Sheaves, Quasimaps, Maps, Covers

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Abstract

Let X be a smooth variety, let S be a smooth surface, let $C_{g,N} \to \overline{M}_{g,N}$ be a universal curve over a moduli space of stable marked curves and let $(C, \mathbf{x})$ be a marked nodal curve.

In the first part of the thesis, comprised of two chapters, we develop the theory of quasimaps to a smooth proper moduli space of stable sheaves M on S. A quasimap is a map to the moduli space of all sheaves (not necessarily stable), generically mapping to M. For each $\epsilon \in \mathbb{R}_{>0}$, there exists a stability condition for quasimaps, termed ϵ-stability. Moduli spaces of ϵ-stable quasimaps interpolate between moduli spaces of stable maps to M and moduli spaces of stable sheaves of the relative geometry $S \times C_{g,N} \to \overline{M}_{g,N}$, the two being the moduli spaces of ϵ-stable quasimaps for extremal values of ϵ. Using Zhou's theory of calibrated tails, we prove wall-crossing formulas, which therefore relate Gromov–Witten invariants of M and relative Donaldson–Thomas invariants of $S \times C_{g,N} \to \overline{M}_{g,N}$.

In the second part we introduce a stability condition for maps from nodal curves to $X \times C$ relative to $X \times \mathbf{x}$ for each $\epsilon \in \mathbb{R}_{\leq 0}$, termed ϵ-admissibility. Moduli spaces of ϵ-admissible maps interpolate between moduli spaces of twisted stable maps to the orbifold symmetric product $[X^{(n)}]$ and stable maps to the relative geometry $X \times C_{g,N} \to \overline{M}_{g,N}$. Using Zhou's theory of calibrated tails, we prove wall-crossing formulas, which therefore relate orbifold Gromov–Witten invariants of $[X^{(n)}]$ and relative Gromov–Witten invariants of $X \times C_{g,N} \to \overline{M}_{g,N}$.

The main result of the thesis is establishment of correspondences between different enumerative theories, using aforementioned wall-crossings. In particular, we prove the wall-crossing part of Igusa cusp form conjecture; higher-rank/rank-one Donaldson–Thomas wall-crossing for some threefolds $S \times C$; Donaldson–Thomas/Pandharipande–Thomas wall-crossing for some threefolds $S \times C$. We show that the quantum cohomology of $S^{[n]}$ is determined by relative Pandharipande–Thomas theory of $S \times \mathbb{P}^1$ for del Pezzo and K3 surfaces. Finally, we express Crepant resolution conjecture for the pair $S^{[n]}$ and $[S^{(n)}]$ in terms of Gromov–Witten/Pandharipande–Thomas correspondence for $S \times C$, thereby proving 3-point genus-0 Crepant resolution conjecture, if S is a toric del Pezzo surface.

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Contents

Chapter 1

Introduction

1.1 Overview

Quasimaps were first considered in an unpublished work by Drinfeld in early 80's in the context of geometric representation theory, see Braverman [Bra06] for an account of the representation-theoretic side of the theory. Their importance in a different but not unrelated field of enumerative geometry was also already understood. In subsequent years the enumerative side of quasimaps was studied as an alternative to Gromov–Witten theory in the case of certain GIT targets by many people (e.g. [MOP11], [Tod11]), leading to the work of Ciocan-Fontanine–Kim–Maulik [CKM14], where the theory was given the most general treatment.

Moduli spaces of stable quasimaps and stable maps are different compactifications of the moduli space of stable maps with smooth domains. There exists also a mixed theory of ϵ-stable quasimaps that interpolates between the two, thereby giving rise to a wall-crossing, which provides an effective way to compute Gromov–Witten invariants in terms of quasimap invariants, which in many cases are more accessible. Moreover, it turned out that the quasimap wall-crossing is related to enumerative mirror symmetry. For example, in [CK20] it was shown that for a quintic threefold the generating series of quasimap invariants exactly matches the B-model series, while the quasimap wall-crossing is the mirror transformation.

Quasimaps then found their applications beyond numbers in the enumerative geometry of Nakajima quiver varieties (see e.g. [Oko17]), which also brought them back to their roots, since enumerative geometry is inseparable from geometric representation theory in this context. It also brings us to the theme of the thesis. Already for the simplest example of a Nakajima quiver variety - a punctorial Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of the affine plane $\mathbb{C}^2$ - one can consider five enumerative theories, among which there is the GIT quasimap theory:

GW - Gromov–Witten theory of $(\mathbb{C}^2)^{[n]}$;

Q - GIT quasimap theory of $(\mathbb{C}^2)^{[n]}$;

$\mathsf{GW}_{\mathrm{orb}}$ - orbifold Gromov–Witten theory of $[(\mathbb{C}^2)^{(n)}]$;

$\mathsf{GW}_{\mathrm{rel}}$ - relative Gromov–Witten theory of $\mathbb{C}^2 \times C_{g,N}/\overline{M}_{g,N}$;

$\mathsf{PT}_{\mathrm{rel}}$ - relative Pandharipande–Thomas theory of $\mathbb{C}^2 \times C_{g,N}/\overline{M}_{g,N}$,

which are related in the following ways:

- GIT quasimap wall-crossing between GW and Q, [CK14];
- analytic continuation and a change of variables between GW and $\mathsf{GW}_{\mathrm{orb}}$ provided by Crepant resolution conjecture (C.R.C.), [BG09], [Rua06];
- analytic continuation and a change of variables between $\mathsf{GW}_{\mathrm{rel}}$ and $\mathsf{PT}_{\mathrm{rel}}$ provided by PT/GW correspondence, [MNOP06];
- the moduli spaces of Q and $\mathsf{PT}_{\mathrm{rel}}$ are naturally isomorphic and virtual fundamental classes coincide, [Oko17, Excercise 4.3.22].

Moreover, all of those correspondences are equivalences - the generating series of invariants of the theories above are equal up to a change of a variable. The above correspondences were established in a series of papers - [OP10d], [OP10b], [OP10a], [BP08] - the culmination of which was [PT19].

Similar correspondences can be formulated for an arbitrary smooth surface S with one exception - the theory of the type Q does not make sense in the form it is stated for $\mathbb{C}^2$, because in general $S^{[n]}$ does not admit a natural GIT presentation[1], despite being constructed with the help of GIT techniques. On the other hand, the moduli space $S^{[n]}$ is naturally embedded into a *rigidified*[2] stack of coherent sheaves $\mathfrak{Coh}(S)/\!\!/\mathbb{C}^*$. More generally, any moduli space M of Gieseker-stable sheaves on S in a class[3] $\mathbf{v}$ is naturally

[1] There is no natural choice of a GIT stack, whose stable locus is $S^{[n]}$, apart from $S^{[n]}$ itself, which is not interesting for our purposes.

[2] Rigidification amounts to taking quotient of the usual stack $\mathfrak{Coh}(S)$ by the scaling $\mathbb{C}^*$-action, the quotient affects the automorphisms of the objects but not the isomorphism classes of the objects. We refer to Section 2.1.1 for more details.

[3] By which we mean that sheaves have Chern character $\mathbf{v} \in H^*(S, \mathbb{Q})$.

embedded into a rigidified stack of all coherent sheaves in the class $\mathbf{v}$,

$$M \subset \mathfrak{Coh}_r(S)_{\mathbf{v}} := \mathfrak{Coh}(S)_{\mathbf{v}} /\!\!/ \mathbb{C}^*.$$

In Chapter 2 and Chapter 3 of the thesis we will be interested in quasimaps to a pair

$$(M, \mathfrak{Coh}_r(S)_{\mathbf{v}}),$$

which we define to be maps from nodal curves to $\mathfrak{Coh}_r(S)_{\mathbf{v}}$ which generically map to M, see Definition 2.2.1. It will be shown that our quasimap theory is naturally equivalent to the theory of the type $\mathsf{PT}_{\mathrm{rel}}$ already on the level of moduli spaces. These chapters are based on two preprint articles, [Nesa] and [Nesb] respectively.

We introduce the notion of ϵ-stability for quasimaps, which depends on a parameter $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$. Moduli spaces of ϵ-stable maps interpolate between theories of the types GW and $\mathsf{PT}_{\mathrm{rel}}$. We prove that their moduli spaces are proper and admit a perfect obstruction theory. Using the theory of calibrated tails of Zhou introduced in [Zho22], we establish a wall-crossing formula relating the invariants for different values of $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$. The result is an equivalence of the theories of type GW and $\mathsf{PT}_{\mathrm{rel}}$ in a general context: for all surfaces, all positive ranks and all curve classes.

Such relation was already studied on the level of invariants, e.g. for $(\mathbb{C}^2)^{[n]}$ in [OP10c] and more recently for $(\mathcal{A}_m)^{[n]}$ in [Liu21]. It was also expected to hold in a more general context. In particular, the conjectures of Oberdieck–Phandharipande [OP16, Conjecture A] and Oberdieck [Obe19, Conjecture 1] regarding such relation for K3 surfaces served as our main motivating goal. In Chapter 3 we will focus on the case of K3 surfaces, as it requires some extra treatment due to the presence of holomorphic symplectic form, which makes the standard invariants trivial.

In Chapter 4 of the thesis we study a correspondence between $\mathsf{GW}_{\mathrm{orb}}$ and $\mathsf{GW}_{\mathrm{rel}}$ for an arbitrary smooth target X. Influenced by the ideas from the theory of quasimaps, we introduce a notion of ϵ-admissibility for maps from nodal curves to $X \times C_{g,N}/\overline{M}_{g,N}$, which depends on a parameter $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$. Moduli space of ϵ-admissible maps interpolate between theories $\mathsf{GW}_{\mathrm{orb}}$ and $\mathsf{GW}_{\mathrm{rel}}$ for arbitrary smooth target X. Using Zhou's theory of calibrated tails, we establish a wall-crossing formula relating the invariants for different values of $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$, which is completely analogous to quasimap wall-crossing formulas. The result is an equivalence of the theories of type $\mathsf{GW}_{\mathrm{orb}}$ and $\mathsf{GW}_{\mathrm{rel}}$ for arbitrary smooth target X. This wall-crossing can be termed Gromov–Witten/Hurwtiz (GW/H) wall-crossing, because if

X is a point, the moduli spaces of ϵ-admissible maps interpolates between Gromov–Witten and Hurtwitz spaces.

Together these wall-crossings can be represented by the square in Figure 1.1. Vertical sides of the square also hold in more general settings - quasimap wall-crossing applies for any higher-rank moduli spaces of sheaves; GW/H applies to a target of an arbitrary dimension.

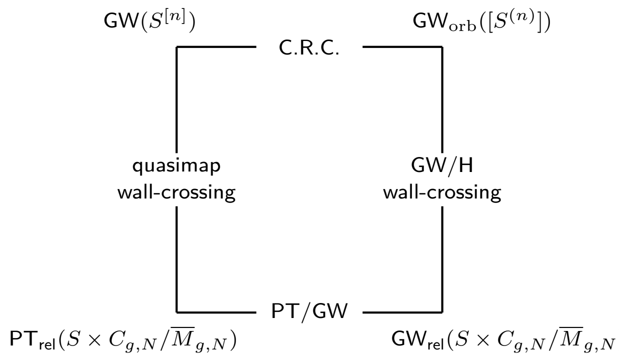

Figure 1.1: The Square

Using the wall-crossings, we prove the following results:

- the wall-crossing part of Igusa cusp form conjecture, conjectured in [OP10c];
- quantum cohomology of $S^{[n]}$ is determined by relative Pandharipande–Thomas theory of $S \times \mathbb{P}^1$, if S is a K3 or del Pezzo surface, conjectured in [Obe19] for K3 surfaces and by Davesh Maulik for del Pezzo surfaces;
- relative higher-rank/rank-one Donaldson–Thomas correspondence for $S \times \mathbb{P}^1$, if S is a K3 surface;
- relative Pandharipande–Thomas/Donaldson–Thomas correspondence for $S \times \mathbb{P}^1$, if S is a K3 surface;
- 3-point genus-0 Crepant resolution conjecture for the pair $S^{[n]}$ and $[S^{(n)}]$, if S is a toric del Pezzo surface.
- a geometric origin of the variable change $y = -e^{iu}$ in PT/GW through C.R.C.

1.2 Quasimaps and sheaves

Let us explain the correspondence between quasimaps to a moduli space of sheaves on a surface and sheaves on threefolds. For simplicity, let the moduli space be $S^{[n]}$ for a smooth surface S over the field of complex numbers $\mathbb{C}$, satisfying $q(S) := h^{1,0}(S) = 0$. Then, as before, we have a natural embedding

$$S^{[n]} \subset \mathfrak{Coh}_r(S)_{\mathbf{v}},$$

such that the complement of $S^{[n]}$ is the locus of non-torsion-free sheaves with Chern character $\mathbf{v} = (1, 0, -n) \in H^*(S, \mathbb{Q})$.

For the choice of $\mathbf{v}$ as above, the stack $\mathfrak{Coh}_r(S)_{\mathbf{v}}$ has a canonical universal family. For a smooth curve C the relation between torsion-free sheaves on a threefold $S \times C$ and quasimaps from C to the pair $(S^{[n]}, \mathfrak{Coh}_r(S)_{\mathbf{v}})$ then becomes apparent. Indeed, by the moduli problem of sheaves on S the later is given by a family of sheaves on S over C, i.e. a sheaf on $S \times C$,

$$f \colon C \to \mathfrak{Coh}_r(S)_{\mathbf{v}} \Leftrightarrow F \in \mathrm{Coh}(S \times C),$$

where F is flat over C. The rigidification of the stack amounts to requiring the determinant of F to be trivial. The general fiber of F over C is torsion-free by the definition of a quasimap. Therefore F is torsion-free itself. Being of rank 1 and having a trivial determinant, F is, in fact, an ideal sheaf of 1-dimension subscheme. Conversely, any ideal sheaf of 1-dimensional subscheme defines a quasimap in the above sense.

The degree of a quasimap to a pair $(S^{[n]}, \mathfrak{Coh}_r(S)_{\mathbf{v}})$ is defined by evaluating it at determinant line bundles over $\mathfrak{Coh}_r(S)_{\mathbf{v}}$. In this way the degree is determined by the Chern character of the corresponding family and vice versa,

$$\text{degree } \beta \text{ of } f \Leftrightarrow \mathrm{ch}(F) = (n, \check{\beta}),$$

for more about the notation on the right we refer to Section 2.2 and Section 2.3.1.

1.2.1 Stability

We import ϵ-stability from the GIT set-up to ours, Definition 2.2.8. This will allow us to interpolate between Gromov–Witten theory and stable[4] quasimap theory. The idea of ϵ-stability can be summarised as follows. In the stable quasimap theory we trade rational tails (which are allowed in Gromov–Witten theory) for *base points*[5](which are prohibited in Gromov–

[4]By which we mean 0^+-stable quasimaps.

[5]Those points that are mapped outside of the stable locus.

Witten theory) for the sake of properness of the moduli space. On the other hand, ϵ-stability allows both rational tails and base points, putting numerical restrictions on their degrees. The value $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$ is the measure of that degree, see Definition 2.2.8. When $\epsilon = 0^+$, quasimaps do not have any rational tails but have base points of all degrees. When $\epsilon = \infty$, quasimaps do not have any base points but have rational tails of all degrees.

In the language of one-dimensional subschemes on threefolds, ϵ-stability controls non-flatness of a subscheme on $S \times C$ over C. Non-flatness arises due to the presence of non-dominant components or floating points. Then ϵ-stability requires that a weighted[6] sum of the degree and the Euler characterstics of either floating points or non-dominant components must not exceed $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$. If it becomes larger than ϵ in the limit, then a curve sprouts a rational tail, like in relative Donaldson–Thomas theory. In addition, ϵ-stability also controls the degree and the Euler charactesrtics of components of the subscheme which lie on rational tails. See Corollary 2.3.1 for more precise statements.

1.2.2 Properness

Having defined ϵ-stability, we then use the relation between sheaves and quasimaps to prove Proposition 2.2.17, where it is shown that the moduli spaces of ϵ-stable quasimaps are proper for fine projective moduli spaces of sheaves. The stack $\mathfrak{Coh}_r(S)_{\mathbf{v}}$ is not bounded, but the stability of quasimaps suffices to guarantee the boundedness of moduli spaces of ϵ-stable quasimaps. However, it is essential to consider the entire stack $\mathfrak{Coh}_r(S)_{\mathbf{v}}$, because with the increase of the degree the more of the stack becomes relevant for the properness of the moduli space. This is one of the reasons why GIT point-of-view breaks[7] down here, at least for a projective surface. Nevertheless, we closely follow the proof of properness in the GIT set-up, and it roughly consists of two steps.

The first step is to prove that the number of components of the domain of a quasimap is bounded after fixing the degree and the genus. This is achieved with a line bundle on the stack $\mathfrak{Coh}_r(S)_{\mathbf{v}}$ which is positive with respect to quasimaps, see Section 2.2.1. Our construction of such line bundle crucially exploits the geometry of coherent sheaves, e.g. Langton's semistable

[6] The degree is weighted more than the Euler characterstics.

[7] More precisely, the stack $\mathfrak{Coh}(S)$ is locally a GIT stack. However, it is unbounded and (very) singular. Moreover, those GIT charts, through which our quasimaps factor for a fixed degree, are not stacky quotients of affine schemes. Therefore results from the theory of GIT quasimaps are not applicable.

reduction.

The second step is to show that quasimaps are bounded for a fixed curve. To achieve this, we reverse Langton's semistable reduction and prove that there is bounded family of choices to obtain a stable quasimap of a fixed degree from a stable map in Lemma 2.2.9. Boundedness also implies that families of sheaves corresponding to quasimaps are stable for a *suitable* stability, see Lemma 2.2.13. In Appendix 5.1 the converse is shown to be true for moduli of slope stable sheaves with $\mathrm{rk} \leq 2$. We also expect it to be true in general. In the case of $S^{[n]}$ it is not difficult to see, as sheaves are of rank one, therefore being stable is equivalent to being torsion-free.

We then prove a variant of *Hartog's property* for sheaves on families of nodal curves over a discrete valuation ring (DVR), Lemma 2.2.15, which allows us to conclude the proof of properness of the moduli spaces in the same way as it is done in the GIT case, [CKM14, Section 4.3].

On the way we establish a precise relation between quasimaps and sheaves. Namely, in the case of $S^{[n]}$ the moduli space of stable quasimaps is naturally isomorphic to a relative Hilbert scheme

$$Q_{g,N}(S^{[n]}, \beta) \cong \mathrm{Hilb}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}).$$

More specifically, the moduli space on the right parametrises triples $(I, S \times C, \mathbf{x})$, where I is an ideal on $S \times C$ and $\mathbf{x}$ is a marking of C. Stability of such triples consists of the following data:

- the curve $(C, \mathbf{x})$ is *prestable*, in particular, it does not have rational tails;
- the subscheme corresponding to the ideal is flat over nodes and marked points [8];
- the ideal is fixed only by finitely many automorphisms of the curve $(C, \mathbf{x})$.

A moduli space of ϵ-stable quasimaps $Q^{\epsilon}_{g,N}(S^{[n]}, \beta)$ similarly admits a purely ideal-theoretic formulation, such that some rational tails are allowed and some subschemes with vertical components are prohibited. We refer to Corollary 2.3.1 for more details. These moduli spaces therefore provide an interpolation,

$$\overline{M}_{g,N}(S^{[n]}, \beta) \overset{\epsilon}{\dashleftarrow\dashrightarrow} \mathrm{Hilb}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}) \tag{1.1}$$

[8]This is a usual stability condition in relative Donaldson–Thomas theory, referred to as *predeformability*.

The higher-rank case admits a similar identification with relative moduli spaces of sheaves, see Definition 2.2.12.

1.2.3 Perverse quasimaps

A variant of the quasimap theory for a moduli space of sheaves is given by considering the stack of objects in a perverse heart $\mathrm{Coh}(S)^\sharp$. For $S^{[n]}$ we have a perverse pair,

$$S^{[n]} \subset \mathfrak{Coh}_r^\sharp(S)_{\mathbf{v}},$$

see Section 2.3.2 for precise definitions. The moduli space of stable perverse quasimaps is then isomorphic to the relative moduli stack of stable pairs,

$$Q_{g,N}(S^{[n]},\beta)^\sharp \cong \mathrm{P}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}).$$

In Section 2.3.3 we discuss the case of $S = \mathbb{C}^2$, for which the moduli stack of perverse sheaves with a framing is naturally isomorphic to the GIT stack associated to $(\mathbb{C}^2)^{[n]}$ (including the unstable part) viewed as Nakajima quiver variety, thereby making GIT quasimaps and moduli-of-sheaves quasimaps equivalent in this case. This provides a conceptual geometric explanation for the equivalences of different enumerative theories that were previously observed numerically, e.g. in [OP10c].

1.2.4 Obstruction theory

An obstruction theory of $Q^\epsilon_{g,N}(S^{[n]},\beta)$ is given by the deformation theory of maps from curves to a derived enhancement $\mathbb{R}\mathfrak{Coh}_r(S)_{\mathbf{v}}$ of $\mathfrak{Coh}_r(S)_{\mathbf{v}}$. The former exists by [TV08]. In fact, we consider a slight modification of the standard derived enhancement - we take a derived fiber over derived Picard stack, as it is explained in [STV15], to obtain the enhancement, whose virtual tangent bundle is given by the traceless obstruction theory of sheaves. The relative sheaf-theoretic obstruction theory of $\mathrm{Hilb}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N})$ can be easily shown to be isomorphic to the map-theoretic obstruction theory of $Q_{g,N}(S^{[n]},\beta)$ given by the pull-back of the virtual tangent bundle on $\mathfrak{Coh}_r(S)_{\mathbf{v}}$, as is explained in Section 2.4.2. With this comparison result we can relate Gromov–Witten theory and relative Donaldson–Thomas theory via quasimap wall-crossing.

The moduli space of ϵ-stable quasimaps has all the necessary structure, such as the evaluation maps, to define invariants via the virtual fundumental class in the spirit of Gromov–Witten theory,

$$\langle \tau_{m_1}(\gamma_1),\ldots,\tau_{m_N}(\gamma_N)\rangle^{M,\epsilon}_{g,\beta} := \int_{[Q^\epsilon_{g,N}(M,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{i=N} \psi_i^{m_i}\mathrm{ev}_i^*\gamma_i,$$

where $\gamma_1, \dots, \gamma_N$ are classes in $H^*(S^{[n]})$ and $\psi_1, \dots, \psi_N$ are ψ-classes associated to the markings. In the language of ideals on threefolds the primary quaismap insertions correspond exactly to relative DT insertions. We similarly define perverse invariants ${}^\sharp\langle \tau_{m_1}(\gamma_1), \dots, \tau_{m_N}(\gamma_N)\rangle^\epsilon_{g,N,\beta}$ associated to the pair $(S^{[n]}, \mathfrak{Coh}^\sharp_r(S)_{\mathbf{v}})$.

1.2.5 Wall-crossing

Invoking recent results of [Zho22], we then establish the quasimap wall-crossing. However, for this part of the present work we mostly refer to Zhou's paper, as his arguments carry over to our case almost verbatim. We now briefly explain what is meant by the quasimap wall-crossing.

The space $\mathbb{R}_{>0} \cup \{0^+, \infty\}$ of ϵ-stabilities is divided into chambers, in which the moduli space $Q^\epsilon_{g,N}(M, \beta)$ stays the same, and as ϵ crosses a wall between chambers the moduli space changes discontinuously. The quasimap wall-crossing relates invariant for different values of ϵ, it involves certain DT-type invariants that are defined via the virtual localisation $S \times \mathbb{P}^1$ with respect to the $\mathbb{C}^*$-action on $\mathbb{P}^1$,

$$t[x, y] = [tx, y], \quad t \in \mathbb{C}^*.$$

These invariants are assembled together in so-called *I-functions*, which is defined in Section 2.5.1. By convention we set

$$e_{\mathbb{C}^*}(\mathbb{C}_{\text{std}}) = -z,$$

where $\mathbb{C}_{\text{std}}$ is the standard representation of $\mathbb{C}^*$ on $\mathbb{C}$. Then in the case of $S^{[n]}$, the I-function is

$$I(q, z) = 1 + \sum_{\beta > 0} q^\beta \mathsf{ev}_* \left(\frac{[F_\beta]^{\text{vir}}}{e_{\mathbb{C}^*}(N^{\text{vir}})} \right) \in A^*(S^{[n]})[z^\pm] \otimes_{\mathbb{Q}} \mathbb{Q}[\![q^\beta]\!],$$

where $F_\beta \subset \text{Hilb}_{n,\check{\beta}}(S \times \mathbb{P}^1)$ is the distinguished $\mathbb{C}^*$-fixed component consisting of subschemes which are supported on fibers of $S \times \mathbb{P}^1 \to S$ and over $0 \in \mathbb{P}^1$, and which are non-flat only over $0 \in \mathbb{P}^1$. The evaluation

$$\mathsf{ev}\colon F_\beta \to S^{[n]}$$

is defined by taking the fiber of the subscheme over $\infty \in \mathbb{P}^1$. We define

$$\mu(z) := [zI(q, z) - z]_+ \in A^*(S^{[n]})[z],$$

where $[\dots]_+$ is the truncation by taking only non-negative powers of z. To state the wall-crossing formula in the most efficient way, we assemble invariants in the following generating series

$$F_g^{\epsilon}(\mathbf{t}(z)) = \sum_{n=0}^{\infty}\sum_{\beta}\frac{q^{\beta}}{N!}\langle \mathbf{t}(\psi_1),\dots,\mathbf{t}(\psi_N)\rangle^{\epsilon}_{g,N,\beta},$$

where $\mathbf{t}(z) \in H^*(X^{[n]},\mathbb{Q})[z]$ is a generic element, and the unstable terms are set to be zero.

Theorem. *For all $g \geq 1$ we have*

$$F_g^{0^+}(\mathbf{t}(z)) = F_g^{\infty}(\mathbf{t}(z)+\mu(-z)).$$

For $g=0$, the same equation holds modulo constant and linear terms in $\mathbf{t}$.

The change of variables above is the consequence of a wall-crossing formula across each wall between extremal values of ϵ, see Theorem 2.5.3. Moreover, by evoking the identification $\mathbb{C}^*$-equivariant sheaves on $S \times \mathbb{C}^*$ with flags of sheaves on S, one can express the wall-crossing invariants in terms of integrals on moduli spaces of flags of sheaves. For more details on this relation we refer to [Obec], where the case of K3 surfaces is treated, leading to a beautiful connection between different enumerative theories.

1.2.6 Reduced wall-crossing

The case of K3 surfaces requires a special treatment due to the presence of a holomorphic symplectic form and, consequently, the vanishing of the standard virtual fundamental class of relevant enumerative theories. Hence one has to construct a surjective cosection of the obstruction theory of ϵ-stable quasimaps.

Let S be a K3 surface. To give a short motivation for our forthcoming considerations, let us recall the origin of reduced perfect obstruction theory of Gromov–Witten theory of M. Since M is hyper-Kähler, for any algebraic curve class $\beta \in H_2(M,\mathbb{Z})$ there exists a first-order *twistor family*

$$\mathcal{M} \to \operatorname{Spec}\mathbb{C}[\epsilon]/\epsilon^2$$

of M, for which the horizontal lift of β is of type (k,k) only at the central fiber. In particular, standard GW invariants vanish. To get a non-trivial enumerative theory, we have to remove obstructions that arise via such deformations of M. However, in the case of ϵ-stable quasimaps we need twistor families not only of the moduli space M but of the pair $(M, \mathfrak{Coh}_r(S)_{\mathbf{v}})$. Such twistor families can be given by non-commutative deformations of S. Let us now elaborate on this point.

Non-commutative deformations

For simplicity assume $M = S^{[1]} = S$. A map $f\colon C \to S$ of degree β is determined by its graph on $S \times C$. Let I be the associated ideal sheaf of this graph. The deformation theories of I and f are equivalent. Assuming C is smooth and $\beta \neq 0$, the existence of a first-order twistor family associated to the class β is equivalent to the surjectivity of the following composition

$$H^1(T_S) \hookrightarrow H^1(T_{S\times C}) \xrightarrow{\cdot \mathrm{At}(I)} \mathrm{Ext}^2(I,I)_0 \xrightarrow{\sigma_I} H^3(\Omega^1_{S\times C}) \cong \mathbb{C}, \tag{1.2}$$

i.e. to the existence of a class $\kappa_\beta \in H^1(T_S)$ whose image is non-zero with respect to the composition above, where $\sigma_I := \mathrm{tr}(* \cdot -\mathrm{At}(I))$ for the Atiyah class $\mathrm{At}(I) \in \mathrm{Ext}^1(I, I \otimes \Omega^1_{S\times C})$. To see this, recall that the second map gives the obstruction to deform I along a first-order deformation $\kappa \in H^1(T_S)$, while the third map, called *semiregularity map* [BF03], relates obstructions of deforming I to the obstructions of

$$\mathrm{ch}_2(I) = (-\beta, 1) \in H^4(S \times C, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}$$

to stay (k,k). With these interpretations in mind it is not difficult to grasp that κ_β is indeed our first-order twistor family associated to β.

The semiregularity map σ_I globalises, i.e. there exists a cosection

$$\sigma\colon \mathbb{E}^\bullet \to \mathcal{O}$$

of the obstruction theory complex of the moduli space of ideals on $S \times C$. This cosection σ is surjective by the existence of first-order twistor families if the second Chern character of ideals is equal to (β, n) for $\beta \neq 0$. By localisation-by-cosection technique introduced by Kiem–Li the standard virtual fundamental class therefore vanishes, as shown in [KL13]. To make the enumerative theory non-trivial, we have to consider the reduced obstruction theory complex $\mathbb{E}_{\mathrm{red}} := \ker(\sigma)$. Proving that $\mathbb{E}_{\mathrm{red}}$ is an obstruction theory is sometimes difficult, instead [KL13] provides a construction of the reduced virtual fundamental class without an obstruction theory.

Let us come back to the case of a general moduli space M. By the moduli problem of M the deformation theory of quasimaps to M is equivalent to the one of sheaves on threefolds of the type $S \times C$. The obstruction theory of sheaves on $S \times C$ also admits a cosection given by the semiregularity map. We want to show it is surjective. However, already for $M = S^{[n]}$ with $n > 1$ there is a problem with the argument presented above. If the degree of

$f\colon C \to S^{[n]}$ is equal to a multiple of the exceptional curve class[9], then (1.2) is zero. Indeed, in this case $\mathrm{ch}_2(I) = (0, n)$ and the composition (1.2) is equal to the contraction $\langle -, \mathrm{ch}_2(I)\rangle$, which therefore pairs trivially with classes in $H^1(T_S)$. The geometric interpretation of this issue is that the exceptional curve class of $S^{[n]}$ stays Hodge along the commutative deformations of S, because punctorial Hilbert schemes deform to punctorial Hilbert schemes under commutative deformations of S. To fix the argument, we have to consider classes not only in $H^1(T_S)$ but in a larger space

$$H^0(\wedge^2 T_S) \oplus H^1(T_S) \oplus H^2(\mathcal{O}_S),$$

i.e. we have to consider non-commutative first-order twistor families to prove the surjectivity of the semiregularity map.

Strategy

Since we consider possibly non-normal threefolds $S \times C$, where C is a nodal curve, we have to take Atiyah classes valued in $\Omega^1_S \boxplus \omega_C$,

$$\mathrm{At}_\omega(F) \in \mathrm{Ext}^1(F, F \otimes (\Omega^1_S \boxplus \omega_C)),$$

instead of $\Omega^1_S \boxplus \mathbb{L}_C = \Omega^1_S \boxplus \Omega^1_C$, as the latter does not behave well under degenerations. Chern characters of sheaves are then defined via the Atiyah class of the form as above. After establishing an expected correspondence between degrees of quasimaps and Chern characters of sheaves, which generalises (2.3), we closely follow [BF03, Section 4] with an exception that we allow contractions with classes in $H^0(\wedge^2 T_S) \oplus H^1(T_S) \oplus H^2(\mathcal{O}_S)$ instead of only $H^1(T_S)$. We deduce surjectivity of the semiregularity map from Proposition 3.1.3.

Having constructed a surjective cosection of the obstruction theory, ideally one would like to reduce it. However, due to the involvement of non-commutative geometry in our considerations, we can reduce the obstruction theory only under a certain assumption, which is nevertheless not unnatural, see Proposition 5.2.1 for the details. This part of the paper is presented in the attempt to complete the story of reduced obstruction theory for $S \times C$. However, we do not use our reduced obstruction theory for the construction of the reduced virtual fundamental class due to the limitations of our assumption. We instead choose to work with the reduced class of [KL13].

[9] The curve class dual to a multiple of the exceptional divisor associated to the resolution of singularities $S^{[n]} \to S^{(d)}$.

1.3 Gromov–Witten/Hurwitz wall-crossing

1.3.1 Analogy

ϵ-stable quasimaps

Let us now illustrate how the theory of quasimaps sheds light on a seemingly unrelated theme of admissible covers. A map from a nodal curve C,

$$f: C \to S^{[n]},$$

is determined by its graph

$$\Gamma_f \subset S \times C.$$

If the curve C varies, the pair (C, Γ_f) can degenerate in two ways:

(i) the curve C degenerates;

(ii) the graph Γ_f degenerates.

By a degeneration of Γ_f we mean that it becomes non-flat[10] over C as a subscheme of $S \times C$, which is due to

- floating points;
- non-dominant components.

Two types of degenerations of a pair (C, Γ_f) are related. Gromov–Witten theory proposes that C sprouts out a rational tail (C degenerates), whenever non-flatness arises (Γ_f degenerates). Donaldson–Thomas theory, on the other hand, allows non-flatness, since it is interested in arbitrary 1-dimensional subschemes, thereby restricting degenerations of C to semistable ones (no rational tails).

A non-flat graph Γ does not define a map to $S^{[n]}$, but it defines a quasimap to $S^{[n]}$. Hence the motto of quasimaps:

Trade rational tails for non-flat points and vice versa.

The idea of ϵ-stability is to allow both rational tails and non-flat points, restricting their degrees. The moduli spaces involved in (1.1) are given by the extremal values of ϵ.

[10] A 1-dimensional subscheme $\Gamma \subset S \times C$ is a graph, if and only if it is flat.

ϵ-admissible maps

The motto of Gromov–Witten/Hurwitz wall-crossing is the following one:

Trade rational tails for branching points and vice versa.

Let us explain what we mean by making a complete analogy with quasimaps. Let

$$f\colon P \to C$$

be an admissible cover, defined in [HM82, Chapter 4], with simple branching[11]. If the curve C varies, the pair (C, f) can degenerate in two ways:

(i) the curve C degenerates;

(ii) the cover f degenerates.

The degenerations of f arise due to

- ramifications of higher order;
- contracted components and singular points mapping to smooth locus.

As previously, these two types of degenerations of a pair (C, f) are related. Hurwtiz theory of a varying curve C proposes that C sprouts out rational tails, whenever f degenerates in the sense above. On the other hand, Gromov–Witten theory of a varying curve C allows f to degenerate and therefore restricts the degenerations of C to semistable ones.

The purpose of ϵ-admissible maps is to interpolate between these Hurwtiz and Gromov–Witten cases. Let $f\colon P \to C$ be a degree-n map between nodal curves, such that it is admissible at nodes and $g(P) = \mathsf{h}$, $g(C) = g$. We allow P to be disconnected, requiring that each connected component maps non-trivially. Following [FP02], we define the branching divisor

$$\mathsf{br}(f) \in \mathrm{Div}(C),$$

it is an effective divisor which measures the degree of ramification away from nodes and the genera of contracted components of f. If C is smooth, then $\mathsf{br}(f)$ can be given by associating to the 0-dimensional complex

$$f_*[f^*\Omega_C \to \Omega_P]$$

its support weighted by Euler characteristics. Otherwise, we need to take the part of the support which is contained in the smooth locus of C.

[11] The fiber of every regular point in C has at most one ramification point, which is of the form $z \mapsto z^2$.

Remark 1.3.1. To establish that branching divisor behaves well in families for maps between singular curves, we have to go through an auxiliary (at least for the purposes of this paper) notion of twisted ϵ-admissable map. The construction of a map br in (4.2) and (4.3) is essentially the only place where we use twisted maps.

Definition. *Let $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$. A map f is ϵ-admissible, if*

- $\omega_C(e^{-1/\epsilon} \cdot \mathsf{br}(f))$ *is ample;*
- $\forall p \in C$, $\mathrm{mult}_p(\mathsf{br}(f)) \leq e^{-1/\epsilon}$.

Remark 1.3.2. Note that the presence of exponential $e^{-1/\epsilon}$ in the definition above is mostly conventional, we could also make the definition with ϵ instead of $e^{-1/\epsilon}$. The reason is that we would like ϵ-admissibility to be defined for $\epsilon \in \mathbb{R}_{<0} \cup \{-\infty\}$, because ϵ-stability of quasimaps is defined for $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$. In this way we can view both theories as a part of one theory which is defined for $\epsilon \in \mathbb{R}$. This is useful for the purposes of Crepant resolution conjecture.

Remark 1.3.3. One can also trade contracted components for higher order singularities of the source curve P, the branching divisor can be defined in this setting. The analogous one-parameter stability condition of such maps was studied in [Deo14]. However, the moduli spaces that one obtains don not have a perfect obstruction theory.

One can readily verify that for $\epsilon = -\infty$, an ϵ-admissible map is an admissible cover with simple branching. On the other hand, if $\epsilon = 0$, an ϵ-admissible map is a stable[12] map, such that the target curve C is semistable. Hence ϵ-admissibility provides an interpolation between the moduli space of admissible covers with simple branching, $Adm_{\mathsf{h},g,n}$, and the moduli space of stable maps, $\overline{M}^{\bullet}_{\mathsf{h}}(C_g/\overline{M}_g, n)$,

$$Adm_{\mathsf{h},g,n} \xleftarrow{\ \epsilon\ } \dashrightarrow \overline{M}^{\bullet}_{\mathsf{h}}(C_g/\overline{M}_g, n)$$

After introducing markings $\mathbf{x} = (x_1, \dots, x_N)$ on C and requiring the map to be admissible[13] over these markings, ϵ-admissibility interpolates between admissible covers with fixed ramifications over markings and relative stable maps. Sometimes it is more convenient to consider normalisation of the moduli space of admissible covers - the moduli space of stable twisted maps

[12] When the target curve C is singular, by a stable map we will mean a stable map which is admissible at nodes.

[13] We require the map to be a ramified cover over the markings with fixed ramifications.

to BS_n, denoted by $\mathcal{K}_{g,N}(BS_n, \mathsf{h})$. The interpolation above can therefore be equally considered as the following one

$$\mathcal{K}_{g,N}(BS_n, \mathsf{h}) \xleftarrow{\epsilon}\!\!\dashrightarrow \overline{M}^{\bullet}_{\mathsf{h}}(C_{g,N}/\overline{M}_{g,N}, n)$$

In fact, this point of view is more natural, if one wants to make an analogy with quasimaps.

Higher-dimensional case

We can upgrade the set-up even further by adding a map $f_X \colon P \to X$ for some target variety X. This leads us to the study of ϵ-admissibility of the data

$$(P, C, \mathbf{x}, f_X \times f_C),$$

which can be represented as follows

$$\begin{array}{ccc} P & \xrightarrow{f_X} & X \\ {\scriptstyle f_C}\downarrow & & \\ (C, \mathbf{x}) & & \end{array}$$

In this case, ϵ-admissibility also takes into account the degree of the components of P with respect to the map f_X, cf. Definition 4.1.4. If X is an point, we get the set-up discussed previously.

Let $\beta = (\gamma, \mathsf{h}) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$ be an *extended* degree[14]. For $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$, we then define

$$Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$$

to be the moduli space of ϵ-admissible data

$$(P, C, \mathbf{x}, f_X \times f_C),$$

such that $g(P) = \mathsf{h}$; $g(C) = g$, $|\mathbf{x}| = N$ and the map $f_X \times f_C$ is of degree (γ, n). The notation is slightly misleading, as ϵ-admissible maps are not maps to $X^{(n)}$. The notation is justified by the analogy with quasimaps and is more natural with respect to our notions of degrees of ϵ-admissible maps.

As in the case of X is a point, we obtain the following description of these moduli spaces for extremal values of ϵ,

$$\begin{aligned} &\overline{M}^{\bullet}_{\mathsf{h}}(X \times C_{g,N}/\overline{M}_{g,N}, (\gamma, n)) = Adm^{0}_{g,N}(X^{(n)}, \beta), \\ &\mathcal{K}_{g,N}([X^{(n)}], \beta) \xrightarrow{\rho} Adm^{-\infty}_{g,N}(X^{(n)}, \beta), \end{aligned}$$

[14]By a version of Riemann-Hurwtiz formula, Lemma 4.1.9, the degree of the branching divisor $\mathsf{br}(f) = \mathsf{m}$ and the genus h determine each other, latter we will use m instead of h.

such that the map ρ is a virtual normalisation in the sense of (4.12), which makes two spaces equivalent from perspective of enumerative geometry. We therefore get an interpolation,

$$\mathcal{K}_{g,N}([X^{(n)}],\beta) \xleftarrow{\ \epsilon\ } \!\!\!\dashrightarrow \overline{M}^{\bullet}_{\mathsf{h}}(X \times C_{g,N}/\overline{M}_{g,N},(\gamma,n))$$

which is completely analogous to (1.1).

1.3.2 Wall-crossing

The invariants of $\overline{M}^{\bullet}_{\mathsf{h}}(X \times C_{g,N}/\overline{M}_{g,N},(\gamma,n))$ that can be related to orbifold invariants of $\mathcal{K}_{g,N}([X^{(n)}],\beta)$ are the *relative* GW invariants taken with respect to the markings of the target curve C. More precisely, for all ϵ, there exist natural evaluations maps

$$\mathrm{ev}_i \colon Adm^{\epsilon}_{g,N}(X^{(n)},\beta) \to \bar{\mathcal{I}}X^{(n)},\ i=1,\ldots,N,$$

where $\bar{\mathcal{I}}X^{(n)}$ is a rigidified version of the inertia stack $\mathcal{I}X^{(n)}$. We define

$$\langle \tau_{m_1}(\gamma_1),\ldots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon}_{g,N,\beta} := \int_{[Adm^{\epsilon}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}}} \prod_{i=1}^{i=N} \psi_i^{m_i}\mathrm{ev}_i^*(\gamma_i,),$$

where $\gamma_1,\ldots,\gamma_N$ are classes in the orbifold cohomology $H^*_{\mathrm{orb}}(X^{(n)})$; $\psi_1,\ldots,\psi_N$ are ψ-classes associated to the markings of the sources curve. By Lemma 4.1.17, these invariants specialise to orbifold GW invariants associated to space $\mathcal{K}_{g,N}([X^{(n)}],\beta)$ and relative GW invariants associated to space $\overline{M}^{\bullet}_{\mathsf{h}}(X\times C_{g,N}/\overline{M}_{g,N},(\gamma,n))$ for corresponding choices of ϵ.

To relate invariants for different values of ϵ, we also use the master space technique developed by Zhou in [Zho22] for the purposes of quasimaps. We establish the properness of the master space in our setting in Section 4.2, following the strategy of Zhou.

As in Section 1.2.5, to state compactly the wall-crossing formula, we define

$$F^{\epsilon}_g(\mathbf{t}(z)) = \sum_{n=0}^{\infty}\sum_{\beta}\frac{q^{\beta}}{N!}\langle \mathbf{t}(\psi_1),\ldots,\mathbf{t}(\psi_N)\rangle^{\epsilon}_{g,N,\beta},$$

where $\mathbf{t}(z) \in H^*_{\mathrm{orb}}(X^{(n)},\mathbb{Q})[z]$ is a generic element, and the unstable terms are set to be zero. There exists an element

$$\mu(z) \in H^*_{\mathrm{orb}}(X^{(n)})[z] \otimes \mathbb{Q}[\![q^{\beta}]\!],$$

defined in Section 4.3.1 as a truncation of an I-function. The I-function is in turn defined via the virtual localisation on the space of stable maps to $X \times \mathbb{P}^1$ relative to $X \times \{\infty\}$. This element $\mu(z)$ provides the change of variables, which relates generating series for extremal values of ϵ.

Theorem. *For all $g \geq 1$ we have*

$$F_g^0(\mathbf{t}(z)) = F_g^{-\infty}(\mathbf{t}(z) + \mu(-z)).$$

For $g = 0$, the same equation holds modulo constant and linear terms in $\mathbf{t}(z)$.

The change of variables above is the consequence of a wall-crossing formula across each wall between extremal values of ϵ, see Theorem 4.3.3.

1.4 Applications

1.4.1 Applications of the quasimap wall-crossing

Higher-rank/rank-one DT wall-crossing for $K3 \times C$

Since smooth projective moduli spaces of higher-rank sheaves M on S are deformation equivalent to $S^{[n]}$, we can prove certain higher-rank/rank-one DT wall-crossing statements for threefolds $S \times C$, using the quasimap wall-crossing on both sides, as it is represented in Figure 1.2.

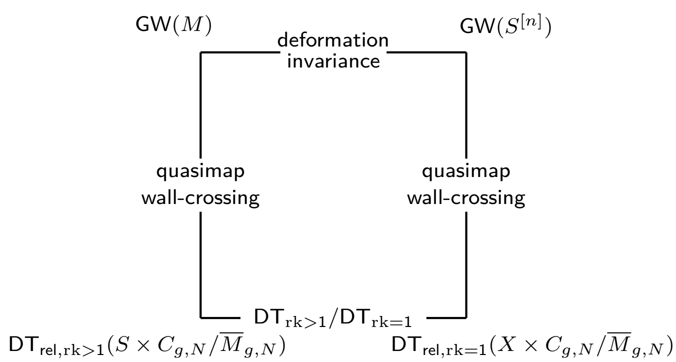

Figure 1.2: Higher-rank/rank-one DT

If $g = 0, N = 3$, the wall-crossing is trivial. We therefore obtain that higher-rank invariants with three relative vertical insertions associated to moduli spaces of sheaves, which are stable at a general fiber, exactly match rank-one invariants on $S \times \mathbb{P}^1$,

$$\mathsf{DT}_{\mathrm{rk}=1}(S \times \mathbb{P}^1/S_{0,1,\infty}) = \mathsf{GW}_{0,3}(S^{[n]}) = \mathsf{DT}_{\mathrm{rk}>1}(S \times \mathbb{P}^1/S_{0,1,\infty}),$$

where $S_{0,1,\infty} = S \times \{0, 1, \infty\} \subset S \times \mathbb{P}^1$. However, the result is not optimal, since stability of a sheaf at a general fiber over a curve is shown to be equivalent to stability of the sheaf itself only under some assumptions. Namely, we require $\mathrm{rk} \leq 2$ and M to be a projective moduli of *slope* stable sheaves, see Proposition 5.1.4. In particular, they are satisfied for threefold invariants that arise from a moduli space of sheaves on a K3 surface with Chern character

$$(2, \alpha, -2k - 1) \in H^*(S, \mathbb{Q})$$

for $k > 0$ and a polarisation such that $\deg(\alpha)$ is odd (or a generic polarisation which is close to a polarisation for which $\deg(\alpha)$ is odd). Note that we are in the setting of non-Calabi–Yau relative geometry, hence the techniques of wall-crossings in derived categories cannot be applied to prove the statement as above. The case of $S \times E$ for an elliptic curve E is also discussed.

DT/PT correspondence for $K3 \times C$

Using both standard and perverse quasimap wall-crossings for $S^{[n]}$, we obtain rank-one Donaldson–Thomas theory/rank-one Pandharipande–Thomas theory (DT/PT) correspondence, as it is illustrated in Figure 1.3.

