ON A NON-LOCAL FREE BOUNDARY PROBLEM MODELING CELL POLARIZATION

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Dedicated to my beloved family and
to the memory of Anna Blana and Konstantinos Kritikos

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Summary

The amplification of an external signal is a key step in direction sensing of biological cells, a process that contributes significantly in the regulation of cell shape. This thesis is concerned with a simple model for cell polarization as a response to a time-depending signal, which was previously proposed in [51].

The model consists of a bulk-surface reaction-diffusion system of partial differential equations for different variants of a protein on the cell surface and interior respectively. The coupling is by a nonlinear Robin-type boundary condition for the bulk variable and a corresponding source term on the cell surface. We study solutions of this model in certain parameter regimes in which several reaction rates on the membrane as well as the diffusion coefficient within the cell are large.

It turns out that in suitable scaling limits solutions converge to solutions of some obstacle type problems. A distinguishing feature of these limiting problems is the presence of a term that depends in a non-local way on the support of the solution itself and makes the analysis quite challenging.

First, we justify the well-posedness of these obstacle type problems. Moreover, we show an L^1-contraction property of solutions, by means of which, we further prove that the steady states are globally stable. It is worth pointing out that, this first part of the thesis complements to a certain extent the analysis in [51] while it also provides a more advanced insight on the limiting problems through this innovative L^1-contraction property.

In the second part of this thesis, we investigate qualitative properties of the free boundary. We conclude that there are necessary and sufficient conditions for the initial data that imply continuity of the support at $t = 0$. If one of these assumptions fail, then jumps of the support take place. We provide a complete characterization of the jumps for a large class of initial data as well. The continuity results concerning the set $\{u(\cdot, t) > 0\}$ can be further improved by imposing some additional assumptions on the initial data. In fact, restricting our analysis to the case of the unit sphere, we prove global in time continuity for the support of the solution.

The latter part of this thesis allows for a better understanding of the evolution of the special non-local term that is involved in these limiting problems and depends on the set $\{u(\cdot, t) > 0\}$, which can be useful for future analysis.

Contents

CHAPTER 1

Introduction

1.1 Outline

This thesis is concerned with the study of a simplified, yet realistic model that contributes to the composite process of cell polarization. More specifically, we consider a minimal model for cell polarization as a response to an external chemical signal. In the following sections of this introductory chapter, we elucidate in depth how the the ability of cells to react in external signals forwards the regulation of their shape.

The model under investigation belongs to the broad family of reaction-diffusion systems. In principle, such systems arise naturally in the mathematical description of a large number of real-life phenomena, spread through chemistry, physics (neutron diffusion theory), geology, ecology and biology. Reaction-diffusion systems are especially appealing to study not only due to their importance in applications, for instance collective phenomena in life sciences, but also because they give rise to a rich variety of complex behaviours, such as pattern formation.

To be more precise, the model that we consider in this thesis (cf. (1.3.1)-(1.3.4)) is associated with the class of bulk-surface partial differential equations (PDEs) that appears in a variety of different applications and has attracted quite some attention over the last years, see for example [13], [41],[48], [58] and the references therein for applications to cell biology. In particular, we consider a bulk-surface reaction-diffusion system of equations for different variants of a protein on the outer plasma membrane and in the cytosol. The cytosolic volume of a cell is represented by a bounded, connected, open domain $\Omega \subset \mathbb{R}^3$ and the cell membrane by the boundary of Ω that is assumed to be a smooth surface Γ.

It is worth noticing that, coupled bulk-surface systems are not covered by the standard theory of reaction-diffusion systems.

The most striking feature of this particular model is that for an appropriate choice of parameters, it is possible to approximate the model by a specific class of free boundary problems, namely obstacle problems. In this thesis we study various features of these obstacle type problems, filling several gaps in the literature. For a certain part of this analysis, the answer had previously been known for the corresponding stationary model [51], and we extend these results to the evolutionary case.

In this introductory chapter we will lay the foundation for these results and discuss the necessary background, and we will give a more detailed description of the results in the following chapters and appendices.

The aim in Section 1.2 is to familiarize non-expert readers with all the necessary biological setting that underlines the present analysis. More specifically, in Subsection 1.2.1 we begin with the description of cell polarization, a process which is principally tightly associated with the symmetry breaking in cells. We further provide briefly various stimuli that are capable of initiating the polarity of cells, emphasizing especially on the chemical ones. To highlight the contribution of polarization in a large number of biological processes, we discuss the well known example of wound healing.

Polarization in response to some external chemical signal motivates the directional movement of the cell in the surrounding environment. This ability of the cells to migrate is of great importance and pertains to the more wide and compound process of chemotaxis. In Subsection 1.2.2 we provide insight on this concept, with the example of the social amoeba Dictyostelium discoideum being the most suitable for serving this purpose. To conclude this first part of the introduction, we present in Subsection 1.2.3 the evolution of mathematical modeling of cell polarity throughout the years and we summarize the critical features that respective models must possess. Starting from Turing, most of the content of this subsection is an exposition of well-known modeling treatments in the literature.

Section 1.3 introduces the particular model for cell polarization that we investigate in this thesis and constitutes the origin of our analysis. Here we provide an in depth description of the bulk-surface reaction-diffusion system of partial differential equations and its components, in accordance with the notions from cell biology described earlier. The stationary solutions of this model have been studied in [51]. It is proven that in suitable scaling limits, steady states converge to solutions of elliptic obstacle type problems. The first part of the present thesis, complements this work by considering the full time-dependent problem. Hence, in the following Subsection 1.3.1 we address the evolutionary obstacle problems that we obtain in the respective limits and which later motivated our first publication [44]. This is a joint work with Barbara Niethammer, Matthias Röger and Juan J. L. Velázquez and has appeared in the **SIAM Journal on Mathematical Analysis**. We outline this work in Chapter 3 while a thorough derivation of these results can be found in Appendix A.

The parabolic obstacle type problems that we obtain in the limit belong to the broad class of free boundary problems. In Section 1.4 we clarify this correlation, providing alternative formulations of the classical obstacle problem and justifying the in between equivalence. Furthermore, since we mostly address these problems as non-local free boundary problems, we explain in Subsection 1.4.1 the reasoning behind this characterization.

In the remaining part of this thesis, namely Chapter 4 and Chapter 5, we focus our attention on the non-local free boundary problem that we obtain in the case of infinite cytosolic diffusivity, described in Subsection 1.3.1. Although we prove in [44] that the problem is well-posed, these results imply only the global existence and uniqueness of solutions to (1.3.1.1)-(1.3.1.3) with the non-local term $\alpha \in L^\infty(0,T)$ without providing much insight about the evolution of the support of the solution. The difficulties that we addressed throughout our former work due to the lack of regularity of the set $\{u(\cdot,t) > 0\}$ as well as of the functional $\alpha(t)$ inspired the second part of this thesis.

To be more precise, the remaining part is related to the qualitative properties of the free boundary and the relevant introduction to that topic is described in Section 1.5 and its follow-up Subsection 1.5.1. In the beginning we mention several prominent results in the literature of regularity of the free boundary for the classical parabolic obstacle problem. Yet, the particular structure of our limit obstacle problem, owing to the fact that the term $\alpha(t)$ depends in a non-local way on the support of the solution, does not allow us to employ already well known techniques in our analysis. Therefore, in this section we describe the necessary and sufficient conditions that are required in order to derive regularity results similar to the ones obtained for the classical obstacle problem.

Section 1.5 serves as an introduction to our latest publication [46] which is a joint work with Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and thus far can be found in the **arXiv**. We summarize the corresponding results in Chapter 4 while a rigorous exposition is presented in Appendix B. In Subsection 1.5.1, we address our ongoing work (cf. [45]) which can be considered as a sequel of [46]. A detailed description of the framework as well as of the respective results are given in Chapter 5.

At the very end of this introductory chapter, we also provide an overview of the structure of this thesis, cf. Section 1.6 and we further highlight the novelty of our research in the field compared with previous work.

1.2 The biological framework

1.2.1 Cell polarization

Cell polarization is in many instances a dynamic, time-dependent process and the term itself pertains to the asymmetric cell shape due to reorganization of several chemicals, some of them attached to the cell membrane and others contained in the cytoplasm. Roughly speaking, polarized patterns are described by the property in which heterogeneous distributions of chemical substances emerge in polarized cells, in contrast to their former existent uniform distribution before polarization occurs.

In [58], the authors address several causes of symmetry breaking in cells that lead to polarization. Although the onset of polarity is often induced by cells' response to extracellular chemical sources, other non-chemical factors are also essential in the regulation of cell shape. A noteworthy example of such non-chemical cues that enable cells to polarize is membrane tension. This particular physical stimulus is correlated to the spontaneous polarization of, especially elongated, cells. In principle, cells can polarize in response to chemical, electrical, mechanical, or other physical stimuli.

In the present thesis, we focus our attention in polarization which is influenced only by chemical factors. For cell polarity occurring from the coupling of both mechanical and chemical cues, see for instance [38], [73]. Cell polarization in response to some external chemical stimulus has a fundamental contribution in numerous biological processes, such as migration, development and organization of eukaryotic cells [58]. A prominent example that illustrates the role of cell polarization and its regulation is tissue repair (wound healing).

Tissues are composed of coordinated cells that form a particular shape with a specific structure and organisation. The ability of a tissue to maintain its shape and structure throughout the lifespan of an organism, as well as to withstand and overcome damage, is of vital importance [27]. In the general case of wounded skin, the repairing process (known as re-epithelialisation) is driven by multiple polarized cell behaviours. It has been shown that a proliferating ring forms surrounding the damaged area but away from the wound edge. Within this ring, cell division is increased and oriented towards the wound in order to replenish the lost population of cells [55]. Simultaneously, to restore the initial stratified architecture of the skin, cells rearrange, proliferate, flatten, elongate and migrate in the direction of injury [cf. [27],[55],[61]].

1.2.2 The concept of chemotaxis

Throughout the process of wound healing, the ability of cells to not only polarize but also migrate, plays a significant role in driving gap closure after injury and re-establishing tissue integrity and architecture. Actually, in a wide variety of biological processes, cells have developed the capacity to polarize in response to extracellular chemical sources and then migrate toward chemo-attractants or away from chemorepellants [40]. This process is referred to as *chemotaxis* and tissue repair constitutes only one among a myriad biological functions, in which chemotaxis is critical.

Cells respond to continuously changing environmental conditions by means of receptors which are embedded in the cell membrane. Many eukaryotic cells can detect both the magnitude and direction of external signals, due to plasma membrane receptors. In case a spatially nonuniform extracellular signal is detected, the term *taxis* refers to the directional motion of the cells, either up or down the gradient of the signal. Whenever this signal is a diffusible molecule, the directed motion of cells is called chemotaxis [12].

As indicated in [12], the concept of chemotaxis can be comprised of three interdependent processes, namely direction sensing, polarization and random motility. The term direction sensing concerns the mechanism by which chemical gradients are detected and amplified, providing an internal compass for the cell. During the process of polarization, the cell establishes its asymmetric shape with a well-defined front and back, a state that is prerequisite for organizing the machinery that powers cell motility in the direction of the polarity axis without an external stimulus. This axis is often referred to as the long-axis of the cell. In the absence of an external signal, cells can extend random pseudopodia and 'diffuse' locally, a process known as random motility.

The link between the former mentioned processes can be described briefly as follows. The step of direction sensing proceeds by the transduction of a signal by receptors on the plasma membrane and its adaption by intracellular signaling cascades, which involve the activation and deactivation of specific proteins and the translation of possibly shallow gradients in the outer signal to large amplitude intracellular gradients in protein distributions. Once such polarity of the cell in form a of a spatial asymmetry in chemical concentrations has been established, changes in cell shape and the movement of the cell in the surrounding environment can be initiated (cf. [44]).

One striking and well-studied example over the years is the chemotaxis of the social amoeba Dictyostelium discoideum which belongs to a wide range of eukaryotic cells that are capable of moving individually, contributing particularly to many biological functions. For instance, fibroblasts, responsible for the process of tissue repair, are also included in the same selection of cells.

Dictyostelium amoebae, upon starvation, initiate the spontaneous production of specific chemo-attractant centers that spread spatially as reaction-diffusion travelling chemical waves. The amoebae, in the surrounding area of these initiation centers, sense the chemo-attractant and direct their chemotactic movements towards them. Near these centers, Dictyostelium amoebae rotate around a spontaneously formed hole, where the cell density increases locally to form aggregates. The developmental program of the amoebae culminates with the formation of terminally differentiated fruiting bodies [cf. [14], [36], [68]].

1.2.3 On the mathematical modeling of polarization

Concentration of chemical substances that differ by only $\sim 2\%$ across the cell body can be sufficient in order for cells to initiate their directed motion [76]. Since cell polarization is essential for the process of chemotaxis, a question that is naturally raised is what mechanism drives relatively weak chemical gradients to yield large spatial changes of the concentrations of chemicals at the cellular level. This is a basic issue that polarization models must address and several modeling treatments of varying complexity have been suggested to analyze the spatial and temporal processes associated with the polarity of cells.

The spark that initiated the development of a significant number of models used to describe cell polarization over the past years is pattern formation theory. Turing, whose name is synonymous with this theory due to his seminal work in [72], discovered the possibility of generating patterns by the interaction of two substances, driven by diffusion instability. More specifically, Turing suggested that under certain conditions, chemicals with different diffusion rates can interact in such a way as to produce spatially heterogeneous stable patterns starting from a homogeneous, uniform initial state. In his work, it is further indicated that no pattern formation occurs in the absence of diffusion.

Turing derived his innovative theory by more abstract mathematical considerations. However, it can be shown, see [[40], [47]], that his principle postulates the generation of stable inhomogeneous patterns by means of a self-enhanced local production of an activator that also regulates the production of a long-range inhibitor. Gierer and Meinhardt managed to adapt this idea in [25], providing one of the first modeling treatments of cell polarization, the concept of which, is correlated to the theory of biological pattern formation for the first time.

Their theory relied on the interaction between a short range activator and a long range inhibitor which leads to pattern formation from an almost homogenous initial condition. In particular, they proposed a minimal, yet molecularly plausible model of reaction-diffusion type, which consists of local activation for the amplification of a small external chemical signal and long-range inhibition in order to prevent the activation from spreading throughout the whole domain unchecked. Since signal amplification is the key element in the process of direction sensing which contributes to the establishment of polarized patterns in cells, the model in [25] was considered suitable for the description of cell polarization.

The former modeling idea fueled the curiosity of many developmental biologists who tried to explain how polarity of cells is formed in the following years. Shortly they discovered though, that there is a lack of practicability in the model suggested in [25]. This stems from the fact that the main underlying theory, influenced by Turing, predicted the formation of only stable activation patterns. Such a feature cannot serve efficiently the process of gradient sensing in cells and by extension cell polarization, since the continuously changing environment of cells requires for a signaling machinery capable of readjustment.

The need of destabilizing the polarized patterns once they had been generated in cells, motivated Meinhardt to revisit the proposed activator-inhibitor model. To this end, he introduced in [47] a second inhibitor to the system that acts locally with a longer activation time. The upgraded mechanism that he suggested, widely known as local excitation-global inhibition (LEGI), accounts for one of the most popular models for cell polarization and led to a proliferation of more comprehensive models which yield the specific chemical patterns associated to cell polarization [cf.[12], [40], [53]].

Numerous mathematical models proposed for cell polarity, have been motivated by small GTPase biology. GTP-binding proteins (GTPases) run a continuous cycle between activation/deactivation and membrane-attachment/membrane detachment, that is quite widespread in diverse families of GTPases. The particular ability of activated GTPase proteins to localize in parts of the membrane, contributes significantly in the regulation of cell shape [50], for instance in the budding of yeast [54] and in the formation of microdomains in continuous membranes [56], [71] .

Typically polarization is achieved by the combination of an internal pattern forming system, a response to an external signal, usually from the outer cell membrane, that imposes some directional preference to the pattern and the amplification of small concentration differences through transport processes and interacting networks both within the cell and on the cell membrane. We focus on a minimal model for the amplification step that has been proposed in [51].

1.3 A bulk-surface reaction-diffusion system

This section includes parts of the introduction of the paper [44], written jointly by Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and the author.

The proposed bio-chemical model consists of a system of PDEs, motivated by the GTPase cycle model presented in [59, 60]. We consider a protein that can be in an active or an inactive state, where the inactive protein moreover can be bound to the cell membrane or be in a cytosolic state, i.e. contained in the cells interior. We denote the surface concentration of the active and incative form by u and v, respectively, and the volume concentration of the inactive cytosolic state by w. The model has only a few ingredients. It accounts for lateral diffusion on the cell membrane, for diffusion inside the cell, for activation and deactivation processes on the cell membrane and for attachment to and detachment from the cell membrane.

More specifically, our model contains three types of activation processes of the proteins which all take place on the cell membrane. First there is an intrinsic activation with rate a_1. Second there is an activation by a positive feedback mechanism and a rate law given by a Michaelis-Menten law. Third, there is an activation induced by an external chemical signal. We assume here that this signal has already been processed and has lead to a concentration field c on the membrane of a chemical that catalyzes the activation (the function c could be also interpreted as the surface concentration of some activated receptors). This concentration in general may vary with space and time. For the deactivation we again prescribe a Michaelis-Menten rate law. The use of Michaelis-Menten laws stems from the fact that the corresponding processes require some catalyzation, as the intrinsic activation and deactivation of GTPase proteins is typically very slow [4].

To give a mathematical formulation, we represent the cell and its outer cell membrane by a domain $\Omega \subset \mathbb{R}^3$ and its boundary $\Gamma := \partial\Omega$. Moreover we fix a time interval $(0,T)$ of observation and a signal concentration $c : \Gamma \times (0,T) \to \mathbb{R}$. The assumptions described above give rise to the following bulk-surface reaction diffusion system.

$$\partial_t u = \Delta_\Gamma u + \Big(a_1 + \frac{a_2 u}{a_3 + u} + c\Big)v - \frac{a_4 u}{1+u} \qquad \text{on } \Gamma \times (0,T)\,, \tag{1.3.1}$$

$$\partial_t v = \Delta_\Gamma v - \Big(a_1 + \frac{a_2 u}{a_3 + u} + c\Big)v + \frac{a_4 u}{1+u} - a_5 v + a_6 w \qquad \text{on } \Gamma \times (0,T)\,, \tag{1.3.2}$$

$$\partial_t w = D\Delta w \qquad \text{in } \Omega \times (0,T)\,, \tag{1.3.3}$$

$$-D\frac{\partial w}{\partial \nu} = -a_5 v + a_6 w \qquad \text{on } \Gamma \times (0,T)\,. \tag{1.3.4}$$

Here $\Delta_\Gamma u$ and $\Delta_\Gamma v$ denote the Laplace-Beltrami operator on the surface Γ and $a_1, \ldots, a_6$ are nonnegative constants while D denotes the quotient of the cytosolic diffusion and the lateral membrane diffusion constants, which typically is very large. Throughout our analysis we assume that both active and inactive proteins diffuse on the membrane with the same rate. However we stress that having different diffusion rates for u and v would not affect the subsequent analysis, since the diffusion of the inactive protein on the membrane vanished in our scaling limit (cf. (3.3.3)). Furthermore, setting $f_1(u) := a_1 + \frac{a_2 u}{a_3+u}$ and $f_2(u) := \frac{u}{1+u}$, we note that in principle both f_1, f_2 could be replaced by any continuously differentiable increasing functions such that $f_1(0) \geq 0, f_2(0) = 0$ with f_1 becoming constant and f_2 having a positive limit as u becomes large. In that case, the rescaling in (cf. (3.3.1)) is then adapted to $(f_1, f_2) \rightsquigarrow (f_1, \epsilon^{-1} f_2)$.

We complement the system with initial conditions:

$$u(\cdot,0) = u_{\text{in}}\,, \quad v(\cdot,0) = v_{\text{in}} \quad \text{on } \Gamma\,, \quad w(\cdot,0) = w_{\text{in}} \quad \text{in } \Omega\,, \tag{1.3.5}$$

where $u_{\text{in}}, v_{\text{in}} : \Gamma \to [0,\infty)$ and $w_{\text{in}} : \Omega \to [0,\infty)$ are given nonnegative data.

The system (1.3.1)-(1.3.4) contains two parts. On the one hand, we have a reaction-diffusion system on the membrane for the variables u and v, with a w-dependent source term. On the other hand, there is a diffusion equation for w in the interior of the cell with a nonlinear Robin-type boundary condition that depends on u and v. Solutions of (1.3.1)-(1.3.5) satisfy the mass conservation property

$$\int_\Omega w(\cdot,t)\,dx + \int_\Gamma \Big(u(\cdot,t) + v(\cdot,t)\Big)\,dS = \int_\Omega w_{\text{in}}\,dx + \int_\Gamma \Big(u_{\text{in}} + v_{\text{in}}\Big)\,dS \tag{1.3.6}$$

for all $t \in (0,T)$.

In addition to (1.3.1)-(1.3.4) we will study a reduced system that is obtained in the limit of infinite cytosolic diffusivity, which is motivated by the fact that cytosolic diffusion within the cell is by a factor of hundred larger than the lateral diffusion on the membrane [34]. In this limit the cytosolic concentration becomes spatially constant and $w = w(t)$ is determined by the total mass conservation, i.e.

$$|\Omega| w(t) = m - \int_\Gamma \Big(u(\cdot,t) + v(\cdot,t)\Big)\,dS\,, \tag{1.3.7}$$

where m is the total amount of protein. The reduction for $D = \infty$ leads to a nonlocal reaction-diffusion system on $\Gamma \times (0,T)$, given by (1.3.1), (1.3.2) and (1.3.7), complemented by initial conditions for u and v. This reduction can be viewed as a kind of shadow system. Such systems have been analyzed intensively in the case of two-variable reaction-diffusion systems in open domains [28, 33, 42], and in the context of obstacle problems in [64].

1.3.1 Large reaction rate limit

This section is basen on the introduction of the paper [44] written jointly by Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and the author.

We parametrize the reaction rates a_4, a_5 and a_6, the diffusion coefficient D and the total mass of proteins by a large parameter $\varepsilon^{-1} > 0$. A distinctive attribute of the model (1.3.1)-(1.3.5) is that a suitably rescaled version of the solution converges as in the *large reaction rate limit* $\varepsilon \to 0$ to solutions of certain reduced systems. An almost immediate and natural question arising is how these fast reactions are justified for extremely large rates a_4, a_5 and a_6. Yet, there are no experimental data that could provide insight in this direction. However, the assistance of these particular rate parameters in the clear distinction of regions in which the concentrations of some chemicals have different orders of magnitude is crucial. First, we will investigate the limit of infinite cytosolic diffusion $D \to \infty$. After appropriate rescaling and renaming (cf. (3.3.1)), taking the limit $\varepsilon \to 0$, yields the following parabolic obstacle-type problem

$$\partial_t u - \Delta_\Gamma u = -a_4(1-g)\xi + \alpha g \qquad \text{on } \Gamma \times (0,T)\,, \tag{1.3.1.1}$$

$$u \geq 0\,, \quad u\xi = u\,, \quad 0 \leq \xi \leq 1 \qquad \text{on } \Gamma \times (0,T)\,, \tag{1.3.1.2}$$

$$u(\cdot, 0) = u_0 \qquad \text{on } \Gamma\,, \tag{1.3.1.3}$$

where u_0 is the limit of suitably rescaled versions of u_{in}, the function $g : \Gamma \times (0,T) \to (0,1)$ is given by

$$g(x,t) = \frac{c(x,t)}{c(x,t) + a_5}\,, \tag{1.3.1.4}$$

and $\alpha : (0,T) \to \mathbb{R}$ only depends on time and is determined by a solvability condition for (3.2.2), see (3.2.5). This function α plays the role a Lagrange multiplier associated to the mass conservation property

$$\int_\Gamma u(\cdot,t)\,dS = \int_\Gamma u_0\,dS \qquad \text{for all } t \in (0,T)\,,$$

that is satisfied in the limit.

In the case $D < \infty$ equation (1.3.1.1) changes and we obtain the system

$$\partial_t u = \Delta_\Gamma u - a_4(1-g)\xi + a_6 g w \qquad \text{on } \Gamma \times (0,T)\,, \tag{1.3.1.5}$$
$$0 = \Delta w \quad \text{in } \Omega_T, \qquad \text{on } \Gamma \times (0,T)\,, \tag{1.3.1.6}$$
$$D\frac{\partial w}{\partial n} = a_4(1-g)\xi - a_6 g w \qquad \text{on } \Gamma \times (0,T)\,, \tag{1.3.1.7}$$
$$u \geq 0\,, \quad u\xi = u\,, \quad 0 \leq \xi \leq 1 \qquad \text{on } \Gamma \times (0,T)\,, \tag{1.3.1.8}$$
$$u(\cdot,0) = u_0 \qquad \text{on } \Gamma\,. \tag{1.3.1.9}$$

The analogy to $D = \infty$ is even more apparent if one expresses w as a non-local operator of u. A particularly convenient form is presented in [44, Proposition 2.7].

The problems (1.3.1.1)-(1.3.1.3) and (1.3.1.5)-(1.3.1.9) provided the foundation of our first publication [44], a detailed outline of which is presented in Chapter 3. The main contributions are a rigorous justification of the asymptotic reduction, the well-posedness of the evolutionary obstacle-type problem that we obtain in the limit, an L^1-contraction property of solutions, and the global stability of steady states.

1.4 Obstacle type problems

This section is based on both introductions of papers [44] and [46] written jointly by Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and the author.

Obstacle type problems are still an active field of research. In particular, parabolic obstacle type problems appear in various applications and have been studied thoroughly over the past decades [24].

For the first time, Duvaut in [15] suggests a suitable transformation for the one-phase Stefan problem, by means of which, he obtains a formulation in terms of a parabolic obstacle problem. Another well known instance arises in the context of fluid flows in porous media, where the Baiocchi transform [2] also leads to an obstacle problem.

Obstacle problems are tightly correlated with free boundary problems. More precisely, obstacle problems belong to a class of free boundary problems that can be formulated as variational inequalities, that is inequalities for bilinear functionals which are satisfied for functions u and test functions in a space satisfying inequalities of the form $u \geq \psi$. On the other hand, under appropriate regularity assumptions it is possible to reformulate the same class of free boundary problems as partial differential equations in which an unknown function ξ satisfies an inequality almost everywhere in the set in which the partial differential equations are solved. Both formulations can be found for example in [37, 62]. The equivalence between both approaches can be seen using the so-called Stampacchia Lemma [62, Section 5:3, Theorem 5:4.3].

In principle, a classical parabolic obstacle problem is given by

$$\begin{cases} u \geq 0, \\ \partial_t u - \Delta u \geq f, \\ \partial_t u - \Delta u = f \quad \text{in } \{u > 0\} \end{cases} \tag{1.4.1}$$

(see for example [66, Section 3.1]). However, as we mentioned earlier, it can always be rephrased as a parabolic variational inequality [5]

$$u \geq 0, \quad (\partial_t u - \Delta u)(v - u) \geq f(v - u) \text{ a.e., for any } v \geq 0. \tag{1.4.2}$$

We notice that under suitable (parabolic Sobolev) regularity assumptions the system (1.4.1) generalizes the classical formulation

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \{u > 0\}\,, \\ f \leq 0 & \text{in } \{u = 0\}\,, \\ u = \nabla u = 0 & \text{on } \partial\{u > 0\}\,. \end{cases} \tag{1.4.3}$$

Throughout this thesis we will only use the equivalent to having a variational inequality approach. On this account, an auxiliary function $\xi \in [0, 1]$ such that $\xi = 1$ in $\{u > 0\}$ must be determined along with the unknown u.

In order to understand better how the problems that we obtain in the *large reaction rate limit* are associated with the parabolic obstacle problems, we focus on the reduced model (1.3.1.1)-(1.3.1.2). In [44, Remark 2.3] we derive the following characterization of solutions,

$$\partial_t u - \Delta u + \Big(a_4(1-g) - \alpha g\Big) = \Big(a_4(1-g) - \alpha g\Big)_+ \mathcal{X}_{\{u=0\}}, \qquad u \geq 0, \tag{1.4.4}$$

with $\alpha = \alpha(t)$ given as a non-local function of u, more precisely

$$\alpha(t) = \frac{a_4 \int_{\{u(\cdot,t)>0\}} \Big(1 - g(\cdot, t)\Big)\, dS}{\int_{\{u(\cdot,t)>0\}} g(\cdot, t)\, dS}\,.$$

In the formulation (1.4.4) the problem corresponds to the classical parabolic obstacle model, where $a_4(1-g) - \alpha g$ is replaced by some given function f independent of u. Defining $\mathcal{F}(u) := \partial_t u - \Delta u + \Big(a_4(1-g) - \alpha g\Big)$, the problem (1.4.4) can be written as

$$u\mathcal{F}(u) = 0, \qquad \mathcal{F}(u) \geq 0, \qquad u \geq 0,$$

and can be expressed as a variational inequality, see for example [43, Section II.9.1].

1.4.1 Non-local free boundary problems

This section is based on the introduction of paper [44] written jointly by Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and the author.

A distinguished feauture of the free boundary problems that we obtain in the limit of both infinite and finite cytosolic diffusivity, is the presence of some terms in the equations that depend in a non-local way on the solution u itself. To be more precise, the problem (1.3.1.1)-(1.3.1.3) establishes the non-locality through the dependence of α in (1.4.4) on the positivity set $\{u > 0\}$. On the other hand, in the case of the bulk-surface problem (1.3.1.5)-(1.3.1.9), the non-local dependence takes place through the function w which solves the elliptic problem (1.3.1.6), (1.3.1.7).

1.5 Regularity of the free boundary

This section is based on the introduction of paper [46] written jointly by Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and the author.

Over the years, many mathematicians engaged with either evolutionary or stationary obstacle problems, have been allured by the study of fine regularity properties of the free boundary. In the context of the present thesis, free boundary is described by the boundary between the support of the solution and the coincidence set $\{u = 0\}$.

Brezis and Friedman prove in [6] the existence of solutions to problem (1.4.1) for $f \in L^\infty(\mathbb{R}^n \times (0,T))$ with $\partial_t f \in L^\infty(\mathbb{R}^n \times (0,T))$ and prescribed initial data u_0 given by a finite, positive measure. Moreover, if f is strictly negative, more precisely $f \leq -\nu$ for some constant $\nu > 0$, and u_0 has compact support then the solution has compact support for all times. Under more restrictive conditions on u_0 it is further shown that the support of $u(\cdot,t)$ has distance at most of order $\sqrt{t}$ from the support of u_0. The proof relies on careful comparison arguments, see also [18] for an alternative (and simpler) approach. Slightly weaker estimates on the support for a larger class of obstacle problems have been derived by probabilistic methods in [3].

The origin of respective analysis appears for the first time in the context of the classical one-phase Stefan problem. In particular, a specific parabolic obstacle problem [16] of the form (1.4.1), where $f = -1$ and the additional sign restriction $\partial_t u \geq 0$ arise. In this case, the free boundary consists of two set of points, namely the regular and the singular ones. The set of regular points includes all points where the coincidence set $\{u = 0\}$ has positive density. Whereas cusps may occur in singular points, in regular points the free boundary is locally smooth [7], see also the exposition in [23, Section 2.9].

In the case $f = -1$ but without any additional assumptions on the sign of $\partial_t u$ it is proved that the boundary is $C^{1,1}$-regular in space and $C^{0,1}$-regular in time [11]. Moreover, around regular points (defined in terms of a lower density function) the free boundary is C^∞-regular in space and time.

Essential part of the proof of such regularity properties relies on local monotonicity formulas and blow ups. We refer to [10], [20], [21], [66] and the references therein for more recent developments and extensions to more general operators.

Half of this thesis is concerned with the study of qualitative properties of the solutions to the non-local free boundary problem that is obtained in Section 1.3.1 as an asymptotic reduction for a cell polarization model. In particular, we restrict our analysis to a simplified version of the problem (1.3.1.1)-(1.3.1.3), in which g is assumed to be a time-independent function.

Without loss of generality, rescaling u and α accordingly in (1.3.1.1), we can further set $a_4 = 1$.

We then consider a triplet (u, ξ, α) of functions $u, \xi : \Gamma_T \to \mathbb{R}$ and $\alpha : (0, T) \to \mathbb{R}$ that solve the following problem in an almost everywhere sense,

$$\partial_t u - \Delta_\Gamma u = -(1-g)\xi + \alpha g \qquad \text{on } \Gamma_T\,, \tag{1.5.1}$$

$$u \geq 0\,, \quad u\xi = u\,, \quad 0 \leq \xi \leq 1 \qquad \text{on } \Gamma_T\,, \tag{1.5.2}$$

$$u(\cdot, 0) = u_0 \qquad \text{on } \Gamma\,. \tag{1.5.3}$$

and that satisfy the following compatibility condition (guaranteeing mass conservation for u)

$$\alpha(t) = \frac{\int_{\{u(\cdot,t)>0\}}(1-g)\,dS}{\int_{\{u(\cdot,t)>0\}} g\,dS} \qquad \text{for } t \in (0, T)\,. \tag{1.5.4}$$

In (1.5.1) the notation Δ_Γ stands for the Laplace-Beltrami operator associated to the surface Γ.

It turns out that there are two conditions that play a crucial role in proving either continuity or jumps of the set $\{u(\cdot, t) > 0\}$ as $t \to 0^+$. More precisely, we introduce the following assumptions on the initial data. We assume that for some fixed $\theta > 0$ it holds a first non-degeneracy condition

$$(1-g)-\alpha_0 g \geq \theta > 0 \quad \text{in } \{u_0 = 0\} \;\text{ where }\; \alpha_0 := \frac{\int_{\{u_0>0\}}(1-g)\,dS}{\int_{\{u_0>0\}} g\,dS} \tag{1.5.5}$$

and further we prescribe a second non-degeneracy condition

$$\mathcal{H}^2\Big(\partial\{u_0 > 0\}\Big) = 0\,, \tag{1.5.6}$$

where $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure in $\mathbb{R}^3$.

The motivation behind the choice of (1.5.5) and (1.5.6) stems from the following observation. The problem (1.5.1), (1.5.2) can be reformulated in the form

$$\begin{cases} u \geq 0, \\ \partial_t u - \Delta u \geq -(1-g) + \alpha g, \\ \partial_t u - \Delta u = -(1-g) + \alpha g \quad \text{in } \{u > 0\} \end{cases} \tag{1.5.7}$$

As a consequence, it can be compared with the classical obstacle problem given by (1.4.1) and its generalized formulation given by (1.4.3). In the case of problem (1.5.7), the term $(1-g)-\alpha g$ plays the role of $-f$ in (1.4.1). The first condition is therefore clearly related to the condition $f \leq -\nu < 0$ for the classical obstacle problem that has been present in all the regularity results stated above. The necessity of such a condition for the regularity of solutions and its free boundary is well known and can for example be seen from an application of the Hopf boundary point lemma [9]. The same condition also appears as a stability condition for the free boundary and estimates on the symmetric difference of the support of different solutions (see [8] and the exposition in [63, Chapter 6]).

Regarding the problem (1.5.7), if $(1-g)-\alpha_0 g \leq -\theta < 0$, it follows from the strong maximum principle that $u(\cdot,t)$ becomes strictly positive for small positive times. Then, the interface $\partial\{u(\cdot,t) > 0\}$ experiences a jump and the same is true for α. Moreover, by the differential inequality in (1.5.7) we obtain at least formally in $\{u(\cdot,t) = 0\}$ for almost all t that

$$0 \geq -(1-g) + \alpha g.$$

It is worth pointing out though that in general the right-hand side of (1.5.1) does not have a sign, since its integral over the support of u vanishes.

Regarding the second nondegeneracy condition, we prove in [46, Appendix B, Lemma B.2] that (1.5.6) is equivalent to

$$\mathcal{H}^2\Big(\big(\{u_0 > 0\}\big)_{+\delta} \setminus \big(\{u_0 > 0\}\big)_{-\delta}\Big) \to 0 \quad \text{as } \delta \to 0\,, \tag{1.5.8}$$

where

$$\big(\{u_0 > 0\}\big)_{+\delta} := \{x \,|d(x, \big(\{u_0 > 0\}\big)) \leq \delta\} \,, \quad \big(\{u_0 > 0\}\big)_{-\delta} := \{x \,|d(x, \big(\{u_0 = 0\}\big)) \geq \delta\} \,.$$

In our analysis in [46] mainly the formulation (1.5.8) is used.

Such a condition seem not to be required for problem (1.4.3) but appears to be quite significant for problem (1.5.7). In future work (cf. [45]), we will provide an example of initial data u_0 such that (1.5.5) holds while (1.5.8) is not satisfied. We expect that in this case, the function $\alpha(t)$ can not be continuous at $t = 0$ and the positivity set $\{u(\cdot, t) > 0\}$ is oscillatory as $t \to 0^+$.

The necessity of this second condition in our analysis is a consequence of the particular structure of the right-hand side in (1.5.7) and its dependence on the positivity set $\{u(\cdot, t) > 0\}$ through the non-local functional $\alpha(t)$, whereas f in (1.4.3) is a general function of space and time.

Under assumptions (1.5.5) and (1.5.6) our first major result is that the set $\{u(\cdot, t) > 0\}$ and therefore the non-local term $\alpha(t)$ are continuous at $t = 0$. That was the starting point for our second publication in [46], a detailed summary of which can be found in Chapter 4. Besides continuity, we further investigate the jump of the support at $t = 0$ in case (1.5.5) is violated but (1.5.6) is valid.

1.5.1 Global continuity of the interface

Thus far, we have proved that $\{u(\cdot, t) > 0\}$ converges as $t \to 0^+$ to $\{u_0 > 0\}$ in a suitable topology, under assumptions (1.5.5) and (1.5.8). Since the non-local term $\alpha(t)$ depends on the support of the solution (cf. (1.5.4)), this convergence implies in particular that the functional $\alpha(t)$ is continuous at $t = 0$.

Nevertheless, the question of whether the former continuity results can be improved arises naturally now. The answer to that question is positive and a detailed proof lies in Chapter 5. In fact, Chapter 5 closely follows the forthcoming article, namely [45]. It consists of though, only a part of it, since the rigorous construction of a counterexample of initial data for which (1.5.5) is valid but (1.5.8) fails, is not included here.

For the purpose of our analysis, we restrict ourselves to the special case of the unit sphere $\mathcal{S}^2$ and to axisymmetric data and axisymmetric solutions. We further assume that (1.5.5) and (1.5.8) are valid. It turns out that, if we impose some additional assumptions on the initial data u_0, we can deduce global continuity of both the positivity set $\{u(\cdot, t) > 0\}$ and the non-local term α.

1.6 Overview

This section contains parts of both introductions of papers [44] and [46] written jointly by Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and the author.

The layout of the current thesis is as follows. In Chapter 2 we introduce the notation that is used throughout this work and we further provide some lemmas that will play a crucial role throughout our analysis.

As it has been already stated, the significance of the model that has been investigated in this thesis stems from the fact that in a suitable parameter regime an asymptotic reduction leads to a generalized obstacle-type problem that allows for a clear and mathematically tractable characterization of polarized states. In [51] the authors have analyzed stationary states and the onset of polarization. In Appendix A which coincides with our first publication [44], we continue this analysis to some extent, by considering the time-dependent problem. More precisely, we prove that in a suitable asymptotic limit the system (1.3.1)-(1.3.5) converges to a bulk-surface parabolic obstacle type problem. For this model and a reduction to a non-local surface equation we show an L^1-contraction property and, in the case of time-constant signals, the stability of stationary states. We outline thoroughly the content of Appendix A in Chapter 3.

The parabolic obstacle problem that is obtained in [44] involves a specific nonlinear non-local term that can be seen as a (time dependent) Lagrange multiplier ensuring mass conservation. The particular dependence of this term on the solution (more specifically on the *support* of the solution) makes the analysis quite challenging. In Appendix B which coincides with our second publication [46], we focus on this specific parabolic obstacle problem and we investigate continuity properties of the Lagrange multiplier and of the (compact) support of the solutions. We present necessary and sufficient conditions for the initial data that imply continuity of the support at $t = 0$. If one of these assumptions fail, then jumps of the support take place. In addition we provide a complete characterization of the jumps for a large class of initial data. We summarize the content of Appendix B in detail in Chapter 4.

In Chapter 5 we improve the continuity results obtained in Chapter 4 and hence, the last chapter of this thesis can be considered as a continuation of the analysis in Chapter 4. We prove that under some additional assumptions on the initial data, along with the necessary and sufficient conditions introduced in Chapter 4, the support of the solution is continuous for all $t \geq 0$.

Let us conclude now this section by underlining the features that distinguish this work from any previous existent literature and hence advance the knowledge in the field.

With regard to the mathematical model (1.3.1)-(1.3.5) proposed for cell polarization that we study in the beginning of this thesis (cf. Section 1.3), we should point out that it is quite different from LEGI-type models (cf. Subsection 1.2.3), and rather describes the signal amplification following a first polarization of the cell, expressed by a heterogeneous distribution c. Indeed, in contrast to the classical Gierer-Meinhardt models [cf. [25]] solutions of the proposed model, under suitable scaling limits, do not exhibit spontaneous polarized patterns if the signal c is constant. Yet, in the absence of an external chemical signal, the model under consideration is closely related to models for spontaneous cell polarization considered in [30], [60].

We have already mentioned that the respective stationary model has been studied in [51]. In particular, in addition to well-posedness for the elliptic obstacle problems which the authors obtain in corresponding scaling limits (cf. Section 1.3.1), the onset of polarization is studied in [51] for sufficiently small (rescaled) mass of protein. In the interpretation of a cell polarization model as described in Section 1.3, the positivity set $\{u(\cdot,t) > 0\}$ corresponds to regions where the concentration of a chemical is high, while the set $\{u(\cdot,t) = 0\}$ indicates those regions where the concentration of such a chemical is low.

Although the initial purpose of [44] was to complement the analysis in [51] by extending most of the results, especially the ones concerning the well-posedness of the limit obstacle problems to the parabolic case, significant innovations have been developed in addition to the former knowledge. More precisely, compared to [51] the main novelty of [44] is to introduce some monotonicity formulas which allow us to prove uniqueness of solutions and also uniqueness and stability of steady states of the problems (1.3.1.1)-(1.3.1.3) and (1.3.1.5)-(1.3.1.8).

Uniqueness of the steady states associated to the problem (1.3.1.1)-(1.3.1.3) has been proved in [51] using a completely different approach. Similar uniqueness results have been obtained in [51] for the stationary states of (1.3.1.5)-(1.3.1.8) in the particular case in which the domain Ω is a ball.

We recall that the second part of this thesis is concerned with the qualitative properties of solutions to the non-local obstacle problem that we derive in [44] for $D = \infty$. In Section 1.4 we provide a reformulation of (1.5.1)-(1.5.2) in terms of (1.5.7). Compared to the subject of [46], there is a key difference between problem (1.5.7) and the classical obstacle problem (1.4.1) (cf. Section 1.4), which makes the analysis in this paper quite challenging and leads in turn to some interesting results. More specifically, the right-hand side of (1.5.7) differs from the corresponding right-hand side of (1.4.1) in the sense that in the first, $\alpha(t)$ depends on the positivity set $\{u(\cdot,t)>0\}$. It is worth noticing that, in principle, problems with a nonlinear and non-local dependence $f := f(u)$ have already been studied. However, in most works this dependence is either local or only includes a dependence on $\int u$, which leads to much better continuity properties and is much easier to control than a dependence on the support of u. A notable exception is the work [65], where also a dependence on the size of the support is present. Yet, the particular non-local dependence of the functional $\alpha(t)$ on the solution $u(\cdot,t)$, does not allow us to use already existent techniques similar to the ones used for problem (1.4.3) or in the more general cases just mentioned.

CHAPTER 2

Notation and Preliminaries

To begin with, let us summarize here the notation that is relevant for all of the thesis.

- For an open, bounded and connected set $\Omega \subset \mathbb{R}^3$ we denote by $|\Omega| = \mathcal{L}^3(\Omega)$ the Lebesgue measure and by $\int_\Omega \cdot \, d\mathcal{L}^3$ the corresponding volume integral. For a smooth orientable hypersurface $\Gamma \subset \mathbb{R}^3$ we denote by $|\Gamma| = \mathcal{H}^2(\Gamma)$ its area (i.e. the 2-dimensional Hausdorff measure) and by $\int_\Gamma \cdot \, dS$ the corresponding surface integral.

- For subsets $A \subset \Gamma$ we denote by $\mathcal{X}_A$ the standard characteristic function of the set A. For $x_0 \in \Gamma$ and $\rho > 0$ we denote by $B_\rho(x_0)$ the ball on the hypersurface Γ with respect to the extrinsic (Euclidean) distance in $\mathbb{R}^3$. We remark that for a smooth hypersurface Γ, the intrinsic (geodesic) and the extrinsic distances induce equivalent metrics.

- For the sake of convenience, Ω_T and Γ_T stand for $\Omega \times (0, T)$ and $\Gamma \times (0, T)$ respectively.

- We denote the usual Sobolev spaces by $W^{k,p}(U)$ and the parabolic Sobolev spaces by $W_p^{k,k/2}(U_T)$, where $U = \Omega$ or $U = \Gamma$, $k \in \mathbb{N}_0$, $1 \leq p \leq \infty$. The Hölder and parabolic Hölder spaces are denoted by $C^\alpha(U)$ and $C^{\alpha,\alpha/2}(U_T)$, respectively, for $0 < \alpha < 1$. Moreover, $C^m(U)$ denotes the space of all functions $f : U \to \mathbb{R}$ which admit continuous partial derivatives $\partial^\alpha f$ in U for each multi-index α with $|\alpha| \leq m$. The weak parabolic solution spaces are denoted by $V_2(U_T) := L^2(0, T; H^1(U)) \cap$

$H^1(0,T;H^1(U)^*)$.

