Stochastic calculus in Riemannian polyhedra and martingales in metric spaces

Dissertation

zur Erlangung des Doktorgrades (Dr. rer. nat.)
der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von Tom Christiansen
aus Flensburg

Bonn, August 2006

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Referent: Prof. Dr. Karl-Theodor Sturm
2. Referent: Prof. Dr. Jean Picard

Tag der Promotion: 16. Februar 2007

Diese Dissertation ist auf dem Hochschulschriftenserver der ULB Bonn
http://hss.ulb.uni-bonn.de/diss_online
elektronisch publiziert

Erscheinungsjahr: 2007

Für meine Eltern

Abstract

The classical stochastic calculus of semimartingales is generalized to semimartingales in polyhedra. The main tool is a local Itô formula for piecewise smooth functions which is given in terms of so-called directional local times. As an example, Brownian motion on a Riemannian polyhedron is constructed and shown to be a semimartingale.

In the case of Euclidean polyhedra, the notion of a martingale is discussed, including a kind of Darling's characterization. In a Euclidean polyhedron of nonpositive curvature, this is shown to be also equivalent to the notion of a strong martingale. The latter is based on the concept of iterated nonlinear conditional expectations and leads to a rich theory of strong martingales in general metric spaces of nonpositive curvature. As an application, a broad characterization of harmonic maps is presented.

Contents

Introduction

The class of Riemannian polyhedra provides a lot of interesting examples in geometry. For instance, they appear naturally as limits of Riemannian manifolds or in the theory of Bruhat-Tits buildings.
Riemannian polyhedra are extremely useful for generalizing concepts of Riemannian geometry towards singular spaces. On one hand, they are sufficiently regular (at least on a considerably large set), so one can use many analytic tools. On the other hand, the presence of singularities allows one to easily construct spaces with properties that do not appear in smooth differential geometry. For example, the $k-$*star* (or $k-$pod) has infinite negative curvature at its origin. In chapter 1 we study local structures in polyhedra, were we try to show the parallels to classical differential geometry. Furthermore, we study the regularity of geodesics in a Riemannian polyhedron in detail.

For a probabilist who works on Riemannian geometry, one of the central tools is the theory of semimartngales and stochastic calculus, in particular Itô's formula. So for analogous results in polyhedra, one should generalize the stochastic calculus to that setting. Picard has developed a stochastic calculus in trees, i.e. in one-dimensional Euclidean polyhedra, cf. [Pic05]. The crucial technique here is the theory of local times for real-valued semimartingales.
In chapter 2 we extend this technique to general polyhedra. The central result is a *local Itô formula*, cf. Theorem 2.1.13. For a semimartingale X and piecewise smooth function f, the semimartingale decomposition of $f(X)$ consists of three parts: The first two terms are the same as in the classical Itô formula, namely the Itô integral and the quadratic variation term. The third part is a process of bounded variation and is given in terms of the *directional local times*. These are nondecreasing processes that describe the behavior of X at a singularity S (i.e. at a simplex of the triangulation).
With the help of the local Itô formula, one can define stochastic integrals in an analogous way as in manifolds (cf. [Éme89]), as we show in sections 2.2 and 2.3. Moreover, it is shown that the discretized squared increments of a semimartingale converge to the quadratic variation. Such an approximation result can be regarded as a direct link between the stochastic calculus (which is given in terms of the differentiable structure) and the language of metric spaces.
Clearly, the main example of a semimartingale should be *Brownian motion*. But so far there were only very few partial approaches towards Brownian motion in a Riemannian polyhedron, and so we study this process in great detail. We define Brownian motion to be the process that is associated to the canonical energy and then show that this is a strong Feller diffusion, in particular defined for *every* starting point. Then we relate this theory to the theory of harmonic functions in

[EF01]. At last it is shown that Brownian motion is indeed a semimartingale.

Actually, the initiating question for this work was if one can define a reasonable theory of martingales in metric spaces that are more general than Riemannian manifolds. So chapter 3 and chapter 4 are concerned with this question.
Our approach is to look at three characterizations of martingales in Euclidean space that may serve as definitions in more general metric spaces: First, one may use *Darling's characterization*. Namely, one can call a process in a metric space M a local martingale if $\varphi(X)$ is a local submartingale for a certain set of convex test function $\varphi : M \to \mathbb{R}$ (to be precise, one should use a localized version of this). This definition is very simple and can be applied to arbitrary geodesic spaces in which there is a notion of convex functions. But as simple it is to write down the definition, as hard it is to derive reasonable results from it.
The second approach is to use stochastic calculus, i.e. to find a suitable definition of a local martingale that extends the notion of a $\nabla-$martingale in Riemannian manifolds. We will do this in the case of Euclidean polyhedra by formulating a martingale condition $\mathbf{M}(S)$ which is given in terms of the local times at a simplex S. With this condition one can prove a Darling characterization in Euclidean complexes, cf. Theorem 3.1.5 and Theorem 3.3.4.
The third approach is to define martingales in terms of generalized conditional expectations. In a certain class of metric spaces (basically spaces of nonpositive curvature) one can define the notion of *barycenter* or *expectation*. From this it is possible to develop a theory of discretized martingales, cf. [Stu02]. We will define a *strong martingale* to be a limit of discretized martingales. Strong martingales feature useful properties such as non-confluence of martingales.
One of the central results is Theorem 3.4.7, which says that in a Euclidean polyhedron of nonpositive Alexandrov curvature all three notions of martingales are equivalent. As an application of this Theorem, we present a characterization of harmonic maps $h : K \to N$, where K is a compact Riemannian polyhedron and N is a Euclidean polyhedron (of arbitrary dimension) of nonpositive curvature (Theorem 3.5.4), which also includes *Ishihara's characterization*.

Acknowledgements:

I am extremely grateful to my supervisor, Karl-Theodor Sturm, for his inspiration and support during the last years. I am also grateful to Jean Picard, first for his inspiring work on martingales in trees and second for his patience when reading this work, to Anton Thalmaier for reading my thesis, too, and finally to Sergio Albeverio for all his kind support.
I would like to thank all members of the stochastics group in Bonn for the nice

working atmosphere, in particular Fabrice Blache, Anca Bonciocat, Martin Hesse, Alexander Lytchak, Gustav Paulik, Robert Philipowski and Max von Renesse for interesting discussions. Special thanks go to all my friends, in particular Jimi Heinrich, for giving me a beautiful time in Bonn.
I am so grateful to Lucía. With her love and patience she gave me the power to finish this work.
At last, I want to thank my family: My brother Kim, my sister Kea, and most of all my parents Frauke and Hans-Hermann, without whose love and support I would never have arrived at this point.

Chapter 1

Local structures in Polyhedra

This chapter is devoted to developing a theory for local analysis in (Riemannian) polyhedra. One aim is to point out the similarities to (Riemannian) manifolds, so that this theory can be regarded as a generalization of classical differential calculus in manifolds. Unfortunately, there has been no formalism for local analysis on polyhedra in literature so far, and so we have to introduce a lot of new notations.
In section 1.1 we start with the model spaces (in analogy to model spaces of manifolds, which are linear spaces), so-called *simplicial cone complexes*.
In section 1.2 we introduce a piecewise differentiable structure on a polyhedron M by mapping it locally to a simplicial cone complex ('simplicial chart') such that we have local coordinates, and we introduce some vector bundles over M (such as tangent, cotangent and bilinear bundle) as direct generalizations of the corresponding objects in differential geometry. In this setting, simplicial cone complexes appear naturally as tangent spaces.
Section 1.3 treats the Riemannian case. Here one can see the limitations of the quite general concept of Riemannian polyhedra: While in a Riemannian polyhedron one can describe first-order (derivative) phenomena quite well, the singularities cause difficulties if one wants to define objects of second order calculus such as the Hessian, cf. section 1.3.2.
A Riemannian polyhedron becomes a complete geodesic space when equipped with the *intrinsic distance* associated to the Riemannian tensor, just as in the classical case of Riemannian manifolds. We study these metric structures in section 1.3.3.
If one investigates the properties of geodesics (such as smoothness), the second order calculus causes trouble again and makes a general investigation of regularity of geodesics a tedious business. However, one can show the existence of a 'generalized inverse exponential map' (which is an important link between differential and metric structure) and show some Taylor-like expansion of this map, cf. Proposition 1.3.17.
In section 1.4, we treat the simpler case of Euclidean polyhedra and study some

properties of convex functions that are defined on such spaces.

1.1 Simplicial cone complexes

In this section we will introduce the class of simplicial cone complexes. A simplicial cone complex can be regarded as a special case of a simplicial complex, namely it is a space that is obtained by gluing together simplicial cones at their boundaries via linear isomorphisms.
Simplicial cone complexes are worth studying because of two reasons: First, in comparison to simplicial complexes, the notations (that are quite complicated anyway) are simpler, and second, they serve as *model spaces* for simplicial complexes in the sense that every simplicial complex is locally equivalent to a simplicial cone complex (cf. Proposition 1.2.4).

1.1.1 Preliminaries

Definition 1.1.1 (i) Let V be an $N-$dimensional real vector space. An $n-$dimensional simplicial cone S with origin $0 \in V$ is a closed convex cone spanned by n linearly independent vectors $u_1, \dots u_n$. Namely,

$$S = \left\{ \sum_{i=1}^{n} \nu^i u_i : \nu^i \geq 0 \right\} \tag{1.1}$$

Such a set of spanning vectors is called a *scaffold* of S. By definition, the $0-$dimensional cone is the set $\{0\}$ and $\mathrm{scaff}(\{0\}) = \emptyset$.
A *face* of S is a simplicial cone that is spanned by a subset of $\mathrm{scaff}(S)$.

(ii) A *simplicial cone complex* in V is a subset $M \subset V$ together with a finite collection $\mathcal{S} = \mathcal{S}(M)$ of simplicial cones

- $M = \bigcup_{S \in \mathcal{S}(M)} S$
- If $S \in \mathcal{S}(M)$ and T is a face of S, then $T \in \mathcal{S}(M)$.
- If $S, \widetilde{S} \in \mathcal{S}(M)$, then $S \cap \tilde{S}$ is a face of both S and $\widetilde{S}$.

$\mathcal{S}$ is called *triangulation* of M. A scaffold of M is a set $\mathrm{scaff}(M)$ of vectors in M such that for all $S \in \mathcal{S}(M)$, $\mathrm{scaff}(M) \cap S$ is a scaffold of S.
For $m \in \mathbb{N}$ denote by $\mathcal{S}^{(m)}(M)$ the set of all $m-$dimensional cones of $\mathcal{S}(M)$. The

dimension of M is defined by $\dim M := \max\{m : \mathcal{S}^{(m)}(M) \neq \emptyset\}$. Let $0 \leq m \leq \dim M$. The m*−skeleton* of M is defined by

$$M^{(m)} := \bigcup\{S : k \leq m, S \in \mathcal{S}^{(k)}(M)\}. \tag{1.2}$$

Remark 1.1.2 (i) Note that by definition, S is always assumed to be a *closed* cone. This differs from other literature, but is most suitable to our applications.
(ii) The notation scaff(M) might be somewhat misleading, since there is not only one scaffold for M. Indeed, if $\{u_1, \dots, u_m\}$ is a scaffold of S, then so is $\{\delta_1 u_1, \dots, \delta_m u_m\}$ for arbitrary $\delta_i > 0$, $i = 1, \dots, m$. However, unless stated otherwise, we assume that M is equipped with a fixed scaffold and keep the notation. Besides, in Euclidean complexes (see below) there is a canonical choice of a scaffold (which then consists of unit vectors).

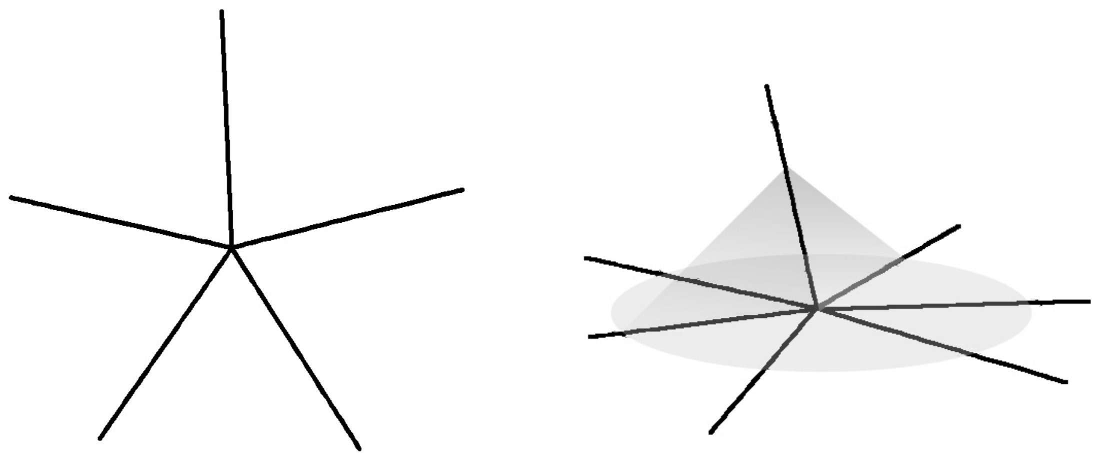


Figure 1.1: examples of simplicial cone complexes

Example 1.1.3 (i) $M = \mathbb{R}^n$ has a 'natural' triangulation into orthants. Namely, let $\{e_1 \dots, e_n\}$ be the standard basis. For $A \subset \{1, \dots, n\}$ and $a \in \{0,1\}^A$, put $\alpha := (A, a)$ and

$$S_\alpha := \left\{ \sum_{i \in A} \nu^i (-1)^{a_i} e_i : \nu^i \geq 0 \right\} \tag{1.3}$$

Then $\{(-1)^{a_i} e_i : i \in A\}$ is a scaffold of S_α, the *standard scaffold*. In particular, $\dim S_\alpha = |A|$ and $|\mathcal{S}^{(m)}| = 2^m \binom{n}{m}$.

(ii) A $1-$dimensional cone complex is a $k-$*star* or $k-$*pod*. It is obtained by gluing together k copies of $\mathbb{R}_+$ at 0. A $k-$pod has a 'natural' symmetric embedding into $\mathbb{R}^2 \cong \mathbb{C}$. Namely, let $u_j := e^{i\frac{j}{n}}$, $j = 0 \ldots, k-1$ (note that here $i := \sqrt{-1}$). Then $M = \{ru_j : 0 \leq j \leq k-1, r \geq 0\}$ is a $k-$pod and every other $k-$pod can easily be mapped to M.

(iii) Every simplicial cone complex can be mapped to a a 'cubical cone complex' $\widehat{M}$ (i.e. a cone complex whose cones are orthants) in the following way: Let $\mathrm{scaff}(M) = u_1 \ldots u_N$. For $S \in \mathcal{S}^{(m)}(M)$, let $\mathrm{scaff}(S) = \{u_{k_1}, \ldots, u_{k_m}\}$. Let $\widehat{S} \subset \mathbb{R}^n$ be the cone (i.e. the orthant) in R^N generated by $\{e_{k_1}, \ldots e_{k_m}\}$ and set $\widehat{M} := \bigcup_{S \in \mathcal{S}(M)} \widehat{S}$. The map $\Phi : u_i \mapsto e_i$ extends naturally to a simplicial linear isomorphism $\Phi : M \to \widehat{M}$ in the sense of Definition 1.1.6. This construction is closely related to the one in [EF01], Lemma 4.3.

Local coordinates and tangent spaces

Let M be an $n-$dimensional simplicial cone complex and $x \in M$. Then x lies in a simplicial cone $S \subset M$ and hence.

$$x = \sum_{u \in \mathrm{scaff}(S)} \nu_S^u(x) u$$

Let now $u \in \mathrm{scaff}(M)$. Then we define a function $\nu^u : M \to \mathbb{R}_+$ by

$$\nu^u(x) := \begin{cases} \nu_S^u(x) & \text{if } \{x, u\} \subset S \\ 0 & \text{else} \end{cases} \tag{1.4}$$

In other words, if x lies in a simplicial cone S that is adjacent to u (hence if $u \in \mathrm{scaff}(S)$), then $\nu^u(x)$ is defined to be the uth coordinate w.r.t. $\mathrm{scaff}(S)$. Note that the cone in which x is contained is not unique, because S is closed and hence contains its faces. However, ν^u is well-defined since on the faces of S the coordinate functions coincide. Thus every $x \in M$ has a unique representation

$$x = \sum_{u \in \mathrm{scaff}(M)} \nu^u(x) u \tag{1.5}$$

In general, it will be convenient to consider also local coordinates around sub-cones of M. First we will introduce some more notations that are basically taken from [EF01] and [BH99].

Definition 1.1.4 Let $(M, \mathcal{S})$ be a simplicial cone complex and $S \in \mathcal{S}$.
(i) The *interior* of S is defined by

$$S^\circ = \left\{ \sum_{u \in \mathrm{scaff}(S)} \nu^u u : \nu^u > 0 \right\} \tag{1.6}$$

(ii)The *star* of S, denoted by $\mathrm{st}(S)$, is the set of all cones $T \in \mathcal{S}$ such that $T \cap S^\circ \neq \emptyset$. The star of a point x is defined by $\mathrm{st}(x) := \mathrm{st}(S_x)$, where S_x is the unique $S \in \mathcal{S}$ such that $x \in S^\circ$. We put $\mathrm{St}(S) := \bigcup_{T \in \mathrm{st}(S)} T$.
For $m \leq \dim M$, we define $\mathrm{st}^{(m)}(S) := \mathrm{st}(S) \cap \mathcal{S}^{(m)}$.

(iii) A neighborhood $O \subset M$ is called *local at* S if O is connected, $O \cap S \neq \emptyset$ and $O \subset \mathrm{St}^\circ(S)$

Remark 1.1.5
(i) If $S \in \mathcal{S}^{(m)}$, then (1.6) means that S° is the interior of S w.r.t. the relative topology of U, where $U \subset V$ is the $m-$dimensional linear subspace generated by S.
(ii) We have that $M = \dot{\bigcup}_{S \in \mathcal{S}} S^\circ$ (a disjoint union). In particular, for any $x \in M$ there is a unique $S \in \mathcal{S}$ such that $x \in S^\circ$.
(iii) Note that he notations $\mathrm{st}(S)$ and $\mathrm{St}(S)$ differ from the notations in [EF01] and [BH99].
(iv) Let $\mathrm{St}^\circ(S)$ be the interior of $\mathrm{St}(S)$ w.r.t. the topology of M. Then $\mathrm{St}^\circ(S) = \bigcup_{T \in \mathrm{st}(S)} T^\circ$. Moreover, $\mathrm{St}^\circ(S)$ itself is local at S and hence is the *maximal* local neighborhood at S.

Let $S \in \mathcal{S}^{(m)}(M)$ and $x \in S^\circ$. Then x has a neighborhood O that is local at S. Denote $\widehat{O} := O - x$. The *tangent space* of M at x is defined by

$$T_x M := \{\lambda y : \quad y \in \widehat{O}, \lambda \geq 0\}$$

$T_x M$ does not depend on the choice of $x \in S^\circ$, i.e. if $\tilde{x} \in S^\circ$, too, then $T_x M = T_{\tilde{x}} M$. Moreover, $T_x M$ has the following structure: Let U be the vector space generated by S (i.e., spanned by scaff(S)) and let $U^\perp$ be a *linear complement* of U, i.e $V = U \oplus U^\perp$. Then $\perp S := T_x M \cap U^\perp$ is an $(n-m)-$dimensional simplicial cone complex and

$$T_x M = U \oplus \perp S. \tag{1.7}$$

Consequently, every $y \in O$ has a unique representation

$$y = y^\top + y^\perp = \sum_{u \in \mathrm{scaff}(S)} \nu^u(y) u + \sum_{u \in \mathrm{scaff}(\perp S)} \nu^u(y) u. \tag{1.8}$$

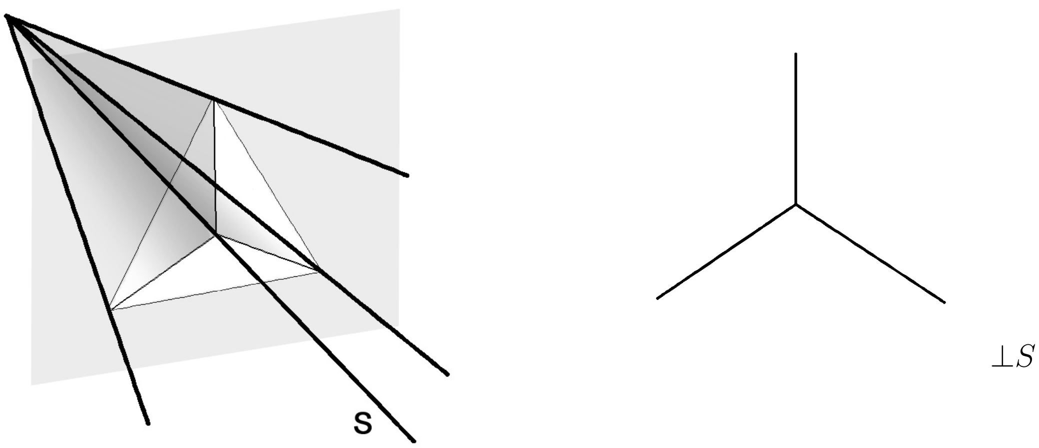


Figure 1.2: the transversal part

We call $y^\top$ the *tangential* part of y, and $y^\perp$ the *transversal* part of y. Note that (1.5) is a special case of (1.8), regarding $\{0\}$ as a 0-dimensional cone of M.
Let O be local at S and let $f : O \to \mathbb{R}$ be a function. Then we can decompose into a *tangential* and a *transversal* part, i.e. we can write $f = f^\top + f^\perp$, where

$$f^\top(y) := f(y^\top) \text{ and } f^\perp := f - f^\top \tag{1.9}$$

Definition 1.1.6 (i) Let $(M, \mathcal{S})$ be an $n-$dimensional simplicial cone complex. A function $f : M \to \mathbb{R}$ is called piecewise smooth (affine) if $f_{|S}$ is the restriction of a smooth (affine) function to S for all $S \in \mathcal{S}(M)$. f is called piecewise linear if it is piecewise affine and $f(0) = 0$.

(ii) Let $(\widetilde{M}, \widetilde{\mathcal{S}})$ be another simplicial cone complex. A map $f : M \to \widetilde{M}$ is called *simplicial* if $f(S) \in \widetilde{\mathcal{S}}$ for all $S \in \mathcal{S}$.

Note that since $S \in \mathcal{S}(M)$ is a closed simplex, if a piecewise smooth function f is well-defined, it is automatically continuous.

The next Lemma is trivial, but very useful:

Lemma 1.1.7 *Let $f : M \to \mathbb{R}$ be piecewise linear. Then*

$$f = \sum_{u \in \mathrm{scaff}(M)} f(u)\nu^u$$

1.1.2 Extending functions

In many applications in the sequel we will be faced with the following problem: Given a sub-cone-complex $L \subset M$ and a piecewise smooth function $f : L \to \mathbb{R}$. Then we need a piecewise smooth extension $\tilde{f} : M \to \mathbb{R}$ such that $\tilde{f}_{|L} \equiv f$. There are many extensions of this type. However, we will present a special extension procedure.

Example 1.1.8 1) We first show how to extend a piecewise smooth function that is defined on the boundary of a simplicial cone to the whole cone. Let S be a $k-$dimensional simplicial cone with a fixed scaffold. Then every $x \in S$ has a unique representation $x = \sum_{u \in \mathrm{scaff}(S)} \nu^u(x)u$. If we set $S_u := \{x \in S : \nu^u(x) = 0\}$ for $u \in \mathrm{scaff}(S)$, then S_u is a $k-1-$dimensional simplicial cone and $\partial S = \bigcup_{u \in \mathrm{scaff}(S)} S_u$. Let $\pi_u : S \to S_u$, be the projection onto S_u, i.e. $\pi_u(x) = \sum_{v \neq u} \nu^v(x)v$. Moreover, for $\emptyset \neq A = \{u_1, \dots u_i\} \subset \mathrm{scaff}(S)$, we define

$$\pi_A := \pi_{u_i} \circ \dots \circ \pi_{u_1}.$$

This is well-defined since $\pi_u \circ \pi_v = \pi_v \circ \pi_u$ for all $u, v \in \mathrm{scaff}(S)$.
Let now $f : \partial S \to \mathbb{R}$. We define a function $\tilde{f} : S \to \mathbb{R}$ by

$$\tilde{f}(x) := \sum_{A \in \mathbb{P}_*(\mathrm{scaff}(S))} (-1)^{|A|+1} f(\pi_A(x)) \tag{1.10}$$

where for an arbitrary set E, $\mathbb{P}_*(E)$ denotes the collection of all non-empty subsets of E. Then $\tilde{f}_{|\partial S} = f$ by Lemma 1.1.9. Moreover, if f is piecewise smooth, then $\tilde{f}$ is smooth.

2) Let now L be an $n-$dimensional simplicial cone complex, let $K \subset L$ be a sub-complex and $f : K \to \mathbb{R}$ piecewise smooth. Fix a scaffold of L. First assume that $f(0) = 0$. We set $\tilde{f}_{|K} := f$. For $x \in L \setminus K$ we will define $\tilde{f}(x)$ by induction on the dimension of S where $S \in \mathcal{S}(L)$ is the unique simplex such that $x \in S^\circ$, as follows: First put $\tilde{f}(x) := 0$ for all $x \in S^\circ$ where $S \in \mathcal{S}^{(1)}(L \setminus K)$. Let now $2 \leq k \leq n$ and assume that $\tilde{f}(x)$ is already defined for all $x \in \bigcup \mathcal{S}^{(k-1)}(L)$. By (1.10), we can define $\tilde{f}(x)$ for all $x \in \bigcup \mathcal{S}^{(k)}(L)$ and so on. At last, if $f(0) = c \neq 0$, then we put $\tilde{f} := \widetilde{(f-c)} + c$.
This extension has the following properties: If f is piecewise affine (linear), then so is $\tilde{f}$. Moreover, if $\partial_u f(0)$ for all $u \in \mathrm{scaff}M$, then $\tilde{\partial}_v \tilde{f} = 0$ for all $v \in V$, where $\tilde{\partial}_v \tilde{f}$ is defined in (2.4).

Lemma 1.1.9 *In the situation of Example 1.1.8 1), $\tilde{f}_{|\partial S} = f$.*

Proof : For an arbitrary set E and $e \in E$, we have the decomposition

$$\mathbb{P}_*(E) = \mathbb{P}_*(E \setminus \{e\}) \cup \{A \in \mathbb{P}_*(E) : e \in A, A \neq \{e\}\} \cup \{\{e\}\}.$$

Indeed, the map $\theta_e : A \mapsto A \cup \{e\}$ is a bijection from the first to the second set and for all $A \in \mathbb{P}_*(E \setminus \{e\})$ we have $|\theta_e(A)| = |A| + 1$ (of course, the latter holds provided E is finite). Thus for any $u \in \text{scaff}(S)$, we can rewrite (1.10) as

$$\tilde{f}(x) = f(\pi_u(x)) + \sum_{A \in \mathbb{P}_*(\text{scaff}(C) \setminus \{u\})} (-1)^{|A|+1} [f(\pi_A(x)) - f(\pi_A(\pi_u(x)))]$$

Let now $x \in \partial S$, so $x = \pi_u(x)$ for some $u \in \text{scaff}(S)$. Then the last sum cancels out and so $\tilde{f}(x) = f(\pi_u(x)) = f(x)$. $\square$

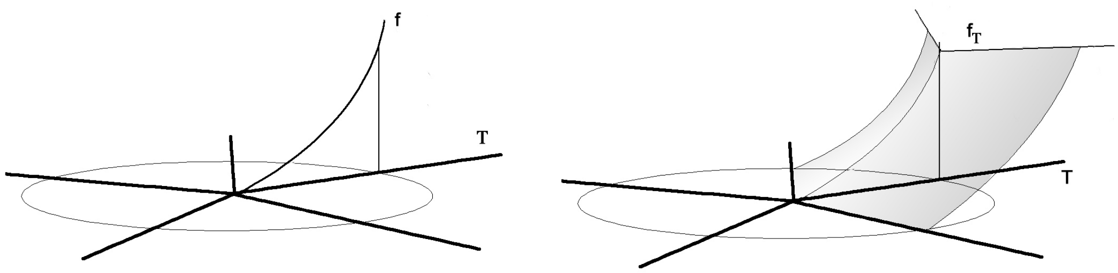


Figure 1.3: extension of a function from a simplex

Cutting and extending

Now we consider a very useful special case, namely when the subcomplex from which we extend a function is a *simplex* $T \in \mathcal{S}(M)$ (cf. figure 1.3). Let $f : M \to \mathbb{R}$ and let $T \in \mathcal{S}(M)$. Then T can be regarded as a sub-complex of M. Let f_T be the extension of $f_{|T}$ to M described in Example 1.1.8. Can we recover f as the sum of all f_T?. The answer is given in the following

Lemma 1.1.10 *There are integer numbers $(a_T)_{T \in \mathcal{S}(M)}$ such that for all functions $f : M \to \mathbb{R}$,*

$$f = \sum_{T \in \mathcal{S}(M)} a_T f_T \tag{1.11}$$

Moreover, the coefficients can be chosen such that $a_T = 1$ for all $T \in \mathcal{S}^{(n)}(M)$.

Proof : We will prove the Lemma by induction on $n = \dim M$. Without loss of generality we may assume that $f(0) = 0$. For $n = 1$, M is a star and we have $f = \sum_{T \in \mathcal{S}^{(1)}(M)} f_T$.
Assume now that the Lemma is proved for all cone complexes whose dimension is strictly less than n. Again we assume that $f(0) = 0$. Consider $M^{(n-1)}$, the $(n-1)-$skeleton of M. Let $\tilde{f}$ be the extension of $f_{|M^{(n-1)}}$ from $M^{(n-1)}$ to M and put $g := f - \tilde{f}$. Then $g_{|M^{(n-1)}} \equiv 0$ and hence $g = \sum_{S \in \mathcal{S}^{(n)}(M)} g_S = \sum_{S \in \mathcal{S}^{(n)}(M)} (f_S - \tilde{f}_S)$. By induction hypothesis, we can write $f_{|M^{(n-1)}} = \sum_{T \in \mathcal{S}(M^{(n-1)})} a_T^{(n-1)} f_T$ (where here f_T is the 'cut and extend' function in $M^{(n-1)}$). Thus $\tilde{f} = \sum_{T \in \mathcal{S}(M^{(n-1)})} a_T^{(n-1)} f_T$ (now f_T is the 'cut and extend' function in M). Moreover, for all $S \in \mathcal{S}^{(n)}(M)$ and all $T \in \mathcal{S}(M^{(n-1)})$ with $S \notin \mathrm{st}(T)$ we have $(f_T)_S \equiv 0$ and hence

$$\tilde{f}_S = \sum_{T \in \mathcal{S}(M^{(n-1)})} a_T^{(n-1)} (f_T)_S = \sum_{\substack{T \in \mathcal{S}(M^{(n-1)}) \\ S \in \mathrm{st}(T)}} a_T^{(n-1)} f_T.$$

Consquently,

$$\begin{aligned} f = g + \tilde{f} &= \sum_{S \in \mathcal{S}^{(n)}(M)} (f_S - \tilde{f}_S) + \sum_{T \in \mathcal{S}(M^{(n-1)})} a_T^{(n-1)} f_T \\ &= \sum_{S \in \mathcal{S}^{(n)}(M)} f_S + \sum_{T \in \mathcal{S}(M^{(n-1)})} a_T^{(n-1)} \left(1 - |\mathrm{st}^{(n)}(T)|\right) f_T \end{aligned}$$

Thus the Lemma is proved. □

1.2 Differentiable structures in Polyhedra

After treating the special case of simplicial cone complexes, we will now come to the class of simplicial complexes, or slightly more general, the class of polyhedra. In the first part we will discuss the piecewise differentiable structure of polyhedra as a generalization of differentiable manifolds. In particular, we will introduce notions of the bundle of tangent spaces or (bi)linear functions and their sections, namely vector fields and forms.
In the second part, we treat the case of Riemannian polyhedra, which are geometric objects.

Let us start with the notion of a simplicial complex, which is defined analogously to a cone complexe with cones replaced by simplices:

Definition 1.2.1 (i) Let V be an $N-$dimensional real vector space. An $n-$dimensional simplex in V is the convex hull of $n+1$ affinely independent vectors.

(ii) A (locally finite) *simplicial complex* in V is a subset $M \subset V$ together with a finite collection $\mathcal{S} = \mathcal{S}(M)$ of closed subsets of M such that

- $M = \bigcup_{S \in \mathcal{S}(M)} S$
- S is a simplex for all $S \in \mathcal{S}(M)$
- If $S \in \mathcal{S}(M)$, and F is a face of S, then $F \in \mathcal{S}(M)$.
- If $S, \widetilde{S} \in \mathcal{S}(M)$ and $S \cap \tilde{S} \neq \emptyset$, then $S \cap \tilde{S}$ is a face of both S and $\widetilde{S}$.

We will also assume in the sequel that M is *dimensionally homogeneous*, i.e. for all $S \in \mathcal{S}(M)$, S is the face of an $n-$dimensional simplex.

A survey on simplicial complexes is given in [EF01] or [BH99].

Definition 1.2.2 A *polyhedron* is a topological space M together with a homeomorphism $\theta : M \to \widetilde{M}$, where $\widetilde{M} \subset V$ is a simplicial complex. θ is called a *triangulation* of M. The set of simplices of M defined by

$$\mathcal{S}(M) := \{\theta^{-1}(S) : S \in \mathcal{S}(\widetilde{M})\} \tag{1.12}$$

The *boundary of* M, denoted by ∂M, is the union of all non-maximal simplices that are contained in only one maximal simplex. The *interior* of M is defined by $M^\circ := M \setminus \partial M$.

Definition 1.2.3 *Let M be a separable topological Hausdorff space. A (piecewise smooth) $n-$ dimensional* simplicial atlas *is a family of homeomorphisms $\xi_\alpha : O_\alpha \to \widehat{O}_\alpha$ ($\alpha \in A$, A some index set) such that*

- *$\widehat{O}_\alpha$ is a connected open neighborhood in some finite $n-$dimensional simplicial cone complex.*
- $M = \bigcup_\alpha O_\alpha$
- *For $\alpha, \beta \in A$ such that $O_\alpha \cap O_\beta \neq \emptyset$, $\xi_\beta \circ \xi_\alpha^{-1} : \widehat{O}_{\alpha\beta} \to \widehat{O}_{\beta\alpha}$ is a simplicial diffeomorphism, where $\widehat{O}_{\alpha\beta} := \xi_\alpha(O_\alpha \cap O_\beta)$ and $\widehat{O}_{\beta\alpha}$ is defined similarly.*

ξ_α *is called a* simplicial chart*.*

So if M is equipped with an $n-$dimensional simplicial atlas, M could be called an $n-$dimensional *simplicial manifold.* In particular, every simplicial complex is a simplicial manifold as stated in the following

Proposition 1.2.4 *Let M be an $n-$dimensionally homogeneous simplicial complex. Then for all $S \in \mathcal{S}(M)$ and all $x \in S^\circ$ there is a chart $\xi : O \to \widehat{O}$ which is local at S (i.e. O is local at S) such that the chart changes are piecewise affine isomorphisms. If $S \in \mathcal{S}^{(m)}$, then by translation $\widehat{O}$ can be regarded as a neighborhood of* 0 *in $U \oplus \perp S$, where U is an $m-$dimensional linear subspace and $\perp S$ is an $(n-m)-$dimensionally homogeneous simplicial cone complex.*

Proof : In [EF01], Lemma 4.3, there is constructed explicitly a simplicial chart from a neighborhood that is local at a corner into the standard orthant. If more generally $x \in S^\circ$ where $S \in \mathcal{S}^{(m)}(M)$, denote by U the linear subspace generated by S and choose a linear complement $U^\perp$. Then $\perp S + x = M \cap (U^\perp + x)$ is locally a simplicial cone complex (in order to be precise, we should define $\perp S = \{\lambda(y-x) : \lambda \geq 0, y \in O \cap (U^\perp + x)\}$). Moreover, since M is dimensionally homogeneous, it is $(n-m)-$dimensionally homogeneous and $M \cong U \oplus \perp S$, locally around x, where this identification is just a translation. This neighborhood can now be translated onto a neighborhood in an $n-$dimensionally homogeneous simplicial cone complex. $\square$

Remark 1.2.5 If $\widetilde{M}$ is a polyhedron with a triangulation $\theta : \widetilde{M} \to M$, then $\tilde{\xi} := \xi \circ \theta$ is a chart for $\widetilde{M}$ (with the corresponding neighborhood $\widetilde{O}$). So $\widetilde{M}$ receives its piecewise differentiable structure from M through θ.
In our sense, a polyhedron is identified with its image under the homeomorphism θ, which is a simplicial complex. This is a very general concept. For instance, in the setting of smooth manifolds, the surface of the unit cube carries a smooth differentiable structure because it can be mapped homeomorphically to the two-dimensional unit sphere. Clearly, one often a priori has a natural piecewise differentiable structure, as e.g. when a polyhedron is obtained by gluing together smooth manifolds. In this case, the homeomorphism θ should be chosen to be a *simplicial diffeomorphism*.

For $x \in M$, the tangent space T_xM can now be defined in the spirit of differentiable manifolds in several ways (cf. e.g. [BJ73]). One can also define T_xM directly via charts (with suitable equivalence relations). We will skip the details of the construction. However, for $x \in S^\circ \in \mathcal{S}^{(m)}(M)$, T_xM is of the form

$$T_xM = T_xS \oplus \perp_x S \tag{1.13}$$

where T_xS is a $m-$dimensional vector space and $\perp_x S$ is an $(n-m)-$dimensional simplicial cone complex. More precisely, let $\xi : O \to \widehat{O}$ be a chart local at S. Let $T_{\xi(x)}\widehat{M} = \widehat{U} \oplus \perp \widehat{S}$ according to (1.7) (so we assume that there was made a choice of a linear complement for all $\widehat{S} \in \mathcal{S}(M)$). If we put $T_xS := d\xi^{-1}_{\xi(x)}(\widehat{U})$ and $\perp_x S := d\xi^{-1}_{\xi(x)}(\perp \widehat{S})$, then $d\xi^{-1}_{\xi(x)}$ is a simplicial linear isomorphism from $T_{\xi(x)}\widehat{M}$ to

T_xM.
Denote by $TM := \bigcup_{x\in M} T_xM$ the tangent bundle over M with natural projection $\pi : TM \to M$. A *vector field* is a section of TM, i.e. a map $F : M \to TM$ with $\pi \circ F = Id$. The set of all vector fields is denoted by $\Gamma(TM)$.
Denote by T_x^*M the set of all piecewise linear functions on T_xM and put $T^*M := \bigcup_{x\in M} T_x^*M$ (the bundle of linear functions). A *linear form* is a section of T^*M.
A *piecewise bilinear function* on T_xM is a function

$$b : \bigcup_{C\in\mathcal{C}(\perp_x S)} (T_xS \oplus\ C) \times (T_xS \oplus\ C)$$

such that for all $C \in \mathcal{C}(\perp_x S)$, $b_{|(T_xS\oplus\ C)\times(T_xS\oplus\ C)}$ is a bilinear function. The vector space of all piecewise bilinear functions on T_xM is denoted by $T_x^*M \otimes T_x^*M$ and the bundle of piecewise bilinear functions by $T^*M \otimes T^*M$. A *bilinear form* is an element of $\Gamma(T^*M \otimes T^*M)$, the set of all sections of $T^*M \otimes T^*M$.
We denote by $\mathcal{C}^\infty(M)$ the set of all piecewise smooth functions on M (and accordingly, by $\mathcal{C}_c^\infty(M)$ the set of functions in $\mathcal{C}^\infty(M)$ with compact support and by $\mathcal{C}_0^\infty(M)$ the set of functions in $\mathcal{C}^\infty(M)$ that vanish at infinity. A linear form is called *piecewise smooth* if for all $S \in \mathcal{S}(M)$, $\alpha_{|T^*S}$ is the restriction of a smooth linear form to S. Likewise, $b \in \Gamma(T^*M \otimes T^*M)$ is called *piecewise smooth* if $b_{|T^*S\otimes T^*S}$ is the restriction of a smooth bilinear form to S.

Local Coordinates
Let $S \in \mathcal{S}(M)$ and let $\xi : O \to \widehat{O}$ be a simplicial chart that is local at S. Put $\hat{x} := \xi(x)$. Due to (1.8), for $x \in O$ we can write

$$\hat{x} = \hat{x}^\top + \hat{x}^\perp = \sum_{u\in\operatorname{scaff}(\widehat{S})} \hat{x}^u + \sum_{u\in\operatorname{scaff}(\perp\widehat{S})} \hat{x}^u \tag{1.14}$$

with $\hat{x}^u := \widehat{\nu}^u \circ \xi$ and $\widehat{\nu}^u : \widehat{O} \to \mathbb{R}$ defined by (1.8). Then $\hat{x}^u$ is piecewise smooth on O and hence $\partial\hat{x}^u = \partial(\widehat{\nu}^u \circ \xi)$ is a piecewise smooth linear form on O.
To the chart $\xi : x \mapsto \hat{x}$ we can associate a 'frame' of vector fields as follows: For $u \in \operatorname{scaff}(\widehat{S})$ set

$$\frac{\partial}{\partial\hat{x}^u}(x) := d\xi_{\hat{x}}^{-1}(u) \quad \in T_xM \tag{1.15}$$

and for $u \in \operatorname{scaff}(\perp\widehat{S})$ set

$$\frac{\partial}{\partial\hat{x}^u}(x) := \begin{cases} d\xi_{\hat{x}}^{-1}(u) & \text{if } \hat{x}^\perp \in \operatorname{St}(u) \\ 0 & \text{else} \end{cases} \tag{1.16}$$

Note that (at least for $u \in \operatorname{scaff}(\perp\widehat{S})$) $\frac{\partial}{\partial\xi^u}$ is not continuous. However, these vector fields are useful for a local representation of forms. Namely, for $b \in \Gamma(T^*M \otimes T^*M)$

we have

$$b = \sum_{u,v \in \mathrm{scaff}(\widehat{S}) \cup \mathrm{scaff}(\perp \widehat{S})} b^{uv} \partial \hat{x}^u \otimes \partial \hat{x}^v \tag{1.17}$$

with

$$b^{uv}(x) := \begin{cases} b_x(\frac{\partial}{\partial \hat{x}^u}(x), \frac{\partial}{\partial \hat{x}^v}(x)) & \text{if } x \sim u \text{ and } x \sim v \\ \text{arbitrary} & \text{else} \end{cases} \tag{1.18}$$

where by definition, $x \sim u$ either if $u \in \mathrm{scaff}(\widehat{S})$ or if $u \in \mathrm{scaff}(\perp \widehat{S})$ and $\hat{x}^\perp \in \mathrm{St}(u)$. Note that if $x \nsim v$, then $\frac{\partial}{\partial \hat{x}^u}(x) = 0$ and hence b^{uv} can indeed be defined arbitrarily if $x \nsim u$ or $x \nsim v$.
For instance, one can extend b^{uv} from $K^{uv} := \{x \in O : x \sim u \text{ and } x \sim v\}$ (which is a neighborhood in a simplicial cone complex) to a piecewise smooth function on O as in Example 1.1.8.
In many situations, we will only be interested in the *tangential part* of a bilinear form. Namely, let $\xi : O \to \widehat{O}$ be a chart that is local at $S \in \mathcal{S}$ and let $x \in O$. For $w = \sum_{u \in \mathrm{scaff}(\widehat{S}) \cup \mathrm{scaff}(\perp \widehat{S})} w^u \frac{\partial}{\partial \hat{x}^u} \in T_x M$, set

$$w^\top = \sum_{u \in \mathrm{scaff}(\widehat{S})} w^u \frac{\partial}{\partial \hat{x}^u} \tag{1.19}$$

Thus we get a decomposition $T_x M = T_x M^\top \oplus T_x M^\perp$. Now we can define

$$b_x^\top(w, \tilde{w}) := b(w^\top, \tilde{w}^\top) = \sum_{u,v \in \mathrm{scaff}(\widehat{S})} b^{uv} \partial \hat{x}^u \otimes \partial \hat{x}^v (w, \tilde{w}). \tag{1.20}$$

Likewise, for a linear form $\alpha \in \Gamma(T^* M)$ we can write

$$\alpha = \alpha^\top + \alpha^\perp = \sum_{u \in \mathrm{scaff}(\widehat{S})} \alpha^u \partial \hat{x}^u + \sum_{u \in \mathrm{scaff}(\perp \widehat{S})} \alpha^u \partial \hat{x}^u, \tag{1.21}$$

where

$$\alpha^u(x) := \begin{cases} \alpha_x(\frac{\partial}{\partial \xi^u}(x)) & \text{if } x \sim u \\ \text{arbitrary} & \text{else} \end{cases} \tag{1.22}$$

and clearly, if α is piecewise smooth, the α^u can be extended to piecewise smooth functions.

1.3 Riemannian polyhedra

1.3.1 Preliminaries

Definition 1.3.1 *A (piecewise smooth)* Riemannian polyhedron *is a polyhedron (M, θ) together with a piecewise smooth positive definite symmetric bilinear form $g \in \Gamma(T^*M \otimes T^*M)$.*

In other words, for all $S \in \mathcal{S}^{(m)}(M)$, $(S, g_{|S})$ is a closed subset of an $m-$dimensional smooth Riemannian manifold $(\widetilde{S}, g^{\widetilde{S}})$. The fact that g is a bilinear form on M means that if S_1 is a face of S_2, then $g^{\widetilde{S_2}}_{|TS_1} \equiv g^{\widetilde{S_1}}_{|TS_1}$ (i.e. for all $x \in S_1$ and all $u, v \in T_xS_1$, $g_x^{\widetilde{S_2}}(u,v) = g_x^{\widetilde{S_1}}(u,v)$). So a Riemannian polyhedron can be obtained by gluing together $n-$dimensional Riemannian simplices along the faces via isometries.

There are lots of examples of Riemannian polyhedra, cf. e.g. [EF01] or [BH99].

Example 1.3.2 (i) The simplest Riemannian complexes are Euclidean complexes, i.e. every simplex endowed with a Euclidean metric, cf. the section below on Euclidean cone complexes.
(ii) The next general class consists of $\mathbb{M}_\kappa-$simplicial complexes. These are Riemannian simplicial complexes such that any simplex is endowed with a metric of constant curvature κ, cf. [BH99]. In other words, an $n-$dimensional $\mathbb{M}_\kappa-$simplicial complex is obtained by gluing together geodesic simplices in $\mathbb{M}^n_\kappa$, the $n-$dimensional model space of constant curvature κ.
(iii) Every paracompact Riemannian manifold (with or without boundary) is a Riemannian polyhedron, i.e. it has a triangulation (cf. [Whi40] for a $\mathcal{C}^1-$version).
(iv) An *orbifold* is a Riemannian polyhedron, cf. [EF01], Examples 8.12. and 8.13.

Let $x \in S^\circ$ for $S \in \mathcal{S}^{(m)}$. Recall from (1.13) that T_xM is of the form

$$T_xM = T_xS \oplus \perp_x S \tag{1.23}$$

where T_xS is an $m-$dimensional vector space and $\perp_x S$ is an $(n-m)-$dimensional cone complex. While in the situation of (1.13), $\perp_x S$ depended on the choice of a linear complement (and the chart), we now have a canonical choice of $\perp_x S$, namely the *orthogonal complement* of T_xS w.r.t. g_x: Set

$$\perp_x S := (T_xS)^\perp := \{v \in T_xM : g_x(u,v) = 0 \quad (\forall u \in T_xS)\}. \tag{1.24}$$

Then $\perp_x S$ is a simplicial cone complex and (1.23) holds. More precisely, there is a unique *orthogonal projection* $\pi_S : T_xM \to S$, defined by $g_x(v - \pi_S(v), u) = 0$ for

all $u \in T_xS$ and $v \in T_xM$, cf. section 1.4. Put $v^\top := \pi_S(v)$ and $v^\perp := v - \pi_S(v)$. Then $v^\perp \in \perp_x S$.
Note that since $\perp_x S$ is a Euclidean simplicial cone complex, there is a *canonical scaffold* , namely the unique scaffold of $\perp_x S$ that consists of unit vectors. Let $\mathrm{scaff}(\perp_x S) = \{u^1(x), \ldots u^k(x)\}$. Keeping the right enumeration, we obtain smooth vector fields $u^i \in \Gamma(\perp S)$, $i = 1, \ldots k$. This set of vector fields is denoted by $\mathrm{scaff}(\perp S)$. To $\mathrm{scaff}(\perp S)$ we associate piecewise smooth linear forms $\nu^u \in \Gamma(\perp^* S)$, defined by $\nu^u : \perp_x S \ni v \mapsto \sum \nu^u_x(v)u(x)$.

Since we have this orthogonal decomposition on the level of tangent spaces, we may use it to define a sort of *normal coordinates*. By this we mean a simplicial chart whose derivative respects (1.23).

Lemma 1.3.3 (Normal chart) *Let $S \in \mathcal{S}$. For any $x_0 \in S^\circ$, there is a simplicial chart $\xi := O \to \widehat{O} \subset \widehat{S} \oplus \perp\widehat{S}$ around x_0 with the property that for all $u \in \mathrm{scaff}(\perp S)$ there is a $\hat{u} \in \mathrm{scaff}(\perp\widehat{S})$ such that for all $x \in O \cap S$, $\partial\xi_x(u(x)) \equiv \hat{u}$, where $\mathrm{scaff}(\perp\widehat{S})$ is a fixed scaffold of $\perp\widehat{S}$. In particular, $(\partial\xi_x)^{-1}(\perp\widehat{S}) = \perp_x S$.*

Proof : Let $\theta := O \to \widetilde{O}$ be a simplicial chart local at S. For $x \in O \cap S$, put $\Theta_x(y) := (\partial\theta_x)^{-1}(\theta(y) - \theta(x)) \in T_xM$. As a consequence of the implicit function theorem, for all $y \in O$, there is a unique $\pi(y) = \pi_S(y) \in S$ such that $\Theta_{\pi(y)}(y) \in \perp_{\pi(y)} S$ (of course, O should be made smaller if necessary). Set

$$\rho^u(y) := \nu^u(\Theta_{\pi(y)}(y)) \tag{1.25}$$

Then ρ^u is piecewise smooth, $\rho^u_{|S\cap O} \equiv 0$ and hence $\partial\rho^u_x = \nu^u_x$ for all $x \in S \cap O$. Thus $\xi : O \to \widehat{O}$ does the job, where

$$\xi(y) := \theta(\pi(y)) + \sum_{u \in \mathrm{scaff}(\perp S)} \rho^u(y)\hat{u} \tag{1.26}$$

and $\widehat{O} := \xi(O)$. $\square$

Remark 1.3.4 (i) If M is a manifold and S a submanifold, then in the proof of Lemma 1.3.3 one usually takes $\Theta_x(y) := \exp^{-1}(y)$. This cannot be done in general polyhedra because exp does not respect the triangulation in general. Even more, exp might not be a simplicial diffeomorphism.
(ii) If especially $S \in \mathcal{S}^{(n-1)}(M)$, then $\perp C$ is a one-dimensional cone complex. Thus by Example 1.1.3 (ii) we can assume that $\perp C \subset \mathbb{R}^2$ is the symmetric $k-$pod for some $k \in \mathbb{N}$.
(iii) For $S \in \mathcal{S}^{(n-1)}(M)$ one also has *special normal coordinates* at S. Namely, let $\xi^1, \ldots, \xi^{n-1}$ be coordinates on S°. Then extend these to functions on a local

neighborhood O to be constant on geodesics that intersect S normally. For any $T \in \mathcal{S}^{(n)}$ and $y \in T \cap O$ let $\xi_T^n(y) := d(S, y)$ if $y \in O \cap T$ and $\xi_T^n(y) := 0$ if $y \in O \backslash T$. Then ξ is a simplicial chart and we have $\langle \frac{\partial}{\partial \hat{x}^i}, \frac{\partial}{\partial \hat{x}_T^n} \rangle \equiv 0$.

We conclude this section with another important object: The *Link*:

Definition 1.3.5 Let (M, g) be a Riemannian polyhedron and let $x \in M$. The *link* of x in M is defined by

$$\mathrm{Lk}_x M := \{v \in T_x M : g_x(v, v) = 1\} \tag{1.27}$$

Regarding $\mathrm{Lk}_x M$ as a subset of the Euclidean cone complex $(T_x M, g_x)$, the induced Riemannian tensor makes $\mathrm{Lk}_x M$ an $(n-1)-$dimensional spherical polyhedron. More precisely, assume that $T_x S$ has a triangulation into a simplicial cone complex, so $T_x M = T_x S \oplus \perp_x S$ is a simplicial cone complex whose set of simplicial cones is denoted by $\mathcal{S}(T_x M)$. Then for all $C \in \mathcal{S}^{(m)}(T_x M)$, $\widetilde{C} := \{v \in C : g_x(v, v) = 1\}$ is an $(m-1)-$dimensional spherical simplex (as a subset of a Euclidean sphere) and $\mathrm{Lk}_x M = \bigcup_{C \in \mathcal{S}(M)} \widetilde{C}$. Let $u_1, \ldots, u_k$ be a scaffold of $T_x S$ consisting of unit vectors. Together with the canonical scaffold of unit vectors for $\perp_x S$ defined above, we get a scaffold for the whole $T_x M$, which is equal to the set of corners of $\mathrm{Lk}_x M$.
At last, $(T_x M, g_x)$ is isometric to $C_0(\mathrm{Lk}_x M)$, the Euclidean cone over $\mathrm{Lk}_x M$, cf. Proposition 1.4.4 and also [BH99].

1.3.2 Christoffel symbols and Hessian

Let $S \in \mathcal{S}^{(m)}(M)$ and let $T \in \mathrm{st}^{(n)}(S)$. Then S is an $m-$dimensional Riemannian submanifold (with corners) of T. Since S is a Riemannian manifold itself, we have the intrinsic Levi-Civita-connection ∇^S on S. The relation between ∇^S and ∇^T (the Levi-Civita-connection on T) is the following:

$$\nabla_Y^S X = (\nabla_Y^T X)^\top := \pi^S(\nabla_Y^T X), \qquad X, Y \in \Gamma(TS), \tag{1.28}$$

where for $x \in S^\circ$, $\pi^S : T_x M \to T_x S$ denotes the orthogonal projection[1] onto $T_x S$, cf. [Jos02], Theorem 3.6.1.
Let us study the local description of the Levi-Civita-connections, namely the Christoffel symbols. Can one define Christoffel symbols on a face $S \in \mathcal{S}^{(m)}$? The answer is: 'Yes' for the tangential part and 'No' for the normal part. In general, if $T, \widetilde{T} \in \mathrm{st}^{(n)}(S)$, then the Christoffel symbols coming from T and $\widetilde{T}$ do not coincide on S. However, in normal coordinates, the tangential parts coincide, as we will

[1]Note that for $v \in T_x M$, the notation $v^\top$ is also used in terms of a local chart, cf. (1.47), so this could be ambiguous here. However, if we choose our chart to be *normal at S*, both notations coincide.

show now:
Let $\xi : O \to \widehat{O}$ be a normal coordinate system at S. Again, we will restrict our attention to T and the submanifold S. On S we have two kinds of Christoffel symbols: $\Gamma^w_{uv}(S)$, $u,v,w \in \mathrm{scaff}(S)$ and $\Gamma^w_{uv}(T)$, $u,v,w \in \mathrm{scaff}(S) \cup \mathrm{scaff}(\perp S)$. The former belong to ∇^S, the latter to ∇^T. Since the coordinates are normal at S, it follows from (1.28) that

$$\Gamma^w_{uv}(S) = \Gamma^w_{uv}(T) \qquad (\forall u,v,w \in \mathrm{scaff}(S)). \tag{1.29}$$

So the 'tangential' Christoffel symbols w.r.t. normal coordinates are well-defined on S.

Remark 1.3.6 One has to be careful: The tangential Christoffel symbols may not be well-defined outside of S, since the chart ξ is no longer normal at other simplices in general.

Let us now come to the Hessian. On a smooth Riemannian manifold (M,g), the *Hessian* of a smooth function f is the bilinear form on M defined by

$$\mathrm{Hess} f_x(u,v) = g_x(\nabla_u \nabla f(x), v) \qquad u,v \in T_xM \tag{1.30}$$

where $F := \nabla f$ is the *gradient* of f and $\nabla_v F$ is the covariant derivative (coming from the Levi-Civita-connection of g) of the vector field F in direction v.
Let now $f : M \to \mathbb{R}$ be a piecewise smooth function and let $x \in S^\circ$ for some $S \in \mathcal{S}$. If $S \in \mathcal{S}^{(n)}$, then the definition of $\mathrm{Hess} f_x$ is clear by (1.30). But if $S \in \mathcal{S}^{(m)}$ for some $m < n$, then the situation is more complicated. For instance, let $x \in S^\circ$. Every $T \in \mathrm{st}^{(n)}(S)$ induces a Hessian on $T_xS \subset T_xT$, but they may not coincide (cf. Remark 1.3.8). However, $(S, g_{|T^*S \otimes T^*S})$ is a Riemannian manifold itself (as above, it is a closed subset of an $m-$dimensional smooth Riemannian manifold $(\widetilde{S}, g^{\widetilde{S}})$) and so for $u,v \in T_xS$ we define $(\mathrm{Hess} f_x)^\top(u,v) := \mathrm{Hess}^{\widetilde{S}}_x(u,v)$, being the *Hessian* of f at x w.r.t. $g^{\widetilde{S}}$. In terms of a local chart we have

$$(\mathrm{Hess} f_x)^\top = \sum_{u,v \in \mathrm{scaff}(S)} \left(\partial_{uv} \hat{f}(\hat{x}) - \sum_{w \in \mathrm{scaff}(S)} \Gamma^w_{uv}(x) \partial_w \hat{f}(\hat{x}) \right) (\partial \hat{x}^u \otimes \partial \hat{x}^v)_x \tag{1.31}$$

where Γ^w_{uv} are the Christoffel symbols of the Levi-Civita-connection for $g^{\widetilde{S}}$ w.r.t. the chart $\xi : O \cap S \to \widehat{O} \cap \widehat{S}$.

Remark 1.3.7 Note that we have defined only the tangential part of the Hessian. This is enough for our purposes, since the stochastic integral of a bilinear form only sees the tangential part, cf. (2.34). If one wants a bilinear form one the whole T_xM, one can for example extend $(\mathrm{Hess} f_x)^\top$ to be 0 on the orthogonal (w.r.t. g) complement of T_xS.

Remark 1.3.8 In general, Hess$f$ is not piecewise smooth, not even continuous. The same holds for the Christoffel symbols Γ^w_{uv}. This is due to the fact that on a face $S_1 \subset S_2$, $\mathrm{Hess}^{S_1}_{|TS_1}$ and $\mathrm{Hess}^{S_2}_{|TS_1}$ may not coincide. In general we have

$$\mathrm{Hess}^{S_2} f_x = \mathrm{Hess}^{S_1} f_x + \sum_{w \in \mathrm{scaff}(\perp S_1)} \partial_w f(x) l^w_x \tag{1.32}$$

where for $w \in \perp_x S_1$ and $u, v \in T_x S_1$

$$l^w_x(u,v) := l^w_x(S_1, S_2)(u,v) := \pi^{S_1}(\nabla^{S^2}_u w) \tag{1.33}$$

is the second fundamental form[2] at x of the submanifold $S_1 \subset S_2$ in direction w (which is orthogonal to S_1), cf. [Jos02], Definition 3.6.2. Thus Hess$f^\top$ is piecewise smooth for all piecewise smooth functions f if and only if the second fundamental form vanishes, i.e. if and only if for any simplex $S_2 \in \mathcal{S}(M)$ and any face S_1, S_1 is a totally geodesic submanifold of S_2 (cf. [Jos02], Theorem 3.6.3), as e.g. in the case where M is an $\mathbb{M}_\kappa-$simplicial complex.
Let now $S_1 \subset S_2 \subset S_3 \in \mathcal{S}$ such that S_1 is a face of S_2 and S_2 is a face of S_3. Let $u, v \in T_x S_1$ and let $w \in \perp_x S_1 \cap T_x S_2$. It follows from (1.28) and (1.33) that $l^w_x(S_1, S_2)(u,v) = l^w_x(S_1, S_3)(u,v)$. In particular, if $w \in \mathrm{scaff}(\perp_x S_1)$, then $l^w_x = l^w_x(S_1, M)$ is a well-defined symmetric bilinear form. Thus we may define

$$\overline{\mathrm{Hess}} f^\top_x := \mathrm{Hess} f^\top_x + \sum_{w \in \mathrm{scaff}(\perp_x S)} \partial_w f(x) l^w_x \tag{1.34}$$

1.3.3 Metric structures and geodesics

Let M be a polyhedron. There are several ways to define a distance on M. We have already seen the easiest way: Assume that M is a simplicial complex embedded into a vector space V. Let $\langle \cdot, \cdot \rangle$ be a Euclidean scalar product on V with corresponding norm $|\cdot|$. Let d_0 be the induced distance on M, i.e. $d_0(x,y) = |x - y|$.
If (M, g) is a Riemannian polyhedron, then the natural way to define an *intrinsic distance* $d = d_g$ on M is analogous to the case of Riemannian manifolds: For a Lipschitz continuous curve[3] $\varphi : [a,b] \to M$ define the *length* of φ by

$$L(\varphi) := L_g(\varphi) := \int_a^b \sqrt{g_{\varphi(\tau)}(\dot\varphi(\tau), \dot\varphi(\tau))} d\tau. \tag{1.35}$$

[2] In order to be precise, one should write $\pi^{S_1}(\nabla^{S^2}_u W)$, where W is a normal vector field with $W(x) = w$. But it is known, that this only depends on w, not on the whole vector field W, cf. [Jos02], Lemma 3.6.1.

[3] here we use the term Lipschitz w.r.t. to the metric which is induced by the ambient vector space V. Note that the actual Lipschitz constant depends on the embedding, while the property of being Lipschitz continuous does not.

For details we refer to [EF01], section I.4. Now define

$$d_g(x,y) := \inf_{\varphi(a)=x,\varphi(b)=y} L(\varphi).$$

Proposition 1.3.9 *(M, d_g) is a complete geodesic space. The metrics $d = d_g$, and d_0 are locally equivalent. In particular, M is proper*[4]*. Moreover, d is equal to the* Caratheodory distance d_{Car}*:*

$$d(x,y) = d_{\text{Car}}(x,y) := \max\{|f(x) - f(y)| : \text{Lip}(f) \leq 1\}. \tag{1.36}$$

Proof : [EF01], Lemma 4.2 and Proposition 4.1, cf. also [BBI01]. Note that in our setting, M is locally a finite union of closed simplices and therefor complete w.r.t. d_0. □

Remark 1.3.10 Let (M, g) be a Riemannian polyhedron with intrinsic distance $d = d^g$. If $(\widetilde{M}, \widetilde{d})$ is a metric space and $\theta : \widetilde{M} \to M$ is an isometry, then $\widetilde{M}$ becomes a Riemannian polyhedron itself by pulling back the metric tensor g. θ is then called an *isometric triangulation*.

