $f_1\times f_2$, where $f_1\colon\mathbb{S}^1\to\mathbb{S}^1$ is the identity and $f_2\colon\mathbb{S}^1\to\mathfrak{M}^1_{1,1}$ is $\mathbf{ad}$; hence it is the homology cross product of the two generators of $H_1(\mathfrak{M}^1_{1,1})$ and $H_1(\mathbb{S}^1)$. □

If we consider instead homology over $\mathbb{F}_2$, then we encounter an occurrence of a vertical Dyer–Lashof square, applied to the fundamental Dehn twist $\mathbf{d}_2$:

**Proposition 5.3.12.** *The $\mathbb{F}_2$-homology of $\mathfrak{M}_{1,2}$ is generated by the following classes:*

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\mathfrak{M}_{1,2}$ | $\mathbf{c}_2\mathbf{c}$ | $\mathbf{c}_2\mathbf{d}, \mathbf{d}_2\mathbf{c}$ | $\mathbf{d}_2\mathbf{d}, \mathbf{g}_2$ | $\hat{T}\mathbf{e}, Q\mathbf{d}_2$ | $\hat{T}E\mathbf{b}$ |

*Proof.* Nearly each generator can be taken, via the universal coefficient theorem, from the previous integral result; we only need a second generator for $H_3(\mathfrak{M}_{1,2};\mathbb{F}_2)$ and a generator of $H_4(\mathfrak{M}_{1,2};\mathbb{F}_2)$. The latter generator is easy to find, as we know that $s^1\hat{T}E\mathbf{b} = TE\mathbf{b} \neq 0$ in $H_3(\mathfrak{M}_{2,1};\mathbb{F}_2)$. To identify the second generator of $H_3(\mathfrak{M}_{1,2};\mathbb{F}_2)$ with the vertical Dyer–Lashof square $Q\mathbf{d}_2$, it suffices to show that $s^1Q\mathbf{d}_2$ is neither 0 nor $s^1\hat{T}\mathbf{e}$ inside $H_3(\mathfrak{M}_{2,1};\mathbb{F}_2)$.

To this aim, we start by showing that $s^1[\mathbf{d}_2, \mathbf{a}] = \mathbf{e}$ holds (even integrally): note that $\mathfrak{M}^1_{0,2}$ fibres over $\mathfrak{M}_{0,2}$ as the universal surface bundle, with fibre a cylinder. In this description, we have $\mathfrak{M}_{0,2} \simeq \mathbb{S}^1$, where the fundamental loop corresponds to the Dehn twist on the cylinder, and $\mathfrak{M}^1_{0,2}$ is the mapping torus of the Dehn twist $D$ on $\mathbb{S}^1 \times [0;1]$. Since $D$ is isotopic to the identity (though not isotopic relative the boundary), $\mathfrak{M}^1_{0,2}$ is equivalent to a 2-torus, whose two fundamental loops are given by the loop of the base, i.e. the Dehn twist, and the loop of the fibre, i.e. the single puncture rotating once inside the cylinder. This is exactly what the 2-torus of $[\mathbf{d}_2, \mathbf{a}]$ parametrises up to sign. Hence, $[\mathbf{d}_2, \mathbf{a}]$ is a generator of $H_2(\mathfrak{M}^1_{0,2}) \cong \mathbb{Z}$, and thus $\mathbf{e}_2 = k \cdot [\mathbf{d}_2, \mathbf{a}]$ for some $k \in \mathbb{Z}$. Since $k \cdot s^1[\mathbf{d}_2, \mathbf{a}] = s^1\mathbf{e}_2 = \mathbf{e} \neq 0$, we conclude that $s^1[\mathbf{d}_2, \mathbf{a}] \neq 0$ inside $H_2(\mathfrak{M}^1_{1,1}) \cong \mathbb{Z}_2\langle \mathbf{e} \rangle$, which proves the subclaim. From that, it follows that $T\mathbf{e} = s^1\hat{T}s^1[\mathbf{d}_2, \mathbf{a}] = s^1s^1\hat{T}[\mathbf{d}_2, \mathbf{a}] = s^1s^1[\mathbf{d}_2, \mathbf{d}_2]$; here we have used that the relation $\hat{T} \circ f = (f \sqcup \mathbb{1}_1) \circ \hat{T}$ holds by construction.

Coming back to the statement, we can employ the relations from Proposition 4.4.25 and obtain that $s^1Q\mathbf{d}_2 = Qs^1\mathbf{d}_2 + s^1s^1[\mathbf{d}_2, \mathbf{d}_2] = Q\mathbf{d} + T\mathbf{e}$ holds inside $H_2(\mathfrak{M}_{2,1}; \mathbb{F}_2)$. Finally, we use from Table 5.2 that $Q\mathbf{d}$ is neither $T\mathbf{e}$ nor 0, whence $s^1Q\mathbf{d}_2$ is neither $T\mathbf{e} = s^1\hat{T}\mathbf{e}$ nor 0. □

Let us close this chapter again with a few open questions to the reader:

1. I am convinced that the factor in the relation $\mathbf{e}_2 = k \cdot [\mathbf{d}_2, \mathbf{a}]$ is a unit, or, in other words, that $\mathbf{e}_2$ is a generator of $H_2(\mathfrak{M}^1_{0,2})$ as well. So far, we only know that $k$ is odd, as $s^1\mathbf{e}_2 \neq 0$ holds inside $\mathbb{Z}_2$.

2. Some of these explicit calculations are useful in order to improve the slopes for higher-order homological stability [GKR19]. One aim is to calculate $H_3(\mathfrak{M}_{4,1}; \mathbb{Z})$: rationally, this group vanishes by calculations of [Tom05]; however, it is integrally unknown. If we could show that it vanishes, we would be able to improve the slope for integral secondary homological stability for the sequence $(\Gamma_{g,1})_{g \geqslant 0}$.

3. Similarly, it would be useful to determine $H_4(\mathfrak{M}_{4,1}; \mathbb{Q})$ or $H_4(\mathfrak{M}_{5,1}; \mathbb{Q})$: understanding these two vector spaces would help us to improve the slope for rational secondary homological stability for $(\Gamma_{g,1})_{g \geqslant 0}$. Maybe it is possible to construct a vertical Browder bracket that does not vanish in $H_4(\mathfrak{M}_{4,1}; \mathbb{Q})$.

Chapter 6

# Parametrised moduli spaces of surfaces as infinite loop spaces

> *One would like to characterize its homotopy type,*
> *but in reality one must settle for less.*
>
> Ib Madsen, on the moduli space $\mathfrak{M}_g$

The methods which we developed so far can be applied to a problem of a different kind: for a space $A$, we study the $E_2$-algebra $\mathfrak{M}_{\bullet,1}[A] := \coprod_g \mathfrak{M}_{g,1}[A]$ of surface bundles over $A$ and compute the homotopy type of its group completion $\Omega B\mathfrak{M}_{\bullet,1}[A]$: it turns out to be the product of the infinite loop space associated with the two-dimensional oriented tangential Thom spectrum $\mathbf{MTSO}(2)$ as in the Madsen–Weiss theorem, with a certain free infinite loop space depending on the family of all '$\partial$-irreducible' maps from $\pi_1(A)$ into mapping class groups $\Gamma_{g,n}$ with $g \geqslant 0$ and $n \geqslant 1$.

This is joint work with Andrea Bianchi and Jens Reinhold which culminated in the eponymous article [BKR21]. I decided to summarise the key steps of [BKR21, § 2+3+6] in Section 6.2. However, as the main result heavily relies on operadic techniques which are based on my own work and are deeply related to the previous chapters of this thesis, I decided to give [BKR21, § 4+5] a full treatise in Section 6.3.

## 6.1. Formulation of the problem

In this foundational first section, we give a short survey on classical results and formulate the generalisation we aim to make. Let us start with recalling the concept of group completing an H-monoid, and the theorem of Madsen and Weiss.

### 6.1.1. *The Madsen–Weiss theorem*

Very often, we work with $E_d$-algebras whose homotopy type or homology is hard to understand, but its *group completion* is more accessible.

**Construction 6.1.1** (Group completion)**.** Let $M$ be an $E_1$-algebra. We can form its bar construction $BM$, which comes with a natural map $\Sigma M \to BM$, and we are interested in the adjoint $M \to \Omega BM$.

If the submonoid $\pi_0(M)$ of the Pontrjagin ring $H_\bullet(M;\mathbb{Z})$ satisfies the Ore condition [Ore31] (which is e.g. satisfied if $M$ is H-commutative), then this map behaves like a *group completion* in the sense of [MS76]: the induced map in integral homology is isomorphic to the localisation of the Pontrjagin ring $H_\bullet(M;\mathbb{Z})$ at the submonoid $\pi_0(M) \subseteq H_0(M;\mathbb{Z})$ of path components.

By basic obstruction theory, being a group completion determines the weak homotopy type uniquely among simple spaces. In particular, if $M$ itself is grouplike (which means that the monoid $\pi_0(M)$ is a group) then the map $M \to \Omega BM$ is a weak equivalence.

Secondly, we need a concise definition of the two-dimensional oriented tangential Thom spectrum [Gal+09].

**Definition 6.1.2** (Oriented tangential Thom spectrum)**.** We consider on $\mathbb{R}^\infty$ the standard Euclidean scalar product; then all $\mathbb{R}^n \subseteq \mathbb{R}^\infty$ are isometrically included. For each dimension $d \geqslant 0$, we have the *orthogonal tautological bundle* $\mathbb{R}^n \to \gamma_{d,n}^\perp \to G_{d,n}$ over the oriented Graßmannian $G_{d,n} := \mathrm{Gr}_d^+(\mathbb{R}^{d+n})$.

Now if we consider the inclusion $\iota_n \colon G_{d,n} \hookrightarrow G_{d,n+1}$ which is induced by $\mathbb{R}^{d+n} = \mathbb{R}^{d+n} \times \{0\} \hookrightarrow \mathbb{R}^{d+n+1}$, then we have an isomorphism of $(n+1)$-dimensional vector bundles $\iota_n^*\gamma_{d,n+1}^\perp \cong \gamma_{d,n}^\perp \oplus \underline{\mathbb{R}}$, where $\underline{\mathbb{R}}$ denotes the trivial line bundle. In particular, we have a morphism of vector bundles

$$\begin{tikzcd}
\gamma_{d,n}^\perp \oplus \underline{\mathbb{R}} \arrow[r] \arrow[d] & \gamma_{d,n+1}^\perp \arrow[d] \\
G_{d,n+1} \arrow[r] & G_{d,n},
\end{tikzcd}$$

yielding a map of Thom spaces $\mathrm{Th}(\gamma_{d,n}^\perp) \wedge S^1 \cong \mathrm{Th}(\gamma_{d,n}^\perp \oplus \underline{\mathbb{R}}) \to \mathrm{Th}(\gamma_{d,n+1}^\perp)$. Thus, we obtain a spectrum $\mathbf{MTSO}(d)$, with $\mathbf{MTSO}(d)_{d+n} = \mathrm{Th}(\gamma_{d,n}^\perp)$, which we call the *d-dimensional oriented tangential Thom spectrum*.

With these facts at hand, the Madsen–Weiss theorem [MW07] can be formulated as follows: we have seen that the collection $\mathfrak{M}_{\bullet,1} := \coprod_g \mathfrak{M}_{g,1}$ carries[1] the structure of an $E_2$-algebra, more precisely an algebra over the little 2-discs operad $\mathscr{D}_2$. Secondly, there is a scanning map $\mathfrak{M}_{\bullet,1} \to \Omega^\infty \mathbf{MTSO}(2)$ which enjoys the property of being a group completion. Thus, the group completion of $\mathfrak{M}_{\bullet,1}$ can be identified with the infinite loop space $\Omega^\infty \mathbf{MTSO}(2)$:

$$\Omega B\, \mathfrak{M}_{\bullet,1} \simeq \Omega^\infty \mathbf{MTSO}(2).$$

This theorem was the key ingredient for the proof of the Mumford conjecture: if we fix a binary operation $\mu \in \mathscr{D}_2(2)$ and a conformal class $\mathcal{C}_1 \in \mathfrak{M}_{1,1}$, then we get a *stabilisation map* $\mu(\mathcal{C}_1, -)\colon \mathfrak{M}_{g,1} \to \mathfrak{M}_{g+1,1}$, and we define the *stable moduli space* $\mathfrak{M}_{\infty,1}$ as the homotopy colimit of the tower of stabilisation maps. Then $\mathfrak{M}_{\infty,1}$ is a classifying space for the *stable mapping class group* $\Gamma_{\infty,1}$, and hence its cohomology encodes stable characteristic classes for surface bundles.

There is a family of rational cohomology classes $\kappa_i \in H^{2i}(\mathfrak{M}_{\infty,1}; \mathbb{Q})$ for $i \geqslant 1$, called *Mumford–Miller–Morita classes* [Mum83; Mil86; Mor87], which was known for a long time to be algebraically independent [Mil86, Thm. 1.1]; in particular, there is an inclusion of $\mathbb{Q}$-algebras $\mathbb{Q}[\kappa_1, \kappa_2, \dots] \to H^\bullet(\mathfrak{M}_{\infty,1}; \mathbb{Q})$. The famous Mumford conjecture states that this map is an isomorphism.

This statement was provable by means of the Madsen–Weiss theorem: it is not hard to see that the homology of the group completion $\Omega B\, \mathfrak{M}_{\bullet,1}$ coincides with the homology of the product space $\mathbb{Z} \times \mathfrak{M}_{\infty,1}$; one copy of the stable moduli space for each integer. Therefore, we obtain an isomorphism in homology $H_\bullet(\mathfrak{M}_{\infty,1}) \cong H_\bullet(\Omega^\infty_0 \mathbf{MTSO}(2))$, where $\Omega^\infty_0 \mathbf{MTSO}(2)$ denotes the path component of $\Omega^\infty \mathbf{MTSO}(2)$ containing the constant loop. The right side, in turn, is accessible by standard methods, see e.g. [Hat11, Apx. C].

### 6.1.2. *Our generalisation of the problem*

We consider the analogous problem with $\mathfrak{M}_{\bullet,1}$ replaced by the space $\mathfrak{M}_{\bullet,1}[A]$ of surface bundles over some path connected space $A$, called *parametrised moduli space*, which is topologised by being identified with the space of maps

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[1] In this chapter, we decided to skip decorations indicating a regular collaring. The reader is free to choose their preferred model from the previous chapter.

from $A$ to $\mathfrak{M}_{\bullet,1}$. This parametrised moduli space is again an $E_2$-algebra by extending the action pointwise, and it is our goal to understand its group completion $\Omega B\mathfrak{M}_{\bullet,1}[A]$

**Remark 6.1.3.** There are two immediate facts which we can observe:

1. Note that $\mathfrak{M}_{\bullet,1}$ is not only a $E_2$-algebra, but also an algebra over the (monochromatic) surface operad $\mathscr{M}|_1$, so the same applies to $\mathfrak{M}_{\bullet,1}[A]$ by pointwise action. The main theorem of [Til00] then tells us that the homotopy type we wish to understand is an infinite loop space.

2. We have a map $\mathfrak{M}_{\bullet,1} \to \mathfrak{M}_{\bullet,1}[A]$ which assigns to each conformal class $\mathcal{C}$ the constant map with value $\mathcal{C}$. Conversely, each choice of basepoint in $A$ gives rise to an evaluation map $\mathfrak{M}_{\bullet,1}[A] \to \mathfrak{M}_{\bullet,1}$, which is a left inverse of the former one. Moreover, both maps are morphisms of $\mathscr{M}|_1$-algebras, so when passing to group completions, we recover the infinite loop space $\Omega B\mathfrak{M}_{\bullet,1}$ as a rectract of the infinite loop space $\Omega B\mathfrak{M}_{\bullet,1}[A]$. This shows that $\Omega^\infty\mathbf{MTSO}(2)$ is a direct factor of $\Omega B\mathfrak{M}_{\bullet,1}[A]$.

Thus, there is a spectrum $E[A]$ such that $\Omega B\mathfrak{M}_{\bullet,1}[A]$ has the homotopy type of $\Omega^\infty\mathbf{MTSO}(2) \times \Omega^\infty E[A]$, and we aim to understand $E[A]$.

**Remark 6.1.4.** For any discrete group $\Gamma$ and each path connected space $A$, one can describe the homotopy of the mapping space as

$$B\Gamma[A] \simeq \coprod_{[\omega]\in\mathrm{Conj}(\pi\to\Gamma)} BZ(\mathrm{im}(\omega),\Gamma),$$

where $\pi := \pi_1(A)$ and $\mathrm{Conj}(\pi \to \Gamma)$ is the set of conjugacy classes of homomorphisms $\omega\colon \pi \to \Gamma$, where two homomorphisms $\omega$ and $\omega'$ are conjugate if there exists a $\varphi \in \Gamma$ with $\omega' = \varphi \cdot \omega \cdot \varphi^{-1}$, and where $Z(\Pi,\Gamma)$ is the centraliser of a subgroup $\Pi \subseteq \Gamma$. Note that the isomorphism type of the group $Z(\mathrm{im}(\omega),\Gamma)$ only depends on the conjugacy class of $\omega$.

Therefore, the problem we address is strongly related to analysing the structure of centralisers of elements of mapping class groups: recall that if we fix a surface $\mathcal{S}_{g,1}$ of type $\Sigma_{g,1}$, then $\mathfrak{M}_{g,1}$ is a classifying space for the mapping class group $\Gamma(\mathcal{S}_{g,1})$, whence we have a homotopy equivalence

$$\mathfrak{M}_{\bullet,1}[A] \simeq \coprod_{g\geqslant 0} B\Gamma(\mathcal{S}_{g,1})[A] \simeq \coprod_{g\geqslant 0}\ \coprod_{[\omega]\in\mathrm{Conj}(\pi\to\Gamma(\mathcal{S}_{g,1}))} BZ(\mathrm{im}(\omega),\Gamma(\mathcal{S}_{g,1})).$$

Therefore, the answer to our problem will necessarily involve a structure result for centralisers of mapping classes.

### 6.1.3. *Results and outline*

It may be surprising at first glance that for the advertised structure result for centralisers of mapping classes, we have to widen our view and pass to mapping class groups of surfaces with *multiple* boundary curves.

