UFR S.T.M.I.A.
École Doctorale IAE + M
Université Henri Poincaré - Nancy I
D.F.D. Mathématiques

Thèse

présentée pour l'obtention du titre de

Docteur de l'Université Henri Poincaré, Nancy-I

en Mathématiques

par

Lars SCHÄFER

Géométrie tt* et applications pluriharmoniques

Soutenue publiquement le 12 Mai 2006.

Membres du jury :

Lionel Bérard Bergery	Examinateur	Professeur, Nancy I
Hans-Peter Nilles	Examinateur	Professeur, Bonn
Jan Sokolowski	Examinateur	Professeur, Nancy I
Vicente Cortés	Directeur de Thèse	Professeur, Nancy I
Werner Ballmann	Directeur de Thèse	Professeur, Bonn
Helga Baum	Rapporteur	Professeur, Berlin
Jost-Hinrich Eschenburg	Rapporteur	Professeur, Augsburg
Vladimir Roubtsov	Rapporteur et président de jury	Professeur, Angers

Institut Élie Cartan Nancy, Laboratoire de Mathématiques, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex

tt*-geometry

and

pluriharmonic maps

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Lars Schäfer

aus

Adenau

Bonn 2006

Angefertigt als gemeinsame Promotion mit der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn und der Faculté des Sciences der Université Henri Poincaré, Nancy I und mit der Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn und der Faculté des Sciences der Université Henri Poincaré, Nancy I.

1. Referent: Professor Dr. W. Ballmann
2. Referent: Professor Dr. V. Cortés

Tag der Promotion: 12.05.2006
Erscheinungsjahr: 2006

Diese Dissertation ist auf dem Hochschulschriftenserver der ULB Bonn http://hss.ulb.uni-bonn.de/diss_online elektronisch publiziert.

Contents

Acknowledgements

The author thanks his advisors Prof. Dr. V. Cortés and Prof. Dr. W. Ballmann for the support and the interest in this work.

Further, he is grateful to Prof. Dr. H. Baum, Prof. Dr. J.-H. Eschenburg and Prof. Dr. V. Roubtsov, who agreed to be referees of my thesis. Moreover I thank Prof. Dr. V. Roubtsov that he accepted to be the president of the 'jury'.

Likewise, he thanks Prof. Dr. L. Bérard Bergery, Prof. Dr. H.-P. Nilles and Prof. Dr. J. Sokolowski who accepted to be member of the 'jury' of this thesis.

The author wants to thank Prof. Dr. B. Dubrovin for explaining some questions related to his paper [D]. He is grateful to Prof. Dr. C. Hertling for the disussions concerning his work [Her], for pointing out the result of Simpson [Sim] and for the further interest in the author's works [Sch4, Sch6], namely by using these papers in his seminar. Moreover, he thanks M. Krahe for the discussions about para-complex geometry and associated families of (para-)pluriharmonic maps. In this list the author should not forget to thank M.-A. Lawn for the enlightening discussions about all kind of mathematics and in particular about the common works. Finally I am very gratefull to D. Lawn for sharing his knowledge in computer science.

The author was supported by the 'Schwerpunktprogramm Stringtheorie' of the 'Deutsche Forschungsgemeinschaft'. The author is indepted to the 'Deutscher Akademischer Austauschdienst', the CROUS Nancy-Metz and the 'Studienstiftung des Deutschen Volkes' for their grants. He thanks them all, since they have enabled this work in form of a 'Co-tutelle de thèse'. He thanks the 'mathematischem Institut' at Bonn for the good working conditions and the IECN at Nancy for the good working conditions and the acceptance as ATER.

Je remercie toute l'équipe de géométrie différentielle de Nancy: Oussama Hijazi, Bernd Amman, Jean-François Grosjean, Emmanuel Humbert, Julien Maubon, Georges Habib, Bertrand Morel, Marie-Amélie Paillusseau, Simon Raulot et Julien Roth pour le bon esprit de travail et les discussions autour de la géométrie. En particulier, je tiens compte de la bonne ambiance entre les doctorants de l'équipe.

L'institut Élie Cartan fut un agréable lieu pour préparer cette thèse. J'en suis très reconnaissant.

Je voudrais remercier les doctorants pour l'atmosphère conviviale qu'ils ont créée à Nancy. Commençons avec les filles du bureau 111: Anne, Manon et Marie-Amélie ont réussi à bien soutenir le moral de la troupe. Le bureau 109 de Pierre L.G. et Simon fut une place de la vie commune des doctorants. Pierre L.G. et Marie-Amélie ont tourjours été le point central des activités sociales. Je remercie ceux qui ont compris ma dépendance au café: Georges, Julien, Manon, Marie-Amélie, les Pierre, Sergio, Sylvain, Simon et assimilés... Un grand merci aussi aux co-bureaux: le grand Christophe, qui est devenu un bon copain et le petit François qui n'arrêtera pas de se battre pour de bonnes causes...

Il ne faut pas oublier la chorale avec tous les choristes: Bertrand, Olivier et Jean-François qui ont eu le courage d'être les dirigents. Florent et Christophe, les piliers de la première génération, Alex et Guillaume, les inséparables biologistes et les petits jeunes en basse: Pierre D. et Pierre E., les piliers de la prochaine génération.

J'ai bien apprécié les activités sportives avec Pierre E., Pierre L.G. et Sergio, qui ont eu leur acmée dans le succés à l'Ekidem... on n'a pas perdu!

Il reste à citer Herrmann qui n' appartient à aucune des classes citées ci-dessus et avec lequel nous avons passé des moments plein d'esprit.

Pour le soutien moral de Mme. et M. Lawn je suis très reconnaissant.

Merci à tous ceux qui n'ont pas cessé d'avoir la patience et l'endurance de m'apprendre le français. Ici il faut nommer aussi les mains de Manon et Marie-Amélie qui ont corrigé les parties françaises de ce travail, et tous ceux qui ont bien lu et critiqué mon exposé de soutenance.

J'envoie une forte pensée aux vieux amis et aux amis rencontrés dans ces dernières années de thèse, car sans eux la vie serait beaucoup moins riche....

Et si d'aventure, quelque oubli s'était glissé entre ces lignes, qu'on veuille bien m'en excuser.

An dieser Stelle möchte ich nicht zuletzt den Deutschsprachigen danken: Beginnen möchte ich mit den Bonner Kommilitonen, welche von Beginn an mein Studium begleitet haben. Ich denke hier an alle, die an den zahlreichen Nachbesprechungen zur Quantenmechanik teilgenommen haben. Moritz, der mit seinem Witz und langen Doppelkopfabenden - den Friesen zu Ehr' - eine Bereicherung war. Dem Bonner 'Dreigestirn' Christoph, Gunther und Ryad, die nicht nur zu Mittag stets eine angenehme und diskusionsfreudige Gesellschaft waren. Gunther danke ich ferner als Sponsor dieser Arbeit in Form von Kaffee und Gummibären. Desweiteren bin ich ihm dafür dankbar, daß er stets bereit war seine Kompetenz der Informatik, insbesondere im Web-Design, mit mir zu teilen. Sehr dankbar bin ich ihm auch für die Hilfe bei der Erstellung kommutativer Diagramme. Zu erwähnen ist auch Marc, dessen Freundschaft seit der Schulbank andauert. Meinem Bruder danke ich für seine Solidarität in allen Situationen. Schließlich spreche ich meinen Eltern, ohne die diese Arbeit nie entstanden wäre, meinen Dank aus. Es verbleibt Marie-Amélie, der ich in jederlei Hinsicht und für den Beistand in allen Bereichen danke.

Introduction

Topological-antitopological fusion or tt^*-geometry is a topic of mathematical and physical interest. In about 1990 physicists studied topological-field-theories and their moduli spaces, in particular $N = 2$ supersymmetric field-theories and found a special geometric structure called topological-antitopological fusion (see the works of Cecotti and Vafa [CV] and Dubrovin [D]). These geometries are realized on the tangent bundle of some manifold and part of their data is a Riemannian metric. One can replace the tangent bundle by an abstract vector bundle. This step allows to consider tt^*-bundles as a generalization of variations of Hodge structures, as it was done in Hertling's paper [Her].

The starting point of this thesis is a correspondence between tt^*-bundles and pluriharmonic maps into the space of positive definite metrics found by Dubrovin [D]. The aim is to find a version of this correspondence for tt^*-bundles on abstract vector bundles. The obvious interest of this correspondence is either to construct pluriharmonic maps or to construct tt^*-geometries. We first analyze pluriharmonic maps which are associated to solutions of tt^*-bundles coming from already known geometries. Famous examples of such solutions are harmonic bundles, variations of Hodge structures, special complex and special Kähler manifolds and flat nearly Kähler manifolds. In the last two cases indefinite metrics appear. This means one needs to understand the above-mentioned correspondence for tt^*-geometries with pseudo-Riemannian metrics.

Recently, special para-Kähler geometry was introduced in [CMMS]. It arises as one of the special geometries of Euclidean supersymmetry. This motivates us to search for para-complex versions of tt^*-geometries and for a correspondence to the para-complex analogue of pluriharmonic maps. In fact, we introduce the para-complex notion of tt^*-geometry, which we call para-tt^*-geometry and establish a correspondence to the para-complex analogue of pluriharmonic maps. This result leads to the question, if there exist para-complex versions of the above-mentioned solutions of tt^*-geometry: harmonic bundles, variations of Hodge structures, special complex and special Kähler manifolds and flat nearly Kähler manifolds and if they are solutions of para-tt^*-geometry. We answer positively to this question in this thesis, since we show, that one can generalize all these geometries to the para-complex category and that these generalizations supply solutions of para-tt^*-geometry.

Let us describe the results of the work. It is a compilation of already published results and of newer unpublished ones. To compress this work we treat at places, where it does not impose too much confusion, the complex case and the para-complex case at the same time. The needed notions of para-complex geometry are introduced in chapter 1. For the rest of the introduction we may ask the reader who is not familiar with these notations to skip the prefix 'para' if it disturbs him.

We explain the structure of a (para-)tt^*-bundle: A (para-)tt^*-bundle (E, D, S) consists of a (real) vector bundle E over a (para-)complex manifold (M, J) endowed with a connection D and a section S of $T^*M \otimes \mathrm{End}(E)$, such that the family of connections

$$D^\theta := \begin{cases} D + \cos(\theta)S + \sin(\theta)S_J, \theta \in \mathbb{R}, & \text{for } M \text{ complex,} \\ D + \cosh(\theta)S + \sinh(\theta)S_J, \theta \in \mathbb{R}, & \text{for } M \text{ para-complex,} \end{cases}$$

is flat. A metric (para-)tt^*-bundle (E, D, S, g) is a (para-)tt^*-bundle (E, D, S) endowed with a D-parallel metric g, such that S is symmetric with respect to g.
First, we establish the correspondence between (para-)tt^*-bundles on abstract vector bundles over simply connected manifolds and (para-)pluriharmonic maps, generalizing Dubrovin [D]. In fact we show the following result in theorem 4.1 and in theorem 4.2.:

Theorem 1

(i) A metric (para-)tt^*-bundle with a metric of signature (p, q) over a simply connected (para-)complex manifold (M, J) gives (after fixing a D^0-parallel frame) rise to an admissible (para-)pluriharmonic map f from M to $GL(r, \mathbb{R})/O(p, q)$.

(ii) An admissible (para-)pluriharmonic map f from a simply connected (para-)complex manifold (M, J) to $GL(r, \mathbb{R})/O(p, q)$ gives rise to a metric (para-)tt^*-bundle $(E = M \times \mathbb{R}^r, D, S)$.

For the definition of admissible (para-)pluriharmonic maps we refer to definition 2.9.
In other words we could roughly say, that our construction defines a bijection

$$\Phi : \left\{ \begin{array}{c} \text{framed metric (para-)}tt^*\text{-bundles} \\ \text{of rank } r \text{ and sign. } (p,q) \end{array} \right\} \to \left\{ \begin{array}{c} \text{admissible (para-)pluriharmonic maps} \\ \text{into } GL(r,\mathbb{R})/O(p,q) \end{array} \right\}. \qquad (0.0.1)$$

from the space of framed metric (para-)tt^*-bundles of rank r over a simply connected (para-)complex manifold (M, J) to the space of (para-)pluriharmonic maps from (M, J) to $GL(r, \mathbb{R})/O(p, q)$. The case of a metric tt^*-bundle of rank r with metric of signature $(r, 0)$ follows from this theorem, since in this case the pluriharmonic maps are shown to be admissible using a result of Sampson [Sam]. Our correspondence contains the classical correspondence shown by Dubrovin [D]. If the manifold M is not simply connected, one has to replace the (para-)pluriharmonic maps by twisted (para-)pluriharmonic maps. We also show a version for unimodular oriented metric tt^*-bundles. The target space of the (para-)pluriharmonic maps is for unimodular oriented metric tt^*-bundles the space $SL(r, \mathbb{R})/SO(p, q)$.
Adapting a rigidity result of Gordon [G] about harmonic maps to pluriharmonic maps we are able to prove a rigidity result for tt^*-bundles with a positive definite metric over a compact Kähler manifold (cf. theorem 4.6 and [Sch5]). Further we apply this to special Kähler manifolds and obtain a new proof Lu's theorem [Lu] in the case of a simply connected compact special Kähler manifold (cf. theorem 4.7 and [Sch5]).

We now shortly discuss the above-mentioned classes of tt^*-bundles:

From [Her] and [Sch1, Sch2] we knew, that harmonic bundles are objects, which are closely related to tt^*-bundles. A correspondence between these bundles and harmonic maps from compact Kähler manifolds to $GL(r,\mathbb{C})/U(r)$ was given in Simpson's paper [Sim]:

$$\Psi : \left\{ \begin{array}{c} \text{harmonic bundles} \\ \text{with pos. def. metric over} \\ \text{comp. Kähler manifolds } M \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{harmonic maps} \\ \text{from} \\ M \text{ to } GL(r,\mathbb{C})/U(r) \end{array} \right\}.$$

From Sampson's theorem [Sam] it follows that in this case the notion of harmonic map and pluriharmonic map coincide. In other words there exists a correspondence between harmonic bundles and pluriharmonic maps from compact Kähler manifolds to $GL(r,\mathbb{C})/U(r)$. This correspondence can be recovered from a more general result, discussed in this thesis, which is an application of our theorem 1. This is described briefly in the next paragraph and was published in [Sch4].
We generalize the notion of a harmonic bundle by admitting indefinite metrics. With this definition we construct metric and symplectic tt^*-bundles from harmonic bundles and we apply the correspondence of theorem 1 to prove that the target space of the admissible pluriharmonic maps can be restricted to the totally geodesic subspace $\mathrm{GL}(r,\mathbb{C})/\mathrm{U}(p,q)$ of $\mathrm{GL}(2r,\mathbb{R})/\mathrm{O}(2p,2q)$. This means, that the application of our construction roughly gives rise to a map:

$$\Psi : \left\{ \begin{array}{c} \text{framed harmonic bundles} \\ \text{over complex manifolds } M \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{admissible pluriharmonic maps} \\ \text{from } M \text{ to } GL(r,\mathbb{C})/U(p,q) \end{array} \right\}.$$

Simpson's result for positive definite signature is recovered, since for positive definite signature the above map Φ (cf. equation (0.0.1)) is essentially bijective. Our result is a generalization of Simpson's work (for more information compare section 5.5), as arbitrary signature of the bundle metric is admitted and the compactness and the Kähler condition are not needed. We restrict to simply connected manifolds M, since the case with non-trivial fundamental group can be obtained by utilizing the corresponding theorems in chapter 4. The pluriharmonic maps are then replaced by twisted pluriharmonic maps.
Moreover, we introduce the notion of para-harmonic bundles, i.e. harmonic bundles in para-complex geometry (cf. [Sch9]). We use the same recipe as in complex geometry to relate these bundles to para-pluriharmonic maps into $GL(r,C)/U^\pi(C^r)$, where $GL(r,C)$ is the para-complex analogue of the general complex linear group and $U^\pi(C^r)$ is the para-complex version of the unitary group. Hence we extend the map Ψ to para-harmonic bundles:

$$\Psi : \left\{ \begin{array}{c} \text{framed para-harmonic bundles} \\ \text{over para-complex manifolds } M \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{admissible para-pluriharmonic maps} \\ \text{from } M \text{ to } GL(r,C)/U^\pi(C^r) \end{array} \right\}.$$

The next class of solutions are variations of Hodge structures (VHS). These are by Hertling's work [Her] tt^*-geometries. Locally VHS are described by their period map, i.e. a holomorphic map into the so-called period domain, which is an open set in a flag manifold. We weaken the second Riemannian bilinear relation. Then we relate the pluriharmonic map associated to a tt^*-bundle, which comes from a given VHS of odd weight, to the period map of this VHS. Likewise we introduce a para-complex version of VHS and associate to this a kind of period map. The para-complex version of a VHS carries a metric para-tt^*-bundle. For odd weight we express the para-pluriharmonic map associated to this para-tt^*-bundle in terms of the para-complex period map.

In all these examples of (para-)tt^*-geometries the (para-)complex structure of the base manifold (M, J) has been integrable. However, in the study of (para-)tt^*-bundles (TM, D, S) on the tangent bundle TM it is reasonable to consider almost (para-)complex manifolds, since like this nearly (para-)Kähler manifolds with flat Levi-Civita connection arise as solutions of (para-)tt^*-geometry. We give a constructive classification of Levi-Civita flat nearly (para-)Kähler manifolds in a common work with V. Cortés [CS2].
Let us explain the structure of this part of the thesis, which is also subject of [Sch7, Sch8]. Part of the tt^*-bundle (TM, D, S) is now a one-parameter family of flat connections D^θ on the tangent bundle TM. Every almost (para-)complex manifold (M, J) endowed with a flat connection ∇ carries a natural one-parameter family of flat connections given by

$$\nabla^\theta = \exp(\theta J) \circ \nabla \circ \exp(-\theta J), \text{ with } \theta \in \mathbb{R}.$$

We study (para-)tt^*-bundles for which the families D^θ and ∇^θ are equivalent in the sense of the following:

Definition 1 *Two one-parameter families of connections ∇^θ and D^θ on some vector bundle E with $\theta \in \mathbb{R}$ are called* (linearly) equivalent with factor $\alpha \in \mathbb{R}$ *if they satisfy the equation $\nabla^\theta = D^{\alpha\theta}$.*

To consider such one-parameter families of connections is motivated by our previous study of special (para-)Kähler solutions of (para-)tt^*-bundles. Like this we obtain a duality between Levi-Civita flat nearly (para-)Kähler manifolds and special (para-)Kähler manifolds, which are both of importance in mathematics and theoretical physics.

Afterwards we restrict to (para-)tt^*-bundles (TM, D, S) as above such that the connection D is (para-)complex, i.e. satisfies $DJ = 0$. These are recovered uniquely from the (para-)complex structure J and the connection ∇. In addition compatibility conditions on the pair (∇, J) are given and it is shown that for special (para-)complex and nearly (para-)Kähler manifolds these compatibility conditions on (∇, J) hold.
More precisely, we give a class of tt^*-bundles (TM, D, S), which corresponds to special (para-)complex manifolds with torsion and non integrable almost (para-)complex structure J and a class of solutions which corresponds to flat almost (para-)complex manifolds satisfying the nearly Kähler condition (with torsion).
In the sequel we study whether the above (para-)tt^*-bundles (TM, D, S) (over almost (para-)complex manifolds) provide *metric* and *symplectic* (para-)tt^*-bundles, respectively. Solutions of the first type are, for example, given by special (para-)Kähler manifolds and solutions of the second kind arise on flat nearly (para-)Kähler manifolds. Otherwise, neither the nearly (para-)Kähler condition is compatible with metric (para-)tt^*-bundles nor the condition to be special (para-)complex is compatible with symplectic (para-)tt^*-bundles.
Finally it remains to analyze if one can transfer the relation between (para-)pluriharmonic maps and (para-)tt^*-geometry to the case of (non-integrable) almost (para-)complex structure of the base (M, J).
Since the (para-)complex structures are no longer integrable, we generalize the notion

of (para-)pluriharmonic maps to the case of a source manifold (M, J) with an (non-integrable) almost complex structure J: This is done by using the (para-)pluriharmonic map equation, where a nice connection (cf. definition 2.6) on M is chosen. Then we introduce $\mathbb{S}^1_\epsilon$-pluriharmonic maps which generalize the notion of associated families of pluriharmonic maps from complex manifolds (see for example [ET]) to maps from almost (para-)complex manifolds into pseudo-Riemannian manifolds. We give conditions for an $\mathbb{S}^1_\epsilon$-pluriharmonic map to be (para-)pluriharmonic and a result, which relates generalized (para-)pluriharmonic maps to harmonic maps. With these notions and results we associate pluriharmonic maps into $\mathrm{Sp}(\mathbb{R}^{2n})/U(p,q)$, respectively $SO_0(2p,2q)/U(p,q)$, to the above metric and symplectic tt^*-bundles. Similiarly we associate para-pluriharmonic maps into $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$, respectively into $SO_0(n,n)/U^\pi(C^n)$, to the above metric and symplectic para-tt^*-bundles.

As already mentioned, special (para-)complex and special (para-)Kähler manifolds are an interesting class of ϵtt^*-bundles, respectively metric ϵtt^*-bundles. In the complex case this follows from the results of Hertling [Her] who associated a VHS of weight 1 to any special complex manifold. We give a direct differential geometric approach and a characterization of the tangent bundles of special complex and special Kähler manifolds as special tt^*-bundles (cf. [CS1]). The associated pluriharmonic map is expressed in terms of the dual Gauß map, which is a holomorphic map into the pseudo-Hermitian symmetric space $\mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k,l)$, where $n = k + l$. These results are generalized to para-complex geometry. This is done in [Sch3] and is detailled in this thesis. The approach, which uses a VHS of weight 1, has also been successfully transferred to (para-)complex geometry and is part of this thesis.

Introduction

La fusion topologique-antitopologique est un sujet d'interêt en physique comme en mathématique. Dans les années 1990, les physiciens ont étudié les espaces modules au sein des théories des champs topologiques et plus particulièrement dans celle des champs $N{=}2$-supersymétriques (cf. les travaux de Cecotti et Vafa [CV] et de Dubrovin [D]). Au cours de leur étude, ils ont découvert une structure géométrique, appelée fusion topologique-antitopologique, ou géométrie tt^*. À la base, cette géométrie était réalisée sur le fibré tangent à une variété et une métrique Riemannienne faisait partie des données géométriques. Mais on peut également remplacer le fibré tangent par un fibré vectoriel abstrait. Ce point de vue, que l'on trouve dans l'article de Hertling [Her], permet de considérer la géométrie tt^* comme une généralisation des variations de structures de Hodge.

Le point de départ de cette thèse est la correspondance existant entre les fibrés tt^* sur le fibré tangent et les applications pluriharmoniques dans l'espace des métriques définies positives. Cette correspondance a été découverte par Dubrovin [D]. Son intérêt est d'une part de construire des applications pluriharmoniques, d'autre part des fibrés tt^*. Nous analyserons les applications pluriharmoniques associées à des fibrés tt^* provenant de solutions déjà connues de la géométrie tt^*. Des exemples célèbres sont: les fibrés harmoniques, les variations de structures de Hodge, les variétés spéciales Kählériennes et les variétés approximativement Kählériennes plates. Dans les deux derniers cas les métriques peuvent être indéfinies. Il est donc nécessaire de généraliser la correspondance précédente pour le cas des fibrés tt^* avec des métriques indéfinies.

Plus récemment, la géométrie spéciale para-Kählérienne a été introduite par [CMMS] comme une des géométries spéciales de la supersymétrie Euclidienne. Notre motivation est d'étudier s'il existe des versions para-complexes des géométries tt^* et si l'on peut trouver une correspondance entre ces versions et les analogues para-complexes des applications pluriharmoniques. Nous répondons par l'affirmative à ces deux problèmes. Se pose alors la question de savoir s'il existe des versions para-complexes des solutions citées ci-dessus: des fibrés harmoniques, des variations de structures de Hodge, des variétés spéciales Kählériennes et des variétés approximativement Kählériennes plates, et de savoir si leurs généralisations sont des solutions de la géométrie para-tt^*. Nous démontrons en effet que ces exemples peuvent être généralisés dans le cadre para-complexe et qu'ils sont des solutions de la géométrie para-tt^*.

Décrivons les résultats de cette thèse. Ce travail rassemble des résultats publiés et des résultats plus récents. Pour comprimer le texte, nous avons, lorsque le risque de confusion n'est pas trop grand, traité les cas complexe et para-complexe en même temps. Les notions de base de géométrie para-complexe sont détaillées dans le premier chapitre.

Pour l'introduction, le lecteur peu habitué aux notions de la géométrie para-complexe, pourra supprimer le préfixe 'para'.

Expliquons la structure d'un fibré (para-)tt^* : un fibré (para-)tt^* (E, D, S) est la donnée d'un fibré vectoriel sur une variété (para-)complexe (M, J), muni d'une connexion D, et d'une section S dans $T^*M \otimes \mathrm{End}(E)$, pour lesquels les connexions de la famille

$$D^\theta := \begin{cases} D + \cos(\theta)S + \sin(\theta)S_J, \theta \in \mathbb{R}, & \text{pour } M \text{ complexe,} \\ D + \cosh(\theta)S + \sinh(\theta)S_J, \theta \in \mathbb{R}, & \text{pour } M \text{ para-complexe,} \end{cases}$$

sont plates. Un fibré (para-)tt^* métrique (E, D, S, g) est un fibré (para-)tt^* (E, D, S) muni d'une métrique parallèle pour D et pour laquelle la section S est g-symétrique.

Généralisant Dubrovin [D], nous établissons d'abord la correspondance entre les fibrés (para-)tt^*, définis sur des fibrés vectoriels abstraits et des applications (para-)pluriharmoniques. En fait, nous démontrons dans les théorèmes 4.1 et 4.2 le résultat suivant :

Théorème 1

(i) Un fibré (para-)tt^* métrique (E, D, S, g) sur une variété (para-)complexe simplement connexe (M, J) induit (après avoir choisi un répère D^0-plat de E) une application (para-)pluriharmonique admissible de la variété M dans $GL(r, \mathbb{R})/O(p, q)$.

(ii) Une application (para-)pluriharmonique admissible d'une variété (para-)complexe simplement connexe (M, J) dans $GL(r, \mathbb{R})/O(p, q)$ induit un fibré (para-)tt^* métrique (E, D, S, g).

Pour la définition des applications (para-)pluriharmoniques admissibles, nous faisons référence à la définition 2.9.
Pour résumer, on pourrait dire que nous avons trouvé une bijection

$$\Phi : \left\{ \begin{array}{l} \text{fibrés (para-)}tt^* \text{ métriques} \\ \text{de rang } r \text{ et sign. } (p,q) \end{array} \right\} \to \left\{ \begin{array}{l} \text{applications pluriharmoniques admissibles} \\ \text{dans } GL(r, \mathbb{R})/O(p,q) \end{array} \right\}. \quad (0.0.2)$$

entre l'espace des fibrés (para-)tt^* métriques (avec repère fixé) de rang r et signature (p, q) sur une variété (para-)complexe (M, J) et l'espace des applications pluriharmoniques admissibles de (M, J) dans $GL(r, \mathbb{R})/O(p, q)$. Le cas d'un fibré (para-)tt^* métrique avec une métrique de signature $(r, 0)$ ou $(0, r)$ est une conséquence de notre théorème, puisque dans ce cas, on peut montrer, en utilisant un théorème de Sampson [Sam], que les applications pluriharmoniques sont admissibles. Si la variété M n'est pas simplement connexe, il faut remplacer les applications (para-)pluriharmoniques par des applications (para-)pluriharmoniques twistées. Nous établissons également une version de ce résultat pour des fibrés (para-)tt^* métriques orientés unimodulaires. Pour des fibrés (para-)tt^* métriques orientés unimodulaires, l'espace cible des applications (para-)pluriharmoniques est donné par l'espace symétrique $SL(r, \mathbb{R})/SO(p, q)$.

En adaptant au cas des applications pluriharmoniques un résultat de rigidité de Gordon [G] concernant les applications harmoniques, nous sommes capables d'obtenir un résultat de rigidité pour des fibrés tt^* métriques sur des variétés Kählériennes compactes (cf. théorème 4.6 et [Sch5]). Nous appliquons ensuite ce résultat au cas spécial Kählérien et nous obtenons une nouvelle preuve du théorème de Lu [Lu] dans le cas d'une variété spéciale Kählérienne compacte simplement connexe (cf. théorème 4.7 et [Sch5]).

Nous allons à présent examiner les classes de fibrés (para-)tt^* citées ci-dessus :

Grâce au travail de Hertling [Her] et en utilisant [Sch1, Sch2], nous savions jusqu'alors que les fibrés harmoniques étaient des objets reliés à la géométrie tt^*. Une correspondance entre les fibrés harmoniques et les applications harmoniques des variétés Kählériennes compactes dans $GL(r,\mathbb{C})/U(r)$ était donnée par Simpson [Sim]:

$$\Psi : \left\{ \begin{array}{c} \text{fibrés harmoniques} \\ \text{avec métrique pos. déf . sur} \\ \text{des var. Kähler. comp. } M \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{applications harmoniques} \\ \text{de } M \text{ dans} \\ GL(r,\mathbb{C})/U(r) \end{array} \right\}.$$

Le théorème de Sampson [Sam] implique que dans ce cas, les notions d'harmonicité et de pluriharmonicité coincident. Ainsi, il existe une correspondance entre les fibrés harmoniques et les applications pluriharmoniques des variétés Kählériennes compactes dans $GL(r,\mathbb{C})/U(r)$. On peut également déduire cette correspondance d'un résultat plus général, qui est une application de notre théorème 1. Cette correspondance est explicitée brièvement dans le paragraphe suivant et a été publiée dans [Sch4].

Dans cette thèse, nous généralisons la notion de fibré harmonique en incluant le cas des métriques indéfinies. À partir de cette généralisation, nous construisons des fibrés tt^*. En appliquant alors notre correspondance donnée dans le théorème 1, nous demontrons que l'on peut restreindre les applications pluriharmoniques au sous-espace totalement géodésique $GL(r,\mathbb{C})/U(p,q)$ de $GL(r,\mathbb{R})/O(2p,2q)$. Ainsi, notre construction induit, sans détailler, une application

$$\Psi : \left\{ \begin{array}{c} \text{fibrés harmoniques} \\ \text{avec métrique de sign. } (p,q) \\ \text{sur des var. complexe } M \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{applications harmoniques admissibles} \\ \text{de } M \text{ dans} \\ GL(r,\mathbb{C})/U(p,q) \end{array} \right\}.$$

Notre résultat est une généralisation du travail de Simpson (plus d'informations se trouvent dans la section 5.5.) : en effet, d'une part, on peut retrouver son résultat avec une application Φ essentiellement bijective, et d'autre part, nous admettons des métriques à signature arbitraire et nous n'avons besoin ni de la condition de compacité ni de la condition Kählérienne. Nous traitons le cas des variétés simplement connexes, le cas général pouvant être obtenu facilement en utilisant les théorèmes correspondants dans le chapitre 4 et en remplaçant les applications pluriharmoniques par des applications pluriharmoniques twistées.

Nous introduisons de plus la notion de fibré para-harmonique, c'est-à-dire de fibré harmonique en géométrie para-complexe (cf. [Sch9]). Nous utilisons par la suite une technique analogue afin d'obtenir une correspondance entre les fibrés para-harmoniques et des applications para-pluriharmoniques à valeurs dans $GL(r,C)/U^\pi(C^r)$. On désigne par $GL(r,C)$ la version para-complexe du groupe linéaire général complexe et par $U^\pi(C^r)$ l'analogue para-complexe du groupe unitaire. Nous généralisons ainsi l'application Ψ aux fibrés para-harmoniques:

$$\Psi : \{ \text{ fibrés para-harmoniques sur des var. para-complexes } M \} \to \{ \text{ appl. para-pluriharm. admissibles de } M \text{ dans } GL(r, C)/U^{\pi}(C^r) \}.$$

La prochaine classe de solutions est celle des variations de structures de Hodge (VHS). On sait d'après les travaux de Hertling [Her] qu'elles sont en effet des géométries tt^*. Localement, une VHS est décrite par son application de périodes, qui est une application holomorphe sur le domaine des périodes, sous-ensemble ouvert dans une variété de drapeaux. Nous affaiblissons la deuxième relation riemannienne bilinéaire. Nous donnons ensuite l'expression explicite de l'application pluriharmonique associée à la géométrie tt^* donnée par une VHS, en termes de l'application de périodes de cette VHS. De la même manière, nous introduisons une version para-complexe des variations de structures de Hodge, appelée les para-VHS. Nous associons une application de périodes à ces para-VHS. Les para-VHS sont des solutions de la géométrie para-tt^*. L'application para-pluriharmonique associée à une géométrie para-tt^*, qui provient d'une para-VHS, est exprimée à l'aide de l'application de périodes.

Dans tous les exemples des géométries (para-)tt^* discutés ci-dessus, la structure (para-)complexe de la variété (M, J) était intégrable. Dans l'étude des fibrés (para)-tt^* (TM, D, S) sur le fibré tangent TM il était nécessaire d'analyser des variétés presque (para-)complexes, car des variétés approximativement Kählériennes plates apparaissaient alors comme solutions de la géométrie (para-)tt^*. Une classification constructive des variétés approximativement Kählériennes plates est le sujet d'un travail en commun avec V. Cortés [CS2]. Expliquons la structure de cette partie de la thèse, dont le sujet est également développé dans [Sch7, Sch8]. La donnée d'un fibré (para-)tt^* induit une famille à un paramètre de connexions plates D^θ. D'autre part, chaque variété presque complexe (M, J) munie d'une connexion plate porte une famille naturelle à un paramètre de connexions défini par

$$\nabla^\theta = \exp(\theta J) \circ \nabla \circ \exp(-\theta J), \text{ avec } \theta \in \mathbb{R}.$$

Nous étudions les fibrés (para-)tt^* pour lesquels les deux familles à un paramètre de connexions sont equivalentes dans le sens de la définition suivante:

Définition 1 *Deux familles à un paramètre de connexions sont dites équivalentes linéaires avec facteur $\alpha \in \mathbb{R}$, si elles satisfont à l'équation $\nabla^\theta = D^{\alpha\theta}$.*

Nos études précédentes des solutions des fibrés (para-)tt^* provenant des variétés spéciales (para-)Kählériennes ont motivé l'examen de ces familles à un paramètre de connexions. De cette manière, nous avons obtenu une dualité entre des variétés approximativement Kählériennes plates et des variétés spéciales (para-)Kählériennes. Dans les deux cas, il s'agit de géométries importantes en mathématique et en physique théorique.
Nous considérons ensuite comme ci-dessus la restriction du problème aux fibrés (para-)tt^* du type (TM, D, S) pour lesquels la connexion D est (para-) complexe, c'est-à-dire vérifie $DJ = 0$. Ces fibrés sont donnés de façon unique par la structure (para-)complexe J et la connexion ∇. De plus, on trouve des conditions de compatibilité pour (∇, J) et on

peut montrer que dans le cas des variétés spéciales (para-)complexes et approximativement (para-)Kählériennes ces conditions sont remplies. Plus précisément, nous donnons deux classes de solutions correspondant respectivement à des variétés spéciales (para-)complexes avec torsion d'une part, et à des variétés plates presque (para-)complexes satisfaisant la condition approximativement Kählérienne (avec torsion) d'autre part. Par la suite, nous étudions si ces fibrés (para-)tt^* (TM, D, S) peuvent donner des fibrés (para-)tt^* métriques (TM, D, S, g) ou symplectiques (TM, D, S, ω). Les solutions du premier type proviennent par exemple des variétés spéciales (para-)Kählériennes et celles du second type de variétés approximativement (para-)Kählériennes. En effet, les variétés spéciales (para-)Kählériennes (M, J, ∇, g) n'admettent pas de fibré tt^* symplectique $(TM, D, S, \omega = g(J\cdot, \cdot))$, de même que les variétés approximativement (para-)Kählériennes (M, J, g) n'admettent aucun fibré (para-)tt^* métrique (TM, D, S, g). Plus précisément, la condition pour une variété d'être approximativement (para-)Kählérienne n'est pas compatible avec des fibrés (para-)tt^* métriques, et celle d'être spéciale (para-)complexe n'est pas compatible avec des fibrés symplectiques (para-)tt^*.
En conclusion, il reste à analyser si l'on peut obtenir la même relation entre les applications (para-)pluriharmoniques et la géométrie (para-)tt^* dans le cas où l'on a comme base une variété presque (para-)complexe (M, J).
Comme les structures (para-)complexes ne sont alors plus intégrables, il faut généraliser la notion d'application pluriharmonique au cas d'une variété de départ (M, J) avec une structure presque complexe J : on y parvient en choisissant une connexion idoine (cf. définition 2.6) sur la variété M et en utilisant l'équation (para-)pluriharmonique. Nous introduisons ensuite la notion d'application $\mathbb{S}^1_\epsilon$-pluriharmonique qui généralise la notion de famille associée à une application pluriharmonique (cf. [ET]) dans le cas des applications de variétés presque (para-)complexes vers des variétés pseudo-Riemanniennes. Nous donnons des conditions pour lesquelles une application $\mathbb{S}^1_\epsilon$-pluriharmonique est (para-)pluriharmonique et nous trouvons des conditions d'harmonicité pour des applications (para-)pluriharmoniques. Ces notions nous permettent d'associer des applications pluriharmoniques vers $\mathrm{Sp}(\mathbb{R}^{2n})/U(p,q)$ (respectivement vers $SO_0(2p, 2q)/U(p, q)$) aux fibrés tt^* métriques (respectivement symplectiques) du dernier paragraphe. Nous associons également des applications pluriharmoniques vers $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ (respectivement vers $SO_0(n, n)/U^\pi(C^n)$) aux fibrés para-tt^* métriques (respectivement symplectiques) décrits ci-dessus.

Comme nous l'avons déjà remarqué, les variétés spéciales (para-)complexes et spéciales (para-)Kählériennes forment une classe intéressante de fibrés (para-)tt^*, respectivement de fibrés (para-)tt^* métriques. Dans le cas complexe, c'est une conséquence d'un travail de Hertling [Her], qui associe une VHS de poids 1 à chaque variété spéciale Kählérienne. Dans [CS1], nous donnons une approche utilisant la géométrie différentielle et une caractérisation des fibrés tangents des variétés spéciales complexes et spéciales Kählériennes comme des fibrés tt^*. L'application pluriharmonique associée peut être exprimée avec l'application de Gauß duale, qui est une application holomorphe dans l'espace symétrique pseudo-Hermitien $\mathrm{Sp}(\mathbb{R}^{2n})/U(k,l)$ avec $n = k + l$. Ces résultats ont été généralisés à la géométrie para-complexe et publiés dans [Sch3]. Également, nous avons généralisé l'approche avec des VHS de poids 1. Plus précisement, on peut construire des para-VHS de poids 1 à partir d'une variété spéciale para-complexe. Les détails se trouvent dans cette thèse.

Chapter 1

Differential geometry on ϵcomplex manifolds

In this chapter we recall some definitions and results of special ϵcomplex geometry given in [CMMS] and prove some results which are analogous to those proven for special Kähler manifolds in [ACD]. We give here just a sketch of the results needed in this thesis. The interested reader can find further information in [CMMS].

1.1 ϵcomplex manifolds

Definition 1.1

(i) A complex structure *on a (real) finite dimensional vector space V is a linear map $J \in End(V)$ satisfying $J^2 = -Id_V$. A* complex vector space *(V, J) is a vector space endowed with a complex structure J. A* complex subspace *of the complex vector space V is a subspace W of the real vector space V, such that the restriction of J to W is a complex structure, i.e. W is J-invariant.*

(ii) A para-complex structure *on a (real) finite dimensional vector space V is a non-trivial involution $\tau \in End(V)$, i.e. $\tau^2 = Id_V$ and $\tau \neq Id_V$, such that the two eigenspaces $V^{\pm} := \ker(Id \mp \tau)$ of τ have the same dimension. A* para-complex vector space *(V, τ) is a vector space endowed with a para-complex structure τ. A* para-complex subspace *of the para-complex vector space V is a subspace W of the real vector space V, such that the restriction of τ to W is a para-complex structure.*

Remark 1.1 *It is well-known, that the eigenspaces of a complex structure have the same dimension. We remark, that for para-complex structures the condition on the eigenspaces to have the same dimension is not trivial. This condition can also be restated by requiring that the para-complex structure τ is trace-free.*

In the rest of this work we want to enrich our language by the following ϵ-notation:
If a word has a prefix ϵ with $\epsilon \in \{\pm 1\}$, i.e. is of the form ϵX, this expression is replaced

by

$$\epsilon X := \begin{cases} \text{X, for } \epsilon = -1, \\ \text{para-X, for } \epsilon = 1. \end{cases}$$

Using this construction we denote an ϵcomplex structure on the vector space V by the symbol J^ϵ, where J^ϵ satisfies ${J^\epsilon}^2 = \epsilon Id_V$.

Definition 1.2

(i) An almost ϵcomplex structure *on a smooth manifold M is an endomorphism field $J^\epsilon \in \Gamma(End(TM)), p \mapsto J^\epsilon_p$, such that J^ϵ_p is an ϵcomplex-structure for all $p \in M$.*

(ii) An almost para-complex structure is called integrable *if the eigendistributions $T^\pm M$ are both integrable.*

(iii) An integrable almost ϵcomplex structure is called ϵcomplex structure*. A manifold with an ϵcomplex structure is called* ϵcomplex manifold.

We remark, that the integrability of an almost ϵcomplex structure J^ϵ is equivalent to the vanishing of the Nijenhuis[1] tensor of J^ϵ defined by

$$N_{J^\epsilon}(X,Y) := [J^\epsilon X, J^\epsilon Y] + \epsilon[X,Y] - J^\epsilon[X, J^\epsilon Y] - J^\epsilon[J^\epsilon X, Y],$$

where $X, Y \in \Gamma(TM)$.
This is a well-known result in complex geometry. More information can be found in [KN] chapter IX. The para-complex case is done in [CMMS].

Definition 1.3 *A smooth map $f : (M, J^\epsilon) \to (N, \tilde{J}^\epsilon)$ from an ϵcomplex manifold (M, J^ϵ) to an ϵcomplex manifold $(N, \tilde{J}^\epsilon)$ is called ϵholomorphic if $df \circ J^\epsilon = \tilde{J}^\epsilon \circ df$ and anti-ϵholomorphic if $df \circ J^\epsilon = -\tilde{J}^\epsilon \circ df$.*

To go further we introduce the algebra $\mathbb{C}_\epsilon$ of ϵcomplex numbers. This is the real algebra generated by 1 and the symbol $\hat{i}$ subject to the relation $\hat{i}^2 = \epsilon$. As one observes for $\epsilon = -1$ this algebra coincides with the complex numbers $\mathbb{C}$. For $\epsilon = 1$ the symbol $\hat{i}$ is also denoted by e. We use the notation

$$\mathbb{C}_\epsilon = \begin{cases} \mathbb{C}, \text{ for } \epsilon = -1, \\ C, \text{ for } \epsilon = 1. \end{cases}$$

If one regards e as a unit vector in a one-dimensional $\mathbb{R}$-vector space with negative definite scalar product, then C is the (real) Clifford algebra $Cl_{0,1} = \mathbb{R} \oplus \mathbb{R}$. In the same manner we obtain $Cl_{1,0} = \mathbb{C}$ by considering the complex unit i as a unit vector in a one-dimensional $\mathbb{R}$-vector space with positive definite scalar product (Here we used the sign convention of [LM].).
As for complex numbers we define the ϵcomplex conjugation by

$$\bar{\cdot} : \mathbb{C}_\epsilon \to \mathbb{C}_\epsilon, \ x + \hat{i}y \mapsto x - \hat{i}y, \text{ for } x, y \in \mathbb{R}, \tag{1.1.1}$$

[1] In [KN] the Nijenhuis tensor was defined with a factor 2.

which is a $\mathbb{C}_\epsilon$-anti-linear involution, i.e. $\overline{\hat{i}z} = -\hat{i}\bar{z}$.
Real and imaginary parts are defined as

$$x = Re\, z := (z + \bar{z})/2 \text{ and } y = Im\, z := \epsilon\hat{i}(z - \bar{z})/2. \tag{1.1.2}$$

One has $z\bar{z} = x^2 - \epsilon y^2$ where $z \in \mathbb{C}_\epsilon$. Therefore the algebra C is sometimes called the hypercomplex numbers.
The circle

$$\mathbb{S}^1_{-1} := \mathbb{S}^1 = \{z = x + iy \in \mathbb{C} \,|\, x^2 + y^2 = 1\}$$

is replaced by the four hyperbola

$$\{z = x + ey \in C \,|\, x^2 - y^2 = \pm 1\}.$$

We define $\mathbb{S}^1_1$ to be the hyperbola given by the one parameter group $z(\theta) = \cosh(\theta) + e\sinh(\theta), \theta \in \mathbb{R}$:

$$\mathbb{S}^1_1 := \{z(\theta) = \cosh(\theta) + e\sinh(\theta) \,|\, \theta \in \mathbb{R}\}$$

and use the notation

$$\mathbb{S}^1_\epsilon = \begin{cases} \mathbb{S}^1_{-1} = \mathbb{S}^1, \text{ for } \epsilon = -1, \\ \mathbb{S}^1_1, \text{ for } \epsilon = 1. \end{cases}$$

In addition we define

$$\cos_\epsilon(x) := \begin{cases} \cos(x), \text{ for } \epsilon = -1, \\ \cosh(x), \text{ for } \epsilon = 1 \end{cases}$$

and

$$\sin_\epsilon(x) := \begin{cases} \sin(x), \text{ for } \epsilon = -1, \\ \sinh(x), \text{ for } \epsilon = 1 \end{cases}$$

to obtain with $z_\epsilon(\theta) = \cos_\epsilon(\theta) + \hat{i}\sin_\epsilon(\theta)$:

$$\mathbb{S}^1_\epsilon = \begin{cases} z_\epsilon(\theta) \text{ with } \theta \in [0, 2\pi], \text{ for } \epsilon = -1, \\ z_\epsilon(\theta) \text{ with } \theta \in \mathbb{R}, \text{ for } \epsilon = 1. \end{cases}$$

Every ϵcomplex vector space V is isomorphic to a trivial free $\mathbb{C}_\epsilon$-module $\mathbb{C}^k_\epsilon$ for some k. Obviously ϵcomplex subspaces $W \subset V$ correspond to free submodules of V.