Remark 1.3.11 Throughout this section we will use the following notation: For $u, v \in T_xM$ set

$$\langle u, v\rangle_x := g_x(u,v) \quad \text{and} \quad \|u\| := \|u\|_x := \sqrt{g_x(u,u)}. \tag{1.37}$$

Note that we may regard x and u as vectors in V (where $V \supset M$ is an ambient vector space). We will always assume that V is equipped with a Euclidean scalar product and by $|x|$ and $|v|$ we mean the norm w.r.t. this fixed scalar product.

As we will see in the sequel, d_g can be quite nasty (for instance if one investigates the regularity of geodesics). So it will be useful to approximate d_g locally by a simpler intrinsic distance, as follows: Let $S \in \mathcal{S}(M)$, $\xi : O \to \widehat{O}$ be a simplicial chart local at S and $x_0 \in S \cap O$. Set

$$g_0 :\equiv g_{x_0} \tag{1.38}$$

More precisely, if $\widetilde{S} \in \text{st}(x_0)$, denote by $U^{\widetilde{S}} \subset V$ the linear subspace generated by $\widetilde{S}$[5]. Then $g_{x_0|T_{x_0}\widetilde{S}\times T_{x_0}\widetilde{S}}$ extends to a Euclidean scalar product $g^{\widetilde{S}}$ on $U^{\widetilde{S}} \cong T_{x_0}\widetilde{S}$. Now for $x \in O$, there is a unique $\widetilde{S} \in \text{st}(x_0)$ such that $x \in \widetilde{S}^\circ$. So if and $v, w \in T_xM \cong U^{\widetilde{S}}$, we set $g_0(v,w) := g^{\widetilde{S}}(v,w)$.

[4]A metric space (M, d) is called *proper* if closed balls in M are compact

[5]due to the chart ξ, we can assume that O is a subset of V

(O, g_0) is a neighborhood in a Euclidean complex. Euclidean complexes are much better understood than general Riemannian polyhedra, and we will quote some features of them in section 1.4.
Let $\varphi : [a, b] \to M$ be a Lipschitz continuous curve with $\mathrm{Lip}(\varphi) \le 1$ and $\varphi(a) = x_0$. Because g is Lipschitz continuous (say, with Lipschitz constant C), we have

$$\begin{aligned} |L_g(\varphi) - L_{g_0}(\varphi)| &\le \int_a^b |(g - g_0)(\dot\varphi(\tau), \dot\varphi(\tau))| d\tau \\ &\le C \int_a^b \tau |\dot\varphi(\tau)| d\tau \le C(b-a)^2, \end{aligned} \tag{1.39}$$

and consequently

$$\lim_{t \searrow a} \frac{L_g(\varphi_{|[a,t]})}{L_{g_0}(\varphi_{|[a,t]})} = 1. \tag{1.40}$$

In particular, if $y \in O$, we can apply this to $\gamma : [0, t] \to O$ and $\gamma_0 : [0, t] \to O$, where γ is a geodesic (w.r.t. g) from x_0 to y and γ_0 is a geodesic (w.r.t. g_0) from x_0 to y, in order to find a suitable constant C such that for all $y \in O$,

$$|d_g(x_0, y) - d_{g_0}(x_0, y)| \le C|y - x_0|^2 \tag{1.41}$$

and consequently

$$\lim_{y \to x_0} \frac{d_g(x_0, y)}{d_{g_0}(x_0, y)} = 1 \tag{1.42}$$

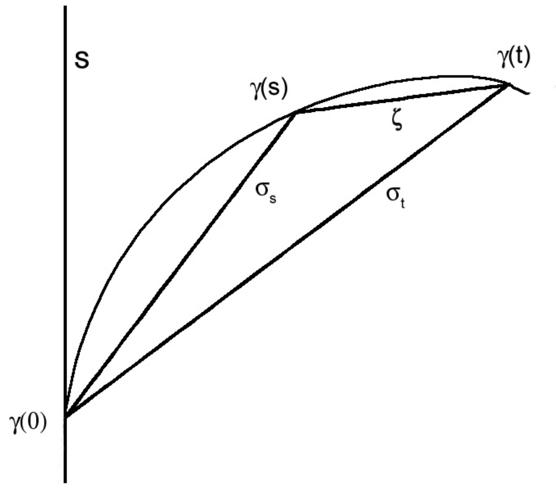
and

$$\begin{aligned} |d_g^2(x_0, y) - d_{g_0}^2(x_0, y)| &= |d_g(x_0, y) - d_{g_0}(x_0, y)||d_g(x_0, y) + d_{g_0}(x_0, y)| \\ &\le \widetilde{C}|y - x_0|^3. \end{aligned} \tag{1.43}$$

Many geometric statements in smooth Riemannian manifolds rely on the fact that geodesics are smooth and that locally a geodesic connecting two points is unique and depends smoothly on its endpoints. This is false in general Riemannian polyhedra. Even in a Riemannian manifold with boundary, geodesics are not smooth anymore. Consider for instance the Euclidean plane with the unit disc removed: A geodesic may enter the boundary (i.e. the unit circle), stay in the boundary some time and then peel into the interior. At the points where the geodesic switches from the interior to the boundary (and vice versa), it is not $\mathcal{C}^2$ anymore. More precisely, the acceleration has a jump at these 'switch points'.
Now we will show that geodesics in a Riemannian polyhedron have one-sided derivatives in the following sense: We may regard a geodesic $\gamma : [a, b] \to M \subset V$ as a curve in V. Note that the property of having a one-sided derivative in V does not depend on the choice of the embedding into V.

Lemma 1.3.12 *Let $\gamma : [a,b] \to M$ be a geodesic. Then the right-hand derivative $\dot\gamma(a+) := \lim_{s\searrow a}(s-a)^{-1}(\gamma(s)-\gamma(a))$ exists and $\|\dot\gamma(a+)\| = 1$.*

Proof : 1. We may assume that $a = 0$. Let $\gamma(0) \in S^\circ$ and let t be so small that $\gamma_{|[0,t]}$ is contained in a neighborhood O that is local at S. Let $g_0 :\equiv g_{\gamma(0)}$. Denote by σ_t the geodesic[6] w.r.t. g_0 from $\gamma(0)$ to $\gamma(t)$ and by $\beta_{s,t}$ the positive angle between σ_s and σ_t $(s \le t)$.



We first claim that there is a $D > 0$ such that for all t

$$\beta_{s,t} \le Dt^{1/2} \qquad (\forall t << 1,\ s \in [t/2,t]). \tag{1.44}$$

Indeed, put $a_s := d_{g_0}(\gamma(0),\gamma(s))$, $b_{s,t} := d_{g_0}(\gamma(s),\gamma(t))$ and $c_t := d_{g_0}(\gamma(0),\gamma(t))$. Let $\zeta : [s,t] \to M$ be a geodesic (w.r.t. g_0) from $\gamma(s)$ to $\gamma(t)$ (so $b_{s,t} = L_0(\zeta)$). The set $\{\gamma(0) + \lambda(\zeta(\tau) - \gamma(0)) : 0 \le \lambda \le 1, s \le \tau \le t\}$, regarded as a subspace of (O, g_0), is isometric to a Euclidean triangle with edges of length a_s, $b_{s,t}$ and c_t by Proposition 1.4.4 (ii). Let $h_{s,t}$ be the distance between $\gamma(s)$ and σ_t. A little computation in Euclidean trigonometry shows that $a_s + b_{s,t} \ge \sqrt{4h_{s,t}^2 + c_t^2}$ (with equality iff $a_s = b_{s,t}$). Thus (1.41) yields

$$t = d(\gamma(0),\gamma(t/2)) + d(\gamma(t/2),\gamma(t)) \ge a_{s,t} + b_{s,t} - Ct^2 \ge \sqrt{4h_{s,t}^2 + c_t^2} - Ct^2$$

and hence $4h_t^2 + c_t^2 \le t^2 + 2Ct^3 + C^2t^4$. Now $c_t^2 \ge t^2 - Ct^3$ by (1.43) and hence $4h_{s,t}^2 \le 3Ct^3 + C^2t^4$. Taking into account that $t \le 1$, this yields

$$h_{s,t} \le \frac{\widetilde{C}}{4} t^{3/2}. \tag{1.45}$$

[6]in fact, σ_t is unique and a straight segment, since (O, g_0) is isometric to a neighborhood of the origin $0 \cong \gamma(0)$ in the Euclidean cone $C_0(\mathrm{Lk}(\gamma(0)))$, cf. Proposition 1.4.4 (i).

At last, since t is small and $s \geq \frac{t}{2}$, we deduce from (1.41) that $a_s \geq \frac{t}{2} - Ct^2 \geq \frac{t}{4}$ and hence $\sin \beta_{s,t} = \frac{h_{s,t}}{a_s} \leq \widetilde{C} t^{1/2}$, and because arcsin is Lipschitz continuous around 0 with $\arcsin 0 = 0$, we obtain (1.44).
Let $k \in \mathbb{N}$. Then the triangle inequality for angles and (1.44) yield

$$\beta_{s,t} \leq \sum_{l=1}^{k} \beta_{2^{-l}t, 2^{-(l-1)}t} \leq Dt^{1/2} \sum_{l=1}^{k} (\sqrt{2})^{-l} \leq \frac{D}{1 - \frac{1}{2}\sqrt{2}} t^{1/2}.$$

for all $s \in [2^{-k}t, t]$. So letting $k \to \infty$, we obtain

$$\beta_{s,t} \leq \widetilde{D} t^{1/2} \qquad (\forall s \leq t << 1,). \tag{1.46}$$

2. Let t_n be a sequence of points converging to 0 from the right. Because γ is Lipschitz, we may assume that $\frac{1}{t_n}(\gamma(t_n) - \gamma(0))$ converges to some vector $v \in V$ by passing to a subsequence. First note that $t = d(\gamma(t), \gamma(0))$ and hence by (1.42),

$$\|v\| = \|v\|_0 = \lim_{t \to 0} \frac{1}{t} \|\gamma(t) - \gamma(0)\|_0 = \lim_{t \to 0} \frac{d_{g_0}(\gamma(t), \gamma(0))}{d(\gamma(t), \gamma(0))} = 1.$$

Thus in order to prove the Lemma, we have to show that whenever s_n is a sequence of points converging to 0 from the right such that $\frac{1}{s_n}(\gamma(s_n) - \gamma(0))$ converges to w, then $v = w$. So let $m \in \mathbb{N}$. Then

$$\angle(v, \sigma_{t_m}) = \lim_{n \to \infty} \beta_{t_n, t_m} \leq \widetilde{D} t_m^{1/2}$$

and

$$\angle(w, \sigma_{t_m}) = \lim_{n \to \infty} \beta_{s_n, t_m} \leq \widetilde{D} t_m^{1/2}$$

So using the triangle inequality for angles and letting $m \to \infty$ we obtain $\angle(v, w) = 0$ and hence $v = w$. $\square$

Clearly, γ is not differentiable in general since the space M is not. But what about the tangential part of γ in a local chart? More precisely, let $\xi : O \to \widehat{O}$ be a simplicial chart local at some $S \in \mathcal{S}$. Let now $\xi : O \to \widehat{O}$ be a chart local at $S \in \mathcal{S}$. If $\varphi : [a, b] \to O$ is a curve, then we can split φ into a tangential and a transversal part. Namely, $\widehat{\varphi} = \varphi^\top + \varphi^\perp$, where $\varphi^\top = \sum_{u \in \mathrm{scaff}(\widehat{S})} \hat{\varphi}^u u$ and $\varphi^\perp = \sum_{u \in \mathrm{scaff}(\perp \widehat{S})} \hat{\varphi}^u u$. Likewise, for $x \in O$, we may split $T_x M$ into a tangential and a transversal part (cf. (1.19)):

$$T_x M = T_x M^\top \oplus T_x M^\perp, \tag{1.47}$$

where $T_x M^\top$ is the subspace generated by $\{\frac{\partial}{\partial \hat{x}^u} : u \in \mathrm{scaff}(\widehat{S})\}$ and $T_x M^\perp$ is the subspace generated by $\{\frac{\partial}{\partial \hat{x}^u} : u \in \mathrm{scaff}(\perp \widehat{S})\}$. So if φ is differentiable at $t \in [a, b]$, then $\dot{\varphi} = \dot{\varphi}^\top + \dot{\varphi}^\perp$, where $\dot{\varphi}^\top = \sum_{u \in \mathrm{scaff}(\widehat{S})} \dot{\varphi}^u \frac{\partial}{\partial \hat{x}^u}$ and $\dot{\varphi}^\perp = \sum_{u \in \mathrm{scaff}(\perp \widehat{S})} \dot{\varphi}^u \frac{\partial}{\partial \hat{x}^u}$.

Remark 1.3.13 We may point out another time that even if the chart ξ is normal at S and $x \in O$, the decomposition of T_xM in (1.47) is in general *not* an orthogonal decomposition. The fact that ξ is normal at S means that (1.47) is an orthogonal decomposition for all $x \in S \cap O$, but it may not if $x \in O \setminus S$ (unless we have *special normal coordinates* at S, as e.g. if $S \in \mathcal{S}^{(n-1)}$, cf. Remark 1.3.4 (iii)). However, if x is close to S, it follows from the Lipschitz continuity of the derivatives of ξ that (1.47) is nearly an orthogonal decomposition. We will make this argument precise in the proof of Lemma 1.3.15.

We will prove some regularity of $\gamma^\top$ with the help of calculus of variations, or more precisely, a Lagrangian argument. Note that $\gamma^\top$ is a minimizer of the functional

$$\varphi \mapsto \int_a^b \langle \dot\varphi(\tau) + \dot\gamma^\perp(\tau), \dot\varphi(\tau) + \dot\gamma^\perp(\tau) \rangle_{\varphi(\tau)+\gamma^\perp(\tau)} \tag{1.48}$$

where we minimize over all Lipschitz curves $\varphi : [a,b] \to \widehat{O} \cap \widehat{S}$ with $\varphi(a) = \gamma^\top(a)$ and $\varphi(b) = \gamma^\top(b)$.
So let us recall some notations and facts from calculus of variations. Let $U \subset \mathbb{R}^m$ be open and let $F : [a,b] \times U \times \mathbb{R}^m \to \mathbb{R}$, $(t,x,v) \mapsto F(t,x,v)$ be a function ('Lagrangian') that is $\mathcal{C}^1$ w.r.t. x and v. On the space of Lipschitz continuous curves $\varphi : [a,b] \to U$ define the functional

$$\Phi(\varphi) := \Phi^F(\varphi) := \int_a^b F(\tau, \varphi(\tau), \dot\varphi(\tau)) d\tau. \tag{1.49}$$

A local minimum γ of Φ satisfies the *Euler-Lagrange equation*, cf. [GH96]. We will state the integrated version. Namely, there is a $c \in \mathbb{R}$ such that

$$\partial_v F(t, \gamma(t), \dot\gamma(t)) = c + \int_a^t \partial_x F(\tau, \gamma(\tau), \dot\gamma(\tau)) d\tau \tag{1.50}$$

for Lebesgue-almost all $t \in [a,b]$. Note that sometimes we will regard $\partial_v F$ and $\partial_x F$ as linear forms and sometimes as vector fields (i.e. as gradients), which makes no essential difference. In either case, we write $|\partial_v F|$ for a suitable norm.
There are a lot of results that say that if F is regular (in some sense), then every local minimizer is also regular. For instance, if F is $\mathcal{C}^1$, then a local minimizer is $\mathcal{C}^2$, cf. [GH96], Chapter 1.3.1, Proposition 4. We will use a similar technique (basically an appliciation of the implicit function theorem) in order to prove a similar result in the case that F satisfies a much weaker regularity assumption.

Lemma 1.3.14 *Let $C > 0$ and let $F(t,x,v)$ be a Lagrangian with the following properties:*

- *F is uniformly elliptic in v in the sense that $\mathrm{Hess}_v F(t,x,v) \geq C^{-1}$ for all $(t,x,v) \in [a,b] \times U \times \mathbb{R}^m$*
- *$|\partial_x F(t,x,v)| \leq C$ for all $(t,x,v) \in [a,b] \times U \times B_1(0)$*

Let $\phi : [a,b] \to U$ be a local minimizer of Φ^F with $\mathrm{Lip}(\phi) \leq 1$. Then the following holds:

(i) There is a nullset $N \subset [a,b]$ such that whenever $s,t \in [a,b] \setminus N$ and $|\partial_v F(t,\phi(t),v) - \partial_v F(s,\phi(s),v)| \leq \epsilon$ for all $v \in B_1(x)$, then $|\dot\phi(t) - \dot\phi(s)| \leq C(\epsilon + C|t-s|)$.

(ii) If $|\partial_v F(t,x,v) - \partial_v F(s,\tilde x,v)| \leq C\left(|t-s| + |x-\tilde x|\right)$ for all $v \in B_1(0)$ and $(t,x),(s,\tilde x) \in [a,b]\times U$, then ϕ is differentiable and $\dot\phi$ is Lipschitz continuous with $\mathrm{Lip}(\dot\phi) \leq 3C^2$.

Proof : Let $\rho(t) := c + \int_a^t \partial_x F(\tau,\phi(\tau),\dot\phi(\tau))d\tau$ (the right hand side of (1.50)) and let $G_t(v) := \partial_v F(t,\phi(t),v) - \rho(t)$. Then $\partial G_t(v) = \mathrm{Hess}_v F(t,x,v)$ is invertible for all v and it follows from the uniform ellipticity of F and the inverse mapping theorem that G_t is a $\mathcal{C}^1-$diffeomorphism from $\mathbb{R}^m$ onto $\mathbb{R}^m$ with $\|\partial G_t^{-1}\| \leq C$. In particular, $\mathrm{Lip}(G_t^{-1}) \leq C$.
Put $v(t) := G_t^{-1}(0)$. Then for all $t \in [a,b]$ we have $G_t(v) = 0$ if and only if $v = v(t)$, and by (1.50), $\dot\phi(t) = v(t)$ for almost every $t \in [a,b]$. First note that for all $v \in B_1(0)$

$$\begin{aligned}|G_t(v) - G_s(v)| &\leq |\partial_v F(t,\phi(t),v) - \partial_v F(s,\phi(s),v)| + |\rho(t) - \rho(s)| \\ &\leq \epsilon + C|t-s|.\end{aligned}$$

Since $\mathrm{Lip}(\phi) \leq 1$, $|v(t)| = |\dot\phi(t)| \leq 1$ for almost all t and hence

$$\begin{aligned}|v(t) - v(s)| &= |G_t^{-1}(0) - G_t^{-1}(G_t(G_s^{-1}(0)))| \\ &\leq C|0 - G_t(G_s^{-1}(0))| \\ &= C|G_s(G_s^{-1}(0)) - G_t(G_s^{-1}(0))| \\ &\leq C(\epsilon + C|t-s|),\end{aligned}$$

proving (i).

(ii) From the assumtions of (ii) and the above calculations we deduce that v is Lipschitz continuous with $\mathrm{Lip}(v) \leq 3C^2$. Because ϕ is absolutely continuous, $\phi(t) - \phi(t_0) = \int_{t_0}^t \dot\phi(\tau)d\tau = \int_{t_0}^t v(\tau)d\tau$. Thus we see that ϕ is differentiable and

$\dot{\phi} \equiv v$. This proves the Lemma.□

Now we will apply this result to our situation of the tangential part of a geodesic. Note that the right hand side of (1.48) is equal to Φ^F for

$$F(t,x,v) := \langle v + \dot{\gamma}^{\perp}(t), v + \dot{\gamma}^{\perp}(t)\rangle_{x+\gamma^{\perp}(t)} = \sum_{i,j} g_{ij}(x+\gamma^{\perp}(t))v^i\dot{\gamma}^j(t) \qquad (1.51)$$

Lemma 1.3.15 *Let $S \in \mathcal{S}$, O be a neighborhood that is local at S and let $\xi : O \to \widehat{O}$ be a simplicial chart that is normal at S. Then there is a $C > 0$ such that whenever $x \in S^{\circ}$, $r > 0$ and $\gamma : [a,b] \to B_r(x) \cap O$ is a unit-speed geodesic , then*

(i) There is a Lebesgue-nullset $N \subset [a,b]$ such that for all $s,t \in [a,b] \setminus N$, $|\dot{\gamma}^{\top}(t) - \dot{\gamma}^{\top}(s)| \leq C(r + |t-s|)$.

(ii) For all $s \leq t \in [a,b]$, $|\gamma^{\top}(t) - \gamma^{\top}(s) - \dot{\gamma}^{\top}(s+)(t-s)| \leq Cr(t-s))$.

(iii) For all $s \leq t \in [a,b]$, $(t-s)|\dot{\gamma}^{\perp}(s+)| \leq C\left(|\gamma^{\perp}(t) - \gamma^{\perp}(s)| + \sqrt{r}(t-s)\right)$.

Proof : (i) Consider the Lagrangian $F(t,x,v)$, defined in (1.51). Then

$$\partial_v F(t,x,v)(h) = 2\langle v + \pi(\dot{\gamma}^{\perp}(t)), h\rangle_{x+\gamma^{\perp}(t)}, \qquad (1.52)$$

where for $y \in O$ and $w \in T_yM$, $\pi(w)$ is the orthogonal projection of w onto $T_yM^{\top}$, cf. (1.47).
Note that in general, $\pi(w^{\perp}) \neq 0$ (cf. Remark 1.3.13). However, since ξ is normal at S and the derivatives of ξ are Lipschitz continuous, we have that $\|\pi(w^{\perp})\| \leq \widetilde{C}r$ whenever $w \in T_yM$ with $y \in B_r(x)$ and $\|w\| = 1$. Consequently, $\|\partial_v F(t,x,v) - \partial_v F(s,x,v)\| \leq \widetilde{C}(r + |t-s|)$. Now $\gamma^{\top}$ is a local minimizer of Φ^F and F satisfies the assumptions of Lemma 1.3.14 with $\epsilon = \widetilde{C}(r + |t-s|)$. Thus (i) follows from Lemma 1.3.14 (i).

(ii) Let N be the nullset of (i). It is contained in the formulation of (i) that $\dot{\gamma}(s)$ exists for all $s \in [a,b] \setminus N$. So let $s \in [a,b] \setminus N$. Integrating the inequality in (i) yields that

$$\left|\frac{1}{t-s}(\gamma^{\top}(t) - \gamma^{\top}(s)) - \dot{\gamma}^{\top}(s)\right| \leq C(r + |t-s|) \qquad (1.53)$$

for *all* $t \in [a,b]$ with $t > s$.
Let now $s \in N$ and let s_n be a sequence with $s_n \in [a,b] \setminus N$ and $s_n \searrow s$. Then there is a subsequence, again denoted by s_n, such that $\dot{\gamma}^{\top}(s_n) \to v$. So by (1.53)

it follows that $|\frac{1}{t-s}(\gamma^\top(t)-\gamma^\top(s))-v|\leq C(r+|t-s|)$ for all $t>s$, and letting $t\searrow s$, we obtain $|\dot\gamma^\top(s+)-v|\leq Cr$. Consequently,

$$|\frac{1}{t-s}(\gamma^\top(t)-\gamma^\top(s))-\dot\gamma^\top(s+)|\leq C(r+|t-s|)\leq 3Cr,$$

where the last inequality follows from the fact that $t-s=d(\gamma(t),\gamma(s))\leq 2r$. Thus (ii) is proved.

(iii) Let $x\in S\cap O$ and set $g_0:\equiv g_x$, cf. (1.38). But contrary to that situation, we do not assume that $x=\gamma(0)$ (because $\gamma(0)$ need not be in S). So instead of (1.39) we only get that whenever $\varphi:[s,t]\to B_r(x)$ is a Lipschitz curve with $\mathrm{Lip}(\varphi)\leq 1$, then

$$|L_g(\varphi)-L_{g_0}(\varphi)|\leq Cr(t-s) \tag{1.54}$$

and consequently

$$|(t-s)^2-d^2_{g_0}(\gamma(t),\gamma(s))|=|d^2_g(\gamma(t),\gamma(s))-d^2_{g_0}(\gamma(t),\gamma(s))|\leq Cr|t-s|^2.$$

Because the derivatives of ξ are Lipschitz continuous, $|g_0(v^\top,v^\perp)|\leq Cr$ for all $y\in B_r(x)$ and $v\in T_yM$ with $\|v\|_0\leq 1$. In particular[7], $|g_0(\dot\gamma^\top(s+),\dot\gamma^\perp(s+))|\leq Cr$, which implies $\|\dot\gamma^\perp(s+)\|_0^2\leq\|\dot\gamma(s+)\|_0^2-\|\dot\gamma^\top(s+)\|_0^2+2Cr$. Moreover, $|\,\|\dot\gamma(s+)\|-\|\dot\gamma(s+)\|_0\,|\leq Cr$, and since $\|\dot\gamma(s+)\|\equiv 1$, we obtain

$$\|\dot\gamma^\perp(s+)\|_0^2\leq 1-\|\dot\gamma^\top(s+)\|_0^2+3Cr. \tag{1.55}$$

Now from (ii) we deduce

$$\begin{aligned} d_{g_0}(\gamma^\top(t),\gamma^\top(s)) &- (t-s)\|\dot\gamma^\top(s+)\|_0 \\ &= \|\gamma^\top(t)-\gamma^\top(s))\|_0-(t-s)\|\dot\gamma^\top(s+)\|_0 \\ &\leq C_1(t-s)^2+r(t-s) \\ &\leq C_2 r(t-s) \end{aligned}$$

and hence

$$d^2_{g_0}(\gamma^\top(t),\gamma^\top(s))-(t-s)^2\|\dot\gamma^\top(s+)\|_0^2\leq C_3 r(t-s)^2. \tag{1.56}$$

At last, note that since ξ is normal at S (and in particular normal at x), we have $d^2_{g_0}(\gamma(t),\gamma(s))=d^2_{g_0}(\gamma^\top(t),\gamma^\top(s))+d^2_{g_0}(\gamma^\perp(t),\gamma^\perp(s))$. So combining this with

[7] note that $\|\dot\gamma(s+)\|_0$ is uniformly bounded in $s\in[a,b]$ because $\|\dot\gamma(s+)\|\equiv 1$, and by scaling, we may assume that $\|\dot\gamma(s+)\|_0\leq 1$.

(1.55) and (1.56), we obtain

$$\begin{aligned}(t-s)^2\|\dot\gamma^\perp(s+)\|_0^2 &\le (t-s)^2\left(1-\|\dot\gamma^\top(s+)\|_0^2+3Cr\right)\\ &\le d_{g_0}^2(\gamma(t),\gamma(s))-(t-s)^2\|\dot\gamma^\top(s+)\|_0^2+4Cr(t-s)^2\\ &= d_{g_0}^2(\gamma^\top(t),\gamma^\top(s))+d_{g_0}^2(\gamma^\perp(t),\gamma^\perp(s))\\ &\qquad -(t-s)^2\|\dot\gamma^\top(s+)\|_0^2+4Cr(t-s)^2\\ &\le d_{g_0}^2(\gamma^\perp(t),\gamma^\perp(s))+(4C+C_3)r(t-s)^2\\ &\le \widetilde{C}\left(|\gamma^\perp(t)-\gamma^\perp(s)|^2+r(t-s)^2\right)\end{aligned}$$

and hence

$$\begin{aligned}(t-s)|\dot\gamma^\perp(s+)| &\le C(t-s)\|\dot\gamma^\perp(s+)\|_0\\ &\le C\sqrt{\widetilde{C}}\sqrt{|\gamma^\perp(t)-\gamma^\perp(s)|^2+r(t-s)^2}\\ &\le C\sqrt{\widetilde{C}}\left(|\gamma^\perp(t)-\gamma^\perp(s)|^2+\sqrt{r}(t-s)\right),\end{aligned}$$

which shows (iii). □

What we have proved now is a Taylor-like expansion for geodesics in a normal chart. We can reformulate our results in terms of a kind of *inverse exponential map*:

Definition 1.3.16 A *generalized inverse exponential map* is a measurable map $(x,y)\mapsto e_x(y): M^2\to TM$ with the property that for all $(x,y)\in M^2$ there is a geodesic $\gamma:[0,1]\to M$ from x to y such that $e_x(y)=\dot\gamma(0+)\in T_xM$.

Proposition 1.3.17 *A generalized inverse exponential map exists. For all $x,y\in M$, $\|e_x(y)\|=d(x,y)$. Moreover, let $S\in\mathcal{S}$, O be a neighborhood that is local at S and let $\xi:O\to\widehat{O}$ be a simplicial chart that is normal at S. Then there is a $C>0$ such that whenever $x_0\in S^\circ$, $r>0$ and $x,y\in B_r(x_0)$, then*

(i) $|y^\top-x^\top-e_x(y)^\top|\le Cr|y-x|$

(ii) $|e_x(y)^\perp|\le C\left(|y^\perp-x^\perp|+\sqrt{r}|y-x|\right)$.

Proof : Denote by $\mathcal{C}([0,1],M)$ the space of continuous curves $\varphi:[0,1]\to M$, equipped with the uniform distance. Let $G(x,y):=\{\gamma:[0,1]\to M:\gamma(0)=x,\gamma(1)=y\}\subset\mathcal{C}([0,1],M)$. So G can be regarded as a set-valued function $G: M^2\to\mathcal{P}(\mathcal{C}([0,1],M))$. The graph of G is closed by Proposition 2.5.17 of [BBI01]. In particular, G is closed-valued and measurable in the sense of [Wag77][8]. Thus by

[8] the measurability can easily be shown using the fact that M^2 is proper, which implies that the set of geodesics whose endpoints are contained in a bounded set is compact.

Theorem 4.1 of [Wag77], there exists a measurable selection $g : M^2 \to \mathcal{C}([0,1], M)$ with $g(x,y) \in G(x,y)$. In other words, there exists a *measurable choice of geodesics* in M. Now the map $\gamma \to \dot{\gamma}(0+)$ is measurable as the limit of the continuous maps $\gamma \to \frac{1}{n}(\gamma(\frac{1}{n}) - \gamma(0))$. Moreover, the fact that $\|e_x(y)\| = d(x,y)$ is a consequence Lemma 1.3.12, noting that in the definition of $e_x(y)$, the corresponding geodesic has constant speed equal to $d(x,y)$. At last, (i) and (ii) follow from Lemma 1.3.15 (ii) and (iii). $\square$

More regularity of geodesics

Until now we have treated the very general case, where γ was a geodesic in an arbitrary Riemannian polyhedron. Moreover, the regularity result of γ in terms of a local chart that is normal at some $S \in \mathcal{S}$ (Lemma 1.3.15) holds for any simplex S of arbitrary dimension. However, the technique used there, namely the Lagrangian method (Lemma 1.3.14), can be used to derive much more regularity in some special cases.

Let us first consider the case where γ is near a simplex with codimension one, i.e. a simplex $S \in \mathcal{S}^{(n-1)}$. As we have seen, the essential tool are *normal coordinates* at some $S \in \mathcal{S}$, and if S is arbitrary, the main difficulty in the analysis of geodesics arises from the fact such a normal chart $\xi : O \to \widehat{O}$ is only normal at S, not in the whole neighborhood O. But the situation is different when S has codimension 1, i.e. $S \in \mathcal{S}^{(n-1)}$. Then we have *special normal coordinates* at S, cf. Remark 1.3.4 (iii). In this case, the situation is much better, and in fact it is quite similar to the case where M is a Riemannian manifold with boundary. So although we will not need this in the sequel, we present how our variational technique yields a better regularity results for geodesics that are in a neighborhood which is local at some $S \in \mathcal{S}^{(n-1)}(M)$. This case is very similar to the situation where S is the boundary of a smooth Riemannian manifold.
A systematic investigation of the regularity of geodesics in a Riemannian manifold with boundary started quite lately, in the eighties. Alexander, Berg and Bishop published a series of papers in which they investigated regularity questions with geometric methods (cf.[AA81], [ABB87], [ABB93]; for other authors see the references quoted therein).
Let $\gamma : [a,b] \to M$ be a geodesic in M, parametrized by arclength (so we can take $a = 0$, $b = d(\gamma(0), \gamma(b))$). Let $S \in \mathcal{S}$. We say that a non-empty interval $]s,t[\subset [a,b]$ is an $S-$*segment* if $\gamma_{|]s,t[}$ lies entirely in S°. The union of all $S-$segments $(S \in \mathcal{S})$ is a dense open subset of $[a,b]$. Due to [ABB87], the remaining points $[a,b]$ are divided into two classes:

- *Switch points*, i.e. points where γ changes from an $S-$segment to an $\widetilde{S}-$segment.

- *Intermittend points* which are accumulation points of switch points.

Clearly, on an $S-$segment $]s,t[$, γ must be a geodesic in S (for the intrinsic geometry of S) and hence smooth. In local coordinates we have

$$\ddot{\gamma}^u(\tau) = -\sum_{v,w\in \mathrm{scaff}(S)} \Gamma^u_{vw}(S)(\gamma(\tau))\dot{\gamma}^v(\tau)\dot{\gamma}^w(\tau) \tag{1.57}$$

for all $\tau \in]s,t[$. Moreover, for all $T \in \mathrm{st}^{(n)}(S)$, the second derivative of γ in T (i.e. $\nabla^T_{\dot{\gamma}}\dot{\gamma}$) is normal to S and points outward T[9].

A switch point is a common endpoint of an $S-$segment and an $\widetilde{S}-$segment for some $S, \widetilde{S} \in \mathcal{S}$. Thus one-sided accelerations (w.r.t. any ambient simplex) exist. The bad points are the intermittend points. Unfortunately, this set can be rather large. For instance, in [AB91] is indicated how to construct a subset of the Euclidean space with $\mathcal{C}^\infty-$boundary such that the set of intermittend points of certain geodesics is a Cantor set having positive measure.

Proposition 1.3.18 *Let $S \in \mathcal{S}^{(n-1)}$ and O be a neighborhood that is local at S. Let $\xi : O \to \widehat{O}$ be special normal coordinates at S as in Remark 1.3.4 (iii). Then there is a $C > 0$ such that whenever $\gamma : [a,b] \to O$ is a unit-speed geodesic, then $\gamma^\top$ is differentiable and* $\mathrm{Lip}(\dot{\gamma}^\top) \leq C$.

Proof : Recall the definition of F in (1.51). As in (1.52) we have

$$\partial_v F(t,x,v)(h) := 2\langle v + \pi(\dot{\gamma}^\perp(t)), h\rangle_{x+\gamma^\perp(t)}$$

But since ξ is a special normal chart, $\pi(\dot{\gamma}^\perp(t)) \equiv 0$, and hence

$$\partial_v F(t,x,v)(h) := 2\langle v, h\rangle_{x+\gamma^\perp(t)}.$$

Consequently, there is a $\widetilde{C} > 0$ such that $|\partial_v F(t,x,v) - \partial_v F(s,\tilde{x},v)| \leq \widetilde{C}\,(|t-s| + |x-\tilde{x}|)$. Thus by Lemma 1.3.14 (ii), the Proposition is proved. □

Let us conclude this section with a remark about other possibilities to ensure some more regularity of geodesics.

Remark 1.3.19 (i) As we have seen in the proofs of Lemma 1.3.15 and Proposition 1.3.18, the key to regularity of geodesics is the regularity of the Lagrangian F. This regularity can be improved for example by requiring that the second fundamental form of S (cf. Remark 1.3.8) vanishes. In this case, S is totally geodesic,

[9] note that whenever S is a face of T, then we can consider the acceleration of γ in T. However, if S is also a face of $\widetilde{T}$, then we get a different acceleration in $\widetilde{T}$.

and one can show that there are no intermittend points, which makes an analysis of geodesics much simpler. Examples of these spaces are Euclidean polyhedra (cf. Proposition 1.4.2) or more generally $\mathbb{M}_\kappa-$complexes, cf. [BH99], Corollary I.7.29.
(ii) Another possibility is to impose *curvature bounds in the sense of Alexandrov* on M. In the special case when M is a space that is obtained by gluing together two Riemannian manifolds at their boundary[10] S, Kosovskii has given a characterization of Alexandrov curvature bounds in terms of the second fundamental form at S and then proved some nice regularity results for geodesics in such a space, cf. [Kos02b], [Kos02a] and [Kos04].

1.4 Euclidean polyhedra

We will treat the case of Euclidean complexes because of two reasons: First, they are much easier to handle than general Riemannianian polyhedra and second, tangent spaces of Riemannian polyhedra are Euclidean cone complexes.
A Euclidean cone complex is a Riemannian simplicial cone complex such that on all cones C, the metric $g_{|C}$ is Euclidean. In order to keep this section self-contained, we will give a formal

Definition 1.4.1 A *Euclidean simplicial (cone) complex* is a simplicial (cone) complex $(M, \mathcal{S}(M)) \subset V$ together with a family $(g_S)_{S\in\mathcal{S}(M)}$ with the following properties:

- g_S is a Euclidean scalar product on $U^S \times U^S$, where $U^S \subset V$ is the subspace generated by S.
- If $S = S_1 \cap S_2$, then $g_S = g_{S_1|U^S\times U^S} = g_{S_2|U^S\times U^S}$

A *Euclidean polyhedron* is a metric space (M, d) together with an isometry (=isometric triangulation) $\theta : M \to \widehat{M}$ such that $\widehat{M}$ is a Euclidean simplicial complex.
A *Euclidean conical polyhedron* is a metric space (M, d) together with an isometry (=isometric triangulation) $\theta : M \to \widehat{M}$ such that $\widehat{M}$ is a Euclidean simplicial cone complex.

First we will introduce the canonical local coordinates that we will use whenever we deal with a Euclidean simplicial complex M. Let O be local at $S \in \mathcal{S}$. Then there is a unique *orthogonal projection* $\pi = \pi^S : O \to S$ (O should be made smaller if necessary[11]), i.e. for all $x \in O$, $x - \pi(x)$ is orthogonal to S w.r.t. $g_{\widetilde{S}}$, where $\widetilde{S}$ is

[10] in the case of upper curvature bounds, one can take an arbitray finite number instead of two, cf. [Kos04]. In our setting, this is locally the situation when $S \in \mathcal{S}^{(n-1)}(M)$
[11] in fact, every S has a neighborhood O_S with $S^\circ \subset O_S \subset \mathrm{St}^\circ(S)$ such that $\pi : O_S \to S^\circ$.

the unique simplex such that $x \in \widetilde{S}^\circ$. We set

$$\perp S := \{\lambda(x - \pi(x)) : x \in O, \lambda \geq 0\} \tag{1.58}$$

$\perp S$ is a simplicial cone complex and

$$O = S \cap O \oplus \perp S \cap (O - x) \tag{1.59}$$

So $\perp S$ is orthogonal to S, i.e. $g(u, v) = 0$ whenever $u \in U^S$ and $v \in \perp S$. $\perp S$ is called the *orthogonal complement of* S *in* M. Note that $\perp S$ as a *canonical scaffold*, namely the unique scaffold that consists of unit vectors.

Let us now come to metric structures on Euclidean polyhedra. Note that the definition of a Euclidean simplicial complex is given in terms of an embedding into a vector space V. The crucial point of this embedding is that M canonically becomes a Riemannian polyhedron in the following sense: If $x \in S$ for some $S \in \mathcal{S}(M)$, then T_xS is naturally isomorphic to U^S (by inclusion)[12]. So if we set $g_{x|T_xS \times T_xS} := g^S$, then (M, g) is a Riemannian polyhedron. In the corresponding intrinsic distance, every simplex $S \in \mathcal{S}(M)$ is isometric to a Euclidean simplex (cf. also the definition of a Euclidean complex in [BH99]). Moreover, its geodesics are finite concatenations of straight lines, which is the content of the next

Proposition 1.4.2 *Let* M *be a Euclidean conical polyhedron. Then* (M, d) *is a complete geodesic space. For each geodesic* $\gamma : [a, b] \to M$ *there is a partition* $a = t_0 < \cdots < t_m = b$ *such that for all* $i = 0, \ldots m - 1$ *there is a* $C_i \in \mathcal{C}$ *such that* $\gamma_{|[t_i, t_{i+1}]}$ *is a straight segment in* C_i.
Proof : [BH99], Corollary I.7.29. □

An important construction in the theory of tangent spaces of metric spaces is the cone over a metric space Y. Consider for instance a set $Y \subset V$, where V is a vector space. Then the *cone over* Y is the set $\{\lambda y : y \in Y, \lambda \geq 0\}$.

Definition 1.4.3 Let (Y, ρ) be a metric space. Put $C := ([0, \infty[\times Y)_{/\sim}$, where $(\lambda, y) \sim (\tilde{\lambda}, \tilde{y}) \Leftrightarrow \lambda = \tilde{\lambda} = 0$. Define a distance d on C by

$$d^2((\lambda, y), (\tilde{\lambda}, \tilde{y})) := \lambda^2 + \tilde{\lambda}^2 - 2\lambda\tilde{\lambda}\cos(\min\{\rho(y, \tilde{y}), \pi\}). \tag{1.60}$$

Then $C_0(Y) := (C, d)$ is called the *Euclidean cone over* (Y, ρ).

The name "Euclidean cone" comes from the fact that in the definition of d, the Euclidean law of cosines is used. The 0 in the notation $C_0(Y)$ stands for "curvature

[12] in the language of differential geometry, we have a natural parallel transport

equals 0“, since Euclidean space is the model space with constant curvature equal to 0. One can also use the $\kappa-$hyperbolic or the $\kappa-$spherical law of cosines (i.e. the law of cosines in the model space of constant curvature equal to κ) in order to obtain a distance d_κ. The resulting space $C_\kappa(Y) := (C, d_\kappa)$ is called the $\kappa-$cone over Y. For details we refer to [BH99]), Definition I.5.6.

Proposition 1.4.4 *Let O be local at S and $x \in S^\circ$. Then the following holds:*
(i) There is an isometry $\theta : O \to \widehat{O} \subset C_0(\mathrm{Lk}(x))$ with $\theta(x) = 0 \in C_0(\mathrm{Lk}(x))$. In particular, For all $T \in \mathrm{st}(\perp S)$, $T \cap O$ is isometric to a neighborhood of 0 in the Euclidean simplicial cone $T \subset U^T$ (U^T being the linear subspace generated by T, equipped with the induced inner product g_T).
(ii) Let $\gamma : [a, b] \to O$ be a geodesic such that $\angle_x(\gamma(a), \gamma(b)) < \pi$. Then the set $\{x + \lambda(\gamma(\tau) - x) : \tau \in [a, b], 0 \leq \lambda \leq 1\}$ is isometric to a Euclidean triangle with side lengths $d(x, \gamma(a))$, $d(x, \gamma(b))$ and $d(\gamma(a), \gamma(b))$.

Proof : (i) [BH99], Theorem I.7.16.
(ii) follows from (i) and [BH99], Proposition I.5.10. □.

Convex functions

Let $\varphi : M \to \mathbb{R}$ be a convex function and let $\gamma : [a, b] \to M$ be a geodesic with $\gamma(a) = x$ and $\dot\gamma(a+) = v \in T_xM$. Then the difference quotient $\frac{1}{t}[\varphi(\gamma(t)) - \varphi(\gamma(a))]$ is nondecreasing, and we may define the one-sided derivative of φ in direction v by

$$\begin{aligned}\partial\varphi_x(v) := \partial_v\varphi(x) : &= \lim_{t\searrow a} \frac{\varphi(\gamma(t)) - \varphi(\gamma(a))}{t} \\ &= \inf_{t\in]a,b]} \frac{\varphi(\gamma(t)) - \varphi(\gamma(a))}{t} \qquad \in \mathbb{R} \cup \{-\infty\} \qquad (1.61)\end{aligned}$$

Note that $\frac{1}{t}\varphi(\gamma(t)) - \varphi(x) \geq \partial\varphi_x(v)$ for all $t \in]a, b]$

Lemma 1.4.5 *Let $\varphi : M \to \mathbb{R}$ be convex. Then $\partial\varphi_x$ is convex on T_xM.*

Proof : Let $r > 0$ be so small that $B_r(x)$ is local at S, where $S \in \mathcal{S}$ is the unique simplex such that $x \in S^\circ$. Define $\theta_x^r : B_1(0_{T_xM}) \to B_r(x) \subset M$ by $\theta_x^r(y) := x + ry$ and set $\varphi_x^r := \varphi \circ \theta_x^r$, i.e. $\varphi_x^r(y) = \varphi(x + ry)$. Because θ_x^r maps geodesics in $B_1(0_{T_xM})$ to geodesics in $B_r(x)$ (of course, their length is decreased by the factor r), φ_x^r is convex. Since $\varphi_x^r \to \partial\varphi_x$ pointwise, $\partial\varphi_x$ is convex on $B_1(0_{T_xM})$ as a limit of convex functions. At last, because $\partial\varphi_x$ is *radial*[13] *at* 0, it is convex on the whole T_xM. □

[13] if M is a conical polyhedron, then a function $f : M \to \mathbb{R}$ is called *radial* if $f(rx) = rf(x)$ for all $x \in M$,$r \geq 0$.

Chapter 2

Stochastic calculus in Polyhedra

Important note: Throughout this text, we will be given a filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, P)$ satisfying the usual conditions. Moreover, all assertions that are made about random variables are understood to hold *almost surely*.

In this chaper we develop a theory of stochastic calculus and stochastic integration in polyhedra as an analogue of stochastic calculus in manifolds.
As a motivation, consider a semimartingale X in $\mathbb{R}^n$. Then by Itô's formula, $f(X)$ is a semimartingale for all smooth functions $f : \mathbb{R}^n \to \mathbb{R}$. But what happens if f has some singularites?
For $n = 1$ (i.e. X is a real semimartingale), the theory of local times can be used to generalize Itô's formula for certain non-smooth functions. Consider a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is smooth on $\mathbb{R}_+$ and (i.e., the restriction of a smooth function to the closed set $\mathbb{R}_+$) and on $\mathbb{R}_-$, but whose derivative has a jump in 0[1], such as the function $f(x) = |x|$. Then $f(X)$ is a semimartingale and its local behavior at 0 can be described in terms of local times, cf. [RY99], chapter VI.
As a direct consequence, one shows an analogous result when X is a semimartingale in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a function whose differential has a jump (in transversal direction) on a hypersurface, cf. [GP03].
In section 2.1 we generalize this technique to the case of a *piecewise smooth* function whose set of singularities is a simplicial cone complex: Assume that $\mathbb{R}^n$ has a triangulation $\mathcal{S}$ into a simplicial cone complex and let f be a piecewise smooth function. We show that $f(X)$ is a semimartingale and give a local desription of $f(X)$ at the simplicial cones in terms of *directional local times* ('Local Itô formula', Theorem 2.1.13). Note that this is a generalization of equation (3.1.8) in [Pic05]
This piecewise smooth stochastic calculus can now be generalized to polyhedra (section 2.2) by using the differentiable structures developed in section 1.2. In

[1]in our terminology, f is *piecewise smooth*.

particular, we present coordinate-free definitions of stochastic integrals in polyhedra such as in the case of manifolds.
In section 2.3 we first introduce the notion of an *Itô integral* in a Riemannian polyhedron and reformulate the local Itô formula in an intrinsic language. It uses the terminology of section 1.3. Then we use our results from section 1.3.3 about the generalized inverse exponential maps in order to prove a discrete approximation result for the $b-$quadratic variation. In particular, a semimartingale has a finite *quadratic variation*, i.e. the squared discretized increments converge to a nondecreasing process $\langle X \rangle$.
At last, we study Brownian motion (more precisely, *isotropic* Brownian motion) in a Riemannian polyhedron and show that it is a semimartingale.

2.1 Semimartingales in simplicial cone complexes

Let $(M, \mathcal{C})$ be simplicial cone complex in V and let $X : \Omega \times \mathbb{R}_+ \to M$ be an adapted continuous process. As in the spirit of manifolds, we could say that X is a semimartingale if $f(X)$ is a semimartingale for all piecewise smooth functions $f : M \to \mathbb{R}$.
On the other hand, $X \in M \subset V$ and we could also say that X is a semimartingale if it is a semimartingale w.r.t. the linear structure of V. However, it turns out that both possible definitions are equivalent, cf. Proposition 2.1.8. In order to keep the arguments simple, we will first prove this for the special case that M is a vector space which is divided into a simplicial cone complex in subsection 2.1.1 below, cf. Proposition 2.1.2. In subsection 2.1.2 we will treat the general case by regarding $M \subset V$ as a sub-complex of V (where V has a triangulation that extends the one of M) and using the extension procedure from Example 1.1.8 in order to prove Proposition 2.1.8.
If X is a semimartingale (in either definition) and $f : M \to \mathbb{R}$ is piecewise smooth, then we can decompose $f(X)$ into a sum: $f(X_t) - f(X_0) = \sum_{S \in \mathcal{S}(M)} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} df(X_\tau)$, cf. (2.25). $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} df(X_\tau)$ is a continuous semimartingale. The main Theorem of this section is a local Itô formula at S (Theorem 2.1.13), which decomposes $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} df(X_\tau)$ into a first and second order part (as in the classical case) and a third term that comes from the singularity of f at S. This third term, a process of bounded variation, will be given in terms of the *directional local times* of X at S, cf. Definition 2.1.9. The directional local times are nondecreasing continuous processes that describe the behavior of X at S and will be an important tool in the sequel.

2.1.1 The case $M = V$

Let V be an $N-$dimensional vector space with a triangulation $\mathcal{S}(V)$ into a simplicial cone complex. Let X be a continuous semimartingale (for the linear structure in V). By Itô's formula, $f(X)$ is a semimartingale for all smooth $f : V \to \mathbb{R}$. We want to prove that $f(X)$ is a semimartingale for any *piecewise smooth* function $f : V \to \mathbb{R}$. This will be done by a smoothing procedure.
Consider the family of standard mollifiers on V, defined in the following way: Let $\varrho : \mathbb{R} \to \mathbb{R}_+$ be given by

$$\varrho(t) := \begin{cases} \exp(\frac{-1}{1-t^2}) & \text{if } |t| < 1 \\ 0 & \text{if } |t| \geq 1 \end{cases} \tag{2.1}$$

Now let $b_1, \ldots, b_N$ be a basis of V and let $x^i = db_i(x)$ be the coordinates of x w.r.t. $b_1, \ldots, b_N$. Let $\|x\| := (\sum_{i=1}^N (x^i)^2)^{1/2}$ (so we regard V as a Euclidean space where $b_1, \ldots, b_N$ is an orthonormal basis). Finally, let $\psi_k(x) := \frac{1}{I_k}\varrho(k\|x\|)$, where $I_k := \int_V \varrho(k\|x\|)dx = k^N \int_V \varrho(\|x\|)dx$. Put $f^k := f*\psi^k$, i.e. $f^k(x) = \int_V f(y)\psi^k(y-x)dy$. Note that with this definition, we have for a multi-index $\alpha \in \bigcup_{i\in\mathbb{N}}\{1, \ldots, N\}^i$,

$$\partial_\alpha f^k(x) = (-1)^{|\alpha|} \int f(y)\partial_\alpha \Psi_k(y-x)dy \tag{2.2}$$

and if $\partial_\alpha f$ exists and is continuous, then integrating by parts yields

$$\partial_\alpha f^k(x) = \int \partial_\alpha f(y)\Psi_k(y-x)dy. \tag{2.3}$$

Let now $f : V \to \mathbb{R}$ be piecewise smooth w.r.t. $\mathcal{S}$. For $i = 1, \ldots N$ we define the ith partial derivative $\tilde{\partial}_i f(x)$ by

$$\tilde{\partial}_i f(x) := \begin{cases} \partial_i f(x) & \text{if } x \in S^\circ \text{ for some } S \in \mathcal{S}^{(N)} \\ \sum_{S\in \mathrm{st}^{(N)}(x)} \partial_i f_{|S}(x)\mu_x(S) & \text{else} \end{cases} \tag{2.4}$$

where μ_x is the normalized n-dimensional Lebesgue measure on $B_{1/k_0}(x)$, i.e. $\mu_x(A) = (\lambda(B_{1/k_0}(x)))^{-1}\lambda(A \cap B_{1/k_0}(x))$[2].

Lemma 2.1.1 *As $k \to \infty$, $f^k \to f$ uniformly on compact sets and for all $i = 1\ldots, N$, $\partial_i f^k \to \tilde{\partial}_i f$ pointwise.*

Proof : The first claim is clear because f is continuous. Moreover, if $x \in S^\circ$ for some $S \in \mathcal{S}^{(N)}$ and if k is large enough such that $B_{1/k}(x) \subset S^\circ$, then by (2.3), $\partial_i f^k(x) = \int_{B_{1/k}(x)} \partial_i f(y)\psi^k(y-x)dy \to \partial_i f(x)$.

[2] k_0 is taken so large that $B_{1/k_0}(x) \subset \mathrm{St}(x)$

For the other case, let $x \in T^\circ$ for some $T \in \mathcal{S}$ and let $k \geq k_0$, where k_0 is so large that $B_{1/k_0}(x) \subset \mathrm{St}(x)$. Since f is piecewise smooth, $\partial_i f(x)$ exists as a one-sided derivative for all x. Thus we have

$$\begin{aligned}\partial_i f^k(x) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_{B_{1/k}(x+\epsilon b_i)} f(y)\Psi_k(y-x-\epsilon b_i)dy - \int_{B_{1/k}(x)} f(y)\Psi_k(y-x)dy\right]\\ &= \int_{B_{1/k}(x)} \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[f(y+\epsilon b_i)-f(y)\right]\Psi_k(y-x)dy\\ &= \int_{B_{1/k}(x)} \partial_i f(y)\Psi_k(y-x)dy.\end{aligned}$$

For every $S \in \mathrm{st}(x))$ we have $\int_S \Psi_k(y-x)dy = \mu_0(S)$ because of the rotational invariance of Ψ_k. Moreover, since $\partial_i(f_{|S})$ is continuous on S, we have that $\int_S \partial_i f(y)\Psi_k(y-x)dy \to \partial_i(f_{|S})(x)\mu_x(S)$ and hence

$$\partial_i f^k(x) = \sum_S \int_S \partial_i f(y)\Psi_k(y-x)dy \to \sum_S \partial_i(f_{|S})(x)\mu_x(S) = \tilde{\partial}_i f(x)\Box$$

Note that for second order derivatives, the case of piecewise smooth functions is more delicate. For instance, let $V = \mathbb{R}$ with $\mathcal{S}(\mathbb{R}) = \{\{0\}, \mathbb{R}_+, \mathbb{R}_-\}$ and $f(x) = |x|$. Clearly, f is piecewise smooth. But $(f^k)''(0) \simeq k \to \infty$.

Proposition 2.1.2 *Let V be an $N-$dimensional vector space and $X : \Omega\times\mathbb{R}_+ \to V$ a continuous semimartingale. Let $\mathcal{S}$ be triangulation such that $(V, \mathcal{S})$ is a simplicial cone complex. Then for all piecewise smooth functions $f : V \to \mathbb{R}$, $f(X)$ is a semimartingale. More precisely, if $b_1 \dots b_N$ is a basis of V, then $A(f)$ is locally of finite variation, where*

$$A_t(f) := f(X_t) - f(X_0) - \sum_{i=1}^N \int_0^t \tilde{\partial}_i f(X_\tau)dX^i_\tau, \tag{2.5}$$

where X^i is the $i-th$ coordinate process of X w.r.t. $b_1 \dots b_N$.

Proof : Let now X be a semimartingale (for the linear structure in V). By stopping, we can assume that X has only values in a compact set $K \subset M$. Since f^k is smooth, $\tilde{\partial}_i f^k \equiv \partial_i f^k$ for all $i = 1, \dots, N$. Moreover $f^k(X)$ is a semimartingale and the Itô formula yields

$$\begin{aligned}A_t(f^k) &= f^k(X_t) - f^k(X_0) - \sum_{i=1}^N \partial_i f^k(X_\tau)dX^i_\tau\\ &= \frac{1}{2}\sum_{i,j=1}^N \int_0^t \partial_{ij} f^k(X_\tau)d\langle X^i, X^j\rangle_\tau.\end{aligned}$$

Now $f^k \to f$ locally uniformly and so $f^k(X) \to f(X)$ uniformly in probability. Moreover, by Lemma 2.1.1 and [RY99], Theorem IV (2.12),

$$\int \partial_i f^k(X_\tau)dX^i_\tau \to \int \tilde{\partial}_i f(X_\tau)dX^i_\tau,$$

locally uniformly in probability. Consequently, $A(f^k)$ converges locally uniformly in probability to $A(f)$. By Lemma 2.1.3, $A(f)$ is locally of finite variation. This proves the proposition. □

Lemma 2.1.3 *Let f be piecewise smooth. Then $A(f)$, defined in (2.5), is locally of bounded variation.*

Proof : 1. By stopping, we may assume that $\langle X^i\rangle_\infty(\omega) \le \gamma$ for all i, where $\gamma > 0$ is some constant. Moreover, we may assume that X lives in some compact set $K \subset V$ and hence we can also assume that the $|\partial_{ij} f|$ are uniformly bounded by γ. For a smooth function h put

$$B_t(h) := \frac{1}{2}\sum_{i,j=1}^{N}\int_0^t |\partial_{ij}h(X)|\,|d\langle X^i, X^j\rangle|_\tau. \tag{2.6}$$

Then $|dA(h)|_\tau \le dB(h)$. We will show that for all ω, $B_\infty(f^k)(\omega)$ is uniformly bounded in k. First note that if $(O_l)_{1\le l\le m}$ is a finite open cover of K, then it suffices to prove that $\int_0^\infty \mathbf{1}_{\{X_\tau\in O_l\}}dB_\tau(f^k)$ is uniformly bounded in k for all l. The idea is the following: Let O be an open set and let γ^k_{ij} be a uniform bound for $|\partial_{ij} f^k|$ on O. Then

$$\int_0^\infty \mathbf{1}_{\{X_\tau\in O_l\}}dB_\tau(f^k) \le \sum_{i,j=1}^{N}\gamma^k_{ij}\int_0^\infty \mathbf{1}_{\{X_\tau\in O\}}d\left|\langle X^i, X^j\rangle\right|_\tau$$

So if γ^k_{ij} is 'large', then $\int_0^\infty \mathbf{1}_{\{X_\tau\in O\}}d\left|\langle X^i, X^j\rangle\right|_\tau$ must be 'small' in order to keep the right hand side bounded. This will be achieved by the right choice of O:
For an arbitrary set $S \subset V$ and $r > 0$, let $B_r(S) := \{x \in V : \|x-y\| < r \text{ for some } y \in S\}$. Let $k \in \mathbb{N}$, $r > 0$ and $S \in \mathcal{S}$. An open set $O \subset V$ is called $(q,r)-$*local at* S if $B_q(O) \subset \mathrm{St}^\circ(S)$ and $O \subset B_r(S)$ (cf. Definition 2.1.6). Roughly speaking, the condition $B_q(O) \subset \mathrm{St}^\circ(S)$ ensures that the mollified function f^k does not 'feel' other singularities of f except S, and we will make this precise in steps 2 and 3 below.
By definition, if O is $(1/k,r)-$local at S, then $B_{1/k}(O)$ is local at S and so we can write $f = f^\top + f^\perp$.

2. Let $m = \dim S$ and let $b_1 \dots, b_N$ be a basis of V with the property that $b_1, \dots, b_m$ is a basis of U^S, the linear subspace generated by S. Because $f_{|S\cap O}$ is smooth, $f^\top$ is smooth on $B_{1/k}(O)$ and hence there is a $\gamma > 0$ such that $|\partial_{ij}(f^\top)(x)| \leq \gamma$ for all $x \in B_{1/k}(O)$ and $1 \leq i, j \leq N$[3]. Consequently,

$$\partial_{ij}(f^\top)^k(x) = \int_{B_{1/k}(x)} \partial_{ij} f^\top(y)\Psi_k(y-x)dy \leq \gamma \tag{2.7}$$

for all $x \in O$, $k \in \mathbb{N}$ and $1 \leq i, j \leq N$. Thus if we enlarge γ, we get that $\int_0^\infty \mathbf{1}_{\{X_\tau \in O\}} dB_\tau((f^\top)^k) \leq \gamma$ for all k.