- For two vectors $a, b \in \mathbb{R}^3$ we denote by $a \cdot b$ the Euclidean inner product in $\mathbb{R}^3$ and by $a \otimes b$ the corresponding tensor product in $\mathcal{B}(\mathbb{R}^3)$, where $\mathcal{B}(\mathbb{R}^3)$ is the space of all bounded linear operators from $\mathbb{R}^3$ to $\mathbb{R}^3$. We further denote by $a_\perp$ the perpendicular vector to a.

- Let ν be the smooth outer unit normal field on Γ. We denote by $P = \mathrm{Id} - \nu \otimes \nu$ the tangential projections. Since any map $f \in C^1(\Gamma)$ can be extended to a map $\hat{f} \in C^1(\mathbb{R}^3)$, we define the tangential derivative in i-th coordinate direction and the gradient of f by
$$\overline{D}_i f = P_{ij}\partial_j \hat{f}, \qquad \nabla_\Gamma f = \left(\overline{D}_1 f, \ldots, \overline{D}_3 f\right)^T = P\nabla \hat{f}. \tag{2.0.1}$$
It turns out that $\overline{D}_i f$ does not depend on the choice of continuation (cf. [17, Lemma 2.4]). The shape operator of Γ is given by
$$\mathcal{H} : \Gamma \to \mathbb{R}^{3\times 3}, \quad \mathcal{H}_{ij} = -\overline{D}_i \nu_j, \quad \mathcal{H} = -P(D\hat{\nu})^T. \tag{2.0.2}$$
It is easily shown that the matrix $\mathcal{H}$ is symmetric with a zero eigenvalue on direction of ν. For the mean curvature H of Γ we obtain
$$H = \operatorname{tr}\mathcal{H} = -\nabla_\Gamma \cdot \nu = -\nabla \cdot \hat{\nu}. \tag{2.0.3}$$
For the second tangential derivatives of a function $f \in C^2(\Gamma)$ we obtain
$$\overline{D}\,\overline{D} f = PD^2\hat{f}P + \mathcal{H}\nabla\hat{f} \cdot \nu + (\mathcal{H}\nabla_\Gamma f) \otimes \nu \tag{2.0.4}$$
and therefore, the Laplace-Beltrami operator of f is defined as
$$\Delta_\Gamma f := \nabla_\Gamma \cdot \nabla_\Gamma f = \operatorname{tr}\overline{D}\,\overline{D} f = \Delta\hat{f} - \nu \cdot D^2\hat{f}\nu + H\nabla\hat{f} \cdot \nu. \tag{2.0.5}$$

- For the Laplace-Beltrami operator on Γ we just write Δ instead of Δ_Γ if there is no reason for confusion. We recall that the relevant diffusion operator on Γ is the corresponding Laplace-Beltrami operator, see for example [57]. In local coordinates the Laplace-Beltrami operator corresponds to an elliptic operator in divergence form (with C^2-regular coefficients in our case). One can deduce parabolic maximum principles in analogy to [22, Chapter 2] for evolution problems on Γ involving the Laplace-Beltrami operator.

In the remaining part of this chapter, we collect some lemmas, which appear to be useful for the analysis in both Chapters 4 and 5.

We define $H : \mathbb{R} \to (0,1)$, $H = \mathcal{X}_{(0,\infty)}$ as the characteristic function of the positive real numbers. By means of [44, Remark 2.3], there is an equivalent and more convenient way to write (1.5.1), given by the following lemma. The practicality of this formulation will be apparent later.

Lemma 2.0.1. *Let $\lambda \in L^\infty(0,T)$ be defined by*

$$\lambda(t) = \fint_{\{u(\cdot,t)>0\}} g \, dS \; . \tag{2.0.6}$$

Then equations (1.5.1), (1.5.2) *are equivalent to*

$$\partial_t u - \Delta u = -\left(1 - \frac{g}{\lambda}\right) H(u) \quad \text{on } \Gamma_T \; , \tag{2.0.7}$$

$$g \leq \lambda \quad \text{almost everywhere in } \{u = 0\} \; . \tag{2.0.8}$$

The proof of this lemma follows by a a straightforward computation and can be found in [46]. Several of the regularity results that we have obtained in the second half of this thesis, are due to this equivalent formulation.

By (2.0.6), it is obvious that the function λ introduces the non-locality in our problem. Actually, it is correlated to the non-local term α by

$$\alpha(t) = \frac{1}{\fint_{\{u(\cdot,t)>0\}} g \, dS} - 1 = \frac{1}{\lambda(t)} - 1 \; . \tag{2.0.9}$$

The former characterization can be easily seen using (1.5.4).

A key result that we often use is the following *nondegeneracy* lemma. This lemma plays an essential role in the analysis of free boundary problems and states that if a solution to (1.4.3) is small in a sufficiently large open set, then it vanishes in a smaller set (cf. [6]). More precisely,

Lemma 2.0.2. *Let $t_1 \in [0,T)$, $t_2 \in (t_1,T]$ and $U_1 := \{u(\cdot,t_1) = 0\}$. Suppose that for some nonnegative function $u \geq 0$ it holds that $\partial_t u - \Delta u \leq -\theta$ for some $\theta > 0$ in $U_1 \times [t_1,t_2]$. Let $x_0 \in \mathring{U}_1$ and $\rho \in (0, \rho_{\max}(\Gamma))$ such that $B_{2\rho}(x_0) \subset \Gamma \cap U_1$. Then there exists $A > 0$ such that the following holds. Suppose that we have*

$$u \leq \frac{\theta}{A}\rho^2 \qquad \text{in } B_{2\rho}(x_0) \times [t_1,t_2]$$

then

$$u = 0 \qquad \text{in } B_\rho(x_0) \times [t_1,t_2] \, .$$

This lemma as well as a regularized version of it, given by [46, Lemma 3.3], contributes significantly in the continuity results that we obtain in [46] (cf. Chapter 4, Section 4.3 for more details) and Chapter 5 for the support of the solution. The proof of Lemma 2.0.2 relies mostly on careful comparison arguments and follows analogously to Lemma 3.3 in [46].

Moreover, in Chapter 5 we restrict ourselves to the spherical case $\Gamma = \mathcal{S}^2$ and to the case that all data and the solution are axisymmetric with respect to the first coordinate axis. To this end, we provide here some explicit calculations concerning axisymmetric functions on the unit sphere $\mathcal{S}^2$ which will be useful in the later analysis.

Lemma 2.0.3. *Let $f : \mathcal{S}^2 \to \mathbb{R}$ be an axisymmetric function on the unit sphere $\mathcal{S}^2$, $f(p) = \tilde{f}(x)$, $p = (x, y, z)$. Then the following properties hold.*

(i) *$f \in C^1(\mathcal{S}^2)$ if and only if $\tilde{f} \in C^1((-1, 1))$ with $\lim_{r\downarrow 0} r\tilde{f}'(\pm\sqrt{1-r^2}) = 0$. In this case*

$$\nabla f(p) = \begin{pmatrix} 1-x^2 \\ -xy \\ -xz \end{pmatrix} \tilde{f}'(x)$$

and $\nabla_{\mathcal{S}^2} f\Big((\pm 1, 0, 0)\Big) = 0$.

(ii) *$f \in C^2(\mathcal{S}^2)$ if and only if $\tilde{f} \in C^2((-1, 1)) \cap C^1([-1, 1])$ with*

$$\lim_{r\downarrow 0} r^2 \tilde{f}''(\pm\sqrt{1-r^2}) = 0. \tag{2.0.10}$$

(iii) *For $f \in C^2(\mathcal{S}^2)$ we have*

$$\overline{D}\,\overline{D} f(p) = \tilde{f}''(x)(\vec{e}_1 - xp) \otimes (\vec{e}_1 - xp) - \tilde{f}'(x)(x \operatorname{Id} + \vec{e}_1 \otimes p) + 2x\tilde{f}'(x) p \otimes p.$$

and

$$\overline{D}\,\overline{D} f(\pm\vec{e}_1) = \mp\tilde{f}'(\pm 1)(\operatorname{Id} - \vec{e}_1 \otimes \vec{e}_1).$$

(iv) *The Laplace-Beltrami operator on $\mathcal{S}^2$ is for $f \in C^2(\mathcal{S}^2)$ given by*

$$\Delta_{\mathcal{S}^2} f(p) = \frac{d}{dx}\Big((1-x^2)\tilde{f}'(x)\Big), \quad p = (x, y, z) \in \mathcal{S}^2\,. \tag{2.0.11}$$

(v) *For $f \in L^1(\mathcal{S}^2)$ the integral over $\mathcal{S}^2$ can be computed as*

$$\int_{\mathcal{S}^2} f\, dS\; = 2\pi \int_{-1}^{1} \tilde{f}\, dx\,. \tag{2.0.12}$$

(vi) *It holds $f \in W^{1,q}(\mathcal{S}^2)$ if and only if $\tilde{f} \in L^q(-1, 1) \cap W^{1,q}_{\text{loc}}(-1, 1)$ with $x \mapsto \sqrt{1-x^2}\,\tilde{f}'(x) \in L^q((-1, 1))$. Then*

$$\|\nabla f\|^q_{L^q(\mathcal{S}^2)} = \int_{-1}^{1} (1-x^2)^{\frac{q}{2}} |\tilde{f}'(x)|^q\, dx.$$

(vii) It holds $f \in W^{2,q}(\mathcal{S}^2)$ if and only if $\tilde{f} \in W^{1,q}(-1,1) \cap W^{2,q}_{\mathrm{loc}}(-1,1)$ with $x \mapsto (1-x^2)\tilde{f}''(x) \in L^q((-1,1))$.

Proof. The outer unit normal field of $B(0,1)$ on $\mathcal{S}^2$ is given by $\nu(p) = p$ and can be extended to $\mathbb{R}^3$ by $\hat{\nu}(p) = p$. Therefore, the tangential projection is given by $P(p) = \mathrm{Id} - p \otimes p$, the shape operator by $-P$ and the mean curvature with respect to ν by -2.

We can extend f to $[-1,1] \times \mathbb{R}^2$ by setting $\hat{f}(x,y,z) = \tilde{f}(x)$. Therefore $f \in C^k(\mathcal{S}^2 \setminus \{(\pm 1, 0, 0)\})$ if and only $\tilde{f} \in C^k((-1,1))$. For $f \in C^1(\mathcal{S}^2)$ and $p = (x,y,z) \in \mathcal{S}^2$, $x \in (-1,1)$ we in particular have

$$\nabla f(p) = (\mathrm{Id} - p \otimes p)\begin{pmatrix} \tilde{f}'(x) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-x^2 \\ -xy \\ -xz \end{pmatrix} \tilde{f}'(x).$$

We next consider the parametrization $\psi : B^2(0,1) \to \mathcal{S}^2_+$, $w \mapsto (\sqrt{1-|w|^2}, w)$ with $\mathcal{S}^2_+ = \{p \in \mathcal{S}^2, p_1 > 0\}$. It holds $f \in C^2(\mathcal{S}^2_+)$ if and only if $f \circ \psi = \tilde{f}(\sqrt{1-|w|^2}) \in C^2(B^2(0,1))$. For $0 < |w| < 1$ we compute

$$\nabla(f \circ \psi)(w) = \tilde{f}'(\sqrt{1-|w|^2})\frac{-w}{\sqrt{1-|w|^2}}.$$

Since f has a local extremum at $\pm\vec{e}_1$ we deduce the properties stated in the first claim.

Similarly we have for $0 < |w| < 1$

$$D^2(f \circ \psi)(w) = \tilde{f}''(\sqrt{1-|w|^2})\frac{w \otimes w}{1-|w|^2} - \tilde{f}'(\sqrt{1-|w|^2})\frac{1}{\sqrt{1-|w|^2}^3}\Big((1-|w|^2)\,\mathrm{Id} + w \otimes w\Big).$$

and in particular

$$\Big(D^2(f \circ \psi)(w) - A\Big)\frac{w_\perp}{|w|} = -\Big(\tilde{f}'(\sqrt{1-|w|^2})\frac{1}{\sqrt{1-|w|^2}}\,\mathrm{Id} + A\Big)\frac{w_\perp}{|w|}.$$

Hence, if $f \in C^2(\mathcal{S}^2)$ then the left-hand side with $A = D^2(\tilde{f} \circ \psi)(0)$ converges to zero as $w \to 0$, which first shows the existence of $\tilde{f}'(1) = \lim_{r \downarrow 0} \tilde{f}'(\sqrt{1-r^2})$ with

$$-\tilde{f}'(1)\,\mathrm{Id} = D^2(f \circ \psi)(0)$$

and then

$$-2\tilde{f}'(1) = \lim_{r \downarrow 0} r^2 \tilde{f}''(\sqrt{1-r^2}) - 2\tilde{f}'(1),$$

hence $\lim_{r \downarrow 0} r^2 \tilde{f}''(\sqrt{1-r^2}) = 0$.

Vice versa we obtain that if $\tilde{f} \in C^2((-1,1))$ and (2.0.10) is satisfied, then $f \in C^2(\mathcal{S}^2)$ holds.

Using the extension $\tilde{f}$ of f we compute in $p = (x, y, z)$, $-1 < x < 1$

$$\begin{aligned}\overline{D}\,\overline{D}f(p) &= (\mathrm{Id} - p\otimes p)\tilde{f}''(x)\vec{e}_1\otimes\vec{e}_1(\mathrm{Id} - p\otimes p)\\ &\qquad - (\mathrm{Id} - p\otimes p)\tilde{f}'(x)\vec{e}_1\cdot p - (\mathrm{Id} - p\otimes p)\tilde{f}'(x)\vec{e}_1\otimes p\\ &\qquad = \tilde{f}''(x)(\vec{e}_1 - xp)\otimes(\vec{e}_1 - xp) - \tilde{f}'(x)(x\,\mathrm{Id} + \vec{e}_1\otimes p) + 2x\tilde{f}'(x)p\otimes p.\end{aligned}$$

In particular, with $p \to \pm\vec{e}_1$, $p \in \mathcal{S}^2$ and (2.0.10) we obtain

$$\overline{D}\,\overline{D}f(\pm\vec{e}_1) = \mp\tilde{f}'(\pm 1)(\mathrm{Id} - \vec{e}_1\otimes\vec{e}_1).$$

For the Laplace-Beltrami operator we compute

$$\begin{aligned}\Delta_{\mathcal{S}^2} f &= \mathrm{tr}\,\overline{D}\,\overline{D}f(p) = [(1-x^2)^2 + (xy)^2 + (xz)^2]\tilde{f}''(x) - 4x\tilde{f}'(x) + 2x(x^2+y^2+z^2)\tilde{f}'(x)\\ &= (1-x^2)\tilde{f}''(x) - 2x\tilde{f}'(x)\\ &= \Big((1-x^2)\tilde{f}'(x)\Big)'.\end{aligned}$$

At this point, we show the representation of the integral. Using the Gauss divergence theorem, Fubini and a partial integration we obtain

$$\begin{aligned}\int_{\mathcal{S}^2} f\,dS &= \int_{B(0,1)} \nabla\cdot\Big(f(p)p\Big)\,d\mathcal{L}^3\\ &= \int_{-1}^{1}\int_{B^2(0,\sqrt{1-x^2})} \frac{1}{x^2}\Big(x^3\tilde{f}(x)\Big)'\,d\mathcal{L}^2\,dx\\ &= \pi\int_{-1}^{1}\frac{1-x^2}{x^2}\Big(x^3\tilde{f}(x)\Big)'\,dx \;=\; 2\pi\int_{-1}^{1}\tilde{f}(x)\,dx.\end{aligned}$$

Finally, due to the first item we estimate

$$\begin{aligned}|\nabla f(p)|^2 &= \Big((1-x^2)^2 + (xy)^2 + (xz)^2\Big)|f'(x)|^2 =\\ &= \Big((1-x^2)^2 + x^2(1-x^2)\Big)|f'(x)|^2\\ &= (1-x^2)|f'(x)|^2\,,\end{aligned}$$

which in particular yields that

$$|\nabla f(p)| = (1-x^2)^{1/2}|f'(x)|\,.$$

Combining this and (2.0.12) the sixth item follow. For the very last item we proceed in a similar way by means of (2.0.11).

■

Chapter 3

A parabolic free boundary problem modeling cell polarization

This chapter outlines the results obtained in [44], written jointly by Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and the author.

This peer-reviewed paper is given in the first part of the appendix, Chapter A and it has appeared in the **SIAM Journal on Mathematical Analysis**.

3.1 Motivation

A minimal model for cell polarization as a response to some external chemical gradients has been proposed in [51]. This model, given by (1.3.1)-(1.3.5), consists of a system of partial differential equations for different variants of a protein on the cell membrane and the cytosol respectively. We discuss the model under consideration in detail in the introductory Section 1.3.

The most remarkable feature of model (1.3.1)-(1.3.5) is that for a suitable choice of parameters, an asymptotic reduction leads to a generalized obstacle-type problem by means of which, polarized patterns can be mathematically fully characterized.

The authors in [51] study stationary states of this model for a time-constant external chemical signal. Under the assumption that several reaction rates on the membrane as well as the diffusion coefficient within the cell are large (particularly of order $\varepsilon^{-1} > 0$), they prove that steady states converge as $\varepsilon \to 0$ to solutions of some elliptic obstacle type problems. Besides the well-posedness of these limit problems, the onset of polarization for sufficiently small (rescaled) mass of protein is studied in [51].

Polarization however is in many cases a dynamic process and therefore the investigation of the time-dependent problem arises almost naturally after the study in [51]. More specifically, we compliment the former analysis by extending a part of the results to the parabolic case while we further prove an L^1-contraction property for solutions that appears to be essential in proving uniqueness of solutions and the fact that steady states are globally stable.

3.2 Main results

To begin with, we consider the model (1.3.1)-(1.3.5) where $c = c(c,t)$ corresponds to the time-dependent external chemical signal that plays an important role in the activation processes of the GTP-ase proteins. We study solutions of (1.3.1)-(1.3.5) in certain parameter regimes similar to the ones introduced in [51], in which the reaction rates a_4, a_5 and a_6, the diffusion coefficient D as well as the total mass of proteins are of order $\varepsilon^{-1} > 0$.

In the *large reaction rate limit* $\varepsilon \to 0$, we prove that solutions of this model converge to solutions of parabolic obstacle type problems analogous to the ones obtained in [51], where the stationary case was investigated. Actually, we consider two different types of scaling limits for solutions of (1.3.1)-(1.3.5). In the first one we assume that $D = \infty$ before taking the limit $\varepsilon \to 0$. That is motivated by the fact that the cytosolic diffusion is typically much larger than the lateral diffusion over the membrane [35].

Although we introduce these limit problems in Section 1.3.1, let us also recall them here for the sake of completeness. We define for simplicity the function $g : \Gamma_T \to (0,1)$ as

$$g(x,t) = \frac{c(x,t)}{c(x,t) + \alpha_5} \tag{3.2.1}$$

and we further assume that the signal c and hence the function g are smooth functions bounded from below by a strictly positive constant. After appropriate rescaling and renaming, we prove the existence of solutions to the following parabolic obstacle-type problems in the limit $\varepsilon \to 0$.

Theorem 3.2.1 (The parabolic obstacle-type problem for $D = \infty$)**.** *There exists a triplet (u, ξ, α) of functions $u \in V_2(\Gamma_T)$ with $u \in W_p^{2,1}(\Gamma \times (\delta, T))$ for any $\delta > 0$, $\xi \in L^\infty(\Gamma_T)$ and $\alpha \in L^\infty(0,T)$ that solve the following problem in an almost everywhere sense*

$$\partial_t u - \Delta_\Gamma u = -a_4(1-g)\xi + \alpha g \qquad \text{on } \Gamma_T\,, \tag{3.2.2}$$

$$u \geq 0\,, \quad u\xi = u\,, \quad 0 \leq \xi \leq 1 \qquad \text{on } \Gamma_T\,, \tag{3.2.3}$$

$$u(\cdot,0) = u_0 \qquad \textit{on } \Gamma\,. \tag{3.2.4}$$

The function α is determined by a solvabilility condition for (3.2.2)*, that is*

$$\alpha(t) = \frac{\int_\Gamma a_4(1-g)(\cdot,t)\xi(\cdot,t)\,dS}{\int_\Gamma g\,dS} = \frac{\int_{\{u(\cdot,t)>0\}} a_4(1-g)(\cdot,t)\,dS}{\int_{\{u(\cdot,t)>0\}} g(\cdot,t)\,dS} \tag{3.2.5}$$

for almost all $t \in (0,T)$.

Theorem 3.2.2 (The parabolic obstacle-type problem for $D < \infty$)**.** *There exists a triplet (u,ξ,w) of functions $u \in V_2(\Gamma_T)$ with $u \in W_p^{2,1}(\Gamma \times (\delta,T))$ for any $\delta > 0$, $\xi \in L^\infty(\Gamma_T)$ and $w \in L^2(0,T;H^1(\Omega))$ that solve the following problem in an almost everywhere sense*

$$\partial_t u = \Delta_\Gamma u - a_4(1-g)\xi + a_6 g w \qquad \textit{on } \Gamma_T\,, \tag{3.2.6}$$

$$0 = \Delta w, \qquad \textit{in } \Omega_T\,, \tag{3.2.7}$$

$$D\frac{\partial w}{\partial n} = a_4(1-g)\xi - a_6 g w \qquad \textit{on } \Gamma_T\,, \tag{3.2.8}$$

$$u \geq 0\,, \quad u\xi = u\,, \quad 0 \leq \xi \leq 1 \qquad \textit{on } \Gamma_T\,, \tag{3.2.9}$$

$$u(\cdot,0) = u_0 \qquad \textit{on } \Gamma\,. \tag{3.2.10}$$

Remark 3.2.3. We highlight that there is in fact a clear correspondence between problems (3.2.2)-(3.2.4) and (3.2.6)-(3.2.10). In [44, Proposition 2.7], we provide a representation formula for w in terms of a non-local operator that depends on the positivity set $\{u(\cdot,t) > 0\}$. As a matter of fact, the function w in the case of finite diffusion, plays the role of α.

The former theorems not only extend the elliptic obstacle-type problems presented in [51, Theorem 3.2] and [51, Theorem 4.2] to the parabolic case, but also serve as a springboard for the rest of the analysis in [44] which differs from [51] and rather advances the previous results. To be more precise, we justify for both infinite and finite diffusion, some monotonicity formulas that allow us to prove uniqueness of solutions and also uniqueness and global stability of steady states of the problems (3.2.2)-(3.2.4) and (3.2.6)-(3.2.10).

Theorem 3.2.4 (L^1-contraction for $D = \infty$ and $D < \infty$)**.** *Let (u_1,ξ_1,α_1) and (u_2,ξ_2,α_2) be two different solutions of* (3.2.2)-(3.2.4) *while (u_1,ξ_1,w_1) and (u_2,ξ_2,w_2) are two different solutions of* (3.2.6)-(3.2.10)*. Then,*

$$t \mapsto \int_\Gamma (u_1 - u_2)_+(\cdot,t)\,dS \textit{ is decreasing on } [0,T].$$

In particular, given $u_0 \geq 0$, there exists at most one solution (u,ξ,α), (u,ξ,w) of (3.2.2)-(3.2.4) *and* (3.2.6)-(3.2.10) *respectively.*

Remark 3.2.5. The monotonicity formula implies that the evolution semigroup associated to the problems (3.2.2)-(3.2.4) and (3.2.6)-(3.2.9) is contractive in the L^1 norm.

Combining the well-posedness results that we first prove for (3.2.2)-(3.2.4) and (3.2.6)-(3.2.9) with Theorem 3.2.4, we show that for any given initial data there exists a unique solution of the parabolic obstacle-type problems for all times $t \geq 0$.

To conclude our study in [51], we restrict our analysis to a time-independent external signal $c := c(x)$ and hence by (3.2.1) to a function $g := g(x)$ that does not depend on time as well. By means of Theorem 3.2.4, we prove convergence of the parabolic obstacle-type problems for both infinite and finite diffusion to the stationary state with the same mass. Before proceeding to the last result of this chapter, we recall the definition of steady states for both $D = \infty$ and $D < \infty$.

Definition 3.2.6. In the case $D = \infty$, we let (u_*, ξ_*, α_*) be the unique stationary solution of the obstacle-type problem given by

$$-\Delta u_* = -(1-g)\xi_* + \alpha_* g\,, \qquad \text{on } \Gamma_T \tag{3.2.11}$$

$$u_* \geq 0\,, \quad 0 \leq \xi_* \leq 1\,, \quad \xi_* u_* = u_*\,, \qquad \text{on } \Gamma_T\,. \tag{3.2.12}$$

In the case $D < \infty$, (u_*, ξ_*, w_*) denotes the unique stationary solution of the following problem

$$\partial_t u_* = \Delta u_* - (1-g)\xi_* + g w_*, \quad u_* \xi_* = u_*, \quad u_* \geq 0 \qquad \text{on } \Gamma_T \tag{3.2.13}$$

$$0 = \Delta w_* \text{ in } \Omega, \quad \frac{\partial w_*}{\partial n} = (1-g)\xi_* - g w_* \qquad \text{on } \Gamma_T\,. \tag{3.2.14}$$

Remark 3.2.7. For the case of infinite diffusion, we stress that existence and uniqueness of steady states for any prescribed mass was proved in [51].

For the case of finite diffusion, we can also show uniqueness of steady states for given mass m with similar arguments as in the proof of Theorem 3.2.4. This result has been shown in [51] only in the case that Γ is a sphere. In [44, Theorem 4.2] we prove even more, namely a monotonicity result from which uniqueness of steady states follows.

The next theorem is concerned with the fact that the unique steady states (u_*, ξ_*, α_*) and (u_*, ξ_*, w_*) as given in Definition 3.2.6 are globally stable.

Theorem 3.2.8 (Global stability of steady states for $D = \infty$ and $D < \infty$)**.** *The unique solutions (u, ξ, α) and (u, ξ, w) of problems (3.2.2)-(3.2.4) and (3.2.6)-(3.2.9) respectively, converge as $t \to \infty$, to the unique stationary solutions (u_*, ξ_*, α_*) and (u_*, ξ_*, w_*) with $\int_\Gamma u_* \, dS = m$.*

3.3 Comments on proofs

We now shortly outline the ideas behind the proofs of the previously mentioned theorems.

We begin with the convergence of system (1.3.1)-(1.3.5) to the obstacle-type problems for infinite and finite diffusion, cf. Theorem 3.2.1 and Theorem 3.2.2. Since the proofs for both theorems follow along the same lines, we only discuss here the underlying ideas behind the proof of Theorem 3.2.1.

To this end, we consider the case of infinite diffusion. As we have already explained in the introductory Section 1.3, in the limit $D \to \infty$, (1.3.1)-(1.3.5) reduces to a non-local reaction-diffusion system on Γ_T, given by (1.3.1),(1.3.2) and (1.3.7), complemented by initial conditions for u and v. Then for small $\varepsilon > 0$ we introduce the following rescalings

$$a_4 \rightsquigarrow \frac{a_4}{\varepsilon}, \quad a_5 \rightsquigarrow \frac{a_5}{\varepsilon}, \quad a_6 \rightsquigarrow \frac{a_6}{\varepsilon}, \quad c \rightsquigarrow \frac{c}{\varepsilon} \quad \text{and} \quad m \rightsquigarrow \frac{m}{\varepsilon}\,. \tag{3.3.1}$$

The corresponding solutions are denoted by $u_\varepsilon, v_\varepsilon$ and w_ε and let $U_\varepsilon := \varepsilon u_\varepsilon$. Hence, the system (1.3.1),(1.3.2) and (1.3.7) can be rewritten as

$$\partial_t U_\varepsilon = \Delta U_\varepsilon + \Big(\varepsilon a_1 + \frac{\varepsilon a_2 U_\varepsilon}{\varepsilon a_3 + U_\varepsilon} + c\Big) v_\varepsilon - \frac{a_4 U_\varepsilon}{\varepsilon + U_\varepsilon} \qquad \text{on } \Gamma_T\,, \tag{3.3.2}$$

$$\varepsilon \partial_t v_\varepsilon = \varepsilon \Delta v_\varepsilon - \Big(\varepsilon a_1 + \frac{\varepsilon a_2 U_\varepsilon}{\varepsilon a_3 + U_\varepsilon} + c\Big) v_\varepsilon + \frac{a_4 U_\varepsilon}{\varepsilon + U_\varepsilon} - a_5 v_\varepsilon + a_6 w_\varepsilon \qquad \text{on } \Gamma_T\,, \tag{3.3.3}$$

$$\varepsilon |\Omega| w_\varepsilon(t) = m - \int_\Gamma (U_\varepsilon(x,t) + \varepsilon v_\varepsilon(x,t))\, dS \qquad \text{for a.a. } t \in (0,T)\,. \tag{3.3.4}$$

In [44, Theorem 2.1] we obtain some suitable uniform estimates for the triplet $(U_\varepsilon, v_\varepsilon, w_\varepsilon)$ independent of ε. More precisely, we prove that

$$\|U_\varepsilon\|_{V_2(\Gamma_T)} + \|v_\varepsilon\|_{L^\infty(0,T;L^2(\Gamma))} + \|w_\varepsilon\|_{L^\infty(0,T)} \le C\,, \tag{3.3.5}$$

where C denotes a constant that depends on the data of the problem but not on ε. These estimates ensure the existence of weak convergent subsequences. Therefore, we can pass to the limit $\varepsilon \to 0$ in (3.3.2),(3.3.3) and (3.3.4) and obtain problem (3.2.2)-(3.2.4).

We now proceed to the proof of Theorem 3.2.4, which refers to the monotonicity formulas that we obtain for both obstacle problems. Let (u_1, ξ_1, α_1) and (u_2, ξ_2, α_2) be two different solutions of (3.2.2)-(3.2.4) while (u_1, ξ_1, w_1) and (u_2, ξ_2, w_2) are two different solutions of (3.2.6)-(3.2.10).

We set $\mathcal{X}_+ := \mathcal{X}_{\{u_1>u_2\}}$. Integrating then the equation for the difference $u_1 - u_2$ over $\{u_1 > u_2\}$ yields

$$\frac{d}{dt}\int_\Gamma (u_1-u_2)_+\,dS \le -\int_\Gamma \mathcal{X}_+(1-g)(\xi_1-\xi_2)\,dS + (\alpha_1-\alpha_2)\int_\Gamma \mathcal{X}_+ g\,dS\,, \tag{3.3.6}$$

$$\frac{d}{dt}\int_\Gamma (u_1-u_2)_+\,dS \le -\int_\Gamma \mathcal{X}_+(1-g)(\xi_1-\xi_2)\,dS + \int_\Gamma \mathcal{X}_+(w_1-w_2)g\,dS\,. \tag{3.3.7}$$

The existence of the monotonicity formulas in Theorem 3.2.4 rely on a delicate balance of the terms $-a_4(1-g)\xi$ and αg, $a_5 gw$ in (3.2.2), (3.2.6). The term $-a_4(1-g)\xi$ has a stabilizing effect, which is similar to the analogous term arising in the study of the reformulation of the one-phase Stefan problem due to Duvaut [15]. On the other hand the terms αg, $a_5 gw$ in (3.2.2), (3.2.6) depend on functions determined as a non-local functional of u (namely α and w respectively). These terms have a destabilizing effect on the solutions of the problems (3.2.2)-(3.2.4) and (3.2.6)-(3.2.9), but some cancellations between the contributions of both terms in the derivative of the L^1 norm of the difference of two solutions of these problems yield an overall stabilizing effect.

To conclude this section, we emphasize once more the significance of the L^1-contraction property that we infer in Theorem 3.2.4. This particular property plays an essential role not only in the proof of uniqueness of solutions but also in stability of the stationary states. Thus, Theorem 3.2.8 can be as well considered a corollary of Theorem 3.2.4, since the monotonicity formulas are key points in the proof of this theorem.

Chapter 4

Qualitative properties of solutions to a non-local free boundary problem modeling cell polarization

This chapter outlines the results obtained in [46], written jointly by Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and the author.

This paper is given in the second part of the appendix, Chapter B and for the moment appears in the **arXiv**.

4.1 Motivation

In [44], which has been outlined in Chapter 3, we obtain particular parabolic obstacle-type problems as an asymptotic reduction for a cell polarization model in response to some external chemical signal. Throughout this work, the signal has been assumed to be a smooth function. However, as indicated by laboratory experiments, an oscillatory external signal, would be much more realistic and natural to consider.

To this end, we restricted our attention to the resulting parabolic obstacle problem obtained for infinite cytosolic diffusion in Theorem 3.2.1, considering this time an external chemical source which is periodic in time. More specifically, we focused on the obstacle problem (3.2.2)-(3.2.4) that involves the non-local term α given by(3.2.5). Applying standard homogenization techniques, the main difficulty that arose almost immediately in the homogenized equation was the control of the function α which depends on the support of the solution.

In principle, several of the technical difficulties that we addressed even in our first work in [44], were due to the fact that the function α changes in a discontinuous manner if the positivity set $\{u(\cdot,t) > 0\}$ changes discontinuously in time. Therefore, this lack of regularity for the non-local term α motivated our most recent work in [46].

To be more precise, in our second publication we investigate time continuity properties of the function $t \mapsto \alpha(t)$. We prove that under suitable assumptions on the initial data, the set $\{u(\cdot,t) > 0\}$ and hence the non-local term α, change continuously in time. Moreover, we justify potential jumps of the support of the solution and of the non-local function α if the former assumptions fail.

4.2 Main results

For the purpose of the present analysis, we consider a simplified version of the parabolic obstacle problem (3.2.2)-(3.2.4) that we obtain in the limit $D \to \infty$, in which the external chemical signal does not depend on time. We conclude that there are two assumptions on the initial data that play an essential role in proving either continuity or jumps of the support of the solution.

A thorough description of the setting of [46] as well as the motivation behind the choice of these assumptions has been introduced in Section 1.5. However, to make this chapter comprehensive, we will recall once again the problem under consideration along with the imposed assumptions.

Let g be as in (3.2.1) but this time is assumed to be a smooth time-independent function. We investigate the following problem

$$\partial_t u - \Delta_\Gamma u = -(1-g)\xi + \alpha g \qquad \text{on } \Gamma_T\,, \tag{4.2.1}$$

$$u \geq 0\,, \quad u\xi = u\,, \quad 0 \leq \xi \leq 1 \qquad \text{on } \Gamma_T\,, \tag{4.2.2}$$

$$u(\cdot,0) = u_0 \qquad \text{on } \Gamma\,, \tag{4.2.3}$$

where the function $\alpha : (0,T) \to \mathbb{R}$ satisfies the compatibility condition given by

$$\alpha(t) = \frac{\int_{\{u(\cdot,t)>0\}} (1-g)\, dS}{\int_{\{u(\cdot,t)>0\}} g\, dS} \qquad \text{for } t \in (0,T)\,. \tag{4.2.4}$$

We assume that the initial data u_0 are smooth and we impose also some further assumptions on them. In particular, for some fixed $\theta > 0$, a first non-degeneracy condition holds, namely

$$(1-g) - \alpha_0 g \geq \theta > 0 \quad \text{in } \{u_0 = 0\} \text{ where } \alpha_0 := \frac{\int_{\{u_0>0\}} (1-g)\, dS}{\int_{\{u_0>0\}} g\, dS}\,. \tag{4.2.5}$$

Moreover we prescribe a second non-degeneracy condition, given by (1.5.6). In [46, Appendix B, Lemma B.2] we prove an equivalent formulation of (1.5.6), that we mainly use throughout our analysis and is given by

$$\mathcal{H}^2\Big(\big(\{u_0>0\}\big)_{+\delta}\setminus\big(\{u_0>0\}\big)_{-\delta}\Big)\to 0 \quad \text{as } \delta\to 0\ , \tag{4.2.6}$$

where

$$\big(\{u_0>0\}\big)_{+\delta}:=\{x\ |d(x,\big(\{u_0>0\}\big))\le\delta\}\ ,\quad \big(\{u_0>0\}\big)_{-\delta}:=\{x\ |d(x,\big(\{u_0=0\}\big))\ge\delta\}\ . \tag{4.2.7}$$

Assuming that both (4.2.5) and (4.2.6) hold, we prove that the function $\alpha(t)$ is continuous at $t=0$. In addition, we show that the positivity set $\{u(\cdot,t)>0\}$ changes continuously as $t\to 0^+$. In fact, we prove the following theorem.

Theorem 4.2.1. *Suppose that* (4.2.5) *and* (4.2.6) *hold true for some* $\theta>0$. *Then for any arbitrary small* $\eta>0$, *there exists* $\bar t=\bar t(\eta;\theta,u_0,g)>0$ *such that*

$$\big(\{u_0>0\}\big)_{-\eta}\subset\{u(\cdot,t)>0\}\subset\big(\{u_0>0\}\big)_{+\eta} \tag{4.2.8}$$

and

$$|\alpha(t)-\alpha_0|\le\eta \tag{4.2.9}$$

for all $t\in[0,\bar t]$.

Remark 4.2.2. We notice that in Chapter 2, Lemma 2.0.1, we provide an equivalent formulation for (4.2.1), (4.2.2) given by (2.0.7), (2.0.8). Hence, all continuity properties that have been justified for the function α in Theorem 4.2.1, hold for the function λ as well (cf. (2.0.9)).

For the sake of convenience, in the following we choose to study instead of (4.2.1)-(4.2.3) the equivalent problem

$$\partial_t u-\Delta u=-\Big(1-\frac{g}{\lambda(t)}\Big)H(u) \qquad \text{on } \Gamma_T\ , \tag{4.2.10}$$

$$\lambda\le g \qquad \text{a.e in } \{u=0\}\ , \tag{4.2.11}$$

$$u(\cdot,0)=u_0 \qquad \text{on } \Gamma\ . \tag{4.2.12}$$

The rest of the analysis in this chapter, highlights the necessity of (4.2.5) for the continuity of the function λ and of the support of the solution u. More precisely, we show that if (4.2.5) fails, then one cannot expect continuity of the function λ nor of the set $\{u(\cdot,t) > 0\}$ at $t = 0$. To this end, we assume that (4.2.6) holds true, while (4.2.5) is violated in the sense that

$$|\{u_0 = 0\} \cap \{(1-g) - \alpha_0 g < 0\}| > 0 \ , \tag{4.2.13}$$

where $\alpha_0 = \frac{\int_{\{u_0>0\}}(1-g)\,dS}{\int_{\{u_0>0\}} g\,dS}$. By (2.0.9), this assumption is equivalent to

$$|\{u_0 = 0\} \cap \{g > \lambda_0\}| > 0 \ , \text{ where } \lambda_0 = \fint_{\{u_0>0\}} g\ dS. \tag{4.2.14}$$

We prove that under assumption (4.2.14) the function λ and the positivity set $\{u(\cdot,t) > 0\}$ both will jump at $t = 0$. We can characterize this jump in terms of a variational principle. More specifically, we define $\Lambda[u_0]$ as follows.

Definition 4.2.3. For any open, measurable set $S \subset \Gamma$, we set

$$\Lambda[u_0] := \sup\left\{\fint_A g\ dS \ : \ A \subset \Gamma \text{ measurable with } \{u_0 > 0\} \subset A\right\} . \tag{4.2.15}$$

Moreover, the maximum in (4.2.15) is attained by the set A_*^0, given by

$$A_*^0 := \Big\{g \geq \Lambda[u_0]\Big\} \cup \{u_0 > 0\} \ , \tag{4.2.16}$$

where $u_0 : \Gamma \to \mathbb{R}$ denotes a given nonnegative continuous function.

We then prove the following.

Theorem 4.2.4. *Suppose that* (4.2.6) *holds true and that g is smooth. For any $\eta > 0$ there exists $\bar{t} = \bar{t}(\eta) > 0$ such that the positivity set $\{u(\cdot,t) > 0\}$ satisfies for all $0 < t \leq \bar{t}(\eta)$*

$$\Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big)_{-\eta} \subset \{u(\cdot,t) > 0\} \subset \Big(\{u_0 > 0\} \cup \{g \geq \Lambda[u_0]\}\Big)_{+\eta}. \tag{4.2.17}$$

Furthermore,

$$|\lambda(t) - \Lambda[u_0]| \leq \eta \quad \textit{for all } 0 < t \leq \bar{t}(\eta) \ . \tag{4.2.18}$$

In particular, $\lambda(t) \to \Lambda[u_0]$ as $t \searrow 0$. If in addition (4.2.14) *holds, then*

$$\Lambda[u_0] > \lambda_0 \quad \textit{and} \quad |A_*^0 \setminus \{u_0 > 0\}| > 0 \ .$$

Remark 4.2.5. The inclusions in (4.2.17) imply that there exists a set $B(t) \subset \{g = \Lambda[u_0]\}$ such that $\{u(\cdot,t) > 0\} \cup B(t) \to A_*^0$ with respect to the L^1-convergence of sets. The set $B(t)$ could in principle be oscillatory.

It is worth noticing that Theorem 4.2.4 can be considered as a generalization of Theorem 4.2.1. Indeed, if along with the assumptions in Theorem 4.2.4, we further assume that (4.2.5) holds, Theorem 4.2.4 reduces to Theorem 4.2.1 (cf. [46, Remark 4.7]).

4.3 Comments on proofs

The proofs of the previously mentioned theorems are long and rely on several auxiliary technical lemmas and propositions. Hence, in this section, we will either briefly sketch these proofs or emphasize the underlying ideas.

Sketch Proof of Theorem 4.2.1

Step 1: Recalling (4.2.4), we estimate

$$|\alpha(t)-\alpha_0| \le C\Big|\{u_0>0\}\Delta\{u(\cdot,t)>0\}\Big| \,. \tag{4.3.1}$$

Therefore, we can control $|\alpha(t)-\alpha_0|$ by the symmetric difference between the positivity sets at the initial time $t=0$ and $t>0$.

Step 2: We claim that for all $\delta>0$ sufficiently small there exists $t^\dagger(\delta)>0$ such that

$$\{u_0\ge\delta\}\subset\{u(\cdot,t)>0\}\subset\Big(\{u_0=0\}_{-\delta}\Big)^{\mathsf{c}} \quad \text{for all } 0<t<t^\dagger(\delta)\,. \tag{4.3.2}$$

The inclusions in (4.3.2) follow immediately for $t=0$.

Step 3: The previous step yields that for all $0<t<t^\dagger(\delta)$

$$\{u(\cdot,t)>0\}\Delta\{u_0>0\}\subset\Big(\{u_0=0\}_{-\delta}\Big)^{\mathsf{c}}\setminus\{u_0\ge\delta\}\,. \tag{4.3.3}$$

Step 4: Using [46, Lemma A.2 and Lemma A.4], we infer that for any $\delta>0$ we can choose $\bar\delta:=\bar\delta(\delta)\ge\delta$ with $\bar\delta\to0$ as $\delta\to0$ such that

$$\Big(\{u_0=0\}_{-\delta}\Big)^{\mathsf{c}}\setminus\{u_0\ge\delta\}\subset\{u_0>0\}_{+\bar\delta}\setminus\{u_0>0\}_{-\bar\delta}\,. \tag{4.3.4}$$

Moreover, (4.3.2) implies that

$$\{u_0>0\}_{-\bar\delta}\subset\{u(\cdot,t)>0\}\subset\{u_0>0\}_{+\bar\delta} \quad \text{for all } 0<t<t^\dagger(\delta)\,. \tag{4.3.5}$$

Step 5: Hence, due to (4.3.3), (4.3.4), the convergence $\bar\delta\to0$ as $\delta\to0$ and (4.2.6) we deduce that

$$|\{u(\cdot,t)>0\}\Delta\{u_0>0\}|\le|\{u_0>0\}_{+\bar\delta}\setminus\{u_0>0\}_{-\bar\delta}|\to0 \quad \text{as } \delta\to0\,.$$

Step 6: Therefore, for given $\eta > 0$, we can choose $0 < \delta \leq \eta$, $\delta = \delta(\eta)$ sufficiently small, such that the right-hand side of (4.3.1) is less or equal than η for all $0 < t < \bar{t}(\eta)$, with $\bar{t}(\eta) = t^\dagger(\delta)$. Thus, we obtain (4.2.9). Moreover, by (4.3.5) and $\delta \leq \eta$ we obtain (4.2.8).

Yet, the proof is not complete. To conclude the proof of Theorem 4.2.1 it remains to justify the claim in (4.3.2). A key estimate for the left-hand side inclusion is the uniform convergence $u(\cdot,t) \to u_0$, which follows from the regularity of the solution u (cf. [44]). To show however the right-hand side inclusion in (4.3.2) is not that easy. In fact, it is the most challenging part of this proof.

Considering the complements in the right-hand side in (4.3.2), an equivalent way to to write this inclusion is the following

$$\{u_0 = 0\}_{-\delta} \subset \{u(\cdot,t) = 0\} \quad \text{for all } 0 < t < t^\dagger(\delta)\ . \tag{4.3.6}$$

At this point, we recall the so-called *nondegeneracy* Lemma 2.0.2. This lemma is the main core of the proof. If we could obtain a sufficiently small uniform estimate for the solution u in a large open set, then (4.3.6) would follow by a comparison argument, provided that the right-hand side in (4.2.1) has the correct sign. Although (4.2.5) serves this purpose at $t = 0$, we cannot claim that the assumption would still be valid even for sufficiently small $t > 0$. The fact that we have no control on α and more specifically knowing that $\alpha \in L^\infty(0,T)$, yields that the limit $\lim_{t\to 0^+} \alpha(t)$ might not exist or might be different from α_0.

