André-Quillen Homology of Global Power Functors and Ultra-Commutative Ring Spectra

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Abstract

In this dissertation, we study multiplications in globally equivariant stable homotopy theory. Concretely, this means studying ultra-commutative ring spectra, which encode a rich multiplicative structure on a family of compatible equivariant homotopy types for all compact Lie groups. The associated algebraic structure is that of a global power functor. This structure formalizes a collection of compatible G-Tambara functors for all compact Lie groups G, encoding the norms present in the homotopy groups of an ultra-commutative ring spectrum. These norms are alternatively encoded as power operations, facilitating the calculations.

The main tool we introduce to study these multiplicative structures is globally equivariant André-Quillen theory. This is a generalization of the cohomology theory introduced independently by André and Quillen in 1967 for commutative rings, and the corresponding topological André-Quillen cohomology for commutative ring spectra by Basterra. This cohomology theory is well-adapted to the study of multiplicative structures, and it is able to detect interesting properties of commutative algebras, like smoothness and étaleness. Moreover, the topological André-Quillen theory has proven to be very useful in the construction of obstruction theories for commutative ring spectra.

In the construction of the globally equivariant generalization of algebraic André-Quillen cohomology, a main aspect is the handling of the power operations. This is mainly done by considering the power operations as a twisted version of the usual power $x \mapsto x^n$, and adapting the classical formulas accordingly. As the André-Quillen cohomology is a non-abelian derived functor of derivations, and the homology is a derived functor of the Kähler differentials, we first introduce derivations and Kähler differentials of global power algebras. Moreover, we study the related notion of square-zero extensions. For the derivations, the new aspect is a twisted version of the Leibniz rule for the derivation of a power operation.

From the definition of derivation and Kähler differentials, we obtain André-Quillen cohomology and homology by resolving global power algebras by simplicial polynomial algebras, and applying the described functors. We observe that also this global version of André-Quillen cohomology is well-suited to detect properties like smoothness and étaleness of global power algebras, and can be used to study extensions. The low-dimensional groups can efficiently be calculated by a short naive cotangent complex. However, the categories of global functors and modules over a global power functor exhibit homological anomalies, such as the phenomenon that projective objects do not need to be flat. This leads to the failure of a long exact cohomology sequence beyond degree $n = 1$.

We emphasize which kind of modules we use for this thesis: Classically, the definitions of modules as objects with an action or as abelian group objects in a category of augmented algebras coincide. However, Strickland has shown that these two notions define different structures in an equivariant setting by defining an abelian group object that does not come from a module. We study an example of this phenomenon extended to the globally equivariant context, and show that it arises as a free abelian group object. Moreover, we discuss an approach followed by Hill to resolve this discrepancy by studying global functor objects instead of abelian group objects. For the main part of this thesis, we use the usual definition of modules as global functors with an action by the global power functor. This definition in fact does not depend on the power operations of the global power functor.

Finally, we also consider topological André-Quillen homology and cohomology for ultra-commutative ring spectra. This is an interesting topic, as we expect this to give rise to an obstruction theory for such ring spectra. As an ultra-commutative ring spectrum is a very rich type of structure, this obstruction theory would be desirable to provide more examples. We lay the foundation for such a theory by providing an André-Quillen homology and cohomology. This starts by a technical consideration of model categories of modules, algebras and non-unital algebras over such ultra-commutative algebras. We show that the commutative monoid axiom of White allows one to put a model category structure on non-unital

commutative monoids in a general model category, and then use this as an intermediate step to build topological André-Quillen objects in general model categories.

When we specialize to ultra-commutative ring spectra, we obtain a generalization of the topological André-Quillen cohomology for commutative ring spectra of Basterra. We show that this comes equipped with a transitivity long exact sequence, which distinguishes the topological from the algebraic situation, where such a sequence does not exist. Moreover, as a first step into a usage of this theory for obstruction theory, we build Postnikov towers of ultra-commutative ring spectra using André-Quillen cohomology.

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Contents

Introduction

Motivation In algebraic topology, we exploit algebraic invariants of topological objects in order to extract more information of the objects of study. Classical invariants are homology and cohomology theories as well as homotopy groups. An invariant is more useful the more structure it comes endowed with. One of the first instances of such an additional structure is the cup product on singular cohomology with ring coefficients, which allows to distinguish more spaces (like $\mathbb{C}P^2$ and $S^2 \vee S^4$, leading to the conclusion that the Hopf map $\eta\colon S^3 \to S^2$ is not null-homotopic). This also allows to formulate structural results like Poincaré duality. When we endow singular cohomology with even more structure, it becomes an even more useful invariant. For example, the Steenrod squares on singular cohomology with $\mathbb{F}_2$-coefficients can be used to show that even the suspension of the Hopf map is not null-homotopic, yielding the calculation of the first stable homotopy group of spheres as $\pi_1^{\text{stable}}(\mathbb{S}) \cong \mathbb{Z}/2$. Furthermore, calculations over the Steenrod algebra are a key part in analyzing the Adams spectral sequence.

Both the cup product and the Steenrod operations come from a multiplicative structure on the representing spectrum $H\mathbb{F}_2$ for singular cohomology. Such multiplicative structures come in different variants and strengths. The weakest multiplicative structure is that of a commutative ring spectrum in the homotopy category of spectra. Such a structure induces a multiplication on the represented cohomology, but no further structure. The richest structure is that of an E_∞-ring spectrum. Those spectra support power operations on the represented cohomology and also on their homotopy groups. In modern homotopy theory, E_∞-ring spectra have become an important object of study, and hence it is important to be able to detect such structures and endow spectra with E_∞-multiplications.

In some cases, putting a structured multiplication on spectra is easy, for example Eilenberg-MacLane spectra for rings carry E_∞-multiplications obtained by their functoriality. In other cases, the existence of multiplications can be a subtle question. For instance on Moore spectra, the observations that $\mathbb{S}/2$ does not support a unital multiplication, $\mathbb{S}/3$ does not support an associative multiplication and more generally, $\mathbb{S}/p$ supports an A_p, but not an A_{p+1}-multiplication, emphasize that structured multiplications can be hard to come by. The non-existence of a unital multiplication on $\mathbb{S}/2$ can be shown by analyzing Steenrod squares [4, Theorem 1.1], and for the multiplications on $\mathbb{S}/p$ for $p \geq 3$ one uses (generalized) Toda brackets [3, Example 3.3]. Further non-existence results can be obtained by studying the power operations induced by hypothetical E_∞-ring structures. This approach is utilized by Lawson and Senger to show that the Brown-Peterson spectra BP do not support E_∞-structures [40, 57].

In order to endow spectra with structured multiplication, a useful tool is obstruction theory. The idea of obstruction theory is to start with a multiplicative structure on an algebraic invariant associated to a spectrum, and then to lift this structure to a multiplication of the spectrum itself. This process aims to iteratively construct approximations to the spectrum in question, with these approximations carrying the desired structure. Whether the construction of the next approximation from the last is possible is governed by an obstruction, an element in an algebraically defined obstruction group, which has to vanish in order to perform the new approximation step. If all obstructions vanish, then the spectrum can be endowed with the desired structure. For commutative ring spectra, the first such obstruction theory was introduced in unpublished work by Kriz [39], who tried to use Postnikov towers of ring spectra in order to show that BP is a commutative ring spectrum. This approach was later carried out formally by Basterra and Mandell in [12], who show that BP can be given an E_4-multiplication. Another successful obstruction theory was introduced by Goerss and Hopkins [26], who utilized this theory to show that the Morava E-theory spectra carry a unique E_∞-ring structure. This obstruction theory was also successfully generalized to other contexts, most recently by Patchkoria and Pstrągowski [51, 50] in order to prove Franke's algebraicity conjecture, and by Burklund [20] to show that Moore spectra $\mathbb{S}/p^k$ indeed posses E_n-structures for sufficiently large k. Thus, we see that obstruction theory is an important tool for studying commutative ring spectra.

Another branch of homotopy theory that recently receives attention is equivariant homotopy theory. Equivariant homotopy theory aims at incorporating symmetries of topological objects into their study. These symmetries are encoded as group actions. Many homotopy theoretic objects come equipped with natural group actions, which makes equivariant methods an important topic of research. Also for non-equivariant problems, taking an equivariant perspective can prove fruitful. This has been illustrated most prominently in the solution of the Kervaire invariant one problem by Hill-Hopkins-Ravenel [29]. In that work, equivariant commutative ring spectra are used, which posses a very rich structure. In addition to the usual multiplication, they also come endowed with so-called norm maps, which can be thought of as products twisted by the group action.

Often, equivariant phenomena occur not only for one fixed group, but work uniformly for a whole family of groups. Such phenomena are studied by global homotopy theory. This approach has been formalized recently by Schwede [55], using orthogonal spectra to model global homotopy types. The approach to study different group actions simultaneously can lead to better structural properties of the involved objects and hence simplify the analysis considerably. For example, Hausmann [28] was able to use this global perspective to solve a long-standing conjecture about an equivariant generalization of Quillen's theorem, identifying equivariant complex bordism with the equivariant Lazard ring, which carries the universal formal group law.

One aim of this dissertation is to bring together these fields of homotopy theory and to study globally equivariant commutative ring spectra. These are a very rich type of structured ring spectra, coming endowed with norm maps for all groups simultaneously. To emphasize this rich structure, Schwede calls these "ultra-commutative ring spectra" in [55], and we follow that terminology here. In particular, we want to be able to put such commutative ring spectrum structures on global spectra. In [55], Schwede constructs global versions of K-theory and Thom spectra and studies ultra-commutative ring structures on these. This can be done by exhibiting explicit models of these as orthogonal spectra and specifying point-set multiplications. On the other hand, in previous work [61], I studied G_∞-multiplications on global Moore spectra. A G_∞-structure is an up-to-homotopy version of an ultra-commutative ring spectrum that also induces power operations on the homotopy groups of such a ring spectrum. It is a global version of the notion of H_∞-ring spectra as studied in [19]. I identified G_∞-ring structures on Moore spectra with the presence of power operations on the underlying commutative ring, and related this algebraic structure to that of a β-ring. In particular, this shows that classical examples like $\mathbb{S}/2$ or $\mathbb{S}[i]$, the Moore spectra for $\mathbb{Z}/2$ and the Gaussian integers $\mathbb{Z}[i]$ respectively, do not support G_∞-ring structures, whereas for example the Moore spectrum for $\mathbb{Z}[i, \frac{1}{2}]$ does posses a G_∞-structure.

To further understand the structure of ultra-commutative ring spectra, and to exhibit more examples, an obstruction theory for ultra-commutative ring spectra is desirable. A main step in building an obstruction theory for any topological structure is to choose an appropriate cohomology theory for the obstructions to take values in. This cohomology theory should reflect the type of structure we are trying to detect. Classically, in the case of commutative ring spectra, the cohomology theory of choice is topological André-Quillen cohomology. As mentioned before, this cohomology theory was used by Basterra-Mandell [12] and Goerss-Hopkins [26] to construct multiplications on BP and Morava E-theory, respectively.

Topological André-Quillen cohomology has its origins in (algebraic) André-Quillen cohomology, a cohomology theory for commutative rings. The latter was introduced and studied independently by André and Quillen [1, 2, 52, 53]. This cohomology theory can detect useful properties of commutative rings, like being smooth or étale, and classifies extensions of commutative algebras. This makes it an apt choice for a cohomology theory for commutative rings. For this reason, our first objective in this dissertation is to generalize André-Quillen cohomology to global power functors, which are the globally equivariant analogues of commutative rings. As mentioned, the homotopy groups and cohomology groups arising from an ultra-commutative ring spectrum come equipped with norms, equivalently encoded as power operations, and the algebraic structure encoding such power operations is called a global power functor. After studying this André-Quillen cohomology for global power functors, we introduce topological André-Quillen cohomology for ultra-commutative ring spectra, and use it to build Postnikov towers of ultra-commutative ring spectra.

Results In the first part of the dissertation, we study algebraic André-Quillen cohomology of global power functors. Global power functors are the analogue of commutative rings in the context of globally equivariant algebra, incorporating power operations. They are a generalization of Tambara functors introduced in [63] for a fixed group G, and were considered in different forms in [65, 18, 24, 55]. An interesting feature that distinguishes the theory of global power functors (and similarly the theory of Tambara functors) from the situation for commutative rings is that they are not monoid objects for a symmetric monoidal structure on the category of global functors, the global analogue of abelian groups. The difference is exactly described by the presence of power operations. Hence, a sensible generalization of André-Quillen cohomology specifically needs to take these power operations into account.

André-Quillen cohomology is a non-abelian derived functor of derivations, and the corresponding homology theory is a derived functor of Kähler differentials. Hence, the first step to generalize these theories to global power functors is to define derivations and Kähler differentials in this context. Both of these notions are moreover closely connected to square-zero extensions of a global power functor R by an R-module M. In defining these notions, we mimic the classical definitions from the case of commutative rings, and additionally have to handle the power operations. We generally work relatively to a base global power functor, introducing the notion of global power algebras and augmented global power algebras.

Definition (Construction 1.2.1 and Definitions 1.2.8 and 1.2.12)**.** Let R be a global power functor, S be a global R-algebra and M be an S-module. The square-zero extension $S \ltimes M$ is a global power functor with power operations defined by

$$P^m(s,m) = (P^m(s), \mathrm{tr}^{\Sigma_m \wr G}_{(\Sigma_{m-1} \wr G)\times G}(P^{m-1}(s) \times m)).$$

This is a global R-algebra with an augmentation map to S.
An R-derivation is a morphism $d\colon S \to M$ of R-modules satisfying the usual Leibniz rule as well as the twisted Leibniz rule

$$d(P^m(s)) = \mathrm{tr}^{\Sigma_m \wr G}_{(\Sigma_{m-1} \wr G)\times G}(P^{m-1}(s) \times d(s))$$

for power operations. This forms a functor $\mathrm{Der}\colon \mathrm{Mod}_S \to \mathrm{Ab}$.
The module of Kähler differentials Ω^1 is a representing module for the functor Der, ie R-linear derivations from S to M correspond to morphisms $\Omega^1_{S/R} \to M$ of S-modules.

These constructions are linked in the usual way.

Theorem (Theorems 1.2.7 and 1.2.11)**.** *Let R be a global power functor, S be an R-algebra and M be an S-module. There is a chain of natural isomorphisms*

$$\mathrm{Hom}_S(\Omega^1_{S/R}, M) \cong \mathrm{Der}_R(S, M) \cong \mathrm{Alg}_R /S(S, S \ltimes M)$$

In the theory of commutative rings, a key observation for the definition of André-Quillen homology is that R-modules can be identified as abelian group objects in the category of augmented R-algebras. In this way, André-Quillen homology is an instance of the slogan "Homology is a derived abelianization" that also applies to singular homology and can be seen in [52, Chapter II.5]. Strickland [62] gives an example that this is no longer the case for $\mathbb{Z}/2$-Tambara functors. In this work, we upgrade this to an example of an abelian group object in augmented global power algebras that does not arise from a square-zero extension of a module. This can be detected from the non-vanishing of the power operations on the kernel of the augmentation map. Our example improves upon Strickland's example in a number of ways: we show that it is freely generated by a single element as an abelian group object, our example is defined over the Burnside ring global power functor $\mathbb{A}$ instead of the constant global power functor $\underline{\mathbb{Z}}$, and this example persists after rationalization.

Example (Proposition 1.3.4)**.** A naive module of Kähler differentials for a polynomial algebra $R[x_G]$ defines the free abelian group object in augmented R-algebras. This abelian group object has non-trivial norms on the kernel of the augmentation ideal and hence does not arise as the square-zero extension of an R-module.

Based on work of Hill [30], we conjecture an alternative description of modules via global functor objects in augmented algebras. The definition of global functor objects is given in Definition 1.3.18.

Conjecture (Conjecture 1.3.21)**.** *Let S be a global power functor. Then the square-zero extension functor*

$$S \ltimes \_\colon \mathrm{Mod}_S \to \mathcal{GF}(\mathrm{Alg}_S /S)$$

is an equivalence between S-modules and global functor objects in augmented S-power algebras.

Having defined derivations and Kähler differentials, we derive these functors in order to define André-Quillen homology and cohomology groups. This can be done since the category of global power functors satisfies the properties needed in order to admit well-behaved non-abelian derived functors, as exhibited by Quillen in [52, 53]. This derivation procedure works by using polynomial resolutions of global power algebras.

Definition (Definition 1.5.6)**.** Let R be a global power functor and S be a global R-power algebra. A simplicial polynomial resolution of S as an R-power algebra is a simplicial R-power algebra $P_\bullet$ with an augmentation $\varepsilon\colon P_\bullet \to S$, such that ε is a weak equivalence of simplicial R-power algebras, each P_n is a polynomial R-algebra and the degeneracies send generators to generators.
Let now $P_\bullet \to S$ be a simplicial polynomial resolution of S over R and M be an S-module. The André-Quillen cohomology of S over R is defined as $D^q(S, R, M) = H^q(\mathrm{Der}_R(P_\bullet, M))$.
The André-Quillen homology of S over R is defined as $D_q(S, R, M) = H_q(S\Box_{P_\bullet}\Omega^1_{P_\bullet/R})$.

This cohomology in fact can detect properties of global power algebras.

Theorem (Proposition 1.5.14 and Definition 1.4.1)**.** *Let R be a global power functor and S be a global R-power algebra, and let M be an S-module. Then there is a natural isomorphism*

$$D^1(S, R, M) \cong \mathrm{Exalcomm}_R(S, M),$$

where $\mathrm{Exalcomm}_R(S, M)$ *classifies R-power algebra extensions of S by M, ie exact sequences $0 \to M \to E \to S \to 0$, where M is square-zero in E.*

The term Exalcomm was introduced by Grothendieck in [27, §18] and is an abbreviation for the french "Extensions des algèbres commutatives".

Theorem (Proposition 1.4.15)**.** *Let R be a global power functor and S be a global R-power algebra. Then S is formally smooth over R if and only if for all S-modules M, the first cohomology $D^1(S, R, M)$ vanishes. Moreover, S is formally unramified over R if and only if for all S-modules M, the zeroth cohomology $S^0(S, R, M)$ vanishes, and it is formally étale if and only if both the zeroth and the first cohomology with arbitrary coefficients vanish.*

In fact, the low-dimensional terms can be computed explicitly by means of a naive cotangent complex, constructed in Construction 1.5.17. For a global power functor R and an R-power algebra S, this takes the form

$$\mathbb{L}^{\mathrm{naive}}_{S/P/R} = I/I^{\geq 2} \to \Omega^1_{P/R} \Box_P S,$$

where $P \to S$ is a surjection of global R-power algebras with P a polynomial R-power algebra, and where $I = \ker(P \to S)$. The choice of P does not influence the (co-)homology of this complex up to isomorphism.

Theorem (Proposition 1.5.21)**.** *Let R be a global power functor and S be a global R-power algebra. Then the cohomology groups $D^0(S, R, M)$ and $D^1(S, R, M)$ can be calculated as*

$$D^0(S, R, M) \cong H^0(\mathbb{L}^{\mathrm{naive}}_{S/P/R}) \qquad \text{and} \qquad D^1(S, R, M) \cong H^1(\mathbb{L}^{\mathrm{naive}}_{S/P/R}).$$

These results suggest that this global André-Quillen cohomology is a useful theory for the study of global power functors. However, homological anomalies present in the category of global functors and inherited by R-modules, as explained in [44], imply that the transitivity and base change results for classical André-Quillen cohomology do not carry over to the global setting.

Proposition (Theorem 1.5.27)**.** *Global André-Quillen cohomology does not satisfy base change, and the transitivity sequence is not exact for general global power algebras.*

This result is obtained from explicit calculations using the naive cotangent complex, further highlighting the utility of an explicit calculational tool.

After setting up the algebraic André-Quillen theory, we pass to topological André-Quillen homology, a theory for ultra-commutative ring spectra. We follow the approach by Basterra in [10] and define this homology theory as the derived functor of Kähler differentials, mimicking the algebraic construction. In order to make the derived functors precise, we use model categorical techniques. For this, we use the (positive) global model categories on orthogonal spectra and orthogonal commutative ring spectra introduced in [55], and extend this by a model structure on non-unital ultra-commutative ring spectra. We denote the category of non-unital S-algebras by Alg^+_S. The definition proceeds in the following steps:

Proposition (Propositions 2.1.3 and 2.1.6)**.** *Let R be an ultra-commutative ring spectrum, and S be an R-algebra spectrum. The sequence*

$$\mathrm{Alg}_R/S \underset{\text{forget}}{\overset{S\wedge_R \_}{\rightleftarrows}} \mathrm{Alg}_S/S \underset{I}{\overset{K}{\leftrightarrows}} \mathrm{Alg}_S^+ \underset{Z}{\overset{Q}{\rightleftarrows}} \mathrm{Mod}_S$$

consists of Quillen adjunctions, with left adjoints on the top. Here, I assigns to an augmented S-algebra the fibre of the augmentation morphism, and K takes a non-unital S-algebra M to the extension $S \vee M$. The functor Q takes the module of indecomposables, by forming the cofiber of the multiplication map $N \wedge_S N \to N$ for a non-unital commutative S-algebra N, and Z endows a module with the zero-multiplication.

Definition (Definitions 2.1.7, 2.1.9 and 2.2.2)**.** Let R be an ultra-commutative ring spectrum and S be a commutative R-algebra. We define the cotangent complex of S over R to be

$$\Omega_{S/R} = (\mathbf{L}Q)(\mathbf{R}I)(S \wedge_R^{\mathbf{L}} S).$$

We call the homology and cohomology theories represented by this S-module spectrum topological André-Quillen (co-)homology.

This definition comes with the desirable properties of a cohomology theory, in that it has cofiber sequences and also satisfies base change. We highlight the transitivity cofiber sequence here.

Theorem (Theorem 2.1.20)**.** *Let $R \to S \to T$ be a sequence of ultra-commutative ring spectra. Then we have a cofibre sequence*

$$\Omega_{S/R} \wedge_S^{\mathbf{L}} T \to \Omega_{T/R} \to \Omega_{T/S}$$

of T-module spectra.

These properties are not satisfied by the algebraic theory, as highlighted before. This suggests that we might also study topological André-Quillen cohomology for global power functors by considering Eilenberg-MacLane ring spectra. This may give a better-behaved theory, in exchange for more complicated calculations.

We end this dissertation by giving a first application of our global topological André-Quillen cohomology and construct Postnikov towers for ultra-commutative ring spectra.

Theorem (Theorem 2.2.9)**.** *Let R be a connective ultra-commutative ring spectrum. Then there is a sequence $R_0, \ldots, R_n, \ldots$ of commutative R-algebras, equipped with maps $R_{n+1} \to R_n$ of commutative R-algebras, and elements $k_n \in \mathrm{TAQ}_e^{n+2}(R_n, R; H\underline{\pi}_{n+1}(R))$, such that the following properties are satisfied:*

i) *$R_0 \cong H\underline{\pi}_0(R)$, and R_n is extended from R_{n-1} using k_{n-1},*

ii) *$\underline{\pi}_k(R_n) = 0$ for $k > n$,*

iii) *the unit maps $\eta_n \colon R \to R_n$ are $(n+1)$-equivalences.*

The k-invariants k_n are lifts of the k-invariants for the Postnikov tower of R as a global spectrum.

Organization This dissertation is organized in two parts: in Chapter 1, we study the algebraic André-Quillen homology and cohomology of global power functors. We start in Section 1.1 with a recollection of the notions of global functors and global power functors, and discuss which notion of R-modules for a global power functor R we consider. We also introduce global power algebras here, and define polynomial algebras. After this, we introduce square-zero extensions, derivations and Kähler differentials for global power algebras in Section 1.2. In Section 1.3, we dive into more details regarding our notion of modules, and show that they cannot be described as abelian group objects in augmented global power algebras. Instead, we conjecture a description in terms of global functor objects. The next section, Section 1.4, studies extensions of global power algebras by modules, and uses these to extend the exact sequence of derivations. Moreover, in this section we show that formally smooth, unramified and étale global power algebras can be detected by considering derivations and extensions. Finally, in Section 1.5 we introduce André-Quillen homology and cohomology of global power algebras. We identify the low-degree terms with derivations, Kähler differentials and extensions, and exhibit the naive cotangent complex as a calculational tool for these low-degree terms. We finish this section with an explicit calculation of some first homology groups, which shows that the transitivity sequence fails to be exact in this global context.

The second part of this dissertation in Chapter 2 is concerned with topological André-Quillen homology and cohomology of ultra-commutative ring spectra. In Section 2.1, we study the definition of (topological) André-Quillen homology in a general model category, and show that this general construction comes equipped with a transitivity cofibre sequence and a base-change result. We use this theory in Section 2.2 for ultra-commutative ring spectra, modelled as commutative orthogonal ring spectra with the positive global model structure, and thus obtain a topological André-Quillen theory for these. We then exhibit a Hurewicz theorem for this homology theory, and use it to construct Postnikov towers of ultra-commutative ring spectra.

In the appendix, we include a discussion of two ingredients for the main body of this work. In Appendix A, we recall the definition of the wreath product $\Sigma_m \wr G$, needed in the formulation of the power operations of a global power functor. We moreover explicitly describe polynomial global power functors, using a classification of conjugacy classes in such wreath products. In Appendix B, we show that the commutative monoid axiom allows to lift a model structure on a symmetric monoidal model category to the category of non-unital commutative monoids. This is an ingredient in the construction of topological André-Quillen homology and cohomology in Section 2.1.

Chapter 1

André-Quillen (Co-)Homology of Global Power Functors

In this first part of the dissertation, we generalize algebraic André-Quillen cohomology to the context of globally equivariant stable homotopy theory. In this globally equivariant algebraic context, we are concerned with global functors and global power functors. These are generalizations of abelian groups and commutative rings. The connection to globally equivariant stable homotopy theory lies in the fact that these algebraic structures describe the structure present in the homotopy groups of globally equivariant spectra and ultra-commutative ring spectra, respectively. They are also generalizations of the notions of Mackey and Tambara functors for a fixed group G. The generalization consists of considering all (finite or compact Lie) groups simultaneously, instead of only subgroups of a fixed group G. The idea of considering global (power) functors originated in relation to the study of Mackey and Tambara functors, for example Webb defines a notion of globally defined Mackey functors in [65]. In his study of homological properties of Mackey functors in [44], Lewis also considers globally defined Mackey functors and shows that they exhibit some homological anomalies, for example that projectivity does not imply flatness in this context. This observation has implications on the cohomology theory we introduce in this work. Also global power functors have been considered in various places, for example by Ganter [24] in the study of λ-rings. On a related note, the author of this dissertation has shown that global power functors exhibit a close relation to β-rings in previous work [61].

In order to define André-Quillen cohomology for global power functors, we first define the notions of square-zero extensions, derivations and Kähler differentials for global power functors in Section 1.2. These notions are well-known for commutative rings. When adapting them to global power functors, we take additional care regarding the behaviour of power operations. We show that our definitions are compatible with the definitions given by Strickland [62] and Hill [30] in terms of norms. In my opinion, however, the formulation in terms of power operations is considerably clearer and easier to work with.

We observe that Kähler differentials define a left adjoint to the square-zero extension functor from modules to augmented algebras, see Theorem 1.2.13. This adjunction is classically used to interpret Kähler differentials as an abelianization functor, since R-modules for a commutative ring are equivalent to abelian group objects in augmented R-algebras. In the case of $\mathbb{Z}/2$-Tambara functors, Strickland [62] gives an example that the analogous statement is no longer true. We upgrade this example to an abelian group object in augmented global power algebras over the Burnside ring global power functor in Section 1.3.a. The presence of power operations on this example shows that also for global power functors, modules and abelian group objects in augmented algebras are different objects. We expect that modules can instead be described as global functor objects in augmented global power functors, transferring the result of Hill [30, Corollary 3.23] to the global context. We give the background to this conjecture in Section 1.3.b.

For a triple of global power functors $R \to S \to T$, and a T-module M, we obtain exact sequences

$$0 \to \mathrm{Der}_S(T, M) \to \mathrm{Der}_R(T, M) \to \mathrm{Der}_R(S, M)$$

and

$$T\Box_S\Omega^1_{S/R} \to \Omega^1_{T/R} \to \Omega^1_{T/S} \to 0$$

for derivations and Kähler differentials. One goal of defining André-Quillen (co-)homology is to extend these sequences indefinitely into long exact sequences. A first step into this direction can be made by using extensions of global power algebras, generalizing Grothendieck's functor Exalcomm [27] to global power functors. We give the details in Section 1.4.a. Moreover, this functor of global power algebra extensions can be used to classify formally smooth global power algebras, defined by a lifting property, as we show in Section 1.4.b. All in all, this shows that the theory of derivations and global power algebra extensions nicely generalizes from classical commutative algebra to the globally equivariant algebra of global power functors.

Finally, in Section 1.5, we define and study André-Quillen homology and cohomology for global power algebras. The homology is defined as a non-abelian derived functor of Kähler differentials, and the cohomology as a non-abelian derived functor of derivations. These derived functors are defined using simplicial resolutions of global power algebras by polynomial algebras. We are able to identify the low-dimensional terms of this cohomology theory in terms of derivations and global power algebra extensions, and exhibit the naive cotangent complex as a calculational tool for the low-degree terms. However, we also show by an explicit calculation of the first André-Quillen homology group of an example that this theory does not come with a long exact sequence extending the sequence of Kähler differentials. In fact, this example shows that it is not possible to extend the six-term exact sequence of derivations and global power algebra extensions to a long exact sequence. This defect is caused by the homological anomalies of the category of global functors exhibited by Lewis in [44], namely that projectivity does not imply flatness.

Nevertheless, the fact that the low-degree terms of this theory detect interesting properties of global power algebras makes this an interesting object of study. Furthermore, the topological André-Quillen theory for ultra-commutative ring spectra does not suffer from the same problem, but indeed comes equipped with a transitivity long exact sequence. Hence, studying the interplay between these theories could also provide valuable insights.

1.1 Global Functors, Global Power Functors, Modules and Algebras

This chapter introduces the basic notions we use throughout the algebraic part of this work. Concretely, we first recall the definitions of global functors and global power functors from [55]. Moreover, we introduce the notions of modules and (augmented) algebras over a global power functor R. These are necessary for the definition of André-Quillen (co-)homology, since this is best described as a relative theory for morphisms of global power functors $R \to S$, and it takes coefficients in S-modules M. Of particular interest is the definition of R-modules, since there are two possible definitions: either as modules over the underlying Green functor of R, or as Beck modules, ie abelian group objects in augmented R-algebras. These notions are indeed different, as first observed by Strickland for $\mathbb{Z}/2$-Tambara functors in [62]. We go into more detail on this difference in Section 1.3.a. For our work, we use the definition as modules over the underlying Green functor of R, and explain this choice in Remark 1.1.26.

1.1.a Recollections on Global Functors and Global Power Functors

In this section, we review the theory of global functors and global power functors needed throughout this dissertation. We follow the definitions given by Schwede in [55]. We assume the basics of global orthogonal spectra and equivariant homotopy groups as known, in particular for the description of the indexing category for global functors in Definition 1.1.1 and the justification of our definition of modules in Remark 1.1.26. The algebraically inclined reader can, however, work with the description of the Burnside category in Remark 1.1.2 and follow the remaining exposition without in-depth knowledge of global orthogonal spectra.

Global Functors We start by introducing the indexing category on which we define global functors.

Definition 1.1.1. The global Burnside category $\mathbb{A}$ has as objects all compact Lie groups, and as morphisms from G to K we set

$$\mathbb{A}(G, K) = \mathrm{Nat}(\pi_0^G, \pi_0^K).$$

This is the abelian group of natural transformations of equivariant homotopy group functors π_0^G to π_0^K from orthogonal spectra to sets. Composition in $\mathbb{A}$ is given as composition of natural transformations.

This definition is topological in nature. However, the relevant set of natural transformations between equivariant homotopy group functors can be described explicitly, as is demonstrated in [55, Theorem 4.2.6]. We summarize the results:

Remark 1.1.2. The morphism sets in the global Burnside category can be described as follows: For compact Lie groups G and K, the morphism group $\mathbb{A}(G, K)$ is free abelian on $(K \times G)$-conjugacy classes of pairs (L, α), where $L \leq K$ is a closed subgroup of K with finite Weyl group $W_K L$, and $\alpha\colon L \to G$ is a continuous homomorphism.

We write the transformation indexed by a pair (L, α) as $\mathrm{tr}_L^K \circ \alpha^*$ and call tr_L^K the transfer from L to K, and α^* the restriction along α. If α is surjective, we also call restriction along α an inflation. Hence, we see that all morphisms in the global Burnside category can be described by transfers and restrictions. The composition is governed by the following rules (as explained after [55, Theorem 4.2.6]):

i) restriction maps are contravariantly functorial,

ii) transfers are covariantly functorial,

iii) transfers along inclusions of subgroups with infinite Weyl group vanish,

iv) restrictions along inner automorphisms are the identity,

v) transfers commute with inflations, in the following sense: Let $H \leq G$ be a closed subgroup and $\alpha\colon K \to G$ be a surjective continuous homomorphism. Let $L = \alpha^{-1}(H) \leq K$. Then $\alpha^* \circ \mathrm{tr}_H^G = \mathrm{tr}_L^K \circ (\alpha_{|L})^*$.

vi) transfers and restrictions along subgroup inclusions compose according to the double coset formula. For general compact Lie groups, the double coset formula is formulated in [55, Theorem 3.4.9]. For transfers along codimension 0 inclusions, the double coset formula takes the usual form, which agrees with the double coset formula for Mackey functors:
Let $H, K \leq G$ be two closed subgroups of G such that H has codimension 0 in G (or

equivalently, finite index). Then the homogeneous space G/H is discrete and K acts on it by left translation. We consider the double coset space, ie the space of K-orbits in G/H, denoted by $K\backslash G/H$. Then

$$\mathrm{res}^G_K \circ \mathrm{tr}^G_H = \sum_{[g]\in K\backslash G/H} \mathrm{tr}^K_{K\cap gHg^{-1}} \circ g_* \,\mathrm{res}^H_{g^{-1}Kg\cap H} \,.$$

Here g_* denotes the restriction along the conjugation with g as a morphism $g^{-1}Kg\cap H \to K\cap gHg^{-1}$.

Definition 1.1.3. A global functor is an additive functor $\mathbb{A} \to \mathrm{Ab}$. The category of global functors is the additive functor category $\mathcal{GF} = \mathrm{Fun}^{\mathrm{add}}(\mathbb{A}, \mathrm{Ab})$. Explicitly, a morphism of global functors is a natural transformation.

Remark 1.1.4. By Remark 1.1.2, the structure of a global functor F is equivalent to the datum of an abelian group $F(G)$ for any compact Lie group G, restriction maps $\alpha^*\colon F(G) \to F(K)$ for any continuous homomorphism $\alpha\colon K \to G$ of compact Lie groups, and transfers $\mathrm{tr}^G_H\colon F(H) \to F(G)$ for any inclusion $H \leq G$ of a closed subgroup. These morphisms have to satisfy the relations explained in Remark 1.1.2.

Remark 1.1.5. In the case of a fixed finite group G, the category of G-Mackey functors can be described in multiple ways. The definition of global functors via the Burnside category used above is akin to the definition of Mackey functors using the G-equivariant stable homotopy category, where transformations between the equivariant homotopy groups may be used for an indexing category for Mackey functors. The description given in Remark 1.1.4 is equivalent to the definition of G-Mackey functors using explicit formulas for restriction and transfers.

There is another way to package the information in a G-Mackey functor, by using an indexing category built from spans of finite G-sets. A similar description can be given for global functors (for the family of finite groups). One way to generalize from a given finite group G to a global setting is by identifying a finite G-set with its translation groupoid. Using this point of view, one observes that an adequate way to formalize actions by arbitrary finite groups on finite sets is by considering all finite groupoids. This approach is for example carried out in [24]. Global functors for the family of finite groups can also be formalized as so-called biset functors, as explained in [18].

Example 1.1.6. The easiest example of a global functor is a represented global functor. In particular, we consider the Burnside ring global functor $\mathbb{A} := \mathbb{A}(e, \_)$, whose value $\mathbb{A}(G)$ at a finite group G can be identified with the Burnside ring of finite G-sets for any finite group G. A general represented global functor $\mathbb{A}(K, \_)$ is a free global functor, in the sense that evaluation at the identity is an isomorphism

$$\mathrm{Hom}_{\mathcal{GF}}(\mathbb{A}(K,\_), F) \xrightarrow{\simeq} F(K)$$

for any global functor F.

Monoidal Structure and Global Green Functors Since we are interested in multiplicative structures in equivariant homotopy theory, we now study analogues of commutative rings in the category of global functors. A possible generalization of commutative rings is obtained by introducing a symmetric monoidal structure on the category $\mathcal{GF}$, generalizing the tensor product of abelian groups, and considering commutative monoids for this monoidal structure. This leads to the notion of global Green functors, which we introduce in this section.

Global functors are defined as a functor category, and hence come equipped with a Day convolution product [21], which we call the box product.

Construction 1.1.7. Recall that the category of global functors is $\mathcal{GF} = \mathrm{Fun}^{\mathrm{add}}(\mathbb{A}, \mathrm{Ab})$. The category Ab of abelian groups carries a symmetric monoidal structure given by the tensor product. The category $\mathbb{A}$ carries a symmetric monoidal structure described in [55, Theorem 4.2.15]. It takes two compact Lie groups G and G' to the cartesian product $G \times G'$, and morphisms $\mathrm{tr}_L^K \circ \alpha^*$ and $\mathrm{tr}_{L'}^{K'} \circ (\alpha')^*$ to $\mathrm{tr}_{L \times L'}^{K \times K'} \circ (\alpha \times \alpha')^*$.

From these two symmetric monoidal structures, we obtain the box product of global functors as a Day convolution product [21]. Its value $F \Box F'$ on global functors F and F' can be described as an Ab-enriched left Kan extension

$$\begin{array}{ccccc} \mathbb{A} \times \mathbb{A} & \xrightarrow{F \times F'} & \mathrm{Ab} \times \mathrm{Ab} & \xrightarrow{\otimes} & \mathrm{Ab}\,. \\ \times \big\downarrow & & & \nearrow_{F \Box F'} & \\ \mathbb{A} & & & & \end{array}$$

Proposition 1.1.8 ([21, Theorem C.10, Remark C.12])**.** *The box product $\Box\colon \mathcal{GF} \times \mathcal{GF} \to \mathcal{GF}$ is part of a closed symmetric monoidal structure on the category of global functors.*

We denote the internal hom-global functor by $\underline{\mathrm{Hom}}$.

Remark 1.1.9. The box product comes with a bimorphism $(F, F') \to F \Box F'$, which at compact Lie groups G and G' is given as a morphism $\boxtimes\colon F(G) \otimes F'(G') \to (F \Box F')(G \times G')$. In particular, we obtain elements $x \boxtimes y \in (F \Box F')(G \times G')$ for elements $x \in F(G)$ and $y \in F'(G')$. We call the morphism $\boxtimes$ an external product.

The Day convolution product of F and F' can be described as a coequalizer of

$$\bigoplus_{K,K',G,G'} \mathbb{A}(K \times K', \_) \otimes \mathbb{A}(G, K) \otimes \mathbb{A}(G', K') \otimes F(G) \otimes F(G') \rightrightarrows \bigoplus_{G,G'} \mathbb{A}(G \times G', \_) \otimes F(G) \otimes F(G').$$

From this description, we observe that as a global functor, the box product is in fact generated by elements of the form $x \boxtimes y \in (F \Box F')(G \times G')$ as described above.

Definition 1.1.10. A global Green functor is a commutative monoid for the box product. The category of global Green functors is defined to be the category of commutative monoids in $\mathcal{GF}$.

Remark 1.1.11. The structure of a global Green functor can be made explicit in a number of equivalent ways. The Day convolution product is universal for bimorphisms, so the multiplication $R \Box R \to R$ required for a global Green functor is equivalent to multiplication morphisms $\times\colon R(G) \otimes R(K) \to R(G \times K)$ for all compact Lie groups G and K, satisfying unitality, associativity, commutativity, and compatibility with restrictions and transfers.

Another way to describe the multiplication of a global Green functor is by diagonal products. These are of the form $\_ \cdot \_\colon R(G) \otimes R(G) \to R(G)$ for all compact Lie groups G. They have to turn each $R(G)$ into a unital, associative and commutative ring such that the restrictions are ring homomorphisms and transfers satisfy the Frobenius reciprocity relation

$$\mathrm{tr}_H^G(x \cdot \mathrm{res}_H^G(y)) = \mathrm{tr}_H^G(x) \cdot y \tag{1.1.12}$$

for $x \in R(H)$ and $y \in R(G)$, where $H \leq G$ is a closed subgroup.

The relation between the two products is as follows. Given the product $\times$, we recover the diagonal product by restricting along the diagonal $\Delta\colon G \to G \times G$, as $x \cdot y = \Delta^*(x \times y)$ for $x, y \in R(G)$. Conversely, from the diagonal product we obtain the usual product by using the projections $G \xleftarrow{\mathrm{pr}_G} G \times K \xrightarrow{\mathrm{pr}_K} K$. Concretely, $x \times y = \mathrm{pr}_G^*(x) \cdot \mathrm{pr}_K^*(y)$ for $x \in R(G)$ and $y \in R(K)$. This correspondence is explained in [55, Remark 4.2.20]

Remark 1.1.13. Using this translation between diagonal and usual products, we also can give an alternative set of generators for the box product, similar to the one given in Remark 1.1.9. For this, we consider for all compact Lie groups G the morphism

$$\boxdot\colon F(G) \otimes F'(G) \xrightarrow{\boxtimes} (F\Box F')(G \times G) \xrightarrow{\Delta^*} (F\Box F')(G).$$

Then $F\Box F'$ is generated as a global functor by the elements $x \boxdot y \in (F\Box F')(G)$ for $x \in F(G)$ and $y \in F'(G)$. In fact, the previous set of generators can be recovered as $x \boxtimes y = \mathrm{pr}_G^*(x) \boxdot \mathrm{pr}_{G'}^*(y)$ for $x \in F(G)$ and $y \in F'(G')$.

We also describe the internal homomorphism global functor explicitly, by using shifts of global functors.

Definition 1.1.14. Let F be a global functor and G be a compact Lie group. We define $F[G]$ to be the global functor with $F[G](K) = F(K \times G)$, where the structure maps are defined via

$$\mathbb{A}(K_1, K_2) \otimes F(K_1 \times G) \xrightarrow{(\_\times G)\otimes F(K_1\times G)} \mathbb{A}(K_1 \times G, K_2 \times G) \otimes F(K_1 \times G) \to F(K_2 \times G).$$

Lemma 1.1.15. *Let F be a global functor and G be a compact Lie group. Then we have an isomorphism*

$$\underline{\mathrm{Hom}}(\mathbb{A}(G, \_), F) \cong F[G]$$

of global functors.

Proof. This is a consequence of a formal manipulation, using the Yoneda Lemma and the tensor-hom adjunction in a closed monoidal category, and that the convolution product of representables is representable. For any compact Lie group K, we have the chain

$$\begin{aligned}\underline{\mathrm{Hom}}(\mathbb{A}(G, \_), F)(K) &\cong \mathrm{Hom}(\mathbb{A}(K, \_), \underline{\mathrm{Hom}}(\mathbb{A}(G, \_), F)) \\ &\cong \mathrm{Hom}(\mathbb{A}(K, \_)\Box\mathbb{A}(G, \_), F) \\ &\cong \mathrm{Hom}(\mathbb{A}(K \times G, \_), F) \\ &\cong F(K \times G) \cong F[G](K)\end{aligned}$$

of isomorphisms. These are natural in K, hence these assemble into isomorphisms of global functors. □

As a result of this statement, we see that any $\theta \in \mathbb{A}(G, K)$ induces a morphism $\theta_*\colon F[G] \to F[K]$ of global functors in a functorial way by precomposition with $\theta^*\colon \mathbb{A}(K, \_) \to \mathbb{A}(G, \_)$. We can also define these operations directly, by defining for $\theta \in \mathbb{A}(G, K)$ and any compact Lie group H the map

$$R[\theta](H)\colon F[G](H) = F(H \times G) \xrightarrow{F(H\times\theta)} F(H \times K) = F[K](H).$$

That these definitions agree follows by the above chain of isomorphisms.

Proposition 1.1.16. *Let E and F be global functors and G be a compact Lie group. Then there is an isomorphism*

$$\underline{\mathrm{Hom}}(E, F)(G) \cong \mathrm{Hom}(E, F[G]).$$

Proof. This is a similar formal manipulation as in the previous lemma:

$$\begin{aligned}\underline{\mathrm{Hom}}(E,F)(G) &\cong \mathrm{Hom}(\mathbb{A}(G,\_),\underline{\mathrm{Hom}}(E,F))\\ &\cong \mathrm{Hom}(\mathbb{A}(G,\_)\Box E,F)\\ &\cong \mathrm{Hom}(E,\underline{\mathrm{Hom}}(\mathbb{A}(G,\_),F))\\ &\cong \mathrm{Hom}(E,F[G])\end{aligned}$$

□

Remark 1.1.17. The above description of the internal Hom object for global functors is analogous to the description of the Hom object for G-Mackey functors for a finite group G given in [43, Definitions 1.2 and 1.3]. Also the description of $F[G]$ as a Hom object from a representable global functor is analogous to the corresponding statement given in [43, Lemma 1.6]. The reader familiar with the theory of G-Mackey functors should be aware, however, that the other descriptions of the shift $F[G]$ via a box product with a representable functor given in the same lemma are not valid in the context of global functors. This comes from the fact that the indexing category $\mathbb{A}$ of global functors is not self-dual under exchanging transfers and restrictions, unlike the G-Burnside category. This comes from the additional presence of inflation morphisms.

Global Power Functors In Definition 1.1.10, we define commutative monoids for the box product of global functors. However, it is well-known that a completely equivariant analogue of commutative rings should come equipped with more structure, namely with power operations or norms. These can be thought of as twisted versions of products, just as transfers can be thought of as twisted versions of sums. This can nicely be observed in the fixed point Mackey functor $\underline{R}$ of a ring R with an action of a finite group G. The value of $\underline{R}$ at a subgroup H are the fixed points R^H. Here, the total transfer $\underline{R}(e) \to \underline{R}(G)$ takes the form

$$\mathrm{tr}_e^G(x) = \sum_{g\in G} gx.$$

We also can define a norm map

$$N_e^G(x) = \prod_{g\in G} gx.$$

A G-Green functor for a (finite) group G equipped with this additional structure is called a G-Tambara functor. For a thorough treatment of Tambara functors, see for example the original source [63] or the survey article [62]. We call the global analogue a global power functor, emphasizing that we choose to formalize this additional structure by power operations rather than by norms. We elaborate on the differences in Remark 1.1.20. This chapter closely follows the discussion of global power functors in [55, Chapter 5.1].

The following definition is given in [55, Definition 5.1.6]. The wreath product $\Sigma_m \wr G$ as well as the various comparison morphisms $\Phi_{i,j}$, $\Psi_{k,m}$ and Δ_m are defined in Appendix A.

Definition 1.1.18. A global power functor is a global Green functor R together with maps $P^m\colon R(G) \to R(\Sigma_m \wr G)$ for all $m \geq 1$ and all compact Lie groups G, satisfying the following properties:

i) $P^m(1) = 1$ for the unit $1 \in R(e)$.

ii) $P^1 = \mathrm{Id}$ as maps $R(G) \to R(\Sigma_1 \wr G) \cong R(G)$ under the identification $G \cong \Sigma_1 \wr G$, $g \mapsto (1; g)$.

iii) For every continuous homomorphism $\alpha\colon K \to G$ between compact Lie groups and all $m \geq 1$, we have

$$P^m \circ \alpha^* = (\Sigma_m \wr \alpha)^* \circ P^m$$

as maps $R(G) \to R(\Sigma_m \wr K)$.

iv) For all compact Lie groups G and all $m \geq 1$, and all $x, y \in R(G)$, we have

$$P^m(x \cdot y) = P^m(x) \cdot P^m(y)$$

in $R(\Sigma_m \wr G)$.

v) For all compact Lie groups G, all $i, j \geq 1$ and all $x \in R(G)$, we have

$$\Phi_{i,j}^*(P^{i+j}(x)) = P^i(x) \times P^j(x)$$

in $R((\Sigma_i \wr G) \times (\Sigma_j \wr G))$.

vi) For all compact Lie groups G, all $k, m \geq 1$ and all $x \in R(G)$, we have

$$\Psi_{k,m}^*(P^{km}(x)) = P^k(P^m(x))$$

in $R(\Sigma_k \wr (\Sigma_m \wr G))$.

vii) For all compact Lie groups G, all $m \geq 1$ and all $x, y \in R(G)$, we have

$$P^m(x + y) = \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}(P^k(x) \times P^{m-k}(y))$$

in $R(\Sigma_m \wr G)$, where $\mathrm{tr}_{k,m-k} = \mathrm{tr}_{(\Sigma_k \wr G)\times(\Sigma_{m-k}\wr G)}^{\Sigma_m \wr G}$ is the transfer associated to the inclusion $\Phi_{k,m-k} \colon (\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G) \to \Sigma_m \wr G$, and $P^0(x) = 1$ is the multiplicative unit.

viii) For every closed subgroup $H \subset G$ of a compact Lie group and for every $m \geq 1$, we have

$$P^m \circ \mathrm{tr}_H^G = \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G} \circ P^m$$

as maps $R(H) \to R(\Sigma_m \wr G)$.