We regard further the ϵcomplexification

$$TM^{\mathbb{C}_\epsilon} = TM \otimes_{\mathbb{R}} \mathbb{C}_\epsilon$$

of the tangent bundle TM of an almost ϵcomplex manifold (M, J^ϵ) and extend

$$J^\epsilon : TM \to TM$$

$\mathbb{C}_\epsilon$-linearly to

$$J^\epsilon : TM^{\mathbb{C}_\epsilon} \to TM^{\mathbb{C}_\epsilon}.$$

Then for all $p \in M$ the free $\mathbb{C}_\epsilon$-module $T_pM^{\mathbb{C}_\epsilon}$ decomposes as $\mathbb{C}_\epsilon$-module into the direct sum of two free $\mathbb{C}_\epsilon$-modules

$$T_pM^{\mathbb{C}_\epsilon} = T_p^{1,0}M \oplus T_p^{0,1}M, \tag{1.1.3}$$

where

$$T_p^{1,0}M := \{X + \epsilon\hat{i}J^\epsilon X | X \in T_pM\} \text{ and } T_p^{0,1}M := \{X - \epsilon\hat{i}J^\epsilon X | X \in T_pM\}.$$

The subbundles $T_p^{1,0}M$ and $T_p^{0,1}M$ can be characterized as the $\pm\hat{i}$-eigenbundles of the linear map $J^\epsilon : TM^{\mathbb{C}_\epsilon} \to TM^{\mathbb{C}_\epsilon}$, i.e. $J^\epsilon = \hat{i}$ on $T^{1,0}M$ and $J^\epsilon = -\hat{i}$ on $T^{0,1}M$.
In the same manner we decompose

$$T^*M^{\mathbb{C}_\epsilon} = \Lambda^{1,0}T^*M \oplus \Lambda^{0,1}T^*M$$

into the $\pm\hat{i}$-eigenbundles of the dual ϵcomplex structure

$$(J^\epsilon)^* : T^*M^{\mathbb{C}_\epsilon} \to T^*M^{\mathbb{C}_\epsilon}.$$

This decomposition induces a bi-grading on the $\mathbb{C}_\epsilon$-valued exterior forms

$$\Lambda^k T^*M^{\mathbb{C}_\epsilon} = \bigoplus_{k=p+q} \Lambda^{p,q}T^*M.$$

We remark that the vector bundles $\Lambda^{p,0}T^*M$ are ϵholomorphic vector bundles in the sense of the following definition (cf. [AK] for $\epsilon = -1$ and [LS] for $\epsilon = 1$):

Definition 1.4

*(i) Let (M, J^ϵ) be an ϵcomplex manifold. An ϵ*complex vector bundle *of rank r is a smooth real vector bundle $\pi : E \to M$ of rank $2r$ where the total space E is endowed with a fiberwise ϵcomplex structure $J^{\epsilon E} \in \Gamma(\mathrm{End}\,(E))$. We will denote it by $(E, J^{\epsilon E})$.*

*(ii) An ϵ*holomorphic vector bundle *is an ϵcomplex vector bundle $\pi : E \to M$ whose total space E is an ϵcomplex manifold, such that the projection π is an ϵholomorphic map and admits local ϵholomorphic trivializations.*
An (local) ϵholomorphic *section of an ϵholomorphic vector bundle $\pi : E \to M$ is a (local) section of E which is an ϵholomorphic map. The set of ϵholomorphic sections of E will be denoted by $\mathcal{O}(E)$.*

Finally we obtain a bi-grading on the $\mathbb{C}_\epsilon$-valued differential forms on M

$$\Omega^k_{\mathbb{C}_\epsilon}(M) = \bigoplus_{k=p+q} \Omega^{p,q}(M).$$

In para-complex geometry there exists another bi-grading:
The decomposition of TM over a para-complex manifold M in T^+M and T^-M induces a bi-grading on exterior forms

$$\Lambda^k T^*M = \bigoplus_{k=p+q} \Lambda^{p+,q-}T^*M. \tag{1.1.4}$$

We remark that for the cases $(1,1)$ and $(1+,1-)$ the two bi-gradings coincide in the sense that

$$\Lambda^{1,1}\,T^*M = \big(\Lambda^{1+,1-}\,T^*M\big) \otimes \mathbb{C}_\epsilon.$$

In complex geometry it is well-known, that every complex manifold admits a complex torsion-free connection (see for example [KN] chapter IX). We generalize this theorem to the ϵcomplex case, which was done in [Sch3]:

Theorem 1.1 *Every almost ϵcomplex manifold (M, J^ϵ) admits an almost ϵcomplex affine connection with torsion T satisfying*

$$N_{J^\epsilon} = -4\epsilon T,$$

where N_{J^ϵ} is the Nijenhuis-tensor of the almost ϵcomplex structure J^ϵ.

Proof: Let ∇ be a torsion-free connection on M. We define $Q \in \Gamma((T^*M)^2 \otimes TM)$ as

$$4Q(X,Y) := [(\nabla_{J^\epsilon Y} J^\epsilon)X + J^\epsilon((\nabla_Y J^\epsilon)X) + 2J^\epsilon((\nabla_X J^\epsilon)Y)]$$

and further

$$\tilde{\nabla}_X Y = \nabla_X Y + \epsilon Q(X,Y).$$

Now we compute

$$\begin{aligned}(\tilde{\nabla}_X J^\epsilon)Y &= \tilde{\nabla}_X J^\epsilon Y - J^\epsilon \tilde{\nabla}_X Y = \nabla_X J^\epsilon Y + \epsilon Q(X, J^\epsilon Y) - J^\epsilon \nabla_X Y - \epsilon J^\epsilon Q(X,Y)\\ &= (\nabla_X J^\epsilon)Y + \epsilon\,\underbrace{(Q(X,J^\epsilon Y) - J^\epsilon Q(X,Y))}_{=:A(X,Y)}.\end{aligned}$$

Hence we have to show $\epsilon A(X,Y) = -(\nabla_X J^\epsilon)Y$. It is

$$\begin{aligned}4Q(X, J^\epsilon Y) &= \epsilon(\nabla_Y J^\epsilon)X + J^\epsilon((\nabla_{J^\epsilon Y} J^\epsilon)X) + 2J^\epsilon((\nabla_X J^\epsilon)\,J^\epsilon Y),\\ 4J^\epsilon Q(X,Y) &= J^\epsilon(\nabla_{J^\epsilon Y} J^\epsilon)X + \epsilon((\nabla_Y J^\epsilon)X) + 2\epsilon((\nabla_X J^\epsilon)Y).\end{aligned}$$

With ${J^\epsilon}^2 = \epsilon\mathbb{1}$ we get $J^\epsilon[(\nabla_X J^\epsilon)\,J^\epsilon Y] = -J^\epsilon[J^\epsilon(\nabla_X J^\epsilon)Y] = -\epsilon(\nabla_X J^\epsilon)Y$ and we obtain finally

$$4A = 4(Q(X, J^\epsilon Y) - J^\epsilon Q(X,Y)) = -4\epsilon(\nabla_X J^\epsilon)Y.$$

It remains to compute the torsion of $\tilde{\nabla}$:

$$T^{\tilde{\nabla}}_{X,Y} = T^{\nabla}_{X,Y} + \epsilon(Q(X,Y) - Q(Y,X)) = \epsilon(Q(X,Y) - Q(Y,X)).$$

With the definition of Q we find

$$\begin{aligned}4\epsilon T^{\tilde{\nabla}}_{X,Y} &= (\nabla_{J^\epsilon Y} J^\epsilon)X + J^\epsilon((\nabla_Y J^\epsilon)X) + 2J^\epsilon((\nabla_X J^\epsilon)Y)\\ &\quad - ((\nabla_{J^\epsilon X} J^\epsilon)Y + J^\epsilon((\nabla_X J^\epsilon)Y) + 2J^\epsilon((\nabla_Y J^\epsilon)X))\\ &= (\nabla_{J^\epsilon Y} J^\epsilon)X - (\nabla_{J^\epsilon X} J^\epsilon)Y + J^\epsilon((\nabla_X J^\epsilon)Y) - J^\epsilon((\nabla_Y J^\epsilon)X)\\ &= (\nabla_{J^\epsilon Y} J^\epsilon X) - (\nabla_{J^\epsilon X} J^\epsilon Y) - J^\epsilon(\nabla_{J^\epsilon Y} X - \nabla_{J^\epsilon X} Y)\\ &\quad + J^\epsilon[\nabla_X(J^\epsilon Y) - J^\epsilon \nabla_X Y] - J^\epsilon[\nabla_Y(J^\epsilon X) - J^\epsilon \nabla_Y X]\\ &= [J^\epsilon Y, J^\epsilon X] + \epsilon[Y,X] + J^\epsilon[\nabla_X(J^\epsilon Y) - \nabla_{J^\epsilon Y} X] + J^\epsilon[\nabla_{J^\epsilon X} Y - \nabla_Y J^\epsilon X]\\ &= [J^\epsilon Y, J^\epsilon X] + \epsilon[Y,X] - J^\epsilon[J^\epsilon Y, X] - J^\epsilon[Y, J^\epsilon X] = N_{J^\epsilon}(Y,X) = -N_{J^\epsilon}(X,Y).\end{aligned}$$

□

If the ϵcomplex structure is integrable we get a usefull corollary:

Corollary 1.1 *Every ϵcomplex manifold (M, J^ϵ) admits an ϵcomplex torsion-free affine connection.*

An important question in ϵcomplex geometry is which kind of connections on a given ϵcomplex vector bundle give rise to an ϵholomorphic structure. To answer to this question we first need the definition of adapted connections, which can be found in [AK] for $\epsilon = -1$ and in [LS] for $\epsilon = 1$.

Definition 1.5

1. *A connection ∇ on an ϵcomplex vector bundle $(E, J^{\epsilon E})$ is called* ϵcomplex *if it commutes with the ϵcomplex structure on E, i.e. $J^{\epsilon E}$ is ∇-parallel. The set of all such connections will be denoted by $\mathcal{P}(E, J^{\epsilon E})$.*

2. *Let $(E, J^{\epsilon E})$ be an ϵholomorphic vector bundle over an ϵcomplex manifold (M, J^ϵ) and $U \subset M$ be an arbitrary open set. Let ∇ be a connection on the vector bundle $(E, J^{\epsilon E})$.*
   Then ∇ is called adapted *if the following equation*
   $$\nabla_{J^\epsilon X}\xi = J^{\epsilon E}\nabla_X\xi \tag{1.1.5}$$
   is satisfied for all $X \in \Gamma(TM|_U)$, $\xi \in \mathcal{O}(E_{|U})$.

Conversely, let $(E, J^{\epsilon E})$ be an ϵholomorphic vector bundle over an ϵcomplex manifold (M, J^ϵ) endowed with an adapted connection $\nabla \in \mathcal{P}(E, J^{\epsilon E})$, then a section $\xi \in \Gamma(E_{|U})$, where $U \subset M$ is an open set, is ϵholomorphic if and only if it satifies equation (1.1.5) for all $X \in \Gamma(TM|_U)$ (cf. Lemma 3 of [LS]).

The following proposition is well-known in complex geometry, compare for example the work of Atiyah, Hitchin and Singer [AHS] theorem 5.2 or proposition 3.7 in the book of Kobayashi [K]. The variety of its applications in complex geometry motivated us to search for a generalization.
For vector bundles over real surfaces this proposition was generalized to para-complex geometry by Erdem [E]. We gave in [LS] a different proof and more general result by adapting the methods of complex geometry to the para-complex setting.

Proposition 1.1 *Let $(E, J^{\epsilon E})$ be an ϵcomplex vector bundle over an ϵcomplex manifold (M, J^ϵ) and ∇ be a connection in $\mathcal{P}(E, J^{\epsilon E})$ with vanishing $(0,2)$-curvature then there exists a unique ϵholomorphic vector bundle structure on $(E, J^{\epsilon E})$ such that ∇ is adapted to this ϵholomorphic vector bundle structure.*

1.2 ϵKähler manifolds

The notion of a (pseudo-)Kähler manifold is classical and the notion of a para-Kähler manifold can be found in [CMMS].

Definition 1.6 *Let (V, J^ϵ) be an ϵcomplex vector space. An ϵ*hermitian *scalar product g on V is a pseudo-Euclidean scalar product for which J^ϵ is an ϵ-isometry, i.e.*

$$(J^\epsilon)^* g = g(J^\epsilon \cdot, J^\epsilon \cdot) = -\epsilon g(\cdot, \cdot).$$

*An ϵ*hermitian vector space *(V, J^ϵ, g) is an ϵcomplex vector space (V, J^ϵ) endowed with an ϵhermitian scalar product g. The pair (J^ϵ, g) is called ϵ*hermitian structure *on the vector space V.*

Definition 1.7 *Let (V, τ, g) be a para-hermitian vector space. The para-unitary group of V is the automorphism group*

$$U^\pi(V) := Aut(V, \tau, g) = \{L \in GL(V) \,|\, [L, \tau] = 0 \text{ and } L^* g = g\}.$$

Its Lie-algebra will be denoted by $\mathfrak{u}^\pi(V)$.

Definition 1.8 *An* almost ϵhermitian manifold *(M, J^ϵ, g) is an almost ϵcomplex manifold (M, J^ϵ) endowed with a pseudo-Riemannian metric g such that $(J^\epsilon)^* g = -\epsilon g$. If J^ϵ is integrable, we call (M, J^ϵ, g) an ϵ*hermitian manifold*. The two-form $\omega := g(J^\epsilon \cdot, \cdot)$ is called the* fundamental two-form *of the almost ϵhermitian manifold (M, J^ϵ, g).*

Definition 1.9 *An ϵ*Kähler manifold *(M, J^ϵ, g) is an ϵhermitian manifold such that J^ϵ is parallel with respect to the Levi-Civita-connection D of g, i.e. $DJ^\epsilon = 0$.*

Remark 1.2 *The fundamental two-form ω satisfies $(J^\epsilon)^* \omega = -\epsilon \omega$ and hence is of type $(1, 1)$ (considered as $\mathbb{C}_\epsilon$-valued two-form).*
Since $DJ^\epsilon = 0$ implies $N_{J^\epsilon} = 0$ and $d\omega = 0$, any ϵKählerian manifold is an ϵhermitian manifold with closed fundamental two-form.
*On an ϵKähler manifold the fundamental two-form ω is called ϵ*Kähler-form*. In fact, ϵKähler manifolds are characterized to be ϵhermitian manifolds with closed fundamental two-form (compare [CMMS] for the para-complex case).*

1.3 Nearly ϵKähler manifolds

In this section we introduce some notions and results of nearly ϵKähler geometry. The complex case is due to Gray in his classical papers [G1, G2, G3]. Recent studies are the works Friedrich and Ivanov [FI] and Nagy [N1, N2]. The para-complex version is very recent and to our knowledge first appeared in the paper of Ivanov and Zamkovoy [IZ].

Definition 1.10 *An almost ϵhermitian manifold (M, J^ϵ, g) is called* nearly ϵKähler *manifold, if its Levi-Civita connection $\nabla = \nabla^g$ satisfies the equation*

$$(\nabla_X J^\epsilon) Y = -(\nabla_Y J^\epsilon) X, \quad \forall\, X, Y \in \Gamma(TM). \tag{1.3.1}$$

A nearly ϵKähler manifold is called strict*, if $\nabla J^\epsilon \neq 0$.*

We recall that the tensor ∇J^ϵ defines two tensors A and B by

$$A(X,Y,Z) := g((\nabla_X J^\epsilon)Y, Z) \text{ and } B(X,Y,Z) := -\epsilon g((\nabla_X J^\epsilon)Y, J^\epsilon Z) \text{ with } X,Y,Z \in TM,$$

which are both (real) three-forms of type $(3,0)+(0,3)$.
A connection of particular importance in nearly ϵKähler geometry is the connection $\bar{\nabla}$ defined by

$$\bar{\nabla}_X Y := \nabla_X Y - \frac{1}{2}\epsilon(\nabla_X J^\epsilon)J^\epsilon Y, \text{ for all } X,Y \in \Gamma(TM). \tag{1.3.2}$$

The torsion of the connection $\bar{\nabla}$ is given by

$$T^{\bar{\nabla}}(X,Y) = -\epsilon(\nabla_X J^\epsilon)J^\epsilon Y, \text{ for all } X,Y \in \Gamma(TM) \tag{1.3.3}$$

and it vanishes if and only if (M, J^ϵ, g) is an ϵKähler manifold.
We remark, that the connection $\bar{\nabla}$ can be characterized to be the unique connection with totally skew-symmetric torsion (cf. Friedrich and Ivanov [FI] for case $\epsilon = -1$ with a Riemannian metric.). In [CS2] we give a self-contained proof of this result using direct arguments for nearly pseudo-Kähler and nearly para-Kähler manifolds.

Proposition 1.2 *Let (M, J^ϵ, g) be a nearly ϵKähler manifold. Then there exists a unique connection $\bar{\nabla}$ with totally skew-symmetric torsion $T^{\bar{\nabla}}$ satisfying $\bar{\nabla} g = 0$ and $\bar{\nabla} J^\epsilon = 0$.*
More precisely, it holds

$$T^{\bar{\nabla}} = -2S \text{ with } S = -\frac{1}{2}\epsilon J^\epsilon \nabla^g J^\epsilon \tag{1.3.4}$$

and $\{S_X, J^\epsilon\} = 0$, for all vector fields X.

1.4 Affine special ϵcomplex and special ϵKähler manifolds

Definition 1.11 *An* affine special ϵcomplex manifold *(M, J^ϵ, ∇) is an ϵcomplex manifold (M, J^ϵ) endowed with a torsion-free flat connection ∇ such that ∇J^ϵ is a symmetric $(1,2)$-tensor field, i.e. $(\nabla_X J^\epsilon)Y = (\nabla_Y J^\epsilon)X$ for all $X, Y \in TM$.*
An affine special ϵKähler manifold *$(M, J^\epsilon, g, \nabla)$ is an affine special ϵcomplex manifold (M, J^ϵ, ∇), such that (M, J^ϵ, g) is an ϵKähler manifold and ∇ is symplectic, i.e. $\nabla\omega = 0$, where ω is the ϵKähler-form.*

Since projective special ϵcomplex and projective special ϵKähler manifolds do not occur in this thesis, we omit the adjective affine. The definition of a special ϵKähler manifold can be found in [ACD, F] for $\epsilon = -1$. Special para-Kähler manifolds were first considered in [CMMS] and special para-complex manifolds in [Sch3].
In the following part of this subsection we are going to generalize some results to ϵcomplex geometry, which are known from the affine special and the affine special Kähler case (see [ACD]). The para-complex results were published in [Sch3].

Remark 1.3 *Given a linear connection ∇ on the tangent bundle TM of a manifold M and an invertible endomorphism field $A \in \Gamma(End(TM))$ we define the connection*

$$\nabla^{(A)} X = A \nabla (A^{-1} X).$$

This connection is flat if and only if the connection ∇ is flat, since

$$\nabla X = 0 \Leftrightarrow \nabla^{(A)}(A\, X) = 0,$$

where X is a local vector field on M.
Again, given a linear flat connection on the real tangent bundle TM of an ϵcomplex manifold (M, J^ϵ), we define a one-parameter family of flat connections by

$$\nabla^\theta = \nabla^{(e^{\theta J^\epsilon})} = \nabla^{(\cos_\epsilon(\theta)\, Id + \sin_\epsilon(\theta)\, J^\epsilon)} \text{ for } \theta \in \mathbb{R}. \tag{1.4.1}$$

Lemma 1.1 *Let ∇ be a flat connection with torsion T on an ϵcomplex manifold (M, J^ϵ). Then it is*

$$\nabla^\theta = \nabla + A^\theta, \text{ where } A^\theta = e^{\theta J^\epsilon} \nabla (e^{-\theta J^\epsilon}) = -\sin_\epsilon(\theta)\, e^{\theta J^\epsilon} \nabla J^\epsilon$$

and the torsion T^θ of the connection ∇^θ is given by

$$T^\theta = T + alt(A^\theta) = T - \sin_\epsilon(\theta)\, e^{\theta J^\epsilon}\, d^\nabla J^\epsilon. \tag{1.4.2}$$

Proposition 1.3 *Let ∇ be a torsion-free flat connection on an ϵcomplex manifold (M, J^ϵ). Then the triple (M, J^ϵ, ∇) defines a special ϵcomplex manifold if and only if one of the following conditions holds:*

a) $d^\nabla J^\epsilon = 0$.

b) The flat connection ∇^θ is torsion-free for some θ with

$$\begin{cases} \theta \neq 0, & \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \text{ mod } \pi, & \text{for } \epsilon = -1. \end{cases}$$

b') The flat connection ∇^θ is torsion-free for all θ with

$$\begin{cases} \theta \neq 0, & \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \text{ mod } \pi, & \text{for } \epsilon = -1. \end{cases}$$

c) There exists an element θ with

$$\begin{cases} \theta \neq 0, & \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \text{ mod } \pi, & \text{for } \epsilon = -1, \end{cases}$$

such that $[e^{\theta J^\epsilon} X, e^{\theta J^\epsilon} Y] = 0$ for all ∇-parallel local vector fields X and Y on M.

c') It holds $[e^{\theta J^\epsilon}X, e^{\theta J^\epsilon}Y] = 0$ *for all* θ *with*

$$\begin{cases} \theta \neq 0, \ \textit{for } \epsilon = 1, \\ \theta \not\equiv 0 \ \textit{mod } \pi, \ \textit{for } \epsilon = -1 \end{cases}$$

and for all ∇*-parallel local vector fields* X *and* Y *on* M*.*

d) There exists an element θ *with*

$$\begin{cases} \theta \neq 0, \ \textit{for } \epsilon = 1, \\ \theta \not\equiv 0 \ \textit{mod } \pi, \ \textit{for } \epsilon = -1, \end{cases}$$

such that $d(\eta \circ e^{-\theta J^\epsilon}) = 0$ *for all* ∇*-parallel local one-forms on* M*.*

d') It holds $d(\eta \circ e^{-\theta J^\epsilon}) = 0$ *for all* θ *with*

$$\begin{cases} \theta \neq 0, \ \textit{for } \epsilon = 1, \\ \theta \not\equiv 0 \ \textit{mod } \pi, \ \textit{for } \epsilon = -1 \end{cases}$$

and for all ∇*-parallel local one-forms on* M*.*

Proof: The property a) defines special ϵcomplex manifolds.
As ∇ is torsion-free, the torsion of ∇^θ is by equation (1.4.2):

$$T^\theta = -\sin_\epsilon(\theta)\, e^{\theta J^\epsilon}\, d^\nabla J^\epsilon.$$

Since $\sin_\epsilon(\theta) \neq 0$ for θ with

$$\begin{cases} \theta \neq 0, \ \text{for } \epsilon = 1, \\ \theta \not\equiv 0 \ \text{mod } \pi, \ \text{for } \epsilon = -1, \end{cases}$$

we get the equivalence of a) and b) respectively b').
Let X and Y be ∇-parallel local vector fields. Then $e^{\theta J^\epsilon}X$ and $e^{\theta J^\epsilon}Y$ are ∇^θ-parallel, by the definition of ∇^θ. Therefore

$$T^\theta(e^{\theta J^\epsilon}X, e^{\theta J^\epsilon}Y) = [e^{\theta J^\epsilon}X, e^{\theta J^\epsilon}Y].$$

This gives $b) \Leftrightarrow c)$ and $b') \Leftrightarrow c')$.
For a ∇-parallel one-form η and X, Y as before we compute:

$$\begin{aligned} & d(\eta \circ e^{-\theta J^\epsilon})(e^{\theta J^\epsilon}X, e^{\theta J^\epsilon}Y) \\ = \ & e^{\theta J^\epsilon}X\eta(Y) - e^{\theta J^\epsilon}Y\eta(X) - \eta(e^{-\theta J^\epsilon}[e^{\theta J^\epsilon}X, e^{\theta J^\epsilon}Y]) \\ = \ & -\eta(e^{-\theta J^\epsilon}[e^{\theta J^\epsilon}X, e^{\theta J^\epsilon}Y]), \end{aligned}$$

as the functions $\eta(X)$ and $\eta(Y)$ are constant. This proves $c) \Leftrightarrow d)$ and $c') \Leftrightarrow d')$. $\square$

Proposition 1.4 *If* (M, J^ϵ, ∇) *is a special* ϵ*complex manifold, then* $(M, J^\epsilon, \nabla^\theta)$ *is a special* ϵ*complex manifold for any* θ*.*
If $(M, J^\epsilon, g, \nabla)$ *is a special* ϵ*Kähler manifold, then* $(M, J^\epsilon, g, \nabla^\theta)$ *is a special* ϵ*Kähler manifold for any* θ*.*

Proof: From above we know, that the connection ∇^θ is torsion-free and flat.
In order to prove this proposition we compute $\nabla^\theta J^\epsilon$ and $\nabla^\theta\omega$.
Let $X, Y, Z \in \Gamma(TM)$:

$$\begin{aligned}(\nabla^\theta_X J^\epsilon)Y &= \nabla^\theta_X(J^\epsilon Y) - J^\epsilon\nabla^\theta_X Y = e^{\theta J^\epsilon}\nabla_X(e^{-\theta J^\epsilon}J^\epsilon Y) - J^\epsilon e^{\theta J^\epsilon}\nabla_X(e^{-\theta J^\epsilon}Y)\\ &= e^{\theta J^\epsilon}\nabla_X(J^\epsilon e^{-\theta J^\epsilon}Y) - e^{\theta J^\epsilon}J^\epsilon\nabla_X(e^{-\theta J^\epsilon}Y)\\ &= e^{\theta J^\epsilon}(\nabla_X J^\epsilon)e^{-\theta J^\epsilon}Y \overset{(*)}{=} e^{2\theta J^\epsilon}(\nabla_X J^\epsilon)Y.\end{aligned}$$

At $(*)$ we have used $J^\epsilon(\nabla J^\epsilon) = -(\nabla J^\epsilon)J^\epsilon$, which follows from $J^{\epsilon 2} = \epsilon Id$.
This shows $d^{\nabla^\theta}J^\epsilon = e^{2\theta J^\epsilon}d^\nabla J^\epsilon = 0$.
Further we find utilizing $\omega(\cdot, e^{\theta J^\epsilon}\cdot) = \omega(e^{-\theta J^\epsilon}\cdot, \cdot)$, which is a consequence of $(J^\epsilon)^*\omega = -\epsilon\omega$:

$$\begin{aligned}\nabla^\theta_Z\omega(X,Y) &= Z\omega(X,Y) - \omega(\nabla^\theta_Z X, Y) - \omega(X, \nabla^\theta_Z Y)\\ &= Z\omega(X,Y) - \omega(e^{\theta J^\epsilon}\nabla_Z e^{-\theta J^\epsilon}X, Y) - \omega(X, e^{\theta J^\epsilon}\nabla_Z e^{-\theta J^\epsilon}Y)\\ &= Z\omega(X,Y) - \omega(\nabla_Z e^{-\theta J^\epsilon}X, e^{-\theta J^\epsilon}Y) - \omega(e^{-\theta J^\epsilon}X, \nabla_Z e^{-\theta J^\epsilon}Y)\\ &= Z\omega(X,Y) - Z\omega(e^{-\theta J^\epsilon}X, e^{-\theta J^\epsilon}Y) = 0.\end{aligned}$$

□

Given an ϵcomplex manifold with a flat connection ∇, we define the conjugate connection via

$$\nabla^c_X Y = \nabla^{(J^\epsilon)}_X Y = \epsilon J^\epsilon(\nabla_X J^\epsilon Y) = \nabla_X Y + \epsilon J^\epsilon(\nabla_X J^\epsilon)Y \text{ for } X, Y \in \Gamma(TM).$$

Proposition 1.5 *Let (M, J^ϵ) be an ϵcomplex manifold with a torsion-free flat connection ∇. Then the following statements are equivalent:*

a) (M, J^ϵ, ∇) is a special ϵcomplex manifold.

b) The conjugate flat connection ∇^c is torsion-free.

Proof: The torsion of the connection ∇^c is

$$T^{\nabla^c} = T^\nabla + \epsilon\, alt(J^\epsilon(\nabla J^\epsilon)) = \epsilon J^\epsilon d^\nabla J^\epsilon.$$

Therefore ∇^c is torsion-free if and only if $d^\nabla J^\epsilon = 0$. □

Proposition 1.6 *Let (M, J^ϵ, ∇) be a special ϵcomplex manifold. Then $D := \frac{1}{2}(\nabla + \nabla^c)$ defines a torsion-free ϵcomplex connection, i.e. a torsion-free connection such that $DJ^\epsilon = 0$.*

Proof: As it is a convex combination of torsion-free connections, D is a torsion-free connection. For any $X \in \Gamma(TM)$ we compute:

$$D_X J^\epsilon = \nabla_X J^\epsilon + \frac{1}{2}\epsilon[J^\epsilon\nabla_X J^\epsilon, J^\epsilon] = \nabla_X J^\epsilon - \nabla_X J^\epsilon = 0.$$

□

Proposition 1.7 *Let $(M, J^\epsilon, g, \nabla)$ be a special ϵKähler manifold and ∇^g the Levi-Civita connection of g. Then the following hold:*

(i) $\nabla^g = \frac{1}{2}(\nabla + \nabla^c) = D$.

(ii) The conjugate connection ∇^c is g-dual, i.e.:

$$Xg(Y,Z) = g(\nabla^c_X Y, Z) + g(Y, \nabla_X Z).$$

Equivalently

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^c_X Z)$$

for all vector fields $X, Y, Z \in \Gamma(TM)$.

(iii) The tensor ∇g is completely symmetric.

Proof: (i) follows immediately from (ii) and proposition 1.6.
(ii) follows from a direct calculation which only uses the fact that ω is ∇-parallel and J^ϵ-ϵ-anti-invariant: With $X, Y, Z \in \Gamma(TM)$ one finds

$$\begin{aligned} Xg(Y,Z) &= X(\epsilon\omega(J^\epsilon Y, Z)) = \epsilon\omega(\nabla_X J^\epsilon Y, Z) + \epsilon\omega(J^\epsilon Y, \nabla_X Z) \\ &= -\omega(J^\epsilon \nabla_X J^\epsilon Y, J^\epsilon Z) + g(Y, \nabla_X Z) \\ &= \omega(J^\epsilon Z, J^\epsilon \nabla_X J^\epsilon Y) + g(Y, \nabla_X Z) \\ &= g(Z, \epsilon J^\epsilon \nabla_X (J^\epsilon Y)) + g(Y, \nabla_X Z) \\ &= g(\nabla^c_X Y, Z) + g(Y, \nabla_X Z). \end{aligned}$$

Finally we show (iii): From part (ii) it follows

$$\begin{aligned} (\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) &= Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &\quad - Yg(X,Z) + g(\nabla_Y X, Z) + g(X, \nabla_Y Z) \\ &\overset{(ii)}{=} -g(\nabla_X Y, Z) + g(\nabla^c_X Y, Z) \\ &\quad + g(\nabla_Y X, Z) - g(\nabla^c_Y X, Z) \\ &= g(-[X,Y] + [X,Y], Z) = 0. \end{aligned}$$

The symmetry of g finishes the proof. □

Proposition 1.8 *Let $(M, J^\epsilon, g, \nabla)$ be a special ϵKähler manifold and D the Levi-Civita connection of g. Define the endomorphism field S as*

$$S := \nabla - D = \nabla - \frac{1}{2}(\nabla + \nabla^c) = \frac{1}{2}(\nabla - \nabla^c) = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon).$$

Then S is

(i) symmetric, i.e. $S_X Y = S_Y X;\ \forall X, Y \in \Gamma(TM)$,

(ii) ω-skew-symmetric, i.e. $\omega(S_X \cdot, \cdot) = -\omega(\cdot, S_X \cdot)$,

(iii) g-symmetric, i.e. $g(S_X \cdot, \cdot) = g(\cdot, S_X \cdot)$ for all $X \in \Gamma(TM)$ and

(iv) anti-commutes with J^ϵ, *i.e.*

$$\{S_X, J^\epsilon\} := S_X J^\epsilon + J^\epsilon S_X = 0 \text{ for all } X \in \Gamma(TM). \tag{1.4.3}$$

Proof: Let $X, Y, Z \in \Gamma(TM)$.
(i) For a special ϵcomplex manifold ∇ and ∇^c are torsion-free (by definition and proposition 1.5), so $\nabla - \nabla^c = -\epsilon J^\epsilon(\nabla J^\epsilon) = 2S$ is symmetric.
(ii) In fact $Dg = 0$ (proposition 1.7) and $DJ^\epsilon = 0$ (proposition 1.6) imply $D\omega = 0$. In addition $\nabla\omega = 0$ yields

$$\omega(S_X Y, Z) + \omega(Y, S_X Z) = \omega((\nabla - D)_X Y, Z) + \omega(Y, (\nabla - D)_X Z) = (\nabla - D)_X \omega(Y, Z) = 0.$$

(iii) Using $Xg(Y,Z) - g(\nabla_X Y, Z) = g(Y, \nabla^c_X Z)$ we prove the g-symmetry of S

$$\begin{aligned} 2g(S_X Y, Z) = g((\nabla - \nabla^c)_X Y, Z) &= g(\nabla_X Y, Z) - g(\nabla^c_X Y, Z) \\ &= Xg(Y,Z) - g(Y, \nabla^c_X Z) - Xg(Y,Z) + g(Y, \nabla_X Z) \\ &= g(Y, (\nabla - \nabla^c)_X Z) = 2g(Y, S_X Z). \end{aligned}$$

(iv) Now we need only the ω-skew-symmetry of S, the g-symmetry of S and $\omega = g(J^\epsilon\cdot, \cdot) = -g(\cdot, J^\epsilon\cdot)$ to get for all $X, Y, Z \in \Gamma(TM)$

$$g(S_X J^\epsilon Y, Z) = g(J^\epsilon Y, S_X Z) = \omega(Y, S_X Z) = -\omega(S_X Y, Z) = -g(J^\epsilon S_X Y, Z)$$

and consequently $\{S_X, J^\epsilon\} = 0$. □

1.5 The extrinsic construction of special ϵKähler manifolds

Now we shortly explain the extrinsic construction of special ϵKähler manifolds given in [ACD, CMMS].

1.5.1 The special Kähler case

We consider the complex vector space $V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$ with canonical coordinates $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ endowed with the standard complex symplectic form

$$\Omega = \sum_{i=1}^n dz^i \wedge dw_i$$

and the standard real structure $\kappa : V \to V$ with fixed points $V^\kappa = T^*\mathbb{R}^n$. These define a hermitian form $\gamma := i\Omega(\cdot, \kappa\cdot)$.
Let (M, J) be a complex manifold of complex dimension n. We call a holomorphic immersion $\phi : M \to V$ non-degenerate (respectively Lagrangian) if $\phi^*\gamma$ is non-degenerate (respectively, if $\phi^*\Omega = 0$). If ϕ is non-degenerate it defines a (possibly indefinite) Kähler metric $g = Re\,\phi^*\gamma$ on the complex manifold (M, J) and the corresponding Kähler form $g(J\cdot, \cdot)$ is a J-invariant symplectic form.

The following theorem gives a description of simply connected special Kähler manifolds in terms of the above data:

Theorem 1.2 *[ACD] Let (M, J, g, ∇) be a simply connected special Kähler manifold of complex dimension n, then there exists a holomorphic non-degenerate Lagrangian immersion $\phi : M \to V = T^*\mathbb{C}^n$ inducing the Kähler metric g, the connection ∇ and the symplectic form $\omega = g(J\cdot, \cdot) = 2\phi^* \left(\sum_{i=1}^n dx^i \wedge dy_i\right)$ on M. Moreover, ϕ is unique up to an affine transformation of V preserving the complex symplectic form Ω and the real structure κ. The flat connection ∇ is completely determined by the condition $\nabla\phi^* dx^i = \nabla\phi^* dy_i = 0$, $i = 1, \dots, n$, where $x^i = Re\, z^i$ and $y_i = Re\, w_i$.*

1.5.2 The special para-Kähler case

First we have to introduce a canonical non-degenerate exact C-valued two-form Ω of type (2,0) on the cotangent bundle $N = T^*M$ of an arbitrary para-complex manifold (M, τ), which is para-holomorphic, i.e. it is a para-holomorphic section of the para-holomorphic vector bundle $\Lambda^{2,0}T^*N$. Its explicit form is given by the following consideration:

We take local para-holomorphic coordinates $(z^1, \dots, z^n)$ on an open subset $U \subset M^n$. Any point of $T_p^*M \cong \mathrm{Hom}(T_p^*M, \mathbb{R}) \cong \mathrm{Hom}_C(T_p^*M, C)$, $p \in U$, where $\mathrm{Hom}_C(T_p^*M, C)$ are the homomorphisms from the para-complex vector space (T_p^*M, τ_p) to C, can be expressed as $\sum w_i\, dz^i_{|p}$. The coordinates z^i and w_i can be regarded as local para-holomorphic coordinates of the bundle $T^*M_{|U}$. The coordinates w_i induce linear para-holomorphic coordinates on each fiber T_p^*M for $p \in U$. In these coordinates the two form Ω is given by

$$\Omega = \sum_{i=1}^{n} dz^i \wedge dw_i = -d\left(\sum_{i=1}^{n} w_i dz^i\right).$$

We observe, that $\sum_{i=1}^n w_i dz^i$ does not depend on the choice of coordinates and hence Ω does not depend on the choice of coordinates, too. The form Ω will be called the symplectic form of T^*M.
In the following, we denote by V the para-holomorphic vector space $T^*C^n = C^{2n}$, endowed with its standard para-complex structure τ_V, its symplectic form Ω and the para-complex conjugation $\bar{\cdot} : V \to V$, $v \mapsto \bar{v}$ with fixed point set $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$. On this space we take a system of para-holomorphic linear coordinates (z^i, w_i) which are real-valued on $T^*\mathbb{R}^n$. The algebraic data (Ω, τ_V) defines a para-hermitian scalar product on V via

$$g_V(v, w) = \mathrm{Re}\,\gamma(v, w) = \mathrm{Re}(e\Omega(v, \bar{w})), \quad \forall v, w \in V \text{ with } \gamma(v, w) = e\Omega(v, \bar{w})$$

and (V, τ_V, g_V) is a flat para-Kähler manifold, whose para-Kähler form is given by

$$\omega_V(v, w) := g_V(\tau_V v, w) = \mathrm{Im}(e\Omega(v, \bar{w})), \quad \forall v, w \in V.$$

Let (M, τ) be a para-complex manifold. We call a para-holomorphic immersion $\phi : M \to V$ para-Kählerian if $g = \phi^* g_V$ is non-degenerate and Lagrangian if $\phi^*\Omega = 0$. Any para-Kählerian immersion $\phi : M \to V$ induces on M the structure of a para-Kähler manifold (M, τ, g) with para-Kähler form $\omega(\cdot, \cdot) = g(\tau\cdot, \cdot) = \phi^*\omega_V$. For a para-Kählerian Lagrangian immersion the para-Kähler form $\omega = g(\tau\cdot, \cdot)$ of M is given by

$$\omega = 2\sum_{i=1}^{n} d\tilde{x}^i \wedge d\tilde{y}_i,$$

where $\tilde{x}^i = \mathrm{Re}(\phi^* z^i)$ and $\tilde{y}^i = \mathrm{Re}(\phi^* w_i)$. Additionally, a para-Kählerian Lagrangian immersion $\phi : M \to V$ induces a canonical flat torsion-free connection ∇ on M which is characterized by the condition, that $\nabla(\mathrm{Re}\,\phi^* df) = 0$ for all para-complex affine functions f on V.
With this information we now can give the extrinsic description of para-Kähler manifolds:

Theorem 1.3 *[CMMS] Let $\phi : M \to V$ be a para-Kählerian immersion with induced geometric data (τ, g, ∇). Then (M, τ, g, ∇) is a special para-Kähler manifold. Conversely, any simply connected special para-Kähler manifold (M, τ, g, ∇) admits a para-Kählerian Lagrangian immersion inducing the special geometric data (τ, g, ∇) on M. The para-Kählerian Lagrangian immersion ϕ is unique up to an affine linear transformation of V whose linear part belongs to the group $Aut(V, \Omega, \bar{\cdot}) = Aut_{\mathbb{R}}(V, \tau_V, \Omega, \bar{\cdot}) = Sp(\mathbb{R}^{2n})$.*

1.6 Variations of ϵHodge structures

In this section we introduce the notion of variations of ϵHodge structures in para-complex geometry and recall variations of Hodge structures which are classical objects in complex geometry. We follow the notations of [CMP] which is a reference for further study of variations of Hodge structures. The para-complex version seems to be new.

1.6.1 ϵHodge structures and their variations

Definition 1.12

(a) *A* real ϵHodge structure of weight $w \in \mathbb{N}$ *is a real vector space H on the ϵcomplexification of which there is a decomposition into ϵcomplex vector spaces*

$$H^{\mathbb{C}_\epsilon} = \bigoplus_{w=p+q} H^{p,q} \text{ with } p, q \in \mathbb{N} \tag{1.6.1}$$

and where

$$\overline{H^{p,q}} = H^{q,p} \text{ with } p, q \in \mathbb{N}. \tag{1.6.2}$$

The ϵcomplex conjugation $\bar{\cdot}$ is relative to the real structure on $H^{\mathbb{C}_\epsilon} = H \otimes \mathbb{C}_\epsilon$.

(b) *Suppose, that an ϵHodge structure of weight w carries a bilinear form $b : H \times H \to \mathbb{R}$ which satisfies the following Riemannian bilinear relations*

(i) *The $\mathbb{C}_\epsilon$-linear extension of the bilinear form b, also denoted by b, satisfies $b(x, y) = 0$ if $x \in H^{p,q}$ and $y \in H^{r,s}$ for $(r, s) \neq (w - p, w - q) = (q, p)$,*

(ii) *The bilinear form b defines an ϵhermitian sesquilinear scalar product (compare definition 2.10) on $H^{p,q}$ by*

$$h(x, y) = (-1)^{w(w-1)/2} \hat{i}^{p-q} b(x, \bar{y}).$$

Then we call this ϵHodge structure weakly polarized.

(c) *Suppose, that a Hodge structure of weight w carries a bilinear form $b : H \times H \to \mathbb{R}$ which satisfies the following Riemannian bilinear relations*

(i) *The $\mathbb{C}$-linear extension of the bilinear form b, also denoted by b, satisfies $b(x, y) = 0$ if $x \in H^{p,q}$ and $y \in H^{r,s}$ for $(r, s) \neq (w - p, w - q) = (q, p)$,*

(ii) *The bilinear form b defines a positive definite hermitian sesquilinear form on $H^{p,q}$ by*

$$h(x, y) = (-1)^{w(w-1)/2} \hat{i}^{p-q} b(x, \bar{y}).$$

Then we call this Hodge structure strongly polarized.