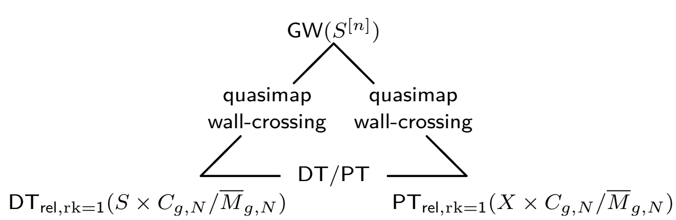

Figure 1.3: DT/PT

As before, for $g = 0, N = 3$ the wall-crossing is trivial. We therefore obtain the following

$$\mathsf{DT}_{\mathrm{rk}=1}(S \times \mathbb{P}^1/S_{0,1,\infty}) = \mathsf{GW}_{0,3}(S^{[n]}) = \mathsf{PT}_{\mathrm{rk}=1}(S \times \mathbb{P}^1/S_{0,1,\infty}).$$

Equality can reasonably be expected due to the nature of the reduced virtual fundamental class.

Enumerative geometry of $S^{[n]}$

The quasimap wall-crossing can be used to study the enumerative geometry of $S^{[n]}$ by relating it to Pandharipande–Thomas theory of $S \times C$ (and, consequently, to Gromov–Witten theory of $S \times C$ by PT/GW correspondence). The latter is much simpler to deal with, as $S \times C$ is a 3-dimensional product variety. By using the virtual localisation on $S \times \mathbb{P}^1$, one can show that genus-0 3-point invariants of $S^{[n]}$ are determined in a certain precise sense by the classical geometry of S and some universal polynomials. This will be addressed in a subsequent paper.

In Section 2.5.4 the wall-crossing invariants of $S^{[n]}$ for a del Pezzo surface S are explicitly computed. In particular, using Nakajima operators, for $n > 1$ we have the following identification

$$H_2(S^{[n]}, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \oplus \mathbb{Z} \cdot \mathsf{A},$$

where A is the exceptional curve class of the Hilbert–Chow morphism

$$S^{[n]} \to S^{(n)}.$$

With respect to the above decomposition we then define

$$^{\sharp}\langle \gamma_1, \ldots, \gamma_N \rangle_{0,\gamma}^{S^{[n]},\epsilon} := \sum_{\mathsf{m}} {}^{\sharp}\langle \gamma_1, \ldots, \gamma_N \rangle_{0,(\gamma,\mathsf{mA})}^{S^{[n]},\epsilon} y^{\mathsf{m}}.$$

Assuming $2g - 2 + N \geq 0$, the quasimap wall-crossing then gives us

$$^{\sharp}\langle \gamma_1, \ldots, \gamma_N \rangle_{0,\gamma}^{S^{[n]},0^+} = (1+y)^{\mathrm{c}_1(S)\cdot\gamma} \cdot {}^{\sharp}\langle \gamma_1, \ldots, \gamma_N \rangle_{0,\gamma}^{S^{[n]},\infty}.$$

After applying the identification of the moduli space of genus-zero 0^+-stable quasimaps with three marked points with the moduli space of stable pairs on $S \times \mathbb{P}^1$ relative to three vertical divisors, the above result relates the quantum cohomology of $S^{[n]}$ to the ring whose structure constants are given by Pandharipande–Thomas theory of $S \times \mathbb{P}^1$. The change of variables as above was predicted[15] by Davesh Maulik.

[15]Communicated to the author by Georg Oberdieck.

Enumerative geometry of $K3^{[n]}$

Let S be a K3 surface. In [Obec] the wall-crossing terms are shown to be virtual Euler numbers of certain Quot schemes, which are computed for $S^{[n]}$. Therefore, using the results of [Obec] together with reduced quasimap wall-crossing for $S^{[n]}$, we obtain the wall-crossing part of the *Igusa cusp form conjecture* [OP16, Conjecture A], thereby completing the proof of the conjecture along with [OS20] and [OP18].

Genus-0 3-point Gromov–Witten theory of $S^{[n]}$ is shown to be equivalent to Pandharipande–Thomas theory of $S \times \mathbb{P}^1$ with three relative vertical insertions. Together with $\mathsf{PT/GW}$ correspondence of [Obeb], this confirms the conjecture proposed in [Obe19].

In [Obea] a holomorphic anomaly equation is established for $S^{[n]}$ for genus-0 GW invariants with at most 3 markings. The proof crucially uses the quasimap wall-crossing, which relates genus-0 GW invariants $S^{[n]}$ to PT invariants of $S \times \mathbb{P}^1$. The later can be in turn related to GW invariants of $S \times \mathbb{P}^1$ by $\mathsf{PT/GW}$ correspondence of [Obeb], thereby reducing the problem to the one of a product threefold.

1.4.2 Applications of Gromov-Witten/Hurwitz wall-crossing

The Square

For a del Pezzo surface S we compute the wall-crossing invariants in Section 4.4. A computation for analogous quasimap wall-crossing invariants is given in Proposition 2.5.10.

The wall-crossing invariants can be easily shown to satisfy $\mathsf{PT/GW}$. Hence when both quasimap wall-crossing and $\mathsf{GW/H}$ wall-crossing are applied, $\mathsf{C.R.C.}$ becomes equivalent to $\mathsf{PT/GW}$. For the precise statements of both we refer to Section 4.5.1. This is expressed in terms of the square of theories, see Figure 1.1.

For a toric surface S [PP17] proves $\mathsf{PT/GW}$ for $S \times \mathbb{P}^1$ relative to $S_{0,1,\infty}$. The square therefore gives us the following result.

Theorem. *If S is a toric del Pezzo surface, $g = 0$ and $N = 3$, then* $\mathsf{C.R.C.}$ *holds for all $S^{[n]}$ in all classes.*

Previously, the theorem above was established for $n = 2$ and $S = \mathbb{P}^2$ in [Wis11, Section 6]; for an arbitrary n and an arbitrary toric surface but only for an exceptional curve class in [Che13]. If $S = \mathbb{C}^2$, $\mathsf{C.R.C.}$ was proven for all genera and any markings on the level of Cohomological field theories in [PT19].

The theorem can also be restated as an isomorphism of quantum cohomologies,

$$QH^*_{\mathrm{orb}}(S^{(n)}) \cong QH^*(S^{[n]}),$$

we refer to Section 4.5.2 for more details. The result is very appealing, because the underlying cohomologies with classical multiplications are not isomorphic for surfaces with $\mathrm{c}_1(S) \neq 0$, but the quantum cohomologies are. In particular, the classical multiplication on $H^*_{\mathrm{orb}}(S^{(n)})$ is a non-trivial quantum deformation of the classical multiplication on $H^*(S^{[n]})$.

We want to stress that $\mathsf{C.R.C.}$ should be considered as a more fundamental correspondence than $\mathsf{PT/GW}$, because it relates theories which are closer to each other. Moreover, as [BG09] points out, $\mathsf{C.R.C.}$ explains the origin of the change of variables,

$$y = -e^{iu}, \tag{1.3}$$

it arises due to the following features of $\mathsf{C.R.C.}$,

(i) analytic continuation of generating series from 0 to -1;

(ii) factor $i = \sqrt{-1}$ in the identification of cohomologies of $S^{[n]}$ and $S^{(d)}$, cf. Remark 4.5.2;

(iii) the divisor equation in $\mathsf{GW}(S^{[n]})$;

(iv) failure of the divisor equation in $\mathsf{GW}_{\mathrm{orb}}([S^{(n)}])$.

More precisely, (i) is responsible for the minus sign in (1.3); (iii) and (iv) are responsible for the exponential; (ii) is responsible for i in the exponential. More conceptual view on $\mathsf{C.R.C.}$ is presented in works of Iritani, e.g. [Iri09].

LG/CY vs C.R.C.

We will now draw certain similarities between $\mathsf{C.R.C.}$ and the theory of Landau–Ginzburg/Calabi–Yau correspondence ($\mathsf{LG/CY}$). For all the details and the notations of $\mathsf{LG/CY}$ we refer to [CIR14].

$\mathsf{LG/CY}$ consists of two types of correspondences - A-model and B-model correspondences. The B-model correspondence is the statement of equivalence of two categories - matrix factorisation categories and derived categories of coherent sheaves. While the A-model correspondence is the statement of equality of generating series of certain curve-counting invariants after an analytic continuation and a change of variables. Moreover, there exists a whole family of enumerative theories depending on a stability parameter $\epsilon \in \mathbb{R}$. For $\epsilon \in \mathbb{R}_{>0}$ it gives the theory of GIT quasimaps, while for $\epsilon \in \mathbb{R}_{<0}$ it gives FJRW (Fan–Jarvis–Ruan–Witten) theory. GLSM (Gauged Linear Sigma Model) formalism, defined mathematically in [FJR18], allows to unify quasimaps and FJRW theory. The analytic continuation occurs, when one crosses the wall at $\epsilon = 0$.

In the case of $\mathsf{C.R.C.}$ we have a similar picture. B-model correspondence is given by an equivalence of categorises, $\mathrm{D}^b(S^{[n]})$ and $\mathrm{D}^b([S^{(n)}])$. A-model correspondence is given by an analytic continuation of generating series and subsequent application of a change of variables, as it is stated in Section 4.5. There also exist a family of enumerative theories depending on a parameter $\epsilon \in \mathbb{R}$. For $\epsilon \in \mathbb{R}_{>0}$, it is given by quasimaps to a moduli space of sheaves, while for $\epsilon \in \mathbb{R}_{\leq 0}$ it is given by ϵ-admissable maps. It would be interesting to know, if a unifying theory exists in this case (like GLSM in $\mathsf{LG/CY}$).

	B-model	A-model
$\mathsf{LG/CY}$	$\mathrm{D^b}(X_W) \cong \mathrm{MF}(W)$	$\mathsf{GW}(X_W) \xleftarrow{\epsilon\leq 0} \|_0 \xrightarrow{\epsilon>0} \mathsf{FJRW}(\mathbb{C}^n, W)$
$\mathsf{C.R.C.}$	$\mathrm{D^b}(S^{[n]}) \cong \mathrm{D^b}([S^{(n)}])$	$\mathsf{GW}(S^{[n]}) \xleftarrow{\epsilon\leq 0} \|_0 \xrightarrow{\epsilon>0} \mathsf{GW}_{\mathrm{orb}}([S^{(n)}])$

Table 1.1: $\mathsf{LG/CY}$ vs $\mathsf{C.R.C}$

The above comparison is not a mere observation about structural similarities of two correspondences. In fact, both correspondences are instances of the same phenomenon. Namely, in both cases there should exist *Kähler moduli spaces*, $\mathcal{M}_{\mathsf{LG/CY}}$ and $\mathcal{M}_{\mathsf{C.R.C.}}$, such that two geometries in question correspond to two different cusps of these moduli spaces (e.g. $S^{[n]}$ and $[S^{(n)}]$ correspond to two different cusps of $\mathcal{M}_{\mathsf{C.R.C.}}$). B-models do not vary across these moduli spaces, hence the relevant categories are isomorphic. On the other hand, A-models vary in the sense that there exist non-trivial global quantum D-modules, $\mathcal{D}_{\mathsf{LG/CY}}$ and $\mathcal{D}_{\mathsf{C.R.C.}}$, which specialise to relevant enumerative invariants around cusps. For more details on this point of view we refer to [CIR14] in the case of $\mathsf{LG/CY}$, and to [Iri10] in the case of $\mathsf{C.R.C.}$

1.4.3 Notation and conventions

We work over the field of complex numbers $\mathbb{C}$. Given a variety X, by $[X^{(n)}]$ we denote the stacky symmetric product by $[X^n/S_n]$ and by $X^{(n)}$ its coarse quotient. By $X^{[n]}$ we denote the Hilbert scheme of length-n points. For a partition μ of n, let $\ell(\mu)$ denote the length of μ and $\mathrm{age}(\mu) = n - \ell(\mu)$.

By convention we set $e_{\mathbb{C}^*}(\mathbb{C}_{\mathrm{std}}) = -z$, where $\mathbb{C}_{\mathrm{std}}$ is the standard representation of $\mathbb{C}^*$ on a vector space $\mathbb{C}$.

After fixing an ample line bundle $\mathcal{O}_S(1)$ on a surface S, for a sheaf F we define $\deg(F)$ to be the degree of F with respect to the $\mathcal{O}_S(1)$. By a *general fiber* of a sheaf F on $S \times C$ we will mean a fiber of F over a point in some dense open subset of C.

Let N be a semigroup and $\beta \in N$ its element. By $\mathbb{Q}[\![q^\beta]\!]$ we will denote the (completed) semigroup algebra $\mathbb{Q}[\![N]\!]$. In our case, N will be a semigroup of various effective curve classes.

For a possibly disconnected curve C, we define $g(C) = 1 - \chi(\mathcal{O}_C)$.

Chapter 2

Quasimaps to a moduli space of sheaves

2.1 Stack of coherent sheaves

2.1.1 Rigidification

Let S be a smooth projective surface. Let $\mathcal{O}_S(1) \in \mathrm{Pic}(S)$ be a very ample line bundle and $\mathbf{v} \in K_{\mathrm{num}}(S)$ be a class, such that

- $\mathrm{rk}(\mathbf{v}) > 0$;
- there are no strictly Gieseker semistable sheaves.

We will frequently identify $\mathbf{v}$ with its Chern character. Let

$$\mathfrak{Coh}(S)_{\mathbf{v}} \colon (Sch/\mathbb{C})^{\circ} \to (Grpd).$$

be the stack of coherent sheaves on S in the class $\mathbf{v}$. We will usually drop $\mathbf{v}$ from the notation, as we will be working with a fixed class, unless we want to emphasise some particular choice of the class. There is a locus of Gieseker $\mathcal{O}_S(1)$-stable sheaves in the class $\mathbf{v}$,

$$\mathcal{M} \hookrightarrow \mathfrak{Coh}(S),$$

which is a $\mathbb{C}^*$-gerbe over a scheme M, where the $\mathbb{C}^*$-automorphisms come from multiplication by scalars. In fact, we can quotient out $\mathbb{C}^*$-automorphisms of the entire stack $\mathfrak{Coh}(S)$, as explained in [AGV08, Appendix C], thereby obtaining a *rigidified* stack

$$\mathfrak{Coh}_r(S) := \mathfrak{Coh}(S) /\!\!/ \mathbb{C}^*.$$

A B-valued point of $\mathfrak{Coh}_r(S)$ can be represented by a pair $(\mathcal{G}, \phi)$, where $\mathcal{G}$ is a $\mathbb{C}^*$-gerbe over B and $\phi\colon \mathcal{G} \to \mathfrak{Coh}(S)$ is a $\mathbb{C}^*$-equivariant map (here we will ignore 2-categorical technicalities, see [AGV08, Appendix C.2] for more details). The moduli space M canonically embeds into the stack $\mathfrak{Coh}_r(S)$, giving rise to the following square

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[d, "\mathbb{C}^*-\text{gerbe}"] & \mathfrak{Coh}(S) \arrow[d, "\mathbb{C}^*-\text{gerbe}"] \\ M \arrow[r, hook] & \mathfrak{Coh}_r(S) \end{tikzcd}$$

Now let $(X, \mathfrak{X})$ be one of the following pairs, $(M, \mathfrak{Coh}_r(S))$ or $(\mathcal{M}, \mathfrak{Coh}(S))$. Abusing the notation, we define

$$\operatorname{Pic}(\mathfrak{X}) := \lim_{\mathcal{U} \subset \mathfrak{X}} \operatorname{Pic}(\mathcal{U}),$$

where the limit is taken over substacks of finite type. The stack $\mathfrak{Coh}(S)$ is not of finite type, therefore this definition of Picard group might not agree with the standard one. However, for our purposes it is the most suitable one. We will refer to the elements of $\operatorname{Pic}(\mathfrak{X})$ as *line bundles*. The need for this definition of the Picard group is justified in Remark 2.1.1.

2.1.2 Determinant line bundles

Let $\mathcal{F}$ be the universal sheaf on $S \times \mathfrak{Coh}(S)$, then for each $\mathcal{U} \subset \mathfrak{Coh}(S)$ of finite type we have naturally defined maps

$$\lambda_{|\mathcal{U}}\colon K_0(S) \xrightarrow{p_S^!} K_0(S \times \mathcal{U}) \xrightarrow{\cdot[\mathcal{F}_{|\mathcal{U}}]} K_0(S \times \mathcal{U}) \xrightarrow{p_{\mathcal{U}!}} K_0(\mathcal{U}) \xrightarrow{\det} \operatorname{Pic}(\mathcal{U})$$

which are compatible with respect to inclusions $\mathcal{U}' \subset \mathcal{U}$, hence we have the induced map

$$\lambda\colon K_0(S) \to \operatorname{Pic}(\mathfrak{Coh}(S)).$$

Remark 2.1.1. The construction of $\lambda_{|\mathcal{U}}$ requires a locally free resolutions of $\mathcal{F}_{|\mathcal{U}}$, the ranks of terms of the resolution grow with $\mathcal{U}$. Hence determinant line bundles cannot be easily defined as honest line bundles on $\mathfrak{Coh}(S)$, but only as elements of $\operatorname{Pic}(\mathfrak{Coh}(S))$ in the sense of our definition of $\operatorname{Pic}(\mathfrak{Coh}(S))$.

In general, the $\mathbb{C}^*$-weight of the line bundle $\lambda(u)$ is equal to $\chi(\mathbf{v} \cdot u)$,

$$w_{\mathbb{C}^*}(\lambda(u)) = \chi(\mathbf{v} \cdot u),$$

so there are two types of classes that we will be of interest to us. A class $u \in K_0(S)$, such that $\chi(\mathbf{v} \cdot u) = 1$, gives a trivilisation of the $\mathbb{C}^*$-gerbe

$$\mathfrak{Coh}(S) \to \mathfrak{Coh}_r(S)$$

over each substack of finite type, or, in other words, a universal family on $\mathfrak{Coh}_r(S)$. While for a class $u \in K_0(S)$, such that $\chi(u, \mathbf{v}) = 0$, the line bundle $\lambda(u)_{|\mathcal{U}}$ descends to $\mathcal{U} /\!\!/ \mathbb{C}^*$. Let

$$K_{\mathbf{v}}(S) := \mathbf{v}^{\perp} \subset K_0(S),$$

then λ restricted to $K_{\mathbf{v}}(S)$ descends to a map to $\mathrm{Pic}(\mathfrak{Coh}_r(S))$,

$$\lambda_{\mathbf{v}} \colon K_{\mathbf{v}}(S) \to \mathrm{Pic}(\mathfrak{Coh}_r(S)).$$

The class $\mathbf{v}$ will be frequently dropped from the notation in $\lambda_{\mathbf{v}}$, when it is clear what stack is considered. We define

$$\mathrm{Pic}_{\lambda}(\mathfrak{Coh}(S)) := \mathrm{Im}(\lambda), \quad \mathrm{Pic}_{\lambda}(\mathfrak{Coh}_r(S)) := \mathrm{Im}(\lambda_{\mathbf{v}}).$$

There exists a particular class of elements in $K_{\mathbf{v}}(S)$, which deserve a special mention and will be used extensively later,

$$u_i := -\mathrm{rk}(\mathbf{v}) \cdot h^i + \chi(\mathbf{v} \cdot h^i) \cdot [\mathcal{O}_x],$$

$$\mathcal{L}_i := \lambda(u_i),$$

where $\mathcal{O}_x$ is a structure sheaf of a point $x \in S$, and $h = [\mathcal{O}_H]$ for a hyperplane $H \in |\mathcal{O}_S(1)|$. More generally, let us fix a $\mathbb{Q}$-basis $\{L_1, \dots, L_{\rho(S)}\}$ of $NS(S)$ consisting of ample $\mathbb{Q}$-line bundles, such that L_i's and $\mathcal{O}_S(1)$ are in the same chamber of Gieseker stabilities. Let $\{\mathcal{L}_{1,1}, \dots, \mathcal{L}_{1,\rho(S)}\}$ be the corresponding determinant $\mathbb{Q}$-line bundles defined in the same way as $\mathcal{L}_1$. The importance of these classes is due to the following theorem.

Theorem 2.1.2. *The line bundles $\mathcal{L}_1$ and $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ are nef and ample respectively on fibers of $M \to \mathrm{Pic}(S)$ for all $m \gg 0$. The same holds for $\mathcal{L}_{1,\ell}$. Moreover, their restrictions to the fibers are independent of a point $x \in S$.*

Proof. See [HL97, Chapter 8]. $\square$

2.2 Quasimaps

For the sake of simplicity of the exposition we assume that $q(S) := h^{1,0}(S) = 0$ from now on. See Section 2.2.5 for the discussion about the theory for surfaces with $q(S) \neq 0$.

Definition 2.2.1. A map $f\colon (C, \mathbf{x}) \to \mathfrak{X}$ is a *quasimap* to $(X, \mathfrak{X})$ of genus g and of degree $\beta \in \mathrm{Hom}(\mathrm{Pic}_\lambda(\mathfrak{X}), \mathbb{Z})$, if

- $(C, \mathbf{x})$ is a nodal connected curve of genus g with n marked points;
- $\mathcal{L} \cdot_f C := \deg(f^*\mathcal{L}) = \beta(\mathcal{L})$ for all $\mathcal{L} \in \mathrm{Pic}_\lambda(\mathfrak{X})$;
- $|\{p \in C|\ f(p) \in \mathfrak{X} \setminus X\}| < \infty$.

We will refer to the set $\{p \in C|\ f(p) \in \mathfrak{X} \setminus X\}$ as *base* points. A quasimap f is *prestable* if

- $\{t \in C|\ f(t) \in \mathfrak{X} \setminus X\} \cap \{\text{nodes}, \mathbf{x}\} = \emptyset$.

We define

$$\Lambda := \bigoplus_p H^{p,p}(S).$$

For a smooth connected curve C we then have a Küneth's decomposition of (p,p)-part of the cohomology of the threefold $S \times C$,

$$\bigoplus_p H^{p,p}(S \times C) = \Lambda \otimes H^0(C, \mathbb{C}) \oplus \Lambda \otimes H^2(C, \mathbb{C}) = \Lambda \oplus \Lambda. \tag{2.1}$$

Let $f\colon C \to \mathfrak{Coh}(S)$ be of degree β. By the moduli problem of sheaves, a map f is given by a sheaf F on $S \times C$ which is flat over C. The Chern character of F has two components with respect to the decomposition in (2.1),

$$\mathrm{ch}(F) = (\mathrm{ch}(F)_{\mathrm{f}}, \mathrm{ch}(F)_{\mathrm{d}}) \in \Lambda \oplus \Lambda,$$

where the subscripts "f" and "d" stand for *fiber* and *degree* respectively. As the notation suggests,

$$\mathrm{ch}(F)_{\mathrm{f}} = \mathbf{v},$$

which can be seen by pulling back $\mathrm{ch}(F)$ to a fiber over C and using the flatness of F. Consider now the linear extension

$$\mathrm{Eff}(\mathcal{M}, \mathfrak{Coh}(S)) \to \Lambda, \quad \beta \mapsto \mathrm{ch}(F)_{\mathrm{d}} \tag{2.2}$$

of the map given by associating the degree part of the Chern character to the degree of the quasimap for smooth C . By relating β to $\mathrm{ch}(F)_{\mathrm{d}}$ in more explicit terms in the following lemma, we show that the association above is indeed well-defined, i.e. a degree β cannot have a presentation by two different $\mathrm{ch}(F)_{\mathrm{d}}$'s. Later in Corollary 2.2.11 it will be shown that the map is even injective, i.e. the degree of f and the Chern character of the corresponding family F determine each other, thereby justifying the subscript "d" in $\mathrm{ch}(F)_{\mathrm{d}}$.

Lemma 2.2.2. *The map (2.2) is well-defined.*

Proof. By the functoriality of the determinant line bundle construction

$$\beta(\lambda(u)) = \deg(\lambda_F(u)),$$

where $\lambda_F(u)$ is the determinant line bundle associated to the family F and a class $u \in K_0(S)$. Using Grothendieck–Riemann–Roch and projection formulas we obtain

$$\begin{aligned} \deg(\lambda_F(u)) &= \int_C \mathrm{ch}(p_{C!}(p_S^! u \cdot [F])) \\ &= \int_{S\times C} \mathrm{ch}(p_S^! u \cdot [F]) \cdot p_S^* \mathrm{td}_S \\ &= \int_S \mathrm{ch}(u) \cdot p_{S*}\mathrm{ch}(F) \cdot \mathrm{td}_S \\ &= \int_S \mathrm{ch}(u) \cdot \mathrm{ch}(F)_{\mathrm{d}} \cdot \mathrm{td}_S. \end{aligned}$$

Now let $\beta_\Lambda \colon \Lambda \to \mathbb{Q}$ be the descend of $(\beta \circ \lambda)_{\mathbb{Q}} \colon K_0(S)_{\mathbb{Q}} \to \mathbb{Q}$ to Λ via Chern character,

$$\begin{tikzcd} \Lambda \arrow[r, dashed, "\beta_\Lambda"] & \mathbb{Q} \\ K_0(S)_{\mathbb{Q}} \arrow[u, "\mathrm{ch}"] \arrow[ur, "(\beta\circ\lambda)_{\mathbb{Q}}"'] & \end{tikzcd}$$

which exists by the above formula for the degree of a determinant line bundle. The formula also shows that the descend β_Λ and β determine each other. We thereby obtain an expression of $\mathrm{ch}(F)_{\mathrm{d}}$ in terms of β_Λ,

$$\mathrm{ch}(F)_{\mathrm{d}} = \beta_\Lambda^\vee \cdot \mathrm{td}_S^{-1},$$

where $\beta_\Lambda^\vee$ is the dual of β_Λ with respect to the cohomological intersection pairing on Λ. Using non-degeneracy of the intersection pairing over algebraic classes and the above expression of $\mathrm{ch}(F)_{\mathrm{d}}$, we obtain that (2.2) is indeed well-defined. Moreover, if $\mathrm{ch}(F)_{\mathrm{d}} = 0$, then $\beta = 0$. $\square$

We define
$$\check{\beta} := \mathrm{ch}(F)_{\mathrm{d}} = \beta_\Lambda^\vee \cdot \mathrm{td}_S^{-1}. \tag{2.3}$$
By the preceding discussion we therefore obtain a neat expression of the Chern character of the family F,
$$\mathrm{ch}(F) = (\mathbf{v}, \check{\beta}) \in \Lambda \oplus \Lambda.$$

Remark 2.2.3. Another justification for the use of $\mathrm{Pic}_\lambda(\mathfrak{X})$ is the following one: $\lambda_{\mathbb{Q}|M}$ is surjective for Hilbert schemes of points of surfaces with $q(S) = 0$, all projective moduli of stable sheaves on a K3 surface and expected to be surjective for all projective moduli of stable sheaves over surfaces with $q(S) = 0$ (see e.g. [HL97, Theorem 8.1.6]). Since we care really only about curve classes on M, we can throw away some obscure classes on $\mathfrak{X}$, leaving $\mathrm{Hom}(\mathrm{Pic}_\lambda(\mathfrak{X}), \mathbb{Z})$, which is good enough for our purposes.

2.2.1 Positivity

The aim of this section is prove the positivity for certain line bundles - Proposition 2.2.6. We start with the following result, which is inspired by [BM14, Proposition 4.4].

Lemma 2.2.4. *Let F be the sheaf on $S \times C$ associated to a map $f\colon C \to \mathfrak{Coh}(S)$, then*
$$\mathcal{L}_1 \cdot_f C = \deg(\mathbf{v})\mathrm{rk}(p_{S*}F) - \mathrm{rk}(\mathbf{v})\deg(p_{S*}F),$$
$$\mathcal{L}_0 \cdot_f C = \chi(\mathbf{v})\mathrm{rk}(p_{S*}F) - \mathrm{rk}(\mathbf{v})\chi(p_{S*}F),$$
where $\deg(\mathbf{v})$ *is the degree of* $\mathbf{v}$ *with respect to* $\mathcal{O}_S(1)$.

Proof. By the proof of Lemma 2.2.2,
$$\mathcal{L}_i \cdot_f C = \chi(u_i \cdot p_{S!}[F]) \text{ for i} = 0, 1.$$
The claim then follows from the following computation
$$\begin{aligned}\chi(u_1 \cdot B) &= -\mathrm{rk}(\mathbf{v})\chi(B \cdot h) + \chi(\mathbf{v} \cdot h)\chi([\mathcal{O}_{\mathrm{pt}}] \cdot B)\\ &= -\mathrm{rk}(\mathbf{v})(\deg(B) - \frac{\mathrm{rk}(B)}{2}H^2 - \frac{\mathrm{rk}(B)}{2}H \cdot \mathrm{c}_1(S))\\ &+ (\deg(\mathbf{v}) - \frac{\mathrm{rk}(\mathbf{v})}{2}H^2 - \frac{\mathrm{rk}(\mathbf{v})}{2}H \cdot \mathrm{c}_1(S))\mathrm{rk}(B)\\ &= \mathrm{rk}(B)\deg(\mathbf{v}) - \mathrm{rk}(\mathbf{v})\deg(B);\\ \chi(u_0 \cdot B) &= -\mathrm{rk}(\mathbf{v})\chi(B) + \chi(\mathbf{v})\chi([\mathcal{O}_{\mathrm{pt}}] \cdot B)\\ &= \chi(\mathbf{v})\mathrm{rk}(B) - \mathrm{rk}(\mathbf{v})\chi(B).\end{aligned}$$

□

The relation between quasimaps to $\mathfrak{Coh}(S)$ and sheaves on $S \times C$ is the central for our study of quasimaps. Since we are interested in quasimaps to the rigidified stack $\mathfrak{Coh}_r(S)$, we would also like to extend that relation to this setting, which is done in the following lemma.

Lemma 2.2.5. *Any quasimap $f\colon C \to \mathfrak{Coh}_r(S)$ admits a lift to $\mathfrak{Coh}(S)$. Different lifts are related by tensoring the corresponding sheaf on $S \times C$ with a line bundle from C.*

Proof. By [AGV08, Appendix C.2] a map $C \to \mathfrak{Coh}_r(S)$ is given by a $B\mathbb{C}^*$-gerbe $\mathcal{G}$ over C with an $\mathbb{C}^*$-equivariant map $\phi\colon \mathcal{G} \to \mathfrak{Coh}(S)$. It can be checked that

$$H^2_{\mathrm{fppf}}(C, \mathcal{O}^*_C) = H^2_{\text{ét}}(C, \mathcal{O}^*_C) = 0$$

by passing to the normalisation of C and using the exponential sequence. Therefore $\mathcal{G}$ is a trivial gerbe. Choose some trivialisation

$$\mathcal{G} \cong C \times B\mathbb{C}^*.$$

By the moduli problem of sheaves a $\mathbb{C}^*$-equivariant map $\phi\colon C \times B\mathbb{C}^* \to \mathfrak{Coh}(S)$ is given by a $\mathbb{C}^*$-equivariant sheaf F on $S \times C$ such that the $\mathbb{C}^*$-equivariant structure is the one given by multiplication by scalars applied to the sheaf F viewed as a sheaf on $S \times C$. In particular, $\mathbb{C}^*$-equivariant structure is unique and determined by F alone. The sheaf F defines a lift $f\colon C \to \mathfrak{Coh}(S)$. Given another lift $f'\colon C \to \mathfrak{Coh}(S)$ with an associated sheaf F' on $S \times C$, then by the properties the rigidification (see [AGV08, Appendix C.2]) there exists an automorphism of the trivial gerbe

$$\psi\colon C \times B\mathbb{C}^* \cong C \times B\mathbb{C}^*,$$

such that $f \cong f' \circ \psi$, therefore $(\mathrm{id}_S \times \psi)^* F' \cong F$. Automorphisms of a trivial gerbe admit the following description

$$\mathrm{Aut}_C(\mathcal{G}) \cong \mathrm{Pic}(C), \quad \psi \mapsto L_\psi,$$

which can be easily proven after recalling that maps to $B\mathbb{C}^*$ are given by line bundles. Moreover, the pullback of a sheaf by ψ is isomorphic to the sheaf tensored by L_ψ. Hence we obtain that

$$F \cong (\mathrm{id} \times \psi)^* F' \cong F' \otimes p_C^* L_\psi.$$

□

Proposition 2.2.6. *Let $f\colon C \to \mathfrak{Coh}_r(S)$ be a prestable quasimap. There exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ the quasimap is non-constant, if and only if*

$$\mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_f C > 0.$$

The constant m_0 depends only on $\mathbf{v}$ and $\mathcal{L}_{1,\ell} \cdot_f C$ for all ℓ. The same holds for all subcurves C' and the induced maps for the same choice of m.

For the illustration of the method, which will be used to prove the claim, we will firstly prove that

$$\mathcal{L}_1 \cdot_f C \geq 0 \tag{2.4}$$

under the same assumption. The proof of the inequality (2.4) also contains the essential ingredients for the proof of the proposition.

Warm-up for Proposition 2.2.6. By Lemma 2.2.5 any $f\colon C \to \mathfrak{Coh}_r(S)$ can be lifted to $\mathfrak{Coh}(S)$ and intersections with $\mathcal{L}_i$'s are independent of the lift. Let F be a family of sheaves on $S \times C$ associated to a lift of f. Assume for simplicity that f has one base point $b \in C$, then by Langton's semistable reduction the sheaf F can be modified at a point b to a sheaf which is stable over b and is isomorphic to F away from $S \times b \subset S \times C$. The modification is given by a finite sequence of short exact sequences

$$\begin{gathered} 0 \to F^1 \to F^0 \to Q^1 \to 0, \\ \vdots \\ 0 \to F^k \to F^{k-1} \to Q^k \to 0, \end{gathered}$$

where $F^0 = F$, the sheaf F^k is stable over $b \in C$ and Q^i is the maximally destabilising quotient sheaf of F^{i-1}_b (if F^{i-1}_b has torsion, then Q^i is the quotient by the maximal torsion subsheaf). In particular, for all i

$$\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) \geq 0. \tag{2.5}$$

Applying derived pushforward p_{S*} to each sequence we get distinguished triangles

$$p_{S*}(F^i) \to p_{S*}(F^{i-1}) \to Q^i \to .$$

By Lemma 2.2.4 we obtain that

$$\mathcal{L}_1 \cdot_{f^{i-1}} C = \mathcal{L}_1 \cdot_{f^i} C + \deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i), \tag{2.6}$$

where f^i is the quasimap associated to F^i. The line bundle $\mathcal{L}_1$ is nef on M by Theorem 2.1.2 and the assumption $q(S) = 0$, therefore

$$\mathcal{L}_1 \cdot_{f^k} C \geq 0,$$

because f^k does not have base points. The property of $\mathcal{L}_1$ stated in (2.4) now follows from (2.5) and (2.6). $\square$

Proof of Proposition 2.2.6. We now deal with the claim in the proposition. By Lemma 2.2.4

$$\begin{aligned}\mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_f C = {} & \mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_{f^k} C \\ & + m \sum_i \deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) + \sum_i \chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i),\end{aligned} \tag{2.7}$$

therefore for $\mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_f C$ to be positive for some big enough m, the terms

$$\chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i)$$

have to be bounded from below. We will now split our analysis, depending on whether (2.5) is positive or zero.

Consider firstly the case of Q^i's, such that

$$\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) > 0.$$

We plan to use Lemma 2.2.7. The sheaves Q^i sit in filtrations (see e.g. [HL97, Theorem 2.B.1]) inside F_b^m,

$$Q^1 \subset Q^2 \subset \cdots \subset Q^m \subset F_b^m. \tag{2.8}$$

Since F_b^m is stable, we have a bound for $\mu_{\max}(Q^i)$,

$$\mu_{\max}(Q^i) \leq \mu(\mathbf{v}).$$

By (2.5) and (2.6) the degrees of such Q^i can be bounded,

$$\frac{\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathcal{L}_1 \cdot_f C}{\mathrm{rk}(\mathbf{v})} \leq \deg(Q^i) < \frac{\deg(\mathbf{v})\mathrm{rk}(Q^i)}{\mathrm{rk}(\mathbf{v})}, \tag{2.9}$$

we therefore get a uniform bound on $\deg(Q^i)$ for all such Q^i depending on the sign of $\deg(\mathbf{v})$,

$$\begin{aligned} -\frac{\mathcal{L}_1 \cdot_f C}{\mathrm{rk}(\mathbf{v})} \leq \deg(Q^i) < \deg(\mathbf{v}), & \quad \text{if } \deg(\mathbf{v}) \geq 0 \ , \\ \deg(\mathbf{v}) - \frac{\mathcal{L}_1 \cdot_f C}{\mathrm{rk}(\mathbf{v})} < \deg(Q^i) < \frac{\deg(\mathbf{v})}{\mathrm{rk}(\mathbf{v})}, & \quad \text{if } \deg(\mathbf{v}) < 0. \end{aligned}$$

If $\rho(S) > 1$, then we can get the similar bounds for $\mathcal{L}_{1,\ell}$'s for all ℓ, thereby bounding $\mathrm{c}_1(Q^i)$. Hence by Lemma 2.2.7 we obtain that

$$\mathrm{ch}_2(Q^i) < \mathsf{A}',$$

where the constant A' depends only on $\mathrm{rk}(\mathbf{v}), \deg(\mathbf{v})$ and $\mathcal{L}_{1,\ell} \cdot_f C$, we therefore can also uniformly bound $\chi(Q^i)$,

$$\chi(Q^i) < \mathsf{A}.$$

We conclude that

$$\begin{aligned} \chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i) &> \chi(\mathbf{v}) - \mathsf{A} \cdot \mathrm{rk}(\mathbf{v}), \quad \text{if } \chi(\mathbf{v}) \geq 0; \\ \chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i) &> \chi(\mathbf{v})\mathrm{rk}(\mathbf{v}) - \mathsf{A} \cdot \mathrm{rk}(\mathbf{v}), \quad \text{if } \chi(\mathbf{v}) < 0. \end{aligned}$$

Consider now the case of Q^i's, such that

$$\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) = 0.$$

By (2.8) and stability of F_b^m it must be that

$$\chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i) > 0.$$

Now let $m_0 \in \mathbb{N}$ be such that $\mathcal{L}_0 \otimes \mathcal{L}_1^{m_0}$ is ample on M (possible by Theorem 2.1.2) and

$$m_0 \cdot (\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i)) > \mathsf{A} \cdot \mathrm{rk}(\mathbf{v}) - \chi(\mathbf{v})$$

for all Q^i, such that $\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) > 0$, if $\chi(\mathbf{v}) \geq 0$, and similarly for $\chi(\mathbf{v}) < 0$. By (2.7) the proposition then follows for quasimaps with one base point. Note that all the bounds do not depend on a base point $b \in C$ and therefore are the same for all base points, hence we can safely drop the assumption that there is one base point.

The dependence of m_0 on $\mathbf{v}$ and $\mathcal{L}_{1,\ell} \cdot_f C$ follows from bounds presented in (2.9). The fact that positivity of the line bundle $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ holds for all subcurves for the same choice of m follows from the proof itself. $\square$

Lemma 2.2.7. *Let F be a torsion-free sheaf of rank r on a smooth projective surface S with Picard rank $\rho(S) = 1$, such that $\mu_{\max}(F) < \mathsf{B}$. Then $\mathrm{ch}_2(F)$ is bounded from above by a number that depends only on $r, \mathrm{c}_1(F)$ and B.*

If $\rho(S) \neq 1$, then $\mu_{\max}(F)$ is considered as a linear function on a neighbourhood $U \subset \mathrm{Amp}(S)$ around $\mathcal{O}_S(1)$ and B is some function in the same neighbourhood. We require the inequality $\mu_{\max}(F) < \mathsf{B}$ to be satisfied pointwise. Then the same conclusion holds.

Proof. We present the proof for $\rho(S) = 1$, the case of $\rho(S) \neq 1$ follows from the same argument. Let

$$0 = HN_0(F) \subset HN_1(F) \subset \cdots \subset HN_k(F) = F$$

be the Harder-Narasimhan filtration of F. Slopes of the graded pieces of the filtration satisfy

$$\mu_{\max}(F) = \mu(gr_1^{HN}) \geq \cdots \geq \mu(gr_k^{HN}),$$

therefore

$$\deg(gr_i^{HN}) < \mathsf{B} \cdot \mathrm{rk}(gr_i^{HN})$$

and

$$\begin{aligned} \deg(gr_i^{HN}) = \deg(F) - \sum_{j \neq i} \deg(gr_j^{HN}) > \deg(F) - \mathsf{B} \cdot \sum_{j \neq i} \mathrm{rk}(gr_j^{HN}) \\ = \deg(F) - \mathsf{B} \cdot (r - \mathrm{rk}(gr_i^{HN})). \end{aligned}$$

Hence we get a uniform bound for all i,

$$\begin{aligned} \deg(F) - \mathsf{B} \cdot r < \deg(gr_i^{HN}) < \mathsf{B} \cdot r, \quad & \text{if } \mathsf{B} \geq 0; \\ \deg(F) < \deg(gr_i^{HN}) < 0, \quad & \text{if } \mathsf{B} < 0. \end{aligned}$$

Which implies that $\mathrm{c}_1(gr_i^{HN})$ is uniformly bounded, since $\rho(S) = 1$. So there exists $\mathsf{A}' \in \mathbb{Z}$, which depends only on B, r and $\mathrm{c}_1(F)$, such that

$$\mathrm{c}_1(gr_i^{HN})^2 < \mathsf{A}', \text{ for all } i,$$

then by semistablity of gr_i^{HN} and Bogomolov-Gieseker inequality

$$\mathrm{ch}_2(gr_i^{HN}) \leq \mathrm{c}_1(gr_i^{HN})^2 / 2\mathrm{rk}(gr_i^{HN}),$$

so we get

$$\mathrm{ch}_2(gr_i^{HN}) < \mathsf{A} = \begin{cases} \mathsf{A}' & \text{if } \mathsf{A}' \geq 0 \\ \mathsf{A}'/2r & \text{if } \mathsf{A}' < 0 \end{cases}$$

Finally, by $\mathrm{ch}_2(F) = \sum \mathrm{ch}_2(gr_i^{HN})$ and by the fact that there are at most r pieces in the filtration we get the desired bound

$$\mathrm{ch}_2(F) < r \cdot \mathsf{A}.$$

$\square$

2.2.2 Stable quasimaps

For all $\beta \in \mathrm{Eff}(X, \mathfrak{X})$ we fix once and for ever a line bundle[1]

$$\mathcal{L}_\beta := \mathcal{L}_0 \otimes \mathcal{L}_1^m \in \mathrm{Pic}_\lambda(\mathfrak{X})$$

for some $m \in \mathbb{N}$ such that $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ satisfies the conclusion of Proposition 2.2.6.

Given a quasimap $f \colon C \to \mathfrak{X}$ of a degree β and a point $p \in C$, we define the *length* of t to be

$$\ell(p) := \mathcal{L}_\beta \cdot_f C - \mathcal{L}_\beta \cdot_{f_p} C,$$

where f_p is the *stabilisation* of f at p, which is defined by viewing f as a rational map to M with an indeterminacy at p and removing the indeterminacy by evoking the properness of M. By the proof of Proposition 2.2.6 we have that $\ell(p) \geq 0$; and $\ell(p) = 0$, if and only if p is not a base point.

Definition 2.2.8. Given $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$, a prestable quasimap $f \colon (C, \mathbf{x}) \to \mathfrak{X}$ of degree β is ϵ-stable, if

(i) $\omega_C(\mathbf{x}) \otimes f^*\mathcal{L}_\beta^\epsilon$ is positive;

(ii) $\epsilon\ell(p) \leq 1$ for all $p \in C$.

We will refer to 0^+-stable and ∞-stable quasimaps as stable quasimaps and stable maps respectively.

A *family* of quasimap over a base B is a family of nodal curves $\mathcal{C}$ over B with a map $f \colon \mathcal{C} \to \mathfrak{Coh}_r(S)$ such that the geometric fibers of f over B are quasimaps.

Let

$$\begin{aligned} Q^\epsilon_{g,N}(M, \beta) \colon (Sch/\mathbb{C})^\circ &\to (Grpd) \\ B &\mapsto \{\text{families of } \epsilon\text{-stable quasimaps over B}\} \end{aligned}$$

be the moduli space of ϵ-stable quasimaps of genus g and the degree β with n marked points.

[1] Such line bundle indeed depends on β, because the conclusions of Proposition 2.2.6 depend on β via the intersection numbers $\mathcal{L}_{1,\ell} \cdot_f C$ for all ℓ.

2.2.3 Properness

The first step on the way to proving properness of the moduli space is the following lemma.

Lemma 2.2.9. *Let $\beta \in \mathrm{Eff}(M, \mathfrak{Coh}_r(S))$ and a nodal curve C be fixed. The moduli space of quasimaps of degree β from C to M is quasi-compact.*

Proof. Choose a lift of f to $\mathfrak{Coh}(S)$, let F^0 be the associated family. The semistable reduction applied to all base points at once gives a sequence of short exact sequences

$$
\begin{gathered}
0 \to F^1 \to F^0 \to Q^1 \to 0, \\
\vdots \\
0 \to F^k \to F^{k-1} \to Q^k \to 0,
\end{gathered}
$$

such that F^k defines a map $f^k \colon C \to M$. To establish the claim of the lemma, we plan to reverse the semistable reduction, i.e. we start with some map from C to M and take consecutive extensions of the corresponding families of sheaves by sheaves supported scheme-theoretically on fibers. For that we have to show that there is bounded number of possibilities. In particular, we have to show that

(i) the number of steps in the semistable reduction is bounded, i.e. k is uniformly bounded;

(ii) the family of possible $f^k \colon C \to M$ is bounded;

(iii) the family of possible Q^i's is bounded.

To be more precise, different lifts of a quasimap are related by tensoring a sheaf with a line bundle coming from C, hence a lift of f^k also determines a lift of f. Therefore if we fix lifts of maps to M, there will always be a lift of f, such that the lift of f^k is the one that we fixed, this will eliminate a potential unboundedness coming from different lifts.

(i) By Proposition 2.2.6 and its proof there are at most $\beta(\mathcal{L}^1)$ steps with

$$\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) > 0$$

and there are at most $\beta(\mathcal{L}_0 \otimes \mathcal{L}_1^m)$ steps with

$$\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) = 0,$$

therefore

$$k \leq \beta(\mathcal{L}_1) + \beta(\mathcal{L}_0 \otimes \mathcal{L}_1^m).$$

(ii) By the proof of Proposition 2.2.6 the numerical degree of possible f^k's with respect to an ample line bundle $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ is bounded in the following way

$$\beta(\mathcal{L}_0 \otimes \mathcal{L}_1^m) = \mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_f C > \mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_{f^k} C \geq 0.$$

Since the family of maps with a fixed domain of a given degree is bounded, the family of possible f^k's must be bounded.

(iii) By the semistable reduction sheaves Q^i's are subsheaves of stable sheaves in the class $\mathbf{v}$ (see [HL97, Theorem 2.B.1]). Chern classes of Q^i's are bounded by Lemma 2.2.4 and by the proof of Proposition 2.2.6. Therefore by boundedness of Quot schemes and stable sheaves, the family of possible Q^i's is also bounded. □

Corollary 2.2.10. *The moduli space $Q^{\epsilon}_{g,N}(M, \beta)$ is quasi-compact.*

Proof. The restriction of a stable quasimap to an unstable component (a rational bridge or a rational tail) must be non-constant by stability and it must pair positively with $\mathcal{L}_\beta$ by Proposition 2.2.6. Therefore the number of unstable components of the domain curve of a stable quasimap is bounded in terms of β. Therefore the projection $Q^{\epsilon}_{g,N}(M, \beta) \to \mathfrak{M}_{g,N}$ factors through a substack of finite type. By Lemma 2.2.9 the projection is quasi-compact, therefore $Q^{\epsilon}_{g,N}(M, \beta)$ is quasi-compact. □

To continue further exploiting the geometry of sheaves, we need to be able to relate quasimaps to sheaves in families (Lemma 2.2.5 permits us to do it only pointwise). For that we have to narrow down our scope. If the $\mathbb{C}^*$-gerbe $\mathfrak{Coh}(S) \to \mathfrak{Coh}_r(S)$ is trivial such that a trivialisation is given by a section

$$s\colon \mathfrak{Coh}_r(S) \to \mathfrak{Coh}(S),$$

then by composing quasimaps with s we can lift quasimaps from $\mathfrak{Coh}_r(S)$ to $\mathfrak{Coh}(S)$ in families. More generally, in order to lift quasimaps of fixed degree in families, the $\mathbb{C}^*$-gerbe has to be trivial only over any substack of finite type $\mathcal{U} \subset \mathfrak{Coh}_r(S)$, since the moduli of quasimaps of fixed degree is quasicompact, hence factors through a substack of finite type. A $\mathbb{C}^*$-gerbe is trivial, if and only if there exists a line bundle of $\mathbb{C}^*$-weight 1. In particular, if there exists a class $u \in K_0(S)$, such that $\chi(u \cdot \mathbf{v}) = 1$, then there is a section

$$s_{u|\mathcal{U}} \colon \mathcal{U} \to \mathfrak{Coh}(S),$$

which is given by the descend of the family $\mathcal{F} \otimes \lambda(u)^{-1}_{|\mathcal{U}}$ to $S \times \mathcal{U}$, note that the section s_u is defined only over substacks of finite type, because $\lambda(u)$ is defined this way. We will only consider trivialisations that arise through determinant line bundles. In any case, they are the only ones that can be checked to exist in practice.