A morphism of global power functors is a morphism of global Green functors that also commutes with the power operations.

Remark 1.1.19. In the above definition of power operations, the multiplicativity relation 1.1.18 *iv)* is expressed in terms of the diagonal product. We can reformulate it in terms of the product $\times \colon R(G) \otimes R(K) \to R(G \times K)$ as follows: For all compact Lie groups G and K, $m \geq 1$ and elements $x \in R(G)$ and $y \in R(K)$, the relation

$$P^m(x \times y) = \Delta_m^*(P^m(x) \times P^m(y))$$

holds in $R(\Sigma_m \wr (G \times K))$, where $\Delta_m \colon \Sigma_m \wr (G \times K) \to (\Sigma_m \wr G) \times (\Sigma_m \wr K)$ is defined via the diagonal on Σ_m. This can be concluded by a straight-forward calculation from the relation for the diagonal product, using the relations explained in Remark 1.1.11.
This is also explained in [60, Remark 2.31].

Remark 1.1.20. Classically, a G-Tambara functor is defined as a G-Green functor R together with norm maps $N_H^K \colon R(H) \to R(K)$ for subgroups $H \leq K \leq G$. These have to satisfy functoriality, unitality and multiplicativity, and the composition of norm and restriction is governed by a multiplicative double coset formula. Finally, the norm of a transfer is calculated by a distributivity law involving analysis of a so-called exponential diagram. This formalization was originally given in [63]. In my opinion, this distributivity law is quite complicated, making proofs in the realm of Tambara functors often rather involved. As an example, the additivity relation considered in [31, Theorem 2.4 – Corollary 2.9] is rather complicated and inexplicit. In comparison, the axioms for the power operations are easy to use, since the compatibility with transfers is straightforward and the additivity is a generalization of the binomial formula. Also in the context of André-Quillen homology and cohomology, a comparison of this dissertation with the work of Leeman [42, eg Lemma 3.3.6 and Proposition 3.3.8] shows that the possibility to work with power operations simplifies calculations considerably. However, since the G-equivariant framework is restricted to working with subgroups of G, it is not possible to formalize Tambara functors in terms of power operations. Thus the presence of power operations is a major advantage of the global setting

In fact, the formulation of a global power functor is equivalent to a formulation in terms of norms satisfying the above-mentioned axioms. The comparison is explained in [55, Remark 5.1.7] and works as follows:
Suppose we are given power operations $P^m\colon R(G) \to R(\Sigma_m \wr G)$ for all compact Lie groups G. We construct norm maps $N_H^G\colon R(H) \to R(G)$ for all subgroups $H \leq G$ of finite index. Suppose H has index m in G. We choose an ordered H-basis $\Gamma = (g_1, \dots, g_m)$ of G, ie an H-equivariant bijection $\coprod_{i=1}^m H \to G$ (for the right H-actions), and observe that $\Sigma_m \wr H$ acts freely and transitively on the set of such bases from the right by multiplication and permutation. Moreover, G acts freely on the left on Γ by multiplication, and this defines an injective homomorphism $\Psi_\Gamma\colon G \to \Sigma_m \wr H$ by sending $g \in G$ to the element $(\sigma; h_1, \dots h_m)$ such that $g\Gamma = \Gamma \cdot (\sigma; h_1, \dots, h_m)$. We then define the norm N_H^G as the composite

$$R(H) \xrightarrow{P^m} R(\Sigma_m \wr H) \xrightarrow{\Psi_\Gamma^*} R(G).$$

We observe that another choice of basis Γ' changes Ψ only by a conjugation, and since conjugated homomorphisms induce the same restrictions by Remark 1.1.2 *iv)*, the definition of the norm does not depend on this choice of basis.

In the other direction, we can recover the power operations from the norms as follows: For a compact Lie group G, we observe that the power operation $P^m\colon R(G) \to R(\Sigma_m \wr G)$ is given as the composite

$$R(G) \xrightarrow{q^*} R((\Sigma_{m-1} \wr G) \times G) \xrightarrow{N^{\Sigma_m \wr G}_{(\Sigma_{m-1} \wr G) \times G}} R(\Sigma_m \wr G).$$

Here, $q\colon (\Sigma_{m-1} \wr G) \times G \to G$ is the projection to the second factor. It is a straight-forward calculation to see that these are inverse constructions.

We use this translation between norms and power operations in Remark 1.2.6 to relate our constructions of square-zero extensions and derivations for global power functors to the constructions proposed by Strickland [62] and Hill [30] for G-Tambara functors for fixed G.

Example 1.1.21. The Burnside ring global functor $\mathbb{A}$ has the structure of a global power functor, and this structure can be made very explicit in the case of finite groups. We already noticed that the values $\mathbb{A}(G)$ are rings, which are the Burnside rings of finite G-sets for finite groups G. These ring structures make $\mathbb{A}$ into a global Green functor. We also describe the power operations in terms of finite G-sets for finite groups G: If X is a finite G-set, then the m-th power operation on X is $P^m(X) = X^m$, with the natural $\Sigma_m \wr G$-action. This definition can be extended by additivity to the entirety of $\mathbb{A}(G)$.

The Burnside ring global power functor is the initial global power functor. This can for example be seen by using that global power functors are comonadic over global Green functors by [55, Section 5.2], and hence the initial global Green functor is also initial as a global power functor. The Burnside ring global functor is initial as a global Green functor, since it is the unit for the box product.

Polynomial Power Functors An important class of global power functors are polynomial global power functor, also called free global power functor in [55, Example 5.1.19]. We recall their definition here.

Definition 1.1.22. Let R be a global power functor and K be a compact Lie group. The polynomial algebra over R generated by an indeterminate x in degree K is

$$R[x_K] = \bigoplus_{n\geq 0} R\Box\mathbb{A}(\Sigma_n \wr K, \_).$$

The multiplication and power operations are described in [55, Example 5.1.19]. The map $R \to R[x_K]$ is the inclusion as the summand indexed by $n = 0$, and the n-th summand is a free R-module on $P^n(x_K)$ in degree $\Sigma_n \wr K$, where by x_K we denote the identity in $\mathbb{A}(K, K)$. The power operation $P^n(x_K)$ is then represented by the identity in $\mathbb{A}(\Sigma_n \wr K, \Sigma_n \wr K)$. This polynomial algebra has the usual universal property, in that

$$\mathrm{Alg}_R(R[x_K], S) \xrightarrow{\cong} S(K),\ f \mapsto f(x_K) \tag{1.1.23}$$

is a bijection for any R-algebra S. This follows from [55, Proposition 5.2.6 (ii)] and the extension of scalars adjunction.

Remark 1.1.24. The category of global power functors can also be described as the category of algebras for a multi-sorted algebraic theory. The basic notion of an algebraic theory was introduced by Lawvere [41] as a means to formalize varying algebraic structures. Such an algebraic theory consists of a category $\mathbb{T}$ with finite products together with a distinguished object t such that any other object s of T is a finite power of t, ie $s = t^{\times n}$ for some $n \in \mathbb{N}$. A model, or algebra, for such a theory in a category $\mathcal{C}$ with finite products is then a product-preserving functor $T \colon \mathbb{T} \to \mathcal{C}$. Examples for such algebraic theories include the theories of groups, abelian groups, (commutative) rings and modules over a fixed ring, and for any of these theories, the models in sets are exactly the given objects. For such theories, the category $\mathbb{T}$ can be interpreted as the full subcategory of free objects on finitely many generators.

A multisorted theory generalizes the above structure to incorporate distinguishable types of structure. A multisorted theory over a set S of sorts is a category $\mathbb{T}$ with finite products containing a set $\{x_s \mid s \in S\}$ of distinguished objects, such that any object of $\mathbb{T}$ is isomorphic to a finite product of such distinguished elements. Again, a model, or algebra, of $\mathbb{T}$ in a category $\mathcal{C}$ with finite products is a product-preserving functor $\mathbb{T} \to \mathcal{C}$. A classical example is the category of (commutative) rings and modules over these, which can be represented as a category of algebras for an algebraic theory with two sorts. Multisorted algebraic theories are introduced in [16, Definition 2.3] and studied for example in [14].

Global functors, global Green functors and global power functors can all be formalized by multisorted algebraic theories. The sorts in this case are (isomorphism classes of) compact Lie groups, and we take algebras in the category of global sets, ie collections of sets indexed by (isomorphism classes of) compact Lie groups. The morphisms between the free objects encode the group and ring structures of $F(G)$, the restrictions, transfers and power operations, as well as the various relations between compositions of these. This formalization allows one to use general results for algebras over algebraic theories, such as completeness and cocompleteness of these categories, that limits and certain colimits are calculated underlying, and that effective epimorphisms are detected on underlying sets. For such results, we refer to [17, Chapter 3].

1.1.b Modules and Algebras over Global Power Functors

In Section 1.1.a, we recall the basic algebraic notions we use in this dissertation, as formulated by Schwede in [55]. This includes global functors and global power functors, which act as our generalizations of abelian groups and commutative rings. These notions are sufficient to perform the equivariant algebra relative to the Burnside ring global power functor, which is initial as a global power functor. However, we mainly work relative to an arbitrary base global power functor R in this thesis, and thus we introduce the notions of modules and algebras over a given global power functor. These notions have already been considered in the context of G-Mackey and Tambara functors, for example in [62, 30]. We follow these definitions and transfer them to the context of global equivariant algebra. One point that deserves special attention is the definition of an R-module for a global power functor R. Since a global power functor comes equipped with power operations, the question arises whether an R-module M should itself have some flavour of power operations, and how these should interact with those of R if they are defined. After being introduced by Strickland [62, Chapter 14], the names *genuine* and *naive* modules have persisted in the literature on G-Tambara functors for modules with and without power operations, respectively. As we show in Section 1.3.a, this distinction is also necessary for modules over a global power functor. We are mostly concerned with the so-called "naive" modules in this dissertation, ie modules that do not come equipped with power operations. This is compatible with the work of Hill in [30], and we elaborate on the reasons for this choice in Remark 1.1.26.

In addition to the definitions of modules and algebras, we also introduce ideals of global power algebras in Definition 1.1.30. This definition as a non-unital sub-global power functor is analogous to the definition of an ideal in a G-Tambara functor given by Nakaoka in [48]. It is used to describe the module of Kähler differentials in Section 1.2.c and the naive cotangent complex in Section 1.5.c.

Definition 1.1.25. Let R be a global power functor. Then an R-module is a module over the underlying global Green functor of R in the category of global functors. Morphisms of R-modules are morphisms of modules for the global Green functor R.

Explicitly, this definition says that an R-module M is a global functor M together with an action map $\alpha\colon R\Box M \to M$, satisfying the usual unitality and associativity relations. The category of R-modules is the categories of modules over a commutative monoid in the symmetric monoidal category $\mathcal{GF}$ with the box product.

Remark 1.1.26. As mentioned in the introduction to this section, there is another feasible definition of an R-module: The definition given above makes no mention of the power operations on R, and in particular, M is not required to have any additional structure of power operations. It would be possible to give a definition of an R-module that encompasses power operations on M and a compatibility with the power operations of R. A way to formalize this is by considering what is classically called a Beck module after its usage in [13, Definition 5], and is called a genuine module by Strickland [62, Chapter 14] in the context of G-Tambara functors. A Beck module is an abelian group object in the category of augmented R-algebras, as defined in our context in Definition 1.1.29. Classically, such a Beck module for a commutative ring R is equivalent to a usual R-module, as is shown in [13, Example 8]. In the context of G-Tambara functors, Strickland proves that this equivalence no longer holds, and that a Beck module is more data than a (Green) module [62, Proposition 14.7]. The same is true for modules over global power functors, as we show in Section 1.3.a.

Since we have two possible choices of a notion of R-modules, we are faced with choosing one of these notions. As described in Definition 1.1.25, we choose to work with modules over the underlying Green functor, ie without power operations on the module. In my opinion, this definition has several advantages over the definition as Beck modules:

i) It recovers global functors as $\mathbb{A}$-modules. This is due to the fact that $\mathbb{A}$ is the initial global Green (and hence also global power) functor. It is desirable to have this property, since this means we work with the classical notions when we take $\mathbb{A}$ as the base global power functor.

ii) It mirrors the topological notion. When we consider an ultra-commutative (orthogonal) ring spectrum R, the category of R-modules is defined as the category of modules over R for the wedge product. The homotopy groups $\underline{\pi}_0(R)$ form a global power functor, and for an R-module M, the global functor $\underline{\pi}_0(M)$ comes equipped with the structure of an R-Green module. However, there are no power operations present on this module. This hold, since the power operations are induced from the multiplications $R^{\wedge m} \to R$. Such products do not exist for M, and hence no power operations are induced.
This point also links back to the first point, since in the context of global spectra, $\mathbb{S}$-modules are equivalent to global spectra.

iii) Since power operations are twisted versions of classical powers, we should not expect an R-module M to have power operations, since on M, there are no products or powers.

iv) Even though the notions of derivations and Kähler differentials need to be adjusted to account for power operations, the definitions for our chosen notion of modules are clean generalizations of the classical definitions, and much of the theory can be developed in parallel to the classical theory. In other words, it is convenient to work with Green modules.

Also in the work by Hill [30], where derivations and Kähler differentials for G-Tambara functors are defined, the notion of Green modules is used. In that work, it is also shown that these "naive" modules can still be interpreted appropriately as *equivariant* Beck modules: Hill shows that R-Green modules are equivalent to Mackey functor objects in the category of augmented algebras over R. This gives further evidence that the notion of Green modules is the correct one to work with, since it also properly generalizes the notion of a Beck module to an equivariant context. We expect that a similar interpretation of R-modules for R a global power functor as *global* Beck modules is possible, and explain the necessary constructions in Section 1.3.b.

In the above discussion, we focussed on the question whether a module M should be equipped with power operations. This is the main question when we consider working with Beck modules. Since power operations are unary operations, power operations on an augmented R-algebra are determined by power operations on R and on the augmentation ideal, which corresponds to the module. However, when posing the question of how to define a module over a global power functor R, another conceivable approach would be to symmetrize expressions of the type $r \times \ldots \times r \times m$ for $r \in R$ and $m \in M$. However, since the M-factor is distinguishable from the R-factors, the maximal symmetry group of such an expression is $\Sigma_{k-1} \times \Sigma_1$, and symmetrization with respect to this group should yield $P^{k-1}(r) \times m$. This is already determined by the power operations on R and the Green-module structure of M. This shows that our definition above is reasonable also under this perspective.

We note that as a category of modules over a commutative monoid R in the symmetric monoidal category $\mathcal{GF}$, the category of R-modules inherits a closed symmetric monoidal structure via the relative box product $\Box_R$. This is defined for R-modules M and N as a coequalizer

$$M \Box R \Box N \rightrightarrows M \Box N \dashrightarrow M \Box_R N. \tag{1.1.27}$$

A proof that this relative tensor product (together with the appropriately defined internal Hom object of R-linear morphisms) defines a closed symmetric monoidal structure can be found in [37, 38]. These references study the more general case of algebras over a commutative monad in a closed symmetric monoidal category, and we use the monad $R\Box\_$ on $\mathcal{GF}$ here.

We now define the notion of global power algebras.

Definition 1.1.28. Let R be a global power functor. Then a global R-power algebra is a global power functor S together with a morphism $\eta\colon R \to S$ of global power functors. The morphism η is called the unit map of the R-power algebra S. We denote the category of R-power algebras by Alg_R.

As usual, the notion of global power algebras can equivalently be described as a global power functor based in the category of R-modules. The category of R-modules is symmetric monoidal for the relative box product $\Box_R$, and commutative monoids for this structure are global Green algebras over R, without power operations. A global power algebra S then needs additional power operations, satisfying the axioms specified in Definition 1.1.18. Additionally, the compatibility axiom $P^m(r\cdot s) = P^m(r)\cdot P^m(s)$ for $r \in R(G)$, $s \in S(G)$ needs to be satisfied.

At various points, we need to consider *augmented* algebras.

Definition 1.1.29. Let R be a global power functor and let T be a global R-power algebra. Then a global R-power algebra S augmented to T is a global R-power algebra S together with a morphism $\varepsilon\colon S \to T$ of global R-power algebras, called the augmentation. The category of R-power algebras augmented to T is denoted Alg_R/T, since it is an over-category.
In the case $R = T$, with unit map given by the identity, we call S an augmented R-power algebra. The category of augmented R-algebras is denoted Alg_R/R.

Finally, we also introduce the notion of a global power ideal, analogous to the definition of an ideal in a G-Tambara functor in [48]. For completeness, we also mention the corresponding notion of a Green ideal.

Definition 1.1.30. Let R be a global power functor. Then a global power ideal in R is a non-unital global power subfunctor. Explicitly, it is a sub-global functor $I \subset R$ that is closed under the product with elements of R and power operations of R.
A global Green ideal is a non-unital global Green subfunctor.

Hence, a global Green ideal does not need to be closed under power operations.

Example 1.1.31. Let R be a global power functor, G be a compact Lie group, and $f \in R(G)$ be an element of R. Then we denote the smallest global power ideal of R containing f by $\langle f\rangle$. We observe that the value of this ideal at a compact Lie group K takes the form

$$\langle f\rangle(K) = \left\{ \sum_{i=1}^{n} \mathrm{tr}_{L_i}^{K}\, \alpha_i^*(r_i \times P^{m_i}(f)) \right\}.$$

Here, n is any natural number, which is allowed to vary for different elements of this set, $L_i \le K$ are closed subgroups, $\alpha_i\colon G_i \times (\Sigma_{m_i} \wr G) \to L_i$ are continuous homomorphisms, and $r_i \in R(G_i)$ for compact Lie groups G_i. The fact that this set is closed under addition, transfer and restriction, multiplication with elements in R and power operations is a straight-forward application of the relations present in a global power functor.

As for ideals in commutative rings, it is possible to take the quotient of a global power functor by a global power ideal.

Proposition 1.1.32. *Let R be a global power functor and $I \subset R$ be a global power ideal. Then the quotient R/I uniquely inherits the structure of a global power functor from R such that the quotient map $R \to R/I$ is a morphism of global power functors.*

Proof. We focus on the power operations here, since the multiplication works completely analogous to the case of commutative rings.
Since $R \to R/I$ is surjective, we have to define the power operations on R/I via $P^m([r]) = [P^m(r)]$. We check that this is well-defined. For this, let $r \in R(G)$ and $i \in I(G)$ for a compact Lie group G. Then we calculate

$$P^m(r+i) = \sum_{k=0}^{m} \operatorname{tr}_{m-k,k}(P^{m-k}(r) \times P^k(i)) = P^m(r) + \sum_{k=1}^{m} \operatorname{tr}_{m-k,k}(P^{m-k}(r) \times P^k(i)).$$

In the last expression, the entire second sum is contained in $I(\Sigma_m \wr G)$, since I is a global power ideal. Hence, the power operations are well-defined. The properties of these power operations are easily deduced from those for R. □

Conversely, any kernel of a morphism of global power functors is a global power ideal.

Lemma 1.1.33. *Let R and S be global power functors and $f\colon R \to S$ be a map of global power functors. Then the kernel* $\ker(f)$ *is a power ideal of R.*

Proof. The kernel of any map of global functors is again a global functor. As f is a map of global Green functors, for any compact Lie group G the map $f\colon R(G) \to S(G)$ is a map of rings, and hence the kernel is an ideal in $R(G)$. Thus $\ker(f)$ is a Green ideal in R. Moreover, for $x \in \ker(f)$, $f(P^k(x)) = P^k(f(x)) = 0$, thus $\ker(f)$ is closed under power operations. Thus, it is a power ideal. □

1.2 Square-Zero Extensions, Derivations and Kähler Differentials

In algebraic geometry, a main objective is to use algebraic techniques to study geometric phenomena of varieties. One important geometric property is smoothness of an algebraic variety. Translating this to the algebraic side, an important property of a commutative algebra is being (formally) smooth. In order to formulate this property, one first needs to consider square-zero extensions, derivations and Kähler differentials. All these notions are intimately linked by various adjunctions and representability results, and they constitute a theory of infinitesimal deformations of commutative rings. These constructions also are the first steps towards constructing André-Quillen (co-)homology, which is defined as a derived functor of Kähler differentials or derivations, respectively.

In this section, we introduce the notions of square-zero extensions, Kähler differentials and derivations for global power functors. In these definitions, we are required to specify how to handle the additional power operations present in a global power functor. The underlying idea is to consider the power operations as a twisted version of the multiplication. The newly defined concepts of square-zero extension, derivation and Kähler differentials satisfy the same relations as the classical notions, and this facilitates calculations on some easy global power functors such as polynomial global power functors. Later, when studying exact sequences related to derivations and Kähler differentials, we also are able to calculate Kähler differentials for quotients of polynomial global power functors.

The definitions in this section are related to the definitions for G-Tambara functors for fixed finite groups G, defined by Strickland [62, Definition 14.8] in the case of square-zero extensions and by Hill [30, Definitions 4.1 and 5.4] in the case of derivations and Kähler differentials. We show that our definitions in terms of power operations recover the definitions by Strickland and Hill in terms of norms, but emphasize that the definitions for the power operations are formulated more concisely and hence are easier to work with. Moreover, our definition is compatible with the additional functoriality present in a global power functor in the form of inflations, and also works for compact Lie groups.

1.2.a Square-Zero Extensions of Global Power Functors

We define a square-zero extension of a global power functor R by an R-module M.

Construction 1.2.1. Let R be a global power functor and M be a module over R. Then we define the global power functor $R \ltimes M$ as follows: As a global functor, we set $R \ltimes M = R \oplus M$. Explicitly, for every compact Lie group G the value of $R \ltimes M$ at G is given as $(R \ltimes M)(G) = R(G) \oplus M(G)$, for any homomorphism $\alpha\colon K \to G$ of compact Lie groups, the restriction map takes the form

$$\alpha^*\colon (R \ltimes M)(G) \to (R \ltimes M)(K),\ \alpha^*(r, m) = (\alpha^* r, \alpha^* m),$$

and for any closed subgroup $H \subset G$, we have the transfer

$$\operatorname{tr}_H^G\colon (R \ltimes M)(H) \to (R \ltimes M)(G),\ \operatorname{tr}_H^G(r, m) = (\operatorname{tr}_H^G r, \operatorname{tr}_H^G m).$$

Then, we endow this global functor with a unit map

$$\mathbb{A} \to R \ltimes M$$

using the unit map of R and the zero-map on M. This is equivalent to defining the unit element of $R \ltimes M$ to be $(1, 0)$, where 1 is the unit in R. Moreover, we define a product by

$$(R \ltimes M)\Box(R \ltimes M) \cong (R\Box R) \oplus (R\Box M) \oplus (M\Box R) \oplus (M\Box M) \to R \oplus M = R \ltimes M,$$

where the middle map is the multiplication of R on the first summand, the module structure of M on the second and third summand and the zero-map on the fourth summand. Explicitly, for every compact Lie group G, the multiplication is given via

$$(r, m) \cdot (r', m') = (rr', rm' + r'm) \tag{1.2.2}$$

for $(r, m), (r', m') \in (R \ltimes M)(G)$. This is equivalent to the external multiplication being given for elements $(r, m) \in (R \ltimes M)(G)$, $(r', m') \in (R \ltimes M)(K)$ by

$$(r, m) \times (r', m') = (r \times r', r \times m' + r' \times m). \tag{1.2.3}$$

Finally, we define power operations on $R \ltimes M$ by the formula

$$P^k(r, m) = (P^k(r), \operatorname{tr}_{(\Sigma_{k-1}\wr G)\times G}^{\Sigma_k \wr G}(P^{k-1}(r) \times m)) \tag{1.2.4}$$

for $(r, m) \in (R \ltimes M)(G)$. For notational ease, we write $\operatorname{tr}_{i,j}^G = \operatorname{tr}_{(\Sigma_i \wr G)\times(\Sigma_j \wr G)}^{\Sigma_{i+j}\wr G}$, which is the transfer along the inclusion $\Phi_{i,j}$ from Remark A.1.3 *i)*. Then, the transfer in the above formula is $\operatorname{tr}_{k-1,1}^G$.

Remark 1.2.5. We offer some context for the above formula: We think about the pair (r, m) as a sum $r + m$ of elements $r \in R(G)$ and $m \in M(G)$, and assume that the multiplication of any two elements in M vanishes. Now, the power operations are a Σ_k-equivariant version of the map $x \mapsto x^k$, so we compare the above formula to the calculation

$$(r+m)^k = r^k + kr^{k-1}m$$

obtained in a non-equivariant square-zero extension by the binomial formula. The equivariant version of this binomial formula is the additivity relation for a global power functor given in Definition 1.1.18 *vii)*. If we set $P^k(m) = 0$ for $m \in M(G)$, $k \geq 2$, motivated by the definition that modules are not equipped with power operations, then this additivity relation recovers the above definition, as

$$P^k(r+m) = P^k(r) + \operatorname{tr}^G_{k-1,1}(P^{k-1}(r) \times m).$$

Remark 1.2.6. The definition of the power operations on the square-zero extension is analogous to the definition used by Strickland for a square-zero extension of a Tambara functor in [62, Definition 14.8]. The relation to our definition can be made precise by the translation between power functors and Tambara functors explained in Remark 1.1.20. We restrict to finite groups G here, since this is the context in which Strickland works. We first show how our formula for the power operations is obtained from the formula for the norm:

Let G be a finite group, and $(r,m) \in (R \ltimes M)(G)$ be an element of the square-zero extension. Then by the translation between norms and power operations, we have $P^k(r,m) = N^{\Sigma_k \wr G}_{(\Sigma_{k-1}\wr G)\times G} q^*(r,m)$, where $q\colon (\Sigma_{k-1} \wr G) \times G \to G$ is the projection to the second factor. By the formula for the norm on a square-zero extension given by Strickland in [62, Definition 14.8], we thus have

$$P^k(r,m) = (N^{\Sigma_k\wr G}_{(\Sigma_{k-1}\wr G)\times G}(q^*r), \operatorname{tr}^{\Sigma_k\wr G}_{(\Sigma_{k-1}\wr G)\times G}((N_{\mathrm{pr}_1}\operatorname{res}_{\mathrm{pr}_2}(q^*r)) \cdot q^*m)).$$

In this formula, the two projections pr_1 and pr_2 are those from the $\Sigma_k \wr G$-set $(\Sigma_k \wr G/((\Sigma_{k-1} \wr G)\times G)) \times (\Sigma_k \wr G/((\Sigma_{k-1}\wr G)\times G)) \setminus \Delta$, where Δ denotes the diagonal. We now observe that this $\Sigma_k \wr G$-set is isomorphic to $\Sigma_k \wr G/(G \times (\Sigma_{k-2}\wr G)\times G)$, and the two projections are associated to the two inclusions $G \times (\Sigma_{k-2}\wr G) \times G \subset (\Sigma_{k-1}\wr G)\times G$ using either the left two or the right two factors. Moreover, the first entry in the description of $P^k(r,m)$ is equal to $P^k(r)$, thus we only need to consider the second component. In the following calculation, we denote with $q_i\colon (\Sigma_{i-1}\wr G)\times G \to G$ the projection to the second factor, and by $p_i\colon (\Sigma_{i-1}\wr G)\times G \to \Sigma_{i-1}\wr G$ the projection to the first factor. We have

$$\begin{aligned}
&\operatorname{tr}^{\Sigma_k\wr G}_{(\Sigma_{k-1}\wr G)\times G}((N_{\mathrm{pr}_1}\operatorname{res}_{\mathrm{pr}_2}(q_k^*r)) \cdot q_k^*m) \\
&\quad= \operatorname{tr}^{\Sigma_k\wr G}_{(\Sigma_{k-1}\wr G)\times G}((N^{(\Sigma_{k-1}\wr G)\times G}_{G\times(\Sigma_{k-2}\wr G)\times G}\operatorname{res}^{G\times(\Sigma_{k-1}\wr G)}_{G\times(\Sigma_{k-2}\wr G)\times G} q_k^*r)\cdot q_k^*m) \\
&\quad= \operatorname{tr}^{\Sigma_k\wr G}_{(\Sigma_{k-1}\wr G)\times G}((N^{(\Sigma_{k-1}\wr G)\times G}_{G\times(\Sigma_{k-2}\wr G)\times G}(G\times p_{k-1})^* q_{k-1}^*r)\cdot q_k^*m) \\
&\quad= \operatorname{tr}^{\Sigma_k\wr G}_{(\Sigma_{k-1}\wr G)\times G}((p_k^* N^{\Sigma_{k-1}\wr G}_{G\times(\Sigma_{k-2}\wr G)} q_{k-1}^*r)\cdot q_k^*m) \\
&\quad= \operatorname{tr}^{\Sigma_k\wr G}_{(\Sigma_{k-1}\wr G)\times G}((N^{\Sigma_{k-1}\wr G}_{G\times(\Sigma_{k-2}\wr G)} q_{k-1}^*r)\times m) \\
&\quad= \operatorname{tr}^{\Sigma_k\wr G}_{(\Sigma_{k-1}\wr G)\times G}(P^{k-1}(r)\times m).
\end{aligned}$$

Hence, we exactly recover our definition of the power operations.

Conversely, we show how to obtain the definition of the norm on the square-zero extension from our formulation of the power operations. For a compact Lie group G and a closed subgroup H of index k, we choose an ordered H-basis $\Gamma = (\gamma_1, \ldots, \gamma_k)$ of G. We assume that $\gamma_k = e$ is the trivial element of G. As explained in Remark 1.1.20, this defines a monomorphism $\Psi_\Gamma\colon G \to \Sigma_k \wr H$. We have for any element $(r, m) \in (R \ltimes M)(H)$ the formula

$$N_H^G(r,m) = \Psi_\Gamma^*(P^k(r,m)) = \Psi_\Gamma^*(P^k(r), \mathrm{tr}_{k-1,1}^H(P^{k-1}(r) \times m)).$$

The first entry in this exactly recovers the norm $N_H^G(r)$, hence we focus on the M-component of this term. First, we calculate the composition $\Psi_\Gamma^* \circ \mathrm{tr}_{k-1,1}^H$ by a double coset formula. For this, we observe that the coset space $(\Sigma_k \wr H)/((\Sigma_{k-1} \wr H) \times H)$ is in bijection to $\{1, \dots, k\}$ by identifying $[(\sigma; h_1, \dots, h_k)]$ with $\sigma(k)$. As left multiplication by G is transitive on G/H and Γ identifies $G/H \cong \{1, \dots, k\}$, we observe that G acts transitively on $(\Sigma_k \wr H)/((\Sigma_{k-1} \wr H) \times H)$ through Ψ_Γ, and hence the double coset formula consists of a single summand. Moreover, the corresponding stabilizer occurring in the double coset formula is given as $H \subset G$, by the choice of taking $\gamma_k = e$ in Γ. Equivalently, $(\Psi_\Gamma)_{|H}\colon H \to \Sigma_k \wr H$ takes values in $(\Sigma_{k-1} \wr H) \times H$ and exhibits H as the preimage of this subgroup. Thus, we calculate

$$\begin{aligned}\Psi_\Gamma^*(\mathrm{tr}_{k-1,1}^H(P^{k-1}(r) \times m)) &= \mathrm{tr}_H^G((\Psi_\Gamma)_{|H}^*(P^{k-1} \times m)) \\ &= \mathrm{tr}_H^G((\Psi_\Gamma)_{|H}^*(p_k^*(P^{k-1}) \cdot q_k^*(m))) \\ &= \mathrm{tr}_H^G((p_k \circ (\Psi_\Gamma)_{|H})^*(P^{k-1}) \cdot (q_k \circ (\Psi_\Gamma)_{|H})^*(m))\end{aligned}$$

In this last line, we observe that $q_k \circ (\Psi_\Gamma)_{|H} = id_H$ is the identity on H, and hence the second factor evaluates to m. On the other hand, to calculate $(p_k \circ (\Psi_\Gamma)_{|H})^*(P^{k-1})$, we have to understand $p_k \circ (\Psi_\Gamma)_{|H}$. This depends on the left action of H on $(G/H) \setminus \{eH\}$. We decompose this set into H-orbits. For each of these H-orbits, which we index by j, we obtain the corresponding sub-basis Γ_j consisting of those γ_i in Γ such that $\gamma_i H$ is part of this specific H-orbit. The stabilizer inside H for this orbit can be written as $H \cap g_j H g_j^{-1}$ for some $g_j \in G$, and we fix a g_j for each orbit. Then, performing the right multiplication $\Gamma_j^H := \Gamma_j g_j^{-1}$ defines an ordered $(H \cap g_j H g_j^{-1})$-basis for H. Finally, we denote by k_j the order of this orbit. Using this, we observe that we can factor $p_k \circ (\Psi_\Gamma)_{|H}$ via

$$H \xrightarrow{\times \Psi_{\Gamma_j^H}} \bigtimes \Sigma_{k_j} \wr (H \cap g_j H g_j^{-1}) \subset \bigtimes \Sigma_{k_j} \wr H \xrightarrow{\Phi_{(k_j)}} \Sigma_{k-1} \wr H.$$

Applying the restriction along this composition to the power operation $P^{k-1}(r)$, we obtain

$$(p_k \circ (\Psi_\Gamma)_{|H})^*(P^{k-1}) = \prod \Psi_{\Gamma_j^H}^*(\mathrm{res}_{\Sigma_{k_j} \wr (H \cap g_j H g_j^{-1})}^{\Sigma_{k_j} \wr H}(P^{k_j}(r))) = \prod N_{H \cap g_j H g_j^{-1}}^H(\mathrm{res}_{H \cap g_j H g_j^{-1}}^H(r)).$$

Finally, we observe that this H-orbit decomposition of $(G/H) \setminus \{eH\}$ is equivalent to the G-orbit decomposition in $(G/H) \times (G/H) \setminus \Delta$, the G-set used by Strickland. Hence, this product of norms of restrictions is exactly $N_{\mathrm{pr}_2} \mathrm{res}_{\mathrm{pr}_1}(r)$, which recovers the formula for the norm in the square-zero extension from our formula for the power operations.

In my opinion, the description of the square-zero extension is another case in which the power operations simplify exposition and calculations compared to the norms. In particular, as explained in Remark 1.2.5, the formula for the power operations in $R \ltimes M$ is a direct generalization of the binomial formula, and benefits from the easy additivity relation of the power operations. In comparison, the norms have to utilize the $\Sigma_k \wr G$-set $(\Sigma_k \wr G/((\Sigma_{k-1} \wr G) \times G)) \times (\Sigma_k \wr G/((\Sigma_{k-1} \wr G) \times G)) \setminus \Delta$, which makes the new norms hard to work with. This is a consequence of the inexplicit additivity formula for general norms.

In addition to the structure required for a global power functor, $R \ltimes M$ comes equipped with maps $\eta\colon R \to R \ltimes M, r \mapsto (r, 0)$ and $\varepsilon\colon R \ltimes M \to R, (r, m) \mapsto r$.

Theorem 1.2.7. *Let R be a global power functor and M be an R-module. Then the square-zero extension $R \ltimes M$, endowed with multiplication and power operations as in Construction 1.2.1, and endowed with the maps $\eta\colon R \to R \ltimes M$ and $\varepsilon\colon R \ltimes M \to R$ defined above, is an augmented R-power algebra. Moreover, for a morphism $f\colon M \to N$ of R-modules, the induced morphism $R \ltimes f\colon R \ltimes M \to R \ltimes N$ is a morphism of augmented R-power algebras.*

We obtain a functor

$$R \ltimes (\_)\colon \mathrm{Mod}_R \to \mathrm{Alg}_R /R$$

from the category of R-modules into the category of augmented R-algebras.

When we consider a global power functor R, an R-power algebra S and an S-module M, we can also consider $S \ltimes M$ as an R-power algebra augmented to S via the functor $\mathrm{Alg}_S /S \to \mathrm{Alg}_R /S$ that composes the unit map of an S-algebra with the unit map $R \to S$ of the R-algebra S.

Proof. By definition as a direct sum of two global functors, $R \ltimes M$ is a global functor. To check that $R \ltimes M$ is a global Green functor, we need to show that for any compact Lie group G, $(R \ltimes M)(G)$ is a commutative ring such that restrictions are ring homomorphisms and transfers satisfy reciprocity, as explained in Remark 1.1.11. By the definition of multiplication in (1.2.2) and the usual arguments in algebra, $(R \ltimes M)(G)$ is a commutative ring. We then calculate for any homomorphism $\alpha\colon K \to G$ and $(r, m), (r', m') \in (R \ltimes M)(G)$:

$$\begin{aligned}\alpha^*((r, m)(r', m')) =&(\alpha^*(rr'), \alpha^*(rm' + r'm))\\ =&(\alpha^*(r)\alpha^*(r'), \alpha^*(r)\alpha^*(m') + \alpha^*(r')\alpha^*(m)) = \alpha^*(r, m)\alpha^*(r', m')\end{aligned}$$

Moreover, for a closed subgroup $H \subset G$ and $(r, m) \in (R \ltimes M)(H), (r', m') \in (R \ltimes M)(G)$, we have

$$\begin{aligned}\mathrm{tr}_H^G((r, m) \cdot \mathrm{res}_H^G(r', m')) =&(\mathrm{tr}_H^G(r \cdot \mathrm{res}_H^G(r')), \mathrm{tr}_H^G(r \cdot \mathrm{res}_H^G(m')) + \mathrm{tr}_H^G(\mathrm{res}_H^G(r') \cdot m))\\ =&(\mathrm{tr}_H^G(r)r', \mathrm{tr}_H^G(r)m' + r' \,\mathrm{tr}_H^G(m)) = \mathrm{tr}_H^G(r, m)(r', m').\end{aligned}$$

This shows that $R \ltimes M$ is a global Green functor.
Now we check the properties for the power operations given in Definition 1.1.18. In the following, $k \geq 1$ and G is a compact Lie group.

i) The unit in $(R \ltimes M)(G)$ is $(1, 0)$. We calculate

$$P^k(1, 0) = (P^k(1), \mathrm{tr}_{k-1,1}^G(P^{k-1}(1) \times 0)) = (1, 0).$$

ii) For any compact Lie group G and $(r, m) \in (R \ltimes M)(G)$, we have

$$P^1(r, m) = (P^1(r), \mathrm{tr}_{\Sigma_0 \wr G \times G}^{\Sigma_1 \wr G}(P^0(r) \times m)) = (r, m),$$

where the transfer is the identity under the identification $\Sigma_1 \wr G \cong G \cong \Sigma_0 \wr G \times G$, and $P^0(r) = 1$ by definition.

iii) Let $\alpha\colon K \to G$ be a homomorphism of compact Lie groups, and $(r, m) \in (R \ltimes M)(G)$. Then

$$\begin{aligned}P^k(\alpha^*(r, m)) &= (P^k(\alpha^*r), \mathrm{tr}_{k-1,1}^K(P^{k-1}(\alpha^*r) \times \alpha^*(m)))\\ &= ((\Sigma_k \wr \alpha)^*P^k(r), \mathrm{tr}_{k-1,1}^K(((\Sigma_{k-1} \wr \alpha) \times \alpha)^*(P^{k-1}(r) \times m)))\\ &= ((\Sigma_k \wr \alpha)^*P^k(r), (\Sigma_k \wr \alpha)^*(\mathrm{tr}_{k-1,1}^G(P^{k-1}(r) \times m))) = (\Sigma_k \wr \alpha)^*P^k(r, m).\end{aligned}$$

Here, to obtain the last line, we use that in this specific case, transfer and restriction commute. This is proved in [55, 5.2.9].

iv) Let $(r,m),(r',m') \in (R \ltimes M)(G)$. Then we calculate

$$\begin{aligned}
P^k((r,m)\cdot(r',m')) &= (P^k(r\cdot r'), \operatorname{tr}^G_{k-1,1}(P^{k-1}(r\cdot r')\times(r\cdot m' + r'\cdot m)))\\
&= (P^k(r)\cdot P^k(r'), \operatorname{tr}^G_{k-1,1}((P^{k-1}(r)\cdot P^{k-1}(r'))\times(r\cdot m')+\\
&\qquad (P^{k-1}(r)\cdot P^{k-1}(r'))\times(r'\cdot m)))\\
&= (P^k(r)\cdot P^k(r'), \operatorname{tr}^G_{k-1,1}((P^{k-1}(r)\times r)\cdot(P^{k-1}(r')\times m')+\\
&\qquad (P^{k-1}(r')\times r')\cdot(P^{k-1}(r)\times m)))\\
&= (P^k(r)\cdot P^k(r'), \operatorname{tr}^G_{k-1,1}(\Phi^*_{k-1,1}P^k(r)\cdot(P^{k-1}(r')\times m'))+\\
&\qquad \operatorname{tr}^G_{k-1,1}(\Phi^*_{k-1,1}P^k(r')\cdot(P^{k-1}(r)\times m)))\\
&= (P^k(r)\cdot P^k(r'), P^k(r)\cdot\operatorname{tr}^G_{k-1,1}(P^{k-1}(r')\times m')+\\
&\qquad P^k(r')\cdot\operatorname{tr}^G_{k-1,1}(P^{k-1}(r)\times m))\\
&= P^k(r,m)\cdot P^k(r',m').
\end{aligned}$$

Here, in the third line, we used an exchange property between the interior product $\cdot$ and the exterior product $\times$ for a global Green functor. This is explicitly stated and proved in [60, Diagram 2.22]. Here, we also have a Green module M in the formula, but the arguments work exactly the same. In the fourth line, the map $\Phi_{k-1,1}\colon (\Sigma_{k-1}\wr G)\times G \to \Sigma_k \wr G$ is the one from Remark A.1.3 *i)*, along which $\operatorname{tr}^G_{k-1,1}$ is the transfer. Thus, the next equation follows by reciprocity.

v) Let $i,j \geq 1$ such that $i+j=k$, and let $(r,m)\in(R\ltimes M)(G)$. Then

$$\Phi^*_{i,j}(P^k(r,m)) = (\Phi^*_{i,j}(P^k(r)), \Phi^*_{i,j}(\operatorname{tr}^G_{k-1,1}(P^{k-1}(r)\times m))).$$

To commute the restriction $\Phi^*_{i,j}$ past the transfer $\operatorname{tr}^G_{k-1,1}$, we need to apply the double coset formula. The double cosets are calculated in Lemma A.1.4 *i)*. Hence, for the double coset formula, we get

$$\begin{aligned}
&\Phi^*_{i,j}(\operatorname{tr}^G_{k-1,1}(P^{k-1}(r)\times m)) =\\
&\quad = \sum_{\varepsilon=0,1}\operatorname{tr}^{i,j,G}_{i-\varepsilon,\varepsilon,j-1+\varepsilon,1-\varepsilon}(\chi(\varepsilon)_\star(\Phi^*_{i-\varepsilon,j-1+\varepsilon}P^{k-1}(r)\times\Phi^*_{\varepsilon,1-\varepsilon}(m)))\\
&\quad = \operatorname{tr}^{i,j,G}_{i,0,j-1,1}(P^i(r)\times P^{j-1}(r)\times m) + \operatorname{tr}^{i,j,G}_{i-1,1,j,0}(P^{i-1}(r)\times m\times P^j(r))\\
&\quad = P^i(r)\times\operatorname{tr}^G_{j-1,1}(P^{j-1}(r)\times m) + P^j(r)\times\operatorname{tr}^G_{i-1,1}(P^{i-1}(r)\times m).
\end{aligned}$$

Here, we abbreviated

$$\operatorname{tr}^{i,j,G}_{i-\varepsilon,\varepsilon,j-1+\varepsilon,1-\varepsilon} = \operatorname{tr}^{(\Sigma_i\wr G)\times(\Sigma_j\wr G)}_{(\Sigma_{i-\varepsilon}\wr G)\times(\Sigma_\varepsilon\wr G)\times(\Sigma_{j-1+\varepsilon}\wr G)\times(\Sigma_{1-\varepsilon}\wr G)}.$$

Then, the total expression can be written as

$$\begin{aligned}
\Phi^*_{i,j}(P^k(r,m)) &= (\Phi^*_{i,j}(P^k(r)), \Phi^*_{i,j}(\operatorname{tr}^G_{k-1,1}(P^{k-1}(r)\times m)))\\
&= (P^i(r)\times P^j(r), P^i(r)\times\operatorname{tr}^G_{j-1,1}(P^{j-1}(r)\times m) + P^j(r)\times\operatorname{tr}^G_{i-1,1}(P^{i-1}(r)\times m))\\
&= (P^i(r), \operatorname{tr}^G_{i-1,1}(P^{i-1}(r)\times m))\times(P^j(r), \operatorname{tr}^G_{j-1,1}(P^{j-1}(r)\times m))\\
&= P^i(r,m)\times P^j(r,m).
\end{aligned}$$

vi) Let $k, l \geq 1$ and $(r, m) \in (R \ltimes M)(G)$. Then we need to calculate

$$\Psi^*_{k,l}(P^{kl}(r,m)) = (\Psi^*_{k,l} P^{kl}(r), \Psi^*_{k,l} \operatorname{tr}^G_{kl-1,1}(P^{kl-1}(r) \times m)).$$

Again, to switch the order of restriction and transfer, we need to apply the double coset formula, this time for the subgroups $\Sigma_k \wr (\Sigma_l \wr G)$ and $(\Sigma_{kl-1} \wr G) \times (\Sigma_1 \wr G)$ in $\Sigma_{kl} \wr G$. There is a single double coset, as described in Lemma A.1.4 *ii)*, and the double coset formula takes the form

$$\begin{aligned}
&\Psi^*_{k,l}(\operatorname{tr}^G_{kl-1,1}(P^{kl-1}(r) \times m)) \\
&\quad = \operatorname{tr}^{\Sigma_k \wr (\Sigma_l \wr G)}_{(\Sigma_{k-1} \wr (\Sigma_l \wr G)) \times (\Sigma_{l-1} \wr G) \times G}(\operatorname{res}^{(\Sigma_{kl-1} \wr G) \times G}_{(\Sigma_{k-1} \wr (\Sigma_l \wr G)) \times (\Sigma_{l-1} \wr G) \times G}(P^{kl-1}(r) \times m)) \\
&\quad = \operatorname{tr}^{\Sigma_k \wr (\Sigma_l \wr G)}_{(\Sigma_{k-1} \wr (\Sigma_l \wr G)) \times (\Sigma_l \wr G)}(P^{k-1}(P^l(r)) \times \operatorname{tr}^{\Sigma_l \wr G}_{(\Sigma_{l-1} \wr G) \times G}(P^{l-1}(r) \times m)).
\end{aligned}$$

Here, in the last line, we used that transfers are transitive and compatible with $\times$ and that

$$\operatorname{res}^{\Sigma_{kl-1} \wr G}_{(\Sigma_{k-1} \wr (\Sigma_l \wr G)) \times (\Sigma_1 \wr (\Sigma_{l-1} \wr G))} P^{kl-1}(r) = P^{k-1}(P^l(r)) \times P^1(P^{l-1}(r)).$$

Thus, we calculate

$$\begin{aligned}
\Psi^*_{k,l}(P^{kl}(r,m)) &= (\Psi^*_{k,l} P^{kl}(r), \Psi^*_{k,l} \operatorname{tr}^G_{kl-1,1}(P^{kl-1}(r) \times m)) \\
&= (P^k(P^l(r)), \operatorname{tr}^{\Sigma_l \wr G}_{k-1,1}(P^{k-1}(P^l(r)) \times \operatorname{tr}^G_{l-1,1}(P^{l-1}(r) \times m))) \\
&= P^k(P^l(r), \operatorname{tr}^G_{l-1,1}(P^{l-1}(r) \times m)) = P^k(P^l(r,m)).
\end{aligned}$$

vii) Let $(r, m), (r', m') \in (R \ltimes M)(G)$. Then

$$\begin{aligned}
P^k((r,m) + (r',m')) &= (P^k(r+r'), \operatorname{tr}^G_{k-1,1}(P^{k-1}(r+r') \times (m+m'))) \\
&= \Big(\sum_{i=0}^{k} \operatorname{tr}^G_{i,k-i}(P^i(r) \times P^{k-i}(r')), \\
&\qquad \operatorname{tr}^G_{k-1,1}(\sum_{j=0}^{k-1} \operatorname{tr}_{j,k-j-1}(P^j(r) \times P^{k-j-1}(r')) \times (m+m')) \Big) \\
&= \Big(\sum_{i=0}^{k} \operatorname{tr}^G_{i,k-i}(P^i(r) \times P^{k-i}(r')), \\
&\qquad \sum_{j=0}^{k-1} \operatorname{tr}^G_{k-1,1}(\operatorname{tr}^G_{j,k-j-1}(P^j(r) \times P^{k-j-1}(r')) \times m) + \\
&\qquad\qquad \operatorname{tr}^G_{k-1,1}(\operatorname{tr}^G_{j,k-j-1}(P^j(r) \times P^{k-j-1}(r')) \times m') \Big) \\
&= \Big(\sum_{i=0}^{k} \operatorname{tr}^G_{i,k-i}(P^i(r) \times P^{k-i}(r')), \\
&\qquad \sum_{j=0}^{k-1} \operatorname{tr}^G_{j+1,k-j-1}(\operatorname{tr}^G_{j,1}(P^j(r) \times m) \times P^{k-j-1}(r')) + \\
&\qquad\qquad \operatorname{tr}^G_{j,k-j}(P^j(r) \times \operatorname{tr}^G_{k-j-1,1}(P^{k-j-1}(r') \times m')) \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\sum_{i=0}^{k} \mathrm{tr}_{i,k-i}^{G}(P^i(r) \times P^{k-i}(r')), \\
&\qquad \sum_{i=0}^{k} \mathrm{tr}_{i,k-i}^{G}(\mathrm{tr}_{i-1,1}^{G}(P^{i-1}(r) \times m) \times P^{k-i}(r')) + \\
&\qquad\qquad \mathrm{tr}_{i,k-i}^{G}(P^i(r) \times \mathrm{tr}_{k-i-1,1}^{G}(P^{k-i-1}(r') \times m')) \Big) \\
&= \sum_{i=0}^{k} \mathrm{tr}_{i,k-i}^{G}(P^i(r,m) \times P^{k-i}(r',m')).
\end{aligned}
$$

Here, in the passage to the penultimate line, we applied an index shift to the first half of the sum. Moreover, observe that in this reindexed sum, we have added two terms. However, both of these evaluate to 0. Hence, we have shown additivity.

viii) Let $H \subset G$ be a closed subgroup and $(r,m) \in (R \ltimes M)(H)$. Then

$$
\begin{aligned}
P^k(\mathrm{tr}_H^G(r,m)) &= (P^k(\mathrm{tr}_H^G(r)), \mathrm{tr}_{k-1,1}^{G}(P^{k-1}(\mathrm{tr}_H^G(r)) \times \mathrm{tr}_H^G(m))) \\
&= (\mathrm{tr}_{\Sigma_k \wr H}^{\Sigma_k \wr G}(P^k(r)), \mathrm{tr}_{k-1,1}^{G}(\mathrm{tr}_{(\Sigma_{k-1}\wr H)\times H}^{(\Sigma_{k-1}\wr G)\times G}(P^{k-1}(r) \times m))) \\
&= (\mathrm{tr}_{\Sigma_k \wr H}^{\Sigma_k \wr G}(P^k(r)), \mathrm{tr}_{\Sigma_k \wr H}^{\Sigma_k \wr G}(\mathrm{tr}_{k-1,1}^{H}(P^{k-1}(r) \times m))) = \mathrm{tr}_{\Sigma_k \wr H}^{\Sigma_k \wr G} P^k(r,m).
\end{aligned}
$$

Here, in the last line, we used the transitivity of the transfer to commute the two transfers.