3. $f^\perp$ is piecewise smooth and $f^\perp_{|S\cap O} \equiv 0$, which implies that

$$\partial_i f^\perp_{|S\cap B_{1/k}(O)} \equiv 0 \text{ for all } i \leq m. \tag{2.8}$$

In particular, for $i, j \leq m$, $\partial_{ij} f^\perp$ exists and is continuous on $B_{1/k}(O)$ and there is a function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to 0} \sigma(t) = 0$ such that $|\partial_{ij} f^\perp(y)| \leq \sigma(r + 1/k)$ for all $i, j \leq m$ and all $y \in B_{1/k}(O)$, so with (2.7) we conclude that $|\partial_{ij}(f^\perp)^k(x)| \leq \sigma(r + 1/k)$ for all $i, j \leq m$, $k \in \mathbb{N}$ and $x \in O$.
By (2.8) and the Taylor formula, there is a $\gamma > 0$ such that $|\partial_i f^\perp(y)| \leq (r+1/k)\gamma$ for all $y \in B_{1/k}(O)$ and hence for all $i \leq m$ and $j \geq m+1$ we have $|\partial_{ij}(f^\perp)^k(x)| = |\int_{B_{1/k}(x)} \partial_i f^\perp(y)\partial_j \Psi_k(y-x)dy| \leq (r+1/k)\gamma k$.
At last, again by Taylor's formula, $|f^\perp(y)| \leq (r + 1/k)\gamma$ for all $y \in B_{1/k}(O)$ and hence we have for all $i, j \geq m+1$ and $x \in O$,
$|\partial_{ij}(f^\perp)^k(x)| = |\int_{B_{1/k}(x)} f^\perp(y)\partial_{ij}\Psi_k(y-x)dy| \leq (r+1/k)\gamma k^2$.
Let us summarize the estimates: There is a $\gamma > 0$ and a function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to 0} \sigma(t) = 0$ such that for all $x \in O$ and $k \in \mathbb{N}$

$$|\partial_{ij}(f^\perp)^k(x)| \leq \begin{cases} \sigma(r + \frac{1}{k}) & \text{if } i, j \leq m \\ \gamma k(r + \frac{1}{k}) & \text{if } i \leq m, j \geq m+1 \\ \gamma k^2(r + \frac{1}{k}) & \text{if } i, j \geq m+1 \end{cases} \tag{2.9}$$

4. Now we will show that the condition $O \subset B_r(S)$ yields a bound on $\int_0^\infty \mathbf{1}_{\{X_\tau \in O\}} d\,|\langle X^i, X^j\rangle|_\tau$.
Since $X \in K$ for a compact set K, there is a compact set $\widetilde{K} \subset \mathbb{R}$ such that all coordinate processes X_i live in $\widetilde{K}$. Let $\mathcal{L}^i_t := \sup_{y\in\widetilde{K}} L^i(y,t)$, where $L^i(y,t)$ is the local time of the real-valued process X^i, cf. (5.5). Moreover, put $\mathcal{L}_t = \max_i \mathcal{L}^i_t$. By stopping, we can assume that $\mathcal{L}_\infty$ is bounded[4]. By Corollary 5.1.6,

$$\int_0^\infty \mathbf{1}_{\{X_\tau \in O\}} d\langle X^i\rangle_\tau \leq \begin{cases} \mathcal{L}_\infty \delta & \text{if } i \leq m \\ \mathcal{L}_\infty r & \text{if } i \geq m+1 \end{cases}$$

[3] in fact, $\partial_{ij} f^\top \equiv 0$ for all $i, j \geq m+1$, but we do not need this here

[4] Since $\mathcal{L}_t$ is continuous, there is a sequence T_n of stopping times increasing to ∞ such that $\mathcal{L}_{T_n}$ is bounded for all n

where $\delta := \mathrm{diam} K$ and K is the compact set in which X lives. Together with the Kunita Watanabe inequality this gives

$$\int_0^s \mathbf{1}_{\{X_\tau \in O\}} d\left|\langle X^i, X^j\rangle\right|_\tau \leq \begin{cases} \mathcal{L}_\infty \delta & \text{if } i,j \leq m \\ \mathcal{L}_\infty \sqrt{\delta r} & \text{if } i \leq m, j \geq m+1 \\ \mathcal{L}_\infty r & \text{if } i,j \geq m+1 \end{cases} \tag{2.10}$$

So combining (2.9) and (2.10), we find a $\gamma = \gamma(\omega) > 0$[5] such that for all $k \in \mathbb{N}$,

$$\int_0^\infty \mathbf{1}_{\{X_\tau \in O\}} B((f^\perp)^k)_\tau \leq \sum_{i,j=1}^N \gamma_{ij} \tag{2.11}$$

with

$$\gamma_{ij} = \gamma_{ij}(k,r,\omega) := \begin{cases} \gamma\delta & \text{if } i,j \leq m \\ \gamma k(r+\frac{1}{k})r^{1/2} & \text{if } i \leq m, j \geq m+1 \\ \gamma k^2(r+\frac{1}{k})r & \text{if } i,j \geq m+1 \end{cases} \tag{2.12}$$

5. By Lemma 2.1.7 there is a $\gamma_1 = \gamma_1(\mathcal{S}(V)) > 0$ such that for all $k \in \mathbb{N}$ and all $S \in \mathcal{S}(V)$ there is a neighborhood O_S that is $(1/k, \gamma_1/k)-$local at S and $V = \bigcup_S O_S$. So in (2.12) we can take $r = \gamma_1/k$, and if we sum up over all i,j, we find for (almost) all $\omega \in \Omega$ a $\gamma(\omega) > 0$ (independent of k) such that for all $S \in \mathcal{S}(V)$, $\int_0^\infty \mathbf{1}_{\{X_\tau \in O_S\}} dB_\tau(f^k)(\omega) \leq \gamma(\omega)$. Now since $V = \bigcup_{S \in \mathcal{S}(V)} O_S$ is a finite union, we take $\gamma(\omega)$ so large that

$$\int_0^\infty |dA(f^k)|_\tau(\omega) \leq B_\infty(f^k)(\omega) \leq \sum_{S \in \mathcal{S}(V)} \int_0^\infty \mathbf{1}_{\{X_\tau \in O_S\}} dB_\tau(f^k)(\omega) \leq \gamma(\omega).$$

Consequently, $A(f^k)(\omega)$ is of bounded variation, uniformly in k. Thus the limit $A(f)$ must be of bounded variation by Lemma 5.1.1 (i). This proves the Lemma. □

Remark 2.1.4 Clearly, the constant γ was adjusted (i.e. enlarged if necessary) during each step of the proof, while we kept the letter γ throughout the proof in order to keep the notations as simple as possible. Actually, it turns out that as soon as X is stopped in order to ensure that $\|X\|$ is finite, γ only depends on the triangulation $\mathcal{S}(V)$ and on f (more precisely, on the first and second order derivatives of f).

We also may point out that $B(f^k)$ was defined in terms of the first and second partial derivatives of f^k w.r.t. *a fixed basis* $b_1, \ldots, b_N$ of V, while during the

[5] we may take $\gamma(\omega) := N^2\gamma\mathcal{L}_\infty(\omega)$ with γ from (2.9), where N^2 comes from summing over all $1 \leq i,j \leq N$

proof we estimated $B(f^k)$ (more precisely, $\int \mathbf{1}_{\{X_\tau \in O_S\}} dB_\tau(f^k)$) in terms of *a basis adapted to the cone* C. However, this does not affect the estimates (except changing the constant) since every coordinate system is equivalent to another via a linear isomorphism.

Before proving Lemma 2.1.7, we will refine the arguments of the preceding proof in order to get some more information that will be useful later.

Lemma 2.1.5 *Let $S \in \mathcal{S}(V)$. Let $b_1, \dots b_N$ be a basis of V such that $b_1, \dots, b_m$ is a basis of U, the linear subspace generated by S, and let $f = f^\top + f^\perp$ locally around S. If $\partial_i f_{|S^\circ} \equiv 0$ for all $i \geq m+1$, then*

$$\int \mathbf{1}_{\{X_\tau \in S^\circ\}} df^\perp(X_\tau) = \frac{1}{2} \int \mathbf{1}_{\{X_\tau \in S^\circ\}} dA_\tau(f^\perp) \equiv 0. \tag{2.13}$$

Proof : 1. The first equality is clear because $\mathbf{1}_{S^\circ} \tilde{\partial}_i f^\perp \equiv 0$.

2. We will start analogously to the proof of Lemma 2.1.3. Let O be $(1/k, r)-$local at S. By assumption above, in addition to (2.8) we have

$$\partial_i f^\perp_{|S \cap B_{1/k}(O)} \equiv 0 \text{ for all } i. \tag{2.14}$$

So using Taylor's formula again and repeating the arguments of the proof of Lemma 2.1.3, we get

$$|\partial_{ij}(f^\perp)^k(x)| \leq \begin{cases} \sigma(r + \frac{1}{k}) & \text{if } i, j \leq m \\ \gamma k (r + \frac{1}{k})^2 & \text{if } i \leq m, j \geq m+1 \\ \gamma k^2 (r + \frac{1}{k})^2 & \text{if } i, j \geq m+1 \end{cases} \tag{2.15}$$

and again a combination of (2.15) and (2.10) shows that there is a $\gamma = \gamma(\omega) > 0$ such that

$$\int_0^\infty \mathbf{1}_{\{X_\tau \in O\}} |\partial_{ij}(f^\perp)^k(X_\tau)| \left| d\langle X^i, X^j \rangle \right|_\tau \leq \gamma_{ij}(k, r) \tag{2.16}$$

with

$$\gamma_{ij}(k, r) = \gamma_{ij}(k, r, \omega) := \begin{cases} \gamma\sigma(r + \frac{1}{k}) & \text{if } i, j \leq m \\ \gamma k (r + \frac{1}{k})^2 r^{1/2} & \text{if } i \leq m, j \geq m+1 \\ \gamma k^2 (r + \frac{1}{k})^2 r & \text{if } i, j \geq m+1 \end{cases} \tag{2.17}$$

3. Now fix $k_0 \in \mathbb{N}$ and $r_0 > 0$ and let O be $(k_0, r_0)-$local at S. For $k \geq k_0$, let $O_k := O \cap B_{2/k}(S)$. Then O_k is $(1/k, 2/k)-$local at S and hence in (2.17) we can take $r = 2/k$. Thus after enlarging γ, (2.16) yields

$$\int_0^\infty \mathbf{1}_{\{X_\tau \in O_k\}} |\partial_{ij}(f^\perp)^k(X_\tau)| d \left| \langle X^i, X^j \rangle \right|_\tau \leq \begin{cases} \gamma\sigma(\frac{3}{k}) & \text{if } i, j \leq m \\ \gamma k^{-3/2} & \text{if } {\scriptstyle i \leq m, \atop j \geq m+1} \\ \gamma k^{-1} & \text{if } i, j \geq m+1 \end{cases} \tag{2.18}$$

So summing over all $1 \leq i, j \leq N$ we conclude that there is a function $\tilde{\sigma} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to 0} \tilde{\sigma}(t) = 0$ such that

$$\int \mathbf{1}_{\{X_\tau \in O_k\}} dB((f^\perp)^k)_\tau \leq \tilde{\sigma}(\frac{1}{k}). \tag{2.19}$$

On the other hand, $f^\perp$ is smooth on $B_{1/k}(O \backslash O_k)$. Thus $\partial_{ij}(f^\perp)^k(x) = \int_{B_{1/k}(x)} \partial_{ij} f(y) \Psi_k(y-x)dy$ for all $x \in O \setminus O_k$ and hence (2.15) yields

$$\int_0^\infty \mathbf{1}_{\{X_\tau \in O\backslash O_k\}} |\partial_{ij}(f^\perp)^k(X_\tau)| \, \left|d\langle X^i, X^j\rangle\right|_\tau \leq \gamma_{ij}(k_0, r_0) \tag{2.20}$$

for all $k \geq k_0$ (γ_{ij} is the same as in (2.17)). So taking a rough estimate, we find a $\gamma > 0$ such that

$$\int \mathbf{1}_{\{X_\tau \in O\backslash O_k\}} dB((f^\perp)^k)_\tau \leq \gamma \left[\sigma(r_0 + \frac{1}{k_0}) + k_0^2(r_0 + \frac{1}{k_0})^2 r_0^{1/2}\right]. \tag{2.21}$$

for all $k \geq k_0$, and with (2.21) and (2.19) we get

$$\int \mathbf{1}_{\{X_\tau \in O\}} dB((f^\perp)^k)_\tau \leq \gamma \left[\sigma(r_0 + \frac{1}{k_0}) + k_0^2(r_0 + \frac{1}{k_0})^2 r_0^{1/2}\right] + \tilde{\sigma}(\frac{1}{k}).$$

for all $k \geq k_0$. Now since O is open, by Lemma 5.1.1 (ii) we have

$$\begin{aligned} \int_0^\infty \mathbf{1}_{\{X_\tau \in O\}} |dA(f^\perp)|_\tau &\leq \liminf_{k\to\infty} \int_0^\infty \mathbf{1}_{\{X_\tau \in O\}} dB_\tau((f^\perp)^k) \\ &\leq \gamma \left[\sigma(r_0 + \frac{1}{k_0}) + k_0^2(r_0 + \frac{1}{k_0})^2 r_0^{1/2}\right] \end{aligned} \tag{2.22}$$

4. At last, we let $k_0 \to \infty$ and $r_0 \to 0$ in a suitable way. Namely, let $U_{k_0} := S \setminus B_{2/k_0}(\partial S)$ and put $\widetilde{O}_{k_0} := B_{1/k_0^2}(U_{k_0})$. Then $\widetilde{O}_{k_0}$ is $(1/k_0, 1/k_0^2)-$local at S, provided k_0 is large enough, and so in (2.22) we can take $r_0 := k_0^{-2}$. Moreover, $\mathbf{1}_{\widetilde{O}_{k_0}} \to \mathbf{1}_{S^\circ}$ pointwise as $k_0 \to \infty$, and the right hand side of (2.22) goes to 0. Thus we get

$$\int_0^\infty \mathbf{1}_{\{X_\tau \in S^\circ\}} |dA(f^\perp)|_\tau = \lim_{k_0\to\infty} \int_0^\infty \mathbf{1}_{\{X_\tau \in \widetilde{O}_{k_0}\}} |dA(f^\perp)|_\tau = 0$$

and the Lemma is proved. □

Definition 2.1.6 *An open set $O \subset V$ is called $(q, r)-$*local *at $S \in \mathcal{S}$ if*

$$B_q(O) \subset \mathrm{St}^\circ(S) \text{ and } O \subset B_r(S) \tag{2.23}$$

Figure 2.1: a neighborhood that is $(q,r)-$local at S

Lemma 2.1.7 *Let $\mathcal{S}$ be a triangulation of M. Then there is a $\gamma = \gamma(\mathcal{S}) > 0$ such that for all $k \in \mathbb{N}$ and $S \in \mathcal{S}$ there is a neighborhood $O_S = O_S(k)$ that is $(1/k, \gamma/k)-$local at S and $V = \bigcup_{S \in \mathcal{S}} O_S$.*

Proof : We may assume that $M \subset V$, where V is equipped with a Euclidean scalar product, and that all $S \in \mathcal{S}(M)$ are orthants (i.e. scaff(S) is orthogonal). This is possible because by Example 1.1.3 (iii), every simplicial cone complex can be mapped to such a complex with a simplicial linear isomorphism which is Lipschitz, so only the constant γ changes. $M \subset V$ is equipped with the induced distance. We start with the $n-$dimensional cones. Let $k \in \mathbb{N}$. For $S \in \mathcal{S}^{(n)}$ set $O_C(k) := S \setminus B_{1/k}(\partial S) = S \setminus B_{1/k}(M^{(n-1)})$. Note that $M = \bigcup_{S \in \mathcal{S}^{(n)}} O_S \cup B_{2/k}(M^{(n-1)})$.
For $S \in \mathcal{S}^{(n-1)}$, set $U_S(k) := S \setminus B_{1/k}(\partial S) = S \setminus B_{1/k}(M^{(n-2)})$ and let $O_S := \{y + x : x \in U_S, y \in B_{2/k}(S) \cap \perp S\}$ be the $2/k-$cylinder around U_S. Then $M = \bigcup_{S \in \mathcal{S}^{(n)} \cup \mathcal{S}^{(n-1)}} O_S \cup B_{2/k}(M^{(n-2)})$ (we could take $B_{r/k}(M^{(n-2)})$ instead of $B_{2/k}(M^{(n-2)})$ for any $r > \sqrt{2}$).
This procedure can be continued. At the end, if O_S constructed as the cylindrical neighborhood of $U_S(k) := S \setminus B_{1/k}(\partial S)$ for all $S \in \mathcal{S} \setminus \{0\}$ (with a radius at most n/k), then $M \setminus \bigcup_{S \in \mathcal{S} \setminus \{0\}} O_S$ is the union of $n-$ dimensional cubes around 0 with side length $1/k$. So if we set $O_0 := B_{n/k}$, then every O_S is $(k, n/k)-$local at S and $M = \bigcup_{S \in \mathcal{S}} O_S$. $\square$.

2.1.2 The general case $M \subset V$

Let now $M \subset V$ be a simplicial cone complex, equipped with a triangulation $\mathcal{S}(M)$. From Proposition 2.1.2 we derive easily the next Proposition which says that all possible definitions of a semimartingale are equivalent.

Proposition 2.1.8 *Let $M \subset V$ and let $X : \Omega \times \mathbb{R}_+ \to M$ be a continuous adapted process. Let $\mathcal{S} = \mathcal{S}(M)$ be any triangulation of M into a simplicial cone complex. Then the following are equivalent:*

(i) X *is a semimartingale in* V

(ii) $f(X)$ *is a semimartingale for all piecewise smooth functions* $f : M \to \mathbb{R}$

(iii) X^u *is a semimartingale for all* $u \in \operatorname{scaff}(M)$

Proof : (i)⇒(ii): Let $f : M \to \mathbb{R}$ be piecewise smooth and let $\tilde{f}$ be its natural extension from M to V described in Example 1.1.8. Then $f(X) = \tilde{f}(X)$ is a semimartingale by Proposition 2.1.2.
(ii)⇒(iii) is trivial.
(iii) ⇒ (i) follows from $X = \sum_{u\in\operatorname{scaff}(M)} X^u u$. □

By (2.5) we have the decomposition

$$\tilde{f}(X_t) - \tilde{f}(X_0) = \sum_{i=1}^{N} \int_0^t \tilde{\partial}_i \tilde{f}(X_\tau) dX^i_\tau + A_t(\tilde{f}) \tag{2.24}$$

which unfortunately is not very useful. In order to give an intrinsic description, we have to introduce some notation.
First we note that if X is a semimartingale and f a piecewise smooth function, then

$$f(X_t) - f(X_0) = \sum_{S\in\mathcal{S}(M)} \int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}} df(X_\tau). \tag{2.25}$$

$\int \mathbf{1}_{\{X_\tau\in S^\circ\}} df(X_\tau)$ is a continuous semimartingale that describes the behavior of $f(X)$ on $\{X_\tau \in S^\circ\}$. In order to investigate this process, we introduce the notion of a *directional local time*:

Definition 2.1.9 *Let* $u \in \operatorname{scaff}(\perp S)$. *We set* $X^u := \nu^u(X)$. *The local time of* X *at* S *in direction of* u *is defined by*

$$L^{S,u}_t(X) := 2\int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}} dX^u_\tau$$

In particular, for $S = \{0\}$ *and* $u \in \operatorname{scaff}(M)$, *the local time of* X *at* 0 *in direction of* u *is defined by* $L^{0,u}_t(X) := 2\int_0^t \mathbf{1}_{\{X_\tau=0\}} dX^u_\tau$

Remark 2.1.10 (i) $L^{S,u}(X)$ is nondecreasing. Indeed, X^u is a nonnegative semimartingale. Moreover, $\{X_\tau \in S^\circ\} \subset \{X^u_\tau = 0\}$ and hence from Lemma 5.1.5 it follows that

$$L^{S,u}_t(X) := 2\int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}} dX^u_\tau = \int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}} dL^{X^u}_\tau \tag{2.26}$$

(ii) Note that for $u \in \text{scaff}(\perp S)$, ν^u is only defined locally on a neighborhood O of S°. However, ν^u is the restriction of a piecewise smooth function on M to O. Consequently, $\int \mathbf{1}_{\{X_\tau \in O\}} dX^u_\tau$ and hence also $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} dX^u_\tau$ are well-defined, cf. also Lemma 5.2.5.
(iii) If we write $X = X^\top + X^\perp$, then for any $u \in \text{scaff}(\perp S)$,

$$\begin{aligned} L^{S,u}_t(X) &= 2\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} dX^u_\tau = 2\int_0^t \mathbf{1}_{\{X^\perp_\tau = 0\}} d(X^\perp)^u_\tau \\ &= L^{0,u}_t(X^\perp) \end{aligned} \tag{2.27}$$

Example 2.1.11 Let $M = \mathbb{R}^n$ be equipped with the standard orthogonal triangulation from Example 1.1.3 (i). As usual, let $x^i = de_i(x)$ be the $i-$th coordinate function. Let $X = (X^1, \dots, X^n)$ be an $n-$dimensional Brownian motion.
We first calculate $L^{S,e_1}(X)$, where $S = \{x \in \mathbb{R}^n : x^1 = 0, x^i \geq 0, i = 2, \dots, n\}$ is the first $n-1-$dimensional positive orthant . Now X^1 is a one-dimensional Brownian motion and $\nu^{e_1} = (x^1)^+$, so $X^{e_1} = \nu^{e_1}(X) = (X^1)^+$ and hence by Lemma 5.1.5,

$$L^{S,e_1}_t(X) = \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} \mathbf{1}_{\{(X^1_\tau)^+ = 0\}} d(X^1)^+_\tau = \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} dL^{X^1}(0, \tau), \tag{2.28}$$

where $L^{X^1}(0,t)$ is the local time of X^1 at $0 \in \mathbb{R}$, the so-called *Brownian local time*. This process is well-known, cf. e.g. [RY99] VI.§2. By symmetry, for every $S \in \mathcal{S}^{(n-1)}$ and all $e \in \text{scaff}(\perp S)$, $L^{S,e}$ has the same form.
Consider now the first $n-2-$dimensional positive orthant $T = \{x \in \mathbb{R}^n : x^1 = x^2 = 0, x^i \geq 0, i = 3, \dots, n\}$ Then T is *polar*, i.e. there is a $P-$nullset out of which $\mathbf{1}_{\{X_\tau \in T^\circ\}} \equiv 0$ for all $\tau \in \mathbb{R}_+$, and hence $L^{T,e_1} \equiv 0$. By symmetry, we conclude that at all $n-2-$dimensional orthants, all directional local times are identically 0. Moreover, the same argument shows that for all orthants S of dimension less than $n-2$, the directional local times at S are also identically 0, and so we get a full description of the behavior of X at the orthants.

As a first application of local times, we will present an Itô formula on $\{X_\tau \in S^\circ\}$ for a special class of piecewise smooth functions that can be regarded as 'linear forms' over S° whose tangential part is 0:

Lemma 2.1.12 *Let $S \in \mathcal{S}$ and let O be local at S. For $u \in \text{scaff}(\perp S)$ let $g^u : S^\circ \to \mathbb{R}$ be a piecewise smooth function. Define a function $g : O \to \mathbb{R}$, $g(x) := \sum_{u \in \text{scaff}(\perp S)} g^u(x^\top)\nu^u(x^\perp)$. Then*

$$\int \mathbf{1}_{\{X_\tau \in S^\circ\}} dg(X_\tau) = \frac{1}{2} \sum_{u \in \text{scaff}(\perp S)} \int_0^t g^u(X_\tau) dL^{S,u}_\tau(X)$$

In particular, if $f : M \to \mathbb{R}$ is piecewise linear, then

$$\int_0^t \mathbf{1}_{\{X_\tau=0\}} df(X)_\tau = \frac{1}{2} \sum_{u\in \mathrm{scaff}(M)} f(u) L_t^{0,u}(X).$$

Proof : By the usual product formula,

$$\begin{aligned}
&\int \mathbf{1}_{\{X_\tau\in S^\circ\}} dg(X_\tau) \\
&\quad = \sum_{u\in\mathrm{scaff}(\perp S)} \int \mathbf{1}_{\{X_\tau\in S^\circ\}} g^u(X_\tau) dX_\tau^u \\
&\qquad + \int \mathbf{1}_{\{X_\tau\in S^\circ\}} X_\tau^u d(g^u(X_\tau)) + \int \mathbf{1}_{\{X_\tau\in S^\circ\}} d\langle g^u(X), X^u\rangle_\tau \\
&\quad = \frac{1}{2} \sum_{u\in\mathrm{scaff}(\perp S)} \int_0^t g^u(X_\tau) dL_\tau^{S,u}(X)
\end{aligned}$$

The last equality holds because $\{X_\tau \in S^\circ\} \subset \{X_\tau^u = 0\}$ and consequently, $\mathbf{1}_{\{X_\tau\in S^\circ\}} X_\tau^u \equiv 0$. Moreover,

$$\int \mathbf{1}_{\{X_\tau\in S^\circ\}} d\langle g^u(X), X^u\rangle_\tau = \langle g^u(X), L^{S,u}\rangle \equiv 0$$

and hence the second and third stochastic integrals vanish. □

Now we come to the main Theorem of this section. Recall the decomposition (2.25). For any $S \in \mathcal{S}$, we will describe $\int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}} df(X_\tau)$ in terms of a generalized Itô formula with local times at S.

Theorem 2.1.13 (Local Itô Formula) *Let $f : M \to \mathbb{R}$ be piecewise smooth. Then*

$$\begin{aligned}
\int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}} df(X_\tau) = &\sum_{u\in\mathrm{scaff}(S)} \int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}} \partial_u f(X_\tau) dX_\tau^u \\
&+ \frac{1}{2} \sum_{u,v\in\mathrm{scaff}(S)} \int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}} \partial_{uv} f(X_\tau) d\langle X_\tau^u, X_\tau^v\rangle \\
&+ \frac{1}{2} \sum_{u\in\mathrm{scaff}(\perp S)} \int_0^t \partial_u f(X_\tau) dL_\tau^{S,u}(X).
\end{aligned}$$

In particular,

$$\int_0^t \mathbf{1}_{\{X_\tau=0\}} df(X)_\tau = \frac{1}{2} \sum_{u\in\mathrm{scaff}(M)} \partial_u f(0) L_t^{0,u}(X).$$

Proof : 1. Assume that $\dim S = m$. Let $\mathrm{scaff}(S) = \{b_1, \dots b_m\}$, let U be the vector subspace generated by $\{b_1, \dots b_m\}$ and let $\{b_{m+1}, \dots b_N\}$ be a basis of $U^\perp$, where $U^\perp$ is an arbitrarily chosen linear complement of U in V. Let $O \subset\subset M$ be a neighborhood that is local at S. Let $\tilde{f}$ be the natural extension of f from M to V. On O we write $\tilde{f} = \tilde{f}^\top + \tilde{f}^\perp$ as in (1.9). Then $\tilde{f}^\top$ is smooth on O and for all $x \in O \cap S$ we have

$$\partial_i \tilde{f}^\top(x) = \begin{cases} \partial_i f(x) & \text{if } i \leq m \\ 0 & \text{if } i \geq m+1 \end{cases} \tag{2.29}$$

Thus the usual Itô formula yields

$$\begin{aligned} \int \mathbf{1}_{\{X_\tau \in S \cap O\}} d\tilde{f}^\top(X_\tau) &= \sum_{i=1}^m \int \mathbf{1}_{\{X_\tau \in S \cap O\}} \partial_i f(X_\tau) dX^i_\tau \\ &+ \frac{1}{2} \sum_{i,j=1}^m \int_0^t \mathbf{1}_{\{X_\tau \in S \cap O\}} \partial_{ij} f(X_\tau) d\langle X^i_\tau, X^j_\tau \rangle \end{aligned} \tag{2.30}$$

2. We now show that

$$\int \mathbf{1}_{\{X_\tau \in S \cap O\}} d\tilde{f}^\perp(X_\tau) = \frac{1}{2} \sum_{u \in \mathrm{scaff}(\perp S)} \int_0^t \mathbf{1}_{\{X_\tau \in S \cap O\}} \partial_u f(X_\tau) dL^{S,u}_\tau(X). \tag{2.31}$$

Define a function $g : O \cap M \to \mathbb{R}$, $g(x) := \sum_{u \in \mathrm{scaff}(\perp S)} \partial_u f(x^\top) \nu^u(x)$ and let $\tilde{g}$ be its natural extension from M to V. Then $\tilde{g}^\top \equiv 0$ and hence by Lemma 2.1.12,

$$\begin{aligned} \int \mathbf{1}_{\{X_\tau \in S \cap O\}} d\tilde{g}^\perp(X_\tau) &= \int \mathbf{1}_{\{X_\tau \in S \cap O\}} d\tilde{g}(X_\tau) \\ &= \int \mathbf{1}_{\{X_\tau \in S \cap O\}} d\tilde{g}(X_\tau) \\ &= \frac{1}{2} \sum_{u \in \mathrm{scaff}(\perp S)} \int_0^t \mathbf{1}_{\{X_\tau \in S \cap O\}} \partial_u f(X_\tau) dL^{S,u}_\tau(X). \end{aligned} \tag{2.32}$$

Let now $h := f - g$. Then $\partial_u h(0) = 0$ for all $u \in \mathrm{scaff}(\perp S)$ and $x \in S \cap O$. Let $\tilde{h} = \tilde{f} - \tilde{g}$ be the natural extension of h from M to V. Then $\partial_u \tilde{h}(x) = 0$ for all $u \in \mathrm{scaff}(\perp S)$ and $x \in S^\circ$ and hence $\tilde{\partial}_i \tilde{h}(x) = 0$ for all $m+1 \leq i \leq N$. Thus by Lemma 2.1.5,

$$\begin{aligned} 0 &= \int \mathbf{1}_{\{X_\tau \in S \cap O\}} d\tilde{h}^\perp(X_\tau) \\ &= \int \mathbf{1}_{\{X_\tau \in S \cap O\}} d\tilde{f}^\perp(X_\tau) - \int \mathbf{1}_{\{X_\tau \in S \cap O\}} d\tilde{g}^\perp(X_\tau) dX^i_\tau \end{aligned}$$

which, together with (2.32), shows (2.31). Thus the Theorem is proved by taking a sequence O_l of neighborhoods that are local at S such that $\mathbf{1}_{O_l} \to \mathbf{1}_{S^\circ}$. $\square$.

Remark 2.1.14 We shall point out another time that $\perp S$ was defined as the 'intersection' of $U^\perp$ with M, where $U^\perp$ is an arbitrary *linear* complement of U, the vector subspace generated by S (although the notation suggests that this is the orthogonal complement). At this moment, the choice of $U^\perp$ does not have any geometric meaning, rather than providing a choice of scaff($\perp S$). Actually, when we deal with Euclidean cone complexes, we will always choose $U^\perp$ to be the orthogonal complement of U.

2.2 Stochastic integration in Polyhedra

Definition 2.2.1 Let M be a polyhedron. A continuous process X in M is called a *semimartingale* if $f(X)$ is a semimartingale for all piecewise smooth functions $f : M \to \mathbb{R}$.

Remark 2.2.2 Note that if $M \subset V$ is a simplicial complex embedded in a vector space V, then a simple localization procedure and an application of Proposition 2.1.8 show that X is a semimartingale in M *if and only if* X is a semimartingale in V.

Let us now come to the theory of stochastic integration. As in the case of manifolds (cf. [Éme89] or [HT94]) one can define the stochastic integral of bilinear forms via their local coordinates, based on the the following observation in the linear case: Let V be an $n-$dimensional vector space. Let X be a continuous semimartingale in V and $b : V \to V^* \otimes V^*$ a bounded measurable bilinear form. If $b_1, \dots, b_N$ is a basis of V, we can write $b = \sum_{i,j} b^{ij} db^i \otimes db^j$ and we define

$$\int b(dX_\tau, dX_\tau) = \sum_{i,j=1}^{N} \int b^{ij}(X_\tau) d\langle X^i, X^j\rangle_\tau. \tag{2.33}$$

This definition is independent of the basis (proven below), so the left-hand side is well-defined. Moreover, note that $\int b(dX, dX)$ only depends on the symmetric part of b.
Denote by $\Gamma_X(T^*M \otimes T^*M)$ the set of all progressively measurable bilinear forms over X, i.e. of processes $b : \Omega \times \mathbb{R}_+ \to T^*M \otimes T^*M$ with $\pi \circ b = X$, where $\pi : T^*M \otimes T^*M \to M$ is the natural projection from the bundle of bilinear forms to M.

Proposition 2.2.3 *There is a unique linear map from $\Gamma_X(T^*M \otimes T^*M)$ to the set of continuous adapted processes of finite variation, $b \mapsto \int b(dX, dX)$ such that for all $b \in \Gamma(T^*M \otimes T^*M)$ and $f, g \in \mathcal{C}^\infty(M)$*

(i) $\int (fb)(dX, dX) = \int (f(X))d(\int b(dX, dX))$

(ii) $\int (df \otimes dg)(X)(dX, dX) = \langle f(X), g(X)\rangle$.

$\int b(dX, dX)$ *is called the* b-quadratic variation *of* X*. For any* $S \in \mathcal{S}(M)$ *and any chart* $\xi : O \to \widehat{O}$ *local at* S *we have*

$$\int 1_{\{X_\tau \in S^\circ\}} b(dX_\tau, dX_\tau) = \sum_{u,v \in \mathrm{scaff}(S)} \int 1_{\{X_\tau \in S^\circ\}} b^{uv}(X_\tau) d\langle \widehat{X}^u, \widehat{X}^v\rangle_\tau$$
$$=: \int 1_{\{X_\tau \in S^\circ\}} b^\top(dX_\tau, dX_\tau) \tag{2.34}$$

In particular, $\int 1_{\{X_\tau \in S^\circ\}} b(dX_\tau, dX_\tau)$ *depends only on* $b_{|TS\times TS}$*.*

Proof : The proof is completely analogous to the one in [Éme89] or [HT94]. By (1.17), we can write b as a finite linear combination in the following way:

$$b_t(\omega) = \sum_k b_t^k(\omega) \partial f^k_{X_t(\omega)} \otimes \partial g^k_{X_t(\omega)}.$$

Then (i) and (ii) force us to define

$$\int_0^t b(dX, dX) := \sum_k \int_0^t b_\tau^k d\langle f^k(X), g^k(X)\rangle_\tau \tag{2.35}$$

Of course, the above representation of b is not unique. So, in order to show that $\int b(dX, dX)$ is well-defined, we have to show that whenever $b = \sum_k b^k df^k \otimes dg^k = 0$, then the right hand side of (2.35) is also 0. So let $\xi : O \to \widehat{O}$, $x \mapsto \hat{x} := \xi(x)$ be a simplicial chart, local at S. By stopping, we can assume that X has only values in O. So by the Itô formula,

$$\begin{aligned}
&\sum_k \int 1_{\{X_\tau \in S^\circ\}} b_\tau^k d\langle f^k(X), g^k(X)\rangle_\tau \\
&\quad = \sum_k \int 1_{\{X_\tau \in S^\circ\}} b_\tau^k d\langle \hat{f}^k(\widehat{X}), g^k(\widehat{X})\rangle_\tau \\
&\quad = \sum_k \left[\sum_{u,v \in \mathrm{scaff}(\widehat{S})} \int 1_{\{X_\tau \in S^\circ\}} b_\tau^k \partial_u \hat{f}^k(\widehat{X}_\tau) \partial_v \hat{g}^k(\widehat{X}_\tau) d\langle \widehat{X}^u, \widehat{X}^v\rangle_\tau \right] \\
&\quad = \sum_{u,v \in \mathrm{scaff}(\widehat{S})} \int 1_{\{X_\tau \in S^\circ\}} \sum_k b_\tau^k (\partial f^k \otimes \partial g^k)(\frac{\partial}{\partial \hat{x}^u}(X_\tau), \frac{\partial}{\partial \hat{x}^v}(X_\tau) d\langle \widehat{X}^u, \widehat{X}^v\rangle_\tau \\
&\quad = 0.
\end{aligned}$$

Moreover, (2.34) follows from the local description above. □

As a next step, one may ask if there is a notion of a *Stratonovich integral* $\alpha \mapsto \alpha(*X) =: \int \alpha_{X_\tau}(*dX_\tau)$, where $\alpha \in \Gamma(T^*M)$ is a piecewise smooth linear form.

Proposition 2.2.4 *There is a unique linear map from the set of piecewise smooth linear forms on M to the set of real continuous semimartingales, $\alpha \mapsto \alpha * X =: \int \alpha(*dX)$, such that for all piecewise smooth $\alpha \in \Gamma(T^*M)$ and $f \in \mathcal{C}^\infty(M)$*

(i) $(f\alpha) * X = \int f(X_\tau) * d(\alpha * X)_\tau$

(ii) $\partial f * X = f(X) - f(X_0)$

$\int \alpha(*dX_\tau)$ *is called the stochastic* Stratonovich integral *of α along X.*

Sketch of the Proof: Let $\alpha_x = \sum_k \alpha^k(x)\partial f^k_x$ with α^k, f^k piecewise smooth, so $\alpha^k(X), f^k(X)$ are real semimartingales. Then we set

$$\begin{aligned}\int_0^t \alpha(*dX_\tau) &:= \sum_k \int_0^t \alpha^k(X_\tau) * df^k(X_\tau)\\ &= \sum_k \left[\int_0^t \alpha^k(X_\tau)df^k(X_\tau) + \frac{1}{2}\langle \alpha^k(X), f^k(X)\rangle_t\right]\end{aligned}$$

Again one shows with help of Itô's formula that this is well-defined (cf. also the Proof of Proposition 2.3.1. □

2.3 Geometric stochastic calculus in Riemannian polyhedra

2.3.1 Itô integral

Now we will introduce the notion of an *Itô integral* of a linear form on a Riemannian polyhedron. As in the case of bilinear forms, denote by $\Gamma_X(T^*M)$ the set of all progressively measurable linear forms over X, i.e. of processes $\alpha : \Omega \times \mathbb{R}_+ \to T^*M$ with $\pi \circ \alpha = X$, where $\pi : T^*M \to M$ is the natural projection from the bundle of piecewise linear functions to M.

Proposition 2.3.1 *There is a unique linear map from $\Gamma_X(T^*M)$ to the set of real continuous semimartingales, $\alpha \mapsto \alpha \bullet X =: \int \alpha(dX)$, such that for all $\alpha \in \Gamma_X(T^*M)$ and $f \in C^\infty(M)$*

(i) $(f\alpha) \bullet X = \int f(X_\tau)d(\alpha \bullet X)_\tau$

(ii) $\partial f \bullet X = f(X) - f(X_0) - \int \mathrm{Hess} f(dX_\tau, dX_\tau)$

$\int \alpha(dX)$ *is called the stochastic* Itô integral *of* α *along* X.

Proof : By (1.21), we can write α as a linear combination in the following way:

$$\alpha_t(\omega) = \sum_k \alpha_t^k(\omega) \partial f^k(X_t(\omega))$$

Then we define

$$\int_0^t \alpha(dX) := \sum_k \left[\int_0^t \alpha_\tau^k df^k(X_\tau) - \frac{1}{2} \int_0^t \alpha_\tau^k \mathrm{Hess} f^k(X_\tau)(dX_\tau, dX_\tau) \right] \tag{2.36}$$

As above, we have to show that this is well-defined. So assume that $\alpha = \sum_k \alpha_t^k \partial f^k(X_t) \equiv 0$. First, by (2.34) and (1.31) we have

$$\int_0^t 1_{\{X_\tau \in S^\circ\}} \mathrm{Hess} f(X_\tau)(dX_\tau, dX_\tau) = \int_0^t 1_{\{X_\tau \in S^\circ\}} (\mathrm{Hess} f)^\top (X_\tau)(dX_\tau, dX_\tau)$$
$$= \sum_{u,v \in \mathrm{scaff}(S)} \int_0^t 1_{\{X_\tau \in S^\circ\}} \left(\partial_{uv} \hat{f}(\widehat{X}_\tau) - \sum_{w \in \mathrm{scaff}(\widehat{S})} \Gamma_{uv}^w (X_\tau) \partial_w \hat{f}(\widehat{X}_\tau) \right) d\langle \widehat{X}^u, \widehat{X}^v \rangle_\tau.$$

Together with the Itô formula we get

$$\sum_k \left[\int 1_{\{X_\tau \in S^\circ\}} \alpha_\tau^k df^k(X_\tau) - \frac{1}{2} \int_0^t \alpha_\tau^k \mathrm{Hess} f^k(X_\tau)(dX_\tau, dX_\tau) \right]$$
$$= \sum_k \left[\sum_{u \in \mathrm{scaff}(\widehat{S})} \int 1_{\{X_\tau \in S^\circ\}} \alpha_\tau^k \partial f^k \left(\frac{\partial}{\partial \hat{x}^u}(X_\tau)\right) d\widehat{X}_\tau^u \right.$$
$$+ \sum_{u \in \mathrm{scaff}(\perp \widehat{S})} \int \alpha_\tau^k \partial f^k \left(\frac{\partial}{\partial \hat{x}^u}(X_\tau)\right) dL_\tau^{S,u}(\widehat{X})$$
$$\left. + \sum_{u,v,w \in \mathrm{scaff}(\widehat{S})} \int 1_{\{X_\tau \in S^\circ\}} \Gamma_{uv}^w (X_\tau) \alpha_\tau^k \partial f^k \left(\frac{\partial}{\partial \hat{x}^w}(X_\tau)\right) d\langle \widehat{X}^u, \widehat{X}^v \rangle_\tau \right]$$
$$= 0 \square$$

Remark 2.3.2 Note that the Itô integral was defined in terms of the *Hessian*. Recall from Remark 1.3.8 that there is another concept of a Hessian, denoted by $\overline{\text{Hess}}$ and defined by

$$\overline{\text{Hess}} f_x^\top := \text{Hess} f_x^\top + \sum_{w \in \text{scaff}(\perp_x S)} \partial_w f(x) l_x^w$$

cf. (1.34). From this form we deduce that one can also define an Itô integral associated to $\overline{\text{Hess}}$, just in the same way as in the proof of Proposition 2.3.1. Although it is out of the scope of this work, we believe that $\overline{\text{Hess}}$ is the right object for a consistent theory of Itô integrals in general Riemannian polyhedra, especially for the theory of martingales. However, in chapter 3, where we introduce the notion of martingales, we only work in Euclidean complexes, where $\overline{\text{Hess}} f = \text{Hess} f$ because every Euclidean simplex is totally geodesic and hence the second fundamental form l vanishes.

We conclude this section with an intrinsic description of $\int \alpha(dX)$. Let $S \in \mathcal{S}(M)$ and $x \in S^\circ$. Recall the intrinsic orthogonal decomposition $T_x M = T_x S \oplus \perp_x S$, where

$$\perp_x S := (T_x S)^\perp := \{v \in T_x M : g_x(u, v) = 0 \quad (\forall u \in T_x S)\},$$

cf. (1.24). Denote by $\text{scaff}(\perp_x S)$ the unique scaffold of $\perp_x S$ that consists of unit vectors. So by varying x, we may regard $u \in \text{scaff}(\perp S)$ as a smooth unit vector field. To $u \in \text{scaff}(\perp S)$ there is associated a piecewise smooth linear forms $\nu^u \in \Gamma(\perp^* S)$, defined by $\perp_x S \ni v = \sum \nu_x^u(v) u(x)$.

Remark 2.3.3 It is important to point out that in the preceding section $\perp_x S$ was defined in terms of a local chart and therefore depended on the choice of the chart, cf. (1.13). In the situation now, the definition of $\perp_x S$ in (1.24) is *independent* of the chart, and so the notations might be ambiguous. But by Lemma 1.3.3 we have normal coordinates at S in which both notations are the same.

For u scaff$(\perp S)$, we set

$$L_t^{S,u} := \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} \nu^u(dX_\tau) \tag{2.37}$$

Proposition 2.3.4 *$L^{S,u}$ is a continuous nondecreasing process. Moreover,*

$$\int 1_{\{X_\tau \in S^\circ\}} \alpha_\tau^\perp(dX_\tau) = \sum_{u \in \text{scaff}(\perp S)} \int \alpha_\tau(\frac{\partial}{\partial \hat{x}^u}(X_\tau)) dL_\tau^{S,u} \tag{2.38}$$

If $\alpha_t = \alpha_{X_t}$ for a piecewise smooth linear form α, then this is also equal to $\int 1_{\{X_\tau \in S^\circ\}} \alpha_\tau^\perp(*dX_\tau)$.

Proof : Let ρ^u defined in (1.25). Then $\rho^u_{|S\cap O} \equiv 0$ and hence $\partial\rho^u_x = \nu^u_x$ and $(\text{Hess}\rho^u_x)^\top \equiv 0$ for all $x \in S\cap O$. Thus by the definition of the Itô-Integral we have

$$\begin{aligned}\int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}}\nu^u(dX_\tau) &= \int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}}d\rho^u(X_\tau)\\ &= \int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}}\mathbf{1}_{\{\rho^u(X_\tau)=0\}}d\rho^u(X_\tau)\end{aligned}$$

which is a nondecreasing process since ρ^u is nonnegative. □.

We will state all we have proved so far in the following intrinsic version of the Itô formula:

Theorem 2.3.5 *Let M be a Riemannian polyhedron, X a semimartingale in M and $f : M \to \mathbb{R}$ a piecewise smooth function. Then $f(X)$ is a semimartingale and for all $S \in \mathcal{S}(M)$ we have*

$$\int \mathbf{1}_{\{X_\tau\in S^\circ\}}df(X_\tau) = \int \mathbf{1}_{\{X_\tau\in S^\circ\}}\partial f(dX_\tau) + \frac{1}{2}\int \mathbf{1}_{\{X_\tau\in S^\circ\}}\text{Hess}f_{X_\tau}(dX_\tau, dX_\tau)$$

and

$$\int \mathbf{1}_{\{X_\tau\in S^\circ\}}\partial f(dX_\tau) = \int \mathbf{1}_{\{X_\tau\in S^\circ\}}(\partial f)^\top(dX_\tau) + \sum_{u\in\text{scaff}(\perp S)}\int \partial_u f(X_\tau))dL^{S,u}_\tau.$$

where $L^{S,u}$ is a nondecreasing process.

2.3.2 Discrete approximation and quadratic variation

In this section we prove a classical discrete approximation result for the $b-$quadratic variation[6].
Let M be a Riemannian polyhedron and let $e : M\times M \ni (x,y) \mapsto e_x(y) \in T_xM$ be a generalized inverse exponential map[7]. Let Δ^k be a sequence of locally finite partitions of $\mathbb{R}_+$ such that $\|\Delta^k\| \to 0$. If $\Delta^k = \{0 = t_0 < t_1 < \dots\}$, we set

$$\Delta X_l := e_{X_{t_l}}(X_{t_{l+1}}) \in T_{X_{t_l}}M \tag{2.39}$$

ΔX_l is called the *increment of X at t_l* w.r.t Δ^k. For $b \in \Gamma(T^*M\otimes T^*M)$, consider the process $\Delta^k B$, defined by

$$\Delta^k B_t := \sum_{t_l\in\Delta^k_t} b_{X_{t_l}}(\Delta^k X_l, \Delta^k X_l), \tag{2.40}$$

[6] see [Éme89], Proposition (3.23) for a similar result in manifolds
[7] cf. Definition 1.3.16. Such a map always exists, cf. Proposition 1.3.17.

where for $t \geq 0$, $\Delta^k_t := \Delta^k \cap [0,t]$ is the partition up to time t. We will prove in the Theorem below that if X is a semimartingale, then $\Delta^k B$ converges to the $b-$quadratic variation of X.

First we introduce some notation: Let $M = \bigcup_{S\in\mathcal{S}} O_S$, where for each $S \in \mathcal{S}$, O_S is local at S and there is a simplicial chart $\xi_S : O_S \to \widehat{O_S}$ that is normal at S. Recall that we can split every tangent vector $w \in T_xM$ into a tangential and transversal part w.r.t. ξ. Namely, $w = w^\top + w^\perp$ with $w^\top := \sum_{u\in\mathrm{scaff}(\widehat{S})} \partial\hat{x}^u(w)\frac{\partial}{\partial\hat{x}^u}$ and $w^\perp := \sum_{u\in\mathrm{scaff}(\perp\widehat{S})} \partial\hat{x}^u(w)\frac{\partial}{\partial\hat{x}^u}$. Moreover, recall the definition $b^\top_x(w,\tilde{w}) := b_x(w^\top,\tilde{w}^\top)$, so $b^\top = \sum_{u,v\in\mathrm{scaff}(\widehat{S})} b^{uv}\partial\hat{x}^u \otimes \partial\hat{x}^v$ and $b - b^\top = \sum_{\{u,v\}\cap\mathrm{scaff}(\perp\widehat{S})\neq\emptyset} b^{uv}\partial\hat{x}^u \otimes \partial\hat{x}^v$.

Let $r > 0$. Then there are $0 < r_1 < r$ and a family $(O^r_S)_{S\in\mathcal{S}}$ such that

$$M = \bigcup_{S\in\mathcal{S}} O^r_S \quad \text{and} \quad B_{r_1}(O^r_S) \subset O_S \cap B_r(S). \tag{2.41}$$

We will make use of the Taylor-like expansion in Proposition 1.3.17 (i) and (ii). Namely, we use the normal chart ξ_S and deduce from Proposition 1.3.17 that there is a $C > 0$ such that whenever $x, y \in B_r(x_0) \subset O_S$ for some $x_0 \in S$, then

$$|b_x(e_x(y)^\top, e_x(y)^\top) - b_x(y^\top - x^\top, y^\top - x^\top)| \leq Cr|y-x|^2 \tag{2.42}$$

and

$$|b_x(e_x(y)^\top, e_x(y)^\perp)| \leq C\left(|y-x||y^\perp - x^\perp| + \sqrt{r}|y-x|^2\right) \tag{2.43}$$

and

$$|b_x(e_x(y)^\perp, e_x(y)^\perp)| \leq C\left(|y-x||y^\perp - x^\perp| + \sqrt{r}|y-x|^2\right). \tag{2.44}$$

Indeed, (2.43) and (2.44) directly follow from Proposition 1.3.17 (ii), and (2.42) can be shown with help of the identity

$$\begin{aligned} b_x(u^\top,u^\top) - b_x(v^\top,v^\top) &= b_x(u^\top - v^\top, u^\top - v^\top) \\ &\quad + b_x(u^\top - v^\top, v^\top) + b_x(v^\top, u^\top - v^\top) \end{aligned}$$

for $u, v \in T_xM$[8]: Set $u^\top := e_x(y)$ and $v^\top := y^\top - x^\top$ and then use Proposition 1.3.17 (i).
Let $\varrho = \varrho_\zeta$ be a distance on the set of real-valued processes (modulo indistinguishability) on $[0,\zeta]$ that metrizes uniform convergence (up to ζ) in probability.

[8] note that we only take the tangential parts in order to ensure that we can use the bilinearity of b_x

Theorem 2.3.6 *Let X be a semimartingale and let Δ^k be a sequence of locally finite partitions of $\mathbb{R}$ such that $\|\Delta^k\| \to 0$ as $k \to \infty$. Then*

$$\sum_{t_k \in \Delta^k} b_{X_{t_k}}(\Delta^k X_l, \Delta^k X_l) \to \int b(dX_\tau, dX_\tau) \tag{2.45}$$

locally uniformly in probability.

Proof : 1. Since we are dealing with convergence in probability, we may neglect an arbitrarily small event. So by stopping, we may assume that X only has values in a compact set $K \subset M$ and that there is some $\zeta > 0$ such that $X_t \equiv X_\zeta$ for all $t \geq \zeta$.
For $r > 0$, consider a family $(O^r_S)_{S \in \mathcal{S}}$ that satisfies (2.41). Again, we may neglect an arbitrarily small event, and because X is continuous (and hence uniformly continuous on $[0, \zeta]$), we may assume that there is some $\delta > 0$ such that whenever $s < t < s+\delta$, then $d(X_s, X_t) < r_1$. Thus we may assume that if k is large enough[9], whenever $t_l \in \Delta^k$, and $X_{t_l} \in O^r_T$, then every geodesic γ connecting X_{t_l} and $X_{t_{l+1}}$ lies entirely in $B_{r_1}(O^r_T)$.
Let $(\lambda^r_T)_{T \in \mathrm{st}(S)}$ be a partition of unity subordinated to $(O^r_T)_{T \in \mathrm{st}(S)}$. Because K only hits a finite number of simplices S, it suffices to show that for all S and all $\epsilon > 0$ there are $r > 0$ and $k_r \in \mathbb{N}$ such that for all $k \geq k_r$,

$$\varrho\left(\sum_{t_k \in \Delta^k} \lambda^r_S(X_{t_l}) b_{X_{t_k}}(\Delta^k X_l, \Delta^k X_l), \int \lambda^r_S(X_\tau) b(dX_\tau, dX_\tau)\right) < \epsilon \tag{2.46}$$

For this purpose, we will treat the tangential and the transversal parts separately. Namely, we have

$$\begin{aligned}
&\sum_{t_l \in \Delta^k} \lambda^r_S(X_{t_l}) b_{X_{t_l}}(\Delta^k X_l, \Delta^k X_l) \\
&\qquad = \sum_{t_l \in \Delta^k} \lambda^r_S(X_{t_l}) \left[b_{X_{t_l}}(\Delta^k X_l^\top, \Delta^k X_l^\top) + b_{X_{t_l}}(\Delta^k X_l^\top, \Delta^k X_l^\perp) \right. \\
&\qquad\qquad\qquad \left. + b_{X_{t_l}}(\Delta^k X_l^\perp, \Delta^k X_l^\top) + b_{X_{t_l}}(\Delta^k X_l^\perp, \Delta^k X_l^\perp) \right] \\
&\qquad =: \Delta^k B^{\top\top} + \Delta^k B^{\top\perp} + \Delta^k B^{\perp\top} + \Delta^k B^{\perp\perp},
\end{aligned}$$

where

$$\Delta^k B^{\top\top}_t := (\Delta^k B^{\top\top}_S)_t := \sum_{t_l \in \Delta^k_t} \lambda^r_S(X_{t_l}) b_{X_{t_l}}(\Delta^k X_l^\top, \Delta^k X_l^\top) \tag{2.47}$$

[9] to be precise, for $r_1 > 0$ and $\epsilon > 0$ there are a $\delta > 0$ and a set $\Omega_0 \subset \Omega$ with $P(\Omega_0) > 1 - \epsilon$ such that $d(X_s(\omega), X_t(\omega)) < r_1$ for all $\omega \in \Omega_0$ and all $s, t \leq T$ with $|s-t| < \delta$. Then take $k_0 \in \mathbb{N}$ so large that $|t_{l+1} - t_l| < \delta$ for all $k \geq k_0$ and all $t_l \in \Delta^k$.

and so on.

2. Let $\epsilon > 0$. We first show that if r is small and k is large enough[10], then

$$\varrho\left(\Delta^k B^{\top\top}, \int \lambda_S^r(X_\tau) b^\top(dX_\tau, dX_\tau)\right) < \epsilon. \tag{2.48}$$

Indeed, by (2.42) we have for all $0 \leq t \leq \zeta$,

$$\begin{aligned}
&\left| \Delta^k B_t^{\top\top} - \sum_{t_l \in \Delta_t^k} \lambda_S^r(X_{t_l}) b_{X_{t_l}}((X_{t_{l+1}} - X_{t_l})^\top, (X_{t_{l+1}} - X_{t_l})^\top) \right| \\
&\leq \sum_{t_l \in \Delta_t^k} \lambda_S^r(X_{t_l}) \left| b_{X_{t_l}}(\Delta^k X_l^\top, \Delta^k X_l^\top) - b_{X_{t_l}}((X_{t_{l+1}} - X_{t_l})^\top, (X_{t_{l+1}} - X_{t_l})^\top) \right| \\
&\leq Cr \sum_{t_l \in \Delta_t^k} \lambda_T^r(X_{t_l}) |X_{t_{l+1}} - X_{t_l}|^2 \leq Cr \Delta^k V_\zeta,
\end{aligned}$$

where $\Delta^k V_t := \sum_{t_l \in \Delta_t^k} |X_{t_{l+1}} - X_{t_l}|^2$. So we may take r so small that for all $k \in \mathbb{N}$,

$$\varrho\left(\Delta^k B^{\top\top}, \sum_{t_l \in \Delta^k} \lambda_S^r(X_{t_l}) b_{X_{t_l}}((X_{t_{l+1}} - X_{t_l})^\top, (X_{t_{l+1}} - X_{t_l})^\top)\right) < \epsilon/2$$

Now $\sum_{t_l \in \Delta^k} \lambda_S^r(X_{t_l}) b_{X_{t_l}}((X_{t_{l+1}} - X_{t_l})^\top, (X_{t_{l+1}} - X_{t_l})^\top)$ converges to $\int \lambda_S^r(X_\tau) b^\top(dX_\tau, dX_\tau)$ as $k \to \infty$ (discrete approximation of the b-quadratic variation for real-valued processes), and so if k is large enough, then

$$\begin{aligned}
&\varrho\left(\sum_{t_l \in \Delta^k} \lambda_T^r(X_{t_l}) b_{X_{t_l}}((X_{t_{l+1}} - X_{t_l})^\top, (X_{t_{l+1}} - X_{t_l})^\top), \int \lambda_T^r(X_\tau) b^\top(dX_\tau, dX_\tau)\right) \\
&< \epsilon/2
\end{aligned}$$

and hence (2.48) holds.
3. Now we show that if r is small and k is large enough, then

$$\varrho(\Delta^k B^{\top\perp}, 0) < \epsilon. \tag{2.49}$$

[10] more precisely, we first fix some $r > 0$ that is sufficiently small and then find a $k_r \in \mathbb{N}$ such that (2.48) holds for all $k \geq k_r$

From (2.43) it follows that for all $0 \le t \le \zeta$,

$$
\begin{aligned}
|\Delta^k B_t^{\top\perp}| &\le \sum_{t_l \in \Delta_t^k} \lambda_S^r(X_{t_l}) |b_{X_{t_l}}((\Delta^k X_l)^\top, (\Delta^k X_l)^\perp)| \\
&\le C \sum_{t_l \in \Delta^k} \lambda_S^r(X_{t_l}) |X_{t_{l+1}} - X_{t_l}| |X_{t_{l+1}}^\perp - X_{t_l}^\perp| + \sqrt{r}\Delta^k V_t \\
&\le C(\Delta^k V_t)^{1/2} \left(\sum_{t_l \in \Delta^k} \lambda_S^r(X_{t_l}) |X_{t_{l+1}}^\perp - X_{t_l}^\perp|^2 \right)^{1/2} + \sqrt{r}\Delta^k V_t
\end{aligned}
$$

Assume that $S \in \mathcal{S}^{(m)}$. Let $b_1, \ldots, b_m$ be a basis for S and let $b_{m+1}, \ldots, b_N$ be a basis for the orthogonal complement of S in V. We know that if $r \to 0$, then

$$
\begin{aligned}
\sum_{j=m+1}^N \int \lambda_S^r(X_\tau) d\langle X^j \rangle_\tau &\le \sum_{j=m+1}^N \int \mathbf{1}_{\{X_\tau \in B_r(S) \cap O_S\}} d\langle X^j \rangle_\tau \\
&\to \sum_{j=m+1}^N \int \mathbf{1}_{\{X_\tau \in S \cap O_S\}} d\langle X^j \rangle_\tau \equiv 0.
\end{aligned}
$$

Thus we may choose $r > 0$ so small that

$$
\varrho\left(\sum_{j=m+1}^N \int \lambda_S^r(X_\tau) d\langle X^j \rangle_\tau, 0 \right) < \epsilon/3 \tag{2.50}
$$

and that $\varrho\left(\sqrt{r}\Delta^k V, 0\right) < \epsilon/3$ for all $k \in \mathbb{N}$.
At last, since $\sum_{t_l \in \Delta^k} \lambda_T^r(X_{t_l}) |X_{t_{l+1}}^\perp - X_{t_l}^\perp|^2 \to \sum_{j=m+1}^N \int \lambda_T^r(X_\tau) d\langle X^j \rangle_\tau$ (discrete approximation of the Euclidean quadratic variation), we have that

$$
\varrho\left(\sum_{t_l \in \Delta^k} \lambda_T^r(X_{t_l}) |X_{t_{l+1}}^\perp - X_{t_l}^\perp|^2, \sum_{j=m+1}^N \int \lambda_T^r(X_\tau) d\langle X^j \rangle_\tau \right) < \epsilon/3 \tag{2.51}
$$

provided k is large enough. This shows (2.49).

4. At last, we can proceed as in 3. in order to show that if r is small and k is large enough, then $\varrho(\Delta^k B^{\perp\top}, 0) < \epsilon$ and $\varrho(\Delta^k B^{\perp\perp}, 0) < \epsilon$, and putting all parts together, this yields (2.46) and the Theorem is proved.□

Definition 2.3.7 The process

$$
\langle X \rangle := \int g(dX_\tau, dX_\tau) \tag{2.52}
$$

is called the *quadratic variation* of X.

Note that $\|\Delta X_l\|_{X_{t_l}} = \|e_{X_{t_l}}(X_{t_{l+1}})\|_{X_{t_l}} = d(X_{t_l}, X_{t_{l+1}})$ by Proposition 1.3.17 and hence we get the following Corollary, showing that $\langle X \rangle$ deserves the name *quadratic variation*:

Corollary 2.3.8 *Let X be a semimartingale and let Δ^k be a sequence of locally finite partitions of $\mathbb{R}$ such that $\|\Delta^k\| \to 0$ as $k \to \infty$. Set*

$$V_t^k := \sum_{t_l \in \Delta^k} d^2(X_{t_l \wedge t}, X_{t_{l+1} \wedge t}). \tag{2.53}$$

Then $V^k \to \langle X \rangle$ locally uniformly in probability as $k \to \infty$.

2.4 Example: Brownian motion

Although the theory of harmonic functions on Riemannian polyhedra has already made a lot of progress (cf. for instance [EF01]), there are only known very few partial results about Brownian motion.

Walsh ([Wal78]) constructed a family of diffusions in a *star*, which is a (not necessarily locally finite) one-dimensional Riemannian polyhedron. Such a diffusion is referred to as *Walsh's Brownian motion*, cf. [BPY89] for a comprehensive study of these processes.

In [BK95] a Feller process (Brownian motion) is constructed in two-dimensional Euclidean polyhedra by writing down the semigroup explicitely.

In [Bou05] a Donsker's principle approach is used, i.e. Brownian motion is defined as a scaling limit of a suitable sequence of geodesic Random walks. But first, this process is only defined for *almost every* starting point, and second, it is not uniquely determined (there is only shown existence by a compactness argument). Moreover, this paper contains some nebulous arguments. For example, in Remark 2.8 a result of [Dyn65] I is used, but according to this one also has to check the Feller property of the semigroup. Besides, the process Y^η, defined in section 3.1 of [Bou05], is not Markov because it is defined by geodesic interpolation. So this paper has to be read carefully.

Our approach is to fill in the gap between the potential theory developed in [EF01] and probability theory. More precisely, we consider the Markov process X that is associated to the 'canonical' energy $\mathcal{E}(f) := \int_M \|\nabla f(x)\|^2 dx$, where $dx := \mu(dx)$ is the Riemannian volume measure. We will show that X is a strong Feller diffusion, in particular it is defined for *every* starting point $x \in M$. In section 2.4.2 we describe the harmonic structure associated to X. In section 2.4.3 it is shown that X is a semimartingale and an explicite description of the semimartingale decomposition in the local Itô formula is given (Theorem 2.4.17).

2.4.1 Preliminaries

Let (M, g) be a piecewise smooth $n-$dimensional Riemannian polyhedron. Throughout this section we will assume that M is *admissible* in the sense of [EF01]:

- M is dimensionally homogeneous, i.e. every simplex is a face of an $n-$dimensional simplex.
- M is locally $(n-1)-$chainable

The $n-$dimensional Riemannian volume measure μ is defined by

$$\mu(A) := \sum_{S \in \mathcal{S}^{(n)}} \mu_S(A \cap S), \tag{2.54}$$

where μ_S is the $n-$dimensional Riemannian volume measure on S. We will also write dx for μ.

Proposition 2.4.1 *Let M be an $n-$dimensional admissible Riemannian polyhedron. Then the following holds:*
*(i)***(Ball volume growth)** *Let $R > 0$ and $x_0 \in M$. Then there is a $C = C(R, x_0) > 0$ such that $C^{-1}r^n \leq \mu(B_r(x)) \leq Cr^n$ for all $x \in M$ and $r > 0$ with $B_{2r}(x) \subset B_R(x_0)$.*

(ii) **(Ball volume doubling)** *Let $R > 0$ and $x_0 \in M$. Then there is a $C = C(R, x_0) > 0$ such that $\mu(B_{2r}(x)) \leq C2^n\mu(B_r(x))$ for all $x \in M$ and $r > 0$ with $B_{2r}(x) \subset B_R(x_0)$.*

Proof : (i) This follows from [EF01], Lemma 4.4.
(ii) easily follows from (i), cf. [EF01], Corollary 4.1. □

As before we will write $\|v\| := \sqrt{g_x(v, v)}$ if $v \in T_xM$. For a piecewise smooth function $f : M \to \mathbb{R}$ we define the *energy* of f by $\mathcal{E}(f) := \int_M \|\nabla f(x)\|^2 dx$ and the Sobolev (1,2)-norm by

$$\|f\|_{W^{1,2}(M)} := \|f\|_{L^2(M,dx)} + \mathcal{E}(f) \tag{2.55}$$

Let $W^{1,2}$ be the completion of $\{f \in \mathcal{C}^\infty(M) : \|f\|_{W^{1,2}(M)} < \infty\}$ w.r.t. $\|\cdot\|_{W^{1,2}}(M)$ and denote by $W_0^{1,2}$ the completion of $\mathcal{C}_c^\infty(M)$ w.r.t. $\|\cdot\|_{W^{1,2}}(M)$, where $\mathcal{C}_c^\infty(M)$ is the set of all piecewise smooth functions on M with compact support. Note that these definitions coincide with the definitions of $W^{1,2}(M)$ and $W_0^{1,2}(M)$ in [EF01], section 5, since every Lipschitz continuous function can be approximated

by piecewise smooth functions in the Sobolev (1,2)-norm[11].
Consider now the bilinear form

$$\mathcal{E}(f,h) := \int_M g_x(\nabla f(x), \nabla h(x))dx, \qquad \mathcal{D}(\mathcal{E}) := W_0^{1,2}(M). \tag{2.56}$$

This form has some nice properties. For a general approach to Dirichlet forms on glued spaces, see [Pau04]. Fortunately, we can quote some results from [EF01].