The parametrised moduli space $\mathfrak{M}_{g,n}[A]$ of surfaces of genus $g$ with $n$ parametrised boundary circles admits an action by the isometry group of the disjoint union of $n$ oriented circles, i.e. by the topological group $\mathbb{T}^n \rtimes \mathfrak{S}_n$. We introduce, for each $n \geqslant 1$ and $g \geqslant 0$, a conjugation-invariant irreducibility criterion for subgroups of $\Gamma(\mathcal{S}_{g,n})$, and consider the subspace $\mathfrak{C}_{g,n}[A] \subseteq \mathfrak{M}_{g,n}[A]$ of connected components whose corresponding conjugacy classes of maps $\pi_1(A) \to \pi_1(\mathfrak{M}_{g,n}) \cong \Gamma_{g,n}$ have an irreducible image. Then the pointwise action of $\mathbb{T}^n \rtimes \mathfrak{S}_n$ on $\mathfrak{M}_{g,n}[A]$ restricts to an action on $\mathfrak{C}_{g,n}[A]$, and our main result is the following identification, where $/\!\!/$ denotes the homotopy quotient:

Theorem 6.4.1. *For every path connected space $A$, there is a weak equivalence of loop spaces*

$$\Omega B\mathfrak{M}_{\bullet,1}[A] \simeq \Omega^\infty \mathbf{MTSO}(2) \times \Omega^\infty \Sigma^\infty_+ \coprod_{n \geqslant 1} \coprod_{g \geqslant 0} \mathfrak{C}_{g,n}[A] /\!\!/ (\mathbb{T}^n \rtimes \mathfrak{S}_n).$$

The proof of this theorem relies on two ingredients: in a first step, which constitutes Section 6.2, we describe centralisers of subgroups of mapping class groups. As a result, we will obtain that $\mathfrak{M}_{\bullet,1}[A]$ is the first level of a relatively free $\mathscr{M}$-algebra, relative to a suboperad built out of the groups $\mathbb{T}^n \rtimes \mathfrak{S}_n$, and the generators being precisely the aforementioned spaces $\mathfrak{C}_{g,n}[A]$, which assemble into an algebra over this suboperad.

Secondly, we develop in Section 6.3 an operadic machinery to compute the homotopy type of group completions of relatively free algebras over coloured operads, provided that the operad in question is homologically stable in a meaningful sense. This part of the work is a generalisation of [Til00; Bas+17] to the coloured and relative case.

## 6.2. Parametrised moduli spaces as free algebras

The aim of this section is to study centralisers of subgroups of mapping class groups of surfaces, culminating in a description of $\mathfrak{M}_{\bullet,1}[A]$ as the first level of a relatively free $\mathscr{M}$-algebra. As already announced in the introduction, we only highlight the main steps and refer to [BKR21, § 2+3+6] for details.

### 6.2.1. *Fixed-arc complexes and ∂-irreducibility*

Here we introduce the notion of ∂-irreducibility for subgroups of mapping class groups. This summarises the content of [BKR21, § 2.2+2.3] and uses the notation from Subsection 5.1.1.

**Idea 6.2.1.** Let us fix a surface $\mathcal{S}$ of type $\Sigma_{g,n}$, with $g \geqslant 0$ and $n \geqslant 1$, and focus on the mapping class group $\Gamma(\mathcal{S})$. Given a subgroup $\Pi \subseteq \Gamma(\mathcal{S})$, we construct a system of simple closed curves on $\mathcal{S}$ which cuts $\mathcal{S}$ into two subsurfaces $W$ and $Y$, where the subsurface $W \subset \mathcal{S}$ is, up to isotopy, the maximal subsurface of $\mathcal{S}$ satisfying the following conditions:

1. all connected components of $W$ touch $\partial\mathcal{S}$;
2. each $\varphi \in \Pi$ can be represented by a diffeomorphism of $\mathcal{S}$ fixing $W$ pointwise.

We start by recalling some standard facts about embedded arcs in surfaces. The material of this subsection is taken, up to minor changes, from [FM12]; for the following definition see [FM12, § 1.2.7].

**Definition 6.2.2.** An *arc* in $\mathcal{S}$ is a smooth embedding $\alpha\colon [0;1] \hookrightarrow \mathcal{S}$ such that we have $\alpha^{-1}(\partial\mathcal{S}) = \{0,1\}$ and $\alpha$ is transverse to $\partial\mathcal{S}$. Two arcs are *disjoint* if their images are disjoint (also at the endpoints). An arc is *essential* if it does not cut $\mathcal{S}$ in two parts, one of which is a disc.

Two arcs $\alpha$ and $\alpha'$ are *directly isotopic* if $\alpha(0) = \alpha'(0)$, $\alpha(1) = \alpha'(1)$, and there is an isotopy of embeddings $[0;1] \hookrightarrow \mathcal{S}$ which is stationary on $\{0,1\}$ and connects $\alpha$ to $\alpha'$. Two arcs are *inversely isotopic* if the previous holds after reparametrising one of the two arcs in the opposite direction. Two arcs are *isotopic* if they are directly or inversely isotopic: we then write $\alpha \sim \alpha'$.

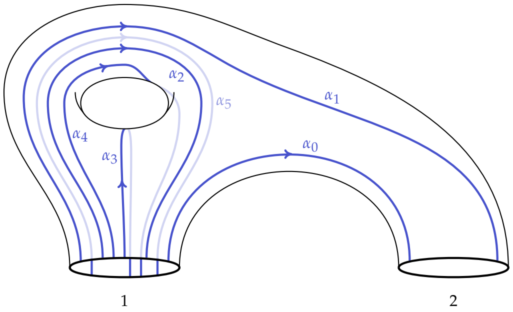

**Figure 6.1.** A maximal collection of six pairwise non-parallel arcs $\alpha_0, \dots, \alpha_5$ on a surface of type $\Sigma_{1,2}$.

Two arcs $\alpha$ and $\beta$ are in *minimal position* if they are disjoint at their endpoints, intersect transversely, and the number of intersection points in $\alpha \cup \beta$ is minimal among all choices of $\alpha' \sim \alpha$ and $\beta' \sim \beta$ with $\alpha'$ and $\beta'$ disjoint at the endpoints and transverse.

Note that we only consider isotopy classes of arcs relative to their endpoints; two arcs sharing one endpoint are never considered in minimal position (and, by convention, cannot be isotoped to be in minimal position).

If $\chi(\mathcal{S}) \leqslant 0$, then the following holds, see [FM12, § 1.2.7]: given a collection of essential arcs $\alpha_1, \dots, \alpha_k$ in $\mathcal{S}$ which are pairwise non-isotopic and have all distinct endpoints, one can replace each $\alpha_i$ with an arc $\alpha'_i \sim \alpha_i$ so that $\alpha'_1, \dots, \alpha'_k$ are pairwise in minimal position. Among all connected surfaces with non-empty boundary, the only one with positive Euler characteristic is the disc $\Sigma_{0,1}$, for which the statement holds vacuously, as there are no essential arcs in $\Sigma_{0,1}$.

**Definition 6.2.3.** Two arcs $\alpha$ and $\alpha'$ in $\mathcal{S}$ are *directly parallel* if they are disjoint and there is an *embedding* $[0;1] \times [0;1] \hookrightarrow \mathcal{S}$ restricting to $\alpha$ on $[0;1] \times \{0\}$ and to $\alpha'$ on $[0;1] \times \{1\}$, and restricting to an embedding $\{0,1\} \times [0;1] \hookrightarrow \partial\mathcal{S}$.

Two arcs $\alpha$ and $\alpha'$ are *inversely parallel* if the previous holds after reparametrising one of the two arcs in the opposite direction. Two arcs $\alpha$ and $\alpha'$ are *parallel* if they are directly or inversely parallel.

Let us fix a subgroup $\Pi \subseteq \Gamma(\mathcal{S})$ and study the isotopy classes of arcs and curves that it fixes.

**Definition 6.2.4.** The *fixed-arc complex of* $\Pi$ is an abstract simplicial complex whose vertices are isotopy classes of essential arcs $\alpha$ in $\mathcal{S}$ which are fixed by all $\varphi \in \Pi$. A collection of isotopy classes of arcs $\alpha_0, \dots, \alpha_h$ spans an $h$-simplex if $\alpha_0, \dots, \alpha_h$ can be isotoped to disjoint, pairwise non-parallel arcs $\alpha'_0, \dots, \alpha'_h$.

The subgroup $\Pi$ is called $\partial$*-irreducible* if its fixed-arc complex is empty and if $\mathcal{S}$ is not of type $\Sigma_{0,1}$. Being $\partial$-irreducible is a conjugation invariant.

**Example 6.2.5.** Here are some examples of fixed-arc complexes:

1. Every isotopy class of essential arcs in $\mathcal{S}$ is fixed (up to isotopy) by the identity $\mathbb{1} \in \Gamma(\mathcal{S})$. Therefore, the trivial subgroup is not $\partial$-irreducible, provided that $\mathcal{S}$ admits some essential arc; otherwise, $\mathcal{S}$ is a disc $\Sigma_{0,1}$ and in this case, $\{\mathbb{1}\} = \Gamma_{0,1}$ is not $\partial$-irreducible by definition.

2. The fixed-arc complex of $\langle \varphi \rangle \subseteq \Gamma_{0,2} \cong \mathbb{Z}$ is empty if $\varphi \neq \mathbb{1}$, and consists of uncountably many vertices, joined by no higher simplex if $\varphi = \mathbb{1}$.

3. For $g \geqslant 1$ the subgroup spanned by the boundary Dehn twist $D_\partial \in \Gamma_{g,1}$ is $\partial$-irreducible, though there are plenty of isotopy classes of simple closed curves in $\Sigma_{g,1}$ that are fixed by $D_\partial$. Nevertheless, *no* isotopy class of essential arcs is fixed by $D_\partial$: here it is crucial to consider isotopy classes of arcs relative to their endpoints, which lie at the boundary.

Now we turn to the decomposition of $\mathcal{S}$ into $Y$ and $W$ from Idea 6.2.1, which we formalise as follows:

**Construction 6.2.6.** Let $\mathcal{S}$ be a connected surface which is not of type $\Sigma_{0,1}$, let $\Pi \subseteq \Gamma(\mathcal{S})$, and let $\alpha_0, \dots \alpha_h$ be disjoint, essential, pairwise non-parallel arcs in $\mathcal{S}$, representing a maximal simplex in the fixed-arc complex of $\Pi$.

Let $U$ be a closed, small neighbourhood of the union $\alpha_0 \cup \dots \cup \alpha_h \cup \partial\mathcal{S}$, and let $W \subset \mathcal{S}$ be the union of $U$ and all components of $\mathcal{S} \setminus U$ that are discs. Then $W$ is a closed, possibly disconnected subsurface of $\mathcal{S}$, and we denote by $Y$ the closure of $\mathcal{S} \setminus W$.

If $\partial W$ denotes the union of all boundary components of $W$, and $\partial Y$ denotes the union of all boundary components of $Y$, then $\partial W = \partial\mathcal{S} \cup c_1 \cup \dots \cup c_k$, for some $k \geqslant 0$ and some curves $c_1, \dots, c_k \subset \mathcal{S}$; similarly $\partial Y = c_1 \cup \dots \cup c_k$. The curves $c_1, \dots, c_k$ inherit a canonical boundary orientation from $Y$, which is oriented as subsurface of $\mathcal{S}$.

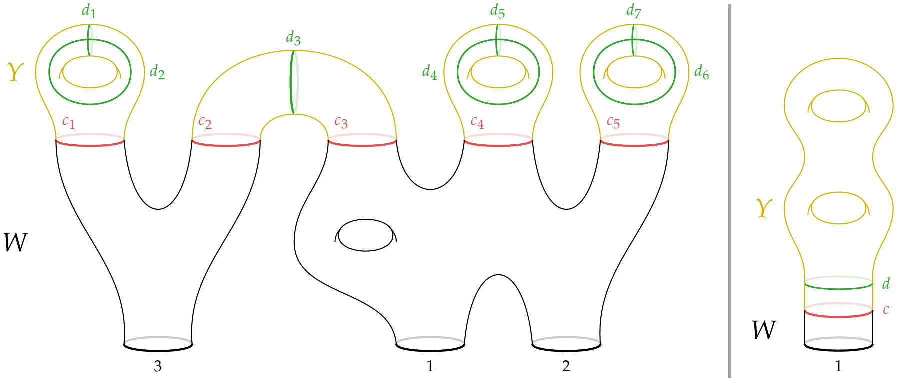

**Figure 6.2.** Two examples of a decomposition of $\mathcal{S}$ into $Y$ and $W$, according to the subgroup spanned by a single mapping class $\varphi$. In the first case, $\varphi$ is given by the product of the Dehn twists along the curves $d_1, \ldots, d_7$, and in the second case, $\varphi$ is just the Dehn twist along the single green curve $d$. In the second case, $\langle\varphi\rangle$ is $\partial$-irreducible, the cut locus consists only of the isotopy class of $d$, oriented as an outgoing boundary $Y$, while $W$ is merely a collar neighbourhood of $\partial\mathcal{S}$. The cut loci are drawn in red.

**Definition 6.2.7.** For $\Pi$ and $\alpha_0, \ldots \alpha_h$ as above, we define the *cut locus* of $\Pi$ as the isotopy class of the multicurve $c_1, \ldots, c_k$, denoted $[c_1, \ldots, c_k]$. Here and in the following, a *multicurve* is an unordered collection of disjoint and *oriented* simple closed curves, and an isotopy moves all curves simultaneously. We declare the cut locus of $\Gamma_{0,1}$ to be empty and $W$ to be the entire surface $\Sigma_{0,1}$.

A priori, the cut locus depends on both a choice of a maximal simplex in the fixed-arc complex of $\Pi$, and on a choice of arcs $\alpha_0, \ldots, \alpha_h$ representing it. In [BKR21, Lem. A.5], we show that the cut locus only depends on $\Pi$. When referring to the *cut locus* of a subgroup $\Pi \subseteq \Gamma(\mathcal{S})$ we mean the cut locus of $\Pi$ with respect to *any* maximal simplex in the fixed-arc complex of $\Pi$.

**Remark 6.2.8.** The cut locus of behaves well under conjugation: let $\psi \in \Gamma(\mathcal{S})$ be a mapping class, $\Pi \subseteq \Gamma(\mathcal{S})$ be a subgroup, and let $[c_1, \ldots, c_k]$ be the cut locus of $\Pi$; then $[\psi(c_1), \ldots, \psi(c_k)]$ is the cut locus of $\psi \cdot \Pi \cdot \psi^{-1}$.

Thus, if $\psi$ lies in the centraliser of $\Pi$, then $\psi$ preserves the cut locus of $\Pi$ as an unordered collection of isotopy classes of oriented simple closed curves.

### 6.2.2. *Centralisers of mapping classes*

In this subsection, we want to use the decomposition of $\mathcal{S}$ which we assigned to a subgroup $\Pi \in \Gamma(\mathcal{S})$ in order to describe the centraliser $Z(\Pi, \Gamma(\mathcal{S}))$ of $\Pi$. By doing so, we summarise the content of [BKR21, § 3]. As a first ingredient, we need the notion of an extended mapping class group.

**Definition 6.2.9.** Let $(\mathcal{S}, \Theta)$ be a surface with parametrised boundary, of type $\Sigma\colon k \to n$ and let $\mathfrak{H} \subseteq \mathfrak{S}_k \times \mathfrak{S}_n$ be a subgroup. We let $\mathrm{Diff}^{\mathfrak{H}}(\mathcal{S}, \Theta)$ be the topological group of orientation-preserving diffeomorphisms $\Phi\colon \mathcal{S} \to \mathcal{S}$ such that there is a permutation $(\sigma, \tau) \in \mathfrak{H}$ with

$$\begin{aligned} \Phi \circ \Theta^{\mathrm{in}} &= \Theta^{\mathrm{in}} \circ (\sigma \times \mathrm{id}_{\mathbb{S}^1}), \\ \Phi \circ \Theta^{\mathrm{out}} &= \Theta^{\mathrm{out}} \circ (\tau \times \mathrm{id}_{\mathbb{S}^1}), \end{aligned}$$

and we define the *extended mapping class group* $\Gamma^{\mathfrak{H}}(\mathcal{S}, \Theta) := \pi_0(\mathrm{Diff}^{\mathfrak{H}}(\mathcal{S}, \Theta))$. If the parametrisation is clear from the context, we write $\Gamma^{\mathfrak{H}}(\mathcal{S})$.

We have a map $\Gamma^{\mathfrak{H}}(\mathcal{S}) \to \mathfrak{H}$, whose kernel is given by the usual mapping class group $\Gamma(\mathcal{S})$.

Next, we fix a surface $\mathcal{S}$ of type $\Sigma_{g,n}$, a subgroup $\Pi \in \Gamma(\mathcal{S})$ and oriented simple closed curves $c_1, \dots, c_k \subset \mathcal{S}$ representing the cut locus of $\Pi$. Moreover, we let $W \cup Y$ be the associated decomposition of $\mathcal{S}$.

Recall from Construction 6.2.6 that the curves $c_1, \dots, c_k$ inherit an orientation from $Y$, and in this way, $c_1, \dots, c_k$ become incoming curves for $W$ and outgoing curves for $Y$. In fact, we have $\partial Y = \partial^{\mathrm{out}} Y = c_1 \cup \dots \cup c_k = \partial^{\mathrm{in}} W$, whereas $\partial^{\mathrm{out}} W = \partial \mathcal{S}$.

For each $\varphi \in \Pi$, we choose a representative $\Phi\colon \mathcal{S} \to \mathcal{S}$ which fixes $W$ pointwise. In particular, $\Phi$ decomposes into diffeomorphisms $\Phi|_Y\colon Y \to Y$ and $\Phi|_W\colon W \to W$ that fix the respective boundaries pointwise.

**Construction 6.2.10.** Let $\varphi_Y \in \Gamma(Y)$ be the mapping class represented by $\Phi|_Y$ and let $Z(\Pi_Y, \Gamma^{\mathfrak{S}_k}(Y))$ be the centraliser of $\Pi_Y := \{\varphi_Y;\ \varphi \in \Pi\}$ inside $\Gamma^{\mathfrak{S}_k}(Y)$. Secondly, we can regard $\mathfrak{S}_k$ as a subgroup of $\mathfrak{S}_k \times \mathfrak{S}_n$, and consider the extended mapping class group $\Gamma^{\mathfrak{S}_k}(W)$.

The crucial observation is the fact that, given mapping classes $\psi_W \in \Gamma^{\mathfrak{S}_k}(W)$ and $\psi_Y \in Z(\Pi_Y, \Gamma^{\mathfrak{S}_k}(Y))$ inducing the same boundary curve permutation in

$\mathfrak{S}_k$, we can build out of it an element in the desired centraliser $Z(\Pi, \Gamma(\mathcal{S}))$ as follows: we pick representatives $\Psi_W \colon W \to W$ and $\Psi_Y \colon Y \to Y$; then the fact that $\psi_W$ and $\psi_Y$ project to the same permutation of $\pi_0(\partial^{\mathrm{in}} W) = \pi_0(\partial Y) = \underline{k}$, together with the fact that both $\Psi_W$ and $\Psi_Y$ preserve the boundary parametrisation (up to permutation), implies that $\Psi_Y|_{\partial Y} = \Psi_W|_{\partial^{\mathrm{in}} W}$, and hence we can glue the two diffeomorphisms together to a diffeomorphism of $\mathcal{S}$.

It is easy to see [BKR21, Lem. 3.5] that the resulting mapping class commutes with each $\varphi \in \Pi$. Thus, we have constructed a homomorphism

$$\varepsilon \colon \Gamma^{\mathfrak{S}_k}(W) \times^{\mathfrak{S}_k} Z(\Pi_Y, \Gamma^{\mathfrak{S}_k}(Y)) \to Z(\Pi, \Gamma(\mathcal{S})),$$

where '$\times^{\mathfrak{S}_k}$' denotes the fibre product of groups. It turns out [BKR21, § 3.4] that $\varepsilon$ is surjective, and the kernel of $\varepsilon$ has the following description:

**Construction 6.2.11.** For each $1 \leqslant \jmath \leqslant k$, we can consider the Dehn twist $D_\jmath$ about the $\jmath^{\mathrm{th}}$ curve $c_\jmath$ of the cut locus. Then $D_\jmath$ is clearly an element in $\Gamma^{\mathfrak{S}_k}(W)$, but it also lies in $Z(\Pi_Y, \Gamma^{\mathfrak{S}_k}(Y))$, since it is central in the non-extended mapping class group $\Gamma(Y)$, to which each $\varphi_Y$ belongs.