That the maps $\eta\colon R \to R \ltimes M$ and $\varepsilon\colon R \ltimes M \to R$ are morphisms of global power functors is an easy calculation. Thus $R \ltimes M$ is an augmented R-algebra.
We now check functoriality. Let $f\colon M \to N$ be a morphism of R-modules, we need to show that $R \ltimes f$ is a morphism of augmented R-algebras. First, we check that $R \ltimes f$ is a map of global functors. This is clear, since additively, $R \ltimes f = R \oplus f$, which is a morphism of global functors. It is a map of global Green functors, since we have $(R \ltimes f)(1,0) = (1,0)$, and for $(r,m),(r',m') \in (R \ltimes M)(G)$, we calculate

$$
\begin{aligned}
(R \ltimes f)((r,m)\cdot(r',m')) &= (rr', f(rm' + r'm)) = (rr', rf(m') + r'f(m)) \\
&= (R \ltimes f)(r,m) \cdot (R \ltimes f)(r',m').
\end{aligned}
$$

Moreover, $R \ltimes f$ commutes with the power operations, since for $(r,m) \in (R \ltimes M)(G)$,

$$
\begin{aligned}
(R \ltimes f)(P^k(r,m)) &= (P^k(r), f(\mathrm{tr}_{k-1,1}^{G}(P^{k-1}(r) \times m))) = (P^k(r), \mathrm{tr}_{k-1,1}^{G}(P^{k-1}(r) \times f(m))) \\
&= P^k((R \ltimes f)(r,m)).
\end{aligned}
$$

Finally, $R \ltimes f$ also commutes with the unit η and the augmentation ε, since it is the identity on the R-summand. Thus $R \ltimes f$ is a morphism of augmented R-algebras. Functoriality is then clear. □

1.2.b Derivations of Global Power Functors

Definition 1.2.8. Let R be a global power functor, S be an R-algebra with unit map $\eta\colon R \to S$ and let M be an S-module. Then, an R-derivation of S with values in M is a map $d\colon S \to M$ of global functors such that the following properties are satisfied:

i) For all compact Lie groups G and $s, s' \in S(G)$,

$$d(ss') = sd(s') + s'd(s).$$

ii) For all compact Lie groups G and $s \in S(G)$,

$$d(P^k(s)) = \mathrm{tr}^G_{k-1,1}(P^{k-1}(s) \times d(s)).$$

iii) The composition $d \circ \eta = 0$ vanishes.

We denote the set of R-derivations of S with values in M by $\mathrm{Der}_R(S, M)$. This is an abelian group by addition of morphisms of global functors.

Remark 1.2.9. As for derivations of commutative rings, the property in Definition 1.2.8 *iii)* is equivalent to d being R-linear. Indeed, if $r \in R(G)$ and $s \in S(G)$, then we calculate

$$d(\eta(r)s) = \eta(r)d(s) + d(\eta(r))s = \eta(r)d(s).$$

Conversely, if d is R-linear, using the Leibniz-rule for $\eta(r) \cdot 1$ for any $r \in R(G)$ gives $d \circ \eta = 0$.

Moreover, the Leibniz rule in Definition 1.2.8 *i)* is formulated in terms of the internal product of a global Green functor. It can equivalently be given using the external product, and then takes the form

$$d(s \times s') = s \times d(s') + d(s) \times s'$$

for $s \in S(G)$ and $s' \in S(K)$.

Remark 1.2.10. The formula given in Definition 1.2.8 *ii)* for the behaviour of a derivation on a power operation can be thought of as a twisted version of the classical Leibniz rule. On a usual power, a derivation satisfies $d(x^k) = kx^{k-1}d(x)$. The power operations are an equivariant refinement of the usual powers in a commutative ring. The index of $(\Sigma_{k-1} \wr G) \times G$ in $\Sigma_k \wr G$ is k, and hence the transfer $\mathrm{tr}^G_{k-1,1}$ can be considered as an equivariantly twisted version of multiplication by k. In particular, for a constant global power algebra $\underline{S}$ over $\underline{R}$ for an R-algebra S, the above notion of a derivation recovers the usual notion of a derivation on a commutative algebra.

Moreover, this formula for the twisted Leibniz rule is a generalization of the formula for derivations of G-Tambara functors present implicitly in Strickland's definition of square-zero extensions in [62, Definition 14.8] and made explicit by Hill in [30, Definition 4.1]. This formula is

$$d(N^K_H(s)) = \mathrm{tr}^K_H(N_{\mathrm{pr}_1}\,\mathrm{res}_{\mathrm{pr}_2}(s) \cdot d(s))$$

for $s \in S(K)$ and $K \leq H \leq G$ a sequence of subgroups in G. This definition can be translated into our context of power operations using Remark 1.1.20, and the calculations are exactly the same as in Remark 1.2.6.

The main property of derivations is that they control maps into square-zero extensions.

Theorem 1.2.11. *Let R be a global power functor, S and T be R-algebras and $f\colon T \to S$ be a map of R-algebras, making T into an R-algebra augmented over S. Let M be an S-module, considered as a T-module via f. Then, the map*

$$\mathrm{Der}_R(T, M) \to \mathrm{Hom}_{\mathrm{Alg}_R/S}(T, S \ltimes M)$$
$$d \mapsto f \ltimes d$$

is bijective.

Proof. We first prove that for an R-derivation $d\colon T \to M$, the map $f \ltimes d$ is indeed a map of R-algebras augmented to S. It is a map of global functors as the direct sum of two maps of global functors. Moreover, for a compact Lie group G and $t, t' \in T(G)$, we have

$$(f \ltimes d)(t \cdot t') = (f(tt'), d(tt')) = (f(t)f(t'), f(t)d(t') + f(t')d(t)) = (f \ltimes d)(t) \cdot (f \ltimes d)(t'),$$

where in the second equation, we note that the T-module structure on M is via the map $f\colon T \to S$. Together with $(f \ltimes d)(1) = (f(1), 0) = (1, 0)$, this proves that $f \ltimes d$ is a map of global Green functors.
Let now G be again a compact Lie group, $t \in T(G)$ and $k \geq 1$. Then

$$(f \ltimes d)(P^k(t)) = (f(P^k(t)), d(P^k(t))) = (P^k(f(t)), \mathrm{tr}^G_{k-1,1}(f(P^{k-1}(t)) \times d(t))) = P^k(f(t), d(t)),$$

thus $f \ltimes d$ is a map of global power functors. Moreover, if $\eta\colon R \to T$ is the unit of the R-algebra T, we have the identity

$$(f \ltimes d)(\eta(r)) = (f(\eta(r)), d(\eta(r))) = (f(\eta(r)), 0)$$

for any compact Lie group G and $r \in R(G)$, since $d \circ \eta = 0$ by the property that d is an R-derivation. Thus, by definition of the algebra structure of $S \ltimes M$, we see that $f \ltimes d$ is a map of R-algebras. As the map $f \ltimes d$ agrees with f on the S-summand and the augmentation of $S \ltimes M$ is the projection onto S, we see that $f \ltimes d$ is a map of R-algebras augmented to S.

Now let $g\colon T \to S \ltimes M$ be a map of R-algebras augmented to S. We denote the two projections of g to S and M by $g_S = \pi_S \circ g$ and $g_M = \pi_M \circ g$, such that $g(t) = (g_S(t), g_M(t))$. As g commutes with the augmentations and the augmentation on $S \ltimes M$ is π_S, we see that $g_S = f$. We now show that $g_M\colon T \to M$ is an R-derivation. As both g and π_M are morphisms of global functors, also g_M is a morphism of global functors. Let G be a compact Lie group and $t, t' \in T(G)$. Then,

$$\begin{aligned} g_M(tt') &= \pi_M(g(tt')) = \pi_M(g(t) \cdot g(t')) = \pi_M(f(t)f(t'), f(t)g_M(t') + f(t')g_M(t)) \\ &= f(t)g_M(t') + f(t')g_M(t) \end{aligned}$$

and for any $k \geq 1$, we have

$$\begin{aligned} g_M(P^k(t)) =& \pi_M(g(P^k(t))) = \pi_M(P^k(g(t))) = \pi_M((P^k(f(t)), \mathrm{tr}^G_{k-1,1}(P^{k-1}(f(t)) \times g_M(t)))) \\ =& \mathrm{tr}^G_{k-1,1}(P^{k-1}(f(t)) \times g_M(t)). \end{aligned}$$

Lastly, for any $r \in R(G)$, we have $g_M(\eta(r)) = \pi_M(g(\eta(r))) = \pi_M(f(\eta(r)), 0) = 0$, since g is a map of R-algebras. Hence, g_M is indeed an R-derivation.
Then we consider the assignment

$$\mathrm{Hom}_{\mathrm{Alg}_R/S}(T, S \ltimes M) \to \mathrm{Der}_R(T, M), g \mapsto g_M.$$

Since for any $g \in \mathrm{Hom}_{\mathrm{Alg}_R/S}(T, S \ltimes M)$, we have $g_S = f$, we see that $g = f \ltimes g_M$, and for any R-derivation $d\colon T \to M$, we have $d = (f \ltimes d)_M$. Thus, these assignments are inverse to each other. □

1.2.c Kähler Differentials of Global Power Functors

Theorem 1.2.11 shows that morphisms into a square-zero extension are characterized by derivations. In the case of commutative rings, the functor of derivations is representable. The same statement is also true for the functor $\mathrm{Der}_R(S, \_)\colon \mathrm{Mod}_S \to \mathrm{Ab}$, where S is a global power functor. We call the representing object the module of Kähler differentials. Most of the relevant properties of this S-module can be deduced purely from this description, and we follow this line of thought in order to give proofs that follow the classical ones without needing to explicitly handle the power operations. We show the existence of the module of Kähler differentials by considering the multiplication map $S\Box_R S \to S$ of an R-power algebra S. Here, the classical recipe of taking the kernel I of the multiplication and forming the indecomposables I/I^2 has to be slightly modified to account for the power operations. In the case of a fixed group G, this has already been carried out for Tambara functors by Hill in [30, Chapter 5]. The construction used here also becomes useful when studying the naive cotangent complex as a means to calculate the low-degree André-Quillen (co-)homology in Section 1.5.c.

Definition 1.2.12. Let R be a global power functor and S be an R-power algebra. Suppose the functor

$$\mathrm{Der}_R(S, \_)\colon \mathrm{Mod}_S \to \mathrm{Ab}$$

is representable. Then we denote a representing object by $(\Omega^1_{S/R}, d)$ and call $\Omega^1_{S/R}$ the module of Kähler differentials. The R-derivation $d\colon S \to \Omega^1_{S/R}$ is called the universal derivation.

From this, we directly observe that if these modules of Kähler differentials exist for every R-power algebra S, they are automatically functorial in the R-algebra S. Since $\Omega^1_{S/R}$ is naturally an S-module for any R-power algebra S, a natural target for a functor of Kähler differentials is $\mathcal{M}od^R$, the category of pairs of an R-power algebra and modules over it, which arises as the Grothendieck construction from the 2-functor $\mathrm{Alg}_R \to \mathrm{Cat}, S \mapsto \mathrm{Mod}_S$. However, we do not need this formulation here and hence do not provide any details on this construction. We only need the resulting functor

$$R\Box_{(\_)}\Omega^1_{(\_)/R}\colon \mathrm{Alg}_R/R \to \mathrm{Mod}_R$$

and the relative version

$$S\Box_{(\_)}\Omega^1_{(\_)/R}\colon \mathrm{Alg}_R/S \to \mathrm{Mod}_S$$

for a fixed R-power algebra S.

This functoriality is also compatible with a change in the base global power functor R. Explicitly, if we are given a commutative square

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

of global power functors, then we obtain an induced morphism $\Omega^1_{S/R} \to \Omega^1_{S'/R'}$ of S-modules, or equivalently by extension of scalars, a morphism $S'\Box_S\Omega^1_{S/R} \to \Omega^1_{S'/R'}$ of S'-modules.

We prove the existence of Kähler differentials for any global power functor R and R-power algebra S in Theorem 1.2.24. This existence allows us to formulate the following adjunction, which is a consequence from the definition of Kähler differentials and Theorem 1.2.11:

Theorem 1.2.13. *Let R be a global power functor and S be an R-algebra. Then the functor*

$$S\Box_{(\_)}\Omega^1_{(\_)/R}\colon \mathrm{Alg}_R/S \to \mathrm{Mod}_S$$

is left adjoint to

$$S \ltimes (\_)\colon \mathrm{Mod}_S \to \mathrm{Alg}_R/S.$$

Proof. Let M be an S-module and T be an R-algebra augmented to S. Then we can consider M as a T-module via the map $\varepsilon\colon T \to S$. Then we have the chain

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Alg}_R/S}(T, S \ltimes M) &\cong \mathrm{Der}_R(T, M)\\ &\cong \mathrm{Hom}_T(\Omega^1_{T/R}, M)\\ &\cong \mathrm{Hom}_S(S\Box_T\Omega^1_{T/R}, M)\end{aligned}$$

of bijections. This proves that the functors are adjoint. □

Example 1.2.14. As an example, we calculate the global functor of Kähler differentials on a polynomial global power functor from Definition 1.1.22. Let R be a global power functor and $R[x_K]$ be a polynomial R-power algebra on a generator at level K. We claim that

$$\Omega^1_{R[x_K]/R} \cong R[x_K]\{dx_K\} := R[x_K]\Box\mathbb{A}(K, \_)$$

is a free $R[x_K]$-module on one generator in degree K, which we call dx_K. For the proof, we show that this satisfies the universal property of the module of Kähler-differentials: Let M be any $R[x_K]$-module. Then we have the chain

$$\mathrm{Der}_R(R[x_K], M) \cong \mathrm{Hom}_{\mathrm{Alg}_R/R[x_K]}(R[x_K], R[x_K]\ltimes M) \cong M(K) \cong \mathrm{Hom}_{R[x_K]}(R[x_K]\Box\mathbb{A}(K, \_), M)$$

of bijections, where the second bijection uses $\mathrm{Hom}_{\mathrm{Alg}_R}(R[x_K], R[x_K] \ltimes M) \cong (R[x_K] \ltimes M)(K)$, and that such a morphism is a morphism over $R[x_K]$ iff the first component is x_K. The composite bijection above exhibits the universal property of $\Omega^1_{R[x_K]/R}$, thus we have shown that we have $\Omega^1_{R[x_K]/R} \cong R[x_K]\{dx_K\}$.
We can also describe the universal derivation in this context, using the above isomorphism to translate the identity on the right side into a derivation $d\colon R[x_K] \to R[x_K]\Box\mathbb{A}(K, \_)$. This yields $d(x_K) = 1 \boxdot id_K$, which justifies the name $R[x_K]\{dx_K\}$ for this module. Generally, a power $P^n(x_K)$ satisfies

$$d(P^n(x_K)) = \mathrm{tr}^K_{n-1,1}(P^{n-1}x_K \times dx_K).$$

By the same representability arguments or using Proposition 1.2.30 (at least for finitely many generators), we get the following result for general polynomial algebras:

Proposition 1.2.15. *Let R be a global power functor and K_i be (not necessarily distinct) compact Lie groups for $i \in I$, with I any indexing set. Let*

$$T = R[x_{i,K_i}, i \in I] := \mathop{\Box_R}_{i\in I} R[x_{i,K_i}]$$

be a polynomial algebra over R in generators indexed by I. Then

$$\Omega^1_{T/R} \cong T\{dx_{i,K_i}, i \in I\} := \bigoplus_{i\in I} T\Box\mathbb{A}(K_i, \_)$$

is a free T-module on generators indexed by I.

Another case in which Kähler differentials are easily computed using the universal property is for surjections $R \to S$. A morphism of global power functors is called a surjection if it is levelwise surjective. This is equivalent to being an epimorphism in the category of global power functors, and all epimorphisms are effective. This can be deduced as a consequence of the description of global power algebras as a multisorted algebraic theory in Remark 1.1.24.

Proposition 1.2.16. *Let R be a global power functor and S be an R-algebra such that the unit map $\eta\colon R \to S$ is surjective. Then $\Omega^1_{S/R} = 0$.*

Proof. Since any R-derivation $d\colon S \to M$ vanishes on the image of the unit map $\eta\colon R \to S$, surjectivity of η implies that $\mathrm{Der}_R(S, \_) = 0$ is trivial. Hence, this functor is represented by the zero module, so $\Omega^1_{S/R} = 0$. □

By exhibiting a construction, we now show that Kähler differentials exist for all R-power algebras S. Classically, we have the formula $\Omega^1_{S/R} \cong I/I^2$, where I denotes the kernel of the multiplication map $S \otimes_R S \to S$. In our context, the kernel of the multiplication map is a global power ideal as defined in Definition 1.1.30 and is still the correct object to consider, but in the presence of power operations, we have to replace I^2 by an object that also contains powers of elements in I. We follow the definitions given by Hill in [30, Definitions 5.1 and 5.4].

Definition 1.2.17. Let R be a global power functor and I be a power ideal in R. Then we define the ideal $I^{\geq 2}$ of decomposables in I as the submodule of R generated by elements of the form $P^{k_1}(x_1) \times \ldots \times P^{k_n}(x_n)$ with $k_1 + \ldots + k_n \geq 2$ and all $x_i \in I$.
Explicitly, the value of $I^{\geq 2}$ at a compact Lie group G is given by

$$I^{\geq 2}(G) = \left\{ \sum_{\substack{j \in J \\ J \text{ finite}}} \mathrm{tr}^G_{H_j} \alpha_j^*(r_j \times P^{k_{j,1}}(x_{j,1}) \times \ldots \times P^{k_{j,n_j}}(x_{j,n_j})) \mid k_{j,1} + \ldots + k_{j,n_j} \geq 2 \right\}, \tag{1.2.18}$$

where K_j and $K_{j,i}$ are compact Lie groups, $H_j \subset G$ are closed subgroups of G, $\alpha_j\colon H_j \to K_j \times \left(\bigtimes_{i=1}^{n_j} \Sigma_{k_{j,i}} \wr K_{j,i}\right)$ are continuous homomorphisms and $r_j \in R(K_j)$, $x_{j,i} \in I(K_{j,i})$.

Lemma 1.2.19. *With the above definition* (1.2.18)*, $I^{\geq 2}$ is a power ideal in R.*

Proof. By the definition of $I^{\geq 2}$ in (1.2.18) and the relations satisfied by compositions of transfers and restrictions given in Remark 1.1.2, we see that $I^{\geq 2}$ is closed under restrictions and transfers. Moreover, it is a Green ideal in R, since for $s \in R(L)$, $r \in R(K)$ and $x_i \in I(K_i)$ and natural numbers $k_i \geq 1$ with $k_1 + \ldots + k_n \geq 2$, and H and α as above, we calculate

$$s \times \mathrm{tr}^G_H(\alpha^*(r \times P^{k_1}(x_1) \times \ldots \times P^{k_n}(x_n))) = \mathrm{tr}^{L\times G}_{L \times H}((L \times \alpha)^*((s \times r) \times P^{k_1}(x_1) \times \ldots \times P^{k_n}(x_n))).$$

This again is an element in $I^{\geq 2}$.
Finally, we check that $I^{\geq 2}$ is closed under power operations. With the same notation as before, we have

$$\begin{aligned} P^k(\mathrm{tr}^G_H \alpha^*(r \times P^{k_1}(x_1) \times \ldots \times P^{k_n}(x_n))) &= \mathrm{tr}^{\Sigma_k \wr G}_{\Sigma_k \wr H}(\Sigma_k \wr \alpha)^*(P^k(r \times P^{k_1}(x_1) \times \ldots \times P^{k_n}(x_n))) \\ &= \mathrm{tr}^{\Sigma_k \wr G}_{\Sigma_k \wr H}((\Sigma_k \wr \alpha)^* \circ \Delta_k^* \circ (\bigtimes \Psi_{k,k_i})^*)(P^k(r) \times P^{kk_1}(x_1) \times \ldots \times P^{kk_n}(x_n)) \\ &= \mathrm{tr}^{\Sigma_k \wr G}_{\Sigma_k \wr H}((\bigtimes \Psi_{k,k_i}) \circ \Delta_k \circ (\Sigma_k \wr \alpha))^*(P^k(r) \times P^{kk_1}(x_1) \times \ldots \times P^{kk_n}(x_n)), \end{aligned}$$

where Δ_k is an iterated version of the diagonal embedding from Remark A.1.3 *iii)*, which is used in the external multiplicativity of power operations, as explained in Remark 1.1.19. The closure of multiplication and power operations on arbitrary sums of the above elements then follows from additivity. This completes the proof that $I^{\geq 2}$ is a power ideal in R. □

We consider the following construction that turns out to calculate the module of Kähler differentials:
Let R be a global power functor and S be an R-algebra, with unit $\eta\colon R \to S$. Let

$$\mu\colon S \square_R S \to S$$

be the multiplication map of S. Let $I = \ker(\mu)$. Then we consider the global functor $I/I^{\geq 2}$. Moreover, we define the morphism

$$d\colon S \xrightarrow{id\Box\eta-\eta\Box id} I/I^{\geq 2} \tag{1.2.20}$$

as the difference of the inclusion as the left and right factor, using $S \cong R\Box_R S \cong S\Box_R R$.

Proposition 1.2.21. *The S-module structure on $S\Box_R S$ by multiplication in the left factor defines the structure of an S-module on the global functor $I/I^{\geq 2}$. Moreover, the map $d\colon S \to I/I^{\geq 2}$ is an R-derivation.*

Proof. We already know that I is an $S\Box_R S$-module, since it is an ideal in $S\Box_R S$. Moreover, $I^{\geq 2}$ is a subideal in I, and hence $I/I^{\geq 2}$ is an $S\Box_R S$-module. Now, using the left inclusion $S \xrightarrow{id\Box\eta} S\Box_R S$, we see that the left multiplication on $S\Box_R S$ induces an S-module structure on $I/I^{\geq 2}$.
As the difference of two maps of global functors, the map $d = id\Box\eta - \eta\Box id$ is a map of global functors. Let $s, s' \in S(G)$, then we have

$$\begin{aligned} d(s\cdot s') =&(s\cdot s')\boxdot 1 - 1\boxdot(s\cdot s') \\ =&(s\cdot s')\boxdot 1 - s\boxdot s' + (s\cdot s')\boxdot 1 - s'\boxdot s \\ =&(s\boxdot 1)\cdot(s'\boxdot 1 - 1\boxdot s') + (s'\boxdot 1)\cdot(s\boxdot 1 - 1\boxdot s) = s\cdot d(s') + s'\cdot d(s). \end{aligned}$$

Here, $\boxdot\colon S(G)\otimes S(G) \to (S\Box_R S)(G)$ is the universal diagonal product explained in Remark 1.1.13. In the second line, we added $(s\boxdot 1 - 1\boxdot s)\cdot(s'\boxdot 1 - 1\boxdot s') \in I^{\geq 2}$, which is trivial in $I/I^{\geq 2}$. Moreover, for $s \in S(G)$ and $k \geq 1$, we have

$$\begin{aligned} d(P^k(s)) =&P^k(s)\boxdot 1 - 1\boxdot P^k(s) \\ =&P^k(s\boxdot 1) - P^k(1\boxdot s) = P^k(s\boxdot 1) - P^k(s\boxdot 1 - d(s)) \\ =&P^k(s\boxdot 1) - \sum_{i=0}^{k} \mathrm{tr}_{k-i,i}(P^{k-i}(s\boxdot 1)\times P^i(-d(s))) \\ =&\,\mathrm{tr}_{k-1,1}(P^{k-1}(s)\times d(s)) - \sum_{i=2}^{k}(P^{k-i}(s)\times P^i(-d(s))). \end{aligned}$$

Here, we used the power operations on the boxproduct $S\Box_R S$, making it into the coproduct in global power functors, as described in [55, Example 5.2.15]. Moreover, we again used that the S-module structure on I is given by the left inclusion of S into $S\Box_R S$, in order to identify the action of $P^k(s)$ with that of $P^k(s\boxdot 1)$. In this formula, the last sum lies in $I^{\geq 2}$, hence we see that in $I/I^{\geq 2}$, we have the relation

$$d(P^k(s)) = \mathrm{tr}_{k-1,1}(P^{k-1}(s)\times d(s)).$$

Finally, since the boxproduct $S\Box_R S$ is taken over R, we see that for $r \in R(G)$, we have $d(\eta(r)) = r\boxdot 1 - 1\boxdot r = 0$, and thus d is indeed an R-derivation of S with values in $I/I^{\geq 2}$. □

Another way to see that d is a derivation is to identify the square-zero extension $S \ltimes (I/I^{\geq 2})$ with $(S\Box_R S)/I^{\geq 2}$ and use the characterization of morphisms into a square-zero extension from Theorem 1.2.11. This in fact amounts to the same calculations as in the previous proof, but since the result is interesting on its own, we state it explicitly.

Lemma 1.2.22. *The map*

$$\psi_{S/R} = (id\Box\eta, -\operatorname{incl})\colon S \ltimes (I/I^{\geq 2}) \to (S\Box_R S)/I^{\geq 2},\ (s,x) \mapsto (s \boxdot 1 - x)$$

is an isomorphism of global power functors, such that the diagram

$$\begin{array}{ccccc} S & \xrightarrow{id\ltimes d} & S \ltimes (I/I^{\geq 2}) & \xrightarrow{\mathrm{pr}_S} & S \\ & \searrow^{\eta\Box id} & \downarrow{\scriptstyle \psi_{S/R}} & \nearrow_{\mu} & \\ & & (S\Box_R S)/I^{\geq 2} & & \end{array}$$

commutes.

Proof. Since as global functors, $S \ltimes I/I^{\geq 2} = S \oplus I/I^{\geq 2}$ and both the left inclusion $id\Box\eta$ and the inclusion of $I/I^{\geq 2}$ into $(S\Box_R S)/I^{\geq 2}$ are maps of global functors, we see that $\psi_{S/R}$ is a morphism of global functors. To see that it is a map of global power functors, we have to do a similar calculation as in the proof of Proposition 1.2.21 as follows. Let G be a compact Lie group and $(s,x),(s',x') \in (S \ltimes I/I^{\geq 2})(G)$. Then,

$$\begin{aligned} \psi_{S/R}((s,x)\cdot(s',x')) =& \psi_{S/R}(ss', sx' + s'x) \\ =& ss' \boxdot 1 - sx' - s'x \\ =& \psi_{S/R}(s,x)\cdot \psi_{S/R}(s',x') - \underbrace{xx'}_{\in I^{\geq 2}}, \end{aligned}$$

and for $k \geq 1$ we have

$$\begin{aligned} P^k(\psi_{S/R}(s,x)) =& P^k(s\boxdot 1 - x) \\ =& \sum_{i=0}^{k} \mathrm{tr}_{k-i,i}(P^{k-i}(s\boxdot 1)\times P^i(-x)) \\ =& P^k(s)\boxdot 1 - \mathrm{tr}_{k-1,1}(P^{k-1}(s)\times x) + \sum_{i=2}^{k}\mathrm{tr}_{k-i,i}(P^{k-i}(s)\times P^i(-x)) \\ =& \psi_{S/R}(P^k(s,x)) + \underbrace{\sum_{i=2}^{k}\mathrm{tr}_{k-i,i}(P^{k-i}(s)\times P^i(-x))}_{\in I^{\geq 2}}. \end{aligned}$$

Thus, the map $\psi_{S/R}$ is a map of global power functors. By the explicit formulas, we directly see that $\psi_{S/R}\circ(id\ltimes d) = \eta\Box id$ is the right inclusion of S in $S\Box_R S$. Moreover, since for any element $x \in I/I^{\geq 2}$ the multiplication $\mu(x) = 0$ vanishes, also the augmentations agree.
To see that $\psi_{S/R}$ is an isomorphism, we explicitly construct an inverse. We consider

$$\tilde{\psi}_{S/R}\colon (S\Box_R S)/I^{\geq 2} \to S \ltimes I/I^{\geq 2},\ \xi \mapsto (\mu(\xi), \mu(\xi)\boxtimes 1 - \xi).$$

This is well-defined on the quotient by $I^{\geq 2}$, since for $\xi \in I^{\geq 2}$, we have both $\mu(\xi) = 0$ and $\xi = 0 \in I/I^{\geq 2}$. By an easy calculation, it is inverse to $\psi_{S/R}$. Thus, $\psi_{S/R}$ is an isomorphism of global power functors. ☐

Now, we show that the derivation $d\colon S \to I/I^{\geq 2}$ is the universal R-derivation on S and $I/I^{\geq 2}$ is a model for the Kähler differentials. First, we need the following lemma:

Lemma 1.2.23. *The S-module $I/I^{\geq 2}$ is generated by the image of $d\colon S \to I/I^{\geq 2}$.*

Proof. Since $I/I^{\geq 2}$ is a quotient of an ideal in $S\Box_R S$, we first consider this box product. By Remark 1.1.9, the elements $s \boxtimes s'$ for $s \in S(H), s' \in S(H')$ generate the global functor $S\Box_R S$, where H and H' run through the pairs of representatives of isomorphism classes of compact Lie groups.
Thus, we may write any element of $I(G) \subset (S\Box_R S)(G)$ as $\sum \mathrm{tr}_{H_i}^G(\alpha_i^*(x_i \boxtimes y_i))$ for some finite index set J, compact Lie groups K_i, K_i', closed subgroups $H_i \subset G$, continuous homomorphisms $\alpha_i \colon H_i \to K_i \times K_i'$ and elements $x_i \in S(K_i), y_i \in S(K_i')$. Then, we can calculate

$$\sum \mathrm{tr}_{H_i}^G(\alpha^*(x_i \boxtimes y_i)) = \sum \mathrm{tr}_{H_i}^G(\alpha^*((x_i \times y_i) \boxtimes 1 - x_i \times d(y_i))).$$

Since we started with an element in the kernel of the multiplication map, we see that the first summands sum to zero. Thus, we see that the elements $d(y_i)$ generate $I/I^{\geq 2}$ as an S-module. □

This allows us now to show that Kähler differentials exist by showing the universal property for $(I/I^{\geq 2}, d)$.

Theorem 1.2.24. *Let R be a global power functor and S be an R-algebra. Let M be an S-module, and $d\colon S \to I/I^{\geq 2}$ be the derivation from* (1.2.20)*. Then the map*

$$\mathrm{Hom}_S(I/I^{\geq 2}, M) \to \mathrm{Der}_R(S, M),\ \varphi \mapsto \varphi \circ d$$

is an isomorphism of abelian groups natural in M. In particular, Kähler differentials exist for all R-power algebras, and we have

$$\Omega^1_{S/R} \cong I/I^{\geq 2}.$$

Proof. First, we need to check that $\varphi \circ d$ is indeed a derivation. This is easily seen from the definition in 1.2.8, since φ is a morphism of S-modules. Moreover, the map $\varphi \mapsto \varphi \circ d$ is clearly additive.
We now construct an inverse map: Let $e \in \mathrm{Der}_R(S, M)$ be an R-derivation. Then, by 1.2.11, we obtain a morphism $id \ltimes e\colon S \to S \ltimes M$ of R-algebras augmented to S. Then, the extension-of-scalars adjunction between R- and S-algebras gives a map

$$\overline{id \ltimes e}\colon S\Box_R S \to S \ltimes M$$

of S-algebras augmented to S, where on the left we have as augmentation the multiplication. Hence, this induces a morphism $\tilde{\varphi}_e\colon I \to M$ between the augmentations ideals as power ideals. Since $S \ltimes M$ is a square-zero extension, all products and power operations vanish on the augmentation ideal M, so this factors over a morphism

$$\varphi_e\colon I/I^{\geq 2} \to M$$

of S-modules.
We now claim that these two assignments are inverse to each other. Let $e\colon S \to M$ be an R-derivation. Then, we consider the diagram

$$\begin{array}{ccc} S\Box_R S & \xrightarrow{\quad \overline{id\ltimes e} \quad} & S \ltimes M \\ {\scriptstyle \mathrm{proj}\cong\overline{id\ltimes d}} \searrow & & \nearrow {\scriptstyle id\ltimes\varphi_e} \\ & (S\Box_R S)/I^{\geq 2} \cong S \ltimes I/I^{\geq 2} & \end{array}$$

of morphisms of augmented S-algebras, where the identification at the bottom and the translation of the projection morphism follow from Lemma 1.2.22. This diagram commutes by the definition of φ_e. Thus, we obtain the equation $e = \varphi_e \circ d$ by restricting to elements of the form $1 \boxdot s$ in $S \Box_R S$. This proves that one composition of the above morphisms is the identity. Moreover, this implies for a morphism $\varphi\colon I/I^{\geq 2} \to M$ and $e = \varphi \circ d$ that $\varphi \circ d = \varphi_{\varphi \circ d} \circ d$. By Lemma 1.2.23, we can cancel precomposition with d in the category of S-modules, hence $\varphi_{\varphi \circ d} = \varphi$, which proves that the other composition is also the identity. Naturality of the morphism is clear. Hence, we have proven the theorem. □

Remark 1.2.25. Theorem 1.2.24 shows that the functor of derivations is representable. Thus, the abelian group of R-derivations on S can be described as the group of homomorphisms between two S-modules. By Proposition 1.1.8, the category of global functors is closed symmetric monoidal, and this transfers to the category of S-modules. Hence, we observe that we can upgrade the abelian group structure on $\mathrm{Der}_R(S, M)$ to a global functor of derivations. This may be done by defining

$$\underline{\mathrm{Der}}_R(S, M)(G) = \mathrm{Der}_R(S, M[G]). \tag{1.2.26}$$

This definition uses the description of the internal function object in Proposition 1.1.16 in terms of shifted modules.

This enrichment to a global-functor-valued functor of derivations later leads to a possible interpretation of S-modules as global functor objects in augmented S-algebras, as we explain in Section 1.3.b. It also emphasizes the duality between Kähler differentials and derivations, which later leads to the duality between André-Quillen homology and cohomology.

1.2.d Base Change and Transitivity

In this section, we prove the basic properties of Kähler differentials and derivations, namely a transitivity exact sequence and a base change result. These results are important for the calculation of Kähler differentials and derivations. Moreover, the existence of right and left exact sequences of Kähler differentials and derivations, respectively, raises the question of existence of derived functors, which then leads to the development of André-Quillen homology and cohomology. We start with the base change.

Proposition 1.2.27. *Let R be a global power functor, S and R' be R-algebras and $S' = R' \Box_R S$. Then, the morphism*

$$\varphi\colon R' \Box_R \Omega^1_{S/R} \cong S' \Box_S \Omega^1_{S/R} \to \Omega^1_{S'/R'}$$

of S'-modules, arising from functoriality of the Kähler differentials, is an isomorphism. Moreover, for any S'-module M, restriction along $S \to S'$ gives an isomorphism

$$\varphi^*\colon\ \mathrm{Der}_{R'}(S', M) \to \mathrm{Der}_R(S, M).$$

Proof. We start with the statement for the functor Der, the statement for Ω^1 follows from representability, since the restriction morphism is dual to φ.
We define an inverse morphism to φ^* as

$$\begin{aligned} \psi^*\colon\ \mathrm{Der}_R(S, M) &\to \mathrm{Der}_{R'}(S', M) \\ d &\mapsto (S' = R' \Box_R S \xrightarrow{R' \Box d} R' \Box_R M \xrightarrow{\mu} M), \end{aligned}$$

where $\mu\colon R' \Box_R M \to M$ is the restriction of the S'-module structure of M. We need to check that for any R-derivation $d\colon S \to M$, the morphism $\psi^*(d)$ indeed is an R'-derivation. It is a morphism of global functors, and we can verify the equations from 1.2.8 on elements of the form $r \boxdot s$ with $r \in R'(G)$ and $s \in S(G)$, since these elements generate $R'\Box_R S$. We have, for $r \boxdot s, r' \boxdot s' \in (R'\Box_R S)(G)$ and $k \geq 1$,

$$\begin{aligned}
\psi^*(d)((r \boxdot s) \cdot (r' \boxdot s')) =& \psi^*(d)((rr') \boxdot (ss')) \\
=& (rr') \cdot d(ss') = (rr') \cdot (sd(s') + s'd(s)) \\
=& (r \boxdot s) \cdot (r'd(s')) + (r' \boxdot s') \cdot (rd(s)) \\
=& (r \boxdot s) \cdot \psi^*(d)(r' \boxdot s') + (r' \boxdot s') \cdot \psi^*(d)(r \boxdot s) \\
\psi^*(d)(P^k(r \boxdot s)) =& \psi^*(d)(P^k(r) \boxdot P^k(s)) \\
=& P^k(r) \cdot \mathrm{tr}_{k-1,1}(P^{k-1}(s) \times d(s)) \\
=& \mathrm{tr}_{k-1,1}(\Phi^*_{k-1,1}(P^k(r)) \cdot (P^{k-1}(s) \times d(s))) \\
=& \mathrm{tr}_{k-1,1}((P^{k-1}(r) \boxdot P^{k-1}(s)) \times (r \cdot d(s))) \\
=& \mathrm{tr}_{k-1,1}(P^{k-1}(r \boxdot s) \times \psi^*(d)(r \boxdot s)).
\end{aligned}$$

Moreover, the composition $R' \to R'\Box_R S \xrightarrow{\psi^*(d)} M$ is the extension of scalars of the morphism $R \to S \xrightarrow{d} M$, which is trivial since d is an R-derivation. So $\psi^*(d)$ vanishes on R' and hence is an R'-derivation. Now we have to check that ψ^* is indeed inverse to φ^*.
The composition $\varphi^* \circ \psi^*$ is the morphism

$$\mathrm{Der}_R(S, M) \to \mathrm{Der}_R(S, M), d \mapsto (S \to S' = R'\Box_R S \xrightarrow{R' \cdot d} M).$$

On any element $s \in S(G)$, we see that the right derivation takes the value $s \mapsto 1 \boxdot s \mapsto 1 \cdot d(s) = d(s)$, so this composition is the identity.
The other composition is the map

$$\mathrm{Der}_{R'}(S', M) \to \mathrm{Der}_{R'}(S', M), d \mapsto (S' = R'\Box_R S \xrightarrow{R' \cdot d_{|S}} M).$$

This maps any element $r \boxdot s$ to $r \cdot d(s)$. Since d is R'-linear, we see that this agrees with $d(r \boxdot s)$. Since elements of this form generate S', we see that also this composition is the identity. This finishes the proof. □

Remark 1.2.28. The inverse morphism $\psi^* \colon \mathrm{Der}_R(S, M) \to \mathrm{Der}_{R'}(S', M)$ constructed in the proof is induced by a morphism $\psi \colon \Omega^1_{S'/R'} \to S'\Box_S \Omega^1_{S/R}$ of S'-modules. This morphism can be constructed by universality from the differential

$$\tilde{d} \coloneqq R'\Box_R d_{S/R} \colon S' \cong R'\Box_R S \to R'\Box_R \Omega^1_{S/R} \cong S'\Box_S \Omega^1_{S/R}.$$

The calculations that this is indeed a differential and that the resulting map ψ is inverse to φ can be done by similar arguments as in the given proof, but also follow formally by representability.

Next, we show that we have the transitivity exact sequences:

Proposition 1.2.29. *Let $R \to S \to T$ be a triple of global power functors and M be a T-module. Then the sequences*

$$T\Box_S \Omega^1_{S/R} \to \Omega^1_{T/R} \to \Omega^1_{T/S} \to 0$$

of T-modules and

$$0 \to \mathrm{Der}_S(T, M) \to \mathrm{Der}_R(T, M) \to \mathrm{Der}_R(S, M)$$

of abelian groups are exact, where the morphisms are the natural ones induced by the restrictions of derivations.

Proof. We again start with the statement for derivations. There, exactness is clear, since by definition, $\mathrm{Der}_S(T, M)$ is the subgroup of $\mathrm{Der}_R(T, M)$ consisting of exactly those derivations which vanish on S.
That the sequence of representing objects is exact is then a consequence of the Yoneda Lemma. □

Finally, we also describe how the Kähler differentials and derivations interact with coproducts. Note that the coproduct in the category of R-power algebras is given by the relative box product $\Box_R$ defined in (1.1.27). This can be deduced from the description of coproducts in the slice category of R-power algebras and since colimits in global power functors are calculated in the category of global Green functors. This last statement holds since global power functors are comonadic over global Green functors, as shown in [55, Section 5.2].

Proposition 1.2.30. *Let R be a global power functor and let S and T be R-algebras. Then the inclusions $S \to S\Box_R T$ and $T \to S\Box_R T$ induce an isomorphism*

$$(\Omega^1_{S/R}\Box T) \oplus (S\Box\Omega^1_{T/R}) \to \Omega_{S\Box_R T/R},$$

and dually, restriction along these inclusions induces an isomorphism

$$\mathrm{Der}_R(S\Box_R T, M) \to \mathrm{Der}_R(S, M) \oplus \mathrm{Der}_R(T, M)$$

for any $S\Box_R T$-module M.

Proof. As before, we only give the proof of the fact for derivations, the claim for the Kähler differentials follows by representability.
We need to define an inverse map $\mathrm{Der}_R(S, M) \oplus \mathrm{Der}_R(T, M) \to \mathrm{Der}_R(S\Box_R T, M)$. So let $d\colon S \to M$ and $d'\colon T \to M$ be R-derivations. Then we define $\bar{d}\colon S\Box_R T \to M$ on $S\Box_R T$ via the bimorphism

$$S(G) \otimes T(K) \to M(G \times K),\ s \boxtimes t \mapsto d(s) \times t + s \times d'(t).$$

This descends to a well-defined map on $S\Box_R T$ by R-linearity of the derivations, and it is easy to check that this is indeed a derivation. Thus, we have defined a morphism $\mathrm{Der}_R(S, M) \oplus \mathrm{Der}_R(T, M) \to \mathrm{Der}_R(S\Box_R T, M)$. Moreover, it is clearly inverse to the morphism

$$\mathrm{Der}_R(S\Box_R T, M) \to \mathrm{Der}_R(S, M) \oplus \mathrm{Der}_R(T, M)$$

defined by restriction. This finishes the proof. □

1.3 The Comparison to Beck Modules

In the classical theory of commutative rings, it is a well-known fact that the category of modules over a commutative ring R can equivalently be described as the category of abelian group objects in the category of augmented algebras over R.

Theorem 1.3.1. *Let R be a commutative ring. Then the functors*

$$\mathrm{Mod}_R \underset{\ker}{\overset{R\ltimes(\_)}{\rightleftarrows}} \mathrm{Ab}(\mathrm{Alg}_R/R)$$

are inverse equivalences.

This theorem amounts to saying that the abelian group structure in an abelian group object $T \to R$ of augmented algebras is determined by the addition on T and on the kernel all products vanish.

This result is used by Quillen to define his cohomology theory for commutative rings in [53]. It goes back to Beck's work on cohomology theories for triples[1] [13], where he considers modules over commutative rings in Example 8 and gives the argument why the products on the kernel vanish for associative algebras in Example 6. After his influential work, abelian group objects in various algebraic categories are also called "Beck modules". The fact that the abelian group structure is determined by the addition on T is an application of the Eckmann-Hilton argument and can be found in [22], where also the notion of (abelian) group objects in general categories is introduced. The argument for vanishing products on the kernel is also spelled out by Strickland in [62, after Proposition 14.7] in the context of Tambara functors.

However, in our situation of modules over global power functors, the square-zero extension is no longer an equivalence between modules and Beck modules. In fact, the abelian group structure is not sufficient to conclude that all power operations on the augmentation ideal vanish. We give an example of this in Section 1.3.a. Based on the work of Hill [30], we propose an alternative identification of modules over a global power functor as "global Beck modules", ie global functor objects in the category of augmented R-algebras. The details of this conjecture are explained in Section 1.3.b.

We also shortly comment on the topological version of this identification, which is integral to the understanding of topological André-Quillen homology of commutative ring spectra. In this context, Basterra and Mandell have shown in [11] that the category of R-modules for a commutative ring spectrum R can be identified with the category of spectra of augmented R-algebras. This result follows the slogan that in stable homotopy theory, the correct analogue of an abelian group is a spectrum, and abelianization should be replaced by stabilization. The result of Basterra-Mandell shows that topological André-Quillen homology and cohomology indeed arise from a stabilization procedure.
In the context of ultra-commutative ring spectra, which we study in Chapter 2, we should follow this slogan and replace the global functor objects from Conjecture 1.3.21 by a stable homotopy theoretic notion. This leads to studying "global stabilizations", constructed by a model category of global spectra over a base model category. Then, we aim at comparing R-modules for an ultra-commutative ring spectrum R with global spectra over augmented R-algebras. This line of study is currently investigated in joint work with Tobias Lenz.

1.3.a Abelian Group Objects that are not Modules

In the case of Tambara functors for a fixed group, there is an example by Strickland [62, Example 14.13, 7.8] of an abelian group object in the category of augmented C_2-Tambara algebras over the constant Tambara functor $\underline{\mathbb{Z}}$ that is not a square-zero extension of a $\underline{\mathbb{Z}}$-module. Indeed, there are non-trivial norms on elements in the augmentation ideal. A priori, it is not clear whether a similar example of an abelian group object with power operations can exist in the context of global power functors, because the presence of the additional inflation maps could prevent this. However, it turns out that Strickland's example can (after a minor adjustment to accommodate for inflations) be generalized to a global functor, and we supply an explicit description for its value at finite groups. In fact, we demonstrate that this example can be obtained from a free example of an abelian group object in augmented global power algebras.

We first recall a modified version of Strickland's example from [62, Example 14.13, 7.8].