From now one we assume that

$$\exists u \in K_0(S), \text{ such that } \chi(u \cdot \mathbf{v}) = 1.$$

A more general case is discussed in the end of the section.

We will identify a class β with its image with respect to the pushforward by the section s_u (more precisely, by the system of sections over substacks of finite type),

$$s_{u*} \colon \mathrm{Eff}(M, \mathfrak{Coh}_r(S)) \hookrightarrow \mathrm{Eff}(\mathcal{M}, \mathfrak{Coh}(S)).$$

Using (2.2), we can identify $\mathrm{Eff}(M, \mathfrak{Coh}_r(S))$ with classes Λ, as shown in the following corollary.

Corollary 2.2.11. *The map*

$$\check{(\ldots)} \colon \mathrm{Eff}(M, \mathfrak{Coh}_r(S)) \to \Lambda$$

defined as the restriction of (2.2) *to* $\mathrm{Eff}(M, \mathfrak{Coh}_r(S))$ *is injective.*

Proof. We need to show that $\beta \neq 0$ implies $\check{\beta} \neq 0$. By Proposition 2.2.6 a non-zero β intersects positively with a line bundle $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ for some m. Hence by the definition of $\check{\beta}$ in (2.3) it also intersects positively with the corresponding class in Λ. Therefore it cannot be zero. $\square$

Consider now the following composition

$$\begin{aligned} Q^{\epsilon}_{g,N}(M,\beta) \hookrightarrow & Q^{\epsilon}_{g,N}(\mathcal{M},\beta) \hookrightarrow \mathfrak{Coh}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}), \\ & f \mapsto s_u \circ f \mapsto F, \end{aligned} \tag{2.10}$$

where $\mathfrak{C}_{g,N} \to \mathfrak{M}_{g,N}$ is the universal curve over the moduli stack of nodal curves and $\mathfrak{Coh}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N})$ is the relative moduli stack of sheaves on $S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}$.

Definition 2.2.12. Let $M^{\epsilon}_{\mathbf{v},\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$ be the image of the composition (2.10). By $M^{\epsilon}_{\mathbf{v},\check{\beta},u}(S \times C/S_{p_i})$ we will denote[2] a fiber of

$$M^{\epsilon}_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N}) \to \overline{M}_{g,N}$$

over a $\mathbb{C}$-valued point $[(C, \mathbf{x})] \in \overline{M}_{g,N}$, here $\overline{M}_{g,N}$ is the moduli of stable curves. Similarly, we define $Q^{\epsilon}_{(C,\mathbf{x})}(M, \beta)$ to be the fiber of

$$Q^{\epsilon}_{g,N}(M, \beta) \to \overline{M}_{g,N}$$

over a $\mathbb{C}$-valued point $[(C, \mathbf{x})] \in \overline{M}_{g,N}$. We will frequently drop $\mathbf{v}$ from the notation, and in the case of $\epsilon = 0^+$ we will drop 0^+.

In (2.10) the first map is a closed immersion (as it is given by composition with a section), while the second is an open immersion. In particular,

$$Q^{\epsilon}_{g,N}(M, \beta) \cong M^{\epsilon}_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N}). \tag{2.11}$$

The $\mathbb{C}$-valued points of the moduli space $M^{\epsilon}_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$ are just families of sheaves associated to quasimaps via the section s_u.

We will now study the moduli on the right in (2.11) in more detail. By the construction the section s_u is given by the descend of $\mathcal{F} \otimes \lambda(u)^{-1}$, therefore a sheaf $F \in M^{\epsilon}_{\check{\beta},u}(S \times C)(\mathbb{C}) \subset M^{\epsilon}_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})(\mathbb{C})$ satisfies[3]

$$\det(p_{C*}(p_S^* u \otimes F)) = (s_u \circ f)^* \lambda(u) = \mathcal{O}_C.$$

Moreover, by the definition of a quasimap, a general fiber of F over C is stable. The stability of a general fiber can be related to the stability of the sheaf F itself, as is shown in the following lemma.

Lemma 2.2.13. *There exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the moduli $M^{\epsilon}_{\check{\beta},u}(S \times C)$ is an open sublocus of a moduli of Gieseker $\mathcal{O}_{S\times C}(1, k)$-stable[4] sheaves on $S \times C$ satisfying the condition $\det(p_{C*}(p_S^* u \otimes F)) = \mathcal{O}_C$.*

We will refer to the stability in the lemma as *suitable*. The converse of the lemma is more subtle, in Appendix 5.1 it is proven in the rank-2 case for slope stabilities, rank-1 case holds trivially. Note that a sheaf which is $\mathcal{O}_{S\times C}(1, k)$-stable for *all* $k \gg 0$ is stable at a general fiber. Hence proving

[2] The notation is similar to the one of Donaldson–Thomas theory relative to divisors.
[3] Later this will be important for the deformation theory.
[4] $\mathcal{O}_{S\times C}(1, k)$ stands for $\mathcal{O}_S(1) \boxtimes \mathcal{O}_C(k)$.

converse amounts to proving that there are no walls between $\mathcal{O}_{S\times C}(1,k)$-stabilities for $k \gg 0$.

Proof of Lemma 2.2.13. Given a sheaf $F \in M^{\epsilon}_{\check{\beta},u}(S\times C)$, a general fiber of F over C is stable, in particular, it is torsion-free, therefore F is torsion-free itself by Lemma 2.2.14. It also must be $\mathcal{O}_{S\times C}(1,k)$-stable for all $k \gg 0$, this can be seen as follows. Since a general fiber of F is stable, the difference between $\mathcal{O}_{S\times C}(1,k)$-Hilbert polynomials of F and of its subsheaves increases as k increases, because $\mathrm{c}_1(\mathcal{O}_C(k))$ pairs only with $\mathrm{ch}_{\mathrm{f}}(F)$. Since the family of $\mathcal{O}_{S\times C}(1,k)$-destabilising subsheaves of F is bounded, for $k \gg 0$ no subsheaves of F will be $\mathcal{O}_{S\times C}(1,k)$-destabilising, therefore F is $\mathcal{O}_{S\times C}(1,k)$-stable for $k \gg 0$.

The moduli space $M^{\epsilon}_{\check{\beta},u}(S\times C)$ is quasi-compact, therefore there exists a uniform choice of k_0 for which the statement holds for all sheaves in $M^{\epsilon}_{\check{\beta},u}(S\times C)$. The fact that it is open follows from openness of stability of fibers. $\square$

Lemma 2.2.14. *Let F be a sheaf on $S\times C$ flat over C, such that F_t is torsion-free for a general $t\in C$, then F is torsion-free.*

Proof. Let $T(F)\subset F$ be the maximal torsion subsheaf. Firstly, $T(F)\neq F$, because $\mathrm{rk}(F)\neq 0$. It also cannot be supported on fibers of $S\times C\to C$ due to flatness of F over C, therefore $\mathrm{Supp}(T(F))$ intersects a general fiber. Since $F/T(F)$ is generically flat, restricting $T(F)\subset F$ to a general fiber $t\in C$, we get a torsion subsheaf of F_t for a general $t\in C$, which is zero, therefore $T(F)$ is zero. $\square$

The final ingredient for the proof of properness of the moduli space is the following lemma, *Hartog's property* for families of nodal curves over a DVR (however, for a general surface Hartog's property fails).

Lemma 2.2.15. *Let $\mathcal{C}\to\Delta$ be a family of nodal curves over a DVR Δ and $\{p_i\}\subset\mathcal{C}$ be finitely many closed points in the regular locus of the central fiber. If there exists a class $u\in K_0(S)$, such that $\chi(u\cdot\mathbf{v})=1$, then any quasimap $\tilde{f}\colon\tilde{\mathcal{C}}=\mathcal{C}\setminus\{p_i\}\to\mathfrak{Coh}_r(S)$ extends to $f\colon\mathcal{C}\to\mathfrak{Coh}_r(S)$, which is unique up to unique isomorphism.*

Proof. Let $\tilde{F}$ be the family on $S\times\tilde{\mathcal{C}}$ corresponding to the lift of $\tilde{f}$ by s_u, we then extend $\tilde{F}$ to a coherent sheaf F on $S\times\mathcal{C}$, quotienting the torsion, if necessary. The sheaf F is therefore flat over Δ. The central fiber F_k of F defines a quasimap, if it is torsion-free, because $\mathcal{C}_k$ is regular at p_i. If

F_k is not torsion-free, we can remove the torsion inductively as follows. Let $F^0 = F$ and F^i be defined by short exact sequences,

$$0 \to F^i \to F^{i-1} \to Q^i \to 0,$$

such that Q^i is the quotient of F_k^{i-1} by the maximal torsion subsheaf. It is not difficult to check, that at each step the torsion of F_k^i is supported at slices $S \times p_i$, therefore all F^i's are isomorphic to F^0 over $S \times \tilde{\mathcal{C}}$. By the standard argument (see e.g. [HL97, Theorem 2.B.1]), this process terminates, i.e. $F^i = F^{i+1}$ and F_k^i is torsion-free for $i \gg 0$. Let us redefine the sheaf F, we set $F = F^i$ for some $i \gg 0$, then the sheaf F induces a quasimap to $\mathfrak{Coh}(S)$, and composing it with the projection to $\mathfrak{Coh}_r(S)$, we thereby obtain an extension $f \colon \mathcal{C} \to \mathfrak{Coh}_r(S)$ of $\tilde{f}$.

Consider now another extension $f' \colon \mathcal{C} \to \mathfrak{Coh}_r(S)$, we lift both f and f' to $\mathfrak{Coh}(S)$ with s_u, then let F' and F be the corresponding families on $S \times \mathcal{C}$ (notice, F might differ from the previous F by a tensor with a line bundle), by Lemma 2.2.13 they define a family of stable sheaves relative to Δ in some relative moduli of sheaves $\mathcal{M}(S \times \mathcal{C}/\Delta)$, hence they must be isomorphic up to tensoring with a line bundle by separateness of the relative moduli of stable sheaves. The isomorphism becomes unique after projection to $\mathfrak{Coh}_r(S)$. $\square$

Remark 2.2.16. In general, Hartog's property fails for sheaves on a surface. Hence the assumption that our surface is given by a family of curves $\mathcal{C} \to \Delta$ is necessary. This form of Hartog's property is good enough for proving Theorem 2.2.17 in the spirit of [CKM14, Section 4].

Theorem 2.2.17. *If there exists a class $u \in K_0(S)$, such that $\chi(u \cdot \mathbf{v}) = 1$, then $Q^{\epsilon}_{g,N}(M, \beta)$ is a proper Deligne–Mumford stack.*

Proof. The morphism $S \times \mathfrak{C}_{g,N} \to \mathfrak{M}_{g,N}$ is locally of finite type. Relative moduli spaces of sheaves are known to be locally of finite type and quasi-separated, therefore by Lemma 2.2.10 and (2.11) the moduli space $Q^{\epsilon}_{g,N}(M, \beta)$ is of finite type and quasi-separated. By (**i**) of the quasimaps' stability (see Definition 2.2.8), ϵ-stable quasimaps have only finitely many automorphisms given by automorphisms of curves which fix the sheaves. The moduli space $Q^{\epsilon}_{g,N}(M, \beta)$ is therefore a quasi-separated Deligne–Mumford stack. Using the valuative criteria of properness for quasi-separated Deligne–Mumford stacks and Lemma 2.2.15, the proof of properness then proceeds as in the GIT case [CKM14, Section 4.3]. $\square$

2.2.4 Stable sheaves

Let us now concentrate further on the moduli space of sheaves associated to quasimaps. Since torsion-free is equivalent to flatness for a smooth curve, the moduli space $M_{\check{\beta},u}(S \times C)$ is a component (it is open by Lemma 2.2.13 and closed by properness) of the moduli of stable sheaves, such that a general fiber of a sheaf over C is stable. On the other hand, the moduli space $Q_C(M, \beta)$ is the moduli of stable quasimaps from a fixed curve C. By definition we have

$$M_{\check{\beta},u}(S \times C) \cong Q_C(M, \beta), \tag{2.12}$$

this identification will be the origin of the relation between Gromov–Witten theory of M and Donladson–Thomas theory of $S \times C$. For slope-stable sheaves with $\mathrm{rk} \leq 2$ the moduli $M_{\check{\beta},u}(S \times C)$ is exactly the moduli of stable sheaves by Proposition 5.1.4, which is also expected to be the case for any rank.

Similarly, quasimaps from a fixed curve C with a marking without identifications by automorphisms is isomorphic to a moduli of sheaves on $S \times C$ relative to a vertical divisor $S_p := S \times p \subset S \times C$,

$$M_{\check{\beta},u}(S \times C/S_p) \cong Q_{(C,p)}(M, \beta),$$

where the additional stability conditions of a sheaf are the ones of the corresponding quasimap, in particular, restriction of a sheaf to the relative divisor and the singular locus of expanded degenerations is stable.

We will now relate the moduli space $M^{\epsilon}_{\check{\beta},u}(S \times C)$ to a more familiar one - a moduli space of sheaves with a fixed determinant. Let $M^{\epsilon}_{\check{\beta},L}(S \times C)$ be the moduli of sheaves stable at a generic fiber with a fixed determinant L such that the associated quasimaps satisfy ϵ-stability, where $L = \det(G)$ for some $G \in M^{\epsilon}_{\check{\beta},u}(S \times C)$. Then there exists projection that relates two moduli spaces,

$$p\colon M^{\epsilon}_{\check{\beta},L}(S \times C) \to M^{\epsilon}_{\check{\beta},u}(S \times C), \quad F \mapsto F \boxtimes \det(p_{C*}(p_S^* u \otimes F))^{-1}, \tag{2.13}$$

which is, in fact, an étale cover, if C is smooth.

Lemma 2.2.18. *Assume C is smooth, then the map p is étale of degree* $\mathrm{rk}(\mathbf{v})^{2g}$.

Proof. The surjectivity can be seen as follows. Consider a sheaf $F \in M^{\epsilon}_{\check{\beta},u}(S \times C)$, then

$$L_0 := \det(F) \otimes L^{-1} \in \mathrm{Pic}_0(S \times C) = \mathrm{Pic}_0(C).$$

Now let $L_0^{\frac{1}{\mathrm{rk}(\mathbf{v})}}$ be a $\mathrm{rk}(\mathbf{v})^{\mathrm{th}}$ root of L_0, then

$$\det(F \otimes L_0^{-\frac{1}{\mathrm{rk}(\mathbf{v})}}) = \det(F) \otimes L_0^{-1} = L,$$

therefore

$$F \otimes L_0^{-\frac{1}{\mathrm{rk}(\mathbf{v})}} \in M^{\epsilon}_{\check{\beta},L}(S \times C)$$

and it can be easily checked that it maps to F via the map (2.13). It is of degree $\mathrm{rk}(\mathbf{v})^{2g}$, because $\det(F \otimes A) = \det(F) \otimes A^{\otimes \mathrm{rk}(F)}$ for a line bundle $A \in \mathrm{Pic}(S \times C)$. Therefore only $\mathrm{rk}(\mathbf{v})$-torsion line bundles preserve $M_{\check{\beta},L}(S \times C)$. Moreover, sheaves in an orbit of the action of $J(C)[\mathrm{rk}(\mathbf{v})]$ map to the same sheaf via (2.13), where $J(C)[\mathrm{rk}(\mathbf{v})] \cong (\mathbb{Z}/\mathrm{rk}(\mathbf{v})\mathbb{Z})^{2g}$ is the subgroup of $\mathrm{rk}(\mathbf{v})$-torsion points of the Jacobian $J(C)$. The action is free, because

$$\det(p_{C*}(p_S^* u \otimes F \boxtimes A) \cong \det(p_{C*}(p_S^* u \otimes F) \otimes A,$$

for a line bundle A, which is due to $\chi(u \cdot \mathbf{v}) = 1$. In particular,

$$M^{\epsilon}_{\check{\beta},L}(S \times C)/J(C)[\mathrm{rk}(\mathbf{v})] \xrightarrow{\sim} M^{\epsilon}_{\check{\beta},u}(S \times C),$$

hence the claim follows. □

2.2.5 More general cases

Non-trivial gerbe

The proof of properness of the moduli $Q^{\epsilon}_{g,N}(M, \beta)$ crucially relies on the identification of the space with the relative moduli of sheaves $M^{\epsilon}_{\check{\beta},u}(S \times \mathfrak{C}/\mathfrak{M}_{g,N})$. To make it work in the case when $\mathbb{C}^*$-gerbe $\mathfrak{Coh}(S) \to \mathfrak{Coh}_r(S)$ is not trivial, one needs to consider twisted universal families. Given any $u \in K^0(S)$ such that

$$\mathrm{w} = \chi(u \cdot \mathbf{v}) \neq 0,$$

then over each finite type open substacks $\mathcal{U} \subset \mathfrak{Coh}(S)$ we can take a w^{th}-root stack associated to $\lambda(u)$ with the universal w^{th}-root $\lambda(u)^{\frac{1}{\mathrm{w}}}$ of $\lambda(u)$,

$$\mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}} \to \mathcal{U}, \quad \lambda(u)^{\frac{1}{\mathrm{w}}} \in \mathrm{Pic}(\mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}}).$$

Then $w_{\mathbb{C}^*}(\lambda(u)^{\frac{1}{\mathrm{w}}}) = 1$, therefore $\lambda(u)^{\frac{1}{\mathrm{w}}}$ defines a trivialisation of the $\mathbb{C}^*$-gerbe

$$\mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}} \to \mathfrak{Coh}_r(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}},$$

given by the descend of the twisted family $\mathcal{F} \otimes \lambda(u)^{-\frac{1}{\mathrm{w}}}$, where

$$\mathfrak{Coh}_r(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}} := \mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}} /\!\!/ \mathbb{C}^*.$$

Thereby we obtain the desired section

$$s_{\frac{u}{\mathrm{w}}} : \mathfrak{Coh}_r(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}} \to \mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}}.$$

The price we pay for this section is that the stable locus becomes a $\mathbb{Z}/\mathrm{w}\mathbb{Z}$-gerbe of M, which we denote by $M^{\frac{u}{\mathrm{w}}}$. In particular, we have to consider orbifold quasimaps for the sake of properness of the moduli space. All the definitions carry over to this setting verbatim, so let us consider now the quasimap theory of the pairs,

$$(M^{\frac{u}{\mathrm{w}}}, \mathfrak{Coh}_r(S)^{\frac{u}{\mathrm{w}}}) \quad \text{and} \quad (\mathcal{M}^{\frac{u}{\mathrm{w}}}, \mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}).$$

As in the case of untwisted case we can consider the following composition

$$Q^{\epsilon}_{g,N}(M^{\frac{u}{\mathrm{w}}}, \beta) \hookrightarrow Q^{\epsilon}_{g,N}(\mathcal{M}^{\frac{u}{\mathrm{w}}}, \beta) \to \mathfrak{Coh}(S \times \mathfrak{C}^{\mathrm{tw}}_{g,N}/\mathfrak{M}^{\mathrm{tw}}_{g,N}),$$

$$f \mapsto s_{\frac{u}{\mathrm{w}}} \circ f \mapsto F,$$

where $\mathfrak{M}^{\mathrm{tw}}_{g,N}$ is the moduli of twisted nodal curves with the universal family $\mathfrak{C}^{\mathrm{tw}}_{g,N}$. The second map is no longer an embedding, because the moduli problem of $\mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}$ is now a pair

$$(F, \det(p_{C*}(p_S^* u \otimes F))^{\frac{1}{\mathrm{w}}}),$$

a sheaf F and a w^{th}-root of $\det(p_{C*}(p_S^* u \otimes F))$. However, by the definition of the section $s_{\frac{u}{\mathrm{w}}}$, the w^{th}-root is fixed

$$(s_u \circ f)^* \lambda(u)^{\frac{1}{\mathrm{w}}} = \det(p_{C*}(p_S^* u \otimes F))^{\frac{1}{\mathrm{w}}} = \mathcal{O}_C,$$

hence the composition above is an embedding and $\det(p_{C*}(p_S^* u \otimes F)) = \mathcal{O}_C$. Let $M^{\epsilon}_{\check{\beta},u}(S \times C^{\mathrm{tw}}_{g,N}/\overline{M}^{\mathrm{tw}}_{g,N})$ be its image. We therefore have the desired identification,

$$Q^{\epsilon}_{g,N}(M^{\frac{u}{\mathrm{w}}}, \beta) \cong M^{\epsilon}_{\check{\beta},u}(S \times C^{\mathrm{tw}}_{g,N}/\overline{M}^{\mathrm{tw}}_{g,N}).$$

the rest goes as in the untwisted case. However, to give a full treatment of twisted invariants, we have to add many modifications and remarks here and there, making the presentation more obscure. In principle, there are no obstacles for extension of all results including wall-crossing formulas. Using [AJT], we then can relate the twisted invariants to untwisted ones.

Non-trivial Jacobian

The case of a surface with $q(S) \neq 0$ can be tackled in the same manner. However, we need to adjust some definitions. Firstly, instead of the stack $\mathfrak{Coh}_r(S)_{\mathbf{v}}$ we have to take its fiber over $\mathrm{Pic}(S)$ with respect to the determinant morphism

$$\det\colon \mathfrak{Coh}_r(S)_{\mathbf{v}} \to \mathrm{Pic}(S),$$

where we slightly abuse the notation, because the morphism det is only defined over substacks of finite type.

Then for the definition of a degree we have to take care of an extra summands in Künneth's decomposition of (p,p)-part of the cohomology on $S \times C$,

$$\bigoplus_i \bigoplus_{\substack{p+p'=i \\ q+q'=i}}^{p\neq q} H^{p,q}(S) \otimes H^{p',q'}(C).$$

The classes $\mathrm{Hom}(\mathrm{Pic}_\lambda(\mathfrak{X}), \mathbb{Z})$ are not sensitive to the piece of Küneth decomposition as above in the sense that the Chern character $\mathrm{ch}(F)$ of a family F is not determined by the degree $\beta \in \mathrm{Hom}(\mathrm{Pic}_\lambda(\mathfrak{X}), \mathbb{Z})$ of the corresponding quasimap. On the Gromov–Witten side of M this extra piece corresponds to extra classes that are not given by determinant line bundles. One could make the definition of the degree finer by defining it as a class in $H_2(\mathfrak{X}, \mathbb{Z})$, but then we loose a direct connection of the degree with the Chern characters of sheaves on threefolds. One could also leave the definition as it is, thereby making the degree slightly coarser than it could be. For genus-0 invariants this, however, does not matter. Indeed the extra piece in Künneth decomposition of cohomology is not present, because $H^{1,0}(\mathbb{P}^1) = 0$.

Similarly, in the case of punctorial Hilbert schemes and the fixed-curve invariants one can define the degree of a quasimap by the Chern character of the corresponding subscheme on a threefold after contracting rational tails and projecting the subscheme to the component corresponding to the fixed curve.

2.3 Hilbert schemes

2.3.1 Relative Hilbert schemes

We now restrict to $\mathbf{v} = (1, 0, -n)$, i.e. $M = S^{[n]}$. Punctorial Hilbert schemes are special, because there exists a canonical trivialisation of $\mathfrak{Coh}(S)_{\mathbf{v}} \to$

$\mathfrak{Coh}_r(S)_{\mathbf{v}}$ over any $\mathcal{U} \subset \mathfrak{Coh}_r(S)_{\mathbf{v}}$ of finite type. It is given by the determinant

$$\det(\mathcal{F}) \in \mathrm{Pic}(S \times \mathfrak{Coh}(S)_{\mathbf{v}})$$

of the universal sheaf $\mathcal{F}$ on $S \times \mathfrak{Coh}(S)_{\mathbf{v}}$. It is indeed a line bundle of weight 1, because $\mathcal{F}$ is of rank 1. Hence the family $\mathcal{F} \otimes \det(\mathcal{F}_{|\mathcal{U}})^{-1}$ descends to $S \times \mathcal{U}$, giving the canonical section

$$s_{\det|\mathcal{U}} \colon \mathcal{U} \to \mathfrak{Coh}(S)_{\mathbf{v}}$$

of the gerbe $\mathfrak{Coh}(S)_{\mathbf{v}} \to \mathfrak{Coh}_r(S)_{\mathbf{v}}$. By Corollary 2.2.10, there exists $\mathcal{U}$ of finite type through which the universal quasimap factors. Therefore the section $s_{\det|\mathcal{U}}$ gives us the embedding

$$Q^{\epsilon}_{g,N}(S^{[n]}, \beta) \hookrightarrow \mathfrak{Coh}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}),$$

which is defined as the one in (2.10). By the construction of the section, the sheaves in the image of the embedding satisfy

$$\det(F) = (\mathrm{id}_S \times f)^* \det(\mathcal{F}) = \mathcal{O}_{S \times \mathcal{C}}$$

over any base scheme B. Therefore the embedding factors through a relative Hilbert scheme,

$$Q^{\epsilon}_{g,N}(S^{[n]}, \beta) \hookrightarrow \mathrm{Hilb}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}).$$

Indeed, the above embedding factors through the relative moduli of sheaves of rank 1 with trivial determinant by the construction of the section $s_{\det}$. This moduli is in turn isomorphic to the moduli of ideals, because there exits a natural embedding $F \hookrightarrow F^{\vee\vee} \cong \mathcal{O}_{S \times C}$. It is a stack but not a scheme, because $S \times \mathfrak{C}_{g,N} \to \mathfrak{M}_{g,N}$ is a stack.

We denote the image of the embedding above by $\mathrm{Hilb}^{\epsilon}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N})$, where the subscript "$n, \check{\beta}$" is the shortening of

$$((1, 0, -n), \check{\beta}) \in \Lambda \otimes \Lambda.$$

The image can be described more explicitly in terms of ideals, or, equivalently, in terms of the corresponding one-dimensional subschemes. Firstly, the automorphisms of a quasimap f admit the following description

$$\mathrm{Aut}(f) = \mathrm{Aut}_{(C,\mathbf{x})}(I) = \mathrm{Aut}_{(C,\mathbf{x})}(\Gamma),$$

where I is the corresponding ideal sheaf, $\Gamma \subset S \times C$ is the associated subscheme and

$$\mathrm{Aut}_{(C,\mathbf{x})}(I) = \{\psi \colon (C,\mathbf{x}) \cong (C,\mathbf{x}) | (\mathrm{id}_S \times \psi)^* I = I\},$$

similarly for $\mathrm{Aut}_{(C,\mathbf{x})}(\Gamma)$. The quasimap ϵ-stability therefore requires the group $\mathrm{Aut}_{(C,\mathbf{x})}(I)$ to be finite.

The part (**ii**) of ϵ-stability in Definition 2.2.8 can be rephrased in terms of Γ as follows. A sheaf I_t is an ideal for all $t \in C$, if and only if all irreducible components of the subscheme Γ are dominant over a component of C and there are not embedded points, if and only if Γ if flat over C. We call non-dominant components without embedded points *vertical*. Let

$$\Gamma^{h+v} \subseteq \Gamma$$

be the maximal subscheme without embedded ponts, then $\Gamma^{h+v} = \Gamma^h \cup \Gamma^v$, where Γ^h is *horizontal* part of Γ, which is dominant over C and therefore is the subscheme associated to the stabilisation of I, and Γ^v is the vertical part of Γ. We have the following equality

$$I^{h+v} = I^h \cap I^v,$$

because there are no embedded points. Therefore there is an exact sequence

$$0 \to I^{h+v} \to I^h \oplus I^v \xrightarrow{+} I_{\Gamma^h \cap \Gamma^v} \to 0, \tag{2.14}$$

such that I^h is stable over all $t \in C$. Now let $\Gamma_i^u \subset \Gamma$ be the maximal non-dominant subscheme (with embedded points) supported on $S \times b_i$ for a given base point b_i and Γ_i^v be its vertical component without embedded points, then by the part (**ii**) of Definition 2.2.8, Lemma 2.2.4 and the sequence above, these Γ_i^u's must satisfy

$$m \cdot \deg(\Gamma_i^u) + \chi(\Gamma_i^u) - \chi(I_{\Gamma^h \cap \Gamma_i^v}) \leq 1/\epsilon,$$

for some fixed m for which Proposition 2.2.6 holds.

Apart from the usual condition on finiteness of automorphisms, the part (**i**) of Definition 2.2.8 can be similarly translated into restriction of the 'size' of Γ on rational tails in terms of its degree and Euler characteristic: given a rational tail R_j of C, let $\deg(\Gamma_{|R_j}) := \deg(\mathrm{ch}(\Gamma_{|R_j})_{\mathrm{d}})$, then for all rational tails the following must be satisfied

$$m \cdot \deg(\Gamma_{|R_j}) + \chi(\Gamma_{|R_j}) > 1/\epsilon.$$

Finally, by stability of quasimaps, Γ has to be flat over nodes and marked points.

Hilbert schemes clearly satisfies the assumption of Theorem 2.2.17, hence summing up the discussion above we obtain the following result.

Corollary 2.3.1. *The moduli stack $Q^{\epsilon}_{g,N}(S^{[n]},\beta)$ is a proper Deligne–Mumford stack. For some fixed $m \gg 0$, there exists a natural isomorphism of the moduli spaces*

$$Q^{\epsilon}_{g,N}(S^{[n]},\beta) \cong \mathrm{Hilb}^{\epsilon}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}),$$

where the stack on the right is the relative moduli stack of 1-dimensional subschemes, satisfying the following properties

- $|\mathrm{Aut}_{(C,\mathbf{x})}(\Gamma)| < \infty$*;*
- Γ *is flat over nodes and marked points;*
- $m \cdot \deg(\Gamma^u_i) + \chi(\Gamma^u_i) - \chi(\Gamma^s \cap \Gamma^v_i) \leq 1/\epsilon$ *for a component* Γ^u_i*;*
- $m \cdot \deg(\Gamma_{|C_j}) + \chi(\Gamma_{|R_j}) > 1/\epsilon$ *for a rational tail* R_j*.*

Remark 2.3.2. It is worth mentioning that a moduli of 0^+-stable quasimaps from a fixed smooth non-rational curve C without identifications by automorphisms in a non-zero class is just a Hilbert scheme on a threefold $S \times C$,

$$Q_C(S^{[n]},\beta) \cong \mathrm{Hilb}_{n,\check{\beta}}(S \times C),$$

while the moduli of 0^+-stable quasimaps with one fixed marked point $p \in C$ is a moduli of ideals relative to a vertical divisor $S_p \subset S \times C$,

$$Q_{(C,p)}(S^{[n]},\beta) \cong \mathrm{Hilb}_{n,\check{\beta}}(S \times C/S_p),$$

see also Remark 2.4.4. Moreover, pulling back a class with a marking on the left is equivalent ot pulling back the class from a relative divisor on the right. In particular, as soon as the obstruction theories are defined and shown to match in Proposition 2.4.5, the equality of corresponding invariants with insertions will follow.

2.3.2 Changing the t-structure

Consider the following *torsion pair* in $\mathrm{Coh}(S)$,

$$\mathcal{T} = \{A \in \mathrm{Coh}(S) \mid \dim(A) = 0\},$$

$$\mathcal{T}^{\perp} = \{B \in \mathrm{Coh}(S) \mid \mathrm{Ext}^{\bullet}(A, B) = 0, \forall A \in \mathcal{T}\}.$$

Let $\mathrm{Coh}^{\sharp}(S) = \langle \mathcal{T}^{\perp}, \mathcal{T}[-1]\rangle$ be the corresponding perverse heart. Punctorial Hilbert schemes sit inside the rigidification of the corresponding moduli stack,

$$S^{[n]} \subset \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}} := \mathfrak{Coh}^{\sharp}(S)_{\mathbf{v}} /\!\!/ \mathbb{C}^*.$$

Before proceeding further we firstly introduce some terminology from [AP06]. Let $\mathcal{A} := \mathrm{Coh}^{\sharp}(S)$ and $\mathcal{A}_C$ be the Abramovich-Polishchuk heart in $D_{\mathrm{perf}}(S \times C)$. An object $F \in \mathcal{A}_C$ is called *torsion*, if it is a pushforward of an object from $D_{\mathrm{perf}}(S \times T)$, where $T \subset C$ is some proper subscheme. The object F is *flat*, if $F_t := Li_t^* F \in \mathcal{A}$ for all $t \in C$, and it is *torsion-free*, if it does not contain any torsion subobjects.

Let $f\colon C \to \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$ be a quasimap to the pair $(S^{[n]}, \mathfrak{Coh}^{\sharp}_r(S))$, then as in the case of the standard heart, we can lift it to $\mathfrak{Coh}^{\sharp}(S)_{\mathbf{v}}$ by the determinant section

$$s_{\det} : \mathcal{U} \to \mathfrak{Coh}^{\sharp}(S)_{\mathbf{v}}$$

over some $\mathcal{U} \subset \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$ of finite type. We now prove the following.

Proposition 2.3.3. *Let F be the family on $S \times C$ associated to the lift of $f \to \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$ via $s_{\det}$, then F is stable pair, i.e. $F \in \mathrm{P}(S \times C)$. Conversely, given a stable pair $I^{\bullet} \in \mathrm{P}(S \times C)$, then $I^{\bullet} \in \mathcal{A}_C$.*

We firstly need the following lemma.

Lemma 2.3.4. *A flat object $F \in \mathcal{A}_C$ is torsion-free.*

Proof of Lemma 2.3.4. Let $\tilde{F}$ be the pullback of F to the normalisation $S \times \tilde{C}$. Let $T \subset \tilde{F}$ be the maximal torsion object, then $\tilde{F}' := \tilde{F}/T$ is a torsion-free object, hence it is flat by [AP06, Corollary 3.1.3]. Restricting to a fiber over some $t \in C$ we get an exact sequence

$$0 \to T_t \to \tilde{F}_t \to \tilde{F}'_t \to 0,$$

because $\tilde{F}$ is flat. Thus $T_t \in \mathrm{Coh}^{\sharp}(S)$ and $\mathrm{ch}(T_t) = 0$ for all $t \in C$, since $\mathrm{ch}(\tilde{F}_t) = \mathrm{ch}(\tilde{F}'_t)$, which implies that $T_t = 0$ for all $t \in C$, which in turn implies that $T = 0$. If F had torsion, it would produce torsion in $\tilde{F}$, hence F is torsion-free. □

Proof of Proposition 2.3.3. Now let be F be an object corresponding to the lift of a quasimap $f\colon C \to \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$, by definition it is family of objects in $\mathcal{A}$, hence $F \in \mathcal{A}_C$ by [AP06] and F is flat. It is also clear that F is of rank 1, and that $\det(F) = \mathcal{O}_{S \times C}$ by the choice of the lift. By [Tod10, Lemma 3.11] to show that $F \in \mathrm{P}(S \times C)$, we have to establish the following properties:

(i) $\mathcal{H}^i(F) = 0$, for $i \neq 0, 1$;

(ii) $\mathcal{H}^0(F)$ is a rank-1 torsion-free sheaf and $\mathcal{H}^1(F)$ is 0-dimensional;

(iii) $\mathrm{Hom}(Q[-1], F) = 0$ for any 0-dimensional sheaf Q.

(**i**) Since F is a family of objects with amplitude $[0,1]$, F cannot be of amplitude wider than $[0,1]$. To see this, consider the two triangles the object F fits in

$$\tau_{<0}F \to F \to \tau_{\geq 0}F \to,$$
$$\tau_{<2}F \to F \to \tau_{\geq 2}F \to,$$

where the truncation is taken with respect to the standard t-structure. Taking fibers over $t \in C$ and considering long exact sequences of cohomologies in the standard heart we conclude that $\tau_{<0}F = 0$ and $\tau_{\geq 2}F = 0$.

(**ii**) Let $T(\mathcal{H}^0(F)) \subseteq \mathcal{H}^0(F)$ be the maximal torsion subsheaf, composition $T(\mathcal{H}^0(F)) \hookrightarrow \mathcal{H}^0(F) \to F$ is zero, because F is torsion-free, but in the standard heart the second map is just an inclusion of 0-th cohomology, hence the whole composition must be zero, therefore $T = 0$ and $\mathcal{H}^0(F)$ is torsion-free. Due to fact that F_t is an ideal for a general $t \in C$ and $F_t \in \mathcal{A}$ for all $t \in C$, $\mathcal{H}^1(F)$ must be 0-dimensional by the definition of $\mathcal{A}$.

(**iii**) The last property follows trivially, because F is torsion-free.

Conversely, given now a stable pair $I^\bullet \in \mathrm{P}(S \times C)$, by definition it sits in a triangle

$$\mathcal{H}^0(I^\bullet) \to I^\bullet \to \mathcal{H}^1(I^\bullet)[-1] \to,$$

such that $\mathcal{H}^0(I^\bullet)$ is an ideal sheaf and $\mathcal{H}^1(I^\bullet)$ is 0-dimensional. Applying $p_{S*}(- \otimes \mathcal{O}_C(m))$ for $m \gg 0$ to the triangle, we obtain that $p_{S*}(\mathcal{H}^0(I^\bullet) \otimes \mathcal{O}_C(m))$ is a torsion-free sheaf and $p_{S*}(\mathcal{H}^1(I^\bullet) \otimes \mathcal{O}_C(m))$ is 0-dimensional, therefore $p_{S*}(I^\bullet \otimes \mathcal{O}_C(m)) \in \mathcal{A}$ for $m \gg 0$, hence by the definition $I^\bullet \in \mathcal{A}_C$.

With a bit more work, one should be able to prove that

$$\mathcal{A}_C = \langle \mathcal{T}_C^\perp, \mathcal{T}_C[-1] \rangle,$$

where $\mathcal{T}_C = \{A \in \mathrm{Coh}(S \times C) \,|\, \dim(A) = 0\}$. $\square$

The determinant line bundle construction in this setting also defines the map $\lambda\colon K_0(S) \to \mathrm{Pic}(\mathfrak{Coh}^\sharp(S)_\mathbf{v})$. The line bundles $\mathcal{L}_0$ and $\mathcal{L}_1$ satisfy the same properties as in the case of the standard heart.

Lemma 2.3.5. *Let $f\colon C \to \mathfrak{Coh}^\sharp(S)$ be a prestable quasimap. There exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ the quasimap is non-constant, if and only if*

$$\mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_f C > 0.$$

The constant m_0 depends only on $\mathbf{v}$ and $\mathcal{L}_{1,\ell} \cdot_f C$ for all ℓ. The same holds for all subcurves C' and the induced maps for the same choice of m.

Proof. The proof is similar to the one of Proposition 2.2.6, but with one exception - the unstable locus of $\mathfrak{Coh}^\sharp(S)$ now contains objects which sit in a distinguished triangle

$$\mathcal{H}^0(A) \to A \to \mathcal{H}^1(A)[-1] \to$$

such that $\mathcal{H}^1(A)$ is a 0-dimension sheaf. When we apply semistable reduction to such objects the corresponding term $\chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i)$ is strictly negative. To get around this problem, for a pair $I^\bullet \in \mathrm{P}(S \times C)$ we firstly take its zeroth cohomology

$$\mathcal{H}^0(I^\bullet) \to I^\bullet \to \mathcal{H}^1(I^\bullet)[-1] \to$$

where $\mathcal{H}^0(I^\bullet)$ is an ideal sheaf and $\mathcal{H}^1(I^\bullet)$ is zero dimensional, and then run the Langton's semistable reduction for $\mathcal{H}^0(I^\bullet)$. $\square$

Fixing a positive line bundle $\mathcal{L}_\beta$ from the Lemma 2.3.5 once and for ever for all $\beta \in \mathrm{Eff}(S^{[n]}, \mathfrak{Coh}_r^\sharp(S)_\mathbf{v})$, we can define the length of base point as previously. The definition of ϵ-stability carries over to this case verbatim. Given $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$ let

$$Q^\epsilon_{g,N}(S^{[n]}, \beta)^\sharp \colon (Sch/\mathbb{C})^\circ \to (Grpd)$$

be a moduli of ϵ-stable perverse quasimaps to the pair $(S^{[n]}, \mathfrak{Coh}_r^\sharp(S)_\mathbf{v})$ for some $\beta \in \mathrm{Eff}(S^{[n]}, \mathfrak{Coh}_r^\sharp(S))$. The proof of boundedness of the moduli is exactly the same as in the case of the standard heart. And using Lemma 2.3.3, we obtain an immersion,

$$Q^\epsilon_{g,N}(S^{[n]}, \beta)^\sharp \hookrightarrow \mathrm{P}(S \times \mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}),$$

where is the space on the right is the relative moduli stack of *stable pairs*. The image of above embedding we denote by $\mathrm{P}^\epsilon_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N})$, which can also be described more explicitly in terms of stable pairs just as in the case of a relative Hilbert scheme, Section 2.3.1. For properness we need the following lemma, whose proof is, however, different from the one of the standard heart.

Lemma 2.3.6. *Let $\mathcal{C} \to \Delta$ be a family of nodal curves and $\{p_i\} \subset \mathcal{C}$ be finitely many closed in the regular locus of the central fiber. Then any quasimap $\tilde{u}\colon \tilde{\mathcal{C}} = \mathcal{C} \setminus \{p_i\} \to \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$ extends to $u\colon \mathcal{C} \to \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$, which is unique up to unique isomorphism.*

Proof. Employing the similar proof as the one of Lemma 2.2.15 is problematic in this case, as we do not know how to extend the objects, so we follow a different strategy.

Restricting $\tilde{u}$ to the generic fiber $\mathcal{C}^{\circ}$ of $\mathcal{C}$ over Δ, we obtain a relative family F° on $S \times \mathcal{C}^{\circ}$, which by properness of the relative moduli of stable pairs $\mathrm{P}(S \times \mathcal{C}/\Delta)$ can be completed to a family F on $S \times \mathcal{C}$, it may only be non-flat over nodes of the central fiber, therefore it defines a rational quasimap $u\colon \mathcal{C} \dashrightarrow \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$ possibly with indeterminacies only at the nodes of the central fiber. It also defines a rational map $u_{rat}\colon \mathcal{C} \dashrightarrow S^{[n]}$, so does $\tilde{u}$, $\tilde{u}_{rat}\colon \mathcal{C} \dashrightarrow S^{[n]}$, the corresponding graphs in $\mathrm{Hilb}(S^{[n]} \times \mathcal{C})$ agree generically, therefore by separateness of Hilbert schemes they are equal, i.e. $u_{rat} = \tilde{u}_{rat}$. If p_i is not a limit of base points of $\tilde{u}$, then there is a neighbourhood $U \subset \mathcal{C}$ around p_i, where

$$\tilde{u}_{|U/p_i} = \tilde{u}_{rat|U/p_i} = u_{rat|U/p_i} = u_{|U/p_i},$$

we then define $\tilde{u}_{|U} = u_{|U}$ (u is defined at p_i, because p_i is in the regular locus), since quasimaps to $\mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$ do not have any internal automorphisms we can glue maps in a unique way thereby extending $\tilde{u}$ to p_i. If p_i is an limit of base points of $\tilde{u}$, let $B_i \subset \mathcal{C}$ be the section corresponding to these base points, then there is some neighbourhood U around B_i, such that

$$\tilde{u}_{|U/B_i} = \tilde{u}_{rat|U/B_i} = u_{rat|U/B_i} = u_{|U/B_i},$$

but since $\tilde{u}_{|\mathcal{C}^{\circ}} = u_{|\mathcal{C}^{\circ}}$, we conclude that $\tilde{u}_{U/p_i} = u_{U/p_i}$, again because quasimaps to $\mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$ do not have any internal automorphisms and therefore glue in a unique away, we then proceed as before. Let $u'\colon \mathcal{C} \to \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$ be the resulting extension and F' be the associated family, then separateness of relative moduli of stable pairs implies that $F' = F$ and that the extension is unique. $\square$

Summing up the discussion above, we obtain the following result.

Corollary 2.3.7. *The moduli stack $Q^{\epsilon}_g(S^{[n]}, \beta)^{\sharp}$ is a proper Deligne–Mumford stack, and there exists a natural isomorphism of the moduli stacks*

$$Q^{\epsilon}_{g,N}(S^{[n]}, \beta)^{\sharp} \cong \mathrm{P}^{\epsilon}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}),$$

the stack on the right is the relative moduli stack of stable pairs, satisfying exactly the same conditions as in the case of the standard heart.

Proof. For the properness we use Lemma 2.3.6 and the proof presented in [CKM14, Proposition 4.3.1.]. $\square$

Remark 2.3.8. As in the case of the standard heart, the moduli space of perverse 0^+-stable quasimaps from a fixed smooth non-rational curve C without identifications by automorphisms in a non-zero class is just the moduli of all stable pairs on a threefold $S\times C$, as torsion-free implies flatness in this setting too by [AP06],

$$Q_C(S^{[n]},\beta)^\sharp \cong \mathrm{P}_{n,\check{\beta}}(S\times C),$$

while the moduli of perverse 0^+-stable quasimaps with one fixed marked point $p \in C$ is a moduli of stable pairs relative to a vertical divisor $S_p \subset S\times C$,

$$Q_{(C,p)}(S^{[n]},\beta)^\sharp \cong \mathrm{P}_{n,\check{\beta}}(S\times C/S_p).$$

2.3.3 Affine plane

A punctorial Hilbert scheme of the affine plane $\mathbb{C}^2$ admits two equivalent descriptions, one is a Nakajima variety of a quiver, which is a GIT construction,

$$(\mathbb{C}^2)^{[n]} = [\mu^{-1}(0)/\mathrm{GL}_n]^s \subset [\mu^{-1}(0)/\mathrm{GL}_n],$$

for the notation see [Gin12]. Another one is a moduli of framed sheaves on $\mathbb{P}^2$. Both descriptions sit in some bigger stack, but to match the unstable loci, one has to consider a non-standard heart of $D^b(\mathbb{P}^2)$, namely $\mathrm{Coh}^\sharp(\mathbb{P}^2)$, then

$$(\mathbb{C}^2)^{[n]} \subset \mathfrak{Coh}^\sharp(\mathbb{P}^2, l^\infty)_{\mathbf{v}},$$

where on the right we consider framings with respect to the line at infinity, which in this case just kills $\mathbb{C}^*$-automorphisms. By [BFG06, Theorem 5.7] we have a canonical isomorphism

$$[\mu^{-1}(0)/\mathrm{GL}_n] \cong \mathfrak{Coh}^\sharp(\mathbb{P}^2, l^\infty)_{\mathbf{v}},$$

which identifies stable loci on both sides. Therefore the GIT quasimap moduli and perverse-coherent-sheaves quasimap moduli of $(\mathbb{C}^2)^{[n]}$ are isomorphic,

$$Q^{0+}_{g,N}((\mathbb{C}^2)^{[n]},\beta)^{\mathrm{GIT}} \cong Q^{0+}_{g,N}((\mathbb{C}^2)^{[n]},\beta)^\sharp.$$

Moreover, since $[\mu^{-1}(0)/GL_n]$ is l.c.i, an easy check of virtual dimensions shows that the obstruction theory on $\mathfrak{Coh}^\sharp(\mathbb{P}^2, l^\infty)$, given by

$$\mathcal{H}om_\pi(\mathcal{F}, \mathcal{F})_0[1]^\vee \to \mathbb{L}_{\mathfrak{Coh}^\sharp(\mathbb{P}^2, l^\infty)},$$

is an isomorphism, where $\mathcal{F}$ is the universal complex and $\mathbb{L}_{\mathfrak{Coh}^\sharp(\mathbb{P}^2, l^\infty)}$ is the *truncated* cotangent complex. Therefore the obstruction theories of both quasimap theories also match, (see Section 2.4.2 for the construction of the obstruction theory for perverse quasimaps). To match other ϵ-stabilities, one would need to check that the naturally defined line bundles of both stacks agree. However, we will not be concerned with it here, since ϵ-stability is mostly an auxiliary tool to do the wall-crossing between $\epsilon = 0^+$ and $\epsilon = \infty$ chambers, and the identification above is enough to conclude that the wall-crossing is the same in both cases.