In order to tackle this difficulty, we consider a regularized version of (4.2.1)-(4.2.3), for which the analogon of the function α is smooth. Arguments similar to those in [44] imply that a unique smooth solution of the regularized problem exists for all positive times, and that solutions approximate (4.2.1)-(1.5.3) as $\varepsilon \to 0$. In [46, Lemma 3.3] we prove a corresponding nondegeneracy result for this regularized problem. It is worth noticing that in the limit $\varepsilon \to 0^+$, this result would "converge" to the standard nondegeneracy result for the Stefan problem, cf. [6, Theorem 3.1] and Lemma 2.0.2.

Therefore, owing to [46, Lemma 3.3] we prove an estimate for the solution u_ϵ of the regularized problem, namely

$$\{u_0 = 0\}_{-\delta} \subset \{u_\varepsilon(\cdot,t) \leq L_0\varepsilon\} \quad \text{for all } 0 < t < t^\dagger(\delta)$$

for some L_0 independent of ε. Letting $\varepsilon \to 0$, we obtain (4.3.6) and this concludes the proof.

Regularization though, is the principal idea behind the proof of Theorem 4.2.4 as well. The full proof of the initial jump of the set $\{u(\cdot,t) > 0\}$ and the function $\lambda(t)$ is been rigorously presented in [46, Section 4.2] and consists of various technical steps. Let us skip here the technicalities and discuss instead the foundation of this proof.

In Theorem 4.2.4, we derive precise estimates for the jump of both functions $t \mapsto \{u(\cdot,t) > 0\}$ and $t \mapsto \lambda(t)$, under the assumption that (4.2.6) holds, while (4.2.5) fails, in the sense of (4.2.14). However, as we have already mentioned in the bottom of Section 4.2, if we assume that (4.2.5) holds in addition to (4.2.6), then Theorem 4.2.4 reduces to the continuity Theorem 4.2.1.

Motivated by this observation, our strategy for the proof of Theorem 4.2.4 is to approximate u by a solution to (4.2.10)-(4.2.11) with suitably modified initial data u_n^0. The latter are chosen such that we can recover assumption (4.2.5) for the modified problem. In particular, we choose u_n^0 such that they converge uniformly to u_0 but such their support, on the other hand, approximates the maximal set A_*^0 given by (4.2.16). We describe this class of initial data in [46, Lemma 4.13]. A key property of the modified solutions is that we can apply the continuity results obtained in Theorem 4.2.1.

Thus, for this specific family of initial data u_n^0, we define our main approximation.

Regularization: Let $(u_n)_n$ be the unique solution of the problem

$$\partial_t u_n - \Delta u_n = -\left(1 - \frac{g}{\lambda_n(t)}\right) H(u_n), \qquad \text{in } \Gamma_T \tag{4.3.7}$$

$$g \leq \lambda_n, \qquad \text{a.e in } \{u_n = 0\} \tag{4.3.8}$$

$$u_n(\cdot,0) = u_n^0 \qquad \text{on } \Gamma, \tag{4.3.9}$$

where

$$\lambda_n(t) = \fint_{\{u_n(\cdot,t)>0\}} g \, dS \, .$$

We stress that by [44, Theorem 3.1]

$$u_n \to u \quad \text{in } C^0([0,T], L^1(\Gamma)) \text{ for all } T > 0 \, . \tag{4.3.10}$$

We derive upper and lower estimates for the sequence of functions λ_n and the positivity sets $\{u_n(\cdot,t) > 0\}$. To infer the proof, we consider the limit $n \to \infty$.

CHAPTER 5

Global continuity of the interfaces of a non-local free boundary problem describing cell polarization.

This chapter is based on paper [45], that is a joint work in preparation with Barbara Niethammer, Matthias Röger and Juan J. L. Velázquez.

5.1 Introduction

In this chapter, we complement to a certain extent the analysis in [46], a detailed summary of which is presented in Chapter 4, of the following problem

$$\partial_t u - \Delta_\Gamma u = -(1-g)\xi + \alpha g \qquad \text{on } \Gamma_T\,, \tag{5.1.1}$$

$$u \geq 0\,, \quad u\xi = u\,, \quad 0 \leq \xi \leq 1 \qquad \text{on } \Gamma_T\,, \tag{5.1.2}$$

$$u(\cdot,0) = u_0 \qquad \text{on } \Gamma\,, \tag{5.1.3}$$

where $\alpha : (0,T) \to \mathbb{R}$ depends only on time and is a non-local functional of u that is given by

$$\alpha(t) = \frac{\int_{\{u(\cdot,t)>0\}} (1-g)\, dS}{\int_{\{u(\cdot,t)>0\}} g\, dS}\,. \tag{5.1.4}$$

The function g is assumed to be time-independent and smooth (precise assumptions on the regularity of g will be stated later). The initial data $u_0 : \Gamma \to \mathbb{R}$ is given and nonnegative and Δ_Γ stands for the Laplace-Beltrami operator associated to the surface Γ.

We remark that the identity (5.1.4) guarantees mass conservation of u.

The problem (5.1.1)-(5.1.4) has been derived in Chapter 3 (cf. [44]) as the limit of a bulk-surface reaction diffusion system of equations which models cell polarization as a response to an external chemical signal. The respective time independent problem has been obtained in [51]. From the biological point of view, the positivity set $\{u(\cdot,t) > 0\}$ corresponds to the regions where the concentration of a chemical is high, while the set $\{u(\cdot,t) = 0\}$ indicates those regions where the concentration of such a chemical is extremely low.

Well-posedness and global stability of steady states have been established in Chapter 3 (cf. [44]). Furthermore, in Chapter 4 (cf. [46]) we have deduced necessary and sufficient conditions which ensure that the positivity set $\{u(\cdot,t) > 0\}$ changes continuously as $t \to 0^+$. It is worth noticing that a key difficulty in the analysis of Chapter 4 was to prove continuity of the function α for sufficiently small times, due to its non-local dependence on the positivity set $\{u(\cdot,t) > 0\}$.

By means of Lemma 2.0.1, we obtain an equivalent formulation of problem (5.1.1)-(5.1.2) given by

$$\partial_t u - \Delta u = -\left(1 - \frac{g}{\lambda}\right) H(u) \quad \text{on } \Gamma_T \,, \tag{5.1.5}$$

$$g \leq \lambda \quad \text{almost everywhere in } \{u = 0\} \,. \tag{5.1.6}$$

Here $\lambda : (0,T) \to \mathbb{R}$ is a non-local functional of u and characterized by

$$\lambda(t) = ⨍_{\{u(\cdot,t)>0\}} g \, dS \tag{5.1.7}$$

and we recall that $H = \mathcal{X}_{(0,\infty)}$ denotes the characteristic function of the positive real numbers.

In terms of problem (5.1.5)-(5.1.6), it has been shown in Chapter 4 (cf. [46]) that under specific assumptions on the initial data, namely (4.2.5) and (4.2.6), it holds that $\{u(\cdot,t) > 0\} \to \{u_0(\cdot) > 0\}$ as $t \to 0^+$ in a suitable topology. In particular, we assume that for some fixed $\theta > 0$, it holds

$$g(x) - \lambda(0) \leq -\theta < 0 \qquad \text{for all } x \in \{u_0 = 0\} \tag{5.1.8}$$

and further we prescribe a second non-degeneracy condition, that is

$$\mathcal{H}^2\Big(\big(\{u_0 > 0\}\big)_{+\delta} \setminus \big(\{u_0 > 0\}\big)_{-\delta}\Big) \to 0 \quad \text{as } \delta \to 0 \,, \tag{5.1.9}$$

where

$$\Big(\{u_0>0\}\Big)_{+\delta} := \{x \mid d(x,\Big(\{u_0>0\}\Big)) \le \delta\}\ , \quad \Big(\{u_0>0\}\Big)_{-\delta} := \{x \mid d(x,\Big(\{u_0=0\}\Big)) \ge \delta\}\ . \tag{5.1.10}$$

We stress that (5.1.9) is a more technical formulation of (1.5.6). We justify the equivalence between both statements in [46, Appendix B, Lemma B.2].

However if (5.1.8) fails but the non-degeneracy condition (5.1.9) is valid, we show in [46] that at $t=0$ the positivity set $\{u(\cdot,t)>0\}$ will have a jump discontinuity that can be characterized by a variational principle.

In this work we continue with the study that began in Chapter 4 (cf. [46]). More specifically, in this chapter, that can be considered as a companion chapter of Chapter 4, we prove under additional assumptions on the initial data u_0 that the positivity set $\{u>0\}$ and the function λ is continuous for all $t>0$. To this end, in the rest of this paper we restrict our analysis to the specific case of the unit sphere,

$$\mathcal{S}^2 = \{p \in \mathbb{R}^3 \mid p_1^2+p_2^2+p_3^2 = 1\}$$

and to axisymmetric data and axisymmetric solutions.

The plan of this chapter is the following. First, we collect in Section 5.2 some results from previous work. In Section 5.3 we provide the framework of the current analysis. Finally, we prove in Section 5.4 global in time continuity for the positivity set $\{u(\cdot,t)>0\}$ and the function λ under some further assumptions on the initial data in addition to (5.1.8) and (5.1.9).

5.2 Previous results

Before proceeding to the main analysis of this chapter, we collect here some results from [44] (cf. Chapter 3), which appear to be useful in what follows.

In [44] we have established that problem (5.1.1)-(5.1.3) admits a unique nonnegative solution for rather general (not necessarily axisymmetric) data. In fact, for nonnegative $u_0 \in C^4(\Gamma)$, there exists a unique $u \in L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^1(\Gamma)^*) \cap W_p^{2,1}(\Gamma_T)$ for all $p \in [1,\infty)$ and $\alpha \in L^\infty(0,T)$ that solves (5.1.1)-(5.1.3). Moreover it holds that $u \in C^{1+\beta,\frac{1+\beta}{2}}(\Gamma_T)$ for all $0 < \beta < 1$.

In [44, Remark 2.3], we further justify a representation formula that we infer for ξ, that is

$$\xi(\cdot,t) = \begin{cases} 1 & \text{in } \{u(\cdot,t) > 0\} \\ \frac{\alpha(t)g(\cdot)}{1-g(\cdot)} & \text{in } \{u(\cdot,t) = 0\} \end{cases} \tag{5.2.1}$$

for almost all $t \in (0,T)$. Due to (5.2.1), we also deduce a formula for α which is equivalent to (5.1.4) and is given by

$$\alpha(t) = \frac{\int_\Gamma (1-g)\xi(\cdot,t)\,dS}{\int_\Gamma g\,dS} \ . \tag{5.2.2}$$

We also prove in [44] that the solution to (5.1.1)-(5.1.3) satisfies

$$\int_\Gamma u\,dS = \int_\Gamma u_0\,dS = m \ . \tag{5.2.3}$$

5.3 Assumptions and preliminaries

Let us state the main assumptions that we impose throughout this chapter. In particular, we will restrict to the spherical case $\Gamma = \mathcal{S}^2$ and to the case that all data and the solution are axisymmetric with respect to the first coordinate axis.

For example, we assume that $g : \mathcal{S}^2 \to \mathbb{R}$ is given by a function $\tilde{g} : [-1,1] \to \mathbb{R}$ as

$$g(p) = \tilde{g}(p_1) \quad \text{for all } p = (p_1,p_2,p_3) \in \mathcal{S}^2.$$

For simplicity we will often drop the $\tilde{}$ and write $g = g(p_1)$ (even though this is an abuse of notation). We typically denote the p_1 variable by x and the derivative of g with respect to

p_1 as g'. It will become clear from the context whether we consider g as a function on $\mathcal{S}^2$ or as a function on $[-1,1]$. For example, $\Delta_{\mathcal{S}^2} g$ refers to the Laplace-Beltrami operator of $g : \mathcal{S}^2 \to \mathbb{R}$, whereas g'' refers to the second derivative of $g : [-1,1] \to \mathbb{R}$ (more precisely $\tilde{g} : [-1,1] \to \mathbb{R}$).

Assumption 5.3.1. Let $\Gamma = \mathcal{S}^2$ be the 2-dimensional unit sphere in $\mathbb{R}^3$.

We assume that

$$u_0 \in C^4(\mathcal{S}^2), \quad \text{with } u_0 \geq 0 \quad \text{and} \quad |\{u_0 > 0\}| > 0 \tag{5.3.1}$$

and

$$g \in C^2(\mathcal{S}^2) \qquad \text{and} \qquad 0 < g_0 \leq g \leq g_1 < 1 \text{ on } \mathcal{S}^2 \tag{5.3.2}$$

for some constants $0 < g_0 < g_1 < 1$.

Moreover we assume that both the initial data u_0 and the function g are axisymmetric functions with respect to the first coordinate axis in $\mathbb{R}^3$.

Due to Lemma 2.0.3, we now define the space of axisymmteric solutions as follows

Definition 5.3.2 (Axisymmetric solution spaces)**.** Let

$$X^2 := \bigcap_{1 \leq q < \infty} \Big\{ \tilde{u} \in W^{1,q}(-1,1) \cap W^{2,q}_{\text{loc}}(-1,1) \text{ with } x \mapsto (1-x^2)\tilde{u}''(x) \in L^q((-1,1)) \Big\}.$$

and

$$X_T^{2,1} := \bigcap_{1 \leq q < \infty} \Big\{ \tilde{u} \in L^q(0,T;X^2) \cap \partial_t u \in L^q\Big((-1,1) \times (0,T)\Big) \Big\}.$$

At this point, we observe that due to Assumption 5.3.1 the solution of (5.1.5) reduces to a one-dimensional problem. We provide a justification for this reduction in the next Proposition.

Proposition 5.3.3. *Suppose that Assumption 5.3.1 holds. Then the unique solution $u \in W_q^{2,1}(\Gamma_T)$ to* (5.1.5), (5.1.6) *with initial data u_0 is axisymmetric with respect to the first coordinate axis, i.e. $u(p,t) = \tilde{u}(x,t)$ for any $p = (x,y,z) \in \mathcal{S}^2$, $t \in (0,T)$.*

Moreover, $\tilde{u} \in X_T^{2,1}$ is the unique solution to

$$\partial_t \tilde{u} - \Big((1-x^2)\tilde{u}'\Big)' = -\left(1 - \frac{g}{\lambda}\right) H(\tilde{u}) \quad \text{in } (-1,1) \times (0,T) \tag{5.3.3}$$

with $\tilde{u}(\cdot,0) = \tilde{u}_0$, where $\lambda \in L^\infty((0,T))$ is given by

$$\lambda(t) = \fint_{\{u(\cdot,t)>0\}} \tilde{g}(x)\, dx\,. \tag{5.3.4}$$

Proof. The proof follows by existence and uniqueness (cf. [44, Theorem 3.1]) and invariance of (5.1.5), (5.1.6) with respect to rotations around the first coordinate axis. ■

5.4 Global continuity results

From now on we will only deal with the axisymmetric case and drop the tilde notation for functions depending on the spatial variables $x \in [-1, 1]$. Moreover we denote by g' et cetera the derivative with respect to $x \in [-1, 1]$.

Let u be a solution to (5.1.5), (5.1.6) with initial data u_0 and moreover assume that u is axisymmetric. Recalling Proposition 5.3.3, we obtain that u is also a bounded solution to

$$\partial_t u - \Big((1-x^2)u'\Big)' = -\Big(1 - \frac{g}{\lambda(t)}\Big)H(u) \qquad \text{in } \ (-1,1)\times(0,T], \tag{5.4.1}$$

$$g \le \lambda(t) \qquad \text{in } \ \{u(\cdot,t)=0\}\ , \tag{5.4.2}$$

where $H = \mathcal{X}_{(0,\infty)}$.

Our aim in this section is to provide global continuity results under certain assumptions on the initial data u_0 and the external stimulus g. More precisely, we assume that Assumptions 5.3.1 hold and that for some $\gamma \in (-1, 1)$

$$\{u_0 > 0\} = (\gamma, 1) \quad \text{and} \quad u_0' \ge 0 \quad \text{in } \ (\gamma, 1)\ . \tag{5.4.3}$$

Moreover, besides (5.3.2) we also assume that g satisfies

$$g' \ge \kappa > 0 \qquad \text{in } \ [-1,1]. \tag{5.4.4}$$

For the following we define the boundary of the positivity set of u via

$$p(t) := \inf\{x \,|\, u(x,t) > 0\}\,. \tag{5.4.5}$$

Indeed, we will see in Lemma 5.4.3 that if u_0 is increasing, then so is $u(\cdot, t)$ for any $t > 0$ and $[p(t), 1]$ is indeed the support of $u(\cdot, t)$.

Theorem 5.4.1. *Suppose that Assumption 5.3.1 hold and that* u_0, g *satisfy* (5.4.3), (5.4.4).

Moreover, let u *be an axisymmetric function which solves* (5.1.5), (5.1.6). *Then the following holds:*

(i) If $p(\tilde{t}) > -1$ for some $\tilde{t} \geq 0$, then there exists $\delta_0 = \delta_0(\tilde{t})$ such that for all $\delta \in (0, \delta_0]$ there exists $\omega(\delta) > 0$ with

$$p(\tilde{t}) - \delta \leq p(t) \leq p(\tilde{t}) + \delta \qquad \text{for all } t \in [\tilde{t}, \tilde{t} + \omega(\delta)].$$

(ii) If for given $m > 0$ the corresponding steady state u_* has an interface, then p is continuous for sufficiently large times.

Remark 5.4.2. (i) Theorem 5.4.1 implies in particular that if $p(\tilde{t}) > -1$, then p is continuous in an interval $[\tilde{t}, t_1]$ where $t_1 = \sup\{t > \tilde{t} \,|\, p(t) > -1\}$. In other words, p moves continuously until it might vanish at the boundary $x = -1$, but we cannot exclude that it jumps back discontinuously into the domain at some later time.

(ii) From the proof it will be apparent that $\delta_0(\tilde{t})$ depends on $\tilde{t}$ only through the distance of $p(\tilde{t})$ to the left boundary -1.

5.4.1 Auxiliary lemmas

First we prove that the solution to (5.4.1)-(5.4.2) is non-decreasing assuming that the initial data u_0 is also non-decreasing.

Lemma 5.4.3. *Under the assumptions of Theorem 5.4.1 it holds that $u'(\cdot, t) \geq 0$ in $(-1, 1) \times [0, T]$.*

Proof. We approximate a solution to (5.1.5)-(5.1.6) by considering the solution u_ε to

$$\partial_t u_\varepsilon - \Big((1-x^2)u_\varepsilon'\Big)' = -(1-g)f_\varepsilon(u_\varepsilon) + \alpha_\varepsilon g \quad \text{in } (-1,1) \times (0,T] \tag{5.4.1.1}$$

$$u_\epsilon(\cdot, 0) = u_0 \quad \text{in } (-1,1)\,, \tag{5.4.1.2}$$

where $f_\varepsilon(u) = \frac{u}{u+\varepsilon}$ and α_ε is given by

$$\alpha_\varepsilon(t) = \frac{\int_{-1}^1 (1-g) f_\varepsilon(u_\varepsilon(\cdot,t))\,dx}{\int_{-1}^1 g\,dx}\,. \tag{5.4.1.3}$$

We notice that (5.4.1.1)-(5.4.1.2) has a global smooth solution that becomes strictly positive for all positive times. Moreover it holds $u_\varepsilon \in W^{2,1}_{p,\mathrm{loc}}((-1,1) \times [0,T]) \cap L^\infty([-1,1] \times [0,T])$. Following the analysis in [44] we deduce for any $1 \leq p < \infty$ and $0 < \beta < 1$ that

$$u_\varepsilon \rightharpoonup u \quad \text{in } W^{2,1}_{p,\mathrm{loc}}((-1,1) \times [0,T]), \tag{5.4.1.4}$$

$$u_\varepsilon \to u \quad \text{in } C^{1+\beta,\frac{1+\beta}{2}}_{\text{loc}}((-1,1)\times[0,T]). \tag{5.4.1.5}$$

We will prove first that $u_\varepsilon(\cdot,t)$ is non decreasing for all $t\geq 0$. Since $u_\varepsilon \in W^{2,1}_{p,\text{loc}}((-1,1)\times[0,T])$, by a standard embedding theorem we obtain that $u_\varepsilon \in C^{1+\beta,\frac{1+\beta}{2}}_{\text{loc}}((-1,1)\times[0,T])$. Then, via a bootstrapping argument we deduce that $u_\varepsilon(\cdot,t)\in C^{4,1+\beta}_{\text{loc}}((-1,1))$ for $0\leq t\leq T$ and $u_\varepsilon(x,\cdot)\in C^{2,\frac{1+\beta}{2}}([0,T])$ for $-1<x<1$. Hence, we can differentiate (5.4.1.1) with respect to x. Setting $v_\varepsilon = u_\varepsilon'$ we obtain

$$\partial_t v_\varepsilon - \Big((1-x^2)v_\varepsilon\Big)'' = -(1-g)\frac{\varepsilon v_\varepsilon}{(\varepsilon+u_\varepsilon)^2} + g' f_\varepsilon(u_\varepsilon) + \alpha_\varepsilon g' \ . \tag{5.4.1.6}$$

Due to the properties of g and f_ε we find that $\Big((1-x^2)v_\varepsilon\Big)'' - \partial_t v_\varepsilon \leq C_{g,\varepsilon} v_\varepsilon$ for some positive constant $C = C_{g,\varepsilon}$. For $w_\varepsilon := e^{C_{g,\varepsilon}t} v_\varepsilon$ we conclude:

$$\Big((1-x^2)w_\varepsilon\Big)'' - \partial_t w_\varepsilon \leq 0 \ . \tag{5.4.1.7}$$

To construct a suitable subsolution for (5.4.1.7) we fix $\mu>0$ that eventually will become arbitrarily small, choose a sufficiently small $\eta := \eta(\varepsilon,\mu) > 0$ and define $y_\varepsilon\colon [-1+\eta_\varepsilon, 1-\eta_\varepsilon]\times[0,T]\to\mathbb{R}_+$ via

$$y_\varepsilon(x,t) = -\mu\Big(\frac{1}{1-x^2} + \frac{t}{1-x^2}\Big) \ . \tag{5.4.1.8}$$

One can easily see that $\Big((1-x^2)y_\varepsilon\Big)'' - \partial_t y_\varepsilon \geq 0$. By assumption $v_\varepsilon(\cdot,0)>0$ and thus we obtain $w_\varepsilon(x,0) = v_\varepsilon(x,0) > 0 > y_\varepsilon(x,0)$ for all $x\in[-1+\eta_\varepsilon, 1-\eta_\varepsilon]$. Moreover (5.4.1.6) yields that $v_\varepsilon \in C^{3,3}_{\text{loc}}((-1,1)\times[0,T])$. Hence, there exists some constant $M_\varepsilon>0$ such that $|w_\varepsilon|\leq M_\varepsilon$ in $(-1,1)\times[0,T]$. Having chosen $\eta=\eta(\varepsilon,\mu)>0$ sufficiently small, we conclude

$$w_\varepsilon(-1+\eta_\varepsilon,t) \geq -M_\varepsilon \geq -\frac{\mu}{1-(\eta_\varepsilon-1)^2} - \frac{\mu t}{1-(\eta_\varepsilon-1)^2} = y_\varepsilon(-1+\eta_\varepsilon,t) \ .$$

for all $t\in[0,T]$. Similarly, we find that $w_\varepsilon(1-\eta_\varepsilon,t)\geq y_\varepsilon(1-\eta_\varepsilon,t)$. Thus, a comparison argument yields

$$w_\varepsilon(x,t) \geq -\mu\Big(\frac{1}{1-x^2} + \frac{t}{1-x^2}\Big) \ , \qquad \text{for all } \ (x,t)\in[-1+\eta_\varepsilon, 1-\eta_\varepsilon]\times[0,T] \ .$$

Letting now $\mu\to 0$ implies $w_\varepsilon \geq 0$ in $(-1,1)\times[0,T]$. We conclude the proof of this lemma by letting $\varepsilon\to 0$. Due to (5.4.1.5), it follows that $u(\cdot,t)$ is also increasing in $(-1,1)$. ■

Next, we show that the support of u can not be the whole interval $[-1,1]$. For that we recall that for any $T>0$ we have a bound $\|u\|_\infty := \|u\|_{L^\infty((-1,1)\times[0,T])} \leq C_T$.

Lemma 5.4.4. *Under the assumptions of Theorem 5.4.1 it holds*

$$p(t) \leq 1 - \frac{m}{2\pi\|u\|_\infty} \qquad \text{for all } \; t \in [0,T]\,. \tag{5.4.1.9}$$

Furthermore we have

$$g(x) \leq \lambda(t) - \frac{\kappa m}{4\pi\|u\|_\infty} \qquad \text{for all } x \in (-1, p(t)) \text{ and } t \in [0,T]\,. \tag{5.4.1.10}$$

Proof. We observe that due to (5.2.3), (2.0.12) estimate (5.4.1.9) follows from

$$m = 2\pi \int_{p(t)}^{1} u(x,t)\; dx \leq 2\pi\|u\|_\infty(1-p(t))\;.$$

By (2.0.12), Taylor's Theorem and (5.4.4), we obtain that

$$\begin{aligned}\lambda(t) = \frac{1}{1-p(t)} \int_{p(t)}^{1} g\; dx &\geq \frac{1}{1-p(t)} \int_{p(t)}^{1} g(p(t)) + \kappa(x - p(t))\; dx \\ &\geq g(p(t)) + \frac{\kappa}{2}(1-p(t)) \geq g(p(t)) + \frac{\kappa m}{4\pi\|u\|_\infty}\,.\end{aligned}$$

Due to the monotonicity of g we deduce (5.4.1.10). ■

Next we prove that $p(t)$ is sufficiently separated from the value $s(t)$ where $g(s(t)) = \lambda(t)$.

Lemma 5.4.5. *Assume that* (5.4.3), (5.4.4) *are valid. Then, there exists a unique* $s(t) \geq p(t)$ *with*

$$g(s(t)) = \lambda(t) \tag{5.4.1.11}$$

and it holds

$$p(t) \leq s(t) - c_0 \qquad \text{for all } t \in [0,T] \tag{5.4.1.12}$$

for some $c_0 = c_0(g, m, \|u\|_\infty) > 0$.

Proof. We notice due to (5.1.7), (5.3.2) and (5.4.4) that $g(-1) < \lambda(t) < g(1)$ for all $t \in [0,T]$. Thus, there exists $s(t) \in (-1,1)$ such that (5.4.1.11) holds true. Furthermore, due to (5.4.4), the function $g^{-1} : [g(-1), g(1)] \to [-1,1]$ is well defined. Then, recalling that $g(p(t)) \leq \lambda(t)$, it follows that $p(t) \leq s(t)$. Inequality (5.4.1.12) then follows from (5.4.1.10) and (5.4.4). ■

Finally, we formulate the degeneracy Lemma 2.0.2 for our particular setting.

Corollary 5.4.6. *Let* $t_1 > 0$, $x_0 < p(t_1)$ *and* $\rho \in (0, \rho_{\max}]$ *such that* $x_0 + 2\rho \leq p(t_1)$. *Then there exists* $A > 0$ *such that if* $u \leq \frac{1}{A}\frac{\kappa m}{4\pi\|u\|_\infty}\rho^2$ *in* $(-1, x_0 + 2\rho) \times (t_1, t_2)$ *then* $u = 0$ *in* $(-1, x_0 + \rho) \times (t_1, t_2)$. *This in particular implies that* $p(t) \geq x_0 + \rho$ *in* $[t_1, t_2]$.

5.4.2 Proof of Theorem 5.4.1

We consider now a time $\tilde{t} \geq 0$ such that $p(\tilde{t}) > -1$ and define $\delta_0 := \frac{1}{8}\min\Big(\frac{1}{2}(p(\tilde{t}) + 1), c_0, \frac{m}{2\pi\|u\|_\infty}, \frac{\kappa m}{4\pi\|u\|_\infty}\Big)$, with c_0 as in (5.4.1.12)

Step 1: $p(\tilde{t}) - \delta \leq p(t)$

We fix any $\delta \in (0, \delta_0]$ and define U as the solution of

$$\partial_t U - \Big((1-x^2)U'\Big)' = 0 \qquad \text{in } [-1+\tfrac{\delta}{4}, p(\tilde{t})+\tfrac{\delta}{4}] \times (\tilde{t}, T)\,, \tag{5.4.2.1}$$

$$U(x,\tilde{t}) = \|u\|_\infty \chi_{\{u(\cdot,\tilde{t})>0\}} \qquad \text{in } [-1+\tfrac{\delta}{4}, p(\tilde{t})+\tfrac{\delta}{4}]\,. \tag{5.4.2.2}$$

$$U(x,\tilde{t}) = \|u\|_\infty \chi_{\{u(\cdot,\tilde{t})>0\}} \qquad \text{in } \{-1+\tfrac{\delta}{4}, p(\tilde{t})+\tfrac{\delta}{4}\} \times (\tilde{t}, T)\,. \tag{5.4.2.3}$$

By (5.4.1.10) and the choice of δ_0 it follows that $g \leq \lambda(t)$ in $[-1, p(\tilde{t})+\delta_0] \times [\tilde{t}, T]$ and thus

$$\partial_t u - \Big((1-x^2)u'\Big)' = -\Big(1 - \frac{g}{\lambda(t)}\Big)H(u) \leq 0 \qquad \text{in } [-1, p(\tilde{t})+\delta_0] \times [\tilde{t}, T].$$

Hence, the maximum principle principle implies that $u(\cdot,\tilde{t}) \leq U(\cdot,\tilde{t})$ in $[-1+\frac{\delta}{4}, p(\tilde{t})+\frac{\delta}{4}]$.

In fact, there is an explicit formula in [22] for the solution U, that is

$$\begin{aligned} U(x,t) &= \int_{p(\tilde{t})}^{p(\tilde{t})+\frac{\delta}{4}} \Gamma(x,t,\xi,\tilde{t})U(x,\tilde{t})\,d\xi = \|u\|_\infty \int_{p(\tilde{t})}^{p(\tilde{t})+\frac{\delta}{4}} \Gamma(x,t,\xi,\tilde{t})\,d\xi \\ &\leq C\Big[\operatorname{erf}\Big(\frac{p(\tilde{t})+\frac{\delta}{4}-x}{\sqrt{C(t-\tilde{t})}}\Big) + \operatorname{erf}\Big(\frac{x-p(\tilde{t})}{\sqrt{C(t-\tilde{t})}}\Big)\Big]\,. \end{aligned}$$

Here $\Gamma(x,t,\xi,\tilde{t})$ denotes the fundamental solution of (5.4.2.1) and $\operatorname{erf}(y)$ stands for the error function. Due to the previous estimate and the monotonicity of u we obtain that

$$\sup_{x\in[-1,p(\tilde{t})-\frac{\delta}{2}]} u(x,t) \leq u(p(\tilde{t})-\tfrac{\delta}{2}, t) \leq \sup_{x\in[-1,p(\tilde{t})-\frac{\delta}{2}]} U(x,t) \to 0 \quad \text{for } |t-\tilde{t}| \to 0\,.$$

Hence, using Lemma 5.4.6, it follows that $p(\tilde{t}) - \delta \leq p(t)$ for $t - \tilde{t} \leq \omega(\delta)$.

Step 2: $p(t) \leq p(\tilde{t}) + \delta$

We are going to construct a subsolution, more precisely, we are going to show that there exists a nonnegative, continuous function W that depends only on the initial data u_0, with $W(\xi) > 0$ for all $\xi > 0$ and $W(0) = 0$ such that

$$u(x,t) \geq W(x-p(t)), \qquad \text{for } x \in (p(t), p(t)+\delta_0(t))\,, t \in [0,T]\,. \tag{5.4.2.4}$$

Indeed, if (5.4.2.4) holds we define for any $\tilde{t} \geq 0$ the function w as the solution to

$$\partial_t w - \Big((1-x^2)w'\Big)' = 0 \qquad \text{in } (p(\tilde{t}), p(\tilde{t})+\delta_0) \times (\tilde{t}, T] \tag{5.4.2.5}$$

$$w(x,\tilde{t}) = W(x - p(\tilde{t})) \qquad \text{in } (p(\tilde{t}), p(\tilde{t})+\delta_0) \tag{5.4.2.6}$$

$$w(p(\tilde{t}), t) = w(p(\tilde{t})+\delta_0, t) = 0 \qquad \text{in } (\tilde{t}, \tilde{t}+\omega(\delta_0)] \tag{5.4.2.7}$$

and let $\tilde{w} := w - (t - \tilde{t})$. Equation (5.4.2.5) yields

$$\partial_t \tilde{w} - \Big((1-x^2)\tilde{w}'\Big)' = -1 \leq \partial_t u - \Big((1-x^2)u'\Big)'$$

in $(p(\tilde{t}), p(\tilde{t})+\delta_0) \times (\tilde{t}, T]$. Furthermore, $u(x,\tilde{t}) \geq W(x - p(\tilde{t}))$ for all $x \in (p(\tilde{t}), p(\tilde{t})+\delta_0)$ by (5.4.2.4) and $u(p(\tilde{t}), t), u(p(\tilde{t})+\delta_0, t) > 0 = w(p(\tilde{t}), t) = w(p(\tilde{t})+\delta_0, t)$ for all $t \in (\tilde{t}, T]$. Hence, we obtain by a comparison principle argument that

$$u(x,t) \geq \tilde{w} \quad \text{in } [p(\tilde{t}), p(\tilde{t})+\delta_0] \times [\tilde{t}, T]. \tag{5.4.2.8}$$

If we consider any $0 < \delta \leq \frac{\delta_0}{2}$ it holds

$$u \geq \tilde{w} > 0 \quad \text{in } [p(\tilde{t})+\delta, p(\tilde{t})+\delta_0-\delta) \times [\tilde{t}, \tilde{t}+\omega(\delta)]$$

redefining $\omega(\delta)$ if necessary. Moreover, the monotonicity of u yields that

$$u(x,t) > 0 \quad \text{in } [p(\tilde{t})+\delta, 1] \times [\tilde{t}, \tilde{t}+\omega(\delta)]$$

which proves the claim.

We now proceed to the proof of (5.4.2.4). Due to Step 1, we find for any $\tilde{t} \geq 0$ a δ_0 such that for $\delta \in (0, \delta_0]$ and for all $t \in [\tilde{t}, \tilde{t}+\omega(\delta)]$ it holds

$$p(\tilde{t}) \leq p(t) + \delta \quad \text{for all } \tilde{t} \leq t \leq \tilde{t}+\omega(\delta)\,. \tag{5.4.2.9}$$

For any $\tau \geq 0$ we set $t^*_\delta := \max\{\tau - \omega(\delta), 0\} \geq 0$ and we infer by (5.4.2.9) that $p(t^*_\delta) \leq p(\tau) + \delta$. In particular, we obtain that

$$u > 0 \quad \text{in } [p(\tau)+\delta, 1) \times [t^*_\delta, \tau]\,. \tag{5.4.2.10}$$

Since $\tau \geq 0$ is arbitrary, we distinguish in the following the cases $\tau - \omega(\delta_0(\tau)) > 0$ and $\tau - \omega(\delta_0(\tau)) \leq 0$. Moreover, in the case of $\tau \leq \omega(\delta_0(\tau))$, we fix $0 < \delta^* \leq \delta_0$ such that $\tau = \omega(\delta^*)$.

Our goal is to construct suitable subsolutions. In the first case we will find an explicit subsolution, while in the second we will work with the initial data.

Case 1: Suppose that $\tau - \omega(\delta_0) > 0$ and consider any arbitrary $\delta \in (0, \frac{\delta_0}{2}]$. Due to (5.4.2.10), u is a solution of

$$\partial_t u - \Big((1-x^2)u'\Big)' = -1 + \frac{g}{\lambda(t)} \,, \qquad \text{in } [p(\tau)+\delta, p(\tau)+2\delta] \times (t^*_\delta, \tau] \,.$$

We observe that, by regularity of u and standard embedding theorems, we can differentiate the above equation with respect to x. Then for $v = u'$ we obtain, using (5.4.4), that

$$\partial_t v - \Big((1-x^2)v\Big)'' = \frac{g'}{\lambda(t)} \geq M \qquad \text{in } [p(\tau)+\delta, p(\tau)+2\delta] \times (t^*_\delta, \tau] \,,$$

for some constant $M := M(||g||_\infty, ||g'||_\infty) > 0$. To construct a suitable subsolution let V satisfy

$$\partial_t V - \Big((1-x^2)V\Big)'' \leq M, \qquad \text{in } [p(\tau)+\delta, p(\tau)+2\delta] \times (t^*_\delta, \tau] \,, \quad (5.4.2.11)$$

$$V(x, t^*_\delta) = 0, \qquad \text{in } [p(\tau)+\delta, p(\tau)+2\delta] \,, \quad (5.4.2.12)$$

$$V(p(\tau)+\delta, t) = V(p(\tau)+2\delta, t) = 0, \qquad t \in (t^*_\delta, \tau] \,. \quad (5.4.2.13)$$

It is easy to verify that the function

$$V(x,\tau) = \frac{M}{2}(t - t^*_\delta) e^{-\mu \frac{t - t^*_\delta}{\delta^2}} \cos\left(\frac{\pi(x - (p(\tau) + \frac{3}{2}\delta))}{\delta}\right) \,, \quad (5.4.2.14)$$

satisfies (5.4.2.11)-(5.4.2.13) if $\mu > 0$ is sufficiently large.

In particular we find that $u'(x,\tau) \geq C\omega(\delta) e^{-\mu \frac{\omega(\delta)}{\delta^2}} =: F(\delta)$ in $[p(\tau) + \frac{5\delta}{4}, p(\tau) + \frac{7\delta}{4}]$. Therefore, since $\delta \leq \delta_0$ but otherwise arbitrary, we deduce

$$u'(x,\tau) \geq \tilde{W}(x - p(\tau)) \qquad \text{in } \Big(p(\tau), p(\tau) + \delta_0\Big]$$

for some function $\tilde{W}$ which is positive on $\mathbb{R}_+$. Integrating this equation we find indeed that (5.4.2.4) holds for $\tau > \omega(\delta_0)$.

Case 2: Next we investigate the case $\tau \leq \omega(\delta_0)$, that is $t^*_\delta = 0$. We find as above that $v = u'$ solves

$$\partial_t v - \Big((1-x^2)v\Big)'' = \frac{g'}{\lambda(t)} \geq M \quad [p(\tau)+\delta, p(\tau)+2\delta] \times (0, \tau] \,.$$

We construct a subsolution V by solving the problem

$$\partial_t V - \Big((1-x^2)V\Big)'' = M, \qquad \text{in } [p(\tau)+\delta, p(\tau)+2\delta] \times (0, \tau] \,, \quad (5.4.2.15)$$

$$V(x, 0) = u'_0, \qquad \text{in } [p(\tau)+\delta, p(\tau)+2\delta] \,, \quad (5.4.2.16)$$

$$V(p(\tau)+\delta,t) = V(p(\tau)+2\delta,t) = 0, \qquad \text{in } (0,\tau]\ . \tag{5.4.2.17}$$

Since $u_0' \geq 0$ by assumption, we conclude that $V(x,\tau) > \phi(\delta, u_0)$ in $[p(\tau)+\frac{5}{4}\delta, p(\tau)+\frac{7}{4}\delta]$ and as above we finally conclude that there exists a positive function W_2 such that

$$u(x,\tau) \geq W_2(x-p(\tau)), \quad \text{in } \ [p(\tau), p(\tau)+\delta_0] \tag{5.4.2.18}$$

for $\tau \in [0, \omega(\delta_0)]$ from which (5.4.2.4) follows with $W := \min\{W_1, W_2\} > 0$.

The second part of Theorem 5.4.1 follows from the fact that $u(\cdot,t) \to u_*$ uniformly as $t \to \infty$ (see Theorem 3.2 in [46]), the monotonicity of $u(\cdot,t)$ and the degeneracy Lemma 5.4.6 which implies that $p(t)$ needs for large times be close to the interface of u_*.

Appendix A

A parabolic free boundary problem modeling cell polarization

This appendix coincides with the paper [44], written jointly by Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and the author.

A.1 Introduction

Cell polarization in response to some external chemical stimulus contributes significantly in numerous biological processes, such as the migration, development, and organization of eukaryotic cells [58]. Roughly speaking, the process of cell polarity is correlated to the reorganization of several chemicals within a cell and on a cell membrane. Typically polarization is achieved by the combination of an internal pattern forming system, a response to an external signal that imposes some directional preference to the pattern, and the amplification of small concentration differences [70].

A key step in the polarization process is the direction sensing [12], where chemical gradients are detected and amplified. This step proceeds by the transduction of a signal by receptors on the plasma membrane and its adaption by intracellular signaling cascades, which involve the activation and deactivation of specific proteins and the translation of possibly shallow gradients in the outer signal to large amplitude intracellular gradients in protein distributions. Once such polarity of the cell in form a of a spatial asymmetry in chemical concentrations has been established, changes in cell shape and the movement of the cell in the surrounding environment can be initiated.

Polarization is in many instances a dynamic, time-dependent process and a tight regulation of the response to changing environmental conditions is key for many biological functions. One prominent and well-studied example is the chemotaxis of the social amoeba Dictyostelium that migrate to the source of waves of chemoattractant, which exposes the cell to a pulsatile gradient [49, 69].

Several mathematical models of varying complexity have been suggested to analyze the spatial and temporal processes associated with cell polarization. One of the most popular models is the local excitation, global inhibition (LEGI) mechanism which was suggested in the seminal paper about cell polarization [47], see also [40, 53], and is often part of more comprehensive models [12].

We focus on a minimal model for the amplification step that has been proposed in [51]. The significance of the suggested model stems from the fact that in a suitable parameter regime an asymptotic reduction leads to a generalized obstacle-type problem that allows for a clear and mathematically tractable characterization of polarized states. In [51] we have analyzed stationary states and the onset of polarization. The present paper continues this analysis by considering the time-dependent problem.

The model proposed in [51] consists of a system of PDEs, motivated by the GTPase cycle model presented in [59, 60]. We consider a protein that can be in an active or an inactive state, where the inactive protein moreover can be bound to the cell membrane or be in a cytosolic state, i.e. contained in the cells interior. We denote the surface concentration of the active and incative form by u and v, respectively, and the volume concentration of the inactive cytosolic state by w. The model has only a few ingredients. It accounts for lateral diffusion on the cell membrane, for diffusion inside the cell, for activation and deactivation processes on the cell membrane and for attachment to and detachment from the cell membrane. One contribution to the activation depends on a concentration c of a protein that characterizes an external signal (possibly after a first processing step). This concentration in general may vary with space and time.

Most of these processes are modeled by linear kinetic laws, except for parts of the activation and deactivation processes that need the catalyzation by enzymes and are described by simple Michaelis-Menten type rate laws, see [51] for more details on the model derivation.

To give a mathematical formulation, we represent the cell and its outer cell membrane by a domain $\Omega \subset \mathbb{R}^3$ and its boundary $\Gamma := \partial\Omega$. Moreover we fix a time interval $(0,T)$ of observation, a signal concentration $c : \Gamma \times (0,T) \to \mathbb{R}$, and request that $u, v : \Gamma \times (0,T)$ and $w : \Omega \times (0,T)$ solve the following coupled system of bulk and surface partial differential equations

$$\partial_t u = \Delta_\Gamma u + \Big(a_1 + \frac{a_2 u}{a_3 + u} + c\Big) v - \frac{a_4 u}{1+u} \qquad \text{on } \Gamma \times (0,T)\,, \tag{A.1.1}$$

$$\partial_t v = \Delta_\Gamma v - \Big(a_1 + \frac{a_2 u}{a_3 + u} + c\Big) v + \frac{a_4 u}{1+u} - a_5 v + a_6 w \qquad \text{on } \Gamma \times (0,T)\,, \tag{A.1.2}$$

$$\partial_t w = D \Delta w \qquad \text{in } \Omega \times (0,T)\,, \tag{A.1.3}$$

$$-D \frac{\partial w}{\partial \nu} = -a_5 v + a_6 w \qquad \text{on } \Gamma \times (0,T)\,. \tag{A.1.4}$$

Here $\Delta_\Gamma u$ and $\Delta_\Gamma v$ denote the Laplace-Beltrami operator on the surface Γ and $a_1, \ldots, a_6$ are nonnegative constants while D denotes the quotient of the cytosolic diffusion and the lateral membrane diffusion constants, which typically is very large. Throughout the whole paper we assume that both active and inactive proteins diffuse on the membrane with the same rate. However we stress that having different diffusion rates for u and v would not affect the subsequent analysis, since the diffusion of the inactive protein on the membrane vanished in our scaling limit (cf. (A.2.1.3)). Furthermore, setting $f_1(u) := a_1 + \frac{a_2 u}{a_3 + u}$ and $f_2(u) := \frac{u}{1+u}$, we note that in principle both f_1, f_2 could be replaced by any continuously differentiable increasing functions such that $f_1(0) \geq 0, f_2(0) = 0$ with f_1 becoming constant and f_2 having a positive limit as u becomes large. In that case, the rescaling in (A.2.1.1) below is then adapted to $(f_1, f_2) \rightsquigarrow (f_1, \epsilon^{-1} f_2)$.

We complement the system with initial conditions:

$$u(\cdot, 0) = u_{\text{in}}\,, \quad v(\cdot, 0) = v_{\text{in}} \quad \text{on } \Gamma\,, \quad w(\cdot, 0) = w_{\text{in}} \quad \text{in } \Omega\,, \tag{A.1.5}$$

where $u_{\text{in}}, v_{\text{in}} : \Gamma \to [0, \infty)$ and $w_{\text{in}} : \Omega \to [0, \infty)$ are given nonnegative data.