[1] Here, triples is used synonymously to the word monad

Example 1.3.2. We define an algebra T over the constant C_2-Tambara functor $\underline{\mathbb{Z}}$ via the following Lewis diagram:

$$\begin{array}{cc} T(C_2) = & \mathbb{Z}[\beta,\gamma]/(\beta^2, \beta\gamma, \gamma^2, 2\gamma) \\ & {\scriptstyle \substack{1\mapsto 2 \\ \alpha\mapsto 2\beta}} \left(\nearrow \quad {\scriptstyle \substack{1\mapsto 1 \\ \alpha\mapsto \beta}} \uparrow \,\Big|\, \downarrow {\scriptstyle \substack{1\mapsto 1 \\ \beta\mapsto\alpha \\ \gamma\mapsto 0}} \quad \nwarrow \right) {\scriptstyle i+j\alpha\mapsto i^2+2ij\beta+j^2\gamma} \\ T(e) = & \mathbb{Z}[\alpha]/(\alpha^2) \end{array}$$

Here, the downwards arrow is the restriction, the leftmost arrow is the transfer and the rightmost arrow is the norm. We have also included the inflation along the surjection $C_2 \to e$, which is present in our generalization to a global power algebra, as the straight upwards arrow. We emphasize that in comparison to Strickland's original example, we shifted one coefficient 2 from the restriction map to the transfer. This is necessary since the presence of the inflation forces the restriction to be surjective. For me, the above placement of the 2 is more natural, since this way the sub-Mackey functor on the elements α and β is itself a constant Mackey functor. This change does not influence the relevant calculations to check that T is a C_2-Tambara functor.
We observe that the maps $\underline{\mathbb{Z}} \to T$ and $T \to \underline{\mathbb{Z}}$ given by including as the summand generated on 1 and projecting to this summand, respectively, give T the structure of an augmented $\underline{\mathbb{Z}}$-algebra. Moreover, as explained in [62, Example 14.13], T is an abelian group object in the category of augmented algebras. Indeed, the augmentation ideal is generated by the elements α, β and γ, on which all products vanish. However, the norm of α is γ and thus not trivial.

We now generalize this example to the global setting. This implies the existence of such examples as G-Tambara functors for all finite G since any global power functor can be restricted to give G-Tambara functors for all G. To carry out the generalization, we observe that the augmentation ideal of T above is essentially a "free $\underline{\mathbb{Z}}$-module with norms" on the generator α, or in the language introduced by Strickland, a free genuine $\underline{\mathbb{Z}}$-module. We first consider a free global functor with power operations, which we model by a naive version of a module of Kähler differentials over $\mathbb{A}$. The generalization of Strickland's example is then obtained by taking the box product with the constant global power functor $\underline{\mathbb{Z}}$. We start by motivating the construction.

As we have seen in Proposition 1.2.15, the module of Kähler differentials of a polynomial global power functor is a free module on a single generator. Moreover, by Lemma 1.2.22, we have a nice description of the square-zero extension $R[x] \ltimes \Omega^1_{R[x]/R} \cong (R[x]\Box_R R[x])/I^{\geq 2}$ for this free module. In this description, we take the quotient by $I^{\geq 2}$ instead of by the naive square I^2 exactly to enforce that all powers on the module of Kähler differentials are trivial. Hence, in order to obtain a free "naive" module on a single generator and allow for norms on the augmentation ideal, we should instead take the naive module of Kähler differentials by taking the quotient by I^2. We carry this out over $\mathbb{A}$:

Construction 1.3.3. We consider the augmented algebra $\mathbb{A}[x_G] \xrightarrow{x_G \mapsto 0} \mathbb{A}$, a polynomial algebra over $\mathbb{A}$ on a single generator at the compact Lie group G. In this algebra, we consider the (Green) ideal I^2 generated by elements of the form $P^k(x_G) \times P^l(x_G)$ for $k, l \geq 1$. In fact, this ideal even is a power ideal in $\mathbb{A}[x_G]$, and it is the square of the kernel $I = \ker(\mathbb{A}[x] \to \mathbb{A})$. We consider the quotient $\widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\} = \mathbb{A}[x_G]/I^2$. This inherits the structure of a global power functor from $\mathbb{A}[x_G]$, and comes equipped with the structure of an augmented $\mathbb{A}$-algebra by sending x_G to 0.

The augmentation ideal of this augmented algebra is isomorphic to I/I^2, and it is generated as an $\mathbb{A}$-module by the elements $P^k(x_G)$ for $k \geq 1$. In particular, the power operations on this augmentation ideal are non-trivial. We denote this augmentation ideal by $\mathbb{A}^{\mathrm{pow}}\{x_G\}$. However, since all products in this augmentation ideal vanish, we see that $\widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\}$ is an abelian group object in the category of augmented $\mathbb{A}$-algebras.

As indicated previously, we can interpret this $\mathbb{A}$-module as a "naive module of Kähler differentials" of $\mathbb{A}[x_G]$ over $\mathbb{A}$, base-changed to $\mathbb{A}$. Here, in contrast to the calculation of the actual, genuine Kähler differentials from Theorem 1.2.24, we only take the quotient by actual decomposables, omitting the power operations of degree ≥ 2 on single elements.

Proposition 1.3.4. *The augmented $\mathbb{A}$-algebra $\widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\}$ is an abelian group object in the category of augmented $\mathbb{A}$-algebra. It is free as an abelian group object, ie for any abelian group object T in augmented $\mathbb{A}$-algebras, the morphism*

$$\mathrm{ev}_x\colon\ \mathrm{Hom}_{\mathrm{Ab}(\mathrm{Alg}_{\mathbb{A}}/\mathbb{A})}(\widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\}, T) \to \ker(T \to \mathbb{A})(G),\ \varphi \mapsto \varphi_G(x)$$

is an isomorphism of abelian groups.
For this free abelian group object, the augmentation ideal $\mathbb{A}^{\mathrm{pow}}\{x_G\}$ has non-trivial norms.

In the description of freeness, we used that an abelian group object is in fact totally determined by its augmentation ideal.

Proof. The augmented algebra $\widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\}$ is an abelian group object since all products on the augmentation ideal vanish. Explicitly, we can check that the morphisms

$$(\widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\} \times_{\mathbb{A}} \widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\})(K) \to \widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\}(K),$$
$$\left(a_0 + \sum_{k\geq 0} a_k P^k(x), a_0 + \sum_{k\geq 0} b_k P^k(x)\right) \mapsto a_0 + \sum_{k\geq 0}(a_k + b_k)P^k(x)$$

and the inclusion

$$\mathbb{A} \to \widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\}$$

as the constant summand define the structure of an abelian group object on $\widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\}$. Here, $a_k, b_k \in \mathbb{A}(\Sigma_k \wr G, K)$ are the Burnside-ring coefficients of a polynomial in $\mathbb{A}[x_G]$. It is clear that on the augmentation ideal $\mathbb{A}^{\mathrm{pow}}\{x_G\}$, non-trivial power operations exist, as $P^k(x) \notin I^2$ for $k \geq 2$. Hence, it remains to show freeness of this construction.
By freeness of the polynomial global power functors (1.1.23), we know that

$$\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{A}}}(\mathbb{A}[x_G], T) \cong T(G),\ \varphi \mapsto \varphi(x)$$

is a bijection. A morphism $\varphi\colon \mathbb{A}[x_G] \to T$ is a morphism of augmented global power algebras if and only if $\varphi(x) \in \ker(T \to \mathbb{A})$, hence we obtain

$$\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{A}}/\mathbb{A}}(\mathbb{A}[x_G], T) \cong \ker(T \to \mathbb{A})(G),\ \varphi \mapsto \varphi(x).$$

As T is an abelian group object, all products on this augmentation ideal vanish, and hence we observe that $I^2 \subset \mathbb{A}[x_G]$ is sent to 0 in T. Thus any such φ factors uniquely over $\mathbb{A}[x_G]/I^2 = \widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\}$. Finally, by the Eckmann-Hilton argument, the abelian group structures on $\widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\}$ and T are completely determined by the addition, and hence any morphism of augmented algebras is automatically a morphism of abelian group objects. In total, we obtain the isomorphism

$$\mathrm{ev}_x\colon\ \mathrm{Hom}_{\mathrm{Ab}(\mathrm{Alg}_{\mathbb{A}}/\mathbb{A})}(\widetilde{\mathbb{A}}^{\mathrm{pow}}\{x_G\}, T) \to \ker(T \to \mathbb{A})(G),\ \varphi \mapsto \varphi_G(x). \qquad \square$$

Corollary 1.3.5. *The functor*

$$\mathbb{A} \ltimes (\_)\colon \mathcal{GF} \cong \mathrm{Mod}_{\mathbb{A}} \to \mathrm{Ab}(\mathrm{Alg}_{\mathbb{A}}/\mathbb{A})$$

is not essentially surjective.

We obtain a generalization of Strickland's example by forming the base change $\underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\} := \underline{\mathbb{Z}}\Box\mathbb{A}^{\mathrm{pow}}\{x_e\}$. We provide explicit descriptions of $\mathbb{A}^{\mathrm{pow}}\{x\}$ and $\underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}$ on finite groups, which highlight the connection to Strickland's example. To formulate this calculation, we introduce the following notation:

Notation 1.3.6. Let $m \geq 1$ and G be a finite group. Then we define the set

$$\mathcal{G}_m(G) = \{(H, X) \mid H \leq G,\ X \text{ a transitive } H\text{-set of cardinality } m\}/\sim.$$

The equivalence relation $\sim$ is conjugacy, ie two pairs (H, X) and (H', X') are equivalent if there is $g \in G$ such that $H' = gHg^{-1}$, and there is a bijection $f\colon X \xrightarrow{\cong} X'$ such that $hx = ghg^{-1}f(x)$ for all $h \in H$ and $x \in X$. We also define $\mathcal{G}(G) = \coprod_{m\geq 1} \mathcal{G}_m(G)$.
On this set, we define a partial ordering by setting $(H, X) \leq (L, Y)$ if H is conjugate to a subgroup $H' \leq L$ and the restriction of Y to H' is equivalent to the H-set X as above. Then, we define $\mathcal{M}_m(G) \subset \mathcal{G}_m(G)$ and $\mathcal{M}(G) \subset \mathcal{G}(G)$ as the sets of all maximal elements.

Proposition 1.3.7. *Let G be a finite group.*

i) We have that

$$\mathbb{A}^{\mathrm{pow}}\{x_e\}(G) \cong \bigoplus_{\mathcal{G}(G)} \mathbb{Z} = \mathbb{Z}\{\mathrm{tr}_H^G X \mid [(H, X)] \in \mathcal{G}(G)\}$$

is a free abelian group on transfers of generators indexed by conjugacy classes of pairs (H, X), where $H \leq G$ is a subgroup of G and X is a transitive H-set.

ii) There is a surjection

$$\bigoplus_{\mathcal{M}_1(G)} \mathbb{Z} \oplus \bigoplus_{\substack{p \text{ a prime} \\ k\geq 1}} \bigoplus_{\mathcal{M}_{p^k}(G)} \mathbb{Z}/p \twoheadrightarrow \underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}(G).$$

Explicitly, we have a set of generators given by conjugacy classes of pairs (H, X), with H a subgroup of G and X a transitive H set with cardinality a prime power, which are maximal with respect to the order defined in Notation 1.3.6. Moreover, $\mathcal{M}_1(G) = \{[(G, *)]\}$ has only one element.

Proof. We start by analysing $\mathbb{A}^{\mathrm{pow}}\{x_e\}$:

i) We know that $\mathbb{A}^{\mathrm{pow}}\{x_e\} = I/I^2$, where I is the kernel of the augmentation $\mathbb{A}[x] \to \mathbb{A}$. Thus, $I(G) = \bigoplus_{m\geq 1} \mathbb{A}(\Sigma_m, G)$. By the description of the Burnside category in Remark 1.1.2, we see that $\mathbb{A}(\Sigma_m, G)$ is free abelian on generators of the form $\mathrm{tr}_H^G \circ \alpha^*$, where $H \leq G$ is a subgroup of G and $\alpha\colon H \to \Sigma_m$ is a group homomorphism. Such a group homomorphism is equivalent to an H-set X of cardinality m, and hence we denote this generator by $\mathrm{tr}_H^G(X)$. The conjugacy relation in $\mathbb{A}(\Sigma_m, G)$ is the same as the one used to define $\mathcal{G}_m(G)$.
We now claim that $I^2(G)$ is the submodule generated by exactly those $\mathrm{tr}_H^G(X)$ where X is not transitive. This proves the desired description of $\mathbb{A}^{\mathrm{pow}}\{x_e\}$. For this, we consider the product on I. This is given on components by the map

$$\mathbb{A}(\Sigma_m, G) \otimes \mathbb{A}(\Sigma_n, K) \xrightarrow{\times} \mathbb{A}(\Sigma_m \times \Sigma_n, G \times K) \xrightarrow{\mathbb{A}(\Phi^*_{m,n}, G\times K)} \mathbb{A}(\Sigma_{m+n}, G \times K)$$

that sends $\operatorname{tr}_H^G(X) \otimes \operatorname{tr}_L^K(Y)$ to $\operatorname{tr}_{H\times L}^{G\times K}(X \amalg Y)$. Here, $H \times L$ acts on $X \amalg Y$ by letting H act as given on X and trivially on Y, and conversely for L. With this description, we see that if the H-set X is not transitive, using a decomposition $X = X_1 \amalg X_2$ as H-sets, we can write $\operatorname{tr}_H^G(X) = \operatorname{tr}_H^G(\operatorname{tr}_H^H(X_1) \cdot \operatorname{tr}_H^H(X_2)) \in I^2(G)$. Here, we used the diagonal product $I(H) \otimes I(H) \to I(H)$, which arises from the product above by applying a restriction along the diagonal. Hence, any generator $\operatorname{tr}_H^G(X)$ with X not transitive is contained in I^2.
Conversely, I^2 is generated as a global functor by elements $\operatorname{tr}_{H\times L}^{G\times K}(X \amalg Y)$. We observe that $X \amalg Y$ is a non-transitive $G \times H$-set, as both X and Y are not empty. It is thus sufficient to show that for a non-transitive H-set X and any homomorphism $\alpha\colon K \to G$, also $\alpha^* \operatorname{tr}_H^G(X)$ is a sum of elements of the form $\operatorname{tr}_L^K(Y)$, where Y is a non-transitive L-set. This follows easily from the double coset formula, since any restriction of a non-transitive set is still not transitive. We omit the exact calculation here.
In total, we have shown $I^2(G) \cong \mathbb{Z}\{\operatorname{tr}_H^G X \mid H \leq G,\ X \text{ a non-transitive } H\text{-set}\}/\!\sim$, and hence we conclude

$$\mathbb{A}^{\mathrm{pow}}\{x_e\}(G) \cong \mathbb{Z}\{\operatorname{tr}_H^G X \mid H \leq G,\ X \text{ a non-empty transitive } H\text{-set}\}/\!\sim\ \cong \bigoplus_{\mathcal{G}(G)} \mathbb{Z}.$$

ii) Observe that the morphism $\mathbb{A} \to \underline{\mathbb{Z}}$, $X \to |X|$ is a quotient map with kernel generated by $\operatorname{tr}_H^G(\operatorname{res}_H^G) - [G : H]$ for all pairs $H \leq G$. Since the box product is right exact, we hence may calculate $\underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}(G)$ as a quotient of $\mathbb{A}^{\mathrm{pow}}\{x_e\}(G)$. We obtain a surjection

$$\bigoplus_{\mathcal{G}(G)} \mathbb{Z} \twoheadrightarrow \underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}(G).$$

Thus, it suffices to show that we may restrict to maximal generators, that generators indexed by a set of non-prime-power order vanish in the quotient, and that generators indexed by sets of prime power order have corresponding prime torsion.
Let $[(H, X)] \leq [(K, Y)]$ be two elements in $\mathcal{G}_m(G)$. Then there is $g \in G$ such that ${}^gH = gHg^{-1} \leq K$. We calculate that in $\underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}(G)$, the relation

$$\operatorname{tr}_H^G(X) = \operatorname{tr}_K^G(\operatorname{tr}_{{}^gH}^K(\operatorname{res}_{{}^gH}^K(Y))) = [K : {}^gH] \operatorname{tr}_K^G(Y)$$

holds. Thus, we may reduce to $\mathcal{M}_m(G)$ as a generating set.
For the other two relations, we consider that for a given $m \geq 1$, any action on a set X with m elements arises by pulling back along a morphism $\alpha\colon H \to \Sigma_m$. Thus, we consider the Σ_m-set X. We already know that restriction to any non-transitive subgroup of Σ_m annihilates X in $\mathbb{A}^{\mathrm{pow}}\{x_e\}$. Any non-transitive subgroup is subconjugate to one of the form $\Sigma_i \times \Sigma_{m-i}$ with $1 \leq i \leq m-1$. We obtain that

$$\binom{m}{i} X = [\Sigma_m : \Sigma_i \times \Sigma_{m-i}] X = \operatorname{tr}_{\Sigma_i\times\Sigma_{m-i}}^{\Sigma_m}(\operatorname{res}_{\Sigma_i\times\Sigma_{m-i}}^{\Sigma_m}(X)) = 0.$$

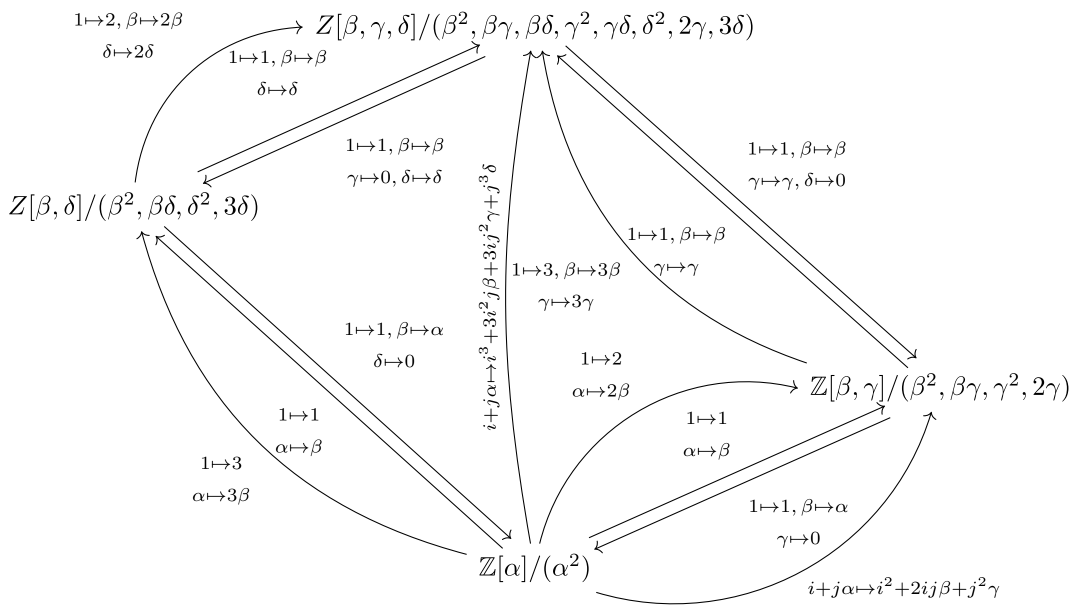
Thus, X is torsion also for the greatest common divisor of these indices. It is well-known that for binomial coefficients, we have

$$\gcd([\Sigma_m : \Sigma_{m-i} \times \Sigma_i] \mid 1 \leq i \leq m-1) = \begin{cases} 0 & \text{if } m = 1 \\ p & \text{if } m = p^k \\ 1 & \text{else.} \end{cases}$$

For a reference, see [54]. This shows both that we can reduce to generators of cardinality $m = p^k$ for p a prime and that these generators are p-torsion. This finishes the calculations. □

Remark 1.3.8. We warn the reader that the usage of H-sets in the description above should not be confused with the usage of G-sets in the classical description of the Burnside ring $\mathbb{A}(G) = \mathbb{A}(e, G)$, which is the Grothendieck ring of finite G-sets. In this description, the disjoint union of G-sets corresponds to addition, and a set of generators may be given by considering transitive G-sets, which are of the form G/H and correspond to tr_H^G in our description of the Burnside category from Remark 1.1.2. Hence, in this description, we use the fact that any G-set decomposes into a disjoint union of orbits in order to describe the free abelian group. Here, the product is given by the Cartesian product of G-sets, equipped with either the diagonal action for the diagonal product $\cdot\colon \mathbb{A}(G)\otimes\mathbb{A}(G) \to \mathbb{A}(G)$, or with the $G\times K$-action for the product $\times\colon \mathbb{A}(G)\otimes\mathbb{A}(K) \to \mathbb{A}(G\times K)$. The power operation is given by the m-th Cartesian power with the natural action of $\Sigma_m \wr G$ by acting on each factor and permuting the factors.
By contrast, in the description of I and I/I^2 in Proposition 1.3.7 *i)* above, H-sets are purely used as a means to conveniently index the generators of the free abelian group and describe the indecomposables. The sum is not given by disjoint union, but as the formal sum in a free abelian group. Also the transfer and restriction are calculated by formal manipulation. The product is induced by disjoint union, and thus also the power operations are defined by iterated disjoint unions. Hence, care should be taken when working with this description.

Example 1.3.9. We observe that restricted to subgroups of the cyclic group C_2, the surjection above is an isomorphism, and this description of $\underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}$ recovers the example given by Strickland, as recalled in Example 1.3.2 (up to the adjusted occurrence of one 2). We also give the structure of the global power functor $\underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}$ on subgroups of Σ_3. These subgroups are Σ_3 (top), C_3 generated by a 3-cycle (left), C_2 generated by a 2-cycle (right) and the trivial subgroup e (bottom).



Here, the downward-pointing morphisms are restrictions, and the straight upward-pointing morphisms are inflations. The linear curved morphisms are transfers, and the non-linear morphisms are the power operations. On $\underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}(C_3)$, we also have an automorphism given by restricting along the (outer) automorphism of C_3 given by squaring all elements. This morphism is the identity and omitted from the diagram. Moreover, there are various compositions of the given transfers and restrictions, which are not depicted for the sake of readability. One such composition of particular interest is $\mathrm{res}^{\Sigma_3}_{C_3} \circ P^3 \colon \underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}(e) \to \underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}(C_3)$, which is the norm $N_e^{C_3}$. One calculates that $N_e^{C^3}(\alpha) = \delta$ is non-trivial.

For these small groups, we actually only see actual transitive G-sets themselves at all levels, and all transfers from lower levels are multiples of generators named by transitive G-sets. Framed differently, all maximal elements in $\mathcal{M}(G)$ are of the form $[(G, X)]$. The necessity to include transfers of transitive sets for proper subgroups becomes apparent eg when considering A_5. Here, we observe that the alternating group A_5 cannot act transitively on a set with 2 elements, since it is simple. However, there is a transitive set with 2 elements for the Klein 4-group $K_4 \leq A_5$, and thus we obtain an element named $\mathrm{tr}^{A_5}_{K_4}(X)$ in $\underline{\mathbb{Z}}^{\mathrm{pow}}\{x_e\}(A_5)$.

One benefit of lifting the example from $\underline{\mathbb{Z}}$ to an example over $\mathbb{A}$ is that at each level G, $\mathbb{A}^{\mathrm{pow}}\{x_K\}(G)$ is torsion-free. Thus, we also obtain that the phenomenon that abelian group objects and modules differ persists after rationalization. Explicitly, we obtain an example over $\mathbb{A} \otimes \mathbb{Q}$, where the tensor product is taken levelwise. This is a global power functor by [55, Example 5.2.19]. Such an example for rational Tambara functors was not known previously, as Strickland's example only supports a norm that is 2-torsion. Also in our global context, it is not yet clear whether there exists an abelian group object in augmented $\underline{\mathbb{Q}}$-algebras with non-trivial power operations.

Corollary 1.3.10. *For the rational global power functors $R = \mathbb{A} \otimes \mathbb{Q}$, the functor*

$$R \ltimes (\_) \colon \mathrm{Mod}_R \to \mathrm{Ab}(\mathrm{Alg}_R / R)$$

is not essentially surjective.

1.3.b Modules as Global Functor Objects

In Section 1.3.a, we exhibit an example of an abelian group object in the category of augmented $\mathbb{A}$-power algebras that does not arise from a module via square-zero extension. This shows that the mismatch between modules and Beck modules in the context of equivariant algebra for a fixed finite group G exhibited by Strickland in [62] persists in the case of global power functors, and our example clarifies some aspects left open by Strickland's example. In the case of modules over a G-Tambara functor R, subsequent work of Hill [30] gives a new perspective on this discrepancy: Instead of being abelian group objects in augmented R-algebras, R-modules can be interpreted as Mackey functor objects in this category. In this fashion, the notion of a Beck module is also suitably generalized to a G-equivariant world.

This interpretation of modules as Mackey functor objects needs some preparations, which are carried out in [30, Section 3]. In particular, it is necessary to define what a Mackey functor object should be. For this, it is convenient to recall a possible definition of abelian group objects: For a category $\mathcal{C}$, the structure of an abelian group object in $\mathcal{C}$ on X is a lift of the functor

$$\mathrm{Hom}_{\mathcal{C}}(\_, X) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$$

through the forgetful functor $\mathrm{Ab} \to \mathrm{Sets}$. In a category with finite products, this is equivalent to the description by the usual maps and diagrams of an abelian group. Hill showed that the category of augmented G-Tambara algebras over R comes with an enrichment in coefficient systems of sets, ie for any subgroup H of G, there is an associated morphism set, and we have restrictions between these. The structure of a Mackey functor object thus is a lift of

$$\mathrm{Hom}(\_, X)\colon (\mathrm{Alg}_R/R)^{\mathrm{op}} \to \mathrm{CoefficientSystems}$$

through the forgetful functor from Mackey functors to coefficient systems. In [30, Theorem 3.22, Corollary 3.23], Hill proved that with this notion of Mackey functor objects, the square-zero-extension functor is indeed an equivalence between R-modules and Mackey functor objects in augmented R-algebras.

One of the ingredients in this interpretation of square-zero extensions as Mackey functor objects is that, in fact, the functor $\mathrm{Hom}_{\mathrm{Alg}_R/R}(\_, R \ltimes M)\colon \mathrm{Alg}_R/R^{\mathrm{op}} \to \mathrm{Sets}$ of morphisms into a square-zero extension can be upgraded to a Mackey functor. This works using the natural bijection

$$\mathrm{Hom}_{\mathrm{Alg}_R/R}(S, R \ltimes M) \cong \mathrm{Der}_R(S, M) \cong \mathrm{Hom}_S(\Omega^1_{S/R}, M)$$

and the closed monoidal structure on S-modules. Thus, the square-zero extension $R \ltimes M$ is indeed a Mackey functor object. We observe that similarly, in the context of global power functors, $\mathrm{Hom}_{\mathrm{Alg}_R/R}(S, R \ltimes M)$ can be upgraded to a global functor. Hence, we consider whether it is possible to interpret R-modules as global functor objects in augmented R-algebras.

As mentioned before, a first step towards this interpretation is to show that the category of augmented R-power algebras is in fact enriched over global coefficient systems. Recall from Proposition 1.1.16 that the internal Hom functor for global functors can be described using shifts of global functors. We consider the same construction in the context of global power functors and algebras.

Construction 1.3.11. Let R be a global power functor, and G be a compact Lie group. We endow the global functor $R[G]$ with the structure of a global power functor as follows:
First, we define the structure of a global Green functor on $R[G]$. For this, we notice that $R[G](H) = R(H \times G)$ has the structure of a commutative ring, since R is a global Green functor. Moreover, we define power operations on $R[G]$ as follows: Let H be a compact Lie group and let $m \geq 1$. Then we define

$$P[G]^m\colon R[G](H) = R(H \times G) \xrightarrow{P^m} R(\Sigma_m \wr (H \times G)) \xrightarrow{(\Delta_G^m)^*} R((\Sigma_m \wr H) \times G) = R[G](\Sigma_m \wr H).$$

Here, $\Delta_G^m\colon (\Sigma_m \wr H) \times G \to \Sigma_m \wr (H \times G)$ is the diagonal on G, sending $((\sigma; h_1, \ldots, h_m), g)$ to $(\sigma; (h_1, g), \ldots, (h_m, g))$.

Proposition 1.3.12. *Let R be a global power functor and G be a compact Lie group. Then $R[G]$ with the multiplication and power operations described in 1.3.11 is a global power functor. Moreover, the external product*

$$\times\colon R[G](H) \times R[G](K) \to R[G](H \times K)$$

is obtained as the composite

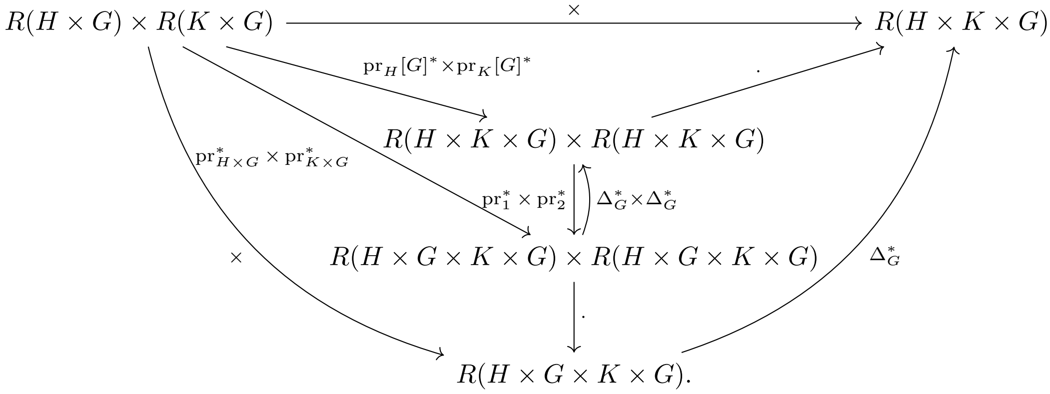
$$R(H \times G) \times R(K \times G) \xrightarrow{\times} R(H \times G \times K \times G) \xrightarrow{\Delta_G^*} R(H \times K \times G)$$

of the external product of R and the restriction along the diagonal of G.

Proof. We first check that the multiplications on $R[G](H) = R(H \times G)$ make $R[G]$ into a global Green functor. For this, notice that for a group homomorphism $\varphi\colon K \to H$, the morphism

$$\varphi[G]^* = (\varphi \times G)^*\colon R[G](H) = R(H \times G) \to R(K \times G) = R[G](K)$$

is a homomorphism of commutative rings. Also, the transfer $\mathrm{tr}[G]_L^H = \mathrm{tr}_{L\times G}^{H\times G}$ for a closed subgroup $L \subset H$ satisfies Frobenius reciprocity for $\mathrm{res}_{L\times G}^{H\times G} = \mathrm{res}[G]_L^H$. Thus, $R[G]$ has the structure of a global Green functor induced from R.
Next, we check the description of the multiplication $\times$ on $R[G]$. For this, consider the diagram



In this diagram, we used the description of $\times$ in the terms of $\cdot$, that the compositions of projections is a projection, and that the restriction along the projections is a section for the restriction along the diagonal. Thus, this diagram commutes and we have proven the claimed description of the external multiplication.
For the power operations, we need to check the properties from Definition 1.1.18:

i) We calculate $P[G]^m(1) = (\Delta_G^m)^* P^m(1) = 1$.

ii) Since $\Delta_G^1 \colon (\Sigma_1 \wr H) \times G \to \Sigma_1 \wr (H \times G)$ is the identity after identifying both sides with $H \times G$, we see $P[G]^1 = \mathrm{Id}$.

iii) Let $\alpha \colon K \to H$ be a homomorphism of compact Lie groups. Then we have

$$\begin{aligned} P[G]^m \alpha[G]^* =& (\Delta_G^m)^* P^m (\alpha \times G)^* = (\Delta_G^m)^* (\Sigma_m \wr (\alpha \times G))^* P^m \\ =& ((\Sigma_m \wr \alpha) \times G)^* (\Delta_G^m)^* P^m = (\Sigma_m \wr \alpha)[G]^* P[G]^m. \end{aligned}$$

iv) Let $x, y \in R[G](H)$. Then

$$P[G]^m(x \cdot y) = (\Delta_G^m)^* P^m(x \cdot y) = (\Delta_G^m)^* P^m(x) \cdot (\Delta_G^m)^* P^m(y) = P[G]^m(x) \cdot P[G]^m(y).$$

v) Let $x \in R[G](H)$ and $i, j \geq 1$. Then

$$\begin{aligned} \Phi_{i,j}[G]^* P[G]^{i+j}(x) =& (\Phi_{i,j} \times G)^* (\Delta_G^m)^* P^{i+j}(x) \\ =& \Delta_G^* (\Delta_G^i \times \Delta_G^j)^* \Phi_{i,j}^* P^{i+j}(x) \\ =& \Delta_G^* ((\Delta_G^i)^* P^i(x) \times (\Delta_G^j)^* P^j(x)) = P[G]^i(x) \times P[G]^j(x). \end{aligned}$$

vi) Let $x \in R[G](H)$ and $k, m \geq 1$. Then

$$\begin{aligned} \Psi_{k,m}[G]^* P[G]^{km}(x) =& (\Psi_{k,m} \times G)^* (\Delta_G^{km})^* P^{km}(x) \\ =& (\Delta_G^k)^* (\Sigma_k \wr \Delta_G^m)^* \Psi_{k,m}^* P^{km}(x) \\ =& (\Delta_G^k)^* (\Sigma_k \wr \Delta_G^m)^* P^k(P^m(x)) \\ =& (\Delta_G^k)^* P^k((\Delta_G^m)^* P^m(x)) = P[G]^k(P[G]^m(x)). \end{aligned}$$

vii) Let $x, y \in R[G](H)$ and $m \geq 1$. Then

$$\begin{aligned} P[G]^m(x+y) = (\Delta_G^m)^* P^m(x+y) &= \sum_{k=0}^{m} (\Delta_G^m)^* \operatorname{tr}_{k,m-k}(P^k(x) \times P^{m-k}(y)) \\ &= \sum_{k=0}^{m} \operatorname{tr}_{k,m-k}[G] \Delta_G^* (\Delta_G^k \times \Delta_G^{m-k})^* (P^k(x) \times P^{m-k}(y)) \\ &= \sum_{k=0}^{m} \operatorname{tr}_{k,m-k}[G] (P[G]^k(x) \times P[G]^{m-k}(y)). \end{aligned}$$

Here, we used that the double coset formula for the composition $(\Delta_G^m)^* \operatorname{tr}_{k,m-k}$ has only one summand, and the intersection of the corresponding subgroups is $(\Sigma_k \wr H) \times (\Sigma_{m-k} \wr H) \times G$.

viii) Let $L \subset H$ be a closed subgroup and $m \geq 1$. Then we calculate

$$\begin{aligned} P[G]^m \operatorname{tr}_L^H[G] &= (\Delta_G^m)^* P^m \operatorname{tr}_{L \times G}^{H \times G} \\ &= (\Delta_G^m)^* \operatorname{tr}_{\Sigma_m \wr (L \times G)}^{\Sigma_m \wr (H \times G)} P^m \\ &= \operatorname{tr}_{(\Sigma_m \wr L) \times G}^{(\Sigma_m \wr H) \times G} (\Delta_G^m)^* P^m = \operatorname{tr}_{\Sigma_m \wr L}^{\Sigma_m \wr H}[G] P[G]^m. \end{aligned}$$

Thus $R[G]$ is a global power functor. □

We also consider whether the morphisms $R[\vartheta] \colon R[G] \to R[K]$ for $\vartheta \in \mathbb{A}(G, K)$ are morphisms of global power functors. This is the case for restrictions.

Proposition 1.3.13. *Let R be a global power functor. Let $\varphi \colon K \to G$ be a homomorphism of compact Lie groups. Then the restriction*

$$R[\varphi^*] \colon R[G] \to R[K]$$

is a morphism of global power functors.

Proof. We already know that $R[\varphi^*]$ is a morphism of global functors. Hence it suffices to prove that it is compatible with the multiplication and the power operations. For the multiplication, we see that $R[\varphi^*](H) \colon R[G](H) \to R[K](H)$ is given as $(H \times \varphi)^*$, and this is a morphism of commutative rings, since R is a global Green functor.
For the power operations, we calculate

$$\begin{aligned} R[\varphi^*] P[G]^m &= ((\Sigma_m \wr H) \times \varphi)^* (\Delta_G^m)^* P^m \\ &= (\Delta_K^m)^* (\Sigma_m \wr (H \times \varphi))^* P^m = (\Delta_K^m)^* P^m (H \times \varphi)^* = P[K]^m R[\varphi^*]. \end{aligned}$$

Thus $R[\varphi^*] \colon R[G] \to R[K]$ is a morphism of global power functors. □

Remark 1.3.14. We could also try to consider transfers $R[\operatorname{tr}_L^G] \colon R[L] \to R[G]$. However, these will in general not be morphisms of global power functors. One reason is that transfers are not homomorphisms of rings in a global Green functor, but only satisfy Frobenius reciprocity. Moreover, in the definition of the power operations, there is the diagonal restriction $(\Delta_G^m)^*$. To commute this with the transfer, we have to consider the double coset formula, and hence calculate the double coset space $(\Sigma_m \wr H) \times G \backslash \Sigma_m \wr (H \times G) / \Sigma_m \wr (H \times L)$. This, however, does not only consist of a single double coset, hence in general we cannot commute these restrictions and transfers without introducing additional summands. Thus, the transfers do not in general define morphisms of global power functors.

Construction 1.3.15. In the definition of derivations and Kähler differentials, we consider not only global power functors, but also global power algebras and augmented power algebras for a global power functor R and an R-algebra S. We generalize the above definition of $R[G]$ to these contexts.
For a global power functor R, the inflation $R[p_G^*]\colon R = R[e] \to R[G]$ along the unique morphism $p_G\colon G \to e$ makes $R[G]$ into an R-power algebra. Using this and functoriality of the construction $(\_)[G]$, we see that for a global R-power algebra S, also $S[G]$ is a global R-power algebra via the unit map $R \xrightarrow{R[p_G^*]} R[G] \to S[G]$.
Moreover, if we have given an R-power algebra S and an R-power algebra T augmented to S, then we define $F_S(G,T)$ as the pullback

$$\begin{array}{ccc} F_S(G,T) & \longrightarrow & T[G] \\ \downarrow & & \downarrow \\ S & \xrightarrow[S[p_G^*]]{} & S[G]. \end{array}$$

This is then also a global R-power algebra augmented to S. Also, the restrictions give morphisms

$$F_S(\varphi^*, T)\colon F_S(G,T) \to F_S(K,T)$$

of global R-power algebras augmented to S by functoriality, using Proposition 1.3.13.

We call the structure given in the above definition a global coefficient system.

Definition 1.3.16. A global coefficient system F (of sets) is a functor $F\colon \mathrm{Rep}^{\mathrm{op}} \to \mathrm{Sets}$, where Rep is the category of compact Lie groups and conjugacy classes of continuous homomorphisms between those. Explicitly, a global coefficient system consists of sets $F(G)$ for all isomorphism classes of Lie groups and restrictions $\varphi^*\colon F(G) \to F(K)$ for any continuous homomorphism $\varphi\colon K \to G$, such that restrictions are functorial and conjugated morphisms induce the same restrictions.
Generally, for a category $\mathcal{C}$, a global coefficient system in $\mathcal{C}$ is a functor $F\colon \mathrm{Rep}^{\mathrm{op}} \to \mathcal{C}$. We denote the category of global coefficient systems in $\mathcal{C}$ as $\mathrm{Coeff}(\mathcal{C}) = \mathrm{Fun}(\mathrm{Rep}^{\mathrm{op}}, \mathcal{C})$.

There is a forgetful functor $\mathcal{GF} \to \mathrm{Coeff}(\mathrm{Sets})$ by forgetting the abelian group structure on $F(G)$ and all transfers for a global functor F.
With this definition, we can reformulate the above results.

Proposition 1.3.17. *Let S be a global power functor. Then for any augmented S-power algebra T, the collection of augmented S-power algebras $F_S(G,T)$ for compact Lie groups G, together with the restrictions $F_S(\varphi^*,T)$, defines a global coefficient system in Alg_S/S.*
The category Alg_S/S is enriched over global coefficient systems by defining

$$\underline{\mathrm{Hom}}_{\mathrm{Alg}_S/S}(T,T')(G) = \mathrm{Hom}_{\mathrm{Alg}_S/S}(T, F_S(G,T')).$$

However, we know that the square-zero extensions provide even more structure, given by transfers and abelian group object structures at every compact Lie group. We formalize this structure as follows.

Definition 1.3.18. Let $\mathcal{C}$ be a category enriched in global coefficient systems of sets. A global functor object in $\mathcal{C}$ is an object X of $\mathcal{C}$ together with a lift of the functor $\underline{\mathrm{Hom}}(\_,X)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Coeff}(\mathrm{Sets})$ through the forgetful functor $\mathcal{GF} \to \mathrm{Coeff}(\mathrm{Sets})$. We denote the category of global functor objects by $\mathcal{GF}(\mathcal{C})$.

Using the isomorphism

$$\mathrm{Hom}_{\mathrm{Alg}_S/S}(T, S \ltimes M) \cong \mathrm{Der}_S(T, M) \cong \mathrm{Hom}_T(\Omega^1_{T/S}, M)$$

for a global power functor S, an augmented S-algebra T and an S-module M, we consider the lift

$$\underline{\mathrm{Hom}}_{\mathrm{Alg}_S/S}(T, S \ltimes M) = \underline{\mathrm{Der}}_S(T, M) \cong \underline{\mathrm{Hom}}_T(\Omega^1_{T/S}, M).$$

This proves that any S-module defines a global functor object.

Proposition 1.3.19. *Let S be a global power functor and M be an S-module. Then $S \ltimes M$ has the structure of a global functor object in Alg_S/S.*

We can also make the structure of the global functor object $S \ltimes M$ more explicit by identifying $(S \ltimes M)(G) = S \ltimes (M[G])$, and for $\theta \in \mathbb{A}(G, K)$ considering $S \ltimes (M[\theta])\colon S \ltimes (M[G]) \to S \ltimes (M[K])$. This uses the following result:

Lemma 1.3.20. *Let S be a global power functor and M be an S-module. Then there is an isomorphism*

$$F_S(G, S \ltimes M) \cong S \ltimes (M[G])$$

of augmented S-power algebras, natural in restrictions $\varphi\colon K \to G$.

Proof. The definition of $(\_)[G]$ is additive, thus we see $(S \ltimes M)[G] = S[G] \oplus M[G]$ as global functors. Moreover, also as global power functors, we have $(S \ltimes M)[G] = S[G] \ltimes M[G]$. To check that the power operations agree, we use the same double coset formula as for additivity of the power operations $P[G]^m$.
We consider the diagram

$$\begin{array}{ccc} S \ltimes (M[G]) & \xrightarrow{S[p_G^* \ltimes id]} & S[G] \ltimes M[G] \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon[G]} \\ S & \xrightarrow[S[p_G^*]]{} & S[G]. \end{array}$$

This is easily seen to be a pull-back diagram of S-algebras, and thus we have the isomorphism $F_S(G, S \ltimes M) \cong S \ltimes (M[G])$. □

Now, we can formulate an interpretation of modules as global functor objects.

Conjecture 1.3.21. *Let S be a global power functor. Then the square-zero extension functor*

$$S \ltimes \_\colon \mathrm{Mod}_S \to \mathcal{GF}(\mathrm{Alg}_S/S)$$

is an equivalence.

We comment here on the proof of the corresponding statement in the context of G-Tambara functors for a finite group G given by Hill in [30, Corollary 3.23]. This proof consists of two steps: showing that the transfers on a Mackey functor object are already determined by the transfers of the underlying augmented algebra [30, Lemma 3.20], and showing that the norms on the augmentation ideal vanish [30, Theorem 3.22]. We reformulate his proof in a diagrammatic way, which highlights the similarity in handling both parts of the argument. Then, we highlight the situation in the case of global power functors, and how it compares to the G-equivariant case.

Remark 1.3.22. The proof in [30, Corollary 3.23] of the G-Tambara version of Conjecture 1.3.21 proceeds as follows (adapted to our notation): Suppose we are given a G-Mackey functor object T in the category of augmented R-algebras, where R is a G-Tambara functor. We need to show that T is isomorphic to a square-zero extension $R \ltimes M$ for an R-module M, as a G-Mackey functor object. Choosing $M = \ker(T \to R)$, we obtain an isomorphism $T \cong R \ltimes M$ as Green functors by the analogous arguments to the classical case. Hence, we only need to check that the norms on M vanish, and that the "external" transfers, ie the transfers of the structure as a Mackey functor object, agree with the given transfers of M. For both of these, it suffices to consider a subgroup inclusion $H \leq G$ to transfer or norm along.

For both of these statements, we consider the diagrams

$$\begin{array}{ccc} M(G/H \times G/H) = M[G/H](G/H) & \xrightarrow{\operatorname{tr}_H^{G,\mathrm{ext}}(G/H)} & M(G/H \times G/G) = M[G/G](G/H) \\ {\scriptstyle \operatorname{tr}_H^G[G/H]}\downarrow\downarrow{\scriptstyle N_H^G[G/H]} & & {\scriptstyle \operatorname{tr}_H^G[G/G]}\downarrow\downarrow{\scriptstyle N_H^G[G/G]} \\ M(G/G \times G/H) = M[G/H](G/G) & \xrightarrow{\operatorname{tr}_H^{G,\mathrm{ext}}(G/G)} & M(G/G \times G/G) = M[G/G](G/G). \end{array} \tag{1.3.23}$$

The two diagrams, one for the transfers and one for the norms, both commute, as by assumption T is a Mackey functor object, so the external transfers are morphisms of G-Tambara functors. Here, we used the shifts $M[X] = M(\_ \times X)$ for M a G-Mackey (or Tambara) functor and X a finite G-set, as considered in [30, Proposition 3.9] similarly to our Proposition 1.3.12.

The main observation here is that $M(G/H \times G/H)$ decomposes as $\bigoplus_{[g]\in H\backslash G/H} M(G/(H \cap gHg^{-1}))$ by the double coset formula. In this decomposition, a special role is played by the diagonal $\Delta \in G/H \times G/H$, which is a G-orbit isomorphic to G/H and thus splits off this decomposition.

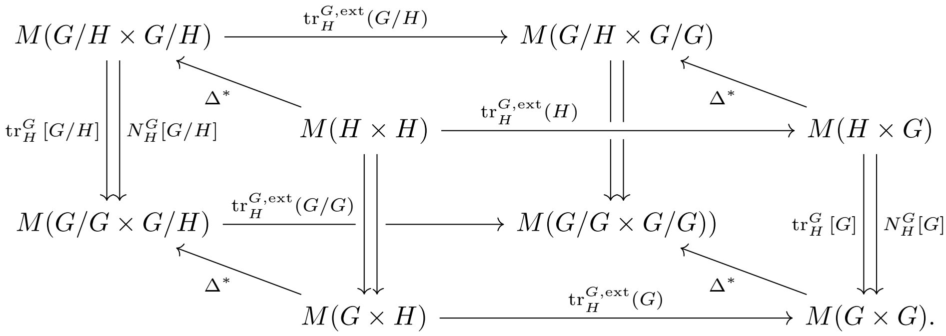
We now analyse these two diagrams, and start with the one for the transfer. Let $x \in M(G/H) \cong M[G/H](G/G)$. Then, the element $(x, 0) \in M(G/H) \oplus \bigoplus_{[e]\neq[g]\in H\backslash G/H} M(G/(H \cap gHg^{-1}))$ is a preimage of x under the transfer $\operatorname{tr}_H^G[G/H]$. We then observe that $\operatorname{tr}_H^{G,\mathrm{ext}}(G/H)(x, 0) = x \in M[G/G](G/H) \cong M(G/H)$. From this and the commutativity of the diagram (1.3.23), we observe that transfer and external transfer agree.
Similarly for the norm, we consider an element $x \in M(G/H) \cong M[G/G](G/H)$. Then, the element $(x, 0) \in M(G/H) \oplus \bigoplus_{[e]\neq[g]\in H\backslash G/H} M(G/(H \cap gHg^{-1}))$ is a preimage of x under the external transfer $\operatorname{tr}_H^{G,\mathrm{ext}}(G/H)$, since this agrees with the usual transfer. But on this element, the norm $N_H^G[G/H]$ vanishes. Hence, also $N_H^G\colon M(G/H) \to M(G/G)$ is trivial.

For the global version, suppose now that T is a global functor object in augmented R-power algebras, where R is a global power functor. Then, for any pair of (finite) groups $H \leq G$, we have the commutative diagrams

$$\begin{array}{ccc} M(H \times H) = M[H](H) & \xrightarrow{\operatorname{tr}_H^{G,\mathrm{ext}}(H)} & M(H \times G) = M[G](H) \\ {\scriptstyle \operatorname{tr}_H^G[H]}\downarrow\downarrow{\scriptstyle N_H^G[H]} & & {\scriptstyle \operatorname{tr}_H^G[G]}\downarrow\downarrow{\scriptstyle N_H^G[G]} \\ M(G \times H) = M[H](G) & \xrightarrow{\operatorname{tr}_H^{G,\mathrm{ext}}(G)} & M(G \times G) = M[G](G). \end{array} \tag{1.3.24}$$

In this, we cannot use a decomposition of $M(H \times H)$ as before. Hence, it is not clear how to adapt the arguments of Hill to our situation. One possible approach is to compare this diagram (1.3.24) to (1.3.23) by means of a diagonal restriction. This uses the interpretation of global functors as functors on spans of finite groupoids, see for example [24]. Then, for any subgroups $K, L \leq G$ and a global functor M, we have a diagonal morphism $\Delta\colon G/K \times G/L \to (G \times G)/(K \times L)$, where the left is the action groupoid of a G-set and the right the action groupoid of a $G \times G$-set.