2.4 Obstruction theory

2.4.1 Preparation

From now on we fix a class $u \in K_0(S)$, such that

$$\chi(\mathbf{v} \cdot u) = 1,$$

to lift quasimaps with a section s_u. For punctorial Hilbert schemes we use the determinant section $s_{\det}$. By a family associated to a quasimap $f: C \to \mathfrak{Coh}_r(S)$ we will mean the one that is obtained from the lift by this fixed section. The content of this chapter applies to the pair $(S^{[n]}, \mathfrak{Coh}_r(S))$ as well as to the pair $(S^{[n]}, \mathfrak{Coh}^\sharp_r(S))$, the arguments are exactly the same for both pairs, hence we will just state and prove everything for $(S^{[n]}, \mathfrak{Coh}_r(S))$.

Lemma 2.4.1. *Let $f: C \to \mathfrak{Coh}(S)$ be a quasimap, then the corresponding family F on $S \times C$ is perfect.*

Proof. Since F is a family of sheaves on a smooth S over C, which is of finite type, there exists a locally free resolution of finite length. □

Let

$$\mathrm{tr}: \mathcal{H}om(F, F \otimes L) \to L$$

be the trace morphism. We define

$$\mathrm{Ext}^i(F, F \otimes L)_0 := \ker H^i(\mathrm{tr}) \text{ for all i.}$$

Proposition 2.4.2. *Let $f\colon C \to \mathfrak{Coh}_r(S)$ be a prestable quasimap. Assume any of the following holds*

(i) $(M, \mathfrak{Coh}_r(S)) = (S^{[n]}, \mathfrak{Coh}_r(S))$ *or*

(ii) *S is a del Pezzo surface or*

(iii) *S is a K3 surface and $g(C) \leq 1$,*

then the corresponding family F satisfies the following

$$\mathrm{Ext}^i(F,F)_0 = 0 \text{ for } i \neq 1, 2.$$

Proof. By Lemma 2.4.1 and by Serre duality we get

$$\mathrm{Ext}^i(F,F) = \mathrm{Ext}^{n-i}(F, F \otimes \omega_{S\times C}),$$

therefore $\mathrm{Ext}^i(F,F) = 0$ for $i \notin [0,3]$, because $S \times C$ is l.c.i. ($\omega_{S\times C}$ is a locally free sheaf). Since F is stable, it is simple, hence $\mathrm{Hom}(F,F)_0 = 0$. We therefore have to show that

$$\mathrm{Hom}(F, F \otimes \omega_{S\times C})_0 = 0.$$

And since the trace morphism has a section given by

$$\mathrm{id}_{\otimes}\colon \omega_{S\times C} \to \mathcal{H}om(F, F \otimes \omega_{S\times C}), \quad s \mapsto \mathrm{id}_F \otimes s$$

after taking cohomology, it is enough to show that $H^0(\mathrm{tr})$ is injective.

(i) Assume that $(M, \mathfrak{Coh}_r(S)) = (S^{[n]}, \mathfrak{Coh}_r(S))$, then F is an ideal sheaf I of a curve $\Gamma \subset X$. Let U be the complement of Γ and

$$\pi\colon S \times \tilde{C} \to S \times C, \quad D \subset S \times C$$

be the normalisation and the singular locus of $S \times C$ respectively, then by applying $H^0(S \times C, -)$ and $H^0(U, -)$ to the exact sequence

$$0 \to \omega_{S\times C} \to \pi_*\omega_{S\times\tilde{C}} \to \omega_{S\times C|D} \to 0$$

we obtain

$$\begin{array}{ccccc}
H^0(S \times C, \omega_{S\times C}) & \hookrightarrow & H^0(S\times C, \pi_*\omega_{S\times\tilde{C}}) & \longrightarrow & H^0(D, \omega_{S\times C|D}) \\
\downarrow & & \| & & \| \\
H^0(U, \omega_{S\times C}) & \longhookrightarrow & H^0(U, \pi_*\omega_{S\times\tilde{C}}) & \longrightarrow & H^0(D\cap U, \omega_{S\times C|D})
\end{array}$$

The last two vertical arrows are bijective, because $\omega_{S\times C}$ is locally free, Γ is of codimension 2, and D intersects properly with Γ. We conclude that

$$H^0(S\times C, \omega_{S\times C}) = H^0(U, \omega_{S\times C}).$$

Finally, since I is torsion free, the restriction of global sections

$$\begin{aligned}\mathrm{Hom}(I, I\otimes\omega_{S\times C}) \to \mathrm{Hom}(I_{|U}, I_{|U}\otimes\omega_{S\times C}) &= H^0(U, \omega_{S\times C})\\ &= H^0(S\times C, \omega_{S\times C})\end{aligned}$$

is injective. Moreover, it is equal to $H^0(\mathrm{tr})$ by the construction of tr, hence the claim follows.

(ii) Assume now that S is a del Pezzo surface, then the degree of a general fiber of $F\otimes\omega_{S\times C}$ is strictly smaller than the degree of a general fiber of F by ampleness of the anti-canonical line bundle of S. Therefore by the stability of a general giber of F we have that

$$\mathrm{Hom}(F, F\otimes\omega_{S\times C}) = 0.$$

(iii) Finally, assume S is a K3 surface. We will show that

$$H^0(\mathrm{id}_\otimes) : H^0(S\times C, \omega_{S\times C}) \to \mathrm{Hom}(F, F\otimes\omega_{S\times C})$$

is surjective. By assumption $\omega_{S\times C}\cong p_C^*\omega_C$, hence we have to show that all morphisms $\phi\colon F\to F\boxtimes\omega_C$ are of the form $\mathrm{id_F}\boxtimes s$ for some $s\in H^0(\omega_C, C)$. By the normalisation sequence it is enough to show it for

$$\pi^*F\to\pi^*F\boxtimes\pi_C^*\omega_C,$$

where

$$\pi = \mathrm{id}\times\pi_C\colon S\times\tilde{C}\to S\times C$$

is the normalisation map. We firstly establish the following result.

Lemma 2.4.3. *Let C be smooth. Given a sheaf F on $S\times C$, that defines a quasimap, and an effective divisor $D=\sum p_i$ on C, then any non-zero morphism*

$$\phi\colon F\to F(D) := F\boxtimes\mathcal{O}_C(D)$$

is an inclusion. Moreover, if all p_i's are distinct, $\mathrm{supp}(\mathrm{coker}(\phi)) = S\times D$ *and F is stable over D, then $\phi=\mathrm{id}_F\boxtimes s$ for some $s\in H^0(\mathcal{O}(D), C)$.*

Proof of Lemma 2.4.3. Assume ϕ is not an inclusion, then the difference of Hilbert polynomials

$$p_{\mathcal{O}_{S\times C}(1,m)}(\mathrm{Im}(\phi)) - p_{\mathcal{O}_{S\times C}(1,m)}(F)$$

increases as n increases, because a general fiber of F is stable. Therefore $\mathrm{Im}(\phi)$ becomes $\mathcal{O}_{S\times C}(1,m)$-destabilising for $F(D)$ for some $m \gg 0$. However, by Lemma 2.2.13 the sheaf $F(D)$ is $\mathcal{O}_{S\times C}(1,m)$-stable for some $m \gg 0$, which is a contradiction. Hence ϕ must be an inclusion.

We now deal with the second part of the lemma. Consider the sequence

$$0 \to F \xrightarrow{\phi} F(D) \to \mathrm{coker}(\phi) \to 0,$$

restricting it to $S \times D$, we obtain

$$0 \to \mathrm{coker}(\phi) \to F_{|D} \xrightarrow{\phi_{|D}} F(D)_{|D} \to \mathrm{coker}(\phi) \to 0,$$

where we used that schematic support of $\mathrm{coker}(\phi)$ is $S \times D$. Since F_{p_i}'s are stable and $F_{|D} \cong F(D)_{|D}$, the map $\phi_{|D}$ must be zero. Therefore the map $F(D)_{|D} \to \mathrm{coker}(\phi)$ is an isomorphism. Consider now the following diagram

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\mathrm{id}_F \boxtimes s_D"] \arrow[d, dashed, "\psi"] & F(D) \arrow[r] \arrow[d, equal] & F(D)_{|D} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r, "\phi"] & F(D) \arrow[r] & \mathrm{coker}(\phi) \arrow[r] & 0,
\end{tikzcd}$$

where $s_D \in H^0(\mathcal{O}(D), C)$ is a defining section of D. The right square of the diagram is commutative, and the last two vertical arrows are isomorphisms, so we have

$$\phi = \psi \boxtimes s_D \text{ for some } \psi \in \mathrm{Aut}(F).$$

But F is stable and therefore simple, hence $\psi = c \cdot \mathrm{id}_F$ for some $c \in \mathbb{C}^*$. The claim now follows. $\square$

Continuation of the proof of Proposition 2.4.2. Recall that there is a natural isomorphism $\pi_C^*\omega_C \cong \omega_{\tilde{C}}(\sum q_i + q_i')$, where q_i and q_i' are preimages of a node of C. Given now a rational component $\tilde{C}_j$ of $\tilde{C}$ with at most two special points, then $\pi_j^*\omega_C \cong \mathcal{O}_{\mathbb{P}^1}(k)$ for $k \leq 0$. Both π_j^*F and $F_{|\tilde{C}_j} \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)$ are $\mathcal{O}_{S\times C}(1,m)$-stable for some $m \gg 0$ by Lemma 2.2.13. If $k < 0$, then Hilbert polynomials satisfy

$$p_{\mathcal{O}_{S\times C}(1,m)}(\pi_j^*F) > p_{\mathcal{O}_{S\times C}(1,m)}(\pi_j^*F \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)),$$

hence

$$\mathrm{Hom}(\pi_j^* F, \pi_j^* F \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)) = 0.$$

If $k = 0$, then $\pi_j^* F \cong \pi_j^* F \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)$. By induction we then conclude that the restriction of ϕ to all rational trees must be zero, and by the previous lemma the restriction of ϕ to their complement comes from box-tensoring a section. $\square$

Remark 2.4.4. All quasimaps are prestable in the case of punctorial Hilbert schemes, since an ideal I of a curve on a threefold $S \times C$ is stable over a node $s \in C$, if and only if it is flat over the node [5]. This can be seen as follows. The sheaf I_s is stable, if and only if it is torsion-free, which is equivalent to the injectivity on the left of the exact sequence

$$I_s \to \mathcal{O}_{S \times s} \to \Gamma_s \to 0,$$

which in turn is equivalent to $\mathrm{Tor}^1_{S \times C}(\mathcal{O}_\Gamma, \mathcal{O}_{S \times s}) = 0$, but by standard periodic resolution of a structure sheaf of a node,

$$\mathrm{Tor}^k_{S \times C}(\mathcal{O}_\Gamma, \mathcal{O}_{S \times s}) = \mathrm{Tor}^1_{S \times C}(\mathcal{O}_\Gamma, \mathcal{O}_{S \times s}) \text{ for all } k \geq 1.$$

If I is flat, then I is perfect, hence $\mathcal{O}_\Gamma$ is also perfect, so

$$\mathrm{Tor}^k_{S \times C}(\mathcal{O}_\Gamma, \mathcal{O}_{S \times s}) = 0 \text{ for some } k \gg 0,$$

which therefore implies that $\mathrm{Tor}^1_{S \times C}(\mathcal{O}_\Gamma, \mathcal{O}_{S \times s}) = 0$.

2.4.2 Obstruction theory

In what follows all the functors are derived. We have the following perfect obstruction theory over a substack of finite type $\mathcal{U} \subset \mathfrak{Coh}_r(S)$,

$$(\mathbb{T}^{\mathrm{vir}})^\vee := (\mathcal{H}om_\pi(\mathcal{F}_{r|\mathcal{U}}, \mathcal{F}_{r|\mathcal{U}})_0[1])^\vee \to \mathbb{L}_\mathcal{U},$$

where $\mathcal{F}_r$ is the universal family on $S \times \mathfrak{Coh}_r(S)$, note that the complex $(\mathbb{T}^{\mathrm{vir}})^\vee$ is of amplitude [-1,1] due to the presence of non-discrete automorphisms of the unstable part of $\mathfrak{Coh}_r(S)$. Let

$$\pi_1 \colon \mathcal{C}_{g,N} \to Q^\epsilon_{g,N}(M, \beta)$$

$$\mathbb{f} \colon \mathcal{C}_{g,N} \to \mathfrak{Coh}_r(S),$$

[5] In Donaldson–Thomas theory this condition is referred to as *predeformable*.

be the canonical projection from the universal curve and the universal map. The universal map $\mathbb{f}$ factors through some substack of finite type, hence we can define the obstruction complex $(\pi_*\mathbb{f}^*\mathbb{T}^{\mathrm{vir}})^\vee$. Let us show how it is related to obstruction-theory complex of a relative moduli of stable sheaves. Let

$$\pi_2 \colon S \times \mathcal{C}_{g,N} \times Q^\epsilon_{g,N}(M,\beta) \to Q^\epsilon_{g,N}(M,\beta),$$
$$\mathbb{F} \in \mathrm{Coh}(S \times \mathcal{C}_{g,N} \times Q^\epsilon_{g,N}(M,\beta))$$

be the canonical projection and the universal sheaf, which is defined via the identifation $Q^\epsilon_{g,N}(M,\beta) \cong M^\epsilon_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$. We then take the traceless part of the relative derived self-hom complex

$$\mathcal{H}om_{\pi_2}(\mathbb{F},\mathbb{F})_0[1],$$

and prove the following.

Proposition 2.4.5. *The complex* $(\pi_{1*}\mathbb{f}^*\mathbb{T}^{\mathrm{vir}})^\vee$ *is canonically isomorhpic to the complex* $(\mathcal{H}om_{\pi_2}(\mathbb{F},\mathbb{F})_0[1])^\vee$.

Proof. Consider the following diagram

$$\begin{tikzcd}
S \times \mathcal{C}_{g,N} \times Q^\epsilon_{g,N}(M,\beta) \arrow[r, "\mathrm{id}\times\mathbb{f}"] \arrow[d] \arrow[dd, bend right, "\pi_2"'] & S \times \mathcal{U} \arrow[d, "\pi_{\mathcal{U}}"] \\
S \times \mathcal{C}_{g,N} \times Q^\epsilon_{g,N}(M,\beta) \arrow[r, "\mathbb{f}"] \arrow[d, "\pi_1"] & \mathcal{U} \\
Q^\epsilon_{g,N}(M,\beta) &
\end{tikzcd}$$

the trace map $\mathrm{tr}\colon \mathcal{H}om(\mathcal{F}_{r|\mathcal{U}},\mathcal{F}_{r|\mathcal{U}}) \to \mathcal{O}_{\mathcal{U}}$ has a section given by the inclusion of identity $\mathcal{O}_{\mathcal{U}} \to \mathcal{H}om(\mathcal{F}_{r|\mathcal{U}},\mathcal{F}_{r|\mathcal{U}})$, therefore

$$\mathcal{H}om(\mathcal{F}_{r|\mathcal{U}},\mathcal{F}_{r|\mathcal{U}}) = \mathcal{H}om(\mathcal{F}_{r|\mathcal{U}},\mathcal{F}_{r|\mathcal{U}})_0 \oplus \mathcal{O}_{\mathcal{U}},$$

and by the moduli problem of $\mathfrak{Coh}_r(S)$ we get

$$(\mathbb{f} \times \mathrm{id})^*\mathcal{F}_r = \mathbb{F},$$

hence by functoriality of the trace and the splitting above we obtain that

$$(\mathbb{f} \times \mathrm{id})^*\mathcal{H}om(\mathcal{F}_{r|\mathcal{U}},\mathcal{F}_{r|\mathcal{U}})_0 = \mathcal{H}om(\mathbb{F},\mathbb{F})_0,$$

and by base change theorem

$$\mathcal{H}om_{\pi_2}(\mathbb{F},\mathbb{F})_0 = \pi_{1*}\mathbb{f}^*\mathcal{H}om_{\pi_{\mathcal{U}}}(\mathcal{F}_{r|\mathcal{U}},\mathcal{F}_{r|\mathcal{U}})_0.$$

$\square$

Corollary 2.4.6. *There exists an obstruction theory*

$$\phi\colon (\pi_{1*}\mathfrak{f}^*\mathbb{T}^{\mathrm{vir}})^\vee \to \mathbb{L}_{Q^{\epsilon}_{g,N}(M,\beta)/\mathfrak{M}_{g,N}},$$

which is is perfect under the assumptions of Proposition 2.4.2. Moreover, for $\epsilon = 0^+$ the corresponding virtual fundamental classes coincide with those of Donaldson–Thomas theory.

Proof. Using the results of [TV08] and [STV15], the stack $\mathfrak{Coh}_r(S)$ can be naturally upgraded to a derived stack $\mathbb{R}\mathfrak{Coh}_r(S)$ whose truncation is $\mathfrak{Coh}_r(S)$,

$$\tau_{\leq 0}\mathbb{R}\mathfrak{Coh}_r(S) = \mathfrak{Coh}_r(S),$$

and

$$\mathbb{L}_{\mathbb{R}\mathfrak{Coh}_r(S)} = (\mathbb{T}^{\mathrm{vir}})^\vee.$$

The obstruction theory

$$\phi\colon (\pi_{1*}\mathfrak{f}^*\mathbb{T}^{\mathrm{vir}})^\vee \to \mathbb{L}_{Q^{\epsilon}_{g,N}(M,\beta)/\mathfrak{M}_{g,N}}$$

is therefore given by the obstruction theory of maps to the derived stack $\mathbb{R}\mathfrak{Coh}_r(S)$, using the construction of a derived mapping stack of [TV08]. It is perfect by Proposition 2.4.5 and Proposition 2.4.2 .

By [Sie04] a virtual fundamental class depends only on Chern characters of the corresponding obstruction-theory complex. The second part of the claim therefore follows from Proposition 2.4.5. □

Let

$$[Q^{\epsilon}_{g,N}(M,\beta)]^{\mathrm{vir}} \in A_{\mathrm{vdim}}(Q^{\epsilon}_{g,N}(M,\beta))_{\mathbb{Q}}$$

be the associated virtual fundamental class. Invoking the identification presented in Lemma 2.2.2, the virtual dimension can be computed via the virtual dimension of the relative moduli stack of sheaves,

$$\begin{aligned}\mathrm{vdim} &= \sum (-1)^i \dim \mathrm{Ext}^i(F,F)_0 + (3g-3) + N \\ &= \int_{S\times C} (\mathrm{ch}(F)\cdot \mathrm{ch}(F)^\vee - 1)\cdot \mathrm{td}_{S\times C} + (3g-3) + N \\ &= \mathrm{rk}(\mathbf{v})\mathrm{c}_1(\check{\beta})\cdot \mathrm{c}_1(S) - \mathrm{rk}(\check{\beta})\mathrm{c}_1(\mathbf{v})\cdot \mathrm{c}_1(S) + (\dim(M)-3)(1-g) + N,\end{aligned}$$

where $\mathrm{rk}(\check{\beta})$ and $\mathrm{c}_1(\check{\beta})$ are the components of $\check{\beta} \in \Lambda$ of cohomogical degrees 0 and 2 respectively.

By our definition of a degree β, it can only pair with determinant line bundles on the stack $\mathfrak{Coh}_r(S)$, and it is unclear, if the virtual anti-canonical line bundle is a determinant line bundle, even though it is the case over the stable locus in some very special instances. Therefore the above formula for the virtual dimension is the most reasonable one. We will treat the first two summands as the degree with respect to the virtual anti-canonical line bundle,

$$\beta(\det(\mathbb{T}^{\mathrm{vir}})) := \mathrm{rk}(\mathbf{v})\mathrm{c}_1(\check{\beta}) \cdot \mathrm{c}_1(S) - \mathrm{rk}(\check{\beta})\mathrm{c}_1(\mathbf{v}) \cdot \mathrm{c}_1(S).$$

The above formula is, however, dependent upon presentation of $Q^{\epsilon}_{g,N}(M,\beta)$ as a relative moduli space of sheaves, the virtual dimension itself is not though.

2.4.3 Invariants

The moduli $Q^{\epsilon}_{g,N}(M,\beta)$ has the usual canonical structures to define the enumerative invariants:

- evaluation maps at marked points

$$ev_i \colon Q^{\epsilon}_{g,N}(M,\beta) \to M, \quad i = 1,\dots,N$$

- cotangent line bundles

$$\mathcal{L}_i := s_i^*(\omega_{\mathcal{C}_{g,N}/Q^{\epsilon}_{g,N}(M,\beta)}), \quad i = 1,\dots,N$$

where $s_i \colon Q^{\epsilon}_{g,N}(M,\beta) \to \mathcal{C}_{g,N}$ are universal markings. We denote

$$\psi_i := \mathrm{c}_1(\mathcal{L}_i), \quad i = 1,\dots,N$$

Definition 2.4.7. The *descendent ϵ-invariants* are

$$\langle \tau^{m_1}(\gamma_1),\dots,\tau^{m_N}(\gamma_N)\rangle^{\epsilon}_{g,N,\beta} := \int_{[Q^{\epsilon}_{g,N}(M,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{i=N} \psi_i^{m_i}\mathrm{ev}_i^*(\gamma_i,),$$

where $\gamma_1,\dots,\gamma_N \in H^*(M,\mathbb{Q})$ and $m_1,\dots m_N$ are non-negative integers. We similarly define the perverse invariants $\langle \tau^{m_1}(\gamma_1),\dots,\tau^{m_N}(\gamma_N)\rangle^{\sharp,\epsilon}_{g,N,\beta}$.

Remark 2.4.8. We can also define another kind of invariants by the identification of quasimaps with the relative moduli of sheaves - relative Donaldson–Thomas descendent invariants (do not confuse with invariants relative to divisors), consider

$$\begin{tikzcd} & S \times \mathcal{C}_{g,N} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S \times \overline{M}_{g,N+1} & & Q^{\epsilon}_{g,N}(M,\beta) \end{tikzcd}$$

where for the π_1 we stabilise the curves and used the identification of $\overline{M}_{g,N+1}$ with the universal curve of $\overline{M}_{g,N}$, for the unstable values of g and N we set the product $S \times \overline{M}_{g,N+1}$ to be S. For a class $\bar{\gamma} \in H^*(S \times \overline{M}_{g,N+1}, \mathbb{Q})$ define the following operation on cohomology,

$$\mathrm{ch}_{k+2}(\bar{\gamma})\colon H_*(Q^{\epsilon}_{g,N}(M,\beta),\mathbb{Q}) \to H_{*-2k+2-\ell}(Q^{\epsilon}_{g,N}(M,\beta),\mathbb{Q}),$$

$$\mathrm{ch}_{k+2}(\bar{\gamma})(\xi) = \pi_{2*}\left(\mathrm{ch}_{k+2}(\mathbb{F}) \cdot \pi_1^*(\bar{\gamma}) \cap \pi_2^*(\xi)\right).$$

The relative descendent invariants are then defined by

$$\begin{aligned}\langle \tilde{\tau}_{k_1}(\bar{\gamma}_1), \dots, \tilde{\tau}_{k_r}(\bar{\gamma}_r)\rangle^{\epsilon}_{g,n,\beta} \\ = (-1)^{k_1}\mathrm{ch}_{k_1+2} \ \circ \ \dots \ \circ \ (-1)^{k_r}\mathrm{ch}_{k_r+2}\left([Q^{\epsilon}_{g,N}(M,\beta)]^{\mathrm{vir}}\right),\end{aligned}$$

here we just transferred the definitions from rank-1 story, note that for higher ranks $\tilde{\tau}_{-1}(-)$ in the notation above might also be non-trivial. We can also define the mix of descendent GW invariants and relative DT invariants,

$$\langle \tilde{\tau}_{k_1}(\bar{\gamma}_1), \dots, \tilde{\tau}_{k_r}(\bar{\gamma}_r) \mid \tau_{m_1}(\gamma_1), \dots, \tau_{m_N}(\gamma_N)\rangle^{\epsilon}_{g,N,\beta},$$

which are essentially a mix of relative and absolute DT invariants of the relative geometry

$$S \times C_{g,N} \to \overline{M}_{g,N}$$

for different ϵ-stabilities. However, we will not be concerned with any of the DT-type invariants defined above in the present work.

The discussion in [CKM14, Section 6] also applies to ϵ-invariants in our setting. In particular, ϵ-invariants satisfy an analogue of the Splitting Axiom in Gromov–Witten theory, and there exists a projection to the moduli of stable nodal curves

$$p\colon Q^{\epsilon}_{g,N}(M,\beta) \to \overline{M}_{g,N}$$

by taking stabilisation of the domain of a quasimap, so that the classes

$$p_*(\prod_{i=1}^{i=N} \psi_i^{m_i} ev_i^*(\gamma_i,)) \in H^*(\overline{M}_{g,N},\mathbb{Q})$$

gives rise to Cohomological Field theory on $H^*(M,\mathbb{Q})$.

2.5 Wall-crossing

2.5.1 Graph space

As previously, all the results of this section apply both to standard and perverse quasimaps, if $M = S^{[n]}$. In the latter case all the notations obtain the superscript '$\sharp$'.

Given $\beta \in \mathrm{Eff}(M, \mathfrak{Coh}_r(S))$, let $\epsilon \in \mathbb{R}_{>0}$ and $k \in \mathbb{Z}_{>0}$ be such that $1/k < \epsilon < 1/\deg \beta$, then we define the *graph space*

$$QG_{0,1}(M,\beta) := Q^{\epsilon}_{0,1}(M \times \mathbb{P}^1, \beta + [\mathbb{P}^1]),$$

where we consider quasimaps to $\mathfrak{Coh}_r(S) \times \mathbb{P}^1$ and ϵ-stability on the right is given with respect to $\mathcal{L}_\beta \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)$. This is the moduli space of genus-0 quasimaps, whose domain has a unique parametrised rational tail, such that the restriction of the quasimap to its complement satisfies ϵ-stability, which is equivalent to 0^+-stability by the choice of ϵ. The definition is independent of ϵ and k, as long as they satisfy the inequality above.

The obstruction theory of $QG_{0,1}(M,\beta)$ is given by

$$(R\pi_* \mathfrak{f}^*(\mathbb{T}^{\mathrm{vir}} \boxplus T_{\mathbb{P}^1}))^\vee \to \mathbb{L}_{QG_{0,1}(M,\beta)/\mathfrak{M}_{0,1}}.$$

There is a $\mathbb{C}^*$-action on $\mathbb{P}^1$ given by

$$t[x,y] = [tx,y], \quad t \in \mathbb{C}^*,$$

the fixed points of this action must have their entire degrees with the marking lie over either 0 or ∞ in the form of rational components or base points . Assuming the marking is over ∞, there are two distinguished extremal fixed components

$$F_\beta \text{ and } F^{0,0}_{1,\beta} \cong Q^{0^+}_{0,1+\bullet}(M,\beta).$$

The former is the locus of quasimaps with entire degree β over 0 as a base point, while the latter is the locus of quasimaps with entire degree over ∞ in the form of rational components. If the degree splits non-trivially between 0 and ∞, then the fixed components are of the following form

$$F^{0,\beta_1}_{1,\beta_2} := F_{\beta_1} \times_M F^{0,0}_{1,\beta_2}, \tag{2.15}$$

where $\beta = \beta_1 + \beta_2$ and the fibered product is taken with respect to distinguished markings. The description of fixed components $F^{1,\beta_1}_{0,\beta_2}$ with the marking over 0 is exactly the same. The virtual fundamental classes $[F^{0,\beta_1}_{1,\beta_2}]^{\mathrm{vir}}$

and the virtual normal bundles $N^{\mathrm{vir}}_{F^{0,\beta_1}_{1,\beta_2}/QG_{0,1}(M,\beta)}$ are defined by fixed and moving parts of the obstruction theory of $QG_{0,1}(M,\beta)$. They are compatible with respect to the product expression above,

$$[F^{0,\beta_1}_{1,\beta_2}]^{\mathrm{vir}} = [F_{\beta_1}]^{\mathrm{vir}} \times_M [F^{0,0}_{1,\beta_2}]^{\mathrm{vir}},$$
$$N^{\mathrm{vir}}_{F^{0,\beta_1}_{1,\beta_2}/QG_{0,1}(M,\beta)} = N^{\mathrm{vir}}_{F_{\beta_1}/QG_{0,1}(M,\beta)} \boxtimes_M N^{\mathrm{vir}}_{F^{0,0}_{1,\beta_2}/QG_{0,1}(M,\beta)}.$$

Let

$$\mathsf{ev}\colon F_\beta \to M$$

be the evaluation map at the unique marking at $\infty \in \mathbb{P}^1$.

Definition 2.5.1. We define *I-function*

$$I(q,z) = 1 + \sum_{\beta>0} -zq^\beta \mathrm{ev}_* \left(\frac{[F_\beta]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta/QG_{0,1}(M,\beta)})} \right) \in A^*(M)[z^{\pm}] \otimes_{\mathbb{Q}} \mathbb{Q}[\![q^\beta]\!],$$

by convention $e_{\mathbb{C}^*}(\mathbb{C}_{\mathrm{std}}) = -z$, where $\mathbb{C}_{\mathrm{std}}$ is the standard representation of $\mathbb{C}^*$. We also define

$$\mu(z) := [zI(q,z) - z]_+ \in A^*(M)[z] \otimes_{\mathbb{Q}} \mathbb{Q}[\![q^\beta]\!]$$

where $[\dots]_+$ is the truncation by taking only non-negative powers of z. Let

$$\mu_\beta(z) \in A^*(M)[z]$$

be the coefficients of q^β in $\mu(z)$.

2.5.2 Graph space and sheaves

There is a forgetful morphism

$$QG_{0,1}(M,\beta) \to \overline{M}_{0,1}(\mathbb{P}^1, 1) \tag{2.16}$$

which is given by projecting a quasimap to its parametrised component, the graph space $QG_{0,1}(M,\beta)$ then admits a relative perfect obstruction

$$(R\pi_* \mathbb{f}^* \mathbb{T}^{\mathrm{vir}})^\vee \to \mathbb{L}_{QC_{0,1}(M,\beta)/\overline{M}_{0,1}(\mathbb{P}^1,1)},$$

which sits in a distinguished triangle

$$\mathbb{L}_{\overline{M}_{0,1}(\mathbb{P}^1,1)} \to \mathbb{E}_{QG_{0,1}(M,\beta)} \to (R\pi_* \mathbb{f}^* \mathbb{T}^{\mathrm{vir}})^\vee \to .$$

Restricting the sequence above to the fixed component $F_{\check{\beta}}$, we obtain that the morphism

$$\mathbb{E}^{\mathrm{f}}_{QG_{0,1}(M,\beta)} \to (R\pi_* \mathbb{f}^* \mathbb{T}^{\mathrm{vir}})^{\vee,\mathrm{f}}$$

between fixed parts is an isomorphism and

$$e_{\mathbb{C}^*}((R\pi_* \mathbb{f}^* \mathbb{T}^{\mathrm{vir}})^{\vee,\mathrm{mv}}) = -z e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta/QG_{0,1}(M,\beta)}),$$

because the restriction of $\mathbb{L}_{\overline{M}_{0,1}(\mathbb{P}^1,1)}$ is a trivial line bundle with the fiber being the cotangent space of $\mathbb{P}^1$ at ∞, which is not fixed and whose Euler class is equal to $-z$. Consider now the component

$$QG_{0,p_\infty}(M,\beta) \subset QG_{0,1}(M,\beta)$$

of quasimaps, whose marking is over ∞. In other words, this is the fiber of (2.16) over ∞. Then applying the identification of quasimaps with sheaves, we obtain

$$QG_{0,p_\infty}(M,\beta) \cong M_{\check{\beta},u}(S\times\mathbb{P}^1/S_\infty),$$

such that obstruction theory $(R\pi_* F^* \mathbb{T}^{\mathrm{vir}})^\vee_{|QG_{0,p_\infty}(M,\beta)}$ matches the relative Donaldson–Thomas obstruction theory. By the discussion above for all purposes the graph space can be replaced by $M_{\check{\beta},u}(S\times\mathbb{P}^1/S_\infty)$. The fixed component $F_\beta \subset M_{\check{\beta},u}(S\times\mathbb{P}^1/S_\infty)$ can then be expressed in terms of flags of sheaves on S by invoking the identifications between flags of sheaves and $\mathbb{C}^*$-equivariant sheaves on $S\times\mathbb{C}$.

2.5.3 Master space and wall-crossing

For the material discussed in this section we refer the reader to [Zho22]. Here we just glide over the machinery developed there, adjusting some minor details to our needs.

The space $\mathbb{R}_{>0}\cup\{0^+,\infty\}$ of ϵ-stabilities is divided into chambers, in which the moduli $Q^\epsilon_{g,N}(M,\beta)$ stays the same, and as ϵ crosses the a wall between chambers, the moduli changes discontinuously. Let $\epsilon_0 = 1/d_0$ be a wall for a given $\beta\in\mathrm{Eff}(M,\mathfrak{Coh}_r(S))$ and ϵ^-, ϵ^+ be some values that are close to ϵ_0 from left and right of the wall respectively. Assuming $2g-2+N+\epsilon_0\deg(\beta) > 0$, let

$$MQ^{\epsilon_0}_{g,N}(M,\beta) \to M\widetilde{\mathfrak{M}}_{g,N}$$

be the master space with the projection to the moduli of curves with calibrated tails constructed in [Zho22], the construction is carried over to our set-up varbatim. The space $M\widetilde{\mathfrak{M}}_{g,N}$ is a $\mathbb{P}^1$-bundle over $\widetilde{\mathfrak{M}}_{g,N}$, the latter is obtained by a series of blow-ups of a moduli space of semistable curves weighted by degree, $\mathfrak{M}^{\mathrm{ss}}_{g,N,d}$, with total degree $d = \deg(\beta)$. As in GIT case the following holds.

Theorem 2.5.2. *$MQ^{\epsilon_0}_{g,N}(M,\beta)$ is a proper Deligne–Mumford stack.*

Proof. With Lemma 2.2.15 the proof is exactly the same as in GIT case, we therefore refer to [Zho22, Section 5]. □

The master space also carries a perfect obstruction theory, which is obtained in the same way as the one for $Q^{\epsilon}_{g,N}(M,\beta)$. Let

$$\mathbb{f}\colon MQ^{\epsilon_0}_{g,N}(M,\beta)\times_{M\widetilde{\mathfrak{M}}_{g,N,d}}\mathfrak{C}\to \mathfrak{Coh}_r(S),$$

$$\pi\colon MQ^{\epsilon_0}_{g,N}(M,\beta)\times_{M\widetilde{\mathfrak{M}}_{g,N,d}}\mathfrak{C}\to MQ^{\epsilon_0}_{g,N}(M,\beta)$$

be the universal quasimap and the canonical projection, then we have a relative perfect obstruction theory over $M\widetilde{\mathfrak{M}}_{g,n}$

$$\phi\colon \mathbb{E}^\bullet = (\pi_*\mathbb{f}^*\mathbb{T}^{\mathrm{vir}})^\vee \to \mathbb{L}_{MQ^{\epsilon_0}_{g,N}(M,\beta)/M\widetilde{\mathfrak{M}}_{g,N}},$$

which is constructed via the same identification as in Proposition 2.4.5. Using the master space we can establish the following result.

Theorem 2.5.3. *Assuming $2g-2+N+\epsilon_0\deg(\beta)>0$, we have*

$$\begin{aligned}&\langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon^-}_{g,N,\beta}-\langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon^+}_{g,N,\beta}\\ &=\sum_{k\geq 1}\sum_{\vec{\beta}}\frac{1}{k!}\int_{[Q^{\epsilon^+}_{g,N+k}(M,\beta')]^{\mathrm{vir}}}\prod_{i=1}^{i=N}\psi_i^{m_i}ev_i^*(\gamma_i)\cdot\prod_{a=1}^{a=k}ev^*_{N+a}\mu_{\beta_a}(z)|_{z=-\psi_{N+a}}\end{aligned}$$

where $\vec{\beta}$ runs through all the $(k+1)$-tuples of effective curve classes

$$\vec{\beta}=(\beta',\beta_1,\dots,\beta_k),$$

such that $\beta=\beta'+\beta_1+\dots+\beta_k$ and $\deg(\beta_i)=d_0$ for all $i=1,\dots,k$, and ϵ_+-stability for the class β' is given by $\mathcal{L}_\beta$. The same holds for perverse quasimap invariants $\langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\sharp,\epsilon}_{g,N,\beta}$.

Sketch of the proof. Here we will sketch the proof, for all the details we refer to [Zho22, Section 6], as the proof in our case is exactly the same as the one for GIT.

The master space $MQ^{\epsilon_0}_{g,N}(M,\beta)$ carries a natural $\mathbb{C}^*$-action, such that up to finite coverings the fixed loci are following three types of spaces:

- $Q^{\epsilon^-}_{g,N}(M,\beta)$;
- $\widetilde{Q}^{\epsilon^+}_{g,N}(M,\beta)$, base change of $Q^{\epsilon^+}_{g,N}(M,\beta)$ from $\mathfrak{M}_{g,N,d}$ to $\widetilde{\mathfrak{M}}_{g,N}$;
- $Y \times_{M^k} \prod_{i=1}^k F_{\beta_i}$, a finite gerbe over $\widetilde{Q}^{\epsilon^+}_{g,N+k}(M,\beta') \times_{M^k} \prod_{i=1}^k F_{\beta_i}$.

Applying the virtual localisation formula and the taking equivariant residue, we obtain certain relations between the classes associated to the spaces above. Projecting everything to a point, we get the wall-crossing formula. All the effort goes into the careful construction of the master space and the analysis of moving and fixed parts of the obstruction theories at fixed loci. The latter task can be separated into two independent parts by splitting the restriction of the absolute obstruction theory $\mathbb{E}^\bullet_{MQ|F}$ of the master space to a fixed locus F (one of the spaces above) into the relative obstruction theory $\mathbb{E}^\bullet_{|F}$ and the restriction cotangent complex $\mathbb{L}_{M\widetilde{\mathfrak{M}}_{g,N,d}|F}$ of the moduli of calibrated curves,

$$\mathbb{L}_{M\widetilde{\mathfrak{M}}_{g,N,d}|F} \to \mathbb{E}^\bullet_{MQ|F} \to \mathbb{E}^\bullet_{|F} \to,$$

the analysis of $\mathbb{L}_{M\widetilde{\mathfrak{M}}_{g,N,d}|F}$ presented in [Zho22] is completely independent of what kind of quasimaps one considers, while the analysis of $\mathbb{E}^\bullet_{|F}$ does not use any special feature of the GIT set-up. For more details we refer the reader to [Zho22, Section 6]. □

Remark 2.5.4. In the GIT set-up there are naturally defined maps $[W/G] \to [\mathbb{C}^{n+1}/\mathbb{C}^*]$, which induce $Q^{\epsilon}_{g,N}(W/G,\beta) \to Q^{\epsilon}_{g,N}(\mathbb{P}^n,d)$. This allows to give a more refined class-valued wall-crossing by pushforwarding the classes on $MQ^{\epsilon_0}_{g,N}(W/G,\beta)$ to $Q^{\epsilon^-}_{g,N}(\mathbb{P}^n,d)$ instead of a point. In our case this seems to be less natural. Even though $\mathfrak{Coh}_r(S)$ is Zariski-locally a GIT stack, we do not have these naturally defined maps, because it is unclear, if line bundles $\mathcal{L}_\beta$'s are actually ample on any of the GIT loci through which the universal quasimap factors. Moreover, for different β these loci change.

It is also possible to pushforward the classes to $\overline{M}_{g,N}$ instead of $Q^{\epsilon}_{g,N}(\mathbb{P}^n,d)$. The problem with this approach is that the projection

$$Q^{\epsilon}_{g,N+k}(M,\beta) \to \overline{M}_{g,N}$$

involves stabilisation of a curve, which implies that ψ-classes do not pullback to ψ-classes. Consequently, the wall-crossing formula becomes inefficient to state.

Since our ϵ-stability depends on a class β, there are only two universally defined values - 0^+ and ∞, i.e. the values that correspond to stable quasimaps and stable maps. Let $\epsilon \in \{0^+, \infty\}$, we define

$$F_g^\epsilon(\mathbf{t}(z)) = \sum_{N=0}^{\infty} \sum_{\beta \geq 0} \frac{q^\beta}{N!} \langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle_{g,N,\beta}^\epsilon,$$

where $\mathbf{t}(z) \in H^*(M, \mathbb{Q})[\![z]\!]$ is a generic element, and the unstable terms are set to be zero. By repeatedly applying Theorem 2.5.3 we obtain.

Corollary 2.5.5. *For all $g \geq 1$ we have*

$$F_g^{0^+}(\mathbf{t}(z)) = F_g^\infty(\mathbf{t}(z) + \mu(-z)).$$

For $g = 0$, the same equation holds modulo constant and linear terms in $\mathbf{t}$.

For $g = 0$ the relation holds only moduli linear terms in $\mathbf{t}(z)$, because the moduli space $Q_{0,1}^{\epsilon^-}(M, \beta)$ is empty, if $\epsilon^- \deg(\beta) \leq 1$. The wall-crossing formula takes a different form in this case.

Theorem 2.5.6. *For $\epsilon \in (\frac{1}{\deg(\beta)}, \frac{1}{\deg(\beta)-1})$ we have*

$$\mathsf{ev}_* \left(\frac{[Q_{0,1}^{\epsilon^-}(M, \beta)]^{\mathrm{vir}}}{z(z - \psi_1)} \right) = [I(q, z)]_{z^{\leq -2}, q^\beta},$$

where $[\dots]_{z^{-2}, q^\beta}$ means that we take a truncation up to z^{-2} and the coefficient of q^β.

Proof. See [Zho22, Lemma 7.2.1]. □

To express the wall-crossing formula above in terms of change of variables, we do the following. Let $\{B^i\}$ be a basis of $H^*(M, \mathbb{Q})$ and $\{B_i\}$ be its dual basis with respect to intersection pairing. Let

$$\begin{aligned} J^{0^+}(\mathbf{t}(z), q, z) &= \frac{\mathbf{t}(-z)}{z} + I(q, z) \\ &\quad + \sum_{\beta \geq 0, N \geq 0} \frac{q^\beta}{N!} \sum_p B_i \langle \frac{B^i}{z(z - \psi)}, \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle_{0,1+N,\beta}^{M,0^+}, \end{aligned}$$

where unstable terms are set to be zero, and let

$$\begin{aligned} J^{\infty}(\mathbf{t}(z), q, z) &= \frac{\mathbf{t}(-z)}{z} + 1 \\ &\quad + \sum_{\beta \geq 0, N \geq 0} \frac{q^\beta}{N!} \sum_p B_i \langle \frac{B^i}{z(z - \psi)}, \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle_{0,1+N,\beta}^{M,\infty}, \end{aligned}$$

then genus-0 case admits more refined wall-crossing formula, which also incorporates some of the unstable contributions.

Theorem 2.5.7. *We have*

$$J^{\infty}(\mathbf{t}(z) + \mu(-z)) = J^{0^+}(\mathbf{t}(z)).$$

Proof. We again refer to [Zho22, Section 7.4]. □

2.5.4 Semi-positive targets

I-function.

Using the virtual localisation on the graph space, we can obtain a more explicit expression for I-functions for *semi-positive* moduli of sheaves.

Definition 2.5.8. A pair $(M, \mathfrak{Coh}_r(S))$ is *semi-positive*, if for all classes $\beta \in \mathrm{Eff}(M, \mathfrak{Coh}_r(S))$ the following holds

$$\beta(\det(\mathbb{T}^{\mathrm{vir}})) \geq 0.$$

The virtual dimension of $QG_{0,1}(M, \beta)$ is equal to $\dim(M)+1+\beta(\det(\mathbb{T}^{\mathrm{vir}}))$. Therefore by the virtual localisation we can establish the degrees of the classes involved in the definition of I-function,

$$-z\mathsf{ev}_* \left(\frac{[F_{\check{\beta}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_{\check{\beta}}/QG_{0,1}(M,\beta)})} \right) \in A^{-\beta(\det(\mathbb{T}^{\mathrm{vir}}))}_{\mathbb{C}^*}(M)_Q,$$

where we made the identification $A^*_{\mathbb{C}^*}(M) \cong A^*(M)[z]$ for a trivial $\mathbb{C}^*$-action on M, and $A^*_{\mathbb{C}^*}(M)_Q$ is the localised equivariant Chow group. Consider now the expansion

$$[zI(q,z) - z]_+ = I_1(q) + (I_0(q) - 1)z + I_{-1}(q)z^2 + I_{-2}(q)z^3 + \dots,$$

by the dimension constraint

$$-\beta(\det(\mathbb{T}^{\mathrm{vir}})) \geq 0$$

all terms I_k with $k \geq -1$ therefore vanish for a semi-positive target. Hence in this case we have

$$[zI(q,z) - z]_+ = I_1(q) + (I_0(q) - 1)z.$$

These terms in turn can be given a more explicit expression.

Proposition 2.5.9. *For a semi-positive pair $(M, \mathfrak{Coh}_r(S))$ the following holds*

(i)

$$I_0(q)^{-1} = 1 + \sum_{\beta \neq 0} \sum_i q^\beta \langle \gamma_i, \mathbb{1}, \gamma^i \rangle^{0+}_{0,\beta};$$

(ii)

$$I_1(q) = f_0(q)\mathbb{1} + \sum_j f_j(q) D_j,$$

where $\{D_j\}$ is a basis of $H^2(M, \mathbb{Q})$, and

$$\frac{f_0(q)}{I_0(q)} = \sum_{\beta \neq 0} q^\beta \langle [\mathrm{pt}], \mathbb{1} \rangle^{0+}_{0,\beta} \quad \frac{f_j(q)}{I_0(q)} = \sum_{\beta \neq 0} \sum_j q^\beta \langle D^j, \mathbb{1} \rangle^{0+}_{0,\beta}.$$

Proof. The proof is exactly the same as in [CK14, Section 5.5]. □

An example of a semi-positive target would be a moduli of sheaves on a del Pezzo surface, e.g. $\mathbb{P}^2$. However, even a pair $(\mathbb{P}^2, \mathfrak{Coh}_r(S))$ is not Fano in the sense of quasimaps, i.e. there exists class a $\beta \in \mathrm{Eff}(\mathbb{P}^2, \mathfrak{Coh}_r(S))$ for which

$$\beta(\det(\mathbb{T}^{\mathrm{vir}})) = 0.$$

These are just the classes such that $\mathrm{c}_1(\check{\beta}) = 0$. In fact, for all punctorial Hilbert schemes of del Pezzo surfaces $S^{[n]}$ there are no classes with $\beta(\det(\mathbb{T}^{\mathrm{vir}})) = 1$, therefore there is no $f_0(q)$ term. Moreover, in the case of punctorial Hilbert schemes of del Pezzo surfaces we can explicitly determine the terms of the perverse I-function, $I_0^\sharp$ and $I_1^\sharp$. Let us firstly do some notational preparations, and from now on we assume that $M = S^{[n]}$.