(d) *An ϵHodge structure of weight w is called* polarized *if it is weakly polarized or strongly polarized.*

Closely related to the ϵHodge decomposition is the ϵHodge filtration

$$F^p = \bigoplus_{a \geq p} H^{a,b}, \ p = 0, \dots, w, \tag{1.6.3}$$

which satisfies for an ϵHodge structure of weight w the relation

$$H^{\mathbb{C}_\epsilon} = F^p \oplus \overline{F^{w-p+1}}, \ p = 1, \dots, w. \tag{1.6.4}$$

Any filtration which obeys equation (1.6.4) is called an ϵHodge filtration.
Such as an ϵHodge decomposition induces an ϵHodge filtration we obtain from an ϵHodge filtration an ϵHodge decomposition by

$$H^{p,q} = F^p \cap \overline{F^q}, \text{ with } p + q = w.$$

This ϵHodge decomposition satifies the relation (1.6.3).
We remark further, that the first Riemannian bilinear relation (cf. definition 1.12) is equivalent to

$$(F^p)^\perp = F^{w-p+1}, \ p = 1, \dots, w,$$

where $\perp$ is taken with respect to the bilinear from b.

Now we are going to consider deformations of these structures:

Definition 1.13 *A* (real) variation of ϵHodge structures (ϵVHS) *is a triple (E, ∇, F^p), where E is a real vector bundle over an (connected) ϵcomplex base manifold (M, J^ϵ), ∇ is a flat connection and F^p is a filtration of $E^{\mathbb{C}_\epsilon}$ by ϵholomorphic subbundles of $E^{\mathbb{C}_\epsilon}$, which is a point-wise ϵHodge structure satisfying the* infinitesimal period relation *or the* Griffiths tranversality

$$\nabla_\chi F^p \subset F^{p-1}, \ \forall \chi \in T^{1,0}M. \tag{1.6.5}$$

A polarization *of a variation of ϵHodge structures (E, ∇, F^p) consists of a non-degenerate bilinear form*

$$b \in \Gamma(E^* \otimes E^*) \tag{1.6.6}$$

having the following properties

(i) b induces a polarization on each fiber obeying the first and the second bilinear relation.

(ii) b is parallel with respect to ∇.

1.6.2 ϵVHS and special ϵKähler manifolds

Each fiber of the ϵcomplex tangent bundle

$$TM^{\mathbb{C}_\epsilon} = T^{1,0}M \oplus T^{0,1}M$$

carries a natural ϵHodge structure of weight 1 :

$$0 = F_x^2 \subset F_x^1 = T_x^{1,0}M \subset F_x^0 = T_x^{\mathbb{C}_\epsilon}M. \tag{1.6.7}$$

The complex version of the next lemma and proposition was proven in [Her] and we generalize it to the para-complex case.

Lemma 1.2 *Let ∇ be a torsion-free flat connection on the ϵcomplex manifold (M, J^ϵ). Then $F^1 = T^{1,0}M$ is an ϵholomorphic subbundle of $F^0 = T^{\mathbb{C}_\epsilon}M$ with respect to the ϵholomorphic structure defined by ∇ (compare proposition 1.1) if and only if (∇, J^ϵ) is special (see definition 1.11).*

Proof: The condition of F^1 to be ϵholomorphic is equivalent to

$$\nabla_{\bar{Y}}X = 0 \text{ for all } X, Y \in \mathcal{O}(T^{1,0}M)$$

and the condition of (∇, J^ϵ) to be special is equivalent to

$$(\nabla_X J^\epsilon)(\bar{Y}) = (\nabla_{\bar{Y}} J^\epsilon)(X) \text{ for all } X, Y \in \mathcal{O}(T^{1,0}M),$$

due to the following short argument :
Let $X, Y \in \Gamma(T^{1,0}M)$

$$(\nabla_X J^\epsilon)(Y) = \nabla_X J^\epsilon Y - J^\epsilon \nabla_X Y = \hat{i}\nabla_X Y - J^\epsilon \nabla_X Y,$$

which is symmetric as one sees by choosing vector fields X and Y such that $[X, Y] = 0$.
Let $X, Y \in \Gamma(T^{0,1}M)$

$$(\nabla_X J^\epsilon)(Y) = \nabla_X J^\epsilon Y - J^\epsilon \nabla_X Y = -\hat{i}\nabla_X Y - J^\epsilon \nabla_X Y,$$

which is again symmetric as one sees by choosing vector fields X and Y such that $[X, Y] = 0$.
Let now $X, Y \in \Gamma(T^{1,0}M)$ be ϵholomorphic vector fields, i.e. $\mathcal{L}_X(J^\epsilon) = 0$ where $\mathcal{L}$ is the Lie-derivative. Then it holds

$$\begin{aligned} 0 &= \mathcal{L}_X(J^\epsilon)\bar{Y} = [X, J^\epsilon \bar{Y}] - J^\epsilon[X, \bar{Y}] \\ &= \nabla_X J^\epsilon \bar{Y} - \nabla_{J^\epsilon \bar{Y}} X - J^\epsilon \nabla_X \bar{Y} + J^\epsilon \nabla_{\bar{Y}} X \\ &= (\nabla_X J^\epsilon)\bar{Y} - (\nabla_{\bar{Y}} J^\epsilon)X + \nabla_{\bar{Y}} J^\epsilon X - \nabla_{J^\epsilon \bar{Y}} X \\ &= [(\nabla_X J^\epsilon)\bar{Y} - (\nabla_{\bar{Y}} J^\epsilon)X] + 2\hat{i}\nabla_{\bar{Y}} X. \end{aligned}$$

This finishes the proof. □

From the lemma we obtain:

Proposition 1.9 *Let* (M, J^ϵ) *be an* ϵ*complex manifold,* ∇ *be a torsion-free flat connection and* $F^\bullet$ *defined as in equation* (1.6.7).

1. *Then* (M, J^ϵ, ∇) *is an affine special* ϵ*complex manifold if and only if* ∇ *and* $F^\bullet$ *give a variation of* ϵ*Hodge structures of weight* 1 *on* $TM^{\mathbb{C}_\epsilon}$.
2. *Then* $(M, J^\epsilon, \nabla, g)$ *is an affine special* ϵ*Kähler manifold if and only if* ∇, $F^\bullet$ *and* $\omega(\cdot,\cdot) = g(J^\epsilon\cdot,\cdot)$ *give a variation of polarized* ϵ*Hodge structures of weight* 1 *on* $TM^{\mathbb{C}_\epsilon}$.

Chapter 2

Harmonic and ϵpluriharmonic maps

In this chapter we introduce the notion of harmonic maps and ϵpluriharmonic maps. We discuss the relation between them and we give a generalization of ϵpluriharmonic maps and of associated families of pluriharmonic maps to maps from almost ϵcomplex manifolds into pseudo-Riemannian manifolds. Afterwards we discuss the target spaces which are of importance in the context of ϵplurihamonic maps associated to ϵtt^*-geometry.

2.1 Harmonic maps

First we recall the notion of a harmonic map.

Definition 2.1 *Let (M, g) and (N, h) be pseudo-Riemannian manifolds and $f : M \to N$ be a C^2-map.*

(i) One defines the energy density *of f by*

$$e(f) = \frac{1}{2} G(df, df), \tag{2.1.1}$$

where df is seen as a section in $T^*M \otimes f^*TN$ and G is the metric on $T^*M \otimes f^*TN$ induced by the metrics g and h.

(ii) If the energy density $e(f)$ is integrable we define the energy *$E(f)$ of f as*

$$E(f) = \int_M e(f) \operatorname{vol}_g . \tag{2.1.2}$$

(iii) The critical points of $E(f)$ with respect to compact supported C^∞-variations are called harmonic maps *where the variation of $E(f)$ with respect to the family of maps f_t with $t \in (-\epsilon, \epsilon)$ is defined by*

$$\delta E(f) = \int_M \partial_t e(f_t) \operatorname{vol}_g .$$

The following proposition states the well-known harmonic map equations, which are the Euler-Lagrange equations of the harmonic functional.

Proposition 2.1 *Let (M, g) and (N, h) be pseudo-Riemannian manifolds and $f : M \to N$ be a C^2-map. Denote by ∇^g the Levi-Civita connection of g, by ∇^h the Levi-Civita connection of h and by ∇ the connection induced by ∇^g and ∇^h on $T^*M \otimes f^*TN$. Then f is harmonic if and only if it satisfies the equation*

$$\mathrm{tr}_{\,g} \nabla df = 0. \tag{2.1.3}$$

First we recall a result about a special class of harmonic morphisms which is needed later:

Proposition 2.2 *Let M, X and Y be pseudo-Riemannian manifolds and $\Psi : X \to Y$ be a totally geodesic immersion. Then a map $f : M \to X$ is harmonic if and only if $\Psi \circ f : M \to Y$ is harmonic.*

Proof: Note $\tau(f) = \mathrm{tr}_{\,g} \nabla df$ and let $\Psi : X \to Y$ be an arbitrary map. Then we calculate

$$\mathrm{tr}\,(\nabla^X d(\Psi \circ f)) = \mathrm{tr}_{\,g}(\nabla^X d\Psi \circ df) = \mathrm{tr}_{\,g}(d\Psi(\nabla^Y df)) + tr_g(II(df, df)),$$

where II is the second fundamental form of Ψ, which vanishes, if Ψ is totally geodesic. This shows

$$\tau(\Psi \circ f) = d\Psi \circ \tau(f).$$

The proof is finished, since Ψ is an immersion and therefore has maximal rank. □

We now restrict to compact source manifolds and to Riemannian metrics to obtain a theorem which is due to Gordon [G]. First we need a definition:

Definition 2.2 *A subset U of a manifold Y is said to be* convex supporting *if and only if every compact subset of U has a Y-open neighborhood admitting a strictly convex C^2-function F. The function F is called* support function *and it is in general not globally defined.*

Theorem 2.1 *(cf. [G] p. 434.) Let M and N be Riemannian manifolds with M compact and connected.*

(A) The image of any harmonic map $f : M \to N$ cannot be contained in any convex supporting subset of N unless it is constant. Hence, any harmonic map from M to N is necessarily constant if N is convex supporting.

(B) If $\pi_1(M)$ is finite and N has a covering space which is convex supporting with respect to the lifted metric of N, then every harmonic map from M to N is necessarily constant.

Remark 2.1 *As also discussed in Example (a) in [G] (p. 434) a complete simply connected Riemannian manifold M with non-positive Riemannian sectional curvature is convex supporting. In fact, for a fixed $p_0 \in M$ the squared geodesic distance from p_0 to p is strictly convex and hence a support function. We are especially interested in the space $GL(r,\mathbb{R})/O(r)$ and in the space $SL(r,\mathbb{R})/SO(r)$, which is a Riemannian symmetric space of non-compact type. As Riemannian symmetric spaces of non-compact type are non-positively curved, they are convex supporting (compare also [BR] p. 71). For $GL(r,\mathbb{R})/O(r)$ we have the de Rham-decomposition $GL(r,\mathbb{R})/O(r) = \mathbb{R}\times SL(r,\mathbb{R})/SO(r)$, where $\mathbb{R}$ corresponds to the connected central subgroup $\mathbb{R}^{>0} = \{\lambda Id | \lambda > 0\} \subset GL(r,\mathbb{R})$. Therefore $GL(r,\mathbb{R})/O(r)$ is non-positively curved.*

2.2 ϵpluriharmonic maps from ϵcomplex manifolds

In this section we discuss general results about ϵpluriharmonic maps from ϵcomplex manifolds into pseudo-Riemannian manifolds.

Definition 2.3 *An* ϵcomplex curve *or* ϵRiemannian surface *is an ϵcomplex manifold of ϵcomplex dimension one. An* ϵcomplex curve in an ϵcomplex manifold *M is an ϵcomplex curve Σ^ϵ which is an ϵcomplex submanifold of M.*

Definition 2.4 *A map $f : (M, J^\epsilon) \to (N, h)$ from an ϵcomplex manifold (M, J^ϵ) to a pseudo-Riemannian manifold (N, h) is ϵpluriharmonic if and only if the restriction of f to any ϵcomplex curve Σ^ϵ in M is harmonic.*

Remark 2.2 *Notice that the harmonicity of f restricted to Σ^ϵ is independent of the choice of a (pseudo-)Riemannian metric in the conformal class induced by J^ϵ on Σ^ϵ, by conformal invariance of the harmonic map equation for (real) surfaces.*

The following notion was introduced in [AK] for holomorphic and in [LS] for para-holomorphic vector bundles.

Definition 2.5 *Let (M, J^ϵ) be an ϵcomplex manifold. A connection D on TM is called* adapted *if it satisfies*

$$D_{J^\epsilon Y} X = J^\epsilon D_Y X \tag{2.2.1}$$

for all vector fields which satisfy $\mathcal{L}_X J^\epsilon = 0$ (i.e. for which $X + \epsilon \hat{i} J^\epsilon X$ is ϵholomorphic).

On every ϵcomplex manifold (M, J^ϵ) there exists an ϵcomplex torsion-free connection, as we have shown in corollary 1.1. The following proposition ensures now the existence of an adapted connection.

Proposition 2.3 *(cf. [CS1] for $\epsilon = -1$, [Sch3] for $\epsilon = 1$)*

(i) Every ϵcomplex torsion-free connection D on an ϵcomplex manifold (M, J^ϵ) is adapted.

(ii) On every ϵcomplex manifold there exists an adapted connection.

Proof: (i) The conditions $T^D = 0$ and $DJ^\epsilon = 0$ yield

$$D_{J^\epsilon Y}X - J^\epsilon D_Y X = [J^\epsilon Y, X] + D_X(J^\epsilon Y) - J^\epsilon D_Y X = [J^\epsilon Y, X] - J^\epsilon[Y, X] = -(\mathfrak{L}_X J^\epsilon)Y. \tag{2.2.2}$$

The right-hand side vanishes if $\mathfrak{L}_X J^\epsilon = 0$.
(ii) The existence of an ϵcomplex torsion-free connection D on (M, J^ϵ) follows from corollary 1.1. Part (i) implies now the statement (ii). □

Proposition 2.4 *(cf. [CS1] for $\epsilon = -1$ and [Sch3] for $\epsilon = 1$) Let (M, J^ϵ) be an ϵcomplex manifold and (N, h) be a pseudo-Riemannian manifold with Levi-Civita connection ∇^h, D an adapted connection on (M, J^ϵ) and ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^h.*
A map $f : M \to N$ is ϵpluriharmonic if and only if it satisfies the equation

$$\nabla''\partial f = 0\,, \tag{2.2.3}$$

where $\partial f = df^{1,0} \in \Gamma(\bigwedge^{1,0} T^*M \otimes_{\mathbb{C}_\epsilon} (TN)^{\mathbb{C}_\epsilon})$ is the $(1,0)$-component of $(df)^{\mathbb{C}_\epsilon}$ and ∇'' is the $(0,1)$-component of $\nabla = \nabla' + \nabla''$.
Equivalently one regards $\alpha = \nabla df \in \Gamma(T^*M \otimes T^*M \otimes f^*TN)$.
Then f is ϵpluriharmonic if and only if

$$\alpha(X, Y) - \epsilon\alpha(J^\epsilon X, J^\epsilon Y) = 0$$

for all $X, Y \in TM$. This can also be expressed as

$$\alpha^{1,1} = 0.$$

Moreover, the ϵpluriharmonic equation (2.2.3) is independent of the adapted connection chosen on M.

We recall, that in the case (1,1) and (1+,1-) the two gradings defined for differential forms on para-complex manifolds in section 1.1 coincide in the sense that

$$\Lambda^{1,1}\, T^*M = (\Lambda^{1+,1-}\, T^*M) \otimes \mathbb{C}_\epsilon.$$

Proof: The fact that D is adapted implies $D''Z = 0$ for all local ϵholomorphic vector fields Z, i.e. $\Gamma^\gamma_{\bar\alpha\beta} = \Gamma^{\bar\gamma}_{\bar\alpha\beta} = 0$ in terms of Christoffel symbols of D with respect to ϵholomorphic coordinates z^α. This implies that the Christoffel symbols of the connection D do not contribute to the ϵpluriharmonic equation (2.2.3). Therefore the ϵpluriharmonicity is independent of the adapted connection chosen on M. In the rest of the proof we suppose the connection D to be torsion-free (see proposition 2.3).
Let $\Sigma^\epsilon \subset M$ be an ϵcomplex curve in (M, J^ϵ). On Σ^ϵ an ϵhermitian metric g in the ϵconformal class of J^ϵ is chosen. As g is ϵhermitian it is of type $(1,1)$. Hence the trace of $\nabla df_{|\Sigma^\epsilon}$ with respect to g is zero if and only if $\nabla''\partial f_{|\Sigma^\epsilon} = 0$, as ∇df is symmetric. Since this holds for all curves Σ^ϵ in M the proposition is proven. □

From the definition of ϵpluriharmonic maps and proposition 2.2 we obtain:

Corollary 2.1 *Let (M, J^ϵ) be an ϵcomplex manifold, X and Y be pseudo-Riemannian manifolds and $\Psi : X \to Y$ a totally geodesic immersion. Then a map $f : M \to X$ is ϵpluriharmonic if and only if $\Psi \circ f : M \to Y$ is ϵpluriharmonic.*

Applying Theorem 2.1 to pluriharmonic maps we find:

Corollary 2.2 *Let (M, J, g) be a connected compact Kähler manifold and N be a Riemannian manifold.*

(i) The image of any pluriharmonic map $f : M \to N$ cannot be contained in any convex supporting subset of N unless it is constant. Hence, any pluriharmonic map from M to N is necessarily constant if N is convex supporting.

(ii) If $\pi_1(M)$ is finite and N has a covering space which is convex supporting with respect to the lifted metric of N, then every pluriharmonic map from M to N is necessarily constant.

Proof: Since (M, J, g) is Kähler, the metric g is hermitian and the Levi-Civita connection D on M is adapted. Therefore we find

$$\mathrm{tr}_{\,g} \nabla df = \mathrm{tr}_{\,g} (\nabla df)^{1,1} = 0,$$

as $(\nabla df)^{1,1}$ vanishes by the pluriharmonic map equation (2.2.3). $\square$

2.3 A generalization of ϵpluriharmonic maps from almost ϵcomplex manifolds into pseudo-Riemannian manifolds

In this section, which is also subject of [Sch7, Sch8], we generalize the notion of an ϵpluriharmonic map to maps from almost ϵcomplex manifolds into pseudo-Riemannian manifolds. Afterwards we show that maps admitting a generalization of an associated family (compare the paper of Eschenburg and Tribuzy [ET]) give rise to an ϵpluriharmonic map and we give conditions under which an ϵpluriharmonic map is harmonic.
Let (M, J^ϵ) be an almost ϵcomplex manifold of real dimension $2n$. From theorem 1.1 we know that on every almost ϵcomplex manifold there exists a connection with torsion $T = -\frac{1}{4}\epsilon N_{J^\epsilon}$ where N_{J^ϵ} is the Nijenhuis tensor of J^ϵ.

Definition 2.6 *Let (M, J^ϵ) be an almost ϵcomplex manifold. A connection D on the tangent bundle of M is called* nice *if it is ϵcomplex and its torsion satisfies $T = \lambda N_{J^\epsilon}$ for some function $\lambda \in C^\infty(M, \mathbb{R})$.*

We introduce the notion of an ϵpluriharmonic map from an almost ϵcomplex manifold:

Definition 2.7 *Let (M, J^ϵ, D) be an almost ϵcomplex manifold endowed with a nice connection D on TM and N be a smooth manifold endowed with a connection ∇^N. Denote by ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^N.*
A smooth map $f : M \to N$ is ϵpluriharmonic *if and only if it satisfies the equation*

$$(\nabla df)^{1,1} = 0. \tag{2.3.1}$$

We recall, that in the case (1,1) and (1+,1-) the two gradings defined for differential forms on para-complex manifolds in section 1.1 coincide in the sense that

$$\Lambda^{1,1}\, T^*M = (\Lambda^{1+,1-}\, T^*M) \otimes \mathbb{C}_\epsilon.$$

As preparation for associated families we recall an integrability condition satisfied by the differential of a smooth map. Let N be a smooth manifold with a connection ∇^N on its tangent bundle having torsion tensor T^N. Given a second smooth manifold M and a smooth map $f : M \to N$, the differential $F := df : TM \to f^*TN = E$ induces a vector bundle homomorphism between the tangent bundle of M and the pull-back of TN via f. The torsion tensor T^N of N induces a bundle homomorphism $T^E : \Lambda^2 E \to E$ satisfying the identity

$$\nabla^E_V F(W) - \nabla^E_W F(V) - F([V,W]) = T^E(F(V), F(W)), \tag{2.3.2}$$

where $\nabla^E = f^*\nabla^N$ denotes the pull-back connection, i.e. the connection which is induced on E by ∇^N and where $V, W \in \Gamma(TM)$.
In the rest of the section we denote by D a nice connection on the almost ϵcomplex manifold (M, J^ϵ). Under this assumption we restate the condition (2.3.2)

$$\begin{aligned} T^E(F(V), F(W)) &= \nabla^E_V F(W) - \nabla^E_W F(V) - F([V,W]) \\ &= \nabla^E_V F(W) - \nabla^E_W F(V) \\ &- F(D_V W) + F(D_W V) + F(T(V,W)) \\ &= \nabla^E_V F(W) - \nabla^E_W F(V) \\ &- F(D_V W) + F(D_W V) + \lambda\, F(N_{J^\epsilon}(V,W)) \\ &= (\nabla_V F)W - (\nabla_W F)V + \lambda\, F(N_{J^\epsilon}(V,W)), \end{aligned} \tag{2.3.3}$$

where ∇ is the connection induced on $T^*M \otimes E$ by D and ∇^E.
Later in this work we consider the case where N is a pseudo-Riemannian symmetric space with its Levi-Civita connection ∇^N.
Given an element $\alpha \in \mathbb{R}$ we define $\mathcal{R}_\alpha : TM \to TM$ as

$$\mathcal{R}_\alpha(X) = \cos_\epsilon(\alpha)X + \sin_\epsilon(\alpha)J^\epsilon X.$$

This defines a parallel endomorphism field on the tangent bundle TM of M. The eigenvalues of which are $e^{\hat{i}\alpha}$ on $T^{1,0}M$ and $e^{-\hat{i}\alpha}$ on $T^{0,1}M$, as one sees easily.
An associated family for f is a family of maps $f_\alpha : M \to N, \alpha \in \mathbb{R}$, such that

$$\Phi_\alpha \circ df_\alpha = df \circ \mathcal{R}_\alpha, \quad \forall \alpha \in \mathbb{R}, \tag{2.3.4}$$

for some bundle isomorphism $\Phi_\alpha : f_\alpha^*TN \to f^*TN, \alpha \in \mathbb{R}$, which is parallel with respect to ∇^N in the sense that

$$\Phi_\alpha \circ (f_\alpha^*\nabla^N) = (f^*\nabla^N) \circ \Phi_\alpha.$$

One observes, that each map f_α of an associated family itself admits an associated family.

Theorem 2.2 *Let (M, J^ϵ) be an almost ϵcomplex manifold endowed with a nice connection D, N a smooth manifold endowed with a torsion-free connection ∇^N and let $f : (M, D, J^\epsilon) \to (N, \nabla^N)$ be a smooth map admitting an associated family f_α, then f is ϵpluriharmonic. More precisely, each map of the associated family f_α is ϵpluriharmonic.*

Proof: As Φ_α is parallel with respect to ∇^N, ∇^N is torsion free and D is nice, we can apply equation (2.3.3) to the family $df_\alpha = F_\alpha = \Phi_\alpha^{-1} \circ df \circ \mathcal{R}_\alpha$ to obtain

$$(\nabla_V F_\alpha)W - (\nabla_W F_\alpha)V + \lambda\, F(N_{J^\epsilon}(V, W)) = 0.$$

Since $\mathcal{R}_\alpha$ is D-parallel we obtain

$$(\nabla_X F_\alpha) = \Phi_\alpha^{-1} \circ (\nabla_X F) \circ \mathcal{R}_\alpha. \tag{2.3.5}$$

If $Z = X - \epsilon\hat{i}J^\epsilon X$ and $W = Y + \epsilon\hat{i}J^\epsilon Y$ have different type it holds $N_{J^\epsilon}(Z, W) = 0$, where we have extended the Nijenhuis tensor ϵcomplex linearly. This implies

$$(\nabla_Z F_\alpha)W = (\nabla_W F_\alpha)Z,\ \forall \alpha \in \mathbb{R}$$

and using equation (2.3.5) we obtain

$$\begin{aligned}(\nabla_Z F_\alpha)W &= e^{\hat{i}\alpha}\, \Phi_\alpha^{-1}(\nabla_Z F)W \\ (\nabla_W F_\alpha)Z &= e^{-\hat{i}\alpha}\, \Phi_\alpha^{-1}(\nabla_W F)Z = e^{-\hat{i}\alpha}\, \Phi_\alpha^{-1}(\nabla_Z F)W\end{aligned}$$

for all $\alpha \in \mathbb{R}$. Since this should coincide, it follows $(\nabla df)^{(1,1)} = 0$, i.e. $f : (M, D, J^\epsilon) \to (N, \nabla^N)$ is ϵpluriharmonic. The rest follows, since each map of the associated family f_α admits an associated family $g_\beta = f_{(\alpha+\beta)}$. □

This motivates the definition

Definition 2.8 *Let (M, J^ϵ) be an almost ϵcomplex manifold endowed with a nice connection D and N be a smooth manifold endowed with a torsion-free connection ∇^N. A smooth map $f : (M, D, J^\epsilon) \to (N, \nabla^N)$ is said to be* $\mathbb{S}^1_\epsilon$-pluriharmonic *if and only if it admits an associated family.*

Given an ϵhermitian metric g on M then in general a nice connection D is not the Levi-Civita connection ∇^g of g. Therefore the ϵpluriharmonic equation (2.3.1) does not imply the harmonicity of f. But if the tensor $D - \nabla^g$ is trace-free the ϵpluriharmonic equation implies the harmonic equation. This is true in the case of a special ϵKähler manifold $(M, J^\epsilon, g, \nabla)$ and for a nearly ϵKähler manifold (M, J^ϵ, g), where $D = \overline{\nabla}$ and $\overline{\nabla} - \nabla^g$ is skew-symmetric.

Proposition 2.5 *Let (M, J^ϵ, g) be an almost ϵhermitian manifold endowed with a nice connection D and N be a pseudo-Riemannian manifold with its Levi-Civita connection ∇^N. Suppose that the tensor $S = \nabla^g - D$ is trace-free.*
Then an ϵpluriharmonic map $f : (M, D, J^\epsilon) \to N$ is harmonic.

Proof: We consider

$$\begin{aligned}
\mathrm{tr}_g(\nabla df) &= \sum_i g(e_i,e_i)\left[\nabla^E_{e_i} df(e_i) - df(D_{e_i}e_i)\right]\\
&= \sum_i g(e_i,e_i)\left[\nabla^E_{e_i} df(e_i) - df((\nabla^g - S)_{e_i}e_i)\right]\\
&= \sum_i g(e_i,e_i)\left[\nabla^E_{e_i} df(e_i) - df(\nabla^g_{e_i}e_i)\right]\\
&= \mathrm{tr}_g(\tilde{\nabla}^g df)
\end{aligned}$$

where $\tilde{\nabla}^g$ is the connection induced on $T^*M \otimes E$ by ∇^g and ∇^E and e_i is an orthogonal basis for g on TM. From the ϵpluriharmonic equation and since g is ϵhermitian we obtain

$$\mathrm{tr}_g(\nabla df) = \mathrm{tr}_g(\nabla df^{(1,1)}) = 0.$$

□

2.4 Special targets

In this subsection we discuss the manifolds, which are the target spaces of the ϵpluriharmonic maps associated to ϵtt^*-bundles later in this work.

2.4.1 The space of pseudo-Riemannian metrics

To unify the results we use the notations

$$\begin{aligned}
&G_0(r) = GL(r,\mathbb{R}),\ G_1(r) = SL(r,\mathbb{R}),\\
&\mathfrak{g}_0 = \mathfrak{gl}_{\mathbb{R}}(r),\ \mathfrak{g}_1 = \mathfrak{sl}_{\mathbb{R}}(r),\\
&K_0(p,q) = O(p,q),\ K_1(p,q) = SO(p,q),\\
&\mathfrak{k}_0 = \mathfrak{k}_1 = \mathfrak{so}(p,q),\\
&S^0(p,q) = S(p,q) = GL(r,\mathbb{R})/O(p,q),\ S^1(p,q) = SL(r,\mathbb{R})/SO(p,q).
\end{aligned}$$

These objects are also written with an index $i \in \{0;1\}$.

Let $Sym^0_{p,q}(\mathbb{R}^r)$ be the symmetric $r\times r$ matrices of symmetric signature (p,q) in $G_0(r)$ and $Sym^1_{p,q}(\mathbb{R}^r)$ the elements of $Sym^0_{p,q}(\mathbb{R}^r)$ with determinant $(-1)^q$. These define pseudo-scalar products of same symmetric signature (p,q) by

$$\langle\cdot,\cdot\rangle_A = \langle A\cdot,\cdot\rangle_{\mathbb{R}^r},$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^r}$ is the Euclidean standard scalar product. The natural action of an element $g \in G_i(r)$ is given by $\langle g^{-1}\cdot, g^{-1}\cdot\rangle_A = \langle (g^{-1})^t A g^{-1}\cdot,\cdot\rangle_{\mathbb{R}^r}$. This gives an action of $G_i(r)$ $A \mapsto (g^{-1})^t A g^{-1}$ on $Sym^i_{p,q}(\mathbb{R}^r)$ which we use to identify $Sym^i_{p,q}(\mathbb{R}^r)$ with $S^i(p,q)$ in the following proposition:

Proposition 2.6 *(cf. [Sch3, Sch6]) Let Ψ^i be the canonical map*

$$\Psi^i \,:\, S^i(p,q) \tilde{\to} Sym^i_{p,q}(\mathbb{R}^r) \subset G_i(r)$$

where $G_i(r)$ carries the pseudo-Riemannian metric induced by the Ad-invariant trace-form. Then Ψ^i is a totally-geodesic immersion and a map f from an ϵcomplex manifold (M, J^ϵ) to $S^i(p,q)$ is ϵpluriharmonic if and only if the map $\Psi^i \circ f \,:\, M \to G_i(r)$ is ϵpluriharmonic.

Proof: The proof is done by expressing the map Ψ^i in terms of the well-known Cartan immersion. For further information see for example [Hel], [CE], [GHL], [KN].

1) First we study the identification $S^i(p,q) \tilde{\to} Sym^i_{p,q}(\mathbb{R}^r)$.
The group $G_i(r)$ operates on $Sym^i_{p,q}(\mathbb{R}^r)$ via

$$G_i(r) \times Sym^i_{p,q}(\mathbb{R}^r) \to Sym^i_{p,q}(\mathbb{R}^r), \;\; (g, B) \mapsto g \cdot B := (g^{-1})^t B g^{-1}.$$

The stabilizer of the point $I_{p,q} = \mathrm{diag}(\mathbb{1}_p, -\mathbb{1}_q)$ is $K_i(p,q)$ and the above action is transitive by Sylvester's theorem. Therefore by the orbit-stabilizer theorem (compare the book of Gallot, Hulin, Lafontaine [GHL] 1.100) we obtain a diffeomorphism

$$\Psi^i : S^i(p,q) \tilde{\to} Sym^i_{p,q}(\mathbb{R}^r), \;\; g\, K_i(p,q) \mapsto g \cdot I_{p,q} = (g^{-1})^t I_{p,q} g^{-1}.$$

2) We recall some results about symmetric spaces (For more information we refer to [CE] theorem 3.42 and [KN] volume II chapter X and XI and [Lo] to extend the proof of [CE] to non-compact groups G. A further reference is [ON].). Let G be a Lie-group and $\sigma \,:\, G \to G$ a group-homomorphism with $\sigma^2 = Id_G$. Let K denote the subgroup $K = G^\sigma = \{g \in G \,|\, \sigma(g) = g\}$. The Lie-algebra $\mathfrak{g}$ of G decomposes in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with $d\sigma_{Id_G}(\mathfrak{h}) = \mathfrak{h}, \quad d\sigma_{Id_G}(\mathfrak{p}) = -\mathfrak{p}$. Moreover we have the following information: The map $\phi : G/K \to G$ with $\phi \,:\, [gK] \mapsto g\sigma(g^{-1})$ defines a totally geodesic immersion called the Cartan immersion.
We want to utilize this:
Therefore we define

$$\sigma \,:\, G_i(r) \to G_i(r), \;\; g \;\mapsto (g^{-1})^\dagger$$

where $g^\dagger = I_{p,q} g^t I_{p,q}$ is the adjoint with respect to the pseudo-scalar product $\langle \cdot, \cdot \rangle_{I_{p,q}} = \langle \cdot, I_{p,q} \cdot \rangle_{\mathbb{R}^r}$.
σ is obviously a homomorphism and an involution with $G_i(r)^\sigma = K_i(p,q)$. By a direct calculation one gets $d\sigma_{Id_{G_i}} = -h^\dagger$ and hence

$$\begin{aligned}
\mathfrak{h} &= \{h \in \mathfrak{g}_i(r) \,|\, h^\dagger = -h\} = \mathfrak{o}(p,q) = \mathfrak{so}(p,q), \\
\mathfrak{p} &= \{h \in \mathfrak{g}_i(r) \,|\, h^\dagger = h\} =: \mathrm{sym}^i(p,q).
\end{aligned}$$

Thus we end up with

$$\begin{aligned}
\phi^i : S^i(p,q) &\to G_i(r), && (2.4.1) \\
g &\mapsto g\sigma(g^{-1}) = gg^\dagger = (gI_{p,q}g^t)I_{p,q} = R_{I_{p,q}} \circ \Psi^i \circ \Lambda(g). && (2.4.2)
\end{aligned}$$

Here R_h is the right multiplication by h and Λ is the map induced by $\Lambda : G_i \to G_i, h \mapsto (h^{-1})^t$ on G_i/K_i. Both maps are isometries of the invariant metrics. Hence Ψ^i is a totally-geodesic immersion.

3) Using point 1) and 2) corollary 2.1 finishes the proof. □

Remark 2.3 *(cf. [CS1, Sch3, Sch6])*
Above we have identified $G_i(r)/K_i(p,q)$ with $Sym^i_{p,q}(\mathbb{R}^r)$ via Ψ^i.
Let us choose $o = eK_i(p,q)$ as base point and suppose that Ψ^i is chosen to map o to $I = I_{p,q}$. By construction Ψ^i is $G_i(r)$-equivariant. We identify the tangent-space $T_S\mathrm{Sym^i}_{p,q}(\mathbb{R}^r)$ at $S \in \mathrm{Sym}^i_{p,q}(\mathbb{R}^r)$ with the (ambient) vector space of symmetric matrices:

$$T_S\mathrm{Sym}^i_{p,q}(\mathbb{R}^r) = \mathrm{Sym}^i(\mathbb{R}^r) := \{A \in \mathfrak{g}_i(r) | A^t = A\}\,. \tag{2.4.3}$$

For $\Psi^i(\tilde{S}) = S$, the tangent space $T_{\tilde{S}}S^i(p,q)$ is canonically identified with the vector space of S-symmetric matrices:

$$T_{\tilde{S}}S^i(p,q) = \mathrm{sym}^i(S) := \{A \in \mathfrak{g}_i(r) | A^tS = SA\}\,. \tag{2.4.4}$$

Note that $\mathrm{sym^i}(I_{p,q}) = \mathrm{sym^i}(p,q)$.

Proposition 2.7 *The differential of $\varphi^i := (\Psi^i)^{-1}$ at $S \in \mathrm{Sym}^i_{p,q}(\mathbb{R}^r)$ is given by*

$$\mathrm{Sym}^i(\mathbb{R}^r) \ni X \mapsto -\frac{1}{2}S^{-1}X \in S^{-1}\mathrm{Sym}^i(\mathbb{R}^r) = \mathrm{sym}^i(S)\,. \tag{2.4.5}$$

Using this proposition we relate now the differentials

$$df_x : T_xM \to \mathrm{Sym}^i(\mathbb{R}^r) \tag{2.4.6}$$

of a map $f : M \to \mathrm{Sym}^i_{p,q}(\mathbb{R}^r)$ at $x \in M$ and

$$d\tilde{f}_x : T_xM \to \mathrm{sym}^i(f(x)) \tag{2.4.7}$$

of a map $\tilde{f} = \varphi \circ f : M \to S^i(p,q)$: $d\tilde{f}_x = d\varphi\, df_x = -\frac{1}{2}f(x)^{-1}df_x$.

One can interpret the one-form $A := -2d\tilde{f} = f^{-1}df$ with values in $\mathfrak{g}_i(r)$ as connection form on the vector bundle $E = M \times \mathbb{R}^r$. We note, that the definition of A is the pure gauge. This means, that A is gauge-equivalent to $A' = 0$, as for $A' = 0$ one has $A = f^{-1}A'f + f^{-1}df = f^{-1}df$. The curvature vanishes, since it is independent of gauge. Thus we get:

Proposition 2.8 *Let $f : M \to G_i(r)$ be a C^∞-mapping and $A := f^{-1}df : TM \to \mathfrak{g}_i(r)$. Then the curvature of A vanishes, i.e. for $X, Y \in \Gamma(TM)$ it holds*

$$Y(A_X) - X(A_Y) = A_{[Y,X]} + [A_X, A_Y]. \tag{2.4.8}$$

In the next proposition we give the equations for ϵpluriharmonic maps from an ϵcomplex manifold to $G_i(r)$.

Proposition 2.9 *Let (M, J^ϵ) be an ϵcomplex manifold, $f : M \to G_i(r)$ a C^∞-map and A defined as in proposition 2.8.*
The ϵpluriharmonicity of f is equivalent to the equation

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] - \epsilon J^\epsilon Y(A_{J^\epsilon X}) - \epsilon \frac{1}{2}[A_{J^\epsilon Y}, A_{J^\epsilon X}] = 0, \tag{2.4.9}$$

for ϵholomorphic $X, Y \in \Gamma(TM)$.

Proof: Again the ϵpluriharmonicity of the map f does not depend on the adapted connection chosen on M. This means, that we can take it torsion-free and ϵcomplex (compare proposition 2.3 and proposition 2.4). We calculate the tensor

$$\nabla df(X, Y) = \nabla^N_X(df(Y)) - df(D_X Y).$$

for (real parts of) ϵholomorphic vector fields X, Y. The contribution to the (1,1)-part of the second term vanishes for (real parts of) ϵholomorphic X, Y, since

$$D_X Y - \epsilon D_{J^\epsilon X} J^\epsilon Y = D_X Y - \epsilon J^\epsilon D_{J^\epsilon X} Y = D_X Y - \epsilon J^{\epsilon 2} D_X Y = 0.$$

Therefore we only have to regard the pulled back Levi-Civita connection ∇ on $G_i(r)$.
Let $X, Y \in \Gamma(TM)$. To find the ϵpluriharmonic equations we write $df(X)$ and $df(Y)$ that are sections in $f^* T\, G_i(r)$, as linear combination of left invariant vector fields $f^*\tilde{E}_{ij} = \tilde{E}_{ij} \circ f$, with $\tilde{E}_{ij}(g) = gE_{ij}$, $\forall g \in G_i(r)$ and a basis $E_{ij}, i, j = 1 \dots r$ of $\mathfrak{g}_i(r)$.
In this notation we have

$$df(X) = \sum_{ij} a_{ij}\, \tilde{E}_{ij} \circ f = \sum_{ij} a_{ij}\, f E_{ij} \text{ and } df(Y) = \sum_{ij} b_{ij}\, \tilde{E}_{ij} \circ f = \sum_{ij} b_{ij}\, f E_{ij},$$

with functions a_{ij} and b_{ij} on M and further

$$A_X = f^{-1} df(X) = \sum_{ij} a_{ij}\, E_{ij} \text{ and } A_Y = f^{-1} df(Y) = \sum_{ij} b_{ij}\, E_{ij}.$$

With this information we compute

$$\begin{aligned}
(f^*\nabla)_Y df(X) &= (f^*\nabla)_Y \sum_{ij} a_{ij}\, \tilde{E}_{ij} \circ f \\
&= \sum_{ij} Y(a_{ij})\, \tilde{E}_{ij} \circ f + \sum_{ij} a_{ij}\, (f^*\nabla)_Y \tilde{E}_{ij} \circ f \\
&= \sum_{ij} Y(a_{ij})\, \tilde{E}_{ij} \circ f + \sum_{ij} a_{ij}\, \nabla_{df(Y)} \tilde{E}_{ij} \circ f \\
&= \sum_{ij} Y(a_{ij})\, f E_{ij} + \sum_{abij} a_{ij}\, b_{ab}\, \underbrace{(\nabla_{\tilde{E}_{ab}} \tilde{E}_{ij}) \circ f}_{\frac{1}{2} f[E_{ab}, E_{ij}]} \\
&= f\left(Y(A_X) + \frac{1}{2}[A_Y, A_X] \right).
\end{aligned}$$

Therefore the ϵpluriharmonicity is equivalent to the equation

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] - \epsilon J^\epsilon Y(A_{J^\epsilon X}) - \epsilon \frac{1}{2}[A_{J^\epsilon Y}, A_{J^\epsilon X}] = 0$$

for ϵholomorphic vector fields X, Y. □

2.4.2 A remark on the space of Riemannian metrics

In the complex case pluriharmonic maps into locally Riemannian symmetric spaces of non-compact type have a nice property.
Suppose that N is a locally Riemannian symmetric space with universal cover G/K with a non-compact semi-simple Lie group G, a maximal compact subgroup K and an associated Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. In each point one identifies the tangent space of N with $\mathfrak{p}$. This is unique up to right action of K and left action of the fundamental group. All relevant structures are preserved by these actions. Therefore, given a map $f : M \to N$, we can regard $df_x(T^{1,0}_x M),\ x \in M$ as a subspace of $\mathfrak{p}^{\mathbb{C}}$. For the 'complexified' sectional-curvature of N holds using the Killing-form b

$$b(R(X,Y)\bar{Y},\bar{X}) = -b([X,Y],[\bar{Y},\bar{X}]) \leq 0. \tag{2.4.10}$$

It is a well-known result of Sampson [Sam], that a harmonic map of a compact complex manifold to a locally symmetric space of non-compact type is pluriharmonic and that its differential sends $T^{1,0}M$ to an Abelian subspace of $\mathfrak{p}^{\mathbb{C}}$. The second claim, that the image of $T^{1,0}M$ under the differential of a pluriharmonic map is Abelian is true on non-compact manifolds, too. We are going to prove, that the pluriharmonicity implies this property.

First we state a definition in a more general context, i.e. for ϵcomplex manifolds and locally pseudo-Riemannian symmetric spaces:

Definition 2.9 *Let (M, J^ϵ) be an ϵcomplex manifold and N a locally pseudo-Riemannian symmetric space with universal cover G/K and associated Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. A map $f : (M, J^\epsilon) \to N$ is said to be* admissible, *if for all $x \in M$ the ϵcomplex linear extension of its differential maps $T^{1,0}_x M$ (equivalenty $T^{0,1}_x M$) to an Abelian subspace of $\mathfrak{p}^{\mathbb{C}_\epsilon}$.*

Theorem 2.3 *(compare [Sam]) Let (M, J) be a complex manifold and N be a locally Riemannian symmetric space with universal cover G/K and associated Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$.*
Then a pluriharmonic map $f : M \to N$ is admissible.
The differential of a pluriharmonic map $f : M \to N$ obeys the equation

$$[df_x(X), df_x(Y)] = [df_x(JX), df_x(JY)]$$

with $X, Y \in T_x M, x \in M$.