Hence, $(D_\jmath, D_\jmath^{-1})$ lies in the fibre product $\Gamma^{\mathfrak{S}_k}(W) \times^{\mathfrak{S}_k} Z(\Pi_Y, \Gamma^{\mathfrak{S}_k}(Y))$, and its image $\varepsilon(D_\jmath, D_\jmath^{-1})$ is trivial, as the two Dehn twists cancel each other out when they are amalgamated over the cut locus.

In [BKR21, § 3.3] we show that these diagonally embedded Dehn twists generate the kernel of $\varepsilon$. As they mutually commute and satisfy no further relation, they generate a subgroup which is free abelian of rank $k$.

**Corollary 6.2.12.** *We have an isomorphism of groups*

$$Z(\Pi, \Gamma(\mathcal{S})) \cong \frac{\Gamma^{\mathfrak{S}_k}(W) \times^{\mathfrak{S}_k} Z(\Pi_Y, \Gamma^{\mathfrak{S}_k}(Y))}{\mathbb{Z}^k}.$$

We still need to get a deeper understanding of the centraliser $Z(\Pi_Y, \Gamma^{\mathfrak{S}_k}(Y))$. To do so, we start by fixing for each $g \geqslant 0$ and $n \geqslant 1$ a standard surface $\mathcal{S}_{g,n}$, and, for each conjugacy class of subgroups in $\Gamma(\mathcal{S}_{g,n})$, a representative.

**Notation 6.2.13.** For each path component $P \subseteq Y$, the subgroup $\Pi$ restricts to a subgroup $\Pi_P$ of $\Gamma(P)$. Two path components $P$ and $P'$ of $Y$ are called *similar* if there is a diffeomorphism $\Xi \colon P \to P'$ preserving the boundary parametrisation (up to permutation), such that $\Pi_{P'} = \Xi \cdot \Pi_P \cdot \Xi^{-1}$.

We write $Y = \coprod_i \coprod_j Y_{i,j}$, where $Y_{1,1}, \dots, Y_{r,s_r} \subseteq Y$ are the connected components of $Y$, and $Y_{i,j}$ is similar to $Y_{i',j'}$ if and only if $i = i'$ holds. We also define $Y_i := \coprod_j Y_{i,j}$ and abbreviate $\Pi_{i,j} := \Pi_{Y_{i,j}}$.

For each $1 \leqslant i \leqslant r$, there are unique $g_i \geqslant 0$ and $k_i \geqslant 1$ such that $Y_{i,j}$ is of type $\Sigma_{g_i,k_i}$, regardless of $j$. We fix a diffeomorphism $\Xi_{i,j} \colon Y_{i,j} \to \mathcal{S}_{g_i,k_i}$ which respects the boundary parametrisation. This gives rise to an isomorphism $\Gamma(Y_{i,j}) \to \Gamma(\mathcal{S}_{g_i,k_i})$, under which $\Pi_{i,j}$ corresponds to $\bar{\Pi}_{i,j} := \Xi \cdot (\Pi_{i,j}) \cdot \Xi_{i,j}^{-1}$. Up to replacing $\Xi_{i,j}$ by another diffeomorphism, we can assume that $\bar{\Pi}_{i,j}$ coincides with the chosen representative inside the conjugacy class. Under this assumption, we also have $\bar{\Pi}_{i,j} = \bar{\Pi}_{i,j'}$ for each $1 \leqslant i \leqslant r$ and $1 \leqslant j, j' \leqslant s_i$, and we write $\bar{\Pi}_i := \bar{\Pi}_{i,1}$.

**Remark 6.2.14.** It is not hard to see that the subgroup $\Pi_{i,j}$ is $\partial$-irreducible, essentially because $W$ emerged from a maximal simplex; for details, see [BKR21, Lem. 3.3]. Secondly, we show in [BKR21, Lem. 3.6] that the centraliser $Z(\Pi_Y, \Gamma^{\mathfrak{S}^k}(Y))$ splits as the product of wreath products

$$Z(\Pi_Y, \Gamma^{\mathfrak{S}_k}(Y)) \cong \prod_{i=1}^{r} Z(\bar{\Pi}_i, \Gamma^{\mathfrak{S}_{k_i}}(\mathcal{S}_{g_i,k_i})) \wr \mathfrak{S}_{s_i},$$

as each $\psi_Y \in \Gamma^{\mathfrak{S}_k}(Y)$ which commutes with $\Pi_Y$ has to preserve the subsurfaces $Y_1, \dots, Y_r$, and each $\psi_{Y_i} \in \Gamma^{\mathfrak{S}_{k_i \cdot s_i}}(Y_i)$ which commutes with $\Pi_{Y_i}$ can be written as a permutation of similar components, followed by a tuple of automorphisms $\psi_{i,j}$, one for each component $Y_{i,j}$, and $\psi_{i,j}$ has to commute with $\Pi_{i,j}$. Finally, we use the diffeomorphisms $\Xi_{i,j}$ to identify the latter centralisers with the centralisers of $\bar{\Pi}_i$ inside $\Gamma^{\mathfrak{S}_{k_i}}(\mathcal{S}_{g_i,k_i})$.

We finish this subsection by mentioning a technicality: when forming the fibre product $\Gamma^{\mathfrak{S}_k}(W) \times^{\mathfrak{S}_k} Z(\Pi_Y, \Gamma^{\mathfrak{S}_k}(Y))$, only the subgroup of $\mathfrak{S}_k$ containing permutations which are hit by *both* factors is relevant. In general, this subgroup depends on the topological type of $W$, but if $W$ is connected (which e.g. is the case if $\mathcal{S}$ has only a single outgoing boundary curve) then $\Gamma^{\mathfrak{S}_k}(W)$ surjects onto $\mathfrak{S}_k$, so we only have to deal with the left side.

Here we see that if we denote by $\mathfrak{H}_i \subseteq \mathfrak{S}_{k_i}$ the image of $Z(\bar{\Pi}_i, \Gamma^{\mathfrak{S}_{k_i}}(\mathcal{S}_{g_i,k_i}))$ under the projection $\Gamma^{\mathfrak{S}_{k_i}}(\mathcal{S}_{g_i,k_i}) \to \mathfrak{S}_{k_i}$, then the image of $Z(\Pi_Y, \Gamma^{\mathfrak{S}_k}(Y))$ under $\Gamma^{\mathfrak{S}_k}(Y) \to \mathfrak{S}_k$ is given by $\prod_i \mathfrak{H}_i \wr \mathfrak{S}_{s_i}$. This yields the following result:

**Corollary 6.2.15.** *If $W$ is connected, then we have an isomorphism*

$$\mathrm{Z}(\Pi, \Gamma(\mathcal{S})) \cong \frac{\Gamma^{\prod_i \mathfrak{H}_i \wr \mathfrak{S}_{s_i}}(W) \times^{\prod_i \mathfrak{H}_i \wr \mathfrak{S}_{s_i}} \prod_i \mathrm{Z}(\bar{\Pi}_i) \wr \mathfrak{S}_{s_i}}{\prod_i \mathbb{Z}^{k_i \cdot s_i}}.$$

### 6.2.3. *Twisted tori and an operadic description*

In this subsection we use the structure result for centralisers of mapping class groups from the previous subsection in order to describe $\mathfrak{M}_{\bullet,1}[A]$ as the first level of a relatively free $\mathscr{M}$-algebra. By doing so, we use the model for moduli spaces of collared Riemann surfaces from Subsection 5.1.2 and we summarise the results from [BKR21, § 6].

**Construction 6.2.16.** For each $n \geqslant 1$, the Lie group $\mathbb{T}^n \rtimes \mathfrak{S}_n$ is denoted by $\mathbb{T}_n$ and called the *twisted k-torus*; it is the isometry group of $\coprod_n \mathbb{S}^1$.

We can embed $\mathbb{T}_n$ in the endomorphism space $\mathbf{M}_\partial\binom{n}{n}$ as follows: given an element $(w_1, \dots, w_k, \sigma)$, we assign to it the conformal class $\mathcal{C} = [\mathcal{F}, \Theta]$ with

$$\mathcal{F} := \underline{n} \times \{z \in \mathbb{C}; 1 \leqslant |z| \leqslant 2\},$$

together with the inherited complex structure, and the two parametrisations $\Theta^{\mathrm{in}} \colon \underline{n} \times \mathbb{S}^1 \hookrightarrow \mathcal{S}$ and $\Theta^{\mathrm{out}} \colon \underline{n} \times \mathbb{S}^1 \hookrightarrow \mathcal{S}$ given by

$$\begin{aligned} \Theta^{\mathrm{in}}(\ell, z) &:= (\sigma(\ell), z), \\ \Theta^{\mathrm{out}}(\ell, z) &:= (\ell, w_\ell \cdot z). \end{aligned}$$

This assignment in fact embeds $\mathbb{T}_n$ as a group into the automorphisms of $n$; in particular, the identity of $n$ can be described as the image of the unit of $\mathbb{T}_n$ along the embedding. If we put $\mathbb{T} := (\mathbb{T}_n)_{n \geqslant 1}$, considered as an $\overline{\mathbb{N}}$-coloured operad with only unaries, then this gives an inclusion of operads $\jmath \colon \mathbb{T} \hookrightarrow \mathscr{M}$, whose image lies in the connective suboperad $\mathscr{M}^c$, as $\mathbb{T}$ contains only unaries, and all unaries are connective.

**Lemma 6.2.17.** *The group $\mathbb{T}_k$ acts freely on the space $\mathbf{M}_\partial\binom{k}{n}$ by precomposition.*

This statement is shown in [BKR21, Lem. 4.14]; however, as we use a slightly refined model for the conformal cobordism PROP, let us repeat the proof.

$$\jmath\left(e^{\frac{\pi i}{4}}, e^{\frac{4\pi i}{3}}, 1, (1\,2\,3)\right) =$$

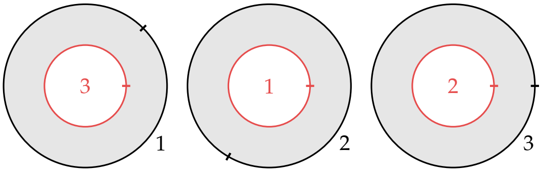

**Figure 6.3.** An instance of the inclusion $\jmath\colon \mathbb{T}^3 \rtimes \mathfrak{S}_3 \hookrightarrow \mathbf{M}_\partial\binom{3}{3}$: the little black ticks show the points $\Theta^{\mathrm{out}}(\ell, 1)$, while the little red ticks show the points $\Theta^{\mathrm{in}}(\ell, 1)$, for $\ell \in \{1, 2, 3\}$.

*Proof.* Let $\mathcal{C} = [\mathcal{F}, \Theta]\colon k \to n$ be a morphism in $\mathbf{M}_\partial$ and $(w_1, \dots, w_k; \sigma) \in \mathbb{T}_k$ be an element in the twisted $k$-torus. The morphism $\mathcal{C} \circ \jmath(w_1, \dots, w_k; \sigma)$ is represented by $(\mathcal{F}, \Theta')$ with $\Theta'^{\mathrm{out}} = \Theta^{\mathrm{out}}$, while

$$\Theta'^{\mathrm{in}}(\jmath, z) = \Theta^{\mathrm{in}}(\sigma(\jmath), w_\jmath^{-1} \cdot z).$$

If $\mathcal{C} \circ \jmath(w_1, \dots, w_k, \sigma) = \mathcal{C}$, then there is a biholomorphism $\Psi\colon \mathcal{S} \to \mathcal{S}$ such that $\Psi \circ \Theta = \Theta'$. The same applies to the unique extension to the standard half-annuli, i.e. $\Psi \circ \bar{\Theta} = \bar{\Theta}'$. In particular, $\Psi$ restricts to the identity of the image of $\bar{\Theta}^{\mathrm{out}}$. Since $\Psi$ is a holomorphic map, it must be the identity on a closed and open subset of $\mathcal{F}$; and since each connected component of $\mathcal{S}$ has non-empty outgoing boundary, and hence intersects the image of $\bar{\Theta}^{\mathrm{out}}$, we conclude that $\Psi$ must be the identity of $\mathcal{F}$.

It follows that the automorphism of $\underline{k} \times S^1$ given by $(\jmath, z) \mapsto (\sigma(\jmath), w_\jmath \cdot z)$ is the identity, and this implies that $(w_1, \dots, w_k, \sigma)$ is the identity of $\mathbb{T}_k$. ☐

Recall from Remark 6.1.4 that since $\mathfrak{M}_{g,n}$ is a classifying space for $\Gamma(\mathcal{S}_{g,n})$, we have, for $\pi := \pi_1(A)$, a decomposition

$$\mathfrak{M}_{g,n}[A] \simeq \coprod_{[\omega] \in \mathrm{Conj}(\pi \to \Gamma_{g,n})} BZ(\mathrm{im}(\omega), \Gamma(\mathcal{S}_{g,n})).$$

**Construction 6.2.18.** For each homomorphism $\omega\colon \pi \to \Gamma(\mathcal{S}_{g,n})$, we denote by $\mathfrak{M}_{g,n}[\omega]$ the corresponding connected component of $\mathfrak{M}_{g,n}[A]$.

Now recall that the colour-$n$ part of the connective surface operad $\mathcal{M}^c|_n$ acts on $\coprod_g \mathfrak{M}_{g,n}[A]$ by extending the action from Subsection 5.1.2 pointwise. By restriction, the twisted $n$-torus $\mathbb{T}_n$ acts on $\coprod_g \mathfrak{M}_{g,n}[A]$, and it clearly preserves

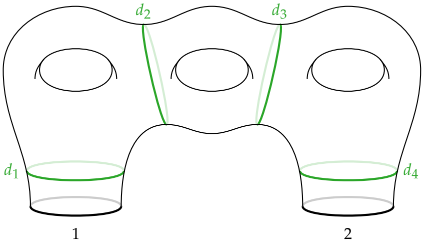

**Figure 6.4.** If we denote by $D_i$ the Dehn twist about the curve $d_i$ for each $i$, then the mapping classes $D_1D_2D_4$ and $D_1D_3D_4$ are both $\partial$-irreducible in $\Gamma(\mathcal{S}_{3,2})$ and not conjugate to each other, but they are conjugate in the extended mapping class group $\Gamma^{\mathfrak{S}_2}(\mathcal{S}_{3,2})$.

the summands $\mathfrak{M}_{g,n}[A]$. Even better, the component $\mathfrak{M}_{g,n}[\omega] \subseteq \mathfrak{M}_{g,n}[A]$ is invariant under the action of $\mathbb{T}^n \subseteq \mathbb{T}_n$, but not necessarily under the action of $\mathfrak{S}_n$. We denote by $\mathfrak{S}_n \cdot \mathfrak{M}_{g,n}[\omega] \subseteq \mathfrak{M}_{g,n}[A]$ the orbit of $\mathfrak{M}_{g,n}[\omega]$ under the action of $\mathfrak{S}_n$, or, equivalently, under the action of $\mathbb{T}_n$. Then we have

$$\mathfrak{S}_n \cdot \mathfrak{M}_{g,n}[\omega] = \bigcup_{\omega'} \mathfrak{M}_{g,n}[\omega'],$$

where $\omega'$ ranges over all conjugates of $\omega$ in the *extended* mapping class group $\Gamma^{\mathfrak{S}_n}(\mathcal{S}_{g,n})$. These conjugates still lie in the subgroup $\Gamma(\mathcal{S}_{g,n})$, which is normal.

Now we have everything at hand to define a $\mathbb{T}$-algebra $\mathfrak{C}[A]$ which enjoys the property that $F^{\mathscr{M}}_{\mathbb{T}}(\mathfrak{C}[A])_1 \simeq \mathfrak{M}_{\bullet,1}[A]$ holds.

**Construction 6.2.19.** For each $n \geqslant 1$ and $g \geqslant 0$, we define the space

$$\mathfrak{C}_{g,n}[A] := \coprod_{\substack{[\omega]\in\mathrm{Conj}(\pi\to\Gamma(\mathcal{S}_{g,n}))\\ \mathrm{im}(\omega)\ \partial\text{-irreducible}}} \mathfrak{M}_{g,n}[\omega].$$

Then the action of $\mathbb{T}_n$ on $\mathfrak{M}_{g,n}[A]$ restricts to an action on those components with constitute $\mathfrak{C}_{g,n}[A]$, simply because for every $\partial$-irreducible subgroup $\Pi \subseteq \Gamma(\mathcal{S}_{g,n})$, all conjugates by elements from the extended mapping class group are still $\partial$-irreducible.

We define $\mathfrak{C}_n[A] := \coprod_g \mathfrak{C}_{g,n}[A]$ and obtain a $\mathbb{T}$-algebra $\mathfrak{C}[A] := (\mathfrak{C}_n[A])_{n\geqslant 1}$. As the sequence $(\mathfrak{M}_{\bullet,n}[A])_{n\geqslant 0}$ is an $\mathscr{M}^c$-algebra, the inclusion of $\mathbb{T}$-algebras $\mathfrak{C}[A] \hookrightarrow (\mathfrak{M}_{\bullet,n}[A])_{n\geqslant 0}$ gives rise to a morphism $\kappa\colon F_{\mathbb{T}}^{\mathscr{M}^c}(\mathfrak{C}[A]) \to (\mathfrak{M}_{\bullet,n}[A])_{n\geqslant 0}$ of $\mathscr{M}^c$-algebras by considering the adjoint. Since $\mathscr{M}^c\binom{K}{1} = \mathscr{M}\binom{K}{1}$, the first level of $\kappa$ is of the form $F_{\mathbb{T}}^{\mathscr{M}}(\mathfrak{C}[A])_1 \to \mathfrak{M}_{\bullet,1}[A]$ and a morphism of $\mathscr{M}|_1$-algebras.

**Proposition 6.2.20.** *The first level* $\kappa_1\colon F_{\mathbb{T}}^{\mathscr{M}}(\mathfrak{C}[A])_1 \to \mathfrak{M}_{\bullet,1}[A]$ *is an equivalence of* $\mathscr{M}|_1$*-algebras.*

This is the content of [BKR21, Thm. 6.5], but we highlight the main steps of the proof to emphasise the connection to the result from Subsection 6.2.2.