By Corollary 2.2.11 we have an embedding

$$-\check{(\ldots)} \colon \mathrm{Eff}(S^{[n]}, \mathfrak{Coh}^\sharp_r(S)) \hookrightarrow H^{1,1}(S) \oplus H^{2,2}(S), \tag{2.17}$$

here we change the sign of the classes, which amounts to considering classes of subschemes instead of classes of ideals on threefolds. Using this embedding, we identify β with its image $-\check{\beta}$. The class β can therefore be decomposed as

$$\beta = (\gamma, \mathsf{m}) \in H^{1,1}(S) \oplus H^{2,2}(S),$$

hence

$$\mathbb{Q}[\![q^\beta]\!] = \mathbb{Q}[\![q^\gamma]\!] \otimes \mathbb{Q}[\![y]\!], \quad q^\beta = q^\gamma \cdot y^{\mathsf{m}}.$$

On the side of $S^{[n]}$ the variable y keeps track of multiples of the exceptional curve class $\mathsf{A} \in H_2(S^{[n]}, \mathbb{Z})$, and the above decomposition corresponds to the one of $H_2(S^{[n]}, \mathbb{Z})$ given by Nakajima basis (images of Nakajima operators applied to classes on S),

$$H_2(S^{[n]}, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \oplus \mathbb{Z} \cdot \mathsf{A}.$$

More precisely, if $\Sigma \subset S$ is a curve, then we can define an associated curve $\Sigma_n \subset S^{[n]}$ given by letting one point move along Σ and keeping $n-1$ other distinct points fixed. Then Σ_n represents a class in $H^2(S, \mathbb{Z}) \subset H_2(S^{[n]}, \mathbb{Z})$ with respect to the identification above. We then define $\mathrm{c}_1(S)_n \in H_2(S^{[n]}, \mathbb{Z})$ to be the class associated to a canonical class $\mathrm{c}_1(S) \in H_2(S, \mathbb{Z})$ as described above.

With the above notation we have the following result, which was kindly communicated to the author by Georg Oberdieck.

Proposition 2.5.10 (Georg Oberdieck)**.** *Assume S is a del Pezzo surface, then for $M = S^{[n]}$ we have*

$$I_0^\sharp(q) = 1$$
$$I_1^\sharp(q) = \log(1+y)\mathrm{c}_1(S)_n.$$

Proof. By dimension constraints and the fact that there are no $\gamma \in \mathrm{Eff}(S)$ such that $\gamma \cdot \mathrm{c}_1(S) = 1$, the non-zero contributions to the I-function come only from classes of the form $\beta = (0, \mathsf{m})$. Let us firstly consider $I_0^\sharp$. Let $P \in S^{[n]}$ be a point, then the preimage $\mathrm{ev}^{-1}(P) \subset F_{\check{\beta}}$ parametrizes stable pairs supported in $U \times \mathbb{P}^1$ where U is a local neighbourhood of the support of P. We can assume that U is the disjoint union's of $\mathbb{C}^2$, hence since $\mathbb{C}^2$ carries a symplectic form, the only non-vanishing contributions are therefore due to $\mathsf{m} = 0$. Hence $\langle I_0^\sharp, P \rangle = 1$, which implies that $I_0 = 1$.

We now consider the term $I_1^\sharp$. With the same argument as above $\langle I_1^\sharp, \mathsf{A} \rangle = 0$. Now let us evaluate $I_1^\sharp$ at the classes in $H_2(S, \mathbb{Z}) \subset H_2(S^{[n]}, \mathbb{Z})$. By the previous argument the $d-1$ fixed points contribute 1 each, so that

$$\langle I_1^{\sharp, S^{[n]}}, [\Sigma_n] \rangle = \langle I_1^{\sharp, S}, [\Sigma] \rangle.$$

Hence we may assume $d = 1$. In this case the moduli space $F_{\bullet,(0,\mathsf{m})}$ is isomorphic to S, parametrizing pairs (F, s) given by $I^\bullet = \mathcal{O}_{\mathbb{P}^1_x} \to \mathcal{O}_{\mathbb{P}^1_x}(D)$ where $\mathbb{P}^1_x = \mathbb{P}^1 \times x$ for a point $x \in S$, and $D = \mathsf{m} \cdot [\infty]$. The local model of $\mathrm{P}_{1,\beta}(S \times \mathbb{P}^1 / S_0)$ near $F_{\check{\beta}}$ is $\mathrm{Sym}^{\mathsf{m}}(\mathbb{P}^1) \times S$. The obstruction theory was computed in [PT09, Section 4.2][6],

$$\mathrm{Def}_{I^\bullet} = H^0(\mathcal{O}_D(D))$$

$$\mathrm{Obs}_{I^\bullet} = H^0(\mathcal{O}_D(D) \otimes \omega_{S\times\mathbb{P}^1})^\vee = H^0(\mathcal{O}_D(D) \otimes \omega_{\mathbb{P}^1})^\vee \otimes \omega_{S|x}^\vee.$$

Consider now the $\mathbb{C}^*$-action on $\mathbb{P}^1$ by $t \cdot (x, y) = (tx, y)$. The coordinate $Y = y/x$ gets scaled by $t \cdot Y = t^{-1}Y$ hence has weight $-z$. Let us analyse the $\mathbb{C}^*$-equivariant structure the obstruction theory. Firstly,

$$H^0(\mathcal{O}(D)|_D) = (Y^{-\mathsf{m}}) \otimes \mathbb{C}[Y]/Y^{\mathsf{m}} = \mathbb{C}Y^{-\mathsf{m}} \oplus \mathbb{C}Y^{-\mathsf{m}+1} \oplus \ldots \oplus \mathbb{C}Y^{-1},$$

which therefore has weights $z, 2z, \ldots, \mathsf{m}z$ as a $\mathbb{C}$-module. Moreover, $\omega_{\mathbb{P}^1} = \mathbb{C}[Y]dY$, so since dY has weight $-z$ we get that $H^0(\mathcal{O}(D)|_D \otimes \omega_{\mathbb{P}^1})$ has weights $0, z, \ldots, (\mathsf{m}-1)z$, therefore its dual has weights $(-\mathsf{m}+1)z, \ldots, -z, 0$. Let $\mathrm{c}_1 = \mathrm{c}_1(S)$, we therefore obtain the following

$$\begin{aligned}
\mathrm{ev}_* \frac{[F_{\check{\beta}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})} &= p_{S*}\left(\frac{e_{\mathbb{C}^*}(\mathrm{Obs}^{\mathrm{mov}}_{I^\bullet})}{e_{\mathbb{C}^*}(\mathrm{Def}^{\mathrm{mov}}_{I^\bullet})} \cdot p_S^*\mathrm{c}_1\right) \\
&= \frac{(-z+\mathrm{c}_1)\cdots((-\mathsf{m}+1)z+\mathrm{c}_1)}{z\cdot 2z\cdots \mathsf{m}z} \cdot \mathrm{c}_1 \\
&= \frac{(-1)^{\mathsf{m}-1}(\mathsf{m}-1)!z^{\mathsf{m}-1}}{\mathsf{m}!z^{\mathsf{m}}} \cdot \mathrm{c}_1 + (\ldots)\cdot \mathrm{c}_1^2 \\
&= \frac{(-1)^{\mathsf{m}-1}}{\mathsf{m}z} \cdot \mathrm{c}_1 + (\ldots)\cdot \mathrm{c}_1^2,
\end{aligned}$$

this proves the claim. □

We now define

$$^\sharp\langle \gamma_1, \ldots, \gamma_N\rangle^{S^{[n]},\epsilon}_{g,\gamma} := \sum_{\mathsf{m}} {}^\sharp\langle \gamma_1, \ldots, \gamma_N\rangle^{S^{[n]},\epsilon}_{g,(\gamma,\mathsf{m})} y^{\mathsf{m}},$$

then using the wall-crossing formula from Theorem 2.5.3, the string and divisor equations, one obtains the following result, which specialises to the result stated in Section 1.4.1 after enumerating the invariants with respect to classes on $S^{[n]}$ instead of $S \times C$.

Corollary 2.5.11. *Assume $2g - 2 + N \geq 0$. If S is a del Pezzo surface, then*

$$^\sharp\langle \gamma_1, \ldots, \gamma_N\rangle^{0^+}_{g,\gamma} = (1+y)^{\mathrm{c}_1(S)\cdot\gamma} \cdot {}^\sharp\langle \gamma_1, \ldots, \gamma_N\rangle^{\infty}_{g,\gamma}.$$

[6] The equivariantly correct obstruction theory is given in the latest arXiv version. The canonical line bundle $\omega_{\mathbb{P}^1}(D)|D = \omega_D$ is equivariantly not trivial.

DT/PT correspondence

Using dilaton equation for GW invariants (see [CK20, Corollary 1.5]) one can restate the wall-crossing formula for $g \neq 1$ (for $g = 1$ there is an extra constant term which we do not want to write down for the clarity of exposition, see [CK20, Corollary 1.5]) as follows

$$(I_0)^{2g-2} \cdot F_g^{0^+}(\mathbf{t}(z)) = F_g^{\infty}\left(\frac{\mathbf{t}(z) + I_1(q)}{I_0(q)}\right),$$

the same holds for the perverse generating series $F_g^{\sharp,\epsilon}(\mathbf{t}(z))$. Since the generating series are related by a change of variable, the above equation is equivalent to

$$(I_0)^{2g-2} \cdot F_g^{0^+}(I_0(q)\mathbf{t}(z) - I_1(q)) = F^{\infty}(\mathbf{t}(z)),$$

therefore perverse and non-perverse generating series are related in the following way

$$(I_0)^{2g-2} \cdot F_g^{0^+}(I_0(q)\mathbf{t}(z) - I_1(q)) = (I_0^{\sharp})^{2g-2} \cdot F_g^{\sharp,0^+}(I_0^{\sharp}(q)\mathbf{t}(z) - I_1^{\sharp}(q)),$$

moving the change of variables to one side we, obtain

$$\frac{(I_0)^{2g-2}}{(I_0^{\sharp})^{2g-2}} \cdot F_g^{0^+}\left(\frac{I_0(q)}{I_0^{\sharp}(q)} \cdot \left(\mathbf{t}(z) + I_1^{\sharp}(q)\right) - I_1(q)\right) = F_g^{\sharp,0^+}(\mathbf{t}(z)).$$

Passing from quasimaps to sheaves and establishing the DT/PT correspondence for wall-crossing terms, we would get the DT/PT correspondence for the relative geometry

$$S \times C_{g,N} \to \overline{M}_{g,N},$$

such that $2g - 2 + N \geq 0$ and $\mathrm{ch}(I)_{\mathrm{d}} \neq 0$. In particular, DT/PT correspondence relative to three vertical divisors on $S \times \mathbb{P}^1$ is reduced to the DT/PT correspondence of wall-crossing invariants.

Chapter 3

Quasimaps to a moduli space of sheaves on a K3 surface

3.1 Surjective cosection

Throughout the chapter we assume S to be a K3 surface. Let F be a sheaf on $S \times C$ flat over a nodal curve C, such that fibers of F have Chern character $\mathbf{v} \in H^*(S, \mathbb{Q})$. We start with some preparations. Consider the Atiyah class

$$\mathrm{At}(F) \in \mathrm{Ext}^1(F, F \otimes \Omega^1_{S\times C}),$$

represented by the canonical exact sequence

$$0 \to F \otimes \Omega^1_{S\times C} \to \mathcal{P}^1(F) \to F \to 0,$$

where $\mathcal{P}^1(F)$ is the sheaf of principle parts. Composing the Atiyah class with the natural map

$$\Omega^1_{S\times C} = \Omega^1_S \boxplus \Omega^1_C \to \Omega^1_S \boxplus \omega_C,$$

we obtain a class

$$\mathrm{At}_\omega(F) \in \mathrm{Ext}^1(F, F \otimes (\Omega^1_S \boxplus \omega_C)).$$

We then define the Chern character of a sheaf F on $S \times C$ for possibly singular C as follows

$$\mathrm{ch}_k(F) := \mathrm{tr}\left(\frac{(-1)^k}{k!}\mathrm{At}_\omega(F)^k\right) \in H^k(\wedge^k(\Omega^1_S \boxplus \omega_C)). \tag{3.1}$$

If C is smooth, it agrees with the standard definition of the Chern character. Using the canonical identification $H^1(\omega_C) \cong \mathbb{C}$ and

$$\wedge^k(\Omega_S^1 \boxplus \omega_C) \cong \Omega_S^k \boxplus (\Omega_S^{k-1} \boxtimes \omega_S),$$

we get a Künneth's decomposition of the cohomology

$$H^k(\wedge^k(\Omega_S^1 \boxplus \omega_C)) \cong H^k(\Omega_S^k) \oplus H^{k-1}(\Omega_S^{k-1}),$$

therefore

$$\bigoplus H^k(\wedge^k(\Omega_S^1 \boxplus \omega_C)) \cong \Lambda \otimes H^0(C, \mathbb{C}) \oplus \Lambda \otimes H^2(C, \mathbb{C}) \cong \Lambda \oplus \Lambda. \tag{3.2}$$

With respect to this decomposition above the Chern character $\mathrm{ch}(F)$ has two components

$$\mathrm{ch}(F) = (\mathrm{ch}(F)_{\mathrm{f}}, \mathrm{ch}(F)_{\mathrm{d}}) \in \Lambda \oplus \Lambda.$$

If C is smooth, it was shown in Lemma 2.2.2 that

$$(\mathrm{ch}(F)_{\mathrm{f}}, \mathrm{ch}(F)_{\mathrm{d}}) = (\mathbf{v}, \check{\beta}),$$

where β is the degree of a quasimap associated to F and $\check{\beta}$ is its dual class in $H^*(S, \mathbb{Q})$, for more details see Section 2.2. We would like to establish the same result in the case of a singular C. Let

$$\pi \colon S \times \tilde{C} \to S \times C$$

be the normalisation morphism and $\pi^* F_i$ be the restriction of $\pi^* F$ to its connected components $\tilde{C}_i$ of $\tilde{C}$. The above decomposition of the Chern character then satisfies the following property.

Lemma 3.1.1. *Under the identification (3.2) the following holds*

$$\mathrm{ch}(F) = (\mathbf{v}, \sum \mathrm{ch}(\pi^* F_i)_{\mathrm{d}}) \in \Lambda \oplus \Lambda.$$

In other words, if the quasimap associated to F is of degree β, then

$$\mathrm{ch}(F)_{\mathrm{d}} = \check{\beta}.$$

Proof. Firstly, there exist canonical maps making the following diagram commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^* F \otimes \pi^* \Omega^1_{S \times C} & \longrightarrow & \pi^* \mathcal{P}^1(F) & \longrightarrow & \pi^* F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \pi^* F \otimes \Omega^1_{S \times \tilde{C}} & \longrightarrow & \mathcal{P}^1(\pi^* F) & \longrightarrow & \pi^* F & \longrightarrow & 0 \end{array}$$

where the first row is exact on the left, because $L\pi^* \cong F$, since[1] F is flat over C. The diagram above implies that the pullback of Atiyah class $\pi^*\mathrm{At}(F)$ is mapped to $\mathrm{At}(\pi^*F)$ with respect to the map

$$\mathrm{Ext}^1(\pi^*F, \pi^*F \otimes \pi^*\Omega^1_{S\times C}) \to \mathrm{Ext}^1(\pi^*F, \pi^*F \otimes \Omega^1_{S\times \tilde{C}}).$$

The same holds for $\pi^*\mathrm{At}^k(F)$. Consider now the following commutative diagram

$$\begin{tikzcd}
R\mathcal{H}om(F, F\otimes \Omega^k_{S\times C}) \arrow[rr] \arrow[d] & & \Omega^k \arrow[dd] \arrow[ddr] & \\
\pi_*R\mathcal{H}om(\pi^*F, \pi^*F\otimes \pi^*\Omega^k_{S\times C}) \arrow[d] & & & \\
\pi_*R\mathcal{H}om(\pi^*F, \pi^*F\otimes \Omega^k_{S\times \tilde{C}}) \arrow[rr] & & \pi_*\Omega^k_{S\times\tilde{C}} \arrow[r] & \wedge^k(\Omega^1_S \boxplus \omega_C)
\end{tikzcd}$$

such that the first vertical map is the composition

$$\begin{aligned}
&R\mathcal{H}om(F, F\otimes \Omega^k_{S\times C}) \to \pi_*L\pi^*R\mathcal{H}om(F, F\otimes \Omega^k_{S\times C}) = \\
&\quad = \pi_*R\mathcal{H}om(\pi^*F, \pi^*F\otimes L\pi^*\Omega^k_{S\times C}) \to \pi_*R\mathcal{H}om(\pi^*F, \pi^*F\otimes \pi^*\Omega^k_{S\times C}),
\end{aligned}$$

where we used that $L\pi^*F \cong \pi^*F$. Taking cohomology of the diagram above and using the exactness of π_*, we can therefore factor the map

$$\mathrm{Ext}^k(F, F\otimes\Omega^k_{S\times C}) \to H^k(\wedge^k(\Omega^1_S \boxplus \omega_C))$$

as follows

$$\begin{aligned}
\mathrm{Ext}^k(F, F\otimes\Omega^k_{S\times C}) &\to \mathrm{Ext}^k(\pi^*F, \pi^*F\otimes\pi^*\Omega^k_{S\times C}) \to \mathrm{Ext}^k(\pi^*F, \pi^*F\otimes\Omega^k_{S\times\tilde{C}}) \\
&\to H^k(\Omega^k_{S\times\tilde{C}}) \cong H^k(\Omega^k_S) \oplus \bigoplus_i H^{k-1}(\Omega^{k-1}_S)\otimes H^1(\omega_{\tilde{C}_i}) \\
&\quad \to H^k(\Omega^k_S) \oplus H^{k-1}(\Omega^{k-1}_S)\otimes H^1(\omega_C) \cong H^k(\wedge^k(\Omega^1_S \boxplus \omega_C)).
\end{aligned}$$

Under the natural identifications $H^1(\omega_{\tilde{C}_i}) \cong \mathbb{C}$ and $H^1(\omega_C) \cong \mathbb{C}$ the last map in the sequence above becomes

$$H^k(\Omega^k_S) \oplus \bigoplus_i H^{k-1}(\Omega^{k-1}_S) \xrightarrow{(\mathrm{id},+)} H^k(\Omega^k_S) \oplus H^{k-1}(\Omega^{k-1}_S).$$

[1] To see that, one can use a standard locally-free resolution for a flat sheaf; these resolutions are functorial with respect to pullbacks.

The claim then follows by tracking the powers of Atiyah class $\mathrm{At}^k(F)$ along the maps above. The fact that

$$\sum \mathrm{ch}(\pi^* F_i)_{\mathrm{d}} = \check{\beta}$$

follows from the definition of $\check{\beta}$, Section 2.2.

3.1.1 Cosection

By pulling back the classes in

$$HT^2(S) := H^0(\wedge^2 T_S) \oplus H^1(T_S) \oplus H^2(\mathcal{O}_S)$$

to $S \times C$, we will treat $HT^2(S)$ as classes on $S \times C$. Let

$$\sigma_i := \mathrm{tr}(* \cdot \frac{(-1)^i}{i!} \mathrm{At}_\omega(F)^i) \colon \mathrm{Ext}^2(F,F) \to H^{i+2}(\wedge^i(\Omega_S^1 \boxplus \omega_C))$$

be the semiregularity map.

Lemma 3.1.2. *The following diagram commutes*

$$\begin{tikzcd}
H^{2-k}(\wedge^k T_S) \arrow[rr, "\cdot \frac{(-1)^k}{k!} \mathrm{At}_\omega(F)^k"] \arrow[dr, "{\langle *, \mathrm{ch}_{k+i}(F) \rangle}"'] & & \mathrm{Ext}^2(F,F) \arrow[dl, "\sigma_i"] \\
& H^{i+2}(\wedge^i(\Omega_S^1 \boxplus \omega_C)) &
\end{tikzcd}$$

Proof. If $i = 0$, then $\sigma_0 = \mathrm{tr}$ and the commutativity is implied by the following statement

$$\langle \kappa, \mathrm{tr}(\mathrm{At}_\omega(F)^k)) \rangle = \mathrm{tr}\langle \kappa, \mathrm{At}_\omega(F)^k \rangle,$$

whose proof is presented in [BF03, Proposition 4.2] for $k = 1$ and is the same for other values of k.

If $i = 1$, then for the commutativity of the digram we have to prove that

$$\langle \kappa, \mathrm{tr}(\frac{\mathrm{At}_\omega(F)^{k+1}}{k+1!})\rangle = \mathrm{tr}(\langle \kappa, \frac{\mathrm{At}^k_\omega(F))}{k!}\rangle \cdot \mathrm{At}_\omega(F)).$$

If $\kappa \in H^2(\mathcal{O}_S)$, the equality follows trivially, since there is no contraction. The case of $\kappa \in H^1(T_S)$ is treated in [BF03, Proposition 4.2]. For $\kappa \in H^0(\wedge^2 T_S)$ we use the derivation property for contraction with a 2-vector field

$$\langle \xi, \mathrm{At}^3_\omega(F) \rangle = 3\langle \xi, \mathrm{At}^2_\omega(F) \rangle \cdot \mathrm{At}_\omega(F),$$

which can be checked locally on a 2-vector field of the form $V \wedge W$. $\square$

Due to the decomposition

$$H^i(\wedge^i(\Omega_S^1 \boxplus \omega_C)) \cong H^i(\Omega_S^i) \oplus H^{i-1}(\Omega_S^{i-1}),$$

there are two ways to contract a class in $H^i(\wedge^i(\Omega_S^1 \boxplus \omega_C))$ with a class in $H^{2-k}(\wedge^k T_S)$: either via the first component of the decomposition above or via the second. Hence due to the wedge degree or the cohomological degree, only one component of $H^i(\wedge^i(\Omega_S^1 \boxplus \omega_C))$ pairs non-trivially with $H^{2-k}(\wedge^k T_S)$ for a fixed k. It is not difficult to check that contraction with the Chern character

$$H^{2-k}(\wedge^k T_S) \xrightarrow{\langle -, \mathrm{ch}_{k+i}(F)\rangle} H^{i+2}(\wedge^i(\Omega_S^1 \boxplus \omega_C))$$

is therefore equal to $\langle -, \mathrm{ch}(F)_{\mathrm{f}}\rangle$ for $i = 0$ and to $\langle -, \mathrm{ch}(F)_{\mathrm{d}}\rangle$ for $i = 1$. Moreover, using the identification

$$H^{i+2}(\wedge^i(\Omega_S^1 \boxplus \omega_C)) \cong H^2(\mathcal{O}_S),$$

contraction $\langle -, \mathrm{ch}(F)_{\mathrm{d/f}}\rangle$ with classes on $S \times C$ is identified with contraction with classes on S.

Proposition 3.1.3. *Assume*

$$\mathrm{ch}(F)_{\mathrm{f}} \wedge \mathrm{ch}(F)_{\mathrm{d}} \neq 0,$$

then there exists $\kappa \in HT^2(S)$*, such that*

$$\langle \kappa, \mathrm{ch}(F)_{\mathrm{f}}\rangle = 0 \quad \textit{and} \quad \langle \kappa, \mathrm{ch}(F)_{\mathrm{d}}\rangle \neq 0.$$

Hence the restriction of the semiregularity map to the traceless part

$$\sigma_1\colon\ \mathrm{Ext}^2(F,F)_0 \to H^3(\Omega_S^1 \boxplus \omega_C)$$

is non-zero.

Proof. Using a symplectic form on S, we have the following identifications

$$\wedge^2 T_S \cong \mathcal{O}_S, \quad T_S \cong \Omega_S^1, \quad \mathcal{O}_S \cong \Omega_S^2.$$

After applying the identifications and taking cohomology, the pairing

$$HT^2(S) \otimes H\Omega_0(S) \to H^2(\mathcal{O}_S), \tag{3.3}$$

which is given by contraction, becomes the intersection pairing

$$H\Omega_0(S) \otimes H\Omega_0(S) \to H^2(\Omega_S^2),$$

where $H\Omega_0(S) = \bigoplus H^i(\Omega^i)$. In particular, the pairing (3.3) is non-degenerate. Hence $\mathrm{ch}(F)_{\mathrm{d}}^{\perp}$ and $\mathrm{ch}(F)_{\mathrm{f}}^{\perp}$ are distinct, if and only if $\mathrm{ch}(F)_{\mathrm{d}}$ is not a multiple of $\mathrm{ch}(F)_{\mathrm{f}}$, therefore there exists a class $\kappa \in HT^2(S)$ with the properties stated in the lemma. By Lemma 3.1.2 and the discussion afterwards the property $\langle \kappa, \mathrm{ch}(F)_{\mathrm{f}} \rangle = 0$ implies that

$$\kappa \cdot \exp(-\mathrm{At}_\omega(F) \in \mathrm{Ext}^2(F, F)_0,$$

while the property $\langle \kappa, \mathrm{ch}(F)_{\mathrm{d}} \rangle = 0$ implies that the restriction of the semiregularity map to $\mathrm{Ext}^2(F, F)_0$ is non-zero, as it is non-zero when applied to the element $\kappa \cdot \exp(-\mathrm{At}_\omega(F))$. $\square$

Remark 3.1.4. From the point of view of quasimaps the condition

$$\mathrm{ch}(F)_{\mathrm{f}} \wedge \mathrm{ch}(F)_{\mathrm{d}} \neq 0,$$

is equivalent to the fact, that the quasimap $f \colon C \to \mathfrak{Coh}_r(S)$ associated to F is not constant.

A geometric interpretation of the above result is the following one. With respect to Hochschild–Kostant–Rosenberg (HKR) isomorphism

$$HT^2(S) \cong HH^2(S)$$

the space $HT^2(S)$ parametrises first-order non-commutative deformations of S, i.e. deformations of $D^b(S)$. Given a first-order deformation $\kappa \in HT^2(S)$, the unique horizontal lift of $\mathrm{ch}(F)_{\mathrm{d/f}}$ relative to some kind of Gauss-Manin connection associated to κ should stay Hodge, if and only if $\langle \kappa, \mathrm{ch}(F)_{\mathrm{d/f}} \rangle = 0$. On the other hand, $\langle \kappa, \exp(-\mathrm{At}_\omega(F))$ gives obstruction for deforming F on $S \times C$ in direction κ. Therefore by Lemma 3.1.2 the semiregularity map σ_i relates obstruction to deform F along κ with the obstruction that $\mathrm{ch}(F)_{\mathrm{d/f}}$ stays Hodge. Proposition 3.1.3 states that there exists a deformation $\kappa \in HT^2(S)$, for which $\mathrm{ch}(F)_{\mathrm{f}}$ stays Hodge, but $\mathrm{ch}(F)_{\mathrm{d}}$ does not. Form the point of view of quasimaps means that the moduli of stable sheaves M on S deforms along κ, but the quasimap associated to F does not, if its degree is non-zero. For example, let S be a K3 surface associated to a cubic 4-fold Y, such that the Fano variety of lines $F(Y)$ is isomorphic to $S^{[2]}$. Then if we deform Y away from the Hasset divisor, $F(Y)$ deforms along, but a point class of S does not. Therefore such deformation of Y will gives us the first-order non-commutative deformation $\kappa \in HT^2(S)$ of S, such that $\mathbf{v} = (1, 0, -2)$ stays Hodge, but $\check{\beta} = (0, 0, k)$ does not. Note that $\check{\beta} = (0, 0, k)$ corresponds to multiplies of the exceptional curve class in $S^{[2]}$. Indeed there are no commutative deformations of S that will make $(0, 0, k)$ non-Hodge, because the exceptional divisor deforms along with $\mathrm{Hilb}^2(S)$.

Applying the identification of $Q^{\epsilon}_{g,N}(M, \beta)$ with the relative moduli stack of stable sheaves $M^{\epsilon}_{\mathbf{v},\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$ and of the corresponding obstructions theories, Proposition 2.4.5, we construct a surjective cosection as follows. Let

$$\mathbb{E}^{\bullet} := (\mathcal{H}om_{\pi}(\mathbb{F}, \mathbb{F})_0[1])^{\vee},$$

then there exists a global relative semiregularity map

$$\mathsf{sr} \colon (\mathbb{E}^{\bullet})^{\vee} \to R^3\pi_*(\Omega^1_S \boxplus \omega_{\mathfrak{C}/\mathfrak{M}_{g,n}})[-1],$$

and since

$$R^3\pi_*(\Omega^1_S \boxplus \omega_{\mathfrak{C}/\mathfrak{M}_{g,N}}) \cong H^2(\mathcal{O}_S) \otimes \mathcal{O}_{Q^{\epsilon}_{g,N}(M,\beta)},$$

we obtain a cosection of the obstruction theory

$$\mathsf{sr} \colon (\mathbb{E}^{\bullet})^{\vee} \to H^2(\mathcal{O}_S) \otimes \mathcal{O}_{Q^{\epsilon}_{g,N}(M,\beta)}[-1] \cong \mathcal{O}_{Q^{\epsilon}_{g,N}(M,\beta)}[-1].$$

Corollary 3.1.5. *Assuming $\beta \neq 0$, the semiregularity map* sr *is surjective.*

Proof. Under the given assumption the surjectivity of sr follows from Proposition 3.1.3 and Lemma 3.1.1. $\square$

Consider now the composition

$$\mathrm{Ext}^1_C(\Omega_C, \mathcal{O}_C(-\sum p_i)) \to \mathrm{Ext}^2(F, F)_0 \xrightarrow{\sigma_1} H^3(\Omega^1_S \boxplus \omega_C), \tag{3.4}$$

where the first map defined by the following composition

$$\mathrm{Ext}^1_C(\Omega_C, \mathcal{O}_C(-\sum p_i)) \to \mathrm{Ext}^1_C(\omega_C, \mathcal{O}_C) \xrightarrow{\cdot - \mathrm{At}_{\omega}(F)} \mathrm{Ext}^2(F, F)_0.$$

The composition (3.4) is zero by the same arguments as the ones which are presented in Lemma 3.1.2. The semiregularity map therefore descends to the absolute obstruction theory

$$\begin{tikzcd} \mathbb{E}^{\bullet}_{\mathrm{abs}} \arrow[r] \arrow[dr, dashed, "\mathsf{sr}"'] & \mathbb{E}^{\bullet} \arrow[r] \arrow[d, "\mathsf{sr}"] & \mathbb{L}_{\mathfrak{M}_{g,n}}[1] \\ & \mathcal{O}_{Q^{\epsilon}_{g,N}(M,\beta)}[-1] & \end{tikzcd}$$

so the results of [KL13] apply.

3.2 Reduced wall-crossing

In what follows we use Kiem–Li construction of reduced classes via localisation by cosection [KL13]. The cosection of the obstruction theory of the master space $MQ^{\epsilon_0}_{g,N}(M,\beta)$ is constructed in the similar way as for $Q^{\epsilon}_{g,N}(M,\beta)$ by viewing it as a relative moduli space of sheaves. Since we need virtual localisation for the proof of the wall-crossing formulas, we refer to [CKL17] for the virtual torus localisation of Kiem–Li reduced classes.

From now on, by a virtual fundamental class we always will mean a *reduced* virtual fundamental class, except for $\beta = 0$, since in this case the standard virtual fundamental class does not vanish. The arguments presented in this section apply both to standards invariants $\langle \tau_{m_1}(\gamma_1), \dots, \tau_{m_N}(\gamma_N)\rangle^{\epsilon}_{g,\beta}$ and perverse invariants $\langle \tau_{m_1}(\gamma_1), \dots, \tau_{m_n}(\gamma_N)\rangle^{\natural,\epsilon}_{g,\beta}$, if $M = S^{[n]}$. We therefore state and prove everything only for the standard invariants.

We start with derivation of a more explicit formula for wall-crossing invariants $\mu_\beta(z)$, Definition 2.5.1, by using localisation on a graph space $GQ_{0,1}(M)$. As it is explained in Section 2.5.1, the $\mathbb{C}^*$-fixed components of $GQ_{0,1}(M)$ are identified with certain products, the reduced class of a product splits as a product of reduced and non-reduced classes on its factors. Assuming the marking is over ∞ the virtual class is therefore non-zero only for $F_{\check{\beta}}$ and $F^{0,0}_{1,\beta}$. Now let

$$\gamma_i p_\infty := \gamma_i \boxtimes p_\infty \in H^*_{\mathbb{C}^*}(M \times \mathbb{P}^1),$$

where $p_\infty \in H^*_{\mathbb{C}^*}(\mathbb{P}^1)$ is the equivariant class of $\infty \in \mathbb{P}^1$. Then by the virtual localisation formula we have the following identity

$$\sum \gamma^i \int_{[GQ_{0,1}(M)]^{\mathrm{vir}}} \gamma_i p_\infty = \sum \gamma^i \int_{[F_{\check{\beta}}]^{\mathrm{vir}}} \frac{-z\mathsf{ev}^*\gamma_i}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})} + \sum \gamma^i \langle \gamma_i, \frac{\mathbb{1}}{-z-\psi}\rangle^{0+}_{0,\beta}, \quad (3.5)$$

where we used that

$$e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{0,0}_{1,\beta}/GQ_{0,1}(M)}) = z(z+\psi)$$

and

$$p_\infty|_0 = 0, \quad p_\infty|_\infty = -z,$$

which also implies that only fixed components with markings over ∞ contribute to the integral 3.5. The virtual dimension of $GQ_{0,1}(M)$ is $\dim(M)+2$, the virtual dimension of $Q^{0+}_{0,2}(M,\beta)$ is $\dim(M)$ and the degree of $\gamma_i p_\infty$ is at most $\dim(M)+1$, hence the left-hand side of (3.5) is zero and the second term on the right-hand side is non-zero only for $\gamma_i = [\mathrm{pt}]$, we therefore get that

$$-z\mathsf{ev}_*\left(\frac{[F_{\check{\beta}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_{\check{\beta}}/QG_{0,1}(M,\beta)})}\right) = \frac{\mathbb{1}}{z}\langle[\mathrm{pt}],\mathbb{1}\rangle^{0+}_{0,\beta} \in A^*(M)[z^{\pm}],$$

in particular, we obtain that

$$\mu_\beta(z) = \mathbb{1}\langle[\mathrm{pt}],\mathbb{1}\rangle^{0+}_{0,\beta} \in A^*(M)[z]. \tag{3.6}$$

Theorem 3.2.1. *Assuming* $2g-2+N+\epsilon_0\deg(\beta)>0$*, we have*

$$\begin{aligned}\langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_n}(\gamma_N)\rangle^{\epsilon-}_{g,\beta} - \langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon+}_{g,\beta}& \\ = \langle[\mathrm{pt}],\mathbb{1}\rangle^{0+}_{0,\beta}\cdot\langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N),\mathbb{1}\rangle_{g,0}&,\end{aligned}$$

if $\deg(\beta)=d_0$*, and*

$$\langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon-}_{g,\beta} = \langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon+}_{g,\beta}$$

otherwise.

Sketch of the proof. As in the case of Theorem 2.5.3 we have to refer mostly to [Zho22, Section 6]. The difference with is that we use reduced classes now.

The fixed components of the master space which contribute to the wall-crossing formula are of the form (up to some finite gerby structure and finite coverings)

$$\widetilde{Q}^{\epsilon^+}_{g,N+k}(M,\beta')\times_{M^k}\prod_{i=1}^{k}F_{\beta_i},$$

where $\beta=\beta'+\beta_1+\dots+\beta_k$ and $\deg(\beta_i)=d_0$. Recall that $\widetilde{Q}^{\epsilon^+}_{g,N+k}(M,\beta')$ is just a base change of $Q^{\epsilon^+}_{g,N}(M,\beta)$ from $\mathfrak{M}_{g,N}$ to $\widetilde{\mathfrak{M}}_{g,N,d}$, where the latter is the moduli space of curves with entangled tails. The reduced class of a product splits as a product of reduced and non-reduced classes on its factors, therefore by Corollary 3.1.5 and [KL13] it vanishes, unless $\beta'=0$ and $k=1$, in which case

$$\widetilde{Q}^{\epsilon^+}_{g,N+1}(M,0) = Q^{\infty}_{g,N+1}(M,0) = \overline{M}_{g,N+1}(M,0),$$

then using the explicit expression of $\mu_\beta(z)$ from (3.6) and the analysis presented in [Zho22, Section 7], we get that contribution of this component to the wall-crossing is

$$\langle [\mathrm{pt}], \mathbb{1}\rangle^{0+}_{0,\beta} \cdot \langle \tau_{m_1}(\gamma_1), \ldots, \tau_{m_N}(\gamma_N), \mathbb{1}\rangle^{\infty}_{g,0},$$

this concludes the argument. $\square$

Corollary 3.2.2. *For all $g \geq 1$ we have*

$$F^{0^+}_g(\mathbf{t}(z)) = F^{\infty}_g(\mathbf{t}(z)) + F_{\mathrm{wall}}(\mathbf{t}(z))$$

where

$$F_{\mathrm{wall}}(\mathbf{t}(z)) = \mu(q) \cdot \left(\sum_{n=0}^{\infty} \frac{1}{N!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi), \mathbb{1}\rangle^{\infty}_{g,N+1,0} \right)$$

and

$$\mu(q) = \sum_{\beta > 0} \langle [\mathrm{pt}], \mathbb{1}\rangle^{0+}_{0,\beta} q^{\beta}.$$

For $g = 0$, the equation holds modulo constant and linear terms in $\mathbf{t}$.

There are invariants that are not covered by the results above and of great interest for us: those of a fixed genus-1 curve. We deal with them now. Let E be a smooth genus-1 curve and $Q^{\epsilon}_{E/E}(M, \beta)$ be the fiber of

$$Q^{\epsilon}_{1,0}(M, \beta) \to \overline{M}_{1,0}$$

over the stacky point $[E]/E \in \overline{M}_{1,0}$. In other words, $Q^{\epsilon}_{E/E}(M, \beta)$ is the moduli space of ϵ-stable quasimaps, whose stabilisation of the domain is equal to E, and such that maps are considered up translations of E (no other automorphisms of E).

Theorem 3.2.3. *Assuming $\beta \neq 0$, we have*

$$\int_{[Q^{\epsilon_-}_{E/E}(M,\beta)]^{\mathrm{vir}}} 1 = \int_{[Q^{\epsilon_+}_{E/E}(M,\beta)]^{\mathrm{vir}}} 1 + \chi(M) \langle [\mathrm{pt}], \mathbb{1}\rangle^{0+}_{0,\beta},$$

if $\deg(\beta) = d_0$*, and*

$$\int_{[Q^{\epsilon_-}_{E/E}(M,\beta)]^{\mathrm{vir}}} 1 = \int_{[Q^{\epsilon_+}_{E/E}(M,\beta)]^{\mathrm{vir}}} 1$$

otherwise.

Sketch of the proof. As in Theorem 3.2.1 the only case when the contribution from the wall-crossing components is non-zero is the one of $\beta' = 0$ and $k = 1$. In this case

$$\widetilde{Q}^{\epsilon_+}_{(E,0)}(M, 0) \cong M,$$

and the obstruction bundle is given by the tangent bundle T_M. Hence the virtual fundamental class is $\chi(M)[\mathrm{pt}]$, then by (3.6) the wall-crossing term is

$$\chi(M)\langle [\mathrm{pt}], \mathbb{1}\rangle^{0^+}_{0,\beta},$$

this concludes the argument. □

3.3 Applications

3.3.1 Enumerative geometry of $S^{[n]}$

Genus-0 invariants

Let us firstly consider genus-0 3-point quasimap invariants of $S^{[n]}$. This case is particularly nice, because the moduli of $\mathbb{P}^1$ with 3 marked points is just a point, hence by fixing markings we get

$$Q^{0^+}_{0,3}(S^{[n]}, \beta)^{\sharp} \cong \mathrm{P}_{n,\check{\beta}}(S \times \mathbb{P}^1 / S_{0,1,\infty}). \tag{3.7}$$

Moreover, by Theorem 3.2.1 there is no wall-crossing, therefore

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle^{\sharp, S^{[n]}, 0^+}_{0,\beta} = \langle \gamma_1, \gamma_2, \gamma_3 \rangle^{S^{[n]}, \infty}_{0,\beta}.$$

Definition 3.3.1. We define

$$\langle \gamma_1, ..., \gamma_N \mid \tilde{\tau}_0(\bar{\gamma}), ..., \tilde{\tau}_0(\bar{\gamma}_{N'}) \rangle^{S \times C}_{\mathbf{v},\check{\beta}} \in \mathbb{Q}$$

to be DT invariants associated to the moduli space of sheaves $M_{\mathbf{v},\check{\beta},u}(S \times C/S_{\mathbf{x}})$ (see Definition 2.10). On the left we put relative primary insertions and on the right the absolute primary ones. Since the moduli spaces can be identified for different choices of $u \in K_0(S)$, we drop u from the notation of the invariants. In rank 1 we add "$\sharp$" in the case of PT invariants.

Applying the identification (3.7), we obtain the following result, which confirms the conjecture proposed in [Obe19] after applying PT/GW of [Obeb].

Corollary 3.3.2.

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle^{S^{[n]}, \infty}_{0,\beta} = \langle \gamma_1, \gamma_2, \gamma_3 \rangle^{\sharp, S \times \mathbb{P}^1}_{n,\check{\beta}}.$$

More generally, the results above can be restated for relative geometries of the type $S \times C_{g,N} \to \mathfrak{M}_{g,N}$ such that $2g-2+N>0$. The marked points play the role of relative divisors in the sense of relative Donaldson–Thomas theory. In this case, the wall-crossing terms are also zero for primary invariants.

Genus-1 invariants, Igusa cusp form conjecture

Let us firstly establish a more precise relation between β and $\check{\beta}$. Given a generic K3 surface S and a elliptic curve E, let

$$\mathrm{Eff}(S) = \langle \beta_h \rangle, \quad \beta_h^2 = 2h-2.$$

Then for $n > 1$, we have

$$\mathrm{Eff}(S^{[n]}) = \langle C_{\beta_h}, \mathsf{A} \rangle,$$

where C_{β_h} and A are the primitive curves classes which are dual to a multiple of $\mathcal{L}_1$ and to a multiple of $\mathcal{L}_0$ respectively with respect to the intersection pairing on $S^{[n]}$. The latter is also the exceptional curve class coming from the Hilbert–Chow morphisms $S^{[n]} \to S^{(n)}$.

Using Corollary 2.2.11, we obtain a correspondence between degrees of quasimaps and classes on the threefold $S \times E$,

$$(n, -\check{\beta})\colon \mathrm{Eff}(S^{[n]}, \mathfrak{Coh}_r^{\sharp}(S)) \hookrightarrow \mathrm{Eff}(S \times E) \oplus H^6(S \times E),$$

such that for $n > 1$ its restriction to $\mathrm{Eff}(S^{[n]}) \subset \mathrm{Eff}(S^{[n]}, \mathfrak{Coh}_r(S))$ is given by

$$k_1 C_{\beta_h} + k_2 \mathsf{A} \mapsto ((n, k_1\beta_h), k_2),$$

and for $n = 1$ the class β_h goes to $((\beta_h, 1), 0)$. Note that we changed the sign of classes on $S \times E$, which amounts to considering the class of the subscheme rather than its ideal. A general class in $\mathrm{Eff}(S^{[n]}, \mathfrak{Coh}_r(S))$ can therefore be identified with $k_1 C_{\beta_h} + k_2 \mathsf{A}$ for possibly negative k_2.

By Corollary 2.3.7 we have the following identification of moduli spaces

$$Q^{0^+}_{E/E}(S^{[n]}, C_{\beta_h} + k\mathsf{A})^{\sharp} \cong [\mathrm{P}_{(n,\beta_h),k}(S \times E)/E],$$

such that the natural obstruction theories match. As before the subscript notation of the moduli on the left indicates that we consider maps up to translations of E, for the same reason we take the quotient by E on the left. On the other hand,

$$Q^{\infty}_{E/E}(S^{[n]}, C_{\beta_h} + k\mathsf{A})^{\sharp} = \overline{M}_{E/E}(S^{[n]}, C_{\beta_h} + k\mathsf{A}).$$

Consider now the following two generating series

$$\mathsf{PT}(p,q,\tilde{q}) := \sum_{n\geq 0}\sum_{h\geq 0}\sum_{k\in\mathbb{Z}} p^k q^{h-1}\tilde{q}^{n-1}\int_{[\mathrm{P}_{(n,\beta_h),k}(S\times E)/E]^{\mathrm{vir}}} 1$$

$$\mathsf{GW}(p,q,\tilde{q}) := \sum_{n>0}\sum_{h\geq 0}\sum_{k\geq 0} p^k q^{h-1}\tilde{q}^{n-1}\int_{[\overline{M}_{E/E}(S^{[n]},C_{\beta_h}+kA)]^{\mathrm{vir}}} 1.$$

The series are well-defined, because (S,β) and (S',β') are deformation equivalent, if and only if

$$\beta^2 = \beta'^2 \quad \text{and} \quad \mathrm{div}(\beta) = \mathrm{div}(\beta'),$$

where $\mathrm{div}(\beta)$ is the divisibility of the class, in our case β_h's are primitive by definition. In [OP18] it was proven that

$$\mathsf{PT}(p,q,\tilde{q}) = \frac{1}{-\chi_{10}(p,q,\tilde{q})},$$

where $\chi_{10}(p,q,\tilde{q})$ is the *Igusa cusp form*, hence the name of the conjecture. By the discussion above, we can view both series as generating series of quasimaps for $\epsilon \in \{0^+,\infty\}$. Using Theorem 3.2.3, we obtain

$$\mathsf{PT}(p,q,\tilde{q}) = \mathsf{GW}(p,q,\tilde{q}) + \sum_{n\geq 0}\sum_{h\geq 0}\sum_{k\in\mathbb{Z}} p^k q^{h-1}\tilde{q}^{n-1}\chi(S^{[n]})\langle[\mathrm{pt}],\mathbb{1}\rangle^{\sharp,S^{[n]},0^+}_{C_{\beta_h}+kA}.$$

Remark 3.3.3. The effective quasimap cone $\mathrm{Eff}(S^{[n]},\mathfrak{Coh}_r(S))$ is strictly bigger than the effective cone $\mathrm{Eff}(S^{[n]})$. For a class, which is not in $\mathrm{Eff}(S^{[n]})$, the moduli space of ∞-stable quasimaps will be just empty. Nevertheless, the wall-crossing formula still applies but with zero contribution from $\epsilon=\infty$.

The invariants $\langle[\mathrm{pt}],\mathbb{1}\rangle^{\sharp,S^{[n]},0^+}_{C_{\beta_h}+kA}$ are just relative rubber PT invariants on $S\times\mathbb{P}^1$. These are invariants associated to the moduli of stable pairs over $S\times\mathbb{P}^1$ relative to two vertical divisors $S_{0,\infty}\subset S\times\mathbb{P}^1$ up to the rescaling $\mathbb{C}^*$-action coming from $\mathbb{P}^1$-factor which fixes $S_{0,\infty}$. These invariants can be re-expressed as standard relative PT invariants with absolute insertions as follows

$$\langle[\mathrm{pt}],\mathbb{1}\rangle^{\sharp,S^{[n]},0^+}_{C_{\beta_h}+kA} = \langle[\mathrm{pt}]\mid\tilde{\tau}_0(D\boxtimes[\omega])\rangle^{\sharp,S\times\mathbb{P}^1}_{(n,\beta_h),k},$$

where $D\in H^2(S,\mathbb{Q})$ is some class such that $D\cdot\beta_h=1$, and $\omega\in H^*(\mathbb{P}^1,\mathbb{Z})$ is the point class. We will prove the above rigidification formula in Lemma 3.3.6 for a general moduli of sheaves M.

The wall-crossing invariants can also be given a different and more sheaf-theoretic interpretation as virtual Euler numbers of Quot schemes, as it is explained in [Obec]. In the same paper the wall-crossing invariants are also explicitly computed for $S^{[n]}$. Therefore we obtain the following corollary, which completes the proof of the Igusa cusp conjecture.

Corollary 3.3.4.

$$\mathsf{PT}(p,q,\tilde{q}) = \mathsf{GW}(p,q,\tilde{q}) + \frac{1}{F^2\Delta} \cdot \frac{1}{\tilde{q}} \prod_{n\geq 1} \frac{1}{(1-(\tilde{q}\cdot G)^n)^{24}}.$$

For the definition of the generating series on the right we refer to [OP16, Section 2].