Hence we can consider the restriction $\Delta^*\colon M(K\times L) = M((G\times G)/(K\times L)) \to M(G/K\times G/L)$. Using these diagonal restrictions, we get the following cube-shaped comparison diagrams



(1.3.25)

In this diagram, the front face commutes, since by assumption T is a global functor object. All side faces involving Δ^* commute by various double coset formulas. However, it is not clear whether the back face, which is a copy of (1.3.23), commutes. If this is the case, then Conjecture 1.3.21 would be proved.

1.4 Extensions of Global Power Algebras

In Section 1.2, we introduce the notions of square-zero extensions, derivations and Kähler differentials. In the classical algebra of commutative rings as introduced by Grothendieck [27], these notions are a starting point to study infinitesimal properties in the framework of algebraic geometry. One introduces infinitesimal extensions of algebras as (non-split) extensions with trivial multiplication on the kernel of the augmentation, and formal smoothness of algebras by testing against such extensions. This allows one to study geometric properties of varieties.

In the context of André-Quillen cohomology, introducing extensions of commutative algebras has another important benefit. This notion can be used to extend the exact sequence of derivations from Proposition 1.2.29 to a six-term sequence, thus providing the first step to a derived functor of derivations. This six-term sequence was later extended to a nine-term sequence by Lichtenbaum and Schlessinger [45] by studying two-term extensions, and finally to a long exact sequence in André-Quillen cohomology by André [1] and Quillen [53] by simplicial methods.

In this section, we introduce infinitesimal extensions of global power algebras in Definition 1.4.1. Collecting all such extensions in an algebraic structure allows us to extend the exact sequence of derivations to a six-term sequence in Theorem 1.4.5. We calculate these extensions for some special cases, and use this to calculate the Kähler differentials for quotients of free global power functors in Example 1.4.11. In Section 1.4.b, we give the definition of formally smooth, unramified and étale global power algebras and show that these notions are characterised by derivations and extensions of global power algebras. Moreover, we show that in the case of formally smooth algebras, the transitivity exact sequences from Proposition 1.2.29 are actually short exact.

1.4.a Infinitesimal Extensions of Global Power Algebras

We have seen in Proposition 1.2.29 that the functor $\mathrm{Der}\_(\_, M)\colon \mathrm{Alg} \to \mathrm{Ab}$ takes triples of global power functors to left exact sequences of abelian groups. It is a classical result that goes back to Grothendieck [27, Théorème 20.2.2] that this exact sequence can be extended by considering extensions of commutative algebras. In fact, maps into such extensions are closely connected to derivations. For the special case of the trivial extension $S \ltimes M$, this reduces to Theorem 1.2.11. We now generalize the theory of extensions of commutative algebras to global power functors.

Definition 1.4.1. Let R be a global power functor, S be an R-algebra and M be an S-module. Then an (infinitesimal) extension of S by M is an exact sequence

$$0 \to M \xrightarrow{i} E \xrightarrow{p} S \to 0$$

of R-modules where E is an R-algebra, p is a map of R-algebras, i is a map of E-modules $p^*(M) \to E$ and on the image of i in E, all products and power operations P^m for $m \geq 2$ vanish. An isomorphism of two extensions $0 \to M \xrightarrow{i} E \xrightarrow{p} S \to 0$ and $0 \to M \xrightarrow{i'} E' \xrightarrow{p'} S \to 0$ is an isomorphism $f\colon E \to E'$ of R-algebras such that the diagram

$$\begin{array}{ccccccc} & & & E & & & \\ & & {\scriptstyle i}\nearrow & & \searrow{\scriptstyle p} & & \\ 0 \longrightarrow & M & & {\scriptstyle f}\downarrow\simeq & & S & \longrightarrow 0 \\ & & {\scriptstyle i'}\searrow & & \nearrow{\scriptstyle p'} & & \\ & & & E' & & & \end{array}$$

commutes. We denote by $\mathrm{Exalcomm}_R(S, M)$ the set of isomorphism classes of extensions of S by M.

Remark 1.4.2. The set $\mathrm{Exalcomm}_R(S, M)$ is endowed in the usual way with the structure of an abelian group using the Baer sum: Given two extensions $0 \to M \to E \to S \to 0$ and $0 \to M \to E' \to S \to 0$, we form

$$0 \to M \to E \times_S E'/((m, -m) \text{ for all } m \in M) \to S \to 0.$$

Here, $E \times_S E'$ is the product in the category of R-power algebras augmented to S. The quotient inherits the power operations, since the power operations on M vanish and hence $((m, -m) \mid m \in M)$ is a global power ideal.
The neutral element in this group $\mathrm{Exalcomm}_R(S, M)$ is given by the trivial square-zero extension $S \ltimes M$ defined in Construction 1.2.1. An extension is equivalent to this trivial extension if and only if there exists an R-algebra section $s\colon S \to E$ to the projection $p\colon E \to S$.
Moreover, Exalcomm is functorial both in the R-algebra S (contravariantly) and in the S-module M (covariantly). The functoriality in the R-algebra is given by the pull-back, ie for a morphism $f\colon S \to T$ of R-power algebras and an extension $M \to E \to T$, we form the new extension as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & S \times_T E & \longrightarrow & S & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0. \end{array}$$

This indeed is an R-power algebra extension of S by M. On the other hand, given a morphism $g\colon M \to N$ of S-modules and an extension $M \to E \to S$, we form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\quad i \quad} & E & \longrightarrow & S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & E \oplus_M N := (E \ltimes N)/((i(m), -g(m)) \text{ for all } m \in M) & \longrightarrow & S & \longrightarrow & 0. \end{array}$$

Here, N is considered as an E-module via the augmentation $E \to S$, and we need to explain why $((i(m), -g(m))$ for all $m \in M)$ is a global power ideal in $E \ltimes N$. It is an ideal by an easy computation explained for example in [27, 18.2.8], hence we only need to consider the power operations. For the power operation $P^1 = id$, there is nothing to prove. For P^2, we calculate

$$\begin{aligned} P^2(i(m), -g(m)) &= (P^2(i(m)), \mathrm{tr}_{1,1}(P^1(i(m)) \times (-g(m)))) \\ &= (0, -\mathrm{tr}_{1,1}(i(m) \times g(m))) \\ &= 0, \end{aligned}$$

using that $p(i(m)) = 0$ and the E-module structure on N is induced from the surjection p. Finally, for all higher power operations, we calculate

$$P^k(i(m), -g(m)) = (P^k(i(m)), \mathrm{tr}_{k-1,1}(P^{k-1}(i(m)) \times (-g(m)))) = 0.$$

Thus, $E \oplus_M N$ is indeed a global power algebra. We alternatively also denote it by $g_* E$.

Using these constructions, we are able to factor any morphism of extensions with possibly different modules and bases. The following lemma the global analogue of [27, 18.2.8] in the case of commutative algebras.

Lemma 1.4.3. *Let R be a global power functor and S and T be global R-power algebras. Moreover let M be an S-module and N be a T-module. Suppose we are given the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & S & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle G} & & \big\downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & T & \longrightarrow & 0, \end{array}$$

in which the rows are extensions of the global R-power algebras S and T by M and N, respectively, γ is a morphism of R-power algebras, g is a morphism of S-modules (using the S-module structure on N induced by γ) and G is a morphism of R-power algebras.

i) Then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & S & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle G_1} & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & g_*E & \longrightarrow & S & \longrightarrow & 0 \\ & & \| & & \big\downarrow{\scriptstyle G_2} & & \big\downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & T & \longrightarrow & 0, \end{array}$$

where the top row of morphisms arises from the construction of functoriality of Exalcomm*, and in the bottom row of morphisms, G_2 is a morphism of R-power algebras.*

ii) Then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & S & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle \tilde{G}_1} & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & F \times_T S & \longrightarrow & S & \longrightarrow & 0 \\ & & \| & & \big\downarrow{\scriptstyle \tilde{G}_2} & & \big\downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & T & \longrightarrow & 0, \end{array}$$

where the bottom row of morphisms arises from the construction of functoriality of Exalcomm*, and in the top row of morphisms, $\tilde{G}_1$ is a morphism of R-power algebras.*

Proof. We only prove the first assertion. The second is similar but easier since $F \times_T S$ is the product in R-power algebras augmented to T.
Recall that $g_*E = E \oplus_M N$, hence we obtain a morphism $G_2 \colon g_*E \to F$ of R-modules, and this makes the diagram commute. Hence, we are left to prove that G_2 is a morphism of global power algebras.
Let (e, n), $(e', n') \in (g_*E)(G)$ for a compact Lie group G, and $k \geq 1$. Then we calculate

$$\begin{aligned} G_2((e,n)\cdot(e',n')) &= G_2(ee', en' + e'n) \\ &= (G(ee'), en' + e'n) = (G(e)G(e'), G(e)n' + G(e')n) \\ &= G_2(e,n)\cdot G_2(e',n') \end{aligned}$$

Here, we used that both the E- and the F-module structure on N are induced from the T-module structure via the maps to T, and thus $en = G(e)n$. Similarly, we calculate

$$\begin{aligned} G_2(P^k(e,n)) &= G_2(P^k(e), \mathrm{tr}_{k-1,1}(P^{k-1}(e)\times n)) \\ &= (P^k(G(e)), \mathrm{tr}_{k-1}(P^{k-1}(G(e))\times n)) \\ &= P^k(G_2(e,n)). \end{aligned}$$

Thus, G_2 is indeed a morphism of R-power algebras. □

Maps to global power algebra extensions are intimately tied to derivations. The following lemma is a generalization of Theorem 1.2.11, where the special case of the trivial extension $S \ltimes M$ is considered.

Lemma 1.4.4. *Let R be a global power functor, S be an R-algebra, M an S-module and*

$$0 \to M \to E \to S \to 0$$

be an extension of S by M. Moreover, let T be an R-algebra augmented to S. We consider M as a T-module by pulling back the S-module structure along the augmentation of T. Then $\mathrm{Hom}_{\mathrm{Alg}_R/S}(T,E)$ *is a torsor over* $\mathrm{Der}_R(T,M)$*. Concretely, given any morphism $\varphi \colon T \to E$ of R-algebras augmented to S, there is a bijection*

$$\begin{aligned} \alpha_\varphi \colon \mathrm{Der}_R(T,M) &\to \mathrm{Hom}_{\mathrm{Alg}_R/S}(T,E) \\ d &\mapsto \varphi + d. \end{aligned}$$

Proof. Since the corresponding statement is classical in the case of commutative algebras and derivations of such (see eg [27, Corollaire 20.1.3]), we focus on the compatibility with power operations here.
Let $\varphi \colon T \to E$ be any morphism of R-algebras augmented to S, and let $d \colon T \to M$ be an R-derivation. We first verify that $\varphi + d$ is a morphism of algebras. For this, let $t \in T(G)$ be an element of T. Then we calculate

$$\begin{aligned} (\varphi+d)(P^m(t)) &= \varphi(P^m(t)) + d(P^m(t)) = P^m(\varphi(t)) + \mathrm{tr}_{m-1,1}(P^{m-1}(t)\times d(t)) \\ &= P^m(\varphi(t)) + \mathrm{tr}_{m-1,1}(P^{m-1}(\varphi(t))\times d(t)) = P^m(\varphi(t)+d(t)). \end{aligned}$$

Here, to get to the last line, we used that the T-module structure on M is via the augmentation of T. Since the S-module structure on M is compatible with that induced from multiplication in E by being square-zero and since φ is a morphism of augmented algebras, we observe that the T-module structure can be described by $t \times m = \varphi(t) \times m$. Moreover, in the last equation we used additivity of the power operations and that power operations on M vanish.

Let now $\psi\colon T \to E$ be another morphism of augmented algebras. We need to check that $d := \psi - \varphi$ is a derivation $T \to M$. Again, we restrict to proving that the corresponding property for the power operations is satisfied.
Let $t \in T(G)$. We calculate

$$\begin{aligned} d(P^m(t)) &= \psi(P^m(t)) - \varphi(P^m(t)) = P^m(\psi(t)) - P^m(\varphi(t)) = P^m(\varphi(t) + d) - P^m(\varphi(t)) \\ &= \sum_{k=1}^{m} \mathrm{tr}_{m-k,k}(P^{m-k}(\varphi(t)) \times P^k(d(t))) = \mathrm{tr}_{m-1,1}(P^{m-1}(\varphi(t)) \times d(t)). \end{aligned}$$

Here, the last equation again uses that all power operations on M vanish. Hence d is a derivation. Since the constructions are inverse to one another, this finishes the proof. □

Using this result, we can show that these groups of global power algebra extensions extend the exact sequence of derivations from Proposition 1.2.29 to the right. This is a global power functor version of [27, Théorème 20.2.2].

Theorem 1.4.5. *Let $R \xrightarrow{f} S \xrightarrow{g} T$ be morphisms of global power functors and M be a T-module. Then the sequence*

$$\begin{aligned} 0 &\to \mathrm{Der}_S(T,M) \to \mathrm{Der}_R(T,M) \to \mathrm{Der}_R(S,M) \to \\ &\xrightarrow{\delta} \mathrm{Exalcomm}_S(T,M) \to \mathrm{Exalcomm}_R(T,M) \to \mathrm{Exalcomm}_R(S,M) \end{aligned}$$

of abelian groups is exact. Here, the map $\delta\colon \mathrm{Der}_R(S,M) \to \mathrm{Exalcomm}_S(T,M)$ associates to an R-derivation $d\colon S \to M$ the class of the extension $0 \to M \to T \ltimes M \to T \to 0$, where we consider $T \ltimes M$ as an S-algebra by the morphism $(g,d)\colon S \to T \ltimes M$. The last two maps are restriction maps induced on Exalcomm *by functoriality.*

Proof. We already proved in Proposition 1.2.29 that the beginning of this sequence is exact. We first examine the map $\delta\colon \mathrm{Der}_R(S,M) \to \mathrm{Exalcomm}_S(T,M)$. For any derivation d, the map $(g,d)\colon S \to T \ltimes M$ indeed defines the structure of an S-algebra extension of T by M on $T \ltimes M$ by Lemma 1.4.4. This extension is a trivial R-algebra extension, since d vanishes on R. It is also S-trivial if and only if there is an S-linear section $T \to T \ltimes M$ of the projection to T. Any such map is of the form (id, e), where $e\colon T \to M$ is an R-linear derivation by Lemma 1.4.4. For the map (id, e) to be an S-algebra map, we need

$$(g(s), e(g(s))) = (g(s), d(s)).$$

Hence we obtain that $d = e \circ g$, so that d lies in the image of the restriction map $\mathrm{Der}_R(T,M) \to \mathrm{Der}_R(S,M)$. Thus we have exactness at $\mathrm{Der}_R(S,M)$.
Let now $M \to E \xrightarrow{p} T$ be an S-algebra extension that is R-trivial. Hence, there is an R-linear isomorphism to the extension $M \to T \ltimes M \to T$. This isomorphism takes the form $(p,d)\colon E \to T \ltimes M$, where $d\colon E \to M$ is an R-derivation. We now consider on $T \ltimes M$ the S-algebra structure defined by $(g, d \circ \eta)$, where $\eta\colon S \to E$ is the unit of the S-algebra E. This is the extension $\delta(d \circ \eta)$. It is obvious that with this structure, the morphism (p,d) is even S-linear, since $p\eta = g$. This shows that E is equivalent to $\delta(d \circ \eta)$.
Finally, let $M \to E \to T$ be an R-algebra extension such that the restricted extension $M \to E \times_T S \to S$ is R-trivial. This means there is a section $S \to E \times_T S$. After projection to E,

this yields a morphism $S \to E$ of R-algebras augmented to T, and thus lifts $M \to E \to T$ to an extension of S-algebras. Conversely, if $M \to E \to T$ is an S-algebra extension, then $M \to E \times_T S \to S$ is a trivial extension, since we have the section $(id, \eta)\colon S \to S \times_T E$. □

Remark 1.4.6. In Remark 1.2.25, we define a lift of the functor of derivations to take values in global functors. This can be extended to the functor of global power algebra extensions, by setting

$$\underline{\mathrm{Exalcomm}}_R(S, M)(G) = \mathrm{Exalcomm}_R(S, M[G]).$$

With this definition, we directly observe that the above Theorem 1.4.5 also provides a six-term exact sequence of global functors by applying it to the modules $M[G]$ for compact Lie groups G.

In general, calculating the groups of algebra extensions might be complicated. If we consider a surjective morphism of global power functors $R \to S$, however, we can identify all extensions in terms of module homomorphisms. This approach can be extended by introducing the naive cotangent complex. We study this in connection with the first André-Quillen cohomology group in Section 1.5.c.

Construction 1.4.7. Let $R \to S$ be a surjective morphism of global power functors with kernel I. Then $I/I^{\geq 2}$ inherits the structure of an S-module since $S \cong R/I$, and we may consider the canonical extension

$$0 \to I/I^{\geq 2} \to R/I^{\geq 2} \to S \to 0.$$

For any S-module M, we compare $\mathrm{Hom}_S(I/I^{\geq 2}, M)$ and $\mathrm{Exalcomm}_R(S, M)$.

To any S-module homomorphism $f\colon I/I^{\geq 2} \to M$, we associate the extension given by functoriality of Exalcomm via

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I/I^{\geq 2} & \longrightarrow & R/I^{\geq 2} & \longrightarrow & S & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \Vert & & \\
0 & \longrightarrow & M & \longrightarrow & (R/I^{\geq 2}) \oplus_{I/I^{\geq 2}} M & \longrightarrow & S & \longrightarrow & 0.
\end{array}$$

Conversely, for a given extension $0 \to M \to E \to S \to 0$, we consider the composition $I \hookrightarrow R \to E$. Since $I = \ker(R \to S)$, we see that this morphism factors over $M = \ker(E \to S)$. Since on M, all products and power operations vanish, this then induces a morphism $I/I^{\geq 2} \to M$. As the map $I \to M \subset E$ is by construction R-linear and all products vanish, this morphism is indeed a morphism of S-modules.

Remark 1.4.8. Via this construction, we are able to interpret the extension

$$0 \to I/I^{\geq 2} \to R/I^{\geq 2} \to R/I \to 0$$

as a universal extension over R/I. This point of view is sometimes useful, for example in the comparison of André-Quillen cohomology with the cohomology of the naive cotangent complex in Section 1.5.c.

In the classical literature, this approach also can be used to identify the second André-Quillen cohomology group with a group of two-term extensions, as shown by Lichtenbaum-Schlessinger [45, Theorem 4.1.2]. There, a universal such two-term extension is identified. We discuss that this approach does not work in our context when discussing identifications of the low-dimensional André-Quillen cohomology groups in Remark 1.5.15.

Proposition 1.4.9. *Let $R \to S$ be a surjective morphism of global power functors with kernel I, and let M be an S-module. Then the assignments defined in Construction 1.4.7 yield inverse isomorphisms*

$$\mathrm{Hom}_S(I/I^{\geq 2}, M) \underset{\longleftarrow}{\overset{\cong}{\longrightarrow}} \mathrm{Exalcomm}_R(S, M).$$

These isomorphisms are natural in surjections $R \to S$ and modules M.

Proof. We study the two compositions of the described morphisms. The composition

$$\mathrm{Hom}_S(I/I^{\geq 2}, M) \to \mathrm{Exalcomm}_R(S, M) \to \mathrm{Hom}_S(I/I^{\geq 2}, M)$$

is the identity since in the defining diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I/I^{\geq 2} & \longrightarrow & R/I^{\geq 2} & \longrightarrow & S & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle f} & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & M & \longrightarrow & (R/I^{\geq 2}) \oplus_{I/I^{\geq 2}} M & \longrightarrow & S & \longrightarrow & 0,
\end{array}$$

the left square commutes. This exactly describes the procedure of factoring $R \to R/I^{\geq 2} \oplus_{I/I^{\geq 2}} M$ through a map $I/I^{\geq 2} \to M$.
Conversely, for any extension $0 \to M \to E \to S$, we obtain a morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & (R/I^{\geq 2}) \oplus_{I/I^{\geq 2}} M & \longrightarrow & S & \longrightarrow & 0 \\
 & & \big\| & & \big\downarrow{\scriptstyle \varphi} & & \big\| & & \\
0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & S & \longrightarrow & 0
\end{array}$$

of extensions. This arises from the defining commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I/I^{\geq 2} & \longrightarrow & R/I^{\geq 2} & \longrightarrow & S & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & S & \longrightarrow & 0
\end{array}$$

by Lemma 1.4.3. By the 5-Lemma, we obtain an isomorphism of extensions. Thus, we see that also the other composition is the identity.

Naturality of these morphisms in the module is straight-forward. We now exhibit the naturality with respect to surjections of global power functors. For this let

$$\begin{array}{ccccc}
I & \dashrightarrow & R & \twoheadrightarrow & S \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
I' & \dashrightarrow & R' & \twoheadrightarrow & S'
\end{array}$$

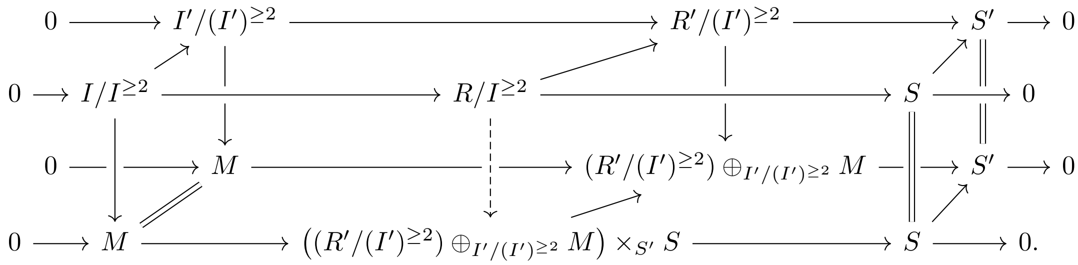
be a commutative diagram of global power functors, with kernels indicated by the dashed arrows, and M be an S'-module. Let moreover $f\colon I'/(I')^{\geq 2} \to M$ be a morphism of S'-modules. Then, we consider the two extensions of S by M defined by the diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I/I^{\geq 2} & \longrightarrow & R/I^{\geq 2} & \longrightarrow & S & \longrightarrow & 0 \\
 & & \big\downarrow & & & & & & \\
 & & I'/(I')^{\geq 2} & & \big\downarrow & & \big\| & & \\
 & & \big\downarrow{\scriptstyle f} & & & & & & \\
0 & \longrightarrow & M & \longrightarrow & (R/I^{\geq 2}) \oplus_{I/I^{\geq 2}} M & \longrightarrow & S & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I'/(I')^{\geq 2} & \longrightarrow & R'/(I')^{\geq 2} & \longrightarrow & S' & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & (R'/(I')^{\geq 2}) \oplus_{I'/(I')^{\geq 2}} M & \longrightarrow & S' & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M & \longrightarrow & \big((R'/(I')^{\geq 2}) \oplus_{I'/(I')^{\geq 2}} M\big) \times_{S'} S & \longrightarrow & S & \longrightarrow & 0.
\end{array}$$

We then observe that the maps $R \to R'$ and $R \to S$ induce the dashed map as indicated in the cube-shaped commutative diagram



But the dashed map as indicated induces an isomorphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & (R/I^{\geq 2}) \oplus_{I/I^{\geq 2}} M & \longrightarrow & S & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & \big((R'/(I')^{\geq 2}) \oplus_{I'/(I')^{\geq 2}} M\big) \times_{S'} S & \longrightarrow & S & \longrightarrow & 0
\end{array}$$

by Lemma 1.4.3. This proves naturality of the isomorphism $\mathrm{Hom}_S(I/I^{\geq 2}, M) \to \mathrm{Exalcomm}_R(S, M)$. □

Combining this with the six-term exact sequence from Theorem 1.4.5, we obtain the following exact sequence, which classically is known under the name "conormal sequence". It extends the exact sequence of Kähler differentials from Proposition 1.2.29 by one term on the left in the case that one morphism of global power functors is a surjection.

Proposition 1.4.10. *Let $R \to S \to T$ be a triple of global power functors where $S \to T$ is surjective with kernel I. Then the sequence*

$$I/I^{\geq 2} \to T\square_S\Omega^1_{S/R} \to \Omega^1_{T/R} \to 0$$

of T-modules is exact. Here, the first morphism arises by composing the universal derivation $d\colon S \to \Omega^1_{S/R}$ with the unit morphism $\Omega^1_{S/R} \to T\square_S\Omega^1_{S/R}$ of the extension-of-scalars adjunction, restricting to I and passing to the quotient $I/I^{\geq 2}$.

Proof. We observing that $\Omega^1_{T/S} = 0$ as explained in Proposition 1.2.16. Moreover, for any T-module M, we identify $\mathrm{Exalcomm}_S(T, M)$ as $\mathrm{Hom}_T(I/I^{\geq 2}, M)$ by Proposition 1.4.9. Then using the exact sequence in Theorem 1.4.5 for derivations into M and extensions by M, we obtain the left exact sequence

$$0 \to \mathrm{Der}_R(T, M) \to \mathrm{Der}_R(S, M) \to \mathrm{Exalcomm}_S(T, M).$$

The right exact sequence of Kähler differentials can thus be detected by mapping out of it by applying $\mathrm{Hom}_T(\_, M)$ for arbitrary T-modules M. □

In the same fashion, we also obtain other three-term sequences when multiple of the morphisms $R \to S \to T$ are surjective.

We can use the above exact sequence for calculations of Kähler differentials on quotients of polynomial algebras.

Example 1.4.11. Let R be a global power functor, and $S = R[x_{i,K_i}]$ be a polynomial algebra over R on generators x_i at level K_i, for $i \in I$. Let moreover $f_j \in S(L_j)$ be polynomials, indexed by $j \in J$. We denote by (f_j) the global power ideal generated by these elements, and consider the global power functor $T = S/(f_j)$.

We calculate the Kähler differentials $\Omega^1_{T/R}$ by means of the exact sequence from Proposition 1.4.10. We know by Proposition 1.2.15 that $\Omega^1_{S/R} \cong S\{dx_{i,K_i}\}$ is a free S-module, and thus also $T\Box_S\Omega^1_{S/R} \cong T\{dx_{i,K_i}\}$ is free. Moreover, the image of $I/I^{\geq 2} \to T\{dx_{i,K_i}\}$ is generated by the elements df_j. Hence, in total we obtain

$$\Omega^1_{(R[x_i]/(f_j))/R} \cong (R[x_i]/(f_j))\{dx_i\}/(df_j).$$

1.4.b Characterizing Smooth, Unramified and Étale Global Power Algebras

In classical commutative algebra, the notion of commutative algebra extensions can be used to define the classes of formally smooth, unramified and étale algebras. We extend these definitions to the context of global power functors.

Definition 1.4.12. Let $R \to S$ be a morphism of global power functors. For any R-algebra extension $M \to E \to B$, we consider the lifting problem

$$\begin{array}{ccc} & & M \\ & & \downarrow \\ R & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ S & \longrightarrow & B \end{array} \qquad (1.4.13)$$

given by morphisms $R \to E$ and $S \to B$ of R-algebras.
The morphism $R \to S$, or the R-algebra S, is called

i) *formally smooth* if for all lifting problems (1.4.13), a lift exists.

ii) *formally unramified* if for all lifting problems (1.4.13), at most one lift exists.

iii) *formally étale* if for all lifting problems (1.4.13), a unique lift exists.

Example 1.4.14. Let R be a global power functor, and P be a polynomial R-algebra. Then P is formally smooth as an R-algebra. The lifting problem can be solved by choosing preimages for all images of the polynomial generators of P. This can be seen using the universal property of polynomial global power functors from (1.1.23).

By the results of Section 1.4.a, we may characterize formally smooth, unramified and étale algebras in terms of derivations and extensions of global power functors.

Proposition 1.4.15. *Let $R \to S$ be a morphism of global power functors. Then $R \to S$ is*

i) formally smooth if and only if for all S-modules M, we have $\mathrm{Exalcomm}_R(S, M) = 0$.

ii) formally unramified if and only if for all S-modules M, we have $\mathrm{Der}_R(S, M) = 0$.

iii) formally étale if and only if for all S-modules M, we have $\mathrm{Der}_R(S, M) = 0$ *and* $\mathrm{Exalcomm}_R(S, M) = 0$.

Proof. *i)* Suppose $R \to S$ is formally smooth, and let $M \to E \to S$ be an extension of R-algebras, representing an element in $\mathrm{Exalcomm}_R(S, M)$. Then the lifting problem

$$\begin{array}{ccc} & & M \\ & & \downarrow \\ R & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ S & = & S \end{array}$$

admits a lift. This lift equivalently describes a section for the projection $E \to S$, and hence the extension $M \to E \to S$ is trivial. This shows $\mathrm{Exalcomm}_R(S, M) = 0$.

Conversely, suppose $\mathrm{Exalcomm}_R(S, M) = 0$ for all S-modules M. Consider a lifting problem

$$\begin{array}{ccc} & & M \\ & & \downarrow \\ R & \longrightarrow & E \\ \downarrow & & \downarrow \\ S & \longrightarrow & B. \end{array}$$

We moreover consider the extension $M \to E \times_B S \to S$ of S by M. Since $\mathrm{Exalcomm}_R(S, M) = 0$, this extension is trivial and hence admits a section $S \to E \times_B S$. Composing this section with the projection to E yields the desired lift. Hence $R \to S$ is formally smooth.

ii) Using Lemma 1.4.4, we see that lifts in the diagram 1.4.13 are a torsor over $\mathrm{Der}_R(S, M)$. Moreover, for any S-module M, the trivial extension $M \to S \ltimes M \to S$ does admit a lift. Hence, $R \to S$ is formally unramified if and only if $\mathrm{Der}_R(S, M) = 0$.

iii) The result for formal étaleness is obtained from combining the characterizations of formal smoothness und unramifiedness. □

Using this characterization of formally smooth algebras, we obtain that the exact sequences of derivations and Kähler differentials are also exact at the other ends in the case of formally smooth algebras.

Proposition 1.4.16. *Let $R \to S \to T$ be a triple of global power functors and let T be a formally smooth S-algebra. Let moreover M be a T-module. Then the sequences*

$$0 \to \Omega^1_{S/R} \Box_S T \to \Omega^1_{T/R} \to \Omega^1_{T/S} \to 0$$

and

$$0 \to \mathrm{Der}_S(T, M) \to \mathrm{Der}_R(T, M) \to \mathrm{Der}_R(S, M) \to 0$$

are split short exact.

Proof. From the six-term exact sequence in Theorem 1.4.5 and since by formal smoothness of T over S, $\mathrm{Exalcomm}_S(T,M) = 0$, it follows that the second sequence is exact. This already implies that the sequence of Kähler differentials is split exact, where a retraction of $\Omega^1_{S/R}\square_S T \to \Omega^1_{T/R}$ is obtained since applying $\mathrm{Hom}_T(\_, \Omega^1_{S/R}\square_S T)$ yields a surjection. Finally, a split of the sequence of Kähler differentials yields a split of the sequence of derivations. □

Proposition 1.4.17. *Let $R \to S \to T$ be a triple of global power functors, let $S \to T$ be surjective with kernel I and let T be a formally smooth R-algebra. Let moreover M be a T-module. Then the sequences*

$$0 \to I/I^{\geq 2} \to \Omega^1_{S/R}\square_S T \to \Omega^1_{T/R} \to 0$$

and

$$0 \to \mathrm{Der}_R(T,M) \to \mathrm{Der}_R(S,M) \to \mathrm{Exalcomm}_S(T,M) \to 0$$

are split short exact.

Proof. The proof is completely analogous to the one for Proposition 1.4.16, using that the six-term sequence from Theorem 1.4.5, the surjectivity of $S \to T$ and the formal smoothness of T over R imply that the second sequence is exact. □

Proposition 1.4.18. *Let R be a global power functor and S be a formally smooth R-algebra. Then $\Omega^1_{S/R}$ is a projective S-module.*

Proof. Let P be a polynomial R-algebra with a surjection $P \to S$ of R-algebras. Then by Proposition 1.4.17, the sequence

$$0 \to I/I^{\geq 2} \to \Omega^1_{P/R}\square_P S \to \Omega^1_{S/R} \to 0$$

is split exact, where I denotes the kernel of the surjection $P \to S$. Since P is polynomial over R, the calculation in Example 1.4.11 show that $\Omega^1_{P/R}$ is a free P-module. Thus the split surjection $\Omega^1_{P/R}\square_P S \to \Omega^1_{S/R}$ exhibits $\Omega^1_{S/R}$ as a direct summand of a free S-module, hence it is a projective S-module. □

1.5 André-Quillen (Co-)Homology

In this section, we describe the André-Quillen (co-)homology theory for global power algebras S over a global power functor R. This is a generalization of the classical cohomology theory for commutative rings described independently by André and Quillen in [1, 52, 53].

André-Quillen cohomology and homology are the non-abelian derived functors of derivations and Kähler differentials, respectively. Hence, they are defined by considering simplicial resolutions of a global power algebra by basic building blocks and then applying derivations or Kähler differentials to this. The basic global power algebras in question are the polynomial global power algebras from Definition 1.1.22, since these form a nice set of projective generators for the category of global power algebras. Moveover, on these polynomial algebras derivations and Kähler differentials behave well.

After giving the definition of the homology and cohomology groups, we identify the low-dimensional terms in Section 1.5.b. As usual, the zeroth terms are given by the functors we derive to obtain the (co-)homology theory, ie Kähler differentials for homology and derivations for cohomology. Furthermore, as the six-term exact sequence from Theorem 1.4.5 suggests, the first André-Quillen cohomology group is given by the group of global power functor extensions. This is a globally equivariant generalization of the corresponding fact in classical commutative algebra due to André [1, Proposition 25.2]. In order to facilitate calculation of these low-dimensional terms, we also introduce the naive cotangent complex in Section 1.5.c. This construction is inspired by the work of Illusie [36, Chapitre III], and allows for explicit calculation of the zeroth and first André-Quillen (co-)homology.

In Section 1.5.d, we use this naive cotangent complex to compute the first homology of certain quotients of polynomial algebras over the Burnside ring global power functors. These calculations show that the exact sequence of the low-dimensional André-Quillen groups cannot be extended beyond a six-term sequence. This failure of the existence of a long exact transitivity sequence is founded in the fact that projective global functors do not need to be flat for the box product, as shown by Lewis [44].

Despite this flaw of the global André-Quillen cohomology, I still believe this to be a useful theory, especially for the interpretation of the low terms and their connection to differential properties of global power functors. Also, the comparison with topological André-Quillen theory as defined in the second part of this dissertation, which in fact does have a transitivity sequence, may provide valuable insights in the algebra of global power algebras.

1.5.a Definition of the (Co-)Homology

In Section 1.2, we generalized the notions of derivations and Kähler differentials to the context of global power algebras. Now, we also generalize André-Quillen cohomology to this context. For this, we have to derive the functors of derivations and Kähler differentials, and hence we recall the necessary definitions and results from homotopical algebra [52, II, 4].

Definition 1.5.1. Let $\mathcal{C}$ be a category closed under finite limits. An effective epimorphism is a morphism $f\colon X \to Y$ which is the coequalizer of its kernel pair $X \times_Y X \rightrightarrows X$. An object P of $\mathcal{C}$ is called projective if for every effective epimorphism $f\colon X \to Y$, the induced map $\mathrm{Hom}(P, f)\colon \mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y)$ is surjective.
We say that $\mathcal{C}$ has enough projectives if for every object X of $\mathcal{C}$, there is an effective epimorphism $p\colon P \to X$ from a projective object P.

In our case, let R be a global power functor and consider the category Alg_R. We claim that in this category, the polynomial algebras $R[x_K]$ for any compact Lie group K, and any box product over R of those are projective. In fact, let $f\colon S \to T$ be an effective epimorphism. Then, we consider the induced morphism $\mathrm{Hom}(R[x_K], f)\colon \mathrm{Hom}(R[x_K], S) \to \mathrm{Hom}(R[x_K], T)$. By the universal property of the polynomial algebra, this is isomorphic to $f_K\colon S(K) \to T(K)$.
As explained in Remark 1.1.24, we can write the category of R-power algebras as a multisorted algebraic theory, where the sorts are given by the compact Lie groups, and we have $R[x_K]$ as the free R-algebra on the group K, left adjoint to evaluation at the group K. By [17, Corollary 3.5.3], for any algebraic theory $\mathcal{T}$, the forgetful functor $U\colon \mathrm{Mod}_{\mathcal{T}} \to \mathrm{Sets}$ preserves and reflects effective epimorphisms, and hence, by the multisorted analogue, the map $f_K\colon S(K) \to T(K)$ is surjective. Thus, the R-algebra $R[x_K]$ is indeed projective. Since any coproduct of projective objects is projective and the box product is the coproduct in Alg_R, we see that any polynomial algebra $R[x_{K_i,i}, i \in I]$ is projective. Thus, we have the following result:

Lemma 1.5.2. *Let R be a global power functor. The category Alg_R of R-power algebras has enough projectives. Explicitly, for any R-power algebra S, the morphism*

$$\varepsilon_S\colon \mathop{\Box}_{\substack{K \\ s \in S(K)}} R[x_{K,s}] \to S,\; x_{K,s} \mapsto s \tag{1.5.3}$$

is an effective epimorphism from a projective object. Here, the box product is indexed over a representing set of isomorphism classes of compact Lie groups K and elements $s \in S(K)$.

By this lemma, we can do homotopical algebra in the category Alg_R. In particular, by [52, II.4, Theorem 4], the category $s\mathrm{Alg}_R$ of simplicial R-algebras has a model category structure and we can talk about cofibrant replacements of an R-algebra.

Definition 1.5.4. Let R be a global power functor and let S be an R-algebra. Then a cofibrant resolution of S is an object $P_\bullet$ of $s\mathrm{Alg}_R$ together with a map $p\colon P_\bullet \to S$ of simplicial R-algebras (where S is considered as a constant simplicial object) such that $P_\bullet$ is cofibrant and p is an acyclic fibration in the model structure on $s\mathrm{Alg}_R$.

Remark 1.5.5. One way to construct cofibrant resolutions is by using polynomial global power functors. We define a polynomial simplicial resolution of an R-power algebra S to be a simplicial R-power algebra $P_\bullet$ with a morphism $P_\bullet \to S$, such that

- $P_n \cong R[x^{(n)}_{i,K_i}]$ is a polynomial R-power algebra, and
- the degeneracies map generators to generators.

In fact, any cofibration is a retract of an inclusion into a polynomial algebra. This procedure is explained in [52, Remark 4, p. II.4.11]. Thus, we can use polynomial resolutions to calculate derived functors.

Now, we can use this formalism to define the derived functors of derivations and Kähler differentials in the category of R-algebras.

Definition 1.5.6. Let R be a global power functor and S be an R-algebra. Let M be an S-module. Let $P_\bullet$ be a cofibrant resolution of S in $s\mathrm{Alg}_R$. Then the André-Quillen cohomology of S over R with coefficients in M is

$$D^q(S, R; M) = H^q(\mathrm{Der}_R(P_\bullet, M)),$$

the cohomology of the simplicial abelian group $\mathrm{Der}_R(P_\bullet, M)$.
Moreover, we define the cotangent complex of S as the simplicial object $\mathbb{L}_{S/R} = S\Box_{P_\bullet}\Omega^1_{P_\bullet/R}$ in Mod_S, where the boxproduct is formed levelwise. Then, the André-Quillen homology of S over R with coefficients in M is the graded global functor

$$D_q(S, R; M) = H_q(\mathbb{L}_{S/R}\Box_S M).$$

These constructions are obviously functorial in squares

$$\begin{array}{ccc} R & \xrightarrow{\eta} & S \\ {\scriptstyle f_R}\downarrow & & \downarrow{\scriptstyle f_S} \\ R' & \xrightarrow[\eta']{} & S' \end{array}$$

of global power functors, induced from the functoriality of Kähler differentials.

Remark 1.5.7. By the universality of the Kähler differentials from Definition 1.2.12, we can rewrite the André-Quillen cohomology as

$$D^q(S, R; M) \cong H^q(\mathrm{Hom}_S(\mathbb{L}_{S/R}, M)).$$

This emphasizes the duality between the simplicial objects defining homology and cohomology.

As in Remark 1.2.25, we can also define a global-functor-valued André-Quillen cohomology by setting

$$\underline{D}^q(S, R; M)(G) = D^q(S, R; M[G]).$$

This satisfies the same duality as the abelian-group-valued cohomology, in that

$$\underline{D}^q(S,R;M) \cong H^q(\underline{\mathrm{Hom}}_S(\mathbb{L}_{S/R},M)).$$

Using this internal definition, the identifications given in Proposition 1.5.10 and Proposition 1.5.14 also lift to give isomorphisms of global functors for the internal cohomology and derivations or global power functor extensions, respectively.

Sometimes it is convenient to have a specific resolution to compute the cotangent complex and André-Quillen homology. We can take a simplicial projective resolution by using the free-forgetful adjunction between global power functors and "global sets", describing global power functors as a multisorted theory. The result is a comparison of Quillen's homotopical algebra approach to cohomology followed in Definition 1.5.6 and a cotriple/comonad cohomology approach as introduced by Beck in [13] and studied by Barr and Beck as well as Ulmer in multiple works [8, 9, 6, 7, 64]:

We consider the functor

$$G\colon \mathrm{Alg}_R \to \mathrm{Alg}_R,\ S \mapsto \mathop{\square}_{\substack{K\\ s\in S(K)}} R[x_{K,s}].$$

This is a comonad, since it arises from the free-forgetful adjunction

$$\mathrm{Alg}_R \underset{\square R[\_]}{\overset{\mathrm{ev}}{\rightleftarrows}} \textstyle\prod_G \mathrm{Sets}\ .$$

In particular, we have the counit $\varepsilon\colon G \to id$ from (1.5.3), and this defines for any R-algebra S a simplicial polynomial R-algebra $G_\bullet S$, where $G_nS = G^{n+1}S$ and the face maps are $d_i = G^{n-i}\varepsilon_{G^iS}\colon G_nS \to G_{n-1}S$. The degeneracy maps are defined using the comultiplication $\nu\colon G \to G^2$. This simplicial R-algebra is augmented to S via $\varepsilon_S\colon GS \to S$.

Proposition 1.5.8. *Let R be a global power functor and S be an R-power algebra. Let M be an S-module. Then we have isomorphisms*

$$D^q(S,R;M) \cong H^q(\mathrm{Der}_R(G_\bullet S,M)) \text{ and } D_q(S,R;M) \cong H_q(S\square_{G_\bullet S}\Omega^1_{G_\bullet S/R}\square_S M).$$

This is part of [52, II.5, Theorem 5], and follows since we may choose any cofibrant resolution in order to compute the cotangent complex.

On polynomial algebras, the André-Quillen cohomology is trivial in positive degrees.

Proposition 1.5.9. *Let R be a global power functor and let $S = R[x_{K_i,i}, i\in I]$ be a polynomial algebra. Then the augmentation $G_\bullet S \to S$ is a homotopy equivalence. In particular, we have*

$$D^q(S,R;M) = \begin{cases} \mathrm{Der}_R(S,M) & \text{for } q=0\\ 0 & \text{else}\end{cases} \quad \text{and } D_q(S,R;M) = \begin{cases} \Omega^1_{S/R}\square_S M & \text{for } q=0\\ 0 & \text{else}\end{cases}$$

for any S-module M.

Proof. We have an additional degeneracy given by

$$S \to GS, x_{K_i} \mapsto x_{x_{K_i}}.$$

This induces the desired homotopy inverse. □

1.5.b Identification of the Low-Degree Terms

We can identify the low-degree terms of the André-Quillen cohomology in terms of previously defined functors, namely derivations and Kähler differentials in degree 0 and Exalcomm in degree 1.

Proposition 1.5.10. *Let R be a global power functor and S be an R-algebra. Let M be an S-module. Then $D^0(S,R;M) = \mathrm{Der}_R(S,M)$ and $D_0(S,R;M) = \Omega^1_{S/R}\Box_S M$.*

Proof. We consider the beginning $P_1 \rightrightarrows P_0 \to S$ of a cofibrant resolution of the R-algebra S. This exhibits S as a reflexive coequalizer in the category of R-algebras of the diagram $P_1 \rightrightarrows P_0$. Since $S\Box_{(\_)}\Omega^1_{(\_)/R}$ is left adjoint to square-zero extension by Theorem 1.2.13, it preserves reflexive coequalizers. Moreover, the functors $(\_)\Box_S M$ and $\mathrm{Hom}_S(\_, M)$ are right respectively left exact and hence send reflexive coequalizers to coequalizers and equalizers respectively. Since $D^0(S,R;M) = H^0(\mathrm{Hom}_S(\mathbb{L}_{S/R}, M))$ and $D_0(S,R;M) = H_0(\mathbb{L}_{S/R}\Box_S M)$, this proves the proposition. □

Classically, the first André-Quillen cohomology group can be described via extensions of commutative algebras. We generalize this into our context. This follows the original connection between derivations and commutative algebra extensions by Grothendieck in [27, §18&20] and the connection to the cotangent complex by Illusie in [36, Chapitre III]. In the context of André-Quillen cohomology of commutative algebras, this is also the content of [1, Proposition 25.2].

Construction 1.5.11. We construct the maps comparing the first André-Quillen cohomology group to the group of global power algebra extensions. Let R be a global power functor, S be an R-algebra and M be an S-module. Let $0 \to M \xrightarrow{i} E \xrightarrow{p} S \to 0$ be an extension of S by M. Moreover, let

$$P_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} P_0 \xrightarrow{\varepsilon} S$$

be the beginning of a simplicial resolution of S via polynomial R-algebras. We define a cohomology class in $D^1(S,R;M)$ as follows.
Choose any R-algebra lift $q\colon P_0 \to E$ of $\varepsilon\colon P_0 \to S$, using that P_0 is a projective R-algebra. Then we consider the map $D = q \circ (d_1 - d_0)\colon P_1 \to E$. By Lemma 1.4.4, this defines an R-derivation $D\colon P_1 \to M$ as the difference of the two power algebra morphisms qd_1 and qd_0. By definition, this clearly is a cocycle and defines a cohomology class $[D] \in D^1(S,R;M)$. Moreover, this is independent of the chosen lift q: again by Lemma 1.4.4, any other lift $q'\colon P_0 \to E$ only differs by an R-derivation $d\colon P_0 \to M$, and the resulting derivation $D'\colon P_1 \to M$ thus differs from D by a coboundary. Hence, we get a well-defined cohomology class in $D^1(S,R;M)$ associated to the extension $0 \to M \to E \to S \to 0$.
Suppose now that conversely we are given a cohomology class in $D^1(S,R;M)$, represented by an R-derivation $D\colon P_1 \to M$. Then we consider the base-change

$$\begin{array}{ccccccc} P_1 & \xrightarrow{d_1 - d_0} & P_0 & \xrightarrow{\varepsilon} & S & \longrightarrow & 0 \\ \downarrow{\scriptstyle D} & & \downarrow & & \| & & \\ M & \longrightarrow & E = P_0 \oplus_{P_1} M & \longrightarrow & S & \longrightarrow & 0. \end{array} \tag{1.5.12}$$

We verify that this indeed defines an algebra extension of S by M in Lemma 1.5.13. Again using Lemma 1.4.4, one may check that the isomorphism class of this extension does not depend on the choice of representative D, since the derivation $d\colon P_0 \to M$ defining the coboundary defines an isomorphism between the two extensions.

Lemma 1.5.13. *Let R be a global power functor and S be an R-power algebra. Let $P_\bullet \to S$ be a simplicial resolution of S by polynomial R-algebras and M be an S-module. Moreover, let $D\colon P_1 \to M$ be a derivation representing a cohomology class in $D^1(S,R;M)$.*
Then $E = P_0 \oplus M/((d_1(x) - d_0(x), -D(x))$ for $x \in P_1)$, as considered in Construction 1.5.11, defines an R-power algebra extension of S by M.