*Proof.* Since $\kappa_1$ is already a map of $\mathscr{M}|_1$-algebras, it suffices to show that it is a weak equivalence on the level of spaces. To do so, let us first compare the path components on both sides. The coend formula for $F_{\mathbb{T}}^{\mathscr{M}}(\mathfrak{C}[A])_1$ is an instance of Example 3.3.26: we have

$$F_{\mathbb{T}}^{\mathscr{M}}(\mathfrak{C}[A])_1 \cong \int^{(k_1,\ldots,k_r)\in\overline{\mathbb{N}}\wr\boldsymbol{\Sigma}} \mathscr{M}\binom{k_1,\ldots,k_r}{1} \times_{\prod_i \mathbb{T}_{k_i}} (\mathfrak{C}_{k_1}[A] \times \cdots \times \mathfrak{C}_{k_r}[A])$$

Given a homomorphism $\omega\colon \pi \to \Gamma(\mathcal{S}_{g,1})$, we consider its image $\Pi := \operatorname{im}(\omega)$ and the decomposition of $\mathcal{S}_{g,1}$ into $W$ and $Y$. We denote by $\bar{\Pi}_i$ the $\partial$-irreducible subgroup in $\Gamma(\mathcal{S}_{g_i,k_i})$ induced by $\Pi$, and write $\mathfrak{H}_i \subseteq \mathfrak{S}_{k_i}$ for the image of $Z(\bar{\Pi}_i, \Gamma^{\mathfrak{S}_{k_i}}(\mathcal{S}_{g_i,k_i}))$ under the projection to $\mathfrak{S}_{k_i}$ as before. We have an induced group homomorphism $\bar{\omega}_i\colon \pi \to \Gamma(\mathcal{S}_{g_i,k_i})$, and $\mathfrak{M}_{g_i,k_i}[\bar{\omega}_i] \simeq BZ(\bar{\Pi}_i, \Gamma(\mathcal{S}_{g_i,k_i}))$.

If we denote by $\Sigma_W\colon k \to 1$ the surface type of $W$, then $\mathfrak{M}(\Sigma_W)$ can be regarded as a connected component of $\mathscr{M}\binom{s_1\times n_1,\ldots,s_r\times n_r}{1}$, using the notation of Construction 3.3.10. If we define

$$\begin{aligned} F[\omega] &:= \mathfrak{M}(\Sigma_W) \times_{\prod_i \mathbb{T}_{k_i}\wr\mathfrak{S}_{s_i}} \textstyle\prod_i (\mathfrak{S}_{k_i} \cdot \mathfrak{M}_{g_i,k_i}[\bar{\omega}_i])^{s_i} \\ &= \mathfrak{M}(\Sigma_W) \times_{\prod_i \mathbb{T}^{k_i\cdot s_i} \rtimes (\mathfrak{H}_i\wr\mathfrak{S}_{s_i})} \textstyle\prod_i \mathfrak{M}_{g_i,k_i}[\bar{\omega}_i]^{s_i}, \end{aligned}$$

then there is an obvious map $F[\omega] \to F_{\mathbb{T}}^{\mathscr{M}}(\mathfrak{C}[A])_1$, which is an isomorphism onto its image [BKR21, Lem. 6.6], which in turn is a connected component. Even more: if we regard $F[\omega]$ as a subspace of $F_{\mathbb{T}}^{\mathscr{M}}(\mathfrak{C}[A])_1$, then we obtain a decomposition $F_{\mathbb{T}}^{\mathscr{M}}(\mathfrak{C}[A])_1 = \coprod_g \coprod_{[\omega]} F[\omega]$, where $[\omega]$ ranges over all conjugacy classes of maps $\pi \to \Gamma(\mathcal{S}_{g,1})$ with $\partial$-irreducible image.

On the other hand, we already know that $\mathfrak{M}_{\bullet,1}[A]$ decomposes into connected components $\mathfrak{M}_{g,1}[\omega]$, where $[\omega]$ ranges over the same index set, and it is not hard to see that $\kappa_1$ decomposes into maps $\kappa[\omega]\colon F[\omega] \to \mathfrak{M}_{g,1}[\omega]$.

In [BKR21, Prop. 6.7], we finally show that these constituents $\kappa[\omega]$ are homotopy equivalences: here we use that the summands $\mathfrak{M}_{g,1}[\omega]$ are classifying spaces for the centraliser $Z(\mathrm{im}(\omega), \Gamma(\mathcal{S}_{g,1}))$, and invoke Corollary 6.2.15. By a careful inspection of the associated fibre sequences, we see that forming the fibre product of extended mapping class groups causes, for the respective classifying spaces, a balancing relation on the product of classifying spaces of *non*-extended mapping class groups, and similarly, quotienting out the diagonally embedded free abelian group $\mathbb{Z}^{k_i \cdot s_i}$ translates to a balancing of the induced toric actions on classifying spaces. □

This last result tells us that we have an equivalence among group completions $\Omega B\mathfrak{M}_{\bullet,1}[A] \simeq \Omega BF^{\mathscr{M}}_{\mathbb{T}}(\mathfrak{C}[A])_1$, and the upcoming section shows that the right side is easier to understand.

## 6.3. Algebras over coloured operads with homological stability

In this section, we would like to study the open problem from the last section more systematically: if $\mathfrak{O}$ is an $N$-coloured operad with homological stability (which is to be made precise soon), $\mathbf{I}$ is a topological category together with a map $\mathscr{B} \odot \mathbf{I} \to \mathfrak{O}$, and if $\boldsymbol{X} = (X_n)_{n\in N}$ a $(\mathscr{B} \odot \mathbf{I})$-algebra, what can we say about the levelwise homotopy type of the relatively free algebra $F^{\mathfrak{O}}_{\mathscr{B}\odot\mathbf{I}}(\boldsymbol{X})$? By answering this question, we extend [Bas+17, § 5], where the monochromatic and non-relative case was treated, i.e. $\mathbf{I} = N = *$, so $\mathscr{B} \odot \mathbf{I} = \mathscr{B}$.

Let us briefly summarise the strategy of [Bas+17, § 5]: in a first step, the authors introduce the notion of a (monochromatic) operad with homological stability. Such an operad $\mathfrak{O}$ comes in particular with a morphism of operads[2] $\iota\colon \mathscr{D}_1 \to \mathfrak{O}$, satisfying the weak homotopy commutativity condition, which demands that $\iota(\mathscr{D}_1(2)) \subseteq \mathfrak{O}(2)$ lies in a single path component; hence it makes sense to consider group completions of $\mathfrak{O}$-algebras.

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[2] In principle, any $A_\infty$-operad would suffice; we restrict to $\mathscr{D}_1$ for simplicity.

In a second step, the authors of [Bas+17] focus on operads with homological stability $\mathcal{O}$ which come with a map $\pi\colon \mathcal{O} \to \mathcal{D}_\infty$ of operads under $\mathcal{D}_1$. Thus, we have, for each based space $X$, two interesting maps of $\mathcal{O}$-algebras:

1. $F^{\mathcal{O}}_{\mathcal{B}}(X) \to F^{\mathcal{O}}_{\mathcal{B}}(*) = \mathcal{O}(0)$ induced by $X \to *$;
2. $F^{\mathcal{O}}_{\mathcal{B}}(X) \to \pi^* F^{\mathcal{D}_\infty}_{\mathcal{B}}(X)$, the unit of the base-change adjunction.

Intuitively, the first map forgets the space $X$, while the second map forgets the operad $\mathcal{O}$. In [Bas+17, Thm. 5.4], it is shown that the product map induces a weak equivalence $\Omega B F^{\mathcal{O}}_{\mathcal{B}}(X) \to \Omega B\mathcal{O}(0) \times \Omega^\infty\Sigma^\infty X$ on group completions.

Finally, an operad with homological stability $\mathcal{O}$ admits a replacement by another operad with homological stability $\mathcal{O}' := \mathcal{O} \times \mathcal{D}_\infty$ which additionally has a comparison map $\pi\colon \mathcal{O}' \to \mathcal{D}_\infty$, and, under mild extra point-set assumptions (e.g. $\mathcal{O}$ is $\mathfrak{S}$-free and $X$ is well-based), the free algebras $F^{\mathcal{O}}_{\mathcal{B}}(X)$ and $F^{\mathcal{O}'}_{\mathcal{B}}(X)$ are equivalent as $A_\infty$-algebras, and hence have equivalent group completions.

### 6.3.1. *Coloured operads with homological stability*

In this subsection, we introduce the notion of an $N$-coloured operad with homological stability for each colour set $N$. Here we heavily use the operadic framework established in Chapter 3.

Recall from Example 3.1.4 the monochromatic operad $\mathcal{B}$ of based spaces, and recall from Example 3.3.25 that for each colour set $N$, the Boardman–Vogt tensor product $\mathcal{B} \odot N$ models $N$-indexed families of based spaces.

**Definition 6.3.1.** A *based N-coloured operad* is an operad map $\mathcal{B} \odot N \to \mathcal{O}$. This is the same as an $N$-coloured operad $\mathcal{O}$ with a choice of nullary operation $\mathfrak{v}_n \in \mathcal{O}\binom{}{n}$ for each colour $n \in N$. A morphism of based operads is an operad morphism $\rho\colon \mathcal{O} \to \mathcal{P}$ with $\rho(\mathfrak{v}^{\mathcal{O}}_n) = \rho(\mathfrak{v}^{\mathcal{P}}_n)$.

The operations $\mathfrak{v}_n$ of a based operad can be used to block inputs by precomposition with them: more precisely, for each input profile $K = (k_1, \dots, k_r)$ and $1 \leqslant i \leqslant r$, we have a face map $d_i\colon \mathcal{O}\binom{K}{n} \to \mathcal{O}\binom{d_iK}{n}$ by $d_i\mu := \mu \circ_i \mathfrak{v}_{k_i}$. In this way, the functors $\mathcal{O}\binom{-}{n}\colon (N \wr \mathbf{\Sigma})^{\mathrm{op}} \to \mathbf{Top}$, which are part of the operadic structure, can be extended to $\mathcal{O}\binom{-}{n}\colon (N \wr \mathbf{Inj})^{\mathrm{op}} \to \mathbf{Top}$ as in Example 3.3.25.

Moreover we have the *capping morphism* which places the preferred nullary at *each* input, and hence produces an element in the initial algebra,

$$\beta\colon \mathcal{O}\tbinom{k_1,\dots,k_r}{n} \to \mathcal{O}\tbinom{}{n}, \quad \mu \mapsto \mu(\mathfrak{v}_{k_1},\dots,\mathfrak{v}_{k_r}).$$

**Example 6.3.2.**

1. The little $d$-discs operad $\mathcal{D}_d$ has precisely one nullary operation and hence is canonically based. The same applies to $\mathcal{D}_d \odot N$ for each colour set $N$.

2. For each small and topologically enriched category $\mathbf{I}$, the tensor product $\mathcal{B} \odot \mathbf{I}$ differs from $\mathbf{I}$ by the single nullary operation $\mathfrak{v}_n \in (\mathcal{B} \odot \mathbf{I})\binom{}{n}$ for each colour $n$, and $(\mathcal{B} \odot \mathbf{I})$-algebras are the same as topologically enriched functors $\mathbf{I} \to \mathbf{Top}_*$ to the category of based spaces.

**Definition 6.3.3.** An *operad under* $\mathcal{D}_1$ is an $N$-coloured operad $\mathcal{O}$, together with an operad morphism $\imath\colon \mathcal{D}_1 \odot N \to \mathcal{O}$ satisfying the weak homotopy commutativity condition levelwise, meaning that $\imath((\mathcal{D}_1 \odot N)\binom{n,n}{n}) \subseteq \mathcal{O}\binom{n,n}{n}$ is contained in a single path component.

If $\mathcal{O}$ is an operad under $\mathcal{D}_1$, then $\mathcal{O}$ is canonically based by $\mathfrak{v}_n := \imath(\mathfrak{v} \odot n)$, where $\mathfrak{v} \in \mathcal{D}_1(0)$ is the unique nullary. Secondly, each $\mathcal{O}$-algebra $(X_n)_{n\in N}$ is levelwise an H-commutative $\mathcal{D}_1$-algebra, so for each $n \in N$, the set $\pi_0(X_n)$ is an abelian monoid, whose (unique) binary operation we denote by '$\curlyvee$'. There is a bar construction of $X_n$ and by the group completion theorem [MS76], we have an isomorphism $H_\bullet(\Omega B X_n) \cong H_\bullet(X_n)[\pi_0(X_n)^{-1}]$.

A morphism $\rho\colon \mathcal{O} \to \mathcal{O}'$ of operads $\imath\colon \mathcal{D}_1 \odot N \to \mathcal{O}$ and $\imath'\colon \mathcal{D}_1 \odot N \to \mathcal{O}'$ under $\mathcal{D}_1$ is a morphism of operads such that $\rho \circ \imath = \imath'$ holds. In that case, for each $\mathcal{O}'$-algebra $\boldsymbol{X}$, the levelwise group completions of the $\mathcal{O}'$-algebra $\boldsymbol{X}$ and of the $\mathcal{O}$-algebra $\rho^*\boldsymbol{X}$ coincide, as they only depend on the $(\mathcal{D}_1 \odot N)$-structure.

**Notation 6.3.4.** Let $\imath\colon \mathcal{D}_1 \odot N \to \mathcal{O}$ be an operad under $\mathcal{D}_1$. Then we fix a binary operation $\mathfrak{p} \in \mathcal{D}_1(2)$ and abbreviate $\mu \curlyvee \mu' := \imath(\mathfrak{p} \odot n) \circ (\mu, \mu') \in \mathcal{O}\binom{K \cdot K'}{n}$ for operations $\mu \in \mathcal{O}\binom{K}{n}$ and $\mu' \in \mathcal{O}\binom{K'}{n}$. If $(X_n)_{n\in N}$ is an $\mathcal{O}$-algebra and if $x, x' \in X_n$, then we also write $x \curlyvee x' := \imath(\mathfrak{p} \odot n)(x, x')$.

The notation '$\curlyvee$' is pictorially inspired by the following example.

**Example 6.3.5.** The map $\mathcal{D}_2 \hookrightarrow \mathcal{M}|_1$ from Construction 5.1.18 gives rise to $\imath_{\mathcal{M}}\colon \mathcal{D}_1 \odot \overline{\mathbb{N}} \to \mathcal{D}_2 \odot \overline{\mathbb{N}} \to \overline{\mathbb{N}}(\mathcal{D}_2) \to \mathcal{M}$, turning $\mathcal{M}$ into an operad under $\mathcal{D}_1$.

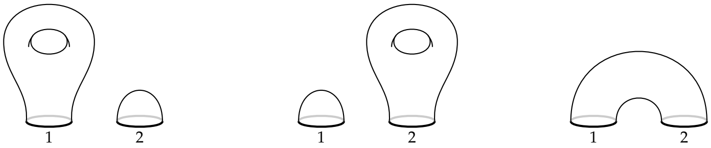

**Figure 6.5.** The three generators $e_2^1$, $e_2^2$ and $e_2^{1,2}$ of $\pi_0(\mathscr{M}\binom{}{2})$.

The capping maps $\beta\colon \mathscr{M}\binom{K}{n} \to \mathscr{M}\binom{}{n}$ are induced by—literally—capping each incoming boundary curve with a disc. The abelian monoid $\pi_0(\mathscr{M}\binom{}{n})$ contains all surface types $\Sigma\colon 0 \to n$, and the addition is defined by gluing $n$ pairs of pants; the neutral element is given by an ordered collection of $n$ discs. Thus, $\pi_0(\mathscr{M}\binom{}{n})$ is finitely generated as an abelian monoid: for instance, it is generated by the following elements $e_n^\ell$ and $e_n^{\ell,\ell'}$, see Figure 6.5:

1. For $1 \leqslant \ell \leqslant n$, we let $e_n^\ell$ be the isomorphism type of surfaces with $n$ path components, such that the component carrying the $\ell^{\text{th}}$ boundary curve has genus 1, whereas all others components are discs.
2. For each $1 \leqslant \ell < \ell' \leqslant n$, we let $e_n^{\ell,\ell'}$ be the isomorphism type of surfaces with $n-1$ path components, all of genus 0, such that one component is a cylinder carrying the $\ell^{\text{th}}$ and the $\ell'^{\text{th}}$ boundary curve.

**Construction 6.3.6** (Stable operation space)**.** We call an operad $\mathfrak{O}$ under $\mathscr{D}_1$ *admissibly graded* if each of the abelian monoids $\pi_0(\mathfrak{O}\binom{}{n})$ is finitely generated and, for each input profile $K$, the *degree map*

$$|\cdot|\colon \mathfrak{O}\binom{K}{n} \xrightarrow{\beta} \mathfrak{O}\binom{}{n} \to \pi_0(\mathfrak{O}\binom{}{n}), \quad \mu \mapsto |\mu| = [\beta(\mu)]$$

that assigns to each operation $\mu$ the path component of its capped version $\beta(\mu)$ is surjective.[3] In this case we write, for each $\delta \in \pi_0(\mathfrak{O}\binom{}{n})$,

$$\mathfrak{O}\binom{K}{n}^\delta := \Big\{\mu \in \mathfrak{O}\binom{K}{n};\, |\mu| = \delta\Big\} \neq \varnothing.$$

---

[3] In the monochromatic setting, the degree map is automatically surjective: by the weak homotopy commutativity condition, we obtain an operad map $\mathscr{C}om \hookrightarrow \pi_0(\mathfrak{O})$ and if we write $\mathscr{C}om(r) = \{\mathfrak{p}_r\}$, then $\beta(\mathfrak{p}_{r+1} \circ_1 \delta) = \mathfrak{p}_1 \circ \delta = \mathbb{1} \circ \delta = \delta$ for each $\delta \in \pi_0(\mathfrak{O}(0))$. For the coloured case, though, it seems necessary to additionally assume this property.

Note that $\mathcal{O}\binom{K}{n}^{\delta}$ is a union of connected components of $\mathcal{O}\binom{K}{n}$. Now we choose, for each $n \in N$, a finite generating set $E_n \subseteq \pi_0(\mathcal{O}\binom{}{n})$ and let $e_n \in \pi_0(\mathcal{O}\binom{}{n})$ be the sum of all elements from $E_n$. If we fix a nullary operation $\tilde{e}_n \in \mathcal{O}\binom{}{n}$ with $|\tilde{e}_n| = e_n$, called *propagator*, then we obtain, for each component $\delta \in \pi_0(\mathcal{O}\binom{}{n})$ and each input profile $K$, a *stabilisation map*

$$\mathrm{stab}\colon \mathcal{O}\binom{K}{n}^{\delta} \to \mathcal{O}\binom{K}{n}^{\delta \curlyvee e_n}, \quad \mu \mapsto \mu \curlyvee \tilde{e}_n.$$

From this, we can form, for each $K$ and $n$, the *stable operation space*

$$\mathcal{O}\binom{K}{n}^{\infty} := \mathrm{hocolim}\Big( \mathcal{O}\binom{K}{n}^{0} \xrightarrow{\mathrm{stab}} \mathcal{O}\binom{K}{n}^{e_n} \xrightarrow{\mathrm{stab}} \mathcal{O}\binom{K}{n}^{2e_n} \xrightarrow{\mathrm{stab}} \cdots \Big).$$

**Definition 6.3.7.** Let $\mathcal{O}$ be an operad under $\mathcal{D}_1$ which is admissibly graded. By the associativity of the operadic composition, capping inputs and stabilisation commute, i.e. for each $\delta$ the square

$$\begin{tikzcd}
\mathcal{O}\binom{K}{n}^{\delta} \arrow[r, "\mathrm{stab}"] \arrow[d, "\beta"'] & \mathcal{O}\binom{K}{n}^{\delta \curlyvee e_n} \arrow[d, "\beta"] \\
\mathcal{O}\binom{}{n}^{\delta} \arrow[r, "\mathrm{stab}"'] & \mathcal{O}\binom{}{n}^{\delta \curlyvee e_n}.
\end{tikzcd}$$

commutes. We therefore obtain a stable capping map $\beta_n^K\colon \mathcal{O}\binom{K}{n}^{\infty} \to \mathcal{O}\binom{}{n}^{\infty}$ for each input profile $K$, which depends, up to homotopy, only on the component from which the propagator is chosen, i.e. on the generating set $E_n$.