Proposition 2.4.2
(i) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a strongly local regular Dirichlet form on $L^2(M, dx)$ and $\mathcal{C}_c^\infty(M)$ is a core for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

*(ii)(***Poincaré inequality***) Let $R > 0$ and $x_0 \in M$. Then there is a $C = C(R, x_0) > 0$ such that*

$$\int_{B_r(x)} |f - f_{x,r}|^2 dx \leq Cr^2 \int_{B_r(x)} \|\nabla f(x)\|^2 dx \tag{2.57}$$

whenever $B_r(x) \subset B_R(x_0)$, where $f_{x,r} := \mu(B_r(x))^{-1} \int_{B_r(x)} f(x)dx$.

Proof : (i) That $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form is proved in [EF01], Proposition 5.1. Moreover, the strong locality follows from [EF01], Remark 5.2.
(ii) follows from [EF01], Theorem 5.1. (see also the remark before that theorem). □

By the general theory of Dirichlet forms, we get a continuous Hunt process on a set $M_0 \subset M$, where $M \setminus M_0$ has zero capacity ([Fukushima], Theorem 7.2.2.). However, we want to define Brownian motion for *every* starting point $x \in M$. This will be done in the context of Feller processes, and consequently we have to do some more work.
We denote by $(A, \mathcal{D}(A))$ the self-adjoint operator on $L^2(M, dx)$ associated to $\mathcal{E}$. A *fundamental solution* p of the parabolic equation $(\frac{\partial}{\partial t} - A)f = 0$ is defined to be a density kernel for the transition semigroup $T_t := e^{-At}$ w.r.t μ. Namely, $p :]0, \infty[\times M \times M \to \mathbb{R}_+$ is a measurable function satisfying

$$T_t f(x) = \int_M f(y)p(t,x,y)dy \tag{2.58}$$

for all $f \in L^2(M, dx)$ and a.e. $x \in M$. p is also called a *heat kernel* of A. Note that T_t is $\mu-$symmetric, p is symmetric in x and y for μ^2-almost all (x, y).

[11]this can easily bee seen by a mollifying argument as in Lemma 2.1.1 or Lemma 3.2.1

Lemma 2.4.3 *A heat kernel $p :]0,\infty[\times M \times M \to \mathbb{R}_+$ of A exists. There is a version of p that is locally Hölder continuous on $]0,\infty[\times M \times M$ (and hence unique) and symmetric in x and y. Moreover, p satisfies locally an 'upper Gaussian estimate': For all $x_0 \in M$ and all $R > 0$ there are constants $C_1 = C_1(R, x_0) > 0$ and $C_2 = C_2(R, x_0) > 0$ such that for all $0 < t < R^2$ and all $x, y \in B_R(x_0)$*

$$p(t,x,y) \le \frac{C_1}{\mu(B_{\sqrt{t}}(x))} e^{-\frac{d^2(x,y)}{C_2 t}}. \tag{2.59}$$

Proof : The proof follows the outline of the proof of [Stu98], Theorem 7.4 which is stated under slightly different circumstances. We are in the following situation: (M, d, μ) is a metric measure space and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a strongly local Dirichlet form on $L^2(M, \mu)$. Moreover, by Proposition 2.4.1 (i), the *intrinsic distance* ρ of $\mathcal{E}$ (which is by definition equal to the Caratheodory distance, cf. [Stu95]) is equal to d. The following properties hold:

(Ia) $(M, d) = (M, \rho)$ is proper by Proposition 1.3.9.

(Ib) Volume doubling holds locally on M in the sense of Proposition 2.4.1 (ii).

(Ic) The Poincare inequality holds locally on M in the sense of Proposition 2.4.2 (ii).

(Ia),(Ib) correspond to conditions (A) and (B) in [Stu95]. Moreover, by [Stu96], Theorem 2.6, there holds a Sobolev inequality locally on M which is condition (C) in [Stu95]. Note that all 'uniform parabolicity' conditions in [Stu95] hold trivially since the operator A does not depend on time. Moreover, conditions (Ia)-(Ic) imply that there holds a parabolic Harnack inequality on X by [Stu96], Theorem 3.5. Thus we can make use of local versions of all the important results in these papers.

By [Stu95], Proposition 2.3., a heat kernel $\tilde{p} = \tilde{p}(t, x, x)$ exists. Moreover, there is a $\mu-$nullset N such that for all $x, y \in M \setminus N$, $\tilde{p}(\cdot, x, \cdot)$ and $\tilde{p}(\cdot, \cdot, y)$ are local weak solutions of the parabolic equation $(\frac{\partial}{\partial t} - A)f = 0$ in the sense of [Stu96], section 3. Now by [Stu96], Proposition 3.1, *every* such local weak solution has a version that is locally Hölder continuous in the sense that for any $x_0 \in M$ and all $R, t_0 > 0$ there are constants $C = C(x_0, R)$ and $\alpha = \alpha(x_0, R) \in]0,1[$ such that whenever (s, y) and (t, z) are points in $Q :=]t_0 - R^2, t_0[\times B_R(x_0)$, then

$$|f(s,y) - f(t,z)| \le C \operatorname*{ess\,sup}_{B_R(x_0)} |f| \left(|s-t|^{1/2} + d(y,z)\right)^{\alpha}. \tag{2.60}$$

We will only show how to get a Hölder continuous version of the function $\tilde{p}(t, \cdot, \cdot)$ on $B_R(x_0)$ for every $t > 0$, since this suffices for our applications in the sequel.

For every $x \in M \setminus N$, let $\hat{p}(t,x,\cdot)$ be the Hölder continuous version of $\tilde{p}(t,x,\cdot)$. Clearly, this is a fundamental solution, too, and hence for every $y \in B_R(x_0) \setminus N$ there is a $\mu-$nullset N_y such that $\hat{p}(t,\cdot,y)$ is Hölder continuous (with the same constants) on $B_R(x_0) \setminus N_y$. So setting $\breve{p}(t,x,y) := \lim_{x_n \to x, x_n \notin N \cup N_y} \hat{p}(t,x_n,y)$, we get a version that is Hölder continuous on $B_R(x_0) \times (B_R(x_0) \setminus N)$.
At last we set $p(t,x,y) := \lim_{y_n \to y, y_n \notin N} \breve{p}(t,x,y_n)$ and obtain a version that is Hölder continuous on $B_R(x_0)$.
The last assertion, namely (2.59), follows from [Stu96] Theorem 4.1. and (4.4). □

Remark 2.4.4 From [Stu96], Theorem 4.8. we can also deduce *lower Gaussian estimates* for the heat kernel. Namely, there is a $C = C(R) > 0$ such that

$$p(t,x,y) \geq \frac{1}{C\mu(B_{\sqrt{t}}(x)}e^{-C\frac{d^2(x,y)}{t}} \tag{2.61}$$

for all $x, y \in B_R(x_0)$ and all $0 < t \leq R^2$.

First we will use the Gaussian estimate (2.59) to show some properties of the heat kernel that are known for the Euclidean heat kernel.

Lemma 2.4.5 *(i) There is a $C = C(x_0, R)$ such that for all $0 < t < R^2$, all $x \in M$ and $r > 0$ with $B_r(x) \subset B_R(x_0)$,*

$$\int_{B_r(x)} d^\alpha(x,y)p(t,x,y)dy \leq Ct^{\frac{\alpha}{2}} \tag{2.62}$$

(ii) Put $g_t(x,y) := \int_0^t p(\tau,x,y)d\tau$. Let $S \in \mathcal{S}^{(m)}$, where $m \leq n-2$. Let O be local at C and put $S_r := B_r(S)$. Then for all $t \leq R^2$

$$\lim_{r \searrow 0} \frac{1}{r}\left(\sup_{x \in B_{R/2}(x_0)} \int_{S_r \cap O} g_t(x,y)dy\right) = 0. \tag{2.63}$$

Proof : (i) By Proposition 1.3.9, there is a $C > 0$ such that $1/C|y-x| \leq d(x,y) \leq C|y-x|$ for all $x, y \in B_R(x_0)$. So from (2.59) we deduce that there are constants $C_1, C_2 > 0$ such that for all $t < R^2$ and $x, y \in B_R(x_0)$

$$d^\alpha(x,y)p(t,x,y) \leq C_1 t^{-\frac{n}{2}}|y-x|^\alpha e^{-\frac{|y-x|^2}{C_2 t}}. \tag{2.64}$$

Let $B_r^e(x)$ be the Euclidean ball around x in V. Then $B_r(x) \subset B_{Cr}^e(x)$. Moreover by Proposition 2.4.1 (i), $\mu(dy) \leq \tilde{C}\lambda^n(dy)$ and hence we can adjust C_1 such that for all $t < R^2$, $r > 0$ and $x \in M$ with $B_r(x) \subset B_R(x_0)$,

$$\int_{B_r(x)} d^\alpha(x,y)p(t,x,y)\mu(dy) \leq C_1 t^{-\frac{n}{2}} \int_{B_{Cr}^e(x)} |y-x|^\alpha e^{-\frac{|y-x|^2}{C_2 t}} \lambda^n(dy). \tag{2.65}$$

In order to estimate the right hand side of (2.65), we remark that $B_R(x_0)$ contains only a finite number of $n-$dimensional simplices of M. So let $T \in \mathcal{S}^{(n)}$. Let U be the $n-$dimensional linear subspace of V generated by T. We consider two cases: First, if $x \in U$, then we can use polar coordinates in U in order to obtain

$$\begin{aligned} C_1 t^{-\frac{n}{2}} \int_{B^e_{Cr}(x)\cap U} |y-x|^\alpha e^{-\frac{|y-x|^2}{C_2 t}} \lambda^n(dy) &= C_3 C_1 t^{-\frac{n}{2}} \int_0^{\tilde{C}r} \rho^{\alpha+n-1} e^{-\frac{\rho^2}{C_2 t}} d\rho \\ &\leq \tilde{C}_1 t^{\frac{\alpha}{2}} \int_0^\infty \rho^{\alpha+n-1} e^{-\frac{\rho^2}{2}} d\rho. \end{aligned}$$

Now $\int_0^\infty \rho^{\alpha+n-1} e^{-\frac{\rho^2}{2}} d\rho < \infty$ and hence (i) is proved in the case $x \in U$. Second, if $x \notin U$, let x_0 be the orthogonal projection of x onto U and let $r_0 := |x - x_0|$.

$$\begin{aligned} &\int_{B^e_{Cr}(x)\cap U} |y-x|^\alpha e^{-\frac{|y-x|^2}{C_2 t}} \lambda^n(dy) \\ &\qquad = \int_{B^e_{Cr}(x)\cap U} (r_0^2 + |y-x|^2)^{\frac{\alpha}{2}} e^{-\frac{r_0^2+|y-x_0|^2}{C_2 t}} \lambda^n(dy) \end{aligned}$$

and we proceed as above in order to obtain (i).

(ii) Let $x \in B_{R/2}(x_0)$. Then by (2.64), for all $t \leq R^2$, $r \leq R/2$ and $y \in B_r(x)$,

$$g_t(x,y) \leq C_1 \int_0^t \tau^{-\frac{n}{2}} e^{-\frac{|y-x|^2}{C_2 \tau}} d\tau. \tag{2.66}$$

Let $S \in \mathcal{S}^{(m)}$, O local at S. One can simplify the situation: First since $\mathrm{st}^{(n)}(S)$ is finite, we may replace $S_r \cap O$ by $S_r \cap O \cap T$, where $T \in \mathrm{st}^{(n)}(S)$ is fixed. Denote by U' the linear subspace generated by T and by U the one generated by S. We may assume that $x \in U$ (otherwise take x_0, the orthogonal projection of x onto U'; then $g_t(x,y) \leq g_t(x_0,y)$ for all $y \in U'$). Thus we may assume that $U' = \mathbb{R}^n$ and $U = \mathbb{R}^m \subset \mathbb{R}^n$. Denote by $p_t^{(k)}(x,y)$ the $k-$dimensional Euclidean heat kernel. Then (2.66) yields

$$\int_{S_r\cap O} g_t(x,y)\mu(dy) \leq C_1 \int_0^t \int_{S_r\cap O} p_\tau^{(n)}(x,y)\lambda^n(dy)d\tau. \tag{2.67}$$

and consequently

$$\begin{aligned}\int_{S_r\cap O} g_t(x,y)\mu(dy) &\le C_1\int_0^t\int_{S_r\cap O} p_\tau^{(n)}(x,y)\lambda^n(dy)d\tau\\ &\le C_1\int_0^t\int_{U_r} p_\tau^{(n)}(x,y)\lambda^n(dy)d\tau\\ &\le C_1\int_0^t\int_{D_r} p_\tau^{(n-m)}(x,y)\lambda^{n-m}(dy)d\tau,\end{aligned}$$

where $D_r := B_r(0)\cap\mathbb{R}^{n-m}$ and $U_r := D_r\times\mathbb{R}^m$ is the $r-$strip around $U=\mathbb{R}^m$. So putting $\delta := \delta(y) := |y-x|$ and substituting $\sigma(\tau)=\frac{\delta^2}{C_2\tau}$, we have

$$\int_{S_r\cap O} g_t(x,y)\mu(dy) \le \int_{D_r} C(\delta(y))\lambda^{n-m}(dy)$$

where

$$\begin{aligned}C(\delta) := C(t,\delta) &:= C_1 C_2^{-\frac{n-m}{2}-1}\delta^{-n+m+2}\int_{\frac{\delta^2}{C_2 t}}^\infty \sigma^{\frac{n-m}{2}-2}e^{-\sigma}d\sigma\\ &\le \begin{cases} C_4\delta^{-n+m+2} & \text{if } n-m\ge 3\\ C_4(|\log\delta|+1) & \text{if } n-m=2\end{cases}\end{aligned}\tag{2.68}$$

Note that up to a constant, the bound of $C(\delta(y))$ in (2.68) is the $(n-m)-$dimensional Newtonian potential with pole x, cf. e.g. [Bas95]. Using polar coordinates (on the $(n-m)-$dimensional subspace) yields

$$\begin{aligned}\int_{D_r} C(|y-x|)\lambda^{n-m}(dy) &\le \int_{D_r} C(|y|)\lambda^{n-m}(dy)\\ &\le C_5\int_0^r \rho^{n-m-1}C(\rho)d\rho.\end{aligned}\tag{2.69}$$

Together with (2.68) and the fact that $n-m\ge 2$, we conclude that the right hand side of (2.69) is an $o(r)$ ('small o of r'). Moreover, the right hand side of (2.69) is independent of $x\in B_{R/2}(x_0)$, showing (ii). □

If we now set $P_tf(x) := \int_M f(y)p(t,x,y)dy$ for $f\in L^2(M,dx)$, then $(P_t)_{t>0}$ is a sub-Markovian semigroup on $L^2(M,dx)$ that is properly associated to A, i.e. P_tf is a version of $e^{-At}f$ for all $f\in L^2(M,dx)$ and all $t\ge 0$.

Proposition 2.4.6 *P_t is a strong Feller semigroup in the sense that $P_t\mathcal{B}_b(M)\subset\mathcal{C}_b(M)$, where $\mathcal{B}_b(M)$ denotes the set of bounded measurable functions on M.*

Proof : Let $f \in \mathcal{B}_b(M)$. Set $f_R := f\mathbf{1}_{B_R(x_0)}$. Then $P_t f_R$ is Hölder continuous on $B_R(x_0)$ since the heat kernel p is Hölder continuous on $B_R(x_0) \times B_R(x_0)$. We have even more: Because $P_t f_R$ a local solution of $Au - \frac{\partial}{\partial t}u = 0$, the Hölder constant can be chosen *independent of* R, cf. (2.60). So letting $R \nearrow \infty$, $P_t f_R \nearrow P_t f$ and hence $P_t f$ is Hölder continuous on $B_R(x_0)$. Thus the proposition is proved. □

Now we will construct the process associated to P_t. By [BG68], Theorem I.(9.4), there is a Hunt process $(\Omega, (P_x)_{x\in M}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0})$ with state space M. In fact, this process is constructed on $M \cup \Delta$, where Δ is the 'Alexandroff point' of M, i.e. Δ is the point ∞ in the 1-point compactification of M if M is not compact and Δ is an isolated point if M is already compact. The *lifetime* ζ of X is the stopping time $\zeta := \inf\{t \geq 0 : X_t \in \Delta\}$. From the properties of a Hunt process (cf. [BG68], Definition I.(9.2) and the Remark after) it follows that for all $x \in M$, X has cadlag paths on $[0, \zeta[$ P^x-a.s. and that X is *quasi-left-continuous*. The latter means that whenever τ_n is an increasing sequence of stopping times with $\tau = \lim \tau_n$, then $X_\tau = \lim X_{\tau_n}$ on $\{\tau < \infty\}$ P^x-a.s.
Note that it follows from the strong Feller property of P_t that for all $t > 0$, $x_0 \in M$ and $R > 0$,

$$\lim_{Z \nearrow \infty} \sup_{x \in B_R(x_0)} P^x(X_t \notin B_Z(x_0), t < \zeta) = 0. \tag{2.70}$$

In the sequel, it will often be convenient to consider a localized version of the process X_t. Namely, fix $x_0 \in M$. For $R > 0$ let

$$\tau_O := \inf\{t > 0 : X_t \notin O\}. \tag{2.71}$$

and for all $R > 0$ set

$$\tau_R := \inf\{t > 0 : X_t \notin B_R(x_0)\} \wedge R^2 = \tau_{B_R(x_0)} \wedge R^2. \tag{2.72}$$

It follows from the quasi-left-continuity of X that $\tau_R \nearrow \zeta$ P^x-a.s. for all $x \in M$. Consider the stopped process X^{τ_R}, given by $X_t^{\tau_R} := X_{t\wedge\tau_R}$.

Lemma 2.4.7 *Let $x_0 \in M$ and $R > 0$.*
(i) For all $r > 0$, set $\sigma_r := \inf\{t \geq 0 : d(X_0, X_t) \geq r\}$. Then there is a $C = C(x_0, R)$ such that for all $x \in B_R(x_0)$ and all $r \leq R$,

$$P^x(t \geq \sigma_r) \leq Ce^{-\frac{r^2}{10t}}. \tag{2.73}$$

(ii) For all $\alpha > 0$ there is a $C = C(x_0, R, \alpha) > 0$ such that for all $x \in B_{R/2}(x_0)$ and all $0 \leq s \leq t$,

$$\mathbf{E}^x[d^\alpha(X_t^{\tau_R}, X_s^{\tau_R})] \leq C(t-s)^{\alpha/2},$$

Proof : (i) Let $K \subset M$ be a compact admissible sub-polyhedron such that $B_{2R}(x_0) \subset K$. Let $(\Omega, (Q^x)_{x\in K}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0})$ be Brownian motion on K (so it is 'reflected' at ∂K). The desired inequality is shown in [Stu96], Lemma 4.6., for Q^x, i.e. $Q^x(t \geq \sigma_r) \leq Ce^{-\frac{r^2}{10t}}$ for all $x \in K$. Denote by $\widetilde{P}^x$ the *part of* P^x *on* $B_R(x_0)$ and by $\widetilde{Q}^x$ the part of Q^x on $B_R(x_0)$, respectively[12]. Note that by Lemma 5.3.3,

$$\begin{aligned} P^x(t < \sigma_r) &= P^x(t < \sigma_r, t < \tau_{2R}) = \widetilde{P}^x(t < \sigma_r) \\ &= \widetilde{Q}^x(t < \sigma_r) = Q^x(t < \sigma_r). \end{aligned}$$

So taking the complements of these events yields $P^x(t \geq \sigma_r) = Q^x(t \geq \sigma_r)$, showing (i).
(ii) Clearly, it suffices to show (ii) for all $s \leq t \leq R^2$. From (i) and the fact that $\sigma_{R/2} \leq \tau_R$ P^x-a.s. for all $x \in B_{R/2}(x_0)$, we deduce that

$$\lim_{t\to 0} \frac{1}{t^{\alpha/2}} P^x(t \geq \tau_R) = 0$$

for all $\alpha > 0$ and hence there is some $C_1 = C_1(x_0, R, \alpha) > 0$ such that $P^x(t \geq \tau_R) \leq C_1 t^{\alpha/2}$ for all $x \in B_R(x_0)$ and all $t \leq R^2$. Moreover, by Lemma 2.4.5 (i), there is a $C_2 > 0$ such that for all $x \in B_{R/2}(x_0)$,

$$\mathbf{E}^x[\mathbf{1}_{\{t<\tau_R\}} d^\alpha(x, X_t)] \leq \mathbf{E}^x[\mathbf{1}_{\{X_t \in B_R(x_0)\}} d^\alpha(x, X_t)] \leq C_2 t^{\alpha/2}.$$

Thus we have for all $x \in B_{R/2}(x_0)$ and all $t \leq R^2$,

$$\begin{aligned} \mathbf{E}^x[d^\alpha(X_0, X_t^{\tau_R})] &= \mathbf{E}^x[d^\alpha(x, X_t^{\tau_R})] \\ &\leq \mathbf{E}^x[\mathbf{1}_{\{t<\tau_R\}} d^\alpha(x, X_t)] + R^\alpha P^x(t \geq \tau_R) \\ &\leq (R^\alpha C_1 + C_2) t^{\alpha/2} =: C t^{\alpha/2}. \end{aligned}$$

So using the Markov property, we obtain

$$\begin{aligned} \mathbf{E}^x[d^\alpha(X_t^{\tau_R}, X_s^{\tau_R})] &= \mathbf{E}^x\left[\mathbf{1}_{\{s<\tau_R\}} \mathbf{E}[d^\alpha(X_t^{\tau_R}, X_s^{\tau_R}) | \mathcal{F}_s]\right] \\ &= \mathbf{E}^x\left[\mathbf{1}_{\{s\leq\tau_R\}} \mathbf{E}^{X_s}[d^\alpha(X_0, X_{t-s}^{\tau_R})]\right] \\ &\leq C(t-s)^{\alpha/2}, \end{aligned}$$

so (ii) is proved. □

It follows from Lemma 2.4.7 (ii) with e.g. $\alpha = 3$ that the assumptions of the

[12] the part of a process X on an open set O is the process obtained by killing it when it reaches the boundary, cf. (5.22)

Kolmogorov-Chentsov-Theorem (cf. e.g. [KS91], Theorem 2.8[13]) are satisfied, and consequently X^{τ_R} has a continuous modification w.r.t. P^x. Then we let $R \nearrow \infty$ in order to obtain a continuous modification on $[0, \zeta[$. We state this result in the following

Corollary 2.4.8 *For all $x \in M$, X has a continuous modification w.r.t. P^x.*

So we can take $\Omega = \mathcal{C}(\mathbb{R}_+, M)$, $X_t(\omega) = \omega(t)$ and $\mathcal{F}_t$ the minimum admissible filtration, i.e. $\mathcal{F}_t := \bigcap_{\mu \in \mathcal{P}(M)} \mathcal{F}^\mu_{t+}$, where $\mathcal{F}^\mu_{t+}$ denotes the completion of $\mathcal{F}^0_{t+}$ w.r.t. P^μ and $\mathcal{F}^0_t := \sigma(X_s : s \leq t)$, $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$.

Definition 2.4.9 The unique strong Feller diffusion

$$(\mathcal{C}(\mathbb{R}_+, M), (\mathcal{F}_t)_{0 \leq t < \zeta}, (X_t)_{0 \leq t < \zeta}, (P^x)_{x \in M})$$

associated to $(P_t)_{t \geq 0}$ is called *Brownian motion on M*.

2.4.2 The harmonic structure(s)

Let M be an admissible Riemannian polyhedron. It is shown in [EF01] that M carries a harmonic structure in the sense of Brelot. We will refer to this as the harmonic structure *in the analytic sense*.
On the other hand, denote by X the Brownian motion on M. We will see below that X also defines a harmonic structure by means of [Dyn65], chapter 12, and we will refer to this as the harmonic structure for X *in the stochastic sense*.
We quote the definitions of (sub-)harmonicity in the analytic and in the stochastic sense, cf. [EF01], Definition 5.2. and [Dyn65], Definition 12.11. In order to avoid technicalities, we restrict ourselves to continuous functions.
Recall the definition of the first exit time from a set O:

$$\tau_O := \inf\{t > 0 : X_t \notin O\}. \tag{2.74}$$

Definition 2.4.10 Let $O \subset M$ be an open set and let $f : O \to \mathbb{R}$ be a continuous function.
f is called *subharmonic in O in the analytic sense* if $f \in W^{1,2}_{loc}$ and $\mathcal{E}(f, g) \leq 0$ for all $g \in \mathcal{C}^\infty_c$ with $g \geq 0$.
f is called *subharmonic in O for X (in the stochastic sense)* if for all relatively open sets $U \subset\subset O$ and all $x \in U$, $f(x) \leq \mathbf{E}^x[f(X_{\tau_U})]$.
In both cases, f is called *harmonic* if f and $-f$ are subharmonic. An open set $O \subset M$ is called *regular* if for any bounded continuous function $f : \partial O \to \mathbb{R}$ there is a solution to the Dirichlet problem, i.e. there is a unique continuous function $h^f : \overline{O} \to \mathbb{R}$ such that h^f is harmonic in O and $h^f_{|\partial O} \equiv f$.

[13] note that this Theorem holds for any complete metric space, cf. [HT94], Satz 2.11

We will see in the sequel that both notions are equivalent. For the proofs we will need the localization procedure for Markov processes, as introduced in section 5.3, namely the concept of the *part* of a Markov process on O. In particular, this concept applies to our situation in Example 5.3.4.

Proposition 2.4.11 *(i) Let $O \subset M$ be an open domain and let $f : O \to \mathbb{R}$ be a bounded continuous function. Then f is subharmonic for X on O if and only if f is subharmonic for X^O, the part of X on O. In this case, $f(X_t)$ is a submartingale on $\{X \in O\}$. Moreover, if $f \in W_0^{1,2}(M)$, then f is subharmonic in O in the analytic sense.*
(ii) Let $h : O \to \mathbb{R}$ be a bounded continuous function. Then f is harmonic for X on O if and only if f is harmonic for X^O, the part of X on O. that is harmonic. In this case, $h(X_t)$ is a martingale. Moreover, h is harmonic in O in the analytic sense.

Proof : (i) The equivalence is shown in Theorem 12.9 of [Dyn65]. If f is subharmonic, then $-f$ is superharmonic and hence f is upper semicontinuous by [Dyn65], Theorem 13.2.
It remains to show that $f(X)$ is a submartingale on $\{X \in O\}$. Because $-f$ is superharmonic, it follows from Corollary 2 of Theorem 12.9 in [Dyn65] that

$$\mathbf{E}^x[f(X_\tau)] \geq f(x) \tag{2.75}$$

whenever $x \in O$ and τ is a stopping time such that $P^x(\tau < \tau_O) = 1$.
Let now $U \subset\subset V_1 \subset\subset V_2 \subset\subset O$. Set $\breve{\sigma} := \inf\{t \geq 0 : X_t \in V_1\}$ and $\breve{\tau} := \inf\{t \geq 0 : X_t \notin V_2\}$. Then define recursively $\sigma_0 := \tau_0 := 0$ and $\sigma_{n+1} := \breve{\sigma} \circ \theta_{\tau_n} = \inf\{t \geq \tau_n : X_t \in V_1\}$ and $\tau_{n+1} := \breve{\tau} \circ \theta_{\sigma_{n+1}} = \inf\{t \geq \sigma_{n+1} : X_t \notin V_2\}$. Note that

$$f(X_{(\sigma_n+s+t)\wedge\tau_n}) = f(X_{t\wedge\breve{\tau}}) \circ \theta_{\sigma_n+s} \quad \text{on} \quad \{\sigma_n + s < \tau_n\}.$$

Thus we can use the strong Markov property and (2.75) in order to obtain

$$\begin{aligned}
\mathbf{1}_{\{\sigma_n+s<\tau_n\}}\mathbf{E}^x[f(X_{(\sigma_n+s+t)\wedge\tau_n})|\mathcal{F}_{\sigma_n+s}] &= \mathbf{1}_{\{\sigma_n+s<\tau_n\}}\mathbf{E}^x[f(X_{t\wedge\breve{\tau}}) \circ \theta_{\sigma_n+s}|\mathcal{F}_{\sigma_n+s}] \\
&= \mathbf{1}_{\{\sigma_n+s<\tau_n\}}\mathbf{E}^{X_{\sigma_n+s}}[f(X_{t\wedge\breve{\tau}})] \\
&\geq \mathbf{1}_{\{\sigma_n+s<\tau_n\}}f(X_{\sigma_n+s}) \\
&= \mathbf{1}_{\{\sigma_n+s<\tau_n\}}f(X_{(\sigma_n+s)\wedge\tau_n}).
\end{aligned}$$

Moreover, $\mathbf{1}_{\{\sigma_n+s\geq\tau_n\}}f(X_{(\sigma_n+s+t)\wedge\tau_n}) = \mathbf{1}_{\{\sigma_n+s\geq\tau_n\}}f(X_{\tau_n})$ is already measurable w.r.t. $\mathcal{F}_{\sigma_n+s}$ and hence

$$\mathbf{1}_{\{\sigma_n+s\geq\tau_n\}}\mathbf{E}[f(X_{(\sigma_n+s+t)\wedge\tau_n})|\mathcal{F}_{\sigma_n+s}] = \mathbf{1}_{\{\sigma_n+s\geq\tau_n\}}f(X_{(\sigma_n+s+t)\wedge\tau_n}).$$

So we deduce that the process $\widetilde{Y}_t := f(X_{(\sigma_n+t)\wedge\tau_n})$ is a submartingale w.r.t. the filtration $\widetilde{\mathcal{F}}_t := \mathcal{F}_{\sigma_n+t}$, which means that $f(X)$ is a submartingale on $[\sigma_n, \tau_n]$. Consequently, $f(X)$ is a submartingale on $\{X \in U\}$, cf. (5.18). Because $U \subset\subset O$ was chosen arbitrarily, $f(X)$ is a submartingale on $\{X \in O\}$.
It remains to show that f is subharmonic in the analytic sense. Note that since $f \in \mathcal{D}(\mathcal{E})$, we have $\mathcal{E}(f,g) = \lim_{t\to 0} \frac{1}{t}(f - P_t f, g)_{L^2}$.
Now fix $g \in \mathcal{C}_c^\infty(O)$ with $g \geq 0$ and let $U \subset\subset O$ be a neighborhood such that $\mathrm{supp}(g) \subset U$. Note that for all $x \in \mathrm{supp}(g)$, $\tau_U < \tau_O$ P^x-a.s. and hence $f(x) \leq P_{t\wedge\tau_U} f(x) := \mathbf{E}^x[f(X_{t\wedge\tau_U})]$, which implies that $\frac{1}{t}(f - P_{t\wedge\tau_U} f, g)_{L^2} \geq 0$ for all $t \geq 0$. Moreover, $r := \inf\{d(x,y) : x \in \mathrm{supp}(g), y \notin U\} > 0$ and hence $\sigma_r := \inf\{t \geq 0 : d(X_0, X_t) \geq r\} \leq \tau_U$ P^x-a.s. for all $x \in \mathrm{supp}(g)$. Thus by Lemma 2.4.7 (i),

$$\begin{aligned}\lim_{t\to 0} \sup_{x\in\mathrm{supp}(g)} \frac{1}{t}|P_t f(x) - P_{t\wedge\tau_U} f(x)| &= \lim_{t\to 0} \sup_{x\in\mathrm{supp}(g)} \frac{1}{t}\mathbf{E}^x[\mathbf{1}_{\{t\geq\tau_U\}}(f(X_t) - f(X_{\tau_U}))] \\ &\leq C \lim_{t\to 0} \sup_{x\in\mathrm{supp}(g)} \frac{1}{t} P^x(t \geq \tau_U) = 0\end{aligned}$$

and hence

$$\mathcal{E}(f,g) = \lim_{t\to 0} \frac{1}{t}(f - P_t f, g)_{L^2} = \lim_{t\to 0} \frac{1}{t}(f - P_{t\wedge\tau_U} f, g)_{L^2} \geq 0.$$

Thus f is subharmonic in O in the analytic sense.
(ii) All assertions follow from (i), noting that if h is harmonic, then h and $-h$ are subharmonic. □

Let us now come to the *Dirichlet problem.* As domain we will take a compact admissible Riemannian polyhedron K with nonempty boundary. We will show that K is regular in both senses and that the notions of harmonic functions are the same. Note that for a Riemannian polyhedron M, the set of compact subpolyhedra K (where also the isometric triangulations may vary with K) form a base of the topology of M, so we can deduce from the following Theorem that the (analytic and stochastic) harmonic structures on M coincide.
Let (K, g) be a compact $n-$dimensional admissible Riemannian polyhedron with nonempty boundary and let $f : \partial K \to \mathbb{R}$ be a continuous function. As in the classical case, the candidate for the stochastic solution for f is the function

$$\widehat{h}^f(x) := \mathbf{E}^x[f(X_{\tau_{K^\circ}})]. \tag{2.76}$$

Theorem 2.4.12 *Let (K, g) be a compact $n-$dimensional admissible Riemannian polyhedron with nonempty boundary. Then $D = K^\circ$ is regular, both in the analytic* *sense and for X (the Brownian motion). Moreover, if $f : \partial D \to \mathbb{R}$ is bounded and continuous, then $h^f \equiv \widehat{h}^f$*[14].

Proof : 1. Without loss of generality we may assume there is some compact $n-$dimensional admissible Riemannian polyhedron (M,g) such that $(K,g) \subset (M,g)$ is a admissible Riemannian sub-polyhedron such that $K \cap \partial M = \emptyset$.[15]. This implies that for any boundary face S of D there is an *adjacent* $n-$dimensional simplex $T \in \mathcal{S}(M) \setminus \mathcal{S}(D)$ containing S. Denote by X the Brownian motion on K and by $\widetilde{X}$ the Brownian motion on M. Note that by (5.28), for all $x \in D$,

$$\widehat{h}^f(x) = \mathbf{E}^x[f(X_\tau)] = \widetilde{\mathbf{E}}^x[f(X_\tau)]. \tag{2.77}$$

Now we shall prove that ∂D is regular for $\widetilde{X}$, i.e $\widetilde{P}^x(\tau_D = 0) = 1$ for all $x \in \partial D$. Indeed, one can prove this with a version of the Poincare cone condition: Let $x \in \partial D$. Then there is some $\epsilon > 0$ and an $n-$dimensional Euclidean cone $\mathcal{C} \subset V$ such that $C \cap B^e_\epsilon(x) \subset \widetilde{M} \setminus D$, where $B^e_\epsilon(x)$ denotes the Euclidean ball around x. Then by (2.4.4) we have for all $0 < t \le 1$,

$$\begin{aligned}
\widetilde{P}^x(\tau_D \le t) \ge \widetilde{P}^x(\widetilde{X}_t \in \mathcal{C}) &\ge C \int_{\mathcal{C}\cap B_\epsilon(x)} t^{-n/2} e^{-\frac{|x-y|^2}{Ct}} \lambda(dy) \\
&= C \int_{\mathcal{C}\cap B_{\epsilon/t}(x)} e^{-\frac{|x-y|^2}{C}} \lambda(dy) \\
&\ge C \int_{\mathcal{C}\cap B_\epsilon(x)} e^{-\frac{|x-y|^2}{C}} \lambda(dy) =: \delta > 0
\end{aligned}$$

Note that δ does not depend on t, and hence letting $t \searrow 0$ yields that $\widetilde{P}^x(\tau_D = 0) > 0$. So it follows from the Blumenthal 0-1 law that $\widetilde{P}^x(\tau_D = 0) = 1$. Thus x is regular for $\widetilde{X}$.

By [Dyn65], Theorem 13.4, $\widehat{h}^f$ is the unique bounded continuous function that is harmonic for $\widetilde{X}$ on D and coincides with f on ∂D. Moreover, by Proposition 2.4.11 (ii), $\widehat{h}^f$ is also harmonic for X and in the analytic sense. It can easily be shown[16] that $\widehat{h}^f$ is the only continuous bounded function that is harmonic in the analytic sense and extends f. Thus D is regular in the analytic sense and $\widehat{h}^f = h^f$.
□

[14] h^f was defined as the unique solution to the Dirichlet problem, cf. Definition 2.4.10. It is part of the Theorem that this can be understood in the analytic and in the stochastic sense, both are equal to $\widehat{h}^f$

[15] If $(K,g) \subset V$ is an admissible compact Riemannian simplicial complex, we can adjoin to K a finite set $\mathcal{S}_0$ of simplices such that $M = \bigcup_{S\in\mathcal{S}_0} S \cup K$ is a simplicial complex with $K \cap \partial M = \emptyset$. Then we can extend g to a piecewise smooth metric tensor on M

[16] cf. e.g. [EF01], Proposition 7.1 or [Fug05a], Theorem 1 (a)

2.4.3 Brownian motion as a semimartingale

Let us now describe the generator[17] $(A, \mathcal{D}(A))$ of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, at least on piecewise smooth functions. For $S \in \mathcal{S}^{(n-1)}(M)$ set

$$\mathcal{A}^S = \{f \in \mathcal{C}_c^\infty(M) : \sum_{T \in \mathrm{st}^{(n)}(S)} \partial_{n_T} f(x) = 0 \text{ for all } x \in S^\circ\}, \tag{2.78}$$

where $n_T(x)$ is the unit normal vector at $x \in S^\circ$ pointing into T. Moreover, put

$$\mathcal{A} := \bigcap_{S \in \mathcal{S}^{(n-1)}(M)} \mathcal{A}^S \tag{2.79}$$

Lemma 2.4.13 *We have*

$$\mathcal{D}(A) \cap \mathcal{C}_c^\infty(M) = \mathcal{A} \tag{2.80}$$

Moreover, for all $f \in \mathcal{D}(A) \cap \mathcal{C}_c^\infty(M)$ and all $T \in \mathcal{S}^{(n)}(M)$ we have

$$Af(x) = \frac{1}{2}\Delta f(x) \qquad \text{for all } x \in T^\circ. \tag{2.81}$$

where $\Delta = \Delta^T$ is the Laplace-Beltrami operator[18] on T.

Proof : 1. Let $f \in \mathcal{C}_c^\infty(M) \subset \mathcal{D}(\mathcal{E})$. Then for all $g \in \mathcal{C}_c^\infty(M)$, Green's formula (applied to every simplex $T \in \mathcal{S}^{(n)}$) yields

$$\begin{aligned}\mathcal{E}(f,g) &= \frac{1}{2}\sum_{T \in \mathcal{S}^{(n)}} \int_T \nabla f(x) \nabla g(x) dx \\ &= \frac{1}{2}\sum_{T \in \mathcal{S}^{(n)}} \left[-\int_T \Delta f(x) g(x) dx - \sum_{S \in \mathcal{N}^{(n-1)}(T)} \int_{S^\circ} \partial_{n_T} f(x) g(x) \sigma(dx) \right],\end{aligned} \tag{2.82}$$

where $\mathcal{N}^{(n-1)}(T) := \{S \in \mathcal{S}^{(n-1)}(M) : T \in \mathrm{st}^{(n)}(S)\}$ is the set of all $(n-1)-$dimensional simplices that belong to the boundary of T. Moreover, for $S \in \mathcal{N}^{(n-1)}(T)$ and $x \in S^\circ$, $n_T(x)$ denotes the unit normal vector pointing into T.

2. Let first $f \in \mathcal{D}(A) \cap \mathcal{C}_c^\infty(M)$. Using test functions $g \in \cap \mathcal{C}_c^\infty(M \setminus S)$ in (2.82),

[17] we mean the infinitesimal generator on $L^2(M, dx)$

[18] strictly speaking, (2.81) only holds for *almost all* $x \in T^\circ$, because Af is only defined in L^2, cf. the discussion around (2.84).

we get (2.81). Moreover, if we use test functions g that are local at $S \in \mathcal{S}^{(n-1)}$, (2.82) boils down to

$$\begin{aligned}\mathcal{E}(f,g) &= \frac{1}{2} \sum_{T \in \mathrm{st}^{(n)}(S)} \int_T \nabla f(x) \nabla g(x) dx \\ &= \frac{1}{2} \sum_{T \in \mathrm{st}^{(n)}(S)} \left[- \int_T \Delta f(x) g(x) dx - \int_{S^\circ} \partial_{n_T} f(x) g(x) \sigma(dx) \right]. \end{aligned} \tag{2.83}$$

Now $f \in \mathcal{D}(A)$ by assumption, which implies $\mathcal{E}(f,g) = -(Af,g)$ for all test functions g. So comparing (2.81) and (2.83), we see that the boundary term in (2.83) (i.e. the integral over S°) must vanish, hence $f \in \mathcal{A}_S$.

3. Conversely, let $f \in \mathcal{A}$. Then (2.82) yields

$$(\frac{1}{2}\Delta f, g) = \frac{1}{2} \sum_{T \in \mathcal{S}(n)} \int_T \Delta f(x) g(x) dx = -\mathcal{E}(f,g) = (Af,g)$$

for all $g \in \mathcal{C}_c^\infty(M)$, which easily extends to $g \in \mathcal{D}(\mathcal{E})$. Thus with $h := \Delta f$ we have $\mathcal{E}(f,g) = -(h,g)$ for all $g \in \mathcal{D}(\mathcal{E})$, which means that $f \in \mathcal{D}(A)$ and $Af = h = \Delta f$. □

The last Lemma has to be read carefully: Note that $(A, \mathcal{A})$ is an operator in $L^2(M,\mu)$ and hence Af is per definition an equivalence class modulo equality a.s. So (2.81) means that for $f \in \mathcal{A}$, the function defined by

$$\widetilde{\Delta} f(x) := \begin{cases} \Delta^T f(x) & \text{if } x \in T^\circ \text{ for some } T \in \mathcal{S}^{(n)} \\ 0 & \text{else} \end{cases} \tag{2.84}$$

is a version of Af and hence

$$(P_t f - f)(x) = (\int_0^t P_\tau \widetilde{\Delta} f d\tau)(x) \tag{2.85}$$

for almost all $x \in M$. But $P_t f(x) = \mathbf{E}^x[f(X_t)]$ defines a strong Feller semigroup, so both sides of (2.85) are continuous in x and hence we have equality for all $x \in M$. If we now regard P_t as a semigroup on the set $\mathcal{B}(M)$ of bounded measurable functions, then $(f, \widetilde{\Delta} f)$ is contained in the *full generator*

$$\widehat{A} := \{(f,g) \in \mathcal{B}(M) \times \mathcal{B}(M) : P_t f - f = \int_0^t P_\tau g d\tau \quad (\forall t > 0)\} \tag{2.86}$$

in the sense of [EK86], Chapter 1, equation (5.5). We can exploit this fact to deduce the following

Corollary 2.4.14 *For any $f \in \mathcal{A}$ and any $x \in M$,*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \Delta f(X_\tau) d\tau \tag{2.87}$$

is a continuous martingale w.r.t. P^x. Moreover, for all $R > 0$ there is a $C(R) > 0$ such that for all $x \in B_{R/2}(x_0)$,

$$\mathbf{E}^x[\langle M^f \rangle_{t\wedge\tau_R} - \langle M^f \rangle_{s\wedge\tau_R}] \leq C(R)\mathrm{Lip}^2(f_{|B_R(x_0)})(t-s), \tag{2.88}$$

where τ_R is defined in (2.72).

Proof : The first assertion follows from [EK86], Chapter 4, Proposition 1.7. In order to prove the second assertion, let $\pi := \{0 = t_0 < t_1 < \dots\}$ be a locally finite partition of $\mathbb{R}_+$. From the theory of continuous semimartingales it is well-known that $V_t(\pi) := \sum (f(X_{t\wedge t_{k+1}}) - f(X_{t\wedge t_k}))^2$ converges to $\langle M^f \rangle_t$ locally uniformly in t in probability as the mesh of π tends to 0. Now applying Lemma 2.4.7 (ii) with $\alpha = 2$, we obtain that

$$\mathbf{E}^x \left[d^2(X_{t_{k+1}}^{\tau_R}, X_{t_k}^{\tau_R}) \right] \leq C(R)(t_{k+1} - t_k)$$

for all $x \in B_R(x_0)$ and hence

$$\mathbf{E}^x \left[(f(X_{t_{k+1}}^{\tau_R}) - f(X_{t_k}^{\tau_R}))^2 \right] \leq C(R)\mathrm{Lip}^2(f_{|B_R(x_0)})(t_{k+1} - t_k).$$

So summing this over all $t_k \in \pi$ and letting the mesh of the partition π tend to 0, we obtain (2.88). □

Remark 2.4.15 The description of the Feller generator (which may be interesting in view of [BK95]) is more complicated. Assume for simplicity that M is compact. According to the theory of Feller semigroups, we set

$$\mathcal{D}_0(A) := \{f \in \mathcal{C}(M) : \lim_{t \searrow 0} \frac{1}{t}(P_t f - f) \text{ exists in } \mathcal{C}(M)\} \tag{2.89}$$

Clearly, $(A, \mathcal{D}(A))$ is an extension of $(A, \mathcal{D}_0(A))$. However, the description of $\mathcal{D}_0(A) \cap \mathcal{C}^\infty(M)$ is more complicated. Namely, for $S \in \mathcal{S}^{(n-1)}(M)$ set

$$\mathcal{A}_0^S = \{f \in \mathcal{A}^S : \Delta^{T_1} f(x) = \Delta^{T_2} f(x) \text{ for all } x \in S^\circ \text{ and } T_1, T_2 \in \mathrm{st}^{(n)}(S)\},$$

where Δ^T is the Laplace-Beltrami operator on T. Clearly, $\mathcal{D}_0(A) \cap \mathcal{C}^\infty(M)$ is contained in $\mathcal{A}_0^S$ for all S, since by definition, Af must be continuous for all $f \in \mathcal{D}_0(A)$.

Now we will show that X is a semimartingale. The proof is lenghty and quite technical. However, one can simplify things a bit: First, we may stop X at τ_R, defined in (2.72). Second, every compact set $K \subset M$, in particular $B_R(x_0)$, can be covered by a finite number of $(O_S)_{S\in\mathcal{S}(M)}$ with $O_S \subset\subset M$ (relatively compact) and O_S is local at S (cf. Lemma 2.1.7. Let $(g_S)_{S\in\mathcal{S}(M)}$ be a partition of unity subordinated to $(O_S)_{S\in\mathcal{S}(M)}$. For any $f \in \mathcal{C}_c^\infty(B_R(x_0))$, let $f_S := fg_S$. Then $f = \sum_S f_S$ (finite sum) and hence if we can reduce the proofs to the case where f is local at S for arbitrary $S \in \mathcal{S}(M)$.
At last, we will use frequently the techniques for localization in space, developed in Section 5.2 below.

Lemma 2.4.16 *Let $f \in \mathcal{C}_c^\infty(M)$. Then $f(X)$ is a uniformly bounded semimartingale on $\{X \in M \setminus M^{(n-2)}\}$ in the sense of Definition 5.2.4.*

Proof : First we localize the problem by stopping X at τ_R, defined in (2.72). Then note that $M \setminus M^{(n-2)} = \bigcup_{S\in\mathcal{S}^{(n-1)}(M)} \mathrm{St}^\circ(S)$ and hence we may assume that $\mathrm{supp}(f) \cap (M \setminus M^{(n-2)}) \subset \mathrm{St}^\circ(S)$ for some $S \in \mathcal{S}^{(n-1)}(M)$. Consider the normal coordinates from Lemma 1.3.3 and Remark 1.3.4 (ii). Namely, let $O \subset\subset \mathrm{St}^\circ(S)$ with $\xi : O \to \widehat{O} \subset C \oplus \perp C$ and $\perp C \subset \mathbb{R}^2$ is the symmetric $k-$pod. Note that $k = |\mathrm{st}^{(n)}(S)|$. More precisely, ξ induces a bijection between $\mathrm{st}^{(n)}(S)$ and $\mathcal{C}^{(1)}(\perp C)$ that induces a bijection between $\{n_T : T \in \mathrm{st}^{(n)}(S)\}$ and $\mathrm{scaff}(\perp C) := \{\hat{u}_1, \dots \hat{u}_k\}$, where $\hat{u}_j = e^{i\frac{j}{k}}$ (with $i = \sqrt{-1}$). Then $\sum_{T\in\mathrm{st}^{(n)}(S)} \partial_{n_T} f(x) = 0$ if and only if $\sum_{j=1}^k \partial_{\hat{u}_j} \hat{f}(\hat{x}) = 0$, where $\hat{f} = f \circ \xi^{-1} : \hat{O} \to \mathbb{R}$.
Let now $b_1, \dots, b_{n-1}$ be a scaffold of C and let b_n, b_{n+1} be a basis of $\mathbb{R}^2$. Denote by $db_j, j = 1, \dots n+1$, the corresponding coordinate functions and let $\beta^j := db_j \circ \xi$ (so $\hat{\beta}^j = db_j$ on $\widehat{O}$). Since $\sum_{l=1}^k u_l = 0$, we have $\sum_{l=1}^k \partial_{\hat{u}_l} \hat{\beta}^j(\hat{x}) = db_j(\sum_{l=1}^k u_l) = 0$ and hence $\beta^j \in \mathcal{D}(A)$. So $\widehat{X} = \sum_{j=1}^{n+1} \beta^j(X) b_j$ is a semimartingale on $\{X \in O\}$. This holds for any $O \subset\subset \mathrm{St}^\circ(S)$, and because all the first and second derivatives of β^j are uniformly bounded on $\mathrm{St}^\circ(S) \subset B_R(x_0)$, $\beta^j(X)$ is a uniformly bounded semimartingale on $\{X \in \mathrm{St}^\circ(S)\}$ by Corollary 2.4.14. Consequently, $f(X) = \hat{f}(\widehat{X})$ is a uniformly bounded semimartingale on $\{X \in \mathrm{St}^\circ(S)\}$ by Proposition 5.2.5 (ii).
□

Theorem 2.4.17 *X is a semimartingale*[19]*. More precisely, let $f \in \mathcal{C}^\infty(M)$. Then for all $T \in \mathcal{S}^{(n)}$,*

$$\int \mathbf{1}_{\{X_\tau \in T^\circ\}} \partial f(dX_\tau) = \int \mathbf{1}_{\{X_\tau \in T^\circ\}} dM_\tau^f \tag{2.90}$$

[19] more precisely, a semimartingale on $[0, \zeta[$, where ζ is the lifetime of X

is a local martingale and

$$\int \mathbf{1}_{\{X_\tau \in T^\circ\}} \mathrm{Hess} f(dX_\tau, dX_\tau) = \int \mathbf{1}_{\{X_\tau \in T^\circ\}} \Delta f(X_\tau) d\tau. \tag{2.91}$$

For all $S \in \mathcal{S}^{(n-1)}$,

$$\int \mathbf{1}_{\{X_\tau \in S^\circ\}} df(X_\tau) = \frac{1}{2} \sum_{u \in \mathrm{scaff}(\perp S)} \int \partial_u f(X_\tau)(dL_\tau^{S,u}), \tag{2.92}$$

i.e. $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} df^\top(X_\tau) \equiv 0$.
At last, for all $S \in \mathcal{S}^{(m)}$ with $m \le n-2$, $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} df(X_\tau) \equiv 0$.

Proof : From Lemma 2.4.16 we already know that $f(X)$ is a locally uniformly bounded semimartingale on $X \in M \setminus M^{(n-2)}$. So (2.90) and (2.91) follow from the properties of Brownian motion on Riemannian manifolds. In order to show (2.92), we may restrict ourselves to the case where f is local at $S \in \mathcal{S}^{(n-1)}$. We write $f = f^\top + f^\perp$. Then $f^\top \in \mathcal{A}$ and $M^f := f^\top(X) - \int \Delta f^\top(X_\tau) d\tau$ is a martingale. Because all derivatives of f (hence of $f^\top$) are uniformly bounded on compact sets, we have

$$\mathbf{E}\left[\int \mathbf{1}_{\{X_\tau \in S^\circ\}} \Delta f^\top(X_\tau) d\tau\right] \le C\mathbf{E}\left[\int \mathbf{1}_{\{X_\tau \in S^\circ\}} d\tau\right] = 0, \tag{2.93}$$

since $\mathbf{E}\left[\mathbf{1}_{\{X_\tau \in S^\circ\}}\right] = 0$ for all τ. Moreover, by (2.88) we have

$$\mathbf{E}\left[\int \mathbf{1}_{\{X_\tau \in S^\circ\}} d\langle M^{f^\top}\rangle_\tau\right] \le C\mathbf{E}\left[\int \mathbf{1}_{\{X_\tau \in S^\circ\}} d\tau\right] = 0 \tag{2.94}$$

and hence $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} dM_\tau^{f^\top} \equiv 0$. Consequently, $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} df^\top(X_\tau) \equiv 0$, or equivalently, (2.92) holds.
It remains to prove the last assertion and the fact that $f(X)$ is a semimartingale. We devide this into several steps:
1. Let first $m = n-2$, i.e. let $S \in \mathcal{S}^{(n-2)}$. As above, we may assume that f is local at S and then write $f = f^\top + f^\perp$. Again, $f^\top \in \mathcal{A}$, and as above one obtains that $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} df^\top(X_\tau) \equiv 0$.
2. Let us now come to $f^\perp$. Let $O \subset\subset \mathrm{St}^\circ(S)$ such that $\mathrm{supp}(f) \subset O$. Consider a function $g \in \mathcal{C}_c^\infty(\mathrm{St}^\circ(S))$ with $0 \le g \le 1$ and $g_{|S\cap O} \equiv 1$. For $0 < r < 1$, put

$$g^r(x) = g^r(x^\top + x^\perp) := g(x^\top + r^{-1}x^\perp) \tag{2.95}$$

Now set $S_r := B_r(S)$ and $f^r := f^\perp(1-g^r)$. Then $f^r_{|S^\circ} \equiv 0$ (since $f^\perp_{|S^\circ} \equiv 0$) and hence $f^r \in \mathcal{C}_c^\infty(M \setminus M^{(n-2)})$. Consequently, $f^r(X)$ is a semimartingale and

$$f^r(X_t) - f^r(X_0) = \int \mathbf{1}_{\{X_\tau \in O \cap S_r\}} df^r(X_\tau) + \int \mathbf{1}_{\{X_\tau \in O \setminus S_r\}} df^r(X_\tau).$$

3. We have that $f^r_{|O\setminus S_r} \equiv f^\perp_{|O\setminus S_r}$ and hence

$$\int \mathbf{1}_{\{X_\tau \in O\setminus S_r\}} df^r(X_\tau) \to \int \mathbf{1}_{\{X_\tau \in O\setminus S^\circ\}} df^\perp(X_\tau)$$

Note that $\int \mathbf{1}_{\{X_\tau \in O\setminus S^\circ\}} df^\perp(X_\tau)$ is well-defined by Proposition 5.2.5 (ii) since X is a uniformly bounded semimartingale on $\{X \in M \setminus M^{(n-2)}\}$ by Lemma 2.4.16.
4. We now show that

$$\int \mathbf{1}_{\{X_\tau \in O\cap S_r\}} df^r(X_\tau) \to 0 \qquad \text{as } r \to 0. \tag{2.96}$$

By the arguments above, we can write $f^r(X_t) - f^r(X_0) = N^{f^r}_t + L^{f^r}_t + A^{f^r}_t$, where $N^{f^r} = \sum_{T\in\mathcal{S}^{(n)}} \mathbf{1}_{\{X_\tau\in T^\circ\}} dM^{f^r}_\tau$, $A^{f^r} = \sum_{T\in\mathcal{S}^{(n)}} \mathbf{1}_{\{X_\tau\in T^\circ\}} \Delta f^r(X_\tau)d\tau$ and

$$L^{f^r} = \sum_{T\in\mathcal{S}^{(n-1)}} \left[\sum_{u\in \mathrm{scaff}(\perp T)} \int \partial_u f^r(X_\tau) dL^{T,u}_\tau \right].$$

Now by Taylor's formula, there is a $C > 0$ such that $|f^\perp| \le Cr$ on $O \cap S_r$ and hence $|\partial_u f^r| \le C$ on $O \cap S_r$ for all $u \in \mathrm{scaff}(S) \cup \mathrm{scaff}(\perp S)$ (after enlarging C if necessary). So $\int \mathbf{1}_{\{X_\tau\in O\cap S_r\}} dL^{f^r}_\tau \to 0$ as $r \to 0$. Moreover,

$$\mathbf{E}\left[\int \mathbf{1}_{\{X_\tau\in O\cap S_r\}} d\langle M^{f^r}\rangle_\tau\right] \le C\mathbf{E}\left[\int \mathbf{1}_{\{X_\tau\in O\cap S_r\}} d\tau\right] \to 0$$

and hence $\int \mathbf{1}_{\{X_\tau\in O\cap S_r\}} dM^{f^r}_\tau \to 0$. At last, $|\partial_{uv} f^r| \le Cr^{-1}$ and hence $|\Delta f^r| \le Cr^{-1}$ on $O \cap S_r$ (of course, after adjusting the constant). Thus by Lemma 2.4.5 (ii),

$$\mathbf{E}\left[\int \mathbf{1}_{\{X_\tau\in O\cap S_r\}} dA^{f^r}_\tau\right] \le C\frac{1}{r}\mathbf{E}\left[\int \mathbf{1}_{\{X_\tau\in O\cap S_r\}} d\tau\right] \to 0$$

showing (2.96).
5. Note that since $f^\perp_{|O\cap S^\circ} \equiv 0$, $f^r \to f^\perp$ uniformly as $r \to 0$. Thus by 3. and 4.,

$$f^\perp(X_t) - f^\perp(X_0) = \lim_{r\to 0} f^r(X_t) - f^r(X_0) = \int_0^t \mathbf{1}_{\{X_\tau\in O\setminus S^\circ\}} df^\perp(X_\tau).$$

Consequently, $f^\perp(X)$ is a semimartingale with $\int \mathbf{1}_{\{X_\tau\in S^\circ\}} df^\perp(X_\tau) \equiv 0$ and together with 1. we get the last assertion of the Theorem for $S \in \mathcal{S}^{(n-2)}$.
6. At last, repeating the arguments above, we can recursively prove the same if $S \in \mathcal{S}^{(n-3)}$, $S \in \mathcal{S}^{(n-4)}$ and so on. Thus the Theorem is proved. □

Remark 2.4.18 (i) The last assertion of the preceding Theorem, namely that $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} df(X_\tau) \equiv 0$ if $S \in \mathcal{S}^{(m)}$ with $m \le n-2$, is not surprising from the potential theoretic point of view: $M^{(n-2)}$ is a polar set ([EF01], Proposition 7.6). In other words, the Brownian motion X never hits $M^{(n-2)}$.
(ii) It can be shown that for $S \in \mathcal{S}^{(n-1)}$, $L^{S,u} \equiv L^{S,v}$ for all $u, v \in \mathrm{scaff}(\perp S)$. More precisely, there is an increasing process l such that $L^{S,u} \equiv \frac{1}{|\mathrm{scaff}(\perp S)|} l$ for all $u \in \mathrm{scaff}(\perp S)$, cf. Example 3.2.8.

Example 2.4.19 Let M be an $n-$dimensional "booklet", i.e. M is the union of a finite number of $n-$dimensional Euclidean half-spaces $H_1, \dots, H_k$ that are glued together at their boundaries. In other words, M is the orthogonal sum $M = S \oplus \perp S$, where S is an $(n-1)-$dimensional Euclidean space and $\perp S$ is a one-dimensional simplicial cone complex with k rays, i.e. a $k-$star.
Let X be Brownian motion in M starting at x. We write $X = X^\top + X^\perp$. Then the following holds:

a) $X^\top$ is an $(n-1)-$dimensional Brownian motion starting at $x^\top$.

b) $X^\perp$ is an isotropic Walsh's Brownian motion[20] starting at $x^\perp$.

c) $r := d(X, S) = d(X^\perp, 0_{\perp S})$ is a reflected Brownian motion.

Proof: a) Let $f \in \mathcal{C}_c^\infty(S)$. Define $\widetilde{f} : M \to \mathbb{R}$ by $\widetilde{f}(x) := f(x^\top)$. Then $\Delta^M \widetilde{f}(x) = \Delta^S f(x^\top)$, where Δ^S is the Euclidean Laplacian on S. Moreover, $\widetilde{f} \in \mathcal{A}^S$[21] and hence $M^f := f(X^\top) - \int \Delta^S f(X^\top) dt = \widetilde{f}(X) - \int \Delta^M \widetilde{f}(X) dt$ is a martingale by Corollary 2.4.14. This means that the law of $X^\top$ solves the martingale problem for Δ^S and hence is the Wiener measure on S.
b) Let the ith ray of $\perp S$ be given by $T_i := \mathbb{R}_+ u_i$, with unit vectors $u_1, \dots u_n$, so $\perp S = \bigcup_{i=1}^k T_i$ (cf. Example 1.1.3 (ii)). Define $g_i : \perp S \to \mathbb{R}$ by

$$g_i(r u_j) := \begin{cases} r\frac{k-1}{k} & \text{if } j = i \\ -\frac{r}{k} & \text{if } j \neq i \end{cases} \tag{2.97}$$

Extend this to a function $\widetilde{g}_i : M \to \mathbb{R}$ by $\widetilde{g}_i(x) := g_i(x^\perp)$. Then $\Delta \widetilde{g}_i \equiv 0$ and $\Delta \widetilde{g}_i^2 \equiv 2$.
For $R > 0$, let $\theta^R : \perp S \to \mathbb{R}$ be a piecewise smooth function with compact support such that $\theta^R_{|B_R(0)} \equiv 1$. Then $\widetilde{\theta^R g_i} \in \mathcal{A}^S$ and hence the processes $g_i(X^\perp_{t \wedge \tau_R})$ and

[20] This is defined in [BPY89]. *Isotropic* means that when $X^\perp$ is in $0_{\perp S}$, it chooses any ray with equal probability $1/k$.

[21] The point is that all normal derivatives of $\widetilde{f}$ are identically 0 on S. Strictly speaking, we only have that $\theta^R f \in \mathcal{A}^S$, where θ^R is a cutoff function as in the proof of b). Then one has to run through a localization argument analogous to the one in the proof of b), which we skip here.