The system (A.1.1)-(A.1.4) contains two parts. On the one hand, we have a reaction-diffusion system on the membrane for the variables u and v, with a w-dependent source term. On the other hand, there is a diffusion equation for w in the interior of the cell with a nonlinear Robin-type boundary condition that depends on u and v. Solutions of (A.1.1)-(A.1.5) satisfy the mass conservation property

$$\int_\Omega w(\cdot, t)\, dx + \int_\Gamma \Big(u(\cdot, t) + v(\cdot, t)\Big)\, dS = \int_\Omega w_{\text{in}}\, dx + \int_\Gamma \Big(u_{\text{in}} + v_{\text{in}}\Big)\, dS \tag{A.1.6}$$

for all $t \in (0, T)$.

In addition to (A.1.1)-(A.1.4) we will study a reduced system that is obtained in the limit of infinite cytosolic diffusivity, which is motivated by the fact that cytosolic diffusion within the cell is by a factor of hundred larger than the lateral diffusion on the membrane

[34]. In this limit the cytosolic concentration becomes spatially constant and $w = w(t)$ is determined by the total mass conservation, i.e.

$$|\Omega| w(t) = m - \int_\Gamma \Big(u(\cdot,t) + v(\cdot,t) \Big)\, dS\,, \tag{A.1.7}$$

where m is the total amount of protein. The reduction for $D = \infty$ leads to a nonlocal reaction-diffusion system on $\Gamma \times (0,T)$, given by (A.1.1), (A.1.2) and (A.1.7), complemented by initial conditions for u and v. This reduction can be viewed as a kind of shadow system. Such systems have been analyzed intensively in the case of two-variable reaction-diffusion systems in open domains [28, 33, 42], and in the context of obstacle problems in [64].

Under the assumption that the reaction rates a_4, a_5 and a_6, the diffusion coefficient D and the total mass of proteins are of order ε^{-1}, we will prove that solutions converge in the *large reaction rate limit* $\varepsilon \to 0$ to solutions of certain reduced systems. Although so far we do not have any experimental data justifying the fast reactions as a_4, a_5 and a_6 become large, these parameters undoubtedly play a crucial role in the clear distinction of regions in which the concentrations of some chemicals have different orders of magnitude. First, we will investigate the limit of infinite cytosolic diffusivity. After appropriate rescaling and renaming (cf. (A.2.1.1)), taking the limit $\varepsilon \to 0$, yields the following parabolic obstacle-type problem

$$\partial_t u - \Delta u = -a_4(1-g)\xi + \alpha g \qquad \text{on } \Gamma \times (0,T)\,, \tag{A.1.8}$$

$$u \geq 0\,, \quad u\xi = u\,, \quad 0 \leq \xi \leq 1 \qquad \text{on } \Gamma \times (0,T)\,, \tag{A.1.9}$$

$$u(\cdot,0) = u_0 \qquad \text{on } \Gamma\,, \tag{A.1.10}$$

where u_0 is the limit of suitably rescaled versions of u_{in} (cf. (A.2.1.6)), the function $g : \Gamma \times (0,T) \to (0,1)$ is given by

$$g(x,t) = \frac{c(x,t)}{c(x,t) + a_5}\,, \tag{A.1.11}$$

and $\alpha : (0,T) \to \mathbb{R}$ only depends on time and is determined by a solvability condition for (A.1.8), see (A.2.1.25). This function α plays the role a Lagrange multiplier associated to the mass conservation property

$$\int_\Gamma u(\cdot,t)\, dS = \int_\Gamma u_0\, dS \qquad \text{for all } t \in (0,T)\,,$$

that is satisfied in the limit.

In the case $D < \infty$ equation (A.1.8) changes and we obtain the system

$$\partial_t u = \Delta u - a_4(1-g)\xi + a_6 g w \qquad \text{on } \Gamma \times (0,T)\,, \tag{A.1.12}$$

$$
\begin{aligned}
0 &= \Delta w \quad \text{ in } \Omega_T, && \text{on } \Gamma \times (0,T)\,, && \text{(A.1.13)}\\
D\frac{\partial w}{\partial n} &= a_4(1-g)\xi - a_6 g w && \text{on } \Gamma \times (0,T)\,, && \text{(A.1.14)}\\
u \geq 0\,, \quad u\xi &= u\,, \quad 0 \leq \xi \leq 1 && \text{on } \Gamma \times (0,T)\,, && \text{(A.1.15)}\\
u(\cdot,0) &= u_0 && \text{on } \Gamma\,. && \text{(A.1.16)}
\end{aligned}
$$

The analogy to $D = \infty$ is even more apparent if one expresses w as a nonlocal operator of u. A particularly convenient form is presented in Proposition A.2.7.

Stationary solutions of model (A.1.1)-(A.1.4) and the corresponding scaling limits have already been studied in [51]. In particular, in addition to well-posedness, the onset of polarization is studied in [51] for sufficiently small (rescaled) mass of protein. The goal of the present paper is to complement this analysis. The main contributions are a rigorous justification of the asymptotic reduction, the well-posedness of the evolutionary obstacle-type problem that we obtain in the limit, an L^1-contraction property of solutions, and the global stability of steady states.

Parabolic obstacle problems appear in various applications and have have been studied intensively over the past decades [24]. For example, the one-phase Stefan problem can be written as a parabolic obstacle problem by a suitable transformation that was first proposed by Duvaut [15]. In the context of fluid flows in porous media the Baiocchi transform [2] also leads to an obstacle problem. Obstacle problems belong to a class of free boundary problems that can be formulated as variational inequalities, i.e. inequalities for bilinear functionals which are satisfied for functions u and test functions in a space satisfying inequalities of the form $u \geq \psi$. Alternatively, under some regularity assumptions it is possible to reformulate the same class of free boundary problems as PDEs in which an unknown function ξ satisfies an inequality almost everywhere in the set in which the PDEs are solved. Both formulations can be found for example in [37, 62]. The equivalence between both approaches can be seen using the so-called Stampacchia Lemma [62, Section 5:3, Theorem 5:4.3]. In this paper we will only use the second approach. Therefore, in addition to the unknown u we must determine also an auxiliary function $\xi \in [0,1]$ such that $\xi = 1$ in $\{u > 0\}$.

The connection of our limit problems with the parabolic obstacle problem is best seen for the reduced model (A.1.8)-(A.1.9). In Remark A.2.3 we derive the following characterization of solutions,

$$
\partial_t u - \Delta u + \Big(a_4(1-g) - \alpha g\Big) = \Big(a_4(1-g) - \alpha g\Big)_+ \mathcal{X}_{\{u=0\}}, \qquad u \geq 0, \qquad \text{(A.1.17)}
$$

with $\alpha = \alpha(t)$ given as a nonlocal function of u, more precisely

$$\alpha(t) = \frac{a_4 \int_{\{u(\cdot,t)>0\}} \Big(1 - g(\cdot,t)\Big)\, dS}{\int_{\{u(\cdot,t)>0\}} g(\cdot,t)\, dS},$$

see (A.2.1.25) and (A.2.1.27). In the formulation (A.1.17) the problem corresponds to the classical parabolic obstacle model, where $a_4(1-g) - \alpha g$ is replaced by some given function f independent of u. Defining $H(u) := \partial_t u - \Delta u + \Big(a_4(1-g) - \alpha g\Big)$, the problem (A.1.17) can be written as

$$uH(u) = 0, \qquad H(u) \geq 0, \qquad u \geq 0,$$

and can be expressed as a variational inequality, see for example [43, Section II.9.1].

One of the features of the free boundary problems considered in this paper is the presence of some terms in the equations that depend in a non-local way on the solution u itself. In the case of problem (A.1.8)-(A.1.10), the non-locality is introduced by the dependence of α in (A.1.17) on the positivity set $\{u > 0\}$. For the bulk-surface problem (A.1.12)-(A.1.16) the non-local dependence takes place through the function w which solves the elliptic problem (A.1.13), (A.1.14). We remark that free boundary problems containing dependences on the positivity set of the solution itself (i.e. $\{u > 0\}$) have been considered in [64].

Several of the technical difficulties that we need to address in this paper are due to the fact that the function α changes in a discontinuous manner if the positivity set $\{u(\cdot,t) > 0\}$ changes discontinuously in time. However, to prove that $\{u(\cdot,t) > 0\}$ changes continuously in time is not an easy task and we expect that jumps of this set are possible in some situations. We will address the continuity properties of $\{u(\cdot,t) > 0\}$ and α in future work, but remark here that possible jumps of the functions $t \mapsto \{u(\cdot,t) > 0\}$ and $t \mapsto \alpha(t)$ are the main reason for several of the most technical points of this paper.

Compared to [51] the main novelty of this paper is to introduce some monotonicity formulas which allow us to prove uniqueness of solutions and also uniqueness and stability of steady states of the problems (A.1.8)-(A.1.10) and (A.1.12)-(A.1.15).

Uniqueness of the steady states associated to the problem (A.1.8)-(A.1.10) has been proved in [51] using a completely different approach. Similar uniqueness results have been obtained in [51] for the stationary states of (A.1.12)-(A.1.15) in the particular case in which the domain Ω is a ball. The monotonicity formulas introduced in this paper (cf. Sections A.3 and A.4) imply that the evolution semigroup associated to the problems (A.1.8)-(A.1.10) and (A.1.12)-(A.1.15) is contractive in the L^1 norm. It is worth to remark that the existence of these monotonicity formulas rely on a delicate balance of the terms

$-a_4(1-g)\xi$ and αg, $a_5 gw$ in (A.1.8), (A.1.12). The term $-a_4(1-g)\xi$ has a stabilizing effect, which is similar to the analogous term arising in the study of the reformulation of the one-phase Stefan problem due to Duvaut [15]. On the other hand the terms αg, $a_5 gw$ in (A.1.8), (A.1.12) depend on functions determined as a non-local functional of u (namely α and w respectively). These terms have a destabilizing effect on the solutions of the problems (A.1.8)-(A.1.10) and (A.1.12)-(A.1.15), but some cancellations between the contributions of both terms in the derivative of the L^1 norm of the difference of two solutions of these problems yield an overall stabilizing effect.

The plan of this paper is the following. Section A.2 is devoted to establishing the convergence of solutions in the fast reaction limit to the limiting obstacle-type problems. In Section A.2.1 we will first investigate the case of infinite cytosolic diffusion $D = \infty$, introduce a suitable rescaled system (A.2.1.2)-(A.2.1.6) and prove the convergence to (A.1.8)-(A.1.10) (cf. Theorem A.2.2). In Section A.2.2 we consider the analogous problem for finite cytosolic diffusion coefficients D. We derive in a scaling limit analogous to the case $D = \infty$ the generalized obstacle-type problem (A.1.12)-(A.1.16) in Theorem A.2.5. Section A.3 focuses on the case $D = \infty$. In Section A.3.1 we justify an L^1-contraction property and the uniqueness of solutions of problem (A.1.8)-(A.1.10) (Theorem A.3.1) while in Section A.3.2 we will show the global stability of the steady states (Theorem A.3.2). In Section A.4 we study the reduced model for finite cytosolic diffusion $D < \infty$. We prove an L^1-contraction property and the uniqueness of solutions of problem (A.1.12)-(A.1.16) in Section A.4.1, see Theorem A.4.1. We also include a monotonicity property and a uniqueness result for solutions of the stationary problem in Theorem A.4.2. This improves the corresponding result from [?] that was only shown for $\Omega = B(0,1)$ there. Along the lines of Section A.3.2, we will further show in Section A.4.2, Theorem A.4.3, that steady states are globally stable.

A.1.1 Notation and Assumptions

Notations: For a set $\Omega \subset \mathbb{R}^3$ we denote by $|\Omega| = \mathcal{L}^3(\Omega)$ the Lebesgue measure. For a surface $\Gamma \subset \mathbb{R}^3$ we denote by $|\Gamma| = \mathcal{H}^2(\Gamma)$ its area (i.e. the 2-dimensional Hausdorff measure) and by $\int_\Gamma \cdot \, dS$ the corresponding surface integral.

For the sake of convenience, Ω_T and Γ_T stand for $\Omega \times (0,T)$ and $\Gamma \times (0,T)$ respectively. For the Laplace-Beltrami operator on Γ we just write Δ instead of Δ_Γ if there is no reason for confusion.

We denote the usual Sobolev spaces by $W^{k,p}(U)$ and the parabolic Sobolev spaces by $W_p^{k,k/2}(U_T)$, where $U = \Omega$ or $U = \Gamma$, $k \in \mathbb{N}_0$, $1 \le p \le \infty$. The Hölder and parabolic Hölder spaces are denoted by $C^\alpha(U)$ and $C^{\alpha,\alpha/2}(U_T)$, respectively, for $0 < \alpha < 1$. The weak parabolic solution spaces are denotes by $V_2(U_T) := L^2(0,T;H^1(U)) \cap H^1(0,T;H^1(U)^*)$.

Assumptions: Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with C^3-regular boundary $\Gamma = \partial\Omega$. Assume $a_1, a_2 \geq 0$, $a_3, a_4, a_5, a_6 > 0$ and $D \geq 1$ and that $c : \Gamma_T \to \mathbb{R}_+$ is smooth and that there exists $c_0 > 0$ with

$$c(x,t) \geq c_0 > 0 \quad \text{for all } (x,t) \in \Gamma_T\,. \tag{A.1.1.1}$$

A.2 The fast reaction limit

A.2.1 Convergence to a parabolic obstacle-type problem for $D = \infty$

In this section we consider the case of infinite cytosolic diffusion coefficient, that is we consider solutions to (A.1.1), (A.1.2) and (A.1.5) together with (A.1.7). It follows from [29] that for given $m > 0$ and for nonnegative data $u_{\text{in}}, v_{\text{in}} \in L^2(\Gamma)$ with $\int_\Gamma (u_{\text{in}} + v_{\text{in}})\, dS \leq m$ there exists a nonnegative solution (u,v,w) with $u, v \in V_2(\Gamma_T)$ and $w \in W^{1,\infty}(0,T)$. In fact, although the analysis in [29] does not consider nonconstant c all arguments easily carry over to the present case.

Our goal in this section is to consider a suitable scaling limit of the system (A.1.1), (A.1.2), (A.1.5) and (A.1.7). More precisely, for small $\varepsilon > 0$ we introduce the following rescalings

$$a_4 = \frac{\hat{a}_4}{\varepsilon}, \quad a_5 = \frac{\hat{a}_5}{\varepsilon}, \quad a_6 = \frac{\hat{a}_6}{\varepsilon}, \quad c = \frac{\hat{c}}{\varepsilon} \quad \text{and} \quad m = \frac{\hat{m}}{\varepsilon} \tag{A.2.1.1}$$

with $\hat{a}_4, \hat{a}_5, \hat{a}_6, \hat{c}$ and $\hat{m}$ being positive and of order one. We denote the corresponding solutions by $u_\varepsilon, v_\varepsilon$ and w_ε and let $U_\varepsilon := \varepsilon u_\varepsilon$. Dropping the hats again, we can rewrite (A.1.1), (A.1.2), (A.1.5) and (A.1.7) as

$$\partial_t U_\varepsilon = \Delta U_\varepsilon + \Big(\varepsilon a_1 + \frac{\varepsilon a_2 U_\varepsilon}{\varepsilon a_3 + U_\varepsilon} + c\Big) v_\varepsilon - \frac{a_4 U_\varepsilon}{\varepsilon + U_\varepsilon} \qquad \text{on } \Gamma_T\,, \tag{A.2.1.2}$$

$$\varepsilon \partial_t v_\varepsilon = \varepsilon \Delta v_\varepsilon - \Big(\varepsilon a_1 + \frac{\varepsilon a_2 U_\varepsilon}{\varepsilon a_3 + U_\varepsilon} + c\Big) v_\varepsilon + \frac{a_4 U_\varepsilon}{\varepsilon + U_\varepsilon} - a_5 v_\varepsilon + a_6 w_\varepsilon \qquad \text{on } \Gamma_T\,, \tag{A.2.1.3}$$

$$\varepsilon |\Omega| w_\varepsilon(t) = m - \int_\Gamma (U_\varepsilon(x,t) + \varepsilon v_\varepsilon(x,t))\, dS \qquad \text{for a.a. } t \in (0,T)\,. \tag{A.2.1.4}$$

For given nonnegative, smooth functions $U_0^\varepsilon, v_0^\varepsilon : \Gamma \to \mathbb{R}$ with $\int_\Gamma \Big(U_0^\varepsilon + \varepsilon v_0^\varepsilon\Big) \leq m$ we prescribe the initial conditions

$$U_\varepsilon(\cdot, 0) = U_0^\varepsilon\,, \quad v_\varepsilon(\cdot,0) = v_0^\varepsilon \quad \text{on } \Gamma\,. \tag{A.2.1.5}$$

In order to obtain a nontrivial limit, we assume for the initial data that

$$U_0^\varepsilon \to u_0 \quad \text{in } L^2(\Gamma) \text{ as } \varepsilon \to 0\,, \quad \sup_{\varepsilon>0}\Big[\int_\Gamma |v_0^\varepsilon|^2\,dS + \frac{1}{\varepsilon}\Big(m - \int_\Gamma \Big(U_0^\varepsilon + \varepsilon v_0^\varepsilon\Big)\,dS\Big)\Big] \le C \tag{A.2.1.6}$$

for some $u_0 \in L^2(\Gamma)$ with $\int_\Gamma u_0\,dS = m$ and some $C > 0$.

We first prove some uniform estimates.

Theorem A.2.1. *For any nonnegative solution* $(U_\varepsilon, v_\varepsilon, w_\varepsilon)$ *of* (A.2.1.2)-(A.2.1.5) *we have*

$$\|U_\varepsilon\|_{V_2(\Gamma_T)} + \|v_\varepsilon\|_{L^\infty(0,T;L^2(\Gamma))} + \|w_\varepsilon\|_{L^\infty(0,T)} \le C\,, \tag{A.2.1.7}$$

where here and in the following C denotes a constant that depends on the data of the problem but not on ε.

Proof. By virtue of (A.2.1.3) we compute

$$\begin{aligned}\frac{d}{dt}\int_\Gamma \frac{\varepsilon a_5 v_\varepsilon^2}{2}\,dS = &-\int_\Gamma \varepsilon a_5|\nabla v_\varepsilon|^2\,dS - \int_\Gamma a_5\Big(\varepsilon a_1 + \frac{\varepsilon a_2 U_\varepsilon}{\varepsilon a_3 + U_\varepsilon} + c\Big)v_\varepsilon^2\,dS\\ &+ \int_\Gamma \frac{a_4 a_5 U_\varepsilon v_\varepsilon}{\varepsilon + U_\varepsilon}\,dS - \int_\Gamma \Big((a_5 v_\varepsilon)^2 - a_5 a_6 v_\varepsilon w_\varepsilon\Big)\,dS\,.\end{aligned} \tag{A.2.1.8}$$

We observe that (A.2.1.2)-(A.2.1.4) imply that

$$\varepsilon|\Omega|\frac{d}{dt}w_\varepsilon = \int_\Gamma (a_5 v_\varepsilon - a_6 w_\varepsilon)\,dS\,, \qquad \varepsilon|\Omega|w_\varepsilon(0) = m - \int_\Gamma \Big(U_0^\varepsilon + \varepsilon v_0^\varepsilon\Big)\,dS \tag{A.2.1.9}$$

and obtain

$$\varepsilon|\Omega|a_6\frac{d}{dt}\frac{1}{2}w_\varepsilon^2 = a_5 a_6\int_\Gamma v_\varepsilon w_\varepsilon\,dS - a_6^2|\Gamma|w_\varepsilon^2. \tag{A.2.1.10}$$

Taking the sum of (A.2.1.8) and (A.2.1.10) and using $c \ge c_0 > 0$ yields

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Gamma \frac{\varepsilon a_5 v_\varepsilon^2}{2}\,dS + \varepsilon|\Omega|a_6\frac{1}{2}w_\varepsilon^2\Big) + \int_\Gamma \varepsilon a_5|\nabla v_\varepsilon|^2\,dS + c_0\int_\Gamma a_5 v_\varepsilon^2\,dS\\ &\le \int_\Gamma a_4 a_5 v_\varepsilon\,dS - \int_\Gamma \Big(a_5^2 v_\varepsilon^2 - 2a_5 a_6 v_\varepsilon w_\varepsilon + a_6^2 w_\varepsilon^2\Big)\,dS\\ &\le \int_\Gamma \frac{c_0 a_5}{4}v_\varepsilon^2 + \frac{1}{c_0}a_4^2 a_5 + \Big(\frac{1}{\delta} - 1\Big)a_5^2 v_\varepsilon^2 - (1-\delta)a_6^2 w_\varepsilon^2\,dS\,,\end{aligned}$$

where we have used Young's inequality and where $\delta > 0$ is arbitrary.

We next choose $\delta < 1$ sufficiently close to one such that $(\frac{1}{\delta} - 1\Big)a_5 < \frac{1}{4}c_0$ and obtain

$$\varepsilon \frac{d}{dt}\Big(\int_\Gamma v_\varepsilon^2 \, dS + w_\varepsilon^2 \Big) \leq C - \Big(\int_\Gamma v_\varepsilon^2 \, dS + w_\varepsilon^2 \Big).$$

Using (A.2.1.6) we deduce

$$\int_\Gamma v_\varepsilon^2(\cdot, t) \, dS + w_\varepsilon^2(t) \leq C \quad \text{for all } 0 \leq t \leq T. \tag{A.2.1.11}$$

This implies the required bounds for $v_\varepsilon, w_\varepsilon$.

Furthermore, by these estimates the reaction-terms on the right-hand side of (A.2.1.2) are uniformly bounded in $L^2(\Gamma_T)$. Parabolic weak solution theory, see [74, Theorem 26.1], and (A.2.1.6) imply the uniform bound for U_ε, which finishes the proof of (A.2.1.7). ■

Theorem A.2.2. *Suppose that* $\{(U_\varepsilon, v_\varepsilon, w_\varepsilon)\}_{\varepsilon>0}$ *is a family of nonnegative solutions of* (A.2.1.2)-(A.2.1.5) *and assume* (A.2.1.6). *Then there exist a subsequence* $\varepsilon \to 0$, *a nonnegative function* $u \in V_2(\Gamma_T)$ *and a measurable function* ξ *such that*

$$U_\varepsilon \rightharpoonup u \quad \text{in } V_2(\Gamma_T)\,, \tag{A.2.1.12}$$

$$\frac{U_\varepsilon}{U_\varepsilon + \varepsilon} \overset{*}{\rightharpoonup} \xi \quad \text{weakly* in } L^\infty(\Gamma_T) \tag{A.2.1.13}$$

as $\varepsilon \to 0$. *Moreover,* $u \in W_p^{2,1}(\Gamma \times (\delta, T))$ *for any* $\delta > 0$, $1 \leq p < \infty$, *with*

$$\|u\|_{W_p^{2,1}(\Gamma\times(\delta,T))} \leq C(p, \delta, T), \tag{A.2.1.14}$$

and $\int_\Gamma u(\cdot, t) \, dS = m$ *holds for all* $t \in [0, T]$.

Finally, there exists a nonnegative function $\alpha \in L^\infty(0, T)$ *such that* (A.1.8)-(A.1.10) *are satisfied.*

Proof. By Theorem A.2.1 there exists a subsequence $\varepsilon \to 0$ (not relabeled) and functions $u \in V_2(\Gamma_T)$, $v \in L^\infty(0, T; L^2(\Gamma))$, $w \in L^\infty(0, T)$ and $\xi \in L^\infty(0, T)$ such that

$$\begin{aligned} U_\varepsilon &\rightharpoonup u \quad \text{in } V_2(\Gamma_T),\\ v_\varepsilon &\overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0, T; L^2(\Gamma)),\\ w_\varepsilon &\overset{*}{\rightharpoonup} w \quad \text{in } L^\infty(0, T),\\ \frac{U_\varepsilon}{U_\varepsilon + \varepsilon} &\overset{*}{\rightharpoonup} \xi \quad \text{in } L^\infty(\Gamma_T). \end{aligned}$$

By the Aubin-Lion's compactness Lemma [1, 67] we also have

$$U_\varepsilon \to u \quad \text{in } L^2(\Gamma_T). \tag{A.2.1.15}$$

With these convergence properties we can pass to the limit in the weak form of (A.2.1.2) and conclude that for any $\phi \in C_c^1(\Gamma \times [0,T))$ it holds

$$\int_{\Gamma_T} \partial_t \phi (u - u_0)\, dS\, dt = \int_{\Gamma_T} \Big(\nabla \phi \cdot \nabla u - \phi\big(cv - a_4 \xi(\cdot, t)\big)\Big)\, dS\, dt\,. \tag{A.2.1.16}$$

In particular, (A.1.8) holds in $H^{-1}(\Gamma)$ for almost all $t \in (0,T)$. Moreover, by [74, Theorem 25.5] we have $u \in C^0([0,T]; L^2(\Gamma))$, and $u(\cdot,0) = u_0$ holds in the sense that $u_0 = \lim_{t\searrow 0} u(\cdot,t)$ in $L^2(\Gamma)$.

Let $\phi \in C_c^{2,1}(\Gamma_T)$ be an arbitrary test function. Multiplying (A.2.1.3) by ϕ and integrating over Γ_T we deduce, after integrating by parts, that

$$\begin{aligned}-\varepsilon \int_0^T \langle v_\varepsilon, \partial_t \phi\rangle\, dt &= \varepsilon \int_0^T \int_\Gamma v_\varepsilon \Delta\phi\, dS\, dt - \int_0^T \int_\Gamma \Big(\varepsilon a_1 + \frac{\varepsilon a_2 U_\varepsilon}{\varepsilon a_3 + U_\varepsilon} + c\Big) v_\varepsilon \phi\, dS\, dt \\ &\quad + \int_0^T \int_\Gamma \frac{a_4 U_\varepsilon \phi}{\varepsilon + U_\varepsilon}\, dS\, dt - \int_0^T \int_\Gamma a_5 v_\varepsilon \phi\, dS\, dt + \int_0^T a_6 w_\varepsilon \int_\Gamma \phi\, dS\, dt\,. \end{aligned} \tag{A.2.1.17}$$

Taking the limit in $(A.2.1.17)$ we obtain

$$0 = -\int_0^T \int_\Gamma (c + a_5) v\phi\, dS\, dt - a_4 \int_0^T \int_\Gamma \xi\phi\, dS\, dt + \int_0^T a_6 w \int_\Gamma \phi\, dS\, dt\,,$$

hence

$$0 = -(c + a_5)v - a_4\xi + a_6 w \quad \text{a.e. in } \Gamma_T. \tag{A.2.1.18}$$

Similarly we deduce from (A.2.1.4) that

$$0 = m - \int_\Gamma u(\cdot,t)\, dS \quad \text{in } (0,T) \tag{A.2.1.19}$$

and from (A.2.1.9)

$$0 = \int_\Gamma (a_5 v(\cdot,t) - a_6 w(t))\, dS \quad \text{a.e. in } (0,T). \tag{A.2.1.20}$$

Finally, we define

$$\alpha(t) = \frac{a_5}{|\Gamma|} \int_\Gamma v(\cdot,t)\, dS$$

such that (A.2.1.18) and (A.2.1.20) imply

$$v = -\frac{a_4}{c + a_5}\xi + \frac{\alpha}{c + a_5} \qquad \text{a.e. in } \Gamma_T\,. \tag{A.2.1.21}$$

Due to the boundedness of g and ξ we can apply parabolic $W_p^{2,1}$-regularity theory to (A.1.8). In fact, fix arbitrary $\delta > 0$ and $p \geq 1$. Choose a smooth cut-off function $\eta \in C_c^\infty((\frac{\delta}{2}, T])$,

$\eta = 1$ in $[\delta, T]$ and use a smooth partition of unity for Γ subordinate to a covering of Γ by parametrized surface patches. In local coordinates we obtain that ηu solves a parabolic equation with bounded continuous coefficients, hence [39, Theorem IV.9.1] yields the $W_p^{2,1}$-regularity of ηu in local coordinates, with an estimate for the corresponding norms only depending on the data. Using the compactness of Γ we finally deduce the $W_p^{2,1}$-regularity of ηu, hence (A.2.1.14) holds.

From (A.2.1.15) we obtain for any test function $\phi \in C^0(\Gamma \times [0,T])$ that

$$\int_0^T \int_\Gamma \phi(\xi u - u)\, dS\, dt = \lim_{\epsilon \to 0} \int_0^T \int_\Gamma \phi\Big(\frac{U_\varepsilon}{U_\varepsilon + \varepsilon} - 1\Big) U_\varepsilon\, dS\, dt = -\lim_{\epsilon \to 0} \int_0^T \int_\Gamma \frac{\varepsilon \phi U_\varepsilon}{\varepsilon + U_\varepsilon}\, dS\, dt = 0\,, \tag{A.2.1.22}$$

which implies $\xi u = u$. ■

Remark A.2.3. By Stampacchia's Lemma [19, Theorem 4.4] and the $W_p^{2,1}(\Gamma \times (\delta, T))$-regularity of u for any $\delta > 0$ one obtains $\partial_t u = \Delta u = 0$ almost everywhere in $\{u = 0\}$. In fact, we can apply the lemma to $W^{1,p}(\Gamma_T)$ and obtain the claim for $\partial_t u$, and then to $W^{2,p}(\Gamma)$ for almost all t to obtain the corresponding property for Δu. This in particular yields the representation formula

$$\xi(\cdot, t) = \begin{cases} 1 & \text{a.e in } \{u(\cdot,t) > 0\} \\ \frac{a_4 \alpha(t) g(\cdot,t)}{1 - g(\cdot,t)} & \text{a.e in } \{u(\cdot,t) = 0\} \end{cases} \tag{A.2.1.23}$$

for almost all $t \in (0,T)$. By $\xi \le 1$ we deduce that

$$\alpha g \le 1 - g \quad \text{almost everywhere in } \{u = 0\}. \tag{A.2.1.24}$$

Moreover, by an integration of (A.1.8) over Γ and by (A.2.1.23) we deduce that

$$\alpha(t) = \frac{\int_\Gamma a_4(1-g)(\cdot,t)\xi(\cdot,t)\, dS}{\int_\Gamma g\, dS} = \frac{\int_{\{u(\cdot,t)>0\}} a_4(1-g)(\cdot,t)\, dS}{\int_{\{u(\cdot,t)>0\}} g(\cdot,t)\, dS} \tag{A.2.1.25}$$

for almost all $t \in (0,T)$. Note that the second equality in (A.2.1.25) shows that α is already determined by u and the data. Similarly, for any measurable set $A \supset \{u(\cdot,t) > 0\}$ we deduce that

$$\alpha(t) = \frac{\int_A a_4(1-g)(\cdot,t)\xi(\cdot,t)\, dS}{\int_A g(\cdot,t)\, dS} \tag{A.2.1.26}$$

holds for almost all $t \in (0,T)$.

We derive a further characterization of solutions. By the properties obtained so far we deduce from (A.1.8), $\xi \leq 1$ and (A.2.1.25) that

$$\partial_t u - \Delta u = -a_4(1-g) + \alpha g + \Big(a_4(1-g) - \alpha g\Big)_+ \mathcal{X}_{\{u=0\}}. \tag{A.2.1.27}$$

Vice versa, this equation implies (A.1.8), with ξ as (A.2.1.23), and the conditions on ξ in (A.1.9).

A.2.2 Convergence to a parabolic obstacle-type problem for $D < \infty$

We now consider the case of finite cytosolic diffusion $D < \infty$. In [29] it is proved that also in this case the system (A.1.1)-(A.1.5) has a unique nonnegative solution (u, v, w) with $u, v \in V_2(\Gamma_T)$ and $w \in V_2(\Omega_T)$, provided that the initial data are such that $u_0^\varepsilon, v_0^\varepsilon \in L^2(\Gamma)$ and $w_0^\varepsilon \in L^2(\Omega)$. Again, this result first only covers the case of constant c. The proof, however, carries over to the present case.

For finite D we use a similar rescaling of the general model (A.1.1)-(A.1.5) as in the previous subsection but consider in addition to (A.2.1.1) that D becomes large with $\varepsilon \to 0$, more precisely $D = \frac{\hat{D}}{\varepsilon}$. This yields, after dropping the hats, the system

$$\partial_t U_\varepsilon = \Delta U_\varepsilon + \Big(\varepsilon a_1 + \frac{\varepsilon a_2 U_\varepsilon}{\varepsilon a_3 + U_\varepsilon} + c\Big) v_\varepsilon - \frac{a_4 U_\varepsilon}{\varepsilon + U_\varepsilon} \qquad \text{on } \Gamma_T\,, \tag{A.2.2.1}$$

$$\varepsilon \partial_t v_\varepsilon = \varepsilon \Delta v_\varepsilon - \Big(\varepsilon a_1 + \frac{\varepsilon a_2 U_\varepsilon}{\varepsilon a_3 + U_\varepsilon} + c\Big) v_\varepsilon + \frac{a_4 U_\varepsilon}{\varepsilon + U_\varepsilon} - a_5 v_\varepsilon + a_6 w_\varepsilon \qquad \text{on } \Gamma_T\,, \tag{A.2.2.2}$$

$$\varepsilon \partial_t w_\varepsilon = D \Delta w_\varepsilon \qquad \text{on } \Omega_T\,, \tag{A.2.2.3}$$

$$-D\frac{\partial w_\varepsilon}{\partial n} = -a_5 v_\varepsilon + a_6 w_\varepsilon \qquad \text{on } \Gamma_T\,, \tag{A.2.2.4}$$

$$U_\varepsilon(\cdot,0) = U_0^\varepsilon\,, \qquad v_\varepsilon(\cdot,0) = v_0^\varepsilon\,, \quad w_\varepsilon(\cdot,0) = w_0^\varepsilon\,, \tag{A.2.2.5}$$

where $\int_\Gamma \Big(U_0^\varepsilon + \varepsilon v_0^\varepsilon\Big)\, dS + \int_\Omega \varepsilon w_0^\varepsilon \, dx = m$.

Similarly as in Section A.2.1 we assume that for some $u_0 \in L^2(\Gamma)$ with $\int_\Gamma u_0 = m$ and some $C > 0$ we have

$$U_0^\varepsilon \to u_0 \quad \text{in } L^2(\Gamma), \qquad \sup_{\varepsilon>0} \Big(\int_\Gamma |v_0^\varepsilon|^2 \, dS + \int_\Omega |w_0^\varepsilon|^2 \, dx\Big) \leq C. \tag{A.2.2.6}$$

We recall that a solution conserves the mass, that is

$$\int_\Omega \varepsilon w_\varepsilon(\cdot,t)\, dx + \int_\Gamma \Big(U_\varepsilon(\cdot,t) + \varepsilon v_\varepsilon(\cdot,t)\Big)\, dS = m \quad \text{ for all } t \in (0,T)\,. \tag{A.2.2.7}$$

We first prove some uniform bounds.

Theorem A.2.4. *For any nonnegative solution* $(U_\varepsilon, v_\varepsilon, w_\varepsilon)$ *of* (A.2.2.1)-(A.2.2.6) *we have*

$$\|U_\varepsilon\|_{V_2(\Gamma_T)} + \|v_\varepsilon\|_{L^\infty(0,T;L^2(\Gamma))} + \|w_\varepsilon\|_{L^\infty(0,T;L^2(\Gamma)))} + \|w_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C\,. \tag{A.2.2.8}$$

Proof. As in the proof of Theorem A.2.1 we test (A.2.2.2) with $a_5 v_\varepsilon$ and obtain

$$\frac{d}{dt}\int_\Gamma \frac{\varepsilon a_5 v_\varepsilon^2}{2}\,dS = -\int_\Gamma \varepsilon a_5 |\nabla v_\varepsilon|^2\,dS - \int_\Gamma a_5\Big(\varepsilon a_1 + \frac{\varepsilon a_2 U_\varepsilon}{\varepsilon a_3 + U_\varepsilon} + c\Big) v_\varepsilon^2\,dS + \int_\Gamma \frac{a_4 a_5 U_\varepsilon v_\varepsilon}{\varepsilon + U_\varepsilon}\,dS$$
$$- \int_\Gamma \Big((a_5 v_\varepsilon)^2 - a_5 a_6 v_\varepsilon w_\varepsilon\Big)\,dS\,.$$

By virtue of (A.2.2.3) and (A.2.2.4) we compute

$$\frac{d}{dt}\int_\Omega \frac{\varepsilon a_6 w_\varepsilon^2}{2}\,dx = -\int_\Omega a_6 D|\nabla w_\varepsilon|^2\,dx + \int_\Gamma (a_5 a_6 w_\varepsilon v_\varepsilon - a_6^2 w_\varepsilon^2)\,dS\,.$$

Combining both inequalities and using $c \ge c_0 > 0$ and $\frac{U_\varepsilon}{\varepsilon + U_\varepsilon} \le 1$ implies

$$\frac{d}{dt}\Big(\int_\Gamma \frac{\varepsilon a_5 v_\varepsilon^2}{2}\,dS + \int_\Omega \frac{\varepsilon a_6 w_\varepsilon^2}{2}\,dx\Big) + \int_\Gamma \varepsilon a_5 |\nabla v_\varepsilon|^2\,dS + \int_\Omega a_6 D|\nabla w_\varepsilon|^2\,dx + \int_\Gamma a_5 c_0 v_\varepsilon^2\,dS$$
$$\le \int_\Gamma a_4 a_5 v_\varepsilon\,dS - \int_\Gamma \Big(a_5^2 v_\varepsilon^2 - 2a_5 a_6 v_\varepsilon w_\varepsilon + a_6^2 w_\varepsilon^2\Big)\,dS$$
$$\le \frac{1}{2}\int_\Gamma a_5 c_0 v_\varepsilon^2\,dS + C - c\int_\Gamma w_\varepsilon^2\,dS, \tag{A.2.2.9}$$

where in the last step we have used a Youngs inequality as in the derivation of (A.2.1.11), and where $C, c > 0$ only depend on the data. Next, applying Poincaré's inequality for functions with mean value zero on the boundary, we deduce

$$\int_\Omega w_\varepsilon^2\,dx \le 2\int_\Omega \Big|w_\varepsilon - \frac{1}{|\Gamma|}\int_\Gamma w_\varepsilon\,dS\Big|^2\,dx + 2\frac{|\Omega|}{|\Gamma|^2}\Big(\int_\Gamma w_\varepsilon\,dS\Big)^2$$
$$\le C\Big(\int_\Omega |\nabla w_\varepsilon|^2\,dx + \int_\Gamma {w_\varepsilon}^2\,dS\Big)$$

and therefore we obtain from (A.2.2.9)

$$\varepsilon\frac{d}{dt}\Big(\int_\Gamma v_\varepsilon^2\,dS + \int_\Omega w_\varepsilon^2\,dx\Big) \le C - c\Big(\int_\Gamma v_\varepsilon^2\,dS + \int_\Omega w_\varepsilon^2\,dx\Big)\,.$$

Hence (A.2.2.6) yields a uniform bound for $\|v_\varepsilon\|_{L^\infty(0,T;L^2(\Gamma))}$ and $\|w_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}$.

By an integration of (A.2.2.9) we in addition obtain

$$\int_0^T \int_\Omega D|\nabla w_\varepsilon|^2\,dx\,dt \le C\,.$$

Finally, weak solution theory for parabolic equations (see [74, Theorem 26.1]), implies a uniform bound also for $\|U_\varepsilon\|_{V_2(\Gamma_T)}$. ■

With these uniform estimates we can pass to the limit $\varepsilon \to 0$ to obtain the following theorem.

Theorem A.2.5. *Consider a sequence* $(U_\varepsilon, v_\varepsilon, w_\varepsilon)$ *of nonnegative solutions to* (A.2.2.1)-(A.2.2.6) *with total mass* $m > 0$ *and under Assumption* (A.2.2.6). *Then there exists a subsequence* $\varepsilon \to 0$, *a function* $u \in V_2(\Gamma_T)$ *with* $u \in W_p^{2,1}(\Gamma \times (\delta, T))$ *for any* $\delta > 0$, $1 \leq p < \infty$, *functions* $w \in L^2(0,T; H^1(\Omega))$ *with* $w(\cdot, t) \in C^\infty(\Omega)$ *for almost all* $t \in (0,T)$ *and* $\xi \in L^\infty(\Gamma_T)$ *with* $0 \leq \xi \leq 1$, *such that*

$$U_\varepsilon \rightharpoonup u \quad \text{in } V_2(\Gamma_T)\,, \quad w_\varepsilon \rightharpoonup w \quad \text{in } L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \frac{U_\varepsilon}{U_\varepsilon + \varepsilon} \overset{*}{\rightharpoonup} \xi \quad \text{in } L^\infty(\Gamma_T)\,.$$

These functions satisfy equations (A.1.12), (A.1.13) *and* (A.1.15) *pointwise almost everywhere and the Robin condition in* (A.1.14) *in a weak sense. Furthermore we have that* $u(\cdot, 0) = u_0$ *on* Γ *in* $L^2(\Gamma)$ *and that* $\int_\Gamma u(\cdot,t)\,dS = m$ *holds for all* $t \in [0,T]$.

Moreover u *and* w *are nonnegative with* $w \in L^\infty(0,T; C^0(\bar{\Omega}))$ *and for all* $\delta > 0$ *and any* $1 \leq p < \infty$ *it holds*

$$\|u\|_{W_p^{2,1}(\Gamma\times(\delta,T))} + \|w\|_{L^\infty(0,T;C^0(\bar{\Omega}))} \leq C(\delta, T, p)\,.$$

Proof. By the uniform bounds provided by Theorem A.2.4 we obtain a subsequence and functions w, u, v, ξ such that

$$w_\varepsilon \rightharpoonup w \quad \text{in } L^2(0,T;H^1(\Omega)) \tag{A.2.2.10}$$
$$U_\varepsilon \rightharpoonup u \quad \text{in } V_2(\Gamma_T) \tag{A.2.2.11}$$
$$v_\varepsilon \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;L^2(\Gamma)) \tag{A.2.2.12}$$
$$\frac{U_\varepsilon}{\varepsilon + U_\varepsilon} \overset{*}{\rightharpoonup} \xi \quad \text{in } L^\infty(\Gamma_T).$$

In particular, we have by the Aubin-Lions Lemma that $U_\varepsilon \to u$ in $L^2(\Gamma_T)$. The continuity of the trace map $H^1(\Omega) \hookrightarrow L^2(\Gamma)$ yields that $w_\varepsilon \rightharpoonup w$ in $L^2(\Gamma_T)$.

We can now multiply (A.2.2.1),(A.2.2.2),(A.2.2.3) and (A.2.2.4) by suitable test functions, integrate and pass to the limit $\varepsilon \to 0$, to deduce that

$$\partial_t u = \Delta u + cv - a_4\xi \qquad \text{on } \Gamma_T\,, \tag{A.2.2.13}$$
$$0 = -cv + a_4\xi - a_5 v + a_6 w \qquad \text{on } \Gamma_T\,, \tag{A.2.2.14}$$
$$0 = D\Delta w \qquad \text{on } \Omega_T\,, \tag{A.2.2.15}$$
$$-D\frac{\partial w}{\partial n} = -a_5 v + a_6 w \qquad \text{on } \Gamma_T\,, \tag{A.2.2.16}$$

are satisfied in a weak sense. Since the arguments are similar to those used in the proof of Theorem A.2.2, we only consider w here. Multiplying (A.2.2.4) with a test function $\phi \in C_c^1(\bar{\Omega} \times (0,T))$ and using (A.2.2.3) we obtain

$$\int_{\Omega_T} \Big(-\varepsilon \partial_t \phi w_\varepsilon + \nabla\phi \cdot \nabla w_\varepsilon \Big)\, dx\, dt = \int_{\Gamma_T} \phi(a_5 v_\varepsilon - a_6 w_\varepsilon)\, dS\, dt\,.$$

Passing to the limit $\varepsilon \to 0$ and using the convergence properties obtained above we deduce that

$$\int_{\Omega_T} \nabla\phi \cdot \nabla w\, dx\, dt = \int_{\Gamma_T} \phi(a_5 v - a_6 w)\, dS\, dt,$$

which implies that (A.2.2.15), (A.2.2.16) holds in a weak sense. In particular $w(\cdot,t)$ is harmonic in Ω for almost all $t \in (0,T)$ and hence smooth inside Ω.

Finally, it follows exactly in the same way as in (A.2.1.22) that $\xi u = u$.