We call $\mathcal{O}$ an *operad with homological stability* if there is a choice of generating sets such that all stable capping maps $\beta_n^K$ induce isomorphisms in integral homology.

**Example 6.3.8.** The coloured surface operad $\mathcal{M}$ is admissibly graded, and we may use the generating sets from Example 6.3.5.

It is even an operad with homological stability: here we use that multiplying with a propagator automatically yields a connected cobordism and increases the genus by at least one, see Figure 6.6, so the stable capping map is given, on each component, by the capping map $\mathfrak{M}_{\infty,n+k_1+\dots+k_r} \to \mathfrak{M}_{\infty,n}$ between stable moduli spaces of Riemann surfaces: this map is a homology equivalence by Harer's stability theorem [Har84].

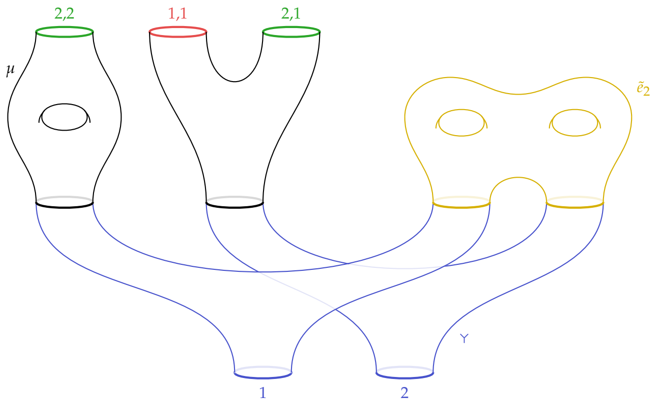

**Figure 6.6.** A single stabilisation step on $\mathcal{M}\binom{1,2}{2}$. Note that $|\tilde{e}_2| = e_2^1 \curlyvee e_2^2 \curlyvee e_2^{1,2}$ is the isomorphism class of surfaces of type $\Sigma_{2,2}$.

### 6.3.2. *Derived base-change and a splitting result*

Recall that we want to establish an analogue of [Bas+17, Thm. 5.4] for the coloured and relative case, i.e. we want to consider relatively free algebras, relative to a map $\mathcal{P} \to \mathcal{O}$ of based operads, where we will soon restrict to the case $\mathcal{P} = \mathcal{B} \odot \mathbf{I}$ for an enriched category $\mathbf{I}$.

The most convenient setting for such a discussion does not use the strict functor $F^{\mathcal{O}}_{\mathcal{P}}$, but a homotopically better behaved one, which we denote by $\tilde{F}^{\mathcal{O}}_{\mathcal{P}}$. This simplifies many point-set issues, and, with regard to our original problem, it will turn out to be equivalent to the space we want to understand.

The functor $\tilde{F}^{\mathcal{O}}_{\mathcal{P}}$ can be constructed by considering the model structure on the categories of $\mathcal{O}$- and $\mathcal{P}$-algebras as in [BM07], but we decided to give an explicit description in the fashion of [Bas+17]. Here we assume that the reader is familiar with monads and their two-sided bar constructions, as it was discussed in [May72].

**Construction 6.3.9.** Let $\mathcal{P} \to \mathcal{O}$ be a map of based $N$-coloured operads. Then we obtain monads $\mathbb{O} := U^{\mathcal{O}}_{\mathcal{B}\odot N} F^{\mathcal{O}}_{\mathcal{B}\odot N}$ and $\mathbb{P} := U^{\mathcal{P}}_{\mathcal{B}\odot N} F^{\mathcal{P}}_{\mathcal{B}\odot N}$ on $\mathbf{Top}^N_*$, and $\mathbb{O}$ is a left $\mathbb{P}$-functor by the transformation $\mathbb{O}\mathbb{P} \Rightarrow \mathbb{O}^2 \Rightarrow \mathbb{O}$. For each $\mathcal{P}$-algebra $\boldsymbol{X}$, we consider the two-sided bar construction $B_\bullet(\mathbb{O}, \mathbb{P}, \boldsymbol{X})$, with $p$-simplices $B_p(\mathbb{O}, \mathbb{P}, \boldsymbol{X}) := \mathbb{O}\mathbb{P}^p U^{\mathcal{P}}_{\mathcal{B}\odot N}\boldsymbol{X}$, which is an $N$-coloured simplicial space, and define the *derived free algebra* $\tilde{F}^{\mathcal{O}}_{\mathcal{P}}(\boldsymbol{X}) := |B_\bullet(\mathbb{O},\mathbb{P},\boldsymbol{X})|$ to be its levelwise[4] geometric realisation. Then $\tilde{F}^{\mathcal{O}}_{\mathcal{P}}(\boldsymbol{X})$ is itself an $\mathcal{O}$-algebra with multiplication

$$\mathbb{O}|B_\bullet(\mathbb{O},\mathbb{P},\boldsymbol{X})| \cong |B_\bullet(\mathbb{O}^2,\mathbb{P},\boldsymbol{X})| \to |B_\bullet(\mathbb{O},\mathbb{P},\boldsymbol{X})|,$$

where the first identification is due to [May72, Lem. 9.7] and the last map is given by $|B_\bullet(\kappa,\mathbb{P},\boldsymbol{X})|$ for the operadic composition $\kappa\colon \mathbb{O}^2 \Rightarrow \mathbb{O}$. Secondly, we have a map $\tilde{F}^{\mathcal{O}}_{\mathcal{P}}(\boldsymbol{X}) \to F^{\mathcal{O}}_{\mathcal{P}}(\boldsymbol{X})$ of $\mathcal{O}$-algebras, by noticing that $F^{\mathcal{O}}_{\mathcal{P}}(\boldsymbol{X})$ is the reflexive coequaliser[5] of $B_1(\mathbb{O},\mathbb{P},\boldsymbol{X}) \rightrightarrows B_0(\mathbb{O},\mathbb{P},\boldsymbol{X})$, see Construction 3.1.15.

Before stating our main theorem, let us fix, once and for all, the point-set requirements we want to assume:

**Setting 6.3.10.** Throughout this section, we consider the following:

1. Let $\mathcal{O}$ be an $N$-coloured $\mathfrak{S}$-free operad (Remark 3.3.11) with homological stability, such that the inclusions $\{\mathbb{1}_n\} \hookrightarrow \mathcal{O}\binom{n}{n}$ are cofibrations.

2. Let $\mathbf{I}$ be a topologically enriched category with object set $N$, such that the inclusions $\{\mathbb{1}_n\} \hookrightarrow \mathbf{I}\binom{n}{n}$ are cofibrations. We assume that there is a map $\mathcal{B} \odot \mathbf{I} \to \mathcal{O}$ of based $N$-coloured operads.

3. Let $\boldsymbol{X} = (X_n)_{n\in N}$ be an $(\mathcal{B}\odot\mathbf{I})$-algebra, or, in other words, an enriched functor $X_\bullet\colon \mathbf{I} \to \mathbf{Top}_*$, and we assume that each $X_n$ is well-based.

Moreover, we assume that all involved spaces are Hausdorff.

Of course, the example we have in mind is $\mathcal{O}$ being the surface operad $\mathcal{M}$, $\mathbf{I}$ being the family $\mathbb{T}$ of twisted tori, and $\boldsymbol{X}$ being the family $\mathfrak{C}[A]_+$ with a levelwise extra basepoint. We want to show the following statement:

**Theorem 6.3.11** (Splitting theorem)**.** *In the above Setting 6.3.10, we have, for each $n \in N$, a weak equivalence of loop spaces*

$$\Omega B\tilde{F}^{\mathcal{O}}_{\mathcal{B}\odot\mathbf{I}}(\boldsymbol{X})_n \simeq \Omega B\mathcal{O}\tbinom{}{n} \times \Omega^\infty\Sigma^\infty \mathrm{hocolim}_{\mathbf{I}}(X_\bullet).$$

---

[4] For us, *levelwise* always refers to the $N$-grading, *not* to the spaces of simplices.

[5] To be pendantic, we saw in Construction 3.1.15 only that $\boldsymbol{X}$ is the reflexive coequaliser of $F^{\mathcal{P}}U^{\mathcal{P}}F^{\mathcal{P}}U^{\mathcal{P}}\boldsymbol{X} \rightrightarrows F^{\mathcal{P}}U^{\mathcal{P}}\boldsymbol{X}$, but the same holds relative to $\mathcal{B}\odot N$, and in fact for each monadic adjunction [GKR18, § 3.2]; here we consider the monadic adjunction $F^{\mathcal{P}}_{\mathcal{B}\odot N} \vdash U^{\mathcal{P}}_{\mathcal{B}\odot N}$.

The proof of Theorem 6.3.11 will occupy the rest of this section. Let us start by establishing a map which compares the two sides.

To do so, we start by constructing an $N$-coloured version of the $E_\infty$-operad $\mathscr{D}_\infty$, and show that we can, without loss of generality, assume that there is a comparison map from $\mathcal{O}$ to it.

**Construction 6.3.12.** For each colour set $N$, recall from Example 3.4.3 the *chaotic category* $EN$ with object set $N$ and morphism spaces $(EN)\binom{k}{n} = *$ for all $k, n \in N$. Then we consider the $N$-coloured operad $\mathscr{D}_\infty \odot EN$.

**Lemma 6.3.13.** *For the proof of Theorem 6.3.11, we can without loss of generality assume a map* $\pi\colon \mathcal{O} \to \mathscr{D}_\infty \odot EN$ *such that the diagram*

$$\begin{tikzcd}
\mathscr{B} \odot N \arrow[r] \arrow[d] & \mathscr{B} \odot \mathbf{I} \arrow[d] \arrow[rdd, bend left] & \\
\mathscr{D}_1 \odot N \arrow[r] \arrow[rrd, bend right] & \mathcal{O} \arrow[rd, "\pi"] & \\
& & \mathscr{D}_\infty \odot EN
\end{tikzcd}$$

*commutes, where all arrows apart from* $\pi$ *are either given or induced by the canonical maps* $\mathscr{B} \to \mathscr{D}_1 \to \mathscr{D}_\infty$ *and* $N \to \mathbf{I} \to EN$.

*Proof.* The commutativity of the square is part of the general setting: recall that we assumed that $\mathscr{B} \odot \mathbf{I} \to \mathcal{O}$ is a map of *based* operads, and $\mathcal{O}$ is canonically based as an operad under $\mathscr{D}_1$.

To establish the map $\pi$, we replace $\mathcal{O}$ by a slightly larger operad: if we consider the product operad $\mathcal{O}' := \mathcal{O} \times (\mathscr{D}_\infty \odot EN)$, together with:

- the diagonal inclusion $\mathscr{D}_1 \odot N \to \mathcal{O}'$,
- the diagonal inclusion $\mathscr{B} \odot \mathbf{I} \to \mathcal{O}'$,
- the second projection $\pi\colon \mathcal{O}' \to \mathscr{D}_\infty \odot EN$,

then the above diagram clearly commutes with $\mathcal{O}$ replaced by $\mathcal{O}'$. Moreover, note that *each* operation space of $\mathscr{D}_\infty \odot EN$ is contractible: hence $\mathcal{O}'$ is again admissibly graded with $\pi_0(\mathcal{O}'\binom{}{n}) = \pi_0(\mathcal{O}\binom{}{n})$ and $\mathcal{O}'$ is again an operad with homological stability, satisfying $\mathcal{O}'\binom{}{n} = \mathcal{O}\binom{}{n}$.

Finally, the first projection $\mathcal{O}' \to \mathcal{O}$ induces a map of monads $\mathbb{O}' \Rightarrow \mathbb{O}$ and hence a map $\tilde{F}^{\mathcal{O}'}_{\mathscr{B}\odot\mathbf{I}}(\boldsymbol{X}) \to U^{\mathcal{O}}_{\mathcal{O}'}\tilde{F}^{\mathcal{O}}_{\mathscr{B}\odot\mathbf{I}}(\boldsymbol{X})$ of $\mathcal{O}'$-algebras, which is in particular a morphism of $A_\infty$-algebras. If we denote the monad for $\mathscr{B}\odot\mathbf{I}$ by $\mathbb{I}$, then, for each $p \geqslant 0$, the map $B_p(\mathbb{O}', \mathbb{I}, \boldsymbol{X}) \to B_p(\mathbb{O}, \mathbb{I}, \boldsymbol{X})$ is an equivalence, since $\mathcal{O}$ is $\mathfrak{S}$-free, each $X_n$ is well-based, and every space is Hausdorff. Secondly, both simplicial spaces are proper in the sense of [May72, § 11], using that the inclusions of the identities are cofibrations. By [May74, Thm. A.4], the induced map $\tilde{F}^{\mathcal{O}'}_{\mathscr{B}\odot\mathbf{I}}(\boldsymbol{X}) \to \tilde{F}^{\mathcal{O}}_{\mathscr{B}\odot\mathbf{I}}(\boldsymbol{X})$ among geometric realisations is an equivalence as well, so their levelwise group completions are equivalent as loop spaces. ☐

Using the lemma, we obtain, as in the monochromatic case, two maps:

1. the map $\boldsymbol{X} \to *$ to $* = (*)_{n\in N}$ induces a map $\tilde{F}^{\mathcal{O}}_{\mathscr{B}\odot\mathbf{I}}(\boldsymbol{X}) \to \tilde{F}^{\mathcal{O}}_{\mathscr{B}\odot\mathbf{I}}(*)$ of $\mathcal{O}$-algebras, which is in particular a map of $A_\infty$-algebras.
2. the morphism $\mathcal{O} \to \mathscr{D}_\infty \odot EN$ induces a map $\tilde{F}^{\mathcal{O}}_{\mathscr{B}\odot\mathbf{I}}(\boldsymbol{X}) \to \tilde{F}^{\mathscr{D}_\infty\odot EN}_{\mathscr{B}\odot\mathbf{I}}(\boldsymbol{X})$ of $A_\infty$-algebras.

The two targets can be identified with the following spaces:

**Lemma 6.3.14.** *For each $n \in N$, we have equivalences of $A_\infty$-algebras*

$$\begin{aligned} \tilde{F}^{\mathcal{O}}_{\mathscr{B}\odot\mathbf{I}}(*)_n &\simeq \mathcal{O}({}_n), \\ \tilde{F}^{\mathscr{D}_\infty\odot EN}_{\mathscr{B}\odot\mathbf{I}}(\boldsymbol{X})_n &\simeq F^{\mathscr{D}_\infty}_{\mathscr{B}}(\mathrm{hocolim}_{\mathbf{I}}(X_\bullet)). \end{aligned}$$

*Proof.* For the first equivalence, we note that, since $(\mathscr{B}\odot\mathbf{I})({}_n) = *$, we have $F^{\mathscr{B}\odot\mathbf{I}}_{\mathscr{B}\odot N}(*)_n = *$ for each $n \in N$. Now consider the natural map

$$\tilde{F}^{\mathcal{O}}_{\mathscr{B}\odot\mathbf{I}}(*) = \tilde{F}^{\mathcal{O}}_{\mathscr{B}\odot\mathbf{I}}(F^{\mathscr{B}\odot\mathbf{I}}_{\mathscr{B}\odot N}(*)) \to F^{\mathcal{O}}_{\mathscr{B}\odot\mathbf{I}}(F^{\mathscr{B}\odot\mathbf{I}}_{\mathscr{B}\odot N}(*)) = F^{\mathcal{O}}_{\mathscr{B}\odot N}(*) = \big(\mathcal{O}({}_n)\big)_{n\in N}.$$

This map arises from the augmentation $B_\bullet := B_\bullet(\mathbb{O}, \mathbb{I}, \mathbb{I}(*)) \to B_{-1} := \mathbb{O}(*)$, which has an extra degeneracy $s_{-1} \colon B_p \to B_{p+1}$ induced by the unit of $\mathbb{I}$, and hence is an equivalence by [Rie14, Cor. 4.5.2].

For the second equivalence, we start with the general observation that, for a sequence $\mathcal{Q} \to \mathcal{P} \to \mathcal{O}$ of $N$-coloured operads, we have a levelwise equivalence of $\mathcal{O}$-algebras among the derived algebras $\tilde{F}^{\mathcal{O}}_{\mathcal{Q}}(\boldsymbol{X}) \simeq \tilde{F}^{\mathcal{O}}_{\mathcal{P}}(\tilde{F}^{\mathcal{P}}_{\mathcal{Q}}(\boldsymbol{X}))$: by construction, the left side is the geometric realisation of the bisimplicial space with $B_{p,q} := \mathbb{O}\mathbb{P}^{p+1}\mathbb{Q}^q\boldsymbol{X}$. If we first realise each $B_{p,\bullet}$, then we obtain a simplicial space $\tilde{B}_\bullet$ with $\tilde{B}_p := |B_\bullet(\mathbb{O}\mathbb{P}^{p+1}, \mathbb{Q}, \boldsymbol{X})|$. Again, we have an augmentation map $\tilde{B}_\bullet \to \tilde{B}_{-1} := |B_\bullet(\mathbb{O}, \mathbb{Q}, \boldsymbol{X})|$, that admits an extra degeneracy by the unit of $\mathbb{P}$, whence the induced map $|B_{\bullet,\bullet}| \simeq |\tilde{B}_\bullet| \to \tilde{B}_{-1}$ is an equivalence, as desired. In our case, we get for each $n \in N$ an equivalence of $\mathcal{D}_\infty$-algebras

$$\begin{aligned} \tilde{F}^{\mathcal{D}_\infty \odot EN}_{\mathcal{B} \odot \mathbf{I}}(\boldsymbol{X})_n &\simeq \tilde{F}^{\mathcal{D}_\infty \odot EN}_{\mathcal{B} \odot EN}(\tilde{F}^{\mathcal{B} \odot EN}_{\mathcal{B} \odot \mathbf{I}}(\boldsymbol{X}))_n \\ &\simeq F^{\mathcal{D}_\infty}_{\mathcal{B}}(\operatorname{hocolim}_{\mathbf{I}}(X_\bullet)), \end{aligned}$$

where for the last equivalence, we use that $\tilde{F}^{\mathcal{B} \odot EN}_{\mathcal{B} \odot \mathbf{I}}(\boldsymbol{X})$ is, when regarded as a functor $EN \to \mathbf{Top}_*$, the constant diagram with value $\operatorname{hocolim}_{\mathbf{I}}(X_\bullet)$, and applying $\tilde{F}^{\mathcal{D}_\infty \odot EN}_{\mathcal{B} \odot EN}$ is the same as postcomposing with $F^{\mathcal{D}_\infty}_{\mathcal{B}}$ objectwise, using that the natural map $\tilde{F}^{\mathcal{D}_\infty}_{\mathcal{B}}(X) \to F^{\mathcal{D}_\infty}_{\mathcal{B}}(X)$ is an equivalence for each based space $X$, as the underlying simplicial space is constant. $\square$

Putting everything together, we get, for each $n \in N$, a map of $A_\infty$-algebras

$$\tilde{F}^{\mathcal{O}}_{\mathcal{B} \odot \mathbf{I}}(\boldsymbol{X})_n \to \mathcal{O}\tbinom{}{n} \times F^{\mathcal{D}_\infty}_{\mathcal{B}}(\operatorname{hocolim}_{\mathbf{I}}(X_\bullet)),$$

which, after group completion, gives us the map from Theorem 6.3.11. In the next subsection, we will show that it is an equivalence.