$g_i^2(X^{\perp}_{t\wedge\tau_R}) - t \wedge \tau_R$ are martingales for $i = 1, \ldots, k$.
Fix $t > 0$. Define the process $N_R := g_i(X^{\perp}_{t\wedge\tau_R})$. Then $(N_R)_{R\geq 0}$ is a martingale with $\langle N\rangle_R = t \wedge \tau_R$. Thus

$$\langle g_i(X^{\perp})\rangle_{t\wedge\zeta} = \lim_{R\to\infty} \langle g_i(X^{\perp})\rangle_{t\wedge\tau_R} = \lim_{R\to\infty} \langle N\rangle_R \leq t,$$

where ζ is the lifetime of $X^{\perp}$. It follows from [RY99], Chapter IV, Proposition 1.26 that $\lim_{R\to\infty} = g_i(X^{\perp}_{t\wedge\tau_R}) = \lim_{s\nearrow t\wedge\zeta} g_i(X^{\perp}_s)$ exists. Consequently, $\zeta \geq t$ (in particular, since t is arbitrary, it follows that $\zeta = \infty$). Moreover, the process $g_i(X^{\perp}_s)_{s\leq t}$ is an L^2-bounded martingale. It follows that the processes $g_i(X^{\perp}_t)_{t\geq 0}$ and $g_i^2(X^{\perp}_t) - t$ are martingales for $i = 1, \ldots, k$. Thus we have shown that the law of $X^{\perp}$ solves the martingale problem (3.3) in [BPY89]. So it must be the law of an isotropic Walsh's Brownian motion by Theorem 3.2 of [BPY89].
c) follows from b) and Lemma 2.2 of [BPY89]. $\square$

Chapter 3

Martingales in Euclidean polyhedra

In a Riemannian manifold M, a semimartingale is a $\nabla-$martingale if and only if $\varphi(X)$ is a local submartingale on $\{X \in O\}$ for all smooth convex functions $\varphi : O \to \mathbb{R}$. This is *Darling's characterization* for martingales. Moreover, in this case one can proof by a standard smoothing procedure that $\varphi(X)$ is a local submartingale on $\{X \in O\}$ for all Lipschitz continuous convex functions.
Picard extended this characterization to the case where M is a metric tree (i.e. a one-dimensional Euclidean polyhedron), cf. [Pic05], Proposition 3.3.4. According to stochastic calculus in polyhedra, there appears a condition on the local time term that reflects the geometry of the Link $\mathrm{Lk}_0 M$ (cf. Example 3.4.5).
We will extend this characterization to the case that M is a Euclidean polyhedron of arbitrary dimension. Moreover, our characterization works without any assumption on curvature bounds.
In section 3.1 we consider Darling's characterization using *piecewise smooth* convex functions. But this is unsatisfactory because it depends on a certain triangulation. In section 3.2 we develop a theory of *local time measure* at a certain simplex S which is a version of the family of directional local times that does not depend on the triangulation[1]. We use this to present a triangulation-free version of Itô's formula, cf. Theorem 3.2.13, which is then the key tool in proving a general version of Darling's characterization (Theorem 3.3.4).
At last, we discuss the special case that M is a Euclidean polyhedron of nonpositive curvature, where one can find a simple description of the martingale condition for the local time term, including the case of Picard's characterization in trees.
Theorem 3.4.7 is one of the central results in this work. It characterizes martingales in terms of the theory of *discretized martingales* that is developed in chapter

[1]we only require that there is some isometric triangulation $\mathcal{S}$ such that $S \in \mathcal{S}$

4.

3.1 Darling's characterization, part I

We start with Darling's characterization for martingales. In manifolds, a semimartingale X is a martingale if and only if $\varphi(X)$ is a local submartingale for every smooth convex function φ (at least locally). So in our case, i.e. where we are given a Euclidean complex $(M, \mathcal{S})$, the test functions should be piecewise smooth convex functions.

Recall the definition of the *convex barycenter* of a probability measure on a geodesic metric space from [ÉM91]. We use a version that is adapted to the present situation.

Definition 3.1.1 Let M be a Euclidean polyhedron and μ a finite nonnegative measure on M.
(i) The *convex barycenter* of μ, denoted by $\mathbb{B}(\mu)$, is the set of all $x \in M$ such that

$$\mu(M)\varphi(x) \leq \int_M \varphi(y)\mu(dy)$$

for all Lipschitz continuous convex functions $\varphi : M \to \mathbb{R}$.
(ii) The *piecewise smooth convex barycenter* of μ (w.r.t. $\mathcal{S}$), denoted by $\mathbb{B}_{\mathcal{S}}(\mu)$, is the set of all $x \in M$ such that

$$\mu(M)\varphi(x) \leq \int_M \varphi(y)\mu(dy)$$

for all piecewise smooth convex functions $\varphi : M \to \mathbb{R}$.

Note that if $\mu \equiv 0$, then $\mathcal{B}(\mu) = M$.

For the rest of this section, we will only be concerned with the *piecewise smooth barycenter* from Definition 3.1.1 (ii). Let $X : \Omega \times \mathbb{R}_+ \to M$ be a semimartingale and let $S \in \mathcal{S}(M)$. The family $(L^{S,u})_{u \in \mathrm{scaff}(\perp S)}$ of local times can be regarded as a process with values in the set of nonnegative measures on $\perp S$ in the following sense: For $t \geq 0$ set

$$\mu_t^S := \mu_t^S(\omega) := \sum_{u \in \mathrm{scaff}(M)} L_t^{S,u} \delta_{\{u\}} \tag{3.1}$$

Then μ_t^S is a nonnegative measure on $\perp S$. Moreover, $\mu_t^S - \mu_s^S$ is nonnegative for all $0 \leq s \leq t$.

Definition 3.1.2 Let (Ω, X, P) be a continuous semimartingale and let $S \in \mathcal{S}$. We say that X satisfies condition $\mathbf{M}_{\mathcal{S}}(S)$ if there is a $P-$nullset out of which $0_{\perp S} \in \mathbb{B}_{\mathcal{S}}(\mu_t^S - \mu_s^S)$ for all $0 \leq s \leq t$.

Remark 3.1.3 Condition $\mathbf{M}_{\mathcal{S}}(S)$ is equivalent to the following: Whenever $\varphi : \perp S \to \mathbb{R}$ is a piecewise smooth convex function with $\varphi(0) = 0$ and $s \leq t$, then

$$\int_{\perp S} \varphi(y)(\mu_t^S - \mu_s^S)(dy) \geq 0. \tag{3.2}$$

Now we know from Lemma 1.4.5 that $f := \partial\varphi_0 = \sum_{u \in \text{scaff}(\perp S)} \partial_u \varphi(0)\nu^u$ is piecewise linear and convex. Moreover, $\varphi \geq f$ and hence $\int_{\perp S} \varphi(y)(\mu_t^S - \mu_s^S)(dy) \geq \int_{\perp S} f(y)(\mu_t^S - \mu_s^S)(dy)$. Thus in order to check $\mathbf{M}_{\mathcal{S}}(S)$, one only has to check (3.2) for *piecewise linear* convex functions $f : \perp S \to \mathbb{R}$.

Lemma 3.1.4 *Let X be a continuous semimartingale and $S \in \mathcal{S}(M)$. If X satisfies condition $\mathbf{M}_{\mathcal{S}}(S)$, then $\sum_{u \in \text{scaff}(\perp S)} \int_0^t \partial_u \varphi(X_\tau) dL_\tau^{S,u}(X)$ is nondecreasing for any piecewise smooth convex function φ.*

Proof : We will approximate $\sum_{u \in \text{scaff}(\perp S)} \int_0^t \partial_u \varphi(X_\tau) dL_\tau^{S,u}(X)$ by discretized integrals. More precisely, since $L^{S,u}$ and $\partial_u \varphi$ are continuous, there is a $P-$nullset out of which

$$\sum_{u \in \text{scaff}(\perp S)} \left[\sum_{t_l \in \Delta^k} \partial_u \varphi(X_{\tau_l})(L_{t_{l+1}}^{S,u} - L_{t_l}^{S,u}) \right] \to \sum_{u \in \text{scaff}(\perp S)} \int_0^t \partial_u \varphi(X_\tau) dL_\tau^{S,u}$$

almost surely whenever Δ^k is a sequence of partitions of $\mathbb{R}_+$ with $\|\Delta^k\| \to 0$ and $\tau_l = \tau_l^k(\omega)$ is a sequence of intermediate points, i.e. $t_l \leq \tau_l^k \leq t_{l+1}$ for all $t_l, t_{l+1} \in \Delta^k$.

Now fix $k \in \mathbb{N}$ and the corresponding partition Δ^k, and let $l \in \mathbb{N}$. Let $\omega \in \Omega$. There are two possible cases:

First, if $X_\tau(\omega) \notin S^\circ$ for all $t_l \leq \tau \leq t_{l+1}$, then $(L_{t_{l+1}}^{S,u} - L_{t_l}^{S,u})(\omega) = 0$ because $L^{S,u}$ only increases on $\{X \in S^\circ\}$. Consequently, $\partial_u \varphi(X_{\tau_l})(L_{t_{l+1}}^{S,u} - L_{t_l}^{S,u}) = 0$ for all $\tau_l \in [t_l, t_{l+1}]$.

In the second case, i.e. if there is some $\tau_l = \tau_l^k(\omega) \in [t_l, t_{l+1}]$ such that $X_{\tau_l}(\omega) \in S^\circ$, then consider the function

$$f := \sum_{u \in \text{scaff}(\perp S)} \partial_u \varphi(X_{\tau_l}(\omega))\nu^u.$$

Then f is convex on $\perp S$ and $f(0_{\perp S}) = 0$. Now by $\mathbf{M}_{\mathcal{S}}(S)$, $0_{\perp S} \in \mathbb{B}(\mu_{t_{l+1}}^S - \mu_{t_l}^S)(\omega)$ and hence by Remark 3.1.3,

$$\sum_{u \in \text{scaff}(\perp S)} \partial_u \varphi(X_{\tau_l})(L_{t_{l+1}}^{S,u} - L_{t_l}^{S,u}) = \int_{\perp S} f(y)(\mu_{t_{l+1}}^S - \mu_{t_l}^S)(dy) \geq f(0) = 0.$$

So we conclude that $\sum_{u\in\text{scaff}(\perp S)}\left[\sum_{t_l\in\Delta^k}\partial_u\varphi(X_{\tau_l})(L^{S,u}_{t_{l+1}}-L^{S,u}_{t_l})\right]$ is a.s. nondecreasing (as a process), and letting $k\to\infty$ proves the Lemma. □

Theorem 3.1.5 *Let M be a Euclidean polyhedron and let $X:\Omega\times\mathbb{R}_+\to M$ be a continuous semimartingale. Then the following are equivalent:*

(i) $\varphi(X)$ is a local submartingale on $\{X\in O\}$ for all piecewise smooth convex functions $\varphi:O\to\mathbb{R}$.

(ii) For all $S\in\mathcal{S}$ and $u\in\text{scaff}(S)$, $\int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}}dX^u_\tau$ is a local martingale and X satisfies $\mathbf{M}_{\mathcal{S}}(S)$.

Proof : $(ii)\Rightarrow(i)$: Let $\varphi:O\to\mathbb{R}$ be piecewise smooth and convex. By localization, we may assume that X has only values in O. Let $S\in\mathcal{S}(M)$. By (ii), $\sum_{u\in\text{scaff}(S)}\int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}}dX^u_\tau$ is a local martingale. Since φ is convex, $\frac{1}{2}\sum_{u,v\in\text{scaff}(S)}\int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}}\partial_{uv}f(X_\tau)d\langle X^u_\tau,X^v_\tau\rangle$ is nondecreasing. At last, $\sum_{u\in\text{scaff}(\perp S)}\int_0^t\partial_u\varphi(X_\tau)dL^{S,u}_\tau(X)$ is nondecreasing by Lemma 3.1.4. Thus (i) follows from (2.25) and Theorem 2.1.13.
$(i)\Rightarrow(ii)$: Let $u\in\text{scaff}(S)$. On a neighborhood $O\supset S$, define the function $\varphi^u(x^\top+x^\perp):=\nu^u(x^\top)$. Then φ^u is piecewise linear and convex. Moreover, $\partial_v\varphi^u(x)=\delta_u(v)$ for all $x\in S^\circ$ and $v\in\text{scaff}(S)\cup\text{scaff}(\perp S)$. Thus by Theorem 2.1.13, $\int_0^t\mathbf{1}_{\{X_\tau\in S^\circ\}}d\varphi^u(X_\tau)=\int_0^t\mathbf{1}_{\{X_\tau\in S^\circ\}}dX^u_\tau$ which is a local submartingale by (i). Moreover, $-\varphi^u$ is convex, too, and consequently $\int_0^t\mathbf{1}_{\{X_\tau\in S^\circ\}}dX^u_\tau$ must be a local martingale.
It remains to show that $\mathbf{M}_{\mathcal{S}}(S)$ holds. By Remark 3.1.3, we have to check that if $f:\perp S\to\mathbb{R}$ is a piecewise linear convex function, then $\int_{\perp S}f(y)\mu^S_t(dy)$ is nondecreasing in t.
Set $\varphi(x):=f(x^\perp)$. Then φ is convex and by (i), $\int\mathbf{1}_{\{X_\tau\in S^\circ\}}d\varphi(X_\tau)$ is a local submartingale. Now $\partial_u\varphi\equiv 0$ for all $u\in\text{scaff}(S)$, and hence by Theorem 2.1.13,

$$\begin{aligned}\int_0^t\mathbf{1}_{\{X_\tau\in S^\circ\}}d\varphi(X_\tau)&=\sum_{u\in\text{scaff}(\perp S)}\int_0^t\partial_u\varphi(X_\tau)dL^{S,u}_\tau\\&=\sum_{u\in\text{scaff}(\perp S)}f(u)dL^{S,u}_t=\int f(y)\mu^S_t(dy)\end{aligned}$$

is nondecreasing in t. Thus $\mathbf{M}_{\mathcal{S}}(S)$ holds. □

3.2 General Convex functions

So far we have used the stochastic calculus for piecewise smooth functions, which is unsatisfactory for Euclidean polyhedra because of two reasons: First, many convex functions, e.g. distance functions if M has nonpositive curvature, are not piecewise smooth in general. Second, all previous results were subject to a given triangulation. But regarding a Euclidean polyhedron as a metric space, there are a lot of isometric triangulations and in general there is no canonical one. Moreover, we already know that the property of being a semimartingale does not depend on the triangulation. The aim of this section is to develop a stochastic calculus that does not depend on a certain triangulation and that also includes (convex) functions that are not piecewise smooth.

In subsection 3.2.1 we show that if X is a semimartingale and $\varphi : O \to \mathbb{R}$ is a Lipschitz continuous function, then $\varphi(X)$ is a semimartingale on $\{X \in O\}$ (Lemma 3.2.1). Then the next question is: How does $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi(X)$ look like? As usual, we will write $\varphi = \varphi^\top + \varphi^\perp$.

Assume for the moment that φ is piecewise smooth w.r.t. some isometric triangulation $\mathcal{S}$. Then by Theorem 2.1.13 we have

$$\begin{aligned}\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi^\top(X_\tau) = &\sum_{u \in \mathrm{scaff}(S)} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} \partial_u \varphi(X_\tau) dX_\tau^u \\ &+ \frac{1}{2} \sum_{u,v \in \mathrm{scaff}(S)} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} \partial_{uv} f(X_\tau) d\langle X_\tau^u, X_\tau^v \rangle\end{aligned}$$

and

$$\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi^\perp(X_\tau) = \frac{1}{2} \sum_{u \in \mathrm{scaff}(\perp S)} \int_0^t \partial_u f(X_\tau) dL_\tau^{S,u}(X). \tag{3.3}$$

As we will see, even if φ is not piecewise smooth, the tangential part does not cause too many problems. Actually, one can approximate $\varphi^\top$ by smooth functions as in the classical linear case.

The hard part is the orthogonal, i.e. the local time term. In subsection 3.2.2 we introduce the notion of the *local time measure at* S, which is a measure on $\mathrm{Lk}_0 \perp S$ and admits a version of (3.3) that does not depend on the triangulation $\mathcal{S}$, cf. Proposition 3.2.4.

In subsection 3.2.3, we examine which convex functions φ admit a *generalized Itô formula* for $\varphi^\perp$. By this we mean a version of (3.3) for certain (not necessarily piecewise smooth) *regular* convex functions, cf. Definition 3.2.11 and Theorem 3.2.13.

3.2.1 Cutting, smoothing and extending

Let $\varphi : M \to \mathbb{R}$ be a Lipschitz continuous convex function, not necessarily piecewise smooth, and let X be a continuous semimartingale. We want to show that

$\varphi(X)$ is a semimartingale. This will be done by a 'cutting, smoothing and extending' procedure, as follows: First for any $T \in \mathcal{S}$, we only consider $\varphi_{|T}$, the restriction of f to T ('cutting'). Then we smoothen $\varphi_{|T}$ with a family of mollifiers that is adapted to T ('smoothing'). At last, we extend the smoothed function to a piecewise smooth function the whole complex as in Example 1.1.8 ('extending'). Let $T \in \mathcal{S}(M)$. For any function $f : M \to \mathbb{R}$, let $f_T : M \to \mathbb{R}$ be the extension of $f_{|T}$ from T to M described in Example 1.1.8 (so in that terminology, $f_T = \tilde{f}$ with $K = T$ and $L = M$). If $f_{|T}$ is smooth, then f_T is piecewise smooth. Note that we have here the special case of extending a function from a *simplex* to the whole complex, cf. Lemma 1.1.10 and figure 1.3.

The main tool of this section is a family of mollifiers adapted to the simplicial structure. Let $0 \neq \theta$ be a nonnegative smooth function with compact support in $[0, \infty[$. For $j \in \mathbb{N}$ and $T \in \mathcal{S}$, put

$$\Psi_T^j(x) := \frac{1}{\alpha^j} \prod_{u \in \mathrm{scaff}(T)} \theta(j\nu^u(x)) \tag{3.4}$$

with

$$\alpha^j := \int_T \prod_{u \in \mathrm{scaff}(T)} \theta(j\nu^u(x)) dx \tag{3.5}$$

So $\Psi_T^j(x)dx$ is a probability measure whose support is contained in T.
For a locally bounded function $f : T \to \mathbb{R}$, put

$$f_T^j(x) := f * \Psi_T^j(x) := \int_T f(y)\Psi_T^j(y-x)dy \tag{3.6}$$

for $x \in T$, and for $x \in M \setminus T$, let $f_T^j(x)$ be defined by the extension procedure described in Example 1.1.8. Then f_T^j is piecewise smooth.

Lemma 3.2.1 *Let φ be a Lipschitz continuous convex function. Then*

(i) $\varphi_T^j \to \varphi_T$ *locally uniformly.*

(ii) $\mathrm{Hess}\varphi_T^j(x)$ *is positive semi-definite for all $x \in M$*

(iii) $\partial_u \varphi_T^j(x) \to \widetilde{\partial}_u \varphi_T(x)$ *for every $x \in T$ and $u \in \mathrm{scaff}(T)$, where*

$$\widetilde{\partial}_u \varphi_T(x) := \int_T \partial_u(\partial\varphi_x)(y-x)\Psi_T^1(y-x)dy = -\int_T \partial\varphi_x(y-x)\partial_u\Psi_T^1(y-x)dy.$$

In particular, if there is some $C > 0$ such that $|\partial_u \varphi| \leq C$ in a neighborhood of x in T , then $|\widetilde{\partial}_u \varphi_T(x)| \leq C$.

Proof : (i) follows from the continuity of φ_T as in the usual case.
(ii) Let $x_0, x_1 \in T$ and put $x_t := (1-t)x_0 + tx_1$. Then

$$\begin{aligned}\varphi_T^j(x_t) &= \int_T \varphi(y)\Psi_T^j(y-x_t)dy = \int_T \varphi(z+x_t)\Psi_T^j(z)dz \\ &\le (1-t)\int_T \varphi(z+x_0)\Psi_T^j(z)dz + t\int_T \varphi(z+x_1)\Psi_T^j(z)dz \\ &= (1-t)\varphi_T^j(x_0) + t\varphi_T^j(x_0)\end{aligned}$$

Thus φ_T^j is convex on T and hence $\mathrm{Hess}\varphi_T^j(x)$ is positive semi-definite for all $x \in T$. Let now $x \in \widetilde{T}^\circ$ for $\widetilde{T} \neq T$, and put $S := T \cap \widetilde{T}$. Let $v \in U^{\widetilde{T}}$. It follows from the special form of φ_T^j that $\mathrm{Hess}\varphi_T^j(x)(v,v) = \mathrm{Hess}\varphi_T^j(\pi_S(x))(\pi_S(v),\pi_S(v)) \ge 0$, where $\pi_S : U^{\widetilde{T}} \to U^S$ is the linear projection[2]. So $\mathrm{Hess}\varphi_T^j(x)$ is positive semi-definite for all $x \in \widetilde{T}$.
(iii) Let $x \in T$ and $u \in \mathrm{scaff}(T)$. Note that

$$\partial_u\varphi_T^j(x) = -\int_T \varphi(y)\partial_u\Psi_T^j(y-x)dy = \int_T \partial_u\varphi(y)\Psi_T^j(y-x)dy. \tag{3.7}$$

Without loss of generality we may assume that $\varphi(x) = 0$. Define

$$\epsilon :]0,1] \times (B_1(x)\cap T) \to \mathbb{R}, \quad (r,y) \mapsto \frac{1}{r}(\varphi(x+r(y-x)) - \partial\varphi_x(r(y-x)))$$

Then $\epsilon(r,y)$ is nondecreasing in r and bounded from below by

$$\lim_{r\downarrow 0}\epsilon(r,y) = \inf_{r>0}\epsilon(r,y) \equiv 0.$$

Moreover, by assumption, ϵ is bounded from above and hence by dominated convergence

$$\begin{aligned}&|\partial_u\varphi^j(x) - \widetilde{\partial}_u\varphi(x)| \\ &= \left|\int_T \varphi(y)\partial_u\Psi_T^j(y-x)dy - \int_T \partial\varphi_x(y-x)\partial_u\Psi_T^1(y-x)dy\right| \\ &= \left|\int_T j\varphi(x+j^{-1}(y-x))\partial_u\Psi_T^1(y-x)dy - \int_T \partial\varphi_x(y-x)\partial_u\Psi_T^1(y-x)dy\right| \\ &\le \int_T \epsilon(j^{-1},y)\partial_u\Psi_T^1(y-x)dy \to 0.\end{aligned}$$

Now since we know that $\partial\varphi_T^j(x)$ converges, the last assertion follows from (3.7), taking into account that $\Psi_T^j(y)dy$ is a probability measure . □

[2] note that S has maximal dimension among all $\widetilde{S} \in \mathcal{S}$ such that $\widetilde{S} \subset T \cap \widetilde{T}$. It can easily be seen that whenever $g = g_T$ is a function of the above form, then $g(x) = g(\pi_S(x))$. Thus the claim above follows by taking the second derivative along straight lines in $\widetilde{T}$.

Lemma 3.2.2 *Let $X : \Omega \times \mathbb{R}_+ \to M$ be a semimartingale and let $\varphi : O \to \mathbb{R}$ be a Lipschitz continuous convex function. Then $\varphi(X)$ is a semimartingale on $\{X \in O\}$.*

Proof : We may assume that X has only values in O. Since φ_T^j is piecewise smooth for all $j \in \mathbb{N}$, Theorem 2.1.13 yields

$$\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi_T^j(X_\tau) = \left[Y_t^S(\varphi_T^j) + A_t^S(\varphi_T^j) + L_t^S(\varphi_T^j)\right]$$

with

$$Y_t^S(\varphi_T^j) := \sum_{u \in \operatorname{scaff}(S)} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} \partial_u \varphi_T^j(X_\tau) dX_\tau^u,$$

$$A_t^S(\varphi_T^j) := \frac{1}{2} \sum_{u,v \in \operatorname{scaff}(S)} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} \partial_{uv} \varphi_T^j(X_\tau) d\langle X_\tau^u, X_\tau^v\rangle$$

and

$$L_t^S(\varphi_T^j) := \sum_{u \in \operatorname{scaff}(\perp S)} \int_0^t \partial_u \varphi_T^j(X_\tau) dL_\tau^{S,u}(X).$$

By Lemma 3.2.1 (i), $\varphi_T^j(X) \to \varphi_T(X)$. Moreover, by Lemma 3.2.1 (iii),

$$Y_t^S(\varphi_T^j) \to \sum_{u \in \operatorname{scaff}(S)} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} \widetilde{\partial}_u \varphi_T(X_\tau) dX_\tau^u =: Y_t^S(\varphi_T)$$

and

$$L_t^S(\varphi_T^j) \to \sum_{u \in \operatorname{scaff}(\perp S)} \int_0^t \widetilde{\partial}_u \varphi_T(X_\tau) dL_\tau^{S,u}(X) := L_t^S(\varphi_T)$$

locally uniformly in probability (note that since φ is locally Lipschitz, $\tilde{\partial}\varphi$ is locally bounded). Consequently, $A_t^S(\varphi_T^j)$ converges to some continuous process $A_t^S(\varphi_T)$ which is nondecreasing since $A_t^S(\varphi_T^j)$is nondecreasing for all j by Lemma 3.2.1 (ii). Thus letting $j \to \infty$, by Lemma 3.2.1 (i) we have

$$\varphi_T(X_t) - \varphi_T(X_0) = \sum_{S \in \mathcal{S}(M)} \left[Y_t^S(\varphi_T) + A_t^S(\varphi_T) + L_t^S(\varphi_T)\right].$$

Plugging this into Lemma 1.1.10, we get

$$\varphi(X_t) - \varphi(X_0) = \sum_{T \in \mathcal{S}} a_T \left[Y_t(\varphi_T) + A_t(\varphi_T) + L_t(\varphi_T)\right] \tag{3.8}$$

with $Y_t(\varphi_T) := \sum_{S \in \mathcal{S}} Y_t^S(\varphi_T)$ and so on. Consequently, $\varphi(X)$ is a semimartingale. □

3.2.2 Local Times revisited

Unfortunately, (3.8) is not very useful for geometric applications as e.g. characterizations of martingales (cf. Theorem 3.1.5). Especially the local time term of (3.8) causes problems since the coefficients a_T may be negative.
In this section we will develop the theory of *local time measure at* $S \in \mathcal{S}$, being an intrinsic and triangulation-free concept of the family of directional local times at S. With help of of the local time measure we will be able to represent the behavior of $\varphi(X)$ at $S \in \mathcal{S}$, even if φ is not piecewise smooth cf. Theorem 3.2.13.

We will start with the special situation where M is a Euclidean conical polyhedron and $S = 0$. Recall that $(\mathrm{Lk}_0 M, \rho)$ is a compact spherical polyhedron (ρ is the spherical intrinsic distance).

Definition 3.2.3 Let (M, d) be an $n-$dimensional Euclidean conical polyhedron and let $\mathcal{S}$ be a triangulation of M. The *mesh* of $\mathcal{S}$ is defined by

$$\|\mathcal{S}\| := \sup\{\mathrm{diam}_\rho(S \cap Lk_0 M) : S \in \mathcal{S}\} \tag{3.9}$$

A sequence $(\mathcal{S}_k)_{k\in\mathbb{N}}$ of isometric triangulations of M is called an *approximating sequence* for M if

- For any $k \le l$, $\mathcal{S}_k \subset \mathcal{S}_l$, i.e. $\mathcal{S}_l$ is finer than $\mathcal{S}_k$.
- $\|\mathcal{S}_k\| \to 0$ as $k \to \infty$

Let $(\mathcal{S}_k)_{k\in\mathbb{N}}$ be an approximating sequence. For $k \in \mathbb{N}$ define a measure-valued process $\mu^k : \Omega \times \mathbb{R}_+ \to \mathcal{M}(\mathrm{Lk}_0 M)$ by

$$\mu^k_t := \sum_{u \in \mathrm{scaff}(M^k)} L^{u,k}_t \delta_{\{u\}} \tag{3.10}$$

where $L^{u,k}_t$ is the local time of X at 0 in direction u w.r.t the triangulation $\mathcal{S}_k$. Then μ^k is continuous w.r.t. to weak convergence of measures on $\mathrm{Lk}_0 M$ and non-decreasing in the sense that $\mu^k_t - \mu^k_s$ is a nonnegative measure for all $s \le t$.
For any function $f : \mathrm{Lk}_0 M \to \mathbb{R}$, we define a piecewise linear function $f^k : (M, \mathcal{S}_k) \to \mathbb{R}$ by $f^k := \sum_{u \in \mathrm{scaff}(\mathcal{S}_k)} f(u)\nu^u$. The crucial observation is that for all $l \ge k$, f^k is also piecewise linear w.r.t. $\mathcal{S}_l$ and hence almost surely

$$\begin{aligned} \int f(y)\mu^k_t(dy) &= \int f^k(y)\mu^k_t(dy) = \sum_{u \in \mathrm{scaff}(M, \mathcal{S}_k)} f(u) L^{u,k}_t \\ &= \int \mathbf{1}_{\{X_\tau = 0\}} df^k(X_\tau) = \int f^k(y)\mu^l_t(dy). \end{aligned} \tag{3.11}$$

Thus the sequence μ^k_t is in some sense 'projective'. So the natural question if there is a 'projective limit' is answered in the following

Proposition 3.2.4 *Let (M,d) be a Euclidean conical polyhedron and $X:\Omega\times\mathbb{R}_+\to M$ a semimartingale. Then there is an almost surely unique*[3] *continuous measure-valued process $L:\Omega\times\mathbb{R}_+\to\mathcal{M}(\mathrm{Lk}_0M)$ such that for all isometric triangulations $(M,\mathcal{S})$ and all piecewise linear functions $f:(M,\mathcal{S})\to\mathbb{R}$ we have*

$$\int_{\mathrm{Lk}_0M} f(y)L_t(dy) = \int_0^t \mathbf{1}_{\{X_\tau=0\}} df(X_\tau) = \sum_{u\in\mathrm{scaff}(M,\mathcal{S})} f(u)L_t^u \ \ a.s. \tag{3.12}$$

Moreover, for any simplicial approximation $(\mathcal{S}_k)_{k\in\mathbb{N}}$ of M, let μ^k defined by (3.11). Then $\mu_t^k\to L_t$ weakly locally uniformly almost surely in the sense that $\sup_{s\le t}\rho_K(\mu_s^k,L_s)\to 0$*, where ρ_K is the* Kantorovich Rubinstein distance *of measures on* Lk_0M*, i.e*

$$\rho_K(\mu,\nu) := \sup\{\int_{\mathrm{Lk}_0M} fd(\mu-\nu) : f\in\mathrm{Lip}_1(\mathrm{Lk}_0M,\mathbb{R})\}.$$

L_t is nondecreasing in the sense that for all $s\le t$, L_t-L_s is a nonnegative measure. $(L_t)_{t\in\mathbb{R}_+}$ is called the local time measure *of X at* 0*.*

Proof : 1. We will show that for any approximating sequence $(\mathcal{S}_k)_{k\in\mathbb{N}}$, the sequence μ_t^k converges weakly locally uniformly almost surely to some continuous measure-valued process μ_t.
First note that if $\mathcal{F}$ is a countable set of functions, then almost surely (3.11) holds for all $f\in\mathcal{F}$ and all $k\le l$.
In particular, with $g\equiv 1$ and g^k defined as above, we have that $\mu_t^k(\mathrm{Lk}(0)) = \int g(y)\mu_t^k(dy)$ for all $k\in\mathbb{N}$. Noting that $g^1_{|\mathrm{Lk}(0)}\ge 1$, we get

$$\mu_s^k(\mathrm{Lk}(0)) \le \mu_t^k(\mathrm{Lk}(0)) \le \int g^1(y)\mu_t^k(dy) = \int g^1(y)\mu_t^1(dy) =: C_t < \infty$$

for all $s\le t$. Let $\mathcal{F}$ be a countable set of functions that is dense in $\mathrm{Lip}_1(\mathrm{Lk}(0),\mathbb{R})$ w.r.t. uniform convergence. Then almost surely, for all $k\le l\in\mathbb{N}$, $s\le t$ and all $f\in\mathcal{F}$ we have

$$\begin{aligned}
|\int f(y)\mu_s^k(dy) - \int f(y)\mu_s^l(dy)| &= |\int f^k(y)\mu_s^k(dy) - \int f^l(y)\mu_s^l(dy)| \\
&= |\int f^k(y)\mu_s^l(dy) - \int f^l(y)\mu_s^l(dy)| \\
&\le \int |f^k(y)-f^l(y)|\mu_s^l(dy) \\
&\le C_t \sup_{u\in\mathrm{scaff}(\mathcal{S}_l)} |f^k(u)-f^l(u)| \le C_t\|\mathcal{S}_k\| \to 0.
\end{aligned}$$

[3] in other words, it is unique up to indistinguishability

Because $\mathcal{F}$ is dense, the above inequality holds almost surely for all $f \in \mathrm{Lip}_1(\mathrm{Lk}(0), \mathbb{R})$). Thus $(\mu_s^k)_{s \le t}$ is a.s. a cauchy sequence w.r.t. the Kantorovich-Rubinstein distance, uniformly in s and has a limit $(\mu_s)_{s \le t}$ since this distance is complete. Moreover, $(\mu_t)_{t \ge 0}$ is continuous and nondecreasing.

2. Let $\mathcal{S}_k$ and $\widetilde{\mathcal{S}}_k$ be two simplicial approximations of M such that $\mu_t^k \to \mu_t$ and $\widetilde{\mu}_t^k \to \widetilde{\mu}_t$. We shall prove that $(\mu_t)_{t \ge 0} = (\widetilde{\mu}_t)_{t \ge 0}$ a.s. Without loss of generality we may assume that $\widetilde{\mathcal{S}}_k$ is finer than $\mathcal{S}_k$, i.e. $\mathcal{S}_k \subset \widetilde{\mathcal{S}}_k$). (Indeed, if this is not the case, we can consider the joint refinement $\widehat{\mathcal{S}}^k := \mathcal{S}_k \cup \widetilde{C}_k$.) In this situation, let $k \in \mathbb{N}$ and let $f : M \to \mathbb{R}$ be piecewise linear w.r.t. $\mathcal{S}_k$, i.e. $f = f^k$. Then f is also piecewise linear w.r.t. $\mathcal{S}_l$ and $\widetilde{\mathcal{S}}_l$ for all $l \ge k$ and hence

$$\begin{aligned} \sum_{u \in \mathrm{scaff}(\mathcal{S}_k)} f(u) L_t^{u,k_0} &= \int_0^t \mathbf{1}_{\{X_\tau = 0\}} df(X_\tau) \\ &= \int_{\mathrm{Lk}(0)} f(y) \mu_t^k(dy) = \int_{\mathrm{Lk}(0)} f(y) \widetilde{\mu}_t^k(dy) \end{aligned}$$

So letting $k \to \infty$ yields almost surely

$$\int_{\mathrm{Lk}(0)} f(y) \mu_t(dy) = \int_{\mathrm{Lk}(0)} f(y) \widetilde{\mu}_t(dy) = \int_0^t \mathbf{1}_{\{X_\tau = 0\}} df(X_\tau) \tag{3.13}$$

So (3.13) holds for every $t \in \mathbb{Q}_+$ a.s., and by continuity of μ_t and $\widetilde{\mu}_t$, (3.13) holds for every $t \in \mathbb{R}_+$.
Let $\mathcal{F}$ be the set of functions $f : M \to \mathbb{R}$ such that f is piecewise linear w.r.t some triangulation $\mathcal{S}_k$ and $f(u) \in \mathbb{Q}$ for all $u \in \mathrm{scaff}(\mathcal{S}_k)$. Then almost surely, (3.13) holds for all $f \in \mathcal{F}$ and all $t \in \mathbb{R}_+$. Moreover, since $\mathcal{F}$ is dense in $C(\mathrm{Lk}_0 M, \mathbb{R})$, $(\mu_t)_{t \ge 0} = (\widetilde{\mu}_t)_{t \ge 0}$ almost surely.

3. Let L_t be a continuous measure-valued process that satisfies (3.12) and let $\mathcal{S}_k$ be a simplicial approximation with corresponding limit measure μ_t. Then one proves in the same way as in 2. that $L = \mu$ a.s., which is the desired uniqueness.

4. In order to finish the proof, we show that if $\mathcal{S}_k$ is a simplicial approximation with corresponding limit measure L, then L satisfies (3.12). Indeed, let $f : M \to \mathbb{R}$ be piecewise linear w.r.t. some triangulation $\mathcal{S}$. If we set $\widetilde{\mathcal{S}}_k := \mathcal{S}_k \cup \mathcal{S}$, then combining (3.13) and (3.11) yields (3.12). $\square$

L_t can be regarded as a random measure on $\mathbb{R}_+ \times \mathrm{Lk}(0)$, which is the content of the following

Lemma 3.2.5 *Let (N,ρ) be a separable metric space and $l:\mathbb{R}_+\to\mathcal{M}(N)$ be a continuous nondecreasing measure-valued map. Then there is a unique nonnegative measure, again denoted by l, such that for all partitions $\Delta=(t_k)$ of $\mathbb{R}_+$ and all 'simple' functions of the form $f(t,\eta)=\sum_{t_k\in\Delta} f_k(\eta)\mathbf{1}_{]t_k,t_{k+1}]}(t)$*

$$\int f(t,\eta)l(d(t,\eta))=\sum_{t_k\in\Delta}\int_N f_k(\eta)(l_{t_{k+1}}-l_{t_k})(d\eta) \tag{3.14}$$

Proof : A standard monotone class argument. □

So far we have only considered the local time at 0. If $S\subset$ is an $m-$dimensional simplex w.r.t. some triangulation, we define L^S analogously to (2.27):

Definition 3.2.6 Let $S\in\mathcal{S}^{(m)}$ for some triangulation $\mathcal{S}$. Then the *local time measure* of X at S is defined by

$$L^S:=L(X^\perp) \tag{3.15}$$

where $L(X^\perp)$ is the local time measure of $X^\perp$ at $0_{\perp S}$.

So L^S is a nonnegative measure on $\mathbb{R}_+\times\mathrm{Lk}_0\perp S$. Note that L^S is a.s. carried on the set $\{X_\tau\in S^\circ\}\times\mathrm{Lk}_0\perp S$, i.e. there is a $P-$nullset out of which $\int\mathbf{1}_{\{X_\tau\notin S^\circ\}}f(\tau,y)L^S(d(\tau,y))=0$ for all integrable $f:\mathbb{R}_+\times\mathrm{Lk}_0\perp S\to\mathbb{R}$.

Example 3.2.7 Assume that $M=\mathbb{R}^2$ is equipped with a triangulation $\mathcal{S}$ into a simplicial cone complex with origin $0\in\mathcal{S}$. Moreover, assume that $\mathcal{S}$ contains $S:=\{x\in\mathbb{R}^2: x_1=0, x_2\geq 0\}$.
(i) Let X be two-dimensional Brownian motion starting at 0. Then $\mathbf{1}_{\{X_\tau=0\}}\equiv 0$ a.s. (because the set $\{0\}$ is polar) and hence

$$L^0_t(\mathrm{Lk}_0 M)=\int_0^t\mathbf{1}_{\{X_\tau=0\}}d\varrho(X_\tau)=0$$

for all t, where $\varrho(x):=\|x\|$. Thus the local time measure of X at 0 is 0.
(ii) Let $a=(a_1,a_2)\in\mathbb{R}^2_+$ with $\|a\|=1$. So there is some $\alpha\in[0,\frac{\pi}{2}]$ such that $a=e^{i\alpha}$. Consider the semimartingale $X=Ba=(a_1B,a_2B)$, where B is a one-dimensional standard Brownian motion. Let l_t be the local time of B_t at $0\in\mathbb{R}$. Then $L^0_t=\frac{1}{2}l_t(\delta_a+\delta_{-a})$. Moreover, locally around S we have $X^\perp=X^1=\cos\alpha B$ and hence $L^S_t(dy)=\frac{1}{2}l_t\cos\alpha(\delta_{e_1}+\delta_{-e_1})(dy)$.
(iii) The preceding examples are extremal in some sense: Two-dimensional Brownian does not see 0, and the process from (ii) is one-dimensional, i.e. it lives on a one-dimensional subspace of $\mathbb{R}^2$. We will now give an example of a process that lives on the whole $\mathbb{R}^2$ but has nontrivial local time measure at 0:

Let $R := |B|$, where B is the Brownian motion from (ii), and θ be a Brownian motion on $\mathbb{S}^1$, independent of B (and hence also of R). Set $X_t := R_t e^{i\theta_t}$. Then $\varrho(X_t) = R_t$ and hence $L_t^0(\mathrm{Lk}_0 M) = \int_0^t \mathbf{1}_{\{X_\tau=0\}} d\varrho(X_\tau) = \int_0^t \mathbf{1}_{\{R_\tau=0\}} dR_t = l_t$.

Example 3.2.8 (i) Let M be a $k-$star. Then $\mathrm{Lk}_0 M = \{u_1, \dots u_k\}$, where $u_i = (1, i)$ is the unit vector on the $i-$th ray. Let X be Brownian motion on M, i.e. X is the isotropic Walsh's Brownian motion, cf. Example 2.4.19 b). For $i = 1 \dots, k$, denote by L^{0,u_i} the directional local time of X at $S = \{0\}$ and consider the function $g_i : M \to \mathbb{R}$, defined in (2.97). Recall from the proof of Example 2.4.19 b) that $g_i(X)$ is a martingale for all $i = 1 \dots, k$, where $g_i : M \to \mathbb{R}$ is defined in (2.97). Moreover, g_i is piecewise linear (more precisely, $g_i = \frac{k-1}{k}\nu^{u_i} - \frac{1}{k}\sum_{j\neq i}\nu^{u_j}$) and hence

$$\begin{aligned} 0 &= \int_0^t \mathbf{1}_{\{g_i(X_\tau)=0\}} dg_i(X_\tau) = \int \mathbf{1}_{\{X_\tau=0\}} dg_i(X_\tau) \\ &= \frac{k-1}{2k} L_t^{0,u_i} - \frac{1}{2k}\sum_{j\neq i} L_t^{0,u_j} = \frac{1}{2k}\sum_{j\neq i}(L_t^{0,u_i} - L_t^{0,u_j}), \end{aligned}$$

for all $i = 1, \dots, k$, all $t \geq 0$ and (almost) all $\omega \in \Omega$. So we conclude that $L^{0,u_i} \equiv L^{0,u_j}$ for all i, j.
Note that the process $r_t := d(X_t, 0)$ is a reflected Brownian motion and hence

$$l_t := 2\int_0^t \mathbf{1}_{\{r_\tau=0\}} dr_\tau = \sum_{i=1}^k L_t^{0,u_i} = kL_t^{0,u_1}$$

is a Brownian local time. So we obtain the representation

$$L_t^0(dy) = \frac{1}{k}\sum_{i=1}^k l_t \delta_{u_i}(dy). \tag{3.16}$$

(ii) Let M be an admissible Euclidean polyhedron and let X be Brownian motion in M. If $S \in \mathcal{S}^{(m)}$ for $m \leq n-2$, then $L^S(X) \equiv 0$, which immediately follows from the last assertion of Theorem 2.4.17. So the only nontrivial case is when $S \in \mathcal{S}^{(n-1)}$. Locally around S, we have the orthogonal decomposition $M = U^S \oplus \perp S$, where U^S is the $(n-1)-$dimensional Euclidean subspace generated by S and $\perp S$ is a $k-$star. In other words, locally around S, M looks like a "booklet" and we are locally in the situation of Example 2.4.19. Recall from that Example that $X^\perp$ is Brownian motion on $\perp S$, and hence from (i) we deduce the representation

$$L_t^S(X)(dy) = L_t^0(X^\perp) = \frac{1}{k}\sum_{i=1}^k l_t \delta_{u_i}(dy),$$

where $l_t^S := 2\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} dr_\tau^S$ and $r_t^S = d(S, X_t)$.

Remark 3.2.9 The representation (3.16) is not new. Indeed, it is known that if a $k-$star M is symmetrically embedded into $\mathbb{R}^2$ (cf. Example 1.1.3 (ii)), then Brownian motion in M (i.e. Walsh's isotropic BM) is a so-called *spider-martingale* in the sense of Definition 17.2 of [Yor97]. Thus one can deduce (3.16) from Proposition 17.5 of [Yor97].
Although out of the scope of this work, we mention another very interesting point implied by (3.16), which may be of particular interest when one wants to establish a theory of stochastic differential equations in polyhedra: For $k > 2$, the natural filtration of X is *not* a Brownian filtration[4]. This was proved by Tsirelson [Tsi97] using the following arguments: If X was a spider-martingale w.r.t. a Brownian filtration, then we would have $dL^{0,u_1} \wedge \cdots \wedge dL^{0,u_k} = 0$ by Theorem 6.1 of [Tsi97], which contradicts (3.16). See also [BÉK⁺98] for questions concerning the filtration of Walsh's Brownian motion.

3.2.3 Itô's formula revisited

Let $\varphi : O \to \mathbb{R}$ be a convex function and X a semimartingale. We already know from Lemma 3.2.2 that $\varphi(X)$ is a semimartingale on $\{X \in O\}$. Assume that O is local at some $S \in \mathcal{S}$ now Let $\mathcal{S}$ be an isometric triangulation containing S. For each $u \in \operatorname{scaff}(\perp S)$ let $g^u : S^\circ \to \mathbb{R}$ be a smooth function. Let O be a neighborhood that is local at S. Consider the function $g : O \to \mathbb{R}$, defined by $g(x) := \sum_{u \in \operatorname{scaff}(\perp S)} g^u(x^\top)\nu^u(x^\perp)$ (actually, g is defined on all $S^\circ \oplus \perp S \cong TM_{|S^\circ}$). Then g is piecewise smooth and for all $x^\top \in O \cap S$ and all $x^\perp \in \perp S$, we have $\partial g_{x^\top}(x^\perp) = g(x^\top + x^\perp) =: g_{x^\top}(x^\perp)$. So g can be regarded as a linear form over S° whose tangential part is 0, and by Lemma 2.1.12 we have

$$\begin{aligned}\int \mathbf{1}_{\{X_\tau \in S^\circ\}} dg(X_\tau) &= \frac{1}{2} \sum_{u \in \operatorname{scaff}(\perp S)} \int_0^t g^u(X_\tau) dL_\tau^{S,u}(X) \\ &= \frac{1}{2} \int \partial g_{X_\tau}(y) L^S(d(\tau, y)). \end{aligned} \tag{3.17}$$

Let now $\varphi : O \to \mathbb{R}$ be Lipschitz continuous and convex. We have the usual decomposition $\varphi = \varphi^\top + \varphi^\perp$. Note that $\varphi^\perp = \varphi - \varphi^\top$ is not necessarily convex. However, $\varphi^\perp$ is *convex in the orthogonal directions*, i.e. $\varphi^\perp_{|(x^\top + \perp S) \cap O}$ is convex for all $x^\top \in S \cap O$. Set

$$\varphi^r(x) := \frac{1}{r} \varphi^\perp(x^\top + r x^\perp). \tag{3.18}$$

[4] by this we mean a filtration which is generated by countably many linear Brownian motions

Then is φ^r convex in the orthogonal directions, i.e. $\varphi^r_{|(x^\top+\perp S)\cap O}$ is convex for all $x^\top \in S\cap O$. For $v \in \mathrm{Lk}_0\perp S$ and $x = x^\top + sv$, set

$$\widehat{\varphi}^r(x) := \frac{s}{r}\varphi^\perp(x^\top + rv) \tag{3.19}$$

So $\widehat{\varphi}^r$ is the unique function that is *radial at* S (i.e. $\widehat{\varphi}^r(x^\top + sx^\perp) = s\widehat{\varphi}^r(x^\top + x^\perp)$ and $\varphi^r \equiv \widehat{\varphi}^r$ on $\{x \in O : d(S,x) = 1\}$. Moreover, $\widehat{\varphi}^r_{|(x^\top+\perp S)\cap O}$ is also convex for all $x^\top \in S\cap O$.
At last, let $\mathcal{S}_k$ be an approximating sequence of isometric triangulations for $\perp S$. Set

$$\widehat{\varphi}^{r,k}(x) := \sum_{u\in\mathrm{scaff}(\perp S,\mathcal{S}_k)} \widehat{\varphi}^r(u)\nu^u = \sum_{u\in\mathrm{scaff}(\perp S,\mathcal{S}_k)} \varphi^r(u)\nu^u \tag{3.20}$$

Then $\widehat{\varphi}^{r,k}$ is piecewise linear w.r.t. $\mathcal{S}_k$ (but not necessarily convex!). Moreover,

$$\widehat{\varphi}^r(x) \leq \widehat{\varphi}^{r,k}(x) \tag{3.21}$$

which is a consequence of Jensen's inequality: Fix $x^\top \in S^\circ$. For each $T \in \mathcal{S}_k$ and $x^\perp \in T^\circ \cap \perp S$ define a probability measure

$$p_x := p_{x^\top+x^\perp} := \sum_{u\in\mathrm{scaff}(T\cap\perp S)} \nu^u(x)\delta_u + (1 - \sum_{u\in\mathrm{scaff}(T\cap\perp S)} \nu^u(x))\delta_0.$$

Then p_x is a probability on the Euclidean simplex T, and because $\widehat{\varphi}^r$ is convex on $x^\top + \perp S$ and $\widehat{\varphi}^r(0) = 0$, Jensen's inequality yields that

$$\widehat{\varphi}^{r,k}(x) = \int \widehat{\varphi}^r(y)p_x(dy) \geq \widehat{\varphi}^r(\int y p_x(dy)) = \widehat{\varphi}^r(x),$$

showing (3.21).

Lemma 3.2.10 *Let O be local at S and let $\varphi : O \to \mathbb{R}$ be a Lipschitz continuous convex function. Assume that $\partial_u\varphi$ is Lipschitz continuous for all $u \in \mathrm{scaff}(S)$. Then*

(i) φ^r is Lipschitz continuous, uniformly in r, and $\partial\varphi^\perp$ is Lipschitz continuous.

(ii) $\varphi^r \to \partial\varphi^\perp$ uniformly on O.

(iii) For every $\epsilon > 0$ there are $0 < r_0 \leq 1$ and $k_0 \in \mathbb{N}$ such that for all $r \leq r_0$, all $k \geq k_0$ and all $x = x^\top + x^\perp \in O \cap B_r(S)$,

$$\partial\varphi^\perp_{x^\top}(x^\perp) \leq \varphi^r(x) \leq \varphi(x) \leq \widehat{\varphi}^r(x) \leq \widehat{\varphi}^{r,k}(x) \leq \partial\varphi^\perp_{x^\top}(x^\perp) + \epsilon d(S,x)$$

Proof : Because φ is convex, $\varphi_{|(x^\top+\perp S)\cap O}$ is convex for all $x^\top \in S \cap O$. Thus $\varphi^r \searrow \partial\varphi^\perp$ pointwise.
Let now $x_1 = x_1^\top + x_1^\perp$ and $x_2 = x_2^\top + x_2^\perp$. Then $d(x_2^\top + rx_1^\perp, x_2^\top + rx_2^\perp) = rd(x_2^\top + x_1^\perp, x_2^\top + x_2^\perp)$ and because φ is Lipschitz continuous (say, with Lipschitz constant C), we obtain $|\varphi^r(x_2^\top + x_1^\perp) - \varphi^r(x_2^\top + x_2^\perp)| \leq Cd(x_2^\top + x_1^\perp, x_2^\top + x_2^\perp)$. On the other hand, $\partial_u\varphi$ is Lipschitz continuous by assumption and hence $\partial_u\varphi^\perp$ is also Lipschitz continuous for all $u \in \mathrm{scaff}(S)$. Now since $\varphi^\perp_{|S\cap O} \equiv 0$, $|\partial_u\varphi^\perp(x^\top + rx^\perp)| \leq Cr$ for all $x = x^\top + x^\perp \in O$. (where C is chosen accordingly[5]). Consequently, $|\varphi^r(x_1^\top + x_1^\perp) - \varphi^r(x_2^\top + x_1^\perp)| \leq Cd(x_1^\top + x_1^\perp, x_2^\top + x_1^\perp)$ (with C adjusted again). So putting this together yields

$$
\begin{aligned}
|\varphi^r(x_1^\top + x_1^\perp) - \varphi^r(x_2^\top + x_2^\perp)| &\leq |\varphi^r(x_1^\top + x_1^\perp) - \varphi^r(x_2^\top + x_1^\perp)| \\
&\quad + |\varphi^r(x_2^\top + x_1^\perp) - \varphi^r(x_2^\top + x_2^\perp)| \\
&\leq Cd(x_1^\top + x_1^\perp, x_2^\top + x_1^\perp) + Cd(x_2^\top + x_1^\perp, x_2^\top + x_2^\perp) \\
&\leq \widetilde{C}d(x_1, x_2).
\end{aligned}
$$

Thus φ^r is Lipschitz continuous, and the Lipschitz constant can be chosen independent of r. So $\partial\varphi^\perp$ is Lipschitz continuous as the pointwise limit of uniformly Lipschitz continuous functions, showing (i).
(ii) follows from Dini's Lemma[6] since $\partial\varphi^\perp$ is continuous by (i).
It remains to show (iii). Because $\varphi_{|(x^\top+\perp S)\cap O}$ is convex for all $x^\top \in S\cap O$ and the curve $t \mapsto x^\top + tx^\perp$ is a geodesic, the first three inequalities follow from the properties of convex functions on $\mathbb{R}$ (in particular, the monotonicity of the difference quotient). The forth inequality was already shown in (3.21).
For the last inequality note that by (i), φ^r is uniformly Lipschitz continuous on $S \cap O \oplus \mathrm{Lk}_0 \perp S$[7], so $(\widehat{\varphi}^{r,k} - \varphi^r)(x^\top + v)$ tends to 0, uniformly in $x^\top \in S \cap O$, $v \in \mathrm{Lk}_0\perp S$ and $r \in]0, r_0]$. Together with (ii) we deduce that if $r \to 0$ and $k \to \infty$, then $\widehat{\varphi}^{r,k} \to \partial\varphi^\perp$ uniformly on $S \cap O \oplus \mathrm{Lk}_0\perp S$. Thus for every $\epsilon > 0$ there are $0 < r_0 \leq 1$ and $k_0 \in \mathbb{N}$ such that for all $r \leq r_0$, all $k \geq k_0$ and all $x = x^\top + v \in S \cap O \oplus \mathrm{Lk}_0\perp S$, $\partial\varphi^\perp_{x^\top}(v) \leq \widehat{\varphi}^{r,k}(x^\top + v) \leq \partial\varphi^\perp_{x^\top}(v) + \epsilon$. Because $\widehat{\varphi}^{r,k}$ and $\partial\varphi^\perp$ are radial at S, it follows by radial interpolation that for all $x = x^\top + x^\perp \in B_r(S) \cap O$, $\partial\varphi^\perp_{x^\top}(x^\perp) \leq \widehat{\varphi}^{r,k}(x) \leq \partial\varphi^\perp_{x^\top}(x^\perp) + \epsilon d(S, x)$, showing (iii). □

[5] note that in a Euclidean complex, a neighborhood O that is local at a simplex is automatically relatively compact. In a Euclidean cone complex, we should assume that O is relatively compact.

[6] Dini's Lemma says that if $K \subset \mathbb{R}^N$ is a compact set and $f_n : K \to \mathbb{R}$ is a sequence of continuous functions such that $f_n \searrow f$ pointwise (important is that the convergence is monotone) and $f : K \to \mathbb{R}$ is continuous, then the convergence is already uniform on K.

[7] here we regard φ^r as a function defined on some neighborhood O^r of $S \cap O$ in $S \cap O \oplus \perp S$, and if r_0 is small enough, then $S \cap O \oplus \mathrm{Lk}_0\perp S \subset O^r$.

Definition 3.2.11 Let $S \in \mathcal{S}$ and O local at S. A Lipschitz continuous function $\varphi : O \to \mathbb{R}$ is called *regular at* S if φ is differentiable in direction S (i.e. the function $x^\top \mapsto \varphi(x^\top + x^\perp)$ is differentiable) and for all $u \in \operatorname{scaff}(S)$, $\partial_u \varphi$ is Lipschitz continuous on O.

Example 3.2.12
(i)If $\varphi(x) \equiv \varphi(x^\perp)$ (i.e. if φ is constant in the directions of S), then φ is regular. In particular, if $S = x_0$ is a corner (i.e. a $0-$dimensional simplex), then every Lipschitz continuous convex function is regular at x_0.
(ii) Let $\varphi : O \to \mathbb{R}$ be a Lipschitz continuous convex function. Consider the mollifier $\Psi_S^j : S \to \mathbb{R}$, defined in (3.4). Let $j_0 \in \mathbb{N}$ and let $\widetilde{O}$ be neighborhood such that $B_{1/j_0}(\widetilde{O}) \subset O$. For $j \geq j_0$ and $x = x^\top + x^\perp \in \widetilde{O}$ define

$$\varphi^j(x) := \varphi^{S,j}(x) := \int_S \varphi(x+y)\Psi_S^j(y)dy. \tag{3.22}$$

So φ is smoothed only *in the tangential directions.* i.e. all directions of S. As in the proof of Lemma 3.2.1 (ii), one can easily see that φ^j is convex, and the smoothness in the tangential directions implies that φ^j is regular at S. We will use this kind of smoothing in many geometric applications, as for instance in Theorem 3.3.4.
(iii) Consider the function $\varrho(x) := d(S,x)$. ϱ is convex on any neighborhood that is local at S. Moreover, ϱ is regular at S by (i).

The crucial point of regularity is that a regular function φ admits an intrinsic local Itô formula for φ at S, which is in particular independent of the triangulation.

Theorem 3.2.13 (Intrinsic local Itô formula) *Let $X : \Omega \times \mathbb{R}_+ \to M$ be a semimartingale. Let $S \in \mathcal{S}$ for some isometric triangulation, O local at S and let $\varphi : O \to \mathbb{R}$ be a Lipschitz continuous convex function that is regular at S. Then on $\{X \in O\}$ we have*

$$\begin{aligned}\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi^\perp(X_\tau) &= \frac{1}{2}\int_{[0,t]\times \mathrm{Lk}_0 \perp S} \partial\varphi^\perp_{X_\tau}(y) L^S(d(\tau,y)) \\ &= \frac{1}{2}\int_{[0,t]\times \mathrm{Lk}_0 \perp S} \partial\varphi_{X_\tau}(y) L^S(d(\tau,y)).\end{aligned}$$

Proof : Let $\mathcal{S}_k$ be an approximating sequence of isometric triangulations of $\perp S$. Recall the definition of $\widehat{\varphi}^{r,k}$ from (3.20). By 3.2.10 (iii), $\widehat{\varphi}^{r,k} \to \partial\varphi^\perp$ uniformly on $(S \cap O) \oplus \mathrm{Lk}_0 \perp S$ as $r \to 0$, $k \to \infty$ and hence

$$\begin{aligned}\lim_{\substack{r\to 0\\ k\to\infty}} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\widehat{\varphi}^{r,k}(X_\tau) &= \frac{1}{2}\lim_{\substack{r\to 0\\ k\to\infty}} \int_{[0,t]\times \mathrm{Lk}_0 \perp S} \widehat{\varphi}^{r,k}_{X_\tau}(y) L^S(d(\tau,y)) \\ &= \frac{1}{2}\int_{[0,t]\times \mathrm{Lk}_0 \perp S} \partial\varphi^\perp_{X_\tau}(y) L^S(d(\tau,y)).\end{aligned}$$

Set $\varrho(x) := d(S,x)$. Then ϱ is convex on O and hence $\varrho(X)$ is a semimartingale on $\{X \in O\}$. Moreover, since $\varrho \geq 0$, $\mathcal{L}_t := \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varrho(X_\tau)$ is a nondecreasing process. Put $h^{r,k} := \widehat{\varphi}^{r,k} - \varphi^\perp$. By Lemma 3.2.10 (iii), $0 \leq h^{r,k} \leq \epsilon\rho$ on $O \cap B_r(S)$ and hence $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} dh^{r,k}(X_\tau)$ is nondecreasing and $\int \mathbf{1}_{\{X_\tau \in S^\circ\}} d(h^{r,k} - \epsilon\varrho)(X_\tau) = \int \mathbf{1}_{\{X_\tau \in S^\circ\}} dh^{r,k}(X_\tau) - \epsilon\mathcal{L}$ is nonincreasing. So letting $\epsilon \to 0$ (and $r \to 0$, $k \to \infty$ accordingly) yields that

$$\begin{aligned} 0 &= \lim_{\substack{r\to 0\\ k\to\infty}} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} dh^{r,k}(X_\tau) \\ &= \lim_{\substack{r\to 0\\ k\to\infty}} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\widehat{\varphi}^{r,k}(X_\tau) - \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi^\perp(X_\tau) \\ &= \frac{1}{2} \int_{[0,t]\times \mathrm{Lk}_0 \perp S} \partial\varphi^\perp_{X_\tau}(y) L^S(d(\tau,y)) - \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi^\perp(X_\tau). \square \end{aligned}$$

As a special case, let $x_0 \in M$. We can assume that $x_0 \in \mathcal{S}^{(0)}$ for some isometric triangulation $\mathcal{S}$, i.e. x_0 is a *corner*. Because every Lipschitz continuous convex function is regular at x_0 (cf. Example 3.2.12 (i)), we get the following

Corollary 3.2.14 *Let $x_0 \in O \subset M$ and let $\varphi : O \to \mathbb{R}$ be a Lipschitz continuous convex function. Then*

$$\int_0^t \mathbf{1}_{\{X_\tau = x_0\}} d\varphi(X_\tau) = \int_{[0,t]\times \mathrm{Lk}_0 \perp S} \partial\varphi_{x_0}(y) L^{x_0}(d(\tau,y)).$$

Example 3.2.15 Although the regularity condition from Definition 3.2.11 is not sharp (i.e. one can establish more general conditions on φ to ensure that Theorem 3.2.13 holds), some condition is necessary, as the following example shows:
Consider the process $X = Ba$ in $M = \mathbb{R}^2$ from Example 3.2.7 (ii). Different from that situation, we choose a different triangulation $\mathcal{S}$ for M: Let $(0,-1) \in \mathcal{S}$ be the origin and assume that $S := \{x = (x_1,x_2) : x_1 = 0, x_2 \geq -1\} \in \mathcal{S}$. Let now $\varphi(x) := d(0,x) = \|x\|$. Then φ is convex and Lipschitz continuous, *but not regular at S*.
Now $\partial\varphi_0(e_1) = \partial\varphi_0(-e_1) = 1$ and hence by Example 3.2.7 (i),

$$\int \partial\varphi_{X_\tau}(y) L^S(d(\tau,y)) = \int \partial\varphi_0(y) L^S(d(\tau,y)) = l_t \cos\alpha$$

(note that the first equality holds since L^S only increases on $\{X \in S^\circ\} = \{X = 0\}$).
On the other hand, $\varphi(X) = |B|$ and $\{X \in S^\circ\} = \{B = 0\}$ and hence $\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi(X_\tau) = l_t$. Moreover, $\varphi^\top(x) = |x_2|$ and hence

$$\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi^\top(X_\tau) = \int_0^t \mathbf{1}_{\{X^2_\tau = 0\}} d|X^2_\tau| = l_t \sin\alpha.$$

Consequently, $\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi^\top(X_\tau) = l_1(1 - \sin\alpha)$.