Proof: The strategy is to show the vanishing of the curvature.
Let $X, Y, Z, W \in \Gamma(T^{1,0}M)$ be holomorphic

$$\begin{aligned} R^N(f_*X, f_*Y)f_*\bar{Z} &= R^{f^*\nabla^N}(X,Y)f_*\bar{Z} \\ &= (f^*\nabla^N)_X(f^*\nabla^N)_Y f_*\bar{Z} - (f^*\nabla^N)_Y(f^*\nabla^N_X)f_*\bar{Z} - (f^*\nabla^N)_{[X,Y]}f_*\bar{Z} \end{aligned}$$

We remark now, that the pluriharmonic equation for holomorphic vector fields does not depend on the adapted connection chosen on the manifold M. Hence it reduces to the equation $(f^*\nabla^N)_X f_*\bar{Y} = 0$, which implies $R^N(f_*X, f_*Y)f_*\bar{Z} = 0$. From equation (2.4.10)

we get $b([f_*X, f_*Y], [f_*\bar{Z}, f_*\bar{W}]) = 0$ and in the end $[f_*X, f_*Y] = 0$ for all X, Y.
Let $Z, W \in \Gamma(T^{1,0}M)$ be of the form $Z = X - iJX$ and $W = Y - iJY$ with $X, Y \in \Gamma(TM)$ and compute $[f_*Z, f_*W] = [f_*X, f_*Y] - [f_*JX, f_*JY] - i([f_*X, f_*JY] + [f_*JX, f_*Y])$. Hence we conclude $[df(X), df(Y)] = [df(JX), df(JY)]$. □

Corollary 2.3 *Let (M, J) be a complex manifold, $f : M \to Sym^i_{r,0}(\mathbb{R}^r) \subset G_i(r)$ a pluriharmonic map induced by a pluriharmonic map to $G_i(r)/K_i(r)$ and A defined as in proposition 2.8. If f is a pluriharmonic map, then the operators A satisfy for all $X, Y \in T_xM$, with $x \in M$, the equation $[A_X, A_Y] = [A_{JX}, A_{JY}]$.*

Proof: First, we apply theorem 2.3 to $A = -2d\tilde{f}$ with a map $\tilde{f} : M \to G_1/K_1$. This yields the corollary for $G_1 = SL(r, \mathbb{R})$.
For $S^0(r, 0) = S(r, 0)$ we have the de Rham decomposition $S(r, 0) = \mathbb{R} \times S^1(r, 0)$, where $\mathbb{R}$ corresponds to the connected central subgroup $\mathbb{R}^{>0} = \{\lambda Id | \lambda > 0\} \subset G_0 = GL(r, \mathbb{R})$. Hence we have the decomposition of $\mathfrak{gl}_{\mathbb{R}}(r) = \mathbb{R} \oplus \mathfrak{sl}_{\mathbb{R}}(r)$, where the $\mathbb{R}$-factor is central. Therefore we are in the situation to apply the result for G_1. □

Remark 2.4 *Since the trace-form on $SL(r, \mathbb{R})$ is a multiple of the Killing-form and on $GL(r, \mathbb{R})$ it corresponds to the metric on the decomposition $S(r, 0) = \mathbb{R} \times S^1(r, 0)$, we can choose the trace-form as metric and obtain the same result as in theorem 2.3 and corollary 2.3.*

2.4.3 The space of hermitian metrics

This subsection is published in [Sch4].
Let $\mathrm{Herm}_{p,q}(\mathbb{C}^r)$ be the complex hermitian $r \times r$ matrices with hermitian signature (p,q) and $I = I_{p,q} = \mathrm{diag}(\mathbb{1}_p, -\mathbb{1}_q)$.
Claim: $GL(r, \mathbb{C})$ operates on $\mathrm{Herm}_{p,q}(\mathbb{C}^r)$ via

$$\begin{aligned} &GL(r, \mathbb{C}) \times \mathrm{Herm}_{p,q}(\mathbb{C}^r) \to \mathrm{Herm}_{p,q}(\mathbb{C}^r), \\ &(g, B) \mapsto g \cdot B := (g^{-1})^H B g^{-1}, \end{aligned}$$

where g^H is the hermitian conjugate of g.
The stabilizer of I is

$$GL(r, \mathbb{C})_I = \{g \in GL(r, \mathbb{C}) \,|\, g \cdot I = (g^{-1})^H I g^{-1} = I\} = U(p, q)$$

and the action is transitive due to Sylvester's theorem. This yields, by identifying orbits and rest classes, a diffeomorphism

$$\begin{aligned} \Psi \quad : \quad &GL(r, \mathbb{C})/U(p, q) \tilde{\to} \mathrm{Herm}_{p,q}(\mathbb{C}^r) \subset GL(r, \mathbb{C}), \\ &gU(p, q) \mapsto g \cdot I = (g^{-1})^H I g^{-1}. \end{aligned}$$

Proposition 2.10 *The map $\Psi : GL(r, \mathbb{C})/U(p, q) \tilde{\to} Herm_{p,q}(\mathbb{C}^r)$ is totally geodesic, where the target-space is carrying the (pseudo-)metric induced by the Ad-invariant trace-form (i.e. $A, B \mapsto \mathrm{tr}\,(AB)$) on $\mathfrak{gl}(r, \mathbb{C})$.*

Let (M, J) be a complex manifold. Then a map $\phi : M \to H(p,q) := GL(r,\mathbb{C})/U(p,q)$ is pluriharmonic if and only if

$$\psi = \Psi \circ \phi \,:\, M \to GL(r,\mathbb{C})/U(p,q) \tilde{\to} Herm_{p,q}(\mathbb{C}^r) \subset GL(r,\mathbb{C})$$

is pluriharmonic.

Proof: Like in the last section the idea is to relate Ψ to the totally geodesic Cartan immersion. Therefore we define

$$\begin{aligned} \sigma \quad : \quad & GL(r,\mathbb{C}) \to GL(r,\mathbb{C}), \\ & g \mapsto (g^{-1})^{\dagger}. \end{aligned}$$

Here $g^{\dagger}$ denotes the adjoint of g with respect to the hermitian scalar product defined by $< \cdot, \cdot > = < I_{p,q} \cdot, \cdot >_{\mathbb{C}^r}$, where $< \cdot, \cdot >_{\mathbb{C}^r}$ is the hermitian standard scalar product on $\mathbb{C}^r$ and $I = I_{p,q}$. Explicitly it is $g^{\dagger} = I g^H I$.
σ is a homomorphism and an involution satisfying $GL(r,\mathbb{C})^{\sigma} = U(p,q)$.
Hence the Cartan immersion can be written as

$$\begin{aligned} i \quad : \quad & GL(r,\mathbb{C})/U(p,q) \to GL(r,\mathbb{C}), \\ & g \mapsto g\sigma(g^{-1}) = gg^{\dagger} = gIg^H I = R_I \circ \Psi \circ \Lambda(g), \end{aligned}$$

where R_h is the right-multiplication with $h \in GL(r,\mathbb{C})$ and Λ the map induced on $GL(r,\mathbb{C})/U(p,q)$ by $\tilde{\Lambda} : GL(r,\mathbb{C}) \to GL(r,\mathbb{C})$, $g \mapsto (g^{-1})^H$. Both maps are isometries of the invariant metrics and therefore Ψ is totally geodesic. Corollary 2.1 finishes the proof.
$\square$

To be complete we mention the related symmetric decomposition:

$$\mathfrak{h} = \{h \in \mathfrak{gl}_r(\mathbb{C}) \,|\, h^{\dagger} = -h\} = \mathfrak{u}(p,q)$$

and

$$\mathfrak{p} = \{h \in \mathfrak{gl}_r(\mathbb{C}) \,|\, h^{\dagger} = h\} =: \mathrm{herm}_{p,q}(\mathbb{C}^r). \tag{2.4.11}$$

Later in this work we need the relation between (pluriharmonic) maps coming from hermitian metrics and these coming from their real part. We are going to study their relation now:
In the rest of this subsection we identify $\mathbb{C}^r$ with $\mathbb{R}^r \oplus i\mathbb{R}^r = \mathbb{R}^{2r}$. In this model the multiplication with i coincides with the automorphism $j = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}$ and $GL(r,\mathbb{C})$ (respectively $\mathfrak{gl}_r(\mathbb{C})$) consists of the elements in $GL(2r,\mathbb{R})$ (respectively $\mathfrak{gl}_{2r}(\mathbb{R})$), which commute with j.
An endomorphism $C \in \mathrm{End}(\mathbb{C}^r)$ decomposes in its real part A and its imaginary part B, i.e. $C = A + iB$ with $A, B \in \mathrm{End}(\mathbb{R}^r)$. In the above model C is identified with a real $2r \times 2r$-matrix. This identification we denote by ι, i.e.

$$\iota(C) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

The complex conjugated of C is identified with

$$\iota(\bar{C}) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

the transpose $C^t = A^t + iB^t$ yields

$$\iota(C^t) = \begin{pmatrix} A^t & -B^t \\ B^t & A^t \end{pmatrix}$$

and consequently the hermitian conjugated is identified with

$$\iota(\bar{C}^t) = \begin{pmatrix} A^t & B^t \\ -B^t & A^t \end{pmatrix}.$$

We observe, that $\iota(\bar{C}^t) = \iota(C)^T$ where $\cdot^T$ is the transpose in $\mathrm{End}(\mathbb{R}^{2r})$.
The hermitian matrices $Herm_{p,q}(\mathbb{C}^r)$ of signature (p,q) are identified with the subset of symmetric matrices $H \in \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r})$, which commute with j, i.e. $[H,j]=0$. Likewise, $T_{I_{p,q}} Herm_{p,q}(\mathbb{C}^r)$ coincides with the symmetric matrices $h \in \mathrm{sym}(\mathbb{R}^{2r})$, which commute with j, i.e. the hermitian matrices in $\mathfrak{gl}_{2r}(\mathbb{R})$ which we denote by $\mathrm{herm}_{p,q}(\mathbb{C}^r)$.
A hermitian scalar product h of signature (p,q) corresponds to a hermitian matrix $H \in Herm_{p,q}(\mathbb{C}^r)$ of hermitian signature (p,q) defined by $h(\cdot,\cdot) = (H\cdot,\cdot)_{\mathbb{C}^r}$. The condition $\bar{C}^t = C$, i.e. C hermitian, means in our model, that C has the form

$$\iota(C) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

with $A = A^t$ and $B = -B^t$.
Finally we find the explicit representation of the map $\mathcal{R}$, which corresponds to taking the real part of a hermitian metric h, i.e. $\mathrm{Re}\, h = (\mathcal{R}(H)\cdot,\cdot)_{\mathbb{R}^{2r}}$:

$$\begin{aligned} \mathcal{R} \quad : \quad & Herm_{p,q}(\mathbb{C}^r) \to \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r}), \\ & H \mapsto \iota(\frac{1}{2}(H + \bar{H}^t)) = \frac{1}{2}(\iota(H) + \iota(H)^T) = \iota(H). \end{aligned}$$

This map has maximal rank and is equivariant with respect to $GL(r,\mathbb{C})$.
Further we claim, that it is totally geodesic: The decomposition

$$\mathfrak{gl}_{2r}(\mathbb{R}) = \mathrm{sym}_{2p,2q}(\mathbb{R}^{2r}) \oplus \mathfrak{o}(2p,2q)$$

is a symmetric decomposition of the symmetric space $GL(2r,\mathbb{R})/O(2p,2q)$ and hence

$$[[\mathrm{sym}_{2p,2q}(\mathbb{R}^{2r}), \mathrm{sym}_{2p,2q}(\mathbb{R}^{2r})], \mathrm{sym}_{2p,2q}(\mathbb{R}^{2r})] \subset \mathrm{sym}_{2p,2q}(\mathbb{R}^{2r}).$$

From $[A,j] = [B,j] = [C,j] = 0$, we conclude with the Jacobi identity $[[A,B],j] = 0$ and $[[[A,B],C],j] = 0$. Consequently $T_{I_{p,q}} GL(r,\mathbb{C})/U(p,q) = \mathrm{herm}_{p,q}(\mathbb{C}^r)$ is a Lie-triple-system[1] in $T_{\mathbb{1}_{2p,2q}} \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r}) = \mathrm{sym}_{2p,2q}(\mathbb{R}^{2r})$, i.e.

$$[[\mathrm{herm}_{p,q}(\mathbb{C}^r), \mathrm{herm}_{p,q}(\mathbb{C}^r)], \mathrm{herm}_{p,q}(\mathbb{C}^r)] \subset \mathrm{herm}_{p,q}(\mathbb{C}^r).$$

[1]We refer to [Hel] Ch. IV.7, [KN] vol. 2, ch. XI.4 and [Lo] ch. III for more information on Lie-triple-systems and totally geodesic subspaces of symmetric spaces and [KN] vol. 2, ch. XI.2 for the (canonical) symmetric decomposition of a symmetric space.

Therefore $GL(r,\mathbb{C})/U(p,q)$ is totally geodesic in $GL(2r,\mathbb{R})/O(2p,2q)$ and we have the commutative diagram:

$$\begin{tikzcd} & \frac{GL(r,\mathbb{C})}{U(p,q)} \arrow[rr, "{[i]}"] \arrow[dd, "\Psi"] & & \frac{GL(2r,\mathbb{R})}{O(2p,2q)} \arrow[dd, "\Psi^0"] \\ M \arrow[ur, "\tilde{h}"] \arrow[dr, "h"'] & & & \\ & \mathrm{Herm}_{p,q}(\mathbb{C}^r) \arrow[rr, "\mathcal{R}"] & & \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r}), \end{tikzcd} \tag{2.4.12}$$

where $[i]$ is induced by the inclusion $i : GL(r,\mathbb{C}) \hookrightarrow GL(2r,\mathbb{R})$. Since all other maps in the square of this diagram are totally geodesic, the map $\mathcal{R} : Herm_{p,q}(\mathbb{C}^r) \to \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r})$ is a totally geodesic map. This gives the proposition:

Proposition 2.11 *A map $h : M \to Herm_{p,q}(\mathbb{C}^r)$ is pluriharmonic, if and only if $g = Re\, h : M \to Sym_{2p,2q}(\mathbb{R}^{2r})$ is pluriharmonic.*
A map $\tilde{h} : M \to H(p,q)$ is pluriharmonic, if and only if $\tilde{g} = [i] \circ h : M \to S(2p,2q)$ is pluriharmonic.

Proof: As discussed above the map $\mathcal{R} : Herm_{p,q}(\mathbb{C}^r) \to \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r})$ is totally geodesic and an immersion. This means that we are in the situation of corollary 2.1.
The second claim follows from the square commutative diagram (2.4.12) and the statements of proposition 2.10 and proposition 2.6, that the composition of a map f from M to $\mathrm{Herm}_{p,q}(\mathbb{C}^r)$ (respectively $\mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r})$) with Ψ^{-1} (respectively $(\Psi^0)^{-1}$) is pluriharmonic, if and only if f is pluriharmonic. $\square$

2.4.4 The space of para-hermitian metrics

In the following subsection we identify C^r with $\mathbb{R}^r \oplus e\mathbb{R}^r = \mathbb{R}^{2r}$. The multiplication with e equals the automorphism $E = \begin{pmatrix} 0 & \mathbb{1}_r \\ \mathbb{1}_r & 0 \end{pmatrix}$ and $GL(r,C)$ (respectively $\mathfrak{gl}_r(C)$) consists of the elements in $GL(2r,\mathbb{R})$ (respectively $\mathfrak{gl}_{2r}(\mathbb{R})$) commuting with E.
First, we introduce the notion of para-hermitian sesquilinear scalar products on para-complex vector spaces

Definition 2.10

1. *A* para-hermitian sesquilinear scalar product *is a non-degenerate sesquilinear form $h : C^r \times C^r \to C$, i.e.*
   - *(i) h is non-degenerate: Given $w \in C^r$ such that for all $v \in C^r$ $h(v,w) = 0$, then it follows $w = 0$,*
   - *(ii) $h(v,w) = \overline{h(w,v)}$, $\forall\, v, w \in C^r$,*
   - *(iii) $h(\lambda v, w) = \lambda h(v,w)$, $\forall\, \lambda \in C$; $v, w \in C^r$.*

2. *Let $z = (z^1, \dots, z^r)$ and $w = (w^1, \dots, w^r)$ be two elements of C^r, then one defines the standard C-bilinear scalar product on C^r by*

$$z \cdot w := \sum_{i=1}^{r} z^i w^i$$

and the standard para-hermitian sesquilinear scalar product *by*

$$(z, w)_{C^r} := z \cdot \bar{w}.$$

3. *Given a matrix C of $End(C^r) = End_C(C^r)$, we define the* para-hermitian conjugation *by $C \mapsto C^h = \bar{C}^t$. We call C* para-hermitian *if and only if $C^h = C$. We denote by $herm(C^r)$ the set of para-hermitian endomorphisms and by $Herm(C^r) = herm(C^r) \cap GL(r, C)$.*

Remark 2.5 *We remark, that there is* **no** *notion of para-hermitian signature, since from $h(v, v) = -1$ for a $v \in C^r$ we obtain $h(ev, ev) = 1$.*

Proposition 2.12

(a) *Given an element C of $End(C^r)$ then it holds $(Cz, w)_{C^r} = (z, C^h w)_{C^r}$, $\forall z, w \in C^r$.*

(b) *The set $herm(C^r)$ is a real vector space.*

(c) *There is a bijective correspondence between $Herm(C^r)$ and para-hermitian sesquilinear scalar products h on C^r given by*

$$H \mapsto h(\cdot, \cdot) := (H\cdot, \cdot)_{C^r}.$$

An endomorphism $C \in \mathrm{End}(C^r)$ decomposes in its real part A and its imaginary part B, i.e. $C = A + eB$ where $A, B \in \mathrm{End}(\mathbb{R}^r)$. In the above identification the endomorphism C is identified via a map, which we denote by ι, with the matrix

$$\iota(C) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The para-complex conjugation of C, i.e. $\bar{C} = A - eB$, corresponds to

$$\iota(\bar{C}) = \begin{pmatrix} A & -B \\ -B & A \end{pmatrix},$$

the transposition $C^t = A^t + eB^t$ yields

$$\iota(C^t) = \begin{pmatrix} A^t & B^t \\ B^t & A^t \end{pmatrix} = \iota(C)^T$$

and the adjoint with respect to $(\cdot, \cdot)_{C^r}$ is $C^h = \bar{C}^t$ which corresponds to

$$\iota(C^h) = \begin{pmatrix} A^t & -B^t \\ -B^t & A^t \end{pmatrix} \overset{(*)}{=} \mathbb{1}_{r,r}\, \iota(C)^T\, \mathbb{1}_{r,r},$$

where $\cdot^T$ is the transposition[2] in $\mathrm{End}(\mathbb{R}^{2r})$. The equality in $(*)$ is due to the calculation:

$$\begin{aligned} \iota(C^h)\mathbb{1}_{r,r} &= \begin{pmatrix} A^t & -B^t \\ -B^t & A^t \end{pmatrix} \mathbb{1}_{r,r} = \begin{pmatrix} A^t & B^t \\ -B^t & -A^t \end{pmatrix} \\ &= \mathbb{1}_{r,r} \begin{pmatrix} A^t & B^t \\ B^t & A^t \end{pmatrix} = \mathbb{1}_{r,r}\, \iota(C)^T = \mathbb{1}_{r,r}\, \iota(C^t) \end{aligned} \tag{2.4.13}$$

with

$$\mathbb{1}_{r,r} = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & -\mathbb{1}_r \end{pmatrix}.$$

A para-hermitian sesquilinear scalar product h corresponds to a para-hermitian matrix $H \in \mathrm{Herm}(C^r)$ (compare with proposition 2.12) defined by $h(\cdot,\cdot) = (H\cdot,\cdot)_{C^r}$. The condition $C^h = C$, i.e. C para-hermitian, means in our model that C is of the form

$$\iota(C) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

with $A = A^t$ and $B = -B^t$.
Using this information we find the explicit representation of the map which corresponds to taking the real part $\mathrm{Re}\, h$ of h. This is the map $\mathcal{R}$ satisfying

$$\mathrm{Re}\, h = (\mathcal{R}(H)\cdot,\cdot)_{\mathbb{R}^{2r}},$$

where $(\cdot,\cdot)_{\mathbb{R}^{2r}}$ is the Euclidean standard scalar product on $\mathbb{R}^{2r}$.
With $z, w \in C^r$ we have

$$\beta(z,w) := \mathrm{Re}\,(z,w)_{C^r} = \frac{1}{2}\,(z\cdot\bar{w} + \bar{z}\cdot w)$$

and

$$\begin{aligned} \mathrm{Re}\, h(z,w) &= \mathrm{Re}\,(Hz,w)_{C^r} \\ &= \frac{1}{2}\left[(Hz)\cdot\bar{w} + (\overline{Hz})\cdot w\right] \\ &= \beta(Hz,w). \end{aligned}$$

Further we remark that $\beta(\cdot,\cdot) = \mathrm{Re}\,(\cdot,\cdot)_{C^r} = (\cdot,\cdot)_{\mathbb{R}^{r,r}}$, where $(\cdot,\cdot)_{\mathbb{R}^{r,r}} = (\mathbb{1}_{r,r}\cdot,\cdot)_{\mathbb{R}^{2r}}$ is the (pseudo-)Euclidean standard scalar product of signature (r,r) on $\mathbb{R}^{2r}$.
This yields

$$\mathrm{Re}\, h(z,w) = (Hz,w)_{\mathbb{R}^{r,r}} = (\mathbb{1}_{r,r}Hz,w)_{\mathbb{R}^{2r}}$$

and for $H = A + eB$ with $A, B \in \mathrm{End}(\mathbb{R}^r)$

$$\mathcal{R}(H) = \mathbb{1}_{r,r}\iota(H) = \mathbb{1}_{r,r}\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}.$$

Since H is para-hermitian, we obtain $\mathcal{R}(H)^T = \mathcal{R}(H)$. The symmetric signature of the

[2] To rest in the same notation as in the last section we use two symbols for the transposition, even if it here seems to be overkill.

symmetric matrix $\mathcal{R}(H)$ is (r,r), as it is the real part of a para-hermitian sesquilinear scalar product.
Summarizing we have

$$\begin{aligned}\mathcal{R} \ : \ & \mathrm{Herm}(C^r) \to \mathrm{Sym}_{r,r}(\mathbb{R}^{2r}),\\ & H \mapsto \mathcal{R}(H) = \mathbb{1}_{r,r}\iota(H).\end{aligned}$$

The map $\mathcal{R}$ has maximal rank.
Claim: $GL(r,C)$ operates on $\mathrm{Herm}(C^r)$ via

$$\begin{aligned}& GL(r,C) \times \mathrm{Herm}(C^r) \to \mathrm{Herm}(C^r),\\ & (g,B) \mapsto g\cdot B := (g^{-1})^h B g^{-1},\end{aligned}$$

$g\cdot B$ is para-hermitian, since one has $g\cdot B = (g\cdot B)^h$.
We now show that $\mathcal{R}$ is equivariant with respect to this $GL(r,C)$-action on $\mathrm{Herm}(C^r)$ and the $GL(2r,\mathbb{R})$-action on $\mathrm{Sym}_{r,r}(\mathbb{R}^{2r})$ given by

$$(g^{-1},S) \mapsto g^{-1}\cdot S = g^T S g$$

with $g\in GL(2r,\mathbb{R})$ and $S \in \mathrm{Sym}_{r,r}(\mathbb{R}^{2r})$. In fact,

$$\begin{aligned}\mathcal{R}(g^{-1}\cdot H) &= \mathcal{R}(g^h H g) = \mathbb{1}_{r,r}\,\iota(g^h H g)\\ &= \mathbb{1}_{r,r}\,\iota(g^h)\,\iota(H)\,\iota(g) \overset{(2.4.13)}{=} \iota(g)^T\,\mathbb{1}_{r,r}\,\iota(H)\,\iota(g)\\ &= \iota(g)^T\,\mathcal{R}(H)\,\iota(g) = \iota(g)^{-1}\cdot\mathcal{R}(H).\end{aligned}$$

Our aim is to show, that this map is totally geodesic:
The decomposition

$$\mathfrak{gl}_{2r}(\mathbb{R}) = \mathrm{sym}_{r,r}(\mathbb{R}^{2r}) \oplus \mathfrak{o}(r,r),$$

where $\mathrm{sym}_{r,r}(\mathbb{R}^{2r})$ are the symmetric matrices with respect to $(\cdot,\cdot)_{\mathbb{R}^{r,r}}$, is a symmetric decomposition associated to the symmetric space $GL(2r,\mathbb{R})/O(r,r)$ and hence

$$[[\mathrm{sym}_{r,r}(\mathbb{R}^{2r}),\mathrm{sym}_{r,r}(\mathbb{R}^{2r})],\mathrm{sym}_{r,r}(\mathbb{R}^{2r})] \subset \mathrm{sym}_{r,r}(\mathbb{R}^{2r}).$$

Let $A,B,C\in \mathrm{herm}(C^r)$. From $[A,E]=[B,E]=[C,E]=0$, we conclude with the Jacobi identity $[[A,B],E]=0$ and $[[[A,B],C],E]=0$. Hence

$$T_{\mathbb{1}_r} GL(r,C)/U^\pi(C^r) = \mathrm{herm}(C^r)$$

is a Lie-triple-system in $T_{\mathbb{1}_{r,r}}\mathrm{Sym}_{r,r}(\mathbb{R}^{2r}) = \mathrm{sym}_{r,r}(\mathbb{R}^{2r})$, i.e.

$$[[\mathrm{herm}(C^r),\mathrm{herm}(C^r)],\mathrm{herm}(C^r)] \subset \mathrm{herm}(C^r)$$

and consequently $GL(r,C)/U^\pi(C^r)$ is a totally geodesic submanifold of $GL(2r,\mathbb{R})/O(r,r)$.

The stabilizer of $\mathbb{1}_r$ under the $GL(r,C)$-action on $\mathrm{Herm}(C^r)$ is

$$GL(r,C)_{\mathbb{1}_r} = \{g\in GL(r,C)\,|\, g\cdot \mathbb{1}_r = (g^{-1})^h\mathbb{1}_r g^{-1} = \mathbb{1}_r\} = U^\pi(C^r).$$

If the operation $\cdot$ is transitive we obtain, by the orbit stabilizer theorem, a diffeomorphism

$$\begin{aligned} \Psi \ : \ & GL(r,C)/U^\pi(C^r) \tilde{\to} \mathrm{Herm}(C^r) \subset GL(r,C), \\ & gU^\pi(C^r) \mapsto g \cdot \mathbb{1}_r = (g^{-1})^h \mathbb{1}_r g^{-1} = (g^{-1})^h g^{-1}. \end{aligned} \tag{2.4.14}$$

The transitivity is due to the following argument: Any para-hermitian sesquilinear scalar product is uniquely determined by its real part, which lies in $\mathrm{Sym}_{r,r}(\mathbb{R}^{2r})$. On this space $GL(2r,\mathbb{R})$ acts transitively.
We claim: $h' = g \cdot h$ with some para-hermitian sesquilinear scalar product h and an element $g \in GL(2r,\mathbb{R})$ is a para-hermitian sesquilinear scalar product if and only if $g \in GL(r,C)$.
Proof: This claim follows from a short calulation: Let $v, w \in C^r$ and $\lambda \in C$:
On the one hand it holds

$$h'(\lambda v, w) = \lambda h'(v,w) = \lambda(g \cdot h)(v,w) = h(\lambda g^{-1} v, g^{-1} w)$$

and on the other hand

$$h'(\lambda v, w) = (g \cdot h)(\lambda v, w) = h(g^{-1}\lambda v, g^{-1} w).$$

Subtracting these two equations yields

$$h((g^{-1}\lambda - \lambda g^{-1})v, g^{-1}w) = 0.$$

Setting $w = gw'$ with arbitrary $w' \in C^r$ we obtain

$$h((g^{-1}\lambda - \lambda g^{-1})v, w') = 0.$$

Since g is invertible and h is non-degenerate we conclude $g^{-1}\lambda v = \lambda g^{-1} v$, which implies the C-linearity of g. $\square$

We are now going to analyze para-pluriharmonic maps into these spaces

Proposition 2.13 *Let (M,τ) be a para-complex manifold and endow $GL(r,C)/U^\pi(C^r)$ with the (pseudo-)metric induced by the trace-form on $GL(r,C)$. Then the map $\Psi : GL(r,C)/U^\pi(C^r) \tilde{\to} Herm(C^r)$ defined in equation (2.4.14) is totally geodesic and a map $\phi : M \to GL(r,C)/U^\pi(C^r)$ is para-pluriharmonic if and only if*

$$\psi = \Psi \circ \phi : M \to GL(r,C)/U^\pi(C^r) \tilde{\to} Herm(C^r) \subset GL(r,C)$$

is para-pluriharmonic.

Proof: To prove this we define

$$\begin{aligned} \sigma \ : \ & GL(r,C) \to GL(r,C), \\ & g \mapsto (g^{-1})^h. \end{aligned}$$

σ is a homomorphism and an involution satisfying $GL(r,C)^\sigma = U^\pi(C^r)$.
Hence the Cartan immersion can be written as

$$\begin{aligned} i \ : \ & GL(r,C)/U^\pi(C^r) \to GL(r,C), \\ & g \mapsto g\sigma(g^{-1}) = gg^h = gg^h = \Psi \circ \Lambda(g), \end{aligned}$$

where Λ is the map induced on $GL(r,C)/U^\pi(C^r)$ by $\tilde{\Lambda} : GL(r,C) \to GL(r,C)$, $g \mapsto (g^{-1})^h$ which is an isometry of the invariant metric, since $g \mapsto g^h = \mathbb{1}_{r,r} g^T \mathbb{1}_{r,r}$ and $g \mapsto g^{-1}$ are isometries of the invariant metric. Therefore Ψ is totally geodesic, since i is totally geodesic. Corollary 2.1 finishes the proof. $\square$

To be complete we mention the related symmetric decomposition:

$$\mathfrak{h} = \{A \in \mathfrak{gl}_r(C) \mid A^h = -A\} = \mathfrak{u}^\pi(C^r)$$

and

$$\mathfrak{p} = \{A \in \mathfrak{gl}_r(C) \mid A^h = A\} = \mathrm{herm}(C^r).$$

Summarizing our knowledge, we have the commutative diagram:

$$\begin{tikzcd}
 & \frac{GL(r,C)}{U^\pi(C^r)} \arrow[r, "{[i]}"] \arrow[dd, "\Psi"] & \frac{GL(2r,\mathbb{R})}{O(r,r)} \arrow[dd, "\Psi^0"] \\
M \arrow[ur, "\tilde{h}"] \arrow[dr, "h"'] & & \\
 & \mathrm{Herm}(C^r) \arrow[r, "\mathcal{R}"] & \mathrm{Sym}_{r,r}(\mathbb{R}^{2r}),
\end{tikzcd} \tag{2.4.15}$$

where $[i]$ is induced by the inclusion $i : GL(r,C) \hookrightarrow GL(2r,\mathbb{R})$. Since all other maps in the square of this diagram are totally geodesic, the map

$$\mathcal{R} : \mathrm{Herm}(C^r) \to \mathrm{Sym}_{r,r}(\mathbb{R}^{2r}), H \mapsto \mathbb{1}_{r,r}\iota(H)$$

is a totally geodesic map.
Using the commutative diagram gives the proposition:

Proposition 2.14 *A map $h : M \to Herm(C^r)$ is para-pluriharmonic, if and only if $g = Re\, h : M \to Sym_{r,r}(\mathbb{R}^{2r})$ is para-pluriharmonic.*
A map $\tilde{h} : M \to H(r) = GL(r,C)/U^\pi(C^r)$ is para-pluriharmonic, if and only if $\tilde{g} = [i] \circ h : M \to S(r,r)$ is para-pluriharmonic.

Proof: As discussed above in this section the map $\mathcal{R} : \mathrm{Herm}(C^r) \to \mathrm{Sym}_{r,r}(\mathbb{R}^{2r})$ is totally geodesic and an immersion. This means that we are in the situation of corollary 2.1.
The second claim follows from the square of the commutative diagram (2.4.15) and from the statements of proposition 2.13 and proposition 2.6, that the composition of a map f from M to $\mathrm{Herm}(C^r)$ (respectively $\mathrm{Sym}_{r,r}(\mathbb{R}^{2r})$) with Ψ^{-1} (respectively $(\Psi^0)^{-1}$) is para-pluriharmonic, if and only if f is para-pluriharmonic. $\square$

Notation: In the following work we use the notation $H(r) = GL(r,C)/U^\pi(C^r)$ and

$$H^\epsilon(p,q) = \begin{cases} H(p,q), & \text{for } \epsilon = -1, \\ H(r), & \text{for } \epsilon = 1. \end{cases} \tag{2.4.16}$$

Further we introduce the notation for the ϵunitary groups

$$U^\epsilon(p,q) = \begin{cases} U(p,q), & \text{for } \epsilon = -1, \\ U^\pi(C^r), & \text{for } \epsilon = 1. \end{cases} \tag{2.4.17}$$

2.5 The Lagrangian Grassmanians

2.5.1 Definition and homogeneous model

Complex version

Like in section 1.5.1 we consider the complex vector space $V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$ with canonical coordinates $(z^1, \dots, z^n, w_1, \dots, w_n)$ endowed with the standard complex symplectic form $\Omega = \sum_{i=1}^n dz^i \wedge dw_i$ and the standard real structure $\kappa = \bar{\cdot} : V \to V$ with fixed points $V^\kappa = T^*\mathbb{R}^n$ and the induced hermitian form $\gamma := i\Omega(\cdot, \kappa\cdot)$.

Definition 2.11 *The subset of the Grassmannian of Lagrangian subspaces $\mathfrak{L}$ of the symplectic vector space (V, Ω), such that γ restricted to $\mathfrak{L}$ defines a hermitian metric of hermitian signature (k, l), with $n = k + l$ is called* the hermitian Lagrangian Grassmannian of signature (k,l) *and is denoted by $Gr_0^{k,l}(\mathbb{C}^{2n})$.*

We remark that $Gr_0^{k,l}(\mathbb{C}^{2n})$ is an open subset of the Grassmannian of Lagrangian subspaces of (V, Ω) and hence a complex submanifold of it.

Proposition 2.15 *The real symplectic group $\mathrm{Sp}(\mathbb{R}^{2n})$ acts transitively on $Gr_0^{k,l}(\mathbb{C}^{2n})$ and we have the following identification:*

$$Gr_0^{k,l}(\mathbb{C}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k,l)\,. \tag{2.5.1}$$

Here $\mathrm{U}(k,l) \subset \mathrm{Sp}(\mathbb{R}^{2n})$ is defined as the stabilizer of

$$W_o = \mathrm{span}\{\frac{\partial}{\partial z^1} + i\frac{\partial}{\partial w_1}, \cdots, \frac{\partial}{\partial z^k} + i\frac{\partial}{\partial w_k}, \frac{\partial}{\partial z^{k+1}} - i\frac{\partial}{\partial w_{k+1}}, \cdots, \frac{\partial}{\partial z^n} - i\frac{\partial}{\partial w_n}\}\,. \tag{2.5.2}$$

The Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$ is a pseudo-hermitian symmetric space and, in particular, a homogeneous pseudo-Kähler manifold.

Proof: Let $\mathfrak{L}, \mathfrak{L}' \in Gr_0^{k,l}(\mathbb{C}^{2n})$.
Since $\gamma^{\mathfrak{L}} = \gamma_{|\mathfrak{L}}$ defines a hermitian sesquilinear form, we obtain from the definition of γ that

$$\mathfrak{L} \cap \overline{\mathfrak{L}} = \{0\}.$$

This means

$$V = \mathfrak{L} \oplus \overline{\mathfrak{L}}$$

defines an orthogonal decomposition with respect to γ.
The same applies to $\mathfrak{L}'$.
Choosing a γ-hermitian base $(f_i)_{i=1}^n$ (respectively $(f_i')_{i=1}^n$) of $\mathfrak{L}$ (respectively $\mathfrak{L}'$) and extending it to a base of V by $(\sqrt{-1}\bar{f}_i)_{i=1}^n$ (respectively $(\sqrt{-1}\bar{f}_i')_{i=1}^n$) we construct two symplectic bases $(f_i, \sqrt{-1}\bar{f}_i)_{i=1}^n$ and $(f_i', \sqrt{-1}\bar{f}_i')_{i=1}^n$ of V and consider the base-change β from $(f_i', \sqrt{-1}\bar{f}_i')_{i=1}^n$ to $(f_i, \sqrt{-1}\bar{f}_i)_{i=1}^n$. β respects Ω and the real structure κ. This means it is an element of $\mathrm{Sp}(\mathbb{R}^{2n})$. Hence the action of $\mathrm{Sp}(\mathbb{R}^{2n})$ on $Gr_0^{k,l}(\mathbb{C}^{2n})$ is transitive.

If one considers the base point W_o one finds by a short calculation (see in the proof of theorem 5.5)

$$(\gamma_{|W_o})_{ij} = \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & -\mathbb{1}_l \end{pmatrix}.$$

□

Para-complex version

In the para-complex setting (compare section 1.5.2) we denote by V the para-holomorphic vector space $T^*C^n = C^{2n}$, endowed with its standard para-complex structure τ_V, its symplectic form Ω, the para-complex conjugation $\kappa = \bar{\cdot} : V \to V,\ v \mapsto \bar{v}$ with fixed point set $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ and with the para-hermitian sesquilinear scalar product on V defined by $\gamma(v,w) := e\Omega(v,\bar{w})$. On this space we take a system of para-holomorphic linear coordinates (z^i, w_i) which are real valued on $T^*\mathbb{R}^n$.

Definition 2.12 *The subset of the Grassmannian of Lagrangian subspaces $\mathfrak{L}$ of the symplectic vector space (V,Ω), such that γ restricted to $\mathfrak{L}$ defines a para-hermitian sesquilinear scalar product is called* the para-hermitian Lagrangian Grassmannian *and is denoted by $Gr_0^n(C^{2n})$.*

Proposition 2.16 *The real symplectic group $Sp(\mathbb{R}^{2n})$ acts transitively on $Gr_0^n(C^{2n})$ and we have the following identification:*

$$Gr_0^n(C^{2n}) = Sp(\mathbb{R}^{2n})/U^\pi(C^n),$$

where $U^\pi(C^n)$ is the stabilizer of

$$W_o = span_C\{\frac{\partial}{\partial z^1} + e\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial z^n} + e\frac{\partial}{\partial w_n}\}. \tag{2.5.3}$$

Proof: Let $\mathfrak{L}, \mathfrak{L}' \in Gr_0^n(C^{2n})$.
Since $\gamma^{\mathfrak{L}} = \gamma_{|\mathfrak{L}}$ defines a para-hermitian sesquilinear scalar product, we obtain from the definition of γ

$$\mathfrak{L} \cap \overline{\mathfrak{L}} = \{0\}.$$

This means

$$V = \mathfrak{L} \oplus \overline{\mathfrak{L}} \tag{2.5.4}$$

defines an orthogonal decomposition with respect to γ.
The same applies to $\mathfrak{L}'$.
The decomposition (2.5.4) and the fact that $\mathfrak{L}$ is Lagrangian implies $\gamma(v,v) = e\Omega(v,\bar{v}) \neq 0$ for all $0 \neq v \in \mathfrak{L}$. This allows us to choose a para-hermitian base $(f_i)_{i=1}^n$ (respectively $(f_i')_{i=1}^n$) of $\mathfrak{L}$ (respectively $\mathfrak{L}'$). We extend this base to a base of V by $(e\bar{f}_i)_{i=1}^n$ (respectively $(e\bar{f}_i')_{i=1}^n$) and obtain in this way two symplectic bases $(f_i, e\bar{f}_i)_{i=1}^n$ and $(f_i', e\bar{f}_i')_{i=1}^n$ of V. Further we consider the base-change β from $(f_i', e\bar{f}_i')_{i=1}^n$ to $(f_i, e\bar{f}_i)_{i=1}^n$. β respects Ω and the real structure κ. This means it is an element of $\mathrm{Sp}(\mathbb{R}^{2n})$. Hence the action of $\mathrm{Sp}(\mathbb{R}^{2n})$ on $Gr_0^n(C^{2n})$ is transitive.

If one considers elements β which leave the base point W_o invariant, one finds $[\beta, \tau_{C^n}] = 0$ and $\beta^* g_{C^n} = g_{C^n}$ with $g_{C^n} = \mathrm{Re}\gamma_{|W_o}$. □

Notation: To unify the notation we introduce

$$Gr_0^{k,l}(\mathbb{C}^{2n}_\epsilon) := \begin{cases} Gr_0^{k,l}(\mathbb{C}^{2n}) \text{ for } \epsilon = -1, \\ Gr_0^{n}(C^{2n}) \text{ for } \epsilon = 1. \end{cases}$$

2.5.2 Holomorphic coordinates on the complex Lagrangian Grassmannian

In this section (cf. [CS1]) we shall introduce a local model for the Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$ and determine the corresponding local expression for the dual Gauß map. This model is a pseudo-Riemannian analog of the Siegel upper half-space

$$\mathrm{Sym}^+(\mathbb{C}^n) := \{A \in \mathrm{Mat}(n, \mathbb{C}) | A^t = A \quad \text{and} \quad \mathrm{Im}\, A \quad \text{is positive definite}\}\,. \tag{2.5.5}$$

Our aim is to construct holomorphic coordinates for the complex manifold $Gr_0^{k,l}(\mathbb{C}^{2n})$ in a Zariski-open neighborhood of a point W_0 of the Grassmannian represented by a Lagrangian subspace $W_0 \subset V$ of signature (k, l). Using a transformation from $\mathrm{Sp}(\mathbb{R}^{2n})$ we can assume that $W_0 = W_o$, see equation (2.5.2). Let $U_0 \subset Gr_0^{k,l}(\mathbb{C}^{2n})$ be the open subset consisting of $W \in Gr_0^{k,l}(\mathbb{C}^{2n})$ such that the projection

$$\pi_{(z)} : V = T^*\mathbb{C}^n = \mathbb{C}^n \oplus (\mathbb{C}^n)^* \to \mathbb{C}^n \tag{2.5.6}$$

onto the first summand (z-space) induces an isomorphism

$$\pi_{(z)}|_W : W \xrightarrow{\sim} \mathbb{C}^n\,. \tag{2.5.7}$$

Notice that $U_0 \subset Gr_0^{k,l}(\mathbb{C}^{2n})$ is an open neighborhood of the base point W_o. For elements $W \in U_0$ we can express w_i as a function of $z = (z^1, \dots, z^n)$. In fact,

$$w_i = \sum C_{ij} z^j\,, \tag{2.5.8}$$

where

$$(C_{ij}) \in \mathrm{Sym}_{k,l}(\mathbb{C}^n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) | A^t = A \text{ and } \quad \mathrm{Im}\, A \text{ has hermitian signature } (k, l)\}. \tag{2.5.9}$$

Proposition 2.17 *The map*

$$C : U_0 \to \mathrm{Sym}_{k,l}(\mathbb{C}^n)\,, \quad W \mapsto C(W) := (C_{ij}) \tag{2.5.10}$$

is a local holomorphic chart for the Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$.

Remark 2.6 *The open subset* $\mathrm{Sym}_{k,l}(\mathbb{C}^n) \subset \mathrm{Sym}(\mathbb{C}^n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) | A^t = A\}$ *is a generalization of the famous Siegel upper half-space* $\mathrm{Sym}_{n,0}(\mathbb{C}^n) = \mathrm{Sym}^+(\mathbb{C}^n)$, *which is a Siegel domain of type I. In the latter case, we have* $U_0 = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(n)$ *and a global coordinate chart*

$$C : Gr_0^{n,0}(\mathbb{C}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(n) \xrightarrow{\sim} \mathrm{Sym}_{n,0}(\mathbb{C}^n)\,. \tag{2.5.11}$$

2.5.3 Para-holomorphic coordinates on the para-complex Lagrangian Grassmannian

In this section (cf. [Sch3]) we introduce a local model of the Grassmannian $\mathrm{Gr}_0^n(C^{2n})$ of para-complex Lagrangian subspaces $W \subset V$ of signature (n,n), i.e. such that $g_V = Re\,\gamma$ restricted to W has signature (n,n).
This model is a para-complex pseudo-Riemannian analog of the Siegel upper half-space

$$\mathrm{Sym}^+(\mathbb{C}^n) := \{A \in \mathrm{Mat}(n,\mathbb{C}) | A^t = A \quad \text{and} \quad \mathrm{Im}\, A \quad \text{is positive definite}\}\,. \tag{2.5.12}$$

Given a point $W \in \mathrm{Gr}_0^n(C^{2n})$ we claim, that $V = T^*C^n$ decomposes into the direct sum

$$V = W \oplus \bar{W} \cong W \oplus W^*. \tag{2.5.13}$$

Let $\gamma^W = \gamma_{|W}$, $\omega^W = (\omega_V)_{|W}$ and $g^W = (g_V)_{|W}$. Then the non-degeneracy of γ^W, g^W and ω^W are equivalent. One sees from the definition of γ^W that it is non-degenerate if and only if $W \cap \bar{W} = \{0\}$. Further it is $\dim_{\mathbb{R}}(W) = \dim_{\mathbb{R}}(\bar{W}) = \frac{\dim_{\mathbb{R}}(V)}{2}$, where the last equation follows since W is Lagrangian. This proves the claim.
One computes easily $\gamma(\bar{v},\bar{w}) = -\gamma(w,v)$, $\forall v,w \in W$. Hence $g^{\bar{W}}$ has signature (n,n), since g^W has signature (n,n). Since $\gamma = e\Omega(\cdot,\bar{\cdot})$ and W is Lagrangian, it follows that the decomposition (2.5.13) is γ-orthogonal. Using the isomorphism induced by the symplectic form Ω on $V = W \oplus \bar{W}$ yields an isomorphism of $W^\perp = \bar{W} \cong W^*$ where $\cdot^\perp$ is the orthogonal complement taken with respect to γ.