By the uniform bounds (A.2.2.8) on w_ε and v_ε we obtain $\int_\Omega \varepsilon w_\varepsilon(\cdot,t)\, dx + \int_\Gamma \varepsilon v_\varepsilon(\cdot,t)\, dS \to 0$, which together with (A.2.2.7),(A.2.2.11) yields $\int_\Gamma u(\cdot,t)\, dS = m$ for almost all t. Since $u \in V_2(\Gamma_T) \hookrightarrow C^0([0,T];L^2(\Gamma))$ this equality even holds for all $t \in [0,T]$. Since $0 \leq \frac{U_\varepsilon}{\varepsilon + U_\varepsilon} \leq 1$ the corresponding bounds for ξ follow. Furthermore, by (A.2.2.8) and (A.2.2.10), (A.2.2.11), (A.2.2.12) we deduce

$$\|u\|_{V_2(\Gamma_T)} + \|v\|_{L^\infty(0,T;L^2(\Gamma))} + \|w\|_{L^2(0,T;H^1(\Omega))} \leq C\,.$$

To improve these bounds, we test for $p > 2$, equation (A.2.2.15) with $(k_p w)^{p-1}$, $k_p := \frac{a_6}{a_5}$ as well as (A.2.2.14) with v^{p-1} and we find almost everywhere in $(0,T)$

$$\begin{aligned}
0 &= -\int_\Omega D{k_p}^{p-1}(p-1) w^{p-2} |\nabla w|^2\, dx + \int_\Gamma {k_p}^{p-1}(a_5 v - a_6 w) w^{p-1}\, dS \\
&\quad - \int_\Gamma \Big((a_5 v - a_6 w) v^{p-1} - a_4 \xi v^{p-1} + c v^p \Big)\, dS \\
&= -\int_\Omega D{k_p}^{p-1}(p-1) w^{p-2} |\nabla w|^2\, dx - \int_\Gamma a_5 (v - k_p w)\Big(v^{p-1} - (k_p w)^{p-1} \Big)\, dS \\
&\quad + \int_\Gamma \Big(a_4 \xi v^{p-1} - c v^p \Big)\, dS \\
&\leq \int_\Gamma \Big(a_4 \xi v^{p-1} - c v^p \Big)\, dS\,.
\end{aligned}$$

Thus, using Young's inequality, $c \geq c_0$ and $|\xi| \leq 1$ we conclude

$$\int_\Gamma v^p\, dS \leq C \quad \text{almost everywhere on } (0,T) \tag{A.2.2.17}$$

and hence v is bounded in $L^\infty(0,T;L^p(\Gamma))$ for any $1 \le p < \infty$. By [52] and (A.2.2.15),(A.2.2.16) we obtain for some $\gamma > 0$ and for almost all $t \in (0,T)$ that $w(t) \in C^{0,\gamma}(\Omega)$, with

$$\|w(t)\|_{C^{0,\gamma}(\Omega)} \le C\Big(\|v(t)\|_{L^p(\Gamma)} + \|w(\cdot,t)\|_{L^2(\Omega)}\Big) .$$

for any $p > 2$. Therefore, this estimate combined with (A.2.2.17) yields that $w \in L^\infty(0,T;C^0(\bar{\Omega}))$. Finally, by parabolic L^p-regularity for (A.2.2.13), see the arguments in the proof of Theorem A.2.2, we deduce that $\|u\|_{W^{2,1}_p(\Gamma\times(\delta,T))} \le C$ for any $\delta > 0$, $1 \le p < \infty$.

Finally, we observe that (A.2.2.14) is equivalent to $v = \frac{1-g}{a_5}\Big(a_4\xi + a_6 w\Big)$. Using this, it is easy to see that (A.2.2.13) - (A.2.2.16) are equivalent to (A.1.12)-(A.1.14). ■

The system (A.1.12)-(A.1.14) can be formulated as an obstacle-type problem in terms of u and ξ only. This formulation will be most convenient for the analysis in Section A.4 and contains a non-local operator that we introduce now. Consider for $s \in L^2(\Gamma)$ and $h \in L^\infty(\Gamma)$, $h \ge 0$, $|\{h > 0\}| > 0$, the solution z of

$$0 = \Delta z \text{ in } \Omega, \qquad \frac{\partial z}{\partial n} + hz = s \text{ on } \Gamma. \tag{A.2.2.18}$$

This defines a linear operator $L_h : L^2(\Gamma) \to H^1(\Omega)$ via $L_h s := z$. We collect some properties of the operator L_h.

Lemma A.2.6. *Let $h \in L^\infty(\Gamma)$, $h \ge 0$, $|\{h > 0\}| > 0$, be given. Then the following hold.*

(i) $L_h : L^2(\Gamma) \to H^1(\Omega)$ *is continuous.*

(ii) $L_h : L^2(\Gamma) \to L^2(\Gamma)$ *is self-adjoint , that is*

$$\int_\Gamma s_1 L_h(s_2)\, dS = \int_\Gamma L_h(s_1) s_2\, dS . \tag{A.2.2.19}$$

(iii) It holds

$$L_h h = 1 . \tag{A.2.2.20}$$

(iv) $h \mapsto L_h$ *is monotone decreasing in the following sense: For any* $h_1, h_2 \in L^\infty(\Gamma)$ *with* $0 \le h_1 \le h_2$ *we have*

$$L_{h_1}(s) \ge L_{h_2}(s) \quad \text{for all } s \in L^2(\Gamma),\ s \ge 0 . \tag{A.2.2.21}$$

(v) L_h *is positive, more precisely there exists a positive constant* $c = c(h,\Omega)$ *such that for all* $s \ge 0$

$$L_h(s) \ge c \int_\Gamma s\, dS \quad \text{in } \overline{\Omega}. \tag{A.2.2.22}$$

Proof. We first have

$$\int_\Omega |\nabla z|^2\,dx = \int_\Gamma \Big(-h|z|^2 + sz\Big)\,dS \le -\int_\Gamma h|z|^2\,dS + \|s\|_{L^2(\Gamma)}\|z\|_{L^2(\Gamma)}.$$

Since there holds a generalized Poincaré inequality in $\{\zeta \in H^1(\Omega) : \int_\Gamma h\zeta^2 \le 1\}$ we deduce

$$\|z\|^2_{H^1(\Omega)} \le C\Big(\int_\Omega |\nabla z|^2 + \int_\Gamma h|z|^2\Big) \le C\|s\|_{L^2(\Gamma)}\|z\|_{H^1(\Omega)},$$

from which $\|z\|_{H^1(\Omega)} \le C\|s\|_{L^2(\Gamma)}$ and the desired continuity of L_h follow.

The second statement is obtained from

$$\int_\Gamma \Big(s_1 L_h(s_2) - L_h(s_1)s_2\Big)\,dS = \int_\Omega \Big(z_2\Delta z_1 - z_1\Delta z_2\Big)\,dx = 0\,.$$

The third property is easily verified from the definition of L_h.

We next prove that L_h is non-negative, i.e.

$$s \ge 0 \implies L_h s \ge 0. \tag{A.2.2.23}$$

In fact, with $z := L_h s$, by a partial integration we deduce

$$0 = -\int_\Omega z_- \Delta z\,dx = -\int_\Omega |\nabla z_-|^2\,dx - \int_\Gamma \Big(sz_- + hz_-^2\Big)\,dS \ge 0.$$

where $z_- = \max\{0, -z\}$. Hence $z_- = 0$ almost everywhere in Ω and $z \ge 0$.

We now verify (A.2.2.21). Let $z_1 = L_{h_1}(s)$, $z_2 = L_{h_2}(s)$. Then

$$0 = \Delta(z_1 - z_2) \quad \text{in } \Omega, \qquad \frac{\partial(z_1 - z_2)}{\partial n} + h_1(z_1 - z_2) = z_2(h_2 - h_1) \ge 0 \quad \text{on } \Gamma.$$

Then (A.2.2.23) ensures that $z_1 \ge z_2$.

We finally prove (A.2.2.22). Therefore fix $h \ge 0$, $s \ge 0$, let $m := \|h\|_{L^\infty(\Gamma)}$ and $\zeta := L_m s$, i.e.

$$\Delta\zeta = 0 \quad \text{in } \Omega, \qquad \frac{\partial\zeta}{\partial n} + m\zeta = s \quad \text{on } \Gamma. \tag{A.2.2.24}$$

Then $z := L_h s \ge L_m s = \zeta$ by (A.2.2.21) and to prove (A.2.2.22) it suffices to show that there exists $\kappa > 0$ with

$$\zeta \ge \kappa \int_\Gamma s\,dS\,. \tag{A.2.2.25}$$

In the first step of the proof of this inequality we show that for any $K \subset\subset \Omega$ there exists a constant $c_1 = c_1(K)$ such that

$$\zeta \geq \frac{c_1}{m} \int_\Gamma s \, dS \qquad \text{in } K. \tag{A.2.2.26}$$

To prove this estimate consider for $x \in K$ the Green's function $G(x, y)$, i.e. the solution of

$$-\Delta G(x, \cdot) = \delta_x \quad \text{in } \mathcal{D}'(\Omega), \qquad G(x, \cdot) = 0 \quad \text{on } \Gamma.$$

By the positivity of G we derive from the Hopf maximum principle that $\frac{\partial}{\partial n} G(x, y) < 0$ for all $x \in K$, $y \in \Gamma$. Since $K \times \Gamma$ is compact and $\frac{\partial}{\partial n} G$ is continuous due to the smoothness of Γ we even obtain the existence of $c_1 = c_1(K, \Omega) > 0$ such that

$$\frac{\partial}{\partial n} G(x, y) \leq -c_1 \quad \text{for all } x \in K, y \in \Gamma. \tag{A.2.2.27}$$

The representation formula in terms of the Green's function implies that for all $x \in K$

$$\zeta(x) = -\int_\Gamma \frac{\partial}{\partial n} G(x, \cdot) \zeta \, dS \geq c_1 \int_\Gamma \zeta \, dS \; = \; \frac{c_1}{m} \int_\Gamma s \, dS \,,$$

where the last equality follows from (A.2.2.24). This proves (A.2.2.26).

We now use (A.2.2.26) to prove a bound from below for ζ in the whole set Ω. By the smoothness of Γ there is a uniform radius $\varrho > 0$ such that for any $y \in \Gamma$ an interior sphere condition is satisfied for a ball $B(z_y, 2\varrho) \subset \Omega$. Moreover ϱ can be chosen such that $\bigcup_{y \in \Gamma} B(z_y, \varrho) \subset\subset \Omega \setminus K$ for some compact set $K \subset \Omega$ such that ∂K is smooth and K has nonempty interior. Denote by K_1 the closure of $\bigcup_{y \in \Gamma} B(z_y, \varrho)$. Then in particular $K_1 \subset\subset \Omega \setminus K$.

We then consider the solution $\tilde{\zeta}$ of

$$\Delta \tilde{\zeta} = 0 \text{ in } \Omega \setminus K, \qquad \tilde{\zeta} = \zeta \text{ on } \partial K, \qquad \frac{\partial \tilde{\zeta}}{\partial n} + m\tilde{\zeta} = 0 \text{ on } \Gamma.$$

As in the proof of (A.2.2.23) we deduce that $\tilde{\zeta} \leq \zeta$ and by the maximum principle that $\tilde{\zeta} \geq 0$.

We claim that

$$\tilde{\zeta} \geq \tilde{\kappa} \int_\Gamma s \, dS \quad \text{in } \Omega \setminus K \tag{A.2.2.28}$$

holds. By (A.2.2.26) and $\zeta \geq \tilde{\zeta}$ this eventually justifies (A.2.2.25).

We consider the Green's function $\tilde{G}$ of $\Omega \setminus K$. Similar as above we obtain that there exists $\tilde{c}_2 > 0$ such that

$$\frac{\partial}{\partial \nu}\tilde{G}(x,y) \leq -\tilde{c}_2 \quad \text{for all } x \in K_1, y \in \partial\Big(\Omega \setminus K\Big), \tag{A.2.2.29}$$

where ν denotes the outer unit normal field of $\Omega \setminus K$. By the representation formula and the non-negativity of $\tilde{\zeta}$ we further deduce that for all $x \in K_1$

$$\begin{aligned}\tilde{\zeta}(x) = -\int_\Gamma \frac{\partial}{\partial \nu}\tilde{G}(x,\cdot)\tilde{\zeta}\, dS - \int_{\partial K} \frac{\partial}{\partial \nu}\tilde{G}(x,\cdot)\tilde{\zeta}\, dS \geq \tilde{c}_2 \int_{\partial K} \tilde{\zeta}\, dS \\ = \tilde{c}_2 \int_{\partial K} \zeta\, dS \;\geq\; c_2 \frac{c_1}{m}\int_\Gamma s\, dS\,,\end{aligned} \tag{A.2.2.30}$$

where $c_2 = \mathcal{H}^{n-1}(\partial K)\tilde{c}_2$ and where we have used (A.2.2.26) in the last step.

Moreover, the harmonic function $\tilde{\zeta}$ attains its minimum on $\partial K \cup \Gamma$. If the minimum is attained on ∂K we have $\tilde{\zeta} \geq c_1 \int_\Gamma s$ by (A.2.2.26) and conclude that (A.2.2.28) holds. If on the other hand the minimum is attained in a point $y_0 \in \partial\Omega$ the Hopf boundary point lemma (cf. the proof of Lemma 3.4 in [26]) imply that

$$\frac{\partial \tilde{\zeta}}{\partial n}(y_0) \leq -c_3\Big(\min_{K_1} \tilde{\zeta} - \tilde{\zeta}(y_0)\Big)$$

for some positive constant $c_3 = c_3(\varrho)$. Using the Robin boundary condition for $\tilde{\zeta}$ we deduce that

$$m\tilde{\zeta}(y_0) \geq c_3\Big(\min_{K_1} \tilde{\zeta} - \tilde{\zeta}(y_0)\Big),$$

hence

$$\inf_{\Omega\setminus K} \tilde{\zeta} \geq \frac{c_3}{m+c_3}\min_{K_1} \tilde{\zeta} \geq \frac{c_1 c_2 c_3}{m(m+c_3)}\int_\Gamma s\, dS\,,$$

where we have used (A.2.2.30) in the last step.

This shows (A.2.2.28) and finishes the proof of (A.2.2.22). ∎

Proposition A.2.7. *Let* (u, w, ξ) *be nonnegative functions with* $\int_\Gamma u = m > 0$*, the same regularity as in Theorem A.2.5 and with* $0 \leq \xi \leq 1$ *almost everywhere in* Γ_T*. Then the following statements are equivalent:*

(i) (u, w, ξ) *satisfies* (A.1.12)-(A.1.14).

(ii) (u, ξ) *satisfies*

$$\partial_t u = \Delta u - a_4(1-g)\xi + \ell g L_{\ell g}\Big(a_4(1-g)\xi\Big), \quad u\xi = u \;\; a.e \text{ on } \Gamma_T, \tag{A.2.2.31}$$

where $\ell = \frac{a_6}{D}$*, and* w *is determined by*

$$w = \frac{\ell}{a_6} L_{\mathcal{X}\ell g}\Big(a_4 \mathcal{X}(1-g)\Big) \qquad a.e.\ on\ \Gamma_T\,, \tag{A.2.2.32}$$

with $\mathcal{X} = \mathcal{X}_{\{u>0\}}$.

Proof. Due to Stampacchia's Lemma and the regularity of u we have that $a_4(1-g)\xi = a_6 g w$ holds almost everywhere in $\{u = 0\}$. Hence $D\frac{\partial w}{\partial n} + \mathcal{X} a_6 g = a_4 \mathcal{X}(1-g)$ and thus (A.2.2.32) follows. ■

Remark A.2.8 (Infinite cytosolic diffusion limit)**.** In (A.2.2.31), (A.2.2.32) the parameter D has been substituted by ℓ. The limit $D \to \infty$ is equivalent to $\ell \to 0$. From the definition of the operator L_h we observe that $z_\ell := \ell L_{\ell g}(s)$ solves

$$0 = \Delta z_\ell \text{ in } \Omega, \qquad \frac{\partial z_\ell}{\partial n} + g\ell z_\ell = \ell s \text{ on } \Gamma.$$

We then obtain an estimate

$$\int_\Omega |\nabla z_\ell|^2\,dx = \ell \int_\Gamma \Big(s z_\ell - g z_\ell^2\Big)\,dS \le \ell \|s\|_{L^2(\Gamma)} \|z_\ell\|_{L^2(\Gamma)} - c\ell \|z_\ell\|^2_{L^2(\Gamma)} \le \frac{\ell}{2c} \|s\|^2_{L^2(\Gamma)}$$

and deduce that $\ell L_{\ell g}(s)$ becomes constant over Γ with $\ell \to 0$. This observation shows that (A.2.2.31) reduces to (A.1.8) in the infinite cytosolic diffusion limit.

Remark A.2.9 (Characterization of ξ, w)**.** In the formulation of (A.2.2.31),(A.2.2.32) we remark that ξ and w are already determined by u. In fact, we have

$$\xi(\cdot,t) = \begin{cases} 1 & \text{a.e. in } \{u(\cdot,t) > 0\}, \\ \frac{a_6 w g}{a_4(1-g)} & \text{a.e. in } \{u(\cdot,t) = 0\} \end{cases} \tag{A.2.2.33}$$

and ξ is determined by u, w. By (A.2.2.32) we see that w is determined by u.

Notice that the characterization (A.2.2.32) is analogous to the second formula in (A.2.1.25). We further remark that we have different representations for the function w in the same manner that we have different characterizations of α (see (A.2.1.26)). In particular we have also the following characterization in terms of an arbitrary measurable set $A \subset \Gamma$ containing $\{u > 0\}$,

$$w = \frac{\ell}{a_6} L_{\mathcal{X}_A \ell g}\Big(a_4(1-g)\mathcal{X}_A \xi\Big) \qquad \text{on } \Gamma\,. \tag{A.2.2.34}$$

From now on we set without loss of generality $a_4 = a_6 = 1$.

A.3 The reduced model for infinite cytosolic diffusion $D = \infty$

A.3.1 Uniqueness of solutions

Theorem A.3.1. *Let* (u_1, ξ_1, α_1) *and* (u_2, ξ_2, α_2) *be two different solutions of* (A.1.8)-(A.1.9) *with* $u_k \in V_2(\Gamma_T)$, $\xi_k \in L^\infty(\Gamma_T)$, $\alpha_k \in L^\infty(0,T)$, $k = 1, 2$. *Then*

$$t \mapsto \int_\Gamma (u_1 - u_2)_+(\cdot, t)\, dS \text{ is decreasing on } [0, T]. \tag{A.3.1.1}$$

In particular, given $u_0 \in L^2(\Gamma)$ *with* $u_0 \geq 0$, *there exists at most one solution* (u, ξ, α) *of* (A.1.8)-(A.1.10) *with* $u \in V_2(\Gamma_T)$, $\xi \in L^\infty(\Gamma_T)$, $\alpha \in L^\infty(0,T)$.

Proof. Any solution satisfies in addition $u \in W_p^{2,1}(\Gamma \times (\delta, T))$ for any $\delta > 0$, $1 \leq p < \infty$.

By the regularity of u_1, u_2 the function $(u_1 - u_2)_+$ belongs to $W^{1,p}(\Gamma_T)$ for any $1 \leq p \leq \infty$ and

$$\partial_t (u_1 - u_2)_+ = \mathcal{X}_{\{u_1 > u_2\}} \partial_t (u_1 - u_2). \tag{A.3.1.2}$$

In particular the weak derivative $\frac{d}{dt} \int_\Gamma (u_1 - u_2)_+ \, dS$ exists as an $L^p(0,T)$ function and hence almost everywhere in $(0,T)$.

Furthermore, for almost all $t \in (0,T)$ we have $(u_1 - u_2)(\cdot, t) \in W^{2,p}(\Gamma)$ and Kato's inequality [32] implies that $\mathcal{X}_{\{u_1 > u_2\}} \Delta(u_1 - u_2) \leq \Delta(u_1 - u_2)_+$ in the sense of distributions. We therefore obtain, with $\mathbf{1}_\Gamma$ denoting the constant function with value 1 on Γ,

$$\int_{\{u_1 > u_2\}} \Delta(u_1 - u_2) \leq \langle \Delta(u_1 - u_2)_+, \mathbf{1}_\Gamma \rangle = 0. \tag{A.3.1.3}$$

This justifies the following computations for almost all $t \in (0,T)$. We drop in the following in most places the argument t.

We let $\mathcal{X}_+ := \mathcal{X}_{\{u_1 > u_2\}}$. Integrating then the equation for the difference $u_1 - u_2$ over $\{u_1 > u_2\}$ and using (A.3.1.2), (A.3.1.3) yields

$$\begin{aligned} \frac{d}{dt} \int_\Gamma (u_1 - u_2)_+ \, dS &= \int_{\{u_1 > u_2\}} \partial_t (u_1 - u_2) \, dS \\ &= \int_{\{u_1 > u_2\}} \Delta(u_1 - u_2) \, dS - \int_{\{u_1 > u_2\}} (1 - g)(\xi_1 - \xi_2) \, dS \\ &\quad + \int_{\{u_1 > u_2\}} g(\alpha_1 - \alpha_2) \, dS \\ &\leq - \int_\Gamma \mathcal{X}_+ (1 - g)(\xi_1 - \xi_2) \, dS + (\alpha_1 - \alpha_2) \int_\Gamma \mathcal{X}_+ g \, dS \, . \end{aligned} \tag{A.3.1.4}$$

We next rewrite the difference $\alpha_1 - \alpha_2$. Almost everywhere in $\{u_1 = 0 = u_2\}$ by Stampacchia's Lemma it holds

$$\Delta u_1 = \Delta u_2 = 0 \quad \text{and} \quad \partial_t u_1 = \partial_t u_2 = 0$$

which yields due to (A.1.8),

$$(\alpha_1 - \alpha_2) g = (1-g)(\xi_1 - \xi_2) \tag{A.3.1.5}$$

almost everywhere in $\{u_1 = u_2 = 0\}$. We use the notation $\mathcal{X} := \mathcal{X}_{\{u_1+u_2>0\}}$ and derive thanks to (A.2.1.25) that

$$(\alpha_1 - \alpha_2)\int_\Gamma g\, dS = \int_\Gamma \mathcal{X}(1-g)(\xi_1 - \xi_2)\, dS + (\alpha_1 - \alpha_2)\int_\Gamma (1-\mathcal{X}) g\, dS$$

and thus

$$(\alpha_1 - \alpha_2)\int_\Gamma \mathcal{X} g\, dS = \int_\Gamma \mathcal{X}(1-g)(\xi_1 - \xi_2)\, dS \tag{A.3.1.6}$$

Plugging (A.3.1.6) into (A.3.1.4) we find

$$\begin{aligned}\frac{d}{dt}\int_\Gamma (u_1 - u_2)_+\, dS \leq & \frac{1}{\int_\Gamma \mathcal{X} g\, dS}\Big(- \int_\Gamma \mathcal{X} g\, dS \int_\Gamma \mathcal{X}_+(1-g)(\xi_1 - \xi_2)\, dS \\ & + \int_\Gamma \mathcal{X}(1-g)(\xi_1 - \xi_2)\, dS \int_\Gamma \mathcal{X}_+ g\, dS \Big) .\end{aligned} \tag{A.3.1.7}$$

For the term on the right-hand side in brackets we further obtain

$$\begin{aligned}\Big(\dots\Big) = & - \int_\Gamma (\mathcal{X} - \mathcal{X}_+) g\, dS \int_\Gamma \mathcal{X}_+(1-g)(1-\xi_2)\, dS \\ & - \int_\Gamma \mathcal{X}_+ g\, dS \int_\Gamma (\mathcal{X} - \mathcal{X}_+)(1-g)(1-\xi_1)\, dS \leq 0 ,\end{aligned} \tag{A.3.1.8}$$

where we have used that $\mathcal{X} - \mathcal{X}_+ \geq 0$, that $\xi_1 = 1$ in $\{\mathcal{X}_+ > 0\}$ and that $\xi_2 = 1$ in $\{\mathcal{X} - \mathcal{X}_+ > 0\}$.

This shows that $t \mapsto \int_\Gamma (u_1 - u_2)_+(\cdot, t)\, dS$ is decreasing in time. Moreover, since $u_1, u_2 \in C^0([0,T]; L^2(\Gamma))$ we deduce that $t \mapsto \int_\Gamma (u_1 - u_2)_+(\cdot, t)\, dS$ is continuous on $[0,T]$, and in particular vanishes at $t = 0$. This proves that $(u_1 - u_2)_+ = 0$ on Γ_T, hence $u_1 \leq u_2$. By the symmetry of the argument, we also have $u_2 \leq u_1$, which gives the desired contraction property and the uniqueness for u. The uniqueness of ξ and α then easily follows from (A.2.1.25) and (A.2.1.23). ■

A.3.2 Global stability of steady states

The results of the previous sections show that for any given initial data with mass $m > 0$ there exists a unique solution of (A.1.8)-(A.1.10) for all times $t \geq 0$. We now consider the case that $c = c(x)$ does not depend on time, hence $g = g(x)$ is time-independent, too. The existence and uniqueness of stationary states for any prescribed mass was proved in [51]. The goal of this section is to prove that $(u, \xi, \alpha)(\cdot, t)$ converge with $t \to \infty$ to the unique steady state (u_*, ξ_*, α_*) with the same mass m.

In the following we consider $\Gamma_1 = \Gamma \times (0, 1)$ and denote by $S_t u : \Gamma_1 \to \mathbb{R}$ the function defined by $(S_t u)(x, s) := u(x, s + t)$. The functions $S_t\xi, S_t\alpha$ are defined analogously. We denote the constant function with value (u_*, ξ_*, α_*) on $(0, 1)$ again by (u_*, ξ_*, α_*).

Theorem A.3.2. *Consider the unique solution (u, ξ, α) of (A.1.8)-(A.1.10) and the stationary solution (u_*, ξ_*, α_*) with the same mass, that is the unique solution of*

$$-\Delta u_* = -(1 - g)\xi_* + \alpha_* g\,, \qquad u_* \geq 0\,, \quad 0 \leq \xi_* \leq 1\,, \quad \xi_* u_* = u_*\,, \tag{A.3.2.1}$$

$$\int_\Gamma u_* \, dS = m\,. \tag{A.3.2.2}$$

Then (u, ξ, α) converges with $t \to \infty$ to (u_*, ξ_*, α_*), more precisely

$$S_t u \rightharpoonup u_* \text{ in } W^{2,1}_p(\Gamma_1)\,, \quad S_t\xi \overset{*}{\rightharpoonup} \xi_* \text{ in } L^\infty(\Gamma_1)\,, \quad S_t\alpha \overset{*}{\rightharpoonup} \alpha_* \text{ in } L^\infty(0, 1)\,. \tag{A.3.2.3}$$

Moreover, $S_t u$ converges with $t \to \infty$ uniformly on Γ to u_*.

Proof. We consider for $k \in \mathbb{N}$ the functions

$$(u_k, \xi_k, \alpha_k) \in W^{2,1}_p(\Gamma_1) \times L^\infty(\Gamma_1) \times L^\infty(0, 1), \qquad u_k = S_k u,\ \xi_k = S_k\xi,\ \alpha_k = S_k\alpha\,.$$

Then these triples are all solutions of (A.1.8), (A.1.9) on Γ_1 and we deduce from Theorem A.2.1 and (A.2.1.14) that they are uniformly bounded in $W^{2,1}_p(\Gamma_1) \times L^\infty(\Gamma_1) \times L^\infty(0, 1)$ for all $p \in [1, \infty)$. Hence, there exists $(u_\infty, \xi_\infty, \alpha_\infty) \in W^{2,1}_p(\Gamma_1) \times L^\infty(\Gamma_1) \times L^\infty(0, 1)$ such that for some subsequence $k \to \infty$

$$u_k \rightharpoonup u_\infty \text{ in } W^{2,1}_p(\Gamma_1)\,, \quad \xi_k \overset{*}{\rightharpoonup} \xi_\infty \text{ in } L^\infty(\Gamma_1)\,, \quad \alpha_k \overset{*}{\rightharpoonup} \alpha_\infty \text{ in } L^\infty(0, 1). \tag{A.3.2.4}$$

By the compact embedding $W^{2,1}_p(\Gamma_1) \hookrightarrow C^{\alpha,\alpha/2}(\Gamma \times [0, 1]) \overset{\text{cpct}}{\hookrightarrow} C^0(\Gamma \times [0, 1])$ for $p > 2$, $0 < \alpha \leq 2 - \frac{4}{p}$ (see [75, Theorem 1.4.1]) we deduce that $\lim_{t\to\infty} u(\cdot, t) = u_\infty$ in $C^0(\Gamma)$.

We therefore can pass in (A.1.8), (A.1.9) (for u replaced by u_k) to the limit and deduce that $(u_\infty, \xi_\infty, \alpha_\infty)$ is again a solution of (A.1.8), (A.1.9) on Γ_1. We would like to show that this solution is time-independent and coincides with (u_*, ξ_*, α_*).

Exactly as in (A.3.1.4)-(A.3.1.8) we can conclude

$$\frac{d}{dt}\int_\Gamma (u-u_*)_+\,dS \le -\int_{\{u>u_*\}} \Big((1-g)(\xi-\xi_*)-(\alpha-\alpha_*)g\Big)\,dS \le 0 \tag{A.3.2.5}$$

and thus $t\mapsto \int_\Gamma (u-u_*)_+(\cdot,t)$ is decreasing.

By (A.3.2.4) and the monotonicity property (A.3.2.5) we deduce that $\lim_{T\to\infty}\int_\Gamma (u-u_*)_+(T,\cdot)\,dS$ exists and that for any $t\in(0,1)$

$$\int_\Gamma (u_\infty(\cdot,t)-u_*)_+\,dS = \lim_{k\to\infty}\int_\Gamma (u_k(\cdot,t)-u_*)_+\,dS = \lim_{T\to\infty}\int_\Gamma (u(\cdot,T)-u_*)_+\,dS \tag{A.3.2.6}$$

is independent of t. Since $(u_\infty,\xi_\infty,\alpha_\infty)$ and (u_*,ξ_*,α_*) are both solutions of (A.1.8), (A.1.9) on Γ_1 we deduce again, as in (A.3.2.5) that

$$0=\frac{d}{dt}\int_\Gamma (u_\infty-u_*)_+\,dS \le -\int_{\{u_\infty>u_*\}} \Big((1-g)(1-\xi_*)-(a_\infty-\alpha_*)\Big)\,dS \le 0 \tag{A.3.2.7}$$

and hence the right-hand side must be zero for almost any $t\in(0,1)$.

Now assume that there exists $t\in(0,1)$ such that $\alpha_\infty(t)<\alpha_*$ and such that (A.3.2.7) holds. Then we deduce that $\{u_\infty(\cdot,t)>u_*\}$ has measure zero and $u_\infty(\cdot,t)\le u_*$ almost everywhere, which implies by the equal mass condition that $u_\infty(\cdot,t)=u_*$. But this further induces $\alpha_\infty(t)=\alpha_*$ by the second equality in (A.2.1.25), a contradiction. Hence $\alpha_\infty(t)\ge\alpha_*$ for almost all $t\in(0,1)$.

In a completely analogous way we can derive that $\alpha_\infty(t)\le\alpha_*$ for almost all $t\in(0,1)$, which finally implies $\alpha_\infty=\alpha_*$ almost everywhere.

Using this information in (A.3.2.7) and the analogous inequality for $\frac{d}{dt}\int_\Gamma (u_*-u_\infty)_+\,dS$ we deduce that $\xi_\infty(\cdot,t)=\xi_*=1$ in $\{u_\infty(\cdot,t)\neq u_*\}$. In addition they also are equal in $\{u_\infty(\cdot,t)=u_*>0\}$ and by (A.2.1.23) also in $\{u_\infty(\cdot,t)=u_*=0\}$. Hence $\xi_\infty=\xi_*$ almost everywhere.

It therefore remains to prove that $(\xi_\infty,\alpha_\infty)=(\xi_*,\alpha_*)$ implies $u_\infty=u_*$. This follows from the following lemma, applied to $u_\infty-u_*$. ■

Lemma A.3.3. *Given $u\in W_2^{2,1}(\Gamma_T)$ with $\partial_t u-\Delta u=0$ almost everywhere and*

$$\int_\Gamma u(\cdot,t)\,dS=0\,,\qquad \frac{d}{dt}\int_\Gamma u(\cdot,t)_+\,dS=0 \qquad \text{for a.a. } t\in(0,T) \tag{A.3.2.8}$$

it follows that $u\equiv 0$.

Proof. Due to the regularity of u the second identity implies

$$\int_\Gamma (u(\cdot,t_1))_+ dS = \int_\Gamma (u(\cdot,t_2))_+ \, dS \qquad \text{for any } 0 < t_1 < t_2 \le T\,. \tag{A.3.2.9}$$

Using standard smoothing effects we can assume that $u \in C^\infty(\Gamma \times (0,T))$. In particular we have that $t \mapsto u(\cdot,t)$ is continuous in $L^q(\Gamma)$ for any $q \in [1,\infty]$. We define ψ as the solution of

$$\psi_t + \Delta\psi = 0\,, \qquad \psi(\cdot,t_2) = \chi_{\{u(\cdot,t_2)>0\}}\,, \qquad t_2 \in (0,T]\,.$$

We notice that the set $\{u(\cdot,t_2) > 0\}$ is well defined since u is smooth. Classical regularity theory for the heat equation implies that $\psi \in C^0([t_1,t_2];L^p(\Gamma))$ with $0 < t_1 < t_2$ and $1 \le p < \infty$. Since $\psi \in C^\infty([t_1,t_2-\delta]\times\Gamma)$ for any arbitrarily small $\delta > 0$ we can use ψ as a test function in the equation for u. Then, integrating by parts we obtain

$$\int_\Gamma u(\cdot,t_1)\psi(\cdot,t_1)\, dS = \int_\Gamma u(\cdot,t_2-\delta)\psi(\cdot,t_2-\delta) dS\,.$$

Using the continuity of the map $t \mapsto u(\cdot,t)$ and $t \mapsto \psi(\cdot,t)$ in $L^2(\Gamma)$ we obtain that $u(\cdot,t_2-\delta)\psi(\cdot,t_2-\delta)$ converges to $u(\cdot,t_2)\psi(\cdot,t_2)$ in $L^1(\Gamma)$ as $\delta \to 0$. Thus

$$\int_\Gamma u(\cdot,t_2-\delta)\psi(\cdot,t_2-\delta)\, dS \to \int_\Gamma u(\cdot,t_2)\psi(\cdot,t_2)\, dS = \int_\Gamma (u(\cdot,t_2))_+\, dS \qquad \text{as } \delta \to 0,$$

whence

$$\int_\Gamma (u(\cdot,t_2))_+\, dS = \int_\Gamma u(\cdot,t_1)\psi(\cdot,t_1)\, dS\,. \tag{A.3.2.10}$$

If $|\{u(\cdot,t_2) > 0\}| > 0$ we have, since $\int_\Gamma u(\cdot,t)\, dS = 0$ for all $t \in (0,T]$, that $|\{u(\cdot,t_2) > 0\}| < |\Gamma|$. Therefore, the strong maximum principle implies that for any $t_1 < t_2$ we have

$$0 < \psi(\cdot,t_1) \le \theta < 1$$

where θ depends on t_1. Then

$$\int_\Gamma u(\cdot,t_1)\psi(\cdot,t_1)\, dS \le \int\limits_{\{u(\cdot,t_1)>0\}} u(\cdot,t_1)\psi(\cdot,t_1)\, dS \le \theta \int\limits_{\{u(\cdot,t_1)>0\}} u(\cdot,t_1)\, dS = \theta \int_\Gamma u(\cdot,t_1)_+\, dS\,.$$

Combining this with (A.3.2.10) we obtain $\int_\Gamma (u(\cdot,t_2))_+\, dS \le \theta \int_\Gamma u(\cdot,t_1)\, dS$ which contradicts (A.3.2.9).

Therefore $|\{u(\cdot,t_2) > 0\}| = 0$. Then, we have that $u(\cdot,t_2) \le 0$, but since $\int_\Gamma u(\cdot,t_2)\, dS = 0$ this implies that $u(\cdot,t_2) \equiv 0$. Since t_2 was arbitrary this proves $u \equiv 0$. ■

A.4 The reduced model for finite cytosolic diffusion $D < \infty$

From now on we choose $D = 1$. All arguments and calculations for the case $D \neq 1$ are analogue. We recall that we have also set $a_4 = a_6 = 1$, which in particular gives $\ell = 1$ in the characterization of Proposition A.2.7.

A.4.1 Uniqueness of solutions

In this section we consider a solution (u, w, ξ) in $V_2(\Gamma_T) \times L^2(0, T; H^1(\Omega)) \times L^\infty(\Gamma_T)$ of

$$\partial_t u = \Delta u - (1-g)\xi + gw, \quad u\xi = u, \quad u \geq 0 \qquad \text{on } \Gamma_T \tag{A.4.1.1}$$

$$0 = \Delta w \text{ in } \Omega, \quad \frac{\partial w}{\partial n} = (1-g)\xi - gw \qquad \text{on } \Gamma_T\,, \tag{A.4.1.2}$$

$$u(\cdot, 0) = u_0 \qquad \text{on } \Gamma\,. \tag{A.4.1.3}$$

We recall that

$$\xi(\cdot, t) = \begin{cases} 1 & \text{a.e. in } \{u(\cdot, t) > 0\} \\ \frac{wg}{1-g}(\cdot, t) & \text{a.e. in } \{u(\cdot, t) = 0\} \end{cases}. \tag{A.4.1.4}$$

In the following we use the operator L_h as defined before Lemma A.2.6, i.e. for given $h \in L^\infty(\Gamma)$, $h \geq 0$ the function $z = L_h s$ solves

$$0 = \Delta z \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n} + hz = s \quad \text{on } \Gamma. \tag{A.4.1.5}$$

We next prove an L^1-contraction property and the uniqueness of solutions.

Theorem A.4.1. *Consider two solutions (u_k, ξ_k, w_k), $k = 1, 2$ of (A.4.1.1)-(A.4.1.2). Then*

$$t \mapsto \int_\Gamma (u_1 - u_2)_+(\cdot, t)\, dS \text{ is decreasing on } (0, T).$$

In particular, given $u_0 \in L^2(\Gamma)$ with $u_0 \geq 0$ and $T > 0$, there exists at most one solution $u \in V_2(\Gamma_T)$, $\xi \in L^\infty(\Gamma_T)$, $w \in L^2(0, T; H^1(\Omega))$ of (A.4.1.1)-(A.4.1.3).

Proof. As above, by parabolic regularity results, we have $u_k \in W^{2,1}_p(\Gamma \times (\delta, T))$ for any $\delta > 0$, $1 \leq p < \infty$.

Letting $s_k = (1-g)\xi_k$ we have

$$w_k = L_g s_k. \tag{A.4.1.6}$$

In the following we let $\mathcal{X}_+ = \mathcal{X}_{\{u_1>u_2\}}$ and $\mathcal{X} = \mathcal{X}_{\{u_1+u_2>0\}}$. As in the proof of (A.2.2.32) we conclude that the difference $w_1 - w_2$ satisfies

$$w_1 - w_2 = L_{\mathcal{X}g}\Big(\mathcal{X}(s_1 - s_2)\Big)\,. \tag{A.4.1.7}$$

Following the arguments in the proof of Theorem A.3.1 we obtain, using also Lemma A.2.6, that

$$\begin{aligned}
\frac{d}{dt}\int_\Gamma (u_1 - u_2)_+\, dS &\leq \int_{\{u_1>u_2\}} \Big(-(s_1 - s_2) + g(w_1 - w_2)\Big)\, dS \\
&= \int_\Gamma \Big(-\mathcal{X}_+(s_1 - s_2) + \mathcal{X}_+ g L_{\mathcal{X}g}\Big(\mathcal{X}(s_1 - s_2)\Big)\Big)\, dS \\
&= \int_\Gamma \Big(-\mathcal{X}_+(s_1 - s_2) L_{\mathcal{X}g}(\mathcal{X}g) + \mathcal{X}_+ g L_{\mathcal{X}g}\Big(\mathcal{X}(s_1 - s_2)\Big)\Big)\, dS \\
&= \int_\Gamma -\mathcal{X} g L_{\mathcal{X}g}\Big(\mathcal{X}_+(s_1 - s_2)\Big) + \mathcal{X}_+ g L_{\mathcal{X}g}\Big(\mathcal{X}(s_1 - s_2)\Big)\Big)\, dS \\
&= -\int_\Gamma \Big(\mathcal{X} - \mathcal{X}_+\Big) g L_{\mathcal{X}g}\Big(\mathcal{X}_+(s_1 - s_2)\Big)\, dS + \int_\Gamma \mathcal{X}_+ g L_{\mathcal{X}g}\Big((\mathcal{X} - \mathcal{X}_+)(s_1 - s_2)\Big)\, dS \leq 0\,.
\end{aligned} \tag{A.4.1.8}$$

In the last line we have used in the first term that $\mathcal{X} - \mathcal{X}_+ \geq 0$ and $\mathcal{X}_+(s_1 - s_2) \geq 0$ and for the second term that $\mathcal{X} - \mathcal{X}_+ = \mathcal{X}_{\{u_2>u_1\}} + \mathcal{X}_{\{u_1=u_2>0\}}$, that $s_1 \leq s_2$ on $\{u_2 > u_1\}$ and $s_1 = s_2$ on $\{u_1 = u_2 > 0\}$.

Applying the same argument to $u_2 - u_1$ we find that $\int_\Gamma |u_1 - u_2|\, dS$ is decreasing in time, and in particular $u_1 = u_2$ since the initial data are the same.

From Remark A.2.9 it follows that $w_1 = w_2$ and $\xi_1 = \xi_2$. ■

With similar arguments as in the proof of Theorem A.4.1 we can also show uniqueness of steady states for given mass m. This result has been shown in [51] only in the case that Γ is a sphere. In the following Theorem we prove even more, namely a monotonicity result from which uniqueness of steady states follows.

Theorem A.4.2 (Monotonicity)**.** *Let (u_1, w_1, ξ_1), $(u_2, w_2, \xi_2) \in H^2(\Gamma) \times H^1(\Omega) \times L^\infty(\Gamma)$ be solutions to*

$$-\Delta u = -(1-g)\xi + gw\,, \qquad u\xi = u, \quad u \geq 0 \qquad \text{on } \Gamma \tag{A.4.1.9}$$

$$0 = \Delta w \text{ in } \Omega\,, \qquad \frac{\partial w}{\partial n} = (1-g)\xi - gw \qquad \text{on } \Gamma\,, \tag{A.4.1.10}$$

with $\int_\Gamma u_1\, dS = m_1$ and $\int_\Gamma u_2\, dS = m_2$. Suppose that $m_1 \geq m_2$, then

$$u_1 \geq u_2, \qquad w_1 \geq w_2, \qquad \xi_1 \geq \xi_2 \qquad \text{on } \Gamma.$$

Proof. Again we let $s_k = (1-g)\xi_k$, $\mathcal{X}_+ = \mathcal{X}_{\{u_1>u_2\}}$ and $\mathcal{X} = \mathcal{X}_{\{u_1+u_2>0\}}$.

We first show that $u_1 \geq u_2$. We integrate the difference of the equations for u_1 and u_2 over the set $\{u_1 > u_2\}$ and obtain, exactly as in (A.4.1.8) that

$$\begin{aligned} 0 &\leq \int_{\{u_1>u_2\}} \Big(-(1-g)(\xi_1-\xi_2) + g(w_1-w_2)\Big)\, dS \\ &= -\int_\Gamma \Big(\mathcal{X}-\mathcal{X}_+\Big) g L_{\mathcal{X}g}\Big(\mathcal{X}_+(s_1-s_2)\Big)\, dS + \int_\Gamma \mathcal{X}_+ g L_{\mathcal{X}g}\Big((\mathcal{X}-\mathcal{X}_+)(s_1-s_2)\Big)\, dS \leq 0\,. \end{aligned} \tag{A.4.1.11}$$

We now exploit that both integrands in the last line of (A.4.1.11) vanish. If $\mathcal{X}_+ = 0$ almost everywhere or $\mathcal{X} - \mathcal{X}_+ = 0$ almost everywhere, then $u_1 \leq u_2$ or $u_1 \geq u_2$, respectively, hence $u_1 \geq u_2$ almost everywhere since we have assumed that $m_1 \geq m_2$.

If $\mathcal{X}_+$ and $\mathcal{X} - \mathcal{X}_+$ are both nontrivial we deduce from the positivity of $L_{\mathcal{X}g}$, see (A.2.2.22), that $s_1 = s_2$ and thus $\xi_1 = \xi_2$ in $\{u_1 + u_2 > 0\}$. By the first line in (A.4.1.11) this in addition implies $w_1 = w_2$ in $\{u_1 > u_2\}$.

Testing the difference equation with $(u_1 - u_2)_+$ yields

$$0 = \int_\Gamma \Big(|\nabla(u_1-u_2)_+|^2 + \Big((s_1-s_2) - g(w_1-w_2)\Big)(u_1-u_2)_+\Big)\, dS = \int_\Gamma |\nabla(u_1-u_2)_+|^2\, dS\,.$$

This implies that $(u_1 - u_2)_+$ is constant, from which we obtain by $m_1 \geq m_2$ that $u_1 \geq u_2$.

The property $u_1 \geq u_2$ implies that $\mathcal{X}(\xi_1 - \xi_2) \geq 0$. Therefore (A.2.2.32) and the positivity of L_h, see (A.2.2.22), imply that

$$w_1 - w_2 = L_{\mathcal{X}g}\Big((1-g)\mathcal{X}(\xi_1-\xi_2)\Big) \geq 0.$$

Then, using (A.2.2.33) we finally deduce $\xi_1 \geq \xi_2$. ■

A.4.2 Global stability of steady states

Again we assume in this section that $c = c(x)$ does not depend on time, hence g has the same property. We prove the convergence of the obstacle-type problem for finite diffusion to the stationary state with the same mass. We again denote the shift operator by S_t, see the definition before Theorem A.3.2.

Theorem A.4.3. *The unique solution* (u, w, ξ) *of* (A.4.1.1)-(A.4.1.3) *converges as* $t \to \infty$ *to the unique stationary solution* (u_*, w_*, ξ_*) *of* (A.4.1.9)-(A.4.1.10) *with* $\int_\Gamma u_*\, dS = m = \int_\Gamma u_0\, dS$*, more precisely*

$$S_t u \rightharpoonup u_* \text{ in } W_p^{2,1}(\Gamma_1)\,, \quad S_t\xi \overset{*}{\rightharpoonup} \xi_* \text{ in } L^\infty(\Gamma_1)\,, \quad S_t\alpha \overset{*}{\rightharpoonup} \alpha_* \text{ in } L^\infty(0,1)\,. \tag{A.4.2.1}$$

In particular, $S_t u$ *converges with* $t \to \infty$ *uniformly on* Γ *to* u_*.