3.3 Darling's characterization, part II

We already know that if X is a semimartingale and $\varphi : M \to \mathbb{R}$ a Lipschitz continuous convex function, then $\varphi(X)$ is a semimartingale. Now we will show that if X is a martingale, then $\varphi(X)$ is a submartingale. This is Darling's characterization of martingales for general Lipschitz continuous convex functions.

Definition 3.3.1 Let (Ω, X, P) be a continuous semimartingale and let $S \in \mathcal{S}$ for some isometric triangulation $\mathcal{S}$. We say that X *satisfies condition* $\mathbf{M}(S)$ if there is a $P-$nullset out of which $0_{\perp S} \in \mathbb{B}(L^S_t - L^S_s)$ for all $0 \leq s \leq t$.

Remark 3.3.2 One can show that X satisfies $\mathbf{M}(S)$ if and only if X satisfies $\mathbf{M}_{\mathcal{S}}(S)$ for all isometric triangulations $\mathcal{S}$ of M. The 'only if' part is trivial. In order to prove the 'if' implication, one should show the following statement, which we state as a conjecture:
Let M be a Euclidean conical polyhedron and let $\varphi : M \to \mathbb{R}$ be a Lipschitz continuous convex function with $\varphi(0) = 0$. Then for every isometric triangulation $\mathcal{S}$, there are a finer triangulation $\widetilde{\mathcal{S}} \supset \mathcal{S}$ and a convex function $\widetilde{\varphi}$ such that $\widetilde{\varphi}(0) = 0$, $\widetilde{\varphi} \leq \varphi$ and $\widetilde{\varphi}$ is piecewise linear w.r.t. $\widetilde{\mathcal{S}}$. Moreover, if $\|\mathcal{S}\| \to 0$, then $\widetilde{\varphi} \to \varphi$.

Lemma 3.3.3 *Let X be a continuous semimartingale and $S \in \mathcal{S}(M)$. If X satisfies condition $\mathbf{M}(S)$, then $\int_{[0,t]\times \mathrm{Lk}_0 \perp S} \partial\varphi_{X_\tau}(y) L^S(d(\tau, y))$ is nondecreasing for any Lipschitz continuous convex function φ that is regular at S.*

Proof : The proof is completely analogous to the one of Lemma 3.1.4. Note that $\partial\varphi$ is Lipschitz continuous and hence the function $(t, y) \mapsto \partial\varphi_{X_t}(y)$ is a.s. continuous. Consequently, there is a $P-$nullset out of which

$$\sum_{t_l \in \Delta^k} \int_{\mathrm{Lk}_0 \perp S} \partial\varphi(X_{\tau_l})(y)(L^S_{t_{l+1}} - L^S_{t_l})(dy) \to \int_{[0,t]\times \mathrm{Lk}_0 \perp S} \partial\varphi_{X_\tau}(y) L^S(d(\tau, y))$$

whenever Δ^k is a sequence of partitions of $\mathbb{R}_+$ with $\|\Delta^k\| \to 0$ and $\tau_l = \tau^k_l(\omega)$ is a sequence of intermediate points, i.e. $t_l \leq \tau^k_l \leq t_{l+1}$ for all $t_l, t_{l+1} \in \Delta^k$.
Moreover, $\partial\varphi_{|x^\top + \perp S}$ is convex for all $x^\top \in S \cap O$, so as in Lemma 3.1.4, for any partition Δ^k we can find a sequence τ_l of intermediate points such that $\sum_{t_l \in \Delta^k} \int_{\mathrm{Lk}_0 \perp S} \partial\varphi(X_{\tau_l})(y)(L^S_{t_{l+1}} - L^S_{t_l})(dy)$ is nondecreasing, proving the Lemma.
$\square$

Theorem 3.3.4 *Let M be a Euclidean polyhedron and let $X : \Omega \times \mathbb{R}_+ \to M$ be a continuous semimartingale. Then the following are equivalent:*

(i) $\varphi(X)$ is a local submartingale on $\{X \in O\}$ for all Lipschitz continuous convex functions $\varphi : O \to \mathbb{R}$.

(ii) *Whenever $\mathcal{S}$ is an isometric triangulation of M, then for all $S \in \mathcal{S}$ and $u \in \operatorname{scaff}(S)$, $\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} dX_\tau^u$ is a local martingale and X satisfies $\mathbf{M}(S)$.*

(iii) *There is an isometric triangulation $\mathcal{S}$ of M such that for all $S \in \mathcal{S}$ and $u \in \operatorname{scaff}(S)$, $\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} dX_\tau^u$ is a local martingale and X satisfies $\mathbf{M}(S)$.*

In either case, X is called a local martingale in M.

Proof : $(iii) \Rightarrow (i)$: Let $\mathcal{S}$ be an isometric triangulation satisfying the assumptions of (iii) and let $\varphi : O \to \mathbb{R}$ be Lipschitz continuous and convex. Without loss of generality we may assume that O is local at some $S \in \mathcal{S}^{(m)}$. We will prove that (i) holds by induction on the codimension $k = n - m$ of S. For $k = 0$ (i.e. $m = n$), this follows from the well-known theory in $\mathbb{R}^n$ (e.g., by Jensen's inequality or by a starndard smoothing procedure). So assume that we have proved the assertion for all $l = 0, \dots, k-1$.
Consider the function $\varphi^j := \varphi^{S,j}$ from Example 3.2.12 (ii). Then φ^j is convex and regular at S. Consequently,

$$\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d(\varphi^j)^\perp(X_\tau) = \frac{1}{2} \int_{[0,t]\times \mathrm{Lk}_0 \perp S} \partial \varphi^j_{X_\tau}(y) L^S(d(\tau, y))$$

is nondecreasing by Lemma 3.3.3. Since $(\varphi^j)^\top$ is smooth,

$$\begin{aligned}\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d(\varphi^j)^\top(X_\tau) = & \sum_{u \in \operatorname{scaff}(S)} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} \partial_u \varphi^j(X_\tau) dX_\tau^u \\ & + \frac{1}{2} \sum_{u,v \in \operatorname{scaff}(S)} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} \partial_{uv} \varphi^j(X_\tau) d\langle X_\tau^u, X_\tau^v \rangle\end{aligned} \tag{3.23}$$

is a local submartingale. Moreover, by induction hypothesis we have that $\int_0^t \mathbf{1}_{\{X_\tau \in O \setminus S\}} d(\varphi^j)(X_\tau)$ is a local submartingale. So putting this together, $\varphi^j(X)$ is a local submartingale on $\{X \in O\}$.
It remains to show that $\varphi(O)$ is a local submartingale on $\{X \in O\}$. Therefor, we may assume that X has only values in some relatively compact set $U \subset\subset O$. Since O is local at S, O is relatively compact. Hence φ is bounded and the sequence φ^j is uniformly bounded on U. So $\varphi^j(X)$ is a (uniformly) bounded submartingale for every j. Now $\varphi^j(X) \to \varphi(X)$ uniformly on $\{X \in U\}$, thus $\varphi(X)$ is a submartingale on $\{X \in U\}$.
$(i) \Rightarrow (ii)$: Let $\mathcal{S}$ be an isometric triangulation and let $S \in \mathcal{S}$. First we show that $\mathbf{M}(S)$ holds. Let $f : \perp S \to \mathbb{R}$ be a Lipschitz continuous convex function. We may assume that $f(0) = 0$. Set $\varphi(x) := f(x^\perp)$. Then φ is convex and by (i),

$\int \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi(X_\tau)$ is a local submartingale. Moreover, φ is regular at S and we have $\varphi = \varphi^\perp$. Thus

$$\begin{aligned}\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi(X_\tau) &= \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} d\varphi^\perp(X_\tau) \\ &= \int_{[0,t]\times \mathrm{Lk}(\perp S)} \partial\varphi_{X_\tau}(y) L^S(d(\tau, y)) \\ &= \int_{[0,t]\times \mathrm{Lk}(\perp S)} \partial f_0(y) L^S(d(\tau, y)) \\ &= \int_{\mathrm{Lk}(\perp S)} \partial f_0(y)(L_t^S - L_0^S)(dy)\end{aligned}$$

must be nondecreasing in t. In particular, for all $s \leq t$,

$$\int_{\mathrm{Lk}(\perp S)} f(y)(L_t^S - L_0^S)(dy) \geq \int_{\mathrm{Lk}(\perp S)} \partial f_0(y)(L_t^S - L_0^S)(dy) \geq 0$$

Thus $\mathbf{M}(S)$ holds.
It remains to show that if $u \in \mathrm{scaff}(S)$, then $\int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} dX_\tau^u$ is a local martingale. But by (i) we have in particular that $\varphi(X)$ is a local submartingale on $\{X \in O\}$ for every piecewise smooth (w.r.t. $\mathcal{S}$) function, and so the assertion follows from Theorem 3.1.5.
$(ii) \Rightarrow (iii)$ is trivial. $\square$

With the same technique as in the proof of the above Theorem one can get some more information about the transformation behavior of convex functions.

Definition 3.3.5 Let M be a geodesic space. A function $\varphi : M \to \mathbb{R}$ is called $\kappa-$*convex* if for all geodesics $\gamma : [0,1] \to M$ and all $t \in [0,1]$,

$$\varphi(\gamma(t)) \leq (1-t)\varphi(\gamma(0)) + t\varphi(\gamma(1)) - \frac{\kappa}{2} t(1-t) d^2(\gamma(0), \gamma(1)) \tag{3.24}$$

For instance, if M is a Riemannian manifold (or if $M = \mathbb{R}^n$), $\kappa-$convexity in this sense means that $\mathrm{Hess}\varphi \geq \kappa g$ in the sense of distributions, where g is the metric tensor on M.

Corollary 3.3.6 *Let X be a local martingale in M and let $\varphi : O \to \mathbb{R}$ be $\kappa-$convex. Then $\varphi(X) - \frac{\kappa}{2}\langle X\rangle$ is a local submartingale on $\{X \in O\}$.*

Proof : We can completely imitate the proof of Theorem 3.3.4, $(ii) \Rightarrow (i)$. The only thing to mention is that because φ is $\kappa-$convex, φ^j is also $\kappa-$convex[8] and

[8] this can easily be shown as in the proof of Lemma 3.2.1 (ii)

hence since $(\varphi^j)^\top$ is smooth, $\mathrm{Hess}(\varphi^j)^\top_x(u,u) \geq \kappa g_x(u,u)$ for all $x \in S^\circ$ and $u \in T_xS$. So in (3.23) we obtain that

$$\frac{1}{2}\sum_{u,v\in \mathrm{scaff}(S)} \int_0^t \mathbf{1}_{\{X_\tau \in S^\circ\}} \partial_{uv}\varphi^j(X_\tau) d\langle X^u_\tau, X^v_\tau\rangle - \frac{\kappa}{2}\int_0^t \mathbf{1}_{\{X_\tau\in S^\circ\}} d\langle X\rangle_\tau$$

is nondecreasing, and consequently, $\varphi^j(X) - \frac{\kappa}{2}\langle X\rangle$ is a local submartingale. So the proof is completed by letting $j \to \infty$ as in the proof of Theorem 3.3.4, $(ii) \Rightarrow (i)$. $\square$.

Example 3.3.7 Let M be an admissible Euclidean polyhedron and let X be Brownian motion in M. Recall Example 3.2.8 (ii). For $S \in \mathcal{S}^{(m)}$, $m \leq n-2$, we have $L^S \equiv 0$ and hence $\mathbf{M}(S)$ is trivially satisfied. Moreover, for $S \in \mathcal{S}^{(n-1)}$, we have $L^S_t(dy) = \frac{1}{k}\sum_{i=1}^k L_t\delta_{u_i}(dy)$ by Example 3.2.8, and it is easily seen that $\mathbf{M}(S)$ is also satisfied (cf. also Example 3.4.5). So X is a local martingale.

3.4 Characterizations in CAT(0) Euclidean complexes

Let us now consider the special case where the Euclidean polyhedron M is an Alexandrov space with curvature bounded from above (i.e. M is $CAT(\kappa)$ for some $\kappa \geq 0$). See e.g. [BH99] for a definition of $CAT(\kappa)$. For the special case $\kappa = 0$, see section 4.1.2. It turns out that in this case, M is automatically a $CAT(0)$ space, or in other words, an NPC space.

Proposition 3.4.1 *Let M be a Euclidean simplicial cone complex. Then the following are equivalent:*

(i) M is a $CAT(\kappa)$ for some $0 \leq \kappa < \infty$.

(ii) M is an NPC space (i.e. $CAT(0)$).

(iii) Lk_0M is a $CAT(1)$ space.

Proof : $(ii) \Leftrightarrow (iii)$: [BH99], Theorem II.5.1.
$(ii) \Rightarrow (i)$ is trivial.
$(i) \Rightarrow (iii)$ It is clear that the *space of directions at* 0[9] is isometric to $\mathrm{Lk}_0 M$. So if M is $CAT(\kappa)$, then $\mathrm{Lk}_0 M$ is $CAT(1)$ by [BH99], Theorem II.3.19. $\square$

[9] the space of directions at 0 is the space of geodesics emanating from 0, equipped with angular metric, cf. [BH99], Definition II.3.18

Lemma 3.4.2 (Minimal convex functions) *Let M be a Euclidean simplicial cone complex of nonpositive curvature.*
(i) Let $\varphi : M \to \mathbb{R}$ be Lipschitz continuous and convex. Let $a \in \mathrm{Lk}_0 M$ be a minimal gradient, i.e. $\partial\varphi_0(a) = \min_{u \in \mathrm{Lk}_0 M} \partial\varphi_0(u)$. Then for all $u \in \mathrm{Lk}_0 M$,

$$\varphi(u) - \varphi(0) \geq \partial\varphi_0(u) \geq \partial\varphi_0(a) \cos\angle_0(a,u). \tag{3.25}$$

(ii) For $a \in \mathrm{Lk}_0 M$, $\varphi^a(x) := -d(0,x)\cos\angle_0(a,x)$ is convex and radial at 0.

Proof : (i) Let $a \in \mathrm{Lk}_0 M$ be a minimal gradient and let $u \in \mathrm{Lk}_0 M$. If $\angle_0(a,u) = \pi$, then the unique geodesic connecting a and u passes through 0 and the assertion follows from the properties of convex functions on $\mathbb{R}$.
If $\angle_0(a,u) < \pi$, let $\varrho : [0,1] \to \mathrm{Lk}(0,M)$ be the geodesic from a to u in $Lk_0 M$ (which is unique since $Lk_0 M$ is CAT(1)). Then the set $\{\lambda\varrho(\tau) : 0 \leq \lambda, \tau \leq 1\}$ is isometric to the two-dimensional Euclidean cone $\mathcal{E}$ with angle $\angle_0(a,u)$. Since a is a minimal gradient for φ on $\mathcal{E}$, we have that $\varphi(x) - \varphi(0) \geq \partial\varphi_0(a)\langle a, x\rangle = \partial\varphi_0(a)\cos\angle_0(a,u)$ for all $x \in \mathcal{E}$, showing the assertion.
(ii) Let $g^a(x) := \frac{1}{2}d^2(a,x)$. Then g^a is convex and hence $\varphi^a = \partial g^a_0$ is convex by Lemma 1.4.5. □

The next Lemma will be crucial for a characterization of $\mathbf{M}(S)$ in a $CAT(0)$ Euclidean polyhedron.

Lemma 3.4.3 *Let M be a Euclidean conical polyhedron of nonpositive curvature and let μ be a nonnegative measure on $\mathrm{Lk}_0 M$. Then the following are equivalent:*

- *(i)* $0 \in \mathbb{B}(\mu)$.
- *(ii) For all $a \in \mathrm{Lk}_0 M$, $\int_{\mathrm{Lk}_0 M} \cos\angle_0(a,y)\mu(dy) \leq 0$.*

Proof : $(i) \Rightarrow (ii)$: Follows from Lemma 3.4.2 (ii).
$(ii) \Rightarrow (i)$: Let $\varphi : M \to \mathbb{R}$ be convex with $\varphi(0) = 0$ and let $a \in \mathrm{Lk}_0 M$ be a minimal gradient. If $\partial\varphi_0(a) \geq 0$, then (i) follows trivially. So assume that $\partial\varphi_0(a) < 0$. Then $\varphi(u) \geq \partial\varphi_0(a)\cos\angle_0(a,u)$ for all $u \in \mathrm{Lk}_0 M$ by Lemma 3.4.2 (i) and hence $\int_{\mathrm{Lk}_0 M} \varphi(y)\mu(dy) \geq \int_{\mathrm{Lk}_0 M} \partial\varphi_0(a)\cos\angle_0(a,y)\mu(dy) \geq 0$. □

Proposition 3.4.4 *Let M be a Euclidean polyhedron of nonpositive curvature and let X be a semimartingale. Let $S \in \mathcal{S}$ for some isometric triangulation. Then the following are equivalent:*

- *(i) X satisfies* $\mathbf{M}(S)$.
- *(ii) For all $a \in \mathrm{Lk}_0{\perp}S$, $\int_{\mathrm{Lk}_0{\perp}S} \cos\angle_0(a,y)L^S_t(dy)$ is nonincreasing in t.*

Proof : A standard approximation argument, using Lemma 3.4.3. □

Example 3.4.5 Let M be a $k-$star, i.e. a one-dimensional simplicial cone complex. We may regard M as a Euclidean simplicial cone complex, where on each ray of M we have the standard Euclidean metric on $[0,\infty[$. Now Lk_0M is a discrete set. More precisely, to each ray R_i of M corresponds exactly one point $u_i \in \mathrm{Lk}_0M$ and for all $u_i, u_j \in \mathrm{Lk}_0M$, $\cos\angle_0(u_i,u_j)$ is equal to 1 if $i=j$ and equal to -1 if $i \neq j$. So we get a very simple version of Lemma 3.4.3: Let μ be a nonnegative measure on Lk_0M. Then the following are equivalent:

(i) $0 \in \mathbb{B}(\mu)$.

(ii) For all $u_i \in \mathrm{Lk}_0M$, $-\mu(u_i) + \sum_{j\neq i}\mu(u_j) \geq 0$.

(iii) For all $u_i \in \mathrm{Lk}_0M$, $\mu(u_i) \leq \frac{1}{2}\mu(\mathrm{Lk}_0M)$.

So we deduce that for a semimartingale X the following are equivalent:

(i) X satisfies $\mathbf{M}(0)$.

(ii) For all $u_i \in \mathrm{Lk}_0M$, $-L^{0,u_i} + \sum_{j\neq i} L^{0,u_j}$ is nondecreasing.

(iii) For all $u_i \in \mathrm{Lk}_0M$, $dL^{0,u_i} \leq \frac{1}{2}dL$, where $L := \sum_{u_j\in\mathrm{Lk}_0M} L^{0,u_j}$.

Note that characterization (iii) is exactly the martingale condition in [Pic05], Proposition 3.3.4. An exhaustive discussion about martingales in stars can be found in that paper.
From the above characterization it is easily seen that a *spider-martingale* (in the sense of [Yor97], Definition 17.2) is a martingale. The reverse is not true in general, cf. [Pic05], section 3.3.

The next Proposition says that if M has nonpositive curvature, we may omit the condition that X is a semimartingale in Theorem 3.1.5 (i), which will be useful for applications as e.g. to harmonic maps.

Proposition 3.4.6 *Let M be a Euclidean polyhedron of nonpositive curvature. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, P)$ be a filtered probability space and let X be a continuous adapted process such that $\varphi(X)$ is a local submartingale on $\{X \in O\}$ for all Lipschitz continuous convex functions $\varphi : O \to \mathbb{R}$. Then X is a semimartingale and hence a local martingale.*

Proof : By localization, we may assume that M is a simplicical cone complex equipped with a fixed triangulation $\mathcal{S}$. Then by Proposition 2.1.8, it suffices to show that for all $u \in \mathrm{scaff}(M)$, $\nu^u(X)$ is a semimartingale. We will do this by representing ν^u in terms of convex functions and certain operations that respect

the semimartingale property.
Namely, let $S \in \mathcal{S}$. For $v \in \mathrm{scaff}(S)$, set $\varphi^u(x) := -d(0,x)\cos\angle_0(v,x)$. Then φ^v is convex by Lemma 3.4.2. We claim that for all $u \in \mathrm{scaff}(S)$ there are (unique) $\xi_S^{uv} \in \mathbb{R}$ ($v \in \mathrm{scaff}(S)$) such that

$$\nu^u_{|S} = \sum_{v\in\mathrm{scaff}(S)} \xi_S^{uv}\varphi^v_{|S}. \tag{3.26}$$

Indeed, we may regard φ^v as a linear function on U^S (the linear subspace generated by S), namely $\varphi^v(x) = -g_S(x,v)$ (cf. Proposition 1.4.4). Then the linear functions $\{\varphi^v : v \in \mathrm{scaff}(S)\}$ are linearly independent and hence a basis of $(U^S)^*$. Thus (3.26) holds.
Let now $u \in \mathrm{scaff}(M)$. For each $S \in \mathrm{st}^{(n)}(u)$, set $f^S := \sum_{v\in\mathrm{scaff}(S)} \xi_S^{uv}\varphi^v$, so $f^S_{|S} = \nu^u_{|S}$. Note that $f^S(X)$ is a semimartingale because f^S is a linear combination of the convex functions φ^v. Moreover, we have $-\nu^u = \min\{\max_{S\in\mathrm{st}^{(n)}(u)} f^S, 0\}$. Thus using the fact that if Y and Z are real semimartingales, then $\max\{Y,Z\}$ (and hence also $\min\{Y,Z\}$) is a semimartingale, we obtain that $\nu^u(X) = -\min\{\max_{S\in\mathrm{st}^{(n)}(u)} f^S(X), 0\}$ is a semimartingale. This proves the Proposition. □

Let us now come to one of the central results in this work, namely a broad characterization of local martingales in a Euclidean complex of nonpositive curvature. To this end, we need the notion of a *strong martingale*, cf. Definition 4.2.6 and Remark 4.2.7: Let Δ^n be a refining sequence of partitions of $\mathbb{R}_+$ and put $\mathbb{T} := \bigcup_{n\in\mathbb{N}} \Delta^n$. a process $(X_t)_{t\in\mathbb{T}}$ is a strong martingale if it can be approximated by a sequence of *discretized* martingales, i.e. if there is a sequence $(\eta^n_{t_k})_{k\in\mathbb{N}}$ of processes such that $(\eta^n_{t_k})_{t_k\in\Delta}$ is a discrete time martingale w.r.t. the filtration $(\mathcal{F}_{t_k})_{t_k\in\Delta^n}$ and $\eta^n_t \to X_t$ in L^1 for all $t \in \mathbb{T}$.

Strong martingales are the subject of chapter 4 and are studied there in great detail. We do not want to give too many details here, but we remark that due to Theorem 4.4.2 we have a characterization of strong martingales that may serve as a definition in our particular setting of semimartingales: Let X be a semimartingale in M and let Δ^k be a sequence of locally finite partitions of $\mathbb{R}$ such that $\|\Delta^k\| \to 0$ as $k \to \infty$. Then by Corollary 2.3.8,

$$V^k_t := \sum_{t_l\in\Delta^k} d^2(X_{t_l\wedge t}, X_{t_{l+1}\wedge t}) \to \langle X\rangle_t$$

locally uniformly in t in probability. This means that $\langle X\rangle := \int g(dX,dX)$ is the *quadratic variation of* X.
In particular, we can find a sequence τ_j of stopping times with $\tau_j \nearrow \infty$ such that

$X_{t\wedge\tau_j}$ is bounded and $V^k_{t\wedge\tau_j} \to \langle X\rangle_{t\wedge\tau_j}$ in L^1 for every $t \in \mathbb{R}_+$.
By Theorem 4.4.2, the stopped process X^{τ_j}, defined by $X^{\tau_j}_t := X_{t\wedge\tau_j}$ is a strong martingale if and only if $d^2(z, X_{t\wedge\tau_j}) - \langle X\rangle_{t\wedge\tau_j}$ is a submartingale for all $z \in M$. If X^{τ_j} is a strong martingale for all j, then X is called a *local strong martingale*, cf. Definition 4.4.5.

Theorem 3.4.7 *Let M be a simply connected Euclidean polyhedron of nonpositive curvature. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, P)$ be a filtered probability space and let X be a continuous adapted process. Then the following are equivalent:*

(i) $\varphi(X)$ is a local submartingale on $\{X \in O\}$ for all Lipschitz continuous convex functions $\varphi : O \to \mathbb{R}$.

(ii) X is a local martingale.

(iii) X is a local strong martingale w.r.t. all sequences (Δ^k) of refining partitions of $[0,\infty[$ such that the mesh goes to 0.

(iv) X is a local strong martingale w.r.t. one sequence (Δ^k) of refining partitions of $[0,\infty[$ such that the mesh goes to 0.

In each of these cases, if X is also bounded, then X is a strong martingale.

Proof : *(i)* $\Rightarrow$ *(ii)*: By Proposition 3.4.6.
(ii) $\Rightarrow$ *(iii)*: Let (Δ^k) be a sequence of refining partitions of $[0,\infty[$ such that $\|\Delta^k\| \to 0$. By Corollary 2.3.8, $\sum_{t_l\in\Delta^k} d^2(X_{t_l}, X_{t_{l+1}}) \to \langle X\rangle$ locally uniformly in probability, and as in the discussion just before this Theorem, we see that there is a sequence $\tau_j \to \infty$ of stopping times such that the stopped process X^{τ_j} fulfills the assumptions of Theorem 4.4.2. Moreover, for any $z \in M$, the function $f^z(x) := d^2(z,x)$ is $2-$convex and hence $f^z(X^{\tau_j}) - \langle X\rangle^{\tau_j}$ is a local submartingale by Corollary 3.3.6. We may also assume that X^{τ_j} is bounded and hence $f^z(X^{\tau_j}) - \langle X\rangle^{\tau_j}$ is really a submartingale. Thus X^{τ_j} is a strong martingale by Theorem 4.4.2 and X is a local strong martingale by Definition 4.4.5.
(iii) $\Rightarrow$ *(iv)*: Trivial.
(iv) $\Rightarrow$ *(i)*: We may localize X by a suitable sequence of stopping times and assume that X is a strong martingale w.r.t. (Δ^k). Moreover, we may assume that X has only values in some convex neighborhood $U \subset\subset O$ (so $\overline{U}$ is an NPC space itself). Let $\mathbb{T} := \bigcup_{k\in\mathbb{N}} \Delta^k$. Then by Theorem 4.2.16, $\varphi(X_t) \leq \mathbf{E}^{\mathcal{F}_s}[\varphi(X_t)]$ for all $s,t \in \mathbb{T}$ with $s \leq t$. By continuity of X we can extend this to all $s \leq t \in [0,\infty[$.
The last assertion is just a reformulation of Proposition 4.4.6. $\square$

3.5 Application to harmonic maps

Now we will apply our results to the theory of harmonic maps, as developed in [EF01]. See also [Hes04] for harmonic maps into trees. Throughout this section, K will denote a compact $n-$dimensional admissible Riemannian polyhedron with nonempty boundary[10].

Let (N,ρ) be a complete separable metric space. Denote by $\mu = dx$ the Riemannian volume measure on K and by $L^2(K,N)$ the set of all maps $f : K \to N$ such that the function $\int_K d^2(f(x),y)dx < \infty$ for some (and hence all) $y \in N$. For $\epsilon > 0$ and $f \in L^2(K,N)$, define the *approximate energy density* of f by

$$e_\epsilon(f)(x) := \int_{B_\epsilon(x)} \frac{\rho^2(f(x),f(y))}{\epsilon^{n+2}} dy \tag{3.27}$$

and the *energy* of f by

$$E(f) := \limsup_{\epsilon \to 0} \int_K e_\epsilon(f)(x)dx. \tag{3.28}$$

The space $W^{1,2}(K,N)$ is by definition the space of all maps $f \in L^2(K,N)$ whose energy is finite.

This concept of nonlinear energy is due to [KS93] and is the basis of the theory of harmonic maps in [EF01], where it is studied in great detail. For instance, it is shown in Theorem 9.1 of [EF01] that if $f \in W^{1,2}(K,N)$, then $e_\epsilon(f)$ converges weakly to some function $e(f) \in L^1(K,\mu)$, and consequently the limsup in (3.28) is actually a limit. Moreover, note that if N is a Riemannian polyhedron itself, then $E(f)$ has a nice local description that generalizes the concept of energy of maps between Riemannian manifolds, cf. [EF01], Definition 9.3. and Theorem 9.3.

Definition 3.5.1 A map $f : K \to N$ is called *harmonic* if f is continuous and a local minimizer of E.

Remark 3.5.2 In order to avoid confusion, we remark the following difference between our notation and the one of Eells and Fuglede: In their setting, harmonicity is only defined for maps $h : M \to N$ (where M is an admissible Riemannian polyhedron), so f is defined *entirely on* M. But in our setting, K is compact and hence if N has nonpositive curvature, then a harmonic map $h : M \to N$ must be constant, cf. [EF01], Remark 12.3. However, K° itself is a noncompact admissible Riemannian polyhedron (cf. [Fug05b], Example 1) and hence the definition of harmonic maps in the sense of Fuglede is coherent with ours.

[10]The assumption that K is compact is not really essential for the following, but it makes things simpler

For a state-of-the-art survey about harmonic maps, see [Fug05a]. We will quote two important results of that paper:

Proposition 3.5.3 *Let K be a compact admissible Riemannian polyhedron and let N be a simply connected complete metric space of nonpositive curvature.*
(i) For every continuous map $f : \partial K \to N$ there is a unique continuous map $h^f : K \to N$ such that h is harmonic in K° and $h^f_{|\partial K} \equiv f$.
(ii) For every continuous convex function $\varphi : O \to \mathbb{R}$ ($O \subset N$), $\varphi \circ f$ is subharmonic in $U := K^\circ \cap f^{-1}(O)$.

Proof : (i): [Fug05a], Theorem 1 (a).
(ii): [Fug05a], Theorem 2 (b). □

Now we can state the main result of this section: In the case that N is a Euclidean polyhedron, we get a broad characterization of harmonic maps, including *Ishihara's characterization*:

Theorem 3.5.4 *Let K be a compact admissible Riemannian polyhedron with nonempty boundary and let N be a simply connected Euclidean polyhedron of nonpositive curvature. Let X be Brownian motion in K and let $h : K \to N$ be a continuous map. Then the following are equivalent:*

(i) h is harmonic in K°

(ii) For every continuous convex function $\varphi : O \to \mathbb{R}$ ($O \subset N$), $\varphi \circ h$ is subharmonic in $U := K^\circ \cap h^{-1}(O)$

(iii) $h(X)$ is a local martingale on $\{X \in K^\circ\}$ w.r.t. P^x for every $x \in K^\circ$.

(iv) $h(X)$ is a strong martingale on $\{X \in K^\circ\}$ w.r.t. P^x for every $x \in K^\circ$.

Proof : $(i) \Rightarrow (ii)$ follows from Proposition 3.5.3
$(ii) \Rightarrow (iii)$: Set $Y := h(X)$. Since $\varphi \circ h$ is subharmonic in U, $\varphi(Y)$ is a submartingale on $\{X \in U\} = \{Y \in O\} \cap \{X \in K^\circ\}$. Thus by definition, Y is a local martingale on $\{X \in K^\circ\}$ (cf. Theorem 3.3.4).
$(iii) \Rightarrow (iv)$: Since $h(X)$ is a local martingale on $\{X \in K^\circ\}$, $h(X)$ is also a strong martingale on $\{X \in K^\circ\}$ by Theorem 3.4.7 (note that $h(X)$ is bounded).
$(iv) \Rightarrow (i)$ Let h as in (iv). By Proposition 3.5.3 (i), there is a continuous function $\widetilde{h} : K \to N$ such that $\widetilde{h}$ is harmonic in K° and $\widetilde{h}_{|\partial K} = h_{|\partial K}$. So in order to show that h is harmonic in K°, it suffices to show that $h \equiv \widetilde{h}$.
So let $x \in K^\circ$. We already know that $\widetilde{h}(X)$ is a strong martingale on $\{X \in K^\circ\}$ w.r.t. P^x. Let $\tau = \tau_{K^\circ}$ be the first exit time from K°. Moreover, let $x \in U_j \subset\subset K^\circ$ be a sequence of relatively compact domains that increases to K° such that

$\tau_{U_j} \nearrow \tau$. Hence $\widetilde{h}(X_{\tau_j}) \to \widetilde{h}(X_\tau)$ and $h(X_{\tau_j}) \to h(X_\tau)$ as $j \to \infty$.
Denote by $D := d(h(X), \widetilde{h}(X))$ the distance process. Now for every $j \in \mathbb{N}$, the stopped processes $\widetilde{h}(X^{\tau_j})$ and $h(X^{\tau_j})$ are strong martingales and hence the stopped distance process D^{τ_j} is a submartingale (w.r.t. P^x) by Proposition 4.2.8. Moreover, because h is bounded, D^{τ_j} is uniformly bounded in j and so D extends to a bounded submartingale on $[0, \tau]$. But $h(X_\tau) = \widetilde{h}(X_\tau)$, which means that $D_\tau = 0$ and hence $D \equiv 0$. In particular, $D_0 = d(h(x), \widetilde{h}(x)) = 0$. Since $x \in K^\circ$ was arbitrary, $h \equiv \widetilde{h}$, and the Theorem is proved. $\square$

Chapter 4

Expectations and Martingales in Metric Spaces

In this chapter we use the approach of generalized expectations (so-called barycenter maps). Section 4.1 gives a general discussion of barycenters and conditional expectations with focus on the definition of barycenter in [Stu03].
In section 4.2 we introduce the notion of strong martingales in terms of iterated (=*filtered*) conditional expectations. First we study the case of time-discrete martingales. In the time-continuous we take a sequence Δ^n of partitions of the time axis whose mesh converges to and define a process to be a *strong martingale* if it can be approximated by a sequence of time-discrete martingales. In both sections we treat separately the 'main example', namely NPC spaces, in which there is a canonical barycenter that features very nice geometric properties such as Jensen's inequality, cf. [Jos97]). In NPC spaces there was developed an exhaustive theory for time-discrete martingales in [Stu02].
Section 4.3 is concerned with the classical problem of finding a martingale with a prescribed terminal value, cf. e.g. [Ken90], [Pic91], [Arn95], [Pic05].
In section 4.4 we present a characterization of strong martingales: A continuous Process (X_t) in an NPC space is a strong martingale if it has a quadratic variation (V_t) (i.e. $\sum_{t_k \in \Delta^n} d^2(X_{t_k \wedge t}, X_{t_{k+1} \wedge t}) \to V_t$ for $n \to \infty$) and $d^2(X_t, z) - V_t$ is a submartingale for all $z \in N$ (Thm. 4.4.2). This characterization has many applications: For example, if N is a Riemannian manifold of nonpositive sectional curvature and X a continuous semimartingale, then X is a strong martingale if and only if X is a martingale in the classical sense. Moreover, Theorem 4.4.2 is the main tool in the proof of our martingale characterization Theorem 3.4.7.

4.1 Expectations and conditional expectations in metric spaces

Let (N, d) be a separable metric space. Sturm ([Stu03]) defined a barycenter map to be a map $b : \mathcal{P}^1(N) \to N$ that is a contraction w.r.t the L^1-Wasserstein distance, where $\mathcal{P}^1(N)$ is the space of probability measures on N with finite mean distance to points. In Alexandrov spaces of nonpositive curvature (for short: NPC spaces) there is a canonical barycenter (cf. [Jos97]).
Let $(\Omega, (\mathcal{F}_k)_{k\in\mathbb{N}}, \mathcal{F}, P)$ be a filtered probability space, N be a separable metric space with a barycenter map b on it and $\xi : \Omega \to N$ be an integrable random variable. Then the regular conditional probability $P_{\xi|\mathcal{F}_k} : \Omega \to \mathcal{P}^1(N)$ of ξ given $\mathcal{F}_k$ is measurable w.r.t. the Borel-$\sigma-$algebra generated by d_W (Prop. 4.1.12). Thus one can define the conditional expectation simply by $E[\xi|\mathcal{F}_k] := b \circ P_{\xi|\mathcal{F}_k}$, and contraction properties of the barycenter easily carry over to the conditional versions.
Although in general metric spaces other barycenter maps are possible (cf. e.g. [ESH99]), in Hilbert spaces there is only one barycenter map, the usual expectation (Rem. 4.1.10).
Let (N, d) be a separable metric space and let $\mathcal{P}(N)$ denote the set of all probability measures p on $(N, \mathcal{B}(N))$. The most common topology on $\mathcal{P}(N)$ is the topology of weak convergence. A sequence p_n is said to converge weakly to p if $\int f dp_n \to \int f dp$ for every continuous bounded function f. This topology is induced by the *Prohorov metric* d^P, which is defined by

$$d^P(p, q) := \inf\{r > 0 : p(A) \leq q(A^r) + r \text{ for all } A \in \mathbb{B}(N)\}$$

where $A^r := \{y \in N : d(x, y) < r \text{ for some } x \in A\}$ is the $r-$neighborhood of A. There is another metric which is equivalent to d^P. For a function $u : N \to \mathbb{R}$ we define the Lipschitz seminorm

$$\mathrm{Lip}(u) := \sup_{x\neq y} \frac{|u(x) - u(y)|}{d(x, y)}.$$

and for $p, q \in \mathcal{P}(N)$

$$d^L(p, q) := \sup\{\int_N udp - \int_N udq : \mathrm{Lip}(u) + \|u\|_\infty \leq 1\}.$$

d^P and d^L are indeed metrics on $\mathcal{P}(N)$, see [Dud89], Theorem 11.3.1 and Proposition 11.3.2.

Given p, $q \in \mathcal{P}(N)$ we say that $\mu \in \mathcal{P}(N^2)$ is a *coupling of p and q* (short: $\mu \in \mathcal{M}(p,q)$) if its marginals are p and q, i.e. if

$$\mu(A \times N) = p(A) \quad \text{and} \quad \mu(N \times A) = q(A) \qquad (\forall A \in \mathcal{B}(N)).$$

Proposition 4.1.1 *(i) d^P and d^L both induce the weak topology on $\mathcal{P}(N)$. If d is a complete metric, then so are d^P and d^L.*
(ii) For $p, q \in \mathcal{P}(N)$,

$$d^P(p,q) = \inf_{\mu \in \mathcal{M}(p,q)} \inf\{r > 0 : \mu\{(x,y) : d(x,y) \geq r\} \leq r\}$$

Proof : (i) See [Dud89], Theorem 11.3.3 and Corollary 11.5.5.
(ii) See [Dud89], Corollary 11.6.4.

The identity in Proposition 4.1.1 (ii) says that $d^P(p,q)$ is the stochastic distance between the coordinate projections under an optimal coupling. Instead of stochastic distance we can consider L^p-distance. For $1 \leq \theta < \infty$, let $\mathcal{P}^\theta(N)$ denote the set of $p \in \mathcal{P}(N)$ with $\int d^\theta(x,y)p(dy) < \infty$ for some (and hence all) $x \in N$, and $\mathcal{P}^\infty(N)$ will denote the set of all $p \in \mathcal{P}(N)$ with bounded supp(p). Obviously, $\mathcal{P}^\infty(N) \subset \mathcal{P}^\theta(N) \subset \mathcal{P}^1(N)$.

We define the (L^θ-) *Wasserstein-distance* d^W_θ on $\mathcal{P}^\theta(N)$ by

$$d^W_\theta(p,q) := \inf\left\{ \left(\int_{N^2} d^\theta(x,y)\mu(d(x,y)) \right)^{1/\theta} : \mu \in \mathcal{M}(p,q) \right\}.$$

We shall quote a well-known result concerning the Wasserstein distance, for a proof see e.g. [Vil03]:

Proposition 4.1.2

(i) d^W_θ is a metric on $\mathcal{P}^\theta(N)$. The set of discrete (i.e. finitely supported) probability measures is dense in $\mathcal{P}^\theta(N)$.
If d is complete, then so is d^θ and for each pair p, $q \in \mathcal{P}^\theta(N)$ there exists an optimal coupling, that is, a coupling μ of p and q for which

$$d^W_\theta(p,q) = \left(\int_{N^2} d^\theta(x,y)\mu(d(x,y)) \right)^{1/\theta}.$$

(ii) (Kantorovich-Rubinstein-duality) For all p, $q \in \mathcal{P}^1(N)$

$$d^W_1(p,q) = \sup\left\{ \int_N u(x)p(dx) - \int_N u(y)q(dy) : \mathrm{Lip}(u) \leq 1 \right\}.$$

Remark 4.1.3 The Wasserstein functional can be considered in a much more general setting. Let $p, q \in \mathcal{P}(N)$ and $h : N \times N \to \mathbb{R}$ be a measurable function such that h is intergrable for some $\mu \in \mathcal{M}(p, q)$. We put

$$h^W(p, q) := \inf\left\{\int_{N^2} h(x, y)\mu(d(x, y) : \mu \in \mathcal{M}(p, q)\right\}.$$

Then if h is symmetric, then so is h^W, and if h satisfies the triangle inequality, then so does h^W.
The Kantorovich-Rubinstein duality does not hold for general functionals. For instance, if M is compact, then Proposition 4.1.2 (ii) holds for h if and only if h is a pseudometric (cf. [Dud89], Lemma 11.8.6; for a more general approach see [Vil03], or [RR98], Section 4.5.).

Lemma 4.1.4 *Let $(M, \mathcal{F})$, $(N, \mathcal{G})$ be measurable spaces such that M is a polish space and $\mathcal{F} \subset \mathbb{B}(M)$. Let $f : M \to N$ and $h : N \times N \to \mathbb{R}$ be measurable maps. Define $h_f(x, y) := h(f(x), f(y))$. Then for all $p^1, p^2 \in \mathcal{P}(M)$ such that h_f is intergrable for some $\mu \in \mathcal{M}(p^1, p^2)$ we have*

$$h_f^W(p^1, p^2) = h^W(p^1 \circ f^{-1}, p^2 \circ f^{-1}).$$

Proof : Let $\mu \in \mathcal{M}(p^1, p^2)$. Then $\mu \circ (f^{-1}, f^{-1}) \in \mathcal{M}(p^1 \circ f^{-1}, p^2 \circ f^{-1})$ and

$$\int_{N\times N} h(y_1, y_2)[\mu \circ (f^{-1}, f^{-1})](d(y_1, y_2)) = \int_{M\times M} h_f(x_1, x_2)\mu(d(x_1, x_2))$$

which implies that $h_f^W(p^1, p^2) \geq h^W(p^1 \circ f^{-1}, p^2 \circ f^{-1})$.
For the other inequality put $\mathcal{A} := f^{-1}(\mathcal{G})$. Then for $i = 1, 2$, the regular conditional probability $p^i_{id|\mathcal{A}}$ exists, since M is polish. Let $A \in \mathbb{B}(M)$. Since $x \mapsto p^i_{id|\mathcal{A}}(x, A)$ is $\mathcal{A}-$measurable, there is a $\mathcal{G}-$measurable map $K^i_A : N \to [0, 1]$ such that $p^i_{id|\mathcal{A}}(x, A) = K_A(f(x))$ for all $x \in M$. Note that K^i is a Markov kernel from $(N, \mathcal{G})$ to Ω. Let now $\nu \in \mathcal{M}(p^1 \circ f^{-1}, p^2 \circ f^{-1})$. We define a probability measure μ on $M \times M$ by

$$\mu(A_1 \times A_2) := \int_{N\times N} K^1_{A_1}(y_1)K^2_{A_2}(y_2)\nu(d(y_1, y_2)).$$

It is easy to see that $\mu \in \mathcal{M}(p^1, p^2)$. Moreover, since $K^i_{f^{-1}(B)}(y) = \mathbf{1}_B(y)$ for $(p^i \circ f^{-1})-$ almost all $y \in N$ and all $B \in \mathcal{G}$, it follows that $\mu = \nu \circ (f^{-1}, f^{-1})$ and hence

$$\int_{M\times M} h(f(x_1), f(x_2))\mu(d(x_1, x_2)) = \int_{N\times N} h(y_1, y_2)\nu(d(y_1, y_2)).$$

Thus, $h_f^W(p^1, p^2) \leq h^W(p^1 \circ f^{-1}, p^2 \circ f^{-1})$. □

From Proposition 4.1.1 (ii) and Proposition 4.1.2 (ii) we see that $d^L \leq d_1^W$. In general, the topologies are different. For example, let $N = \mathbb{R}$. Let $p_n := (n-1)/n\delta_{\{0\}} + 1/n\delta_{\{n\}}$. Then $p_n \to \delta_{\{0\}}$ weakly, while $d^W(p_n, \delta_{\{0\}}) \equiv 1$.
However, if N is bounded, then $d_1^W \leq (\operatorname{diam}(N) + 1)d^L$, and the topologies coincide. The next proposition says that the Borel σ-fields on $\mathcal{P}^1(N)$ induced by the weak topology and d_1^W are the same:

Proposition 4.1.5 *Let $p \in \mathcal{P}^1(N)$. Then the map $q \mapsto d^W(p, q)$ is measurable w.r.t $\mathbb{B}(\mathcal{P}^1(N))$, the Borel σ-field induced by the weak topology.*

Proof : Let

$$d_n(p, q) = \sup \left\{ \int_N u(x)p(dx) - \int_N u(y)q(dy) : \operatorname{Lip}(u) + \|u\|_\infty \leq n \right\}.$$

Then d_n is a metric on $\mathcal{P}^1(N)$ and $d_n \leq (n+1)d^L$. Hence $q \mapsto d_n(p, q)$ is continuous w.r.t weak convergence. By Proposition 4.1.2 (ii) and truncation, $d_1^W(p, q) = \sup d_n(p, q)$, proving the claim. □

From now on, we will concentrate on the case $\theta = 1$. We will write $d^W := d_1^W$. For $p \in \mathcal{P}^1(N)$ we want to define an expectation.

Definition 4.1.6 A *barycenter map* is a map $b : \mathcal{P}^1(N) \to N$ satisfying

(i) $b(\delta_x) = x$ for all $x \in N$

(ii) $d(b(p), b(q)) \leq d^W(p, q)$ for all $p, q \in \mathcal{P}^1(N)$

The point $b(p) \in N$ is called *barycenter* of the probability measure p. If $X : (\Omega, P) \to N$ is a random variable, then $\mathbf{E}[X] := b(P^X)$ is called the *expectation* of X, where $P^X := P \circ X^{-1}$. A triple (N, d, b), where (N, d) is a complete metric space and b is a barycenter map on it, is called *barycentric metric space* or *barycenter space*.

Example 4.1.7
(i) Global NPC spaces are barycenter spaces. For the definition of NPC spaces see section 1.3 below.
(ii) Banach spaces are barycenter spaces. A construction of a barycenter map on Banach spaces is given in [LT91].

We will now quote some geometric properties of barycenter spaces.

Definition 4.1.8 A metric space (N, d) is called *geodesic space* if for all $x, y \in N$ there is a curve $\gamma : [0,1] \to N$ with $\gamma(0) = x, \gamma(1) = y$ and $d(\gamma(s), \gamma(t)) = |t-s| d(x,y)$ for all $s, t \in [0,1]$. Such a γ is called a *geodesic.*

Proposition 4.1.9 *(i) A barycenter space is a geodesic space. For $x, y \in N$, a geodesic from x to y is given by $\gamma(t) := b((1-t)\delta_x + t\delta_y)$.*

(ii) Let $x_i, y_i \in N$ $(i = 1,2)$ and let γ_i be the 'barycentric' geodesic from x_i to y_i, defined as in (i). Then the function $t \mapsto d(\gamma_1(t), \gamma_2(t))$ is convex.

Proof : A simple computation. □

Remark 4.1.10 (i) The existence of a barycenter map on N can be regarded as an upper curvature bound for N: If N is a geodesic space such that geodesics are unique, then Proposition 4.1.9 (ii) implies that N has convex geometry, i.e. the map $(x, y) \mapsto d(x, y)$ is convex on N^2. Hence, N has (globally) nonpositive curvature in the sense of Busemann (see [Jos97]). For instance, if N is a simply connected Riemannian manifold which has a barycenter map, then N has nonpositive sectional curvature. Conversely, a simply connected Riemannian manifold with nonpositive curvature is an NPC space and hence a barycenter space.

(ii) In a Euclidean (or more generally, Hilbert) space E there is only one barycenter, the usual integral. Indeed, let $p \in \mathcal{P}^2(E)$, $x \in E$. Denote by $\pi(p) := \int y p(dy)$ the expectation and by $V(p) := \int |y - \pi(p)|^2 p(dy) = d_2^2(p, \pi(p))$. Then

$$d_2^W(p, x) = \sqrt{|x - \pi(p)|^2 + V(p)}.$$

Now $\sqrt{t^2 + v} - t \to 0$ as $t \to +\infty$ and hence for all $\epsilon > 0$ there is an $R > 0$ such that $d_2^W(p, x) \le |x - \pi(p)| + \epsilon$ and $d_2^W(p, 2\pi(p) - x) \le |x - \pi(p)| + \epsilon$ for all $x \in E \setminus B_R(\pi(p)$. If b is a barycenter map, then $|b(p) - x| \le |x - \pi(p)| + \epsilon$ and $|b(p) - (2\pi(p) - x)| \le |x - \pi(p)| + \epsilon$. Moreover, x can be chosen such that $\pi(p), b(p), x$ and $2\pi(p) - x$ are on one line. Letting $\epsilon \to 0$ yields that $b(p) = \pi(p)$. □

(iii) In general, there may be more than one barycenter map on a metric space. For example, in [ESH99] was constructed a barycenter map in proper metric spaces of nonpositive curvature in the sense of Busemann (cf. [Jos97]).

4.1.1 Conditional probabilities and expectations

Let $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}')$ be to measurable spaces. Recall that a *Markovian transition kernel* (or *Markov kernel*) from $(\Omega, \mathcal{F})$ to $(\Omega', \mathcal{F}')$ is a function $K : \Omega \times \mathcal{F}' \to [0,1]$ such that

(i) For each $\omega \in \Omega$, $K(\omega, \cdot)$ is a probability measure on $\mathcal{F}'$.

(ii) For each $A' \in \mathcal{F}'$, $K(\cdot, A')$ is $\mathcal{F}-$measurable.

By (i), a Markov kernel defines a map $K : \Omega \to \mathcal{P}((\Omega', \mathcal{F}'))$, the set of probability measures on $\mathcal{F}'$. We will prove that if $(\Omega', \mathcal{F}') = (N, \mathbb{B}(N))$ where N is a polish space, the the map $K : (\Omega, \mathcal{F}) \to (\mathcal{P}(N), \mathbb{B}(\mathcal{P}(N)))$ is measurable, where $\mathbb{B}(\mathcal{P}(N))$ is the Borel $\sigma-$algebra induced by the weak topology on $\mathcal{P}(N)$. For this purpose we need a lemma.
Let (N, d) be a separable metric space. It is known (e.g. [Dud89], Thm. 2.8.2) that there is a totally bounded metric $\tilde{d}$ on N, inducing the same topology as d. (Totally bounded means that the completion $\hat{N}$ w.r.t $\tilde{d}$ is compact.) So we can assume that d is totally bounded itself.
Let $\hat{D}$ be a countable topological base of $\hat{N}$. Let $\hat{E} := \{\hat{N} \setminus O : O \in \hat{D}\}$. Then every set $\hat{A}$ that is closed in $\hat{N}$ is of the form $\hat{A} = \bigcap_{n\in\mathbb{N}} \hat{A}_n$ with $\hat{A}_n \in \hat{E}$. Moreover we can assume without restriction that $\hat{A}_{n+1} \subset \hat{A}_n$.
Recall that for a set $A \subset N$, A^r denotes the $r-$neighborhood of A.

Lemma 4.1.11 *Let (N, d) be a metric space such that d is totally bounded. Let $\hat{E}$ be a countable collection of closed sets in $\hat{N}$ (the completion of N) such that every set $\hat{A}$ that is closed in $\hat{N}$ is of the form $\hat{A} = \bigcap_{n\in\mathbb{N}} \hat{A}_n$ with $\hat{A}_n \in \hat{E}$ and $\hat{A}_{n+1} \subset \hat{A}_n$. Let $p, q \in \mathcal{P}(N)$ and $R > 0$.*
Then the following are equivalent:

(i) $d^P(p, q) < R$, i.e. there is some $r < R$ such that $p(A) \leq q(A^r) + r$ for all $A \in \mathbb{B}(N)$.

(ii) there is some $r < R$ such that $p(A) \leq q(A^r) + r$ for all A that are closed in N.

(iii) there is some $r < R$ such that $p(A) \leq q(A^r) + r$ for all $A \in E := \{\hat{A} \cap N : \hat{A} \in \hat{E}$.

Proof : $(i) \Rightarrow (ii) \Rightarrow (iii)$ is trivial.
$(ii) \Rightarrow (i)$ is clear, since $A^r = (\bar{A})^r$ for every set $A \subset N$.
$(iii) \Rightarrow (ii)$: a) Let first $N = \hat{N}$, i.e. N is compact. Let $A = \bigcap_{n\in\mathbb{N}} A_n$, where $A_n \in E$. Let $y \in \bigcap_{n\in\mathbb{N}} A_n^r$. Then there are $x_n \in A_n$ with $d(x_n, y) < r$ for all $n \in \mathbb{N}$. Since N is compact, there is a subsequence converging to some $x \in A$ and $d(x, y) \leq r$. Hence $\bigcap_{n\in\mathbb{N}} A_n^r \subset A^{\tilde{r}}$ for all $\tilde{r} > r$. Thus, if $p(A_n) \leq q(A_n^r) + r$, then

$$p(A) = \inf p(A_n) \leq \inf q(A_n^r) + r = q(\bigcap_{n\in\mathbb{N}} A_n^r) + r \leq q(A^{\tilde{r}}) + \tilde{r}$$

for all $\tilde{r} > r$, which implies that (ii) holds with some $r < \tilde{r} < R$.

b) For the general case define probability measures $\hat{p}, \hat{q}$ on $\mathcal{B}(\hat{N})$ by $\hat{p}(\hat{B}) := p(\hat{B} \cap N)$ and $\hat{q}$ similarly. (Note $\mathcal{B}(N) = \{\hat{B} \cap N : \hat{B} \in \mathcal{B}(\hat{N})\}$.)
By (iii), there is an $r < R$ such that

$$\hat{p}(\hat{A}) = p(\hat{A} \cap N) \leq q((\hat{A} \cap N)^r) + r \leq q(\hat{A}^r \cap N) + r = \hat{q}(\hat{A}^r) + r$$

for all $\hat{A} \in \hat{E}$. (The reader should always care wether the $r-$neighborhood is taken in N or in $\hat{N}$.) Hence, a) implies that there is some $r < R$ such that $\hat{p}(\hat{A}) \leq \hat{q}(\hat{A}^r) + r$ for all sets $\hat{A}$ that are closed in $\hat{N}$.
Let now A be closed in N. Let $\hat{A}$ be its closure in $\hat{N}$. It is easy to see that $A^r = \hat{A}^r \cap N$. Hence

$$p(A) = \hat{p}(\hat{A}) \leq \hat{q}(\hat{A}^r) + r = q(A^r) + r$$

and (ii) follows. □

Proposition 4.1.12 *Let $(\Omega, \mathcal{F})$ be a measurable space, (N, d) a separable metric space and $K : \Omega \to \mathcal{P}(N)$. Then K is a Markov kernel from $(\Omega, \mathcal{F})$ to $(N, \mathbb{B}(N))$ (i.e. the map $\omega \mapsto K(\omega)(A)$ is measurable for all $A \in \mathbb{B}(N)$) if and only if the map $K : (\Omega, \mathcal{F}) \to (\mathcal{P}(N), \mathbb{B}(\mathcal{P}(N)))$, defined by $K(\omega) := K(\omega, \cdot)$, is measurable.*

Proof : First the 'only if'-direction. Without restriction we can assume that d is a totally bounded metric on N. Let $p \in \mathcal{P}(N)$, $\omega \in \Omega$ and $R > 0$. Let E be as in Lemma 4.1.11 (iii). Then $d^P(p, K(\omega)) < R$ iff there is an $r < R, r \in \mathbb{Q}$ such that $p(A) \leq K(\omega)(A^r) + r$ for all $A \in E$. Hence

$$K^{-1}(B_R(p)) = \bigcup_{\substack{r<R \\ r\in\mathbb{Q}}} \bigcap_{A\in E} \{K(A^r) \geq p(A) - r\}$$

is in $\mathcal{F}$. Since $\mathbb{B}(\mathcal{P}(N))$ is generated by those balls, K is measurable.
For the 'if'-direction, let M denote the space of bounded measurable functions $f : N \to \mathbb{R}$ such that the function $p \mapsto \int f dp$ is measurable. Then M is stable under increasing limits. Moreover, by defintion of weak convergence, all continuous bounded functions lie in M. By a monotone class argument, M contains all bounded measurable functions. Thus, the map $p \mapsto p(A) = \int \mathbf{1}_A dp$ is measurable for all $A \in \mathbb{B}(N)$. □

Definition 4.1.13 Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(\Omega', \mathcal{F}')$ a measurable space and $X : \Omega \to \Omega'$ a measurable map. Moreover, let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. A *regular conditional probability* for X given $\mathcal{G}$ is a Markov kernel K from $(\Omega, \mathcal{G})$ to $(\Omega', \mathcal{F}')$ such that for all $B \in \mathcal{F}'$,

$$K(\cdot, B) = P[X \in B|\mathcal{G}] \qquad \text{a.s.}$$

Proposition 4.1.14 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, (N, d) a complete metric space and $X : \Omega \to N$ a random variable with separable support. Then a regular conditional probability for X given $\mathcal{G}$ exists and is unique in the sense that if K and $\tilde{K}$ are two such conditional probabilities, then there is a set $Z \in \mathcal{G}$ with $P(Z) = 0$ such that $K(\omega, A) = \hat{K}(\omega, A)$ for all $\omega \in \Omega \setminus Z$ and $A \in \mathbb{B}(N)$. We write $\mathbf{P}_{X|\mathcal{G}} := K$.*

Proof : See [Bau91], Satz 44.3. □

Let N, d be a metric space and $(\Omega, \mathcal{F}, P)$ a probability space. Let $L^\theta(\mathcal{F}, N)$ be the set of all $\mathcal{F}$-measurable $X : \Omega \to N$ such that $\mathbf{E}[d^\theta(X, z)] < \infty$ for some (and hence all) $z \in N$.

Definition 4.1.15 Let $(\Omega, \mathcal{F}, P)$ be a probability space, (N, d, b) a separable barycenter space and $X \in L^1(\mathcal{F}, N)$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra and let $K : \Omega \to \mathcal{P}(N)$ be the regular conditional probability for X given $\mathcal{G}$. Then $Y := b \circ K$ is called the *conditional expectation* of X, conditioned on $\mathcal{G}$. We write

$$\mathbf{E}^{\mathcal{G}}[X] := \mathbf{E}[X|\mathcal{G}] := Y.$$

Note that the conditional expectation is $\mathcal{G}$- measurable by the Propositions 4.1.5 and 4.1.12 and P-a.s. unique by Proposition 4.1.14.

Example 4.1.16 Let $\mathcal{G} \subset \mathcal{F}$ as above and let $\mathcal{H}$ be another $\sigma-$algebra. Assume that $X \in L^1(\mathcal{F} \cap \mathcal{H}, N)$, i.e. X is measurable w.r.t. $\mathcal{F}$ and $\mathcal{H}$. Then for all $B \in \mathbb{B}(N)$, $P[X \in B|\mathcal{G}] = P[X \in B|\mathcal{G} \cap \mathcal{H}]$, i.e. $\mathbf{P}_{X|\mathcal{G}} = \mathbf{P}_{X|\mathcal{G}\cap\mathcal{H}}$. Thus $\mathbf{E}[X|\mathcal{G}] = \mathbf{E}[X|\mathcal{G}\cap\mathcal{H}]$. This situation is particularly interesting when we are given a filtration $(\mathcal{F}_t)_{t\geq 0}$, a progressively measurable process $(X_t)_{t\geq 0}$ and a stopping time τ. If we set $\mathcal{F} := \mathcal{F}_t$, $\mathcal{G} = \mathcal{F}_s$ and $\mathcal{H}$, then $\mathcal{F}_{s\wedge\tau} = \mathcal{G} \cap \mathcal{H}$ and $\mathcal{F}_{t\wedge\tau} = \mathcal{F} \cap \mathcal{H}$ and hence $\mathbf{E}[X_{t\wedge\tau}|\mathcal{F}_s] = \mathbf{E}[X_{t\wedge\tau}|\mathcal{F}_{s\wedge\tau}]$.

As one may expect, the contraction property of a barycenter carries over to the corresponding conditional expectation, which is the content of the next

Proposition 4.1.17 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, (N, d, b) a separable barycenter space and $X \in L^1(\mathcal{F}, N)$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Then*

$$d(\mathbf{E}^{\mathcal{G}}[X], \mathbf{E}^{\mathcal{G}}[Y]) \leq \mathbf{E}^{\mathcal{G}}[d(X, Y)] \qquad a.s.$$

Proof : We denote by $\mathbf{P}_{X|\mathcal{G}}$ (resp. $\mathbf{P}_{Y|\mathcal{G}}$) the regular conditional probability of X (resp. Y) given $\mathcal{G}$. Moreover, we denote by $\mathbf{P}_{(X,Y)|\mathcal{G}}$ denote the regular conditional probability of (the $N \times N-$ valued map) (X,Y) given $\mathcal{G}$. Let $A, B \in \mathbb{B}(N)$. Then

$$\begin{aligned}\mathbf{P}_{(X,Y)|\mathcal{G}}(\cdot, A \times N) &= \mathbf{E}^{\mathcal{G}}[\mathbf{1}_{A\times N} \circ (X,Y)] \\ &= \mathbf{E}^{\mathcal{G}}[\mathbf{1}_A \circ X] = \mathbf{P}_{X|\mathcal{G}}(\cdot, A) \qquad \text{a.s.}\end{aligned}$$

and

$$\begin{aligned}\mathbf{P}_{(X,Y)|\mathcal{G}}(\cdot, N \times B) &= \mathbf{E}^{\mathcal{G}}[\mathbf{1}_{N\times B} \circ (X,Y)] \\ &= \mathbf{E}^{\mathcal{G}}[\mathbf{1}_B \circ Y] = \mathbf{P}_{Y|\mathcal{G}}(\cdot, B) \qquad \text{a.s.}\end{aligned}$$

By Proposition 4.1.14 we find $Z \in \mathcal{G}$ with $P(Z) = 0$ such that $\mathbf{P}_{(X,Y)|\mathcal{G}}(\omega) \in \mathcal{M}(\mathbf{P}_{X|\mathcal{G}}(\omega), \mathbf{P}_{Y|\mathcal{G}}(\omega))$ for all $\omega \in \Omega \setminus Z$ and hence

$$d(\mathbf{E}^{\mathcal{G}}[X], \mathbf{E}^{\mathcal{G}}[Y]) \le \int_{N\times N} d(x,y)\mathbf{P}_{(X,Y)|\mathcal{G}}(\cdot, d(x,y)) = \mathbf{E}^{\mathcal{G}}[d(X,Y)] \qquad \text{a.s.}\square$$

Example 4.1.18 Let (M, ρ) be a separable metric space. Let $(p_t)_{t>0}$ be a Markovian transition function on M and $(\Omega, (X_t), P^x)$ the corresponding Markov process. For a measurable map $f : M \to N$ such that $p_t(x) \circ f^{-1} \in \mathcal{P}^1(N)$ for all t and x (where $p_t(x)$ is regarded as a probability measure on M), we define the *nonlinear Markov operator* $P_t f : M \to N$ by

$$P_t f(x) := b(p_t(x) \circ f^{-1}). \tag{4.1}$$

If we put $Y_t := f(X_t)$ then for all $x \in M$

$$\mathcal{P}^x_{Y_{s+t}|\mathcal{F}_s}(\omega) = p_t(X_s(\omega)) \circ f^{-1}$$

and hence $\mathbf{E}^x[Y_{s+t}|\mathcal{F}_s] = P_t f(X_s)$. $\square$

4.1.2 The main example: NPC spaces

Curvature bounds in geodesic spaces in the sense of Alexandrov can be defined in terms of comparing triangles. Nonpositive curvature means that triangles are 'slimmer' than in Eucledian space. More precisely, let $z \in N$ and $\gamma : [0,1] \to N$ be a geodesic. z and γ span a triangle. Let $\overline{z}$ and $\overline{\gamma}$ be a comparison triangle in the Eucledian plane ,i.e. $d(\gamma(i), z) = |\overline{\gamma}(i) - \overline{z}|$, $i = 0, 1$ and $d(\gamma(0), \gamma(1)) = |\overline{\gamma}(0) - \overline{\gamma}(1)|$ (of course, $\overline{\gamma}$ is a line). Now $Curv(N) \le 0$ means that $d(\gamma(t), z) \le d(\overline{\gamma}(t), \overline{z})$ for all choices of z and γ and $t \in [0,1]$.
Calculating Eucledian distances yields the following rigorous

Definition 4.1.19 A complete geodesic space (N, d) is called (global) *NPC-space* if

$$d^2(z, \gamma(t)) \leq (1-t)d^2(z, \gamma(0)) + td^2(z, \gamma(1)) - t(1-t)d^2(\gamma(0), \gamma(1)) \qquad (4.2)$$

for any $z \in N$, any geodesic $\gamma : [0,1] \to N$ and any $t \in [0,1]$.