We now construct para-holomorphic coordinates for the para-complex Grassmannian $\mathrm{Gr}_0^n(C^{2n})$ in an open neighborhood of a point W_0 of the Grassmannian represented by a Lagrangian subspace $W_0 \subset V$ of signature (n,n). Using the transitive action of the group $Sp(\mathbb{R}^{2n})$ on $\mathrm{Gr}_0^n(C^{2n})$ we may assume $W_0 = W_o$, see equation (2.5.3). Let $U_0 \subset \mathrm{Gr}_0^n(C^{2n})$ be the open subset consisting of $W \in \mathrm{Gr}_0^n(C^{2n})$ such that the projection

$$\pi_{(z)} : V = T^*C^n = C^n \oplus (C^n)^* \to C^n \tag{2.5.14}$$

onto the first summand (z-space) induces an isomorphism

$$\pi_{(z)|W} : W \xrightarrow{\sim} C^n. \tag{2.5.15}$$

Observe, that $U_0 \subset \mathrm{Gr}_0^n(C^{2n})$ is an open neighborhood of the base point W_o. For elements $W \in U_0$ we can express w_i as a function of $z = (z^1,\ldots,z^n)$. In fact,

$$w_i = \sum C_{ij} z^j \tag{2.5.16}$$

where

$$C_{ij} \in \mathrm{Sym}_{n,n}(C^n) = \{A \in \mathrm{Mat}(n,C) | A^t = A \text{ and } \mathrm{Im}(A) \text{ has sym. signature } (n,n)\}. \tag{2.5.17}$$

Proposition 2.18 *The map*

$$C : U_0 \to Sym_{n,n}(C^n), \quad W \mapsto C(W) := (C_{ij}) \tag{2.5.18}$$

is a local para-holomorphic chart for the Grassmannian $Gr_0^n(C^{2n})$.

2.6 The space of compatible ϵcomplex structures

In this section (cf. [Sch7, Sch8]) we study the differential geometry of the spaces of ϵcomplex structures, which are compatible with a given metric or a given symplectic form. First we recall the definition of these spaces:

Definition 2.13

(i) Let (V, ω) be a real (finite dimensional) symplectic vector space. An ϵcomplex structure J^ϵ is called compatible *if and only if it satisfies*

$$J^{\epsilon *}\omega = -\epsilon\omega. \tag{2.6.1}$$

The set of such ϵcomplex structures is denoted by $\mathcal{J}^\epsilon(V, \omega)$.

(ii) Let $(V, \langle\cdot,\cdot\rangle)$ be a real (finite dimensional) pseudo-Euclidean vector space. An ϵcomplex structure J^ϵ is called compatible *if and only if it satisfies*

$$J^{\epsilon *}\langle\cdot,\cdot\rangle = -\epsilon\langle\cdot,\cdot\rangle. \tag{2.6.2}$$

The set of such ϵcomplex structures is denoted by $\mathcal{J}^\epsilon(V, \langle\cdot,\cdot\rangle)$.

We use with $g = \langle\cdot,\cdot\rangle$ the following notations

$$\mathcal{J}^\epsilon(V, \omega) = \begin{cases} \mathcal{J}(V, \omega), & \text{for } \epsilon = -1, \\ \mathcal{P}(V, \omega), & \text{for } \epsilon = 1 \end{cases}$$

and

$$\mathcal{J}^\epsilon(V, g) = \begin{cases} \mathcal{J}(V, g), & \text{for } \epsilon = -1, \\ \mathcal{P}(V, g), & \text{for } \epsilon = 1. \end{cases}$$

One easily shows the next proposition.

Proposition 2.19 *Let $\omega_{J^\epsilon} := g(J^\epsilon\cdot,\cdot)$ and $g_{J^\epsilon} := \epsilon\omega(J^\epsilon\cdot,\cdot)$. Then it holds:*

(a) Given $J^\epsilon \in \mathcal{J}^\epsilon(V, \omega)$ then it is $J^\epsilon \in \mathcal{J}^\epsilon(V, g_{J^\epsilon})$.

(b) Given $J^\epsilon \in \mathcal{J}^\epsilon(V, g)$ then it is $J^\epsilon \in \mathcal{J}^\epsilon(V, \omega_{J^\epsilon})$.

2.6.1 Differential geometry of the sets of compatible complex structures

The metric case

One can consider $\mathcal{J}(V, \langle\cdot,\cdot\rangle) = \mathcal{J}^{-1}(V, \langle\cdot,\cdot\rangle)$, where $V = \mathbb{C}^n = (\mathbb{R}^{2n}, j_0)$ is endowed with its standard complex structure j_0 and its standard scalar product $\langle\cdot,\cdot\rangle$ of hermitian signature (p, q), as a subset in the vector space $\mathfrak{so}(2p, 2q) = \mathfrak{so}(V) \subset \mathrm{Mat}(\mathbb{R}^{2n})$ characterized by the equations

$$f(j) = -\mathbb{1}_{2n}, \tag{2.6.3}$$

where $f : \mathrm{Mat}(\mathbb{R}^{2n}) \to \mathrm{Mat}(\mathbb{R}^{2n})$ is given by $f : A \mapsto A^2$. The differential of this map is $df_A(H) = \{A, H\}$ for $A, H \in \mathrm{Mat}(\mathbb{R}^{2n})$. In addition, df has constant rank in points j satisfying equation (2.6.3), since one sees

$$\begin{aligned} \ker df_j &= \{A \in \mathfrak{so}(V) \,|\, \{j, A\} = 0\}, \\ \mathrm{im}\, df_j &\cong \{A \in \mathfrak{so}(V) \,|\, [j, A] = 0\} \cong \mathfrak{u}(p, q). \end{aligned}$$

Applying the regular value theorem $\mathfrak{J}(V, \langle\cdot,\cdot\rangle)$ is shown to be a submanifold of $\mathfrak{so}(V)$. Its tangent space at $j \in \mathfrak{J}(V, \langle\cdot,\cdot\rangle)$ is

$$T_j \mathfrak{J}(V, \langle\cdot,\cdot\rangle) = \ker df_j = \{A \in \mathfrak{so}(V) \,|\, \{j, A\} = 0\}. \tag{2.6.4}$$

Moreover, $\mathfrak{J}(V, \langle\cdot,\cdot\rangle)$ can be identified with the pseudo-Riemannian symmetric space $SO_0(2p, 2q)/U(p, q)$, where $SO_0(2p, 2q)$ is the identity component of the special pseudo-orthogonal group $SO(2p, 2q)$ and $U(p, q)$ is the unitary group of signature (p, q), by the map

$$\begin{aligned} \Phi \quad : \quad & SO_0(2p, 2q)/U(p, q) \to \mathfrak{J}(V, \langle\cdot,\cdot\rangle), \\ & gK \mapsto g\, j_0\, g^{-1}, \end{aligned}$$

which maps the canonical base point $o = eK$ to j_0.
Any $j \in \mathfrak{J}(V, \langle\cdot,\cdot\rangle)$ defines a symmetric decomposition of $\mathfrak{so}(V)$ by

$$\begin{aligned} \mathfrak{p}(j) &= \{A \in \mathfrak{so}(V) \,|\, \{j, A\} = 0\}, \\ \mathfrak{k}(j) &= \{A \in \mathfrak{so}(V) \,|\, [j, A] = 0\} \cong \mathfrak{u}(p, q). \end{aligned}$$

In particular $\mathfrak{k}(j_0) = \mathfrak{u}(p, q)$. Moreover, one observes $T_j \mathfrak{J}(V, \langle\cdot,\cdot\rangle) = \mathfrak{p}(j)$.
Let $\tilde{j} \in SO_0(2p, 2q)/U(p, q)$ and $j = \Phi(\tilde{j})$, then $T_{\tilde{j}} SO_0(2p, 2q)/U(p, q)$ is canonically identified with $\mathfrak{p}(j)$. We determine now the differential of the above identification:

Proposition 2.20 *Let* $\Psi = \Phi^{-1} : \mathfrak{J}(V, \langle\cdot,\cdot\rangle) \to SO_0(2p, 2q)/U(p, q)$. *Then it holds at* $j \in \mathfrak{J}(V, \langle\cdot,\cdot\rangle)$

$$d\Psi : T_j \mathfrak{J}(V, \langle\cdot,\cdot\rangle) \ni X \mapsto -\frac{1}{2} j^{-1} X \in \mathfrak{p}(j). \tag{2.6.5}$$

The symplectic case

Now we discuss the differential geometry of $\mathfrak{J}(V, \omega_0) = \mathfrak{J}^{-1}(V, \omega_0)$, where ω_0 is the standard symplectic form on $V = \mathbb{C}^n = (\mathbb{R}^{2n}, j_0)$.
First, we consider $\mathfrak{J}(V, \omega_0)$ as a subset of the vector space $\mathfrak{sp}(\mathbb{R}^{2n}) \subset \mathrm{Mat}(\mathbb{R}^{2n})$ characterized by the set of equations

$$f(j) = -\mathbb{1}_{2n}, \tag{2.6.6}$$

where $f : \mathrm{Mat}(\mathbb{R}^{2n}) \to \mathrm{Mat}(\mathbb{R}^{2n})$ is given as above. Again, df has constant rank in points j satisfying equation (2.6.6), since one sees

$$\begin{aligned} \ker df_j &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, \{j, A\} = 0\}, \\ \mathrm{im}\, df_j &\cong \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, [j, A] = 0\} \cong \mathfrak{u}(p, q). \end{aligned}$$

Applying the regular value theorem we obtain that $\mathcal{J}(V,\omega_0)$ is a submanifold of $\mathfrak{sp}(\mathbb{R}^{2n})$. Its tangent space at $j \in \mathcal{J}(V,\omega_0)$ is

$$T_j\,\mathcal{J}(V,\omega_0) = \ker df_j = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, \{j,A\} = 0\}. \tag{2.6.7}$$

In addition the manifold $\mathcal{J}(V,\omega_0)$ can be identified with the pseudo-Riemannian symmetric space $\mathrm{Sp}(\mathbb{R}^{2n})/U(p,q)$, where (p,q) is the hermitian signature of the hermitian metric $g(\cdot,\cdot) = \omega(J\cdot,\cdot)$, by the map

$$\begin{array}{rcl} \Phi\,:\,\mathrm{Sp}(\mathbb{R}^{2n})/U(p,q) & \to & \mathcal{J}(V,\omega_0), \\ gK & \mapsto & g\,j_0\,g^{-1}, \end{array}$$

which maps the canonical base point $o = eK$ to j_0.
Any $j \in \mathcal{J}(V,\omega_0)$ defines a symmetric decomposition of $\mathfrak{sp}(\mathbb{R}^{2n})$ by

$$\begin{array}{rcl} \mathfrak{p}(j) & = & \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, \{j,A\} = 0\}, \\ \mathfrak{k}(j) & = & \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, [j,A] = 0\} \cong \mathfrak{u}(p,q). \end{array}$$

In particular $\mathfrak{k}(j_0) = \mathfrak{u}(p,q)$. Moreover, one observes $T_j\,\mathcal{J}(V,\omega_0) = \mathfrak{p}(j)$.
Let $\tilde{j} \in \mathrm{Sp}(\mathbb{R}^{2n})/U(p,q)$ and $j = \Phi(\tilde{j})$, then $T_{\tilde{j}}\mathrm{Sp}(\mathbb{R}^{2n})/U(p,q)$ is canonically identified with $\mathfrak{p}(j)$ and for the differential of the identification one obtains:

Proposition 2.21 *Let $\Psi = \Phi^{-1} : \mathcal{J}(V,\omega_0) \to \mathrm{Sp}(\mathbb{R}^{2n})/U(p,q)$. Then it holds at $j \in \mathcal{J}(V,\omega_0)$*

$$d\Psi : T_j\,\mathcal{J}(V,\omega_0) \ni X \mapsto -\frac{1}{2}j^{-1}X \in \mathfrak{p}(j). \tag{2.6.8}$$

2.6.2 Differential geometry of the sets of compatible para-complex structures

The metric case

One can consider $\mathcal{P}(V,\langle\cdot,\cdot\rangle) = \mathcal{J}^1(V,\langle\cdot,\cdot\rangle)$, where $V = C^n = \mathbb{R}^n \oplus e\mathbb{R}^n = (\mathbb{R}^{2n}, j_0)$ is endowed with its standard para-complex structure j_0 and its standard scalar product $\langle\cdot,\cdot\rangle$, as a subset in the vector space $\mathfrak{so}(n,n) = \mathfrak{so}(V) \subset \mathrm{Mat}(\mathbb{R}^{2n})$ characterized by the equations

$$f(j) = \mathbb{1}_{2n}, \tag{2.6.9}$$

where $f : \mathrm{Mat}(\mathbb{R}^{2n}) \to \mathrm{Mat}(\mathbb{R}^{2n})$ is given in the last subsection. We remark, that elements satisfying equation (2.6.9) define automatically para-complex structures, since they are trace-free and hence their eigenspaces to the eigenvalues ± 1 have the same dimension. As before df has constant rank in points j satisfying equation (2.6.9), since one sees

$$\begin{array}{rcl} \ker df_j & = & \{A \in \mathfrak{so}(V) \,|\, \{j,A\} = 0\}, \\ \mathrm{im}\, df_j & \cong & \{A \in \mathfrak{so}(V) \,|\, [j,A] = 0\} \cong \mathfrak{u}^\pi(C^r). \end{array}$$

Applying the regular value theorem $\mathcal{P}(V,\langle\cdot,\cdot\rangle)$ is shown to be a submanifold of $\mathfrak{so}(V)$. Its tangent space at $j \in \mathcal{P}(V,\langle\cdot,\cdot\rangle)$ is

$$T_j\,\mathcal{P}(V,\langle\cdot,\cdot\rangle) = \ker df_j = \{A \in \mathfrak{so}(V) \,|\, \{j,A\} = 0\}. \tag{2.6.10}$$

Moreover, $\mathcal{P}(V, \langle\cdot,\cdot\rangle)$ can be identified with the pseudo-Riemannian symmetric space $SO_0(n,n)/U^\pi(C^n)$, where $SO_0(n,n)$ is the identity component of the special pseudo-orthogonal group $SO(n,n)$ and $U^\pi(C^n)$ is the para-unitary group, by the map

$$\begin{aligned} \Phi \quad : \quad & SO_0(n,n)/U^\pi(C^n) \to \mathcal{P}(V, \langle\cdot,\cdot\rangle), \\ & gK \mapsto g\, j_0\, g^{-1}, \end{aligned}$$

which maps the canonical base point $o = eK$ to j_0.
Any $j \in \mathcal{P}(V, \langle\cdot,\cdot\rangle)$ defines a symmetric decomposition of $\mathfrak{so}(V)$ by

$$\begin{aligned} \mathfrak{p}(j) &= \{A \in \mathfrak{so}(V) \,|\, \{j, A\} = 0\}, \\ \mathfrak{k}(j) &= \{A \in \mathfrak{so}(V) \,|\, [j, A] = 0\} \cong \mathfrak{u}^\pi(C^r). \end{aligned}$$

In particular $\mathfrak{k}(j_0) = \mathfrak{u}^\pi(C^r)$. Moreover, one observes $T_j\, \mathcal{P}(V, \langle\cdot,\cdot\rangle) = \mathfrak{p}(j)$.
Let $\tilde{j} \in SO_0(n,n)/U^\pi(C^n)$ and $j = \Phi(\tilde{j})$, then $T_{\tilde{j}} SO_0(n,n)/U^\pi(C^n)$ is canonically identified with $\mathfrak{p}(j)$. We determine now the differential of the above identification:

Proposition 2.22 *Let $\Psi = \Phi^{-1} : \mathcal{P}(V, \langle\cdot,\cdot\rangle) \to SO_0(n,n)/U^\pi(C^n)$. Then it holds at $j \in \mathcal{P}(V, \langle\cdot,\cdot\rangle)$*

$$d\Psi : T_j\, \mathcal{P}(V, \langle\cdot,\cdot\rangle) \ni X \mapsto -\frac{1}{2} j^{-1} X \in \mathfrak{p}(j). \tag{2.6.11}$$

The symplectic case

Now we discuss the differential geometry of $\mathcal{P}(V, \omega_0) = \mathcal{J}^1(V, \omega_0)$, where ω_0 is the standard symplectic form on $V = C^n = (\mathbb{R}^{2n}, j_0)$.
First, we consider $\mathcal{P}(V, \omega_0)$ as a subset of the vector space $\mathfrak{sp}(\mathbb{R}^{2n}) \subset \mathrm{Mat}(\mathbb{R}^{2n})$ characterized by the set of equations

$$f(j) = \mathbb{1}_{2n}, \tag{2.6.12}$$

where $f : \mathrm{Mat}(\mathbb{R}^{2n}) \to \mathrm{Mat}(\mathbb{R}^{2n})$ is given as above. Again, df has constant rank in points j satisfying equation (2.6.12), since one sees

$$\begin{aligned} \ker df_j &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, \{j, A\} = 0\}, \\ \operatorname{im} df_j &\cong \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, [j, A] = 0\} \cong \mathfrak{u}^\pi(C^r). \end{aligned}$$

Applying the regular value theorem we obtain that $\mathcal{P}(V, \omega_0)$ is a submanifold of $\mathfrak{sp}(\mathbb{R}^{2n})$. Its tangent space at $j \in \mathcal{P}(V, \omega_0)$ is

$$T_j\, \mathcal{P}(V, \omega_0) = \ker df_j = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, \{j, A\} = 0\}. \tag{2.6.13}$$

In addition the manifold $\mathcal{P}(V, \omega_0)$ can be identified with the pseudo-Riemannian symmetric space $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ by the map

$$\begin{aligned} \Phi \,:\, \mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n) &\to \mathcal{P}(V, \omega_0), \\ gK &\mapsto g\, j_0\, g^{-1}, \end{aligned}$$

which maps the canonical base point $o = eK$ to j_0.
Any $j \in \mathcal{P}(V, \omega_0)$ defines a symmetric decomposition of $\mathfrak{sp}(\mathbb{R}^{2n})$ by

$$\begin{aligned} \mathfrak{p}(j) &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, \{j, A\} = 0\}, \\ \mathfrak{k}(j) &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, [j, A] = 0\} \cong \mathfrak{u}^{\pi}(C^r). \end{aligned}$$

In particular $\mathfrak{k}(j_0) = \mathfrak{u}^{\pi}(C^r)$. Moreover, one observes $T_j\, \mathcal{P}(V, \omega_0) = \mathfrak{p}(j)$.
Let $\tilde{j} \in \mathrm{Sp}(\mathbb{R}^{2n})/U^{\pi}(C^n)$ and $j = \Phi(\tilde{j})$, then $T_{\tilde{j}}\mathrm{Sp}(\mathbb{R}^{2n})/U^{\pi}(C^n)$ is canonically identified with $\mathfrak{p}(j)$ and for the differential of the identification one obtains:

Proposition 2.23 *Let $\Psi = \Phi^{-1} : \mathcal{P}(V, \omega_0) \to \mathrm{Sp}(\mathbb{R}^{2n})/U^{\pi}(C^n)$. Then it holds at $j \in \mathcal{P}(V, \omega_0)$*

$$d\Psi : T_j\, \mathcal{P}(V, \omega_0) \ni X \mapsto -\frac{1}{2} j^{-1} X \in \mathfrak{p}(j). \tag{2.6.14}$$

2.6.3 Lagrangian Grassmannians and ϵcomplex structures

We are now going to identify the spaces of compatible ϵcomplex structures $\mathcal{J}^{\epsilon}(V, \omega)$ on $V = \mathbb{C}^n_{\epsilon} = (\mathbb{R}^{2n}, j^{\epsilon}_0)$ with the above discussed Lagrangian Grassmannians.
Given an element $J^{\epsilon} \in \mathcal{J}^{\epsilon}(V, \omega)$ we define $g(\cdot, \cdot) = \epsilon\omega(J^{\epsilon}\cdot, \cdot)$.
The data (g, ω) defines on V an ϵhermitian sesquilinear scalar product by

$$h = g + \hat{i}\omega. \tag{2.6.15}$$

In the complex case, i.e. for $\epsilon = -1$, the hermitian signature (k, l) of which is determined by the symmetric signature $(2k, 2l)$ of g.

Proposition 2.24 *(cf. Woodhouse [W] ch. 5 for $\epsilon = -1$) Let $V^{\mathbb{C}_{\epsilon}} = V \otimes \mathbb{C}_{\epsilon}$ be the ϵcomplexifaction of V. Then there is a bijective correspondence between Lagrangian subspaces $\mathcal{L} \in Gr^{k,l}_0(\mathbb{C}^{2n}_{\epsilon})$ and compatible ϵcomplex structures in $\mathcal{J}^{\epsilon}(V, \omega)$.*

Proof: First, let a compatible ϵcomplex structure J^{ϵ} be given.
The map

$$\begin{aligned} \Gamma \;:\; & V \to V^{\mathbb{C}_{\epsilon}} \\ & X \mapsto \frac{1}{2}(X + \epsilon\hat{i}J^{\epsilon}X) \end{aligned}$$

identifies the ϵcomplex vector space (V, J^{ϵ}) with the Lagrangian subspace

$$\mathcal{L}_{J^{\epsilon}} = \{X + \epsilon\hat{i}J^{\epsilon}X \,|\, X \in V\} \subset V^{\mathbb{C}_{\epsilon}},$$

i.e. the maximal subspace W satisfying $J^{\epsilon}{}_{|W} = \hat{i}$, in such a way, that h coincides with the product

$$\langle Z, Z' \rangle = -2\hat{i}\omega(Z, \bar{Z}').$$

Conversely, we start with a Lagrangian subspace $\mathcal{L} \subset V^{\mathbb{C}_{\epsilon}}$ such that $h_{|\mathcal{L}}$ is non degenerate.
This is equivalent to the condition $\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$.
Claim: $\mathcal{L}$ defines a unique J^{ϵ} such that $\mathcal{L} = \mathcal{L}_{J^{\epsilon}}$.
From $\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$ we get $V^{\mathbb{C}_{\epsilon}} = \mathcal{L} \oplus \overline{\mathcal{L}}$. Therefore $J^{\epsilon}X$ is uniquely given by expressing X as $X = Z + \bar{Z}$ with $Z \in \mathcal{L}$ and defining $J^{\epsilon}(X) = \hat{i}(Z - \bar{Z})$. $\square$

2.7 Period domains of variations of ϵHodge structures

We recall some information about period domains of variations of ϵHodge structures and have a closer look at the description of these either as homogeneous spaces or as flag manifolds, since this is crucial to understand our later results. A reference for the complex case is the book [CMP]. Again the complex case is classical and the para-complex case is new.
We introduce the period domain parameterizing the set of polarized ϵHodge structures on a fixed real vector space H having a fixed weight w and fixed ϵHodge numbers $h^{p,q}$. Such an ϵHodge structure is determined by specifying a flag $F^w \subset F^{w-1} \subset \ldots \subset F^0$ of fixed type satisfying the two bilinear relations. The set of such flags satisfying the first bilinear relation is usually called $\tilde{D}$ and can be described in a homogeneous model $G_{\mathbb{C}_\epsilon}/B$ where $G_{\mathbb{C}_\epsilon}$ is the group of automorphisms of $H^{\mathbb{C}_\epsilon}$ fixing the polarization b and B is the stabilizer of some given reference structure $F_o^\bullet$.

Proposition 2.25 *The set $\tilde{D}$ classifying ϵHodge decompositions of weight w with fixed ϵHodge numbers $h^{p,q}$ which obey the first bilinear relation is a flag manifold of type $(f_w, \ldots, f_v)$, $f_p = \dim F^p, v = \left[\frac{w+1}{2}\right]$, such that*

(i) in the case of even weight $w = 2v$ each F^p, for $p = w, \ldots, v+1$, is isotropic with respect to the bilinear form b.

(ii) in the case of odd weight $w = 2v-1$ each F^p, for $p = w, \ldots, v$, is isotropic with respect to the bilinear form b.

It can also be identified with the homogeneous manifold $G_{\mathbb{C}_\epsilon}/B$.

Proof:

(i) In the case of even weight we recover the spaces F^p, for $p = 0, \ldots, (w-v+1) = v+1$, from F^p, for $p = w, \ldots, v$, by using the decomposition
$$H^{\mathbb{C}_\epsilon} = F^p \oplus_\perp \overline{F^{w-p+1}},$$
where $\perp$ is taken with respect to the non-degenerate ϵhermitian sesquilinear form $b(\cdot, \bar{\cdot})$. The condition on F^p, for $p = w, \ldots, v+1$, to be isotropic is the first Riemannian bilinear relation.

(ii) In fact, for odd weight, one can recover the whole flag from F^p for $p = w, \ldots, v$, by using the decomposition
$$H^{\mathbb{C}_\epsilon} = F^p \oplus_\perp \overline{F^{w-p+1}},$$
where $\perp$ is taken with respect to the non-degenerate ϵhermitian sesquilinear form $b(\cdot, \bar{\cdot})$. The condition on F^p, for $p = w, \ldots, v$, to be isotropic is in the case of odd weight w inherited from the first Riemannian bilinear relation. $\square$

In the complex case B is a parabolic subgroup. There seems to be no equivalent para-complex notion in the literature.
The subset of $\tilde{D}$ classifying ϵHodge structures which also satisfy the second bilinear relation is called D. As a non-degeneracy or a positivity condition the second bilinear relation defines an open subset of $\tilde{D}$.

Proposition 2.26 *The period domain D classifying ϵHodge filtrations $F^\bullet$ of fixed dimension $f^p = \dim F^p$ satisfying both bilinear relations is an open subset of $\tilde{D}$ and it is a homogeneous manifold*

$$D = G/V,$$

where G is the group of linear automorphisms of H preserving b and $V = G \cap B$.

We consider the case of Hodge structures which are strongly polarized. Given the space G/V, we call G/K where K is the maximal compact subgroup of G the 'associated symmetric space' and denote the canonical map by

$$\pi : G/V \to G/K.$$

The case of odd weight

We now have a glance at the groups G, V and K and the associated flag manifolds for Hodge structures of odd weight. Using this we describe for strongly polarized variations of Hodge structures the map π at the level of flag manifolds. This description is needed later to relate the (classical) period map to the ϵpluriharmonic maps appearing in ϵtt^*-geometry.

In the case of odd weight $w = 2l + 1$ for $l = v - 1$ the form b is anti-symmetric due to the first Riemannian bilinear relation. In particular the real dimension of H is even. Hence the group G is the symplectic group $Sp(H, b) \cong Sp(\mathbb{R}^r)$ with $r = \dim_{\mathbb{R}} H \in 2\mathbb{N}$. The maximal compact subgroup of $Sp(\mathbb{R}^r)$ is $K = U(r)$.
We define the b-isotropic ϵcomplex vector space

$$\mathcal{L} = \bigoplus_{p=0}^{l} H^{w-p,p} = F^{w-l} = F^v.$$

One sees by equation (1.6.4)

$$H^{\mathbb{C}_\epsilon} = \mathcal{L} \oplus \overline{\mathcal{L}}. \tag{2.7.1}$$

Since they have the same dimension, $\mathcal{L}$ and $\overline{\mathcal{L}}$ are, by the first bilinear relation, Lagrangian subspaces.
We further fix a reference structure $F_o^\bullet$.
Taking successively ϵunitary bases[3]

$$\{f^i\}_{i=1}^{\dim(\mathcal{L})}$$

[3]This means a basis with $h(f_i, f_j) = \pm\delta_{ij}$.

and

$$\{f_o^i\}_{i=1}^{\dim(\mathfrak{L}_o)} \tag{2.7.2}$$

with respect to the ϵhermitian sesquilinear scalar product

$$h(\cdot,\cdot) = (-1)^{w(w-1)/2}\hat{i}^{p-q} b(\cdot,\bar{\cdot})$$

of the flags

$$H^{w,0} \subset H^{w,0} \oplus H^{w-1,1} \subset \ldots \subset \mathfrak{L}$$

and

$$H_o^{w,0} \subset H_o^{w,0} \oplus H_o^{w-1,1} \subset \ldots \subset \mathfrak{L}_o$$

and extending these with $\{\bar{f}^i\}_{i=1}^{\dim(\mathfrak{L}_o)}$ and $\{\bar{f}_o^i\}_{i=1}^{\dim(\mathfrak{L}_o)}$ to symplectic bases of $\overline{\mathfrak{L}}$ and $\overline{\mathfrak{L}}_o$ one sees that $Sp(\mathbb{R}^r)$ acts transitively by change of the basis from $\{f_o^i\}_{i=1}^{\dim(\mathfrak{L}_o)}$ to $\{f^i\}_{i=1}^{\dim(\mathfrak{L}_o)}$.

(i) First we discuss the complex case.
If we have a strongly polarized variation of Hodge structures, then the stabilizer of $F_o^\bullet$ is the group $V = \Pi_{p=0}^l U(\,h^{w-p,p})$. The map $\pi : G/V \to G/K$ is at this level nothing else than the forgetful map from the flag $H^{w,0} \subset H^{w,0} \oplus H^{w-1,1} \subset \ldots \subset \mathfrak{L}$ to the subspace $\mathfrak{L}$. We remark, that the stabilizer of $\mathfrak{L}_o$ is contained in the group $U(r)$, if we assume the variation of Hodge structures to be strongly polarized.
If we consider a weakly polarized variation of Hodge structures, then the stabilizer of $F_o^\bullet$ is the group $V = \Pi_{p=0}^l U(k_p, l_p)$, where (k_p, l_p), with $h^{p,q} = k_p + l_p$, is the hermitian signature of h restricted to $H^{w-p,p}$ with $q = w - p$.
The stabilizer of $\mathfrak{L}_o$ is in this case an element of the group $U(k,l)$, where $r = 2(k+l)$ and (k,l) is the hermitian signature of h on $\mathfrak{L}_o$, i.e. $k = \sum k_p$ and $l = \sum l_p$.
Given a variation of Hodge structures of odd weight over the complex base manifold (M, J) we denote by L the (holomorphic) map

$$\begin{aligned} L : M &\to Sp(\mathbb{R}^r)/U(k,l), && (2.7.3)\\ x &\mapsto \mathfrak{L}_x. && (2.7.4) \end{aligned}$$

The Grassmannian of Lagrangian subspaces, on which h has signature (k,l) will be denoted by $\mathrm{Gr}_0^{k,l}(\mathbb{C}^r)$ and on which h is positive definite will be denoted by $\mathrm{Gr}_0(\mathbb{C}^r) = \mathrm{Gr}_0^{r,0}(\mathbb{C}^r)$.

(ii) In the para-complex case the stabilizer of $\mathfrak{L}_o$ is the group $U^\pi(C^n)$, with $r = 2n$, compare definition 1.7. As before given a variation of para-Hodge structures of odd weight w over the para-complex base manifold (M, τ) we denote by L the (para-holomorphic) map

$$\begin{aligned} L : M &\to Sp(\mathbb{R}^r)/U^\pi(C^n), && (2.7.5)\\ x &\mapsto \mathfrak{L}_x. && (2.7.6) \end{aligned}$$

The associated Grassmannian of Lagrangian subspaces will be denoted by $\mathrm{Gr}_0^n(C^{2n})$ with $r = 2n$.

Chapter 3

tt*-geometry and some of its solutions

In the first part of this chapter we introduce ϵtt^*-bundles and characterize these in terms of explicit geometric data and equations on this data. In the second section we study ϵtt^*-bundles on the tangent bundle TM of a given almost ϵcomplex manifold (M, J^ϵ). In particular special ϵcomplex, special ϵKähler and Levi-Civita flat nearly ϵKähler manifolds are solutions of tt^*-geometry on the tangent bundle TM. These three classes of solutions are discussed separately. The last two sections of this chapter deal with variations of ϵHodge structures and ϵharmonic bundles as solutions of tt^*-geometry.

3.1 tt*-bundles

In this section we introduce the real differential geometric definition of an ϵtt^*-bundle. For integrable ϵcomplex structures the complex geometric version was given in [CS1, Sch6] and the para-complex geometric version was introduced in [Sch4]. The non-integrable case and the symplectic version were first considered in complex geometry in [Sch7] and in para-complex geometry in [Sch8].

Definition 3.1 *An* ϵtt*-bundle (E, D, S) *over an almost* ϵ*complex manifold* (M, J^ϵ) *is a real vector bundle* $E \to M$ *endowed with a connection* D *and a section* $S \in \Gamma(T^*M \otimes \operatorname{End} E)$ *satisfying the* ϵtt*-equation

$$R^\theta = 0 \quad \textit{for all} \quad \theta \in \mathbb{R}\,, \tag{3.1.1}$$

where R^θ *is the curvature tensor of the connection* D^θ *defined by*

$$D^\theta_X := D_X + \cos_\epsilon(\theta) S_X + \sin_\epsilon(\theta) S_{J^\epsilon X} \quad \textit{for all} \quad X \in TM\,. \tag{3.1.2}$$

A metric ϵtt*-bundle (E, D, S, g) *is an* ϵtt^**-bundle* (E, D, S) *endowed with a possibly indefinite* D*-parallel fiber metric* g *such that* S *is* g*-symmetric, i.e. for all* $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \textit{for all} \qquad X, Y, Z \in T_pM\,. \tag{3.1.3}$$

A unimodular metric ϵtt*-bundle (E, D, S, g) *is a metric ϵtt^*-bundle* (E, D, S, g) *such that* $\operatorname{tr} S_X = 0$ *for all* $X \in TM$.
An oriented unimodular metric ϵtt*-bundle (E, D, S, g, or) *is a unimodular metric ϵtt^*-bundle endowed with an orientation or of the bundle E.*
A symplectic ϵtt^*-bundle (E, D, S, ω) *is an ϵtt^*-bundle (E, D, S) endowed with the structure of a symplectic vector bundle*[1] (E, ω), *such that ω is D-parallel and S is ω-symmetric, i.e. for all $p \in M$*

$$\omega(S_X Y, Z) = \omega(Y, S_X Z) \quad \textit{for all} \qquad X, Y, Z \in T_p M\,. \tag{3.1.4}$$

In the case of moduli spaces of topological quantum field theories [CV, D] and the moduli spaces of singularities [Her], the complexified tt^*-bundle $E^{\mathbb{C}}$ (This means we consider $\epsilon = -1$.) is identified with $T^{1,0}M$ and the metric g is positive definite. The case $E = TM$, and hence $E^{\mathbb{C}} = T^{1,0}M + T^{0,1}M$ includes special complex and special Kähler manifolds, as we have proven in [CS1] and follows from [Her] in the complex situation. This was shown in [Sch4] in the para-complex framework. We discuss this later in more details.

Remark 3.1
1) If (E, D, S) is an ϵtt^*-bundle then (E, D, S^θ) is an ϵtt^*-bundle for all $\theta \in \mathbb{R}$, where

$$S^\theta := D^\theta - D = \cos_\epsilon(\theta) S + \sin_\epsilon(\theta) S_{J^\epsilon}\,. \tag{3.1.5}$$

The same remark applies to metric and symplectic ϵtt^*-bundles.
2) Notice that an oriented unimodular metric ϵtt^*-bundle (E, D, S, g, or) carries a canonical metric volume element $\nu \in \Gamma(\wedge^r E^*)$, $r = \operatorname{rk} E$, determined by g and or, which is D^θ-parallel for all $\theta \in \mathbb{R}$.
Further, a symplectic ϵtt^*-bundle (E, D, S, ω) of rank $2r$ carries a D-parallel volume given by $\underbrace{\omega \wedge \ldots \wedge \omega}_{r\ \textit{times}}$.

The following proposition characterizes ϵtt*-bundles (E, D, S) in form of explicit equations for D and S. These equations are important in the later calculations.

Proposition 3.1 *Let E be a real vector bundle over an (almost) ϵcomplex manifold (M, J^ϵ) endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \operatorname{End} E)$.*
Then (E, D, S) is an ϵtt^*-bundle if and only if D and S satisfy the following equations:

$$R^D + S \wedge S = 0, \tag{3.1.6}$$
$$S \wedge S \textit{ is of type (1,1)}, \tag{3.1.7}$$
$$[D_X, S_Y] - [D_Y, S_X] - S_{[X,Y]} = 0, \quad \forall\, X, Y \in \Gamma(TM), \tag{3.1.8}$$
$$[D_X, S_{J^\epsilon Y}] - [D_Y, S_{J^\epsilon X}] - S_{J^\epsilon [X,Y]} = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{3.1.9}$$

Fixing a torsion-free connection on (M, J^ϵ) the last two equations are equivalent to

$$d^D S = 0 \quad \textit{and} \quad d^D S_{J^\epsilon} = 0. \tag{3.1.10}$$

[1]see D. Mc Duff and D. Salamon [McDS]

Proof: As the attentive reader observes, it is easier to show this proposition after ϵcomplexifying TM. But since one idea of this work was to formulate these results in terms of real differential geometry, we give a real version of the proof.
To prove the proposition, we have to compute the curvature tensor of D^θ.
Let $X, Y \in \Gamma(TM)$ be arbitrary:

$$\begin{aligned}
R^\theta_{X,Y} &= R^D_{X,Y} \\
&+ [D_X, \cos_\epsilon(\theta)S_Y + \sin_\epsilon(\theta)S_{J^\epsilon Y}] \\
&+ [\cos_\epsilon(\theta)S_X + \sin_\epsilon(\theta)S_{J^\epsilon X}, D_Y] \\
&+ [\cos_\epsilon(\theta)S_X + \sin_\epsilon(\theta)S_{J^\epsilon X}, \cos_\epsilon(\theta)S_Y + \sin_\epsilon(\theta)S_{J^\epsilon Y}] \\
&- \cos_\epsilon(\theta)S_{[X,Y]} - \sin_\epsilon(\theta)S_{J^\epsilon[X,Y]} \\
&= R^D_{X,Y} \\
&+ {\sin_\epsilon}^2(\theta)[S_{J^\epsilon X}, S_{J^\epsilon Y}] \\
&+ {\cos_\epsilon}^2(\theta)[S_X, S_Y] \\
&+ \cos_\epsilon(\theta)\sin_\epsilon(\theta)\left([S_X, S_{J^\epsilon Y}] + [S_{J^\epsilon X}, S_Y]\right) \\
&+ \cos_\epsilon(\theta)\big([D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}\big) \\
&+ \sin_\epsilon(\theta)\big([S_{J^\epsilon X}, D_Y] + [D_X, S_{J^\epsilon Y}] - S_{J^\epsilon[X,Y]}\big).
\end{aligned}$$

We recall the theorems of addition

$$\begin{aligned}
\cos_\epsilon(\theta)\sin_\epsilon(\theta) &= \frac{1}{2}\sin_\epsilon(2\theta), \\
{\cos_\epsilon}^2(\theta) &= \frac{1}{2}(1 + \cos_\epsilon(2\theta)) \text{ and} \\
{\sin_\epsilon}^2(\theta) &= \frac{1}{2}\epsilon(\cos_\epsilon(2\theta) - 1)
\end{aligned} \tag{3.1.11}$$

to find

$$\begin{aligned}
R^\theta_{X,Y} &= R^D_{X,Y} + \frac{1}{2}\left([S_X, S_Y] - \epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}]\right) \\
&+ \cos_\epsilon(\theta)\big([D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}\big) \\
&+ \sin_\epsilon(\theta)\big([S_{J^\epsilon X}, D_Y] + [D_X, S_{J^\epsilon Y}] - S_{J^\epsilon[X,Y]}\big) \\
&+ \frac{1}{2}\cos_\epsilon(2\theta)\left([S_X, S_Y] + \epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}]\right) \\
&+ \frac{1}{2}\sin_\epsilon(2\theta)\left([S_X, S_{J^\epsilon Y}] + [S_{J^\epsilon X}, S_Y]\right).
\end{aligned}$$

Taking 'Fourier-coefficients' yields

$$\begin{aligned}
&R^D_{X,Y} + \frac{1}{2}\left([S_X, S_Y] - \epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}]\right) = 0, \\
&[S_X, S_Y] + \epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}] = 0, \\
&[S_X, S_{J^\epsilon Y}] + [S_{J^\epsilon X}, S_Y] = 0, \\
&[D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]} = 0, \\
&[S_{J^\epsilon X}, D_Y] + [D_X, S_{J^\epsilon Y}] - S_{J^\epsilon[X,Y]} = 0.
\end{aligned}$$

The first three equations give

$$R^D_{X,Y} + [S_X, S_Y] = 0,\ S \wedge S(X,Y) = [S_X, S_Y] = -\epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}].$$

Choosing a torsion-free connection on M the last two equations yield

$$d^D S = 0 \text{ and } d^D S_{J^\epsilon} = 0.$$

□

3.2 Solutions on the tangent bundle of an almost ϵcomplex manifold

The following sections are contained in [Sch7, Sch8].

3.2.1 Solutions without metrics

Given an almost ϵcomplex manifold (M, J^ϵ) with a flat connection ∇ it is natural to consider the one-parameter family of connections ∇^θ, which is defined by

$$\nabla^\theta_X Y = \exp(\theta J^\epsilon)\nabla_X(\exp(-\theta J^\epsilon)Y) \text{ for } X, Y \in \Gamma(TM), \tag{3.2.1}$$

where $\exp(\theta J^\epsilon) = \cos_\epsilon(\theta) Id + \sin_\epsilon(\theta) J^\epsilon$.
Recall, that the flatness of ∇ implies the flatness of the family of connections ∇^θ (compare remark 1.3).

Let us recall a definition

Definition 3.2 *Two one-parameter families of connections ∇^θ and D^θ on some vector bundle E with $\theta \in \mathbb{R}$ are called* (linearly) equivalent with factor $\alpha \in \mathbb{R}$ *if they satisfy the equation $\nabla^\theta = D^{\alpha\theta}$.*

We are now going to analyze the form of ϵtt^*-bundles (TM, D, S) on the tangent bundle TM of M for which the family of connections D^θ defined in equation (3.1.2) is linearly equivalent to the family of connections ∇^θ defined in equation (3.2.1).

Proposition 3.2 *Given an almost ϵcomplex manifold (M, J^ϵ) with a flat connection ∇ and a decomposition of $\nabla = D + S$ in a connection D and a section S in $T^*M \otimes \mathrm{End}\,(TM)$. Then (TM, D, S) defines an ϵtt^*-bundle, such that the family of connections D^θ is linearly equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$ if and only if S and D satisfy*

$$S_{J^\epsilon X} = \pm J^\epsilon S_X Y$$

and

$$-(D_X J^\epsilon)Y = J^\epsilon S_X Y + S_X J^\epsilon Y = \{S_X, J^\epsilon\}Y$$

for all $X, Y \in \Gamma(TM)$.