Proof. First, we endow E with the structure of an R-power algebra. For this, we consider the square-zero extension $P_0 \ltimes M$, and check whether the relations imposed on the quotient $E = P_0 \oplus M/((d_1(x) - d_0(x), -D(x))$ for $x \in P_1)$ are compatible with product and power operations. This follows from the fact that D is an R-derivation, as follows:
We show that the sub-global functor $\{(d_1(x) - d_0(x), -D(x)) \mid x \in P_1\} \subset P_0 \oplus M$ is a global power ideal, as defined in Definition 1.1.30. Suppose we are given for a compact Lie group G elements $(y,m) \in (P_0 \oplus M)(G)$ and $(d_1(x) - d_0(x), -D(x))$ for $x \in P_1(G)$. Then we consider the element $x' = s(y) \cdot x \in P_1(G)$, using the degeneracy $s\colon P_0 \to P_1$ of the simplicial polynomial resolution $P_\bullet$, and calculate

$$\begin{aligned}(y,m)\cdot(d_1(x) - d_0(x), -D(x)) &= (yd_1(x) - yd_0(x), -yD(x) + md_1(x) - md_0(x))\\ &= (d_1(s(y))d_1(x) - d_0(s(y))d_0(x), -\varepsilon(y)D(x))\\ &= (d_1(s(y)x) - d_0(s(y)x), -(\varepsilon(d_1(s(y)))D(x) + \varepsilon(d_1(x))D(s(y))))\\ &= (d_1(x') - d_0(x'), -D(x')).\end{aligned}$$

Here we used that the structures of M as a module over P_0 and P_1 are obtained from the S-module structure by pulling back along the augmentation $\varepsilon\colon P_\bullet \to S$, which equalizes all face maps. Moreover, we observe that since $s(y)$ lies in the image of the differential $d_2 - d_1 + d_0\colon P_2 \to P_1$ and D vanishes on the image of this map, $D(s(y)) = 0$.

For the power operations, we employ an inductive argument, starting with $P^1 = id$ as the trivial induction beginning. Let $x \in P_1(G)$. For $k \geq 2$, we calculate by additivity of the power operations

$$\begin{aligned}P^k(d_1(x) - d_0(x), -D(x)) &= (P^k(d_1(x) - d_0(x)), \operatorname{tr}_{k-1,1}(P^{k-1}(d_1(x) - d_0(x)) \times (-D(x))))\\ &= \Big(P^k(d_1(x)) - P^k(d_0(x)) - \sum_{i=1}^{k-1} \operatorname{tr}_{k-i,i}(P^{k-i}(d_1(x) - d_0(x)) \times P^i(d_0(x))), 0\Big)\\ &= (d_1(P^k(x)) - d_0(P^k(x)), -D(P^k(x)))\\ &\quad - (\operatorname{tr}_{1,k-1}(P^1(d_1(x) - d_0(x)) \times P^{k-1}(d_0(x))), -\operatorname{tr}_{1,k-1}(D(x) \times P^{k-1}(x)))\\ &\quad - \Big(\sum_{i=1}^{k-2} \operatorname{tr}_{k-i,i}(P^{k-i}(d_1(x) - d_0(x)) \times P^i(d_0(x))), 0\Big)\\ &= (d_1(P^k(x)) - d_0(P^k(x)), -D(P^k(x)))\\ &\quad - (\operatorname{tr}_{1,k-1}((d_1(x) - d_0(x)) \times P^{k-1}(d_0(x))), -\operatorname{tr}_{1,k-1}(D(x) \times P^{k-1}(d_0(x))))\\ &\quad - \sum_{i=1}^{k-2}(\operatorname{tr}_{k-i,i}(P^{k-i}(d_1(x) - d_0(x)) \times P^i(d_0(x))),\\ &\qquad\quad \operatorname{tr}_{1,k-i-1,i}((-D(x)) \times P^{k-i-1}(d_1(x) - d_0(x)) \times P^i(d_0(x)))).\end{aligned}$$

Here, in the second line, we used that $P^{k-1}(d_1(x) - d_0(x))$ acts trivially on M, since it vanishes upon projection to S. Moreover, we calculated $P^k(d_1(x) - d_0(x))$ by using additivity for $P^k(d_1(x)) = P^k((d_1(x) - d_0(x)) + d_0(x))$ and solving for the relevant term. In the last line, we again used that $d_1(x) - d_0(x)$ acts trivially on M. In total, we observe that each of the summands in the last line lies in $\{(d_1(x) - d_0(x), -D(x)) \mid x \in P_1\}$, the first summand as the image of $P^k(x)$, the second summand since we already established that this set is closed under multiplication with arbitrary elements in $P_0 \oplus M$, and the last $k-2$ summands since we may use induction for $P^{k-i}(d_1(x) - d_0(x), -D(x))$ and again because this set is closed under multiplication. This finishes the verification that we take the quotient by a global power ideal. Thus, we have the structure of an R-algebra on E.

Moreover, the maps $\varepsilon\colon P_0 \to S$ and $0\colon M \to S$ define a surjective morphism $p\colon E \to S$ of R-algebras compatible with ε. We claim that the map $M \to E$ exhibits M as the kernel of the map p. By definition, p vanishes on M, and by exactness of the upper row in (1.5.12), M surjects onto the kernel. If an element in M is sent to 0 in E, then there exists an preimage in $\ker(d_1 - d_0)$ under D. But since D is a cocycle, it has to vanish on the kernel of $d_1 - d_0$. So M in fact is the kernel of $p\colon E \to S$. Finally, by construction, M is square-zero in E and the S-module structure on M induced from this agrees with the given one. Hence, we have constructed an algebra extension of S by M. □

Proposition 1.5.14. *Let R be a global power functor, S be an R-power algebra and M be an S-module. Then the two assignments from Construction 1.5.11 define natural inverse group isomorphisms*

$$D^1(S, R; M) \cong \mathrm{Exalcomm}_R(S, M).$$

Proof. For the proof, it is left to check that the maps defined above are natural, compatible with the group structure and inverse to one another. The naturality and additivity of the maps is straight-forward to check. For the two compositions of the above constructions, we consider the diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{d_1 - d_0} & P_0 & \xrightarrow{\varepsilon} & S & \longrightarrow & 0 \\ \downarrow{\scriptstyle D} & & \downarrow & & \| & & \\ M & \longrightarrow & E = P_0 \oplus_{P_1} M & \longrightarrow & S & \longrightarrow & 0. \end{array}$$

from (1.5.12). This already shows that starting from a derivation, forming the associated extension and then taking the corresponding cohomology class is the identity. Conversely, we consider the diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{d_1 - d_0} & P_0 & \xrightarrow{\varepsilon} & S & \longrightarrow & 0 \\ \downarrow{\scriptstyle D} & & \downarrow & & \| & & \\ M & \longrightarrow & P_0 \oplus_{P_1} M & \longrightarrow & S & \longrightarrow & 0 \\ \| & & \downarrow & & \| & & \\ M & \longrightarrow & E & \longrightarrow & S & \longrightarrow & 0. \end{array}$$

Similar to the arguments in Lemma 1.4.3, the dashed morphism of R-algebras exists and makes the diagram commute, and hence the extensions formed by E and $P_0 \oplus_{P_1} M$ are equivalent. Thus, the two constructions are indeed inverse to one another. □

Remark 1.5.15. In the classical literature, one can also find interpretations of the second André-Quillen cohomology group. Both Gerstenhaber [25] and Lichtenbaum-Schlessinger [45] define a group of *two-term extensions* of an R-algebra S by an S-module M. These are exact sequences of the form $0 \to M \xrightarrow{i} N \xrightarrow{f} E \xrightarrow{p} S \to 0$, where E is an R-algebra and the map p is a surjection of R-algebras, M and N are E-modules and the maps i and f are E-linear, and finally the morphism $N \to E$ satisfies $f(n_1)n_2 = f(n_2)n_1$. Moreover, in [45, Definition 2.1.8], a three-term cotangent complex is defined (a generalization of the naive cotangent complex we transfer to the theory of global power algebras in Section 1.5.c), and it is shown that its second cohomology is given by the group of two-term extension in [45, 4.1.2]. The zeroth and first cohomology indeed recover derivations and extensions. Also, the six-term exact sequence from Theorem 1.4.5 is extended to a nine-term exact sequence in [45, Theorem 2.2.4, 2.3.5-6].

In the proof that these two-term extensions give the second André-Quillen cohomology, one may follow the outline given in the comparison of the first André-Quillen cohomology group and (one-term) extensions of commutative algebras in Construction 1.5.11. Since I was not able to find the explicit comparison in the literature, and since a precise understanding of the construction is helpful to see how the situation changes in the presence of power operations, I decided to give some detail here. We explain the construction of a cohomology class in $D^2(S, R; M)$ associated to a two-term extension. The argument roughly takes the following form:

We consider the diagram

$$\begin{array}{ccccccc} P_2 & \Rrightarrow & P_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & P_0 & \xrightarrow{\varepsilon} & S \\ {\scriptstyle D}\downarrow & & {\scriptstyle \lambda}\downarrow & & {\scriptstyle g}\downarrow & & \| \\ M & \xrightarrow[i]{} & N & \xrightarrow[f]{} & E & \xrightarrow[p]{} & S. \end{array} \tag{1.5.16}$$

Here, we are given a two-term extension $M \to N \to E \to S$ of S by M and a simplicial resolution $P_\bullet \to S$ of S by polynomial R-power algebras. Since P_0 is a projective R-algebra and p is surjective, we find an R-algebra lift $g\colon P_0 \to E$ of ε against p. Then the two maps $g \circ d_0$ and $g \circ d_1$ are also R-algebra maps. Then with respect to the two different P_1-module structures induced on E by pulling back along these maps, the difference $g \circ d_1 - g \circ d_0$ becomes a *biderivation*. This is an analogue of the notion of a derivation in the presence of two module structures, considered as a left and a right module structure. The relevant Leibniz condition becomes

$$(g \circ d_1 - g \circ d_0)(pp') = (g \circ d_1)(p)(g \circ d_1 - g \circ d_0)(p') + (g \circ d_1 - g \circ d_0)(p)(g \circ d_0)(p').$$

The properties that $\mathrm{im}(g \circ d_1 - g \circ d_0) \subset \mathrm{im}(f)$ (by exactness) and that f satisfies $f(n_1)n_2 = n_1 f(n_2)$ allow to construct a lift $\lambda\colon P_1 \to N$, which then also is a biderivation. The proof of this fact is given in [45, Lemma 2.1.6, Definition 2.1.4].

Finally, on the image of $d_2 - d_1 + d_0\colon P_2 \to P_1$, the morphisms $d_1, d_0\colon P_1 \to P_0$ agree, so $\lambda \circ (d_2 - d_1 + d_2)\colon P_2 \to N$ is a derivation with image contained in $\mathrm{im}(i)$. Thus, we obtain a derivation $D\colon P_2 \to M$. This represents a cohomology class in $D^2(S, R; M)$, and this assignment is used to identify this second André-Quillen cohomology group with the group of two-term extensions.

In the global context, however, we need to also consider the twisted Leibniz rule for the derivation of a power operation. This formula requires a symmetry for the module structure on the target, since the power operations exactly capture the additional Σ_m-action available from the symmetry of the multiplication. The notion of a biderivation classically is a relaxation to two different left and right module structures. This does not make sense globally, since this breaks the required symmetry, and hence the above construction cannot be transferred to the situation of global power algebras. It is also unclear what type of structure we should require on N and the morphism $f\colon N \to E$. Here, the dichotomy of R-modules (without power operations) and of abelian group objects in augmented algebras (with power operations) might play a role, and N could be required to have power operations. However, the mentioned lack of symmetry inherent in the classical construction suggests that no sensible definition can lead to a description of the second cohomology group in terms of extensions.

The fact that this comparison is impossible to perform for global power algebras is also suggested by the failure of the transitivity sequence shown in Section 1.5.d. As observed there, the six-term sequence from Theorem 1.4.5 cannot be extended by an additional term to the left while also enforcing that the higher André-Quillen cohomology groups vanish on polynomial power algebras. This shows that no hypothetical interpretation of the second cohomology in terms of extensions can lead to a nine-term exact sequence as constructed by Lichtenbaum-Schlessinger.

1.5.c The Naive Cotangent Complex

After identifying the low-dimensional terms of the André-Quillen cohomology, we exhibit a different way of calculation, by means of the so-called "naive cotangent complex". This is an explicit truncation of the actual cotangent complex, and its construction generalizes the comparison of Exalcomm with $\mathrm{Hom}(I/I^{\geq 2}, M)$ obtained in Proposition 1.4.9. The next results follow [36, III 1.3].

Construction 1.5.17. Let R be a global power functor and S be an R-power algebra. Let moreover P be a polynomial R-algebra and $\varepsilon\colon P \to S$ be a surjection of R-algebras with kernel $I \subset P$. We then define the naive cotangent complex as the two-term complex

$$\mathbb{L}^{\mathrm{naive}}_{S/P/R} = \big(I/I^{\geq 2} \to \Omega^1_{P/R} \Box_P S\big), \tag{1.5.18}$$

concentrated in degrees 0 and 1. Here, the morphism $I/I^{\geq 2} \to \Omega^1_{P/R} \Box_P S$ is induced from the universal derivation $P \to \Omega^1_{P/R}$ and the unit $\Omega^1_{P/R} \to \Omega^1_{P/R} \Box_P S$ of the scalar extension adjunction between P-modules and S-modules. This uses the observation that since $S \cong P/I$, any derivation $I \subset P \to M$ with M an S-module annihilates $I^{\geq 2}$.

We directly observe that $H_0(\mathbb{L}^{\mathrm{naive}}_{S/P/R}) \cong \Omega^1_{S/R}$ by the exact sequence from Proposition 1.4.10. Moreover, in the case that $R \to S$ is already surjective and we choose $P = R$, we see that $\Omega^1_{P/R} = 0$ and the cohomology of this naive cotangent complex recovers $\mathrm{Exalcomm}_R(S, M)$ by Proposition 1.4.9. In fact, the homology and cohomology of the naive cotangent complex are independent of the surjection $P \to S$ we choose:

Lemma 1.5.19. *Let R be a global power functor and S be an R-algebra. Let $\varepsilon\colon P \to S$ and $\eta\colon Q \to S$ be two surjections of R-algebras with P and Q polynomial. Denote the kernels by I and J, respectively. Let moreover $\varphi\colon P \to Q$ be a morphism of R-algebras with $\eta\varphi = \varepsilon$. Then there is an induced morphism*

$$\varphi_*\colon \mathbb{L}^{\mathrm{naive}}_{S/P/R} \to \mathbb{L}^{\mathrm{naive}}_{S/Q/R},$$

which is a quasi-isomorphism. These morphisms make the naive cotangent complex functorial in the choice of the surjection $P \to S$.

Proof. The existence of an induced morphism is clear from functoriality of the Kähler differentials and the universal derivation, and follows from considering the diagram

$$\begin{array}{ccccc} I & \longrightarrow & P & & \\ \downarrow & & \downarrow & \searrow & \\ & & & & S. \\ & & & \nearrow & \\ J & \longrightarrow & Q & & \end{array}$$

We now consider the effect of this morphism on homology. First, we observe that we may factor $\varphi\colon P \to Q$ through $P \to P\Box_R Q \to Q$, where the first morphism exhibits $P\Box_R Q$ as a polynomial P-algebra and the second morphism is surjective. Thus, we consider these cases separately and start with the assumption that Q is polynomial over P.
In this case, we consider the sequence

$$0 \to I/I^{\geq 2} \to J/J^{\geq 2} \to \Omega^1_{Q/P}\Box_Q S \to 0. \tag{1.5.20}$$

Applying $\mathrm{Hom}_S(\_, M)$ for arbitrary S-modules M and using Theorem 1.2.24 and Proposition 1.4.9, we obtain the sequence

$$0 \to \mathrm{Der}_P(Q, M) \to \mathrm{Exalcomm}_Q(S, M) \to \mathrm{Exalcomm}_P(S, M) \to 0.$$

This is exact by Theorem 1.4.5, since $\mathrm{Der}_P(S, M) = 0$ by surjectivity of $P \to S$ and since $\mathrm{Exalcomm}_P(Q, M) = 0$ as Q is polynomial over P. Hence the sequence (1.5.20) is exact.
We also obtain the split short exact sequence

$$0 \to \Omega^1_{P/R}\Box_P Q \to \Omega^1_{Q/R} \to \Omega^1_{Q/P} \to 0$$

by Proposition 1.4.16, since Q is smooth over P. Upon applying $\_\Box_Q S$, we obtain the short exact sequence

$$0 \to \Omega^1_{P/R}\Box_P S \to \Omega^1_{Q/R}\Box_Q S \to \Omega^1_{Q/P}\Box_Q S \to 0.$$

The differential for the naive cotangent complex now defines a morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I/I^{\geq 2} & \xrightarrow{\varphi_1} & J/J^{\geq 2} & \longrightarrow & \Omega^1_{Q/P}\Box_Q S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^1_{P/R}\Box_P S & \xrightarrow{\varphi_0} & \Omega^1_{Q/R}\Box_Q S & \longrightarrow & \Omega^1_{Q/P}\Box_Q S & \longrightarrow & 0
\end{array}$$

of short exact sequences. The long exact homology sequence for this short exact sequence of two-term complexes then shows that the morphism φ_* is a quasi-isomorphism.

We now have to consider the case that $P \to Q$ is surjective. In this case, we denote $K = \ker(P \to Q)$ and consider the diagram

$$\begin{array}{ccccccccc}
 & & K/K^{\geq 2}\Box_Q S & \longrightarrow & I/I^{\geq 2} & \xrightarrow{\varphi_1} & J/J^{\geq 2} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K/K^{\geq 2}\Box_Q S & \longrightarrow & \Omega^1_{P/R}\Box_P S & \xrightarrow{\varphi_0} & \Omega^1_{Q/R}\Box_Q S & \longrightarrow & 0.
\end{array}$$

In this diagram, the bottom row is split exact by Proposition 1.4.17, since Q is polynomial over R. The top sequence is dual to the sequence

$$0 \to \mathrm{Exalcomm}_Q(S, M) \to \mathrm{Exalcomm}_P(S, M) \to \mathrm{Exalcomm}_P(Q, M)$$

from Theorem 1.4.5, which is exact because $\mathrm{Der}_P(Q, M) = 0$, since $P \to Q$ is surjective. Hence the top sequence is right exact. Again, the long exact sequence in homology then shows that φ_* is a quasi-isomorphism. $\Box$

Next, we show that indeed the naive cotangent complex always computes the first two terms of André-Quillen (co-)homology.

Proposition 1.5.21. *Let R be a global power functor and S be an R-algebra. Moreover let $P_\bullet \to S$ be a simplicial polynomial resolution of S as an R-algebra. In particular, we have a surjection $\varepsilon\colon P_0 \to S$ from a polynomial R-algebra onto S. Then there is a morphism*

$$\varphi\colon \mathbb{L}_{S/R} \to \mathbb{L}^{\text{naive}}_{S/P_0/R}$$

inducing a (co-)homology isomorphism in degrees at most 1.

Proof. We first construct the map $\mathbb{L}_{S/R} \to \mathbb{L}^{\text{naive}}_{S/P_0/R}$. It arises from considering the extension

$$0 \to I_0/I_0^{\geq 2} \to P_0/I_0^{\geq 2} \to S \to 0,$$

where I_0 denotes the kernel of the augmentation $P_0 \to S$. Using Construction 1.5.11, we may lift $d_1 - d_0\colon P_1 \to P_0 \to P_0/I_0^{\geq 2}$ to an R-derivation $D\colon P_1 \to I_0/I_0^{\geq 2}$. This derivation is then equivalent to a homomorphism $\tilde{\varphi}_1\colon \Omega^1_{P_1/R} \to I_0/I_0^{\geq 2}$ of P_1-modules. We base-change it to a morphism $\varphi_1\colon \Omega^1_{P_1/R}\Box_{P_1} S \to I_0/I_0^{\geq 2}$ of S-modules. By the construction of φ_1 by lifting $d_1 - d_0$, it is clear that this induces a morphism $\varphi\colon \mathbb{L}_{S/R} \to \mathbb{L}^{\text{naive}}_{S/P_0/R}$ of complexes together with the identity in degree 0. We only show that this induces an isomorphism in homology of degree at most 1, the proof in cohomology is analogous.

To check that this morphism induces a homology isomorphism in degrees at most 1, we build an entire complex of naive cotangent complexes. We observe that the construction of the naive cotangent complex is functorial in morphisms of R-algebras compatible with surjections to S, as shown in Lemma 1.5.19. Hence, the simplicial object $P_\bullet$ of polynomial R-algebras augmented to S defines a simplicial object

$$\mathbb{L}^{\text{naive}}_{S/P_\bullet/R} = (I_\bullet/I_\bullet^{\geq 2} \to \Omega_{P_\bullet/R}\Box_{P_\bullet} S).$$

In particular, we obtain a double complex

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
I_0/I_0^{\geq 2} & \longleftarrow & I_1/I_1^{\geq 2} & \longleftarrow & I_2/I_2^{\geq 2} & \longleftarrow & \cdots \\
\downarrow & \nwarrow & \downarrow & & \downarrow & & \\
\Omega^1_{P_0/R}\Box_{P_0} S & \longleftarrow & \Omega^1_{P_1/R}\Box_{P_1} S & \longleftarrow & \Omega^1_{P_2/R}\Box_{P_2} S & \longleftarrow & \cdots .
\end{array}$$

For later reference, we have included the morphism $\varphi_1\colon \Omega^1_{P_1/R}\Box_{P_1} S \to I_0/I_0^{\geq 2}$ constructed above as the dashed morphism. Associated to this double complex are two spectral sequences converging to the homology of the total complex [47, Theorem 2.15, Homological version]. If we first take the vertical homology, we obtain the homology of the naive cotangent complexes. Since all face maps $d_i\colon P_n \to P_{n-1}$ induce isomorphisms between these homologies by Lemma 1.5.19, the horizontal complexes are now constant chain complexes. Hence we obtain that the E^2-page is concentrated in bidegrees $(0,0)$ and $(0,1)$, and the homology of the total complex is computed as the homology of the naive cotangent complex $\mathbb{L}^{\text{naive}}_{S/P_0/R}$.

We now consider the other spectral sequence and start by computing the horizontal homology. Since the zeroth row is exactly the usual cotangent complex, its homology computes the André-

Quillen homology of S over R. We obtain the following E^1-page:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
H_0(I_\bullet/I_\bullet^{\geq 2}) & & H_1(I_\bullet/I_\bullet^{\geq 2}) & & H_2(I_\bullet/I_\bullet^{\geq 2}) & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
D_0(S,R) & & D_1(S,R) & & D_2(S,R) & & \cdots
\end{array}$$

We now claim that the homology groups $H_0(I_\bullet/I_\bullet^{\geq 2})$ and $H_1(I_\bullet/I_\bullet^{\geq 2})$ vanish. After we show this, we have proved the proposition, since the only groups contributing to the lowest two homology groups of the total complex are $D_0(S,R)$ and $D_1(S,R)$, which thus have to be isomorphic to the homology groups of the naive cotangent complex. The identification of the morphism inducing this isomorphism follows by considering the edge homomorphism of the second spectral sequence, which agrees with the morphism $\varphi_1\colon \Omega_{P_1/R}\Box_{P_1}S \to I_0/I_0^{\geq 2}$ by considering the diagram

$$\begin{array}{ccc}
I_0/I_0^{\geq 2} & \overset{\varphi_1}{\nwarrow} & \\
{\scriptstyle D}\downarrow & & \\
\Omega^1_{P_0/R}\Box_{P_0}S & \xleftarrow{d_1-d_0} & \Omega^1_{P_1/R}\Box_{P_1}S,
\end{array}$$

which is part of the spectral sequence considered above.

We are left to prove that $H_0(I_\bullet/I_\bullet^{\geq 2})$ and $H_1(I_\bullet/I_\bullet^{\geq 2})$ vanish. For this, we consider the simplicial object $I_\bullet$ of non-unital global power functors. Since the augmentation $P_\bullet \to S$ induces isomorphisms on homotopy groups, $I_\bullet$ is acyclic. The following lemma then proves that the first two homology groups of $I_\bullet/I_\bullet^{\geq 2}$ vanish and thus finishes the proof. □

Lemma 1.5.22. *Let $I_\bullet$ be an acyclic simplicial object of non-unital global power functors, then $H_0(I_\bullet/I_\bullet^{\geq 2}) = 0$ and $H_1(I_\bullet/I_\bullet^{\geq 2}) = 0$.*

Proof. We know that $d_1 - d_0\colon I_1 \to I_0$ is surjective by acyclicity. Then also the induced map $I_1/I_1^{\geq 2} \to I_0/I_0^{\geq 2}$ is surjective, and thus $H_0 = 0$.
We now consider H_1. Suppose we have a cycle $[f] \in I_1/I_1^{\geq 2}$. This means that $(d_1 - d_0)(f) \in I_0^{\geq 2}$. If we are able to find a preimage $g \in I_1^{\geq 2}$ of $(d_1 - d_0)(f)$, then the element $f - g$ satisfies $(d_1 - d_0)(f - g) = 0$ and thus lifts to I_2. Since we consider the quotient $I_1/I_1^{\geq 2}$, this yields a lift of $[f]$ to $I_2/I_2^{\geq 2}$. We have thus reduced to showing that also $d_1 - d_0\colon I_1^{\geq 2} \to I_0^{\geq 2}$ is surjective.

By definition, $I_0^{\geq 2}$ is generated by elements $f \cdot g$ and $P^k(f)$ for $f, g \in I_0(G)$, $k \geq 2$, G a compact Lie group. We start by considering $f \cdot g$. Here, we choose a preimage $F \in I_1(G)$ of f and calculate

$$(d_1 - d_0)(F \cdot s(g)) = d_1(F)d_1(s(g)) - d_0(F)d_0(s(g)) = (d_1 - d_0)(F) \cdot g = fg.$$

Thus all products lie in the image. We now turn to the power operations. For this, we again take a preimage $F \in I_1(G)$ of $f \in I_0(G)$, and we calculate

$$\begin{aligned} P^k(f) &= P^k(d_1(F) - d_0(F)) \\ &= d_1(P^k(F)) - d_0(P^k(F)) - \sum_{i=1}^{k-1} \mathrm{tr}_{k-i,i}(P^{k-i}(d_1(F) - d_0(F)) \times P^i(d_0(F))) \\ &= (d_1 - d_0)(P^k(F)) - \sum_{i=1}^{k-1} \mathrm{tr}_{k-i,i}(P^{k-i}(f) \times P^i(d_0(F))). \end{aligned}$$

Here, we used additivity for $P^k((d_1(F) - d_0(F)) + d_0(F))$ for the decomposition in the first line. In the last term, we now observe that the first summand lies in the image of $I_1^{\geq 2}$. All other summands are products of elements in I_0, and hence lie in the image of $I_1^{\geq 2}$ by the previous arguments. This finishes the proof. □

1.5.d Failure of Transitivity and Base Change

In the classical theory for commutative rings, André-Quillen (co-)homology exhibits a long exact sequence for a sequence of rings. This extends the exact sequences of Kähler differentials and derivations given in Proposition 1.2.29. We show that this three-term exact sequence for global power functors is extended to a six-term exact sequence by the Exalcomm-functor in Theorem 1.4.5. For commutative rings, the higher André-Quillen groups extend this to a long exact sequence. Concretely, for morphisms $R \to S \to T$ of commutative rings, the natural maps of cotangent complexes

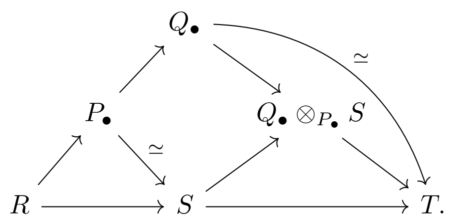
$$T \otimes_S \mathbb{L}_{S/R} \to \mathbb{L}_{T/R} \to \mathbb{L}_{T/S} \tag{1.5.23}$$

form an exact triangle in the triangulated category of chain complexes of T-modules, ie a short exact sequence of chain complexes. We obtain exact sequences in homology and cohomology. The cohomology sequence for example reads

$$\ldots \to D^n(T, S; M) \to D^n(T, R; M) \to D^n(S, R; M) \to D^{n+1}(T, S; M) \to \ldots$$

These results are proved in [53, Theorem 5.1], [1, Proposition 18.2] and [2, Theorem V.1].

We quickly sketch the proof for this transitivity sequence as given by Quillen in [53, Theorem 5.1]. For the sequence of commutative rings $R \to S \to T$, we consider the diagram



(1.5.24)

In this diagram, we choose a resolution $R \to P_\bullet \to S$ of S by a polynomial simplicial R-algebra $P_\bullet$ and a resolution $P_\bullet \to Q_\bullet \to T$ of T by a polynomial simplicial $P_\bullet$-algebra $Q_\bullet$. We then form $Q_\bullet \otimes_{P_\bullet} S$. This receives a map from S, which exhibits it as a polynomial simplicial S-algebra. Moreover, since the polynomial algebra $Q_\bullet$ over $P_\bullet$ is in particular flat, we can conclude that the base-change of the weak equivalence $P_\bullet \to S$ to $Q_\bullet \to Q_\bullet \otimes_{P_\bullet} S$ also is a weak equivalence. By the 2-out-of-3 property, also $Q_\bullet \otimes_{P_\bullet} S \to T$ is a weak equivalence. In total, we have constructed a resolution $S \to Q_\bullet \otimes_{P_\bullet} S \to T$ of T by a polynomial simplicial S-algebra. Using these three resolutions, we obtain the distinguished triangle (1.5.23) by applying the transitivity sequence of Kähler differentials to $R \to P_\bullet \to Q_\bullet$, using base change for Kähler differentials from $Q_\bullet/P_\bullet$ to $Q_\bullet \otimes_{P_\bullet} S/S$, and tensoring with T. Since the involved algebras here are polynomial and thus smooth, the resulting sequence is (levelwise split) short exact by Proposition 1.4.16.

When we try to transfer this result to our André-Quillen theory for global power functors, we may attempt to mimic the above proof. However, this fails as in the category of modules over a global power functor, projective modules might not be flat. This observation goes back to a study of flatness in various categories of equivariant algebra conducted by Lewis in [44]. This then implies that a polynomial algebra might not be flat. Hence, we are no longer able to conclude that $Q_\bullet \otimes_{P_\bullet} S \to T$ is a weak equivalence.

In fact, we can explicitly calculate that the six-term exact sequence of the low-dimensional André-Quillen homology groups as identified in Section 1.5.b does not extend to a long exact sequence. For this, we first recall Lewis's example of a projective global functor that is not flat.

Proposition 1.5.25. *Let p be a prime. Then the global functor $\mathbb{A}(\mathbb{Z}/p, \_)$ is free, hence projective, but not flat.*

Proof. This is the statement of [44, Theorem 6.10]. For this, we observe that the category of Fin-global functor is the category of $(\emptyset, \infty)$-Mackey functors as considered by Lewis. The same calculation also goes through in the case of compact Lie groups. There is a slight mistake in the description in the map $\tilde{\tau}^e_{\mathbb{Z}/p}(\mathbb{Z}/p)$ on page 244, where restriction along the non-surjective endomorphism of $\mathbb{Z}/p$ should be sent to p times the inflation, but this does not change the argument. ☐

The concrete example Lewis uses for this proof is the sequence $\mathbb{A}(\mathbb{Z}/p, \_) \xrightarrow{(\mathrm{tr}_e^{\mathbb{Z}/p})^*} \mathbb{A} \to \mathbb{A}/\operatorname{tr}_e^{\mathbb{Z}/p}$. The proof cited above can be interpreted as considering this sequence as the start of a projective resolution of $\mathbb{A}/\operatorname{tr}_e^{\mathbb{Z}/p}$, on which taking $\Box\mathbb{A}(\mathbb{Z}/p, \_)$ picks up a non-trivial Tor_1-term.

Guided by this example, we state the following comparison result for certain first André-Quillen homology groups that implies that the long transitivity sequence fails to be exact in general. In this statement, the notation $\langle f \rangle$ denotes the power ideal generated by f. We also observe that this statement expresses that the corresponding base-change result [53, Theorem 5.3] fails, which usually can be deduced as a corollary from the transitivity sequence.

Proposition 1.5.26. *Consider the global power functors $S = \mathbb{A}[x_e]/\langle \operatorname{tr}_e^{\mathbb{Z}/2}(x) \rangle$ and $T = S[t_{\mathbb{Z}/2}]$. Then the natural morphism*

$$T \Box_S D_1(S, \mathbb{A}) \to D_1(T, \mathbb{A})$$

is surjective but not injective.

Before we give the proof by explicit calculations, we record the consequence for the transitivity sequence.

Theorem 1.5.27. *For general morphisms $R \to S \to T$ of global power functors and T-modules M, there is no possible choice of connecting homomorphisms $\partial_n \colon D_n(T, S; M) \to T \Box_S D_{n-1}(S, R; M)$ such that the transitivity sequence*

$$\ldots \to T \Box_S D_n(S, R; M) \to D_n(T, R; M) \to D_n(T, S; M) \to T \Box_S D_{n-1}(S, R; M) \to \ldots$$

is exact. In fact, this fails even for $n = 2$.

Proof. We consider the sequence $\mathbb{A} \to S \to T$ from Proposition 1.5.26. Since T is polynomial over S, we see that $D_n(T, S; M) = 0$ for all $n \geq 1$ and all T-modules M. Hence, exactness of the transitivity sequence implies that

$$T \Box_S D_1(S, \mathbb{A}) \to D_1(T, \mathbb{A})$$

is an isomorphism. This is not the case by Proposition 1.5.26. □

Remark 1.5.28. In his book [2], André gives an overview of the homology theory for commutative rings and gives modified arguments for many results from [1]. In particular, in this proof of the transitivity sequence (which he calls Jacobi-Zariski sequence), he shows that it can be deduced from the property that enlarging an R-algebra S to a polynomial S-algebra T only base-changes the higher André-Quillen homology groups over R from S to T ([2, Condition V.9]). This is exactly the property we show to fail for global power algebras in Proposition 1.5.26. That this condition is necessary is the content of the proof of Theorem 1.5.27. André's arguments show that this special case of the transitivity sequence is also sufficient.

The condition mentioned above is always satisfied for polynomial R-algebras S by the calculations in Proposition 1.5.9. This shows that if the middle global power functor in the transitivity sequence is polynomial, we in fact do obtain a transitivity result. In the classical case, André shows in [2, V.11] that one can bootstrap the general case from this special case. However, this argument also uses that polynomial algebras are flat, which is no longer the case for global power functors.

Remark 1.5.29. The reader well versed in the literature on equivariant André-Quillen (co-)homology may find the above statement about failure of the transitivity sequence surprising. The reason for this is that Leeman claims in his PhD-thesis [42, Proposition 3.4.11] that André-Quillen homology and cohomology for Tambara functors do come equipped with a transitivity sequence. On the other hand, Tambara functors also exhibit the property that polynomial algebras are not flat, see [32] for an extensive study (even though this comes about because polynomial algebras are seldom free as Mackey functors, whereas projective Mackey functors are indeed flat). In fact, I believe that Leeman's proof is not correct, and expect that also for Tambara functors, counterexamples for the transitivity sequence can be found. The faulty argument in the cited proof (transcribed into our notation) lies in the analysis of the pushout

$$\begin{array}{ccc} P_\bullet & \rightarrowtail & Q_\bullet \\ \downarrow{\simeq} & & \downarrow \\ S & \longrightarrow & Q_\bullet \Box_{P_\bullet} S, \end{array}$$

which is part of the Diagram 1.5.24. Leeman claims that since the $P_\bullet$ and $Q_\bullet$ are cofibrant in the model category of simplicial R-algebras and $P_\bullet \to Q_\bullet$ is a cofibration, the pushout has to preserve the weak equivalence $P_\bullet \to S$ by a classical argument, see eg [33, Proposition 13.2]. However, this argument actually needs S to be cofibrant over R, which is generally not the case. In fact, this is the reason we need to resolve S by $P_\bullet$ in the first place.

Proof of Proposition 1.5.26. We calculate the first André-Quillen homology modules by using the naive cotangent complex from Proposition 1.5.21. We start by performing the calculations for $S = \mathbb{A}[x_e]/\langle \mathrm{tr}_e^{\mathbb{Z}/2}(x) \rangle$. We calculate the values of the André-Quillen global functors for the groups e and $\mathbb{Z}/2$. At the trivial group, we recover the classical results from the case of commutative rings and the morphism in question is an isomorphism. At $\mathbb{Z}/2$, we obtain that the morphism is not injective.

The naive cotangent complex depends on a surjection $P \to S$ from a polynomial $\mathbb{A}$-algebra P. We choose the canonical surjection $P = \mathbb{A}[x_e] \to \mathbb{A}[x_e]/\langle \mathrm{tr}_e^{\mathbb{Z}/2}(x)\rangle = S$. Then we have that the Kähler differentials $\Omega^1_{P/\mathbb{A}} = P\{dx\} \cong \mathbb{A}[x]\{dx\}$ are a free P-module on a generator at e by Proposition 1.2.15. Thus the 0-th term of the naive cotangent complex is

$$(\mathbb{L}^{\mathrm{naive}}_{S/P/\mathbb{A}})_0 = \Omega^1_{P/\mathbb{A}} \Box_P S = S\{dx\}.$$

We explicitly calculate the values at the groups e and $\mathbb{Z}/2$. For this, we recall that $\mathbb{A}[x_e](G) = \bigoplus_{m\geq 0} \mathbb{A}(\Sigma_m, G)$. Using the description of the Burnside ring global power functor from [55, Theorem 4.2.6], we obtain the description of the polynomial global power functor $\mathbb{A}[x_e]$ as explained in Remark A.2.1, as

$$\begin{aligned} P(e) &= \mathbb{Z}[x] \\ P(\mathbb{Z}/2) &= \mathbb{Z}[P^2(x), p^*(x)] \ltimes \bigoplus_{k\geq 0} \mathbb{Z}\{\mathrm{tr}_e^{\mathbb{Z}/2}(x^k)\}. \end{aligned}$$

Here, $p\colon \mathbb{Z}/2 \to e$ is the unique homomorphism, and $p^*(x)$ is the inflation of x along this homomorphism. The product structure on the left summand of $P(\mathbb{Z}/2)$ is the one of a polynomial ring on the two designated generators, as both power operations and inflations are multiplicative. The symbol $\ltimes$ indicates that $\bigoplus_{k\geq 0} \mathbb{Z}\{\mathrm{tr}_e^{\mathbb{Z}/2}(x^k)\}$ is an ideal in $P(\mathbb{Z}/2)$, with multiplication determined by the formulas $\mathrm{tr}(x^i)\,\mathrm{tr}(x^k) = 2\,\mathrm{tr}(x^{i+k})$, $P^2(x)\,\mathrm{tr}(x^k) = \mathrm{tr}(x^{k+2})$ and $p^*(x)\,\mathrm{tr}(x^k) = \mathrm{tr}(x^{k+1})$. These relations follow from Frobenius reciprocity.
From this description, we obtain that $\Omega^1_{P/\mathbb{A}} \Box_P S \cong S\{dx\}$ has the form

$$\begin{aligned} S\{dx\}(e) &= \mathbb{Z}[x]/(2x)\{dx\} \\ S\{dx\}(\mathbb{Z}/2) &= \left(\mathbb{Z}[P^2(x), p^*(x)]/(2P^2(x), 2p^*(x)) \oplus \mathbb{Z}\{\mathrm{tr}_e^{\mathbb{Z}/2}(1)\}\right)\{dx\}. \end{aligned}$$

For the next term of the naive cotangent complex of S, we denote $I = \ker(P \to S) = \langle \mathrm{tr}_e^{\mathbb{Z}/2}(x)\rangle$ and calculate $I/I^{\geq 2}$. We calculate from the above description of P that I has the form

$$\begin{aligned} I(e) &= (2x) \\ I(\mathbb{Z}/2) &= (2P^2(x), 2p^*(x), \mathrm{tr}_e^{\mathbb{Z}/2}(x^k) \text{ for } k \geq 1), \end{aligned}$$

where each level is considered as an ideal in $P(G)$ for $G = e, \mathbb{Z}/2$ on the specified generators. From this, we calculate $I^{\geq 2}$ as

$$\begin{aligned} I^{\geq 2}(e) &= (4x^2) \\ I^{\geq 2}(\mathbb{Z}/2) &= (2P^2(x) + \mathrm{tr}_e^{\mathbb{Z}/2}(x^2), 4p^*(x)^2, 2\,\mathrm{tr}_e^{\mathbb{Z}/2}(x^k) \text{ for } k \geq 2), \end{aligned}$$

and $I/I^{\geq 2}$ as

$$\begin{aligned} (I/I^{\geq 2})(e) &= 2x \cdot \mathbb{Z}[x]/2x = \mathbb{Z}\{2x\} \oplus \bigoplus_{k\geq 2} \mathbb{Z}/2\{2x^k\} \\ (I/I^{\geq 2})(\mathbb{Z}/2) &= \mathbb{Z}\{2p^*(x)\} \oplus \bigoplus_{k\geq 2} \mathbb{Z}/2\{2p^*(x)^k\} \oplus \mathbb{Z}\{\mathrm{tr}_e^{\mathbb{Z}/2}(x)\} \oplus \bigoplus_{k\geq 2} \mathbb{Z}/2\{\mathrm{tr}_e^{\mathbb{Z}/2}(x^k)\}. \end{aligned}$$

Note that $P^2(2x) \in I^{\geq 2}$ by definition, and $P^2(2x) = \mathrm{tr}_e^{\mathbb{Z}/2}(x^2) + 2P^2(x)$ by additivity of the power operations. Generally, $P^2(2nx)$ is also contained in $I^{\geq 2}(\mathbb{Z}/2)$, which can explicitly be checked with the above description and the $\mathbb{A}$-linearity of the power operations. The fact that this element is contained in $I^{\geq 2}$ allows us to identify $2P^2(x)$ with $-\operatorname{tr}_e^{\mathbb{Z}/2}(x^2)$ in $I/I^{\geq 2}$, and hence we do not introduce a generator $2P^2(x)$ in the above description.

We have now computed the naive cotangent complex $\mathbb{L}^{\mathrm{naive}}_{S/P/\mathbb{A}} = (I/I^{\geq 2} \xrightarrow{d} \Omega^1_{P/\mathbb{A}} \Box_P S)$. The first André-Quillen homology group is now the kernel of the map d. Explicitly, the morphism d comes from the universal derivation and hence takes the form

$$\begin{aligned} \mathbb{Z}\{2x\} \oplus \bigoplus_{k\geq 2} \mathbb{Z}/2\{2x^k\} &\to \mathbb{Z}[x]/(2x)\{dx\} \\ 2x^k &\mapsto 2kx^{k-1}dx \\ \begin{pmatrix} \mathbb{Z}\{2p^*(x), \operatorname{tr}_e^{\mathbb{Z}/2}(x)\} \oplus \\ \bigoplus_{k\geq 2} \mathbb{Z}/2\{2p^*(x)^k, \operatorname{tr}_e^{\mathbb{Z}/2}(x^k)\} \end{pmatrix} &\to \left(\mathbb{Z}[P^2(x), p^*(x)]/(2P^2(x), 2p^*(x)) \oplus \mathbb{Z}\{\operatorname{tr}_e^{\mathbb{Z}/2}(1)\}\right)\{dx\} \\ \operatorname{tr}_e^{\mathbb{Z}/2}(x^k) &\mapsto k \operatorname{tr}_e^{\mathbb{Z}/2}(x^{k-1})dx \\ 2p^*(x^k) &\mapsto 2kp^*(x^{k-1})dx. \end{aligned}$$

The kernel is thus given by

$$\begin{aligned} D_1(S, \mathbb{A})(e) &= \bigoplus_{k\geq 2} \mathbb{Z}/2\{2x^k\} \\ D_1(S, \mathbb{A})(\mathbb{Z}/2) &= \bigoplus_{k\geq 2} \mathbb{Z}/2\{2p^*(x^k)\} \oplus \bigoplus_{k\geq 2} \mathbb{Z}/2\{\operatorname{tr}_e^{\mathbb{Z}/2}(x^k)\}. \end{aligned}$$

Finally, we now calculate $D_1(S, \mathbb{A})\Box_S T \cong D_1(S, \mathbb{A})\Box\mathbb{A}[t_{\mathbb{Z}/2}]$. We describe the polynomial global power functor $\mathbb{A}[t_{\mathbb{Z}/2}]$ in Corollary A.2.3. The classical coend formulation, recalled in Remark 1.1.9, implies that only the levels e and $\mathbb{Z}/2$ of the two box-product factors influence the value of the box product at e and $\mathbb{Z}/2$. Considering all relations, we obtain

$$\begin{aligned} (D_1(S, \mathbb{A})\Box_S T)(e) &= \bigoplus_{k\geq 2} \mathbb{Z}/2[\operatorname{res}_e^{\mathbb{Z}/2}(t)]\{2x^k\} \\ (D_1(S, \mathbb{A})\Box_S T)(\mathbb{Z}/2) &= \begin{pmatrix} \bigoplus_{k\geq 2} \mathbb{Z}/2[t, P^2(\operatorname{res}_e^{\mathbb{Z}/2}(t)), p^*(\operatorname{res}_e^{\mathbb{Z}/2}(t))]\{\operatorname{tr}_e^{\mathbb{Z}/2}(x^k)\} \oplus \\ \bigoplus_{k\geq 2} \mathbb{Z}/2[t, P^2(\operatorname{res}_e^{\mathbb{Z}/2}(t)), p^*(\operatorname{res}_e^{\mathbb{Z}/2}(t))]\{2p^*(x^k)\} \end{pmatrix}. \end{aligned}$$

We now calculate the corresponding first André-Quillen homology global functor at the groups e and $\mathbb{Z}/2$ for $\mathbb{A} \to T = \mathbb{A}[x_e, t_{\mathbb{Z}/2}]/\langle \operatorname{tr}_e^{\mathbb{Z}/2}(x)\rangle$. Here, we use as input for the naive cotangent complex the canonical surjection $Q = \mathbb{A}[x_e, t_{\mathbb{Z}/2}] \to T$. We again obtain that the Kähler differentials $\Omega^1_{Q/\mathbb{A}} = Q\{dx, dt\}$ are a free Q-module on two generators at e and $\mathbb{Z}/2$ by Proposition 1.2.15. Thus the 0-th term of the naive cotangent complex is

$$(\mathbb{L}^{\mathrm{naive}}_{T/Q/\mathbb{A}})_0 = \Omega^1_{Q/\mathbb{A}} \Box_Q T = T\{dx, dt\}.$$

We again start by writing down the polynomial algebra $Q = \bigoplus_{m,n\geq 0} \mathbb{A}((\Sigma_m \wr \mathbb{Z}/2) \times \Sigma_n, \_)$. The computation of $\mathbb{A}(\Sigma_m \wr \mathbb{Z}/2, \mathbb{Z}/2)$ is carried out in Corollary A.2.3. For Q, we obtain

$$\begin{aligned} Q(e) &= \mathbb{Z}[x, \operatorname{res}_e^{\mathbb{Z}/2}(t)] \\ Q(\mathbb{Z}/2) &= \mathbb{Z}[P^2(x), p^*(x), t, P^2(\operatorname{res}_e^{\mathbb{Z}/2}(t)), p^*(\operatorname{res}_e^{\mathbb{Z}/2}(t))] \ltimes \bigoplus_{k,l\geq 0} \mathbb{Z}\{\operatorname{tr}_e^{\mathbb{Z}/2}(x^k \operatorname{res}_e^{\mathbb{Z}/2}(t)^l)\}. \end{aligned}$$

Here, the multiplication on the left summand is again that of a polynomial ring, and the right summand on the transfers is an ideal, with multiplications given by Frobenius reciprocity. In particular, the term $\mathrm{tr}_e^{\mathbb{Z}/2}(x^k \operatorname{res}_e^{\mathbb{Z}/2}(t)^l)$ agrees with $\mathrm{tr}_e^{\mathbb{Z}/2}(x^k)t^l$. This determines the Kähler differentials.