This notion generalizes the characterization of sectional curvature by A. Alexandrov (it can be localized, but we do not need this here). Indeed, a Riemannian Manifold is a (local) NPC space if and only if it has nonpositive sectional curvature. NPC-spaces are also called CAT(0)-spaces or Hadamard spaces. For details see [Jos97], [BH99], [BBI01] or [Bal95].
A condition equivalent to (4.2) is

$$\inf_{z \in N} \int_N d^2(z, x)p(dx) \leq \int_N \int_N d^2(x, y)p(dx)p(dy) \qquad (4.3)$$

for all discrete probability measures p on N. Moreover, Reshetnyak's *quadruple inequality* holds (cf. e.g. [Jos97]): For every quadruple of points $x_1, x_2, x_3, x_4 \in N$, we have

$$d^2(x_1, x_3) + d^2(x_2, x_4) \leq d^2(x_2, x_3) + d^2(x_4, x_1) + 2d(x_1, x_2)d(x_3, x_4). \qquad (4.4)$$

Example 4.1.20 (i) (Trees) An $\mathbb{R}-$tree (for a definition, see e.g. [Pic05]) is a global NPC space. A particular case of a tree is a $k-$star.
(ii) If N is an NPC space and $(\Omega, \mathcal{F}, P)$ a probability space, then $L^2(\mathcal{F}, N)$ is an NPC space, too, where the metric is given by $d(X, Y) := (\mathbf{E}d^2(X, Y))^{1/2}$. For $X, Y \in L^2(\mathcal{F}, N)$, the geodesic from X to Y is given by $\gamma(t)(\omega) := \gamma_\omega(t)$, where γ_ω is the geodesic from $X(\omega)$ to $Y(\omega)$. For details, see e.g. [Stu01]. □

From (4.2) follows that for $z \in N$ the function $f^z(y) := d^2(z, y)$ is strictly convex. Thus, in NPC-spaces, expectations can be defined as minimizers of the mean squared distance in the spirit of C.F. Gauß. For details and proofs of the following Proposition we refer to [Stu02].

Proposition 4.1.21 *Let $p \in \mathcal{P}^2(N)$. Then there is a unique point $b(p) \in N$ such that*

$$\int d^2(x, b(p))p(dx) \leq \int d^2(x, z)p(dx)$$

for all $z \in N$. The map $b : \mathcal{P}^2(N) \to N$ extends to a barycenter map $b : \mathcal{P}^1(N) \to N$. Hence, an NPC space is a barycenter space. This barycenter is called canonical barycenter *and enjoys the following properties:*

(i) (**Variance inequality**) *For all* $p \in \mathcal{P}^2(N)$ *and* $z \in N$,

$$\int d^2(x,z)p(dx) \geq \int d^2(x,b(p))p(dx) + d^2(b(p),z). \tag{4.5}$$

(ii) (**Jensen's inequality**) *For all* $p \in \mathcal{P}^1(N)$ *and all lower semicontinuous convex fuctions* $\varphi : N \to \mathbb{R}$,

$$\varphi(b(p)) \leq \int \varphi(x)p(dx). \Box$$

Remark 4.1.22 Instead of first defining b on $\mathcal{P}^2(N)$ and then extending it to a barycenter map, one can define $b(p)$ as the minimizer of the functional $z \mapsto \int d^2(z,y) - d^2(z_0,y)p(dy)$ for some $z_0 \in N$. Here p is only required to be in $\mathcal{P}^1(N)$, since $|d^2(z,y) - d^2(z_0,y)| \leq (d(z,y) + d(z_0,y))d(z,z_0)$. This definition leads to the same barycenter.$\Box$

In NPC spaces, a conditional expectation can be defined quite generally without any separability assumptions, just by convexity (cf. [Stu02]) . However, if an NPC space is separable, then this conditional expectation coincides with the one from Definition 4.1.15. Hence, for simplicity, *we will assume from now on that all NPC spaces are separable*, so we can define the conditional expectation as in Definition 4.1.15 and from Proposition 4.1.21 immediately follows its 'conditional' version, namely

Proposition 4.1.23 *Let* N *be a separable NPC space,* $(\Omega, \mathcal{F}, P)$ *a probability space and* $\mathcal{G} \subset \mathcal{F}$ *be a sub-*σ*-algebra. Then*

(i) *For all* $X \in L^2(\mathcal{F}, N)$ *and all* $Z \in L^2(\mathcal{G}, N)$,

$$\mathbf{E}^{\mathcal{G}} d^2(X,Z) \geq \mathbf{E}^{\mathcal{G}}[d^2(X, \mathbf{E}^{\mathcal{G}}[X])] + d^2(\mathbf{E}^{\mathcal{G}}[X], Z) \qquad a.s.$$

(ii) *For all* $X \in L^1(\mathcal{F}, N)$ *and for all lower semicontinuous convex fuctions* $\varphi : N \to \mathbb{R}$,

$$\varphi(\mathbf{E}^{\mathcal{G}}[X]) \leq \mathbf{E}^{\mathcal{G}}[\varphi(X)] \qquad a.s. \Box$$

We conclude with a characterization of conditional expectations in NPC spaces. Let $z \in N$. Let

$$f^z(y) := d^2(z,y). \tag{4.6}$$

Then f^z is convex. Hence if $\gamma : I \to N$ is a geodesic, then $\varphi^z_\gamma := f^z \circ \gamma : I \to \mathbb{R}$ is differentiable from the right (and from the left, of course).

Let $x, y \in N$. Then there is a unique geodesic $\gamma : [0,1] \to N$ with $\gamma(0) = x$ and $\gamma(1) = y$. We define

$$\partial f_x^z(y) := (\varphi^z)'_\gamma(0+). \tag{4.7}$$

where $\varphi'(0+)$ denotes the right-hand side derivative in 0.
Note that if $\tilde{\gamma} : [0, d(x,y)] \to N$ is the unit-speed geodesic from x to y, then

$$\begin{aligned}\partial f_x^z(y) &= 2d(x,y)d(x,z) \lim_{t\to 0} \frac{d(\tilde{\gamma}(t), z) - d(x,z)}{t} \\ &= -2d(x,y)d(x,z)\cos\angle_x(y,z)\end{aligned}$$

where the last equality can be found in [BH99], Corollary II.3.6. In particular, $\partial f_x^z(y) = \partial f_x^y(z)$.

Lemma 4.1.24 *Let N be a separable NPC space and $X \in L^2(\mathcal{F}, N)$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-Algebra und $Y \in L^2(\mathcal{G}, N)$. Then the following are equivalent:*

(i) $Y = \mathbf{E}[X|\mathcal{G}]$ *a.s.*

(ii) $\mathbf{E}[\partial f_Y^Z(X)|\mathcal{G}] \geq 0$ *a.s. for all* $Z \in L^2(\mathcal{G}, N)$.

(iii) $\mathbf{E}[\partial f_Y^Z(X)] \geq 0$ *for all* $Z \in L^2(\mathcal{G}, N)$.

(iv) $\mathbf{E}[\partial f_Y^z(X)|\mathcal{G}] \geq 0$ *a.s. for all* $z \in N$

Proof :
$(i) \Rightarrow (ii)$: Let $X_t(\omega), t \in [0,1]$ be the geodesic from $Y(\omega)$ to $X(\omega)$. Then X_t is the geodesic from Y to X in $L^2(\mathcal{F}, N)$. Now first using the triangle ineqality and then (i) yield that

$$d_2(X_t, Z) \geq (1-t)d_2(X,Y) - d_2(X,Z) \geq td_2(X,Y) = d_2(X_t, Y)$$

and hence $\mathbf{E}[X_t|\mathcal{G}] = Y$ $(\forall t \in [0,1])$. Thus

$$\mathbf{E}[\frac{1}{t}(f^Z(X_t) - f^Z(Y))|\mathcal{G}] \geq 0$$

for all $Z \in L^2(\mathcal{G}, N)$. Letting $t \to 0$ yields (ii).
$(iii) \Rightarrow (i)$: First note that $\partial f_y^z(x) = \partial f_y^x(z)$ for all $x, y, z \in N$. Hence

$$0 \leq \mathbf{E}[\partial f_Y^Z(X)] = \mathbf{E}[\partial f_Y^X(Z)] \leq \mathbf{E}[f^X(Z) - f^X(Y)]$$

for all $Z \in L^2(\mathcal{G}, N)$. Thus $Y = \mathbf{E}[X|\mathcal{G}]$.
$(iv) \Rightarrow (ii)$ follows by approximation of $Z \in L^2(\mathcal{G}, N)$ through a sequence Z^k with finite range. $\square$

4.2 Filtered conditional expectations and strong martingales

Since the conditional expectation is in general not projective, i.e. for $k \leq l$ the classical identity

$$E[X|\mathcal{F}_k] = E[E[X|\mathcal{F}_l]|\mathcal{F}_k]$$

from Euclidian space does not hold in general metric spaces, we consider the notion of filtered conditional expectations and martingales as in [Stu02]. In order to define martingales (and filtered conditional expectations) for continuous-time filtrations $(\mathcal{F}_t)_{t\geq 0}$, we fix a refining sequence Δ^n of partitions of $[0, \infty[$ with their mesh converging to 0 and consider the sequence of martingales w.r.t. the discrete-time filtrations $\mathcal{F}^n_k := \mathcal{F}_{t^n_k}$, where $\Delta^n = \{0 = t^n_0, t^n_1, \dots\}$. By constant extrapolation on the intervals $[t^n_k, t^n_{k+1}[$, these martingales can be regarded as time-continuous processes. If this sequence of processes has a limit, it is called the (time-continuous) strong martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ (and the sequence of partitions). This definition has strong connections to the nonlinear semigroup defined in [Stu05].

4.2.1 Discrete Time

Let (N, d, b) be a barycentric metric space. Let $(\Omega, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathcal{F}, P)$ be a filtered probability space, $m \in \mathbb{N}$ and $X \in L^1(\mathcal{F}_m, N)$. Unfortunately, the conditional expectation is in general not projective, i.e. for $k \leq l \leq m$ the classical identity

$$\mathbf{E}^{\mathcal{F}_k}[X] = \mathbf{E}^{\mathcal{F}_k}\mathbf{E}^{\mathcal{F}_l}[X]$$

does not hold in general (c.f. [Stu02], Example 3.2.).
However, we can define the discrete *filtered conditional expectation* (short: *FCE*) w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$ by

$$\mathbf{E}^{(\mathcal{F}_n)_{n\geq k}} := \mathbf{E}[X|(\mathcal{F}_n)_{n\geq k}] := \begin{cases} \mathbf{E}^{\mathcal{F}_k}\mathbf{E}^{\mathcal{F}_{k+1}}\dots\mathbf{E}^{\mathcal{F}_{m-1}}[X] & \text{if } k < m \\ X & \text{if } k \geq m. \end{cases}$$

Clearly,

$$\mathbf{E}^{(\mathcal{F}_n)_{n\geq k}}[X] = \mathbf{E}^{(\mathcal{F}_n)_{n\geq k}}[\mathbf{E}^{(\mathcal{F}_n)_{n\geq l}}[X]] \qquad \text{for all } k \leq l$$

or in words, the discrete FCE is projective.
We will call an adapted process $(X_n)_{n\in\mathbb{N}}$ such that $X_n \in L^1(\mathcal{F}_n, N)$ for all n a *martingale* if $\mathbf{E}^{(\mathcal{F}_n)_{n\geq k}}[X_l] = X_k$ for all $k \leq l$, or equivalently, if $\mathbf{E}^{\mathcal{F}_k}[X_{k+1}] = X_k$ for all $k \in \mathbb{N}$.

From Proposition 4.1.17 immediately follows the next

Proposition 4.2.1 *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two martingales. Then the distance process $(d(X_n, Y_n))_{n\in\mathbb{N}}$ is a submartingale.*

In particular, if $(X_n)_{n\in\mathbb{N}}$ is a martingale, then $(d(X_n, z))_{n\in\mathbb{N}}$ is a submartingale for all $z \in N$. If N has the property that closed balls are compact, then we have a 'martingale' convergence theorem, which is known for quite a long time (c.f. [Dos62]; for a proof see e.g. [Stu02]).

Theorem 4.2.2 (Convergence Theorem) *Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, P)$ be a filtered probability space and N be a complete metric space such that the closed balls in N are compact. Let $(X_n)_{n\in\mathbb{N}}$ be an adapted process such that $d(z, X)$ is a submartingale with $\sup_{n\in\mathbb{N}} \mathbf{E}[d(z, X_n)] < \infty$ for all $z \in N$. Then there is an $\mathcal{F}_\infty-$measurable map $X_\infty : \Omega \to N$ such that*

$$\lim_{n\to\infty} X_n = X_\infty \qquad a.s.$$

If $d(X, z)$ is uniformly $p-$integrable, then we also have convergence in L^p.

Remark 4.2.3 (i) One can prove a corresponding backward martingale convergence theorem, i.e. for decreasing filtrations.
(ii) Of course, the convergence theorem also holds for contiuous-time processes $(X_t)_{t\geq 0}$ (c.f. Corollary 4.2.12).

Corollary 4.2.4 *Let $(X_n)_{n\in\mathbb{N}}$ be a martingale such that $\sup_{n\in\mathbb{N}} \mathbf{E}[d(z, X_n)] < \infty$ for all $z \in N$. Then there is an $\mathcal{F}_\infty-$ measurable map $X_\infty : \Omega \to N$ such that*

$$\lim_{n\to\infty} X_n = X_\infty \qquad a.s.$$

If $d(X, z)$ is uniformly $p-$integrable, then we also have convergence in L^p.

Proposition 4.2.5 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, P)$ be a filtered probability space and (N, d, b) a separable barycenter space. Let $X \in L^1(\mathcal{F}_\infty, N)$. Then there is a martingale $(X_n)_{n\in\mathbb{N}}$ such that X_n converges to X in L^1.*

4.2.2 Continuous Time

Let $0 \leq s < t \leq \infty$ and $(\Omega, (\mathcal{F}_\tau)_{s\leq\tau\leq t}, \mathcal{F}, P)$ be a filtered probability space and $\xi \in L^1(\mathcal{F}_t, N)$. In order to define FCE in continuous time, we take a sequence of partitions of $[s, t]$ with their mesh converging to 0 and consider the limit of the discrete FCE, provided it exists.
In order to formulate this rigorously, we need some notation. A partition of $[0, \infty[$

is a set $\Delta = \{t_k : k \in \mathbb{N}\}$ such that $t_0 = 0$, $t_k < t_{k+1}$ and $t_k \to \infty$ as $k \to \infty$. The mesh of Δ is defined by

$$\|\Delta\| := \sup_{t_k \in \Delta} |t_{k+1} - t_k|.$$

For the sequel we fix a sequence $(\Delta^n)_{n\in\mathbb{N}}$ of partitions of $[0,\infty[$ such that $\Delta^n \subset \Delta^{n+1}$ and $\|\Delta^n\| \to 0$ and put $\mathbb{T} := \bigcup \Delta^n$. $\mathbb{T}$ is a dense subset of $[0,\infty[$.

Let $n \in \mathbb{N}$ and $s,t \in \Delta^n$ such that $s < t$. Then $\Delta^n \cap [s,t] = \{t_0, \dots, t_m\}$ with $s = t_0 < t_1 < \dots < t_m = t$. For $\xi \in L^1(\mathcal{F}_t, N)$ we define

$$\xi^n_k := \mathbf{E}^{\Delta^n}_k[\xi] := \mathbf{E}^{\mathcal{F}_{t_k}}\mathbf{E}^{\mathcal{F}_{t_{k+1}}} \dots \mathbf{E}^{\mathcal{F}_{t_{m-1}}}[\xi], \qquad k = 0 \dots m-1 \tag{4.8}$$

and the elementary process

$$\xi^n_\tau := \xi^n_k \qquad \text{for } \tau \in [t_k, t_{k+1}[. \tag{4.9}$$

Note that $(\xi^n_k)_{0\le k\le m}$ is the martingale w.r.t. the discrete-time filtration $(\mathcal{F}_k) := (\mathcal{F}_{t_k})$ with endpoint $\xi^n_m = \xi$. If ξ^n_s converges to some (ξ_s) in L^1, then ξ_s is called the (continuous-time) *filtered conditional expectation* (short: *FCE*) of ξ w.r.t $(\Delta^n)_{n\in\mathbb{N}}$, conditioned on $\mathcal{F}_s$ and we write

$$\mathbf{E}^{(\mathcal{F}_\tau)_{\tau\ge s}}[\xi] := \mathbf{E}[\xi|(\mathcal{F}_\tau)_{\tau\ge s}] := \xi_s. \tag{4.10}$$

Now we can introduce the notion of a strong martingale.

Definition 4.2.6 Let $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathcal{F}, P)$ be a filtered probability space and $X = (X_t)_{t\in\mathbb{T}}$ be a process such that $X_t \in L^1(\mathcal{F}_t, N)$ for all $t \in \mathbb{T}$. X is called a *strong martingale* w.r.t. $(\Delta^n)_{n\in\mathbb{N}}$ if for all $s,t \in \mathbb{T}$ with $s \le t$, $\mathbf{E}^{(\mathcal{F}_\tau)_{\tau\ge s}}[X_t]$ exists and is equal to X_s.

Remark 4.2.7 Note that the above definition is equivalent to the following one: X is a strong martingale if and only if there is a sequence $(\eta^n_{t_k})_{k\in\mathbb{N}}$ of processes such that $(\eta^n_{t_k})_{t_k\in\Delta}$ is a discrete time martingale w.r.t. the filtration $(\mathcal{F}_{t_k})_{t_k\in\Delta^n}$ and $\eta^n_t \to X_t$ in L^1 for all $t \in \mathbb{T}$.

Proposition 4.2.8 *Let $(X_t)_{t\in\mathbb{T}}$ and $(Y_t)_{t\in\mathbb{T}}$ be two strong martingales. Then the distance process $(d(X_t,Y_t))_{t\in\mathbb{T}}$ is a submartingale. In particular, for all $z \in N$, the process $(d(X_t,z))_{t\in\mathbb{T}}$ is a submartingale.*

Proof : Let $s,t \in \mathbb{T}$. Then $s,t \in \Delta^n$ for n large enough. Let $\xi = X_t$ and $\eta = Y_t$. Using the notation above and applying Proposition 4.2.1, one has

$$\mathbf{E}^{\mathcal{F}_s}[d(\xi,\eta)] \ge d(\xi^n_s, \eta^n_s) \qquad \text{a.s.}$$

Since $d(\xi^n_s,\eta^n_s) \to d(X_s,Y_s)$ in L^1, there is a subsequence n_k such that $d(\xi^{n_k}_s,\eta^{n_k}_s) \to d(X_s,Y_s)$ a.s., and the Proposition is proved. $\square$

Remark 4.2.9 Doss ([Dos62]) defined a martingale to be a process X such that $d(z,X)$ is a submartingale for all $z \in N$. In particular, by the above Proposition, every strong martingale is a martingale in the sense of Doss. However, even in manifolds, a Doss-martingale need not be a $\nabla-$martingale, which follows from the next example.

Example 4.2.10 Let $N = \mathbb{H}^2$, the hyperbolic plane. Let $A_1, A_2 \in \Gamma(TN)$ be an orthonormal frame in TN, i.e A_1, A_2 are vector fields with $\|A_i\| \equiv 1$ and $< A_1, A_2 > \equiv 0$. Let $R \in \Gamma(TN)$ be a vector field with $\|R\| \equiv \frac{1}{2}$ and

$$A_0 := -\frac{1}{2}\sum_{i=1}^{2} \nabla_{A_i} A_i + R.$$

At last, consider a standard filtered probability space $(\Omega, (\mathcal{F}_t), P)$ such that it carries a two-dimensional Brownian motion $B = (B^1, B^2)$ and a random variable $\xi : \Omega \to N$ which is $\mathcal{F}_0-$measurable, independent of B and whose distribution has no atoms (e.g., a uniform distribution on an open ball in N). Let X be the solution of the Stratonovic - SDE

$$dX = A_0(X)dt + \sum_{i=1}^{2} A_i(X) * dB^i$$

with $X_0 = \xi$. Let $f \in C^\infty(M)$. Then

$$df(X) = dM^f + \frac{1}{2}\Delta f(X)dt + df_X^z(R(X))dt,$$

where $M^f := \sum_{i=1}^2 \int df_X^z(A_i(X))dB^i$ is a martingale and Δ denotes the Laplace-Beltrami operator in $\mathbb{H}^2$. Thus X is the solution to the Martingale problem for the operator $\frac{1}{2}\Delta + R$ with initial condition X_0. Let $z \in M$ and put $r(x) := d(z,x)$. Then (cf. [Cra91] or [Pau] , note that $X_0 \neq z$ a.s.) there is a Brownian motion $\hat{B}$ (possibly defined on an extension $(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{P})$) such that

$$dr(X) = d\hat{B} + (\frac{1}{2}\Delta + R)(r)(X)dt.$$

Now $\Delta r(x) = \coth(r(x)) \geq 1$ and $|R(r)(x)| \leq \|R(x)\| = 1/2$ (note that $\|\mathrm{grad} r(x)\| \equiv 1$) and hence $|(\frac{1}{2}\Delta + R)(r)(x)| \geq 0$ for all $x \neq z$. This yields

$$\mathbf{E}^{\mathcal{F}_s}[r(X_t) - r(X_s)] = \mathbf{E}^{\mathcal{F}_s}\mathbf{E}^{\hat{\mathcal{F}}_s}[\hat{B}_t - \hat{B}_s] + \mathbf{E}^{\mathcal{F}_s}[\int_s^t (\frac{1}{2}\Delta + R)(r)(X_\tau)d\tau] \geq 0.$$

Thus $d(X,z)$ is a submartingale for all $z \in N$, but X is not a martingale. $\Box$

Let us quote some immediate consequences of Proposition 4.2.8. The first one is the so-called *non-confluence* of martingales:

Corollary 4.2.11 *Let $(X_t)_{t\in\mathbb{T}}$ and $(Y_t)_{t\in\mathbb{T}}$ be two strong martingales such that $X_{t_0} = Y_{t_0}$ almost surely for some $t_0 \in \mathbb{T}$. Then $X_t = Y_t$ for all $t \le t_0$ almost surely.*

Second, the convergence Theorem 4.2.2 immediately implies an analogous result for the continuous-time process $(X_t)_{t\in\mathbb{T}}$ (the proof is basically the same as the one of Lemma 4.2.13):

Corollary 4.2.12 *Let (N, d, b) be a proper barycentric metric space. Let $(X_t)_{t\in\mathbb{T}}$ be a martingale such that $\sup_{t\in\mathbb{T}} \mathbf{E}[d(z, X_t)] < \infty$ for all $z \in N$. Then there is an $\mathcal{F}_\infty-$ measurable map $X_\infty : \Omega \to N$ such that*

$$\lim_{t\to\infty} X_t = X_\infty \qquad a.s.$$

If $d(X, z)$ is uniformly $p-$integrable, then we also have convergence in L^p.

So far, a strong martingale w.r.t. a sequence Δ^n is only defined on the set $\mathbb{T}$ of all partition points, which is a countable dense subset of $\mathbb{R}_+$. We will now show how to extend it to a process that is defined on the whole $\mathbb{R}_+$. The technique we use is just an adaption of the regularization results known for real-valued (sub-)martingales, cf. [RY99], section II.2.

Lemma 4.2.13 *Let (N, d, b) be a proper barycentric metric space and let $(X_t)_{t\in\mathbb{T}}$ be a strong martingale. Then for almost all $\omega \in \Omega$, $X_{t^-}(\omega) := \lim_{s\nearrow t, s\in\mathbb{T}} X_s(\omega)$ and $X_{t^+}(\omega) := \lim_{s\searrow t, s\in\mathbb{T}} X_s(\omega)$ exist for all $t \in \mathbb{R}_+$. Moreover, if $(\mathcal{F}_t)_{t\ge 0}$ is right continuous, then for almost all $\omega \in \Omega$, $X_t(\omega) = X_{t^+}(\omega)$ for all $t \in \mathbb{T}$.*

Proof : Let $z \in N$ and set $g^z(x) := d(x, z)$. Then $(g^z(X_t))_{t\in\mathbb{T}}$ is a submartingale, and by [RY99], Theorem II.(2.5), there is an $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ and all $t \in [0, T]$, $\lim_{s\nearrow t, s\in\mathbb{T}} g^z(X_s(\omega))$ and $\lim_{s\searrow t, s\in\mathbb{T}} g^z(X_s(\omega))$ exist[1]. Moreover, if N_0 is a countable dense subset in N, then there is some $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$ such that for all $\omega \in \Omega_1$, all $t \in [0, T]$ and all $z \in N_0$, $\lim_{s\nearrow t, s\in\mathbb{T}} g^z(X_s(\omega))$ and $\lim_{s\searrow t, s\in\mathbb{T}} g^z(X_s(\omega))$ exist.
Let now $\omega \in \Omega_1$, $z_0 \in N_0$ and $t \in \mathbb{R}_+$. Let $(s_n)_{n\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$ converge to t from the left (right, respectively). Because $g^{z_0}(X_{s_n}(\omega))$ converges to some $R \ge 0$, $X_{s_n}(\omega)$ is contained in the (relatively compact) ball $B_{R+1}(z_0)$ for sufficiently large $n \in \mathbb{N}$. Thus there is a subsequence along which $X_{s_n}(\omega)$ converges to some $x_t(\omega)$. So in

[1] Revuz and Yor prove this result for a submartingale $(Y)_{t\in\mathbb{R}_+}$ (so it is defined on the whole time axis) and taking limits along rational numbers, but the technique also applies in our setting

order to prove that $x_t(\omega) = \lim_{s \nearrow t, s\in\mathbb{T}} g^z(X_s(\omega))$ (or $x_t(\omega) = \lim_{s\searrow t, s\in\mathbb{T}} g^z(X_s(\omega))$, respectively), we have to show that if $(\tilde{s}_n)_{n\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$ is another sequence converging to t from the left (or right, respectively) such that $X_{\tilde{s}_n}(\omega)$ converges to some $\tilde{x}_t(\omega)$, then $\tilde{x}_t(\omega) = x_t(\omega)$. So assume the contrary, i.e. $\epsilon := d(\tilde{x}_t(\omega), x_t(\omega)) > 0$. Then there is a $z \in N_0$ such that $d(\tilde{x}_t(\omega), z) < \epsilon/2$ and hence $d(x_t(\omega), z) \geq \epsilon/2$. But

$$d(x_t(\omega), z) = \lim_{n\to\infty} d(X_{s_n}(\omega), z) = \lim_{n\to\infty} d(X_{\tilde{s}_n}(\omega), z) = d(\tilde{x}_t(\omega), z) < \epsilon/2,$$

which is a contradiction.
It remains to prove the last assertion. Let $t \in \mathbb{T}$. By [RY99], Proposition II.(2.6), there is some $\Omega_2 \subset \Omega_1$ with $P(\Omega_2) = 1$ such that for all $\omega \in \Omega_2$ and all $z \in N_0$,

$$g^z(X_t(\omega)) \leq \mathbf{E}[g^z(X_{t^+})|\mathcal{F}_t](\omega) = g^z(X_{t^+}(\omega)),$$

where the last equality follows from the right continuity of the filtration. So if we assume that $\epsilon := d(X_t(\omega), X_{t^+}(\omega)) > 0$, we can proceed as above in order to obtain a contradiction. Thus $X_t(\omega) = X_{t^+}(\omega)$ and the Lemma is proved.□

Theorem 4.2.14 (Regularization) *Let (N, d, b) be a proper barycentric metric space and let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, P)$ be a filtered probability space satisfying the usual conditions. Let $(\Delta^n)_{n\in\mathbb{N}}$ be a sequence of partitions such that their mesh tends to 0 and let $(X_t)_{t\in\mathbb{T}}$ be a strong martingale w.r.t $(\Delta^n)_{n\in\mathbb{N}}$. Then there is a cadlag modification $\widetilde{X}$ of X. More precisely, there is a cadlag process $(\widetilde{X}_t)_{t\geq 0}$ such that $X_t = \widetilde{X}_t$ for all $t \in \mathbb{T}$, almost surely. Moreover, $(d(z, \widetilde{X}))_{t\geq 0}$ is a submartingale for all $z \in N$.*

Proof :. For $t \geq 0$, set $\widetilde{X}_t := X_{t^+}$. Then the Theorem follows from Lemma 4.2.13.
□

4.2.3 Martingales in NPC spaces

Let us consider the 'main example' for barycenter spaces, namely NPC spaces with the 'canonical' barycenter from Proposition 4.1.21. As we have seen, this barycenter enjoys certain properties, in particular the variance inequality and Jensen's inequality. In [Stu02] was developed a discrete-time martingale theory. We will shortly quote some results, which can be derived from Proposition 4.1.23:

Proposition 4.2.15 *Let (N, d, b) be an NPC space with canonical barycenter. Let $(X_n)_{n\in\mathbb{N}_0}$ be a discrete-time martingale. Then*

(i) $(\varphi(X_n))_{n\in\mathbb{N}}$ is a submartingale for all lower semicontinuous convex functions $\varphi : N \to \mathbb{R}$ such that $\varphi(X_n) \in L^1$ for all n, in particular for all Lipschitz continuous convex functions.

(ii) Let $X_n \in L^2(\mathcal{F}_n, N)$ for all n. Define

$$V_n := \sum_{k=1}^{n} \mathbf{E}^{\mathcal{F}_{k-1}}[d^2(X_{k-1}, X_k)].$$

Then $f^z(X_n) - V_n := d^2(z, X_n) - V_n$ is a submartingale for all $z \in N$.

An immediate consequence is the fact that a strong martingale satisfies *Darling's characterization*:

Theorem 4.2.16 *Let $X = (X_t)_{t\in\mathbb{T}}$ be a strong martingale. Then $(\varphi(X_t))_{t\in\mathbb{T}}$ is a submartingale for all lower semicontinuous convex functions $\varphi : N \to \mathbb{R}$ such that $\varphi(X_t) \in L^1$ for all t.*

Proof : Let $s, t \in \mathbb{T}$ with $s < t$. Put $\xi := X_t$. We use the notation of (4.9). Then $\xi_s^n \to X_s$ in L^1, and by choosing a subsequence we can assume that $\xi_s^n \to X_s$ $P-$a.s. Now $\varphi(\xi_s^n) \leq \mathbf{E}^{\mathcal{F}_s}[\varphi(X_t)]$ for all n, $P-$a.s. Due to the lower semicontinuity of φ we have

$$\varphi(X_s) \leq \liminf_{n\to\infty} \varphi(\xi_s^n) \leq \mathbf{E}^{\mathcal{F}_s}[\varphi(X_t)] \qquad \text{a.s.}\square$$

With the same technique one obtains the following

Corollary 4.2.17 *Let N be a proper NPC space, let $(X_t)_{t\in\mathbb{T}}$ be a strong martingale and let $(\widetilde{X}_t)t \geq 0$ be its extension from Theorem 4.2.14. Then $(\varphi(\widetilde{X}_t))_{t\geq 0}$ is a submartingale for all lower semicontinuous convex functions $\varphi : N \to \mathbb{R}$ such that $\varphi(X_t) \in L^1$ for all t.*

Another feature of a strong martingale is that it 'respects' the product structure of NPC spaces. For instance, let $(N_1, d_1), (N_2, d_2)$ be two NPC spaces. On $N_1 \times N_2$ define the product distance by $d^2((x_1, x_2), (y_1, y_2)) := d_1^2(x_1, y_1) + d_2^2(x_2, y_2)$. Then $N_1 \times N_2$ is again an NPC space, cf [Jos97].

Proposition 4.2.18 *Let N_1, N_2 be two NPC spaces. Let $(X_t^1)_{t\in\mathbb{T}}$ and $(X_t^2)_{t\in\mathbb{T}}$ be two adapted processes in N_1 and N_2, respectively. Then (X^1, X^2) is a martingale in $N_1 \times N_2$ if and only if X^i is a martingale in N_i for $i = 1, 2$.*

Proof : The definition of the canonical barycenter implies that if $p_i \in \mathcal{P}^2(N_i)$, then $b(p) = (b(p_1), b(p_2))$ for any coupling p of p_1 and p_2. Since $\mathcal{P}^2(N_i)$ is dense in $\mathcal{P}^1(N_i)$, this is also true for $p_i \in \mathcal{P}^1(N_i)$. So if $X^i \in L^1(\mathcal{F}, N_i)$, then $\mathbf{E}^{\mathcal{G}}[(X^1, X^2)] = (\mathbf{E}^{\mathcal{G}}[X^1], \mathbf{E}^{\mathcal{G}}[X^2])$ and consequently the assertion holds for time-discrete martingales. Thus by approximation the Proposition is proved. $\square$

4.3 Existence of FCE and strong martingales

We prove the existence of strong martingales with prescribed limit is established in two basic cases: First if (N, d) is proper with an arbitrary barycenter map and the filtered probability space satisfies certain coupling condition (Thm. 4.3.3). Second, if the target space is an NPC space with an additional lower curvature bound (Thm. 4.3.9).
In both cases our techniques are basically variants of the corresponding ones in [Stu05].

4.3.1 A coupling condition

For the sequel we will be concerned with the existence of continuous-time FCE's, or equivalently, of strong martingales with prescribed limit. We will start with a special situation which is related to Example 4.1.18. For simplicity, we will only consider dyadic partitions. More precisely, let $\Delta^n := \{k2^{-n} : k \in \mathbb{N}\}$ and $\mathbb{T} := \bigcup \Delta^n$.
Let (M, ρ) be a complete separable metric space and (N, d) a locally compact NPC space. Let $\rho_0 : M \times M \to [0, \infty)$ be a nonnegative symmetric measurable function and $(p_t)_{t>0}$ a Markovian transition function on M such that

$$\int \rho(z, y) p_t(x, dy) < \infty \qquad \forall x, z \in M, \quad \forall t > 0$$

and there exists a $\kappa \in \mathbb{R}$ such that

$$\rho^W(p_t(x), p_t(y)) \leq e^{\kappa t} \rho(x, y) \qquad \forall x, y \in M, \forall t > 0 \tag{4.11}$$

Then it follows from [Stu05], Thm. 4.3, that there is a subsequence $(\delta_k) = (2^{-n_k})$ such that for all $f \in Lip(M, N)$, all $x \in M$ and all $t \in \mathbb{T}$

$$P_t^* f(x) := \lim_{k \to \infty} P_{\delta_k}^{t/\delta_k} f(x)$$

exists. In terms of FCE the above result is as follows: Put $Y_t = f(X_t)$ and let $s, t \in \mathbb{T}$. Then the FCE

$$\mathbf{E}^{(\mathcal{F}_\tau)_{\tau \geq s}}[Y_T]$$

w.r.t. the sequence of partitions $(\Delta^{n_k})_{k \in \mathbb{N}}$ exists.

Remark 4.3.1 (i) $(P_t^*)_{t \in \mathbb{T}}$ is a semigroup acting on $Lip(M, N)$. It is called the nonlinear semigroup associated with p_t. In [Stu05] it is studied in great detail.
Geometrically, condition (4.11) can be regarded as a kind of lower curvature bound. For instance, in [vRS04] was shown that if (M, ρ) is a Riemannian manifold and

p_t the heat kernel on M, then (ii) holds with κ if and only if $Ric_M \geq -\kappa$.

(ii) Picard ([Pic05]) uses similar techniques in order to prove the existence of martingales with prescribed terminal value of the form $Y = f(X_T)$, where X is a Markov process on a metric space M satisfying a certain coupling condition and $f : M \to N$ is uniformly continuous and bounded. Although Picard stated his result only for trees, the techniques also apply in our context in order to prove martingales along subsequences in general NPC spaces (or proper metric spaces with barycenter).

Since in general processes need not be images of Markov processes, we want to formulate the above fact in a more general setting. We define a family $(\rho_s)_{s\geq 0}$ of pseudometrics on Ω by $\rho_s(\omega, \tilde{\omega}) := \rho(X_s(\omega), X_s(\tilde{\omega}))$. Moreover, put

$$Q_s(\omega) := P_{id|\mathcal{F}_s}(\omega) \in \mathcal{P}(\Omega). \tag{4.12}$$

Now since X is a Markov process, we have $Q_s(\omega) \circ X_{t+s}^{-1} = p_t(X_s(\omega), \cdot)$, and hence (4.11) together with Lemma 4.1.4 implies that for all $s, t \in \mathbb{T}$ and almost all $\omega_1, \omega_2 \in \Omega$,

$$\rho_{t+s}^W(Q_s(\omega_1), Q_s(\omega_2)) \leq e^{\kappa t}\rho_s(\omega_1, \omega_2). \tag{4.13}$$

Now we can formulate the assumptions of the following theorem in a general situation. Let $\mathbb{T}$ be the set of nonnegative dyadic numbers. Let $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathcal{F}, P)$ be a filtered probability space with a family $(\rho_t)_{t\in\mathbb{T}}$ of symmetric nonnegative functions on $\Omega \times \Omega$. Assume that the Markov kernels Q_s, defined by (4.12), exist for all $s \in \mathbb{T}$ (e.g. if Ω is a Polish space and $\mathcal{F} \subset \mathbb{B}(\Omega)$) and that (4.13) is satisfied for some $\kappa \in \mathbb{R}$.

Let (N, d, b) be a barycenter space and put

$$\mathcal{L}_N := \{Y : \Omega \times \mathbb{T} \to N : \mathbf{E}[d(Y_s, z)] < \infty \text{ for all } s \in \mathbb{T} \text{ and } z \in N\}.$$

For $Y \in \mathcal{L}_N$ we define a new process $P_tY \in \mathcal{L}_N$ by

$$P_tY(\omega, s) := \mathbf{E}^{\mathcal{F}_s}[Y_{t+s}](\omega) = b(Q_s(\omega) \circ Y_{t+s}^{-1}). \tag{4.14}$$

Note that this procedure defines a semigroup on $\mathcal{L}_{\mathbb{R}}$. Moreover, for $Y, \tilde{Y} \in \mathcal{L}_N$ an iterated application of Proposition 4.1.17 yields

$$d(P_t^nY(\omega, s), P_t^n\tilde{Y}(\omega, s)) \leq \mathbf{E}^{\mathcal{F}_s}[d(Y_{s+nt}, \tilde{Y}_{s+nt})](\omega) \tag{4.15}$$

for almost all ω. Note that there is a set $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that (4.13) and (4.15) hold pointwise for all $s, t \in \mathbb{T}$, $n \in \mathbb{N}$ and $\omega \in \Omega_0$.

Lemma 4.3.2 *Let $Y \in \mathcal{L}_N$ be a process such that for all $s \in \mathbb{T}$ and almost all ω_1, ω_2*

$$d(Y_s(\omega_1), Y_s(\omega_2)) \leq C\rho_s(\omega_1, \omega_2). \tag{4.16}$$

Then

$$d((P_t^n Y)_s(\omega_1), (P_t^n Y)_s(\omega_2)) \leq e^{knt} C\rho_s(\omega_1, \omega_2).$$

Proof : Let $n = 1$. From (4.16) and (4.13) follows that

$$\begin{aligned} d^W(Q_s \circ Y_{t+s}^{-1}(\omega_1), Q_s \circ Y_{t+s}^{-1}(\omega_2)) &\leq C\rho_{s+t}^W(Q_s(\omega_1), Q_s(\omega_2)) \\ &\leq e^{\kappa t} C\rho_s(\omega_1, \omega_2) \end{aligned}$$

and hence by the barycenter contraction property we derive the claim for $n = 1$. For arbitrary n, this can be iterated. $\square$

Theorem 4.3.3 *Assume that (N, d) is proper. Moreover, assume that there is a countable set $\Omega_1 \subset \Omega$ such that for all $s \in \mathbb{T}$ and almost all $\omega \in \Omega$*

$$\inf\{\rho_s(\omega, \tilde{\omega}) : \tilde{\omega} \in \Omega_1\} = 0.$$

Let $Y \in \mathcal{L}_N$ be a process satisfying (4.16). Then there is a subsequence n_k such that for all $s, t \in \mathbb{T}$ and almost all $\omega \in \Omega$

$$P_t^* Y(\omega, s) := \lim_{k \to \infty} P_{\delta_k}^{t/\delta_k} Y(\omega, s)$$

exists, where $\delta_k := 2^{-n_k}$. For all $t \geq 0$, the process $((P_{t-s}^* Y)_s)_{s \in \mathbb{T} \cap [0,t]}$ is a strong martingale.

Proof : Fix $t \in \mathbb{T}$. Let $s \in \mathbb{T}$ and $\omega \in \Omega$. Put $z_n(\omega, s) := P_{2^{-n}}^{t2^n} Y(\omega, s) \in N$, where n is assumed to be large enough such that $t2^n \in \mathbb{N}$. Let $z \in N$. From (4.15), applied to the Y and the constant process $\tilde{Y}(\omega, s) \equiv z$ follows that

$$d(z_n(\omega, s), z) \leq \mathbf{E}^{\mathcal{F}_s}[d(Y_{s+t}, z)](\omega) < \infty$$

for all n. In other words, all $z_n(\omega, s)$ are contained in a closed ball, which is compact by assumption. Thus there is a subsequence (n_k) such that $z_{n_k}(\omega, s)$ converges. Since Ω_1 is countable, we find subsequence, again denoted by (n_k), such that $z_{n_k}(\tilde{\omega}, s)$ converges for all $\tilde{\omega} \in \Omega_1$ and $s \in \mathbb{T}$. By Lemma 4.3.2 we have

$$d(P_{\delta_k}^{t/\delta_k} Y(\omega_1, s), P_{\delta_k}^{t/\delta_k} Y(\omega_2, s)) \leq e^{\kappa t} C\rho_s(\omega_1, \omega_2)$$

for all $k \in \mathbb{N}$ and $\omega_1, \omega_2 \in \Omega_0$. Thus a standard $\epsilon/3-$argument yields that $P_{\delta_k}^{t/\delta_k} Y(\omega, s)$ converges for all $s \in \mathbb{T}$ and $\omega \in \Omega_0$. Now for any $t \in \mathbb{T}$, we have $(P_t^* Y)_s = \mathbf{E}^{(\mathcal{F}_\tau)_{\tau \geq s}}[Y_{t+s}]$ by construction, so $((P_{t-s}^* Y)_s)_{s \leq t}$ is a martingale.$\square$

4.3.2 Lower Curvature Bounds

Analogously to the case of upper curvature bounds, lower curvature bounds will be defined by comparing triangles. Let us briefly sketch the definition. Let $z \in N$ and $\gamma : [a,b] \to N$ be a unit-speed geodesic. z and γ span a triangle. Let $\overline{z}$ and $\overline{\gamma}$ be a comparison triangle in $\mathbb{H}_\kappa$ (the Hyperbolic plane of constant curvature $-\kappa$), i.e. $d(\gamma(i), z) = d(\overline{\gamma}(i), \overline{z})$, $i = a, b$ and $d(\gamma(a), \gamma(b)) = d(\overline{\gamma}(a), \overline{\gamma}(b))$ (such a comparison triangle always exists, cf [BH99]). Then $Curv(N) \geq -\kappa$ means nothing else but $d(\gamma(t), z) \geq d(\overline{\gamma}(t), \overline{z})$ for all $z \in N$, all geodesics $\gamma : [a,b] \to N$ and $t \in [a,b]$.
In the above situation define a function $g_{a,b} : [a,b] \to \mathbb{R}$ by $g_{a,b}(t) := d^2(\overline{\gamma}(t), \overline{z})$ (here the distance is taken in $\mathbb{H}_\kappa$). We have $g_{a,b}(t) \leq f^z(\gamma(t))$ for all $t \in [a,b]$ with equality if $t = a$ and $t = b$. Moreover, it follows from Riemannian comparison theorems (c.f. [Jos02], Thm. 4.6.1) that for $R > 0$ there is a $C = C(R)$ such that whenever $\gamma([a,b]) \subset B_R(z)$, then

$$0 \leq \frac{1}{2} g''_{a,b}(t) - 1 \leq C d^2(\overline{\gamma}(t), \overline{z}) \leq C d^2(\gamma(t), z). \tag{4.17}$$

In particular, $g''_{a,b}$ is uniformly bounded for all geodesics which are contained in $B_R(z)$.

Lemma 4.3.4 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function such that for all a, b with $a < b$ there is a smooth function $g_{a,b} : [a,b] \to \mathbb{R}$ such that $g_{a,b}(t) \leq \varphi(t)$ for all $t \in [a,b]$ and $g_{a,b}(a) = \varphi(a)$, $g_{a,b}(b) = \varphi(b)$.*
Let I be an open interval and $c : I \to [0, \infty[$ such that $|g''_{a,b}(t)| \leq c(t)$ for all $a, b \in I$ and all $t \in [a,b]$. Moreover, let

$$D := \sup\{|g'''_{a,b}(t)| : a, b \in I; \quad t \in [a,b]\} < \infty.$$

Then φ is differentiable in I and we have for all $s, t \in I$

$$|\varphi(s) - \varphi(t) - \varphi'(t)(s-t)| \leq \frac{1}{2} c(t)(t-s)^2 + D|t-s|^3$$

Proof : Let $\epsilon > 0$. We can assume that $0 \in I$ and, by adding an affine function if necessary, that $\varphi(0) = 0$ and $\varphi(t) \geq 0$ for all $t \in [-\epsilon, \epsilon]$. We show that φ is differentiable in 0. Since φ is convex, the one-sided derivatives $\varphi'(0+)$ and $\varphi'(0-)$ exist and $a := \varphi'(0+) - \varphi'(0-) \geq 0$. Thus φ is differentiable in 0 if and only if $a = 0$.
Assume that $a > 0$. Then $0 = \varphi(0) \leq \frac{1}{2}(\varphi(-\epsilon) + \varphi(\epsilon)) - \frac{a}{2}\epsilon$. But $g_{-\epsilon,\epsilon}(0) \leq 0$ and hence there is a $\xi \in]-\epsilon, \epsilon[$ such that $g'_{-\epsilon,\epsilon}(\xi) = 0$. Thus by the Taylor formula

$$\varphi(\epsilon) \leq g_{-\epsilon,\epsilon}(\epsilon) - g_{-\epsilon,\epsilon}(\xi) \leq \frac{1}{2} c(\xi)(\epsilon - \xi)^2 \leq 2C\epsilon^2.$$

The same holds for $-\epsilon$. Letting $\epsilon \to 0$ yields a contradiction. Hence, $a = 0$ and so φ is differentiable in 0.
Now we prove the second claim. Let $t = 0$. Again we can add an affine function and can hence assume that $\varphi(0) = 0$ and $\varphi'(0) = 0$. Let $s \in I$. Then Taylor's formula yields

$$\begin{aligned}\varphi(s) = g_{0,\epsilon}(s) - g_{0,\epsilon}(0) &\leq g'_{0,\epsilon}(0)s + \frac{1}{2}c(0)s^2 + \frac{D}{3}|s|^3 \\ &\leq \frac{1}{2}c(0)s^2 + \frac{D}{3}|s|^3\end{aligned}$$

because $g'_{0,\epsilon}(0)s \leq 0$. $\square$

Recall that the functions f^z from (4.6) are convex. Moreover, recall the definition of $\partial f^z_x(y)$ from (4.7).

Corollary 4.3.5 *Let N be a geodesically complete NPC space of lower bounded curvature on all balls, i.e. for all $z \in N$ and all $R > 0$ there is a $\kappa > 0$ such that $Curv(B_R(z)) \geq -\kappa$. Let $z \in N$. Then f^z, is differentiable along geodesics, i.e. for all geodesics $\gamma : \mathbb{R} \to N$ the map $f^z \circ \gamma$ is differentiable. Moreover, for all $z_0 \in N$ and all $R > 0$ there is a $C > 0$ such that for all $x, y, z \in B_R(z_0)$*

$$f^z(y) - f^z(x) - \partial f^z_x(y) - d^2(x, y) \leq Cd^2(x, z)d^2(x, y) + Cd^3(x, y)$$

Proof : Let $x, y, z \in B_R(z_0)$. Let $\gamma : \mathbb{R} \to N$ be the (unit-speed) geodesic with $\gamma(0) = x$ and $\gamma(d(x, y)) = y$. Let $I := \gamma^{-1}(B_R(z_0))$. Then we can apply the preceding Lemma to $\varphi := f^z \circ \gamma$. Note that $\partial f^z_x(y) = \varphi'(0)d(x, y)$. Moreover, by (4.17) we can choose $c(t) = Cd^2(z, \gamma(t)) + 1$. Putting this together proves the Corollary. $\square$

Lemma 4.3.6 *Let N be an NPC space of lower bounded curvature on all balls. Let $X \in L^2(\mathcal{F})$. Then for all $z \in N$*

(i)

$$\mathbf{E}^{\mathcal{G}}[\partial f^Z_{\mathbf{E}^{\mathcal{G}}[X]}(X)] = 0 \qquad a.s.$$

(ii) Let $z_0 \in N$ and $R > 0$ such that $X(\Omega) \subset B_R(z_0)$. Then there is a $C > 0$ such that for all $z \in B_R(z_0)$

$$\begin{aligned}\mathbf{E}^{\mathcal{G}}[d^2(X, z) - &d^2(\mathbf{E}^{\mathcal{G}}[X], z) - d^2(X, \mathbf{E}^{\mathcal{G}}[X])] \\ &\leq C\mathbf{E}^{\mathcal{G}}[d^2(\mathbf{E}^{\mathcal{G}}[X], z)d^2(X, \mathbf{E}^{\mathcal{G}}[X]) + Cd^3(X, \mathbf{E}^{\mathcal{G}}[X])]\end{aligned}$$

Proof : Fix $z \in N$. For $y \in N$ let $\gamma_y : [0,1] \to N$ be the geodesic from y to z. Since N is geodesically complete, we can extend it to a geodesic $\gamma : \mathbb{R} \to N$. Moreover, the map $y \mapsto \gamma_y(t)$ is continuous for all $t \in \mathbb{R}$.
Now put $Y := \mathbf{E}^{\mathcal{G}}[X]$. Then $\gamma_Y(t) \in L^2(\mathcal{G}, N)$ for all t, hence γ_Y is a geodesic in $L^2(\mathcal{G}, N)$. Since $t \mapsto d^2(X(\omega), \gamma_{Y(\omega)}(t))$ is differentiable for all ω by Corollary 4.3.5 (i), so is the map $t \mapsto \varphi_A(t) := \int_A d^2(X(\omega), \gamma_{Y(\omega)}(t))P(d\omega)$ for all $A \in \mathcal{G}$. Moreover, since $\gamma_Y(0) = \mathbf{E}^{\mathcal{G}}[X]$, 0 is the minimizer of φ_A and hence

$$\int_A \partial f_Y^z(X)dP = \int_A \partial f_Y^X(z)dP = \varphi_A'(0) = 0$$

for all $A \in \mathcal{G}$, proving (i).
(ii) follows from (i) and Corollary 4.3.5. □

For the rest of this section let $(\Delta^n)_{n\in\mathbb{N}}$ be a refining sequence of partitions such that the mesh converges to 0 as n tends to infinity. Put $\mathbb{T} := \bigcup_{n\in\mathbb{N}} \Delta^n$. The next Lemma will give a sufficient condition for the existence of continuous-time FCE's. Again we will use the notation of (4.9) and define

$$v_t^{n,m} := \sum_{t_k \in \Delta^n} \mathbf{E}d^2(\xi^m_{t_k\wedge t}, \xi^m_{t_{k+1}\wedge t}). \tag{4.18}$$

Let $n \le m$. Let $\Delta^n = \{t_k : k = 0, \dots K^n\}$. Then an iterated application of the Variance Inequality yields that for all $Z \in L^2(\mathcal{F}_{t_{k-1}})$

$$\mathbf{E}d^2(\xi^{\Delta^m}_{t_k}, Z) - \mathbf{E}d^2(\xi^{\Delta^m}_{t_{k-1}}, Z) \ge v^{m,m}_{t_k} - v^{m,m}_{t_{k-1}}. \tag{4.19}$$

Lemma 4.3.7 *Let $s < t$. If*

$$\limsup_{n\to\infty} \limsup_{m\to\infty} v_t^{n,m} - v_s^{n,m} - (v_t^{m,m} - v_s^{m,m}) \le 0,$$

then $\xi^n_s \to \mathbf{E}^{(\mathcal{F}_\tau)_{\tau\ge s}}[\xi]$ in L^2.

Proof : Let $\Delta^n = \{t_k : k = 0, \dots K\}$. Let $k \in \{0, \dots K\}$. Then (4.19) implies that

$$\mathbf{E}d^2(\xi^n_{t_{k-1}}, \xi^m_{t_{k-1}}) \le \mathbf{E}d^2(\xi^n_{t_{k-1}}, \xi^m_{t_k}) - (v^{m,m}_{t_k} - V^{m,m}_{t_{k-1}})$$

and

$$\mathbf{E}d^2(\xi^n_{t_{k-1}}, \xi^m_{t_{k-1}}) \le \mathbf{E}d^2(\xi^n_{t_k}, \xi^m_{t_{k-1}}) - \mathbf{E}d^2(\xi^n_{t_{k-1}}, \xi^n_{t_k}).$$

Adding up the two inequalities and applying the quadruple inequality (4.4) yields

$$\begin{aligned}2\mathbf{E}d^2(\xi^n_{t_{k-1}}, \xi^m_{t_{k-1}}) &\le \mathbf{E}d^2(\xi^n_{t_{k-1}}, \xi^m_{t_k}) + \mathbf{E}d^2(\xi^n_{t_k}, \xi^m_{t_{k-1}}) \\ &\quad - \mathbf{E}d^2(\xi^n_{t_{k-1}}, \xi^n_{t_k}) - (v^{m,m}_{t_k} - v^{m,m}_{t_{k-1}}) \\ &\le \mathbf{E}d^2(\xi^n_{t_k}, \xi^m_{t_k}) + \mathbf{E}d^2(\xi^n_{t_{k-1}}, \xi^m_{t_{k-1}}) \\ &\quad + \mathbf{E}d^2(\xi^m_{t_{k-1}}, \xi^m_{t_k}) - (v^{m,m}_{t_k} - v^{m,m}_{t_{k-1}})\end{aligned}$$

and hence

$$\mathbf{E}d^2(\xi^n_{t_{k-1}}, \xi^m_{t_{k-1}}) \le \mathbf{E}d^2(\xi^n_{t_k}, \xi^m_{t_k}) + \mathbf{E}d^2(\xi^m_{t_{k-1}}, \xi^m_{t_k}) - (v^{m,m}_{t_k} - v^{m,m}_{t_{k-1}})$$

By iteration we get for all $n \le m$

$$\mathbf{E}d^2(\xi^n_s, \xi^m_s) \le v^{n,m}_t - v^{n,m}_s - (v^{m,m}_t - v^{m,m}_s).$$

Thus, by assumption, ξ^n_s is a Cauchy sequence for all $s \in \mathbb{T}$, converging to some $\xi_s \in L^2(\mathcal{F}_s)$ which is, by definition, equal to $\mathbf{E}^{(\mathcal{F}_\tau)_{\tau \ge s}}[\xi]$. $\square$

Definition 4.3.8 Let $\xi_t \in L^2(\mathcal{F}_t)$. ξ is called *regular* for $(\Delta^n)_{n\in\mathbb{N}}$ if the increments of v^m, m_t are finally controlled by a continuous function, i.e. there is a continuous function v_t such that for all $s, t \in \mathbb{T}$,

$$\limsup_{m\to\infty} v^{m,m}_t - v^{m,m}_s \le v_t - v_s$$

Theorem 4.3.9 *Let N be a geodesically complete NPC space of lower bounded curvature on all balls. Let $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathcal{F}, P)$ be a filtered probability space. Let $t \in \mathbb{T}$ and $\xi \in L^\infty(\mathcal{F}_t, N)$ be regular for $(\Delta^n)_{n\in\mathbb{N}}$. Then $\mathbf{E}^{(\mathcal{F}_\tau)_{\tau\ge s}}[\xi]$ exists for all $s \in \mathbb{T}$.*

Proof : Recall the notations of Lemma 4.3.7. Let $n \le m$. Let $\Delta^n \cap [s,t] = \{t_k : k = 0, \dots K\}$ and $\Delta^m \cap [s,t] = \{s_l : l = 0, \dots L\}$.

$$\begin{aligned}
v^{n,m}_{t_k} - v^{n,m}_{t_{k-1}} &= \mathbf{E}d^2(\xi^m_{t_k}, \xi^m_{t_{k-1}}) = \sum_{t_{k-1}<s_l\le t_k} \mathbf{E}d^2(\xi^m_{s_l}, \xi^m_{t_{k-1}}) - \mathbf{E}d^2(\xi^m_{s_{l-1}}, \xi^m_{t_{k-1}}) \\
&\le \sum_{t_{k-1}<s_l\le t_k} \mathbf{E}d^2(\xi^m_{s_l}, \xi^m_{s_{l-1}}) + C \sum_{t_{k-1}<s_l\le t_k} \mathbf{E}d^2(\xi^m_{s_l}, \xi^m_{t_{k-1}})\mathbf{E}d^2(\xi^m_{s_l}, \xi^m_{s_{l-1}}) \\
&\quad + C \sum_{t_{k-1}<s_l\le t_k} \mathbf{E}[d^3(\xi^m_{s_l}, \xi^m_{s_{l-1}})] \\
&= V^{m,m}_{t_k} - V^{m,m}_{t_{k-1}} + C \sum_{t_{k-1}<s_l\le t_k} \mathbf{E}d^2(\xi^m_{s_l}, \xi^m_{t_{k-1}})\mathbf{E}d^2(\xi^m_{s_l}, \xi^m_{s_{l-1}}) \\
&\quad + C \sum_{s_l} \mathbf{E}[d^3(\xi^m_{s_l}, \xi^m_{s_{l-1}})] \\
&\le \tilde{C}(V^{m,m}_{t_k} - V^{m,m}_{t_{k-1}}).
\end{aligned}$$

Since ξ is regular, $\epsilon^n := \sup_{t_k\in\Delta^n, m\ge n} d^2(\xi^m_{t_k}, \xi^m_{t_{k-1}})$ tends to 0 as $n \to \infty$. So, looking again at the first inequality, we have

$$\begin{aligned}
V^{n,m}_t - V^{n,m}_s \le V^{m,m}_t - V^{m,m}_s &+ C \sum_{t_k} \sum_{t_{k-1}<s_l\le t_k} \mathbf{E}d^2(\xi^m_{s_l}, \xi^m_{t_{k-1}})\mathbf{E}d^2(\xi^m_{s_l}, \xi^m_{s_{l-1}}) \\
&+ C \sum_{s_l} \mathbf{E}[d^3(\xi^m_{s_l}, \xi^m_{s_{l-1}})]
\end{aligned}$$

The second sum tends to 0 for $n \to \infty$ by the preceding considerations. Clearly, the third sum goes to 0, too. Hence the assumptions of Lemma 4.3.7 are satisfied and it follows that $\mathbf{E}^{(\mathcal{F}_\tau)_{\tau\ge s}}[\xi]$ exists. $\square$

Example 4.3.10 Consider the Example at the beginning of section 4.3.1. Namely, we are given on a metric space (M, ρ) a Markov process X with semigroup p_t satisfying (4.11). Let Δ be a partition of $\mathbb{R}_+$ and let $s \le t$. Then $\Delta \cap [s,t] = \{t_0, \dots t_k\}$. For a Lipschitz continuous function $f : M \to N$, set

$$P^{\Delta}_{t,s} f(x) := P_{t_0-s} P_{t_1-t_0} \dots P_{t-t_k} f(x),$$

where $P_\tau f$ is the *nonlinear Markov operator* applied to f, cf. Example 4.1.18. Then $\mathrm{Lip}(P^{\Delta}_{t,s} f) \le \mathrm{Lip}(f) e^{\kappa(t-s)}$ (cf. e.g. Lemma 4.3.2).
Let now $(\Delta^n)_{n\in\mathbb{N}}$ be a refining sequence of partitions. If we put $\xi := f(X_t)$, then in the notation of (4.8) and (4.9) we have $\xi^n_s = P^{\Delta^n}_{t,s} f(X_s)$.
Assume furthermore that there is some $C > 0$ such that for all $x \in M$ and $t > 0$,

$$p_t f^x(x) \le Ct, \tag{4.20}$$

where $f^x(y) := \rho^2(x, y)$. This is in particular fulfilled when X is Brownian motion or the solution of an SDE with sufficiently smooth coefficients.
For $n \in \mathbb{N}$, let $\Delta^n \cap [s,t] = \{t_0, \dots t_k\}$ as above and let $f : M \to N$ be Lipschitz continuous. Fix $j \in \{0, \dots k\}$ and put $g := P^{\Delta}_{t,t_{j+1}} f$. Then

$$\begin{aligned}
\mathbf{E}d^2(\xi^n_{t_j}, \xi^n_{t_{j+1}}) &= \mathbf{E}d^2(P^{\Delta}_{t,t_{j+1}} f(X_{t_{j+1}}), P^{\Delta}_{t,t_j} f(X_{t_j})) = \mathbf{E}d^2(g(X_{t_{j+1}}), P_{t_{j+1}-t_j} g(X_{t_j})) \\
&\leq 2\mathbf{E}\left[d^2(g(X_{t_{j+1}}), g(X_{t_j})) + d^2(g(X_{t_j}), P_{t_{j+1}-t_j} g(X_{t_j}))\right] \\
&= 2\mathbf{E}\left[\mathbf{E}^{X_{t_j}}[d^2(g(X_{t_{j+1}-t_j}), g(X_0))] + d^2(g(X_{t_j}), P_{t_{j+1}-t_j} g(X_{t_j}))\right] \\
&\leq 2\mathbf{E}\left[\mathbf{E}^{X_{t_j}}[d^2(g(X_{t_{j+1}-t_j}), g(X_0))] + \mathbf{E}^{X_{t_j}}[d^2(g(X_{t_{j+1}-t_j}), g(X_0))]\right] \\
&\leq 4C\mathrm{Lip}(g)(t_{j+1} - t_j) \leq 4C\mathrm{Lip}(f) e^{\kappa(t-t_{j+1})}(t_{j+1} - t_j)
\end{aligned}$$

and hence

$$v^{n,n}_t - v^{n,n}_s = \sum_{j=0}^{k} \mathbf{E}d^2(\xi^n_{t_j}, \xi^n_{t_{j+1}}) \leq 4C\mathrm{Lip}(f) \sum_{j=0}^{k} e^{\kappa(t-t_{j+1})}(t_{j+1} - t_j)$$

As $n \to \infty$, the right hand side converges to $v_t - v_s$ with $v_s := 4C\mathrm{Lip}(f) \int_s^t e^{t-\tau} d\tau$. So we deduce that $\xi = f(X_t)$ is regular for $(\Delta^n)_{n\in\mathbb{N}}$, and it follows from the above Theorem that if N is locally of lower bounded curvature and f is bounded and Lipschitz continuous, then $\mathbf{E}^{(\mathcal{F}_\tau)_{\tau\geq s}}[\xi]$ exists for *any* sequence $(\Delta^n)_{n\in\mathbb{N}}$ of refining partitions, not only along a subsequence of certain partitions as in Theorem 4.3.3. Moreover, the target space N need not be locally compact.