### 6.3.3. *Proof of the splitting theorem*

For the proof of Theorem 6.3.11, let us first consider the strict and absolute case: we denote the monads associated with $\mathcal{O}$ and $\mathcal{D}_\infty \odot EN$ by $\mathbb{O}$ and $\mathbb{D}$, respectively, and we also write $\mathbb{O}(\boldsymbol{X})_n$ and $\mathbb{D}(\boldsymbol{X})_n$ for the $n^{\text{th}}$ levels.

Since permuting and capping inputs preserve the degree of the operations, we obtain, for each colour $n \in N$ and each degree $\delta \in \pi_0(\mathcal{O}\tbinom{}{n})$, a functor $\mathcal{O}\tbinom{-}{n}^\delta \colon (N \wr \mathbf{Inj})^{\mathrm{op}} \to \mathbf{Top}$. This gives rise to a decomposition

$$\mathbb{O}(\boldsymbol{X})_n = \coprod_\delta \mathbb{O}(\boldsymbol{X})_n^\delta \quad \text{with} \quad \mathbb{O}(\boldsymbol{X})_n^\delta := \int^{K \in N \wr \mathbf{Inj}} \mathcal{O}\tbinom{K}{n}^\delta \times \boldsymbol{X}^K.$$

We denote by $\tilde{x}_n := [\tilde{e}_n; ()] \in \mathbb{O}(\boldsymbol{X})_n^{e_n}$ the image of the propagator and define, in analogy with Construction 6.3.6,

$$\mathbb{O}(\boldsymbol{X})_n^\infty := \operatorname{hocolim}\Big(\mathbb{O}(\boldsymbol{X})_n^0 \xrightarrow{- \curlyvee \tilde{x}_n} \mathbb{O}(\boldsymbol{X})_n^{e_n} \xrightarrow{- \curlyvee \tilde{x}_n} \mathbb{O}(\boldsymbol{X})_n^{2e_n} \xrightarrow{- \curlyvee \tilde{x}_n} \cdots\Big).$$

Again, we have two relevant maps:

1. The map $f\colon \mathbb{O}(\boldsymbol{X}) \to \mathbb{O}(*)$ decomposes into maps $f_n^\delta\colon \mathbb{O}(\boldsymbol{X})_n^\delta \to \mathbb{O}(*)_n^\delta$ which are compatible with stabilisations. Therefore we obtain a map between the mapping telescopes $f_n^\infty\colon \mathbb{O}(\boldsymbol{X})_n^\infty \to \mathbb{O}(*)_n^\infty$.

2. Similarly, the map of $\mathcal{O}$-algebras $\eta\colon \mathbb{O}(\boldsymbol{X}) \to \mathbb{D}(\boldsymbol{X})$ restricts to maps of spaces $\eta_n^\delta\colon \mathbb{O}(\boldsymbol{X})_n^\delta \to \mathbb{D}(\boldsymbol{X})_n$, and the triangle

$$\begin{tikzcd}
\mathbb{O}(\boldsymbol{X})_n^\delta \arrow[rr, "- \curlyvee \tilde{x}_n"] \arrow[dr, "\eta_n^\delta"'] & & \mathbb{O}(\boldsymbol{X})_n^{\delta \curlyvee e_n} \arrow[dl, "\eta_n^{\delta \curlyvee e_n}"] \\
& \mathbb{D}(\boldsymbol{X})_n &
\end{tikzcd}$$

is H-commutative, whence we obtain a map from the mapping telescope $\eta_n^\infty\colon \mathbb{O}(\boldsymbol{X})_n^\infty \to \mathbb{D}(\boldsymbol{X})_n$.

**Lemma 6.3.15.** *For each colour $n \in N$, the product map*

$$(f_n^\infty, \eta_n^\infty)\colon \mathbb{O}(\boldsymbol{X})_n^\infty \to \mathbb{O}(*)_n^\infty \times \mathbb{D}(\boldsymbol{X})_n$$

*is a homology equivalence.*

Let us first prove Theorem 6.3.11 using Lemma 6.3.15:

*Proof of Theorem 6.3.11.* Let us abbreviate $\tilde{\mathbb{O}} := \tilde{F}^{\mathcal{O}}_{\mathscr{B}\odot\mathbf{I}}$ and $\tilde{\mathbb{D}} := \tilde{F}^{\mathcal{D}_\infty \odot EN}_{\mathscr{B}\odot\mathbf{I}}$. Then we have to show that the map $(f, \eta)\colon \tilde{\mathbb{O}}(\boldsymbol{X}) \to \tilde{\mathbb{O}}(*) \times \tilde{\mathbb{D}}(\boldsymbol{X})$ from the previous section is levelwise an equivalence.

To this aim, we study the stabilisations for $\tilde{\mathbb{O}}$: as before, restricting the operation spaces of $\mathcal{O}$ gives rise to a grading of each level $B_\bullet(\mathbb{O}, \mathbb{I}, \boldsymbol{X})_n$; we denote the components by $B_\bullet(\mathbb{O}, \mathbb{I}, \boldsymbol{X})_n^\delta$ and their realisations by $\tilde{\mathbb{O}}(\boldsymbol{X})_n^\delta$. Adding the propagator gives rise to maps $B_\bullet(\mathbb{O}, \mathbb{I}, \boldsymbol{X})_n^\delta \to B_\bullet(\mathbb{O}, \mathbb{I}, \boldsymbol{X})_n^{\delta \curlyvee e_n}$ of simplicial spaces, and, thus, also to maps $\tilde{\mathbb{O}}(\boldsymbol{X})_n^\delta \to \tilde{\mathbb{O}}(\boldsymbol{X})_n^{\delta \curlyvee e_n}$. We denote the colimits by $B_\bullet(\mathbb{O}, \mathbb{I}, \boldsymbol{X})_n^\infty$ and $\tilde{\mathbb{O}}(\boldsymbol{X})_n^\infty$; then clearly $\tilde{\mathbb{O}}(\boldsymbol{X})_n^\infty \simeq |B_\bullet(\mathbb{O}, \mathbb{I}, \boldsymbol{X})_n^\infty|$.

Again, we obtain a product map $(f_n^\infty, \eta_n^\infty)\colon \tilde{\mathbb{O}}(\boldsymbol{X})_n^\infty \to \tilde{\mathbb{O}}(*)_n^\infty \times \tilde{\mathbb{D}}(\boldsymbol{X})_n$, and we claim that it is a homology equivalence: since we have already seen that the simplicial spaces are proper, we can invoke the spectral sequence for the geometric realisation from [Seg74, Prop. A1], whence it is enough to see that for each dimension $p \geqslant 0$ and each colour $n \in N$, the map of $p$-simplices $B_p(\mathbb{O}, \mathbb{I}, \boldsymbol{X})^\infty_n \to B_p(\mathbb{O}, \mathbb{I}, *)^\infty_n \times B_p(\mathbb{D}, \mathbb{I}, \boldsymbol{X})_n$ is a homology equivalence. As we have $B_p(\mathbb{O}, \mathbb{I}, *)^\infty_n = \mathbb{O}(*)^\infty_n$, the map in question is exactly the map from Lemma 6.3.15 for the sequence $\mathbb{I}^p\boldsymbol{X}$. This shows the subclaim.

The rest of the proof is a combinatorially enhanced variation of the first part of the proof of [Bas+17, Thm. 5.4] which uses the classical group completion theorem: let us denote by $\boldsymbol{e}_n \in \pi_0(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)$ the component of the 0-simplex $(\tilde{e}_n, [\mathfrak{v}; \varnothing])$, i.e. the propagator and the unit, and by $\boldsymbol{x}_n \in \pi_0(\tilde{\mathbb{O}}(\boldsymbol{X})_n)$ the component of the 0-simplex $\tilde{x}_n$. By a classical telescope argument, the subclaim implies that the map

$$H_\bullet(f_n, \eta_n) \colon H_\bullet(\tilde{\mathbb{O}}(\boldsymbol{X})_n)[\boldsymbol{x}_n^{-1}] \to H_\bullet(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)[\boldsymbol{e}_n^{-1}]$$

induced by the map of Pontrjagin rings is an isomorphism.

Next, recall that all our capping morphisms $\beta^K_n \colon \mathfrak{S}\binom{K}{n} \to \pi_0(\mathfrak{S}(_n))$ are assumed to be surjective. Then $(f_n, \eta_n)_* \colon \pi_0(\tilde{\mathbb{O}}(\boldsymbol{X})_n) \to \pi_0(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)$ is surjective as well, so under the above map, the multiplicative submonoid $\pi_0(\tilde{\mathbb{O}}(\boldsymbol{X})_n)$ is sent surjectively onto the submonoid $\pi_0(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)$. Therefore, we can localise further, with respect to the multiplicative submonoids of *all* path components on both sides, still obtaining an isomorphism. We get a diagram

$$\begin{tikzcd}
H_\bullet(\tilde{\mathbb{O}}(\boldsymbol{X})_n)[\boldsymbol{x}_n^{-1}] \arrow[r, "H_\bullet(f_n,\eta_n)", "\cong"'] \arrow[d] & H_\bullet(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)[\boldsymbol{e}_n^{-1}] \arrow[d] \\
H_\bullet(\tilde{\mathbb{O}}(\boldsymbol{X})_n)[\pi_0^{-1}] \arrow[r, "H_\bullet(f_n,\eta_n)", "\cong"'] \arrow[d, equal] & H_\bullet(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)[\pi_0^{-1}] \arrow[d, equal] \\
H_\bullet(\Omega B\tilde{\mathbb{O}}(\boldsymbol{X})_n) \arrow[r, "H_\bullet(\Omega B(f_n,\eta_n))", "\cong"'] & H_\bullet(\Omega B\tilde{\mathbb{O}}(*)_n \times \Omega B\tilde{\mathbb{D}}(\boldsymbol{X})_n),
\end{tikzcd}$$

where the vertical isomorphisms between the second and the third row follow from the group completion theorem [MS76, Prop. 1] for $\mathscr{D}_1$-algebras. This shows that $\Omega B(f_n, \eta_n)$ is a homology equivalence of loop spaces and, thus, a weak equivalence. □

The pending proof of Lemma 6.3.15 requires some further preparation. Recall that, for each tuple $K$, we denote by $N[K] \subseteq N \wr \mathbf{Inj}$ the full subgroupoid spanned by objects of the form $\tau^*K$. For an input profile $K$ and an output

$n \in N$, recall the stable operation spaces $\mathcal{O}\binom{K}{n}^{\infty}$. Since input permutation and precomposition commute with stabilisation, these spaces assemble, for each $n$ and each $K$, into a functor $\mathcal{O}\binom{-}{n}^{\infty} \colon N[K]^{\mathrm{op}} \to \mathbf{Top}$.

Now let $\boldsymbol{Q} \colon N[K] \to \mathbf{Top}$ be any functor. Again, we have two maps: firstly, for each tuple $L = \tau^* K$ and each $n \in N$, we have the stable capping map $\mathcal{O}\binom{L}{n}^{\infty} \times \boldsymbol{Q}(L) \to \mathcal{O}\binom{}{n}^{\infty}$, which in addition discards the factor $\boldsymbol{Q}(L)$. These maps define a natural transformation of functors $N[K]^{\mathrm{op}} \times N[K] \to \mathbf{Top}$, from $\mathcal{O}\binom{-}{n}^{\infty} \times \boldsymbol{Q}(-)$ to the constant functor with value $\mathcal{O}\binom{}{n}^{\infty}$, so we get

$$\alpha_1 \colon \int^{L \in N[K]} \mathcal{O}\binom{L}{n}^{\infty} \times \boldsymbol{Q}(L) \to \mathcal{O}\binom{}{n}^{\infty}.$$

Secondly, the morphism $\pi \colon \mathcal{O} \to \mathcal{D}_\infty \odot EN$ gives, for each $n \in N$, rise to a natural transformation $\mathcal{O}^{\infty}\binom{-}{n} \Rightarrow (\mathcal{D}_\infty \otimes EN)\binom{-}{n}$ of functors $N[K]^{\mathrm{op}} \to \mathbf{Top}$, so we obtain a morphism, where $L$ ranges in $N[K]$,

$$\alpha_2 \colon \int^{L} \mathcal{O}\binom{L}{n}^{\infty} \times \boldsymbol{Q}(L) \to \int^{L} (\mathcal{D}_\infty \odot EN)\binom{L}{n} \times \boldsymbol{Q}(L) \cong \mathcal{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_{L = \tau^* K} \boldsymbol{Q}(L).$$

The following lemma is a coloured version of [Bas+17, Lem. 5.2].

**Lemma 6.3.16.** *The product map $\alpha_{\boldsymbol{Q}} := (\alpha_1, \alpha_2)$ is a homology equivalence:*

$$\alpha_{\boldsymbol{Q}} \colon \int^{L \in N[K]} \mathcal{O}\binom{L}{n}^{\infty} \times \boldsymbol{Q}(L) \to \mathcal{O}\binom{}{n}^{\infty} \times \mathcal{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_{L \in N[K]} \boldsymbol{Q}(L).$$

*Proof.* Since $\mathcal{D}_\infty(r)$ is contractible, the sequence

$$\mathcal{O}\binom{L}{n}^{\infty} \longrightarrow \mathcal{O}\binom{L}{n}^{\infty} \times \boldsymbol{Q}(L) \xrightarrow{\pi \times \mathrm{id}} \mathcal{D}_\infty(r) \times \boldsymbol{Q}(L)$$

induces a split long exact sequence of homotopy groups for each $L = \tau^* K$ and each choice of basepoint. If we take for the total space and the base space the disjoint union over all such $L$, the common fibre for each component is $\mathcal{O}\binom{L}{n}^{\infty} \cong \mathcal{O}\binom{K}{n}^{\infty}$. If we moreover quotient by the free and compatible $\mathfrak{S}_r$-actions on total space and base space, we finally obtain a long exact sequence of homotopy groups which is assigned to

$$\mathcal{O}\binom{K}{n}^{\infty} \to \int^{L} \mathcal{O}\binom{L}{n}^{\infty} \times \boldsymbol{Q}(L) \to \mathcal{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_{L = \tau^* K} \boldsymbol{Q}(L).$$

Now the product map $\alpha_Q$ is the composition of the two middle vertical maps in the following (3×3)-diagram, where we abbreviate $\mathfrak{S} := \mathfrak{S}_r$,

$$\begin{tikzcd}
\mathcal{O}\binom{K}{n}^{\infty} \arrow[r] \arrow[d, equal] & \int^{L} \mathcal{O}\binom{L}{n}^{\infty} \times \boldsymbol{Q}(L) \arrow[r] \arrow[d, "\int^{L}(\mathrm{id},\pi)\times\mathrm{id}"] & \mathcal{D}_{\infty}(r) \times_{\mathfrak{S}_r} \coprod_L \boldsymbol{Q}(L) \arrow[d, equal] \\
\mathcal{O}\binom{K}{n}^{\infty} \arrow[r] \arrow[d, "\beta_n^K"'] & \int^{L} \big(\mathcal{O}\binom{L}{n}^{\infty} \times (\mathcal{D}_{\infty} \odot EN)\binom{L}{n}\big) \times \boldsymbol{Q}(L) \arrow[r] \arrow[d, "\int^{L}(\beta_n^L\times\mathrm{id})\times\mathrm{id}"] & \mathcal{D}_{\infty}(r) \times_{\mathfrak{S}_r} \coprod_L \boldsymbol{Q}(L) \arrow[d, equal] \\
\mathcal{O}\binom{}{n}^{\infty} \arrow[r] & \mathcal{O}\binom{}{n}^{\infty} \times \mathcal{D}_{\infty}(r) \times_{\mathfrak{S}_r} \coprod_L \boldsymbol{Q}(L) \arrow[r] & \mathcal{D}_{\infty}(r) \times_{\mathfrak{S}_r} \coprod_L \boldsymbol{Q}(L).
\end{tikzcd}$$

Here the top left square commutes up to homotopy, and all other squares commute strictly. We have already seen that the top row induces a long exact sequence on homotopy groups, and the second row is clearly a fibration. By the five lemma, the first middle vertical map is a weak equivalence. Similarly, we know that both, the second and the third row, are fibrations, so we obtain a morphism between the associated Serre spectral sequences in homology. Since $\mathcal{O}$ is an operad with homological stability, the map $\beta_n^K$ between the fibres is a homology equivalence, so by a standard argument [Wei94, 5.2.12], also the second middle vertical map is a homology equivalence. □

Now we have everything together to prove Lemma 6.3.15.

*Proof of Lemma 6.3.15.* Recall that, after identifying the stable spaces $\mathbb{O}(*)_n^{\infty}$ and $\mathcal{O}\binom{}{n}^{\infty}$, our aim is to show that the map

$$q := (f_n^{\infty}, \eta_n^{\infty}) \colon \mathbb{O}(\boldsymbol{X})_n^{\infty} \to \mathcal{O}\binom{}{n}^{\infty} \times \mathbb{D}(\boldsymbol{X})_n$$

induces isomorphisms on homology. If we denote by $(N \wr \mathbf{Inj})_{\leqslant r}$ the full subcategory of $N \wr \mathbf{Inj}$ whose objects are tuples of length at most $r$, then the two sides of $q$ are exhaustively filtered by $F_{-1} := F'_{-1} := *$ and

$$F_r := \int^{K \in (N \wr \mathbf{Inj})_{\leqslant r}} \mathcal{O}\binom{K}{n}^{\infty} \times \boldsymbol{X}^K,$$

$$F'_r := \mathcal{O}\binom{}{n}^{\infty} \times \int^{K \in (N \wr \mathbf{Inj})_{\leqslant r}} (\mathcal{D}_{\infty} \odot EN)\binom{K}{n} \times \boldsymbol{X}^K,$$

and the map $q$ is filtration-preserving. Now let us fix a system $S_r \subseteq N^r$ of representatives for unordered tuples and set $\boldsymbol{Q}(K) := X_{k_1} \wedge \cdots \wedge X_{k_r}$. Then the filtration quotients are of the form

$$F_r / F_{r-1} \cong \bigvee_{K \in S_r} \int^{L \in N[K]} \mathscr{O}\binom{L}{n}_+^{\infty} \wedge \boldsymbol{Q}(L),$$

$$F'_r / F'_{r-1} \cong \bigvee_{K \in S_r} \mathscr{O}\binom{}{n}_+^{\infty} \wedge \mathscr{D}_\infty(r)_+ \wedge_{\mathfrak{S}_r} \bigvee_{L = \tau^* K} \boldsymbol{Q}(L),$$

and the map $q_r \colon F_r / F_{r-1} \to F'_r / F'_{r-1}$ between the filtration quotients splits as a bouquet $q_r = \bigvee_{K \in S_r} q_K$. We show that each $q_K$ is a homology equivalence; it then follows that also the map $q_r$ between the filtration quotients is a homology equivalence, so by applying the comparison argument [Wei94, 5.2.12] to the morphism of spectral sequences assigned to the filtration-preserving map $q$, we get that $q$ itself is a homology equivalence.