We now compute the next term of the naive cotangent complex. We denote the kernel of the surjection $Q \to T$ by $J = \langle \mathrm{tr}_e^{\mathbb{Z}/2}(x) \rangle$. We have

$$\begin{aligned} J(e) &= (2x) \\ J(\mathbb{Z}/2) &= (2P^2(x), 2p^*(x), \mathrm{tr}_e^{\mathbb{Z}/2}(x^k) \text{ for } k \geq 1), \end{aligned}$$

where each level is considered as an ideal in $Q(G)$ for $G = e, \mathbb{Z}/2$ on the specified generators. From this, we calculate $J^{\geq 2}$ as

$$\begin{aligned} J^{\geq 2}(e) &= (4x^2) \\ J^{\geq 2}(\mathbb{Z}/2) &= (2P^2(x) + \mathrm{tr}_e^{\mathbb{Z}/2}(x^2), 4p^*(x)^2, 2\,\mathrm{tr}_e^{\mathbb{Z}/2}(x^k) \text{ for } k \geq 2), \end{aligned}$$

and $J/J^{\geq 2}$ as

$$(J/J^{\geq 2})(e) = 2x \cdot \mathbb{Z}[x, \operatorname{res}_e^{\mathbb{Z}/2}(t)]/2x = \mathbb{Z}[\operatorname{res}_e^{\mathbb{Z}/2}(t)]\{2x\} \oplus \bigoplus_{k\geq 2} \mathbb{Z}/2[\operatorname{res}_e^{\mathbb{Z}/2}(t)]\{2x^k\}$$

$$(J/J^{\geq 2})(\mathbb{Z}/2) = \begin{pmatrix} \mathbb{Z}[t, P^2(\operatorname{res}_e^{\mathbb{Z}/2}(t)), p^*(\operatorname{res}_e^{\mathbb{Z}/2}(t))]\{2p^*(x)\} \\ \oplus \bigoplus_{k\geq 2} \mathbb{Z}/2[t, P^2(\operatorname{res}_e^{\mathbb{Z}/2}(t)), p^*(\operatorname{res}_e^{\mathbb{Z}/2}(t))]\{2p^*(x)^k\} \\ \oplus \mathbb{Z}[t]\{\mathrm{tr}_e^{\mathbb{Z}/2}(x)\} \oplus \bigoplus_{k\geq 2} \mathbb{Z}/2[t]\{\mathrm{tr}_e^{\mathbb{Z}/2}(x^k)\} \end{pmatrix}.$$

For the map to $\Omega^1_{Q/\mathbb{A}} \Box_Q T$, we obtain similar formulas as in the calculation for S. Calculating the kernel then gives as André-Quillen homology groups

$$\begin{aligned} D_1(T, \mathbb{A})(e) &= \bigoplus_{k\geq 2} \mathbb{Z}/2[\operatorname{res}_e^{\mathbb{Z}/2}(t)]\{2x^k\} \\ D_1(T, \mathbb{A})(\mathbb{Z}/2) &= \bigoplus_{k\geq 2} \mathbb{Z}/2[t]\{\mathrm{tr}_e^{\mathbb{Z}/2}(x^k)\} \oplus \bigoplus_{k\geq 2} \mathbb{Z}/2[t, P^2(\operatorname{res}_e^{\mathbb{Z}/2}(t)), p^*(\operatorname{res}_e^{\mathbb{Z}/2}(t))]\{2p^*(x^k)\}. \end{aligned}$$

Comparing this term to the calculation of $D_1(S, \mathbb{A}) \Box_S T$ above, we observe that the natural comparison map on

$$\mathbb{Z}/2[t, P^2(\operatorname{res}_e^{\mathbb{Z}/2}(t)), p^*(\operatorname{res}_e^{\mathbb{Z}/2}(t))]\{\mathrm{tr}_e^{\mathbb{Z}/2}(x^k)\} \to \mathbb{Z}/2[t]\{\mathrm{tr}_e^{\mathbb{Z}/2}(x^k)\}$$

identifies polynomials in t, $P^2(\operatorname{res}_e^{\mathbb{Z}/2}(t))$ and $p^*(\operatorname{res}_e^{\mathbb{Z}/2}(t))$ if they have the same restriction to e. Hence, the map is not injective. $\square$

Remark 1.5.30. When carefully going through the calculations in the proof of Proposition 1.5.26, we see that the reason the considered morphism is not injective is that in the two calculations of $D_1(S, \mathbb{A}) \Box_S T$ and $D_1(T, \mathbb{A})$, the Frobenius reciprocity for terms of the form $F \cdot \mathrm{tr}_e^{\mathbb{Z}/2}(x)$ with $F \in \mathbb{Z}[t, P^2(\operatorname{res}_e^{\mathbb{Z}/2}(t)), p^*(\operatorname{res}_e^{\mathbb{Z}/2}(t))]$ is used at different steps of the calculation. In the first one, it is used at a step where this "transfer" is not actually a transfer of anything at level e, since the term it is a transfer of has been killed during the calculation of $D_1(S, \mathbb{A})$. Thus, we cannot apply Frobenius reciprocity for this term. In the calculation of the André-Quillen homology for T, we use the Frobenius reciprocity already in T, and there $\mathrm{tr}_e^{\mathbb{Z}/2}(x)$ still is a transfer of x. Hence we identify terms of the form $F \cdot \mathrm{tr}_e^{\mathbb{Z}/2}(x)$ when the restrictions of F agree.

Remark 1.5.31. As the functor $\_\Box_S T$ is not exact since polynomial algebras need not be flat, there is another candidate for the spot in the transitivity sequence occupied by $D_1(S, \mathbb{A})\Box_S T$, namely $H_1(\mathbb{L}^{\text{naive}}_{S/P/\mathbb{A}}\Box_S T)$. However, also this term does not agree with $D_1(T, \mathbb{A})$. Using the calculation of the naive cotangent complex in the proof of Proposition 1.5.26, we can calculate that this term takes the form

$$\begin{aligned} H_1(\mathbb{L}^{\text{naive}}_{S/P/\mathbb{A}}\Box_S T)(e) &= \bigoplus_{k\geq 2} \mathbb{Z}/2[\operatorname{res}^{\mathbb{Z}/2}_e(t)]\{2x^k\} \\ H_1(\mathbb{L}^{\text{naive}}_{S/P/\mathbb{A}}\Box_S T)(\mathbb{Z}/2) &= \mathbb{Z}/2\{\operatorname{tr}^{\mathbb{Z}/2}_e(x)\} \otimes \ker(\operatorname{res}\colon \mathbb{A}[t](\mathbb{Z}/2) \to \mathbb{A}[t](e)) \\ &\oplus \bigoplus_{k\geq 2} \mathbb{Z}/2[t, P^2(\operatorname{res}^{\mathbb{Z}/2}_e(t)), p^*(\operatorname{res}^{\mathbb{Z}/2}_e(t))]\{\operatorname{tr}^{\mathbb{Z}/2}_e(x^k)\} \\ &\oplus \bigoplus_{k\geq 2} \mathbb{Z}/2[t, P^2(\operatorname{res}^{\mathbb{Z}/2}_e(t)), p^*(\operatorname{res}^{\mathbb{Z}/2}_e(t))]\{2p^*(x^k)\}. \end{aligned}$$

This differs from $D_1(S, \mathbb{A})\Box_S T$ by the additional first summand, which again comes about since we are able to use Frobenius reciprocity one step earlier, namely on the level of the naive cotangent complex. This then picks up additional elements in the kernel. It also differs from $D_1(T, \mathbb{A})$.

Chapter 2

Topological André-Quillen (Co-)Homology of Ultra-Commutative Ring Spectra

In this second part of the dissertation, we study ultra-commutative ring spectra. These are modelled by commutative orthogonal ring spectra in the positive global model category of orthogonal spectra, as defined in [55]. The name “ultra-commutative” is chosen in order to convey that such ring spectra represent a very rich structure, in particular this structure is stronger than that of E_∞-ring spectra in global spectra. Specifically, the homotopy groups and cohomology groups of global spaces represented by such spectra inherit not only the structure of commutative monoids in global functors (called global Green functors), but also additional equivariant power operations. Thus, we obtain a global power functor $\underline{\pi}_0(R)$ from such an ultra-commutative ring spectrum R. These power operations are equivalent to a global version of norm maps, which recently have seen many applications in equivariant homotopy theory. The prime example is the usage of genuine equivariant commutative ring spectra in the solution of the Kervaire invariant one problem by Hill-Hopkins-Ravenel [29].

Hence, the study of ultra-commutative ring spectra is an interesting subject. In this section, we construct a cohomology theory for such ring spectra, which can then be used to study the multiplication in detail. We aim to provide the tools which later might be applied in the construction of an obstruction theory for ultra-commutative ring spectra. A similar program for commutative ring spectra was initiated by work of Kriz [39], who proposed to use a topological version of André-Quillen cohomology to build Postnikov towers of commutative ring spectra. The aim in that project was to study multiplications on the Brown-Peterson spectrum BP. The theoretical groundwork was later carried out by Basterra [10], and applied by Basterra and Mandell to put an E_4-multiplication on BP in [12]. Also using topological André-Quillen cohomology, Goerss and Hopkins [26] built an obstruction theory to show that Morava E-theory spectra have a unique E_∞-ring structure.

Thus, we observe that topological André-Quillen cohomology has proven to be a useful cohomology theory for commutative ring spectra. Hence, in this chapter, we generalize it to topological André-Quillen cohomology of ultra-commutative ring spectra. For this, we use a model-categorical approach, utilizing the model of orthogonal ring spectra with a positive global model structure, as considered in [55, Chapter 5]. Hence, the first step is to study the construction of topological André-Quillen cohomology in a general model category. This is carried out in Section 2.1, and follows the constructions given by Basterra in [10, Sections 2-4]. In this section, we also show that the transitivity sequence of topological André-Quillen cohomology can be deduced in this general context.

In Section 2.2, we then apply this theory to the model category of ultra-commutative ring spectra, and obtain a topological André-Quillen cohomology of ultra-commutative ring spectra. This cohomology theory is well-behaved in that it comes equipped with a transitivity long exact sequence, and satisfies base-change and additivity results. Moreover, we exhibit a Hurewicz theorem for this cohomology theory, treating the behaviour on connective algebras, and construct Postnikov towers of ultra-commutative ring spectra. In these Postnikov towers, the k-invariants lie in topological André-Quillen cohomology, and refine the classical k-invariants as global spectra through a comparison map to the underlying cohomology. In this way, we expect an obstruction theory to arise by studying lifts of the k-invariants along this comparison map.

2.1 André-Quillen Homology in Abstract Model Categories

In this section, we define André-Quillen cohomology for commutative algebra objects in an abstract symmetric monoidal model category $\mathcal{C}$. We follow the outline of [10], generalizing it to a cofibrantly generated symmetric monoidal model category $\mathcal{C}$ that supports a model category structure on categories of commutative algebras and modules. The classical example in [10] is the category of spectra, realized as S-modules. In this category, the commutative monoids are commutative S-algebras, modelling commutative ring spectra, or equivalently E_∞-ring spectra. In Section 2.2, we apply the theory to the model category of orthogonal spectra with the global model structure defined in [55], thus defining topological André-Quillen cohomology for ultra-commutative ring spectra.

Throughout this chapter, we assume that $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category, satisfying the monoid and strong commutative monoid axiom. We moreover assume that $\mathcal{C}$ is pointed and that the symmetric monoidal structure, denoted $\wedge$, distributes over the coproduct $\vee$, and that coproducts $\vee$ are homotopical.
Under these assumptions, we obtain induced model category structures on the categories of algebras and modules. For a commutative monoid R in $\mathcal{C}$, we denote by Mod_R the category of R-modules, by CAlg_R the category of commutative R-algebras and by CAlg_R^+ the category of non-unital commutative R-algebras. Moreover, for a commutative R-algebra B, we denote by CAlg_R/B the category of R-algebras augmented to B. These categories all inherit model structures from $\mathcal{C}$ by the results obtained in [56, 66] and Appendix B.

2.1.a Construction of the Cotangent Complex

We define the functors from which we obtain the cotangent complex and the definition of André-Quillen cohomology. Since usual André-Quillen cohomology is defined in Definition 1.5.6 as the derived functor of derivations, represented by the module of Kähler differentials, we mimic the construction of the module of Kähler differentials in this context. This is accomplished by recalling the construction of Ω^1 as the module of indecomposables for the kernel of the multiplication map in Theorem 1.2.24.

Definition 2.1.1. Let R be a commutative monoid in $\mathcal{C}$. Then for an augmented R-algebra $A \in \mathrm{CAlg}_R/R$ with structure morphisms $\eta\colon R \to A$ and $\varepsilon\colon A \to R$, we define the augmentation ideal $I(A)$ as the pullback

$$\begin{array}{ccc} I(A) & \xrightarrow{i} & A \\ \downarrow & & \downarrow{\scriptstyle \varepsilon} \\ * & \longrightarrow & R \end{array}$$

in the category of R-modules. This comes equipped with a multiplication map $\mu\colon I(A)\wedge_R I(A) \to I(A)$ induced from the diagram

$$\begin{array}{ccccc} I(A)\wedge_R I(A) & \xrightarrow{i\wedge i} & A\wedge_R A & \xrightarrow{\mu_A} & A \\ \downarrow & & \downarrow{\scriptstyle \varepsilon\wedge\varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ * & \longrightarrow & R\wedge_R R & \xrightarrow[\mu_R]{} & R. \end{array}$$

This defines a functor $I\colon \operatorname{CAlg}_R/R \to \operatorname{CAlg}_R^+$.
Moreover, for a non-unital R-algebra J, we define $K(J) = R \vee J$, with multiplication given as

$$(R\vee J)\wedge_R (R\vee J) \cong R \vee (R\wedge_R J) \vee (J \wedge_R R) \vee (J\wedge_R J) \xrightarrow{\text{multiply}} R\vee J.$$

This R-algebra comes equipped with an augmentation $R\vee J \to R$ by projection onto the R-summand. This defines a functor $K\colon \operatorname{CAlg}_R^+ \to \operatorname{CAlg}_R/R$.

Proposition 2.1.2. *The functors*

$$\operatorname{CAlg}_R^+ \underset{I}{\overset{K}{\rightleftarrows}} \operatorname{CAlg}_R/R$$

are adjoint, with K left and I right adjoint. The adjunction unit is given as

$$u\colon \operatorname{Id} \to IK;\ J \to I(R\vee J) \cong J$$

via the pullback square

$$\begin{array}{ccc} J & \xrightarrow{\text{incl}} & R\vee J \\ \downarrow & & \downarrow{\scriptstyle \text{pr}} \\ * & \longrightarrow & R. \end{array}$$

The adjunction counit is given as

$$c\colon KI \to \operatorname{Id};\ R\vee I(A) \xrightarrow{\eta\vee i} A.$$

Proof. We need to check that the triangle equalities hold. Thus, we consider the diagrams

$$\begin{array}{ccc} R\vee J & \xrightarrow{R\vee u} & R\vee I(R\vee J) \\ & \searrow^{=} & \downarrow{\scriptstyle c_{R\vee J}} \\ & & R\vee J \end{array} \quad \text{and} \quad \begin{array}{ccc} I(A) & & \\ {\scriptstyle u_{I(A)}}\downarrow & \searrow^{=} & \\ I(R\vee I(A)) & \xrightarrow[I(c)]{} & I(A). \end{array}$$

Both diagrams commutes, thus the maps c and u indeed exhibit K and I as adjoint. □

Proposition 2.1.3. *Let $\mathcal{C}$ be a cofibrantly generated symmetric monoidal model category satisfying the monoid and the strong commutative monoid axiom. Then the adjunction*

$$\operatorname{CAlg}_R^+ \underset{I}{\overset{K}{\rightleftarrows}} \operatorname{CAlg}_R/R$$

is a Quillen equivalence, for the induced model structures on the categories of non-unital and augmented commutative algebras defined in Appendix B. Moreover, the functor K is homotopical.

Proof. We first prove that K preserves both cofibrations and acyclic cofibrations of non-unital commutative R-algebras. Let I^{gen} be a set of generating cofibrations and J^{gen} be a set of generating acyclic cofibrations for $\mathcal{C}$. Then, the proof of Theorem B.2.6 shows that the sets $R \wedge \mathbb{P}^{+} I^{\text{gen}}$ and $R \wedge \mathbb{P}^{+} J^{\text{gen}}$ are generating cofibrations and generating acyclic cofibrations for the model category CAlg_R^{+}. Moreover, the corresponding sets $R \wedge \mathbb{P} I^{\text{gen}}$ and $R \wedge \mathbb{P} J^{\text{gen}}$ are the generating sets for CAlg_R /R, with augmentations given by projection on the first wedge summand of $\mathbb{P}$. Since K as a left adjoint preserves colimits, it is enough to check that K preserves the generating cofibrations and acyclic cofibrations. But $K(R \wedge \mathbb{P}^{+}(i)) \cong R \wedge \mathbb{P}(i)$, and thus K is a left Quillen functor. Hence the adjunction is a Quillen adjunction.

The functor K is homotopical, since weak equivalences of algebras are defined using the underlying morphisms and in $\mathcal{C}$, the wedge sum is homotopical by assumption.

To check that this Quillen adjunction is indeed a Quillen equivalence, we check the following criterion [34, Definition 1.3.12]: Let $J \in \mathrm{CAlg}_R^{+}$ be cofibrant and $A \in \mathrm{CAlg}_R /R$ be fibrant. Then we need to check that a morphism $f\colon KJ \to A$ is a weak equivalence if and only if its adjoint $\tilde{f}\colon J \to I(A)$ is a weak equivalence. So suppose first that $\tilde{f}$ is a weak equivalence. Its adjoint f is given as the composite $R \vee J \xrightarrow{R \vee \tilde{f}} R \vee I(A) \xrightarrow{c} A$. Now if A is a fibrant object in CAlg_R /R, then by the definition of fibrations in the over-category CAlg_R /R [34, Proposition 1.1.8], the augmentation $\varepsilon\colon A \to R$ is a fibration of commutative R-algebras. Thus the defining square

$$\begin{array}{ccc} I(A) & \xrightarrow{i} & A \\ \big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} \\ * & \longrightarrow & R \end{array}$$

is a homotopy fibre square in Mod_R. Thus, it is in particular also a homotopy fibre square in $\mathcal{C}$. Moreover, the unit map $\eta\colon R \to A$ defines a section of ε. This proves that $R \vee I(A) \xrightarrow{c} A$ is a weak equivalence. Moreover, $R \vee \tilde{f}$ is a weak equivalence whenever $\tilde{f}$ is one.

Suppose conversely that f is a weak equivalence. Its adjoint is given as $\tilde{f}\colon J \xrightarrow{u} I(R \vee J) \xrightarrow{I(f)} I(A)$. By considering the commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{\tilde{f}} & I(A) \\ {\scriptstyle \text{incl}}\big\downarrow & & \big\downarrow{\scriptstyle \text{incl}} \\ R \vee J & \xrightarrow{R \vee \tilde{f}} & R \vee I(A) \\ & {\scriptstyle f}\searrow & {\scriptstyle \simeq}\big\downarrow{\scriptstyle c} \\ & & A, \end{array}$$

we observe that $R \vee \tilde{f}$ is a weak equivalence. Moreover, the inclusions in the above diagram and the associated projections $R \vee J \to J$ and $R \vee I(A) \to I(A)$ prove that $\tilde{f}$ is a retract of $R \vee \tilde{f}$. Since weak equivalences are stable under retracts, we conclude that $\tilde{f}$ is a weak equivalence. Thus K and I form a Quillen equivalence. □

Next, we define the indecomposables functor.

Definition 2.1.4. Let R be a commutative monoid, and J be a non-unital commutative R-algebra. Then we define the R-module $Q(J)$ of indecomposables of J via the pushout diagram

$$\begin{array}{ccc} J \wedge_R J & \xrightarrow{\mu} & J \\ \downarrow & & \downarrow{\scriptstyle q} \\ * & \longrightarrow & Q(J) \end{array}$$

in Mod_R. This defines a functor $Q\colon \mathrm{CAlg}_R^+ \to \mathrm{Mod}_R$.
Moreover, for an R-module M, we define the non-unital R-algebra $Z(M)$ by endowing M with the zero-multiplication. This defines a functor $Z\colon \mathrm{Mod}_R \to \mathrm{CAlg}_R^+$.

Proposition 2.1.5. *The functors*

$$\mathrm{CAlg}_R^+ \underset{Z}{\overset{Q}{\rightleftarrows}} \mathrm{Mod}_R$$

are adjoint, with Q left and Z right adjoint. The adjunction counit is given as

$$c\colon QZ \to \mathrm{Id};\ QZ(M) \xrightarrow{\cong} M$$

via the (strict) pushout square

$$\begin{array}{ccc} M \wedge_R M & \xrightarrow{*} & M \\ \downarrow & & \downarrow{\scriptstyle id} \\ * & \longrightarrow & M. \end{array}$$

The adjunction unit is given as the projection

$$u\colon \mathrm{Id} \to ZQ;\ J \xrightarrow{q} ZQ(J).$$

Proof. We need to check that the triangle equalities hold. Thus, we consider the diagrams

$$\begin{array}{ccc} QJ & \xrightarrow{Q(u)} & Q(ZQJ) \\ & \searrow^{=} & \downarrow{\scriptstyle c_{QJ}} \\ & & QJ \end{array} \quad \text{and} \quad \begin{array}{ccc} ZM & & \\ {\scriptstyle u_{ZM}}\downarrow & \searrow^{=} & \\ ZQ(ZM) & \xrightarrow[Z(c)]{} & ZM. \end{array}$$

Both diagrams commutes by straight-forward considerations, thus the maps c and u indeed exhibit Q and Z as adjoint. $\square$

Proposition 2.1.6. *Let $\mathcal{C}$ be a cofibrantly generated symmetric monoidal model category satisfying the monoid and the strong commutative monoid axiom. Then the adjunction*

$$\mathrm{CAlg}_R^+ \underset{Z}{\overset{Q}{\rightleftarrows}} \mathrm{Mod}_R$$

is a Quillen adjunction, for the induced model structures on the categories of non-unital commutative algebras and R-modules defined in Appendix B and [56, Theorem 4.1]. Moreover, the functor Z is homotopical.

Proof. We only need to prove that the right adjoint Z preserves fibrations and weak equivalences. But since both fibrations and weak equivalences are defined on the underlying objects, and Z is the identity on the underlying objects, Z preserves both fibrations and weak equivalences. $\square$

Finally, for a commutative R-algebra B, we have the usual extension-of-scalars adjunction

$$\mathrm{CAlg}_R /B \underset{\text{forget}}{\overset{B\wedge_R(\_)}{\rightleftarrows}} \mathrm{CAlg}_B /B$$

between R-algebras and B-algebras, descending to augmented algebras. Again, since the forgetful functor from B-algebras to R-algebras is the identity on underlying objects and morphisms, it preserves fibrations and weak equivalences, hence this adjunction is a Quillen adjunction as well. We denote the left derived functor of $B \wedge_R \_$ by $B \wedge^{\mathbf{L}}_R \_$.

Definition 2.1.7. Let R be a commutative monoid in the cofibrantly generated symmetric monoidal pointed model category $\mathcal{C}$, and let B be a commutative R-algebra. Then we define the abelianization functor as

$$\mathrm{Ab}_{B/R} \colon \mathrm{CAlg}_R /B \to \mathrm{Mod}_B,\ A \mapsto (\mathbf{L}Q)(\mathbf{R}I)(B \wedge^{\mathbf{L}}_R A).$$

In algebra, abelian group objects in augmented algebras are equivalent to modules and the square-zero extension functor has a left adjoint given by Kähler differentials. This adjunction is shown for global power functors in Theorem 1.2.13, even though here the interpretation as abelian group objects is not valid anymore, as we show in Section 1.3.a. This adjunction justifies the name *abelianization* for this functor, which we adopt from Basterra [10]. In the case of commutative ring spectra, Basterra and Mandell [11] show that this functor can be interpreted as a stabilization. A corresponding statement for ultra-commutative ring spectra is the topic of a joint project with Tobias Lenz.

Also in this general context over a model category $\mathcal{C}$, we observe that abelianization and square-zero extension are adjoint after passage to the homotopy categories. In this case, the square-zero extension functor is the composite

$$B \ltimes (\_) \coloneqq KZ \colon \mathrm{Mod}_B \to \mathrm{CAlg}_R /B,\ M \mapsto B \ltimes M = B \vee M.$$

Since both Z and K are homotopical, this composite descends to the homotopy categories.

Proposition 2.1.8. *The functors*

$$\mathrm{Ho}(\mathrm{CAlg}_R /B) \underset{\mathrm{Ho}(B\ltimes(\_))}{\overset{\mathrm{Ab}_{B/R}}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Mod}_B)$$

are adjoint, with $\mathrm{Ab}_{B/R}$ *left and* $\mathrm{Ho}(B \ltimes (\_))$ *right adjoint.*

Proof. We use the Quillen adjunctions exhibited in Propositions 2.1.2 and 2.1.5 and calculate

$$\begin{aligned}\mathrm{Ho}(\mathrm{Mod}_B)(\mathrm{Ab}_{B/R}(A), M) &\cong \mathrm{Ho}(\mathrm{CAlg}^+_B)((\mathbf{R}I)(B \wedge^{\mathbf{L}}_R A), (\mathbf{R}Z)(M))\\ &\cong \mathrm{Ho}(\mathrm{CAlg}_B /B)((\mathbf{L}K)(\mathbf{R}I)(B \wedge^{\mathbf{L}}_R A), (\mathbf{L}K)(\mathbf{R}Z)(M))\\ &\cong \mathrm{Ho}(\mathrm{CAlg}_B /B)(B \wedge^{\mathbf{L}}_R A, B \ltimes M) \cong \mathrm{Ho}(\mathrm{CAlg}_R /B)(A, B \ltimes M).\end{aligned}$$

Here, we used that $\mathbf{L}K$ is an equivalence of homotopy categories inverse to $\mathbf{R}I$. □

Using this definition, we define the cotangent complex.

Definition 2.1.9. Let R be a commutative monoid in $\mathcal{C}$ and B be a commutative R-algebra. Then the cotangent complex of B over R is the B-module

$$\Omega_{B/R} = \mathrm{Ab}_{B/R}(B) = (\mathbf{L}Q)(\mathbf{R}I)(B \wedge^{\mathbf{L}}_R B),$$

taking indecomposables of the fibre of the multiplication map $B \wedge^{\mathbf{L}}_B B \to B$.

Definition 2.1.10. Let R be a commutative monoid in $\mathcal{C}$ and B be a commutative R-algebra. Then we call the map

$$d_{B/A} \colon B \to \Omega_{B/R}$$

of R-modules, obtained by composing the unit $B \to B \ltimes \Omega_{B/R}$ of the adjunction from Proposition 2.1.8 with the projection $B \ltimes \Omega_{B/R} \to \Omega_{B/R}$, the *universal derivation* of B over R.

We record the functoriality of the cotangent complex.

Construction 2.1.11. Let

$$\begin{array}{ccc} R & \xrightarrow{f_R} & R' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f_B]{} & B' \end{array}$$

be a commutative diagram of commutative monoids in $\mathcal{C}$. This induces a morphism $B \wedge_R^{\mathbf{L}} B \to B' \wedge_{R'}^{\mathbf{L}} B'$. Moreover, by functoriality of pullbacks and pushouts, we also obtain morphisms

$$\mathbf{R}I_R(B \wedge_R^{\mathbf{L}} B) \to \mathbf{R}I_{R'}(B' \wedge_{R'}^{\mathbf{L}} B')$$

and

$$\mathbf{L}Q_R(\mathbf{R}I_R(B \wedge_R^{\mathbf{L}} B)) \to \mathbf{L}Q_{R'}(\mathbf{R}I_{R'}(B' \wedge_{R'}^{\mathbf{L}} B')).$$

In total, this induces a morphism $f_* \colon \Omega_{B/R} \to \Omega_{B'/R'}$. This construction is functorial.

As a first calculation, we can exhibit the cotangent complex on a free commutative algebra $\mathbb{P}X$ as a free module generated by X. We formulate this relative to a commutative monoid R, where $\mathbb{P}_R \colon \mathrm{Mod}_R \to \mathrm{CAlg}_R$ denotes the free R-algebra functor.

Proposition 2.1.12. *Let R be a commutative monoid in $\mathcal{C}$, and X be an R-module. Then there is a natural weak equivalence*

$$\Omega_{\mathbb{P}_R X/R} \cong \mathbb{P}_R X \wedge_R^{\mathbf{L}} X.$$

Proof. We show this assertion by showing that $\mathbb{P}_R X \wedge_R^{\mathbf{L}} X$ has the universal property of $\Omega_{\mathbb{P}_R X/R}$ in the homotopy category of $\mathbb{P}_R X$-modules, exhibited by the adjunction in Proposition 2.1.8. Hence, we need to show that for any $\mathbb{P}_R X$-module M, we have an isomorphism

$$\mathrm{Ho}(\mathrm{Mod}_{\mathbb{P}_R X})(\mathbb{P}_R X \wedge_R^{\mathbf{L}} X, M) \cong \mathrm{Ho}(\mathrm{CAlg}_R / \mathbb{P}_R X)(\mathbb{P}_R X, \mathbb{P}_R X \ltimes M).$$

This can be seen by the chain of isomorphisms

$$\begin{aligned} \mathrm{Ho}(\mathrm{Mod}_{\mathbb{P}_R X})(\mathbb{P}_R X \wedge_R^{\mathbf{L}} X, M) &\cong \mathrm{Ho}(\mathrm{Mod}_R)(X, M) \\ &\cong \mathrm{Ho}(\mathrm{CAlg}_R^+)(\mathbb{P}_R^+ X, \mathbf{R}Z(M)) \\ &\cong \mathrm{Ho}(\mathrm{CAlg}_R / R)(R \ltimes \mathbb{P}_R^+ X, R \ltimes M) \\ &\cong \mathrm{Ho}(\mathrm{CAlg}_R / \mathbb{P}_R X)(\mathbb{P}_R X, \mathbb{P}_R X \ltimes M). \end{aligned}$$

Here, in the third line, we used that $\mathbf{L}K$ is an equivalence of homotopy categories between CAlg_R^+ and CAlg_R / R. Moreover, in the last line, we used the adjunction of over categories

$$\mathrm{CAlg}_R / \mathbb{P}_R X \underset{\varepsilon^!}{\overset{\varepsilon_*}{\rightleftarrows}} \mathrm{CAlg}_R / R,$$

where $\varepsilon \colon \mathbb{P}_R X \to R$ is the projection to the constant summand of the free algebra, ε_* is postcomposition with ε and $\varepsilon^!$ is pullback along ε. It is straight-forward that this is indeed an adjunction, and since ε_* is the identity on objects and morphisms, it is left Quillen. The last isomorphism then follows from the observation that $R \ltimes \mathbb{P}_R^+ X \cong \varepsilon_*(\mathbb{P}_R X)$. $\square$

2.1.b The Transitivity Sequence

In this section, we establish the basic properties of the cotangent complex, namely the transitivity sequence, flat base change and additivity. For the algebraic cotangent complex, this is carried out in [53, Section 5], and for topological André-Quillen cohomology for S-algebras, this is done in [10, Section 4]. We mimic their arguments for the proof in our context.

First, we need to establish certain compatibilities of the augmentation ideal and the indecomposables functors with change of rings.

Lemma 2.1.13. *Let R be a commutative monoid in $\mathcal{C}$, B be a commutative R-algebra and A be a commutative R-algebra augmented to R. Then the natural map*

$$\mathbf{R}I_R(A) \wedge^{\mathbf{L}}_R B \to \mathbf{R}I_B(A \wedge^{\mathbf{L}}_R B), \tag{2.1.14}$$

induced by the commutative diagram

$$\begin{array}{ccc} I_R(A) \wedge_R B & \longrightarrow & A \wedge_R B \\ \downarrow & & \downarrow \\ * & \longrightarrow & B, \end{array} \tag{2.1.15}$$

is an isomorphism in the homotopy category of non-unital commutative B-algebras.

Proof. Since the functors I and K are inverse Quillen equivalences, it suffices to show that the mate transformation

$$\mathbf{L}K_B(A \wedge^{\mathbf{L}}_R B) \to \mathbf{L}K_R(A) \wedge^{\mathbf{L}}_R B$$

is a weak equivalence (see eg. [58, Lemma 2.2], applied to the adjunction on the level of homotopy categories). However, this transformation is the isomorphism $B \vee (A \wedge_R B) \cong (R \vee A) \wedge_R B$, applied to suitable cofibrant replacements of A and B. □

Lemma 2.1.16. *Let R be a commutative monoid in $\mathcal{C}$, B be a commutative R-algebra and J be a non-unital commutative R-algebra. Then the natural map*

$$\mathbf{L}Q_B(J \wedge^{\mathbf{L}}_R B) \to \mathbf{L}Q_R(J) \wedge^{\mathbf{L}}_R B, \tag{2.1.17}$$

induced by the commutative diagram

$$\begin{array}{ccc} (J \wedge_R J) \wedge_R B & \longrightarrow & J \wedge_R B \\ \downarrow & & \downarrow \\ * & \longrightarrow & Q_R(J) \wedge_R B, \end{array}$$

is an isomorphism in the homotopy category of B-modules.

Proof. As a left adjoint, $\_ \wedge_R B$ commutes with the pushout diagram defining $Q(J)$. Hence, considering suitable cofibrant replacements of B and J shows that the map (2.1.17) is indeed an isomorphism. □

Remark 2.1.18. On first glance, it might not be obvious how the transformation $I_R(\_) \wedge_R B \Rightarrow I_B(\_ \wedge_R B)$, defined by diagram 2.1.15, induces the transformation in (2.1.14) on composites of derived functors. In this transformation, we mix left and right derived functors. This can be handled formally by considering the double category of model categories and left and right Quillen functors as vertical and horizontal morphisms, respectively. Then [58, Theorem 7.6] shows that taking homotopy categories can be made into a double pseudofunctor. This incorporates the definition of derived transformations as in (2.1.14), which are constructed in [58, 7.1].

In the similar situation of Lemma 2.1.16, the situation is easier, since both functors in question are left Quillen. Hence they preserve cofibrant objects, so that the second cofibrant replacement is not necessary. Formally, the natural transformation can be obtained by using the fact that on the 2-category of model categories and left Quillen functors, taking homotopy categories is a pseudo-2-functor by [34, Theorem 1.4.3]. This incorporates the existence of a natural isomorphism $\mathbf{L}F \circ \mathbf{L}G \to \mathbf{L}(F \circ G)$, which together with its inverse can be used to define the transformation in (2.1.17).

Lemma 2.1.19. *Let R be a commutative monoid, S and T be commutative R-algebras with a map $S \to T$ of commutative R-algebras. Assume moreover that $\mathcal{C}$ is left proper or that S is cofibrant as a commutative monoid. Then a homotopy cofiber of $S \wedge_R^{\mathbf{L}} T \to T \wedge_R^{\mathbf{L}} T$ in CAlg_T /T is given by $T \wedge_S^{\mathbf{L}} T$.*

Proof. This statement mainly consists of a consideration of the relevant cofibrant replacements. We consider the morphism $S \to T$. In order to calculate $S \wedge_R^{\mathbf{L}} T$, we consider a cofibrant replacement $\Gamma_R S \to S$ of S as a commutative R-algebra. Moreover, in order to calculate $T \wedge_R^{\mathbf{L}} T$, we decompose the morphism $\Gamma_R S \to S \to T$ into a cofibration followed by a weak equivalence as $\Gamma_R S \to \Gamma_R T \to T$. Obviously, both $\Gamma_R S$ and $\Gamma_R T$ are cofibrant replacements also of commutative R-algebras augmented to T, and since the morphism $\Gamma_R S \to \Gamma_R T$ is a cofibration, $\Gamma_R T$ is also cofibrant as a $\Gamma_R S$-algebra. Then the map we need to take the homotopy cofiber of is

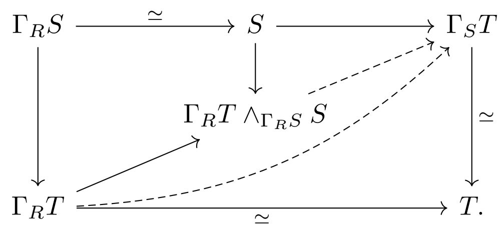
$$\Gamma_R S \wedge_R T \to \Gamma_R T \wedge_R T.$$

Since $\_ \wedge_R T\colon \mathrm{CAlg}_R /T \to \mathrm{CAlg}_T /T$ is a left Quillen functor, we observe that this map is a cofibration between cofibrant objects. Hence, its homotopy cofiber is represented by the actual cofiber of this map. This can be calculated by the usual properties of pushouts as

$$(\Gamma_R T \wedge_R T) \wedge_{\Gamma_R S \wedge_R T} T \cong (\Gamma_R T \wedge_{\Gamma_R S} \Gamma_R S \wedge_R T) \wedge_{\Gamma_R S \wedge_R T} T \cong \Gamma_R T \wedge_{\Gamma_R S} T.$$

We now need to compare this with $\Gamma_S T \wedge_S T$, where $\Gamma_S T$ is a cofibrant replacement of T as a commutative S-algebra. For this, it suffices to establish a weak equivalence $\Gamma_R T \wedge_{\Gamma_R S} S \to \Gamma_S T$ of commutative S-algebras. Since both $\Gamma_R T \wedge_{\Gamma_R S} S$ and $\Gamma_S T$ are cofibrant S-algebras (since $\_ \wedge_{\Gamma_R S} S$ is left Quillen and by definition, respectively), the left Quillen functor $\_ \wedge_S T$ preserves this weak equivalence.

In order to establish this weak equivalence, we consider the diagram



In this diagram, we factored the map $S \to T$ into a cofibration followed by an acyclic fibration as $S \to \Gamma_S T \to T$. Since the left vertical map is a cofibration, the diagonal dashed morphisms exist by the lifting property. Moreover, since either the categories in question are left proper or S is cofibrant, we conclude that the morphism $\Gamma_R T \to \Gamma_R T \wedge_{\Gamma_R S} S$ is a weak equivalence. This now proves that we indeed have a weak equivalence $\Gamma_R T \wedge_{\Gamma_R S} S \to \Gamma_S T$ as desired. ☐

We now have the necessary ingredients to show the existence of the transitivity sequence.

Theorem 2.1.20. *Let $R \to S \to T$ be a sequence of commutative monoids in $\mathcal{C}$. Moreover, assume that either $\mathcal{C}$ is left proper or that S is cofibrant. Then the sequence*

$$\Omega_{S/R} \wedge^{\mathbf{L}}_S T \to \Omega_{T/R} \to \Omega_{T/S},$$

induced from functoriality of Ω, has the structure of a homotopy cofiber sequence of T-modules.

Proof. We consider the morphism $S \wedge^{\mathbf{L}}_R T \to T \wedge^{\mathbf{L}}_R T$ of T-algebras augmented to T. By Lemma 2.1.19, the homotopy cofiber of this morphism in CAlg_T/T is given by $T \wedge^{\mathbf{L}}_S T$. Since I is a Quillen equivalence and Q is left Quillen, applying $\mathbf{L}Q_T \circ \mathbf{R}I_T$ to the resulting cofiber sequence yields a cofiber sequence of T-modules. This takes the form

$$\mathbf{L}Q_T(\mathbf{R}I_T(S \wedge^{\mathbf{L}}_R T)) \to \mathbf{L}Q_T(\mathbf{R}I_T(T \wedge^{\mathbf{L}}_R T)) \to \mathbf{L}Q_T(\mathbf{R}I_T(T \wedge^{\mathbf{L}}_S T)).$$

The last two terms are by definition the cotangent complexes $\Omega_{T/R}$ and $\Omega_{T/S}$. We thus only have to identify the first term as $\Omega_{S/R} \wedge^{\mathbf{L}}_S T$.

For this, we observe that $S \wedge^{\mathbf{L}}_R T \cong (S \wedge^{\mathbf{L}}_R S) \wedge^{\mathbf{L}}_S T$. Then, we use Lemmas 2.1.13 and 2.1.16 to calculate

$$\begin{aligned}\mathbf{L}Q_T(\mathbf{R}I_T(S \wedge^{\mathbf{L}}_R T)) &\cong \mathbf{L}Q_T(\mathbf{R}I_T((S \wedge^{\mathbf{L}}_R S) \wedge^{\mathbf{L}}_S T)) \\ &\cong \mathbf{L}Q_T(\mathbf{R}I_S(S \wedge^{\mathbf{L}}_R S) \wedge^{\mathbf{L}}_S T) \\ &\cong \mathbf{L}Q_S(\mathbf{R}I_S(S \wedge^{\mathbf{L}}_R S)) \wedge^{\mathbf{L}}_S T \\ &\cong \Omega_{S/R} \wedge^{\mathbf{L}}_S T.\end{aligned}$$

In total, we obtain the desired homotopy cofiber sequence. □

Moreover, we also get analogues of the base change and additivity results.

Proposition 2.1.21. *Let R be a commutative monoid in $\mathcal{C}$ and S and T be two commutative R-algebras. Then there are natural equivalences*

$$\begin{aligned}\Omega_{S \wedge^{\mathbf{L}}_R T/T} &\cong \Omega_{S/R} \wedge^{\mathbf{L}}_R T \qquad \text{and} \\ \Omega_{S \wedge^{\mathbf{L}}_R T/R} &\cong (\Omega_{S/R} \wedge^{\mathbf{L}}_R T) \vee (S \wedge^{\mathbf{L}}_R \Omega_{T/R}).\end{aligned}$$

Proof. For the first assertion, we calculate

$$\begin{aligned}\Omega_{S \wedge^{\mathbf{L}}_R T/T} &\cong \mathbf{L}Q_{S \wedge^{\mathbf{L}}_R T}(\mathbf{R}I_{S \wedge^{\mathbf{L}}_R T}((S \wedge^{\mathbf{L}}_R T) \wedge^{\mathbf{L}}_T (S \wedge^{\mathbf{L}}_R T))) \\ &\cong \mathbf{L}Q_{S \wedge^{\mathbf{L}}_R T}(\mathbf{R}I_{S \wedge^{\mathbf{L}}_R T}((S \wedge^{\mathbf{L}}_R S) \wedge^{\mathbf{L}}_S (S \wedge^{\mathbf{L}}_R T))) \\ &\cong \mathbf{L}Q_{S \wedge^{\mathbf{L}}_R T}(\mathbf{R}I_S(S \wedge^{\mathbf{L}}_R S) \wedge^{\mathbf{L}}_S (S \wedge^{\mathbf{L}}_R T)) \\ &\cong \mathbf{L}Q_S(\mathbf{R}I_S(S \wedge^{\mathbf{L}}_R S)) \wedge^{\mathbf{L}}_S (S \wedge^{\mathbf{L}}_R T) \\ &\cong \Omega_{S/R} \wedge^{\mathbf{L}}_R T.\end{aligned}$$

The second assertion follows from observing that the transitivity cofiber sequences for $R \to S \to S \wedge^{\mathbf{L}}_R T$ and $R \to T \to S \wedge^{\mathbf{L}}_R T$ fit together to define splittings for each other, and applying the first assertion to the resulting cotangent complexes. □

2.2 Topological André-Quillen Homology for Ultra-Commutative Ring Spectra

In this section, we apply the general theory of cotangent complexes developed in Section 2.1 to the model category $\mathcal{S}p$ of global spectra established in [55]. Here, the commutative monoids are the ultra-commutative ring spectra, which represent a very rich structure. Hence, the construction of topological André-Quillen cohomology for these ring spectra is a useful step in understanding this structured multiplication better. Classically, topological André-Quillen cohomology has been used successfully in obstruction theory for commutative ring spectra, and we introduce some tools in order to facilitate such an analysis also in the case of ultra-commutative ring spectra. Concretely, after the definition of the cohomology, we observe that we have a base-change result as well as a transitivity long exact sequence for this theory.

2.2.a Construction and Basic Properties of Topological André-Quillen Homology

We denote by $\mathcal{S}p$ the category of orthogonal spectra, equipped with the *positive* global model structure constructed in [55, Proposition 4.3.33]. This model structure satisfies the monoid axiom and the strong commutative monoid axiom by [55, Proposition 4.3.28, Theorem 5.4.1]. In particular, the positive global model structure transfers to the category CAlg of ultra-commutative ring spectra, as is elaborated in [55, Theorem 5.4.3]. Moreover, if R is an ultra-commutative ring spectrum, then also the categories Mod_R of R-modules, CAlg_R of commutative R-algebras and CAlg_R^+ of non-unital commutative R-algebras inherit model structures from the positive global model structure on $\mathcal{S}p$.

We now specialize the results from Section 2.1 to the case of ultra-commutative ring spectra.

Definition 2.2.1. Let R be an ultra-commutative ring spectrum and B be a commutative R-algebra. Then we define the cotangent complex of B over R as the B-module

$$\Omega_{B/R} = (\mathbf{L}Q)(\mathbf{R}I)(B \wedge_R^{\mathbf{L}} B).$$

In the context of B-modules, we can now consider the (co-)homology theory represented by this cotangent complex. This is called topological André-Quillen (co-)homology.

Definition 2.2.2. Let R be an ultra-commutative ring spectrum and B be a commutative R-algebra. Let M be a B-module. Then we define the topological André-Quillen homology of B over R with coefficients in M as

$$\underline{\mathrm{TAQ}}_*(B, R; M) = \underline{\pi}_*(\Omega_{B/R} \wedge_B^{\mathbf{L}} M),$$

and the topological André-Quillen cohomology of B over R with coefficients in M as the B-module

$$\underline{\mathrm{TAQ}}^*(B, R; M) = \underline{\pi}_{-*}(\mathbf{R}F_B(\Omega_{B/R}, M)).$$

Here, F_B denotes the function spectrum in the category of B-modules.

In particular, we obtain a transitivity long exact sequence on topological André-Quillen homology theory by considering the cofiber sequence established in Theorem 2.1.20.

Theorem 2.2.3. *Let $R \to S \to T$ be a sequence of ultra-commutative ring spectra, and M be a T-module. Then there are long exact sequences*

$$\ldots \to \underline{\mathrm{TAQ}}_{n+1}(T, S; M) \to \underline{\mathrm{TAQ}}_n(S, R; M) \to \underline{\mathrm{TAQ}}_n(T, R; M) \to \underline{\mathrm{TAQ}}_n(T, S; M) \to \ldots$$

and

$$\ldots \to \underline{\mathrm{TAQ}}^n(T,S;M) \to \underline{\mathrm{TAQ}}^n(T,R;M) \to \underline{\mathrm{TAQ}}^n(S,R;M) \to \underline{\mathrm{TAQ}}^{n+1}(T,S;M) \to \ldots$$

of global functors.

We moreover obtain base change and additivity results from considering the effect of Proposition 2.1.21 on cohomology.

Proposition 2.2.4. *Let R be an ultra-commutative ring spectrum and S and T be two commutative R-algebras. Let moreover M be an $S \wedge_R^{\mathbf{L}} T$-module and $n \in \mathbb{Z}$. Then there are natural isomorphisms*

$$\underline{\mathrm{TAQ}}_n(S \wedge_R^{\mathbf{L}} T, T; M) \cong \underline{\mathrm{TAQ}}_n(S,R;M) \qquad \text{and}$$
$$\underline{\mathrm{TAQ}}_n(S \wedge_R^{\mathbf{L}} T, R; M) \cong \underline{\mathrm{TAQ}}_n(S,R;M) \oplus \underline{\mathrm{TAQ}}^n(T,R;M)$$

in homology and

$$\underline{\mathrm{TAQ}}^n(S \wedge_R^{\mathbf{L}} T, T; M) \cong \underline{\mathrm{TAQ}}^n(S,R;M) \qquad \text{and}$$
$$\underline{\mathrm{TAQ}}^n(S \wedge_R^{\mathbf{L}} T, R; M) \cong \underline{\mathrm{TAQ}}^n(S,R;M) \oplus \underline{\mathrm{TAQ}}^n(T,R;M)$$

in cohomology.

2.2.b A Hurewicz Theorem for Topological André-Quillen Homology

We now study some structural properties of topological André-Quillen cohomology and the cotangent complex. As a first result, we prove a Hurewicz theorem for topological André-Quillen homology. Since André-Quillen homology is an invariant of R-algebras and thus of morphisms $R \to S$ of ultra-commutative ring spectra, the Hurewicz theorem compares it to the relative homotopy groups of this morphism. The Hurewicz theorem is a key input in the construction of Postnikov towers for ultra-commutative ring spectra.

As a first step, we study how the functor $Q\colon \mathrm{CAlg}_R^+ \to \mathrm{Mod}_R$ behaves on connective spectra.

Lemma 2.2.5. *Let R be a connective ultra-commutative ring spectrum and J be a non-unital commutative R-algebra. Suppose moreover that J is n-connected for $n \geq 0$. Then also $Q(J)$ is n-connected, and the adjunction unit $\eta\colon J \to QJ$ induces an isomorphism on $\underline{\pi}_k$ for $n+1 \leq k < 2n$.*

Proof. We may assume that J is a cofibrant non-unital R-algebra. We consider the defining cofiber sequence

$$J \wedge_R J \xrightarrow{\mu} J \xrightarrow{\eta} Q(J)$$

of R-modules. In this sequence, J is n-connected by assumption. Moreover, by the compatibility of the smash product with the t-structure on global spectra established in [55, Proposition 4.4.15], the spectrum $J \wedge_R J$ is $(2n-1)$-connected.
Using this observation, the claim of the lemma follows from the long exact sequence in homotopy groups. □

Remark 2.2.6. In the above proof, we used [55, Proposition 4.4.15] to use connectivity assumptions on R-modules X and Y in order to obtain connectivity of $X \wedge_R Y$. However, the statement of [55, Proposition 4.4.15] makes no mention of the relative smash product over R and only considers $X \wedge Y$. In order to deduce statements about $\wedge_R$, we observe that we can copy the arguments of the cited proof for the triangulated category of R-modules, replacing the generators

$\Sigma^\infty_+ B_{\mathrm{gl}}G$ by $R \wedge \Sigma^\infty_+ B_{\mathrm{gl}}G$. These are again compact generators of this triangulated category. Since homotopy classes of morphisms out of these generators recover the equivariant homotopy groups of the underlying spectrum of an R-module by adjunction, these compact generators also define a t-structure on the homotopy category of R-modules. In this t-structure, the connective R-modules are those that are connective as underlying spectra, and this class is spanned by the compact generators given above. The proof of these fact can be copied verbatim from [55, Theorem 4.4.9 and Proposition 4.4.13]. In all this, we use that R is connective, so the generators $R \wedge \Sigma^\infty_+ B_{\mathrm{gl}}G$ are connective as well.