Proof. Since (u_*, w_*, ξ_*) is a solution of (A.4.1.1)-(A.4.1.2) we obtain from Theorem A.4.1 that $t \mapsto \int_\Gamma (u(\cdot,t) - u_*)_+ \, dS$ is decreasing and

$$\lim_{T\to\infty} \int_\Gamma (u(\cdot,T) - u_*)_+ \, dS \quad \text{exists.} \tag{A.4.2.2}$$

We consider for $k \in \mathbb{N}$ the functions

$$(u_k, w_k, \xi_k) \in W_p^{2,1}(\Gamma_1) \times L^2(0,1;H^1(\Omega)) \times L^\infty(\Gamma_1),$$
$$\Big(u_k(\cdot,t), w_k(\cdot,t), \xi_k(\cdot,t)\Big) = \Big(u(\cdot,t+k), w(\cdot,t+k), \xi(\cdot,t+k)\Big).$$

Then u_k, w_k, ξ_k are uniformly bounded in $W_p^{2,1}(\Gamma_1) \times L^2(0,1;H^1(\Omega)) \times L^\infty(\Gamma_1)$ for all $p \in [1,\infty)$. Hence, there exists $(u_\infty, w_\infty, \xi_\infty) \in W_p^{2,1}(\Gamma_1) \times L^2(0,1;H^1(\Omega)) \times L^\infty(\Gamma_1)$ such that for some subsequence $k \to \infty$

$$u_k \rightharpoonup u_\infty \text{ in } W_p^{2,1}(\Gamma_1)\,, \quad w_k \rightharpoonup w_\infty \text{ in } L^2(0,1;H^1(\Omega))\,, \quad \xi_k \overset{*}{\rightharpoonup} \xi_\infty \text{ in } L^\infty(\Gamma_1)\,. \tag{A.4.2.3}$$

As in the proof of Theorem A.3.2 we deduce that $\lim_{t\to\infty} u(\cdot,t) = u_*$ in $C^0(\Gamma)$ and that $(u_\infty, w_\infty, \xi_\infty)$ is again a solution of (A.4.1.1),(A.4.1.2). We prove that this solution is time-independent and coincides with (u_*, w_*, ξ_*).

We first deduce from (A.4.2.2) as in (A.3.2.6) that $t \mapsto \int_\Gamma (u_\infty(\cdot,t) - u_*)_+ \, dS$ is independent of $t \in (0,1)$.

Since $(u_\infty, w_\infty, \xi_\infty)$ and (u_*, w_*, ξ_*) are both solutions to (A.4.1.1),(A.4.1.2) we obtain from (A.4.1.8) that

$$\begin{aligned} 0 &= \frac{d}{dt}\int_\Gamma (u_\infty - u_*)_+ \, dS \\ &\le -\int_\Gamma \Big(\mathcal{X} - \mathcal{X}_+)gL_{\mathcal{X}g}\Big(\mathcal{X}_+(s_\infty - s_*)\Big)\, dS + \int_\Gamma \mathcal{X}_+ g L_{\mathcal{X}g}\Big((\mathcal{X} - \mathcal{X}_+)(s_\infty - s_*)\Big)\, dS \le 0\,, \end{aligned}$$

where $s_\infty = (1-g)\xi_\infty$, $s_* = (1-g)\xi_*$, $\mathcal{X}_+ = \mathcal{X}_{\{u_\infty > u_*\}}$ and $\mathcal{X} = \mathcal{X}_{\{u_\infty + u_* > 0\}}$. We therefore deduce as for (A.4.1.8) that both integrals on the right-hand side are zero.

In this situation we can follow the arguments after (A.4.1.11). Since u_∞, u_* have the same mass we obtain that $u_\infty = u_*$ or $s_\infty = s_*$ on $\{u_\infty + u_* > 0\}$. In the first case the claim is proved.

In the second case we have $\xi_\infty = \xi_*$ on $\{u_\infty + u_* > 0\}$ and it remains to examine what holds in the region $\{u_\infty = u_* = 0\}$. To this end, it is more convenient to show first that $w_\infty = w_*$. This follows easily from (A.2.2.34), with $A = \{u_\infty + u_* > 0\}$. Indeed, since $\xi_\infty = \xi_*$ almost everywhere in $\{u_\infty + u_* > 0\}$, we deduce that $w_\infty = w_*$ almost everywhere in $\Gamma \times (0,1)$. This, combined with (A.2.2.33) implies that $\xi_\infty = \xi_*$ almost everywhere in $\Gamma \times (0,1)$.

What is left to prove is that $(\xi_\infty, w_\infty) = (\xi_*, w_*)$ implies $u_\infty = u_*$. We notice that:

$$\partial_t(u_\infty - u_*) = \Delta(u_\infty - u_*).$$

In addition, $\int_\Gamma u_\infty(\cdot,t)\ dS = \int_\Gamma u_*(\cdot,t)\ dS$ for all $t \in (0,1)$ and we recall that $\int_\Gamma (u_\infty - u_*)_+(\cdot,t)\,dS$ is constant for all $t \in (0,1)$. Therefore, it follows from Lemma A.3.3 that $u_\infty = u_*$.

■

Appendix

B

Qualitative properties of solutions to a non-local free boundary problem modeling cell polarization

This appendix coincides with the paper [46], written jointly by Barbara Niethammer, Matthias Röger, Juan J. L. Velázquez and the author.

B.1 Introduction

Obstacle problems appear in various applications and are still an active field of current research.

In this work we are concerned with a particular parabolic obstacle problem that was obtained as an asymptotic reduction for a cell polarization model [44] (see also [51] for the stationary case). As we discuss below, this model involves a specific nonlinear nonlocal term that can be seen as a (time dependent) Lagrange multiplier ensuring mass conservation. The particular dependence of this term on the solution (more specifically on the *support* of the solution) makes the analysis quite challenging.

Whereas existence of solutions and convergence towards a unique stationary state have been discussed in [44], our focus here is on the characterization of qualitative properties of solutions, in particular continuity properties of the Lagrange multiplier and of the (compact) support of the solutions.

The setting: Let Γ be a smooth, compact, two-dimensional manifold without boundary embedded in $\mathbb{R}^3$ and let $T > 0$. We set $\Gamma_T := \Gamma \times (0, T)$. Moreover, let a smooth function $g : \Gamma \to (0,1)$ and nonnegative initial data $u_0 : \Gamma \to \mathbb{R}$ be given (precise assumptions will be stated later).

We then consider a triplet (u, ξ, α) of functions $u, \xi : \Gamma_T \to \mathbb{R}$ and $\alpha : (0,T) \to \mathbb{R}$ that solve the following problem in an almost everywhere sense,

$$\partial_t u - \Delta_\Gamma u = -(1-g)\xi + \alpha g \qquad \text{on } \Gamma_T\,, \tag{B.1.1}$$

$$u \geq 0\,, \quad u\xi = u\,, \quad 0 \leq \xi \leq 1 \qquad \text{on } \Gamma_T\,, \tag{B.1.2}$$

$$u(\cdot, 0) = u_0 \qquad \text{on } \Gamma\,. \tag{B.1.3}$$

and that satisfy the following compatibility condition (guaranteeing mass conservation for u)

$$\alpha(t) = \frac{\int_{\{u(\cdot,t)>0\}}(1-g)\,dS}{\int_{\{u(\cdot,t)>0\}} g\,dS} \qquad \text{for } t \in (0,T)\,. \tag{B.1.4}$$

In (B.1.1) the notation Δ_Γ stands for the Laplace-Beltrami operator associated to the surface Γ. For simplicity, in the following we will drop the subscript Γ in the notation.

In the interpretation of a cell polarization model as described in [44], [51], the positivity set $\{u(\cdot,t) > 0\}$ corresponds to regions where the concentration of a chemical is high, while the set $\{u(\cdot,t) = 0\}$ indicates those regions where the concentration of such a chemical is low.

Context of this work: The problem (B.1.1), (B.1.2) can be reformulated in the form

$$\begin{cases} u \geq 0, \\ \partial_t u - \Delta u \geq -(1-g) + \alpha g, \\ \partial_t u - \Delta u = -(1-g) + \alpha g \quad \text{in } \{u > 0\} \end{cases} \tag{B.1.5}$$

Problem (B.1.5) can be compared with the classical obstacle problem given by

$$\begin{cases} u \geq 0, \\ \partial_t u - \Delta u \geq f, \\ \partial_t u - \Delta u = f \quad \text{in } \{u > 0\} \end{cases} \tag{B.1.6}$$

(see for example [66, Section 3.1]), which again can be rephrased as a parabolic variational inequality [5]

$$u \geq 0, \quad (\partial_t u - \Delta u)(v - u) \geq f(v - u) \text{ a.e., for any } v \geq 0. \tag{B.1.7}$$

Note that under suitable (parabolic Sobolev) regularity assumptions the system (B.1.6) generalizes the classical formulation

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \{u > 0\}\,, \\ f \leq 0 & \text{in } \{u = 0\}\,, \\ u = \nabla u = 0 & \text{on } \partial\{u > 0\}\,. \end{cases} \tag{B.1.8}$$

Classical obstacle problems and variational inequalities have been thoroughly investigated over the past decades. In [6] the existence of solutions to problem (B.1.6) is proved for $f \in L^\infty(\mathbb{R}^n \times (0,T))$ with $\partial_t f \in L^\infty(\mathbb{R}^n \times (0,T))$ and prescribed initial data u_0 given by a finite, positive measure. Moreover, if f is strictly negative, more precisely $f \leq -\nu$ for some constant $\nu > 0$, and u_0 has compact support then the solution has compact support for all times. Under more restrictive conditions on u_0 it is further shown that the support of $u(\cdot, t)$ has distance at most of order $\sqrt{t}$ from the support of u_0. The proof relies on careful comparison arguments, see also [18] for an alternative (and simpler) approach. Slightly weaker estimates on the support for a larger class of obstacle problems have been derived by probabilistic methods in [3].

Fine regularity properties of the (moving) free boundary, which here is described by the boundary between the support of the solution and the coincidence set $\{u = 0\}$, have ever been a key question in evolutionary or stationary obstacle problem.

Corresponding results have first been obtained in the context of the classical one-phase Stefan problem. This leads to a specific parabolic obstacle problem [16] of the form (B.1.6), where $f = -1$ and the additional sign restriction $\partial_t u \geq 0$ holds. In this case, the free boundary splits into regular and singular points. The set of regular points includes all points where the coincidence set $\{u = 0\}$ has positive density. Whereas cusps may occur in singular points, in regular points the free boundary is locally smooth [7], see also the exposition in [23, Section 2.9].

In the case $f = -1$ but without any assumptions on the sign of $\partial_t u$ it is proved that the boundary is $C^{1,1}$-regular in space and $C^{0,1}$-regular in time [11]. Moreover, around regular points (defined in terms of a lower density function) the free boundary is C^∞-regular in space and time.

The main tool in deriving all such regularity properties are local monotonicity formulas and blow ups. We refer to [10], [20], [21], [66] and the references therein for more recent developments and extensions to more general operators.

Compared to the subject of the present work, there is a key difference between problems (B.1.5) and (B.1.6), which makes the analysis in this paper quite challenging and leads in turn to some interesting results. More specifically, the right-hand side of (B.1.5) differs from the corresponding right-hand side of (B.1.6) in the sense that in the first, $\alpha(t)$ depends on the positivity set $\{u(\cdot,t)>0\}$. It is worth noticing that, in principle, problems with a nonlinear and nonlocal dependence $f := f(u)$ have already been studied. However, in most works this dependence is either local or only includes a dependence on $\int u$, which leads to much better continuity properties and is much easier to control than a dependence on the support of u. A notable exception is the work [65], where also a dependence on the size of the support is present. Yet, the particular non-local dependence of the functional $\alpha(t)$ on the solution $u(\cdot,t)$, does not allow us to use already existent techniques similar to the ones used for problem (B.1.8) or in the more general cases just mentioned.

The goal of this work: The well-posedness of (B.1.5) has been investigated in [44] whereas the corresponding steady states have been studied in [51]. The authors in [44] also prove uniqueness and global stability of the steady states. However, the well-posedness results in [44] imply only the global existence and uniqueness of solutions to (B.1.1)-(B.1.4) with $\alpha \in L^\infty(0,T)$ without providing much insight about the evolution of the set $\{u(\cdot,t)>0\}$.

In this paper, we investigate the time continuity of the positivity set $\{u(\cdot,t)>0\}$. To this end, we focus our attention on the study of the function $t \mapsto \alpha(t)$. We find that there are two conditions that play a crucial role in proving either continuity or jumps of the set $\{u(\cdot,t)>0\}$ as $t \to 0^+$. More precisely, we introduce the following assumptions on the initial data. We assume that for some fixed $\theta > 0$ it holds a first non-degeneracy condition

$$(1-g) - \alpha_0 g \geq \theta > 0 \quad \text{ in } \{u_0 = 0\} \text{ where } \quad \alpha_0 := \frac{\int_{\{u_0>0\}}(1-g)\,dS}{\int_{\{u_0>0\}} g\,dS} \tag{B.1.9}$$

and further we prescribe a second non-degeneracy condition

$$\mathcal{H}^2\Big(\partial\{u_0>0\}\Big) = 0\,, \tag{B.1.10}$$

where $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure in $\mathbb{R}^3$.

Let us discuss the assumptions (B.1.9) and (B.1.10). In the case of problem (B.1.5), the term $(1-g)-\alpha g$ plays the role of $-f$ in (B.1.6). The first condition is therefore clearly related to the condition $f \leq -\nu < 0$ for the classical obstacle problem that has been present in all the regularity results stated above. The necessity of such a condition for the regularity of solutions and its free boundary is well known and can for example be seen from an application of the Hopf boundary point lemma [9]. The same condition also appears as a stability condition for the free boundary and estimates on the symmetric difference of the support of different solutions (see [8] and the exposition in [63, Chapter 6]).

Regarding the problem (B.1.5), if $(1-g)-\alpha_0 g \leq -\theta < 0$, it follows from the strong maximum principle that $u(\cdot,t)$ becomes strictly positive for small positive times. Then, the interface $\partial\{u(\cdot,t)>0\}$ experiences a jump and the same is true for α. Moreover, by the differential inequality in (B.1.5) we obtain at least formally in $\{u(\cdot,t)=0\}$ for almost all t that

$$0 \geq -(1-g)+\alpha g.$$

It is worth pointing out though that in general the right-hand side of (B.1.1) does not have a sign, since its integral over the support of u vanishes.

Regarding the second nondegeneracy condition, we prove in Appendix A.3.2.8, Lemma II.2 that (B.1.10) is equivalent to

$$\mathcal{H}^2\Big(\big(\{u_0>0\}\big)_{+\delta} \setminus \big(\{u_0>0\}\big)_{-\delta}\Big) \to 0 \quad \text{as } \delta \to 0\;, \tag{B.1.11}$$

where

$$\big(\{u_0>0\}\big)_{+\delta} := \{x\;|d(x,\big(\{u_0>0\}\big)) \leq \delta\}\;, \quad \big(\{u_0>0\}\big)_{-\delta} := \{x\;|d(x,\big(\{u_0=0\}\big)) \geq \delta\}\;.$$

In our analysis below mainly the formulation (B.1.11) is used.

Such a condition seem not to be required for problem (B.1.8) but appears to be quite significant for problem (B.1.5). In fact, in the companion paper [46], we provide an example of initial data u_0 such that (B.1.9) holds while (B.1.11) is not satisfied. We prove that in this case, the function $\alpha(t)$ can not be continuous at $t=0$.

The necessity of this second condition in our analysis is a consequence of the particular structure of the right-hand side in (B.1.5) and its dependence on the positivity set $\{u(\cdot,t)>0\}$ through the nonlocal functional $\alpha(t)$, whereas f in (B.1.8) is a general function of space and time.

The plan of this paper is the following. First, we collect some results from previous work and formulas that will play a crucial role in the current analysis. Under assumptions (B.1.9) and (B.1.11) we prove in Section B.3 continuity properties for the set $\{u(\cdot,t)>0\}$ and the function α at $t=0$. In Section B.4, we illustrate the importance of (B.1.9) in the proof of continuity. More specifically, we assume that (B.1.9) is violated and only (B.1.11) is valid. We prove that the set $\{u(\cdot,t)>0\}$ and the function α will experience a jump at $t=0$ and we characterize their behavior in the limit $t \searrow 0$.

B.2 Previous results and preliminaries

Let us state the main assumptions that we impose throughout this paper.

Assumption B.2.1. Let $\Gamma \subset \mathbb{R}^3$ be a smooth compact surface without boundary. For subsets $A \subset \Gamma$ we denote by $|A| = \mathcal{H}^2(A)$ its Hausdorff measure and by $\mathcal{X}_A$ the standard characteristic function of the set A.

For $x_0 \in \Gamma$ and $\rho > 0$ we denote by $B_\rho(x_0)$ the ball on the surface Γ with respect to the extrinsic (Euclidean) distance in $\mathbb{R}^3$. We remark that the assumptions on Γ imply that the intrinsic (geodesic) and the extrinsic distances induce equivalent metrics.

By Δ we denote the Laplace-Beltrami operator on Γ, see also the remark below.

We assume that

$$u_0 \in C^2(\Gamma) \quad \text{with } \; u_0 \geq 0 \quad \text{and} \quad |\{u_0 > 0\}| > 0, \tag{B.2.1}$$

as well as

$$g \in C^2(\Gamma) \quad \text{and} \quad 0 < g_0 \leq g \leq g_1 < 1 \;\text{ on } \Gamma \tag{B.2.2}$$

for some $0 < g_0 < g_1 < 1$. We notice that in some cases, assuming only continuity for the function g in the following analysis would be sufficient. However, (B.2.2) simplifies the computations.

Remark B.2.2. We recall that the relevant diffusion operator on Γ is the corresponding Laplace-Beltrami operator, see for example [57]. In local coordinates the Laplace-Beltrami operator corresponds to an elliptic operator in divergence form (with C^2-regular coefficients in our case). One can deduce parabolic maximum principles in analogy to [22, Chapter 2] for evolution problems on Γ involving the Laplace-Beltrami operator.

Before proceeding to the main analysis of this paper, we collect here some results from our previous work in [44], which appear to be useful in what follows.

To begin with, we have established that problem (B.1.1)-(B.1.3) admits a unique non-negative solution. In fact, for nonnegative $u_0 \in C^2(\Gamma)$, there exists a unique nonnegative $u \in W_p^{2,1}(\Gamma_T)$ for all $p \in [1, \infty)$ and $\alpha \in L^\infty(0,T)$ that solves (B.1.1)-(B.1.3). Moreover it holds that $u \in C^{1+\beta, \frac{1+\beta}{2}}(\Gamma_T)$ for all $0 < \beta < 1$.

In [44, Remark 2.3], we further justify a representation formula for ξ, that is

$$\xi(\cdot, t) = \begin{cases} 1 & \text{in } \{u(\cdot,t) > 0\} \\ \frac{\alpha(t) g}{1-g} & \text{in } \{u(\cdot,t) = 0\} \end{cases} \tag{B.2.3}$$

for almost all $t \in (0,T)$. Due to (B.2.3), one obtains a formula for α which is equivalent to (B.1.4) and is given by

$$\alpha(t) = \frac{\int_\Gamma (1-g)\xi(\cdot,t)\, dS}{\int_\Gamma g\, dS} \; . \tag{B.2.4}$$

We also prove in [44] that the solution to (B.1.1)-(B.1.3) conserves the mass, i.e

$$\int_\Gamma u(\cdot,t)\; dS = \int_\Gamma u_0\; dS = m \quad \text{for all } \; t \in (0,T) \; . \tag{B.2.5}$$

Furthermore, we show that for any two solutions u_1, u_2 of (B.1.1)-(B.1.2), the map

$$t \mapsto \int_\Gamma \Big(u_1 - u_2\Big)_+(\cdot,t)\; dS \quad \text{is decreasing in time.} \tag{B.2.6}$$

This property is crucial for the proof of Theorem B.4.3.

Moreover, we observe that due to [44, Remark 2.3] there is an equivalent and more convenient way to write (B.1.1). We provide a straightforward computation in the next lemma.

We define $H : \mathbb{R} \to \{0,1\}$, $H = \mathcal{X}_{(0,\infty)}$ as the characteristic function of the positive real numbers.

Lemma B.2.3. *Let $\lambda \in L^\infty(0,T)$ be defined by*

$$\lambda(t) = \fint_{\{u(\cdot,t)>0\}} g\; dS \; . \tag{B.2.7}$$

Then equations (B.1.1)*,* (B.1.2) *are equivalent to*

$$\partial_t u - \Delta u = -\left(1 - \frac{g}{\lambda}\right) H(u) \quad \text{on } \; \Gamma_T \; , \tag{B.2.8}$$

$$g \le \lambda \quad \text{almost everywhere in } \{u = 0\} \; . \tag{B.2.9}$$

Proof. We stress that λ is well defined due to the continuity of u. Now, the characterization of α in (B.1.4) yields

$$\alpha(t) = \frac{1}{\fint_{\{u(\cdot,t)>0\}} g\; dS} - 1 = \frac{1}{\lambda(t)} - 1 \; . \tag{B.2.10}$$

By [44, Remark 2.3] and (B.2.3) we can rewrite (B.1.1), (B.1.2) equivalently as

$$\partial_t u - \Delta u + \Big((1-g) - \alpha g\Big) = \Big((1-g) - \alpha g\Big)_+ \mathcal{X}_{\{u=0\}} \,, \tag{B.2.11}$$

Due to (B.2.10) this is equivalent to

$$\partial_t u - \Delta u + \left(1 - \frac{g}{\lambda(t)}\right) = \left(1 - \frac{g}{\lambda(t)}\right)_+ \mathcal{X}_{\{u=0\}} \,. \tag{B.2.12}$$

Using once again [44, Remark 2.3] equation (B.2.12) yields (B.2.9). Thus we conclude that (B.2.8) holds.

Vice versa, if u is a solution of (B.2.8), (B.2.9), we can rewrite (B.2.8) as (B.2.12) and hence obtain (B.1.1), (B.1.2). This completes the proof.

■

B.3 Continuity results

Our goal in this section is to prove that the function α is continuous at $t = 0$ assuming that both (B.1.9) and (B.1.11) hold true. From now on $\theta > 0$ will be fixed. Furthermore, we will show that the positivity set $\{u(\cdot, t) > 0\}$ changes continuously as $t \to 0^+$. More specifically, we will show the following.

Theorem B.3.1. *Suppose that* (B.1.9) *and* (B.1.11) *hold true for some* $\theta > 0$*. Then for any arbitrary small* $\eta > 0$*, there exists* $\bar{t} = \bar{t}(\eta; \theta, u_0, g) > 0$ *such that*

$$\Big(\{u_0 > 0\}\Big)_{-\eta} \subset \{u(\cdot, t) > 0\} \subset \Big(\{u_0 > 0\}\Big)_{+\eta} \tag{B.3.1}$$

and

$$|\alpha(t) - \alpha_0| \leq \eta \tag{B.3.2}$$

for all $t \in [0, \bar{t}]$*.*

We will prove this theorem in the following subsections. As we have already mentioned in Section B.2, problem (B.1.1)-(B.1.3) admits a unique nonnegative solution with $\alpha \in L^\infty(0, T)$. However, beyond this regularity we have no control on α and its evolution could have jumps and fast oscillations.

We note that it is sufficient to prove the claim for all sufficiently small $\eta > 0$. In fact, if (B.3.1), (B.3.2) hold for some $\eta > 0$, both properties hold for all $\tilde{\eta} \geq \eta$, with $\bar{t}(\tilde{\eta}; \theta, u_0, g) = \bar{t}(\eta; \theta, u_0, g)$.

A priori, the limit $\lim_{t\to 0^+}\alpha(t)$ might not exist or might be different from α_0. Therefore, it is convenient to consider a regularized version of (B.1.1)-(B.1.3), for which the analogon of the function α is smooth.

B.3.1 Formulation of the regularized problem

For $\varepsilon > 0$ we consider the following problem

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = -(1-g) f_\varepsilon(u_\varepsilon) + \alpha_\varepsilon g \qquad \text{on } \Gamma \times (0,T)\,, \tag{B.3.1.1}$$

$$u_\varepsilon(\cdot,0) = u_0^\varepsilon \qquad \text{on } \Gamma\,, \tag{B.3.1.2}$$

where

$$f_\varepsilon(u) = \frac{u}{u+\varepsilon} \tag{B.3.1.3}$$

describes a standard Michaelis-Menten law and the nonnegative initial data u_0^ε will be suitably constructed later (see (B.3.1.12) below). Arguments similar to those in Chapter 3 imply that a unique smooth solution of the regularized problem exists for all positive times, and that solutions approximate (B.1.1)-(B.1.3) as $\varepsilon \to 0$. More precisely, for any $1 \le p < \infty$ and $0 < \beta < 1$ we have

$$u_\varepsilon \rightharpoonup u \quad \text{in } W_p^{2,1}(\Gamma_T), \tag{B.3.1.4}$$

$$u_\varepsilon \to u \quad \text{in } C^{1+\beta,\frac{1+\beta}{2}}(\Gamma_T). \tag{B.3.1.5}$$

It follows from (B.3.1.1) that α_ε is given by

$$\alpha_\varepsilon(t) = \frac{\int_\Gamma (1-g) f_\varepsilon(u_\varepsilon(\cdot,t))\, dS}{\int_\Gamma g\, dS}\,. \tag{B.3.1.6}$$

We notice that α_ε is Hölder continuous but that a priori there is no uniform bound on its modulus of continuity. Moreover, the solution of (B.3.1.1)-(B.3.1.2) is strictly positive for all positive times. We aim to prove uniform continuity estimates for the function α_ε as $\varepsilon \to 0^+$ for short times.

We stress that solutions to the original problem (B.1.1)-(B.1.3) and to the regularization (B.3.1.1)-(B.3.1.2) exist globally in time. As we are only concerned with the behavior for small times, throughout this chapter we will consider a fixed time interval $(0,T)$ that is independent of ε.

For further reference we also note that due to (B.2.2) we have

$$0 < c(g) \le \alpha_\varepsilon,\, \alpha \le C(g) \qquad \text{in } [0,T]\,. \tag{B.3.1.7}$$

Construction of initial data for the regularized problem: Due to (B.1.9) and the continuity of g, we can choose a sufficiently small $\sigma > 0$ such that

$$(1-g) - \alpha_0 g \geq \frac{\theta}{2} \qquad \text{in } K_\sigma := \{u_0 = 0\}_{+\sigma} \,. \tag{B.3.1.8}$$

The definition of the set $\{\,\cdot\,\}_{+\sigma}$ is given in Definition I.1. Then, using (B.3.1.8), we can define the positive function

$$\hat{u}_0^\varepsilon := \frac{\varepsilon \alpha_0 g}{(1-g) - \alpha_0 g} \qquad \text{in } K_\sigma \,. \tag{B.3.1.9}$$

Moreover, due to (B.3.1.3), we have that

$$-f_\varepsilon(\hat{u}_0^\varepsilon)(1-g) + \alpha_0 g = 0 \quad \text{in } K_\sigma. \tag{B.3.1.10}$$

We observe that

$$\varepsilon \frac{(1-\max g) g}{\max g} \leq \hat{u}_0^\varepsilon \leq \varepsilon \frac{2(1-g) - \theta}{\theta} \quad \text{in } K_\sigma, \tag{B.3.1.11}$$

where we have used $\alpha_0 \geq \frac{1-\max g}{\max g}$ and (B.3.1.8).

We now consider a smooth cut-off function $\zeta \in C_c^1(\Gamma)$ with the following properties. We assume that, $0 \leq \zeta \leq 1$ in Γ, $\zeta = 1$ in $\{u_0 = 0\}$ and $\zeta = 0$ in $\Gamma \backslash K_\sigma$ with $|\nabla \zeta| \leq \frac{\kappa}{\sigma}$ for some $\kappa > 0$ which is independent of σ. Then we take as initial data in (B.3.1.2) the function

$$u_0^\varepsilon = u_0 + \hat{u}_0^\varepsilon \zeta \qquad \text{in } \Gamma \,, \tag{B.3.1.12}$$

in particular $u_0^\varepsilon \geq u_0$ on Γ and $u_0^\varepsilon = \hat{u}_0^\varepsilon$ in $\{u_0 = 0\}$. Moreover, we observe that

$$u_0^\varepsilon \leq m\varepsilon \quad \text{in } \{u_0 = 0\}, \tag{B.3.1.13}$$

for some $m = m(g) > 0$.

Remark. We point out that $\sigma > 0$ is fixed throughout the chapter and does only depend on θ and the modulus of continuity of g. Here its only role is to specify the size of the region in which we can define $\hat{u}_0^\epsilon$ via (B.3.1.9).

We can easily check that $u_0^\varepsilon \to u_0$ uniformly on Γ as $\varepsilon \to 0$. Furthermore we obtain

$$\begin{aligned}\alpha_\varepsilon(0) \int_\Gamma g\, dy = \int_\Gamma (1-g) f_\varepsilon(u_0^\varepsilon)\, dy \to \int_{\{u_0>0\}} (1-g)\, dy + \alpha_0 \int_{\{u_0=0\}} g\, dy \\ = \alpha_0 \int_{\{u_0>0\}} g\, dy + \alpha_0 \int_{\{u_0=0\}} g\, dy = \alpha_0 \int_\Gamma g\, dy\end{aligned}$$

and hence we have that $\alpha_\varepsilon(0) \to \alpha_0$ and

$$(1-g(x)) - \alpha_\varepsilon(0)g(x) \geq \frac{3\theta}{4} \qquad \text{for all } x \in \{u_0 = 0\} \tag{B.3.1.14}$$

for sufficiently small $\varepsilon > 0$.

The continuity of the function $t \mapsto u_\epsilon(\cdot,t)$ at $t=0$ in the uniform topology and (B.3.1.3), imply that α_ε is also continuous at $t=0$. Combining this and the fact that $\alpha_\varepsilon(0) \to \alpha_0$ as $\epsilon \to 0$, we conclude that for any $\eta > 0$ there exists $T_\varepsilon > 0$, in principle depending on ε such that

$$|\alpha_\varepsilon(t) - \alpha_0| \leq \eta \quad \text{for all } t \leq T_\varepsilon. \tag{B.3.1.15}$$

Recall that it is sufficient to prove the claims for all sufficiently small $\eta > 0$. In the following we will assume that $0 < \eta < \frac{\theta}{4\max g}$. Then by (B.1.9) and (B.3.1.15) we conclude that

$$(1-g(x)) - \alpha_\varepsilon(t)g(x) \geq \frac{\theta}{2} \qquad \text{for all } x \in \{u_0 = 0\} \text{ and } t \in [0, T_\varepsilon]\,. \tag{B.3.1.16}$$

Our goal will be to show that there exists a time $\bar{t} > 0$ that is independent of ε such that (B.3.1.15) and hence (B.3.1.16) hold also in $[0, \bar{t}]$. We will use this result to show that for some $L_0 > 0$ and for $t \in [0, \bar{t}]$ the sets $\{u_\varepsilon(\cdot,t) > L_0\varepsilon\}$ and $\{u_0^\varepsilon > L_0\varepsilon\}$ are close in the sense of Lebesgue measure for small t. Then we can pass to the limit $\varepsilon \to 0$ to conclude the same for $\{u(\cdot,t) > 0\}$ and $\{u_0 > 0\}$. Recalling the form of α in (B.2.4) we see that this is the key estimate in order to prove the continuity of α.

B.3.2 Uniform continuity of α_ε at $t = 0$

We start with a few auxiliary lemmas. The dependence of constants on the data u_0 and g, also through the uniform bounds on $\alpha_\varepsilon, \alpha$ in (B.3.1.7), will not be written explicitly in the following. However, any additional dependence will be specified each time.

Furthermore u_ε always denotes a solution to (B.3.1.1)-(B.3.1.2) with data u_0^ϵ as in (B.3.1.12) and $T_\varepsilon > 0$ such that (B.3.1.16) holds.

Our first result yields uniform continuity in ε for short times.

Lemma B.3.2. *Let u_ε be the unique solution to* (B.3.1.1)-(B.3.1.2) *with data u_0^ϵ as in* (B.3.1.12). *Then,*

$$\|u_\varepsilon(\cdot,t) - u_0^\varepsilon\|_{L^\infty(\Gamma)} \leq C_1 t^{\frac{1+\beta}{2}} \qquad \textit{for all } t \leq T \tag{B.3.2.1}$$

and for all $0 < \beta < 1$ *.*

Proof. Since u_ε is a solution to (B.3.1.1)-(B.3.1.2) with data as in (B.3.1.12) and the right-hand side of (B.3.1.1) is uniformly bounded in ε, we deduce from standard Hölder regularity results for parabolic equations, see [22], that $u_\varepsilon \in C^{1+\beta,\frac{1+\beta}{2}}(\Gamma_T)$ for any $0 < \beta < 1$. The claim then follows. ■

To proceed, we define for the sake of simplicity

$$U := \{u_0 = 0\}\ , \quad V := \Gamma\backslash U = \{x \,|\, u_0(x) > 0\} \tag{B.3.2.2}$$

and for any sufficiently small $\delta > 0$ we define

$$U_{-\delta} := \Big(\{u_0 = 0\}\Big)_{-\delta} = \{x\ |d(x, \Big(\{u_0 > 0\}\Big)) \geq \delta\}, \quad V^\delta := \{u_0 \geq \delta\}\ . \tag{B.3.2.3}$$

A key property in the analysis of free boundary problems is the so-called nondegeneracy property.

This property states that if a solution to (B.1.8) is small in a sufficiently large open set, then it vanishes in a smaller set (cf. [6]). A version of this property for the stationary solutions to (B.1.1)-(B.1.3), has been formulated in [51, Proposition 3.9(5)]. The next lemma yields a variation of this nondegeneracy property for the regularized problem (B.3.1.1), (B.3.1.2).

Since the solution to (B.3.1.1), (B.3.1.2) is strictly positive due to maximum principle, the corresponding nondegeneracy result is formulated as follows. If u_ε is smaller than some number independent of ε, in a sufficiently large set with size independent of ε, then $u_\varepsilon \leq L_0\varepsilon$ for some positive constant L_0 which is independent of ε, in a smaller set with size independent of ε as well. It is worth noticing that in the limit $\varepsilon \to 0^+$, this result would "converge" to the standard nondegeneracy result for the Stefan problem, cf. [6, Theorem 3.1].

Lemma B.3.3. *Consider $T_\varepsilon > 0$ such that* (B.3.1.15) *holds. Then there exist positive constants $\rho_{\max} = \rho_{\max}(\Gamma)$, $A = A(\Gamma)$ and $L_0 = L_0(g, \theta)$ such that for any $\tilde{t} \in [0, T_\varepsilon]$ and any $\rho \in (0, \rho_{max})$ the following holds:*

If $B_{2\rho}(x_0) \subset U$ and if

$$u_\varepsilon \leq \frac{\theta}{A}\rho^2 \qquad \text{in } B_{2\rho}(x_0) \times [0, \tilde{t}]\,,$$

then u_ε satisfies

$$u_\varepsilon \leq L_0\varepsilon \qquad \text{in } B_\rho(x_0) \times [0, \tilde{t}].$$

Proof. Without loss of generality $x_0 = 0 \in \Gamma$. Let us also recall (B.3.1.13). Then we can choose $L_0 > 0$ such that

$$L_0 \geq 2m \quad \text{and} \quad (1-g)\frac{1}{L_0+1} \leq \frac{\theta}{4}\,. \tag{B.3.2.4}$$

We proceed by contradiction, that is we suppose that there exists $(y,\tau) \in B_\rho(0) \times [0,\tilde{t}]$ such that $u_\varepsilon(y,\tau) > L_0\varepsilon$. As a candidate for a supersolution we define

$$\tilde{u}(x,t) := L_0\varepsilon + \frac{\theta}{A}|x-y|^2 + \frac{\theta}{8}(\tau - t)$$

and compute that

$$\partial_t\tilde{u} - \Delta\tilde{u} = -\frac{\theta}{8} - \frac{4\theta}{A} - \frac{2\theta}{A}\vec{H}\cdot(x-y) \geq -\frac{\theta}{4}$$

if $\rho \leq \rho_{\max}(\Gamma)$ is sufficiently large. (Here $\vec{H}$ denotes the mean curvature vector on Γ.) By (B.3.1.16) and the choice of L_0 we have in $(B_{2\rho}(0) \times [0,\tau]) \cap \{u_\varepsilon \geq L_0\varepsilon\}$ that

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = -(1-g)f_\varepsilon(u_\varepsilon) + \alpha_\varepsilon(t)g \leq -\frac{\theta}{4} \leq \partial_t\tilde{u} - \Delta\tilde{u}\,.$$

Furthermore, we check that on $(\partial B_{2\rho}(0) \times [0,\tau]) \cap \{u_\varepsilon \geq L_0\varepsilon\}$ we have

$$u_\varepsilon \leq \frac{\theta}{A}\rho^2 \leq \tilde{u}$$

while on $(B_{2\rho}(0) \times [0,\tau]) \cap \partial\{u_\varepsilon \geq L_0\varepsilon\}$ it clearly holds that $u_\varepsilon - \tilde{u} \leq 0$. Furthermore $u_\varepsilon(\cdot,0) \leq \tilde{u}(\cdot,0)$ by our choice of L_0 in (B.3.2.4). Hence the parabolic maximum principle implies $u \leq \tilde{u}$ in $(B_{2\rho}(0) \times [0,\tau]) \cap \{u_\varepsilon \geq L_0\varepsilon\}$ which gives a contradiction. ■

Now, we can prove the left inclusion in (B.3.1) for the solution of the regularized problem (B.3.1.1), (B.3.1.2).

Corollary B.3.4. *Let $\delta > 0$, $\beta \in (0,1)$ be fixed and $T_\varepsilon > 0$ be such that* (B.3.1.15) *holds. Let $A = A(\Gamma)$ be as in Lemma B.3.3, $C_1 > 0$ as in Lemma B.3.2 and set*

$$t^*(\delta) := \left(\frac{\theta\delta^2}{8AC_1}\right)^{\frac{2}{1+\beta}}. \tag{B.3.2.5}$$

Then, there exists $\varepsilon_0 = \varepsilon_0(\delta, u_0, g) > 0$, such that for all $0 < \varepsilon \leq \varepsilon_0$ we have

$$U_{-\delta} \subset \{u_\varepsilon(\cdot,t) \leq L_0\varepsilon\} \quad \textit{for all} \quad 0 \leq t \leq \min\{T_\varepsilon, t^*(\delta)\},$$

where $U_{-\delta}$ is given by (B.3.2.3).

Proof. Due to (B.3.1.13) Lemma B.3.2 implies

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(U)} \le \|u_\varepsilon(\cdot,t) - u_0^\varepsilon\|_{L^\infty(U)} + \|u_0^\varepsilon\|_{L^\infty(U)} \le C_1 t^{\frac{1+\beta}{2}} + m\varepsilon\,.$$

Hence, we deduce by (B.3.2.4) that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(U)} \le 2m\varepsilon \le L_0\epsilon \quad \text{for all } 0 \le t \le \min\left\{T_\varepsilon, t^*(\delta), \left(\frac{m\varepsilon}{C_1}\right)^{\frac{2}{1+\beta}}\right\}.$$

On the other hand, for $\left(\frac{m\varepsilon}{C_1}\right)^{\frac{2}{1+\beta}} < t \le \min\{T_\varepsilon, t^*(\delta)\}$ we obtain by (B.3.2.5) that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(U)} \le 2C_1\Big(\min\{T_\varepsilon, t^*(\delta)\}\Big)^{\frac{1+\beta}{2}} \le \frac{\theta}{A}\left(\frac{\delta}{2}\right)^2.$$

We now apply Lemma B.3.3. To this end we choose $\rho = \min(\rho_{\max}, \delta/2)$ and arbitrary $x_0 \in U_{-\delta}$. Then Lemma B.3.3 implies the claim. ■

As long as (B.3.1.15) is valid, we obtain in the next lemma some detailed pointwise estimates for the function $f_\epsilon(u_\epsilon)$ which is given by (B.3.1.3).

Lemma B.3.5. *Let $\eta \in \left(0, \frac{\theta}{4\max g}\right]$ and $T_\varepsilon > 0$ be such that* (B.3.1.15) *holds. Then for given small $\delta > 0$ there exists a constant $C_\delta > 0$ such that in the set $U_{-2\delta} \times (0, T_\varepsilon)$ we have*

$$f_\epsilon(u_\epsilon)(1-g) - (\alpha_0+\eta)g \le C_\delta\varepsilon \tag{B.3.2.6}$$

and

$$f_\epsilon(u_\epsilon)(1-g) - (\alpha_0-\eta)g \ge -C_\delta\varepsilon\,. \tag{B.3.2.7}$$

Proof. We are going to prove (B.3.2.6), the proof of (B.3.2.7) goes analogously.

Step 1: We first construct a suitable supersolution by defining $\bar{u}_\varepsilon \colon U_{-\delta} \times [0, T_\varepsilon] \to \mathbb{R}_+$ via

$$\partial_t \bar{u}_\varepsilon - \Delta \bar{u}_\varepsilon = -(1-g) f_\varepsilon\Big(\bar{u}_\varepsilon\Big) + (\alpha_0+\eta)g\,, \qquad \text{in } U_{-\delta} \times (0, T_\varepsilon) \tag{B.3.2.8}$$

$$\bar{u}_\varepsilon(\cdot,0) = \varepsilon\frac{(\alpha_0+\eta)g}{(1-g)-(\alpha_0+\eta)g} \qquad \text{in } U_{-\delta}\,, \tag{B.3.2.9}$$

$$\bar{u}_\varepsilon = L_0\varepsilon \qquad \text{on } \partial U_{-\delta} \times (0, T_\varepsilon)\,. \tag{B.3.2.10}$$

Indeed, for $w_\varepsilon := u_\varepsilon - \bar{u}_\varepsilon$ we find, due to (B.3.1.15), that

$$\partial_t w_\varepsilon - \Delta w_\varepsilon \le -(1-g)\Big(f_\varepsilon(u_\varepsilon) - f_\varepsilon(\bar{u}_\varepsilon)\Big)$$

$$= -(1-g)\frac{\varepsilon}{(\varepsilon + u_\varepsilon)(\varepsilon + \bar{u}_\varepsilon)} w_\varepsilon \,.$$

Furthermore, on $U_{-\delta}$ it holds

$$u_\varepsilon(\cdot, 0) \leq \varepsilon \frac{(\alpha_0 + \eta)g}{(1-g) - (\alpha_0 + \eta)g} = \bar{u}_\varepsilon(\cdot, 0)\,,$$

while on $\partial U_{-\delta} \times [0, T_\varepsilon]$ we have $w_\varepsilon \leq 0$ by Corollary B.3.4. This yields $w_\varepsilon \leq 0$, hence

$$u_\varepsilon \leq \bar{u}_\varepsilon \quad \text{in } U_{-\delta} \times [0, T_\varepsilon]. \tag{B.3.2.11}$$

Step 2: We next show that for some $C = C(\delta, L_0)$ it holds

$$|\bar{u}_\varepsilon(x,t) - \bar{u}_\varepsilon(x,0)| \leq C\varepsilon^2 \qquad \text{in } U_{-2\delta} \times [0, T_\varepsilon]\,. \tag{B.3.2.12}$$

Further we consider $U_\varepsilon^0 := \frac{1}{\varepsilon}\bar{u}_0^\varepsilon$, $U_\varepsilon := \frac{1}{\varepsilon}\bar{u}_\varepsilon$ and $v_\varepsilon := U_\varepsilon - U_\varepsilon^0$. We compute, with $f(u) = \frac{u}{u+1}$, that

$$\begin{aligned}
\partial_t v_\varepsilon - \Delta v_\varepsilon &= -\frac{1}{\varepsilon}(1-g) f\Big(U_\varepsilon\Big) + \frac{1}{\varepsilon}(\alpha_0 + \eta)g - \Delta U_\varepsilon^0(x) \\
&= -\frac{1}{\varepsilon}(1-g)\Big(f(U_\varepsilon) - f(U_\varepsilon^0)\Big) - \Delta U_\varepsilon^0(x) \\
&= -\frac{1}{\varepsilon}(1-g)\frac{1}{(1+U_\varepsilon)(1+U_\varepsilon^0)} v_\varepsilon - \Delta U_\varepsilon^0(x)\,.
\end{aligned}$$

We first notice that, since $|\Delta U_\varepsilon^0| \leq C_0$ for some $C_0 = C_0(g)$, we have that $|v_\varepsilon|$ and then also U_ε are uniformly bounded in ε by some constant only depending on δ, L_0. Hence, we have

$$\partial_t v_\varepsilon - \Delta v_\varepsilon \leq -\frac{\mu}{\varepsilon} v_\varepsilon + C_0 \tag{B.3.2.13}$$

for some $\mu = \mu(\delta, L_0) > 0$.