3.3.2 Higher-rank DT invariants

Let M be a smooth projective moduli of sheaves on a K3 surface in a class $\mathbf{v}$ satisfying the assumptions of Theorem 2.2.17. The moduli space M is deformation equivalent to a punctorial Hilbert scheme $S^{[n]}$, where $2n = \dim(M)$. Therefore Gromov–Witten theory of M is equivalent to the one of $S^{[n]}$. Applying quasimap wall-crossing both to M and to $S^{[n]}$, we can express higher-rank DT invariants of a threefold $\mathrm{K3} \times C$ in terms of rank-one DT invariants and corresponding I-functions.

$\mathrm{K3} \times \mathbb{P}^1$

Let us firstly consider invariants on $S \times \mathbb{P}^1$ relative to $S_{0,1,\infty}$. As previously by fixing the markings we obtain

$$Q^{0^+}_{0,3}(M,\beta) \cong M_{\mathbf{v},\beta,u}(S\times\mathbb{P}^1/S_{0,1,\infty}).$$

Moreover, as in the case of $S^{[n]}$ there is no wall-crossing by Theorem 3.2.1, therefore

$$\langle \gamma_1,\gamma_2,\gamma_3\rangle^{M,0^+}_{0,\beta} = \langle \gamma_1,\gamma_2,\gamma_3\rangle^{M,\infty}_{0,\beta}.$$

We deform (M,β) to $(S^{[n]},\beta)$, keeping β as a (1,1)-class and identifying cohomologies via the deformation $H^*(M,\mathbb{Q}) \cong H^*(S^{[n]},\mathbb{Q})$. Under these identifications we have

$$\begin{aligned}\langle \gamma_1,\gamma_2,\gamma_3\rangle^{M,0^+}_{0,\beta} &= \langle \gamma_1,\gamma_2,\gamma_3\rangle^{M,\infty}_{0,\beta}\\ &= \langle \gamma_1,\gamma_2,\gamma_3\rangle^{S^{[n]},\infty}_{0,\beta} = \langle \gamma_1,\gamma_2,\gamma_3\rangle^{S^{[n]},0^+}_{0,\beta}. \quad (3.8)\end{aligned}$$

Passing from quasimaps to sheaves and using (3.8), we obtain the following result.

Corollary 3.3.5. *Given a deformation of* (M,β) *to* $(S^{[n]},\beta)$ *we have*

$$\langle\gamma_1,\gamma_2,\gamma_3\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check\beta}=\langle\gamma_1,\gamma_2,\gamma_3\rangle^{S\times\mathbb{P}^1}_{n,\check\beta}.$$

More generally, the result above can be restated for relative geometries of the type $S\times C_{g,N}\to\overline{M}_{g,N}$ such that $2g-2+N>0$.

K3 $\times E$

Consider now $S\times E$ for an elliptic curve E. Applying the same procedure as for $S\times\mathbb{P}^1$, we obtain

$$\int_{[M_{\mathbf{v},\check\beta,u}(S\times E)/E]^{\mathrm{vir}}}1=\int_{[\mathrm{P}_{n,\check\beta}(S\times E)/E]^{\mathrm{vir}}}1 \\ +\chi(S^{[n]})\left(\langle[\mathrm{pt}],\mathbb{1}\rangle^{S^{[n]},0^+}_{0,\beta}-\langle[\mathrm{pt}],\mathbb{1}\rangle^{M,0^+}_{0,\beta}\right).$$

Now we express the wall-crossing invariants in terms of DT invariants with relative insertions by a standard rigidification-of-rubber argument.

Lemma 3.3.6. *Given a class* $D\in H^2(S,\mathbb{Q})$*, such that* $\mathrm{c}_1(\check\beta)\cdot D=1$*, then*

$$\langle[\mathrm{pt}],\mathbb{1}\rangle^{M,0^+}_{0,\beta}=\langle[\mathrm{pt}],\mathbb{1}\mid\tilde\tau_0(D\boxtimes\omega)\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check\beta}.$$

Proof. There exists a map

$$p\colon M_{\mathbf{v},\check\beta,u}(S\times\mathbb{P}^1/S_{0,\infty})\to Q_{0,2}(M,\beta)$$

which is given by associating to a sheaf the corresponding quasimap and contracting unstable components. Let $\mathfrak{M}_{(\mathbb{P}^1,0,\infty)}$ be the moduli of expanded degenerations of $\mathbb{P}^1$ at 0 and ∞, then the obstruction theory of $Q_{0,2}(M,\beta)$ relative to $\mathfrak{M}_{0,2}$ pullbacks to the obstruction theory of $M_{\mathbf{v},\check\beta,u}(S\times\mathbb{P}^1/S_{0,\infty})$ relative to $\mathfrak{M}_{(\mathbb{P}^1,0,\infty)}$,

$$\mathcal{H}om_\pi(\mathbb{F},\mathbb{F})_0[1]=p^*\mathcal{H}om_{\bar\pi}(\bar{\mathbb{F}},\bar{\mathbb{F}})_0[1].$$

In particular, $p_*[M_{\mathbf{v},\check\beta,u}(S\times\mathbb{P}^1/S_{0,\infty})]^{\mathrm{vir}}=[Q_{0,2}(M,\beta)]^{\mathrm{vir}}$. Consider now the following square

$$\begin{array}{ccc} S\times\mathfrak{C}_{(\mathbb{P}^1,0,\infty)}\times_{\mathfrak{M}_{(\mathbb{P}^1,0,\infty)}}M_{\mathbf{v},\check\beta,u}(S\times\mathbb{P}^1/S_{0,\infty}) & \xrightarrow{\rho} & S\times\mathfrak{C}_{0,2}\times_{\mathfrak{M}_{0,2}}Q_{0,2}(M,\beta) \\ \downarrow\pi & & \downarrow\bar\pi \\ M_{\mathbf{v},\check\beta,u}(S\times\mathbb{P}^1/S_{0,\infty}) & \xrightarrow{p} & Q_{0,2}(M,\beta) \end{array}$$

Let

$$\iota\colon S\times M_{\mathbf{v},\check{\beta},u}(S\times\mathbb{P}^1/S_{0,\infty})\hookrightarrow S\times\mathfrak{C}_{(\mathbb{P}^1,0,\infty)}\times_{\mathfrak{M}_{(\mathbb{P}^1,0,\infty)}}M_{\mathbf{v},\check{\beta},u}(S\times\mathbb{P}^1/S_{0,\infty})$$

be the inclusion of the fiber over $1\in\mathbb{P}^1$. The composition $\rho\circ\iota$ is an isomorphism. Indeed, $S\times C$ can be identified with some expended degeneration $\mathbb{P}^1[k_0,k_\infty]$ by sending $p\in C$ to $1\in\mathbb{P}^1$. Now given $(x,p)\in S\times C$ and a sheaf F on $S\times C$, we send $((x,p),\bar{F})$ to (x,F), where F is sheaf on the corresponding expanded degeneration $\mathbb{P}^1[k_0,k_\infty]$. This defines the inverse of $\rho\circ\iota$. Then given a class $D\in H^2(S,\mathbb{Q})$,

$$\begin{aligned}
&p_*\pi_*(\mathrm{ch}_2(\mathbb{F})\cdot p_S^*(D)\cdot p_{\mathbb{P}^1}^*(\omega)\cap\pi^*[M_{\mathbf{v},\check{\beta},u}(S\times\mathbb{P}^1/S_{0,\infty})]^{\mathrm{vir}})\\
&\quad=\bar{\pi}_*\rho_*\iota_*(\iota^*(\mathrm{ch}_2(\mathbb{F})\cdot p_S^*(D))\cap\pi^*[M_{\mathbf{v},\check{\beta},u}(S\times\mathbb{P}^1/S_{0,\infty})]^{\mathrm{vir}})\\
&\qquad=\bar{\pi}_*(\mathrm{ch}_2(\bar{\mathbb{F}})\cdot p_S^*(D)\cap\bar{\pi}^*[Q_{0,2}(M,\beta)]^{\mathrm{vir}})\\
&\qquad\quad=(c_1(\check{\beta})\cdot D)[Q_{0,2}(M,\beta)]^{\mathrm{vir}}=[Q_{0,2}(M,\beta)]^{\mathrm{vir}}.
\end{aligned}$$

After feeding the virtual class with the relative insertions we obtain the claim of the lemma. □

By degenerating $\mathbb{P}^1$ to $\mathbb{P}^1\cup\mathbb{P}^1$ and applying degeneration formula we obtain

$$\langle[\mathrm{pt}],\mathbb{1}\mid\tilde{\tau}_0(D\boxtimes\omega)\mid\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check{\beta}}=\langle[\mathrm{pt}]\mid\tilde{\tau}_0(D\boxtimes\omega)\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check{\beta}}.$$

Note that we treat sheaves as quasimaps and therefore the stability of sheaves is the one of quasimaps, the rest goes as in the rank-one case. Putting everything together, we get the following wall-crossing expression for higher-rank DT invariants.

Corollary 3.3.7.

$$\begin{aligned}
\int_{[M_{\mathbf{v},\check{\beta},u}(S\times E)/E]^{\mathrm{vir}}}1&=\int_{[\mathrm{P}_{n,\check{\beta}}(S\times E)/E]^{\mathrm{vir}}}1\\
&\quad+\chi(S^{[n]})\left(\langle[\mathrm{pt}]\mid\tilde{\tau}_0(D\boxtimes\omega)\rangle^{S\times\mathbb{P}^1}_{n,\check{\beta}}-\langle[\mathrm{pt}]\mid\tilde{\tau}_0(D\boxtimes\omega)\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check{\beta}}\right).
\end{aligned}$$

Using the same argument as in Lemma 3.3.6, we also get the following rigidification of the genus-1 invariant

$$\int_{[M_{\mathbf{v},\check{\beta},u}(S\times E)/E]^{\mathrm{vir}}}1=\langle\tilde{\tau}_0(D\boxtimes\omega)\rangle^{S\times E}_{\mathbf{v},\check{\beta}},$$

then by degenerating E to $E \cup \mathbb{P}^1$ and applying the degeneration formula to the invariants $\langle \tilde{\tau}_0(D \boxtimes \omega) \rangle^{S \times E}_{\mathbf{v},\check{\beta}}$ we obtain

$$\int_{[M_{\mathbf{v},\check{\beta},u}(S\times E)/E]^{\mathrm{vir}}} 1 = \langle \mathbb{1} \mid \tilde{\tau}_0(D \boxtimes \omega) \rangle^{S\times E}_{\mathbf{v},\check{\beta}} + \chi(M) \langle [\mathrm{pt}] \mid \tilde{\tau}_0(D \boxtimes \omega) \rangle^{S \times \mathbb{P}^1}_{\mathbf{v},\check{\beta}},$$

the second term on the right is the wall-crossing term, therefore

$$\langle \mathbb{1} \mid \tilde{\tau}_0(D \boxtimes \omega) \rangle^{S\times E}_{\mathbf{v},\check{\beta}} = \langle \mathbb{1} \mid \tilde{\tau}_0(D \boxtimes \omega) \rangle^{S\times E}_{n,\check{\beta}}.$$

Using the Igusa cusp conjecture, we therefore can explicitly express these higher-rank relative DT invariants. Using the results of [Obec], we can also express the wall-crossing invariants in terms of virtual Euler numbers of Quot schemes.

By Lemma 2.2.18 the higher-rank invariants associated to the moduli space $M_{\mathbf{v},\check{\beta},u}(S \times C)$ can be related to those of the moduli space of sheaves with a fixed determinant $M_{\mathbf{v},\check{\beta},L}(S \times C)$. In particular, we obtain

$$\int_{[M_{\mathbf{v},\check{\beta},L}(S\times E)/E]^{\mathrm{vir}}} 1 = \int_{[M_{\mathbf{v},\check{\beta},u}(S\times E)/E]^{\mathrm{vir}}} \mathrm{rk}(\mathbf{v})^2.$$

Remark 3.3.8. Recall that there are some limitations for the above results due to the assumptions of Theorem 2.2.17 and Proposition 5.1.4. An ideal example for which the above correspondence between higher-rank and rank-one invariants holds is the invariants that arise from the moduli of sheaves in the class $\mathbf{v} = (2, \alpha, 2k + 1)$ for a polarisation such that $\deg(\alpha)$ is odd (or a generic polarisation that is close to a polarisation for which $\deg(\alpha)$ is odd). Firstly, $\mathrm{rk}(\mathbf{v})$ and $\deg(\mathbf{v})$ are coprime, therefore there are no strictly slope semistable sheaves. For those k for which the expected dimension of the moduli is positive, the moduli is non-empty, and there exists a class $u = [\mathcal{O}_S] - k[\mathcal{O}_{\mathrm{pt}}] \in K_0(S)$ that satisfies $\chi(\mathbf{v} \cdot u) = 1$. Moreover, Proposition 5.1.4 holds in the rank-two case. Therefore the moduli $M_{\mathbf{v},\check{\beta},u}(S \times C)$ is the moduli of all stable sheaves for a suitable polarisation. More specifically, such set-up can be arranged on an elliptic K3 surface.

3.3.3 DT/PT correspondence

Using the wall-crossing for the standard pair $(S^{[n]}, \mathfrak{Coh}_r(S))$ and the perverse pair $(S^{[n]}, \mathfrak{Coh}^{\sharp}_r(S))$, we also can relate rank-one PT invariants to rank-one DT invariants. Unlike for standard virtual fundamental classes, in the reduced case we really have the equivalence of two theories by Corollary 3.2.1,

$$\mathsf{DT}(S \times C_{g,N}/\overline{M}_{g,N}) = \mathsf{PT}(S \times C_{g,N}/\overline{M}_{g,N}),$$

for a relative geometry

$$S \times C_{g,N} \to \overline{M}_{g,N},$$

where $2g - 2 + N > 0$. The wall-crossing terms are zero on both sides, because from quasimap point-of-view we consider only primary invariants. In particular, we obtain that

$$\mathsf{DT}(S \times \mathbb{P}^1/S_{0,1,\infty}) = \mathsf{PT}(S \times \mathbb{P}^1/S_{0,1,\infty}).$$

The equivalence of two theories also holds for $S \times E$, it was shown in [OS19, Theorem 3] and [Obe18].

Chapter 4

Gromov–Witten/Hurwitz wall-crossing

4.1 The moduli problem

Let X be a smooth projective variety, $(\mathcal{C}, \mathbf{x})$ be a twisted[1] marked nodal curve and $\mathcal{P}$ be a possibly disconnected orbifold nodal curve.

Definition 4.1.1. For a map

$$f = f_X \times f_{\mathcal{C}} \colon \mathcal{P} \to X \times \mathcal{C},$$

the data $(\mathcal{P}, \mathcal{C}, \mathbf{x}, f)$ is called a *twisted pre-admissible* map, if

- $f_{\mathcal{C}}$ is étale over marked points and nodes;
- $f_{\mathcal{C}}$ is a representable;
- f is non-constant on each connected component.

We will refer to $\mathcal{P}$ and $\mathcal{C}$ as *source* and *target* curves, respectively. Note that by the first two conditions of pre-admissibility, $\mathcal{P}$ itself must be a twisted nodal curve with orbifold points over nodes and marked points of $\mathcal{C}$.

Consider now the following complex

$$Rf_{\mathcal{C}*}[f_{\mathcal{C}}^* \mathbb{L}_{\mathcal{C}} \to \mathbb{L}_{\mathcal{P}}] \in \mathrm{D}^b(\mathcal{C}),$$

[1] By a twisted nodal curve we will always mean a balanced twisted nodal curve.

which is supported at finitely many points of the non-stacky smooth locus, which we call *branching* points. They arise either due to ramification points or contracted components of the map $f_{\mathcal{C}}$. Following [FP02], to the complex above we can associate a effective Cartier divisor

$$\mathsf{br}(f) \in \mathrm{Div}(\mathcal{C})$$

by taking the support of the complex weighted by its Euler characteristics. This divisor will be referred to as *branching divisor*.

Let us give a more explicit expression for the branching divisor. Let $\mathcal{P}_\circ \subseteq \mathcal{P}$ be the maximal subcurve of $\mathcal{P}$ which contracted by the map $f_{\mathcal{C}}$. Let $\mathcal{P}_\bullet \subseteq \mathcal{P}$ be the complement of $\mathcal{P}_\circ$, i.e. the maximal subcurve which is not contracted by the map $f_{\mathcal{C}}$. By $\widetilde{\mathcal{P}}_\bullet$ we denote its normalisation at the nodes which are mapped into a regular locus of $\mathcal{C}$. Note that the restriction of $f_{\mathcal{C}}$ to $\widetilde{\mathcal{P}}_\bullet$ is a ramified cover, the branching divisor of which is therefore given by points of ramifications.

By $\widetilde{\mathcal{P}}_{\circ,i}$ we denote the connected components of the normalisation $\widetilde{\mathcal{P}}_\circ$ and by $p_i \in \mathcal{C}$ their images in $\mathcal{C}$. Finally, let $N \subset \mathcal{P}$ be the locus of nodal points which are mapped into regular locus of $\mathcal{C}$. The branching divisor $\mathsf{br}(f)$ then has the following explicit expression.

Lemma 4.1.2. *With the notation from above we have*

$$\mathsf{br}(f) = \mathsf{br}(f_{|\widetilde{\mathcal{P}}_\bullet}) + \sum_i (2g(\widetilde{\mathcal{P}}_{\circ,i}) - 2)[p_i] + 2f_*(N).$$

Proof. By the definition of twisted pre-admissibility, all the branching takes place away from orbifold points. We therefore have to determine what are the contributions of contracted components (which are schemes) to the branching divisor.

Given a nodal curve C and its normalisation $v : \widetilde{C} \to C$, let $N_C \subset C$ be the singular locus of C. Recall that $\mathbb{L}_C \cong \Omega_C$, we therefore have the following sequence

$$0 \to \mathcal{O}_{N_C} \to \mathbb{L}_C \to v_*\mathbb{L}_{\widetilde{C}} \to 0, \tag{4.1}$$

which, in particular, implies that

$$\chi(\mathbb{L}_C) = \chi(\omega_C).$$

With the sequence (4.1) the proof of the claim is the same as in [FP02], the difference is that we use (4.1) instead of [FP02, (20)]. $\square$

Remark 4.1.3. The reason we use $\mathbb{L}_{\mathcal{C}}$ instead of $\omega_{\mathcal{C}}$ is that $\pi^*\omega_C \cong \omega_{\mathcal{C}}$, where $\pi\colon \mathcal{C} \to C$ is the projection to the coarse moduli space. Hence $\omega_{\mathcal{C}}$ does not see non-étalness of π. Moreover, it is unclear, if a map $f_{\mathcal{C}}^*\omega_{\mathcal{C}} \to \omega_{\mathcal{P}}$ exists at all in general.

We fix $L \in \mathrm{Pic}(X)$, an ample line bundle on X, such that for all effective curve classes $\gamma \in H_2(X, \mathbb{Z})$,

$$\deg(\gamma) = \beta \cdot \mathrm{c}_1(L) \gg 0.$$

Let $(\mathcal{P}, \mathcal{C}, \mathbf{x}, f)$ be a twisted pre-admissible map. For a point $p \in \mathcal{C}$, let

$$f^*L_p := f^*L_{|f^{-1}(p)},$$

we set $\deg(f^*L_p) = 0$, if $f^{-1}(p)$ is 0-dimensional. For a component $\mathcal{C}' \subseteq \mathcal{C}$, let

$$f^*L_{|\mathcal{C}'} := f^*L_{|f^{-1}(\mathcal{C}')}.$$

Recall that a *rational tail* of a curve $\mathcal{C}$ is a component isomorphic to $\mathbb{P}^1$ with one special point (a node or a marked point). A *rational bridge* is a component isomorphic to $\mathbb{P}^1$ with two special points.

Definition 4.1.4. Let $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$. A twisted pre-admissible map f is twisted ϵ-admissible, if

(**i**) for all points $p \in \mathcal{C}$,

$$\mathrm{mult}_p(\mathsf{br}(f)) + \deg(f^*L_p) \leq e^{-1/\epsilon};$$

(**ii**) for all rational tails $T \subseteq (\mathcal{C}, \mathbf{x})$,

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^*L_{|T}) > e^{-1/\epsilon};$$

(**iii**)

$$|\mathrm{Aut}(f)| < \infty.$$

Lemma 4.1.5. *The condition of twisted ϵ-admissability is an open condition.*

Proof. The conditions of twisted ϵ-admissibility are constructable. Hence we can use the valuative criteria for openness, i.e. we need to show that if a pre-admissible map

$$(\mathcal{P}, \mathcal{C}, \mathbf{x}, f) \in \mathfrak{M}(X \times \mathfrak{C}^{\mathrm{tw}}_{g,N}/\mathfrak{M}^{\mathrm{tw}}_{g,N}, (\gamma, n))(R)$$

is ϵ-admissible at the closed fiber of a discrete valuation ring R with fraction field K, then it is ϵ-admissible at the generic fiber. In fact, each of conditions of ϵ-admissibility is an open condition. For example, let

$$T \subseteq (\mathcal{C}, \mathbf{x})$$

a family of curves such that in the generic fiber $T_{|\operatorname{Spec}(K)}$ is a rational tail that does not satisfy the condition (**ii**). Then the central fiber $T_{|\operatorname{Spec}(\mathbb{C})}$ of T will be a tree of rational curves, whose rational tails don not satisfy the condition (**ii**), because the degree of both $\mathsf{br}(f)$ and f^*L can only decrease on rational tails of $T_{|\operatorname{Spec}(\mathbb{C})}$. Here we need to use that $\mathsf{br}(f)$ is defined for families of pre-admissible twisted maps to conclude that the degree of $\mathsf{br}(f)$ is constant in families. $\square$

A family of twisted ϵ-admissable maps over a base scheme B is given by two families of twisted B-curves $\mathcal{P}$ and $(\mathcal{C}, \mathbf{x})$ and a B-map

$$f = f_X \times f_{\mathcal{C}} \colon \mathcal{P} \to X \times (\mathcal{C}, \mathbf{x}),$$

whose fibers over geometric points of B are ϵ-admissable maps. An isomorphism of two families

$$\Phi = (\phi_1, \phi_2) \colon (\mathcal{P}, \mathcal{C}, \mathbf{x}, f) \cong (\mathcal{P}', \mathcal{C}', \mathbf{x}', f')$$

is given by the data of isomorphisms of the source and target curves

$$(\phi_1, \phi_2) \in \operatorname{Isom}_B(\mathcal{P}, \mathcal{P}') \times \operatorname{Isom}_B((\mathcal{C}, \mathbf{x}), (\mathcal{C}', \mathbf{x}')),$$

which commute with the maps f and f',

$$f' \circ \phi_1 \cong \phi_2 \circ f.$$

Definition 4.1.6. Given an element

$$\beta = (\gamma, \mathsf{m}) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z},$$

we say that a twisted ϵ-admissible map is of degree β to $X^{(n)}$, of genus g with N markings, if

- f is of degree (γ, n) and $\deg(\mathsf{br}(f)) = \mathsf{m}$;
- $g(C) = g$ and $|\mathbf{x}| = N$.

We define

$$\begin{aligned} Adm^{\epsilon}_{g,N}(X^{(n)},\beta)^{\mathrm{tw}} \colon (Sch/\mathbb{C})^{\circ} &\to (Grpd) \\ S &\mapsto \{\text{families of } \epsilon\text{-admissable maps over B}\} \end{aligned}$$

to be the moduli space of twisted ϵ-admissible to $X^{(n)}$ maps of degree β and genus g with N markings. Following [FP02], one can construct the branching divisor for any base scheme B, thereby obtaining a map

$$\mathsf{br} : Adm^{\epsilon}_{g,N}(X^{(n)},\beta)^{\mathrm{tw}} \to \mathfrak{M}_{g,N,\mathsf{m}}. \tag{4.2}$$

The space $\mathfrak{M}_{g,N,\mathsf{m}}$ is an Artin stack which parametrises triples

$$(C, \mathbf{x}, D),$$

where $(C, \mathbf{x})$ is a genus-g curve with n markings; D is an effective divisor of degree m disjoint from markings $\mathbf{x}$. An isomorphism of triples is an isomorphism of curves which preserve markings and divisors.

There is another moduli space related to $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)^{\mathrm{tw}}$, which is obtained by associating to a twisted ϵ-admissible map the corresponding map between the coarse moduli spaces of the twisted curves. This association defines the following map

$$p\colon Adm^{\epsilon}_{g,N}(X^{(n)},\beta)^{\mathrm{tw}} \to \mathfrak{M}(X\times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N},(\gamma,n)),$$

where $\mathfrak{M}(X\times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N},(\beta,n))$ is the relative moduli space of stable maps to the relative geometry

$$X\times \mathfrak{C}_{g,N} \to \mathfrak{M}_{g,N},$$

where $\mathfrak{C}_{g,N} \to \mathfrak{M}_{g,N}$ is the universal curve. By Lemma 4.1.5 the image of p is open.

Definition 4.1.7. We denote the image of p by $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ with its natural open-substack structure.

The closed points of $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ are relative stable maps with restricted branching away from marked points and nodes, to which we refer as *ϵ-admissable maps*. On can similarly define *pre-admissable maps*. As in Definition 4.1.1, we denote the data of a pre-admissible map by

$$(P, C, \mathbf{x}, f).$$

The moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ will be the central objects of our study.

Remark 4.1.8. The difference between the moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ and $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)^{\mathrm{tw}}$ is the same as the one between admissible covers and twisted bundles of [ACV03]. We prefer to work with $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$, because it is more convenient to work with schemes than with stacks for the purposes of deformation theory and of analysis of the basic properties of the moduli spaces. Moreover, the enumerative geometries of these two moduli spaces are equivalent, at least for the relevant values of ϵ. For more details see Section 4.1.3 and Section 4.1.6.

Since $\mathsf{br}(f)$ is supported away from stacky points, the branching-divisor map descends,

$$\mathsf{br} : Adm^{\epsilon}_{g,N}(X^{(n)},\beta) \to \mathfrak{M}_{g,N,\mathsf{m}}. \tag{4.3}$$

The moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ also admit a disjoint-union decomposition

$$Adm^{\epsilon}_{g,N}(X^{(n)},\beta) = \coprod_{\underline{\mu}} Adm^{\epsilon}_{g,N}(X^{(n)},\beta,\underline{\mu}), \tag{4.4}$$

where $\underline{\mu} = (\mu^1,\dots,\mu^N)$ is a N-tuple of ramifications profiles of f_C over the markings $\mathbf{x}$.

Riemann-Hurwitz formula extends to the case of pre-admissible maps.

Lemma 4.1.9. *If $f\colon P \to (C,\mathbf{x})$ is a degree-n pre-admissible map with ramification profiles $\underline{\mu} = (\mu^1,\dots,\mu^N)$ at the markings $\mathbf{x} \subset C$, then*

$$2g(P) - 2 = n \cdot (2g(C) - 2) + \deg(\mathsf{br}(f)) + \sum_i \mathrm{age}(\mu^{\mathrm{i}}).$$

Proof. Using Lemma 4.1.2 and the standard Riemann-Hurwitz formula, one can readily check that the above formula holds for pre-admissible maps. □

4.1.1 Properness

We now establish the properness of $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$, starting with the following result.

Proposition 4.1.10. *The moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ are quasi-separated Deligne–Mumford stacks of finite type.*

Proof. By ϵ-admissibility condition, the map br factors through a quasi-separated substack of finite type. Indeed, $(C,\mathbf{x},\mathsf{br}(f))$ is not stable (i.e. has infinitely many automorphisms), if one of the following holds (**i**) there is a rational tail $T \subseteq (C, \mathbf{x})$, such that $\mathrm{supp}(\mathsf{br}(f)_{|T})$ is at most a point;

(**ii**) there is a rational bridge $B \subseteq (C, \mathbf{x})$, such that $\mathrm{supp}(\mathsf{br}(f)_{|B})$ is empty.

Up to a change of coordinates, the restriction of f_C to T or B must of the form

$$z^{\underline{n}}\colon (\sqcup^k \mathbb{P}^1) \sqcup_0 P' \to \mathbb{P}^1 \tag{4.5}$$

at each connected component of P over T or B. Let us clarify the notation of (4.5). The curve $\sqcup^k \mathbb{P}^1$ is a disjoint union of k distinct $\mathbb{P}^1$. A curve P' is attached to to a disjoint union $\sqcup^k \mathbb{P}^1$ at the points $0 \in \mathbb{P}^1$ at each connected component of the disjoint union; P' is contracted to $0 \in \mathbb{P}^1$ in the target curve $\mathbb{P}^1$, while on i'th $\mathbb{P}^1$ in the disjoint union the map is given by z^{n_i} for $\underline{n} = (n_1, \dots, n_k)$.

The fact that the restriction of f_C is given by a map of such form can be seen as follows. The condition (**i**) or (**ii**) implies that the restriction of f_C to T or B has at most two[2] branching points, which in turn implies that the source curve must be $\mathbb{P}^1$ by Riemann-Hurwitz theorem. A map from $\mathbb{P}^1$ to itself with two branching points is given by $z^{m'}\colon \mathbb{P}^1 \to \mathbb{P}^1$ up to change of coordinates. For a rational tail T, there might also be a contracted component P' attached to the ramification points.

In the case of (**ii**), the ϵ-admissibility condition then says that

$$\deg(f^* L_{|B}) > 0.$$

While in the case of (**i**),

$$\deg(\mathsf{br}(f)_{|T}) = \mathrm{mult}_p(\mathsf{br}(f))$$

for a unique point $p \in T$ which is not a node. Hence ϵ-admissibility says that

$$\deg(f^* L_{|T}) - \deg(f^* L_p) > 0.$$

Since we fixed the class β, the conclusions above bound the number of components T or B by $\deg(\beta)$. Hence the image of br is contained in a quasi-compact substack of $\mathfrak{M}_{g,N,\mathsf{m}}$, which is therefore quasi-separated and of finite type, because $\mathfrak{M}_{g,N,\mathsf{m}}$ is quasi-separated and locally of finite type.

The branching-divisor map br is of finite type and quasi-separated, since the fibers of br are sub-loci of stable maps to $X \times C$ for some nodal curve C. The moduli space $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$ is of finite type and quasi-separated itself, because br is of finite type and quasi-seperated and factors through a quasi-separated substack of finite type. $\square$

[2]Remember that branching might also be present at the nodes.

Lemma 4.1.11. *Given a pre-admissable map* $(P, C, \mathbf{x}, f)$*. Let* $(P', C', \mathbf{x}', f')$ *be given by contraction of a rational tail* $T \subseteq (C, \mathbf{x})$ *and stabilisation of the induced map*

$$f\colon P \to X \times C'.$$

Let $p \in C'$ *be the image of contraction of* T*. Then the following holds*

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^* L_{|T}) = \mathrm{mult}_p(\mathsf{br}(f')) + \deg(f'^* L_p).$$

Proof. By Lemma 4.1.9,

$$2g(P_{|T}) - 2 = -2d + \deg(\mathsf{br}(f)) + d - \ell(p),$$

where $\ell(p)$ is the number of points in fiber above p, from which it follows that

$$\begin{aligned} \deg(\mathsf{br}(f)) &= 2g(P_{|T}) - 2 + 2d - d + \ell(p) \\ &= 2g(P_{|T}) - 2 + d + \ell(p). \end{aligned}$$

By Lemma 4.1.2,

$$\begin{aligned} \mathrm{mult}_p(\mathsf{br}(f)) &= 2g(P_{|T}) - 2 + 2\ell(p) + d - \ell(p) \\ &= 2g(P_{|T}) - 2 + d + \ell(p). \end{aligned}$$

It is also clear by definition, that

$$\deg(f^* L_{|T}) = \deg(f^* L_p),$$

the claim then follows. □

Definition 4.1.12. Let R be a discrete valuation ring. Given a pre-admissible map $(P, C, \mathbf{x}, f)$ over $\operatorname{Spec} R$. A *modification* of $(P, C, \mathbf{x}, f)$ is a pre-admissible map $(\widetilde{P}, \widetilde{C}, \widetilde{\mathbf{x}}, \widetilde{f})$ over $\operatorname{Spec} R'$ such that

$$(\widetilde{P}, \widetilde{C}, \widetilde{\mathbf{x}}, \widetilde{f})_{|\operatorname{Spec} K'} \cong (P, C, \mathbf{x}, f)_{|\operatorname{Spec} K'},$$

where R' is a finite extension of R with a fraction field K'.

A modification of a family of curves C over a discrete valuation ring is given by three operations:

- blow-ups of the central fiber of C;
- contractions of rational tails and rational bridges in the central fiber of C;

- base changes with respect to finite extensions of discrete valuation rings.

A modification of a pre-admissible map is therefore given by an appropriate choice of three operations above applied to both target and source curves, such that the map f can be extended as well.

Theorem 4.1.13. *The moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ are proper Deligne–Mumford stacks.*

Proof. We will now use the valuative criteria of properness for quasi-separated Deligne–Mumford stacks. Let

$$(P^*, C^*, \mathbf{x}^*, f^*) \in Adm^{\epsilon}_{g,N}(X^{(n)},\beta)(K)$$

be a family of ϵ-admissable maps over the fraction field K of a discrete valuation ring R. The strategy of the proof is to separate P^* into two components $P^*_{\circ}$ and $P^*_{\bullet}$, the contracted component and non-contracted one of f^*_C, respectively (as it was done for Lemma 4.1.2). We then take a limit of $f^*_{|P^*_{\bullet}}$ preserving it as a cover over the target curve, and a limit of $f^*_{|P^*_{\circ}}$ as a stable map. We then glue the two limits back and perform a series of modifications to get rid of points or rational components that don not satisfy ϵ-admissibility.

Existence, Step 1. Let

$$(P^*_{\circ}, \mathbf{q}^*_{\circ}) \subseteq P^*$$

be the maximal subcurve contracted by $f^*_{C^*}$, the markings $\mathbf{q}^*_{\circ}$ are given by the nodes of P^* disconnecting $P^*_{\circ}$ from the rest of the curve. By

$$(P^*_{\bullet}, \mathbf{q}^*_{\bullet}) \subseteq P^*$$

we denote the complement of $P^*_{\circ}$ with similar markings. Let

$$(\widetilde{P}^*_{\bullet}, \mathbf{t}^*, \mathbf{t}'^*)$$

be the normalisation of $P^*_{\bullet}$ at nodes which are mapped by $f^*_{C^*}$ to the regular locus of C^*, the markings $\mathbf{t}^*$ and $\mathbf{t}'^*$ are given by the preimages of the those nodes. The induced map

$$\tilde{f}^*_{\bullet,C^*} \colon \widetilde{P}^*_{\bullet} \to C^*$$

is an admissible cover. By properness of admissible covers, there exists, possibly after a finite base change[3], an extension

$$((P_{\bullet}, \mathbf{q}_{\bullet}, \mathbf{t}, \mathbf{t}'), (C, \mathbf{x}), \tilde{f}_{\bullet,C}) \in \mathcal{A}dm(R),$$

[3]For this proof, if we take a finite extension $R \to R'$, we relabel R' by R.

where $\mathcal{A}dm$ is the moduli space of stable admissible covers with fixed ramification profiles, such that both source and target curves are marked, and markings of the source curve are not allowed to map to the markings of the target curve. The ramification profiles are given by the ramification profiles of $\tilde{f}^*_{\bullet,C^*}$. If necessary, we then take a finite base change and modify the central fibers of source and target curves to obtain a map

$$f_\bullet \colon P_\bullet \to X \times C,$$

such that $f_{\bullet,C}$ is still an admissible cover (possibly unstable)[4]. Now let

$$f_\circ \colon (P_\circ, \mathbf{q}_\circ) \to X \times C$$

be the extension of

$$f^* \colon (P^*_\circ, \mathbf{q}^*_\circ) \to X \times C$$

to $\operatorname{Spec} R$. It exists, possibly after a finite base change, by properness of the moduli space of stable marked maps. If necessary, we modify the curve C to avoid contracted components mapping to the markings $\mathbf{x}$. If we do so, we modify $P_\bullet$ accordingly to make $f_{\bullet,C}$ an admissible cover (again, possibly unstable). We then glue back $P_\circ$ and $P_\bullet$ at the markings $(\mathbf{q}_\circ, \mathbf{q}_\bullet)$ and $(\mathbf{t}, \mathbf{t}')$ to obtain a map

$$f \colon P \to X \times C.$$

Let

$$(P, C, \mathbf{x}, f) \tag{4.6}$$

be the corresponding pre-admissible map. We now perform a series of modification to the map above to obtain an ϵ-admissible map.

Existence, Step 2. Let us analyse $(P, C, \mathbf{x}, f)$ in relation to the conditions of ϵ-admissibility.

(**i**) Let $p_0 \in C_{|\operatorname{Spec}\mathbb{C}}$ be a point in the central fiber of C that does not satisfy the condition (**i**) of ϵ-admissibility. There must be a contracted component over p_0, because $f_{\bullet,C}$ was constructed as an admissible cover, preserving the ramifications profiles. We then blow-up the family C at the point $p_0 \in C$. The map f_C lifts to a map $\tilde{f}_C$ to $\mathrm{Bl}_{p_0}(C)$

$$\begin{tikzcd} & P \arrow[dl, "\tilde{f}_C"'] \arrow[d, "f_C"] \\ \mathrm{Bl}_{p_0} C \arrow[r] & C \end{tikzcd}$$

[4]The map $f_\bullet$ can be constructed differently. One can lift $\tilde{f}^*_\bullet \colon \widetilde{P}^*_\bullet \to X \times C^*$ to an element of the moduli of twisted stable map $\mathcal{K}_{g,N}([\mathrm{Sym}^n X])$ after passing from admissible covers to twisted stable maps and then take a limit there.

by the universal property of a blow-up, since the preimage of the point p_0 is a contracted curve (which is a Cartier divisor inside P). The map f_X is left unchanged. Let $T \subset \mathrm{Bl}_{p_0}C$ be the exceptional curve, which is also a rational tail of the central fiber of $\mathrm{Bl}_{p_0}C$ attached at p_0 to $C_{|\operatorname{Spec}\mathbb{C}}$. By Lemma 4.1.11 we obtain that

$$\deg(\mathsf{br}(\tilde{f})_{|T}) + \deg(\tilde{f}^*L_{|T}) = \mathrm{mult}_{p_0}(\mathsf{br}(f)) + \deg(f^*L_{p_0}) \tag{4.7}$$

and that for all points $x \in T$

$$\mathrm{mult}_x(\mathsf{br}(\tilde{f})) + \deg(\tilde{f}^*L_x) < \mathrm{mult}_{p_0}(\mathsf{br}(f)) + \deg(f^*L_{p_0}). \tag{4.8}$$

We repeat this procedure inductively for all points of the central fiber for which the part (**i**) of ϵ-admissibility is not satisfied. By (4.7) and (4.8) this procedure will terminate and we will arrive at the map which satisfies the part (**i**) of ϵ-admissibility. Moreover, the procedure does not create rational tails which don not satisfy the part (**ii**) of ϵ-admissibility.

(**ii**) If a rational tail $T \subseteq (C_{|\operatorname{Spec}\mathbb{C}}, \mathbf{x}_{|\operatorname{Spec}\mathbb{C}})$ does not satisfy the condition (**ii**) of ϵ-admissibility, we contract it

$$\begin{tikzcd} & P \arrow[dl, "\tilde{f}_C"'] \arrow[d, "f_C"] \\ C \arrow[r] & \mathrm{Con}_T C \end{tikzcd}$$

The map f_X is left unchanged. Let $p_0 \in \mathrm{Con}_T C$ be the image of the contracted rational tail T. Since

$$\deg(\mathsf{br}(\tilde{f})_{|P}) + \deg(\tilde{f}^*L_{|P}) = \mathrm{mult}_{p_0}(\mathsf{br}(f)) + \deg(f^*L_{p_0}),$$

the central fiber satisfies the condition (**i**) of ϵ-admissibility at the point $p_0 \in \mathrm{Con}_T C$. We repeat this process until we get rid of all rational tails that don not satisfy the condition (**ii**) of ϵ-admissibility.

(**iii**) By the construction of the family (4.6), it satisfies (**iii**) of ϵ-admissibility. The modifications above do not change this property.

Uniqueness. Assume we are given two families of ϵ-admissible maps over $\operatorname{Spec} R$

$$(P_1, C_1, \mathbf{x}_1, f_1) \text{ and } (P_2, C_2, \mathbf{x}_2, f_2),$$

which are isomorphic over $\operatorname{Spec} K$. Possibly after a finite base change, there exists a family of pre-admissible maps

$$(\widetilde{P}, \widetilde{C}, \tilde{\mathbf{x}}, \tilde{f})$$

which dominates both families in the sense that there exists a commutative square

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\tilde{f}} & X \times \widetilde{C} \\ \downarrow & & \downarrow \\ P_i & \xrightarrow{f_i} & X \times C_i \end{array} \tag{4.9}$$

We take a minimal family $(\widetilde{P}, \widetilde{C}, \tilde{\mathbf{x}}, \tilde{f})$ with such property. The vertical maps are given by contraction of rational tails. Then by the equality

$$\deg(\mathsf{br}(f)_{|P}) + \deg(L_{|P}) = \mathrm{mult}_{p_0}(\mathsf{br}(f)) + \deg(L_{p_0}),$$

those rational tails cannot satisfy the condition (**ii**) of ϵ-admissibility, otherwise the the point to which a rational tail contracted will not satisfy the condition (**i**) of ϵ-admissibility. But $(P_i, C_i, \mathbf{x}_i, f_i)$'s are ϵ-admissible by assumption. Hence the source curves are isomorphic, by separatedness of the moduli space of maps to a fixed target it must be that

$$(P_1, C_1, \mathbf{x}_1, f_1) \cong (\widetilde{P}, \widetilde{C}, \tilde{\mathbf{x}}, \tilde{f}) \cong (P_2, C_2, \mathbf{x}_2, f_2).$$

□

4.1.2 Obstruction theory

The obstruction theory of $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$ is defined via the obstruction theory of relative maps in the spirit of [GV05, Section 2.8] with the difference that we have a relative target geometry $X \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}$. There exists a complex $E^{\bullet}$, which defines a perfect obstruction theory relative to $\mathfrak{M}_{h,N'} \times \mathfrak{M}_{g,N}$,

$$\phi : E^{\bullet} \to \mathbb{L}_{Adm^{\epsilon}_{g,N}(X^{(n)},\beta))/\mathfrak{M}_{h,N'} \times \mathfrak{M}_{g,N}},$$

where $\mathfrak{M}_{h,N'}$ is the moduli space of source curves with markings at the fibers over marked points of the target curves; and $\mathfrak{M}_{g,N}$ is the moduli space of target curves. More precisely, such a complex exists at each connected component $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta, \underline{\mu})$.

Proposition 4.1.14. *The morphism ϕ is a perfect obstruction theory.*

Proof. This is a relative version of [GV05, Section 2.8]. □

4.1.3 Relation to other moduli spaces

Let us now relate the moduli spaces of ϵ-admissible maps for the extremal values of $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$ to more familiar moduli spaces.

$\epsilon = -\infty$

In this case the first two conditions of Definition 4.1.4 are

(**i**) for all points $p \in \mathcal{C}$,

$$\mathrm{mult}_p(\mathsf{br}(f)) + \deg(f^* L_p) \leq 1;$$

(**ii**) for all rational tails $T \subseteq (\mathcal{C}, \mathbf{x})$,

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^* L_{|T}) > 1.$$

Since multiplicity and degree take only integer values, by Lemma 4.1.2 and the choice of L, there is only possibility for which the condition (**i**) is satisfied. Namely, if $f_{\mathcal{C}}$ does not contract any irreducible components and has only simple branching.

To unpack the condition (**ii**), recall that a non-constant ramified map from a smooth curve to $\mathbb{P}^1$ has at least two ramification points; it has precisely two simple ramification points , if it is given by

$$z^2 \colon \mathbb{P}^1 \to \mathbb{P}^1 \tag{4.10}$$

up to a change of coordinates. Hence

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^* L_{|T}) = 1,$$

if and only if $f_C = z^2$ and f_X is constant; in particular, $|\mathrm{Aut}(f)| = \infty$. In the light of the condition (**iii**) of ϵ-admissibility, the condition (**ii**) is therefore empty for $\epsilon = -\infty$.

We obtain that the data of a $-\infty$-admissible map $(P, C, \mathbf{x}, f)$ can be represented by the following correspondence

$$\begin{array}{ccc} P & \xrightarrow{f_X} & X \\ f_C\downarrow & & \\ (C, \mathbf{x}, \mathbf{p}) & & \end{array}$$

where f_C is a degree-n admissible cover with arbitrary ramifications over the marking $\mathbf{x}$ and with simple branching over the unordered marking $\mathbf{p} = \mathsf{br}(f)$. Hence the moduli space $Adm^{-\infty}_{g,N}(X^{(n)}, \beta)$ admits a surjective projection from the moduli space of twisted stable maps with *extended degree* (see [BG09, Section 2.1] for the definition) to the orbifold $[X^{(n)}]$,

$$\rho \colon \mathcal{K}_{g,N}([X^{(n)}], \beta) \to Adm^{-\infty}_{g,N}(X^{(n)}, \beta), \tag{4.11}$$

which is given by passing from twisted curves to their coarse moduli spaces. Indeed, an element of $\mathcal{K}_{g,N}([X^{(n)}], \beta)$ is given by a data of

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f_X} & X \\ {\scriptstyle f_{\mathcal{C}}}\downarrow & & \\ (\mathcal{C}, \mathbf{x}, \mathbf{p}) & & \end{array}$$

where $f_{\mathcal{C}}$ is a representable degree-n étale cover over twisted marked curve $(C, \mathbf{x}, \mathbf{p})$. The additional marking $\mathbf{p}$ is unordered, and over this marking the map $f_{\mathcal{C}}$ must have simple branching after passing to coarse moduli spaces. Hence after passing to coarse moduli spaces, we obtain the data of $-\infty$-admissible maps.

Moreover, the virtual fundamental classes of two moduli spaces are related by the push-forward, as it is shown in the following lemma.