Proof: First one has to analyze the family of connections ∇^θ for $X, Y \in \Gamma(TM)$

$$\begin{aligned}
\nabla^\theta_X Y &= \exp(\theta J^\epsilon)(D_X + S_X)[(\cos_\epsilon(\theta)Id - \sin_\epsilon(\theta)J^\epsilon)Y] \\
&= D_X Y - \exp(\theta J^\epsilon)\sin_\epsilon(\theta)(D_X J^\epsilon)Y \\
&+ (\cos_\epsilon(\theta)Id + \sin_\epsilon(\theta)J^\epsilon)S_X(\cos_\epsilon(\theta)Id - \sin_\epsilon(\theta)J^\epsilon)Y \\
&= D_X Y - (\cos_\epsilon(\theta)\sin_\epsilon(\theta) + \sin_\epsilon{}^2(\theta)J^\epsilon)(D_X J^\epsilon)Y + \cos_\epsilon{}^2(\theta)S_X Y \\
&- \sin_\epsilon{}^2(\theta)J^\epsilon S_X J^\epsilon Y - \cos_\epsilon(\theta)\sin_\epsilon(\theta)[S_X, J^\epsilon]Y,
\end{aligned}$$

which yields with the theorems of addition (see equation (3.1.11)), the identity

$$\begin{aligned}
\nabla^\theta_X Y &= D_X Y - \frac{1}{2}\sin_\epsilon(2\theta)(D_X J^\epsilon)Y - \frac{1}{2}\epsilon(\cos_\epsilon(2\theta) - 1)J^\epsilon(D_X J^\epsilon)Y \\
&+ \frac{1}{2}(1 + \cos_\epsilon(2\theta))S_X Y - \frac{1}{2}\epsilon(\cos_\epsilon(2\theta) - 1)J^\epsilon S_X J^\epsilon Y - \frac{1}{2}\sin_\epsilon(2\theta)[S_X, J^\epsilon]Y \\
&= D_X Y + \frac{1}{2}\left[S_X + \epsilon J^\epsilon S_X J^\epsilon + \epsilon J^\epsilon D_X J^\epsilon\right] Y \\
&+ \frac{1}{2}\sin_\epsilon(2\theta)\left[[J^\epsilon, S_X] - D_X J^\epsilon\right] Y \\
&+ \frac{1}{2}\cos_\epsilon(2\theta)\left[S_X - \epsilon J^\epsilon S_X J^\epsilon - \epsilon J^\epsilon D_X J^\epsilon\right] Y \\
&\overset{!}{=} D_X Y + \cos_\epsilon(\vartheta)T_X Y + \sin_\epsilon(\vartheta)T_{J^\epsilon X} Y \text{ with } \vartheta = \pm 2\theta,
\end{aligned}$$

where we have to determine $T \in \Gamma(T^*M \otimes \text{End}\,(TM))$.
Comparing coefficients of $1, \cos_\epsilon(n\vartheta), \sin_\epsilon(n\vartheta)$ with $n = 1, 2$ yields

$$\begin{aligned}
-\epsilon J^\epsilon(D_X J^\epsilon)Y &= S_X Y + \epsilon J^\epsilon S_X J^\epsilon Y, \text{ or equivalently} &\quad (3.2.2)\\
-(D_X J^\epsilon)Y &= J^\epsilon S_X Y + S_X J^\epsilon Y = \{S_X, J^\epsilon\}Y, \\
T_X Y &= \frac{1}{2}(S_X Y - \epsilon J^\epsilon S_X J^\epsilon Y - \epsilon J^\epsilon(D_X J^\epsilon)Y) \overset{(3.2.2)}{=} S_X Y, &\quad (3.2.3)\\
T_{J^\epsilon X} Y &= \pm\frac{1}{2}([J^\epsilon, S_X]Y - (D_X J^\epsilon)Y) \\
&\overset{(3.2.2)}{=} \pm\frac{1}{2}(J^\epsilon S_X Y - S_X J^\epsilon Y + J^\epsilon S_X Y + S_X J^\epsilon Y) \\
&= \pm J^\epsilon S_X Y. &\quad (3.2.4)
\end{aligned}$$

The last two equations yield the constraint on S

$$S_{J^\epsilon X} = \pm J^\epsilon S_X Y$$

and the first equation the one on D and S. □

Now we suppose the connection D to be ϵcomplex. Such a connection exists on every almost ϵcomplex manifold, as we have shown in theorem 1.1.

Corollary 3.1 *Given an almost ϵcomplex manifold (M, J^ϵ) with a flat connection ∇ and a decomposition of $\nabla = D + S$ in a connection D and a section S in $T^*M \otimes \text{End}\,(TM)$, such that J^ϵ is D-parallel, i.e. $DJ^\epsilon = 0$. Then (TM, D, S) defines an ϵtt^*-bundle, such* *that the family of connections D^θ is linearly equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$ if and only if S satisfies*

$$S_{J^\epsilon X} = \pm J^\epsilon S_X \text{ and } \{S_X, J^\epsilon\} = 0.$$

Proof: The second constraint in proposition 3.2 is for $DJ^\epsilon = 0$ the condition $\{S_X, J^\epsilon\} = 0$. The first constraint of proposition 3.2 is exactly $S_{J^\epsilon X} = \pm J^\epsilon S_X \overset{\{S_X,J^\epsilon\}=0}{=} \mp S_X J^\epsilon$. □

We are going to show some uniqueness result. Therefore we prove the

Lemma 3.1 *Let (M, J^ϵ) be an almost ϵcomplex manifold. Given a connection ∇ on M which decomposes as $\nabla = D + S$, where D is a connection on M and S is a section in $T^*M \otimes \mathrm{End}\,(TM)$, such that J^ϵ is D-parallel, i.e. $DJ^\epsilon = 0$ and S anti-commutes with J^ϵ, i.e. $\{S_X, J^\epsilon\} = 0$ for all $X \in \Gamma(TM)$. Then S and D are uniquely given by*

$$S_X Y = -\frac{1}{2}\epsilon J^\epsilon(\nabla_X J^\epsilon)Y \text{ and } D_X Y = \nabla_X Y - S_X Y \text{ for } X, Y \in \Gamma(TM). \tag{3.2.5}$$

Otherwise, given a connection ∇ and define D and S by equation (3.2.5), then D and S satisfy $DJ^\epsilon = 0$ and $\{S_X, J^\epsilon\} = 0$.

Proof: First we observe $\nabla = D + S$ and

$$S_X J^\epsilon Y = -\frac{1}{2}\epsilon J^\epsilon(\nabla_X J^\epsilon)J^\epsilon Y = \frac{1}{2}\epsilon J^{\epsilon 2}(\nabla_X J^\epsilon)Y = -J^\epsilon S_X Y,$$

where the second equality follows from deriving $J^{\epsilon 2} = \epsilon Id$. Further it is

$$(D_X J^\epsilon)Y = (\nabla_X J^\epsilon)Y - [S_X, J^\epsilon]Y \overset{\{S_X,J^\epsilon\}=0}{=} (\nabla_X J^\epsilon)Y + 2J^\epsilon S_X Y = 0.$$

Now we prove the uniqueness: Suppose there exist D' and S' with the same properties. Thus we get

$$0 = (D'_X J^\epsilon)Y = (\nabla_X J^\epsilon)Y - [S'_X, J^\epsilon]Y = (\nabla_X J^\epsilon)Y + 2J^\epsilon S'_X Y$$

and consequently

$$S'_X Y = -\frac{1}{2}\epsilon J^\epsilon(\nabla_X J^\epsilon)Y = S_X Y \text{ and } D'_X Y = \nabla_X Y - S'_X Y = \nabla_X Y - S_X Y = D_X Y.$$

□

Summarizing corollary 3.1 and lemma 3.1 we find the following uniqueness result:

Theorem 3.1 *Given an almost ϵcomplex manifold (M, J^ϵ) with a flat connection ∇ and a decomposition of $\nabla = D + S$ in a connection D and a section S in $T^*M \otimes \mathrm{End}\,(TM)$, such that J^ϵ is D-parallel, i.e. $DJ^\epsilon = 0$. If (TM, D, S) defines an ϵtt^*-bundle, such that* *the family of connections D^θ is linearly equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$, then D and S are uniquely determined by the equations*

$$S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon) \tag{3.2.6}$$

and

$$D = \nabla - S. \tag{3.2.7}$$

Moreover, (TM, D, S) as given by equation (3.2.6) and (3.2.7) defines an ϵtt^*-bundle, such that the family of connections D^θ is linearly equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$, if and only if J^ϵ satisfies $(\nabla_{J^\epsilon X} J^\epsilon) = \pm J^\epsilon(\nabla_X J^\epsilon)$ and D and S are given by $S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon)$ and $D = \nabla - S$.

In the following propositions we are going to give some classes of examples which satisfy the condition $S_{J^\epsilon X} = \pm J^\epsilon S_X$.

Proposition 3.3 *Given an almost ϵcomplex manifold (M, J^ϵ) with a connection ∇ and let S be the section in $T^*M \otimes \mathrm{End}\,(TM)$ defined by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon). \tag{3.2.8}$$

If the pair (∇, J^ϵ) satisfies one of the following conditions

(i) (∇, J^ϵ) *is* special, *i.e.* $(\nabla_X J^\epsilon)Y = (\nabla_Y J^\epsilon)X$ *for all* $X, Y \in \Gamma(TM)$,

(ii) (∇, J^ϵ) *satisfies the* nearly ϵKähler condition, *i.e.* $(\nabla_X J^\epsilon)Y = -(\nabla_Y J^\epsilon)X$ *for all* $X, Y \in \Gamma(TM)$,

then it holds $S_{J^\epsilon X}Y = -J^\epsilon S_X Y$.

Proof: If the condition (i) or (ii) holds, we obtain the identity

$$\begin{aligned}(\nabla_{J^\epsilon X} J^\epsilon)Y &= \pm(\nabla_Y J^\epsilon)J^\epsilon X = \pm\left[\epsilon\nabla_Y X - J^\epsilon\nabla_Y(J^\epsilon X)\right] \\ &= \mp J^\epsilon\left[\nabla_Y(J^\epsilon X) - J^\epsilon\nabla_Y X\right] = \mp J^\epsilon(\nabla_Y J^\epsilon)X = -J^\epsilon(\nabla_X J^\epsilon)Y.\end{aligned}$$

The following calculation finishes the proof

$$S_{J^\epsilon X}Y = -\frac{1}{2}\epsilon J^\epsilon(\nabla_{J^\epsilon X} J^\epsilon)Y = \frac{1}{2}\epsilon J^{\epsilon 2}(\nabla_X J^\epsilon)Y = -J^\epsilon S_X Y.$$

□

Proposition 3.4 *Given an ϵcomplex manifold (M, J^ϵ) with a connection ∇ and let S be the section in $T^*M \otimes \mathrm{End}\,(TM)$ defined by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon). \tag{3.2.9}$$

If ∇ is (anti-)adapted, *i.e. $\nabla_{J^\epsilon X}Y = \pm J^\epsilon \nabla_X Y$ for all ϵholomorphic vector fields X, Y, then it holds $S_{J^\epsilon X}Y = \pm J^\epsilon S_X Y$.*

Proof: Since ∇ is (anti-)adapted, we obtain for all ϵholomorphic vector fields X, Y

$$(\nabla_{J^\epsilon X} J^\epsilon)Y = \pm J^\epsilon(\nabla_X J^\epsilon)Y.$$

The following computation gives the proof

$$S_{J^\epsilon X}Y = -\frac{1}{2}\epsilon J^\epsilon(\nabla_{J^\epsilon X} J^\epsilon)Y = \mp\frac{1}{2}\epsilon J^{\epsilon 2}(\nabla_X J^\epsilon)Y = \pm J^\epsilon S_X Y.$$

□

Remark 3.2
One sees easily that condition (i) of proposition 3.3 is the symmetry of $S_X Y$ and condition (ii) is its anti-symmetry. We recall that if the connection ∇ is torsion-free, flat and special then (M, J^ϵ, ∇) is a special ϵcomplex manifold*, see subsection 1.4. ϵtt^*-bundles coming from special ϵcomplex manifolds and special ϵKähler manifolds were studied in [CS1, Sch3] and are discussed later in subsection 3.2.4.*
Further we want to remark that the second condition arises in nearly ϵKählerian geometry and therefore is quite natural. These geometries as solutions of tt^*-geometry are discussed in subsection 3.2.3.
Finally, the notion of adapted connections appeared in the study of decompositions on (ϵholomorphic) vector bundles over ϵcomplex manifolds, compare the paper of Abe and Kurosu [AK] for the complex and a common paper with M.-A. Lawn-Paillusseau [LS] for the para-complex case.

3.2.2 Solutions on almost ϵhermitian manifolds

In this section we consider almost ϵcomplex manifolds (M, J^ϵ) endowed with a flat connection ∇ such that (∇, J^ϵ) is special or satisfies the nearly ϵKähler condition and analyze under which assumptions these define symplectic or metric ϵtt^*-bundles.

First, we recall a lemma from tensor-algebra:

Lemma 3.2 *Let V be a vector space $\alpha \in T^3(V^*)$ an element in the third tensorial power of V^*, the dual space of V. Suppose that $\alpha(X, Y, Z)$ is symmetric (resp. anti-symmetric) in X, Y and Y, Z and $\alpha(X, Y, Z)$ is anti-symmetric (resp. symmetric) in X, Z then $\alpha = 0$.*

Proof: It is

$$\alpha(X, Y, Z) = \sigma\alpha(Y, X, Z) = \sigma\alpha(X, Z, Y)$$

with $\sigma \in \{\pm 1\}$ which implies

$$\alpha(X, Y, Z) = \sigma\alpha(Y, X, Z) = \sigma^2\alpha(Y, Z, X) = \sigma^3\alpha(Z, Y, X).$$

But further it holds

$$\alpha(X, Y, Z) = -\sigma\alpha(Z, Y, X)$$

and consequently

$$-\alpha(Z, Y, X) = \sigma^2\alpha(Z, Y, X) = \alpha(Z, Y, X).$$

This shows $\alpha = 0$. $\square$

The subsequent proposition shows that the condition to be special is not compatible with symplectic ϵtt^*-bundles:

Proposition 3.5 *Given an almost ϵhermitian manifold (M, J^ϵ, g) with a flat connection ∇, such that (∇, J^ϵ) is special. Define S, a section in $T^*M \otimes \mathrm{End}\,(TM)$, by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \tag{3.2.10}$$

then $(TM, D = \nabla - S, S)$ defines an ϵtt^*-bundle. Suppose, that $(TM, D, S, \omega = g(J^\epsilon\cdot, \cdot))$ is a symplectic ϵtt^*-bundle, then it is trivial, i.e. $S = 0$.

Proof: In fact we know from theorem 3.1 and proposition 3.3, that (TM, D, S) is an ϵtt^*-bundle.
Suppose, that $(TM, D, S, \omega = g(J^\epsilon\cdot, \cdot))$ is a symplectic ϵtt^*-bundle. To finish the proof, we define the tensor

$$\alpha(X, Y, Z) := \omega(S_X Y, Z) = g(J^\epsilon S_X Y, Z), \text{ with } X, Y, Z \in T_pM.$$

$\alpha(X, Y, Z)$ is symmetric in X, Y, since ∇J^ϵ is special, i.e. is symmetric in X, Y.
Further it holds

$$\begin{aligned} \alpha(X, Y, Z) &= \omega(S_X Y, Z) = -\omega(Z, S_X Y) \\ &= -\omega(Z, S_Y X) = -\omega(S_Y Z, X) = -\omega(S_Z Y, X) = -\alpha(Z, Y, X), \end{aligned}$$

which is the anti-symmetry of $\alpha(X, Y, Z)$ in X, Z. Finally

$$\begin{aligned} \alpha(X, Y, Z) &= \omega(S_X Y, Z) = \omega(Y, S_X Z) \\ &= \omega(Y, S_Z X) = -\omega(S_Z X, Y) = -\alpha(Z, X, Y) = -\alpha(X, Z, Y), \end{aligned}$$

i.e. the anti-symmetry of $\alpha(X, Y, Z)$ in Y, Z.
Hence α vanishes and consequently S. $\square$

Otherwise, the nearly ϵKähler condition is not compatible with metric ϵtt^*-bundles:

Proposition 3.6 *Given an almost ϵhermitian manifold (M, J^ϵ, g) with a flat connection ∇, such that (∇, J^ϵ) satisfies the nearly ϵKähler condition. Define S, a section in $T^*M \otimes \mathrm{End}\,(TM)$, by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \tag{3.2.11}$$

then $(TM, D = \nabla - S, S)$ defines an ϵtt^*-bundle. Suppose, that (TM, D, S, g) is a metric ϵtt^*-bundle, then it is trivial, i.e. $S = 0$.

Proof: In fact we know from theorem 3.1 and proposition 3.3, that (TM, D, S) is an ϵtt^*-bundle.
Suppose, that it is a metric ϵtt^*-bundle. To finish the proof, we define the tensor

$$\alpha(X, Y, Z) := g(S_X Y, Z), \text{ with } X, Y, Z \in T_pM.$$

$\alpha(X,Y,Z)$ is anti-symmetric in X,Y, since ∇J^ϵ is anti-symmetric in X,Y by the nearly ϵKähler condition.
Further it holds

$$\begin{aligned}\alpha(X,Y,Z) &= g(S_XY,Z) = g(Z,S_XY)\\ &= -g(Z,S_YX) = -g(S_YZ,X) = g(S_ZY,X) = \alpha(Z,Y,X)\end{aligned}$$

which is the symmetry of $\alpha(X,Y,Z)$ in X,Z. Finally

$$\begin{aligned}\alpha(X,Y,Z) &= g(S_XY,Z) = g(Y,S_XZ)\\ &= -g(Y,S_ZX) = -g(S_ZX,Y) = -\alpha(Z,X,Y) = \alpha(X,Z,Y),\end{aligned}$$

i.e. the symmetry of $\alpha(X,Y,Z)$ in Y,Z.
Hence α vanishes by the above lemma and so does S. □

The following theorem gives solutions of symplectic ϵtt^*-bundles on the tangent bundle, which are more general then the later discussed nearly ϵKähler manifolds in the sense, that we admit the connection ∇ to have torsion, but more special in the sense, that our connection ∇ has to be flat.

Theorem 3.2 *Given an almost ϵhermitian manifold (M,J^ϵ,g) with a flat metric connection ∇, such that (∇,J^ϵ) satisfies the nearly ϵKähler condition. Define S, a section in $T^*M\otimes \mathrm{End}\,(TM)$, by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \tag{3.2.12}$$

then $(TM, D=\nabla - S, S, \omega = g(J^\epsilon\cdot,\cdot))$ defines a symplectic ϵtt^*-bundle. In addition, it is $DJ^\epsilon = 0$. Moreover, the torsion T^D of D and the torsion T^∇ of ∇ are related by $T^D = T^\nabla - 2S$.

Proof: In fact we know from theorem 3.1 and proposition 3.3, that (TM,D,S) is an ϵtt^*-bundle.
It remains to check $D\omega = 0$ and that S is ω-symmetric.
First we remark, that, since g is ϵhermitian and $\nabla g = 0$, $\nabla_X J^\epsilon$ is skew-symmetric with respect to g. Using this we show by the following calculation, that S is skew-symmetric with respect to g:

$$\begin{aligned}-2\epsilon g(S_XY,Z) &= g(J^\epsilon(\nabla_X J^\epsilon)Y,Z) = -g((\nabla_X J^\epsilon)Y, J^\epsilon Z)\\ &= g(Y,(\nabla_X J^\epsilon)J^\epsilon Z) = -g(Y,J^\epsilon(\nabla_X J^\epsilon)Z) = 2\epsilon g(Y,S_XZ).\end{aligned}$$

The definition of $\omega = g(J^\epsilon\cdot,\cdot)$ and $\{S_X,J^\epsilon\}=0$ yield the ω-symmetry of S_X.
Further it holds $D = \nabla + \frac{1}{2}\epsilon J^\epsilon \nabla J^\epsilon$, which implies

$$DJ^\epsilon = \nabla J^\epsilon + \frac{1}{2}\epsilon[J^\epsilon\nabla J^\epsilon, J^\epsilon] = 0.$$

Hence we see, that $D\omega = 0$ if and only if $Dg = 0$. But $\nabla g = 0$ and S is skew-symmetric with respect to g, so g is parallel for $D = \nabla - S$.

This shows, that $(TM, D = \nabla - S, S, \omega)$ is a symplectic ϵtt^*-bundle.
Calculating the torsion we find $T^D(X,Y) = T^\nabla(X,Y) - S_X Y + S_Y X = T^\nabla(X,Y) - 2S_X Y$.
$\square$

The next theorem gives solutions of metric ϵtt^*-bundles on the tangent bundle, which are more general then special ϵKähler manifolds in the sense, that we admit connections ∇ with torsion.

Theorem 3.3 *Given an almost ϵhermitian manifold (M, J^ϵ, g) with a flat connection ∇, such that (∇, J^ϵ) is special and the fundamental two-form $\omega = g(J^\epsilon\cdot,\cdot)$ is ∇-parallel. Define S, a section in $T^*M \otimes \mathrm{End}\,(TM)$, by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \tag{3.2.13}$$

then $(TM, D = \nabla - S, S, g)$ defines a metric ϵtt^*-bundle. In addition, it is $DJ^\epsilon = 0$ and the torsion T^D of D equals the torsion T^∇ of ∇.
Suppose that ∇ is torsion-free, then D is the Levi-Civita connection of g, (M, J^ϵ, g) is an ϵKähler manifold and $(M, J^\epsilon, g, \nabla)$ is a special ϵKähler manifold.

Proof: In fact we know from theorem 3.1 and proposition 3.3, that (TM, D, S) is an ϵtt^*-bundle.
It remains to check $Dg = 0$ and that S is g-symmetric.
First we remark that $\omega(J^\epsilon X, Y) = -\omega(X, J^\epsilon Y)$ as g is ϵhermitian. This yields using $\nabla\omega = 0$ the ω-skew-symmetry of $\nabla_X J^\epsilon$, which implies that $S_X = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon)$ is ω-skew-symmetric, since $J^\epsilon(\nabla_X J^\epsilon) = -(\nabla_X J^\epsilon)J^\epsilon$. Finally $\{S_X, J^\epsilon\} = 0$ shows the g-symmetry of S_X.
Further it is

$$DJ^\epsilon = \nabla J^\epsilon + \frac{1}{2}\epsilon[J^\epsilon \nabla J^\epsilon, J^\epsilon] = 0$$

and consequently $Dg = 0$ is equivalent to $D\omega = 0$.
From $\nabla\omega = 0$ and the ω-skew-symmetry of S it follows $D\omega = (\nabla - S)\omega = 0$.
The symmetry of ∇J^ϵ, i.e. $(\nabla_X J^\epsilon)Y = (\nabla_Y J^\epsilon)X$ for all $X, Y \in TM$ implies $S_X Y = S_Y X$.
This shows using $D = \nabla - S$ that $T^D = T^\nabla$.
Suppose now that ∇ is torsion-free, then D is torsion-free and consequently the Levi-Civita connection of g. Therefore $DJ^\epsilon = 0$ implies the vanishing of the Nijenhuis tensor. Further the equation $\nabla\omega = 0$ implies $d\omega = 0$ since ∇ is torsion-free. Hence (M, J^ϵ, g) is ϵKähler. In addition (M, J^ϵ, ∇) is special ϵcomplex by the conditions on ∇ and J^ϵ. As it holds $\nabla\omega = 0$, $(M, J^\epsilon, \nabla, g)$ is special ϵKähler. $\square$

In [CS1, Sch3] we studied special ϵKähler solutions of ϵtt^*-geometry in more details. The results are discussed in subsection 3.2.4.

3.2.3 Nearly ϵKähler manifolds

In this section we want to apply the above results to nearly ϵKähler manifolds and we use the notation of subsection 1.3.

Corollary 3.2 *Given a nearly ϵKähler manifold (M, J^ϵ, g) such that its Levi-Civita connection $\nabla = \nabla^g$ is flat and let S be the section in $T^*M \otimes \mathrm{End}\,(TM)$ defined by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \tag{3.2.14}$$

then $(TM, \bar{\nabla}, S)$ defines an ϵtt^*-bundle. Suppose, that $(TM, \bar{\nabla}, S, g)$ is a metric ϵtt^*-bundle, then it is trivial, i.e. $S = 0$ and consequently (M, J^ϵ, g) is ϵKähler.

Proof: By setting $D = \bar{\nabla}$ we are in the situation of proposition 3.6. $\square$

Theorem 3.4 *Given a nearly ϵKähler manifold (M, J^ϵ, g) such that its Levi-Civita connection ∇ is flat. Let S be the section in $T^*M \otimes \mathrm{End}\,(TM)$ defined by*

$$S := -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \tag{3.2.15}$$

then $(TM, \bar{\nabla}, S, \omega := g(J^\epsilon \cdot, \cdot))$ is a symplectic ϵtt^*-bundle. Further it holds

$$B(X, Y, Z) = -2g(S_X Y, Z) \text{ and } \bar{\nabla} J^\epsilon = 0. \tag{3.2.16}$$

Proof: By setting $D = \bar{\nabla}$ we are in the situation of theorem 3.2. In addition it holds

$$2g(S_X Y, Z) \;=\; -\epsilon g(J^\epsilon(\nabla_X J^\epsilon)Y, Z) = \epsilon g((\nabla_X J^\epsilon)Y, J^\epsilon Z) = -B(X, Y, Z).$$

$\square$

Remark 3.3
Nearly Kähler manifolds (M, J, g) such that their Levi-Civita connection ∇^g is flat were characterized in common work with V. Cortés [CS2]. More precisely, a constructive classification of nearly Kähler manifolds with flat Levi-Civita connection was given. We further recall that a Levi-Civita flat nearly Kähler cannot be strict. This means that the more interesting examples appear for non definite signature.

3.2.4 Special ϵcomplex and special ϵKähler manifolds

In this subsection we consider another time ϵtt^*-bundles on the tangent-bundle TM of an ϵcomplex manifold (M, J^ϵ) and discuss the results which were published in [CS1] for $\epsilon = -1$ and [Sch3] for $\epsilon = 1$. More precisely, we analyze solutions coming from special ϵcomplex and special ϵKähler manifolds. In this context it is natural to restrict to ϵtt^*-bundles, such that the family of connections D^θ is torsion-free.

Definition 3.3 *An ϵtt^*-bundle (TM, D, S) over an ϵcomplex manifold (M, J^ϵ) is called* special *if D^θ is torsion-free and special, i.e. $D^\theta J^\epsilon$ is symmetric, for all $\theta \in \mathbb{R}$.*

Proposition 3.7 *An ϵtt^*-bundle (TM, D, S) is special if and only if D is torsion-free and DJ^ϵ, S and S_{J^ϵ} are symmetric.*

Proof: The torsion T^θ of D^θ equals

$$T^\theta(X,Y) = T(X,Y) + \cos_\epsilon(\theta)(S_XY - S_YX) + \sin_\epsilon(\theta)(S_{J^\epsilon X}Y - S_{J^\epsilon Y}X) \qquad (3.2.17)$$

where T is the torsion-tensor of D. This implies, that $T^\theta = 0$ for all $\theta \in \mathbb{R}$ if and only if $T = 0$ and S and S_{J^ϵ} are symmetric.
The equation

$$\begin{aligned} (D^\theta_X J^\epsilon)Y &= (D_X J^\epsilon)Y + \cos_\epsilon(\theta)[S_X, J^\epsilon]Y + \sin_\epsilon(\theta)[S_{J^\epsilon X}, J^\epsilon]Y \\ &\overset{\{S_X,J^\epsilon\}=0}{=} (D_X J^\epsilon)Y - 2\cos_\epsilon(\theta)\, J^\epsilon\, S_XY - 2\sin_\epsilon(\theta)\, J^\epsilon\, S_{J^\epsilon X}Y \end{aligned} \qquad (3.2.18)$$

shows that $D^\theta J^\epsilon$ is symmetric if and only if DJ^ϵ, S and S_{J^ϵ} are symmetric.
Conversely, let $T^\theta = 0$ and $D^\theta J^\epsilon$ be symmetric: Then the first part of the proof yields, that S and S_{J^ϵ} are symmetric and $T = 0$. Equation (3.2.18) implies finally the symmetry of DJ^ϵ. □

Theorem 3.5

(i) Let (M, J^ϵ, ∇) be a special ϵcomplex manifold. Put $S := -\frac{1}{2}\epsilon J^\epsilon \nabla J^\epsilon$ and $D := \nabla - S$. Then (TM, D, S) is a special ϵtt^*-bundle with the following additional properties:

a) $S_X J^\epsilon = -J^\epsilon S_X$ for all $X \in TM$ and

b) $DJ^\epsilon = 0$.

This defines a map Φ from special ϵcomplex manifolds to special ϵtt^*-bundles.

(ii) Let (TM, D, S) be a special ϵtt^*-bundle over an ϵcomplex manifold (M, J^ϵ). Then $(M, J^\epsilon, \nabla := D + S)$ is a special ϵcomplex manifold. This defines a map Ψ from special ϵtt^*-bundles to special ϵcomplex manifolds such that $\Psi \circ \Phi = Id$. If (TM, D, S) is a special ϵtt^*-bundle satisfying the conditions a) and b) of (i), then $\Phi(\Psi(TM, D, S)) = (TM, D, S)$.

(iii) Let $(M, J^\epsilon, g, \nabla)$ be a special ϵKähler manifold with S and D as in (i). Then (TM, D, S, g) defines a special metric ϵtt^*-bundle satisfying a) and b) of (i). This defines a map, also called Φ, from special ϵKähler manifolds to special metric ϵtt^*-bundles.

(iv) Let (TM, D, S, g) be a special metric ϵtt^*-bundle over an ϵhermitian manifold (M, J^ϵ, g) satisfying the conditions a) and b) in (i). Then $(M, J^\epsilon, g, \nabla := D + S)$ is a special ϵKähler manifold. In particular, we have a map Ψ from special metric ϵtt^*-bundles over ϵhermitian manifolds (M, J^ϵ, g) satisfying the conditions a) and b) in (i) to special ϵKähler manifolds. Moreover Ψ is a bijection and $\Psi^{-1} = \Phi$.

(v) Let (TM, D, S, g) be a metric ϵtt^*-bundle over an ϵhermitian manifold (M, J^ϵ, g) satisfying the conditions a) and b) in (i) and such that D is torsion-free. Then it is special if and only if $(M, J^\epsilon, g, \nabla := D + S)$ is a special ϵKähler manifold.

Proof:
(i) From theorem 3.1 and proposition 3.3 we know, that (TM, D, S) is an ϵtt^*-bundle. This ϵtt^*-bundle is special, since the family of connections D^θ and the family of connections ∇^θ are linearly equivalent and since by proposition 1.4 $(M, J^\epsilon, \nabla^\theta)$ is a special ϵKähler manifold. The additional properties hold, as (M, J^ϵ, ∇) is a special ϵcomplex manifold (compare proposition 1.6 and 1.8).
(ii) In order to prove the second statement, let (TM, D, S) be a special ϵtt^*-bundle, i.e. D^θ is flat, torsion-free and special. In particular, $\nabla = D + S = D^0$ is flat, torsion-free and special. Hence (M, J^ϵ, ∇) is a special ϵcomplex manifold. Obviously we have $\Psi \circ \Phi = Id$. Conversely, let (TM, D, S) be a special ϵtt^*-bundle satisfying $DJ^\epsilon = 0$ and $S_X J^\epsilon = -J^\epsilon S_X$ for all $X \in T_pM$. Then we use lemma 3.1 to recover D and S uniquely from $\nabla = D + S$ by the formulas $S = -\frac{1}{2}\epsilon J^\epsilon \nabla J^\epsilon$ and $D = \nabla - S$.
(iii) Let $(M, J^\epsilon, g, \nabla)$ be a special ϵKähler manifold with D and S defined as in (i). Then (TM, D, S) is a special ϵtt^*-bundle satisfying a) and b), due to (i). Proposition 1.7 implies, that $Dg = 0$ and proposition 1.8 implies, that S is g-symmetric and hence that (TM, D, S, g) is a special metric ϵtt^*-bundle.
(iv) Let (TM, D, S, g) be a special metric ϵtt^*-bundle over an ϵhermitian manifold (M, J^ϵ, g) satisfying a) and b) in (i). By (ii), we know already, that $(M, J^\epsilon, \nabla := D + S)$ is a special ϵcomplex manifold. Therefore it remains to prove $\nabla\omega = 0$. This implies $d\omega = 0$, as ∇ is torsion-free. We have $Dg = 0$ and $DJ^\epsilon = 0$ (property b) in (i)) and consequently $D\omega = 0$. As $D\omega = 0$, $\nabla\omega = 0$ is equivalent to the ω-skew-symmetry of S and finally to the g-symmetry of S, since $\{J^\epsilon, S_X\} = 0$. But by the definition of a metric ϵtt^*-bundle S is g-symmetric. Therefore $(M, J^\epsilon, \nabla, g)$ is a special ϵKähler manifold. The rest of part (iv) follows from part (ii).
(v) It remains to show the direction which does not follow from (iv). Let (TM, D, S, g) be a metric ϵtt^*-bundle over an ϵhermitian manifold (M, J^ϵ, g), such that $(M, J^\epsilon, g, \nabla = D + S) = \Psi(TM, D, S, g)$ is a special ϵKähler manifold. If D is torsion-free, then it is the Levi-Civita connection of g, and therefore $D = \nabla + \frac{1}{2}\epsilon J^\epsilon \nabla J^\epsilon$, see proposition 1.7. This shows, that $\Phi(M, J^\epsilon, g, \nabla) = (TM, D, S, g)$ and that (TM, D, S, g) is a special metric ϵtt^*-bundle. □

Corollary 3.3 *A special metric ϵtt^*-bundle (TM, D, S, g) over an ϵhermitian manifold (M, J^ϵ, g) which satisfies a) and b) in theorem 3.5 is oriented and unimodular.*

Proof: By theorem 3.5, $(M, J^\epsilon, g, \nabla = D + S)$ is a special ϵKähler manifold. Hence we can orient it by $\omega \wedge \ldots \wedge \omega$, where ω is its ϵKähler-form. Its ϵKähler-form is parallel with respect to the connections D and ∇ and therefore invariant under $S_X = \nabla_X - D_X$. This shows $\operatorname{tr} S_X = 0$. □

3.3 Variations of ϵHodge structures

In this section we recall the result of Hertling [Her] that variations of Hodge structures give solutions of metric tt^*-bundles and generalize it to para-complex geometry and symplectic ϵtt^*-bundles. Our presentation differs form that of [Her], since we give this result in the language of real differential geometry. Again, the para-complex version seems to be new.

Let (E, ∇, F^p) be a (real) variation of ϵHodge structures of weight w. The ϵcomplexified connection of ∇ on $E^{\mathbb{C}_\epsilon} = E \otimes \mathbb{C}_\epsilon$ will be denoted by ∇^c. Griffiths transversality and the ϵholomorphicity of the subbundles F^p gives

$$\nabla^c \,:\, \Gamma(F^p) \to \Lambda^{1,0}(F^{p-1}) + \Lambda^{0,1}(F^p) \tag{3.3.1}$$

and ϵcomplex conjugation yields

$$\nabla^c \,:\, \Gamma(\overline{F}^p) \to \Lambda^{0,1}(\overline{F}^{p-1}) + \Lambda^{1,0}(\overline{F}^p). \tag{3.3.2}$$

Summarizing one obtains with $H^{p,w-p} = F^p \cap \overline{F}^{w-p}$

$$\nabla^c \,:\, \Gamma(H^{p,w-p}) \to \underbrace{\Lambda^{1,0}(H^{p,w-p}) + \Lambda^{0,1}(H^{p,w-p})}_{D} + \underbrace{\Lambda^{1,0}(H^{p-1,w+1-p}) + \Lambda^{0,1}(H^{p+1,w-1-p})}_{S}. \tag{3.3.3}$$

Using the decomposition induced by the ϵHodge structure and by the bi-degree of differential forms, one can find, that the curvature of ∇^c vanishes if and only if (E^c, D, S) defines an ϵtt^*-bundle. In addition the ϵcomplex conjugation $\kappa = \bar{\cdot}$ respects the ϵHodge decomposition and it is $\nabla^c\kappa = 0$. Again the decomposition induced by the ϵHodge structure and by the bi-degree of differential forms implies that $D\kappa = 0$, i.e. D leaves E invariant and that $S\kappa = \kappa S$, i.e. S leaves E invariant, too.
If b is a polarization of the above variation of ϵHodge structures (E, ∇, F^p), then $\nabla b = 0$ and $\nabla^c\kappa = 0$ yield after decomposing with respect to ϵHodge structure the equations $Dg = 0$ and $g(S\cdot,\cdot) = g(\cdot, S\cdot)$ with $g = \operatorname{Re} h$. Concluding we obtain the proposition

Proposition 3.8 *Let (E, ∇, F^p) be a (real) variation of ϵHodge structures of weight w with a polarization b, then $(E, D, S, g = Re\, h)$ with D and S as defined in equation* (3.3.3) *is a metric ϵtt^*-bundle.*

The above consideration holds for $\Omega = \operatorname{Im} h$, too. This implies $D\Omega = 0$ and $\Omega(S\cdot,\cdot) = \Omega(\cdot, S\cdot)$. Hence we have proven

Proposition 3.9 *Let (E, ∇, F^p) be a (real) variation of ϵHodge structures of weight w with a polarization b, then $(E, D, S, \Omega = Im\, h)$ with D and S as defined in equation* (3.3.3) *is a symplectic ϵtt^*-bundle.*

3.4 Harmonic bundles

In this section (cf. [Sch4] for the complex case, i.e. $\epsilon = -1$) we introduce the notion of an ϵharmonic bundle and show that every such bundle gives two solutions of the ϵtt^*-equations. The first is a metric and the second is a symplectic ϵtt^*-bundle.

To introduce the notion of an ϵharmonic bundle we need a definition:

Definition 3.4 *An ϵhermitian sesquilinear metric h on an ϵcomplex vector bundle E over an ϵcomplex manifold (M, J^ϵ) is a smooth fiberwise ϵhermitian sesquilinear product.*

Definition 3.5 *An ϵ*harmonic bundle *$(E \to M, D, C, \bar{C}, h)$ consists of the following data:*
An ϵcomplex vector bundle E over an ϵcomplex manifold (M, J^ϵ), an ϵhermitian sesquilinear metric h on E, a metric connection D with respect to h and two C^∞-linear maps $C : \Gamma(E) \to \Gamma(\Lambda^{1,0}T^*M \otimes E)$ and $\bar{C} : \Gamma(E) \to \Gamma(\Lambda^{0,1}T^*M \otimes E)$, such that the connection

$$D^{(\lambda)} = D + \lambda C + \lambda^{-1}\bar{C}$$

is flat for all $\lambda \in \mathbb{S}^1_\epsilon$ and $h(C_Z a, b) = h(a, \bar{C}_{\bar{Z}} b)$ for all $a, b \in \Gamma(E)$ and $Z \in \Gamma(T^{1,0}M)$.

Remark 3.4
In the case $\epsilon = -1$ and positive definite metric h, this definition is equivalent to the definition of a harmonic bundle given in Simpson's paper [Sim]. Equivalent structures with metrics of arbitrary signature have been also considered in [Her].

Theorem 3.6 *Let $(E \to M, D, C, \bar{C}, h)$ be an ϵharmonic bundle over the ϵcomplex manifold (M, J^ϵ), then $(E, D, S, g = Re\, h\ [\omega = Im\, h])$ with $S_X := C_Z + \bar{C}_{\bar{Z}}$ for $X = Z + \bar{Z} \in TM$ and $Z \in T^{1,0}M$ is a metric [symplectic] ϵtt^*-bundle.*

Proof: For $\lambda = \cos_\epsilon(\alpha) + \hat{i}\sin_\epsilon(\alpha) \in \mathbb{S}^1_\epsilon$ we have a look at $D^{(\lambda)}$:

$$\begin{aligned} D^{(\lambda)}_X &= D_X + \lambda C_Z + \bar{\lambda}\bar{C}_{\bar{Z}} = D_X + \cos_\epsilon(\alpha)(C_Z + C_{\bar{Z}}) + \sin_\epsilon(\alpha)(\hat{i}C_Z - \hat{i}C_{\bar{Z}}) \\ &= D_X + \cos_\epsilon(\alpha)S_X + \sin_\epsilon(\alpha)(C_{J^\epsilon Z} + C_{J^\epsilon \bar{Z}}) \\ &= D_X + \cos_\epsilon(\alpha)S_X + \sin_\epsilon(\alpha)S_{J^\epsilon X} = D^\alpha_X. \end{aligned}$$

Hence we have

$$D^\alpha = D^{(\lambda)} \tag{3.4.1}$$

and D^α is flat if and only if $D^{(\lambda)}$ is flat.
Further we claim, that S is g-symmetric [ω-symmetric]. With $X = Z + \bar{Z}$ for $Z \in T^{1,0}M$ one finds

$$h(S_X\cdot, \cdot) = h(C_Z + C_{\bar{Z}}\cdot, \cdot) = h(\cdot, C_Z + C_{\bar{Z}}\cdot) = h(\cdot, S_X\cdot)$$

and consequently the symmetry of S with respect to

$$g = \mathrm{Re}\, h$$

and

$$\omega = \mathrm{Im}\, h.$$

Finally we show $Dg = 0$ and $D\omega = 0$

$$\begin{aligned} X(h(e,f) \pm h(f,e)) &= (Z + \bar{Z})(h(e,f) \pm h(f,e)) \\ &= h(D_Z e, f) + h(e, D_{\bar{Z}} f) + h(D_{\bar{Z}} e, f) + h(e, D_Z f) \\ &\pm [h(D_Z f, e) + h(f, D_{\bar{Z}} e) + h(D_{\bar{Z}} f, e) + h(f, D_Z e)] \\ &= h((D_Z + D_{\bar{Z}})e, f) + h(e, (D_{\bar{Z}} + D_Z)f) \\ &\pm [h((D_Z + D_{\bar{Z}})f, e) + h(f, (D_{\bar{Z}} + D_Z)e)] \\ &= h(D_X e, f) + h(e, D_X f) \pm h(D_X f, e) \pm h(f, D_X e). \end{aligned}$$

Summarizing we obtain

$$Xg(e,f) = g(D_X e, f) + g(e, D_X f)$$

and

$$X\omega(e,f) = \omega(D_X e, f) + \omega(e, D_X f).$$

This proves, that $(E, D, S, g = \operatorname{Re} h\ [\omega = \operatorname{Im} h])$ is a metric [symplectic] ϵtt^*-bundle. $\square$

Chapter 4

ϵtt^*-geometry and ϵpluriharmonic maps

In this section we are going to state and prove the central results which give the correspondence between ϵpluriharmonic maps and ϵtt^*-bundles. In the first section we consider ϵtt^*-bundles over simply connected manifolds. The case of non trivial fundamental group is dicussed in the second section. These results are part of [Sch6, Sch3].The third section deals with a kind of rigidity result for tt^*-bundles over compact Kähler manifolds with finite fundamental group. Applying this rigidity result to simply connected compact special Kähler manifolds in the fourth section we obtain a special case of Lu's theorem for simply connected compact special Kähler manifolds.

4.1 The simply connected case

Let (M, J^ϵ) be an ϵcomplex manifold and let f be a map $f : M \to G_i(r)$. Like in section 2.4.1 one regards the mapping $A = f^{-1}df = -2d\tilde{f}$ as a flat connection $A : TM \to \mathfrak{g}_i(r)$ on the bundle $E = M \times \mathbb{R}^r$.

Theorem 4.1 *(cf. [Sch6, Sch3]) Let (M, J^ϵ) be a simply connected ϵcomplex manifold. Let $(E, D, S, g\,[, or]\,)$ be a metric [an oriented unimodular metric] ϵtt^*-bundle where E has rank r and M dimension n.*
Then the matrix representing the metric g in a D^θ-flat frame of E $f : M \to Sym^i_{p,q}(\mathbb{R}^r)$ induces an admissible ϵpluriharmonic map $\tilde{f} : M \xrightarrow{f} Sym^i_{p,q}(\mathbb{R}^r) \tilde{\to} S^i(p,q)$, where $S^i(p,q)$ carries the metric induced by the bi-inariant pseudo-Riemannian trace-form on $\mathfrak{g}_i(r)$.
Let s' be another D^θ-flat frame. Then $s' = s{\cdot}U$ for a constant matrix and the ϵpluriharmonic map associated to S' is $f' = U^t f U$.

Remark 4.1 *(cf. [CS1, Sch6, Sch3])*
Before proving the theorem we make some remarks on the condition that the map $\tilde{f}$ is admissible. Let $x \in M$ and $\tilde{f}(x) = uo$. If $d\tilde{f}(T^{1,0}_x M)$ consist of commuting matrices, then

$dL_u^{-1} d\tilde{f}(T_x^{1,0}M)$ is commutative, too. This follows from the fact, that

$$dL_u \,:\, T_o S^i(p,q) \to T_{uo} S^i(p,q) = T_{\tilde{f}(x)} S^i(p,q)$$

equals

$$Ad_u \,:\, sym^i(p,q) = sym^i(I_{p,q}) \to sym^i(u \cdot I_{p,q}) = sym^i(\tilde{f}(x)),$$

which preserves the Lie-bracket.