We compare v_ε with the solution of the boundary value problem

$$-\varepsilon \Delta w_\varepsilon = -\mu w_\varepsilon + \varepsilon C_0 \quad \text{in } U_{-\delta}, \qquad w_\varepsilon = v_\varepsilon \quad \text{on } \partial U_{-\delta}. \tag{B.3.2.14}$$

By applying maximum principles we find that in $U_{-\delta}$ for some $C_1 = C_1(g, L_0)$

$$0 \leq w_\varepsilon \leq C_1, \qquad v_\varepsilon(\cdot, t) \leq w_\varepsilon \quad \text{for all } 0 < t < T_\varepsilon. \tag{B.3.2.15}$$

We next claim that there exists $\Lambda = \Lambda(\mu, \delta)$ such that

$$w_\varepsilon \leq \Lambda\varepsilon \quad \text{in } U_{-2\delta}. \tag{B.3.2.16}$$

To prove this estimate consider an arbitrary $\Lambda > 0$ (to be chosen sufficiently large later) and assume by contradiction that $w_\varepsilon(x_0) > \Lambda\varepsilon$ for some $x_0 \in U_{-2\delta}$.

Let $\vartheta := \varepsilon\frac{\Lambda}{2C_1} + \varphi^2$ where $\varphi \in C_c^\infty(B_\delta(x_0))$ is chosen such that

$$\vartheta(x_0) = 1, \quad \vartheta \leq 1, \quad \|\vartheta\|_{C^2(B_\delta(x_0))} \leq C(\delta).$$

Observe that $\vartheta \geq \varepsilon\frac{\Lambda}{2C_1}$ and

$$\frac{|\nabla\vartheta|^2}{\vartheta} \leq \frac{4\varphi^2|\nabla\varphi|^2}{\varepsilon\frac{\Lambda}{2C_1} + \varphi^2} \leq C(\delta) \tag{B.3.2.17}$$

for some $C(\delta)$ independent of ε, Λ. Furthermore, we compute

$$\nabla(\vartheta w_\varepsilon) = \vartheta\nabla w_\varepsilon + w_\varepsilon\nabla\vartheta, \qquad \Delta(\vartheta w_\varepsilon) = \vartheta\Delta w_\varepsilon + 2\nabla\vartheta\cdot\nabla w_\varepsilon + w_\varepsilon\Delta\vartheta,$$

and deduce from (B.3.2.14)

$$\begin{aligned} C_0\vartheta &= -\Delta(\vartheta w_\varepsilon) + 2\nabla\vartheta\cdot\nabla w_\varepsilon + w_\varepsilon\Delta\vartheta + \frac{\mu}{\varepsilon}(\vartheta w_\varepsilon) \\ &= -\Delta(\vartheta w_\varepsilon) + \frac{2}{\vartheta}\nabla\vartheta\cdot\nabla(\vartheta w_\varepsilon) + \Big(\frac{\mu}{\varepsilon} - \frac{2|\nabla\vartheta|^2}{\vartheta^2} + \frac{1}{\vartheta}\Delta\vartheta\Big)(\vartheta w_\varepsilon). \end{aligned}$$

Using (B.3.2.17) yields for all $\Lambda \geq \Lambda_0$, $\Lambda_0 = \Lambda_0(\delta, C_1, \mu)$

$$-\frac{2|\nabla\vartheta|^2}{\vartheta^2} + \frac{1}{\vartheta}\Delta\vartheta \geq -\frac{C(\delta)}{\vartheta} \geq -\frac{2C_1C(\delta)}{\varepsilon\Lambda} \geq -\frac{\mu}{2\varepsilon},$$

hence

$$C_0\vartheta \geq -\Delta(\vartheta w_\varepsilon) + \frac{2}{\vartheta}\nabla\vartheta\cdot\nabla(\vartheta w_\varepsilon) + \frac{\mu}{2\varepsilon}(\vartheta w_\varepsilon). \tag{B.3.2.18}$$

By the choice of ϑ we have $(\vartheta w_\varepsilon)(x_0) > \Lambda\varepsilon > (\vartheta w_\varepsilon)|_{\partial B(x_0,\delta)}$. Hence ϑw_ε attains an interior maximum at some $x_1 \in B(x_0, \delta)$. Evaluating (B.3.2.18) we deduce that

$$C_0 \geq C_0\vartheta(x_1) \geq \frac{\mu}{2\varepsilon}(\vartheta w_\varepsilon)(x_1) \geq \frac{\mu}{2\varepsilon}(\vartheta w_\varepsilon)(x_0) > \frac{\mu\Lambda}{2},$$

a contradiction for $\Lambda \geq \Lambda_*$, where $\Lambda_* = \Lambda_*(C_0, C_1, \mu, \delta)$ only depends on δ and L_0.

This completes the proof of (B.3.2.16).

Step 3: Using (B.3.2.11), (B.3.2.9), (B.3.2.12) and the monotonicity and continuity of f we finally obtain

$$\begin{aligned} f_\epsilon(u_\epsilon)(1-g) - (\alpha_0+\eta)g &\leq f_\epsilon(\bar{u}_\epsilon)(1-g) - (\alpha_0+\eta)g \\ &= f_\epsilon(\bar{u}_\epsilon^0)(1-g) - (\alpha_0+\eta)g + \frac{\varepsilon(\bar{u}_\varepsilon - \bar{u}_\varepsilon^0)}{(\varepsilon+\bar{u}_\varepsilon)(\varepsilon+\bar{u}_\varepsilon^0)}(1-g) \\ &\leq C(\delta, L_0)\varepsilon. \end{aligned}$$

Since the choice of L_0 only depends on g, θ this proves (B.3.2.6). ■

The following proposition plays a crucial role in proving Theorem B.3.1. It states that we can choose $T_\varepsilon > 0$ in (B.3.1.15) independent of ε. The key idea in this proof is to consider the maximal time interval where (B.3.1.15) is satisfied and show that it could be extended unless it contains and interval $[0, \bar{t}]$, with $\bar{t}$ independent of ε.

Proposition B.3.6. *Let $\theta > 0$ as in* (B.1.9) *and $0 < \eta < \frac{\theta}{4 \max g}$. Then, for sufficiently small $\varepsilon > 0$, there exists $\bar{t} := \bar{t}(\eta, \theta, u_0, g) > 0$ that is independent of ε such that*

$$|\alpha_\varepsilon(t) - \alpha_0| \leq \eta \quad \textit{for all } t \in [0, \bar{t}] \, .$$

In particular, we can choose $T_\varepsilon = \bar{t}$ in Corollary B.3.4 independent of $\varepsilon > 0$.

Proof. **Step 1:** We observe that Lemma B.3.5 yields that there exists $r_1 \in [-\eta, \eta]$ such that

$$\int_U (1-g) f_\varepsilon\Big(u_\varepsilon(\cdot, t)\Big) \, dS = \Big(\alpha_0 + r_1\Big) \int_U g \, dS \; + O(C_\delta \varepsilon) + \omega(\delta), \tag{B.3.2.19}$$

where $\omega(\delta) \to 0$ as $\delta \to 0$. Moreover, by (B.3.1.12) we have that $u_0^\varepsilon \geq u_0$ on Γ. Using this and Lemma B.3.2 and possibly passing to a smaller value of $t^*(\delta)$, we obtain for all $t \leq t^*(\delta)$ that

$$\int_{V^\delta} (1-g) f_\varepsilon\Big(u_\varepsilon(\cdot, t)\Big) \, dS = \int_{V^\delta} (1-g) \, dS + O\Big(\frac{\varepsilon}{\delta}\Big) \, . \tag{B.3.2.20}$$

Moreover, using (B.1.11), (B.2.2) and (B.3.1.3) we deduce

$$0 \leq \int_{\Gamma \backslash (U \cup V^\delta)} (1-g) f_\varepsilon\Big(u_\varepsilon(\cdot, t)\Big) \, dS \leq |\Gamma \backslash (U \cup V^\delta)| \; \to \; 0 \, , \;\; \text{as } \delta \to 0. \tag{B.3.2.21}$$

Therefore we can write, combining (B.3.2.19), (B.3.2.20),

$$\begin{aligned}
\alpha_\varepsilon(t) \int_\Gamma g \, dS &= \int_{V^\delta} (1-g) f_\varepsilon\Big(u_\varepsilon(\cdot, t)\Big) \, dS + \int_U (1-g) f_\varepsilon\Big(u_\varepsilon(\cdot, t)\Big) \, dS \\
&\qquad + \int_{\Gamma \backslash (U \cup V^\delta)} (1-g) f_\varepsilon\Big(u_\varepsilon(\cdot, t)\Big) \, dS \\
&= \int_{V^\delta} (1-g) \, dS + O\Big(\frac{\varepsilon}{\delta}\Big) + \Big(\alpha_0 + r_1\Big) \int_U g \, dS + r_2 \, ,
\end{aligned}$$

with $|r_2| \leq C_\delta \varepsilon + \omega(\delta)$. This estimate can be further simplified as follows

$$\begin{aligned}
\alpha_\varepsilon(t) \int_\Gamma g \, dS &= \int_V (1-g) \, dS + \Big(\alpha_0 + r_1\Big) \int_U g \, dS + r_3 \\
&= \alpha_0 \int_{\Gamma \backslash U} g \, dS + \alpha_0 \int_U g \, dS + r_1 \int_U g \, dS + r_3
\end{aligned}$$

$$= \alpha_0 \int_\Gamma g\, dS + r_1 \int_U g\, dS + r_3\,,$$

where $|r_3| \le r_2 + O\Big(\frac{\varepsilon}{\delta}\Big) + |V \setminus V^\delta|$. Thus, we obtain for all $0 \le t \le T_\varepsilon$

$$\Big|\alpha_\varepsilon(t) - \alpha_0\Big| \le \kappa |r_1| + C|r_3| \le (1-\kappa)\eta + C\omega(\delta) + C_\delta \varepsilon$$

with $\kappa := \frac{\int_{\{u_0>0\}} g\, dS}{\int_\Gamma g\, dS} > 0$. This is the key estimate in order to complete the proof.

Indeed, let us first fix δ depending on η such that $C\omega(\delta) \le \frac{\kappa}{4}\eta$ and then ε depending on δ sufficiently small such that $C_\delta\varepsilon \le \frac{\kappa}{4}\eta$. Then it holds

$$\Big|\alpha_\varepsilon(t) - \alpha_0\Big| \le \Big(1 - \frac{\kappa}{2}\Big)\eta \quad \text{for all } 0 \le t \le T_\epsilon. \tag{B.3.2.22}$$

Next, define

$$\tilde{T}_\varepsilon := \sup\Big\{0 \le s \le t^*(\delta) \;:\; |\alpha_\varepsilon(t) - \alpha_0| \le \eta \;\text{ for all } t \in [0,s]\Big\}\,.$$

If $\tilde{T}_\varepsilon < t^*(\delta)$ the continuity of α_ε and (B.3.2.22) imply that $|\alpha_\varepsilon(t) - \alpha_0| \le \eta$ holds on an interval $[0, T^\dagger]$ with $\tilde{T}_\varepsilon < T^\dagger \le t^*(\delta)$, a contradiction to the definition of $\tilde{T}_\varepsilon$.

Thus, setting $\bar{t} := \bar{t}(\eta) = t^*(\delta)$, we obtain that $|\alpha_\varepsilon(t) - \alpha_0| \le \eta$ in $[0, \bar{t}]$. ■

Proposition B.3.6 implies that there exists a time $\bar{t} > 0$ which is independent of ε such that (B.3.1.15) and hence (B.3.1.16) holds for all $t \in [0, \bar{t}]$.

B.3.3 Proof of Theorem B.3.1

With the uniform continuity result stated in Proposition B.3.6 we can now prove Theorem B.3.1 by passing to the limit $\varepsilon \to 0$.

Proof of Theorem B.3.1. Recalling (B.1.4) and (B.3.1.7), we estimate

$$\begin{aligned} |\alpha(t) - \alpha_0| &= \left| \frac{\int_{\{u(\cdot,t)>0\}} (1-g)\, dS}{\int_{\{u(\cdot,t)>0\}} g\, dS} - \frac{\int_{\{u_0>0\}} (1-g)\, dS}{\int_{\{u_0>0\}} g\, dS} \right| \\ &\le \frac{1}{\int_{\{u_0>0\}} g\, dS} \Big| \int_{\{u_0>0\}} (1-g)\, dS - \int_{\{u(\cdot,t)>0\}} (1-g)\, dS \Big| \\ &\quad + \frac{\alpha(t)}{\int_{\{u_0>0\}} g\, dS} \Big| \int_{\{u(\cdot,t)>0\}} g\, dS - \int_{\{u_0>0\}} g\, dS \Big| \\ &\le C \Big| \{u_0 > 0\} \Delta \{u(\cdot,t) > 0\} \Big| \end{aligned} \tag{B.3.3.1}$$

Then it is sufficient to show that for all $\delta > 0$ sufficiently small there exists $t^\dagger(\delta) > 0$ such that

$$V^\delta \subset \{u(\cdot,t) > 0\} \subset \Big(U_{-\delta}\Big)^{\mathsf{c}} \quad \text{for all } 0 < t < t^\dagger(\delta)\,, \tag{B.3.3.2}$$

where $V^\delta, U_{-\delta}$ are given by (B.3.2.3).

Indeed, by (B.3.2.2) we observe that

$$V^\delta \subset V \subset \Big(U_{-\delta}\Big)^{\mathsf{c}}\,.$$

This in turn yields that for all $0 < t < t^\dagger(\delta)$

$$\{u(\cdot,t) > 0\}\Delta\{u_0 > 0\} \subset \Big(U_{-\delta}\Big)^{\mathsf{c}} \setminus V^\delta\,. \tag{B.3.3.3}$$

Using Lemma I.4, we infer that for any $\delta > 0$ we can choose $\bar\delta := \bar\delta(\delta) \geq \delta$ with $\bar\delta \to 0$ as $\delta \to 0$ such that $V_{-\bar\delta} \subset V^\delta$. By Lemma I.2 it holds $\Big(U_{-\delta}\Big)^{\mathsf{c}} \subset V_{+\bar\delta}$, which yields

$$\Big(U_{-\delta}\Big)^{\mathsf{c}} \setminus V^\delta \subset V_{+\bar\delta} \setminus V_{-\bar\delta}\,. \tag{B.3.3.4}$$

Moreover, we deduce by (B.3.3.2) that

$$V_{-\bar\delta} \subset \{u(\cdot,t) > 0\} \subset V_{+\bar\delta} \quad \text{for all } 0 < t < t^\dagger(\delta)\,. \tag{B.3.3.5}$$

Finally, (B.3.3.3), (B.3.3.4), the convergence $\bar\delta \to 0$ as $\delta \to 0$ and (B.1.11) yield that

$$|\{u(\cdot,t) > 0\}\Delta\{u_0 > 0\}| \leq |\{u_0 > 0\}_{+\bar\delta} \setminus \{u_0 > 0\}_{-\bar\delta}| \to 0 \quad \text{as } \delta \to 0\,.$$

Therefore, for given $\eta > 0$, we can choose $0 < \delta \leq \eta$, $\delta = \delta(\eta)$ sufficiently small, such that the right-hand side of (B.3.3.1) is less or equal than η for all $0 < t < \bar t(\eta)$, with $\bar t(\eta) = t^\dagger(\delta)$. Thus, we obtain (B.3.2). Moreover, by (B.3.3.5) and $\delta \leq \eta$ we obtain (B.3.1).

This proves that if (B.3.3.2) holds, then Theorem B.3.1 follows. To this end, we now proceed to the proof of (B.3.3.2).

On the one hand, by Corollary B.3.4 and Proposition B.3.6 we obtain that for any $\delta > 0$ and for any $\varepsilon = \varepsilon(\delta)$ sufficiently small, $U_{-\delta} \subset \{u_\varepsilon \leq L_0\varepsilon\}$ for all $t \in [0, t^*(\delta)]$. This in particular yields that for any $x \in U_{-\delta}$ and for all $0 \leq t \leq t^*(\delta)$, $u_\varepsilon(x,t) \leq L_0\varepsilon$. Due to the uniform convergence $u_\varepsilon \to u$, we conclude that $U_{-\delta} \subset \{u(\cdot,t) = 0\}$ for all t in $[0, t^*(\delta)]$. Taking the complements, the right inclusion follows.

On the other hand, $x \in V^\delta$ implies $u_0(x) \geq \delta$ and by uniform convergence $u(\cdot,t) \to u_0$, see Section B.2, and after possibly passing to a lower value of $t^*(\delta)$, we have that $u(x,t) > 0$ for all $t \leq t^*(\delta)$. Thus $V^\delta \subset \{u(\cdot,t) > 0\}$ for all $t \leq t^*(\delta)$.

Hence, choosing $t^\dagger(\delta) = t^*(\delta)$ we deduce (B.3.3.2).

■

B.4 Initial jump for incompatible data

So far, we have proven the continuity of the function α in $t = 0$ assuming (B.1.9) and (B.1.10). We notice that in Lemma B.2.3, we have derived an equivalent formulation for (B.1.1), (B.1.2) given by (B.2.8), (B.2.9). Hence, all continuity properties that have been justified for the function α in Section B.3, hold for the function λ as well. For the sake of convenience, we choose in the following analysis to study the equivalent problem

$$\partial_t u - \Delta u = -\left(1 - \frac{g}{\lambda(t)}\right) H(u) \qquad \text{on } \Gamma_T , \qquad \text{(B.4.1)}$$

$$\lambda \leq g \qquad \text{a.e in } \{u = 0\} , \qquad \text{(B.4.2)}$$

$$u(\cdot, 0) = u_0 \qquad \text{on } \Gamma \qquad \text{(B.4.3)}$$

and investigate the behavior of λ, as defined in (B.2.7).

This section highlights the necessity of (B.1.9) for the continuity of the function λ and of the support of the solution u. More precisely, we will show that if (B.1.9) fails, then one cannot expect continuity of the function λ nor of the set $\{u(\cdot, t) > 0\}$ at $t = 0$. To this end, we assume that (B.1.10) holds true, while (B.1.9) is violated in the sense that

$$|\{u_0 = 0\} \cap \{(1 - g) - \alpha_0 g < 0\}| > 0 , \qquad \text{(B.4.4)}$$

where $\alpha_0 = \frac{\int_{\{u_0>0\}} (1-g)\, dS}{\int_{\{u_0>0\}} g\, dS}$. By (B.2.10), this assumption is equivalent to

$$|\{u_0 = 0\} \cap \{g > \lambda_0\}| > 0 , \text{ where } \lambda_0 = \fint_{\{u_0>0\}} g\, dS. \qquad \text{(B.4.5)}$$

The non-generic case in which $|\{u_0 = 0\} \cap \{g = \lambda_0\}| > 0$ will be addressed below (see Remark B.4.8).

We stress that both (B.1.9) and (B.1.10) are necessary in order to obtain any continuity properties for the function λ. In fact, we will present in Chapter 4 an example of initial data u_0 for which (B.1.9) holds while (B.1.10) is invalid. We will prove that under these assumptions, the function λ is not continuous at $t = 0$ and the positivity set $\{u(\cdot, t) > 0\}$ is oscillatory as $t \to 0^+$.

Our aim is to prove that under assumption (B.4.5) the function λ and the positivity set $\{u(\cdot, t) > 0\}$ both will jump at $t = 0$. It turns out that we can characterize this jump in terms of a variational principle. More specifically, we define $\Lambda[u_0]$ as follows.

Definition B.4.1. For any open, measurable set $S \subset \Gamma$, we set

$$\Lambda_S := \sup \left\{ \fint_A g \, dS \ : \ A \subset \Gamma \text{ measurable with } S \subset A \right\} . \tag{B.4.6}$$

In the particular case $S = \{u_0 > 0\}$, where $u_0 : \Gamma \to \mathbb{R}$ denotes the nonnegative continuous initial data, we will write $\Lambda[u_0] := \Lambda_{\{u_0 > 0\}}$ for the sake of simplicity.

We will prove that under (B.4.5)

$$\lim_{t \searrow 0} \lambda(t) = \Lambda[u_0] \quad \text{and} \quad \Lambda[u_0] > \lambda_0 \, .$$

Moreover, we will show that the positivity set $\{u(\cdot, t) > 0\}$ approximates as $t \to 0^+$, one of the sets, for which the maximum in (B.4.6) is attained. Notice that we can have several maximizers in (B.4.6) differing by sets contained in $\{g = \Lambda_S\}$.

Definition B.4.2. For a given nonnegative continuous function $u_0 : \Gamma \to \mathbb{R}$, we set

$$A^0_* := \Big\{ g \geq \Lambda[u_0] \Big\} \cup \{u_0 > 0\} \, . \tag{B.4.7}$$

We will prove later that the maximum in (B.4.6) is attained by the set A^0_*.

Our main goal in this section is to prove the following result.

Theorem B.4.3. *Suppose that* (B.1.10) *holds true and that g satisfies* (B.2.2). *For any $\eta > 0$ there exists $\bar{t} = \bar{t}(\eta) > 0$ such that the positivity set $\{u(\cdot, t) > 0\}$ satisfies for all $0 < t \leq \bar{t}(\eta)$*

$$\Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\} \Big)_{-\eta} \subset \{u(\cdot, t) > 0\} \subset \Big(\{u_0 > 0\} \cup \{g \geq \Lambda[u_0]\} \Big)_{+\eta} . \tag{B.4.8}$$

Furthermore,

$$|\lambda(t) - \Lambda[u_0]| \leq \eta \quad \textit{for all } 0 < t \leq \bar{t}(\eta) \, . \tag{B.4.9}$$

In particular, $\lambda(t) \to \Lambda[u_0]$ as $t \searrow 0$.

Remark B.4.4. We stress that the inclusions in (B.4.8) imply that there exists a set $B(t) \subset \{g = \Lambda[u_0]\}$ such that $\{u(\cdot, t) > 0\} \cup B(t) \to A^0_*$ with respect to the L^1-convergence of sets. It is worth noticing that $B(t)$ could in principle be oscillatory.

Remark B.4.5. It follows from the proof of Theorem B.4.3 and more specifically from (B.4.2.33) that

$$\{u_0 > 0\} \cup \{g > \Lambda[u_0]\} \subset \{\liminf_{t\downarrow 0} u(\cdot,t) > 0\} \subset \liminf_{t\downarrow 0}\{u(\cdot,t) > 0\} = \bigcup_{\delta>0}\bigcap_{0<t<\delta}\{u(\cdot,t) > 0\}$$

and from (B.4.8) that

$$\limsup_{t\downarrow 0}\{u(\cdot,t) > 0\} = \bigcap_{\delta>0}\bigcup_{0<t<\delta}\{u(\cdot,t) > 0\} \subset \overline{\{u_0 > 0\}} \cup \{g \geq \Lambda[u_0]\}.$$

Note that $|\overline{\{u_0 > 0\}} \setminus \{u_0 > 0\}| = 0$ by the regularity of the set $\{u_0 > 0\}$. On the other hand, if $|\{g = \Lambda[u_0]\}| > 0$, then the upper and lower inclusion may differ by a set of positive measure.

Corollary B.4.6. *Assume in addition that* (B.4.5) *holds. Then*

$$\lim_{t\downarrow 0}\lambda(t) = \Lambda[u_0] > \lambda_0 \quad \text{and} \quad |A_*^0 \setminus \{u_0 > 0\}| > 0\ .$$

Proof. The first equality follows from Theorem B.4.3.

Let us consider the set $A = \{u_0 > 0\} \cup B$, where $B := \{u_0 = 0\} \cap \{g > \lambda_0\}$ satisfies $|B| > 0$ by (B.4.5). We obtain that

$$⨍_A g\, dS = \frac{1}{|A|}\left(\int_{\{u_0>0\}} g\, dS + \int_B g\, dS\right) > \frac{1}{|A|}\Big(\lambda_0|\{u_0 > 0\}| + \lambda_0|B|\Big) = \lambda_0\ ,$$

which yields due to Definition B.4.1 that $\Lambda[u_0] > \lambda_0$.

Moreover,

$$\Big|A_*^0 \setminus \{u_0 > 0\}\Big| = \Big|\{g \geq \Lambda[u_0]\} \cap \{u_0 = 0\}\Big| > 0,$$

otherwise using once again Definition B.4.1 we obtain that $\Lambda[u_0] = \lambda_0$ which contradicts the first statement of the Corollary. ■

Remark B.4.7. If in addition to the assumptions in Theorem B.4.3 also (B.1.9) holds, Theorem B.4.3 reduces to Theorem B.3.1. Indeed, (B.1.9) is by (B.2.10) equivalent to

$$g < \lambda_0 \quad \text{in } \{u_0 = 0\}\ . \tag{B.4.10}$$

Then, for any set A as in Definition B.4.1 with $|A \setminus \{u_0 > 0\}| > 0$, we obtain due to (B.4.10) and (B.2.7) that

$$⨍_A g\, dS = \frac{1}{|A|}\left(\int_{\{u_0>0\}} g\, dS + \int_{A\setminus\{u_0>0\}} g\, dS\right) < \frac{1}{|A|}\Big(\lambda_0|\{u_0 > 0\}| + \lambda_0|A \setminus \{u_0 > 0\}|\Big) = \lambda_0\ .$$

Thus, since $\lambda_0 \leq \Lambda$ by definition of λ_0, we conclude that $\Lambda[u_0] = \lambda_0$ and (B.4.9) reduces to (B.3.2). Furthermore, by (B.4.10) and the fact that $\Lambda[u_0] = \lambda_0$, we observe that the right- as well as the left-hand side of (B.4.8) can be written now equivalently as

$$\Big\{g \geq \Lambda[u_0]\Big\} \cup \{u_0 > 0\} = \Big\{g \geq \lambda_0\Big\} \cup \{u_0 > 0\} = \{u_0 > 0\} = \Big\{g > \Lambda[u_0]\Big\} \cup \{u_0 > 0\} \ .$$

Hence, (B.4.8) reduces to (B.3.1) and therefore, Theorem B.4.3 can be considered as a generalization of Theorem B.3.1.

Remark B.4.8. We will not consider in detail the non-generic case

$$|\{u_0 = 0\} \cap \{g = \lambda_0\}| > 0 \ . \tag{B.4.11}$$

It is worth noticing though, that in this case one can deduce, arguing in a similar way as in Remark B.4.7, that $\lambda_0 = \Lambda[u_0]$ while $|\{u_0 > 0\}\Delta A_*^0| > 0$. Hence, it is possible to choose initial data that satisfy (B.4.11) such that λ will be continuous at $t = 0$, while the interface will jump.

Remark B.4.9. In the case of plateaus

$$|\{g = \Lambda[u_0]\}| > 0 \ , \tag{B.4.12}$$

in which g takes a constant value, it is not possible to decide using only the assumptions of Theorem B.4.3 if the points of the set $\{g = \Lambda[u_0]\}$ are contained in the positivity set $\{u(\cdot,t) > 0\}$ or in the set $\{u(\cdot,t) = 0\}$ as $t \to 0^+$.

The reason for that is that in this case the sign of $\lambda(t) - \Lambda[u_0]$ could depend on the details of the initial data u_0 and as a consequence, the points of $\{g = \Lambda[u_0]\}$ could lie either in the positivity set $\{u(\cdot,t) > 0\}$ or in the set $\{u(\cdot,t) = 0\}$ as $t \to 0^+$. As a matter of fact, a similar situation can occur not only when (B.4.12) holds, but also in any set $A \subset \{g = \Lambda[u_0]\}$ such that $A_{+\delta} \subset \{g \leq \Lambda[u_0]\}$ for any $\delta > 0$.

B.4.1 A variational characterization of Λ

We collect some properties of $\Lambda[u_0]$ and A_*^0 that will be used in the following analysis.

Lemma B.4.10. *(i) The maximum in* (B.4.6) *is attained by the set* A_*^0. *Any maximizer is (up to sets of measure zero) contained in* A_*^0. *If* $|\{g = \Lambda[u_0]\} \cap \{u_0 = 0\}| = 0$ *then* A_*^0 *is unique (up to sets of measure zero).*

(ii) For every $x \in (A_*^0)^{\mathsf{c}}$ *it holds that* $g(x) < \Lambda[u_0]$.

(iii) For every $x \in \partial A_*^0 \cap \{u_0 = 0\}^\circ$ *it holds that* $g(x) = \Lambda[u_0]$.

Proof. We first prove *(i)*. For any $A, B \subset \Gamma$ we observe that

$$\fint_{A\cup B} \Big(\Lambda[u_0] - g\Big)\, dS = \frac{|A|}{|A\cup B|} \fint_A \Big(\Lambda[u_0] - g\Big)\, dS \\ + \frac{|B|}{|A\cup B|} \fint_B \Big(\Lambda[u_0] - g\Big)\, dS - \frac{|A\cap B|}{|A\cup B|} \fint_{A\cap B} \Big(\Lambda[u_0] - g\Big)\, dS. \tag{B.4.1.1}$$

For $A, B \subset \Gamma$ with $\{u_0 > 0\} \subset A \cap B$ (B.4.1.1) equation combined with the definition of $\Lambda[u_0]$ implies

$$\begin{aligned} \Lambda[u_0] - \fint_{A\cup B} g\, dS &\le \frac{|A|}{|A\cup B|}\Big(\Lambda[u_0] - \fint_A g\, dS\Big) + \frac{|B|}{|A\cup B|}\Big(\Lambda[u_0] - \fint_B g\, dS\Big) \\ &\le \Big(\Lambda[u_0] - \fint_A g\, dS\Big) + \Big(\Lambda[u_0] - \fint_B g\, dS\Big). \end{aligned}$$

We next show that the maximum in (B.4.6) is attained. We therefore consider a sequence $(A_k)_k$ of measurable subsets of Γ with $\{u_0 > 0\} \subset A_k$ and

$$\fint_{A_k} g\, dS \ge \Lambda[u_0] - 2^{-k-1}$$

for all $k \in \mathbb{N}$. Let $A^N := \bigcup_{k\ge N}$, then the first item implies that

$$\Lambda[u_0] - \fint_{A^N} g\, dS \le \sum_{k\ge N} \Big(\Lambda[u_0] - \fint_{A_k} g\, dS\Big) \le 2^{-N} \;\to\; 0 \quad \text{as} \;\; N \to \infty.$$

Moreover $(A^N)_{N\in\mathbb{N}}$ is monotonically decreasing and converges to $A^\infty = \limsup_{k\to\infty} A_k$. By monotonicity we conclude

$$\fint_{A^\infty} g\, dS = \lim_{N\to\infty} \fint_{A^N} g\, dS \ge \Lambda[u_0].$$

On the other hand $\{u_0 > 0\} \subset A^\infty$, hence A^∞ is a maximizer in (B.4.6). We next show that A^0_* is also a maximizer. Let A^∞ be an arbitrary maximizer, which exists by the previous item. We first use $A^\infty = (A^\infty \cap A^0_*) \cup (A^\infty \setminus A^0_*)$ and deduce by the first item

$$\begin{aligned} \Lambda[u_0] &= \frac{|A^\infty \cap A^0_*|}{|A^\infty|} \fint_{A^\infty\cap A^0_*} g\, dS + \frac{|A^\infty \setminus A^0_*|}{|A^\infty|} \fint_{A^\infty\setminus A^0_*} g\, dS \\ &\le \frac{|A^\infty \cap A^0_*|}{|A^\infty|}\Lambda[u_0] + \frac{|A^\infty \setminus A^0_*|}{|A^\infty|}\Lambda[u_0] = \Lambda[u_0], \end{aligned}$$

where the inequality follows by the definition of $\Lambda[u_0]$, and since $g < \Lambda$ on $A^\infty \setminus A^0_* \subset \{u_0 = 0\}$. This shows that the inequality needs to be an equality, which implies $|A^\infty \setminus A^0_*| = 0$ and $A^\infty \subset A^0_*$ up to a set of measure zero.

On the other hand, we use $A_*^0 = A^\infty \cup (A_*^0 \setminus A^\infty)$ up to a set of measure zero and obtain by $\{u_0 > 0\} \subset A_*^0$ and the definition of $\Lambda[u_0]$ together with the first item

$$\begin{aligned}\Lambda[u_0] \geq \fint_{A_*^0} g\, dS &= \frac{|A^\infty|}{|A_*^0|} \fint_{A^\infty} g\, dS + \frac{|A_*^0 \setminus A^\infty|}{|A_*^0|} \fint_{A_*^0 \setminus A^\infty} g\, dS \\ &\geq \frac{|A^\infty|}{|A_*^0|} \Lambda[u_0] + \frac{|A_*^0 \setminus A^\infty|}{|A_*^0|} \Lambda[u_0] = \Lambda[u_0],\end{aligned}$$

where we have used the second item and $g \geq \Lambda$ on $A_*^0 \setminus A^\infty \subset \{u_0 = 0\}$. Equality in the inequalities above requires $\fint_{A_*^0} g\, dS = \Lambda[u_0]$ and $g = \Lambda[u_0]$ almost everywhere on $A_*^0 \setminus A^\infty$. This proves the first item of the lemma.

The items *(ii)* and *(iii)* follow from the definition of the set A_*^0. ■

We collect some further properties of the functional Λ.

Proposition B.4.11.

(i) Monotonicity: For any measurable sets S_1, S_2 with $S_1 \subset S_2$ we have $\Lambda_{S_1} \geq \Lambda_{S_2}$. The maximizer

$$S_*^j := S_j \cup \{g \geq \Lambda_{S_j}\},\ j = 1, 2$$

(see Lemma B.4.10) satisfies $S_*^1 \subset S_*^2$.

(ii) Continuity: For any open set $S \subset \Gamma$, let the sets $S_{-\delta}$ be as in Definition I.1. Then $\Lambda_{S_{-\delta}} \searrow \Lambda_S$ as $\delta \searrow 0$.

Proof. (i) The property $\Lambda_{S_1} \geq \Lambda_{S_2}$ follows directly from the definition of Λ_S. This also implies $\{g \geq \Lambda_{S_1}\} \subset \{g \geq \Lambda_{S_2}\}$, hence $S_*^1 \subset S_*^2$ by definition.

(ii) By definition $S_{-\delta_1} \supset S_{-\delta_2}$ for $\delta_1 < \delta_2$ and hence the first item implies that $\delta \mapsto \Lambda_{S_{-\delta}}$ is increasing. Therefore

$$\lambda_0 := \lim_{\delta \downarrow 0} \Lambda_{S_{-\delta}}$$

exists. By the first item the maximizers

$$S_*^{-\delta} := S_{-\delta} \cup \{g \geq \Lambda_{S_{-\delta}}\}$$

are monotone in the sense that $S_*^{-\delta_1} \supset S_*^{-\delta_2}$ for $0 < \delta_1 < \delta_2$. We deduce that $\mathcal{X}_{S_*^{-\delta}}$ is monotonically increasing with $\delta \downarrow 0$ and hence converges to $\mathcal{X}_{S_0}$ with $S_0 := \bigcup_{\delta > 0} S_*^{-\delta}$, in particular

$$\mathcal{X}_{S_*^{-\delta}} \to \mathcal{X}_{S_0} \quad \text{in } L^1(\Gamma)$$

and therefore

$$\fint_{S_0} g = \lim_{\delta \downarrow 0} \Lambda_{S_{-\delta}}.$$

Since S is open and g is continuous we also have

$$S_0 = \bigcup_{\delta>0} S_*^{-\delta} = \bigcup_{\delta>0} S_{-\delta} \cup \bigcup_{\delta>0} \{g \geq \Lambda_{S_{-\delta}}\} = S \cup \{g > \lim_{\delta\downarrow 0} \Lambda_{S_{-\delta}}\},$$

hence

$$S \subset S_0$$

which implies by (B.4.6)

$$\Lambda_S \geq \fint_{S_0} g = \lim_{\delta\downarrow 0} \Lambda_{S_{-\delta}}.$$

On the other hand $S_{-\delta} \subset S$, hence using once again the monotonicity in the first item, we obtain

$$\lim_{\delta\downarrow 0} \Lambda_{S_{-\delta}} \geq \Lambda_S,$$

which proves equality.

■

We next consider a solution u of (B.1.1)-(B.1.3) and connect the functional Λ to the function λ.

Corollary B.4.12. *Recall that* $\Lambda[u(\cdot,t)] = \Lambda_{\{u(\cdot,t)>0\}}$. *Then, it holds*

$$\lambda(t) \leq \Lambda[u(\cdot,t)] \quad \textit{for all } t \in (0,T), \tag{B.4.1.2}$$

$$\lambda(t) = \Lambda[u(\cdot,t)] \quad \textit{for almost all } t \in (0,T). \tag{B.4.1.3}$$

Proof. The inequality (B.4.1.2) follows from the definitions of Λ and λ.

Let $A_*(t) := \{u(\cdot,t) > 0\} \cup \{g \geq \Lambda[u(\cdot,t)]\}$. By (B.4.2) for all $t \in (0,T) \setminus N$, $|N| = 0$ we have $g \leq \lambda(t)$ in $\{u(\cdot,t) = 0\}$. Assume for some $t \in (0,T) \setminus N$ that $\lambda(t) < \Lambda[u(\cdot,t)]$. Then

$$A_*(t) \setminus \{u(\cdot,t) > 0\} = \{g \geq \Lambda[u(\cdot,t)]\} \cap \{u(\cdot,t) = 0\} = \emptyset,$$

which shows $A_*(t) = \{u(\cdot,t) > 0\}$ and

$$\Lambda[u(\cdot,t)] = \fint_{A_*(t)} g = \fint_{\{u(\cdot,t)>0\}} g = \lambda(t).$$

■

B.4.2 Proof of Theorem B.4.3

Throughout this section, let u be a solution to (B.4.1), (B.4.2) with initial data u_0. We also recall that we only assume (B.1.11) but not (B.1.9).

As we have already mentioned in Remark B.4.6, if (B.1.9) fails in a set of positive measure, or equivalently (B.4.5) is valid, one cannot expect any continuity properties for the positivity set $\{u(\cdot,t) > 0\}$ nor for the function λ. Our goal is to derive precise estimates for the corresponding jumps for short times.

Motivated by Remark B.4.7, our strategy is to approximate u by a solution to (B.4.1)-(B.4.2) with suitably modified initial data u_n^0. The latter are chosen such that they, in particular, converge uniformly to u_0 but such their support, on the other hand, approximates A_*^0, i.e.

$$u_n^0 \searrow u_0 \quad \text{uniformly on } \Gamma, \quad A_*^0 = \bigcap_{n\in\mathbb{N}} \{u_n^0 > 0\}.$$

A key property of the modified solutions will be that we can apply the continuity results obtained in Section B.3.

We first fix a suitable family of initial data u_n^0 and describe specific properties that we can obtain. We denote in the following

$$\lambda_n^0 := \fint_{\{u_n^0>0\}} g\, dS.$$

Lemma B.4.13. *Let $g \in C^2(\Gamma)$ and assume that the set $\{u_0 > 0\}$ is regular according to* (B.1.10)*. There exists a non increasing sequence $(\gamma_n)_n$ of numbers with*

$$\gamma_n \searrow 0 \quad \text{as } n \to \infty \tag{B.4.2.1}$$

and a sequence $(u_n^0)_{n\in\mathbb{N}}$ of nonnegative functions in $C^2(\Gamma)$ such that the following properties hold for all $n \in \mathbb{N}$:

(1) $u_0 \le u_{n+1}^0 \le u_n^0$.
(2) $\{u_n^0 > 0\} \supset \{u_{n+1}^0 > 0\}$.
(3) $u_n^0 > 0$ *in* $\{g \ge \Lambda[u_0] - \gamma_n\}$.
(4) $u_n^0 = 0$ *in* $\{g \le \Lambda[u_0] - 2\gamma_n\} \cap \{u_0 = 0\}$.
(5) The set $\{u_n^0 > 0\}$ *is regular.*
(6) $|\lambda_n^0 - \Lambda[u_0]| < \dfrac{\gamma_n}{4}$.
(7) $\|u_n^0 - u_0\|_{C^0(\Gamma)} \le \gamma_n$.

and such that

(8) $\left|\{u_n^0 > 0\} \setminus A_*^0\right| \to 0 \quad$ *as* $n \to \infty$

Moreover, for any $\eta > 0$ *there exists* $n^* = n^*(\eta)$ *such that*

(9) $\{u_n^0 > 0\} \subset \left(A_*^0\right)_{+\eta} \quad$ *for any* $n \geq n^*$.

We prove this lemma in Appendix III. Fixing sequences $(\gamma_n)_n$ and (u_n^0) as in Lemma B.4.13 we now define our main approximation.

Regularization: Let $(u_n)_n$ be the unique solution of the problem

$$\partial_t u_n - \Delta u_n = -\left(1 - \frac{g}{\lambda_n(t)}\right) H(u_n), \qquad \text{in } \Gamma_T \tag{B.4.2.2}$$

$$g \leq \lambda_n, \qquad \text{a.e in } \{u_n = 0\} \tag{B.4.2.3}$$

$$u_n(\cdot, 0) = u_n^0 \qquad \text{on } \Gamma, \tag{B.4.2.4}$$

where

$$\lambda_n(t) = \fint_{\{u_n(\cdot,t)>0\}} g \, dS \; .$$

By (B.2.6) and items (1), (7) of Lemma B.4.13 we deduce that for all $n_1 \geq n_2$ it holds

$$u(\cdot, t) \leq u_{n_1}(\cdot, t) \leq u_{n_2}(\cdot, t) \quad \text{ for all } t \geq 0, \tag{B.4.2.5}$$

and by Theorem A.3.1

$$u_n \to u \quad \text{ in } C^0([0, T], L^1(\Gamma)) \text{ for all } T > 0 \; . \tag{B.4.2.6}$$

Now we proceed to the proof of the main result of this Section, Theorem B.4.3.

Proof of Theorem B.4.3. As we already mentioned, our aim is to approximate (B.4.1)-(B.4.3) by the regularized problem (B.4.2.2)-(B.4.2.4) and then pass to the limit as $n \to \infty$. The strategy of the proof consists of six steps. More specifically, in the first five steps we derive upper and lower estimates for the sequence of functions λ_n and the positivity sets $\{u_n(\cdot, t) > 0\}$. In the sixth step we consider the limit $n \to \infty$.

Step 1: First we will prove that there exists a modulus of continuity $\hat{\delta}_1$ and for any $r > 0$ a $t_1 = t_1(r) > 0$ such that

$$\lambda_n(t) \leq \Lambda\left[u_0\right] + \hat{\delta}_1(r) \; , \tag{B.4.2.7}$$

$$\{u_0 > r\} \subset \left\{u_n(\cdot, t) > \frac{r}{2}\right\}, \tag{B.4.2.8}$$

for all $n \in \mathbb{N}$ and for all $0 \leq t \leq t_1(r)$.

We will derive the uniform inclusion property (B.4.2.8) by means of a suitable subsolution. To this end, we define $\tilde{u}(x,t) := S(t)u_0(x) - t$ where $S(t)$ denotes the heat semigroup on Γ and we calculate

$$\partial_t \tilde{u} - \Delta \tilde{u} = -1 < -\left(1 - \frac{g}{\lambda_n(t)}\right) H(u_n) = \partial_t u_n - \Delta u_n$$

in Γ_T. Furthermore, $\tilde{u}(\cdot, 0) = u_0 \leq u_n(\cdot, 0)$ on Γ by Lemma B.4.13, item (1). Hence, we obtain by a comparison principle argument that

$$u_n(\cdot, t) \geq S(t)u_0 - t \quad \text{ in } \Gamma_T \, . \tag{B.4.2.9}$$

Now, since $S(t)u_0 - t \to u_0$ as $t \to 0^+$, it follows that for all $r > 0$ there exists $t_1(r) > 0$ such that

$$u_n(\cdot, t) > \frac{r}{2} > 0 \quad \text{in } \{u_0 > r\} \quad \text{ for all } n \in \mathbb{N} \text{ and } 0 \leq t \leq t_1(r) \, ,$$

which proves (B.4.2.8). Next, item (1) of Proposition B.4.11 and (B.4.1.2) in Corollary B.4.12 yield

$$\Lambda\Big[\{u_0 > r\}\Big] \geq \Lambda\Big[\{u_n(\cdot, t) > 0\}\Big] \geq \lambda_n(t). \tag{B.4.2.10}$$

By the second item of Proposition B.4.11 and Lemma I.4, we conclude that (B.4.2.7) holds.