In total, we may mimic the arguments given in [55, 4.4.15] and conclude that the analogous statement in Mod_R also holds.

The Hurewicz theorem for topological André-Quillen homology of commutative ring spectra appears as [10, Lemma 8.2]. That version contains an error in that the homotopy group of the cotangent complex is compared to that of the base instead of the cone, which was pointed out in [5, Lemma 1.2].

Theorem 2.2.7 (Hurewicz theorem)**.** *Let R be a connective ultra-commutative ring spectrum and let B be a connective commutative R-algebra such that the unit map $\eta\colon R \to B$ is an n-equivalence, with $n \geq 1$. Then $\Omega_{B/R}$ is n-connected and $\underline{\pi}_{n+1}(\mathrm{Cone}(\eta)) \cong \underline{\pi}_{n+1}(\Omega_{B/R})$.*

Proof. By cofibrantly replacing B, we assume that B is cofibrant as a commutative R-algebra. We consider the universal derivation $d_{B/R}\colon B \to \Omega_{B/R}$ defined in Definition 2.1.10. Restricting this map to R along η, it becomes trivial, since it factors through $\Omega_{R/R} \cong *$. Hence, we obtain an induced map $\tau\colon \mathrm{Cone}(\eta) \to \Omega_{B/R}$. We claim that τ induces an isomorphism on $\underline{\pi}_k$ for $k \leq n+1$, from which the claim follows.

In order to compare the cone of η with the cotangent complex, we consider the diagram

$$\begin{array}{ccccc} R & \xrightarrow{\eta} & B & \longrightarrow & \mathrm{Cone}(\eta) \\ {\scriptstyle\eta}\downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & B \wedge_R B & \longrightarrow & B \wedge_R \mathrm{Cone}(\eta). \end{array}$$

Here, the lower line arises by applying $B \wedge_R \_$ to the top line and the vertical maps are the inclusions of the right factor. Since B is cofibrant over R, both horizontal lines are cofiber sequences. Moreover, we know by Proposition 2.1.3 that the counit

$$B \vee \mathbf{R}I(B \wedge_R B) \to B \wedge_R B$$

is a weak equivalence. Hence, by comparing cofibers, we obtain $B \wedge_R \mathrm{Cone}(\eta) \cong \mathbf{R}I(B \wedge_R B)$. In total, instead of considering τ, we can consider

$$\mathrm{Cone}(\eta) \xrightarrow{\iota} B \wedge_R \mathrm{Cone}(\eta) \cong \mathbf{R}I(B \wedge_R B) \to \Omega_{B/R}.$$

Now, since η is an n-equivalence, $\mathrm{Cone}(\eta)$ is n-connected. Since moreover B is connective, by [55, Proposition 4.4.15], ι induces an isomorphism on $\underline{\pi}_k$ for $k \leq n+1$. Finally, Lemma 2.2.5 shows that also the map $\mathbf{R}I(B \wedge_R B) \to \Omega_{B/R}$ induces isomorphisms on $\underline{\pi}_k$ for $k \leq n+1$. This finishes the proof. $\square$

2.2.c Postnikov Towers of Ultra-Commutative Ring Spectra

In this section, we construct Postnikov towers for ultra-commutative ring spectra. These can be seen in a precise way as lifts of the usual Postnikov towers for global spectra arising from the t-structure established in [55, Section 4.4]. Thus, these Postnikov systems could be used in order to construct ultra-commutative ring spectrum structures on global spectra, by successively lifting the k-invariants in the Postnikov tower of global spectra to ones for ultra-commutative ring spectra, and then studying convergence of the resulting tower of ultra-commutative ring spectra. This idea goes back to Kriz [39] and is for example explained in [12, Section 4, esp. Corollary 4.6]. There, this theory is used in order to show that the Brown-Peterson spectrum BP supports an E_4-multiplication. Moreover, this obstruction theory can also be used to study maps between ring spectra.

In the classical story, a Postnikov tower is determined by its k-invariants, elements of the cohomology of the successively defined stages of the tower, which are used to construct the next stage as a homotopy fiber. We first explain how this step can be carried out in the category of ultra-commutative ring spectra, where the correct cohomology theory is topological André-Quillen cohomology.

Construction 2.2.8. Let R be an ultra-commutative ring spectrum, and B be a commutative R-algebra. Let moreover M be a B-module. Then we have an identification

$$\mathrm{TAQ}^n_e(B,R;M) \cong \mathrm{Ho}(\mathrm{Mod}_B)(\Omega_{B/R},\Sigma^n M) \cong \mathrm{Ho}(\mathrm{CAlg}_R/B)(B, B\ltimes \Sigma^n M)$$

by the defining adjunction for the abelianization functor.

Thus, if we are given a class $k \in \mathrm{TAQ}^n_e(B,R;M)$, we consider it as a morphism $B \to B\ltimes\Sigma^n M$ of commutative R-algebras over B. Then, we form the homotopy pullback

$$\begin{array}{ccc} B[k] & \longrightarrow & B \\ \downarrow & & \downarrow \iota \\ B & \xrightarrow[k]{} & B\ltimes \Sigma^n M \end{array}$$

in the category of R-algebras over B, where the right vertical map is the inclusion of a wedge summand. We call $B[k]$ an extension of B by the element $k \in \mathrm{TAQ}^n_e(B,R;M)$.

We moreover consider the projection map $\mathrm{pr}\colon B\ltimes\Sigma^n M \to \Sigma^n M$. This map is the same as the projection used to define the universal derivation $d_{B/R}\colon B \to \Omega_{B/R}$, and hence we see that the induced map

$$d^*_{B/R}\colon \underline{\mathrm{TAQ}}^n(B,R;M) \to \underline{H}^n(B,R;M)$$

from topological André-Quillen cohomology to usual (Bredon) cohomology, represented by the spectrum B, sends k to $\mathrm{pr}\circ k$. From this, we conclude that the R-module underlying $B[k]$ is the homotopy fiber of the map $\mathrm{pr}\circ k\colon B \to \Sigma^n M$ of R-modules.

Theorem 2.2.9. *Let R be a connective ultra-commutative ring spectrum. Then there is a sequence $R_0,\dots,R_n,\dots$ of commutative R-algebras, equipped with maps $R_{n+1}\to R_n$ of commutative R-algebras, and elements $k_n \in \mathrm{TAQ}^{n+2}_e(R_n,R;H\underline{\pi}_{n+1}(R))$, such that the following properties are satisfied:*

i) $R_0 \cong H\underline{\pi}_0(R)$, and $R_n \cong R_{n-1}[k_{n-1}]$,

ii) $\underline{\pi}_k(R_n)=0$ for $k>n$,

iii) the unit maps $\eta_n\colon R\to R_n$ are $(n+1)$-equivalences.

Proof. We define $R_0 = H\underline{\pi}_0(R)$ as an Eilenberg-MacLane spectrum for the global power functor $\underline{\pi}_0(R)$. By [55, Theorem 5.4.14], this carries the structure of an ultra-commutative ring spectrum.

Moreover, as discussed before the cited theorem, we obtain a map $\eta_0 \colon R \to H\underline{\pi}_0(R)$ of ultra-commutative ring spectra, which is an isomorphism on $\underline{\pi}_0$. Hence this serves as a first stage of the required Postnikov tower.

We now suppose we have constructed the required sequence of R-algebras up to level n. Then, the map $\eta_n \colon R \to R_n$ is an $(n+1)$-equivalence, and $\underline{\pi}_{n+2}(R_n) = \underline{\pi}_{n+1}(R_n) = 0$. Hence, by the Hurewicz theorem 2.2.7 we observe that $\Omega_{R_n/R}$ is $(n+1)$-connected and has $\underline{\pi}_{n+2}(\Omega_{R_n/R}) \cong \underline{\pi}_{n+1}(R)$. Thus, we get a map $\tilde{k}_n \colon \Omega_{R_n/R} \to \Sigma^{n+2} H\underline{\pi}_{n+1}(R)$ of R-modules realizing this equivalence on $\underline{\pi}_{n+1}$. This map corresponds to an element $k_n \in \mathrm{TAQ}_e^{n+2}(R_n, R; H\underline{\pi}_{n+1}(R))$, and by adjunction to a map $k_n \colon R_n \to R_n \ltimes \Sigma^{n+2} H\underline{\pi}_{n+1}(R)$ of R-algebras. We then define $R_{n+1} = R_n[k_n]$. This comes equipped with a map $R_{n+1} \to R_n$ of R-algebras. Since as an R-module, R_{n+1} is the homotopy fiber of the map $R_n \xrightarrow{d_{R_n/R}} \Omega_{R_n/R} \xrightarrow{\tilde{k}_n} \Sigma^{n+2} H\underline{\pi}_{n+1}(R)$, we observe that the morphism $R \to R_n$ is indeed an $(n+2)$-equivalence, and the higher homotopy groups of R_{n+1} vanish. Thus, the theorem follows by induction. □

Remark 2.2.10. We note that the above theorem also shows that the k-invariants of the Postnikov tower of R-modules and the k-invariants of the Postnikov tower of R-algebras are linked by the map

$$d^*_{B/R} \colon \underline{\mathrm{TAQ}}^n(B, R; M) \to \underline{H}^n(B, R; M)$$

induced by the universal derivation on cohomology. As mentioned before, this can then be used to establish an obstruction theory for maps of ultra-commutative ring spectra and for the existence of ultra-commutative ring structures by considering whether the classical k-invariants lift through this universal comparison map.

Appendix A

Symmetric Groups and Wreath Products

In this section, we recall the definition of wreath products of symmetric groups with arbitrary (finite or compact Lie) groups. These wreath products are used in the definition of the power operations of a global power functor, and we exhibit double cosets and conjugacy classes of elements in wreath products needed in calculations with power operations.

A.1 Definition of the Wreath Product

Definition A.1.1. Let G be a group. Then the wreath product $\Sigma_m \wr G$ is the semi-direct product $\Sigma_m \ltimes G^m$ with respect to the permutation action of Σ_m on the factors of G^m. Elements of $\Sigma_m \wr G$ are denoted $(\sigma; g_1, \ldots, g_m)$, with $\sigma \in \Sigma_m$ and $g_i \in G$ for $i = 1, \ldots, m$. Explicitly, the multiplication is given as

$$(\sigma; g_1, \ldots, g_m)(\tau; h_1, \ldots, h_m) = (\sigma\tau; g_{\tau(1)}h_1, \ldots, g_{\tau(m)}h_m).$$

If G is a compact Lie group, then also $\Sigma_m \wr G$ is a compact Lie group.

Remark A.1.2. The wreath product occurs in the definition of power operations in Definition 1.1.18, and one reason for this is that for a G-set X, the wreath product $\Sigma_m \wr G$ naturally acts on the m-power X^m by combining the G-action on each factor with the permutation action of Σ_m. On elements, this takes the form

$$(\sigma; g_1, \ldots, g_m)(x_1, \ldots, x_m) = (g_{\sigma^{-1}(1)}x_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(m)}x_{\sigma^{-1}(m)}).$$

In particular, if V is a G-representation, this action turns V^m into a $\Sigma_m \wr G$-representation. Moreover, for a G-set X, $\Sigma_m \wr G$ similarly also acts on the m-fold disjoint union $X^{\amalg m}$, by using the G-action of the relevant copy of X and the permutation action of Σ_m. On an element (x, i) in the i-th copy of X, with $1 \leq i \leq m$, we thus have $(\sigma; g_1, \ldots, g_m)(x, i) = (g_i x_i, \sigma^{-1}(i))$ considered as an element in the $\sigma^{-1}(i)$-th copy of X.

Observe that for small values of m, we obtain that $\Sigma_0 \wr G = e$ is the trivial group and $\Sigma_1 \wr G \cong G$ is the group G itself. The wreath product $\Sigma_m \wr G$ contains G^m as a normal subgroup, represented by the elements where the permutation is the identity, and the quotient by this subgroup is isomorphic to Σ_m. The quotient map splits by setting all G-coordinates to the neutral element of G.

Remark A.1.3. There are various comparison morphisms between products of wreath products and iterated wreath products. These morphisms are used in the formulation of the properties of the power operations. The first two of these morphisms are also described in [55, 2.2.5-6].

i) Let $i \geq 0$ and $j \geq 0$, and let G be a group. Then we define the injective homomorphism

$$\begin{aligned}\Phi_{i,j} \colon (\Sigma_i \wr G) \times (\Sigma_j \wr G) &\to \Sigma_{i+j} \wr G\\ ((\sigma; g_1, \ldots, g_i), (\sigma'; g'_1, \ldots, g'_j)) &\mapsto (\sigma + \sigma'; g_1, \ldots, g_i, g'_1, \ldots, g'_j)\end{aligned}$$

by concatenating the permutations and the tupels of elements of G. Here $+$ denotes the concatenation of permutations, ie the inclusion $\Sigma_i \times \Sigma_j \hookrightarrow \Sigma_{i+j}$ obtained by identifying $\{1, \ldots, i+j\}$ with $\{1, \ldots, i\} \amalg \{1, \ldots, j\}$.

ii) Let $k \geq 0$ and $m \geq 0$, and let G be a group. Then we define the injective homomorphism

$$\begin{aligned}\Psi_{k,m} \colon \Sigma_k \wr (\Sigma_m \wr G) &\to \Sigma_{km} \wr G\\ (\sigma; (\tau_1; g_{1,1}, \ldots, g_{1,m}), \ldots, (\tau_k; g_{k,1}, \ldots, g_{k,m})) &\mapsto (\sigma\natural(\tau_1, \ldots, \tau_k); g_{1,1}, \ldots, g_{1,m}, \ldots, g_{k,m}).\end{aligned}$$

Here, we use the identification of $\coprod_k \{1, \ldots, m\}$ with $\{1, \ldots, km\}$ by concatenating the k-copies of the set $\{1, \ldots, m\}$. Since $\Sigma_k \wr \Sigma_m$ naturally acts on the first set as described in Remark A.1.2, we thus identify it with a subgroup of "block permutations" in Σ_{km}, and call this inclusion $\natural$.

iii) Let $m \geq 0$ and G and K be two groups. Then we define the injective homomorphism

$$\begin{aligned}\Delta_m \colon \Sigma_m \wr (G \times K) &\to (\Sigma_m \wr G) \times (\Sigma_m \wr K)\\ (\sigma; (g_1, k_1), \ldots, (g_m, k_m)) &\mapsto ((\sigma; g_1, \ldots, g_m), (\sigma; k_1, \ldots, k_m))\end{aligned}$$

using the diagonal on the symmetric group.

In the verification of the properties of the power operations on a square-zero extension in Theorem 1.2.7, we need two double coset formulas. We explain these double coset formulas here. They are also utilized in a more general form in the comonadic description of power operations in [55, Chapter 5.2, Equations 3, 4 and 12].

Lemma A.1.4. *Let $i, j, k, m \geq 0$ and G be a group.*

i) There are exactly two double cosets in $((\Sigma_i \wr G) \times (\Sigma_j \wr G))\backslash\Sigma_{i+j} \wr G/((\Sigma_{i+j-1} \wr G) \times G)$. They may be represented by the permutations $\chi(\varepsilon)$ for $\varepsilon = 0, 1$, defined as

$$\chi(\varepsilon)(t) = \begin{cases} t & \text{for } 1 \leq t \leq i - \varepsilon\\ t + \varepsilon & \text{for } i - \varepsilon + 1 \leq t \leq k - 1\\ t - j & \text{for } t = k \text{ and } \varepsilon = 1\\ t & \text{for } t = k \text{ and } \varepsilon = 0.\end{cases}$$

Thus, $\chi(0) = id$, and $\chi(1)$ is a cyclic permutation on the last $j+1$ elements, permuting k into the i-th position.
The intersections of the two subgroups occurring in the double coset formula are

$$((\Sigma_i \wr G) \times (\Sigma_j \wr G))^{\chi(\varepsilon)} \cap ((\Sigma_{i+j-1} \wr G) \times G) = (\Sigma_{i-\varepsilon} \wr G) \times (\Sigma_{j-1+\varepsilon} \wr G) \times (\Sigma_\varepsilon \wr G) \times (\Sigma_{1-\varepsilon} \wr G)$$

and

$$((\Sigma_i \wr G) \times (\Sigma_j \wr G)) \cap {}^{\chi(\varepsilon)}((\Sigma_{i+j-1} \wr G) \times G) = (\Sigma_{i-\varepsilon} \wr G) \times (\Sigma_\varepsilon \wr G) \times (\Sigma_{j-1+\varepsilon} \wr G) \times (\Sigma_{1-\varepsilon} \wr G).$$

ii) There is a single double coset in $(\Sigma_k \wr (\Sigma_m \wr G))\backslash \Sigma_{km} \wr G/((\Sigma_{km-1} \wr G) \times G)$*. It is thus represented by the identity element.*
The relevant intersection of the two subgroups occurring in the double coset formula is

$$(\Sigma_k \wr (\Sigma_m \wr G)) \cap ((\Sigma_{km-1} \wr G) \times G) = (\Sigma_{k-1} \wr (\Sigma_m \wr G)) \times (\Sigma_{m-1} \wr G) \times G.$$

This is a subgroup of $\Sigma_k \wr (\Sigma_m \wr G)$ *via the inclusion*

$$(\Sigma_{k-1} \wr (\Sigma_m \wr G)) \times (\Sigma_{m-1} \wr G) \times G \xrightarrow{id \times \Phi_{m-1,1}} (\Sigma_{k-1} \wr (\Sigma_m \wr G)) \times (\Sigma_m \wr G) \xrightarrow{\Phi_{k-1,1}} \Sigma_k \wr (\Sigma_m \wr G),$$

and a subgroup of $(\Sigma_{km-1} \wr G) \times G$ *via the inclusion*

$$\begin{aligned}(\Sigma_{k-1} \wr (\Sigma_m \wr G)) \times (\Sigma_{m-1} \wr G) \times G &\xrightarrow{\Psi_{k-1,m} \times id \times id} (\Sigma_{(k-1)m} \wr G) \times (\Sigma_{m-1} \wr G) \times G \\ &\xrightarrow{\Phi_{(k-1)m,m-1} \times id} (\Sigma_{km-1} \wr G) \times G.\end{aligned}$$

The proof is a straight-forward calculation, and we omit the details.

A.2 Polynomial Global Power Functors and Conjugacy Classes in Wreath Products

In the main body of this dissertation, we study polynomial global power functors. As global functors, these take the form $\mathbb{A}[x_G] = \bigoplus_{m \geq 0} \mathbb{A}(\Sigma_m \wr G, \_)$. We now explicitly describe these polynomial global power functors for small groups G and evaluated at small groups. These calculations are for example used in analysing the transitivity sequence for André-Quillen homology of global power functors in Theorem 1.5.27. As described in Remark 1.1.2, for any compact Lie groups G and K, the group $\mathbb{A}(G, K)$ is free abelian on conjugacy classes of pairs consisting of a subgroup $L \leq K$ and a continuous homomorphism $\alpha\colon L \to G$. We describe this data explicitly for small groups.

Remark A.2.1. We first describe the free global power functor generated by an element at the trivial group, for values at small groups. This is

$$\mathbb{A}[x_e](G) = \bigoplus_{m \geq 0} \mathbb{A}(\Sigma_m, G).$$

Recall that we denote the generator of the m-th summand, corresponding to the identity in $\mathbb{A}(\Sigma_m, \Sigma_m)$, by $P^m(x)$, since it indeed is the m-th power of the generator x. We thus also denote all other elements of this free global power functor in terms of power operations, restrictions and transfers of the generator x.
For $G = e$, the group $\mathbb{A}(\Sigma_m, e)$ is free abelian on a single generator, which is the restriction $\operatorname{res}^{\Sigma_m}_e(P^m(x)) = x^m$. For $G = \mathbb{Z}/2$, additionally to the composition $\operatorname{tr}^{\mathbb{Z}/2}_e \operatorname{res}^{\Sigma_m}_e(P^m(x)) = \operatorname{tr}^{\mathbb{Z}/2}_e(x^m)$, which is the only generator associated to the subgroup $e \leq \mathbb{Z}/2$, there is one generator $\alpha^*(P^m(x))$ for any conjugacy class of homomorphism $\alpha\colon \mathbb{Z}/2 \to \Sigma_m$. Such conjugacy classes of homomorphisms are equivalent to conjugacy classes of elements of order at most 2 in Σ_m. In the symmetric group, the conjugacy classes of elements are determined by the cycle type. The trivial element forms one such conjugacy class and corresponds to the composition $p^*_{\mathbb{Z}/2} \operatorname{res}^{\Sigma_m}_e(P^m(x)) = p^*_{\mathbb{Z}/2}(x)^m$ of inflation and restriction. Any element of order 2 in the symmetric group is a product of disjoint transpositions. There are $\lfloor \frac{m}{2} \rfloor$ such cycle types. The restriction associated to such an

element containing k disjoint 2-cycles applied to the generating element $P^m(x)$ gives the element $(P^2(x))^k \cdot (p^*_{\mathbb{Z}/2}(x))^{m-2k}$. We thus observe that

$$\mathbb{A}[x_e](\mathbb{Z}/2) \cong \mathbb{Z}[P^2(x), p^*_{\mathbb{Z}/2}(x)] \ltimes \bigoplus_{m \geq 0} \mathbb{Z}\{\mathrm{tr}_e^{\mathbb{Z}/2}(x^m)\}.$$

The multiplication on the left summand is the one of a polynomial ring, and all multiplications involving the transfer summand are governed by Frobenius reciprocity (1.1.12) and are again a transfer term. The symbol $\ltimes$ is chosen to reflect this action of the polynomial part of $\mathbb{A}[x_e](\mathbb{Z}/2)$ on the transfer part.

The other case we consider is the polynomial global power functor $\mathbb{A}[x_{\mathbb{Z}/2}]$ generated by an element at $\mathbb{Z}/2$. To analyse this, again at the groups e and $\mathbb{Z}/2$, we need to exhibit conjugacy classes of elements of order at most 2 in $\Sigma_m \wr \mathbb{Z}/2$. Conjugacy classes of elements in wreath products are studied using generalized cycle types in [59, 49], and the results are nicely summarized in [15]. These results can be used to study more generally $\mathbb{A}(\Sigma_m \wr G, K)$ for finite groups G and K. We present the relevant special case for $\mathbb{Z}/2$ here for completeness.

Lemma A.2.2. *Let $m \geq 0$. We denote $\mathbb{Z}/2 = \{0, 1\}$ additively.*

i) An element $(\sigma; s_1, \ldots, s_m) \in \Sigma_m \wr \mathbb{Z}/2$ has order at most 2 if and only if $\sigma \in \Sigma_m$ has order at most 2, ie is a (possibly empty) product of disjoint transpositions, and for any transposition (i, j) occurring in the cycle decomposition of σ, we have $s_i = s_j$.

ii) Two such elements $(\sigma; s_1, \ldots, s_m), (\tau; t_1, \ldots, t_m) \in \Sigma_m \wr \mathbb{Z}/2$ of order at most 2 are conjugate if and only if σ and τ have the same cycle type, and the cardinalities of $\{i \in \{1, \ldots, m\} \mid \sigma(i) = i \text{ and } s_i = 1\}$ and $\{j \in \{1, \ldots, m\} \mid \tau(j) = j \text{ and } t_i = 1\}$ agree.

Proof. The results in this lemma are special cases of the decomposition of general wreath products into conjugacy classes, as explained in [59, Satz II] and [49, Theorem 4]. For this classification, any element $(\sigma; s_1, \ldots, s_m)$ is decomposed into a product of elements $(\sigma_i; s_{i,1}, \ldots, s_{i,m})$, where each σ_i is either a cycle occurring in the cycle decomposition of σ or the identity, with every non-trivial cycle occurring exactly once. If σ_i is non-trivial, then $s_{i,j} = 0$ if j is fixed by σ_i, else $s_{i,j} = s_j$. If σ_i is the identity, then there is exactly one j fixed by σ such that $s_{i,j} = s_j \neq 0$, and all other $s_{i,j}$ are trivial. This is the decomposition of an element of the wreath product into so called disjoint wreath cycles.
To any wreath cycle $(\sigma; s_1, \ldots, s_m)$ with $\sigma = (i_1, i_2, \ldots, i_k)$ a k-cycle, we associate the determinant $\Delta = s_{i_1} s_{i_2} \ldots s_{i_k}$. If σ is the identity, then we set the determinant to be the non-trivial entry s_i, or if all entries are trivial, we set the determinant to be the trivial element of G. Then two wreath cycles are conjugate if and only if they have conjugate cycles and their determinants are conjugate in G. The conjugacy class of any element in the wreath product is then determined by its wreath cycle decomposition. The order of a wreath cycle is $|(\sigma; s_1, \ldots, s_m)| = |\sigma| \cdot |\Delta|$, see [49, Theorem 4].
From these statements, we deduce the lemma as the special case for $G = \mathbb{Z}/2$ and elements of order at most 2. □

From this description, we are able to calculate the explicit form of the free global power functor $\mathbb{A}[x_{\mathbb{Z}/2}]$ at the groups e and $\mathbb{Z}/2$.

Corollary A.2.3. *The free global power functor generated at $\mathbb{Z}/2$ is given as follows:*

i) At level e,

$$\mathbb{A}[x_{\mathbb{Z}/2}](e) \cong \mathbb{Z}[\mathrm{res}_e^{\mathbb{Z}/2}(x)],$$

a polynomial ring on the restriction of the generator.

ii) At level $\mathbb{Z}/2$,

$$\mathbb{A}[x_{\mathbb{Z}/2}](\mathbb{Z}/2) \cong \mathbb{Z}[x, P^2(\mathrm{res}_e^{\mathbb{Z}/2}(x)), p^*_{\mathbb{Z}/2}(\mathrm{res}_e^{\mathbb{Z}/2}(x))] \ltimes \bigoplus_{m\geq 0} \mathbb{Z}\{\mathrm{tr}_e^{\mathbb{Z}/2}(\mathrm{res}_e^{\mathbb{Z}/2}(x^m))\},$$

where the left summand is a polynomial ring on three generators, and the multiplications involving the transfers are given by Frobenius reciprocity (1.1.12).

Proof. The first calculation follows from the fact that $\mathbb{A}(\Sigma_m \wr \mathbb{Z}/2, e)$ is a free abelian group generated on the restriction to the trivial group. Applying this restriction to the generator $P^m(x)$ gives $\mathrm{res}_e^{\Sigma_m \wr \mathbb{Z}/2}(P^m(x)) = (\mathrm{res}_e^{\mathbb{Z}/2}(x))^m$. Hence, we obtain the polynomial ring as claimed.

For the value at $\mathbb{Z}/2$, we observe that $\mathbb{A}(\Sigma_m \wr \mathbb{Z}/2, \mathbb{Z}/2)$ is free abelian on the following generators: There is a single generator that is a transfer, which is of the form $\mathrm{tr}_e^{\mathbb{Z}/2} \mathrm{res}_e^{\Sigma_m \wr \mathbb{Z}/2}$. Applied to $P^m(x)$, we obtain the terms $\mathrm{tr}_e^{\mathbb{Z}/2}(\mathrm{res}_e^{\mathbb{Z}/2}(x^m))$. Moreover, we have generators given by conjugacy classes of homomorphisms $\mathbb{Z}/2 \to \Sigma_m \wr \mathbb{Z}/2$. Such conjugacy classes of homomorphisms correspond to conjugacy classes of elements of order at most 2 in $\Sigma_m \wr \mathbb{Z}/2$, which we classify in Lemma A.2.2. Suppose we consider an element $(\sigma; s_1, \dots, s_m) \in \Sigma_m \wr \mathbb{Z}/2$ where σ is a product of k disjoint transpositions, and $|\{i \in \{1, \dots, m\} \mid \sigma(i) = i \text{ and } s_i = 1\}| = l$. Then the corresponding restriction, applied to $P^m(x)$, yields the element $(P^2(\mathrm{res}_e^{\mathbb{Z}/2}(x)))^k \cdot x^l \cdot p^*_{\mathbb{Z}/2}(\mathrm{res}_e^{\mathbb{Z}/2}(x))^{m-2k-l}$. In total, we observe that the part of $\mathbb{A}[x_{\mathbb{Z}/2}](\mathbb{Z}/2)$ generated by the restrictions is polynomial in the generators $P^2(\mathrm{res}_e^{\mathbb{Z}/2}(x))$, x and $p^*_{\mathbb{Z}/2}(\mathrm{res}_e^{\mathbb{Z}/2}(x))$. This finishes the calculations. □

Appendix B

Model Categories of Non-Unital Commutative Monoids

Recently, the study of ring spectra and commutative ring spectra has become an important field of study in algebraic topology and homotopy theory. This made it necessary to endow the category of (commutative) ring spectra with a framework in which to do homotopy theory. Early results in this direction were obtained by Elmendorf-Kriz-Mandell-May in [23], where the smash product on S-modules was constructed and a model category structure was put on the category of S-modules, which lifts to one on the category of ring spectra. An important point in this construction is that any S-module is fibrant in this model structure. Later, the examples of symmetric and orthogonal spectra with their corresponding model structures defined by Hovey-Shipley-Smith and Mandell-May-Schwede-Shipley [35, 46] provided examples of model categories where not every object is fibrant. In this context, the work [56] of Schwede-Shipley provided formal criteria in the form of the monoid axiom for when the category of modules in a monoidal model category inherits a model structure. Recently, this has been generalized to a formal criterion for when the category of commutative monoids inherits a model structure by White in [66], making use of a commutative monoid axiom.

In this section, we adapt the arguments from [66] to show that the commutative monoid axiom also allows one to put a model structure on the category of non-unital commutative monoids. A similar result was obtained by Basterra [10], but there the condition that every object is fibrant is used in the analysis. Since this is often not the case in applications, and especially not true for the category of orthogonal spectra with the positive global model structure we apply this theory to in Section 2.2, we instead use the commutative monoid axiom.

B.1 The Commutative Monoid Axiom and General Lifting Results

We first recall the definitions of the monoid axiom and the commutative monoid axiom and give the basic result for lifting model structures to categories of algebras over a monad. Using this, the proof of the existence of a model category of non-unital commutative monoids reduces to an analysis of a certain pushout in this category, as it is also the case for associative monoids [56] and for commutative monoids [66]. In absence of a unit map, this analysis will look slightly different, but follows along the same lines.

Let $(\mathcal{C}, \wedge, \mathbb{S})$ be a symmetric monoidal cocomplete category. We always assume that the symmetric monoidal category is closed, such that $X \wedge \_$ is a left adjoint for any $X \in \mathcal{C}$. Then the category of non-unital commutative monoids can be described as the category of algebras for the monad $\mathbb{P}^+ \colon \mathcal{C} \to \mathcal{C}$, which is defined as $\mathbb{P}^+(X) = \bigvee_{n \geq 1} X^{\wedge n}/\Sigma_n$, where $\bigvee$ denotes the

coproduct in the category $\mathcal{C}$.
We now assume that $\mathcal{C}$ is a cofibrantly generated model category. Then by lifting the model structure of $\mathcal{C}$ to the category of T-algebras for a monad T, we mean transferring the model structure along the adjunction $\mathcal{C} \underset{U}{\overset{T}{\rightleftarrows}} \mathrm{Alg}_T$, such that weak equivalences and fibrations in Alg_T are precisely the underlying weak equivalences and fibrations respectively. We say that this model structure on Alg_T is inherited by that of $\mathcal{C}$. The following lemma, which is part of [56, Lemma 2.3] and [66, Lemma 2.1], gives a condition when Alg_T inherits a model structure from $\mathcal{C}$.

Here and in the following arguments, if K is any class of morphisms in $\mathcal{C}$, we denote by K-cell the class of all morphisms obtained as transfinite compositions of pushouts of morphisms in $\mathcal{C}$. These morphisms are called regular cofibrations in [56].

Lemma B.1.1. *Let $\mathcal{C}$ be a cofibrantly generated model category with sets I of generating cofibrations and J of generating acyclic cofibrations. Let T be a monad on $\mathcal{C}$ which commutes with filtered colimits. Suppose the domains of $T(I)$ and $T(J)$ are small with respect to $T(I)$*-cell *and $T(J)$*-cell *respectively, and that $T(J)$*-cell *is contained in the weak equivalences. Then* Alg_T *inherits a model structure from $\mathcal{C}$, where weak equivalences and fibrations are the underlying weak equivalences and fibrations, respectively. It is cofibrantly generated, with $T(I)$ as a set of generating cofibrations and $T(J)$ as a set of generating acyclic cofibrations.*

If $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category, then the monads $\mathbb{A}(X) = \bigvee_{n\geq 0} X^{\wedge n}$ and $\mathbb{P}(X) = \bigvee_{n\geq 0} X^{\wedge n}/\Sigma_n$ for associative and commutative monoids commute with filtered colimits, as does $\mathbb{P}^+$. Then, in [56, Definition 3.3], the following condition is given such that the above lemma applies to the associative monad $\mathbb{A}$:

Definition B.1.2. Let $\mathcal{C}$ be a monoidal model category. We say that $\mathcal{C}$ satisfies the monoid axiom if (acyclic cofibrations $\wedge\, \mathcal{C}$)-cell is contained in the weak equivalences.

Similarly, in [66], a condition for $\mathbb{P}$ to satisfy the condition of the above lemma is given:

Definition B.1.3. Let $\mathcal{C}$ be a symmetric monoidal model category. We say that $\mathcal{C}$ satisfies the commutative monoid axiom if for any acyclic cofibration $f\colon X \to Y$, the map $f^{\square n}/\Sigma_n$ is an acyclic cofibration for all $n \geq 0$.

In the above definition, the map $f^{\square n}$ is a generalization of the pushout product map $f\square g\colon A\wedge Y \cup_{A\wedge B} X\wedge B \to X \wedge Y$. Explicitly, we consider the cube

$$W\colon \mathcal{P}(\{1,\dots,n\}) \to \mathcal{C},\ S \mapsto W_1 \wedge \ldots \wedge W_n,$$

where

$$W_i = \begin{cases} X & \text{if } i \notin S \\ Y & \text{if } i \in S. \end{cases}$$

On an inclusion $S \subset T$, we set $W(S \subset T) = \varphi_1 \wedge \ldots \wedge \varphi_n$, with

$$\varphi_i = \begin{cases} id & \text{if } i \notin T \setminus S \\ f & \text{if } i \in T \setminus S. \end{cases}$$

This defines a map $f^{\square n}\colon Q_n = \mathrm{colim}_{\mathcal{P}(\{1,\dots,n\})\setminus\{\{1,\dots,n\}\}} W \to W(\{1,\dots,n\}) = Y^{\wedge n}$. This is Σ_n-equivariant for the action permuting the elements of $\{1,\dots,n\}$, hence induces the map $f^{\square n}/\Sigma_n\colon Q_n/\Sigma_n \to \mathbb{P}^n(Y)$.

We also record here a condition introduced in [66] to study the cofibrations of commutative monoids:

Definition B.1.4. Let $\mathcal{C}$ be a symmetric monoidal model category. We say that $\mathcal{C}$ satisfies the strong commutative monoid axiom if for any cofibration and acyclic cofibration $f\colon X \to Y$, the map $f^{\square n}/\Sigma_n$ is a cofibration or acyclic cofibration for all $n \geq 0$, respectively.

Using these properties, the following results hold (see [56, Theorem 4.1] and [66, Theorem 3.2 and Proposition 3.5]):

Theorem B.1.5. *Let $\mathcal{C}$ be a cofibrantly generated symmetric monoidal model category, such that the domains of the generating cofibrations I are small with respect to $(I \wedge \mathcal{C})$-cell and the domains of the generating acyclic cofibrations J are small with respect to $(J \wedge \mathcal{C})$-cell.*

i) If $\mathcal{C}$ satisfies the monoid axiom, the category Alg *of monoids in $\mathcal{C}$ inherits a cofibrantly generated model structure. Moreover, if $f\colon X \to Y$ is a cofibration in* Alg *such that X is cofibrant in $\mathcal{C}$, then f is a cofibration in $\mathcal{C}$.*

ii) If $\mathcal{C}$ satisfies moreover the commutative monoid axiom, then the category CAlg *of commutative monoids in $\mathcal{C}$ inherits a cofibrantly generated model structure.*

iii) If $\mathcal{C}$ satisfies moreover the strong commutative monoid axiom, then if $f\colon X \to Y$ is a cofibration in CAlg *and X is cofibrant in $\mathcal{C}$, then f is a cofibration in $\mathcal{C}$.*

B.2 Non-Unital Commutative Monoids

We now prove the analogous statement to Theorem B.1.5 in the case of non-unital commutative monoids. To do so, we follow the same path as in the unital cases, so we prove the following analogue of [56, Lemma 6.2] and [66, Lemma B.1]:

Lemma B.2.1. *Let $\mathcal{C}$ be a cofibrantly generated symmetric monoidal model category with sets I of generating cofibrations and J of generating acyclic cofibrations, and let $\mathbb{P}^+\colon \mathcal{C} \to \mathrm{CAlg}^+$ denote the free non-unital commutative monoid functor. If $\mathcal{C}$ satisfies the commutative monoid axiom and the monoid axiom, then any morphism in $\mathbb{P}^+(J)$-*cell *forgets to a weak equivalence in $\mathcal{C}$.*
*If moreover $\mathcal{C}$ satisfies the strong commutative monoid axiom, then any morphism in $\mathbb{P}^+(I)$-*cell *with domain cofibrant in $\mathcal{C}$ forgets to a cofibration in $\mathcal{C}$.*

The proof of this lemma relies on the following analysis of certain pushouts in the category CAlg^+ of non-unital commutative monoids, This is analogous to the statements in the proof of [56, Lemma 6.2] and in [66, Lemma B.2].

Construction B.2.2. Let $\mathcal{C}$ be a cocomplete symmetric monoidal category. Let $h\colon K \to L$ and $p\colon K \to X$ be morphisms in $\mathcal{C}$, where X is a non-unital commutative monoid with multiplication map $\mu\colon X \wedge X \to X$. Consider the pushout

$$\begin{array}{ccc} \mathbb{P}^+K & \xrightarrow{\mathbb{P}^+h} & \mathbb{P}^+L \\ {\scriptstyle\tilde{p}}\downarrow & & \downarrow \\ X & \xrightarrow[f]{} & P \end{array} \tag{B.2.3}$$

in CAlg^+, where $\tilde{p}$ is the adjoint to p under the free-forgetful adjunction.
We claim that the morphism $f\colon X \to P$ factors as

$$X = P_0 \xrightarrow{f_1} P_1 \xrightarrow{f_2} \ldots \to \operatorname*{colim}_{n\geq 0} P_n = P,$$

where we think of P_n as consisting of words on letters in X and L, where we multiply letters in X by the monoid structure, and have at most n letters from L. Formally, P_n and the morphisms $f_n\colon P_{n-1} \to P_n$ are inductively defined via pushouts in $\mathcal{C}$ of the form

$$\begin{array}{ccc} (\mathbb{S} \vee X) \wedge (Q_n/\Sigma_n) & \xrightarrow{(\mathbb{S}\vee X)\wedge h^{\square n}/\Sigma_n} & (\mathbb{S} \vee X) \wedge \mathbb{P}^n L \\ {\scriptstyle t_n}\downarrow & & \downarrow{\scriptstyle T_n} \\ P_{n-1} & \xrightarrow[f_n]{} & P_n. \end{array}$$

Here, Q_n is again the colimit of the punctured n-cube defined by h, as explained after Definition B.1.3.

The vertical morphism

$$t_n\colon (\mathbb{S} \vee X) \wedge (Q_n/\Sigma_n) \to P_{n-1}$$

is defined as follows:

We first describe this for $n = 1$. There, we need to define a morphism $(\mathbb{S}\vee X)\wedge K \cong K\vee(X\wedge K) \to X$. On K, we define this as p, and on $X \wedge K$, we define it as

$$X \wedge K \xrightarrow{X\wedge p} X \wedge X \xrightarrow{\mu} X.$$

For $n > 1$, we write $(\mathbb{S} \vee X) \wedge (Q_n/\Sigma_n) \cong (Q_n/\Sigma_n) \vee (X \wedge (Q_n/\Sigma_n))$, and then define the map to P_{n-1} on both wedge summands inductively. To simplify notation, let $\varepsilon \in \{0, 1\}$, such that $X^{\wedge\varepsilon} \in \{\mathbb{S}, X\}$. Then we define a map on each vertex of the punctured cube and then argue that this descends to the colimit. This induces a morphism on $X^{\wedge\varepsilon} \wedge (Q_n/\Sigma_n)$, since $X^{\wedge\varepsilon} \wedge \_$ preserves colimits.

So let $S \subsetneq \{1, \dots, n\}$ be a proper subset. On the vertex $X^{\wedge\varepsilon} \wedge W_1 \wedge \dots \wedge W_n$ of the cube indexed by S, we define a map t_S to P_{n-1} as

$$\begin{array}{ccccccc} X^{\wedge\varepsilon} \wedge W_1 \wedge \dots \wedge W_n & \xrightarrow{\wedge p} & X^{\wedge\varepsilon} \wedge W_1' \wedge \dots \wedge W_n' & & \xrightarrow{\text{shuffle}} & & X^{\wedge(n-|S|+\varepsilon)} \wedge L^{\wedge|S|} \\ {\scriptstyle t_S}\downarrow & & & & & & \downarrow{\scriptstyle \mu\wedge L^{\wedge|S|}} \\ P_{n-1} & \xleftarrow[f_{n-1}\circ\dots\circ f_{|S|+1}]{} & P_{|S|} & \xleftarrow[T_{|S|}]{} & X \wedge \mathbb{P}^{|S|}L & \xleftarrow[\text{pr}]{} & X \wedge L^{\wedge|S|}. \end{array}$$

Here, the map labelled $\wedge p$ applies the map $p\colon K \to X$ to any factor K inside $W_1 \wedge \dots \wedge W_n$. There is at least one K, since S is a proper subset of $\{1, \dots, n\}$. Then, we denote

$$W_i' = \begin{cases} X & \text{if } i \notin S \\ L & \text{if } i \in S. \end{cases}$$

The map $T_{|S|}$ was already constructed by induction, and we restrict it here to the summand $X \wedge \mathbb{P}^{|S|}L$.

We claim that this defines objects P_n for all $n \geq 0$ and morphisms $f_n\colon P_{n-1} \to P_n$ fitting into pushouts as stated. The arguments are exactly analogous to those carried out in the commutative case at the beginning of [66, Proof of Lemma B.2]. In our case, the factor $X \wedge \_$ may fall away, but this does not change the argument.

Now, we have defined the sequence

$$X = P_0 \xrightarrow{f_1} P_1 \to \dots .$$

We define $P = \operatorname{colim}_{n\geq 0} P_n$. This receives maps $f\colon X = P_0 \to P$ and $L = \mathbb{P}^1 L \xrightarrow{T_1} P_1 \to P$, hence it fits into a diagram

$$\begin{array}{ccc} K & \xrightarrow{h} & L \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \xrightarrow{f} & P. \end{array} \tag{B.2.4}$$

This diagram commutes, since by definition of P_1, the corresponding diagram with P_1 in the lower right corner commutes, and the above diagram is obtained from this by applying $P_1 \to P$.

Lemma B.2.5. *Let $\mathcal{C}$ be a cocomplete symmetric monoidal category. Let $h\colon K \to L$ and $p\colon K \to X$ be morphisms in $\mathcal{C}$, where X is a non-unital commutative monoid. Then the object P defined in Construction B.2.2 can be endowed with the structure of a non-unital commutative monoid, such that the morphism $f\colon X \to P$ is a morphism of commutative monoids and* (B.2.4) *induces the pushout-square*

$$\begin{array}{ccc} \mathbb{P}^+K & \xrightarrow{\mathbb{P}^+h} & \mathbb{P}^+L \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow \\ X & \xrightarrow[f]{} & P \end{array}$$

in CAlg^+.

Proof. We begin by defining the multiplication. It is induced by compatible maps $P_n \wedge P_m \to P_{n+m}$. Such a map is defined inductively by using the pushout diagram

$$\begin{array}{ccc} (X_+ \wedge \tilde{Q}_n) \wedge (X_+ \wedge \mathbb{P}^m L) \cup_{(X_+\wedge\tilde{Q}_n)\wedge(X_+\wedge\tilde{Q}_m)} (X_+ \wedge \mathbb{P}^n L) \wedge (X_+ \wedge \tilde{Q}_m) & \longrightarrow & (X_+ \wedge \mathbb{P}^n L) \wedge (X_+ \wedge \mathbb{P}^m L) \\ \downarrow & & \downarrow \\ (P_{n-1} \wedge P_m) \cup_{P_{n-1}\wedge P_{m-1}} (P_n \wedge P_{m-1}) & \longrightarrow & P_n \wedge P_m, \end{array}$$

where we abbreviated $X_+ = \mathbb{S} \vee X$ and $\tilde{Q}_n = Q_n/\Sigma_n$. The upper left span, of which $P_n \wedge P_m$ is the pushout, admits a morphism to the span in the pushout square

$$\begin{array}{ccc} X_+ \wedge \tilde{Q}_{n+m-1} & \longrightarrow & X_+ \wedge \mathbb{P}^{n+m} L \\ \downarrow & & \downarrow \\ P_{n+m-1} & \longrightarrow & P_{n+m} \end{array}$$

by multiplying the X_+-factors in the top row and using the maps $P_{n-1} \wedge P_m \to P_{n+m-1}$ and $P_n \wedge P_{m-1} \to P_{n+m-1}$ provided by the induction hypothesis. Hence, taking pushouts provides the map $P_n \wedge P_m \to P_{n+m}$.
That this map induces a multiplication map $P \wedge P \to P$ which makes P into a non-unital commutative monoid, that $f\colon X \to P$ is a monoid map and that P is the pushout of the diagram (B.2.3) in CAlg^+ follows analogous to the proof for unital commutative monoids in [66, Proof of Lemma B.2]. □

Using this description of pushouts in non-unital commutative monoids, we can now prove B.2.1. The proof is completely analogous to that of [66, Lemma B.1]

Proof of Lemma B.2.1. Let $\mathcal{C}$ be a cofibrantly generated symmetric monoidal model category which satisfies the monoid and the commutative monoid axiom. Let F be any map in $\mathbb{P}^+(J)$-cell. This means that F is a transfinite composition of pushouts f of the form (B.2.3). By the above Lemma B.2.5, f is a (transfinite) composition of pushouts in $\mathcal{C}$ of maps of the form $X_+ \wedge h^{\square n}/\Sigma_n$, where h is an acyclic cofibration and X is some object in $\mathcal{C}$. By the commutative monoid axiom, $h^{\square n}/\Sigma_n$ is an acyclic cofibration. By the monoid axiom, any transfinite composition of pushouts of morphisms of the form $Y \wedge j$ is a weak equivalence, where j are acyclic cofibrations. Since F is such a transfinite composition, this proves that F is a weak equivalence.

Assume now that $\mathcal{C}$ also satisfies the strong commutative monoid axiom. Let G be a morphism in $\mathbb{P}^+(I)$-cell with cofibrant domain. Then we can write G as a transfinite composition of maps g which arise as pushouts of the form (B.2.3), where h is a cofibration. It suffices to prove that if the domain X of g is cofibrant, then g is a cofibration, since transfinite compositions of cofibrations are cofibrations. We can write g as a transfinite composition of pushouts in $\mathcal{C}$ of maps of the form $X_+ \wedge (h^{\square n}/\Sigma_n) \cong (h^{\square n}/\Sigma_n) \vee (X \wedge (h^{\square n}/\Sigma_n))$. By the strong commutative monoid axiom, $h^{\square n}/\Sigma_n$ is a cofibration, and since X is cofibrant, also $X \wedge (h^{\square n}/\Sigma_n)$ is a cofibration. Thus, g is a transfinite composition of cofibrations, hence itself a cofibration in $\mathcal{C}$. $\square$

Using this result, we can invoke Lemma B.1.1 to obtain the model structure on non-unital commutative monoids. By considering the category of R-modules for any commutative monoid R as the base category, we also obtain by the same arguments a model structure on the category of non-unital R-algebras.

Theorem B.2.6. *Let $\mathcal{C}$ be a cofibrantly generated symmetric monoidal model category, such that the domains of the generating cofibrations I are small with respect to $(I \wedge \mathcal{C})$-*cell *and the domains of the generating acyclic cofibrations J are small with respect to $(J \wedge \mathcal{C})$-*cell*. Let R be a commutative monoid in $\mathcal{C}$.*
Suppose that $\mathcal{C}$ satisfies the monoid axiom and the commutative monoid axiom. Then the category CAlg_R^+ *of non-unital R-algebras inherits a cofibrantly generated model structure from $\mathcal{C}$, in which a morphism is a weak equivalence or fibration if and only if its underlying morphism in $\mathcal{C}$ is.*
If moreover $\mathcal{C}$ satisfies the strong commutative monoid axiom, then any cofibration in CAlg_R^+ *with domain cofibrant in $\mathcal{C}$ forgets to a cofibration in $\mathcal{C}$.*

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