Proof: Using remark 3.1.1) it suffices to prove the case $\theta = \pi$ for $\epsilon = -1$ or $\theta = 0$ for $\epsilon = 1$.
We first consider a metric ϵtt^*-bundle (E, D, S, g).
Let $s := (s_1, \ldots, s_r)$ be a D^θ-flat frame of E (i.e. $Ds = -\epsilon Ss$), f the matrix $g(s_k, s_l)$ and further S^s the matrix-valued one-form representing S in the frame s. For $X \in \Gamma(TM)$ we get:

$$\begin{aligned} X(f) &= Xg(s,s) = g(D_X s, s) + g(s, D_X s) \\ &= -\epsilon(g(S_X s, s) + g(s, S_X s)) \\ &= -2\epsilon g(S_X s, s) = -2\epsilon f \cdot S^s(X) = -2\epsilon f \cdot S^s_X. \end{aligned} \tag{4.1.1}$$

Consequently $A_X = -2\epsilon S^s_X$. We now prove the ϵpluriharmonicity using

$$d^D S(X,Y) = D_X(S_Y) - D_Y(S_X) - S_{[X,Y]} = 0, \tag{4.1.2}$$
$$d^D S_{J^\epsilon}(X,Y) = D_X(S_{J^\epsilon Y}) - D_Y(S_{J^\epsilon X}) - S_{J^\epsilon [X,Y]} = 0. \tag{4.1.3}$$

The equation (4.1.3) implies

$$\begin{aligned} 0 = d^D S_{J^\epsilon}(J^\epsilon X, Y) &= D_{J^\epsilon X}(S_{J^\epsilon Y}) - \underbrace{\epsilon D_Y(S_X)}_{\overset{(4.1.2)}{=} \epsilon(D_X(S_Y) - S_{[X,Y]})} - S_{J^\epsilon [J^\epsilon X, Y]} \\ &= D_{J^\epsilon X}(S_{J^\epsilon Y}) - \epsilon D_X(S_Y) + \epsilon S_{[X,Y]} - S_{J^\epsilon [J^\epsilon X, Y]}. \end{aligned}$$

In local ϵholomorphic coordinate fields X, Y on M we get in the frame s

$$J^\epsilon X(S^s_{J^\epsilon Y}) - \epsilon X(S^s_Y) + [S^s_X, S^s_Y] - \epsilon [S^s_{J^\epsilon X}, S^s_{J^\epsilon Y}] = 0.$$

Now $A = -2\epsilon S^s$ gives equation (2.4.9) and proves the ϵpluriharmonicity of f.
Using $A_X = -2\epsilon S^s_X = -2d\tilde{f}(X)$, we find the property of the differential, as $S \wedge S$ is of type (1,1) by the ϵtt^*-equations, see proposition 3.1.
The last statement is obvious.
In the case of an oriented unimodular metric ϵtt^*-bundle (E, D, S, g, or) we can take the frame s to be oriented and of volume 1, with respect to the canonical D^θ-parallel-metric volume ν. Therefore the map f takes values in $Sym^1_{p,q}(\mathbb{R}^r)$ and the above arguments show the rest. □

Theorem 4.2 *(cf. [Sch6, Sch3]) Let (M, J^ϵ) be a simply connected ϵcomplex manifold and put $E = M \times \mathbb{R}^r$.*

Then an ϵpluriharmonic map $\tilde{f} : M \to S^i(p,q)$ gives rise to an ϵpluriharmonic map $f : M \xrightarrow{\tilde{f}} S^i(p,q) \tilde{\to} Sym^i_{p,q}(\mathbb{R}^r) \subset G_i(r)$.
If the map $\tilde{f}$ is admissible, then the map f induces a metric ϵtt^*-bundle [an oriented unimodular metric ϵtt^*-bundle] $(E, D = \partial - \epsilon S, S = \epsilon d\tilde{f}, g = \langle f\cdot,\cdot\rangle_{\mathbb{R}^r} [, or])$ on M where ∂ is the canonical flat connection on E and or is the canonical orientation on E.

Remark 4.2 *We observe, that for ϵRiemannian surfaces $M = \Sigma$ the condition on the differential holds, since $T^{1,0}\Sigma$ is one-dimensional.*

Proof:
Let $\tilde{f} : M \to S^i(p,q)$ be an ϵpluriharmonic map. Then by proposition 2.6 we know, that $f : M \tilde{\to} Sym^i_{p,q}(\mathbb{R}) \subset G_i(r)$ is ϵpluriharmonic.
Since $E = M \times \mathbb{R}^r$, we can regard sections of E as r-tuples of $C^\infty(M,\mathbb{R})$-functions.
In the spirit of section 2.4.1 we regard the one-form $A = -2d\tilde{f} = f^{-1}df = -2\epsilon S$ with values in $\mathfrak{g}_i(r)$ as a connection on E. We remind, that the curvature of this connection vanishes (proposition 2.8).

a) First, we check the conditions on the metric:

Lemma 4.1 *The connection D is compatible with the metric g and S is symmetric with respect to g.*

Proof: This is a direct computation with $X \in \Gamma(TM)$ and $v, w \in \Gamma(E)$ using the relations $(*)$ $S = -\frac{1}{2}\epsilon f^{-1}df$, $(**)$ $df_x : T_xM \to T_{f(x)}\mathrm{Sym}^i_{p,q}(\mathbb{R}^r) = \mathrm{Sym}^i(\mathbb{R}^r)$ (compare remark 2.3) and $g = \langle f\cdot,\cdot\rangle_{\mathbb{R}^r} = \langle\cdot, f\cdot\rangle_{\mathbb{R}^r}$ which follows from $f : M \to \mathrm{Sym}^i_{p,q}(\mathbb{R}^r)$:

$$\begin{aligned}
X(g(v,w)) &= X(\langle fv, w\rangle_{\mathbb{R}^r}) = \langle X(f)v, w\rangle_{\mathbb{R}^r} + \langle f(\partial_X v), w\rangle_{\mathbb{R}^r} + \langle fv, \partial_X w\rangle_{\mathbb{R}^r} \\
&\overset{(**)}{=} \frac{1}{2}\langle X(f)v, w\rangle_{\mathbb{R}^r} + \frac{1}{2}\langle v, X(f)w\rangle_{\mathbb{R}^r} + \langle f(\partial_X v), w\rangle_{\mathbb{R}^r} + \langle fv, \partial_X w\rangle_{\mathbb{R}^r} \\
&= \frac{1}{2}\langle f\cdot f^{-1}(X(f))v, w\rangle_{\mathbb{R}^r} + \frac{1}{2}\langle v, f\cdot f^{-1}(X(f))w\rangle_{\mathbb{R}^r} \\
&\quad + \langle f\partial_X v, w\rangle_{\mathbb{R}^r} + \langle fv, \partial_X w\rangle_{\mathbb{R}^r} \\
&\overset{(*),(**)}{=} g(X.v - \epsilon S_X v, w) + g(v, X.w - \epsilon S_X w) \\
&= g(D_X v, w) + g(v, D_X w).
\end{aligned}$$

For $x \in M$ $d\tilde{f}_x$ takes by remark 2.3 values in $\mathrm{sym}^i(f(x))$. This shows that $S = \epsilon d\tilde{f}$ is symmetric with respect to $g = \langle f\cdot,\cdot\rangle_{\mathbb{R}^r}$. □

To finish the proof, we have to check the ϵtt^*-equations. The second ϵtt^*-equation

$$-\epsilon[S_X, S_Y] = [S_{J^\epsilon X}, S_{J^\epsilon Y}] \tag{4.1.4}$$

for S follows from the assumption that the image of $T^{1,0}M$ under $(d\tilde{f})^{\mathbb{C}_\epsilon}$ is Abelian. In fact, this is equivalent to $[d\tilde{f}(J^\epsilon X), d\tilde{f}(J^\epsilon Y)] = -\epsilon[d\tilde{f}(X), d\tilde{f}(Y)]$ $\forall X, Y \in TM$.

$$\begin{aligned}
d^D S(X,Y) &= [D_X, S_Y] - [D_Y, S_X] - S_{[X,Y]} \\
&= \partial_X(S_Y) - \partial_Y(S_X) - 2\epsilon[S_X, S_Y] - S_{[X,Y]} = 0
\end{aligned}$$

is equivalent to the vanishing of the curvature of $A = -2\epsilon S$ interpreted as a connection on E (see proposition 2.8).
Finally one has for ϵholomorphic coordinate fields $X, Y \in \Gamma(TM)$

$$\begin{aligned}
d^D S_{J^\epsilon}(J^\epsilon X, Y) &= [D_{J^\epsilon X}, S_{J^\epsilon Y}] - \epsilon[D_Y, S_X] \\
&= [\partial_{J^\epsilon X} - \epsilon S_{J^\epsilon X}, S_{J^\epsilon Y}] - \epsilon[\partial_Y - \epsilon S_Y, S_X] \\
&= \partial_{J^\epsilon X}(S_{J^\epsilon Y}) - \epsilon \partial_Y(S_X) - \epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}] - [S_X, S_Y] \\
&\overset{(4.1.4)}{=} -\frac{1}{2}\epsilon\left(\partial_{J^\epsilon X}(A_{J^\epsilon Y}) - \epsilon\partial_Y(A_X)\right) \\
&\overset{(2.4.8)}{=} -\frac{1}{2}\epsilon\left(\partial_{J^\epsilon X}(A_{J^\epsilon Y}) - \epsilon\partial_X(A_Y) - \epsilon[A_X, A_Y]\right) \\
&\overset{(4.1.4)}{=} -\frac{1}{2}\epsilon\left(\partial_{J^\epsilon X}(A_{J^\epsilon Y}) - \epsilon\partial_X(A_Y) - \frac{1}{2}\epsilon[A_X, A_Y] + \frac{1}{2}[A_{J^\epsilon X}, A_{J^\epsilon Y}]\right) \\
&\overset{(2.4.9)}{=} 0.
\end{aligned}$$

This shows the vanishing of the tensor $d^D S_{J^\epsilon}$.
It remains to show the curvature equation for D. We observe, that $D + \epsilon S = \partial - \epsilon S + \epsilon S = \partial$ and that the connection ∂ is flat, to find

$$0 = R^{D+\epsilon S}_{X,Y} = R^D_{X,Y} + \epsilon d^D S(X,Y) + [S_X, S_Y] \overset{d^D S=0}{=} R^D_{X,Y} + [S_X, S_Y].$$

b) With the same proof as in part a) we get a metric ϵtt^*-bundle. The orientation is given by the orientation of $E = M \times \mathbb{R}^r$.
It remains to check the condition on the trace of S. This property is clear, since in this case $d\tilde{f}_x$ takes values in $\mathrm{sym}^1(f(x))$ for all $x \in M$. □

We want to emphasize the last result in case of metric tt^*-bundles with positive definite metric over a complex manifold (M, J).

Theorem 4.3 *Let (M, J) be a simply connected complex manifold and put $E = M \times \mathbb{R}^r$. Then a pluriharmonic map $\tilde{f} : M \to S^i(r, 0)$ is admissible. Moreover, it induces a second pluriharmonic map $f : M \xrightarrow{\tilde{f}} S^i(r,0) \tilde{\to} Sym^i_{r,0}(\mathbb{R}^r) \subset G_i(r)$ and a metric ϵtt^*-bundle [an oriented unimodular metric ϵtt^*-bundle] $(E, D = \partial + S, S = -d\tilde{f}, g = \langle f\cdot, \cdot\rangle_{\mathbb{R}^r} [, or])$ on M where ∂ is the canonical flat connection on E and or is the canonical orientation of E.*

Proof: In the case of signature $(r, 0)$ corollary 2.3 implies that any pluriharmonic map $\tilde{f} : M \to S^i(r, 0)$ is admissible as required in theorem 4.2. □

In the situation of theorem 4.2 the two constructions are inverse in the following sense:

Proposition 4.1

1. *Let $(E, D, S, g [, or])$ be a metric [an oriented unimodular metric] ϵtt^*-bundle on an ϵcomplex manifold (M, J^ϵ) and let $\tilde{f}$ be the associated ϵpluriharmonic map constructed to a D^θ-flat frame s in theorem 4.1. Then $\tilde{f}$ is admissible and the metric*

[oriented unimodular metric] ϵtt^*-bundle $(M \times \mathbb{R}^r, \tilde{D} = \partial - \epsilon\tilde{S}, \tilde{S}, \tilde{g}, [or])$ associated to $\tilde{f}$ in theorem 4.2 is the representation of $(E, D, S, g\,[, or])$ in the frame s.

2. *Given an ϵpluriharmonic map $\tilde{f}$ from an ϵcomplex manifold (M, J^ϵ) to $S^i(p,q)$, then one obtains via theorem 4.2 a metric [an oriented unimodular metric] ϵtt^*-bundle $(M \times \mathbb{R}^r, D, S, g\,[, or])$. The ϵpluriharmonic map associated to this metric ϵtt^*-bundle is conjugated to the map $\tilde{f}$ by a constant matrix in $G_i(r)$.*

Proof: Using again remark 3.1.1) we can set $\theta = \pi$ for $\epsilon = -1$ or $\theta = 0$ for $\epsilon = 1$.

1. The maps f, $\tilde{f}$ and the metric $\tilde{g} = \langle f\cdot, \cdot\rangle_{\mathbb{R}^r}$ express the metric g in the frame s. In the computations of theorem 4.1 and with theorem 4.2 one finds $2\tilde{S} = -\epsilon A = -\epsilon f^{-1}df = 2S^s$. From $0 = D^\theta s = Ds + \epsilon S s$ we obtain that the connection D in the frame s is just $\partial - \epsilon S^s = \partial + \frac{A}{2} = \partial - \epsilon\tilde{S} = \tilde{D}$.

2. To find the ϵpluriharmonic map associated to $(M \times \mathbb{R}^r, D, S, g\ [, or])$ we have to express the metric g in a D^θ-flat frame s. But $D^\theta = \partial - \epsilon S + \epsilon S = \partial$. Hence we can take s as the standard-basis of $\mathbb{R}^r$ and we get f. Every other basis gives a conjugated result. □

4.2 The general case

In this section we are going to transfer the results in the simply connected case to manifolds with non-trivial fundamental group.

Definition 4.1 *Let $p : \tilde{M} \to M$ be the universal cover of an ϵcomplex manifold (M, J^ϵ) with the pulled back ϵcomplex structure.*
Let (E, D, S) be an ϵtt^*-bundle, then we define the pulled back ϵtt^*-bundle of (E, D, S) to be given by (p^*E, p^*D, p^*S).
Let (E, D, S, g) be a metric ϵtt^*-bundle, then we define the pulled back metric ϵtt^*-bundle of (E, D, S, g) to be given by (p^*E, p^*D, p^*S, p^*g).
Finally, let (E, D, S, g, or) be an oriented unimodular metric ϵtt^*-bundle, then we define the pulled back oriented unimodular metric ϵtt^*-bundle of (E, D, S, g, or) to be given by $(p^*E, p^*D, p^*S, p^*g, p^*or)$.

Remark 4.3 *The pulled back ϵtt^*-bundles, metric ϵtt^*-bundles and oriented unimodular metric ϵtt^*-bundles are ϵtt^*-bundles, metric ϵtt^*-bundles and oriented unimodular metric ϵtt^*-bundles respectively, as one checks easily. This motivates the above definition.*

Theorem 4.4 *Let (M, J^ϵ) be an ϵcomplex manifold.*
Let $(E, D, S, g\,[, or])$ be a [an oriented unimodular] metric ϵtt^*-bundle where E has rank r and M dimension n and $(p^*E, p^*D, p^*S, p^*g\,[, p^*or])$ the corresponding pulled-back [oriented unimodular] metric ϵtt^*-bundle on the universal cover $\tilde{M}$ of M.
Denote by $f^* : \tilde{M} \to S^i(p,q)$ the ϵpluriharmonic map obtained from theorem 4.1 in the p^*D^θ-flat frame p^*s, where s is a D^θ-flat frame and $f : M \to S^i(p,q)$ the map obtained

from the representation of g in the frame s. Then f^* is a $\pi_1(M)$-equivariant map (Here equivariant means by the left-action on M and via the holonomy on $S^i(p,q)$.) and the lift p^*f of f. In other words f is a twisted ϵpluriharmonic map.

Proof: The equivariance follows, since we have pulled back all structures. If s is D^θ-flat, p^*s is p^*D^θ-flat, too.
The map f^* at $\tilde{x} \in \tilde{M}$ with $p(\tilde{x}) = x$ is given by

$$f^*(\tilde{x}) = p^*g(p^*s, p^*s)(x) = g_{p(\tilde{x})}(s \circ p(\tilde{x}), s \circ p(\tilde{x})) = f(x) = f \circ p(\tilde{x}) = p^*f(x).$$

□

Theorem 4.5 *Let (M, J^ϵ) be an ϵcomplex manifold, $p : \tilde{M} \to M$ its universal covering with the pulled back ϵcomplex structure, also called J^ϵ. Set $E = \tilde{M} \times \mathbb{R}^r$.
Let $\tilde{f}^* : \tilde{M} \to S^i(p,q)$ be an admissible ϵpluriharmonic map, which is equivariant with respect to a representation $\rho : \pi_1(M) \to G_i(r)$ and $f^* : \tilde{M} \to Sym^i_{p,q}(\mathbb{R}^r)$ the corresponding map. Then $\tilde{f}^*$ induces by theorem 4.2 a [an unimodular oriented] metric ϵtt^*-bundle $(E, D = \partial - \epsilon S, S = \epsilon d\tilde{f}^*, g =< f^* \cdot, \cdot >_{\mathbb{R}^r})$ on $\tilde{M}$ where ∂ is the canonical flat connection on E. This [oriented unimodular] metric ϵtt^*-bundle induces a [an oriented unimodular] metric ϵtt^*-bundle $(F, D = \partial - \epsilon T, T, h)$ on M, such that the [unimodular oriented] metric ϵtt^*-bundle $(E, D = \partial - \epsilon S, S = \epsilon d\tilde{f}^*, g =< f^* \cdot, \cdot >_{\mathbb{R}^r})$ is its pull back.*

Proof:

a) We want to regard the action of $\pi_1(M)$ on E, given by

$$(\gamma, m, v) \in \pi_1(M) \times E \mapsto (\gamma.m, \rho(\gamma)v) =: \gamma.(m, v) \in E \qquad (4.2.1)$$

which induces the action

$$(\gamma, m, A) \in \pi_1(M) \times \mathrm{End}(E) \mapsto (\gamma.m, \rho(\gamma)A\rho(\gamma)^{-1}) =: \gamma.(m, A) \in \mathrm{End}(E) \quad (4.2.2)$$

of $\pi_1(M)$ on $\mathrm{End}(E)$. The quotient of E by the action of $\pi_1(E)$ gives a vector bundle $F \to M$ over M.
The equivariance of the map $\tilde{f}^* : \tilde{M} \to S(p,q)$ means for $m \in \tilde{M}$:

$$\tilde{f}^*(\gamma.m) = \rho(\gamma)\tilde{f}^*(m)\rho(\gamma)^{-1}, \qquad (4.2.3)$$

which implies for $X \in T_m\tilde{M},\ m \in \tilde{M}$

$$d\tilde{f}^*_{\gamma.m}(d\gamma X) = \rho(\gamma)d\tilde{f}^*_m(X)\rho(\gamma)^{-1}. \qquad (4.2.4)$$

Equation (4.2.3) is the equivariance of g and equation (4.2.4) is the equivariance of S. Hence they descend to a metric h on F and an endomorphism field T on F, which is h-symmetric. Since ∂ is $\pi_1(M)$-invariant, it defines connection on F and since S is equivariant $D = \partial - \epsilon T$ defines connection on F which preserves h. With the same argument the family $D^\theta = D + \cos_\epsilon(\theta)T + \sin_\epsilon(\theta)T_{J^\epsilon}$ defines a family of connections on F which is flat. Hence $(F, D = \partial - \epsilon T, T, h)$ is a metric ϵtt^*-bundle on F over M.

b) One gets the data $(F, D = \partial - \epsilon T, T, h)$ as in part a). The orientation is given by the orientation of $E = \tilde{M} \times \mathbb{R}^r$, since ρ takes values in $SL(r)$. □

4.3 A rigidity result

In [Sch5] we showed a rigidity result, which will be used later to obtain a new proof of Lu's theorem [Lu] in the case of simply connected compact special Kähler manifolds.

Theorem 4.6 *Let M be a compact Kähler manifold of dimension n with finite fundamental group $\pi_1(M)$ (i.e., the universal cover of M is compact). Let (E, D, S, g) be a metric tt^*-bundle, where E has rank r, with positive definite metric g. Then (E, D, S, g) is trivial, i.e. $S = 0$, D is flat and g D^θ-parallel.*

Proof: Pulling back all structures to the universal cover of M we suppose that M is simply connected. $S = 0$ if and only if the same holds for its pull back. Let s be a D^π-flat frame of E. The associated pluriharmonic map $\tilde{f} : M \to GL(r, \mathbb{R})/O(r)$ obtained from theorem 4.1 is constant by corollary 2.2. Hence, the representing matrix G^s of g in the frame s is constant. We recall the relation between the representation S^s of S in the frame s with G^s which we found in equation (4.1.1):

$$X(G^s) \;\; = \;\; 2G^s \cdot S^s_X.$$

This shows $S^s = 0$ and consequently $S = 0$ and $D^\theta = D$ for all $\theta \in \mathbb{R}$. Hence D is flat and $D^\theta g = 0$. □

4.4 A special case of Lu's theorem

As a corollary of our rigidity result, theorem 4.6, we obtain Lu's theorem [Lu] for simply connected compact manifolds. Another proof of Lu's theorem was given in [BC1]. The authors immersed any simply connected special Kähler manifold M^n as a parabolic affine hypersphere into $\mathbb{R}^{n+1}$ and obtained Lu's theorem from a result of Calabi and Pogorelov.

Theorem 4.7 *Let (M, J, g, ∇) be a simply connected compact special Kähler manifold of dimension n. Then M is a point.*

Proof: Using theorem 3.5 the data $(TM, D = \nabla - S, S = \frac{1}{2}J\nabla J, g)$ defines a metric tt^*-bundle. Then theorem 4.6 yields $S = 0$ and hence $D = \nabla$. From $Dg = 0$ and the torsion-freeness of ∇ it follows that D is the Levi-Civita connection. Therefore M is Levi-Civita flat, compact and simply connected, i.e. M is a point. □

Chapter 5

The ϵpluriharmonic maps associated to the above examples of ϵtt^*-bundles

In this chapter we analyze and apply the correspondence between ϵtt^*-bundles and ϵpluriharmonic maps for the classes of solutions which were discussed in chapter 3. In addition we associate generalized ϵpluriharmonic maps to the geometries with non integrable ϵcomplex structures.

5.1 Solutions on the tangent bundle

This subsection is also subject of [Sch7, Sch8].

5.1.1 The classifying map of a flat nearly ϵKähler manifold

In this section we consider simply connected almost ϵhermitian manifolds (M, J^ϵ, g) endowed with a flat metric connection ∇ such that (∇, J^ϵ) satisfies the nearly ϵKähler condition.
In particular, simply connected flat nearly ϵKähler manifolds (M^{2n}, J^ϵ, g), i.e. nearly ϵKähler manifolds (M, J^ϵ, g) with flat Levi-Civita connection ∇^g are of this type.
Since (M, g, ∇) is simply connected and flat, we may identify by fixing a ∇-parallel frame s_0 its tangent bundle TM with $(M \times V, \langle \cdot, \cdot \rangle)$, where $V = \mathbb{C}^n_\epsilon = (\mathbb{R}^{2n}, j_0^\epsilon)$ is endowed with the standard scalar product $\langle \cdot, \cdot \rangle$ of the same hermitian signature (p, q) as the hermitian metric g for $\epsilon = -1$ and of symmetric signature (n, n) for $\epsilon = 1$.
The compatible ϵcomplex structure J^ϵ defines via this identification a map

$$J^\epsilon : M \to \mathcal{J}^\epsilon(V, \langle \cdot, \cdot \rangle),$$

where $\mathcal{J}^\epsilon(V, \langle \cdot, \cdot \rangle)$ is the set of ϵcomplex structures on V which are compatible with $\langle \cdot, \cdot \rangle$ and the orientation of $V = \mathbb{R}^{2n}$. The differential geometry of this set was discussed in section 2.5.

Theorem 5.1 *Let (M, J^ϵ, g) be a simply connected almost ϵhermitian manifold endowed with a flat metric connection ∇ such that (∇, J^ϵ) satisfies the nearly ϵKähler condition,*

then $(TM, D = \nabla - S, S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \omega = g(J^\epsilon \cdot, \cdot))$ *defines a symplectic* ϵtt^**-bundle and the matrix of* J^ϵ *in a* D^θ*-flat frame* $s^\theta = (s_i^\theta)$ *defines an* $\mathbb{S}^1_\epsilon$*-pluriharmonic map* $\tilde{J^\epsilon}^\theta : M \to \mathfrak{J}^\epsilon(V, \langle\cdot,\cdot\rangle) \to SO_0(2p,2q)/U^\epsilon(p,q)$.
In particular, given a nice connection D *on* M *the map*

$$\tilde{J^\epsilon}^\theta : (M, J^\epsilon, D) \to SO_0(2p,2q)/U^\epsilon(p,q)$$

is ϵ*pluriharmonic.*

Proof: We observe $D^\theta g = 0$ since $\nabla g = 0$, $D^0 = \nabla$ and $S^\theta_X := \cos_\epsilon(\theta) S_X + \sin_\epsilon(\theta) S_{J^\epsilon X}$ is skew-symmetric with respect to g. Therefore we can choose for each $\theta \in \mathbb{R}$ the D^θ-flat frame s^θ orthonormal, such that $s^{\theta=0} = s_0$. This yields using $DJ^\epsilon = 0$ (compare theorem 3.1 and lemma 3.1)

$$Xg(J^\epsilon s_i^\theta, s_j^\theta) = g(D^\theta_X(J^\epsilon s_i^\theta), s_j^\theta) = g((D^\theta_X J^\epsilon)s_i^\theta, s_j^\theta) = g([S^\theta_X, J^\epsilon]s_i^\theta, s_j^\theta) = -2g(J^\epsilon S^\theta_X s_i^\theta, s_j^\theta).$$

Let S^{s^θ} and $J^{\epsilon s^\theta}$ be the representation of S and J^ϵ in the frame s^θ, then

$$(J^{\epsilon s^\theta})^{-1} X(J^{\epsilon s^\theta}) = -2S^{s^\theta}$$

or

$$d\tilde{J^\epsilon}^\theta = (s^\theta)^{-1} \circ S^\theta \circ s^\theta,$$

where the frame s^θ is seen as a map $s^\theta : M \times V \to TM$. This shows for $X \in \Gamma(TM)$

$$\begin{aligned} d\tilde{J^\epsilon}^\theta(X) &= (s^\theta)^{-1} \circ S^\theta_X \circ (s^\theta) = (s^\theta)^{-1} \circ S_{\mathcal{R}_\theta X} \circ (s^\theta) \\ &= ((s^\theta)^{-1}s^0) \circ d\tilde{J^\epsilon}(\mathcal{R}_\theta X) \circ ((s^0)^{-1}s^\theta) \\ &= Ad^{-1}_{\alpha_\theta} \circ d\tilde{J^\epsilon}(\mathcal{R}_\theta X) = \Phi^{-1}_\theta \circ d\tilde{J^\epsilon}(\mathcal{R}_\theta X), \end{aligned}$$

where $\alpha_\theta = (s^\theta)^{-1}s^0$ is the frame change from s^0 to s^θ and $\Phi_\theta = Ad_{\alpha_\theta}$ which is parallel with respect to the Levi-Civita connection on $SO_0(2p,2q)/U^\epsilon(p,q)$. This shows, that $\tilde{J^\epsilon}^\theta$ is $\mathbb{S}^1_\epsilon$-pluriharmonic. Given a nice connection D on M theorem 2.2 shows that $\tilde{J^\epsilon}^\theta$ is ϵpluriharmonic. □

We emphasize the nearly ϵKähler setting:

Corollary 5.1 *Let* (M, J^ϵ, g) *be a flat nearly* ϵ*Kähler manifold and* $(TM, \overline{\nabla} = \nabla^g - S, S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \omega(\cdot,\cdot) = g(J^\epsilon\cdot,\cdot))$ *the associated symplectic* ϵtt^**-bundle, then the matrix of* J^ϵ *in a* D^θ*-flat frame* $s^\theta = (s_i^\theta)$ *defines an* $\mathbb{S}^1_\epsilon$*-pluriharmonic map* $\tilde{J^\epsilon}^\theta : M \to \mathfrak{J}^\epsilon(V, \langle\cdot,\cdot\rangle) \to SO_0(2p,2q)/U^\epsilon(p,q)$.

For nearly ϵKähler manifolds we have more precise informations about the map $\tilde{J^\epsilon}^\theta$:

Theorem 5.2 *Let* (M, J^ϵ, g) *be a flat nearly* ϵ*Kähler manifold and* $(TM, \overline{\nabla} = \nabla^g - S, S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), \omega(\cdot,\cdot) = g(J^\epsilon\cdot,\cdot))$ *the associated symplectic* ϵtt^**-bundle. Then the connection* $\overline{\nabla}$ *is nice and the matrix of* J^ϵ *in a* D^θ*-flat frame* $s^\theta = (s_i^\theta)$ *defines an* ϵ*pluriharmonic map* $\tilde{J^\epsilon}^\theta : (M, J^\epsilon, \bar{\nabla}) \to \mathfrak{J}^\epsilon(V, \langle\cdot,\cdot\rangle) \to SO_0(2p,2q)/U^\epsilon(p,q)$. *Moreover, the map* $\tilde{J^\epsilon}^\theta$ *is harmonic.*

Proof: First we show, that $\overline{\nabla}$ is nice. Therefore we rewrite the Nijenhuis tensor

$$\begin{aligned} N_{J^\epsilon}(X,Y) &= (\nabla_{J^\epsilon X} J^\epsilon)Y - (\nabla_{J^\epsilon Y} J^\epsilon)X - J^\epsilon(\nabla_X J^\epsilon)Y + J^\epsilon(\nabla_Y J^\epsilon)X \\ &= -4J^\epsilon(\nabla_X J^\epsilon)Y, \end{aligned}$$

where the second equality follows from the nearly ϵKähler condition and by

$$(\nabla_{J^\epsilon X} J^\epsilon)Y = -(\nabla_Y J^\epsilon)J^\epsilon X = J^\epsilon(\nabla_Y J^\epsilon)X = -J^\epsilon(\nabla_X J^\epsilon)Y.$$

But the torsion of $\overline{\nabla}$ is by equation (1.3.3)

$$T^{\overline{\nabla}}(X,Y) = \epsilon J^\epsilon(\nabla_X J^\epsilon)Y.$$

This shows that $\overline{\nabla}$ is nice.
By corollary 5.1 the map $\tilde{J^\epsilon}^\theta$ is $\mathbb{S}^1_\epsilon$-pluriharmonic. Since $\overline{\nabla}$ is nice, theorem 2.2 implies that $\tilde{J^\epsilon}^\theta$ is ϵpluriharmonic. From the skew-symmetry of S and proposition 2.5 we obtain that $\tilde{J^\epsilon}^\theta$ is harmonic. □

5.2 The dual Gauß map of a special ϵKähler manifold with torsion

In this subsection we consider a simply connected almost ϵhermitian manifold (M, J^ϵ, g) with a flat connection ∇, such that (∇, J^ϵ) is special and the two-form $\omega = g(J^\epsilon\cdot,\cdot)$ is ∇-parallel.
Using the flat connection ∇ we identify by fixing a ∇-parallel symplectic frame s_0 the tangent space (TM, ω) with $(M \times V, \omega_0)$ where $V = \mathbb{R}^{2n}$ and ω_0 is its standard symplectic form.
The compatible ϵcomplex structure J^ϵ is seen as a map

$$J^\epsilon : M \to \mathfrak{J}^\epsilon(V, \omega_0),$$

where $\mathfrak{J}^\epsilon(V,\omega_0)$ is the set of ϵcomplex structures on V which are compatible with ω_0.
The differential geometry of this set was discussed in section 2.5.
Recall, that under the above assumptions $(TM, D = \nabla - S, S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), g)$ defines a metric ϵtt^*-bundle. Analogous to the last section we obtain:

Theorem 5.3 *Let (M, J^ϵ, g) be a simply connected almost ϵhermitian manifold with a flat connection ∇, such that (∇, J^ϵ) is special and the two-form $\omega = g(J^\epsilon\cdot,\cdot)$ is ∇-parallel and let $(TM, D = \nabla - S, S = -\frac{1}{2}\epsilon J^\epsilon(\nabla J^\epsilon), g)$ be the associated metric ϵtt^*-bundle. Then the matrix of J^ϵ in a D^θ-flat frame $s^\theta = (s^\theta_i)$ defines an $\mathbb{S}^1_\epsilon$-pluriharmonic map $\tilde{J^\epsilon}^\theta : M \to \mathfrak{J}^\epsilon(V,\omega_0) \to \mathrm{Sp}(\mathbb{R}^{2n})/U^\epsilon(p,q)$.*
In particular, given a nice connection D on (M, J^ϵ) then the map $\tilde{J^\epsilon}^\theta : (M, J^\epsilon, D) \to \mathrm{Sp}(\mathbb{R}^{2n})/U^\epsilon(p,q)$ is ϵpluriharmonic.

Proof: Since $D^0\omega = \nabla\omega = (D+S)\omega = 0$ and $S^\theta_X := \cos_\epsilon(\theta)S_X + \sin_\epsilon(\theta)S_{J^\epsilon X}$ is skew-symmetric with respect to ω, we obtain $D\omega = 0$ and $D^\theta\omega = 0$. Therefore we can choose for each $\theta \in \mathbb{R}$ the D^θ-parallel frame s^θ as a symplectic frame, such that $s^{\theta=0} = s_0$. This yields using $DJ^\epsilon = 0$ (compare theorem 3.1 and lemma 3.1)

$$X\omega(J^\epsilon s^\theta_i, s^\theta_j) = \omega(D^\theta_X(J^\epsilon s^\theta_i), s^\theta_j) = \omega((D^\theta_X J^\epsilon)s^\theta_i, s^\theta_j) = \omega([S^\theta_X, J^\epsilon]s^\theta_i, s^\theta_j) = -2\omega(J^\epsilon S^\theta_X s^\theta_i, s^\theta_j).$$

Let S^{s^θ} and $J^{\epsilon s^\theta}$ be the representation of S and J^ϵ in the frame s^θ, then

$$(J^{\epsilon s^\theta})^{-1}X(J^{\epsilon s^\theta}) = -2S^{s^\theta}$$

or

$$d\tilde{J^\epsilon}^\theta = (s^\theta)^{-1} \circ S^\theta \circ s^\theta,$$

where the frame s^θ is seen as a map $s^\theta : M \times V \to TM$. This shows for $X \in \Gamma(TM)$

$$\begin{aligned} d\tilde{J^\epsilon}^\theta(X) &= (s^\theta)^{-1} \circ S^\theta_X \circ (s^\theta) = (s^\theta)^{-1} \circ S_{\mathcal{R}_\theta X} \circ (s^\theta) \\ &= ((s^\theta)^{-1}s^0) \circ d\tilde{J}^\epsilon(\mathcal{R}_\theta X) \circ ((s^0)^{-1}s^\theta) \\ &= Ad^{-1}_{\alpha_\theta} \circ d\tilde{J}^\epsilon(\mathcal{R}_\theta X) = \Phi^{-1}_\theta \circ d\tilde{J}^\epsilon(\mathcal{R}_\theta X), \end{aligned}$$

where $\alpha_\theta = (s^\theta)^{-1}s^0$ is the frame change from s^0 to s^θ and $\Phi_\theta = Ad_{\alpha_\theta}$ which is parallel with respect to the Levi-Civita connection on $\mathrm{Sp}(\mathbb{R}^{2n})/U^\epsilon(p,q)$. In other words we have found an associated family. Given a nice connection D on (M, J^ϵ) theorem 2.2 shows that $\tilde{J^\epsilon}^\theta$ is ϵpluriharmonic. $\square$

If the above ϵtt^*-bundle comes from a special ϵKähler manifold we have the

Theorem 5.4 *Let $(M, J^\epsilon, g, \nabla)$ be a special ϵKähler manifold and $(TM, D = \nabla - S, S = -\frac{1}{2}\epsilon J^\epsilon \nabla J^\epsilon, g)$ the associated metric ϵtt^*-bundle, then the matrix of J^ϵ in a D^θ-flat frame $s^\theta = (s^\theta_i)$ defines an ϵpluriharmonic map $\tilde{J^\epsilon}^\theta : (M, J^\epsilon, D) \to \mathrm{Sp}(\mathbb{R}^{2n})/U^\epsilon(p,q)$. Moreover, $\tilde{J^\epsilon}^\theta$ is harmonic.*

Proof: By theorem 5.3 the map $\tilde{J^\epsilon}^\theta$ is $\mathbb{S}^1_\epsilon$-pluriharmonic. In the special ϵKähler case we know that D is the Levi-Civita connection and hence torsion-free. The ϵcomplex structure J^ϵ is integrable and so $N_{J^\epsilon} = 0$. This means, that D is nice and theorem 2.2 shows that $\tilde{J^\epsilon}^\theta$ is ϵpluriharmonic. Since S is trace-free we get from proposition 2.5 that $\tilde{J^\epsilon}^\theta$ is harmonic. $\square$

We remark, that the last result can also be obtained by observing, that the map $\tilde{J^\epsilon}^\theta$ is ϵpluriharmonic and that the manifold M is ϵKähler, as ϵpluriharmonic maps from ϵKähler manifolds are harmonic.

5.3 The ϵpluriharmonic map in the case of a special ϵKähler manifold

The results of this subsection were published in [CS1] for $\epsilon = -1$ and in [Sch3] for $\epsilon = 1$.

5.3.1 The Gauß maps of a special Kähler manifold

Let (M, J, g, ∇) be a special Kähler manifold of complex dimension $n = k + l$ and of hermitian signature (k, l), i.e. g has symmetric signature $(2k, 2l)$. Let $(\tilde{M}, J, g, \nabla)$ be its universal covering with the pullback special Kähler structure, which is again denoted by (J, g, ∇). According to Theorem 1.2, there exists a (holomorphic) Kählerian Lagrangian immersion $\phi : \tilde{M} \to V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$, which is unique up to a complex affine transformation of V with linear part in $\mathrm{Sp}(\mathbb{R}^{2n})$. We consider the *dual Gauß map* of ϕ

$$L : \tilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n}), \quad p \mapsto L(p) := T_{\phi(p)}\tilde{M} := d\phi_p T_p \tilde{M} \subset V \tag{5.3.1}$$

into the Grassmannian of complex Lagrangian subspaces $W \subset V$ of *signature (k,l)*, i.e. such that the restriction of γ to W is a hermitian form of signature (k, l). The map $L : \tilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$ is in fact the dual of the *Gauß map*

$$L^\perp : \tilde{M} \to Gr_0^{l,k}(\mathbb{C}^{2n}), \quad p \mapsto L(p)^\perp = \overline{L(p)} \cong L(p)^* . \tag{5.3.2}$$

Here $L(p)^\perp$ stands for the γ-orthogonal complement of $L(p)$ and the isomorphism $\overline{L(p)} \cong L(p)^*$ is induced by the symplectic form Ω on $V = L(p) \oplus \overline{L(p)}$.

Proposition 5.1

(i) The dual Gauß map $L : \tilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$ is holomorphic.

(ii) The Gauß map $L^\perp : \tilde{M} \to Gr_0^{l,k}(\mathbb{C}^{2n})$ is anti-holomorphic.

Proof: The holomorphicity of L follows from that of ϕ. Part (ii) follows from (i), since $L^\perp = \overline{L} : p \mapsto \overline{L(p)}$. □

The Gauß maps L and $L^\perp$ induce Gauß maps

$$L_M : M \to \Gamma \setminus Gr_0^{k,l}(\mathbb{C}^{2n}) \tag{5.3.3}$$
$$L_M^\perp : M \to \Gamma \setminus Gr_0^{l,k}(\mathbb{C}^{2n}) \tag{5.3.4}$$

into the quotient of the Grassmannian by the holonomy group $\Gamma = \mathrm{Hol}(\nabla) \subset \mathrm{Sp}(\mathbb{R}^{2n})$ of the flat symplectic connection ∇.

Corollary 5.2

(i) The dual Gauß map $L_M : M \to \Gamma \setminus Gr_0^{k,l}(\mathbb{C}^{2n})$ of M is holomorphic.

(ii) The Gauß map $L_M^\perp : M \to \Gamma \setminus Gr_0^{l,k}(\mathbb{C}^{2n})$ is anti-holomorphic.

If $\Gamma \subset \mathrm{Sp}(\mathbb{R}^{2n})$ acts properly discontinuously on $Gr_0^{k,l}(\mathbb{C}^{2n})$ then $\Gamma \setminus Gr_0^{k,l}(\mathbb{C}^{2n})$ is a locally symmetric space of pseudo-hermitian type.

5.3.2 The local expression of the dual Gauß map

We shall now describe the dual Gauß map L in local holomorphic coordinates in neighborhoods of $p_0 \in \tilde{M}$ and $L(p_0) \in Gr_0^{k,l}(\mathbb{C}^{2n})$. Applying a transformation from $\mathrm{Sp}(\mathbb{R}^{2n})$, if necessary, we can assume that $L(p_0) \in U_0$, where U_0 was defined in section 2.5.2. We put $U := L^{-1}(U_0)$. The open subset $U \subset \tilde{M}$ is a neighborhood of p_0.

Let $\phi : \tilde{M} \to T^*\mathbb{C}^n$ be the Kählerian Lagrangian immersion. It defines a system of local (special) holomorphic coordinates

$$\varphi := \pi_{(z)} \circ \phi|_U : U \xrightarrow{\sim} U' \subset \mathbb{C}^n\,, \quad p \mapsto (z^1(\phi(p)), \cdots, z^n(\phi(p))) \tag{5.3.5}$$

with $\pi_{(z)}$ as defined in section 2.5.2.
This yields the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{L} & U_0 \\ \varphi \downarrow & & \downarrow C \\ U' & \xrightarrow{L_U} & \mathrm{Sym}_{k,l}(\mathbb{C}^n)\,, \end{array} \tag{5.3.6}$$

where the vertical arrows are holomorphic diffeomorphisms and L_U at $z = (z^1, \ldots z^n)$ is given by

$$L_U(z) = (F_{ij}(z)) := \left(\frac{\partial^2 F(z)}{\partial z^i \partial z^j}\right)\,. \tag{5.3.7}$$

Here $F = F(z)$ is a holomorphic function on $U' \subset \mathbb{C}^n$, called the prepotential, determined, up to a constant, by the equations

$$w_j(\phi(p)) = \left.\frac{\partial F}{\partial z^j}\right|_{z(\phi(p))}\,. \tag{5.3.8}$$

Summarizing, we obtain the following proposition:

Proposition 5.2 *The dual Gauß map L has the following coordinate expression*

$$L_U = C \circ L \circ \varphi^{-1} = (F_{ij})\,, \tag{5.3.9}$$

where $\varphi : U \to \mathbb{C}^n$ is the (special) holomorphic chart of $\tilde{M}$ associated to the Kählerian Lagrangian immersion ϕ, see equation (5.3.5)*, and $C : U_0 \to \mathrm{Sym}(\mathbb{C}^n)$ is the holomorphic chart of $Gr_0^{k,l}(\mathbb{C}^{2n})$ constructed in equation* (2.5.10).

5.3.3 The special Kähler metric in affine coordinates

As before, let (M, J, g, ∇) be a special Kähler manifold of hermitian signature (k,l), $k + l = n = \dim_{\mathbb{C}} M$, and $(\tilde{M}, J, g, \nabla)$ its universal covering. As in chapter 4, we shall now consider the metric g in a ∇-parallel frame. Such a frame is provided by the Kählerian Lagrangian immersion $\phi : \tilde{M} \to V$. In fact, any point $p \in \tilde{M}$ has a neighborhood in which the functions $\tilde{x}^i := \mathrm{Re}\, z^i \circ \phi$, $\tilde{y}_i := \mathrm{Re}\, w_i \circ \phi$, $i = 1, \ldots, n$, form a system of local ∇-affine coordinates. We recall that the ∇-parallel Kähler form is given by $\omega = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i$.

This implies that the globally defined one-forms $\sqrt{2}d\tilde{x}^i$, $\sqrt{2}d\tilde{y}_i$ constitute a ∇-parallel unimodular frame

$$(e^a)_{a=1,\dots,2n} = (e^1,\dots,e^{2n}) := (\sqrt{2}d\tilde{x}^1,\dots,\sqrt{2}d\tilde{x}^n,\sqrt{2}d\tilde{y}_1,\dots,\sqrt{2}d\tilde{y}_n) \tag{5.3.10}$$

of $T^*\tilde{M}$ with respect to the metric volume form $\nu = (-1)^{n+1}\frac{\omega^n}{n!} = 2^n d\tilde{x}^1 \wedge \ldots \wedge d\tilde{y}_n$. The dual frame (e_a) of $T\tilde{M}$ is also ∇-parallel and unimodular. The metric defines a smooth map

$$G : \tilde{M} \to \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) = \{A \in \mathrm{Mat}(2n,\mathbb{R}) | A^t = A\,,\ \det A = 1 \text{ has signature } (2k,2l)\} \tag{5.3.11}$$

by

$$p \mapsto G(p) := (g_{ab}(p)) := (g_p(e_a,e_b))\,. \tag{5.3.12}$$

We will call $G = (g_{ab})$ *the fundamental matrix* of ϕ. As before, we identify

$$\mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) = \mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(2k,2l)\,. \tag{5.3.13}$$

This is a pseudo-Riemannian symmetric space. For conventional reasons, in this section, $\mathrm{SO}(2k,2l) \subset \mathrm{SL}(2n,\mathbb{R})$ is defined as the stabilizer of the symmetric matrix

$$E_o^{k,l} := \mathrm{diag}(\mathbb{1}_k, -\mathbb{1}_l, \mathbb{1}_k, -\mathbb{1}_l)\,. \tag{5.3.14}$$

The fundamental matrix induces a map

$$G_M : M \to \Gamma \setminus \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) \tag{5.3.15}$$

into the quotient of $\mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$ by the action of the holonomy group $\Gamma = \mathrm{Hol}(\nabla) \subset \mathrm{Sp}(\mathbb{R}^{2n}) \subset \mathrm{SL}(2n,\mathbb{R})$. The target $\Gamma \setminus \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$ is a pseudo-Riemannian locally symmetric space, provided that Γ acts properly discontinuously.