Step 2: Let t_1 be as in Step 1. For any $\sigma > 0$ there exist $r_1(\sigma) > 0$, $0 < t_2(\sigma) \leq t_1(r_1(\sigma))$ and a positive function $\omega_2^* : [0, \infty)^2 \to \mathbb{R}^+$ such that for all $t \leq t_2(\sigma)$ we have

$$\overline{\{g > \Lambda[u_0] + \sigma\}} \subset \Big\{u_n(\cdot, t) > \omega_2^*(\sigma, t)\Big\} \quad \text{ for all } n \in \mathbb{N}. \tag{B.4.2.11}$$

Consider the set $A_\sigma := \{g > \Lambda[u_0] + \gamma\sigma\}$, where $0 < \gamma < 1$ is chosen such that $\Lambda[u_0] + \gamma\sigma$ is a regular value of g. Then A_σ has a C^2-regular boundary and it holds

$$A_\sigma \subset\subset \{g > \Lambda[u_0] + \sigma\}.$$

By (B.4.2.7), we calculate that in $A_\sigma \times [0, t_1(r)]$ it holds

$$\begin{aligned} \partial_t u_n - \Delta u_n = -\left(1 - \frac{g}{\lambda_n(t)}\right) H(u_n) &\geq \left(-1 + \frac{\Lambda[u_0] + \sigma}{\Lambda[u_0] + \hat{\delta}_1(r)}\right) H(u_n) \\ &= \left(\frac{\sigma - \hat{\delta}_1(r)}{\Lambda[u_0] + \hat{\delta}_1(r)}\right) H(u_n) \, . \end{aligned}$$

For sufficiently small $r_1 = r_1(\sigma) > 0$ and $t_1 = t_1(r_1(\sigma))$ from Step 1 we obtain

$$\partial_t u_n - \Delta u_n \geq \frac{1}{2} \frac{\sigma}{\Lambda[u_0]} H(u_n) \quad \text{ in } \; A_\sigma \times [0, t_1].$$

Due to the fact that $u_n^0 > 0$ in $\{g \leq \Lambda[u_0]\}$ by Lemma B.4.13, item (3), there exists a maximal $0 < \hat{t}_n \leq t_1$ such that $u_n > 0$ in $A_\sigma \times [0, \hat{t}_n)$. Therefore, we find that

$$\partial_t u_n - \Delta u_n \geq \frac{1}{2} \frac{\sigma}{\Lambda[u_0]} \quad \text{in } A_\sigma \times [0, \hat{t}_n).$$

We construct a suitable subsolution of u_n. To this end, we let U denote the solution of

$$\begin{aligned} \partial_t U - \Delta U &= \frac{1}{2} \frac{\sigma}{\Lambda[u_0]} && \text{in } A_\sigma \times (0, \hat{t}_n] \,, \\ U &= 0 && \text{in } \partial A_\sigma \times (0, \hat{t}_n] \,, \\ U(\cdot, 0) &= 0 && \text{in } A_\sigma \,. \end{aligned}$$

It follows immediately that $U > 0$ in $A_\sigma \times [0, \hat{t}_n]$. Due to Lemma B.4.13, item (3), we obtain that $u_n^0 > U(\cdot, 0)$ in $\overline{A_\sigma}$ and moreover that $u_n \geq U$ on $\partial A_\sigma \times [0, \hat{t}_n]$. A comparison argument yields then that $u_n \geq U$ in $\overline{A_\sigma} \times [0, \hat{t}_n]$.

Now assume that $\hat{t}_n < t_1$. Then $\min_{\overline{A_\sigma}} u_n(\cdot, \hat{t}_n) = 0$. Since $u_n(\cdot, \hat{t}_n) \geq U(\cdot, \hat{t}_n) > 0$ in A_σ there exists a boundary point $\hat{x} \in \partial A_\sigma$ with

$$u_n(\hat{x}, \hat{t}_n) = 0 = U(\hat{x}, \hat{t}_n).$$

However, using the parabolic Hopf lemma, $U(\cdot, \hat{t}_n) > 0$ in A_σ and $0 = \min_\Gamma u_n$ we deduce that

$$0 < \nabla\Big(U(\hat{x}, \hat{t}_n) - u_n(\hat{x}, \hat{t}_n)\Big) \cdot \nu \leq 0,$$

which is a contradiction.

Therefore, $u_n \geq U$ in $\overline{A_\sigma} \times [0, t_1]$ and in particular we infer a uniform lower estimate for u_n, that is

$$u_n(\cdot, t) \geq \omega_2^*(\sigma, t) \quad \text{in } \overline{\{g > \Lambda[u_0] + \sigma\}} \times (0, t_1] \quad \text{for all } n \in \mathbb{N} \,, \tag{B.4.2.12}$$

where

$$\omega_2^*(\sigma, t) := \inf_{\{g > \Lambda[u_0] + \sigma\}} U(\cdot, t) > 0, \quad t \in (0, t_1]$$

is positive by (B.4.2.10) and $U > 0$ in $A_\sigma \times [0, \hat{t}_n]$.

This yields (B.4.2.11) and finishes Step 2 of the proof.

Step 3: Next we will show that for any $\eta > 0$ there exists $t_3(\eta)$ such that

$$\Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big)_{-\eta} \subset \{u_n(\cdot, t) > 0\} \quad \text{for all } 0 \leq t \leq t_3(\eta) \tag{B.4.2.13}$$

and for all $n \in \mathbb{N}$.

In fact, for $\sigma > 0$ and $r = r_1(\sigma)$ we have by (B.4.2.8) and (B.4.2.11) that

$$\begin{aligned}\Big\{u_0 > r_1(\sigma)\Big\} \cup \Big\{g > \Lambda[u_0] + \sigma\Big\} &= \Big\{(u_0 - r_1(\sigma))_+ + (g - \Lambda[u_0] - \sigma)_+ > 0\Big\} \\ &\subset \{u_n(\cdot, t) > 0\} \quad \text{for all } 0 \le t \le t_2(\sigma). \end{aligned} \tag{B.4.2.14}$$

On the other hand, Lemma I.4 yields for some modulus of continuity δ for any $0 \le t \le t_2(\sigma)$

$$\begin{aligned}\Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big)_{-\eta} &= \Big(\{u_0 + (g - \Lambda[u_0])_+ > 0\}\Big)_{-\eta} \\ &\subset \Big\{u_0 + (g - \Lambda[u_0])_+ > \delta(\eta)\Big\} \\ &\subset \Big(\Big\{u_0 > \delta(\eta)/2\Big\} \cup \{g > \Lambda[u_0] + \delta(\eta)/2\Big\}\Big). \end{aligned} \tag{B.4.2.15}$$

Choosing $\sigma = \sigma(\eta) > 0$ sufficiently small such that $\sigma, r_1(\sigma) < \delta(\eta)/2$ and setting $t_3(\eta) = t_2(\sigma)$ we deduce from (B.4.2.14), (B.4.2.15) the inclusion property (B.4.2.13).

Step 4: Following Step 3, we will show that for any $\eta > 0$ there exists $t_4(\eta)$ such that, with $n^* = n^*(\eta)$ as in item (9) of Lemma B.4.13, for all $n \ge n^*(\eta)$

$$\{u_n(\cdot, t) > 0\} \subset \Big(\{u_0 > 0\} \cup \{g \ge \Lambda[u_0]\}\Big)_{+2\eta} \qquad \text{for all } t \le t_4(\eta)\,. \tag{B.4.2.16}$$

To prove this claim let $\eta > 0$ be given and let $n^* = n^*(\eta)$ be as in item (9) of Lemma B.4.13. Due to the items (1) and (9) of Lemma B.4.13 we obtain that

$$\begin{aligned}&\Big(\{u_0 > 0\}\Big)_{-\eta} \subset \Big(\{u^0_{n^*} > 0\}\Big)_{-\eta} \\ &\Big(\{u^0_{n^*} > 0\}\Big)_{+\eta} \subset \Big(\{u_0 > 0\} \cup \{g \ge \Lambda[u_0]\}\Big)_{+2\eta}.\end{aligned}$$

By the monotonicity property (B.4.2.5) we also have $u_n(\cdot, t) \le u_{n^*}(\cdot, t)$ for any $n \ge n^*$ and for all $t \ge 0$. This in particular implies that for any $n \ge n^*$

$$\{u_n(\cdot, t) > 0\} \subset \{u_{n^*}(\cdot, t) > 0\} \quad \text{for all } t \ge 0. \tag{B.4.2.17}$$

As mentioned in the beginning of this subsection, we will take advantage of the continuity results in Section B.3 for both u_{n^*} and λ_{n^*} at $t = 0$. In order to apply Theorem B.3.1 we need that $\lambda^0_{n^*}$ and the initial data $u^0_{n^*}$ satisfy the conditions (B.1.9) and (B.1.11), that is

$$g < \lambda^0_{n^*} - \theta_{n^*} \ \text{ in } \{u^0_{n^*} = 0\} \qquad \text{and} \qquad \text{the set } \{u^0_{n^*} > 0\} \text{ is regular} \tag{B.4.2.18}$$

for some $\theta_{n^*} > 0$. By Lemma B.4.13, item (8) the second condition is fulfilled. Moreover by Lemma B.4.13, item (3) and item (6) we obtain that

$$g < \Lambda[u_0] - \gamma_{n^*} \leq \lambda^0_{n^*} - \gamma_{n^*} + \left|\lambda^0_{n^*} - \Lambda[u_0]\right| \leq \lambda^0_{n^*} - \frac{3\gamma_{n^*}}{4} \quad \text{in } \{u^0_{n^*} = 0\}\,. \tag{B.4.2.19}$$

Hence, the first statement in (B.4.2.18) is satisfied with $\theta_{n^*} = \frac{3\gamma_{n^*}}{4} > 0$ We deduce by Theorem B.3.1 that there exists $t_4(\eta) := t^*(\eta; \theta_{n^*}, u^0_{n^*}, g) > 0$ such that

$$|\lambda_{n^*}(t) - \lambda^0_{n^*}| < \eta, \tag{B.4.2.20}$$

$$\{u_{n^*}(\cdot, t) > 0\} \subset \Big(\{u^0_{n^*} > 0\}\Big)_{+\eta} \tag{B.4.2.21}$$

for all $t \leq t_4(\eta)$. Combining this and (B.4.2), (B.4.2.17) we conclude that for any $n \geq n^*$ the inclusion (B.4.2.16) holds.

Step 5: We justify a uniform lower bound for λ_n. More precisely, for any $\eta > 0$ and $n^* = n^*(\eta)$, $t_4(\eta)$ as chosen in Step 4 there exists a modulus of continuity $\hat{\omega}$ independent of $n \geq n^*$ such that for any $n \geq n^*$

$$\lambda_n(t) \geq \Lambda[u_0] - \hat{\omega}(\eta) \quad \text{for all } 0 \leq t \leq t_4(\eta) \tag{B.4.2.22}$$

holds.

Let us fix in the following an arbitrary $t \in [0, t_4(\eta)]$. The key idea here is to rewrite $\{u_n(\cdot, t) > 0\}$ as a union of disjoint sets. More precisely, we write

$$\{u_n(\cdot, t) > 0\} = A_n \cup B_n \cup C_n \tag{B.4.2.23}$$

with

$$\begin{aligned} A_n &:= \{u_n(\cdot, t) > 0\} \cap \Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big), \\ B_n &:= \{u_n(\cdot, t) > 0\} \cap \Big(\{u_0 = 0\} \cap \{g = \Lambda[u_0]\}\Big), \\ C_n &:= \{u_n(\cdot, t) > 0\} \cap \Big(\{u_0 = 0\} \cap \{g < \Lambda[u_0]\}\Big). \end{aligned}$$

Furthermore, by (B.4.2.13) and (B.4.2.16) we notice that for all $0 \leq t \leq t_4(\eta)$

$$A_n \supset \Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big)_{-\eta} \quad \text{for all } n\,, \tag{B.4.2.24}$$

$$C_n \subset \Big(\{u_0 > 0\} \cup \{g \geq \Lambda[u_0]\}\Big)_{+2\eta} \cap \Big(\{u_0 = 0\} \cap \{g < \Lambda[u_0]\}\Big) \quad \text{for all } n \geq n^*. \tag{B.4.2.25}$$

Due to (B.4.2.23) ,(B.4.2.24) and the fact that $0 < g < 1$ we derive for all n

$$\begin{aligned}\lambda_n(t) &= \frac{1}{|A_n| + |B_n| + |C_n|}\left(\int\limits_{A_n} g\, dS + \int\limits_{B_n} g\, dS + \int\limits_{C_n} g\, dS\right) \\ &\geq \frac{1}{|A_n| + |B_n| + |C_n|}\left(\int\limits_{A_n} g\, dS + \int\limits_{B_n} g\, dS\right) \\ &\geq \frac{1}{|A_n| + |B_n| + |C_n|}\left(\int_{\left(\{u_0>0\}\cup\{g>\Lambda[u_0]\}\right)_{-\eta}} g\, dS + \Lambda[u_0]|B_n|\right) \end{aligned} \tag{B.4.2.26}$$

Since by definition $A_n \subset \Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big)$, it follows that

$$|A_n| \leq \Big|\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big| \,. \tag{B.4.2.27}$$

Moreover, (B.4.2.25) and Lemma I.3 yield that for all $n \geq n^*$

$$\begin{aligned} C_n &\subset \Big((\{u_0 > 0\})_{+2\eta} \cap \{u_0 = 0\}\Big) \cup \Big((\{g \geq \Lambda[u_0]\})_{+2\eta} \cap \{g < \Lambda[u_0]\}\Big) \\ &= \Big((\{u_0 > 0\})_{+2\eta} \setminus \{u_0 > 0\}\Big) \cup \Big((\{g \geq \Lambda[u_0]\})_{+2\eta} \setminus \{g \geq \Lambda[u_0]\}\Big) \,. \end{aligned}$$

This in turn implies, by (B.1.11) and (II.2) that

$$|C_n| \leq \Big|(\{u_0 > 0\})_{+2\eta} \setminus \{u_0 > 0\}\Big| + \Big|(\{g \geq \Lambda[u_0]\})_{+2\eta} \setminus \{g \geq \Lambda[u_0]\}\Big| \leq \omega_1(\eta) \tag{B.4.2.28}$$

for some $\omega_1(\eta) \to 0$ as $\eta \to 0$.

By (B.4.2.27) and (B.4.2.28) we deduce that for all $n \geq n^*$

$$\frac{1}{|A_n| + |B_n| + |C_n|} \geq \frac{1}{\Big|\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big| + |B_n| + \omega_1(\eta)} \,. \tag{B.4.2.29}$$

In addition, the expression inside the parentheses on the right-hand side of (B.4.2.26) can be estimated below by

$$\Big(\ldots\Big) \geq \int_{\{u_0>0\}\cup\{g>\Lambda[u_0]\}} g\, dS + \Lambda[u_0]|B_n| - (\max g)|D_\eta| \,, \tag{B.4.2.30}$$

where $D_\eta := \Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big) \setminus \Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big)_{-\eta}$. We observe that by (II.3)

$$|D_\eta| \leq \omega_2(\eta) \tag{B.4.2.31}$$

for some modulus of continuity ω_2.

We can plug now (B.4.2.29), (B.4.2.30) and (B.4.2.31) into (B.4.2.26) and obtain

$$\lambda_n(t) \geq \frac{1}{\Big|\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big| + |B_n| + \omega_1(\eta)} \left(\int_{\big(\{u_0>0\}\cup\{g>\Lambda[u_0]\}\big)\cup B_n} g \, dS - \max g \; \omega_2(\eta) \right) .$$

By (B.2.1), there is a positive constant $M > 0$ such that

$$\Big|\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big| + |B_n| \geq M > 0.$$

Then we deduce that

$$\lambda_n(t) \geq ⨍_{\big(\{u_0>0\}\cup\{g>\Lambda[u_0]\}\big)\cup B_n} g \, dS - \hat{\omega}(\eta) \ , \tag{B.4.2.32}$$

for some modulus of continuity $\hat{\omega}$. We set $\tilde{B}_n := \Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big) \cup B_n$ and we write $A_*^0 = \Big(A_*^0 \setminus \tilde{B}_n\Big) \cup \tilde{B}_n$. Due to (B.4.7), we further observe that $P_n := A_*^0 \setminus \tilde{B}_n \subset \{g = \Lambda[u_0]\}$. This in turn implies that

$$\begin{aligned} ⨍_{A_*^0} g \, dS &= \frac{1}{|\tilde{B}_n| + |P_n|} \left(|\tilde{B}_n| ⨍_{\tilde{B}_n} g \, dS + |P_n| ⨍_{P_n} g \, dS \right) \\ &= \frac{1}{|\tilde{B}_n| + |P_n|} \left(|\tilde{B}_n| ⨍_{\tilde{B}_n} g \, dS + |P_n| \Lambda[u_0] \right). \end{aligned}$$

Since the maximum in (B.4.6) is attained by the set A_*^0, it holds that

$$(|\tilde{B}_n| + |P_n|)\Lambda[u_0] = |\tilde{B}_n| ⨍_{\tilde{B}_n} g \, dS + |P_n| \Lambda[u_0]$$

and $⨍_{\tilde{B}_n} g \, dS = \Lambda[u_0]$. Therefore, we conclude by (B.4.2.32) that

$$\lambda_n(t) \geq \Lambda[u_0] - \hat{\omega}(\eta) \ .$$

Step 6: To complete this proof, it remains to show (B.4.8) and (B.4.9).

By (B.4.2.5) we have $u_n \geq u$ for all n. Due to (B.4.2.16), setting $t_5(\eta) = t_4(\eta/2)$ this immediately yields the right inclusion in (B.4.8), that is

$$\{u(\cdot, t) > 0\} \subset \{u_n(\cdot, t) > 0\} \subset \Big(\{u_0 > 0\} \cup \{g \geq \Lambda[u_0]\}\Big)_{+\eta},$$

for all $t \in [0, t_5(\eta)]$.

In order to obtain the left inclusion, we deduce from (B.4.2.6) and (B.4.2.8), (B.4.2.11) that for all $t \in [0, t_4(\eta)]$ we have

$$\Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big)_{-\eta} \subset \left\{u_0 > \frac{\delta(\eta)}{2}\right\} \cup \left\{g > \Lambda[u_0] + \frac{\delta(\eta)}{2}\right\}$$

$$\subset \Big\{u_n(\cdot,t) > \frac{\delta(\eta)}{4}\Big\} \cup \Big\{u_n(\cdot,t) > \omega_2^*\Big(\frac{\delta(\eta)}{2},t\Big)\Big\}.$$

Moreover, using (B.4.2.6) and passing with $n \to \infty$ to the limit gives

$$\Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big)_{-\eta} \subset \Big\{u(\cdot,t) \geq \min\Big\{\frac{\delta(\eta)}{4}, \omega_2^*\Big(\frac{\delta(\eta)}{2},t\Big)\Big\}\Big\} \tag{B.4.2.33}$$

for all $t \in [0, t_4(\eta)]$. This justifies the left inclusion in (B.4.8).

For proving (B.4.9), we argue as in the Step 1 and Step 5 in order to obtain upper and lower bounds for $\lambda(t) - \Lambda[u_0]$ in terms of (B.4.8). More precisely, for the upper bound we deduce by the left inclusion in (B.4.8) and Lemma I.3 that

$$\{u_0 > 0\}_{-\eta} \subset \{u_0 > 0\}_{-\eta} \cup \{g > \Lambda[u_0]\}_{-\eta} \subset \{u(\cdot,t) > 0\} \quad \text{for all} \;\; t \in [0, \bar{t}(\eta)] \ .$$

Similar to Step 1, we deduce using the first item of Proposition B.4.11 and (B.4.1.2) in Corollary B.4.12 that

$$\Lambda\Big[\{u_0 > 0\}_{-\eta}\Big] \geq \Lambda\Big[\{u(\cdot,t) > 0\}\Big] \geq \lambda(t).$$

Then, the second item of Proposition B.4.11 and Lemma I.4 yield the existence of a modulus of continuity ω_3 such that $\lambda(t) - \Lambda[u_0] \leq \omega_3(\eta)$ holds for all $t \in [0, \bar{t}(\eta)]$. After possibly enlarging η and redefining $\bar{t}(\eta)$ we deduce

$$\lambda(t) - \Lambda[u_0] \leq \eta \quad \text{for all } 0 < t \leq \bar{t}(\eta) \ .$$

For the lower bound, we rewrite $\{u(\cdot,t) > 0\}$ as the following union of disjoint sets

$$\{u(\cdot,t) > 0\} = A \cup B \cup C$$

with

$$\begin{aligned} A &:= \{u(\cdot,t) > 0\} \cap \Big(\{u_0 > 0\} \cup \{g > \Lambda[u_0]\}\Big), \\ B &:= \{u(\cdot,t) > 0\} \cap \Big(\{u_0 = 0\} \cap \{g = \Lambda[u_0]\}\Big), \\ C &:= \{u(\cdot,t) > 0\} \cap \Big(\{u_0 = 0\} \cap \{g < \Lambda[u_0]\}\Big). \end{aligned}$$

Following the same line of arguments as in Step 5, we infer by means of (B.4.8) the existence of a modulus of continuity ω_4

$$\lambda(t) - \Lambda[u_0] \geq -\omega_4(\eta) \quad \text{for all } 0 < t \leq \bar{t}(\eta) \ .$$

Again, possibly enlarging η and redefining $\bar{t}(\eta)$ we deduce $\lambda(t) - \Lambda[u_0] \geq -\eta$. This completes the proof of (B.4.9). ■

B.5 Comparison with the Stefan problem

We collect here some respective continuity and jump properties for the interfaces associated to the classical parabolic free boundary problem (B.1.8).

We define $H : \mathbb{R} \to \{0, 1\}$, $H = \mathcal{X}_{(0,\infty)}$ as the characteristic function of the positive real numbers. Let $\mathcal{U}$ be a smooth compact manifold without boundary embedded in $\mathbb{R}^n$ and also let $T > 0$. Then we can reformulate problem (B.1.8) as follows

$$\begin{cases} \partial_t u - \Delta u = f H(u) & \text{on } \mathcal{U} \times (0, T) \\ f \leq 0 & \text{a.e. in } \{u = 0\} \\ u(\cdot, 0) = u_0 & \text{on } \mathcal{U} \end{cases} \tag{B.5.1}$$

where $f \in C(\mathcal{U} \times [0, T])$.

Clearly the only difference between this problem and the problem under consideration (B.1.5), is the absence of the non local term α which depends on the positivity set $\{u(\cdot, t) > 0\}$.

The well-posedness as well as the properties of the interfaces of (B.5.1) has been studied in detail over the past years. For the continuity of the corresponding positivity set $\{u(\cdot, t) > 0\}$, we state the following theorems.

Theorem B.5.1. *Assume that $f \in C(\mathcal{U} \times [0, T])$ for some $T > 0$ and consider nonnegative initial data $u_0 \in C(\mathcal{U})$. If in addition we assume that $f \leq -\theta$ for some fixed $\theta > 0$ in a neighbourhood of $\partial\{u_0 > 0\} \times [0, T]$, then the set $\{u(\cdot, t) > 0\}$ is continuous at $t = 0$ in the sense that for any $\eta > 0$, there exists $t^*(\eta) > 0$ such that*

$$\{u_0 > 0\}_{-\eta} \subset \{u(\cdot, t) > 0\} \subset \{u_0 > 0\}_{+\eta}$$

for all $0 \leq t \leq t^*(\eta)$.

One can prove Theorem B.5.1 adapting the arguments in the proofs of [6, Theorem 4.2] and [18, Theorem 3.2 (ii)] .

Theorem B.5.2. *Suppose that u is a solution to (B.5.1) with initial data $u_0 \in C(\mathcal{U})$. The positivity set $\{u(\cdot, t) > 0\}$ converges to the set $\{u_0 > 0\} \cup \{f(\cdot, 0) > 0\}$ as $t \to 0^+$ in the sense that for any $\eta > 0$, there exists $t^*(\eta) > 0$ such that*

$$\Big(\{u_0 > 0\} \cup \{f(\cdot, 0) > 0\}\Big)_{-\eta} \subset \{u(\cdot, t) > 0\} \subset \Big(\{u_0 > 0\} \cup \{f(\cdot, 0) \geq 0\}\Big)_{+\eta}$$

for all $0 \leq t \leq t^*(\eta)$.

The proof of Theorem B.5.2 can be obtained arguing in a similar way as in the proof of Theorem B.4.3. Now the proof will follow by simpler techniques, since the dependence of the non local term α on the positivity set $\{u(\cdot,t)>0\}$ does not occur here.

Remark. In particular, we notice that in the case of problem (B.5.1) the positivity set $\{u(\cdot,t)>0\}$ as $t\to 0^+$ depends only on the set $\{u_0>0\}$ and the positivity set of $f(\cdot,t)$. The strict inequality $f<0$ in a neighbourhood of $\{u_0=0\}$ plays a similar role to the condition (B.1.9) for the problem (B.1.1)-(B.1.3). However, since $f(\cdot,t)$ in (B.5.1) does not depend on the positivity set $\{u(\cdot,t)>0\}$, we can omit (B.1.11) here. We recall that imposing such a regularity condition for the set $\{u_0>0\}$, was necessary in order to prove continuity properties for the function α that depends on the support of the solution u.

Appendix I On the $\pm\delta$-sets

Throughout this section let $\Gamma\subset\mathbb{R}^3$ be a smooth compact surface without boundary. In the formulation of the nondegeneracy condition for the initial data the following definition of inner and outer approximations where used.

Definition I.1. For a Borel set $A\subset\Gamma$ and $\delta>0$ we set

$$A_{+\delta}:=\{x\;|d(x,A)\le\delta\}\;,\quad A_{-\delta}:=\{x\;|d(x,A^{\mathsf{c}})\ge\delta\}\;. \tag{I.1}$$

First, we are going to justify some properties of the sets $A_{\pm\delta}$, which will be useful in our analysis.

Lemma I.2. *(i) (Monotonicity) For any Borel set $A\subset\Gamma$ and $0<\delta\le r$ we have*

$$A\subset A_{+\delta}\subset A_r\quad\textit{and}\quad A_{-r}\subset A_{-\delta}\subset A.$$

(ii) (Complements) For all $\delta>0$, the following inclusions hold true

$$(A_{+\delta})^{\mathsf{c}}=(A^{\mathsf{c}})_{-\delta-0}:=\bigcup_{r>\delta}(A^{\mathsf{c}})_{-r}\subset(A^{\mathsf{c}})_{-\delta} \tag{I.2}$$

$$(A_{-\delta})^{\mathsf{c}}=(A^{\mathsf{c}})_{+\delta-0}:=\bigcup_{r<\delta}(A^{\mathsf{c}})_{+r}\subset(A^{\mathsf{c}})_{+\delta}. \tag{I.3}$$

In particular, $(A^{\mathsf{c}})_{+\frac{\delta}{2}}\subset(A_{-\delta})^{\mathsf{c}}\subset(A^{\mathsf{c}})_{+\delta}$ and $(A^{\mathsf{c}})_{-2\delta}\subset(A_{+\delta})^{\mathsf{c}}\subset(A^{\mathsf{c}})_{-\delta}$.

Proof. (i) The monotonicity follows immediately using (I.1).

(ii) Due to (I.1), we can write

$$(A_{+\delta})^{\mathsf{c}}=\Big(\{x\;|d(x,A)\le\delta\}\Big)^{\mathsf{c}}=\{x\;|d(x,A)>\delta\}=\bigcup_{r>\delta}\{x\;|d(x,A)\ge r\}=\bigcup_{r>\delta}(A^{\mathsf{c}})_{-r}$$

and by means of the monotonicity obtained in the first item, we deduce that

$$(A^{\mathsf{c}})_{-r} \subset (A^{\mathsf{c}})_{-\delta}\ , \qquad \text{for all} \ \ r > \delta\ .$$

This proves the first inclusion. Arguing in a similar way, we prove also the second inclusion.

■

The significance of the sets $A_{\pm\delta}$, as defined in (I.1), becomes evident in the following lemmas.

Lemma I.3. *For any $\delta > 0$*

$$\Big(A_{-\delta} \cup C_{-\delta}\Big) \subset \Big(A \cup C\Big)_{-\delta} \subset \Big(A_{-\delta} \cup C_{-\delta}\Big) \cup \Big(A_{+\delta} \setminus A_{-\delta}\Big)$$
$$\Big(A \cup C\Big)_{+\delta} = \Big(A_{+\delta} \cup C_{+\delta}\Big).$$

Proof. By definition, it follows easily that $\Big(A_{-\delta} \cup C_{-\delta}\Big) \subset \Big(A \cup C\Big)_{-\delta}$. Hence, we write $\Big(A \cup C\Big)_{-\delta} = \Big(A_{-\delta} \cup C_{-\delta}\Big) \cup \Big[\Big(A \cup C\Big)_{-\delta} \setminus \Big(A_{-\delta} \cup C_{-\delta}\Big)\Big]$ and we fix $x \in \Big(A \cup C\Big)_{-\delta}$. We observe that there exists a ball $B_\delta(x) \subset A \cup C$ such that $B_\delta(x) \cap A \neq \emptyset$ and $B_\delta(x) \cap C \neq \emptyset$. This in turn implies that $d(x, A) \leq \delta$ and in particular that $x \in A_{+\delta}$. Therefore,

$$\Big(A \cup C\Big)_{-\delta} \setminus \Big(A_{-\delta} \cup C_{-\delta}\Big) \subset A_{+\delta} \setminus \Big(A_{-\delta} \cup C_{-\delta}\Big) \subset A_{+\delta} \setminus A_{-\delta}\ .$$

Moreover, for any $x \in A_{+\delta} \cup C_{+\delta}$ we have

$$\begin{aligned} x \in A_{+\delta} \cup C_{+\delta} &\Leftrightarrow d(x, A) \leq \delta \quad \text{or} \quad d(x, C) \leq \delta \\ &\Leftrightarrow d(x, A \cup C) \leq \delta \\ &\Leftrightarrow x \in \Big(A \cup C\Big)_{+\delta}\ . \end{aligned}$$

■

Lemma I.4. *Let $h \in C(\Gamma)$ be given with $\{h > 0\}$ and $\{h \leq 0\}$ both being non-empty. Then there exist positive non-decreasing functions $\delta_1, \delta_2 : (0, 1) \to \mathbb{R}$ with $\lim_{r\to 0} \delta_1(r) = \lim_{r\to 0} \delta_2(r) = 0$ such that*

$$(\{h > 0\})_{-\delta_1(r)} \subset \{h \geq r\} \subset (\{h > 0\})_{-\delta_2(r)}\ .$$

Proof. We first prove the second inclusion. We observe that for any $r > 0$ the sets $\{h \geq r\}$ and $\{h \leq 0\}$ are compact and disjoint. Thus, setting

$$\delta_2(r) := d\Big(\{h \geq r\}, \{h \leq 0\}\Big)$$

yields a positive non-decreasing function. Moreover, by definition of δ_2, in $\{h \geq r\}$ we have $d(\cdot, \{h \leq 0\}) \geq \delta_2(r)$, which implies $\{h \geq r\} \subset (\{h > 0\})_{-\delta_2(r)}$.

Since δ_2 is non-decreasing $\omega := \lim_{r \searrow 0} \delta_2(r)$ exists. Assume $\omega > 0$. Then

$$d\Big(\cdot, \{h \leq 0\}\Big) \geq \omega \quad \text{ in } \{h > 0\} = \bigcup_{r>0} \{h \geq r\},$$

which yields a contradiction since h is continuous and $\partial\{h > 0\} \subset \{h \leq 0\}$ is non-empty.

To prove the first inclusion define for $\delta > 0$ the function

$$m(\delta) := \min\Big\{h(x) \,:\, d(x, \{h \leq 0\}) \geq \delta\Big\}.$$

We observe that $m : (0, 1) \to \mathbb{R}$ is well-defined, positive and non-decreasing.

By definition we have

$$(\{h > 0\})_{-\delta} \subset \{h \geq m(\delta)\}. \tag{I.4}$$

Assume that $\omega := \lim_{\delta \searrow 0} m(\delta) > 0$. Then by continuity of h

$$h \geq \omega \quad \text{ in } \bigcup_{\delta>0} (\{h > 0\})_{-\delta} = \Big\{d(\cdot, \{h \leq 0\}) > 0\Big\} = \{h > 0\},$$

a contradiction.

We next define

$$\bar{\delta}_1(r) := \inf\{\delta > 0 \,:\, m(\delta) \geq r\} \tag{I.5}$$

and we notice that $\bar{\delta}_1$ is positive, non-decreasing and moreover $\lim\limits_{r \to 0} \bar{\delta}_1(r) = 0$ since $\lim_{\delta \searrow 0} m(\delta) = 0$. Hence, if we set $\delta_1(r) := \bar{\delta}_1(r) + r$, we deduce by (I.5) and the monotonicity of m that

$$m(\delta_1(r)) \geq r\ ,$$

for $\delta_1(r) \to 0$ as $r \to 0$. Then we conclude from (I.4) that

$$(\{h > 0\})_{-\delta_1(r)} \subset \{h \geq m(\delta_1(r))\} \subset \{h \geq r\}.$$

■

Appendix II On the regularity of sets

Definition II.1. Assume that $\Gamma \subset \mathbb{R}^3$ is a smooth surface. We will call a Borel set $A \subset \Gamma$ regular if

$$|\partial A| = 0 \,, \tag{II.1}$$

where $|\cdot| = \mathcal{H}^2$ denotes the two-dimensional Hausdorff measure.

To begin with, we prove an equivalent characterization for (II.1).

Lemma II.2. *For some Borel set $A \subset \Gamma$ and $\delta > 0$, we consider the sets $A_{\pm\delta}$ as these are given by* (I.1)*. Then, the set A is regular iff*

$$|A_{+\delta} \setminus A_{-\delta}| \to 0 \quad \textit{as } \delta \to 0 \,.$$

Moreover, we have that

$$\lim_{\delta \downarrow 0} |A_{+\delta} \setminus A| \downarrow 0 \quad \textit{for all } A \subset \Gamma \textit{ closed,} \tag{II.2}$$

$$\lim_{\delta \downarrow 0} |A \setminus A_{-\delta}| \downarrow 0 \quad \textit{for all } A \subset \Gamma \textit{ open.} \tag{II.3}$$

Proof. By the monotonicity properties stated in Lemma I.2, for $\delta_1 > \delta_2$ it holds that $A_{+\delta}$ is decreasing and $A_{-\delta}$ is increasing as $\delta \searrow 0$. Moreover,

$$\bigcap_{\delta>0} A_{+\delta} = \{x \,|d(x,A) = 0\} = \overline{A} \quad \text{and} \quad \bigcup_{\delta>0} A_{-\delta} = \{x \,|d(x,A^c) > 0\} = \mathring{A}.$$

This implies

$$\lim_{\delta \searrow 0} |A_{+\delta} \setminus A| = \Big|\bigcap_{\delta>0} A_{+\delta} \setminus A\Big| = |\overline{A} \setminus A| \,, \tag{II.4}$$

$$\lim_{\delta \searrow 0} |A \setminus A_{-\delta}| = \Big|\bigcap_{\delta>0} A \setminus A_{-\delta}\Big| = |A \setminus \mathring{A}| \,, \tag{II.5}$$

and

$$\lim_{\delta \searrow 0} |A_{+\delta} \setminus A_{-\delta}| = |\bigcap_{\delta>0} \Big(A_{+\delta} \setminus A_{-\delta}\Big)| = |\overline{A} \setminus \mathring{A}| = |\partial A| \,.$$

Therefore, if the set A is regular according to Definition II.1, then $\lim_{\delta \searrow 0} |A_{+\delta} \setminus A_{-\delta}| = 0$ and vice versa.

The other claims follow by (II.4), (II.5). ■

Next, we will show that the union of two regular sets is also regular.

Lemma II.3. *Suppose that both A and C are regular. Then, the union $A \cup C$ is also regular in terms of Definition II.1.*

Proof. It holds that $\partial(A \cup C) \subset \partial A \cup \partial C$. This in particular implies that

$$|\partial(A \cup C)| \leq |\partial A \cup \partial C| \leq |\partial A| + |\partial C| \ .$$

Using Definition II.1 the claim follows immediately.

■

Appendix III Construction of suitable initial data

In Lemma B.4.13 we have claimed that we can construct initial data that enjoy suitable convergence, positivity and monotonicity properties. In this section, we provide a proof of the lemma.

Proof of Lemma B.4.13. Let $(\gamma_n)_n$ be a positive, non increasing sequence with $2\gamma_{n+1} < \gamma_n$ for all $n \in \mathbb{N}$. In particular, $\gamma_n \downarrow 0$ as $n \uparrow \infty$. We are going to construct u_n^0 as follows.

We write $\Gamma = \{g \leq \Lambda[u_0] - 2\gamma_n\} \cup \{\Lambda[u_0] - 2\gamma_n < g < \Lambda[u_0] - \gamma_n\} \cup \{g \geq \Lambda[u_0] - \gamma_n\}$. Since Γ is smooth and $g \in C^2(\Gamma)$, by the Morse-Sard Theorem [31, Theorem 3.1.3] almost all $r \in g(\Gamma)$ are regular values of g. Moreover, for all regular values r the sets $\{g = r\}$ are one-dimensional C^2-submanifolds of Γ. In particular, for such values the sets $\{g > r\}$ are regular in the sense of Definition II.1.

Hence, for all $n \in \mathbb{N}$ we can fix a regular value

$$r_n \in [\Lambda[u_0] - 2\gamma_n, \Lambda[u_0] - \gamma_n]. \tag{III.1}$$

This choice and the fact that $2\gamma_{n+1} < \gamma_n$ for all $n \in \mathbb{N}$ imply that $(r_n)_n$ is a strictly monotone increasing sequence.

Since g is continuous and $(r_n)_n$ is strictly monotone increasing, the sets $\{g > r_n\}$ are open and satisfy $\{g > r_{n+1}\} \subset\subset \{g > r_n\}$. Therefore, applying Lemma III.1 below, we obtain that for all $n \in \mathbb{N}$ there exists $\zeta_n \in C^\infty(\Gamma)$ such that

$$\zeta_n = 1 \ \text{ in } \{g > r_{n+1}\}, \quad \zeta_n \in (0,1] \ \text{ in } \{r_n < g \leq r_{n+1}\}, \quad \zeta_n = 0 \ \text{ in } \{g \leq r_n\}. \tag{III.2}$$

At this point, we define u_n^0 as

$$u_n^0 = u_0 + \gamma_n \zeta_n \ . \tag{III.3}$$

By (III.3) it follows immediately that $u_n^0 \geq u_0$ for all n. Since $\gamma_n \downarrow 0$ and by (III.2), we have that $\gamma_{n+1} \leq \gamma_n$ and further that $\zeta_{n+1} \leq 1 = \zeta_n$ in $\{g > r_{n+1}\}$. Hence, we obtain that $u_{n+1}^0 \leq u_n^0$ and item (1) is justified. Item ((7)) is a direct consequence of (III.3), $0 \leq \zeta_n \leq 1$ and $\gamma_n \downarrow 0$.

We easily observe that item ((2)) follows from item ((1)), and that item ((3)) and ((4)) in Lemma B.4.13 follow by (III.2) and (III.3). Moreover since $\{u_n^0 > 0\} = \{u_0 > 0\} \cup \{\zeta_n > 0\}$, we derive ((5)) by means of Lemma II.3. In order to deduce ((8)), we combine ((4)) with Lebesgue's dominated convergence theorem. Then

$$|\{u_n^0 > 0\} \cap \{u_0 = 0\} \cap \{g < \Lambda[u_0]\}| \leq |\{\Lambda[u_0] - 2\gamma_n < g < \Lambda[u_0]\}| \to 0$$

as $n \to \infty$.

At this point we prove item (6). We recall due to (B.4.1.2) and the fact that $\{u_0 > 0\} \subset \{u_n^0 > 0\}$ that $\lambda_n^0 \leq \Lambda[u_0]$. In particular we have that

$$\lambda_n^0 = \frac{1}{|A_*^0| + |\{u_0 = 0\} \cap \{r_n < g < \Lambda[u_0]\}|} \left(\int_{A_*^0} g \, dS + \int_{\{u_0=0\}\cap\{r_n<g<\Lambda[u_0]\}} g \, dS \right). \tag{III.4}$$

For the sake of simplicity we set $B := \{u_0 = 0\} \cap \{r_n < g < \Lambda[u_0]\}$. By means of Lemma B.4.10, (III.4) implies

$$\begin{aligned}
\lambda_n^0 &= \frac{1}{|A_*^0| + |B|} \left(\Lambda[u_0]|A_*^0| + \int_{\{u_0=0\}\cap\{r_n<g<\Lambda[u_0]\}} g \, dS \right) \\
&= \frac{1}{|A_*^0| + |B|} \left(\Lambda[u_0]|A_*^0| + \Lambda[u_0]|B| + \int_B \Big(g - \Lambda[u_0]\Big) \, dS \right) \\
&= \Lambda[u_0] + \frac{1}{|A_*^0| + |B|} \int_B \Big(g - \Lambda[u_0]\Big) \, dS \, . \qquad \text{(III.5)}
\end{aligned}$$

Employing once again (B.4.1.2) and the fact that $r_n \in [\Lambda[u_0] - 2\gamma_n, \Lambda[u_0] - \gamma_n]$, we obtain that

$$0 \leq \Lambda[u_0] - \lambda_n^0 < \frac{2\gamma_n |B|}{|A_*^0| + |B|} \, .$$

Furthermore, since $B = \{u_n^0 > 0\} \cap \{u_0 = 0\} \cap \{g < \Lambda[u_0]\}$ we deduce by item (8) and $|A_*^0| \geq |\{u_0 > 0\}| > 0$ that $|B| \to 0$ as $n \to \infty$. This in particular yields that there exists $n^* \in \mathbb{N}$ such that

$$|\lambda_n^0 - \Lambda[u_0]| < \frac{\gamma_n}{4}, \quad \text{for all } n > n^* \, .$$

We can assume that $n^* = 1$, otherwise we pass to the sequences $(\gamma_{n+n^*})_{n\in\mathbb{N}}$, $(u^0_{n+n^*})_{n\in\mathbb{N}}$. Therefore, the proof of item (6) is complete.

It remains to prove item (9) of the lemma. First we show the right-hand side inclusion in item. To this end, we notice that due to item (4) it holds

$$\{u_n^0 > 0\} \subset \{g > \Lambda[u_0] - 2\gamma_n\} \cup \{u_0 > 0\}.$$

Furthermore, for any $\eta > 0$ we have that $\{u_0 > 0\} \subset \Big(\{u_0 > 0\}\Big)_{+\eta}$. We claim that it suffices to show that there exists $n^* = n^*(\eta)$ with $n^*(\eta) \to 0$ as $\eta \to 0$ such that

$$\{g > \Lambda[u_0] - 2\gamma_{n^*}\} \subset \Big(\{g \geq \Lambda[u_0]\}\Big)_{+\eta}. \tag{III.6}$$

Indeed, if (III.6) holds, then by virtue of Lemma I.3 and the monotonicity property in item (2) we conclude that

$$\{u_n^0 > 0\} \subset \{u^0_{n^*} > 0\} \subset \Big(\{u_0 > 0\}\Big)_{+\eta} \cup \Big(\{g \geq \Lambda[u_0]\}\Big)_{+\eta} = \Big(\{u_0 > 0\} \cup \{g \geq \Lambda[u_0]\}\Big)_{+\eta}$$

for any $n \geq n^*$. In order to show (III.6), we recall Lemma I.2 and Lemma I.4. More precisely, we compute

$$\Big(\{g \geq \Lambda[u_0]\}\Big)_{+\eta} = \left(\Big(\{g < \Lambda[u_0]\}\Big)^{\mathsf{c}}\right)_{+\eta} \supset \left(\Big(\{g < \Lambda[u_0]\}\Big)_{-\eta}\right)^{\mathsf{c}} \supset \Big(\{g \leq \Lambda[u_0] - 2\gamma_{n^*}\}\Big)^{\mathsf{c}}$$

for some $n^*(\eta) > 0$ such that $n^*(\eta) \to \infty$ as $\eta \to 0$.

The proof is complete.

■

We now prove the claim in (III.7).

Lemma III.1. *For any two open sets $U_1 \subset\subset U_2 \subset \Gamma$, there exists $\zeta \in C^\infty(\Gamma)$ such that*

$$\zeta = 1 \ \textit{in}\ U_1, \quad \zeta > 0 \ \textit{in}\ U_2, \quad \zeta = 0 \ \textit{in}\ \Gamma \setminus U_2, \quad 0 \leq \zeta \leq 1 \ \textit{in}\ \Gamma. \tag{III.7}$$

Proof. In a first step we construct $\psi \in C^\infty(\Gamma)$ with $\psi > 0$ in U_2 and $\psi = 0$ in $\Gamma \setminus U_2$.

We choose a sequence $(x_j)_j$ in U_2 such that $\{x_j : j \in \mathbb{N}\}$ is dense in U_2 and set $\rho_j := \frac{1}{2} d(x_j, \Gamma \setminus U_2) > 0$.

Next, we fix a nonnegative function $\phi \in C^\infty(\mathbb{R}^3)$ that vanishes outside the unit ball $B_1(0)$ and satisfies $0 \leq \phi \leq 1$ in $\mathbb{R}^3$ and $\phi > 0$ in $B_1(0)$. We define $\psi : \Gamma \to \mathbb{R}$ by

$$\psi(x) := \sum_{j\in\mathbb{N}} 2^{-j} c_j \phi\left(\frac{x - x_j}{\rho_j}\right),$$

where $c_j > 0$ is chosen such that

$$\left\|\partial^\alpha \phi\left(\frac{\cdot - x_j}{\rho_j}\right)\right\|_{C^0(\Gamma)} \leq \frac{1}{c_j}$$

for all (covariant) partial derivatives of order $|\alpha| \leq j$.

We observe that $0 \leq \psi \leq 1$ is well-defined and smooth with $\psi = 0$ outside U_2. Next, we claim that $\psi > 0$ in U_2.

In fact, for all $x \in U_2$, there exists $(x_{j(k)})_k$ with $x_{j(k)} \to x$ as $k \to \infty$. This implies

$$\lim_{k\to\infty} \rho_{j(k)} = \frac{1}{2} \lim_{k\to\infty} d(x_{j(k)}, \Gamma \setminus U_2) = \frac{1}{2} d(x, \Gamma \setminus U_2) > 0.$$

Therefore $x \in B(x_{j(k)}, \rho_{j(k)})$ for k sufficiently large, and hence $\psi(x) > 0$.

Next, we choose $U_1 \subset\subset V \subset\subset U_2$ and a bump function $\vartheta \in C^\infty(\Gamma)$, $0 \leq \vartheta \leq 1$ with $\vartheta = 0$ outside V and $\vartheta = 1$ in U_1. Finally, we set

$$\zeta = (1 - \vartheta)\psi + \vartheta$$

and observe that $\zeta = 1$ in U_1, that $\zeta = \psi = 0$ outside U_2 and that $\zeta \geq \psi > 0$ in U_2, and that $\zeta \leq \max\{\psi, 1\} = 1$. ■

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