4.4 Characterization of strong martingales

The next Theorem gives a characterization of martingales in terms of their 'quadratic variation'. The prove will use similar techniques as those in Lemma 4.3.7. Again let $(\Delta^n)_{n\in\mathbb{N}}$ be a refining sequence of partitions such that the mesh converges to 0 as n tends to infinity. Put $\mathbb{T} := \bigcup_{n\in\mathbb{N}} \Delta^n$.

Definition 4.4.1 We say that a process $(X_t)_{t\in\mathbb{T}}$ has a *quadratic variation w.r.t.* $(\Delta^n)_{n\in\mathbb{N}}$ if there is a nondecreasing process $(\langle X\rangle_t)_{t\in\mathbb{T}}$ such that for all $t \in \mathbb{T}$, $X_t \in L^2(\mathcal{F}_t)$ and

$$V^n_t := \sum_{t_k\in\Delta^n} \mathbf{E}^{\mathcal{F}_{t_k}}[d^2(X_{t_k\wedge t}, X_{t_{k+1}\wedge t})] \to \langle X\rangle_t$$

in L^1 as $n \to \infty$ (in particular, $\langle X\rangle_t \in L^1$ for all $t \in \mathbb{T}$).

Theorem 4.4.2 *Let N be a separable NPC space and $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathcal{F}, P)$ be a filtered probability space. Let $(X_t)_{t\in\mathbb{T}}$ be an adapted process with quadratic variation $\langle X\rangle$. Then X is a strong martingale if and only if $d^2(X_t, z) - \langle X\rangle_t$ is a submartingale for all $z \in N$.*

Proof : The 'only if'-implication follows from Proposition 4.2.15.
For the 'if'-implication we first remark that since $d^2(X_t, z)-\langle X\rangle_t$ is a submartingale for all $z \in N$ and N is separable, it follows that for all $s < t$ and $Z \in L^2(\mathcal{F}_s, N)$

$$\mathbf{E}^{\mathcal{F}_s}[d^2(X_t, Z) - d^2(X_s, Z) - (\langle X\rangle_t - \langle X\rangle_s)] \geq 0 \tag{4.21}$$

Let $s, t \in \mathbb{T}$ with $s < t$. Put $\xi := X_t$. We have to prove that $\xi^n_s \to X_s$. Let $\Delta^n \cap [s, t] = \{t_0, \dots, t_m\}$ with $s = t_0 < t_1 < \dots < t_m = t$. Using the notation of (4.9), we have for $k = 1, \dots m$

$$d^2(\xi^n_{t_{k-1}}, X_{t_{k-1}}) \leq \mathbf{E}^{\mathcal{F}_{t_{k-1}}}[d^2(\xi^n_{t_{k-1}}, X_{t_k}) - (\langle X\rangle_{t_k} - \langle X\rangle_{t_{k-1}})]$$

by (4.21), and the variance inequality yields

$$d^2(\xi^n_{t_{k-1}}, X_{t_{k-1}}) \leq \mathbf{E}^{\mathcal{F}_{t_{k-1}}}[d^2(\xi^n_{t_k}, X_{t_{k-1}}) - d^2(\xi^n_{t_{k-1}}, \xi^n_{t_k})]$$

Adding up the two inequalities and applying the quadruple inequality (4.4) yields

$$\begin{aligned}
2d^2(\xi^n_{t_{k-1}}, X_{t_{k-1}}) &\leq \mathbf{E}^{\mathcal{F}_{t_{k-1}}}[d^2(\xi^n_{t_{k-1}}, X_{t_k}) + d^2(\xi^n_{t_k}, X_{t_{k-1}}) \\
&\qquad - d^2(\xi^n_{t_{k-1}}, \xi^n_{t_k}) - (\langle X\rangle_{t_k} - \langle X\rangle_{t_{k-1}})] \\
&\leq \mathbf{E}^{\mathcal{F}_{t_{k-1}}}[d^2(\xi^n_{t_k}, X_{t_k}) + d^2(\xi^n_{t_{k-1}}, X_{t_{k-1}}) \\
&\qquad + d^2(X_{t_k}, X_{t_{k-1}}) - (\langle X\rangle_{t_k} - \langle X\rangle_{t_{k-1}})]
\end{aligned}$$

and hence

$$d^2(\xi^n_{t_{k-1}}, X_{t_{k-1}}) \leq \mathbf{E}^{\mathcal{F}_{t_{k-1}}}[d^2(\xi^n_{t_k}, X_{t_k}) + (V^n_{t_k} - V^n_{t_{k-1}}) - (\langle X\rangle_{t_k} - \langle X\rangle_{t_{k-1}})]. \tag{4.22}$$

Iterating this yields

$$\mathbf{E}[d^2(\xi^n_s, X_s)] \leq \mathbf{E}[(V^n_t - V^n_s) - (\langle X\rangle_t - \langle X\rangle_s)]$$

while the right-hand side tends to 0 as n tends to infinity. □

Remark 4.4.3 From (4.22) follows that the process $S_k := d^2(\xi^n_{t_k}, X_{t_k}) + V^n_{t_k} - \langle X\rangle_{t_k}$ is a submartingale w.r.t. the filtration $(\mathcal{F}_{t_k})_{0\leq k\leq m}$. Let $\epsilon > 0$. Then

$$\begin{aligned}
P(\sup_{0\leq k\leq m} d^2(\xi^n_{t_k}, X_{t_k}) > \epsilon) &\leq P(\sup_{0\leq k\leq m} S_{t_k} > \epsilon) + P(\sup_{0\leq k\leq m} |V^n_{t_k} - \langle X\rangle_{t_k}| > \epsilon) \\
&\leq \frac{1}{\epsilon}\mathbf{E}[|V^n_t - \langle X\rangle_t|] + P(\sup_{0\leq k\leq m} |V^n_{t_k} - \langle X\rangle_{t_k}| > \epsilon)
\end{aligned}$$

where the last inequality follows from Doob's inequality. In particular, if $V^n \to \langle X\rangle$ locally uniformly in L^1, then $\xi^n \to X$ locally uniformly in L^2.

Example 4.4.4 (i) If N is a Riemannian manifold with nonpositive sectional curvature, then Theorem 4.4.2 yields that every continuous $\nabla-$martingale is (locally) a martingale in our sense. Together with Corollary 4.2.16 we deduce that a continuous semimartingale X such that $X_t \in L^2(\mathcal{F}_t, N)$ is a $\nabla-$martingale if and only if it is a martingale.
(ii) Theorem 4.4.2 is also the key to the martingale characterization in Theorem 3.4.7 in the case that N is a Euclidean polyhedron of nonpositive curvature.

Let us conclude this section with some remarks on localization. All results about strong martingales so far were formulated under the assumption that X was a *global* strong martingale. But in stochastic calculus, the most convenient objects are *local* martingales. It is not difficult to define a local martingale in our setting:

Definition 4.4.5 Let $(X_t)_{t\geq 0}$ be an adapted process and let $(\Delta^n)_{n\in\mathbb{N}}$ be a sequence of refining partitions with $\|\Delta^n\| \to 0$. X is called a *local strong martingale* w.r.t. $(\Delta^n)_{n\in\mathbb{N}}$ if there is a sequence τ_j of stopping times such that $\tau_j \nearrow \infty$ and $X_{t\wedge\tau_j}$ is a strong martingale w.r.t. $(\Delta^n)_{n\in\mathbb{N}}$.

However, some results such as non-confluence of martingales do not hold for local martingales, even in the simplest case $N = \mathbb{R}$. So sometimes it is important to know when a local strong martingale is a martingale.

Proposition 4.4.6 *Let $(\Delta^n)_{n\in\mathbb{N}}$ be a sequence of refining partitions with $\|\Delta^n\| \to 0$. Let $(X_t)_{t\geq 0}$ be a continuous local strong martingale w.r.t. $(\Delta^n)_{n\in\mathbb{N}}$. If X is bounded, then X is a strong martingale.*

Proof : Let us go back to the situation of (4.8). For $\xi \in L^1(\mathcal{F}_t)$ set

$$\xi_k^n(\xi) := \mathbf{E}^{\mathcal{F}_{t_k}}\mathbf{E}^{\mathcal{F}_{t_{k+1}}}\ldots\mathbf{E}^{\mathcal{F}_{t_{m-1}}}[\xi], \qquad k = 0\ldots m-1$$

and $\xi_\sigma^n(\xi) := \xi_k^n(\xi)$ for $\sigma \in [t_k, t_{k+1}[$. We have to show that for all $s \in \mathbb{T}$ with $s \leq t$, $d_1(\xi_s^n(X_t), X_s) \to 0$ as $n \to \infty$.
Let now τ_j be a localizing sequence of stopping times as in Definition 4.4.5. Accordingly, we set

$$\xi_k^{n,j}(\xi) := \mathbf{E}^{\mathcal{F}_{t_k\wedge\tau_j}}\mathbf{E}^{\mathcal{F}_{t_{k+1}\wedge\tau_j}}\ldots\mathbf{E}^{\mathcal{F}_{t_{m-1}\wedge\tau_j}}[\xi], \qquad k = 0\ldots m-1$$

and $\xi_\sigma^{n,j}(\xi) := \xi_k^{n,j}(\xi)$ for $\sigma \in [t_k, t_{k+1}[$. Because $X_{t\wedge\tau_j}$ is a strong martingale, $d_1(\xi_s^{n,j}(X_{t\wedge\tau_j}), X_{s\wedge\tau_j}) \to 0$ as $n \to \infty$ for every $j \in \mathbb{N}$. Moreover, from Example 4.1.16 we know that $\xi_k^n(X_{t\wedge\tau_j}) = \xi_k^{n,j}(X_{t\wedge\tau_j})$ for all $k = 0, \ldots, m-1$ and hence $d_1(\xi_s^n(X_{t\wedge\tau_j}), X_{s\wedge\tau_j}) = d_1(\xi_s^{n,j}(X_{t\wedge\tau_j}), X_{s\wedge\tau_j}) \to 0$.
Now because X is continuous by assumption, $X_{t\wedge\tau_j} \to X_t$ as $j \to \infty$, and since X is bounded, this convergence is also in L^1. So we deduce from the contraction property of the (iterated) conditional expectation that $d_1(\xi_s^n(X_t), \xi_s^n(X_{t\wedge\tau_j})) \to 0$ as $j \to \infty$, *uniformly in* $n \in \mathbb{N}$. So noting that also $X_{s\wedge\tau_j} \to X_s$ as $j \to \infty$, a standard $\epsilon/3$ argument yields that $d_1(\xi_s^n(X_t), X_s) \to 0$. $\square$

Chapter 5

Appendix

5.1 Some facts from real stochastic analysis

Standard references for real stochastic analysis are [RY99] or [KS91]. There is also a tight survey in [Bas95]. We will mostly refer to [RY99].

We start with functions of finite variation. Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function of locally finite variation. The *total variation function* $\hat{a}_t$ is defined by

$$\hat{a}_t := \sup\{\sum_{l=0}^{m-1} |a_{t_{l+1}} - a_{t_l}| : 0 = t_0 < t_1 \cdots < t_m = t, m \in \mathbb{N}\}. \tag{5.1}$$

$\hat{a}$ is a nondecreasing continuous function with $\hat{a}_0 = 0$. The *total variation measure* $|da|$ is defined by $|da| := d\hat{a}$, i.e. $\int f(\tau)|da|_\tau := \int f(\tau) d\hat{a}_\tau$.

Lemma 5.1.1 *Let $a^k : [0,T] \to \mathbb{R}$ be a sequence of continuous functions of finite variation such that $\gamma := \sup_k \int_0^T |da^k|_\tau \leq \infty$. Assume that a^k converge uniformly to some function a. Then*

(i) a is of finite variation. More precisely, $\int_0^T |da|_\tau \leq \gamma$.

(ii) For any open set $O \subset \mathbb{R}$,

$$\int \mathbf{1}_O(\tau)|da|_\tau \leq \liminf_{k\to\infty} \int \mathbf{1}_O(\tau)|da^k|_\tau$$

Proof : Let $s < t \in \mathbb{R}_+$ and let $s = t_0 < t_1 \cdots < t_m = t$ be a partition of $[s,t]$. Since $a^k \to a$ uniformly, for all $\epsilon > 0$ there is a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $l = 0, \ldots m$, $|a^k_{t_l} - a_{t_l}| \leq \frac{\epsilon}{2m}$ and hence by the triangle inequality

$$\sum_{l=0}^{m-1} |a_{t_{l+1}} - a_{t_l}| \leq \sum_{l=0}^{m-1} |a^k_{t_{l+1}} - a^k_{t_l}| + \epsilon \leq \gamma + \epsilon \tag{5.2}$$

So letting $s = 0$, $t = T$ yields (i).

In order to prove (ii), we first remark that the sequence $d\hat{a}^k$ is tight as a sequence of measures on $[0, T]$. Let now $O \subset \mathbb{R}_+$ be open and let a^{k_j} be a subsequence such that $\mathbf{1}_O(\tau)|da^{k_j}|_\tau \to \alpha := \liminf_{k\to\infty} \int \mathbf{1}_O(\tau)|da^k|_\tau$. By tightness, there is a sub-subsequence, again denoted by a^{k_j}, such that $|da^{k_j}| \to \mu$ weakly as $j \to \infty$, where μ is a measure on $[0, T]$. We claim that for all $s < t$,

$$\hat{a}_t - \hat{a}_s \leq \mu([s,t]). \tag{5.3}$$

Indeed, let $\epsilon > 0$. Then we can find a partition $s = t_0 < t_1 \cdots < t_m = t$ such that $\hat{a}_t - \hat{a}_s \leq \sum_{l=0}^{m-1} |a_{t_{l+1}} - a_{t_l}| + \epsilon$. By (5.2) we can find a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $\hat{a}_t - \hat{a}_s \leq \sum_{l=0}^{m-1} |a^k_{t_{l+1}} - a^k_{t_l}| + 2\epsilon \leq \hat{a}^k_t - \hat{a}^k_s + 2\epsilon$. Now since $\hat{a}^{k_j} \to \mu$ weakly and $[s, t]$ is closed, $\limsup_j \hat{a}^{k_j}_t - \hat{a}^{k_j}_s \leq \mu([s, t])$. Thus choosing j_0 large enough, we have that $\hat{a}_t - \hat{a}_s \leq \hat{a}^{k_j}_t - \hat{a}^{k_j}_s + 2\epsilon \leq \mu([s, t]) + 3\epsilon$. Letting $\epsilon \to 0$ yields (5.3). With a monotone class argument we immediately get that $|da| \leq \mu$, i.e. $\int \mathbf{1}_B(\tau)|da|_\tau \leq \mu(B)$ for every measurable set $B \subset \mathbb{R}_+$. In particular, $\int \mathbf{1}_O(\tau)|da|_\tau \leq \mu(O) \leq \liminf_{j\to\infty} \int \mathbf{1}_O(\tau)|da^{k_j}|_\tau = \alpha$. This proves (ii). □

Proposition 5.1.2 (Kunita-Watanabe inequality) *Let X and Y be two semimartingales and H and K be two progressively measurable processes. Then*

$$\int_0^t |H_\tau K_\tau| \; |d\langle X, Y\rangle|_\tau \leq \left(\int_0^t H_\tau^2 d\langle X\rangle_\tau\right)^{1/2} \left(\int_0^t K_\tau^2 d\langle Y\rangle_\tau\right)^{1/2} \tag{5.4}$$

Proof : [RY99] Theorem IV (1.15). □

Let Y be a continuous semimartingale and $a \in \mathbb{R}$. The *local time of Y at a* is defined by

$$L(a,t) := L^Y(a,t) := 2\left[(Y_t - a)^+ - (Y_0 - a)^+ - \int_0^t \mathbf{1}_{\{Y_\tau > a\}} dY_\tau\right] \tag{5.5}$$

In the next proposition we quote the basic properties of semimartingale local times. All the proofs can be found in chapter VI §1 of [RY99].

Proposition 5.1.3
(i) For all a, $L(a, t)$ is nondecreasing and continuous in t. In particular, $(Y_t - a)^+$ is a semimartingale. Moreover, for the process $(a, t) \mapsto L(a, t)$ there is a modification that is continuous in t and cadlag in a.

(ii) The positive measure $dL(a, \cdot)$ is carried on the set $\{Y_\tau = a\}$, i.e.

$\int \mathbf{1}_{\{Y_\tau \neq a\}} dL(a,\tau) \equiv 0$. *Moreover, if* $Y = M + A$ *is the semimartingale decomposition, then*

$$L(a,t) - L(a-,t) = 2\int_0^t \mathbf{1}_{\{Y_\tau = a\}} dY_\tau = 2\int_0^t \mathbf{1}_{\{Y_\tau = a\}} dA_\tau. \tag{5.6}$$

In particular, if Y *is a local martingale, the modification from (i) is bicontinuous in* (a,t).

(iii) **(Itô-Tanaka formula)** *If* $f : \mathbb{R} \to \mathbb{R}$ *is a the difference of two convex functions, then* $f(Y)$ *is a semimartingale and*

$$f(Y_t) - f(Y_0) = \int_0^t f'_-(Y_\tau) dY_\tau + \frac{1}{2}\int_{\mathbb{R}} L(a,t) f''(da) \tag{5.7}$$

where f'_- *is the left-hand derivative of* f *and* $f''(da)$ *is the second derivative of* f *in the sense of distributions.*

(iv) **(Occupation times formula)** *For any positive measurable function* $g : \mathbb{R} \to \mathbb{R}_+$ *we have*

$$\int_0^t g(Y_\tau) d\langle Y \rangle_\tau = \int_{\mathbb{R}} g(a) L(a,t) da \tag{5.8}$$

In particular,

$$L(a,t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{\{Y_\tau \in [a, a+\epsilon[\}} d\langle Y \rangle_\tau \tag{5.9}$$

where the $P-$*null set out of which (5.8) and (5.9) hold can be chosen independent of* a, t *and* g.

Remark 5.1.4 Because of the occupation times formula, $L(a,\cdot)$ is also called the *occupation time density* of Y at a.

In this text, we will be particularly interested in the case where Y is a nonnegative semimartingale. Let

$$L_t := L_t^Y := L^Y(0,t). \tag{5.10}$$

Lemma 5.1.5 *Let* Y *be a nonnegative semimartingale. Then*

$$L_t^Y = 2\int_0^t \mathbf{1}_{\{Y_\tau = 0\}} dY_\tau$$

Proof : The Lemma follows immediately from (5.5), noting that $Y = Y^+$ since Y is nonnegative. $\square$

Now we state a consequence of the occupation times formula. Let $K \subset \mathbb{R}$ be a compact set. Put

$$\mathcal{L}_t(\omega) := \mathcal{L}_t^K(\omega) := \sup_{y \in K} L(y,t)(\omega) \tag{5.11}$$

Because K is compact and $y \mapsto L(y,t)$ is cadlag, there is some $z = z_t(\omega)$ such that $\mathcal{L}_t(\omega) = \max\{L(z,t)(\omega), L(z-,t)(\omega)\}$. In particular, $\mathcal{L}_t$ is finite. Moreover, $\mathcal{L}_t$ is continuous, which follows from the fact that $(y,t) \mapsto L(y,t)$ is jointly continuous in t and cadlag in y.

Corollary 5.1.6 *For all continuous real-valued semimartingales Y and all non-negative Borel functions $f : \mathbb{R} \to \mathbb{R}_+$ we have*

$$\int_0^t (\mathbf{1}_K f)(Y_\tau) d\langle Y\rangle_\tau \le \mathcal{L}_t^K \int_K f(y)dy \tag{5.12}$$

Although we will not need this in the sequel, we state the following annealed version of the above Corollary. Let $Y = M + A$ be a continuous semimartingale. We define

$$\|Y\|_0 := \mathbf{E}\left[\langle M\rangle_\infty^{1/2} + \int_0^\infty |dA|_\tau\right] \tag{5.13}$$

and

$$\|Y\| := (\mathbf{E}[\langle M\rangle_\infty])^{1/2} + \mathbf{E}[\int_0^\infty |dA|_t]. \tag{5.14}$$

Both are norms on the space of semimartingales starting at 0 where these values are finite (more precisely, on equivalence classes modulo indistinguishability). Clearly, $\|\cdot\|_0 \le \|\cdot\|$ and for a bounded predictable process H, we have

$$\| \int H_\tau dY_\tau \| \le \|H\|_\infty \|Y\|. \tag{5.15}$$

where $\|H\|_\infty := \sup_t |H_t|$. Moreover, if Y is a continuous semimartingale, there is a sequence T_l of stopping times with $T_l \nearrow \infty$ such that $\|Y^{T_l}\| < \infty$ for all l (Y^{T_l} is the process stopped at T_l).

Lemma 5.1.7 *There is a $\gamma > 0$ such that for all continuous real-valued semi-martingales Y and all nonnegative Borel functions $f : \mathbb{R} \to \mathbb{R}_+$ we have*

$$\mathbf{E}\left[\int_0^\infty f(Y_\tau) d\langle Y\rangle_\tau\right] \le \gamma \|Y\|_0 \int_{\mathbb{R}} f(y)dy \le \gamma \|Y\| \int_{\mathbb{R}} f(y)dy \tag{5.16}$$

Proof : The argument is taken from the proof of Theorem VI (1.7) in [RY99]. For $y \in \mathbb{R}$ we have

$$L(y,t) = 2\left[|Y_t - y| - |Y_0 - y| - \int_0^t \mathbf{1}_{\{Y_\tau > y\}} dY_\tau\right].$$

Now $||Y_t - y| - |Y_0 - y|| \leq |Y_t - Y_0| \leq 2\sup_{s\leq\infty}|Y_s|$. Moreover, by the Burkholder inequality, there is a constant $\gamma_1 > 0$ (independent of Y) such that

$$\begin{aligned}\mathbf{E}\left[\sup_{t\leq\infty}\left|\int_0^t \mathbf{1}_{\{Y_\tau > y\}} dM_\tau\right|\right] &\leq \left(\mathbf{E}\left[\sup_{t\leq\infty}\left|\int_0^t \mathbf{1}_{\{Y_\tau > y\}} dM_\tau\right|\right]^2\right)^{1/2}\\ &\leq \gamma_1 \mathbf{E}\left[\langle M\rangle_\infty^{1/2}\right]\end{aligned}$$

So with $\gamma := 4\gamma_1$ we have for all $y \in \mathbb{R}$

$$\mathbf{E}[L(y,\infty)] \leq \gamma\|Y\|_0. \tag{5.17}$$

Thus the occupation times formula together with the stochastic Fubini theorem yields

$$\begin{aligned}\mathbf{E}\left[\int_0^\infty f(Y_\tau)d\langle Y\rangle_\tau\right] = \mathbf{E}\left[\int_{\mathbb{R}} f(y)L(y,\infty)dy\right] &= \int_{\mathbb{R}} f(y)\mathbf{E}[L(y,\infty)]dy\\ &\leq \gamma\|Y\|_0\int_{\mathbb{R}} f(y)dy.\Box\end{aligned}$$

5.2 Localization in space

A typical situation for stochastic analysis in manifolds (and also in our context) is that one is given a functions that is defined *locally* on a space. Unfortunately, many definitions and theorems of stochastic analysis require functions that are defined *globally* on the whole space (e.g. the Itô formula). Consequently, one has to localize many arguments in space. This can be a tedious business. So in order to keep the proofs simple, we will introduce some notations and tools for localization. In the sequel we will be given a continuous process X with values in some (proper) metric space M and an open subset $O \subset M$. Then $A := \{X \in O\}$ is an 'open' set in $\Omega \times \mathbb{R}_+$ (i.e. the set $\{t : (\omega,t) \in A\}$ is open for all ω). Schwartz (cf. [Sch80]) has studied a localized semimartingale theory on such an open set systematically in great generality.

Definition 5.2.1 Let $X : \Omega \times \mathbb{R}_+ \to M$ be a continuous adapted process. Let $Y, \widetilde{Y}$ be two continuous adapted real-valued. We say that $dY = d\widetilde{Y}$ *on* $\{X \in O\}$ if for all stopping times $S \leq T$ with $X_{|[S,T]\cap\{S<T\}} \in O$, $(Y_T - Y_S) = \widetilde{Y}_T - \widetilde{Y}_S$.
Y is called a *(sub-; semi-)martingale on* $\{X \in O\}$ if for all stopping times $S \leq T$ with $X_{|[S,T]\cap\{S<T\}} \in O$, $Y_{|[S,T]}$ is a (sub-; semi-)martingale.

Note that $Y_{|[S,T]}$ is a (sub-; semi-)martingale if and only if the process $\widetilde{Y}_t := Y_{(S+t)\wedge T}$ is a (sub-; semi-)martingale w.r.t. the filtration $\widetilde{\mathcal{F}}_t := \mathcal{F}_{S+t}$.
The next Lemma characterizes the above definition in the case where Y and $\widetilde{Y}$ are semimartingales (cf. also [Sch80], Proposition 3.7).

Lemma 5.2.2 *Let M be a proper metric space and let $X : \Omega \times \mathbb{R}_+ \to M$ be a continuous adapted process. Let $Y, \widetilde{Y}$ be two continuous real semimartingales. Then the following are equivalent:*

(i) $dY = d\widetilde{Y}$ *on* $\{X \in O\}$

(ii) $\int \mathbf{1}_{\{X_\tau \in U\}} dY = \int \mathbf{1}_{\{X_\tau \in U\}} d\widetilde{Y}$ *for all* $U \subset\subset O$

(iii) $\int \mathbf{1}_{\{X_\tau \in O\}} dY = \int \mathbf{1}_{\{X_\tau \in O\}} d\widetilde{Y}$

Proof : $(i) \Rightarrow (ii)$ Let $U \subset\subset V_1 \subset\subset V_2 \subset\subset O$. We define two sequences of stopping times recursively, as follows: Let $S_0 = T_0 := 0$. Put $S_{n+1} := \inf\{t \geq T_n : X_t \in V_1\}$ and $T_{n+1} := \inf\{t \geq S_{n+1} : X_t \notin V_2\}$. Then, because X is continuous, $S_n < T_n$ on $\{S_n < \infty\}$ and $T_n < S_{n+1}$ on $\{T_n < \infty\}$. for all $n \geq 1$ and $S_n \nearrow \infty$. Moreover, $\{X \in U\} \subset \bigcup_n]S_n, T_n]$ and hence by (i),

$$
\begin{aligned}
\int \mathbf{1}_{\{X_\tau \in U\}} d(Y - \widetilde{Y}) &= \int \mathbf{1}_{\{X_\tau \in U\}} d\left(\sum_n \int \mathbf{1}_{]S_n, T_n]} d(Y - \widetilde{Y})\right) \\
&= \int \mathbf{1}_{\{X_\tau \in U\}} d\left(\sum_n (Y - \widetilde{Y})_{T_n} - (Y - \widetilde{Y})_{S_n}\right) \\
&\equiv 0.
\end{aligned}
$$

$(ii) \Rightarrow (iii)$ Let $U_n \subset\subset O$ a sequence of relatively compact domains with $U_n \nearrow O$. Then $\mathbf{1}_{U_n} \to \mathbf{1}_O$ and (iii) follows from (ii) with the convergence theorem for stochastic integrals.
$(iii) \Rightarrow (i)$ Let $S \leq T$ such that $X_t(\omega) \in O$ for all $\omega \in \{S < T\}$ and all $t \in [S(\omega), T(\omega)]$. Then by (iii),

$$
\begin{aligned}
\left((Y_T - Y_S) - (\widetilde{Y}_T - \widetilde{Y}_S)\right) \mathbf{1}_{\{S<T\}} &= \int \mathbf{1}_{\{S<T\}} \mathbf{1}_{[S,T]} d(Y - \widetilde{Y}) \\
&= \int \mathbf{1}_{\{S<T\}} \mathbf{1}_{[S,T]} \mathbf{1}_{\{X_\tau \in O\}} d(Y - \widetilde{Y}) \\
&= 0,
\end{aligned}
$$

which implies (i). $\square$

Let now Y be a semimartingale on $\{X \in O\}$. For $U \subset\subset O$ we define $\int \mathbf{1}_{\{X_\tau \in U\}} dY_\tau$ in the following way: Let S_n, T_n be the stopping times defined in the proof of Lemma 5.2.2. Set

$$\int_0^t \mathbf{1}_{\{X_\tau \in U\}} dY_\tau := \sum_{n=0}^{\infty} \int_{S_n \wedge t}^{T_n \wedge t} \mathbf{1}_{\{X_\tau \in U\}} dY_\tau. \tag{5.18}$$

Since $T_n \nearrow \infty$ as $n \to \infty$, $\int_0^t \mathbf{1}_{\{X_\tau \in U\}} dY_\tau$ is defined for all $t \in \mathbb{R}_+$. This definition is coherent with the usual integral if Y is a semimartingale. Moreover, it is the unique continuous semimartingale Z such that $\int \mathbf{1}_{\{X_\tau \in U\}} dZ_\tau = Z$ and $dZ = dY$ on $\{X \in U\}$.
So we have shown that for all $U \subset\subset O$ *there is a continuous semimartingale* Z *such that* $dZ = dY$ *on* $\{X \in U\}$. This is a standard assumption in §3 of [Sch80]. More delicate is the question if there is also a continuous semimartingale Z such that $dZ = dY$ on $\{X \in O\}$, or equivalently, if $\int \mathbf{1}_{\{X_\tau \in O\}} dY_\tau$ is well-defined. This is not always the case, as indicated in the next

Example 5.2.3 (i) Consider the deterministic continuous real-valued process $X_t := t(\sin 1/t + 2)$ (with $X_0 = 0$). Let $O :=]0, \infty[$. Then $\{X \in O\} =]0, \infty[$ (more precisely, $\{X \in O\} = \Omega \times]0, \infty[$, but since X is deterministic, we can forget about Ω). Clearly, X is locally of finite variation on $\{X \in O\}$ and hence X is a semimartingale on $\{X \in O\}$. But X is not locally of finite variation on $[0, \infty[$ and hence not a semimartingale. Namely, fix $T > 0$ and let $U_n :=]1/n, T[$. Then $\int \mathbf{1}_{\{X_\tau \in U_n\}} |dX|_\tau \to \infty$.
Assume now that there is a function $Z : [0, \infty[\to \mathbb{R}$ such that Z is locally of finite variation (i.e., Z is a semimartingale) such that $dX = dZ$ on $\{X \in O\}$. Then $\int \mathbf{1}_{\{X_\tau \in U_n\}} |dZ|_\tau$ is uniformly bounded in n. But on the other hand, $\int \mathbf{1}_{\{X_\tau \in U_n\}} |dX|_\tau = \int \mathbf{1}_{\{X_\tau \in U_n\}} |dZ|_\tau$ for all n, which is a contradiction.

(ii) A more interesting example is *Reflecting Brownian motion in a cusp*, as defined in [DT93b]: Consider the symmetric cusp $C := C_\beta := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, -x_1^\beta \leq x_2 \leq x_1^\beta\}$, where $\beta > 1$. The authors construct a diffusion X on C which they call reflecting Brownian motion in C. This process behaves like two-dimensional Brownian motion as long as it is in C°, the interior of C, and hence it is a semimartingale on $\{X \in C^\circ\}$. Let X be reflecting Brownian motion in C starting at 0. In [DT93a] the same authors show that X is a semimartingale in $\mathbb{R}^2$ if and only if $\beta < 2$. In particular, if $\beta \geq 2$, X cannot be a uniformly bounded semimartingale on $\{X \in C^\circ\}$.

The next definition gives a sufficient condition to ensure that $\int \mathbf{1}_{\{X_\tau \in O\}} dY_\tau$ is well-defined, which is proved in Proposition 5.2.5 below. Actually, this condition is also necessary as one can easily see.

Definition 5.2.4 Let $X : \Omega \times \mathbb{R}_+ \to M$ be a continuous adapted process and let $O \subset M$ open. Let Y be a semimartingale on $\{X \in O\}$. We say that Y is *uniformly bounded on* $\{X \in O\}$ if there is a $\gamma > 0$ such that $\| \int \mathbf{1}_{\{X_\tau \in U\}} dY_\tau \| \leq \gamma$ for all $U \subset\subset O$, cf. (5.14).
X is called *locally uniformly bounded on* $\{X \in O\}$ if there is an sequence T_n of stopping times increasing to ∞ such that the stopped process X^{T_n} is uniformly bounded on $\{X \in O\}$ for all n.

Proposition 5.2.5 *(i) Let $X : \Omega \times \mathbb{R}_+ \to M$ be a continuous adapted process and let $O \subset M$ open, where M is a proper metric space. Let Y be a semimartingale on $\{X \in O\}$ that is locally uniformly bounded on $\{X \in O\}$. Then there is a unique continuous semimartingale Z such that $\int \mathbf{1}_{\{X_\tau \in O\}} dZ_\tau = Z$ and $dZ = dY$ on $\{X \in U\}$ (or equivalently, $\int \mathbf{1}_{\{X_\tau \in U\}} dZ_\tau = \mathbf{1}_{\{X_\tau \in U\}} dY_\tau$) for all $U \subset\subset O$. We set*

$$\int \mathbf{1}_{\{X_\tau \in O\}} dY_\tau := Z \tag{5.19}$$

(ii) Let $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ be a continuous adapted process and let $O \subset \mathbb{R}^n$ open. Assume that X is a semimartingale on $\{X \in O\}$ that is locally uniformly bounded on $\{X \in O\}$, i.e. the coordinate processes w.r.t some basis are locally uniformly bounded on $\{X \in O\}$ [1]. Let $f : O \to \mathbb{R}$ smooth such that the first and second derivatives of f are uniformly bounded on O. Then $\int \mathbf{1}_{\{X_\tau \in O\}} df(X_\tau)$ exists in the sense of (5.19).

Proof : The uniqueness follows from Lemma 5.2.2. For the existence, assume first that Y is uniformly bounded on $\{X \in O\}$. Let $U_k \subset\subset O$ be a sequence of domains increasing to O. Put $Y^k := \int \mathbf{1}_{\{X_\tau \in U_k\}} dY_\tau$.
For a continuous semimartingale $S = M + A$, set

$$\lfloor S \rfloor_t := \langle M \rangle_t + \int_0^t |dA|_\tau \tag{5.20}$$

Let $k \leq l$. Then $\int \mathbf{1}_{\{X_\tau \in U_k\}} dY^l_\tau = \int \mathbf{1}_{\{X_\tau \in U_k\}} dY^k_\tau$ and hence

$$\begin{aligned}
\lfloor Y^k \rfloor_t - \lfloor Y^k \rfloor_s &= \lfloor \int \mathbf{1}_{\{X_\tau \in U_k\}} dY^l_\tau \rfloor_t - \lfloor \int \mathbf{1}_{\{X_\tau \in U_k\}} dY^l_\tau \rfloor_s \\
&= \int_s^t 1_{\{X_\tau \in U_k\}} d\langle Y^l \rangle_\tau + \int_s^t 1_{\{X_\tau \in U_k\}} |dA^l|_\tau \\
&\leq \int_s^t 1_{\{X_\tau \in U_l\}} d\langle Y^l \rangle_\tau + \int_s^t 1_{\{X_\tau \in U_l\}} |dA^l|_\tau \\
&= \lfloor Y^l \rfloor_t - \lfloor Y^l \rfloor_s
\end{aligned}$$

[1] This is fulfilled in particular if X is a semimartingale on the whole $\mathbb{R}^n$.

In particular, $\lfloor Y^k \rfloor$ increases to some continuous nondecreasing process K and $d\lfloor Y^k \rfloor \leq dK$. Moreover, $\mathbf{E}[K_\infty] \leq \lim_{k\to\infty} \mathbf{E}\left[\langle Y^k\rangle_\infty + \int_0^\infty |dA^k|_\tau\right] < \infty$ since $\|Y^k\|$ is uniformly bounded by assumption. Now $\mathbf{1}_{\{X_\tau \in U_l \setminus U_k\}} \to 0$ as $k, l \to \infty$ and consequently

$$\begin{aligned}\mathbf{E}\left[\sup_t |Y_t^k - Y_t^l|\right] &= \mathbf{E}\left[\sup_t |\int_0^t \mathbf{1}_{\{X_\tau \in U_l \setminus U_k\}} dY_\tau^l|\right] \\ &\leq \mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{X_\tau \in U_l \setminus U_k\}} dK_\tau\right] \to 0.\end{aligned}$$

Thus $Y^l - Y^k \to 0$ uniformly in probability and consequently there is a continuous semimartingale Z such that $Y^k \to Z$ uniformly in probability and Z has the desired properties. A standard localization procedure shows then the assertion for the case that Y is locally uniformly bounded on $\{X \in O\}$.
(ii) follows from (i) with $Y = f(X)$ and an application of Itô's formula. □

Local-to-global

We first quote a useful Lemma, also known as 'space-time-localization':

Lemma 5.2.6 *Let M be a metric space, $(\Omega, \mathcal{F}_t, P)$ a filtered probability space and $X : \Omega \times \mathbb{R}_+ \to M$ a continuous adapted process. Let $(O_k)_{k\in\mathbb{N}}$ be a countable open covering of M. Then there is a sequence $(T_l)_{l\in\mathbb{N}}$ of stopping times with $T_0 = 0$, $T_l \leq T_{l+1}$, $\sup_l T_l = \infty$ such that for all $l \in \mathbb{N}$, $X_{|[T_l,T_{l+1}]\cap\{S<T\}} \in O_k$ for some $k \in \mathbb{N}$.*

Proof : [Éme89], Lemma 3.5. □

Next we prove a certain 'sheaf property' of stochastic integrals that are localized by a continuous process. Namely, if a stochastic integral is defined locally on a covering family of neighborhoods, then it is defined globally:

Lemma 5.2.7 *Let M be a metric space, $(\Omega, \mathcal{F}_t, P)$ a filtered probability space and $X : \Omega \times \mathbb{R}_+ \to M$ a continuous adapted process. Let $(O_\alpha)_{\alpha\in A}$ be a open covering of M. Let Y^α be a family of semimartingales such that for all $\alpha, \beta \in A$, $dY^\alpha = dY^\beta$ on $\{X \in O_\alpha \cap O_\beta\}$. Then there is a unique continuous semimartingale Y with $Y_0 = 0$ such that for all $\alpha \in A$, $dY = dY^\alpha$ on $\{X \in O_\alpha\}$.*

Proof : Let $(O_k)_{k\in\mathbb{N}}$ be a countable covering subordinated to $(O_\alpha)_{\alpha\in A}$, i.e. for all k, $O_k \subset O_\alpha$ for some α. By Lemma 5.2.6, there is a sequence T_l of stopping times such that $X_{|[T_l,T_{l+1}]\cap\{T_l<T_{l+1}\}} \in O_k \subset O_\alpha$ for some $k = k(l) \in \mathbb{N}$ and some $\alpha = \alpha(l) \in A$. Put

$$Y_t := \sum_l (Y_t^{\alpha(l)} - Y_{T_l}^{\alpha(l)}) \mathbf{1}_{|[T_l,T_{l+1}]}(t). \tag{5.21}$$

Let now $\alpha \in A$ and let $S \leq T$ be two stopping times such that $X_{|[S,T]} \in O_\alpha$. Then $X_{|[(S\vee T_l)\wedge T, S\vee(T\wedge T_{l+1})]} \in O_\alpha \cap O_{\alpha(l)}$ for all $l \in \mathbb{N}$ and hence by assumption, $Y_{(S\vee T_l)\wedge T} - Y_{S\vee(T\wedge T_{l+1})} = Y^\alpha_{(S\vee T_l)\wedge T} - Y^\alpha_{S\vee(T\wedge T_{l+1})}$ for all l. Thus

$$\begin{aligned} Y^\alpha_T - Y^\alpha_S &= \sum_l Y^\alpha_{(S\vee T_l)\wedge T} - Y^\alpha_{S\vee(T\wedge T_{l+1})} \\ &= \sum_l Y_{(S\vee T_l)\wedge T} - Y_{S\vee(T\wedge T_{l+1})} \\ &= Y_T - Y_S. \end{aligned}$$

This shows that $dY^\alpha = dY$ on $\{X \in O_\alpha\}$.
The uniqueness is proved analogously. Namely, for $S = 0$ and $T = t \in \mathbb{R}_+$, repeat the arguments from above in order to show that any semimartingale $\tilde{Y}$ that satisfies the properties stated in the Lemma must be equal to Y, defined in (5.21).
□

5.3 Parts of Markov processes

Let us briefly recall a standard procedure of localization in space for Markov processes. In this section, M denotes a locally compact Hausdorff space. IF X is a Markov process in M and $O \subset M$ an open set, then the *part of* X *on* O will be the process that is obtained by 'killing' X as soon as it reaches $X \setminus O$.

Definition 5.3.1 (i) Let $X = (\Omega, (X_t)_{t\geq 0}, \mathcal{F}, (P^x)_{x\in M_\Delta})$ be a Markov process in M. Let $O \subset M$ be an open set. Set

$$X^O_t(\omega) := \begin{cases} X_t(\omega) & \text{if } 0 \leq t < \tau_O(\omega) \\ \Delta & \text{else} \end{cases} \tag{5.22}$$

The *part of* X *on* O is the process $X^O = (\Omega, (X^O_t)_{0\leq t\leq \zeta}, \mathcal{F}, (P^x)_{x\in O_\Delta})$.
(ii) Let μ be a measure on M, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(M,\mu)$. The *part of* $\mathcal{E}$ *on* O is the Dirichlet form

$$\mathcal{E}^O := \mathcal{E}, \qquad \mathcal{D}(\mathcal{E}^O) := \{f \in \mathcal{D}(\mathcal{E}) : f = 0 \text{ q.e. on } O\}. \tag{5.23}$$

Note that the part of a process or a Dirichlet form on O can also be defined even if O is not open, cf. [FŌT94], chapter 4.4. and appendix A.2. However, we do not need this here in full generality, and if O is open, then taking parts respects many nice properties of processes and forms, which is the subject of the next

Proposition 5.3.2 *(i) X^O is a Markov process on O with transition function*

$$p^O_t(x, B) := P^x(X_t \in B, t < \tau_O). \tag{5.24}$$

If X is a Hunt process, then so is X^O.
(ii) Let $\mathcal{E}$ be a regular Dirichlet form on $L^2(M,\mu)$ such that X is associated to $\mathcal{E}$. Then $\mathcal{E}^O$ is a regular Dirichlet form on $L^2(M,\mu)$ and X^O is associated to $\mathcal{E}^O$.
(iii) Let X be a continuous strong Feller process on a metric space M such that for all relatively compact open subsets $U \subset\subset O$ and all $\epsilon > 0$,

$$\lim_{t\searrow 0} \sup_{x\in U} P(t,x,M\setminus B_\epsilon(x)) = 0. \tag{5.25}$$

Then X^O is strong Feller.

Proof : (i) [FŌT94], Theorem A.2.10.
(ii) [FŌT94], Theorem 4.4.2 and Theorem 4.4.3.
(iii) is proved in the Corollary to Theorem 13.3 in [Dyn65]. □

Denote by $\mathcal{F}^0_{\tau_O^-}$ the smallest $\sigma-$algebra containing all $\mathcal{F}^0_t \cap \{t < \tau_O\}$ $(t \geq 0)$. Then $\mathcal{F}^0_{\tau_O^-}$ is generated by the family of sets

$$\left\{ A = \bigcap_{i=0}^n \{X_{t_i} \in B_i, t_i < \tau_O\} : n \in \mathbb{N}, B_i \in \mathcal{B}(O) \right\}.$$

Thus for any $x \in O$, the restriction of P^x to $\mathcal{F}^0_{\tau_O^-}$ is uniquely determined by its transition function p^O. Likewise, the restriction of P^x to $\overline{\mathcal{F}}^0_{\tau_O^-}$ is uniquely determined by p^O, where $\overline{\mathcal{F}}^0_{\tau_O^-}$ is the completion of $\mathcal{F}^0_{\tau_O^-}$ w.r.t. P^x. This observation leads to the following useful localization Lemma:

Lemma 5.3.3 *Let $X = (\Omega, (X_t)_{t\geq 0}, \mathcal{F}, (P^x)_{x\in M_\Delta})$ and $\widetilde{X} = (\Omega, (X_t)_{t\geq 0}, \mathcal{F}, (\widetilde{P}^x)_{x\in M_\Delta})$ be two Markov processes on M such that $p^O(t,x,B) = \widetilde{p}^O(t,x,B)$ for all $t \geq 0$, all $x \in O$ and all $B \in \mathbb{B}(O)$. Then the following holds:*
(i) For all $x \in O$, P^x and $\widetilde{P}^x$ coincide on $\mathcal{F}^0_{\tau_O^-}$ and on $\overline{\mathcal{F}}^0_{\tau_O^-}$, where $\overline{\mathcal{F}}^0_{\tau_O^-}$ is the completion of $\mathcal{F}^0_{\tau_O^-}$ w.r.t. P^x.
(ii) Let $x \in O$ and let σ be a stopping time such that $\sigma < \tau_O$ P^x-a.s. Let $f : O \to \mathbb{R}$ be a bounded measurable function. Then $\mathbf{E}^x[f(X_\sigma)] = \widetilde{\mathbf{E}}^x[f(X_\sigma)]$.

Proof : (i) was proved just before the Lemma.
(ii) We have that X_σ is $\mathcal{F}_\sigma-$measurable since X and $\widetilde{X}$ are progressively measurable. So let $A \in \mathcal{F}_\sigma$, which means that $A \cap \{\sigma \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. Then

$$A \cap \{\sigma < \tau_O\} = \bigcup_{q\in\mathbb{Q}} A \cap \{\sigma \leq q\} \cap \{q < \tau\} \in \overline{\mathcal{F}}^0_{\tau_O^-}.$$

Now since $P^x(\{\sigma < \tau_O\}) = 1$, $A \in \overline{\mathcal{F}}^0_{\tau_O^-}$ and hence $\mathcal{F}_\sigma \subset \overline{\mathcal{F}}^0_{\tau_O^-}$. Thus (ii) follows from (i). □

Example 5.3.4 Let $(M, g) \subset V$ be an admissible piecewise smooth Riemannian polyhedron and let X be Brownian motion in M. So X is the strong Feller diffusion associated to the canonical Dirichlet form $\mathcal{E}$. Let $O \subset M$ be an open set. It follows from Lemma 2.4.7 (i) that X satisfies condition (5.25). Consequently, X^O is a strong Feller process that is associated to $\mathcal{E}^O$, given by

$$\mathcal{E}^O(f, g) = \int_O \langle \nabla f(x) \nabla g(x) \rangle dx, \qquad \mathcal{D}(\mathcal{E}^O) = \overline{\mathcal{C}_0^\infty(O)}. \tag{5.26}$$

Let now $(\widetilde{M}, \widetilde{g}) \subset V$ be another Riemannian polyhedron with corresponding Brownian motion $\widetilde{X}$ and canonical energy $\widetilde{\mathcal{E}}$. We can assume that X and $\widetilde{X}$ are realized as canonical processes on $\Omega = \mathcal{C}(\mathbb{R}_+, V_\Delta)$, so they only differ by the measures P^x and $\widetilde{P}^x$.
Let $M \cap \widetilde{M} \neq \emptyset$, and let $O \subset M \cap \widetilde{M}$ that is open both in M and $\widetilde{M}$. Clearly, $\mathcal{E}^O = \widetilde{\mathcal{E}}^O$ and hence for every bounded continuous function $f : O \to \mathbb{R}$ and all $t \geq 0$,

$$p_t^O f(x) = \widetilde{p}_t^O f(x) \tag{5.27}$$

for quasi every $x \in O$. Because p^O and $\widetilde{p}^O$ are Feller, (5.27) holds for *all* $x \in O$.
Assume now that O is relatively compact and let $f : \overline{O} \to \mathbb{R}$ be a continuous function. Let O_n be a sequence of relatively compact open sets such that for all n,

$$O \setminus B_{1/n}(\partial O) \subset O_n \subset \overline{O_n} \subset O.$$

Let $n_0 \in \mathbb{N}$. Then for all $x \in O_{n_0}$ and all $n \geq n_0$, $\tau_n < \tau$ P^x-a.s. and hence from Lemma 5.3.3 (ii) it follows that $\mathbf{E}^x[f(X_{\tau_n})] = \widetilde{\mathbf{E}}^x[f(X_{\tau_n})]$. Now $\tau_n \nearrow \tau = \tau_O$ P^x-a.s. and hence it follows from the continuity of X and $\widetilde{X}$ that

$$\mathbf{E}^x[f(X_\tau)] = \widetilde{\mathbf{E}}^x[f(X_\tau)]. \tag{5.28}$$

Now since every $x \in O$ is contained in O_{n_0} for some $n_0 \in \mathbb{N}$, (5.28) holds for all $x \in O$.

Bibliography

[AA81] Ralph Alexander and S. Alexander, *Geodesics in Riemannian manifolds-with-boundary*, Indiana Univ. Math. J. **30** (1981), no. 4, 481–488. MR MR620261 (82j:58040)

[AB91] Felix Albrecht and I. D. Berg, *Geodesics in Euclidean space with analytic obstacle*, Proc. Amer. Math. Soc. **113** (1991), no. 1, 201–207. MR MR1077783 (91k:53050)

[ABB87] Stephanie B. Alexander, I. David Berg, and Richard L. Bishop, *The Riemannian obstacle problem*, Illinois J. Math. **31** (1987), no. 1, 167–184. MR MR869484 (88a:53038)

[ABB93] ______, *Geometric curvature bounds in Riemannian manifolds with boundary*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 703–716. MR MR1113693 (93m:53034)

[Arn95] Marc Arnaudon, *Barycentres convexes et approximations des martingales continues dans les variétés*, Séminaire de Probabilités, XXIX, Lecture Notes in Math., vol. 1613, Springer, Berlin, 1995, pp. 70–85.

[Bal95] Werner Ballmann, *Lectures on spaces of nonpositive curvature*, Birkhäuser Verlag, Basel, 1995.

[Bas95] Richard F. Bass, *Probabilistic techniques in analysis*, Probability and its Applications (New York), Springer-Verlag, New York, 1995. MR MR1329542 (96e:60001)

[Bau91] Heinz Bauer, *Wahrscheinlichkeitstheorie*, fourth ed., Walter de Gruyter & Co., Berlin, 1991. MR 93a:60002

[BBI01] Dimitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, American Mathematical Society, Providence, RI, 2001. MR 1 835 418

[BÉK+98] M. T. Barlow, M. Émery, F. B. Knight, S. Song, and M. Yor, *Autour d'un théorème de Tsirelson sur des filtrations browniennes et non browniennes*, Séminaire de Probabilités, XXXII, Lecture Notes in Math., vol. 1686, Springer, Berlin, 1998, pp. 264–305. MR MR1655299 (2000g:60135)

[BG68] R. M. Blumenthal and R. K. Getoor, *Markov processes and potential theory*, Pure and Applied Mathematics, Vol. 29, Academic Press, New York, 1968. MR MR0264757 (41 #9348)

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.

[BJ73] Theodor Bröcker and Klaus Jänich, *Einführung in die Differentialtopologie*, Springer-Verlag, Berlin, 1973, Heidelberger Taschenbücher, Band 143. MR MR0358848 (50 #11307)

[BK95] M. Brin and Yu. Kifer, *Brownian motion and harmonic functions on polygonal complexes*, Stochastic analysis (Ithaca, NY, 1993), Proc. Sympos. Pure Math., vol. 57, Amer. Math. Soc., Providence, RI, 1995, pp. 227–237. MR MR1335474

[Bou05] Taoufik Bouziane, *Brownian motion in Riemannian admissible complexes*, Illinois J. Math. **49** (2005), no. 2, 559–580 (electronic). MR MR2164353

[BPY89] Martin Barlow, Jim Pitman, and Marc Yor, *On Walsh's Brownian motions*, Séminaire de Probabilités, XXIII, Lecture Notes in Math., vol. 1372, Springer, Berlin, 1989, pp. 275–293. MR MR1022917 (91a:60204)

[Cra91] M. Cranston, *Gradient estimates on manifolds using coupling*, J. Funct. Anal. **99** (1991), no. 1, 110–124. MR 93a:58175

[Dos62] Shafik Doss, *Moyennes conditionnelles et martingales dans un espace métrique*, C. R. Acad. Sci. Paris **254** (1962), 3630–3632.

[DT93a] R. Dante DeBlassie and Ellen H. Toby, *On the semimartingale representation of reflecting Brownian motion in a cusp*, Probab. Theory Related Fields **94** (1993), no. 4, 505–524. MR MR1201557 (94a:60118)

[DT93b] ———, *Reflecting Brownian motion in a cusp*, Trans. Amer. Math. Soc. **339** (1993), no. 1, 297–321. MR MR1149119 (93k:60199)

[Dud89] Richard M. Dudley, *Real analysis and probability*, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1989. MR 91g:60001

[Dyn65] E. B. Dynkin, *Markov processes. Vols. I, II*, Translated with the authorization and assistance of the author by J. Fabius, V. Greenberg, A. Maitra, G. Majone. Die Grundlehren der Mathematischen Wi ssenschaften, Bände 121, vol. 122, Academic Press Inc., Publishers, New York, 1965. MR MR0193671 (33 #1887)

[EF01] J. Eells and B. Fuglede, *Harmonic maps between Riemannian polyhedra*, Cambridge University Press, Cambridge, 2001.

[EK86] Stewart N. Ethier and Thomas G. Kurtz, *Markov processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986, Characterization and convergence. MR MR838085 (88a:60130)

[ÉM91] Michel Émery and G. Mokobodzki, *Sur le barycentre d'une probabilité dans une variété*, Séminaire de Probabilités, XXV, Lecture Notes in Math., vol. 1485, Springer, Berlin, 1991, pp. 220–233.

[Éme89] Michel Émery, *Stochastic calculus in manifolds*, Universitext, Springer-Verlag, Berlin, 1989, With an appendix by P.-A. Meyer. MR MR1030543 (90k:58244)

[ESH99] Aziz Es-Sahib and Henri Heinich, *Barycentre canonique pour un espace métrique à courbure négative*, 355–370.

[FŌT94] Masatoshi Fukushima, Yōichi Ōshima, and Masayoshi Takeda, *Dirichlet forms and symmetric Markov processes*, de Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 1994. MR MR1303354 (96f:60126)

[Fug05a] Bent Fuglede, *The Dirichlet problem for harmonic maps from Riemannian polyhedra to spaces of upper bounded curvature*, Trans. Amer. Math. Soc. **357** (2005), no. 2, 757–792 (electronic). MR MR2095630 (2006a:58017)

[Fug05b] ———, *Dirichlet problems for harmonic maps from regular domains*, Proc. London Math. Soc. (3) **91** (2005), no. 1, 249–272. MR MR2149857 (2006g:31006)

[GH96] Mariano Giaquinta and Stefan Hildebrandt, *Calculus of variations. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 310, Springer-Verlag, Berlin, 1996, The Lagrangian formalism. MR MR1368401 (98b:49002a)

[GP03] R. Ghomrasni and G. Peskir, *Local time-space calculus and extensions of Itô's formula*, High dimensional probability, III (Sandjberg, 2002), Progr. Probab., vol. 55, Birkhäuser, Basel, 2003, pp. 177–192. MR MR2033888 (2005j:60106)

[Hes04] Martin Hesse, *Harmonic maps into trees and graphs – analytical and numerical aspects*, PhD Thesis, Bonner Mathematische Schriften, vol. 370, University of Bonn, 2004.

[HT94] Wolfgang Hackenbroch and Anton Thalmaier, *Stochastische Analysis*, Mathematische Leitfäden. [Mathematical Textbooks], B. G. Teubner, Stuttgart, 1994, Eine Einführung in die Theorie der stetigen Semimartingale. [An introduction to the theory of continuous semimartingales]. MR MR1312827 (96e:60094)

[Jos97] Jürgen Jost, *Nonpositive curvature: geometric and analytic aspects*, Birkhäuser Verlag, Basel, 1997.

[Jos02] ______, *Riemannian geometry and geometric analysis*, third ed., Universitext, Springer-Verlag, Berlin, 2002.

[Ken90] Wilfrid S. Kendall, *Probability, convexity, and harmonic maps with small image. I. Uniqueness and fine existence*, Proc. London Math. Soc. (3) **61** (1990), no. 2, 371–406.

[Kos02a] N. N. Kosovskiĭ, *Gluing of Riemannian manifolds of curvature* $\geq \kappa$, Algebra i Analiz **14** (2002), no. 3, 140–157. MR MR1921991 (2003d:53053)

[Kos02b] ______, *Gluing of Riemannian manifolds of curvature* $\leq \kappa$, Algebra i Analiz **14** (2002), no. 5, 73–86. MR MR1970333 (2004g:53044)

[Kos04] ______, *Gluing with branching of Riemannian manifolds of curvature* $\leq \kappa$, Algebra i Analiz **16** (2004), no. 4, 132–145. MR MR2090854 (2005f:53044)

[KS91] Ioannis Karatzas and Steven E. Shreve, *Brownian motion and stochastic calculus*, second ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR MR1121940 (92h:60127)

[KS93] Nicholas J. Korevaar and Richard M. Schoen, *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom. **1** (1993), no. 3-4, 561–659. MR MR1266480 (95b:58043)

[LT91] Michel Ledoux and Michel Talagrand, *Probability in Banach spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 23, Springer-Verlag, Berlin, 1991.

[Pau] Gustav Paulik, *Anwendungen des Couplings von Flußprozessen auf Riemannschen Mannigfaltigkeiten*, Diplomarbeit.

[Pau04] ______, *Gluing spaces and analysis*, PhD Thesis, Bonner Mathematische Schriften, vol. 372, University of Bonn, 2004.

[Pic91] Jean Picard, *Martingales on Riemannian manifolds with prescribed limit*, J. Funct. Anal. **99** (1991), no. 2, 223–261.

[Pic05] ______, *Stochastic calculus and martingales on trees*, Ann. Inst. H. Poincaré Probab. Statist. **41** (2005), no. 4, 631–683. MR MR2144228 (2006e:60051)

[RR98] Svetlozar T. Rachev and Ludger Rüschendorf, *Mass transportation problems. Vol. I*, Springer-Verlag, New York, 1998, Theory. MR 99k:28006

[RY99] Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, third ed., Springer-Verlag, Berlin, 1999.

[Sch80] Laurent Schwartz, *Semi-martingales sur des variétés, et martingales conformes sur des variétés analytiques complexes*, Lecture Notes in Mathematics, vol. 780, Springer, Berlin, 1980. MR MR575167 (82m:60051)

[Stu95] Karl-Theodor Sturm, *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations*, Osaka J. Math. **32** (1995), no. 2, 275–312. MR MR1355744 (97b:35003)

[Stu96] K. T. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. (9) **75** (1996), no. 3, 273–297. MR MR1387522 (97k:31010)

[Stu98] ______, *Diffusion processes and heat kernels on metric spaces*, Ann. Probab. **26** (1998), no. 1, 1–55. MR MR1617040 (99b:31008)

[Stu01] Karl-Theodor Sturm, *Nonlinear Markov operators associated with symmetric Markov kernels and energy minimizing maps between singular spaces*, Calc. Var. Partial Differential Equations **12** (2001), no. 4, 317–357.

[Stu02] ———, *Nonlinear martingale theory for processes with values in metric spaces of nonpositive curvature*, Ann. Probab. **30** (2002), no. 3, 1195–1222.

[Stu03] K. T. Sturm, *Probability measures on metric spaces of nonpositive curvature*, to appear in Proceedings of 'Heat Kernels, Diffusions and Random Walks on Manifolds and Graphs' IHP 2002. AMS Contemporary Mathematics (2003), SFB-Preprint No. 79.

[Stu05] ———, *A semigroup approach to harmonic maps*, Potential Anal. **23** (2005), 225–277.

[Tsi97] B. Tsirelson, *Triple points: from non-Brownian filtrations to harmonic measures*, Geom. Funct. Anal. **7** (1997), no. 6, 1096–1142. MR MR1487755 (2001g:60199)

[Vil03] Cédric Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003. MR MR1964483 (2004e:90003)

[vRS04] M. K. von Renesse and K. T. Sturm, *Transport inequalities, gradient estimates, entropy and ricci curvature*, Comm. Pure Appl.Math. **58(7)** (2004), 923–940.

[Wag77] Daniel H. Wagner, *Survey of measurable selection theorems*, SIAM J. Control Optimization **15** (1977), no. 5, 859–903. MR MR0486391 (58 #6137)

[Wal78] John B. Walsh, *A diffusion with a discontinuous local time*, Astrisque **52-53** (1978), 37–45.

[Whi40] J. H. C. Whitehead, *On C^1-complexes*, Ann. of Math. (2) **41** (1940), 809–824. MR MR0002545 (2,73d)

[Yor97] Marc Yor, *Some aspects of Brownian motion. Part II*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1997, Some recent martingale problems. MR MR1442263 (98e:60140)

Index