Lemma 4.1.15.

$$\rho_*[\mathcal{K}_{g,N}([X^{(n)}], \beta)]^{\mathrm{vir}} = [Adm^{-\infty}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}}.$$

Proof. Let $\mathfrak{K}_{g,N}(BS_n, \mathsf{m})$ be the moduli stacks of twisted maps to BS_n (not necessarily stable) and $\mathfrak{Adm}_{g,\mathsf{m},n,N}$ be the moduli stack of admissible covers (again not necessarily stable) with corresponding discrete invariants. There exists the following pull-back diagram,

$$\begin{array}{ccc} \mathcal{K}_{g,N}([X^{(n)}], \beta) & \xrightarrow{\rho} & Adm^{-\infty}_{g,N}(X^{(n)}, \beta) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} \\ \mathfrak{K}_{g,N}(BS_n, \mathsf{m}) & \longrightarrow & \mathfrak{Adm}_{g,\mathsf{m},n,N} \end{array} \tag{4.12}$$

The bottom arrow is a normalisation map, therefore it is of degree 1. By [Cos06, Theorem 5.0.1] we therefore obtain the equality of virtual fundamental classes given by the relative perfect obstruction theories.

$$\rho_*[\mathcal{K}_{g,N}([X^{(n)}], \beta)/\mathfrak{K}_{g,N}(BS_n, \mathsf{m})]^{\mathrm{vir}} \\ = [Adm^{-\infty}_{g,N}(X^{(n)}, \beta)/\mathfrak{Adm}_{g,\mathsf{m},n,N}]^{\mathrm{vir}}. \tag{4.13}$$

The moduli space $\mathfrak{K}_{g,N}(BS_n, \mathsf{m})$ is smooth and $\mathfrak{Adm}_{g,\mathsf{m},n,N}$ is a locally complete intersection (see [ACV03, Proposition 4.2.2]), which implies that their naturally defined obstruction theories are given by cotangent complexes. Using virtual pull-backs of [Man12], one can therefore express the virtual fundamental classes given by absolute perfect obstruction theories as follows

$$\begin{aligned}
[Adm^{-\infty}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}} &= (p\circ\pi_2)^{!}[\mathfrak{M}_{g,N}] \\
&= \pi_2^{!}p^{!}[\mathfrak{M}_{g,N}] \\
&= \pi_2^{!}[\mathfrak{Adm}_{g,\mathsf{m},n,N}] \\
&= [Adm^{-\infty}_{g,N}(X^{(n)},\beta)/\mathfrak{Adm}_{g,\mathsf{m},n,N}]^{\mathrm{vir}},
\end{aligned}$$

where

$$p\colon \mathfrak{Adm}_{g,\mathsf{m},n,N}\to\mathfrak{M}_{g,N}$$

is the natural projection; we used that $p^{!}[\mathfrak{M}_{g,N}]=[\mathfrak{Adm}_{g,\mathsf{m},n,N}]$, which is due to the fact that the obstruction theory is given by the cotangent complex. The same holds for $\mathcal{K}_{g,N}(BS_n,\mathsf{m})$, hence we obtain that

$$\rho_*[\mathcal{K}_{g,N}([X^{(n)}],\beta)]^{\mathrm{vir}}=[Adm^{-\infty}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}}.$$

□

$\epsilon=0$

By the first two conditions of Definition 4.1.4 the map f_C can have arbitrary ramifications and contracted components of arbitrary genera (more precisely, the two are only restricted by n, g, N and β). In conjunction with other conditions of Definition 4.1.4 we therefore obtain the following identification of moduli spaces

$$Adm^{0}_{g,N}(X^{(n)},\beta)=\overline{M}^{\bullet}_{\mathsf{m}}(X\times C_{g,N}/\overline{M}_{g,N},(\gamma,n)), \tag{4.14}$$

where the space on the right is the moduli space of relative stable maps with disconnected domains to the relative geometry

$$X\times C_{g,N}\to\overline{M}_{g,N},$$

where $C_{g,N}\to\overline{M}_{g,N}$ is the universal curve and where the markings play the role of relative divisors. Instead of fixing the genus of source curves, we fix the degree m of the branching divisor. At each component $Adm^{0}_{g,N}(X^{(n)},\beta,\underline{\mu})$ of the decomposition (4.4), the genus of the source curve and the degree of the branching divisor are related by Lemma 4.1.9.

The obstruction theories of two moduli spaces are equal, since the obstruction theory of the space $Adm^{0}_{g,N}(X^{(n)},\beta)$ was defined via the obstruction theory of relative stable maps.

4.1.4 Inertia stack

We would like to define evaluation maps of moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$ to a certain rigidification of the inertia stack $\mathcal{I}X^{(n)}$ of $[X^{(n)}]$, for that we need a few observations.

The inertia stack can be defined as follows

$$\mathcal{I}X^{(n)} = \coprod_{[g]} [X^{n,g}/C(g)],$$

where the disjoint union is taken over conjugacy classes $[g]$ of elements of S_n, $X^{n,g}$ is the fixed locus of g acting on X^n and $C(g)$ is the centraliser subgroup of g. Recall that conjugacy classes of elements of S_n are in one-to-one correspondence with partitions μ of n. Let us express a partition μ in terms to repeating parts and their multiplicities,

$$\mu = (\underbrace{\eta_1, \cdots, \eta_1}_{m_1}, \cdots, \underbrace{\eta_s, \cdots, \eta_s}_{m_s}).$$

We define

$$C(\mu) := \prod_{t=1}^{s} C_{\eta_t} \wr S_{m_t}, \tag{4.15}$$

here C_{η_t} is a cyclic group and " $\wr$ " is a *wreath product* defined as

$$C_{\eta_t} \wr S_{m_t} := C_{\eta_t}^{\Omega_t} \rtimes S_{m_t},$$

where $\Omega_t = \{1, \ldots, m_t\}$; S_{m_t} acts on $C_{\eta_t}^{\Omega_t}$ by permuting the factors. There exist two natural subgroups of $C(\mu)$,

$$\mathrm{Aut}(\mu) := \prod_{t=1}^{s} S_{m_t} \quad \text{and} \quad N(\mu) := \prod_{t=1}^{s} C_{\eta_t}^{\Omega_t}, \tag{4.16}$$

as the notation suggests, $\mathrm{Aut}(\mu)$ coincides with the automorphism group of the partition μ. The inclusion $\mathrm{Aut}(\mu) \hookrightarrow C(\mu)$ splits the following the sequence from the right

$$1 \to N(\mu) \to C(\mu) \to \mathrm{Aut}(\mu) \to 1. \tag{4.17}$$

Viewing a partition μ as a partially ordered[5] set, we define X^{μ} as the self-product of X over the set μ. In particular,

$$X^{\mu} \cong X^{\ell(\mu)},$$

[5] $\mu_i \geq \mu_j, \iff j \geq i.$

where $\ell(\mu)$ is the length of the partition μ. The group $C(\mu)$ acts on X^μ as follows. The products of cyclic groups $C^{\Omega}_{\eta_t}$ acts trivially on corresponding factors of X^μ, while S_{m_t} permutes the factors corresponding to the same part η_t. These actions are compatible with the wreath product.

Given an element $g \in S_n$ in a conjugacy class corresponding to a partition μ, we have the following identifications

$$C(g) \cong C(\mu) \text{ and } X^{n,g} \cong X^\mu,$$

such that the group actions match. In particular, with the notation introduced above the inertia stack can be re-expressed,

$$\mathcal{I}X^{(n)} = \coprod_{\mu} [X^\mu / C(\mu)], \tag{4.18}$$

and by the splitting of (4.17) we obtain that

$$\mathcal{I}X^{(n)} = \coprod_{\mu} [X^\mu / \mathrm{Aut}(\mu)] \times BN(\mu). \tag{4.19}$$

We thereby define a rigidified version of $\mathcal{I}X^{(n)}$,

$$\bar{\mathcal{I}}X^{(n)} := \coprod_{\mu} [X^\mu / \mathrm{Aut}(\mu)].$$

Note, however, that this is not a rigidified inertia stack in the sense of [AGV08, Section 3.3], $\bar{\mathcal{I}}X^{(n)}$ is a further rigidifiction of $\mathcal{I}X^{(n)}$.

Recall that as a graded vector space, the orbifold cohomology is defined as follows

$$H^*_{\mathrm{orb}}(X^{(n)}, \mathbb{Q}) := H^{*-2\mathrm{age}(\mu)}(\mathcal{I}X^{(n)}, \mathbb{Q}).$$

By (4.18) we therefore get that

$$H^*_{\mathrm{orb}}(X^{(n)}, \mathbb{Q}) = H^{*-2\mathrm{age}(\mu)}(\mathcal{I}X^{(n)}, \mathbb{Q}) = H^{*-2\mathrm{age}(\mu)}(\bar{\mathcal{I}}X^{(n)}, \mathbb{Q}). \tag{4.20}$$

4.1.5 Invariants

Let $\overrightarrow{Adm}^{\epsilon}_{g,N}(X^{(n)}, \beta)$ be the moduli space obtained from $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$ by putting the *standard order*[6] on the fibers over marked points of the source curve. The two moduli spaces are related as follows

$$\coprod_{\underline{\mu}} [\overrightarrow{Adm}^{\epsilon}_{g,N}(X^{(n)}, \beta, \underline{\mu}) / \prod \mathrm{Aut}(\mu^i)] = \overline{M}^{\epsilon}_{g,N}(X^{(n)}, \beta), \tag{4.21}$$

[6] We order the points in a fiber in accordance with their ramification degrees.

There exist naturally defined evaluation maps at marked points

$$\mathrm{ev}_i \colon \overrightarrow{Adm}^{\epsilon}_{g,N}(X^{(n)}, \beta) \to \coprod_{\mu} X^{\mu}, \quad i = 1, \dots, N.$$

By (4.16), (4.18) and (4.21) we can define evaluation maps

$$\mathrm{ev}_i \colon Adm^{\epsilon}_{g,N}(X^{(n)}, \beta) \to \bar{\mathfrak{I}}X^{(n)}, \quad i = 1, \dots, N, \tag{4.22}$$

as the composition

$$Adm^{\epsilon}_{g,N}(X^{(n)}, \beta) = \coprod_{\underline{\mu}} [\overrightarrow{Adm}^{\epsilon}_{g,N}(X^{(n)}, \beta, \underline{\mu}) / \prod \mathrm{Aut}(\mu^i)] \xrightarrow{\mathrm{ev}_i}$$

$$\xrightarrow{\mathrm{ev}_i} \coprod_{\mu} [X^{\mu}/\mathrm{Aut}(\mu)] = \bar{\mathfrak{I}}X^{(n)}.$$

For universal markings

$$s_i \colon Adm^{\epsilon}_{g,N}(X^{(n)}, \beta) \to \mathcal{C}_{g,N}$$

to the universal *target* curve

$$\mathcal{C}_{g,N} \to Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$$

we also define cotangent line bundles as follows

$$\mathcal{L}_i := s_i^*(\omega_{\mathcal{C}_{g,N}/Adm^{\epsilon}_{g,N}(X^{(n)},\beta)}), \quad i = 1, \dots, N,$$

where $\omega_{\mathcal{C}_{g,N}/Adm^{\epsilon}_{g,N}(X^{(n)},\beta)}$ is the universal relative dualising sheaf. We denote

$$\psi_i := \mathrm{c}_1(\mathcal{L}_i).$$

With above structures at hand we can define ϵ-admissible invariants.

Definition 4.1.16. The *descendent ϵ-admissible invariants* are

$$\langle \tau_{m_1}(\gamma_1), \dots, \tau_{m_N}(\gamma_N) \rangle^{\epsilon}_{g,\beta} := \int_{[Adm^{\epsilon}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}}} \prod_{i=1}^{i=N} \psi_i^{m_i} \mathrm{ev}_i^*(\gamma_i,),$$

where $\gamma_1, \dots, \gamma_N \in H^*_{\mathrm{orb}}(X^{(n)})$ and $m_1, \dots m_N$ are non-negative integers.

4.1.6 Relation to other invariants

We will now explore how ϵ-admissible invariants are related to the invariants associated to the spaces discussed in Section 4.1.3.

Classes

Let $\{\delta_1, \dots \delta_{m_S}\}$ be an ordered basis of $H^*(X, \mathbb{Q})$. Let

$$\vec{\mu} = ((\mu_1, \delta_{\ell_1}), \dots, (\mu_k, \delta_{\ell_k}))$$

be a cohomology-weighted partition of n with the standard ordering, i.e.

$$(\mu_i, \delta_{\ell_i}) > (\mu_{i'}, \delta_{\ell_{i'}}),$$

if $\mu_i > \mu_{i'}$, or if $\mu_i = \mu_{i'}$ and $\ell_i > \ell_{i'}$. The underlying partition will be denoted by μ. For each $\vec{\mu}$ we consider a class

$$\delta_{l_1} \otimes \cdots \otimes \delta_{l_k} \in H^*(X^\mu, \mathbb{Q}),$$

we then define

$$\lambda(\vec{\mu}) := \overline{\pi}_*(\delta_{l_1} \otimes \cdots \otimes \delta_{l_k}) \in H^*_{\mathrm{orb}}(S^{(d)}, \mathbb{Q}),$$

where

$$\overline{\pi}\colon \coprod_\mu X^\mu \to \bar{\mathfrak{I}}X^{(n)}$$

is the natural projection. More explicitly, as an element of

$$H^*(X^\mu, \mathbb{Q})^{\mathrm{Aut}(\mu)} \subseteq H^*_{\mathrm{orb}}(X^{(n)}, \mathbb{Q}),$$

the class $\lambda(\vec{\mu})$ is given by the following formula

$$\sum_{h \in \mathrm{Aut}(\mu)} h^*(\delta_{l_1} \otimes \cdots \otimes \delta_{l_k}) \in H^*(X^\mu, \mathbb{Q})^{\mathrm{Aut}(\mu)}.$$

The importance of these classes is due to the fact they form a basis of $H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{Q})$.

Comparison

Given weighted partitions

$$\vec{\mu}^i = ((\mu^i_1, \delta^i_1), \dots, (\mu^i_{k_i}, \delta^i_{k_i})), \quad i = 1, \dots N,$$

the relative Gromov–Witten descendent invariants associated to the moduli space $\overline{M}^\bullet_{\mathsf{m}}(X \times C_{g,N}/\overline{M}_{g,N}, (\gamma, n))$ are usually[7] defined as

$$\int_{[\overline{M}^\bullet_{\mathsf{m}}(X \times C_{g,N}/\overline{M}_{g,N}, (\gamma, n))]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{m_i} \prod_{j=1}^{k_i} \mathrm{ev}^*_{i,j} \delta^i_j,$$

[7]Note that sometimes the factor $1/|\mathrm{Aut}(\vec{\mu})|$ is introduced, in this case we add such factor for all classes defined previously.

such that the product is ordered according to the standard ordering of weighted partitions and

$$\mathrm{ev}_{i,j} \colon \overline{M}^{\bullet}_{\mathsf{m}}(X \times C_{g,N}/\overline{M}_{g,N}, (\gamma, n)) \to X$$

are evaluation maps defined by sending a corresponding point in a fiber over a marked point.

In the case of $\mathcal{K}_{g,N}([X^{(n)}], \beta)$ we define evaluation maps as the composition

$$\mathrm{ev}_i \colon \mathcal{K}_{g,N}([X^{(n)}], \beta) \to \mathcal{I}X^{(n)} \to \bar{\mathcal{I}}X^{(n)}, \quad i = 1, \ldots N,$$

where we used (4.19).

The next lemma concludes the comparison initiated in Section 4.1.3. In what follows, by a ψ-class on $\mathcal{K}_{g,N}([X^{(n)}], \beta)$ we will mean a *coarse* ψ-class. Orbifold ψ-classes are rational multiples of coarse ones.

Lemma 4.1.17.

$$\langle \tau_{m_1}(\lambda(\vec{\mu}^1)), \ldots, \tau_{m_N}(\lambda(\vec{\mu}^N)) \rangle^{0}_{g,\beta} = \int_{[\overline{M}^{\bullet}_{\mathsf{m}}(X \times C_{g,N}/\overline{M}_{g,N}, (\gamma,n))]^{\mathrm{vir}}} \prod_{i=1}^{N} \psi_i^{m_i} \prod_{j=1}^{k_i} \mathrm{ev}^*_{i,j} \delta^i_j$$

$$\langle \tau_{m_1}(\lambda(\vec{\mu}^1)), \ldots, \tau_{m_N}(\lambda(\vec{\mu}^N)) \rangle^{-\infty}_{g,\beta} = \int_{[\mathcal{K}_{g,N}([X^{(n)}],\beta)]^{\mathrm{vir}}} \prod_{i=1}^{N} \psi_i^{m_i} \mathrm{ev}^*_i \lambda(\vec{\mu}^i).$$

Proof. In the light of our conventions it is a straightforward application of projection and pullback-pushforward formulas. □

4.2 Master space

4.2.1 Definition of the master space

The space $\mathbb{R}_{\leq 0} \cup \{-\infty\}$ of ϵ-stabilities is divided into chambers, inside of which the moduli space $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$ stays the same, and as ϵ crosses a wall between chambers, the moduli space changes discontinuously. Let $\epsilon_0 \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$ be a wall, and ϵ_+, ϵ_- be some values that are close to ϵ_0 from the right and the left of the wall, respectively. We set

$$d_0 = e^{-1/\epsilon_0} \quad \text{and} \quad \deg(\beta) := \mathsf{m} + \deg(\gamma) = d.$$

From now on we assume

$$2g - 2 + N + 1/d_0 \cdot \deg(\beta) \geq 0.$$

and $1/d_0 \cdot \deg(\beta) > 2$, if $(g, N) = (0, 0)$.

Definition 4.2.1. A pre-admissible map $(P, C, f, \mathbf{x})$ is called ϵ_0*-pre-admissible*, if

(i) for all points $p \in \mathcal{C}$,

$$\mathrm{mult}_p(\mathsf{br}(f)) + \deg(f^*L_p) \leq e^{-1/\epsilon_0};$$

(ii) for all rational tails $T \subseteq \mathcal{C}$,

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^*L_{|T}) \geq e^{-1/\epsilon_0};$$

(iii) for all rational bridges $B \subseteq \mathcal{C}$,

$$\deg(\mathsf{br}(f)_{|B}) + \deg(f^*L_{|B}) > 0;$$

We denote by $\mathfrak{M}^{\epsilon_0}_{g,N}(X^{(n)}, \beta)$ the moduli space of ϵ_0-pre-admissible maps. Let $\mathfrak{M}^{ss}_{g,N,d}$ be the moduli space of weighted semistable curves defined in [Zho22, Definition 2.1.2]. There exists a map

$$\mathfrak{M}^{\epsilon_0}_{g,N}(X^{(n)}, \beta) \to \mathfrak{M}^{ss}_{g,N,d}$$

$$(P, C, f, \mathbf{x}) \mapsto (C, \mathbf{x}, \underline{d}),$$

where the value of $\underline{d}$ on a subcurve $C' \subseteq C$ is defined as follows

$$\underline{d}(C') = \deg(\mathsf{br}(f_{|C'})) + \deg(f^*L_{|C'}).$$

By $M\mathfrak{M}^{\epsilon_0}_{g,N}(X^{(n)}, \beta)$ we denote the moduli space of ϵ_0-pre-admissible maps with calibrated tails, defined as the fiber product

$$M\mathfrak{M}^{\epsilon_0}_{g,N}(X^{(n)}, \beta) = \mathfrak{M}^{\epsilon_0}_{g,N}(X^{(n)}, \beta) \times_{\mathfrak{M}^{ss}_{g,N,d}} M\widetilde{\mathfrak{M}}_{g,N,d},$$

where $M\widetilde{\mathfrak{M}}_{g,N,d}$ is the moduli space of curves with calibrated tails introduced in [Zho22, Definition 2.8.2].

Definition 4.2.2. Given a pre-admissible map $(P, C, f, \mathbf{x})$. We say a rational tail $T \subseteq (C, \mathbf{x})$ is of degree d_0, if

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^*L_{|T}) = d_0.$$

We say a branching point $p \in C$ is of degree d_0, if

$$\mathrm{mult}_p(\mathsf{br}(f)) + \deg(f^*L_p) = d_0.$$

Definition 4.2.3. We say a rational tail $T \subseteq (C, \mathbf{x})$ is *constant*, if

$$|\mathrm{Aut}((P, C, f, \mathbf{x})_{|T})| = \infty.$$

In other words, a rational tail $T \subseteq (C, \mathbf{x})$ is constant, if at each connected component of $P_{|T}$ the map $f_{C|T}$ is equal to

$$z^{\underline{n}} \colon (\sqcup^k \mathbb{P}^1) \sqcup_0 P' \to \mathbb{P}^1$$

up to a change of coordinates. The notation is the same as in (4.5).

Definition 4.2.4. A B-family family of ϵ_0-pre-admissible maps with calibrated tails

$$(P, C, \mathbf{x}, f, e, \mathcal{L}, v_1, v_2)$$

is ϵ_0-admissible if

1) any constant tail is an entangled tail;

2) if a geometric fiber C_b of C has tails of degree d_0, then those rational tails contain all the degree-d_0 branching points;

3) if $v_1(b) = 0$, then $(P, C, \mathbf{x}, f)_b$ is ϵ_--admissible;

4) if $v_2(b) = 0$, then $(P, C, \mathbf{x}, f)_b$ is ϵ_+-admissible.

We denote by $MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)$ the moduli space of genus-g, n-marked, ϵ_0-admissable maps with calibrated tails.

4.2.2 Obstruction theory

The obstruction theory of $MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)$ is defined in the same way as the one of $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$. There exists a complex $E^\bullet$, which defines a perfect obstruction theory relative to $\mathfrak{M}_{h,N'} \times M\widetilde{\mathfrak{M}}_{g,N,d}$,

$$\phi : E^\bullet \to \mathbb{L}_{MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)/\mathfrak{M}_{h,N'} \times M\widetilde{\mathfrak{M}}_{g,N}}.$$

The fact that it is indeed a perfect obstruction theory is a relative version of [GV05, Section 2.8].

4.2.3 Properness

Theorem 4.2.5. *The moduli space $MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)$ is a quasi-separated Deligne–Mumford stack of finite type.*

Proof. The proof is the same as in [Zho22, Proposition 4.1.11]. $\square$

We now deal with properness of $MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)$ with the help of valuative criteria of properness. We will follow the strategy of [Zho22, Section 5]. Namely, given a discrete valuation ring R with the fraction field K. Let

$$\xi^* = (P^*, C^*, \mathbf{x}^*, f^*, e^*, \mathcal{L}^*, v_1^*, v_2^*) \in MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)(K)$$

be a family of ϵ_0-admissable map with calibrated tails over $\operatorname{Spec} K$. We will classify all the possible ϵ_0-pre-admissible extensions of ξ^* to R up to a finite base change. There will be a unique one which is ϵ_0-admissible.

$(g, N, d) \neq (0, 1, d_0)$

Assume that $(g, N, d) \neq (0, 1, d_0)$ and η^* does not have rational tails of degree d_0. Let

$$\eta^* = (P^*, C^*, \mathbf{x}^*, f^*) \text{ and } \lambda^* = (e^*, \mathcal{L}^*, v_1^*, v_2^*)$$

be the underlying pre-admissable map and the calibration data of η^*, respectively. Let

$$\xi_- = (\eta_-, \lambda_-) \in M\mathfrak{M}^{\epsilon_0}_{g,N}(X^{(n)}, \beta)(R')$$

be family over degree-r extension R' of R, where the ϵ_--pre-admissible map

$$\eta_- = (P_-, C_-, \mathbf{x}_-, f_-).$$

is constructed according to the same procedure as (4.6). More precisely, we apply modifications of *Step 2* with respect to ϵ_--stability; we leave the degree-d_0 branching points which are limits of degree-d_0 branching points of the generic fiber untouched. The family η_- is the one closest to being ϵ_--admissible limit of η^*. The calibration λ_- is given by a unique extension of λ^* to the curve C_-, which exists by [Zho22, Lemma 5.1.1 (1)].

Let

$$\{p_i \mid i = 1, \ldots, \ell\}$$

be an ordered set, consisting of nodes of degree-d_0 rational tails and degree-d_0 branching points of the central fiber

$$p_i \in C_{-|\operatorname{Spec}\mathbb{C}} \subset C_-.$$

We now define

$$b_i \in \mathbb{R}_{>0} \cup \{\infty\},\ i = 1, \ldots, \ell$$

as follows. Set b_i to be ∞, if p_i is a degree-d_0 branching point. If p_i is a node of a rational tail, then we define b_i via the singularity type of C_- at p_i - if the family C_- has A_{b-1}-type singularity at p_i, we set $b_i = b/r$.

We now classify all ϵ_0-pre-admissible modifications of ξ_- in the sense of Definition 4.1.12. By [Zho22, Lemma 5.1.1 (1)] it is enough to classify the modifications of η_-.

All the modifications of η_- are given by blow-ups and blow-downs around the points p_i after taking base-changes with respect to finite extensions of R. The result of these modifications will be a change of singularity type of η_- around p_i. Hence the classification will depend on an array of rational numbers

$$\underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Q}^\ell_{\geq 0},$$

the nominator of which keeps track of the singularity type around p_i, while the denominator is responsible for the degree of an extension of R. The precise statement is the following lemma.

Lemma 4.2.6. *For each $\underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Q}^\ell_{\geq 0}$, such that $\underline{\alpha} \leq \underline{b}$, there exists a ϵ_0-pre-admissible modification $\eta_{\underline{\alpha}}$ of η_- with following properties*

- *up to a finite base change,*

$$\eta_{\underline{\alpha}} \cong \eta_{\underline{\alpha}'} \iff \underline{\alpha} = \underline{\alpha}';$$

- *given a ϵ_0-pre-admissible modification $\tilde{\eta}$ of η_-, then there exists $\underline{\alpha}$ such that*

$$\tilde{\eta} \cong \eta_{\underline{\alpha}}$$

  up to a finite base change.

- *the central fiber of $\eta_{\underline{\alpha}}$ is ϵ_--stable, if and only if $\underline{\alpha} = \underline{b}$.*

Proof. Let us choose a fractional presentation of $(a_1, \dots, a_\ell)$ with a common denominator

$$(a_1, \dots, a_\ell) = (\frac{a'_1}{rr'}, \dots, \frac{a'_\ell}{rr'}).$$

Take a base change of η_- with respect to a degree-r' extension of R'. We then construct $\eta_{\underline{\alpha}}$ by applying modifications η_- around each point p_i, the result of which is a family

$$\eta_{\alpha_i} = (P_{\alpha_i}, C_{\alpha_i}, \mathbf{x}_{\alpha_i}, f_{\alpha_i}),$$

which is constructed as follows.

Case 1. If p_i is a node of a degree-d_0 rational tail and $a_i \neq 0$, we blow-up C_- at p_i,

$$\mathrm{Bl}_{p_i}(C_-) \to C_-.$$

The map f_{C_-} then defines a rational map

$$f_{C_-} \colon P_- \dashrightarrow \mathrm{Bl}_{p_i}(C_-).$$

We can eliminate the indeterminacies of the map above by blowing-up P_- to obtain an everywhere-defined map

$$f_{\mathrm{Bl}_{p_i}(C_-)} \colon \widetilde{P}_- \to \mathrm{Bl}_{p_i}(C_-),$$

we take a minimal blow-up with such property. The exceptional curve E of $\mathrm{Bl}_{p_i}(C_-)$ is a chain of $r'b_i$ rational curves. The exceptional curve of $\widetilde{P}_-$ is a disjoint union $\sqcup_j E_j$, where each E_j is a chain of rb_i rational curves mapping to E without contracted components. We blow-down all the rational curves but the a'_i-th ones in both E and E_j for all j. The resulting families are C_{α_i} and P_{α_i}, respectively. The family C_{α_i} has an $A_{\alpha'_i - 1}$-type singularity at p_i. The marking $\mathbf{x}_-$ clearly extends to a marking $\mathbf{x}_{\alpha_i}$ of C_{α_i}. The map $f_{\mathrm{Bl}_{p_i}(C_-)}$ descends to a map

$$f_{C_{\alpha_i}} \colon P_{\alpha_i} \to C_{\alpha_i}.$$

The map $f_{-,X}$ is carried along with all those modifications to a map

$$f_{\alpha_i, X} \colon P_{\alpha_i} \to X,$$

because exceptional divisors are of degree 0 with respect to $f_{-,X}$, hence the contraction of curves in the exceptional divisors does not introduce any indeterminacies. We thereby constructed the family η_{α_i}.

Case 2. Assume now that p_i is a node of a degree-d_0 rational tail, but $a_i = 0$. The family C_{α_i} is then given by the contraction of that degree-d_0 rational tail, it is smooth at p_i. The marking $\mathbf{x}_-$ extends to a marking $\mathbf{x}_{\alpha_i}$ of C_{α_i}. The family P_{α_i} is set to be equal to P_-. The map f_{α_i} is the composition of the contraction and f_-.

Case 3. If p_i is a branching point, we blow-up C_- inductively a'_i times, starting with a blow-up at p_i and then continuing with a blow-up at a point of the exceptional curve of the previous blow-up. We then blow-down all rational curves in the exceptional divisor but the last one. The resulting family is C_{α_i}, it has an $A_{a'_i}$-type singularity at p_i. The marking $\mathbf{x}_-$ extends to the marking $\mathbf{x}_{\alpha_i}$ of C_{α_i}. The map f_{C_-} then defines a rational map

$$f_{C_-} \colon P_- \dashrightarrow C_{\alpha_i}.$$

We set

$$f_{C_{\alpha_i}} \colon P_{\alpha_i} \to C_{\alpha_i}$$

to be the minimal resolution of indeterminacies of the rational map above. More specifically, P_{α_i} is obtained by consequently blowing-up P_- and blowing-down all the rational curves in the exceptional divisor but the last one. The map $f_{-,X}$ is carried along, as in *Case 1.*

It is not difficult to verify that the central fiber of $\eta_{\underline{\alpha}}$ is indeed ϵ_0-pre-admissible. Up to a finite base change, the resulting family is uniquely determined by $\underline{\alpha} = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Q}^{\ell}_{\geq 0}$ and independent of its fractional presentation, because of the singularity types at points p_i and the degree of an extension R.

Given now an arbitrary ϵ_0-pre-admissible modification

$$\eta = (P, C, \mathbf{x}, f)$$

of η_-. Possibly after a finite base change, there exists a modification

$$\tilde{\eta} = (\widetilde{P}, \widetilde{C}, \tilde{\mathbf{x}}, \tilde{f})$$

that dominates both η and η_- in the sense of (4.9). We take a minimal modification with such property. The family $\tilde{\eta}$ is given by blow-ups of C_- and P_-. By the assumption of minimality and ϵ_0-pre-admissibility of η, these are blow-ups at p_i. By ϵ_0-pre-admissibility of η, the projections

$$\widetilde{C} \to C \quad \text{and} \quad \widetilde{P} \to P$$

are given by contraction of degree-d_0 rational tails or rational components which don not satisfy ϵ_0-pre-admissibility. These are exactly the operations described in *Steps 1,2,3* of the proof. Uniqueness of of maps follows from seperatedness of the moduli space of maps to a fixed target. Hence we obtain that

$$\eta \cong \eta_{\underline{\alpha}}$$

for some $\underline{\alpha} = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Q}^\ell_{\geq 0}$, where $\underline{\alpha}$ is determined by the singularity types of η at points p_i. $\square$

$(g, N, d) = (0, 1, d_0)$

We now assume that $(g, N, d) = (0, 1, d_0)$. In this case the calibration bundle is the relative cotangent bundle along the unique marking. Moreover, there is no entanglement. Given a family of pre-admissible maps $(P, C, \mathbf{x}, f)$, we will denote the calibration bundle by M_C. Therefore the calibration data λ is given just by a rational section s of M_C.

Let

$$\xi_- = (\eta_-, \lambda_-) \in M\mathfrak{M}^{\epsilon_0}_{0,1}(X^{(n)}, \beta)(R')$$

be the family over degree-r extension R' of R, such that η_- is again given by (4.6), if there is no degree-d_0 branching point in η^*. Otherwise, let η_- be any pre-admissible limit. The calibration data λ_- is given by a rational section s_- which is an extension of the section s^* of M_{C^*} to M_{C_-}.

Given a modification $\widetilde{\eta}$ of η_- over a degree-r' extension of R', the section s^* extends to a rational section $\tilde{s}$ of $M_{\widetilde{C}}$.

Definition 4.2.7. The order of the modification $\widetilde{\eta}$ is defined to be $\mathrm{ord}(\tilde{s})/r$ at the closed point of $\operatorname{Spec} R'$.

We set $b = \mathrm{ord}(s_-)/r$, of there is no degree-d_0 branching point in the generic fiber of η^*. Otherwise set $b = -\infty$.

Lemma 4.2.8. *For each $\alpha \in \mathbb{Q}$, such that $\alpha \leq b$, there exists a ϵ_0-pre-admissible modification η_α of η_- of order α with following properties*

- *up to a finite base change,*

$$\eta_\alpha \cong \eta_{\alpha'} \iff \alpha = \alpha';$$

- *given a ϵ_0-pre-admissible modification $\tilde{\eta}$ of η_-, then there exists α such that*

$$\tilde{\eta} \cong \eta_\alpha$$

up to a finite base change.

- *the central fiber of η_α is ϵ_--stable, if and only if $\alpha = b$.*

Proof. Assume η^* does not have a degree-d_0 branching point. We choose a fractional presentation $a = a'/rr'$. We take a base change of η_- with respect to a degree-r' extension of R'. We blow-up consequently a' times the central fiber at the unique marking. We then blow-down all rational curves in the exceptional divisor but the last one. The resulting family with a marking is $(C_\alpha, \mathbf{x}_\alpha)$. We do the same with the family P_- at the points in the fiber over the marked point to a obtain the family P_α and the map

$$f_{P_\alpha} \colon P_\alpha \to \widetilde{C},$$

the map $f_{-,X}$ is carried along with blow-ups and blow-downs. The resulting family of ϵ_0-pre-admissible maps is of order a.

Assume now that the generic fiber has a degree-d_0 branching point. We take a base change of η_- with respect to a degree-r' extension of R'. After choosing some trivialisation of M_{C^*}, we have that

$$s^* = \pi^{r'a_-} \in K',$$

where a_- is the order of vanishing of s_- before the base-change and π is a uniformiser of R'. Because of automorphisms of $\mathbb{P}^1$ which fix a branching point and a marked point, we have an isomorphisms of ϵ_0-pre-admissible maps with calibrated tails,

$$(\eta^*, s^*) \cong (\eta^*, \pi^c \cdot s^*)$$

for an arbitrary $c \in \mathbb{Z}$. Hence we can multiply the section s_- with $\pi^{a'-r'a_-}$ to obtain a modification of order a.

The fact that these modifications classify all possible modifications follow from the same arguments as in the case $(g, N, d) \neq (0, 1, d_0)$. $\square$

Theorem 4.2.9. *The moduli space $MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)$ is proper.*

Proof. With the classifications of modifications of η_- of Lemma 4.2.6 and Lemma 4.2.8, the proof of properness follows from the same arguments as in [Zho22, Proposition 5.0.1].

$\square$

4.3 Wall-crossing

4.3.1 Graph space

For a class $\beta = (\beta, \mathsf{m}) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$ consider now

$$\overline{M}^{\bullet}_{\mathsf{m}}(X \times \mathbb{P}^1/X_\infty, (\gamma, n)),$$

the space of relative stable maps with disconnected domains of degree (γ, n) to $X \times \mathbb{P}^1$ relative to

$$X_\infty := X \times \{\infty\} \subset X \times \mathbb{P}^1.$$

One should refer to this moduli space as *graph space*, as it will play the same role, as the graph space in the quasimap wall-crossing. Note that we fix the degree of the branching divisor m instead of the genus h, the two are determined by Lemma 4.1.9.

There is a standard $\mathbb{C}^*$-action on $\mathbb{P}^1$ given by

$$t[x, y] = [tx, y], \ t \in \mathbb{C}^*,$$

which induces a $\mathbb{C}^*$-action on $\overline{M}^{\bullet}_{\mathsf{m}}(X \times \mathbb{P}^1/X_\infty, (\gamma, n))$. Let

$$F_\beta \subset \overline{M}^{\bullet}_{\mathsf{m}}(X \times \mathbb{P}^1/X_\infty, (\gamma, n))^{\mathbb{C}^*}$$

be the distinguished $\mathbb{C}^*$-fixed component consisting of maps to $X \times \mathbb{P}^1$ (no expanded degenerations). Said differently, F_β is the moduli space of maps, which are admissible over $\infty \in \mathbb{P}^1$ and whose degree lies entirely over $0 \in \mathbb{P}^1$ in the form of a branching point. Other $\mathbb{C}^*$-fixed components admit exactly the same description as in the case of quasimaps in Section 2.5.1.

The virtual fundamental class of F_β,

$$[F_\beta]^{\mathrm{vir}} \in A_*(F_\beta),$$

is defined via the fixed part of the perfect obstruction theory of

$$\overline{M}^{\bullet}_{\mathsf{m}}(X \times \mathbb{P}^1/X_\infty, (\gamma, n)).$$

The virtual normal bundle $N^{\mathrm{vir}}_{F_\beta}$ is defined by the moving part of the obstruction theory. There exists an evaluation map

$$\mathsf{ev} \colon F_\beta \to \bar{\mathfrak{I}}X^{(n)}$$

defined in the same way as (4.22).

Definition 4.3.1. We define an I-function to be

$$I(q,z) = 1 + \sum_{\beta \neq 0} q^{\beta} \mathsf{ev}_* \left(\frac{[F_\beta]}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta})} \right) \in H^*_{\mathrm{orb}}(X^{(n)})[z^{\pm}] \otimes_{\mathbb{Q}} \mathbb{Q}[\![q^{\beta}]\!].$$

Let

$$\mu(z) \in H^*_{\mathrm{orb}}(X^{(n)})[z] \otimes_{\mathbb{Q}} \mathbb{Q}[\![q^{\beta}]\!]$$

be the truncation $[zI(q,z)-z]_+$ by taking only non-negative powers of z. Let

$$\mu_\beta(z) \in H^*_{\mathrm{orb}}(X^{(n)})[z]$$

be the coefficient of $\mu(z)$ at q^β.

For later it is also convenient to define

$$\mathfrak{I}_\beta := \frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta})} \in A^*(F_\beta)[z^{\pm}].$$

4.3.2 Wall-crossing formula

We will now concentrate on the case

$$2g-2+N+1/d_0 \cdot \deg(\beta) > 0,$$

for $(g,N,d) = (0,1,d_0)$ we refer to [Zho22, Section 6,4]. There exists a natural $\mathbb{C}^*$-action on the master space $MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)$ given by

$$t \cdot (P,C,\mathbf{x},f,e,\mathcal{L},v_1,v_2) = (P,C,\mathbf{x},f,e,\mathcal{L},t\cdot v_1,v_2), \quad t \in \mathbb{C}^*.$$

By arguments presented in [Zho22, Section 6.5], the fixed locus then admits the following expression

$$MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)^{\mathbb{C}^*} = F_+ \sqcup F_- \sqcup \coprod_{\vec{\beta}} F_{\vec{\beta}},$$

we will now explain the meaning of each term in the disjoint union above, giving a description of virtual fundamental classes and virtual normal bundles.

F_+

This is a simplest component,

$$F_+ = Adm^{\epsilon_+}_{g,N}(X^{(n)}, \beta), \quad N^{\mathrm{vir}}_{F_+} = \mathbb{M}^\vee_+,$$

where $\mathbb{M}^\vee_+$ is the dual of the calibration bundle $\mathbb{M}_+$ on $Adm^{\epsilon_+}_{g,N}(X^{(n)}, \beta)$, with a trivial $\mathbb{C}^*$-action of weight -1, cf. [Zho22]. The obstruction theories also match, therefore

$$[F_+]^{\mathrm{vir}} = [Adm^{\epsilon_+}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}}$$

with respect to the identification above.

F_-

We define

$$\widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta) := Adm^{\epsilon_-}_{g,N}(X^{(n)}, \beta) \times_{\mathfrak{M}_{g,N,d}} \widetilde{\mathfrak{M}}_{g,N,d},$$

then

$$F_- = \widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta), \quad N^{\mathrm{vir}}_{F_-} = \mathbb{M}_-,$$

where, as previously, $\mathbb{M}_-$ is the calibration bundle on $\widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta)$ with trivial $\mathbb{C}^*$-action of weight 1. The obstruction theories also match and

$$p_*[\widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}} = [Adm^{\epsilon_-}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}},$$

where

$$p \colon \widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta) \to Adm^{\epsilon_-}_{g,N}(X^{(n)}, \beta)$$

is the natural projection.

$F_{\vec{\beta}}$

These are the wall-crossing components responsible for the non-trivial wall-crossing formulas. Let

$$\overrightarrow{\beta} = (\beta', \beta_1, \dots, \beta_k)$$

be a $k+1$-tuple of classes in $H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$, such that $\beta = \beta' + \beta_1 + \dots + \beta_k$ and $\deg(\beta_i) = d_0$. Then a component $F_{\overrightarrow{\beta}}$ is defined as follows

$$F_{\overrightarrow{\beta}} = \{\xi \mid \xi \text{ has exactly } k \text{ entangled tails,} \\ \text{which are all fixed tails, of degree } \beta_1, \dots, \beta_k\}.$$

Let

$$\mathcal{E}_i \underset{p_i}{\rightleftarrows} F_{\vec{\beta}} \qquad i = 1, \ldots, k,$$

be the universal i-th entangled rational tail with a marking p_i given given by the node. We define $\psi(\mathcal{E}_i)$ to be the ψ-class associated to the marking p_i. Let

$$\widetilde{\mathrm{gl}}_k \colon \widetilde{\mathfrak{M}}_{g,N+k,d-kd_0} \times (\mathfrak{M}^{ss}_{0,1,d_0})^k \to \widetilde{\mathfrak{M}}_{g,N,d}$$

be the gluing morphism, cf. [Zho22, Section 2.4]. Let

$$\mathfrak{D}_i \subset \widetilde{\mathfrak{M}}_{g,N,d}$$

be a divisor defined as the closure of the locus of curves with exactly $i+1$ entangled tails. Finally, let

$$Y \to \widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta')$$

be the stack of k-roots of $\mathbb{M}^{\vee}_{-}$.

Proposition 4.3.2. *There exists a canonical isomorphism*

$$\widetilde{\mathrm{gl}}_k^* F_{\vec{\beta}} \cong Y \times_{(\bar{\mathcal{I}}X^{(n)})^k} \prod_{i=1}^{i=k} F_{\beta_i}.$$

With respect to the identification above we have

$$\begin{aligned} [\widetilde{\mathrm{gl}}_k^* F_{\vec{\beta}}]^{\mathrm{vir}} =& [Y]^{\mathrm{vir}} \times_{(\bar{\mathcal{I}}X^{(n)})^k} \prod_{i=1}^{i=k} [F_{\beta_i}]^{\mathrm{vir}}, \\ \frac{1}{e_{\mathbb{C}^*}(\widetilde{\mathrm{gl}}_k^* N^{\mathrm{vir}}_{F_{\vec{\beta}}})} =& \frac{\prod_{i=1}^k (z/k + \psi(\mathcal{E}_i))}{-z/k - \psi(\mathcal{E}_1) - \psi_{N+1} - \sum_{i=k}^{\infty} \mathfrak{D}_i} \cdot \prod_{i=1}^{k} \mathcal{I}_{\beta_i}(z/k + \psi(\mathcal{E}_i)). \end{aligned}$$

Proof. See [Zho22, Lemma 6.5.6]. □

Theorem 4.3.3. *Assuming $2g - 2 + N + 1/d_0 \cdot \deg(\beta) > 0$, we have*

$$\begin{aligned} &\langle \tau_{m_1}(\gamma_1), \ldots, \tau_{m_n}(\gamma_N) \rangle^{\epsilon_+}_{g,\beta} - \langle \tau_{m_1}(\gamma_1), \ldots, \tau_{m_n}(\gamma_N) \rangle^{\epsilon_-}_{g,\beta} \\ =& \sum_{k \geq 1} \sum_{\vec{\beta}} \frac{1}{k!} \int_{[Adm^{\epsilon_-}_{g,N+k}(X^{(n)}, \beta')]^{\mathrm{vir}}} \prod_{i=1}^{i=N} \psi_i^{m_i} \mathrm{ev}_i^*(\gamma_i) \cdot \prod_{a=1}^{a=k} \mathrm{ev}^*_{n+a} \mu_{\beta_a}(z)|_{z=-\psi_{n+a}} \end{aligned}$$

where $\vec{\beta}$ runs through all the $(k+1)$-tuples of effective curve classes

$$\vec{\beta} = (\beta', \beta_1, \ldots, \beta_k),$$

such that $\beta = \beta' + \beta_1 + \cdots + \beta_k$ and $\deg(\beta_i) = d_0$ for all $i = 1, \ldots, k$.

Sketch of Proof. We will just explain the master-space technique. For all the details we refer to [Zho22, Section 6]. By the virtual localisation formula we obtain

$$[MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}} = \left(\sum \iota_{F_\star *}\left(\frac{[F_\star]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\star})}\right)\right) \in A^{\mathbb{C}^*}_*(MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)) \otimes_{\mathbb{Q}[z]} \mathbb{Q}(z),$$

where $F_\star$ are the components of the $\mathbb{C}^*$-fixed locus of $MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)$. Let

$$\alpha = \prod_{i=1}^{i=N} \psi_i^{m_i} \mathrm{ev}_i^*(\gamma_i) \in A^*(MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta))$$

be the class corresponding to decedent insertions. After taking the residue[8] at $z=0$ of the above formula, capping with α and taking the degree of the class, we obtain the following equality

$$\int_{[Adm^{\epsilon_+}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}}} \alpha - \int_{[Adm^{\epsilon_-}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}}} \alpha = \deg\left(\alpha \cap \mathrm{Res}_{z=0}\left(\sum \iota_{F_\beta *}\left(\frac{[F_\beta]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta})}\right)\right)\right),$$

where we used that there is no $1/z$-term in the decomposition of the class

$$[MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}} \in A^{\mathbb{C}^*}_*(MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)),$$

and that

$$\frac{1}{e_{\mathbb{C}^*}(\mathbb{M}_\pm)} = 1/z + O(1/z^2).$$

The analysis of the residue on the right-hand side presented in [Zho22, Section 7] applies to our case. The resulting formula is the one claimed in the statement of the theorem. $\square$

We define

$$F_g^\epsilon(\mathbf{t}(z)) = \sum_{n=0}^{\infty}\sum_{\beta} \frac{q^\beta}{N!}\langle \mathbf{t}(\psi_1),\dots,\mathbf{t}(\psi_N)\rangle^\epsilon_{g,N,\beta},$$

where $\mathbf{t}(z) \in H^*_{\mathrm{orb}}(S^{(n)},\mathbb{Q})[z]$ is a generic element, and the unstable terms are set to be zero. By repeatedly applying Theorem 4.3.3 we obtain.

[8] i.e. by taking the coefficient of $1/z$ of both sides of the equality.

Corollary 4.3.4. *For all $g \geq 1$ we have*

$$F_g^0(\mathbf{t}(z)) = F_g^{-\infty}(\mathbf{t}(z) + \mu(-z)).$$

For $g = 0$, the same equation holds modulo constant and linear terms in $\mathbf{t}(\mathbf{z})$.

The fact that the change of variables above holds only moduli linear terms in $\mathbf{t}(\mathbf{z})$ is due to the same reasons as in the case of quasimaps, and it is explained in Section 2.5.3. The variants of Theorem 2.5.6 and Theorem 2.5.7 in exactly the same form hold in this case too.

4.4 Del Pezzo

In this section we determine the I-function in the case $X = S$ is a del Pezzo surface. Firstly, consider the expansion

$$[zI(q,z) - z]_+ = I_1(q) + (I_0(q) - 1)z + I_{-1}(q)z^2 + I_{-2}(q)z^3 + \dots,$$

we will show that by the dimension constraint, the terms I_{-k} vanish for all $k \geq 1$.