Theorem 5.5 *The fundamental matrix*

$$G : \tilde{M} \to \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) = \mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(2k,2l) \tag{5.3.16}$$

takes values in the totally geodesic submanifold

$$i : Gr_0^{k,l}(\mathbb{C}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k,l) \hookrightarrow \mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(2k,2l) \tag{5.3.17}$$

and coincides with the dual Gauß map $L : \tilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$ *in the sense that*

$$G = i \circ L.$$

Proof: The proof will follow from a geometric description of the inclusion i. To any Lagrangian subspace $W \in Gr_0^{k,l}(\mathbb{C}^{2n})$ we can associate the scalar product

$$g^W := \mathrm{Re}\,\gamma|_W$$

of signature $(2k,2l)$ on $W \subset V$. The projection onto the real points

$$\mathrm{Re} : V = T^*\mathbb{C}^n \to T^*\mathbb{R}^n = \mathbb{R}^{2n}\,, \quad v \mapsto \mathrm{Re}\,v = \frac{1}{2}(v+\overline{v}) \tag{5.3.18}$$

induces an isomorphism of real vector spaces $W \xrightarrow{\sim} \mathbb{R}^{2n}$ the inverse of which we denote by $\psi = \psi_W$.
We claim that

$$i(W) = \psi^* g^W =: (g^W_{ab}) =: G^W \,. \tag{5.3.19}$$

To check this, it is sufficient to prove that the map

$$Gr_0^{k,l}(\mathbb{C}^{2n}) \ni W \mapsto G^W \in \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) \tag{5.3.20}$$

is $\mathrm{Sp}(\mathbb{R}^{2n})$-equivariant and maps the base point W_o with stabilizer $\mathrm{U}(k,l)$, see (2.5.2), to the base point $E^{k,l}_o$ with stabilizer $\mathrm{SO}(2k,2l)$, see (5.3.14). Let us verify that indeed $G^{W_o} = E^{k,l}_o$.

Using the definition of γ, one finds that in the basis of V given by

$$(e^{\pm}_j) := \left(\frac{\partial}{\partial z^j} \pm i \frac{\partial}{\partial w_j}\right) \tag{5.3.21}$$

the only non-vanishing components of γ are $\gamma(e^{\pm}_j, e^{\pm}_j) = \pm 2$. This shows that $g^{W_o} = \mathrm{Re}\,\gamma|_{W_o}$ is represented by the matrix $2E^{k,l}_o$ with respect to the basis

$$(e^+_1, \ldots, e^+_k, e^-_1, \ldots, e^-_l, ie^+_1, \ldots, ie^+_k, ie^-_1, \ldots, ie^-_l)\,. \tag{5.3.22}$$

In order to calculate $G^{W_o} = (g^{W_o}_{ab}) = (g(\psi e_a, \psi e_b))$, we need to pass from the real basis (5.3.22) of W_o to the real basis (ψe_a).

Recall that the real structure κ is complex conjugation with respect to the coordinates (z^i, w_i). This implies that

$$\psi^{-1}(e^+_j) \;=\; \frac{\partial}{\partial x^j} = \sqrt{2} e_j\,,\ \psi^{-1}(ie^+_j) \;=\; -\frac{\partial}{\partial y_j} = -\sqrt{2} e_{n+j}\,,\ j = 1, \ldots, k, \tag{5.3.23}$$

$$\psi^{-1}(e^-_j) \;=\; \frac{\partial}{\partial x^j} = \sqrt{2} e_j\,,\ \psi^{-1}(ie^-_j) \;=\; \frac{\partial}{\partial y_j} = \sqrt{2} e_{n+j}\,,\ j = 1, \ldots, l \tag{5.3.24}$$

and shows that $G^{W_o} = E^{k,l}_o$.

It remains to check the equivariance of $W \mapsto G^W = \psi^*_W g$. Using the definition of the map $\psi = \psi_W : \mathbb{R}^{2n} \to W$, one easily checks that, under the action of $\Lambda \in \mathrm{Sp}(\mathbb{R}^{2n})$, ψ transforms as

$$\psi_{\Lambda W} = \Lambda \circ \psi_W \circ \Lambda^{-1}|_{\mathbb{R}^{2n}}\,. \tag{5.3.25}$$

From this we deduce the transformation law of G^W:

$$G^{\Lambda W} = \psi^*_{\Lambda W} g^{\Lambda W} = (\Lambda^{-1})^* \psi^*_W \Lambda^* g^{\Lambda W} = (\Lambda^{-1})^* \psi^*_W g^W = (\Lambda^{-1})^* G^W = \Lambda \cdot G^W\,. \tag{5.3.26}$$

The above claim (5.3.19), together with the fact that

$$g^{L(p)} = g_p \quad \text{and} \quad G^{L(p)} = G(p) \tag{5.3.27}$$

for all $p \in \tilde{M}$, implies that

$$i(L(p)) = G^{L(p)} = G(p)\,. \tag{5.3.28}$$

□

Corollary 5.3 *The fundamental matrix*

$$G : \tilde{M} \to \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) \tag{5.3.29}$$

is pluriharmonic.

Proof: The map $G = i \circ L$ is the composition of the holomorphic map $L : \tilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$ with the totally geodesic inclusion $Gr_0^{k,l}(\mathbb{C}^{2n}) \subset \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$. The composition of a holomorphic map with a totally geodesic map is pluriharmonic. □

5.3.4 The Gauß maps of a special para-Kähler manifold

Now we are going to introduce the Gauß maps of a special para-Kähler manifold, which are the para-complex analogue of the Gauß maps introduced in section 5.3.2 and were introduced in [Sch3].
Let (M, τ, g, ∇) be a special para-Kähler manifold of para-complex dimension n. Consequently the metric g has signature (n, n). Let $(\tilde{M}, \tau, g, \nabla)$ be the universal cover of M with the pull-back special para-Kähler structure, which we denote again by (τ, g, ∇). According to Theorem 1.3, there exists a (para-holomorphic) Kählerian Lagrangian immersion $\Phi : \tilde{M} \to V = C^{2n} = T^*C^n$, which is unique up to an affine transformation of V with linear part in $\mathrm{Aut}(V, \Omega, \bar{\cdot}) = \mathrm{Sp}(\mathbb{R}^{2n})$.
We consider the dual Gauß map of ϕ, i.e.

$$L : \tilde{M} \to \mathrm{Gr}_0^n(C^{2n}), \quad p \mapsto L(p) := T_{\phi(p)}\tilde{M} := d\phi_p T_p \tilde{M} \subset V$$

into the Grassmannian $\mathrm{Gr}_0^n(C^{2n})$ of para-complex Lagrangian subspaces $W \subset V$ of signature (n, n), i.e. $g_V = Re\,\gamma$ restricted to W has signature (n, n). The map $L : \tilde{M} \to \mathrm{Gr}_0^n(C^{2n})$ is in fact the dual of the Gauß map

$$L^\perp : \tilde{M} \to \mathrm{Gr}_0^n(C^{2n}), \quad p \mapsto L(p)^\perp = \bar{L}(p) \cong L(p)^*.$$

With $L(p)^\perp$ we mean the γ-orthogonal complement of $L(p)$ and the isomorphism $L(p)^\perp \cong L(p)^*$ is induced by the symplectic form Ω on $V = L(p) \oplus \bar{L}(p)$. The structure of a para-complex manifold on $\mathrm{Gr}_0^n(C^{2n})$ was introduced in section 2.5.3.

Proposition 5.3

(i) The dual Gauß map $L : \tilde{M} \to Gr_0^n(C^{2n})$ is para-holomorphic.

(ii) The Gauß map $L^\perp : \tilde{M} \to Gr_0^n(C^{2n})$ is anti-para-holomorphic.

Proof: The para-holomorphicity of L follows from that of ϕ and part (ii) follows from $L^\perp = \bar{L} : p \mapsto \bar{L}(p)$. □

The Gauß maps L and $L^\perp$ induce Gauß maps

$$L : M \to \Gamma \setminus \mathrm{Gr}_0^n(C^{2n}),$$
$$L^\perp : M \to \Gamma \setminus \mathrm{Gr}_0^n(C^{2n})$$

into the quotient of the Grassmannian by the holonomy group $\Gamma \subset Hol(\nabla) \subset \mathrm{Sp}(\mathbb{R}^{2n})$ of the flat symplectic connection ∇. This yields the

Corollary 5.4

(i) The dual Gauß map $L_M : M \to \Gamma \setminus Gr_0^n(C^{2n})$ *is para-holomorphic.*

(ii) The Gauß map $L_M^{\perp} : M \to \Gamma \setminus Gr_0^n(C^{2n})$ *is anti-para-holomorphic.*

If $\Gamma \subset \mathrm{Sp}(\mathbb{R}^{2n})$ acts properly discontinuously on $Gr_0^n(C^{2n})$ then $\Gamma \setminus Gr_0^n(C^{2n})$ is a locally symmetric space and a para-Kähler manifold.

5.3.5 The local expression of the dual Gauß map

We now describe the dual Gauß map L in local para-holomorphic coordinates of $p_0 \in \tilde{M}$ and $L(p_0) \in \mathrm{Gr}_0^n(C^{2n})$. Utilizing a transformation of $Sp(\mathbb{R}^{2n})$, if necessary, we can assume $L(p_0) \in U_0$. For the definition of U_0 we refer to section 2.5.3. We put $U := L^{-1}(U_0)$. The set $U \subset \tilde{M}$ is an open neighborhood of p_0.
Let $\phi : \tilde{M} \to T^*C^n$ be the para-Kählerian Lagrangian immersion. It defines a system of local (special) para-holomorphic coordinates

$$\varphi := \pi_{(z)} \circ \phi_{|U} \; : \; U \tilde{\to} U' \subset C^n, \quad p \mapsto (z^1(\phi(p)), \ldots, z^n(\phi(p)), \tag{5.3.30}$$

where $\pi_{(z)}$ was introduced in section 2.5.3.
This means that we have the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{L} & U_0 \\ \varphi \downarrow & & \downarrow C \\ U' & \xrightarrow{L_U} & \mathrm{Sym}_{n,n}(C^n)\,, \end{array} \tag{5.3.31}$$

where the vertical arrows are para-holomorphic diffeomorphisms and L_U at $z = (z^1, \ldots, z^n)$ is given by

$$L_U(z) = (F_{ij}(z)) := \left(\frac{\partial^2 F(z)}{\partial z^i \partial z^j} \right). \tag{5.3.32}$$

Here $F(z)$ is a para-holomorphic function on $U' \subset C^n$, called prepotential (see [CMMS]), which is up to a constant determined by the equations

$$w_j((\phi(p)) = \left. \frac{\partial F}{\partial z^j} \right|_{z(\phi(p))}. \tag{5.3.33}$$

Summarizing, we obtain the proposition:

Proposition 5.4 *The dual Gauß map L has the following coordinate expression*

$$L_U = C \circ L \circ \varphi^{-1} = (F_{ij})\,, \tag{5.3.34}$$

where $\varphi : U \to C^n$ *is the (special) para-holomorphic chart of* $\tilde{M}$ *associated to the para-Kählerian Lagrangian immersion* ϕ*, see equation* (5.3.30)*, and* $C : U_0 \to Sym_{n,n}(C^n)$ *is the para-holomorphic chart of* $Gr_0^n(C^{2n})$ *constructed in equation* (2.5.18)*.*

5.3.6 The special para-Kähler metric in an affine frame

In this section we show that the para-pluriharmonic map associated to a para-Kähler manifold coincides with the dual Gauß map.
As above, let (M, τ, g, ∇) be a special para-Kähler manifold of dimension $n = dim_C M$ and $(\tilde{M}, \tau, g, \nabla)$ be its universal covering. Like in chapter 4 we now consider the metric g in a ∇-parallel frame. Such a frame is provided by the para-Kählerian Lagrangian immersion $\phi : \tilde{M} \to V$. In fact, an arbitrary point $p \in \tilde{M}$ has a neighborhood in which the functions $\tilde{x}^i := \mathrm{Re}\, z^i \circ \phi$ and $\tilde{y}_i := \mathrm{Re}\, w_i \circ \phi$, $i = 1, \ldots, n$ form a system of local ∇-affine coordinates. We recall that the ∇-parallel Kähler form is given by $\omega = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i$. Therefore the globally defined one-forms $\sqrt{2} d\tilde{x}^i$ and $\sqrt{2} d\tilde{y}_i$ constitute a ∇-parallel unimodular frame

$$(e^a)_{a=1,\ldots,2n} := (\sqrt{2} d\tilde{x}^1, \ldots, \sqrt{2} d\tilde{x}^n, \sqrt{2} d\tilde{y}_1, \ldots, \sqrt{2} d\tilde{y}_n) \tag{5.3.35}$$

of $T^*\tilde{M}$ with respect to the metric volume form $\nu = (-1)^{n+1}\omega^n/n! = 2^n d\tilde{x}^1 \wedge \ldots \wedge d\tilde{y}_n$. The dual frame e_a of $T\tilde{M}$ is also ∇-parallel and unimodular. The metric g defines a smooth map

$$G : \tilde{M} \to \mathrm{Sym}^1_{n,n}(\mathbb{R}^{2n}) = \{A \in \mathrm{Mat}(2n, \mathbb{R}) | A^t = A,\ \det(A) = (-1)^n \text{ of signature } (n,n)\}$$

by

$$p \mapsto G(p) = (g_{ab}(p)) := (g_p(e_a, e_b)). \tag{5.3.36}$$

We call $G = (g_{ab})$ the $\mathsf{fundamental\ matrix}$ of ϕ. As before, we have the identification

$$\mathrm{Sym}^1_{n,n}(\mathbb{R}^{2n}) = SL(2n, \mathbb{R})/SO(n,n)$$

of $\mathrm{Sym}^1_{n,n}(\mathbb{R}^{2n})$ with a pseudo-Riemannian symmetric space.
The group $SO(n,n) \subset SL(2n, \mathbb{R})$ is in this section considered as the stabilizer of the symmetric matrix

$$E^n_0 = \mathrm{diag}(-\mathbb{1}_n, \mathbb{1}_n). \tag{5.3.37}$$

The fundamental matrix induces a map

$$G_M : M \to \Gamma \setminus \mathrm{Sym}^1_{n,n}(\mathbb{R}^{2n})$$

into the quotient of $\mathrm{Sym}^1_{n,n}(\mathbb{R}^{2n})$ by the action of the holonomy group $\Gamma = Hol(\nabla) \subset \mathrm{Sp}(\mathbb{R}^{2n}) \subset SL(2n, \mathbb{R})$. The target $\Gamma \setminus \mathrm{Sym}^1_{n,n}(\mathbb{R}^{2n})$ is a pseudo-Riemannian locally symmetric space, provided that Γ acts properly discontinuously.

Theorem 5.6 *The fundamental matrix*

$$G : \tilde{M} \to \ Sym^1_{n,n}(\mathbb{R}^{2n}) = SL(2n, \mathbb{R})/SO(n,n) \tag{5.3.38}$$

takes values in the totally geodesic submanifold

$$i : Gr^n_0(C^{2n}) = Sp(\mathbb{R}^{2n})/U^\pi(C^n) \hookrightarrow SL(2n, \mathbb{R})/SO(n,n)$$

and coincides with the dual Gauß map $L : \tilde{M} \to Gr^n_0(C^{2n})$ in the sense that

$$G = i \circ L.$$

Proof: The proof follows from a geometric interpretation of the inclusion i. To any Lagrangian subspace $W \in \mathrm{Gr}_0^n(C^{2n})$ we associate the scalar product

$$g^W := \mathrm{Re}\,\gamma_{|W}$$

of signature (n,n) on $W \subset V$. The projection onto the real points

$$\mathrm{Re} : V = T^*C^n \mapsto T^*\mathbb{R}^n = \mathbb{R}^{2n}, v \mapsto \mathrm{Re}\,v = \frac{1}{2}(v + \bar{v}) \tag{5.3.39}$$

induces an isomorphism of real vector spaces $W \tilde{\rightarrow} \mathbb{R}^{2n}$ with inverse $\psi = \psi_W$.
We claim that

$$i(W) = \psi_W^* g =: (g_{ab}^W) =: G^W. \tag{5.3.40}$$

To check the claim, we have to show the $\mathrm{Sp}(\mathbb{R}^{2n})$-equivariance of the map

$$\mathrm{Gr}_0^n(C^{2n}) \ni W \mapsto G^W \in \mathrm{Sym}_{n,n}^1(\mathbb{R}^{2n})$$

and that it maps the base point W_o, see equation (2.5.3), to E_0^n (equation (5.3.37)).
By the definition of γ we find for the basis

$$e_j^{\pm} := \frac{\partial}{\partial z^j} \pm e \frac{\partial}{\partial w_j} \tag{5.3.41}$$

of V that the only non-vanishing components of γ are $\gamma(e_j^{\pm}, e_j^{\pm}) = \mp 2$. This shows that g^{W_o} is represented by the matrix $2E_0^n$ with respect to the real basis

$$(e_1^+, \ldots, e_n^+, e\, e_1^+, \ldots,\, e\, e_n^+). \tag{5.3.42}$$

In order to calculate $G^{W_o} = (g_{ab}^{W_o}) = (g(\psi e_a, \psi e_b))$, we need to pass from the real basis (5.3.42) to the real basis (ψe_a) of W_o.
Recall that the real structure is the para-complex conjugation with respect to the coordinates (z^i, w_i). This implies that

$$\psi^{-1}(e_j^+) = \frac{\partial}{\partial x^j} = \sqrt{2} e_j, \qquad \psi^{-1}(e\, e_j^+) = \frac{\partial}{\partial y^j} = \sqrt{2} e_{n+j},\ j = 1, \ldots, n, \tag{5.3.43}$$

$$\psi^{-1}(e_j^-) = \frac{\partial}{\partial x^j} = \sqrt{2} e_j, \qquad \psi^{-1}(e\, e_j^-) = -\frac{\partial}{\partial y^j} = -\sqrt{2} e_{n+j},\ j = 1, \ldots, n. \tag{5.3.44}$$

This shows that $G^{W_o} = E_0^n$.
It remains to show the equivariance of $W \mapsto G^W = \psi_W^* g$. Using the definition of the map $\psi = \psi^W : \mathbb{R}^{2n} \to W$, one easily checks that, under the action of $\Lambda \in \mathrm{Sp}(\mathbb{R}^{2n})$, ψ transforms as

$$\psi_{\Lambda W} = \Lambda \circ \Psi_W \circ \Lambda^{-1}_{|\mathbb{R}^{2n}}. \tag{5.3.45}$$

This implies the transformation law for G^W:

$$G^{\Lambda W} = \psi_{\Lambda W}^* g^{\Lambda W} = (\Lambda^{-1})^* \psi_W^* \Lambda^* g^{\Lambda W} = (\Lambda^{-1})^* \psi_W^* g^W = (\Lambda^{-1})^* G^W = \Lambda \cdot G^W. \tag{5.3.46}$$

The above claim (5.3.40) and the fact

$$g^{L(p)} = g_p \text{ and } G^{L(p)} = G(p) \tag{5.3.47}$$

for all $p \in \tilde{M}$ imply

$$i(L(p)) = G^{L(p)} = G(p). \tag{5.3.48}$$

□

Corollary 5.5 *The fundamental matrix $G : \tilde{M} \to Sym^1_{n,n}(\mathbb{R}^{2n})$ is para-pluriharmonic.*

Proof: In fact, $G = i \circ L$ is the composition of a para-holomorphic map $L : \tilde{M} \to \mathrm{Gr}^n_0(C^{2n})$ with the totally geodesic inclusion $\mathrm{Gr}^n_0(C^{2n}) \subset \mathrm{Sym}^1_{n,n}(\mathbb{R}^{2n})$. The composition of a para-holomorphic map with a totally geodesic one is para-pluriharmonic. □

5.4 Variations of ϵHodge structures

5.4.1 The period map of a variation of ϵHodge structures

Like period domains describe ϵHodge structures, ϵholomorphic maps into period domains describe variations of ϵHodge structures, in the sense of the following proposition which is in the complex case due to Griffiths. We only consider the simply connected case:

Proposition 5.5 *Let (M, J^ϵ) be a simply connected ϵcomplex manifold and G/V the period domain classifying polarized ϵHodge structures of given weight and ϵHodge numbers, then giving a variation of ϵHodge structures is equivalent to giving an ϵholomorphic map from M to G/V which satisfies the Griffiths transversality condition. Such maps are called* period maps.

The following result is known for strongly polarized complex variations of Hodge structures and will be generalized for variations of ϵHodge structures of odd weight later in this work.

Theorem 5.7 *(cf. [CMP] Theorem 14.4.1) Let $f : M \to G/V$ be a period mapping and $\pi : G/V \to G/K$, as defined in section 2.7, the canonical map to the associated locally symmetric space. The $\pi \circ f$ is pluriharmonic.*

5.4.2 The period map of a variation of ϵHodge structures from the viewpoint of ϵtt^*-geometry

Let (E, ∇, F^p) be a variation of ϵHodge structures of odd weight w over the ϵcomplex base manifold (M, J^ϵ) endowed with a polarization b where E has rank r and where $f_p = \dim F_p$. Denote by (E, D, S, g) the corresponding ϵtt^*-bundle constructed in proposition 3.8. We suppose, that M is simply connected.

Like in chapter 4 we examine the metric g in a $D^0 = \nabla$-parallel frame s of E. The metric g defines a smooth map

$$G : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r) = \{A \in \mathrm{Mat}(\mathbb{R}^r) \,|\, A = A^t \text{ and } A \text{ has signature } (p, q)\}. \quad (5.4.1)$$

In the complex case $(p, q) = (2k, 2l)$ is the symmetric signature of g. We remark that for a variation of para-Hodge structures the metric g is forced to have split signature $(p, q) = (n, n)$ with $n = \frac{1}{2} \dim_{\mathbb{R}} H$.

The map G will be called the **fundamental matrix** of the variation of ϵHodge structures (E, ∇, F^p) and as above $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ is identified with the pseudo-Riemannian symmetric space $GL(r, \mathbb{R})/O(p, q)$.

We recall that for odd weight each fibre of E has the structure of a symplectic vector space and consequentely it holds $\mathrm{rk}_{\mathbb{R}} E = r = 2n \in 2\mathbb{N}$.

Theorem 5.8 *Let (E, ∇, F^p) be a polarized variation of ϵHodge structures of odd weight w with polarization b over the ϵcomplex base manifold (M, J^ϵ). Let $r = 2n$ be the real rank of E.*
Then the fundamental matrix G takes values in the totally geodesic submanifold

$$i \ : \ Gr_0^{k,l}(\mathbb{C}^{2n}) = Sp(\mathbb{R}^{2n})/U(k,l) \to GL(r,\mathbb{R})/O(2k,2l), \ \text{for } \epsilon = -1, \quad (5.4.2)$$
$$i \ : \ Gr_0^{n}(C^{2n}) = Sp(\mathbb{R}^{2n})/U^{\pi}(C^n) \to GL(r,\mathbb{R})/O(n,n), \ \text{for } \epsilon = 1 \quad (5.4.3)$$

and coincides with the map L, i.e. $G = i \circ L : M \to GL(r, \mathbb{R})/O(p, q)$.

Proof: Given a point $x \in M$ we put $V = H_x^{\mathbb{C}_\epsilon}$ and $V^{\mathbb{R}} = H_x \cong \mathbb{R}^r$. To any polarized ϵHodge structure F^p of odd weight w with polarization b the map L associated a Lagrangian subspace $\mathfrak{L} \in \mathrm{Gr}_0^{k,l}(V)$ in the complex and a Lagrangian subspace $\mathfrak{L} \in \mathrm{Gr}_0^{n}(V)$ in the para-complex case (see section 2.7). We define a scalar product $g^{\mathfrak{L}} = \mathrm{Re}\, h|_{\mathfrak{L}}$ on $\mathfrak{L} \subset V$. The projection onto the real points

$$\mathrm{Re} \, : \, V \to V^{\mathbb{R}} \quad (5.4.4)$$

induces an isomorphism $\mathfrak{L} \cong V^{\mathbb{R}}$. Its inverse we call $\Phi = \Phi_{\mathfrak{L}} : V^{\mathbb{R}} \to \mathfrak{L}$.
We are going to prove

$$i(\mathfrak{L}) = \Phi_{\mathfrak{L}}^* \, g^{\mathfrak{L}} =: G^{\mathfrak{L}}. \quad (5.4.5)$$

We first show the $Sp(\mathbb{R}^r)$ equivariance of the map

$$\mathfrak{L} \mapsto G^{\mathfrak{L}}. \quad (5.4.6)$$

From the definition of $\Phi_{\mathfrak{L}}$ we obtain with $\Lambda \in Sp(\mathbb{R}^r)$:

$$\Phi_{\Lambda\mathfrak{L}} = \Lambda \circ \Phi_{\mathfrak{L}} \circ \Lambda^{-1}_{|\mathbb{R}^r} \quad (5.4.7)$$

and from this the transformation law of $G^{\mathfrak{L}}$

$$G^{\Lambda\mathfrak{L}} = \Phi_{\Lambda\mathfrak{L}}^* g^{\Lambda\mathfrak{L}} = (\Lambda^{-1})^* \Phi_{\mathfrak{L}}^* \Lambda^* g^{\Lambda\mathfrak{L}} = (\Lambda^{-1})^* \Phi_{\mathfrak{L}}^* g^{\mathfrak{L}} = (\Lambda^{-1})^* G^{\mathfrak{L}} = \Lambda \cdot G^{\mathfrak{L}}. \quad (5.4.8)$$

Let F_o^p be the reference flag of $V_o^{\mathbb{C}_\epsilon} = H_o^{\mathbb{C}_\epsilon}$ with $\dim F_o^p = f_p$. We calculate $G^{\mathfrak{L}_o}$ in the basis $\{f_o^i\}_{i=1}^{\dim(\mathfrak{L}_o)}$ constructed in equation (2.7.2)

$$(G^{\mathfrak{L}_0}(\mathrm{Re} f_i, \mathrm{Re} f_j)) = \mathbb{1}_{p,q}, \text{after permutation.} \quad (5.4.9)$$

This yields

$$\Phi_{\mathfrak{L}_0}^* g^{\mathfrak{L}_0} = \mathbb{1}_{p,q}. \quad (5.4.10)$$

The proof is finished, since $G(x) = G^{L(x)} = i(L(x))$. □

Corollary 5.6 *Let (E, ∇, F^p) be a polarized variation of ϵHodge structures of odd weight w with polarization b over the ϵcomplex base manifold (M, J^ϵ). Then the map $L : M \to Gr_0^{k,l}(\mathbb{C}_\epsilon^r) = Sp(\mathbb{R}^r)/U^\epsilon(k,l)$ is ϵpluriharmonic.*

Proof: This follows from the ϵpluriharmonicity of the fundamental matrix $G : M \to \mathrm{GL}(r,\mathbb{R})/O(p,q)$, since $G = i \circ L$, where i is a totally geodesic immersion and consequentely, by corollary 2.1, the ϵpluriharmonicity of L is equivalent to that of G. □

The last theorem and the last corollary can be specialized for variations of Hodge structures (This means $\epsilon = -1$.), which are strongly polarized:

Theorem 5.9 *Let (E, ∇, F^p) be a strongly polarized variation of Hodge structures of odd weight w with polarization b over the complex base manifold (M, J). Then the fundamental matrix G takes values in the totally geodesic submanifold*

$$i \,:\, Gr_0(\mathbb{C}^r) = Gr_0^{r,0}(\mathbb{C}^r) = Sp(\mathbb{R}^r)/U(r) \to GL(r,\mathbb{R})/O(r) \tag{5.4.11}$$

and coincides with the map $L = \pi \circ \mathcal{P} \,:\, M \to G/K$, i.e. $G = i \circ L : M \to GL(r,\mathbb{R})/O(r)$.

With the same argument as before, we obtain the

Corollary 5.7 *Let (E, ∇, F^p) be a strongly polarized variation of Hodge structures of odd weight w with polarization b over the complex base manifold (M, J). Then the map $L : M \to Gr_0(\mathbb{C}^r) = Gr_0^{r,0}(\mathbb{C}^r) = Sp(\mathbb{R}^r)/U(r)$ is pluriharmonic.*

5.5 ϵHarmonic bundles

The complex version of this chapter was published in [Sch4].
Collecting our knowledge from the previous chapters we obtain the corollary:

Corollary 5.8 *Let $(E \to M, D, C, \bar{C}, h)$ be an ϵharmonic bundle of ϵcomplex rank r over the simply connected ϵcomplex manifold (M, J^ϵ), then the representation of $g = Re\, h$ in a $D^{(\lambda)}$-flat frame defines an ϵpluriharmonic map $\Phi_g : M \to S^\epsilon(2p, 2q)$ where we define*

$$S^\epsilon(2p,2q) := \begin{cases} S(2p,2q) \text{ for } \epsilon = -1, \\ S(r,r) \text{ for } \epsilon = 1, \end{cases}$$

where (p,q) with $r = p + q$ is the hermitian signature of h for $\epsilon = -1$.

Proof: By theorem 3.6 the ϵharmonic bundle $(E \to M, D, C, \bar{C}, h)$ induces a metric ϵtt^*-bundle $(E, D, S, g = \mathrm{Re}\, h)$ with $S_X := C_Z + \bar{C}_{\bar{Z}}$ for $X = Z + \bar{Z} \in TM$ and $Z \in T^{1,0}M$. The identity (3.4.1), i.e. $D_X^{(\lambda)} = D_X^\alpha$ for $\lambda = \cos_\epsilon(\alpha) + \hat{\imath}\sin_\epsilon(\alpha) \in \mathbb{S}_\epsilon^1$ and theorem 4.1 prove the corollary. □

With our considerations about ϵpluriharmonic maps we are going to show the next theorem. First we introduce a notion:

$$\mathrm{Herm}^{\epsilon}_{p,q}(\mathbb{C}^r_\epsilon) := \begin{cases} \mathrm{Herm}_{p,q}(\mathbb{C}^r), & \text{for } \epsilon = -1, \\ \mathrm{Herm}(C^r), & \text{for } \epsilon = 1. \end{cases}$$

Theorem 5.10 *Let $(E \to M, D, C, \bar{C}, h)$ be an ϵharmonic bundle over the simply connected ϵcomplex manifold (M, J^ϵ). Then the representation of h in a $D^{(\lambda)}$-flat frame defines an ϵpluriharmonic map $\phi_h : M \to Herm^{\epsilon}_{p,q}(\mathbb{C}^r_\epsilon)$, which itself induces an admissible ϵpluriharmonic map $\tilde{\phi}_h : M \to H^\epsilon(p,q)$ (The space $H^\epsilon(p,q)$ was defined in equation (2.4.16).).*

Proof: The ϵpluriharmonicity of the map ϕ_h follows from corollary 5.8 and propositions 2.11 and 2.14. For the second part we observe, that the differential of $\mathcal{R} : \mathfrak{gl}_r(\mathbb{C}_\epsilon) \to \mathfrak{gl}_{2r}(\mathbb{R})$ is a homomorphism of Lie-algebras and therefore preserves the vanishing of the Lie-bracket. $\square$

The following theorem gives the converse statement:

Theorem 5.11 *Let (M, J^ϵ) be a simply connected ϵcomplex manifold and $E = M \times \mathbb{C}^r_\epsilon$. An admissible ϵpluriharmonic map $\tilde{\phi}_h : M \to H^\epsilon(p,q)$ induces an admissible ϵpluriharmonic map $\tilde{\phi}_g = [i] \circ \tilde{\phi}_h : M \to S^\epsilon(2p, 2q)$ and an ϵharmonic bundle $(E, D = \partial - \epsilon(C + \bar{C}), C = \epsilon(d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{\mathbb{C}^r_\epsilon})$, where ∂ is the ϵcomplex linear extension on $TM^{\mathbb{C}_\epsilon}$ of the flat connection on $E = M \times \mathbb{C}^r_\epsilon$.*
If $M = \Sigma$ is an ϵRiemannian surface, then every ϵpluriharmonic map $\tilde{\phi}_h$ is admissible.
If (M, J) is a complex manifold and the signature is $(r, 0)$ or $(0, r)$, then every pluriharmonic map $\tilde{\phi}_h$ is admissible.

Proof: Due to proposition 2.11 and 2.14 the map $\tilde{\phi}_g$ is ϵpluriharmonic. Hence one obtains from theorem 4.2 an ϵtt^*-bundle $(E = M \times \mathbb{R}^{2r}, D = \partial - \epsilon S, S = \epsilon d\tilde{\phi}_g, g = < \phi_g \cdot, \cdot >_{\mathbb{R}^{2r}})$, since the condition on $d\tilde{\phi}_g|_x$ is obtained as in theorem 5.10. We are now going to use the additional information, we have from the fact, that the map ϕ_g comes from ϕ_h, to show that $(E, D = \partial - \epsilon(C + \bar{C}), C = \epsilon(d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{\mathbb{C}^r})$ is an ϵharmonic bundle.
The ϵhermitian sesquilinear metric h is given by

$$h = g + \hat{i}\,\omega$$

with $\omega = g(j^\epsilon \cdot, \cdot)$. This is the standard relation between ϵhermitian metrics on ϵcomplex vector spaces and the ϵhermitian metrics on the underlying real vector spaces.
We observe $Dj^\epsilon = [\partial - \epsilon S, j^\epsilon] = -\epsilon[S_X, j^\epsilon] = 0$, because S is is the derivation of a map from M to $GL(r, \mathbb{C}_\epsilon)$ and hence commutes with j^ϵ. Therefore $D\omega = 0$ follows from $Dg = 0$ and $Dh = 0$ from $D\omega = 0$ and $Dg = 0$.
From the definition of S and S_{J^ϵ} in theorem 3.6, i.e.

$$S_X = C_Z + \bar{C}_{\bar{Z}},$$

$$S_{J^\epsilon X} = C_{J^\epsilon Z} + \bar{C}_{J^\epsilon \bar{Z}}$$

for $X = Z + \bar{Z}$ and $Z \in T^{1,0}M$ we obtain the definition of

$$2C_Z = S_X + \epsilon j^\epsilon S_{J^\epsilon X},$$

$$2\bar{C}_{\bar{Z}} = S_X - \epsilon j^\epsilon S_{J^\epsilon X}.$$

In addition we have the identity $D^{(\lambda)}_X = D^\alpha_X$ for $\lambda = \cos_\epsilon(\alpha) + \hat{i}\sin_\epsilon(\alpha) \in \mathbb{S}^1_\epsilon$ which again gives the equivalence between the flatness of $D^{(\lambda)}$ and D^α.
It remains to show

$$h(C_Z \cdot, \cdot) = h(\cdot, \bar{C}_{\bar{Z}} \cdot).$$

We recall the relations $j^{\epsilon *} g = -\epsilon g$ and $(*)\, g(j^\epsilon \cdot, \cdot) = -g(\cdot, j^\epsilon \cdot)$, which implies the anti-symmetry of $\omega = g(j^\epsilon \cdot, \cdot)$ and $(*')\, \omega(j^\epsilon \cdot, \cdot) = -\omega(\cdot, j^\epsilon \cdot)$. Further we use the identities $(**)\, [S, j^\epsilon] = [S_{J^\epsilon}, j^\epsilon] = 0$ and that $(***)\, S, S_{J^\epsilon}$ are g-symmetric. Due to $(**)$ and $(***)$ we get $(****)\, S, S_{J^\epsilon}$ ω-symmetric. These identities imply

$$\begin{aligned}
2h(C_Z \cdot, \cdot) &= g(S_X + \epsilon j^\epsilon S_{JX} \cdot, \cdot) + \hat{i}\omega(S_X + \epsilon j^\epsilon S_{J^\epsilon X} \cdot, \cdot) \\
&\overset{(*),(**),(***)}{=} g(\cdot, S_X - \epsilon j^\epsilon S_{J^\epsilon X} \cdot) + \hat{i}\omega(S_X + \epsilon j^\epsilon S_{J^\epsilon X} \cdot, \cdot) \\
&\overset{(*'),(**),(****)}{=} g(\cdot, S_X - \epsilon j^\epsilon S_{J^\epsilon X} \cdot) + \hat{i}\omega(\cdot, S_X - \epsilon j^\epsilon S_{J^\epsilon X} \cdot) \\
&= 2h(\cdot, \bar{C}_{\bar{Z}} \cdot).
\end{aligned}$$

Using $S = \epsilon d\tilde{\phi}_g = \epsilon d([i] \circ \tilde{\phi}_h) = \epsilon d\tilde{\phi}_h$ we find extending S on $TM^{\mathbb{C}_\epsilon}$ to $S^{\mathbb{C}_\epsilon}$ for $Z \in T^{1,0}M$ the equations $C_Z = S^{\mathbb{C}_\epsilon}_Z = \epsilon d\tilde{\phi}_h(Z)$ and $\bar{C}_{\bar{Z}} = \epsilon d\tilde{\phi}_h(\bar{Z})$. □

In [Sim] section 1 Simpson studied Higgs-bundles with harmonic positive definite metrics, i.e. harmonic bundles, over a compact Kähler-manifold M^n and related these to harmonic maps from M in $GL(n,\mathbb{C})/U(n)$. From his results one can find, that a given flat bundle with a harmonic metric induces a harmonic map from M in $GL(n,\mathbb{C})/U(n)$. Conversely, a harmonic map from M in $GL(n,\mathbb{C})/U(n)$ and a flat bundle give rise to a harmonic bundle. From Sampson's theorem [Sam] one obtains, that in the above case the notion of harmonic and pluriharmonic coincide.
Simpson's result follows from the theorems 5.10 and 5.11, since the condition on the differential of $\tilde{\phi}_h$ is satisfied in the case of signature $(r, 0)$ and $(0, r)$. We remark, that the theorems 5.10 and 5.11 are in fact more general, since the compactness of M and the Kähler condition are not needed. Simpson uses Kähler-identities for vector bundles over compact Kähler manifolds in his proof. Further he needs the compactness, since he uses arguments from harmonic map theory, which are developped from Sius Bochner formula for harmonic maps to obtain the vanishing of the object which he calls pseudo-curvature and which is the integrability constraint for a flat bundle to define a Higgs bundle. Dubrovin's work [D] and this thesis deal with pluriharmonic maps. The results are proven by direct calculations using the pluriharmonic and the tt^*-equations, respectively. In the case of signature $(r, 0)$ and $(0, r)$ we needed only the second statement of Sampson's theorem [Sam] and therefore compactness is not needed.

The next theorem gives a rigidity result for harmonic bundles:

Theorem 5.12 *Let (M, J) be a compact Kähler manifold of dimension n with finite fundamental group $\pi_1(M)$ (i.e., the universal cover of M is compact). Let $(E \to M, D, C, \bar{C}, h)$ be a harmonic bundle over (M, J) with positive definite hermitian metric h. Then $(E \to M, D, C, \bar{C}, h)$ is trivial, i.e. $C = \bar{C} = 0$, $D^{(\lambda)} = D$ for all $\lambda \in \mathbb{S}^1$, D is flat and h is $D^{(\lambda)}$-parallel.*

Proof: Pulling back all structures to the universal cover of M we suppose that M is simply connected. $C = \bar{C} = 0$ if and only if the same holds for its pull back.
Let s be a $D^{(1)}$-flat frame of E. The associated pluriharmonic map $\tilde{f} : M \to GL(r, \mathbb{C})/U(r)$ obtained from theorem 5.10 is constant by corollary 2.2. We consider again the representation H^s of h in the frame s to compute the representations C^s and $\bar{C}^s$ of C and $\bar{C}$ in the frame s for $Z \in \Gamma(T^{1,0}M)$:

$$\begin{aligned} Z(H^s) &= h(D_Z s, s) + h(s, D_{\bar{Z}} s) \\ &= -h(C_Z s, s) - h(s, \bar{C}_{\bar{Z}} s) \\ &= -2h(C_Z s, s) = -2H^s \cdot C^s_Z, \\ \bar{Z}(H^s) &= h(D_{\bar{Z}} s, s) + h(s, D_Z s) \\ &= -h(\bar{C}_{\bar{Z}} s, s) - h(s, C_Z s) \\ &= -2h(\bar{C}_{\bar{Z}} s, s) = -2H^s \cdot \bar{C}^s_{\bar{Z}}. \end{aligned}$$

This yields $C^s = \bar{C}^s = 0$. It follows $C = \bar{C} = 0$ and $D^{(\lambda)} = D$. $\square$

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In this work we introduce the real differential geometric notion of a tt^*-bundle (E, D, S), a metric tt^*-bundle (E, D, S, g) and a symplectic tt^*-bundle (E, D, S, ω) on an abstract vector bundle E over an almost complex manifold (M, J). With this notion we construct, generalizing Dubrovin [D], a correspondence between metric tt^*-bundles over complex manifolds (M, J) and *admissible* pluriharmonic maps from (M, J) into the pseudo-Riemannian symmetric space $GL(r, \mathbb{R})/O(p, q)$ where (p, q) is the signature of the metric g. Moreover, we show a rigidity result for tt^*-bundles over compact Kähler manifolds and we obtain as application a special case of Lu's theorem.
In addition we study solutions of tt^*-bundles (TM, D, S) on the tangent bundle TM of (M, J) and characterize an interesting class of these solutions which contains special complex manifolds and flat nearly Kähler manifolds. We analyze which elements of this class admit metric or symplectic tt^*-bundles. Further we consider solutions coming from varitations of Hodge structures (VHS) and harmonic bundles.
Applying our correspondence to harmonic bundles we generalize a correspondence given by Simpson. Analyzing the associated pluriharmonic maps we obtain roughly speaking for special Kähler manifolds the dual Gauß map and for VHS of odd weight the period map. In the case of non-integrable complex structures, we need to generalize the notions of pluriharmonic maps and some results.
Apart from the rigidity result we generalize all above results to para-complex geometry.

Dans cette thèse nous introduisons la notion de fibré tt^* (E, D, S), de fibré tt^* métrique (E, D, S, g) et de fibré tt^* symplectique (E, D, S, ω) sur un fibré vectoriel E au-dessus d'une variété complexe, dans le langage de la géométrie différentielle réelle. Grâce à cette notion on obtient une correspondance entre des fibrés tt^* métriques et des applications pluriharmoniques *admissibles* de (M, J) dans l'espace symétrique pseudo-Riemannien $GL(r, \mathbb{R})/O(p, q)$, avec (p, q) la signature de la métrique g. En utilisant ce résultat on obtient dans le cas où M est compact Kählérienne, un résultat de rigidité, puis un cas particulier du théorème de Lu.
De plus nous étudions des fibrés tt^* sur le fibré tangent TM et caractérisons une classe spéciale qui contient les variétés spéciales complexes et les variétés nearly Kählériennes plates, et la sous-classe qui admet un fibré tt^* métrique ou symplectique. En outre on analyse les fibrés tt^* qui proviennent de variations de structures de Hodge (VHS) et de fibrés harmoniques. Pour les fibrés harmoniques, la correspondance permet de généraliser un résultat de Simpson. L'application pluriharmonique associée à une variété spécialement Kählérienne est reliée à l'application de Gauß duale, et celle associée à une VHS de poids impair est l'application de périodes. Si la structure complexe n'est pas intégrable, on doit généraliser la notion de pluriharmonicité.
Hors la rigidité ces résultats sont généralisés au cas para-complexe.

Discipline: Mathématiques
Mots clés: géométrie tt*, applications pluriharmoniques, fibrés harmoniques, géométrie spéciale complexe et Kählérienne, nearly Kählériennes, espaces symmétriques pseudo-Riemannienes

Institut Élie Cartan Nancy,
Laboratoire de Mathématiques,
B.P. 23, 54506 Vandœuvre-lès-Nancy

Mathematisches Insitut der Universiät Bonn,
Beringstraße 1,
D-53115 Bonn