In order to see that each $q_K$ is indeed a homology equivalence, we use that $\boldsymbol{X}$ is well-based and obtain that the induced maps

$$\int^{L \in N[K]} \mathscr{O}\binom{L}{n}^{\infty} \to \int^{L \in N[K]} \mathscr{O}\binom{L}{n}^{\infty} \times \boldsymbol{Q}(L)$$

$$\text{and} \quad \mathscr{D}_\infty(r) \times_{\mathfrak{S}_r} [K] \to \mathscr{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_{L = \tau^* K} \boldsymbol{Q}(L)$$

are cofibrations, where we write $[K] := \{\tau^* K; \tau \in \mathfrak{S}_r\}$. If we write $\alpha_{\boldsymbol{Q}}$ for the product map from Lemma 6.3.16 and $\alpha_0$ for the analogous one for $* = (*)_{n \in N}$, then we obtain a morphism of cofibre sequences (written vertically)

$$\begin{tikzcd}
\int^{L \in N[K]} \mathscr{O}\binom{L}{n}^{\infty} \arrow[r, "\alpha_0"] \arrow[d] & \mathscr{O}\binom{}{n}^{\infty} \times \mathscr{D}_\infty(r) \times_{\mathfrak{S}_r} [K] \arrow[d] \\
\int^{L \in N[K]} \mathscr{O}\binom{L}{n}^{\infty} \times \boldsymbol{Q}(L) \arrow[r, "\alpha_{\boldsymbol{Q}}"] \arrow[d] & \mathscr{O}\binom{}{n}^{\infty} \times \mathscr{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_L \boldsymbol{Q}(L) \arrow[d] \\
\int^{L \in N[K]} \mathscr{O}\binom{L}{n}_+^{\infty} \wedge \boldsymbol{Q}(L) \arrow[r, "q_K"] & \mathscr{O}\binom{}{n}_+^{\infty} \wedge \mathscr{D}_\infty(r)_+ \wedge_{\mathfrak{S}_r} \bigvee_L \boldsymbol{Q}(L),
\end{tikzcd}$$

where $L$ ranges in $N[K]$. By Lemma 6.3.16, $\alpha_0$ and $\alpha_{\boldsymbol{Q}}$ induce isomorphisms in homology, so by the five lemma applied to the long exact sequence associated to the cofibre sequence, we obtain that also $q_K$ is a homology equivalence. □

## 6.4. The main result and two outlooks

Combining the ingredients of the previous two sections, we have everything together to prove the already announced main theorem of the chapter:

**Theorem 6.4.1.** *For each path connected space $A$, there is a weak equivalence of loop spaces*

$$\Omega B\mathfrak{M}_{\bullet,1}[A] \simeq \Omega^\infty \mathbf{MTSO}(2) \times \Omega^\infty\Sigma^\infty_+ \coprod_{n\geqslant 1}\coprod_{g\geqslant 0} \mathfrak{C}_{g,n}[A] /\!\!/ (\mathbb{T}^n \rtimes \mathfrak{S}_n).$$

*Proof.* The point-set requirements of Setting 6.3.10 are clearly satisfied in the case $\mathcal{O} = \mathcal{M}$, $\mathbf{I} = \mathbb{T}$, and $\boldsymbol{X} = \mathfrak{C}[A]_+ = (\mathfrak{C}_n[A]_+)_{n\geqslant 0}$, where for each colour $n \geqslant 1$, a disjoint basepoint is added, on which the twisted torus acts trivially. Note that the map $\mathcal{B}\odot\mathbb{T} \to \mathcal{M}$ combines the map $\mathcal{B}\odot N \to \mathcal{M}$ from Example 6.3.5 with the map $\mathbb{T} \to \mathcal{M}$ from Definition 6.2.16: these obviously interchange. Secondly, note that $F^{\mathcal{M}}_{\mathcal{B}\odot\mathbb{T}}(\mathfrak{C}[A]_+) \cong F^{\mathcal{M}}_{\mathbb{T}}(\mathfrak{C}[A])$.

If we apply Theorem 6.3.11, then we obtain, for each $n \geqslant 1$,

$$\Omega B\tilde{F}^{\mathcal{M}}_{\mathcal{B}\odot\mathbb{T}}(\mathfrak{C}[A]_+)_n \simeq \Omega B\mathcal{M}\binom{}{n} \times \Omega^\infty\Sigma^\infty \mathrm{hocolim}_{\mathbb{T}}(\mathfrak{C}_\bullet[A]_+).$$

Consider the left side first: we claim that $\varphi\colon \tilde{F}^{\mathcal{M}}_{\mathcal{B}\odot\mathbb{T}}(\mathfrak{C}[A]_+) \to F^{\mathcal{M}}_{\mathcal{B}\odot\mathbb{T}}(\mathfrak{C}[A]_+)$, the map of $\mathcal{M}$-algebras from the derived relatively free algebra to the actual one, is an equivalence: here we use that the basepoints are isolated, whence the map splits into $\coprod_{K\in S}\varphi_K$ for a system $S \subseteq \coprod_r \overline{\mathbb{N}}^r$ of representatives of tuples with respect to coordinate permutation. If we denote again by $r(k) \geqslant 0$ the number of occurrences of $k$ in the sequence $K$, then the compact Lie group $G := \prod_{k\geqslant 0}\mathbb{T}_k \wr \mathfrak{S}_{r_k}$ acts on $Y := \mathcal{M}\binom{K}{n} \times \prod_i \mathfrak{C}_{k_i}[A]$, and $\varphi_K$ is exactly the map that compares the *homotopy* quotient of this action with the *actual* quotient. However, $Y$ is a Hausdorff space and $G$ acts freely on $Y$, since $\mathcal{M}$ is $\mathfrak{S}$-free and $\mathbb{T}_k$ acts freely on $\mathcal{M}$ by precomposition, see Lemma 6.2.17. In this situation, [Kör18, Thm. A.7] tells us that the map comparing the homotopy quotient with the actual quotient is an equivalence. In particular, for $n = 1$, the left side is equivalent, as a loop space, to $\Omega B\mathfrak{M}_{\bullet,1}[A]$ by Proposition 6.2.20.

Let us now look at the right side: here we see that

$$\mathrm{hocolim}_{\mathbb{T}}(\mathfrak{C}_\bullet[A]_+) \simeq \bigvee_{k\geqslant 1} \mathfrak{C}_k[A]_+ /\!\!/ \mathbb{T}_k = \{*\} \sqcup \coprod_{k\geqslant 1} \mathfrak{C}_k[A] /\!\!/ \mathbb{T}_k,$$

where $/\!\!/$ denotes the homotopy quotient. If we focus again on the case $n = 1$, then we saw in Example 5.1.14 that the first level of the initial $\mathscr{M}$-algebra, $\mathscr{M}\binom{}{1}$, coincides with the old $\mathscr{M}|_1$-algebra $\coprod_g \mathfrak{M}_{g,1}$, whose group completion is accessible by the Madsen–Weiss theorem. Hence, we can replace the factor $\Omega B\mathscr{M}\binom{}{1}$ by $\Omega^\infty \mathbf{MTSO}(2)$. This proves the claim. □

Note that both sides of the equivalence depend only on the fundamental group $\pi := \pi_1(A)$. I would like to emphasise two special cases:

1. If $\pi \cong \mathbb{Z}$, then conjugacy classes of maps $\pi \to \Gamma_{g,n}$ are just conjugacy classes of mapping classes, and our irreducibility criterion for subgroups translates to an irreducibility criterion for single mapping classes $\varphi$ by considering the subgroup $\langle\varphi\rangle$ spanned by $\varphi$.
2. If $\pi$ is a torsion group, then there are no non-trivial homomorphisms $\pi \to \Gamma_{g,n}$, since $\Gamma_{g,n}$ is torsion free. As the trivial subgroup will turn out to be *non*-irreducible, we obtain $\Omega B\mathfrak{M}_{\bullet,1}[A] \simeq \Omega^\infty \mathbf{MTSO}(2)$.

Let us close this chapter by showing two directions into which one might wish to go from here:

**Outlook 6.4.2.** We can consider parametrised versions of further well-known $E_d$-algebras. In [BKR21, Apx. B], we studied the following two cases, which are much easier and do not need the coloured operadic setting.

On the one hand, the $E_2$-algebra $\coprod_r B\mathrm{Br}_r[A]$ of parametrised braid spaces: again, we can introduce an irreducibility criterion for subgroups $\Pi \subseteq \mathrm{Br}_r$, where $\Pi$ is irreducible if it is not conjugate to the block sum of subgroups of smaller braid groups. If we put $\mathfrak{C}^{\mathrm{Br}}[A] := \coprod_r \coprod_{[\omega]} BZ(\mathrm{im}(\omega), \mathrm{Br}_r)$, where $[\omega]$ ranges again over all conjugacy classes of maps $\pi \to \mathrm{Br}_r$ with irreducible image, then one can show that $\coprod_r B\mathrm{Br}_r[A]$ is equivalent to the free $E_2$-algebra over $\mathfrak{C}^{\mathrm{Br}}[A]$; thus, we obtain

$$\Omega B \coprod_{r \geqslant 0} B\mathrm{Br}_r[A] \simeq \Omega^2\Sigma^2_+ \mathfrak{C}^{\mathrm{Br}}[A].$$

On the other hand, consider the $E_\infty$-algebra $\coprod_r B\mathfrak{S}_r[A]$: again, we have an irreducibility criterion for subgroups $\Pi \subseteq \mathfrak{S}_r$, where $\Pi$ is irreducible if it is not conjugate to the block sum of subgroups inside smaller symmetric groups, i.e. if $\Pi$ is transitive. If we define $\mathfrak{C}^{\mathfrak{S}}[A] := \coprod_r \coprod_{[\omega]} BZ(\mathrm{im}(\omega), \mathfrak{S}_r)$, where $[\omega]$ ranges again over all conjugacy classes of maps $\pi \to \mathfrak{S}_r$ with irreducible image, then one can show that $\coprod_r B\mathfrak{S}_r[A]$ is equivalent to the free $E_\infty$-algebra over $\mathfrak{C}^{\mathfrak{S}}[A]$; hence we obtain

$$\Omega B \coprod_{r \geqslant 0} B\mathfrak{S}_r[A] \simeq \Omega^\infty \Sigma^\infty_+ \mathfrak{C}^{\mathfrak{S}}[A].$$

In the special case $\pi \cong \mathbb{Z}$, we consider cyclic and transitive subgroups of $\mathfrak{S}_r$; these are spanned by a long cycle. Up to conjugation, there is one such subgroup for each $r \geqslant 0$, and its centraliser is the subgroup itself, and hence isomorphic to $\mathbb{Z}_r$. In this way, we recover the result of [Rei19, Cor. 4.32]: for $\pi_1(A) \cong \mathbb{Z}$, we have $\Omega B \coprod_r B\mathfrak{S}_r[A] \simeq \Omega^\infty \Sigma^\infty_+ \coprod_r B\mathbb{Z}_r$.

As in the case of the Mumford conjecture, one might try to understand the homology of the *stable* parametrised spaces $B\Gamma_{\infty,1}[A]$, $B\mathrm{Br}_\infty[A]$, or $B\mathfrak{S}_\infty[A]$ in terms of the homology of $A$.

One further aim is to understand the group completion of the $E_\infty$-algebra $\coprod_n B\mathrm{GL}_n(R)[A]$ for a ring $R$; this can be seen as the parametrised algebraic K-theory of $R$ over $A$. Once again, the infinite loop space $\Omega B \coprod_n B\mathrm{GL}_n(R)[A]$ contains the classical algebraic K-theory $\Omega B \coprod_n B\mathrm{GL}_n(R)$ as a direct factor, and it would be interesting to understand the remainder.

**Outlook 6.4.3.** In [Gal+09], the authors introduced a topological category $\mathbf{Cob}_d$ of (oriented) cobordisms and identified its homotopy type $B\mathbf{Cob}_d$ with $\Omega^{\infty-1}\mathbf{MTSO}(d)$. For $d = 2$, this result can be related to the Madsen–Weiss theorem, by showing that the inclusion of the positive boundary subcategory $\mathbf{Cob}_{2,\partial} \hookrightarrow \mathbf{Cob}_2$ induces an equivalence on classifying spaces and by identifying the map $\coprod_g \mathfrak{M}_{g,1} = \mathbf{Cob}_{2,\partial}(\substack{\varnothing \\ S^1}) \to \Omega B\mathbf{Cob}_2$ as a group completion.

Similarly, we can consider a category $\mathbf{Cob}_d[A]$ of parametrised cobordisms, formally defined as the topological category of maps from $A$ to $\mathbf{Cob}_d$. This is a special case of the construction from [RS17], where additionally, tangential structures are allowed to vary over $A$. Our result can be seen as a first step towards understanding $B\mathbf{Cob}_2[A]$.

# Appendices

## A. Artin's braid groups

The theory of braids goes back to the works of Artin from 1925 [Art25]. In our topological setting, the most concise definitions of the *pure braid group on r strands*, called $\mathrm{PBr}_r$, and the *braid group on r strands*, called $\mathrm{Br}_r$, are given by

$$\begin{aligned} \mathrm{PBr}_r &:= \pi_1\big(\tilde{C}_r(\mathbb{R}^2)\big), \\ \mathrm{Br}_r &:= \pi_1\big(C_r(\mathbb{R}^2)\big), \end{aligned}$$

where we choose as basepoints the standard configurations $((1,0),\dots,(r,0))$ and $\{(1,0),\dots,(r,0)\}$, respectively. Since $C_r(\mathbb{R}^2)$ arises from $\tilde{C}_r(\mathbb{R}^2)$ by quotienting out the free $\mathfrak{S}_r$-action on $\tilde{C}_r(\mathbb{R}^2)$, these groups assemble into a short exact sequence

$$1 \to \mathrm{PBr}_r \to \mathrm{Br}_r \to \mathfrak{S}_r \to 1.$$

Elements in $\mathrm{Br}_r$ are called *braids*, and a braid is called *pure* if it belongs to $\mathrm{PBr}_r$, i.e. if it projects to the trivial permutation of points. We can visualise braids by drawing the union of the trajectories of the $r$ points, as a subset of $\mathbb{R}^2 \times [0;1]$ for some representing path $\gamma\colon [0;1] \to C_r(\mathbb{R}^2)$, see Figure A.1.

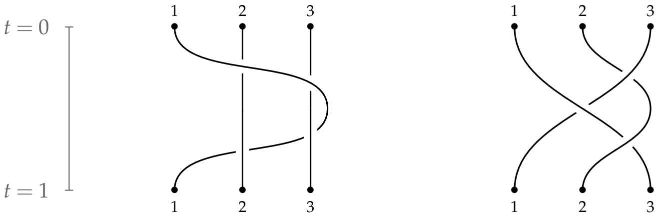

**Figure A.1.** Two braids on three strands. The left one is pure, but the right one is not; its corresponding permutation interchanges 1 and 3.

Thinking of braids as being built out of strands of hair, we prefer to read them from top to bottom, i.e. the time coordinate increases when we move downwards. Consequently, we form the product $[\gamma] \cdot [\gamma']$ of two braids by placing the braid $[\gamma]$ *below* the braid $[\gamma']$.

Let us now restrict ourselves to the *pure* braid group $\mathrm{PBr}_r$. As in Section 1.5, we want to consider, for each $1 \leqslant u < v \leqslant r$, a specific 'elementary' pure braid $\alpha_{u,v}$, which is depicted in Figure A.2.

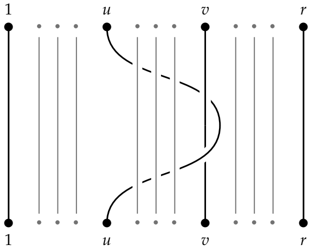

**Figure A.2.** The generator $\alpha_{u,v}$ inside $\mathrm{PBr}_r$.

As done in [Lam00, p. 276], one can give a full presentation of $\mathrm{PBr}_r$ which uses the collection of these braids $\alpha_{u,v}$ for $1 \leqslant u < v \leqslant r$ as generating set, and a sufficient set of relations is given by the following equalities, where '$\star$' abbreviates first factor of the corresponding term: $y^{-1} \cdot h \cdot (\star) = y^{-1} \cdot h \cdot y$. The relations identify the conjugate $\alpha_{u',v'} \cdot \alpha_{u,v} \cdot \alpha^{-1}_{u',v'}$ with

$$\begin{cases} \alpha_{u,v} & \text{if } v < u' \text{ or } u < u' < v' < v, \\ \alpha^{-1}_{u',v} \cdot \alpha_{u,v} \cdot (\star) & \text{if } u' < v' = u < v, \\ \big(\alpha_{u,v'} \cdot \alpha_{u,v}\big)^{-1} \cdot \alpha_{u,v} \cdot (\star) & \text{if } u < v = u' < v', \\ \big(\alpha^{-1}_{u,v'} \cdot \alpha^{-1}_{u,u'} \cdot \alpha_{u,v'} \cdot \alpha_{u,u'}\big)^{-1} \cdot \alpha_{u,v} \cdot (\star) & \text{if } u < u' < v < v'. \end{cases}$$

Lambropoulou uses a 'mirrored' generating set $a_{u,v} = \alpha_{r-u,r-v}$, and she composes braids from bottom to top; hence her formulæ look slightly different.

# B. Discrete Morse theory

In this Appendix, we repeat the basic notions of discrete Morse theory in the sense of [For98], a technique which is very useful for homological calculations.

As we use the theory basically only in Chapter 4, we will be very short; in particular, we will forgo a survey on the geometric background, and refer to [Heß12] for a comprehensive treatment.

**Remark B.1.** Let $R$ be a commutative ring. A finite and non-negative $R$-chain complex $(C_\bullet, \partial)$, together with a choice of bases for each $C_h$, is the same as a sequence $\Omega = (\Omega_0, \Omega_1, \dots)$ of sets such that $\coprod_h \Omega_h$ is finite, together with a labelling $\varepsilon\colon \Omega_h \times \Omega_{h-1} \to R, (\pi, \pi') \mapsto \varepsilon_{\pi,\pi'}$ for each $h \geqslant 1$, such that for each $\pi \in \Omega_h$ and $\pi'' \in \Omega_{h-2}$, we have

$$\sum_{\pi' \in \Omega_{h-1}} \varepsilon_{\pi,\pi'} \cdot \varepsilon_{\pi',\pi''} = 0.$$

We call such a collection $(\Omega, \varepsilon)$ a *based complex*, and we recover the chain complex $(C_\bullet, \partial) = R(\Omega, \varepsilon)$ by linearising $C_h := R\langle \Omega_h \rangle$ and $\partial\pi = \sum_{\pi'} \varepsilon_{\pi,\pi'} \cdot \pi'$. Moreover, we call the elements $\pi \in \Omega_h$ the *h-cells* of $(\Omega, \varepsilon)$ and $\varepsilon_{\pi,\pi'} \in R$ the *incidence number* of $\pi$ and $\pi'$.