In this we consider $H^*_{\mathrm{orb}}(X^{(n)})$ with its *naive*[9] grading. Let z be of cohomological degree 2 in $H^*_{\mathrm{orb}}(X^{(n)})[z^{\pm}]$. The virtual dimension of $\overline{M}^{\bullet}_{\mathsf{m}}(X \times \mathbb{P}^1/X_\infty, (\gamma, n), \mu)$ is equal to

$$\int_{c_1(S)} \beta + n + \ell(\mu).$$

Hence by the virtual localisation, the classes involved in the definition of I-function,

$$\mathsf{ev}_* \left(\frac{[F_{\beta,\mu}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})} \right) \in H^*(S^\mu/\mathrm{Aut}(\mu))[z^{\pm}] \subseteq H^*_{\mathrm{orb}}(S^{(n)})[z^{\pm}],$$

have naive cohomological degree equal to

$$-2\left(\int_{c_1(S)} \beta + n - \ell(\mu) \right). \tag{4.23}$$

Since S is a del Pezzo surface, the above quantity is non-positive, which implies that

$$I_0 = 1 \text{ and } I_{-k} = 0$$

[9] We grade it with the cohomological grading of $H^*(\bar{\mathfrak{I}}S^{(n)}, \mathbb{Q})$.

for all $-k \geq 1$, because cohomology is non-negatively graded. Moreover, the quantity (4.23) is zero, if and only if

$$\mu = (1, \ldots, 1) \text{ and } \beta = (0, \mathsf{m}).$$

Let us now study $F_{\beta,\mu}$ for these values of μ and β. It is more convenient to put an ordering on fibers over $\infty \in \mathbb{P}^1$, so let $\vec{F}_{\beta,\mu}$ be the resulting space. We will not give a full description of $\vec{F}_{\beta,\mu}$, even though it is simple. We will only be interested in one type of components of $\vec{F}_{\beta,\mu}$,

$$\iota_i \colon \overline{M}_{\mathsf{h},p_i} \times S^n \hookrightarrow \vec{F}_{\beta,\mu}, \tag{4.24}$$

where $\overline{M}_{\mathsf{h},p_i}$ is the moduli spaces of stable genus-h curve with *one* marking labelled by p_i, $i = 1, \ldots N$. The embedding ι_i is constructed as follows. Given a point

$$((C, \mathbf{x}), x_1, \ldots, x_n)) \in \overline{M}_{\mathsf{h},p_i} \times S^n,$$

let

$$(\tilde{P}, p_1, \ldots, p_n) = \coprod_{i=1}^{i=n} (\mathbb{P}^1, 0) \tag{4.25}$$

be an ordered disjoint union of $\mathbb{P}^1$ with markings at $0 \in \mathbb{P}^1$. We define a curve P by gluing $(\tilde{P}, p_1, \ldots, p_n)$ with (C, p_i) at the marking with the same labelling. We define

$$f_{\mathbb{P}^1} \colon P \to \mathbb{P}^1$$

to be an identity on $\mathbb{P}^1$'s and contraction on C. We define

$$f_S \colon P \to S$$

by contracting j-th $\mathbb{P}^1$ in P possibly with an attached curve to the point $x_j \in S$. We thereby defined the inclusion

$$\iota_i((C,p), x_1, \ldots, x_n)) = (P, \mathbb{P}^1, 0, f_{\mathbb{P}^1} \times f_S).$$

By Lemma 4.1.9,

$$\mathsf{h} = \mathsf{m}/2, \tag{4.26}$$

in particular, m is even. More generally, any connected component of $\vec{F}_{\beta,\mu}$ admits a similar description with the difference that there might more markings on possibly disconnected C by which it attaches to $\widetilde{P}$, i.e. P has more nodes. These components are not relevant for our needs, as it will be explained below.

Let us now consider the virtual fundamental classes and the normal bundles of these components $\overline{M}_{\mathsf{h},p_i} \times S^n$. By standard arguments we obtain that

$$\iota_i^* \frac{[F_{\beta,\mu}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})} = e(\pi_i^* T_S \otimes p^* \mathbb{E}^\vee) \cdot \frac{e(\mathbb{E}^\vee z)}{z(z-\psi_1)},$$

where $\pi_i \colon \overline{M}_{\mathsf{h},p_i} \times S^n \to S$ is the projection to i-th factor of S^n and $p \colon \overline{M}_{\mathsf{h},p_i} \times S^n \to \overline{M}_{\mathsf{h},p_i}$ is the projection to $\overline{M}_{\mathsf{h},p_i}$; $\mathbb{E}$ is the Hodge bundle on $\overline{M}_{\mathsf{h},p_i}$.

For other components of $\overrightarrow{F}_{\beta,\mu}$ the equivariant Euler classes $e_{\mathbb{C}^*}(N^{\mathrm{vir}})$ acquire factors

$$\frac{1}{z(z-\psi_i)}$$

for each marked point. This makes them irrelevant for purposes of determining the truncation of I-function by non-negative powers of z. We therefore have to determine the following classes

$$\pi_* \left(e(\pi_i^* T_S \otimes p^* \mathbb{E}^\vee) \cdot \frac{e(\mathbb{E}^\vee z)}{z(z-\psi_1)} \right) \in H^*(S^n)[z^{\pm}],$$

where $\pi : \overline{M}_{\mathsf{h},p_i} \times S^n \to S^n$ is the natural projection, which is identified with evaluation map ev via the inclusion (4.24).

Let ℓ_1 and ℓ_2 be the Chern roots of $\pi_i^* T_S$. Then we can rewrite the class above as follows

$$\int_{\overline{M}_{\mathsf{h},1}} \frac{\mathbb{E}^\vee(\ell_1) \cdot \mathbb{E}^\vee(\ell_2) \cdot \mathbb{E}^\vee(z)}{z(z-\psi_1)},$$

where

$$\mathbb{E}^\vee(z) := e(\mathbb{E}^\vee z) = \sum_{j=0}^{j=\mathsf{h}} (-1)^{\mathsf{h}-j} \lambda_{\mathsf{h}-j} z^j,$$

and similarly for $\mathbb{E}^\vee(\ell_1)$ and $\mathbb{E}^\vee(\ell_2)$.

By putting these Hodge integrals into a generating series, we obtain an explicit expression for them. Note that below we sum over the degree m of the branching divisor, which in this case is related to the genus h by (4.26).

Proposition 4.4.1.

$$1 + \sum_{\mathsf{h}>0} u^{2\mathsf{h}} \int_{\overline{M}_{\mathsf{h},1}} \frac{\mathbb{E}^\vee(\ell_1) \cdot \mathbb{E}^\vee(\ell_2) \cdot \mathbb{E}^\vee(z)}{z(z-\psi_1)} = \left(\frac{\sin(u/2)}{u/2} \right)^{\frac{\ell_1+\ell_2}{z}}$$

Proof. The claim implicitly follows from the results of [FP00]. Firstly,

$$1+\sum_{\mathsf{h}>0} u^{2\mathsf{h}} \int_{\overline{M}_{\mathsf{h},1}} \frac{\mathbb{E}^\vee(\ell_1)\cdot \mathbb{E}^\vee(\ell_2)\cdot \mathbb{E}^\vee(z)}{z(z-\psi_1)} = 1+\sum_{\mathsf{h}>0} u^{2\mathsf{h}} \int_{\overline{M}_{\mathsf{h},1}} \frac{\mathbb{E}^\vee(\ell_1/z)\cdot \mathbb{E}^\vee(\ell_2/z)\cdot \mathbb{E}^\vee(1)}{1-\psi_1}.$$

Now let

$$a=\ell_1/z, \quad b=\ell_2/z$$

and

$$F(a,b)=1+\sum_{\mathsf{h}>0} u^{2\mathsf{h}} \int_{\overline{M}_{\mathsf{h},1}} \frac{\mathbb{E}^\vee(a)\cdot \mathbb{E}^\vee(b)\cdot \mathbb{E}^\vee(1)}{1-\psi_1}.$$

By using the virtual localisation on a moduli space of stable maps to $\mathbb{P}^1$, we obtain the following identities

$$F(a,b)\cdot F(-a,-b)=1;$$

$$F(a,b)\cdot F(-a,1-b)=F(0,1).$$

These identities with the fact that $F(a,b)$ is symmetric in a and b imply that

$$F(a,b)=F(0,1)^{a+b} \tag{4.27}$$

for integer values of a and b. Each coefficient of a power of u in $F(a,b)$ is a polynomial in a and b, hence the identity (4.27) is in fact a functional identity.

By the discussion in [FP00, Section 2.2] and by [FP00, Proposition 2], we obtain that

$$F(0,1)=\frac{\sin(u/2)}{u/2},$$

the claim now follows. □

Using the commutativity of the following diagram

$$\begin{array}{ccc} \vec{F}_{\beta,\mu} & \xrightarrow{\vec{\mathrm{ev}}} & S^n \\ \downarrow & & \downarrow \overline{\pi} \\ F_{\beta,\mu} & \xrightarrow{\mathrm{ev}} & [S^{(n)}] \end{array}$$

and Proposition 4.4.1, we obtain

$$I_1(q)=\log\left(\frac{\sin(u/2)}{u/2}\right)\cdot \frac{1}{n-1!}\overline{\pi}_*(\mathrm{c}_1(S)\otimes\cdots\otimes 1). \tag{4.28}$$

For $2g-2+N \geq 0$ we define

$$\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma} := \sum_k \langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,(\gamma,\mathsf{m})} u^{\mathsf{m}},$$

setting invariants corresponding to unstable values of g,N and β to zero. By repeatedly applying Theorem 4.3.3, we obtain that

$$\langle \gamma_1, \dots, \gamma_N \rangle^{0}_{g,\beta} = \sum_{k\geq 1} \frac{1}{k!} \left\langle \gamma_1, \dots, \gamma_N, \underbrace{I_1(q), \dots, I_1(q)}_{k} \right\rangle^{-\infty}_{g,\beta}.$$

Applying the divisor equation[10] and (4.28), we get following corollary.

Corollary 4.4.2. *Assuming $2g-2+N \geq 0$,*

$$\langle \gamma_1, \dots, \gamma_N \rangle^{0}_{g,\gamma} = \left(\frac{\sin(u/2)}{u/2}\right)^{\gamma\cdot \mathrm{c}_1(S)} \cdot \langle \gamma_1, \dots, \gamma_N \rangle^{-\infty}_{g,\gamma}.$$

4.5 Crepant resolution conjecture

To a cohomology-weighted partition

$$\vec{\mu} = ((\mu_1, \delta_{\ell_1}), \dots, (\mu_k, \delta_{\ell_k}))$$

we can also associate a class in $H^*(S^{[n]}, \mathbb{Q})$, using Nakajima operators,

$$\theta(\vec{\mu}) := \frac{1}{\prod_{i=1}^k \mu_i} P_{\delta_{\ell_1}}[\mu_1] \cdots P_{\delta_{\ell_k}}[\mu_k] \cdot 1 \in H^*(S^{[n]}, \mathbb{Q}),$$

where operators are ordered according to the standard ordering (see Subsection 4.1.6). For more details on these classes we refer to [Nak99, Chapter 8].

Proposition 4.5.1. *There exists a graded isomorphism of vector spaces*

$$L\colon H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{C}) \cong H^*(S^{[n]}, \mathbb{C}),$$
$$L(\lambda(\vec{\mu})) = (-i)^{\mathrm{age}(\mu)} \theta(\vec{\mu}).$$

Proof. See [FG03, Proposition 3.5]. ☐

Remark 4.5.2. The peculiar choice of the identification with a factor $(-i)^{\mathrm{age}(\mu)}$ is justified by Crepant resolution conjecture - this factor makes the invariants match on the nose. See the next section for more details.

[10] One can readily verify that an appropriate form of the divisor equation holds for classes in $H^*(S^{(d)}, \mathbb{Q}) \subseteq H^*_{\mathrm{orb}}(S^{(d)}, \mathbb{Q})$.

4.5.1 Quasimaps and admissible covers

From now on, we assume that $2g-2+N \geq 0$. Using (2.17), we obtain an identification

$$H_2(S^{[n]}, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \oplus \mathbb{Z}, \tag{4.29}$$

which is given by associating to the degree of a (quasi-)map the Chern character of its graph. Given classes $\gamma_i \in H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{C})$, $i = 1, \dots N$, and a class

$$(\gamma, \mathsf{m}) \in H_2(S, \mathbb{Z}) \oplus \mathbb{Z},$$

for $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$ we set

$$\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,(\gamma,\mathsf{m})} := {}^{\sharp}\langle L(\gamma_1), \dots, L(\gamma_N) \rangle^{\epsilon}_{g,(\gamma,\mathsf{m})} \in \mathbb{C},$$

the invariants on the right are defined in Section 2.4.3 and L is defined in Proposition 4.5.1. We set

$$\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma} := \sum_{\mathsf{m}} \langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,(\gamma,\mathsf{m})} y^{\mathsf{m}}.$$

For $\epsilon = 0^+$, these are the relative PT invariants of the relative geometry $S \times C_{g,N} \to \overline{M}_{g,N}$. The summation over m with respect to the identification (4.29) corresponds to the summation over ch_3 of a subscheme.

Using wall-crossings, we will now show the compatibility of $\mathsf{PT/GW}$ and Crepant resolution conjecture $\mathsf{C.R.C.}$. Let us firstly stress out our conventions.

- We sum over the degree of the branching divisor instead of the genus of the source curve. Assuming γ_i's are homogenous with respect to the age, the genus h and the degree m are related by Lemma 4.1.9,

  $$2\mathsf{h} - 2 = -2n + \mathsf{m} + \sum \mathrm{age}(\gamma_i).$$

  For $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$ let

  $${}'\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma} := \sum_{\mathsf{h}} \langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,(\gamma,\mathsf{h})} u^{2\mathsf{h}-2}$$

  be generating series where the summation is taken over genus instead. Then two two generating series are are related as follows

  $${}'\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma} \mapsto u^{2n - \sum \mathrm{age}(\gamma_i)} \cdot \langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma}.$$

- We sum over Chern character ch_3 instead of Euler characteristics χ. For $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$ let

$$'\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma} := \sum_{\chi} {}^{\sharp}\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,(\gamma,\chi)} y^{\chi}$$

be the generating series where the summation is taken over Euler characteristics instead. Then by Hirzebruch–Riemann–Roch theorem the two generating series are related as follows

$$'\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma} \mapsto y^{(g-1)n} \cdot \langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma}.$$

- The identification of Proposition 4.5.1 has a factor of $(-i)^{\mathrm{age}(\mu)}$.

Taking into account all the conventions above and Lemma 4.1.17, we obtain that [MNOP06, Conjectures 2R, 3R] can be reformulated[11] as follows.

PT/GW. *The generating series* $\langle \gamma_1, \dots, \gamma_N \rangle^{0^+}_{g,\gamma}(y)$ *is a Taylor expansion of a rational function around* 0, *such that under the change of variables* $y = -e^{iu}$,

$$(-y)^{-\gamma \cdot c_1(S)/2} \cdot \langle \gamma_1, \dots, \gamma_N \rangle^{0^+}_{g,\gamma}(y) = (-iu)^{\gamma \cdot c_1(S)} \cdot \langle \gamma_1, \dots, \gamma_N \rangle^{0}_{g,\gamma}(u).$$

Assume now that S is a del Pezzo surface. Let us apply our wall-crossing formulas. Using Corollary 4.4.2, we obtain

$$(-iu)^{\gamma \cdot c_1(S)} \cdot \langle \gamma_1, \dots, \gamma_N \rangle^{-\infty}_{g,\gamma} = (e^{iu/2} - e^{-iu/2})^{\gamma \cdot c_1(S)} \cdot \langle \gamma_1, \dots, \gamma_N \rangle^{0}_{g,\gamma}. \quad (4.30)$$

Using Corollary 2.5.11, we obtain

$$(-y)^{-\gamma \cdot c_1(S)/2} \cdot \langle \gamma_1, \dots, \gamma_N \rangle^{\infty}_{g,\gamma} = (y^{1/2} - y^{-1/2})^{\gamma \cdot c_1(S)} \cdot \langle \gamma_1, \dots, \gamma_N \rangle^{0^+}_{g,\gamma}. \quad (4.31)$$

Combining the two, we obtain the statement of C.R.C.

C.R.C. *The generating series* $\langle \gamma_1, \dots, \gamma_N \rangle^{\infty}_{g,\gamma}(y)$ *is a Taylor expansion of a rational function around 0, such that under the change of variables* $y = -e^{iu}$,

$$\langle \gamma_1, \dots, \gamma_N \rangle^{\infty}_{g,\gamma}(y) = \langle \gamma_1, \dots, \gamma_N \rangle^{-\infty}_{g,\gamma}(u).$$

By both wall-crossings, the statements of PT/GW and C.R.C. in the form presented above are equivalent.

[11] The conjectures of [MNOP06] were formulated for a fixed threefold, but they can be analogously formulated for a moving one, see [PT19].

Corollary 4.5.3. *If S is a del Pezzo surface, then*

$$\mathbf{PT/GW} \iff \mathbf{C.R.C.}$$

4.5.2 Quantum cohomology

Let $g = 0, N = 3$. This is a particularly nice case, firstly, because these invariants collectively are known as *quantum cohomology*. Secondly, the moduli space of genus-0 curves with 3 markings is a point. Hence the invariants $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{0,\gamma}^{-\infty}$ are PT invariants of $S \times \mathbb{P}^1$ relative to the divisor $S \times \{0, 1, \infty\}$. In [PP17] $\mathsf{PT/GW}$ is proven for $S \times \mathbb{P}^1$, if S is toric. Corollary 4.5.3 then implies the following.

Corollary 4.5.4. *If S is a toric del Pezzo surface, $g = 0$ and $N = 3$, then* **C.R.C.** *holds.*

The above result can also be stated as an isomorphism of quantum cohomologies with appropriate coefficient rings. Let

$$\begin{aligned} QH^*(S^{[n]}) &:= H^*(S^{[n]}) \otimes_{\mathbb{C}} \mathbb{C}[\![q^\gamma]\!](y) \\ QH^*_{\mathrm{orb}}(S^{(n)}) &:= H_{\mathrm{orb}}(S^{(n)}) \otimes_{\mathbb{C}} \mathbb{C}[\![q^\gamma]\!](e^{iu}) \end{aligned}$$

be quantum cohomologies, where $\mathbb{C}[\![q^\gamma]\!](y)$ and $\mathbb{C}[\![q^\gamma]\!](e^{iu})$ are rings of rational functions with coefficients in $\mathbb{C}[\![q^\gamma]\!]$ and in variables y and e^{iu}, respectively. The latter we view as a subring of functions in the variable u. The quantum cohomologies are isomorphic by Corollary 4.5.4,

$$QL \colon QH^*_{\mathrm{orb}}(S^{(n)}) \cong QH^*(S^{[n]}),$$

where QL is given by a linear extension of L, defined in Proposition 4.5.1, from $H^*_{\mathrm{orb}}(S^{(n)})$ to $H^*_{\mathrm{orb}}(S^{(n)}) \otimes_{\mathbb{C}} \mathbb{C}[\![q^\gamma]\!]$ and by a change of variables $y = -e^{iu}$. In particular,

$$QL(\alpha \cdot q^\gamma \cdot y^k) = (-1)^n L(\alpha) \cdot q^\gamma \cdot e^{niu}$$

for an element $\alpha \in H_{\mathrm{orb}}(S^{(n)})$. Ideally, one would also like to specialise to $y = 0$ and $y = -1$, because in this way we recover the classical multiplications on $H^*_{\mathrm{orb}}(S^{(n)})$ and $H^*(S^{[n]})$. To do so, a more careful choice of coefficients is needed - we have to take rational functions with no poles at $y = 0$ and $y = -1$.

Chapter 5

Appendix

5.1 Stability of fibers

The aim of this section is to prove Proposition 5.1.4, the converse of Lemma 2.2.13. The proof is inspired by the proof of [Tho00, Proposition 4.2], which, however, contains a mistake in the direction

$$\text{stability} \implies \text{stability of a general fiber},$$

because a sheaf F on a threefold restricts to stable sheaf on the hyperplane section with respect to the stability that defines the hyperplane section, which is not necessarily suitable. If one adds fiber classes to the polarisation to make it suitable, then one has to take a hyperplane section of bigger degree, for which suitable polarisation may be different.

Let $X := S \times C \to C$ be a trivial surface fibration over a connected nodal curve C. Let us fix a very ample line bundle $\mathcal{O}_S(1)$. We denote a line bundle with specified degrees on each irreducible components $\mathcal{O}_S(1) \boxtimes \mathcal{O}_C(k_1, \dots, k_m)$ by L_{k_i}, and the degree of a sheaf F with respect to L_{k_i} by $\deg_{k_i}(F)$. Recall that for a possibly singular scheme X slope of a torsion-free sheaf F can be defined as follows

$$\mu(F) = \frac{a_{\dim(X)-1}(F)}{a_{\dim(X)}(F)},$$

where $a_i(F)$'s are the coefficients in a Hilbert polynomial

$$P(F, m) = \sum a_i(F) \frac{m^i}{i!}.$$

In what follows by stability we will mean *slope* stability.

Proposition 5.1.1. *Assume C is smooth. Fix a class $\beta \in H^*(S \times C, \mathbb{Q})$, such that* $\mathrm{rk}(\beta) = 2$. *There exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ and for all torsion free sheaves F with* $\mathrm{ch}(F) = \beta$ *the following statement holds: F is L_n-stable, if F_t is stable for a general $t \in C$.*

Proof. We will prove the proposition by restricting to a hyperplane section and then applying [HL97, Theorem 5.3.2], see also [Yos99, Lemma 1.2].

Firstly, consider the Küneth's decomposition,

$$H^2(S \times C, \mathbb{Q}) = H^2(S, \mathbb{Q}) \oplus H^1(S, \mathbb{Q}) \otimes H^1(C, \mathbb{Q}) \oplus \mathbb{Q},$$

the first Chern class of a sheaf can be expressed accordingly

$$\mathrm{c}_1(F) = \mathrm{c}_1(F_t) \oplus \alpha \oplus \mathrm{k}(F),$$

where each summand is in a corresponding Küneth component and F_t is a general fiber of F over $t \in C$. The intersection numbers with L_n's take the following form

$$\mathrm{c}_1(F) \cdot L_n \cdot L_m = d \cdot \mathrm{k}(F) + (n+m) \cdot \deg(F)_{\mathrm{f}}, \tag{5.1}$$

where $d = \mathcal{O}_S(1)^2$ and $\deg(F)_{\mathrm{f}} = \deg(F_t)$. In particular, slope-stability with respect to a curve class $L_1 \cdot L_{2n-1}$ coincides with slope-stability with respect to a curve class $L_n \cdot L_n$.

Consider now a general hyperplane section $H \in |\mathcal{O}_S(1) \boxtimes \mathcal{O}_C(1)|$, let $2n_0 - 1$ be the smallest odd integer such that [HL97, Theorem 5.3.2] holds for $H \to C$, the class $\beta_{|H}$ and a polarisation $L_{2n_0-1|H}$.

Assume F_t is unstable for all $t \in C$. Let $G \hookrightarrow F$ be a relative destabilising subsheaf (strictly speaking, it exists over some non-empty open subscheme $U \subseteq C$, we then extend over the entire C). Consider now the restriction to a general hyperplane section $G_{|H} \hookrightarrow F_{|H}$, it is destabilising by the proof of [HL97, Theorem 5.3.2] with respect to $L_{2n_0-1|H}$, therefore $G \hookrightarrow F$ is L_{n_0}-destabilising. □

Remark 5.1.2. The reason for the failure of the proof of Proposition 5.1.1 for rk > 2 is already present at the level of fibered surfaces. For a fibred surface the difference between rk $= 2$ and rk > 2 cases is that for the former a suitable polarisation has a stronger property, namely, a subsheaf is destabilising, if and only if it is destabilising on a fiber as it is shown in [HL97, Theorem 5.3.2]. However, the author couldn't establish such property of a suitable polarisation for rk > 2. In this case one can show that there are no walls between the fiber stability and L_n-stability for $n \gg 0$, which is a weaker statement.

Corollary 5.1.3. *Assume we are in the setting of Proposition 5.1.1 and F is unstable at a general fiber, let $G \subset F$ be a relatively destabilising subsheaf, then*

$$\operatorname{rk}(G)\deg_n(F) - \operatorname{rk}(F)\deg_n(G) < 2(n_0 - n),$$

for all $n \geq n_0$, i.e. the difference of slopes can be made arbitrary negative by increasing n.

Proof. By the proof of Proposition 5.1.1 $G \subset F$ is L_n-destabilising for all $n \geq n_0$, therefore

$$\begin{aligned}
&\operatorname{rk}(G)\deg_n(F) - \operatorname{rk}(F)\deg_n(G) \\
&< \operatorname{rk}(G)\deg_n(F) - \operatorname{rk}(F)\deg_n(G) - (\operatorname{rk}(G)\deg_N(F) - \operatorname{rk}(F)\deg_{n_0}(G)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq 2(n_0 - n),
\end{aligned}$$

where for the last inequality we used (5.1). □

Now let C be a connected nodal curve and $\tilde{C}$ be its normalisation, by $\tilde{C}_i$ we will denote its connected components. For a sheaf F on a threefold $S \times C$ we denote its pullback to $X_i := S \times \tilde{C}_i$ by F_i.

Proposition 5.1.4. *Fix classes $\beta_i \in H^*(S \times \tilde{C}_i, \mathbb{Q})$ with the same fiber component, such that $\operatorname{rk}(\beta_i) = 2$. There exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ and for all sheaves F flat over C with $\operatorname{ch}(F_i) = \beta_i$ the following statement holds: F is L_{nk_i}-stable, if F_t is stable for a general $t \in C$.*

We will prove the proposition for the case of C with one node, splitting the proof into two parts depending on whether the node is separating or non-separating. The proof easily generalises to the case of C with more nodes.

Proof (non-separating node). Let C be a connected nodal curve with one non-separating node $s \in C$ and $\pi\colon S \times \tilde{C} \to S \times C$ be the normalization map of the product. The sheaves F and π^*F are related by the normalisation sequence

$$0 \to F \to \pi_*\pi^*F \to F_s \to 0,$$

from which we obtain

$$a_3(F) = a_3(\pi^*F), \quad a_2(F) = a_2(\pi^*F) - a_2(F_s).$$

Now let $G \subset \pi^*F$ be a relatively destabilising subsheaf and $\tilde{G}$ be the kernel of the following composition

$$\pi_*G \hookrightarrow \pi_*\pi^*F \to F_s,$$

by construction $\tilde{G}$ is a subsheaf of F and

$$a_3(\tilde{G}) = a_3(G), \quad a_2(\tilde{G}) \leq a_2(G).$$

The difference of slopes of F and $\tilde{G}$ can then be bounded from above as follows

$$\frac{a_2(F)}{a_3(F)} - \frac{a_2(\tilde{G})}{a_3(\tilde{G})} \geq \frac{a_2(\pi^*F)}{a_3(\pi^*F)} - \frac{a_2(G)}{a_3(G)} - \frac{a_2(F_s)}{a_3(\pi^*F)}.$$

After multiplying by denominators, the right-hand side of the expression above is equal to

$$a_3(G) \cdot a_2(\pi^*F) - a_3(\pi^*F) \cdot a_2(G) - a_3(G) \cdot a_2(F_s) \tag{$*$}$$

Recall that

$$\begin{aligned} a_2(F) &= \deg_k(F) + \mathrm{rk}(F) \cdot a_2(\mathcal{O}_X), \\ a_3(F) &= \mathrm{rk}(F) \cdot a_3(\mathcal{O}_X). \end{aligned}$$

Substituting the above expressions into the summands of $(*)$, we obtain

$$(*) = a_3(\mathcal{O}_X) \cdot (\mathrm{rk}(G) \cdot \deg_k(\pi^*F) - \mathrm{rk}(F) \cdot \deg_k(G) - d \cdot \mathrm{rk}(F) \cdot \mathrm{rk}(G)),$$

where we also used that

$$a_2(F_s) = d \cdot \mathrm{rk}(F_s) = d \cdot \mathrm{rk}(F),$$

because F is flat over C. By Corollary 5.1.3 the term

$$\mathrm{rk}(G) \cdot \deg_k(\pi^*F) - \mathrm{rk}(F) \cdot \deg_k(G)$$

can be made arbitrary negative by taking big enough power of $\mathcal{O}_C(k)$, thereby making the difference of slopes negative. Moreover, the choice of the power is uniform for all F. $\square$

Proof (separating node). Let $C = C_1 \cup C_2$ be a connected nodal curve with one separating node $s \in C$, and let $\mathcal{O}_C(k_1, k_2)$ be the ample line bundle with prescribed degrees on each component. The restrictions of F to $S \times C_i$ are related to F by the normalisation sequence

$$0 \to F \to F_1 \oplus F_2 \to F_s \to 0,$$

from which we obtain

$$a_3(F) = a_3(F_1) + a_3(F_2), \quad a_2(F) = a_2(F_1) + a_2(F_2) - a_2(F_s).$$

Now let $G_i \subset F_i$ be relatively destabilising subsheaves and $\tilde{G}$ be the kernel of the following composition

$$G_1 \oplus G_2 \hookrightarrow F_1 \oplus F_2 \to F_s,$$

by construction $\tilde{G}$ is a subsheaf of F and

$$a_3(\tilde{G}) = a_3(G_1) + a_3(G_2), \quad a_2(\tilde{G}) \leq a_2(G_1) + a_2(G_2).$$

The difference of slopes of F and $\tilde{G}$ then takes the following form

$$\frac{a_2(F)}{a_3(F)} - \frac{a_2(\tilde{G})}{a_3(\tilde{G})} \geq \frac{\sum a_2(F_i)}{\sum a_3(F_i)} - \frac{\sum a_2(G_i)}{\sum a_3(G_i)} - \frac{a_2(F_s)}{\sum a_3(F_i)}.$$

After multiplying by denominators, the right-hand side of the the expression above is equal to

$$\begin{aligned} &a_2(F_1) \cdot (a_3(G_1) + a_3(G_2)) - a_2(G_1) \cdot (a_3(F_1) + a_3(F_2)) \\ +&a_2(F_2) \cdot (a_3(G_1) + a_3(G_2)) - a_2(G_2) \cdot (a_3(F_1) + a_3(F_2)) \\ -&a_2(F_s) \cdot \sum a_3(G_i) \end{aligned}$$

We now group the summands in the following way

$$\begin{aligned} a_2(F_1) \cdot a_3(G_1) - a_2(G_1) \cdot a_3(F_1) + a_2(F_2) \cdot a_3(G_2) - a_2(G_2) \cdot a_3(F_2) \\ - a_2(F_s) \cdot \sum a_3(G_i) \end{aligned} \quad (\mathbf{a})$$

$$+a_2(F_1) \cdot a_3(G_2) - a_2(G_2) \cdot a_3(F_1) + a_2(F_2) \cdot a_3(G_1) - a_2(G_1) \cdot a_3(F_2) \quad (\mathbf{b})$$

We will analyse terms **(a)** and **(b)** separately.

Term (a). The term **(a)** is simple to deal, substituting

$$\begin{aligned} a_2(F_i) &= \deg_{k_i}(F_i) + \mathrm{rk}(F_i) \cdot a_2(\mathcal{O}_{X_i}) \\ a_3(F_i) &= \mathrm{rk}(F_i) \cdot a_3(\mathcal{O}_{X_i}) \end{aligned}$$

into **(a)** we obtain that

$$(\mathbf{a}) = \sum a_3(\mathcal{O}_{X_i}) \cdot \left(\mathrm{rk}(G_i) \cdot \deg_{k_i}(F_i) - \mathrm{rk}(F) \cdot \deg_{k_i}(G_i) - \mathrm{rk}(F) \cdot \mathrm{rk}(G_i)\right),$$

since F is stable at a general fiber, the right-hand side can be made negative taking big enough power of $\mathcal{O}_C(k_1, k_2)$ by Corollary 5.1.3.

Term (b). Making the same substitution into **(b)** we obtain

$$\begin{aligned}
&\mathrm{rk}(G_2)\cdot\deg_{k_1}(F_1)\cdot a_3(\mathcal{O}_{X_2})-\mathrm{rk}(F_2)\cdot\deg_{k_1}(G_1)\cdot a_3(\mathcal{O}_{X_2})\\
&+\mathrm{rk}(G_1)\cdot\deg_{k_2}(F_2)\cdot a_3(\mathcal{O}_{X_1})-\mathrm{rk}(F_1)\cdot\deg_{k_2}(G_2)\cdot a_3(\mathcal{O}_{X_1}) \qquad (\mathbf{b.1})\\
&+\mathrm{rk}(F_1)\cdot\mathrm{rk}(G_2)\cdot a_2(\mathcal{O}_{X_1})\cdot a_3(\mathcal{O}_{X_2})-\mathrm{rk}(F_1)\cdot\mathrm{rk}(G_2)\cdot a_2(\mathcal{O}_{X_2})\cdot a_3(\mathcal{O}_{X_1})\\
&+\mathrm{rk}(F_2)\cdot\mathrm{rk}(G_1)\cdot a_2(\mathcal{O}_{X_2})\cdot a_3(\mathcal{O}_{X_1})-\mathrm{rk}(F_2)\cdot\mathrm{rk}(G_1)\cdot a_2(\mathcal{O}_{X_1})\cdot a_3(\mathcal{O}_{X_2})\\
&\qquad (\mathbf{b.2})
\end{aligned}$$

We again split the analysis in two parts. For the term ($\mathbf{b.1}$) we use that

$$\begin{aligned}
\deg_{k_i}(F_i)&=d\cdot\mathrm{k}(F_i)+2k_i\cdot\deg(F_i)_{\mathrm{f}}\\
a_3(\mathcal{O}_{X_i})&=d\cdot k_i
\end{aligned}$$

to obtain

$$\begin{aligned}
&2d\cdot k_1\cdot k_2\cdot\mathrm{rk}(G_2)\cdot\deg(F_1)_{\mathrm{f}}-2d\cdot k_1\cdot k_2\cdot\mathrm{rk}(F_2)\cdot\deg(G_1)_{\mathrm{f}}\\
&+2d\cdot k_1\cdot k_2\cdot\mathrm{rk}(G_1)\cdot\deg(F_2)_{\mathrm{f}}-2d\cdot k_1\cdot k_2\cdot\mathrm{rk}(F_1)\cdot\deg(G_2)_{\mathrm{f}}\\
&+d\cdot k_2\cdot d\cdot\mathrm{rk}(G_2)\cdot\mathrm{k}(F_1)-d\cdot k_2\cdot d\cdot\mathrm{rk}(F_2)\cdot\mathrm{k}(G_1)\\
&+d\cdot k_1\cdot d\cdot\mathrm{rk}(G_1)\cdot\mathrm{k}(F_2)-d\cdot k_1\cdot d\cdot\mathrm{rk}(F_1)\cdot\mathrm{k}(G_2)
\end{aligned}$$

Let K_i be the smallest integer for which the proposition holds, then by (5.1)

$$d\cdot\mathrm{rk}(F)\cdot\mathrm{k}(G_i)>2K_i\cdot(\cdot\mathrm{rk}(G_i)\cdot\deg(F)_{\mathrm{f}}-\mathrm{rk}(F)\cdot\deg(G_i)_{\mathrm{f}})+d\cdot\mathrm{rk}(G_i)\cdot\mathrm{k}(F_i),$$

where we also used that

$$\mathrm{rk}(F_1)=\mathrm{rk}(F_2)=\mathrm{rk}(F).$$

Regrouping the summands and applying the above inequality, we obtain that

$$\begin{aligned}
(\mathbf{b.1})<\sum d\cdot k_{i+1}\cdot(k_i-K_i)\cdot(\mathrm{rk}(G_i)\cdot\deg(F)_{\mathrm{f}}-\mathrm{rk}(F)\cdot\deg(G_i)_{\mathrm{f}})&\\
+\sum d\cdot k_{i+1}\cdot d\cdot\mathrm{k}(F_i)\cdot(\mathrm{rk}(G_i)-\mathrm{rk}(G_{i+1}))&.
\end{aligned}$$

For the term ($\mathbf{b.2}$) we use that

$$a_2(\mathcal{O}_{X_i})=d\cdot g_i+\frac{k_i\cdot\mathrm{c}_1(\mathcal{O}_S(1))\cdot\mathrm{c}_1(S)}{2},$$

where $g_i=g(C_i)$, then after some cancellations we obtain

$$(\mathbf{b.2})=\sum d\cdot k_i\cdot d\cdot g_{i+1}\cdot\mathrm{rk}(F)\cdot(\mathrm{rk}(G_i)-\mathrm{rk}(G_{i+1})),$$

now putting ($\mathbf{b.1}$) and ($\mathbf{b.2}$) together we see that if

$$(k_i - K_i)\cdot(\mathrm{rk}(G_i)\cdot\deg(F)_{\mathrm{f}} - \mathrm{rk}(F)\cdot\deg(G_i)_{\mathrm{f}}) \\ < d\cdot(\mathrm{rk}(G_i)-\mathrm{rk}(G_{i+1}))\cdot(g_{i+1}\cdot\mathrm{rk}(F)-\mathrm{k}(F_i)),$$

then ($\mathbf{b.1}$) + ($\mathbf{b.2}$) is negative. The right-hand side of the above inequality can be bounded independently of F, therefore by taking high enough power of $\mathcal{O}_C(k_1,k_2)$ the term **(b)** is negative independently of F. $\square$

5.2 Reduced obstruction theory

Let $\mathbb{E}^\bullet_{\mathrm{red}}$ be the cone of the dual of the semiregularity map $\mathsf{sr}^\vee$. The existence of the obstruction-theory morphism

$$\mathbb{E}^\bullet_{\mathrm{red}} \to \mathbb{L}_{Q^\epsilon_{g,N}(M,\beta)/\mathfrak{M}_{g,N}}$$

is slightly problematic from a technical point of view, as one needs to consider Hodge theory for non-commutative spaces to run the same argument as in [KT18] in full generality. Another option would be to use results from [Pri], however, there a singular case is not discussed, and in our case $S\times C$ might be singular due to singularity of C. The optimal result is therefore the following one, if one refrains from going too deeply into non-commutative geometry. The proof closely follows [KT18].

Proposition 5.2.1. *Given $(\mathbf{v},\check{\beta})\in\Lambda\oplus\Lambda$, assume a first-order deformation $\kappa_S\in HT^2(S)\cong HH^2(S)$ from Proposition 3.1.3 is represented by a $\mathbb{C}[\epsilon]/\epsilon^2$-linear admissible subcategory*

$$\mathcal{C}\subseteq \mathrm{D}_{\mathrm{perf}}(\mathcal{Y}),$$

where $\mathcal{Y}\to B=\operatorname{Spec}\mathbb{C}[\epsilon]/\epsilon^2$ is flat. Then there exists an obstruction theory morphism

$$\mathbb{E}^\bullet_{\mathrm{red}} \to \mathbb{L}_{Q^\epsilon_{g,N}(M,\beta)/\mathfrak{M}_{g,N}}.$$

Proof. Firstly, by taking the central fiber, we get that

$$D_{\mathrm{perf}}(S)\subseteq D_{\mathrm{perf}}(Y)$$

is an admissible subcategory, where Y is the central fiber of $\mathcal{Y}$. Therefore there is an isomorphism of moduli stacks

$$\mathfrak{Coh}(S)\cong\mathfrak{D}_{\mathrm{Coh}(S)}(Y), \tag{5.1}$$

where $\mathfrak{D}_{\mathrm{Coh}(S)}(Y)$ is the moduli stack of objects on Y which are contained in the subcategory $\mathrm{Coh}(S)$. This also implies that the quasimap moduli stacks are isomorphic,

$$Q^{\epsilon}_{g,N}(M, \mathfrak{Coh}(S), \beta) \cong Q^{\epsilon}_{g,N}(M, \mathfrak{D}_{\mathrm{Coh}(S)}(Y), \beta).$$

Let

$$M_Y := M^{\epsilon}_{\mathbf{v},\check{\beta}}(Y \times C_{g,N}/\overline{M}_{g,N}) \cong M^{\epsilon}_{\mathbf{v},\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}) =: M_S$$

be the relative moduli of sheaves corresponding to $Q^{\epsilon}_{g,N}(M, \mathfrak{D}_{\mathrm{Coh}(S)}(Y), \beta)$ and $Q^{\epsilon}_{g,N}(M, \mathfrak{Coh}(S), \beta)$ respectively.

Secondly, the inclusion $\mathrm{D^b}(S) \hookrightarrow \mathrm{D_{perf}}(Y)$ induces a map between Hochschild cohomologies

$$HH^2(Y) \to HH^2(S), \tag{5.2}$$

given by restricting the natural transformation of functors

$$\mathrm{id}_{\mathrm{D_{perf}(Y)}} \to [2].$$

This map sends κ_Y to κ_S (see e.g. [Per, Lemma 4.6]), where κ_Y is the class associated to the deformation $\mathcal{Y} \to B$. Moreover, for a complex $F \in \mathrm{D^b}(S \times C)$ the class

$$\kappa(F) \in \mathrm{Ext}^2(F, F),$$

which is given by applying the natural transformation associated to $\kappa \in HH^2(S)$ to F, is the obstruction to deform F in κ-direction, and by [Tod09, Proposition 5.2] and [C05] it agrees with obstruction class given by composing Koidara-Spencer class with Atiyah class

$$\kappa(F) = \kappa \cdot \exp(-\mathrm{At}(F))$$

after applying HKR isomorphism

$$HH^2(S) \cong HT^2(S).$$

We now identify a sheaf $F \in \mathrm{Coh}(S \times C)$ with its image in $\mathrm{D_{perf}}(Y \times C)$, then the following triangle commutes

$$\begin{tikzcd} HH^2(S) \arrow[r] & \mathrm{Ext}^2(F,F) \\ HH^2(Y) \arrow[u] \arrow[ur] & \end{tikzcd}$$

Hence by the choice of κ_S the deformation of sheaves in the class $(\mathbf{v}, \check{\beta})$ viewed as complexes on $Y \times C$ is obstructed in κ_Y-direction, because the obstruction class is non-zero by the construction of κ_S.

We now closely follow [KT18, Section 3.2]. By the above discussion the inclusion of the central fiber over B

$$M_Y \hookrightarrow M_{\mathcal{Y}/B}$$

is an isomorphism. The obstruction complexes of M_Y and M_S are isomorphic under the natural identifications of the moduli spaces

$$\mathcal{H}om_{\pi_S}(\mathbb{F}_S, \mathbb{F}_S) \cong \mathcal{H}om_{\pi_Y}(\mathbb{F}_Y, \mathbb{F}_Y), \tag{5.3}$$

because both complexes can be defined just in terms of $\mathrm{D}^{\mathrm{b}}(S)$, where $\mathbb{F}_{S/Y}$ are universal families of $M_{S/Y}$ with $\pi_{S/Y}$ being the obvious projections. Note that the trace on $Y \times C$ has no effect on Ext^2, since $H^2(\mathcal{O}_{Y\times C}) = 0$, and in certain sense a semiregularity map σ_i on $S \times C$ corresponds to semiregularity map σ_{i+1} on $Y \times C$. In particular, $\mathcal{H}om_{\pi_S}(\mathbb{F}_S, \mathbb{F}_S)_0$ and $\mathcal{H}om_{\pi_Y}(\mathbb{F}_Y, \mathbb{F}_Y)_0$ are not isomorphic. Nevertheless, we *claim* that the following composition

$$\mathbb{E}^\bullet \colon = \mathcal{H}om_{\pi_S}(\mathbb{F}_S, \mathbb{F}_S)_0 \to \mathcal{H}om_{\pi_Y}(\mathbb{F}_Y, \mathbb{F}_Y)_0 \to \mathbb{L}_{M_{\mathcal{Y}/B}/B} \tag{5.4}$$

is a perfect obstruction theory, where the first map is given by identification (5.3), while the second is by Atiyah class on $\mathcal{Y} \times M_{\mathcal{Y}/B}$, in particular, the second map is an obstruction theory. For proof of the claim we plan to use criteria from [BF97, Theorem 4.5].

Since for any B-scheme Z_0 a B-map $Z_0 \to M_{\mathcal{Y}/B}$ factors though the central fiber, the B-structure map $Z_0 \to B$ factors through the closed point of B. Let $\mathcal{F}_0$ be the sheaf associated to the map $Z_0 \to M_{\mathcal{Y}/B}$. The morphism $\mathcal{H}om_\pi(\mathbb{F}_Y, \mathbb{F}_Y)_0 \to \mathbb{L}_{M_{\mathcal{Y}/B}/B}$ is an obstruction theory, therefore [BF97, Theorem 4.5] to prove that (5.4) is an obstruction theory, it suffices to prove that the image of a non-zero obstruction class $\varpi(\mathcal{F}_0) \in \mathrm{Ext}^2_{Y\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_Y^* I)$ with respect to the map

$$\begin{aligned}\mathrm{Ext}^2_{Y\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_Y^* I) &\cong \mathrm{Ext}^2_{S\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_S^* I) \\ &\to \mathrm{Ext}^2_{S\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_S^* I)_0 \end{aligned} \tag{5.5}$$

is non-zero for any square-zero B-extension Z of Z_0 given by an ideal I, where $p_Y \colon Y \times_B Z_0 = Y \times Z_0 \to Z_0$ and $p_S \colon S \times Z_0 \to Z_0$ are the natural projections. Given a square-zero B-extension Z of Z_0 there are two possibilities

(i) the B-structure map $Z \to B$ factors through the closed point;

(ii) the B-structure map $Z \to B$ does not factor through the closed point.

(i) In this case, the obstruction of lifting the map to $Z \to M_{\mathcal{Y}/B}$ coincides with the obstruction of lifting the map to $Z \to M_Y \cong M_S$, hence if $\varpi(\mathcal{F}_0)$ is non-zero, its image with respect (5.5) is non-zero.
(ii) In this case, a lift to $Z \to M_{\mathcal{Y}/B}$ is always obstructed, and the obstruction is already present at a single fiber of p_Y in the following sense. By assumption there exists a section $B \to Z$ which is an immersion (we can find an open affine subscheme $U \subset Z$ such that $U \to B$ is flat, but then $U \cong U_0 \times B$, because first-order deformations of affine schemes are trivial, thereby we get a section). Let $z \in Z$ be image of the closed point of B of the section, then the restriction

$$\mathrm{Ext}^2_{Y\times S_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_Y^* I) \to \mathrm{Ext}^2_{Y\times z}(\mathcal{F}_{0,z}, \mathcal{F}_{0,z} \otimes p_Y^* I_z)$$

applied to the obstruction class $\varpi(\mathcal{F}_0)$ is non-zero and is the obstruction to lift the sheaf $\mathcal{F}_{0,z}$ on Y to a sheaf on $\mathcal{Y}$, hence due to the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^2_{Y\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_Y^* I) & \longrightarrow & \mathrm{Ext}^2_{Y\times z}(\mathcal{F}_{0,z}, \mathcal{F}_{0,z} \otimes p_Y^* I_z) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^2_{S\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_S^* I)_0 & \longrightarrow & \mathrm{Ext}^2_{S\times z}(\mathcal{F}_{0,z}, \mathcal{F}_{0,z} \otimes p_S^* I_z)_0
\end{array}$$

we conclude that the image of $\varpi(\mathcal{F}_0)$ in $\mathrm{Ext}^2_{S\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p^* I)_0$ is non-zero, because the image of $\varpi(\mathcal{F}_{0,z})$ is non-zero in $\mathrm{Ext}^2_{S\times z}(\mathcal{F}_{0,z}, \mathcal{F}_{0,z} \otimes p^* I_z)_0$. This establishes claim.

The absolute perfect obstruction theory $\mathbb{H}^\bullet$ is then defined by taking the cone of $\mathbb{E}^\bullet \to \Omega_B[1]$, so that we have the following diagram

$$\begin{array}{ccccc}
\mathbb{H}^\bullet & \longrightarrow & \mathbb{E}^\bullet & \longrightarrow & \Omega_B[1] \\
\downarrow & & \downarrow & & \| \\
\mathbb{L}_{M_Y} & \longrightarrow & \mathbb{L}_{M_{\mathcal{Y}/B}/B} & \longrightarrow & \Omega_B[1]
\end{array}$$

By the same argument as in [KT18, Section 2.3] the composition

$$\mathbb{H}^\bullet \to \mathbb{E}^\bullet \to \mathbb{E}^\bullet_{\mathrm{red}}$$

is an isomorphism, hence the proposition follows. $\square$

For example, if $M = S^{[n]}$ and $\mathrm{c}_1(\check{\beta}) \neq 0$ (i.e. the curve class is not exceptional), we can use a commutative deformation given by the infinitesimal twistor family $\mathcal{S} = \mathcal{Y} \to B$ with respect to the class $\mathrm{c}_1(\check{\beta})$.

The situation becomes more complicated already in the case of $S^{[n]}$ and $\mathrm{c}_1(\check{\beta}) = 0$ (i.e. an exceptional curve class), a commutative first-order deformation can no longer satisfy the property stated in Proposition 3.1.3. If $d = 2$ and $S^{[2]}$ is isomorphic to a Fano variety of lines of some special cubic fourfold (e.g. see [Has00, Theorem 1.0.3]), then

$$D_{\mathrm{perf}}(Y) = \langle D_{\mathrm{perf}}(S), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$$

and the family $\mathcal{Y} \to B$ is given by deformation of Y away from the Hassett divisor.

Remark 5.2.2. In [Tod09] Toda constructed geometric realisations of infinitesimal non-commutative deformations in $HH^2(X)$ for a smooth projective X. However, it is not clear if they are of the type required by Proposition 5.2.1. In principle, there should be no problem in proving Proposition 5.2.1 dropping the assumption. For that one has to show that Toda's infinitesimal deformations behave well under base change.

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