**Definition B.2.** A *(finite) directed graph* is a tuple $\Gamma = (\Omega, A, d)$ where $\Omega$ and $A$ are finite sets and $d\colon A \to \Omega \times \Omega$ is injective. If $a \in A$ with $d(a) = (\pi, \pi')$, then we interpret $a$ as an arrow from $\pi$ to $\pi'$, write $a\colon \pi \to \pi'$.

For $\pi, \pi' \in \Omega$, a path from $\pi$ to $\pi'$, write $\gamma\colon \pi \rightsquigarrow \pi'$, is a finite sequence of arrows $\pi = \pi_0 \to \cdots \to \pi_t = \pi'$, and we call $t$ the *length* of $\gamma$. A *cycle* is a path $\gamma\colon \pi \rightsquigarrow \pi$ of positive length.

An *R-valued labelling* of $\Gamma$ is a map $\lambda\colon A \to R$, and we call the pair $(\Gamma, \lambda)$ a *labelled directed graph*. For each $a\colon \pi \to \pi'$, we write $\lambda_{\pi,\pi'} := \lambda(a)$, which is well-defined as $d$ was assumed to be injective. If $\gamma\colon \pi \rightsquigarrow \pi'$ is a path, given by $\pi = \pi_0 \to \cdots \to \pi_t = \pi'$, then we let $\lambda(\gamma) := \lambda_{\pi_0,\pi_1} \cdots \lambda_{\pi_{t-1},\pi_t} \in R$.

**Construction B.3.** Each based complex $(\Omega, \varepsilon)$ has an underlying labelled directed graph $(\Gamma, \lambda)$ with vertex set $\Omega := \coprod_h \Omega_h$ and arrows $\pi \to \pi'$ with label $\lambda_{\pi,\pi'} := \varepsilon_{\pi,\pi'}$ whenever $\pi \in \Omega_h$ and $\pi' \in \Omega_{h-1}$ with $\varepsilon_{\pi,\pi'} \neq 0$.

**Definition B.4.** Let $(\Omega, \varepsilon)$ be a based complex. Then a *matching* $\Lambda$ on $(\Omega, \varepsilon)$ is given by declaring, for each $h \geqslant 0$:

1. a decomposition $\Omega_h = \Omega_h^{\mathrm{ess}} \mathbin{\dot\cup} \Omega_h^{\mathrm{red}} \mathbin{\dot\cup} \Omega_h^{\mathrm{coll}}$, and
2. a bijection $(-)^\sharp\colon \Omega_h^{\mathrm{red}} \rightleftarrows \Omega_{h+1}^{\mathrm{coll}} :(-)^\flat$,

such that for each $\pi \in \Omega_h^{\mathrm{red}}$, the incidence number $\varepsilon_{\pi^\sharp,\pi} \in R$ is invertible.

We call a cell $\pi$ *essential*, *redundant*, or *collapsible* if it lies inside $\Omega^{\mathrm{ess}}$, $\Omega^{\mathrm{red}}$, or $\Omega^{\mathrm{coll}}$, respectively. Let $\Gamma$ be the underlying directed graph of $(\Omega, \varepsilon)$ and let $\Gamma^\Lambda$ be the directed graph arising from $\Gamma$ by inverting all arrows of the form $\pi^\sharp \to \pi$. We call the matching $\Lambda$ a *discrete Morse flow* if $\Gamma^\Lambda$ has no cycles.

**Notation B.5.** Inside $\Gamma^\Lambda$, we denote an inverted arrow by '$\nearrow$', as it raises the degree, while we denote a usual arrow by '$\searrow$', as it lowers the degree.

**Construction B.6.** Let $(\Omega, \varepsilon)$ be a based complex and $\Lambda$ be a discrete Morse flow on $(\Omega, \varepsilon)$.

- On the modified directed graph $\Gamma^\Lambda$ as above, we define a labelling $\lambda^\Lambda$ where we put $\lambda^\Lambda_{\pi',\pi} := -\lambda^{-1}_{\pi,\pi'}$ for all inverted arrows $\pi' \nearrow \pi$.
- We define the *derived based complex* $(\Omega, \varepsilon)^\Lambda$ with vertices only the essential cells, and for $\eta \in \Omega_h^{\mathrm{ess}}$ and $\eta' \in \Omega_{h-1}^{\mathrm{ess}}$, we let

  $$\varepsilon^\Lambda_{\eta,\eta'} := \sum_{\gamma\colon \eta \rightsquigarrow \eta'} \lambda^\Lambda(\gamma),$$

  where $\gamma$ ranges over all paths from $\eta$ to $\eta'$ inside $\Gamma^\Lambda$.

One readily checks that we have again $\sum_{\eta'} \varepsilon^\Lambda_{\eta,\eta'} \cdot \varepsilon^\Lambda_{\eta',\eta''} = 0$ for all $\eta \in \Omega_h^{\mathrm{ess}}$ and $\eta'' \in \Omega_{h-2}^{\mathrm{ess}}$, whence $(\Omega, \varepsilon)^\Lambda$ is indeed again a based complex. Moreover, we have chain maps $\varphi^\Lambda\colon R(\Omega, \varepsilon) \leftrightarrows R(\Omega, \varepsilon)^\Lambda :\psi^\Lambda$ by

$$\varphi^\Lambda(\pi) := \sum_{\gamma\colon \pi \rightsquigarrow \eta} \lambda^\Lambda(\gamma) \cdot \eta \quad \text{and} \quad \psi^\Lambda(\eta) := \sum_{\gamma\colon \eta \rightsquigarrow \pi} \lambda^\Lambda(\gamma) \cdot \pi,$$

where the first sum ranges over all paths inside $\Gamma^\Lambda$ from $\pi$ to a critical cell $\eta$ of the same dimension, and the second sum ranges over all paths inside $\Gamma^\Lambda$ from $\eta$ to a cell $\pi$ of the same dimension. Note the following:

- if $\eta$ is essential, then $\varphi^\Lambda(\eta) = \eta$; if $\pi$ is collapsible, then $\varphi^\Lambda(\pi) = 0$.
- $\psi^\Lambda(\eta)$ differs from $\eta$ only by collapsible cells.

The main result of [For98, Thm. 7.3] is the following statement.

**Theorem B.7.** *The chain maps* $\varphi^\Lambda \colon R(\Omega, \varepsilon) \rightleftarrows R(\Omega, \varepsilon)^\Lambda : \psi^\Lambda$ *are chain homotopy inverses of each other.*

Let me make a few remarks about this statement:

1. We immediately see that $\varphi^\Lambda \circ \psi^\Lambda$ is the identity on the derived complex $R(\Omega, \varepsilon)^\Lambda$, as $\psi^\Lambda(\eta)$ differs from $\eta$ only by collapsible cells, and these vanish again when applying $\varphi^\Lambda$.
2. To be historically precise, [For98, Thm. 7.3] only shows that the induced maps in homology are inverses for each other. An explicit chain homotopy $\psi^\Lambda \circ \varphi^\Lambda \Rightarrow \mathrm{id}_{R(\Omega,\varepsilon)}$ is constructed e.g. in [Heß12, Thm. 1.1.13].
3. In many sources, the derived complex $R(\Omega, \varepsilon)^\Lambda$ has a different description: it is defined as a *subcomplex* of $R(\Omega, \varepsilon)$ containing so-called invariant chains. It is then an additional observation that this subcomplex is isomorphic to $R(\Omega, \varepsilon)^\Lambda$, see for example [For98, Thm. 8.2].

Finally, the entire construction behaves nicely with respect to dualising, as the following remark shows.

**Remark B.8.** If $(\Omega, \varepsilon)$ is a based complex of dimension $d$ (i.e. there are no cells of dimension larger than $d$), then we define its *dual* $(\Omega, \varepsilon)^* := (\Omega^*, \varepsilon^*)$ with $\Omega^*_h := \Omega_{d-h}$ and $\varepsilon^*_{\pi,\pi'} := \varepsilon_{\pi',\pi}$ for $\pi \in \Omega^*_h$ and $\pi' \in \Omega^*_{h-1}$.

On the side of chain complexes, this corresponds to the dualised and mirrored chain complex $(C^*_{d-\bullet}, \partial^*)$, together with the dual basis.

If $\Lambda$ is a discrete Morse flow for $(\Omega, \varepsilon)$, then we obtain a discrete Morse flow $\Lambda^*$ for $(\Omega, \varepsilon)^*$ by exchanging the rôles of redundant and collapsible cells. Additionally, the two chain complexes $(R(\Omega, \varepsilon)^\Lambda)^*_{d-\bullet}$ and $(R(\Omega, \varepsilon)^*_{d-\bullet})^{\Lambda^*}$ are isomorphic: clearly, the $h^{\text{th}}$ level of both sides is freely generated by $\Omega^{\text{ess}}_{d-h}$, and for an essential cell $\pi$ of dimension $d-h$, and an essential cell $\pi'$ of dimension $h+1$, we have

$$(\varepsilon^*)^{\Lambda^*}_{\pi,\pi'} = \sum_{\substack{\gamma\colon \pi \rightsquigarrow \pi' \\ \text{inside } \Gamma^{\Lambda^*}}} \lambda^{\Lambda^*}(\gamma) = \sum_{\substack{\gamma\colon \pi' \rightsquigarrow \pi \\ \text{inside } \Gamma^{\Lambda}}} \lambda^{\Lambda}(\gamma) = \varepsilon^\Lambda_{\pi',\pi} = (\varepsilon^\Lambda)^*_{\pi,\pi'}.$$

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# Symbol Index

| | |
|---|---|
| $\mathscr{Ass}$ | associative operad, 78 |
| $[-,-]$ | Browder bracket, 91 |
| $\mathrm{Br}_r$ | braid group on $r$ strands, 269 |
| $\mathscr{B}$ | operad of based sets (or general objects), 77 |
| $\mathscr{O} \odot \mathscr{O}'$ | Boardman–Vogt tensor product of $\mathscr{O}$ and $\mathscr{O}'$, 100 |
| $\mathrm{cat}(\mathscr{C})$ | PROP associated to the monochromatic operad $\mathscr{C}$, 109 |
| $\mathscr{C}_d$ | little $d$-cubes operad, 79 |
| $\mathbf{Ch}_R$ | category of chain complexes of $R$-modules, 76 |
| $C_\bullet^{\mathrm{cell}}$ | cellular chain complex, 29 |
| $C_\bullet^{\mathrm{sing}}$ | singular chain complex, 83 |
| $\mu \circ_i \mu'$ | partial composition inside an operad, 81 |
| $\mathscr{Com}$ | commutative operad, 78 |
| $\tilde{C}_r(E)$ | configuration space of $r$ ordered points in $E$, 2 |
| $C_r(E)$ | configuration space of $r$ unordered points in $E$, 2 |
| $\mathscr{D}_d$ | little $d$-discs operad, 79 |
| $\mathbf{\Delta\Sigma}$ | category of non-commutative finite sets, 111 |
| $\mathbf{\Delta\Sigma}^\times$ | operad with $\mathbf{\Delta\Sigma}^\times\binom{k_1,\ldots,k_r}{n} = \prod_i \mathbf{\Delta\Sigma}\binom{k_i}{n}$, 149 |
| $EN$ | chaotic category with object set $N$, 258 |

| | |
|---|---|
| $\mathbf{Fin}$ | skeletal category of finite sets with all maps, 108 |
| $\mathbf{Fin}_d$ | PROP associated to $\mathscr{C}_d$, 111 |
| $F^{\mathscr{O}}$ | free $\mathscr{O}$-algebra, 82 |
| $F^{\mathscr{O}}_{\mathscr{P}}$ | free $\mathscr{O}$-algebra over a given $\mathscr{P}$-algebra, 82 |
| $\tilde{F}^{\mathscr{O}}_{\mathscr{P}}$ | derived free $\mathscr{O}$-algebra over a given $\mathscr{P}$-algebra, 257 |
| $\Gamma^m_{g,n}$ | mapping class group of surfaces of type $\Sigma^m_{g,n}$, 191 |
| $\boldsymbol{G} \wr \mathbf{Inj}$ | wreath product of $\boldsymbol{G} = (G_n)_{n\in N}$ and $\mathbf{Inj}$, 47 |
| $\boldsymbol{G}[K]$ | connected subgroupoid of $\boldsymbol{G} \wr \mathbf{Inj}$ spanned by objects of the form $\tau^*K$, 47 |
| $\mathbf{hCob}_2$ | 2-dimensional homotopy cobordism category, 108 |
| $\mathscr{I}$ | trivial (monochromatic) operad, 77 |
| $\imath'$ | counterpart of the shuffle $\imath$, 29 |
| $\mathbf{Inj}$ | skeletal category of finite sets with injections, 46 |
| $\int^{k\in\mathbf{I}} H(k,k)$ | coend construction for a functor on $\mathbf{I}^{\mathrm{op}} \times \mathbf{I}$, 77 |
| $\mathbf{M}$ | conformal cobordism category, 196 |
| $\mathscr{M}$ | coloured surface operad, 197 |
| $\mathfrak{M}^m_{g,n}$ | moduli space of surfaces of type $\Sigma^m_{g,n}$, 193 |
| $\mathfrak{M}_{g,1}[A]$ | moduli space of $\Sigma_{g,1}$-bundles over $A$, 237 |
| $\mathbf{MTSO}(d)$ | $d$-dimensional oriented tangential Thom spectrum, 236 |
| $\overline{\mathbb{N}}(\mathscr{C})$ | dyeing of the monochromatic operad $\mathscr{C}$, 113 |
| $\overline{\mathbb{N}}^c(\mathscr{C})$ | connective suboperad of $\overline{\mathbb{N}}(\mathscr{C})$, 117 |
| $\mathscr{O}$-$\mathbf{Alg}$ | category of $\mathscr{O}$-algebras, 81 |

| | |
|---|---|
| $\mathrm{PBr}_r$ | pure braid group on $r$ strands, 269 |
| $\mathfrak{P}^m_{g,n}$ | space of parallel slit domains of type $\Sigma^m_{g,n}$, 204 |
| $\mathcal{P}ois^R_d$ | Poisson $d$-operad in $R$-modules, 90 |
| $Q_c$ | divided power operation induced by the class $c$, 97 |
| $\mathrm{rep}(\mathbf{P})$ | representable operad underlying the PROP $\mathbf{P}$, 112 |
| $R\text{-}\mathbf{Mod}$ | category of $R$-modules, 76 |
| $\Sigma^m_{g,n}$ | isomorphism class of connected surfaces of genus $g \geqslant 0$, $n \geqslant 1$ boundary curves, and $m \geqslant 0$ punctures, 191 |
| $\mathbf{\Sigma}$ | skeletal category of finite sets with bijections, 46 |
| $\sigma_K$ | block permutation, for $\sigma \in \mathfrak{S}_r$ and $K = (k_1, \dots, k_r)$, 78 |
| $\mathfrak{S}$ | sequence $(\mathfrak{S}_n)_{n \geqslant 1}$ of symmetric groups, 47 |
| $\mathfrak{S}_K$ | subgroup of $\mathfrak{S}_{\lvert K \rvert}$ preserving the unordered partition $K$, 3 |
| $\mathfrak{S}^K$ | automorphism group $\mathrm{Aut}_{N \wr \mathbf{\Sigma}}(K)$, 104 |
| $\mathbb{T}_n$ | $n^{\text{th}}$ twisted torus $\mathbb{T}^n \rtimes \mathfrak{S}_n$, 247 |
| $\tilde{V}_K(E)$ | configuration space of ordered vertical clusters in $E$, 2 |
| $V_K(E)$ | configuration space of unordered vertical clusters in $E$, 3 |
| $V^k_r(E)$ | short notation for $V_K(E)$ with $K = (k, \dots, k)$, 4 |
| $\tilde{V}^<_K(\mathbb{R}^{p,1})$ | component of $\tilde{V}_K(\mathbb{R}^{p,1})$ in which points within a cluster are ordered in accordance to their last coordinate, 28 |
| $V(E; \mathbf{X})$ | configuration space of labelled vertical clusters in $E$, 47 |
| $\mathcal{V}_{p,q}$ | vertical operad, 120 |
| $X[n]$ | based sequence $(X_k)_{k \geqslant 1}$ with $X_n = X$ and $X_k = *$ else, 48 |
| $\mathbb{Y}_K$ | tableau of type $K$, 6 |
| $Z(\Pi, \Gamma)$ | centraliser of a subgroup $\Pi \subseteq \Gamma$, 238 |

# Dicta index

*… nos esse quasi nanos gigantum umeris insidentes.*

> Bernard de Chartres, quoted by John of Salesbury (‘Dicebat Bernardus Carnotensis nos esse quasi nanos gigantum humeris insidentes, ut possimus plura eis et remotiora videre, non utique proprii visus acumine, aut eminentia corporis, sed quia in altum subvehimur et extollimur magnitudine gigantea’) in *Metalogicon*, Vol. iv, 1159. Transl. ed. by J. B. Hall, K. S. B. Keats-Rohan, in *Corpus Christianorum Continuatio Mediaevalis* **98**, doi: `10.1484/M.CCT-EB.5.105892`

Ἀϱχὰς εἶναι τῶν ὅλων ἀτόμους καὶ κενὸν, τὰ δ’ ἄλλα πάντα νενομίσϑαι.

> Democritus, quoted by Diog. Laërtius in *Vitae Philosophorum*, 9:44. Transl. *Lives of Eminent Philosophers*, ed. by S. White. Cambridge: Cambridge University Press, 2021. doi: `10.1017/9781139047111`

*Ubi materia, ibi geometria.*

> Johannes Kepler, *De fundamentis astrologiæ certioribus*, Thesis xx, 1601. In: *Joannis Kepleri astronomi opera omnia* **1**. Ed. by C. Frisch. Frankfurt a. M., Erlangen: Heyder & Zimmer, 1858.

*The inputs are numbered red, yellow, and green;*
*and their colours are three, two, and four!*

> Spoken by either Andrea Bianchi or myself in one of our weekly ‘Forschungsseminar’ meetings in Carl-Friedrich Bödigheimer’s office, Bonn, 2019.

*First recall what trees themselves are.*

> Tom Leinster. *Higher Operads, Higher Categories*, §7.3. London Mathematical Society Lecture Note Series **298**. Cambridge: Cambridge University Press, 2004. DOI: `10.1017/CBO9780511525896`.

*Es soll das Innere von Ω auf eine*
*von einem geradlinigen Schlitz begrenzte*
*bene konform abgebildet werden.*

> David Hilbert. 'Zur Theorie der konfirmen Abbildung'. In: *Nachr. Königl. Ges. Wiss.* (1909), pp. 314–323.

*One would like to characterize its homotopy type,*
*but in reality one must settle for less.*

> Ib Madsen. 'Moduli spaces from a topological viewpoint'. In: *Proceedings of the International Congress of Mathematicians 2006* **1**, Ed. by M. Sanz-Solé, J. Soria, J. L. Varona, and J. Verdera. Madrid: European Mathematical Society, 2007, pp. 385–412.

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