Local Gauge Coupling Running in Supersymmetric Gauge Theories on Orbifolds

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Abstract

By extending Feynman's path integral calculus to fields which respect orbifold boundary conditions we provide a straightforward and convenient framework for loop calculations on orbifolds. We take advantage of this general method to investigate supersymmetric Abelian and non-Abelian gauge theories in five, six and ten dimensions where the extra dimensions are compactified on an orbifold. We consider hyper and gauge multiplets in the bulk and calculate the renormalization of the gauge kinetic term which in particular allows us to determine the gauge coupling running. The renormalization of the higher dimensional theories in orbifold spacetimes exhibits a rich structure with three principal effects: Besides the ordinary renormalization of the bulk gauge kinetic term the loop effects may require the introduction of both localized gauge kinetic terms at the fixed points/planes of the orbifold and higher dimensional operators.

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Contents

Chapter 1

Introduction

The Standard Model of Particle Physics

The Standard Model of particle physics describes the fundamtental particles that physicists today believe the world is made of. In this sense, it is a modern analog of Mendeleev's periodic table of the elements in which the atoms were considered to be the indivisible elementary building blocks of nature. The analysis of the Standard Model advances to even smaller structures: Its spectrum contains quarks, leptons, gauge bosons and a Higgs particle. While quarks and leptons are matter particles, the gauge bosons describe the mediation of forces between them and the Higgs particle fulfills a special task that is described below. Quarks appear in six different flavors belonging to two groups: They are either up-type quarks (up, charm, top) or down-type quarks (down, strange, bottom). There are two possibilities for quarks to form stable bound states, namely mesons that are a combination of a quark and an antiquark ($q\bar{q}$) and baryons that are composed of three quarks (qqq) or antiquarks ($\bar{q}\bar{q}\bar{q}$). Well-known examples for baryons are the proton (uud) and the neutron (udd) that form the atomic nuclei. Leptons are also divided into two categories, they either belong to the electron and its heavier counterparts (electron, muon, tauon) or to the neutrinos (electron-neutrino, μ-neutrino, τ-neutrino). Quarks and leptons are the constituents of all matter so far observed, for example all the atoms in the periodic table of the elements are made up from protons and neutrons confined in a nucleus with electrons orbiting around them. The other matter particles are found naturally in radioactive decays or cosmic rays or can be produced in collider experiments.

But the Standard Model is not merely a collection of particles, it also provides the rules how these particles interact with each other. It describes three of the four forces that are realized in nature (electromagnetic, strong and weak force) by the exchange of gauge bosons. The gauge group of the Standard Model is $SU(3)_C \times SU(2)_L \times U(1)_Y$. Here the $SU(3)_C$ factor of the gauge group belongs to the theory called quantum chromodynamics (QCD) that describes the strong force and is responsible for the fact that the quarks can bind together to a proton and that protons (which would repel due to their electromagnetic charge) are bound together in a nucleus. The dimension of $SU(3)_C$ (the number of its generators) is eight, therefore there are eight QCD gauge bosons which are called gluons.

The electromagnetic and the weak force are combined in the $SU(2)_L \times U(1)_Y$ factor of the Standard Model gauge goup. This sophisticated construction is the electroweak theory that was formulated by Glashow, Weinberg and Salam (GWS). The dimension of $SU(2)_L \times U(1)_Y$ is four and hence this theory contains four electroweak gauge bosons (three of them conventionally called W and one called B). This unified description of the electromagnetic and weak interactions is the valid picture at energies above roughly 100 GeV (the weak scale). Below the weak scale one encounters electroweak symmetry breaking which is implemented in the Standard Model via the Higgs effect: It is assumed that a Higgs particle exists which is not in its true potential minimum. Therefore, one has to shift the field by a vacuum expectation value with the result that the electroweak symmetry is broken. In this process three combinations of the electroweak gauge bosons become massive and form the W^+, W^- and Z^0 vector bosons of the weak interaction. A fourth combination is identified as the photon which is the massless gauge boson of the electromagnetic interaction. This is the way how Feynman's quantum electrodynamics (QED), the theory that describes the electromagnetic interaction separately, is embedded in the Standard Model. The Higgs mechanism is also responsible for giving mass to the fermions via Yukawa couplings. This Higgs particle which is of strong conceptual importance has not been observed yet and is thus the last missing piece for the confirmation of Standard Model physics. Collider experiments which have been conducted so far could only show that the Higgs, if it exists, has to be heavier then 114 GeV. On the other hand, electroweak precision data suggests an upper bound on the Higgs mass of around 200 GeV. Particle physicists hope to discover the Higgs in this mass window at the Large Hadron Collider (LHC) that is currently under construction at the European laboratory for particle physics (CERN)[1].

The theoretical framework of the Standard Model is that of a quantum field theory (QFT) which treats particles as point-like objects. The quantum aspect of a theory becomes important when one describes physics at small scales or in other words at high energies. The theory allows the calculation of cross-sections of particle reactions and particle lifetimes and has been tested up to roughly 100 GeV. So far no deviations to the theory have been found and it seems to describe nature extremely well. This is in spite of the fact that the Model has conceptual

[1]Originally, the abbreviation CERN referred to the Conseil Européen pour la Recherche Nucléaire, a council that was founded in 1952 with the aim to establish a fundamental physics research organization in Europe. The council was dissolved two years later, when the European Organization for Nuclear Research came into existence which hosts the European laboratory for particle physics.

drawbacks if one wants to regard it as a universal theory of nature.

One tends to assume that the Standard Model is not the final theory. One reason for this is that the Standard Model has of the order 20 free parameters and one thinks that a fundamental theory of nature should be a unified concept with less parameters. The expectation is that in a unified theory new physics comes into play at some high energy scale (e.g. a GUT scale of the order of 10^{16} GeV or an even higher string scale, see below). The question of why there is such a big difference between this high scale and the weak scale of fourteen orders of magnitude or more and how this can be implemented consistently into a single theory is called the hierarchy problem. One aspect of the hierarchy problem is that if the theory contains two scales, they should not mix with each other. This problem manifests itself in the question of how the Higgs mass can be kept at the order of the weak scale, because the mass receives large loop corrections that are quadratic in the high energy scale and is thus driven to very high values.

The Standard Model describes only three of the four forces that are realized in nature. The missing item here is the gravitational force and it is this force around which Einstein's theory of general relativity is centered. While being a theory that can look back at remarkable successes of its own, general relativity seems to be incompatible with the Standard Model. The reason is that the mathematical framework of Einstein's theory is that of a classical field theory and it is not known how to quantize general relativity consistently in order to combine it with the Standard Model. The consistent quantization of general relativity which would allow to enlarge the Standard Model to a theory which would encompass gravitation together with the other three forces in a unified setup is one of the greatest puzzles of theoretical physics today.

Beyond the Standard Model: Grand Unified Theories, Supersymmetry, Extra Dimensions and Strings

One attempt to reduce the number of parameters of the Standard Model is inspired by the success of the GWS theory of the electroweak interactions which successfully unifies the electromagnetic and weak interactions as we have seen above. With this idea in mind, one can look for a gauge group that contains the Standard Model gauge group as a subgroup and thus describes the three forces within the framework of a Grand Unified Theory (GUT). It is necessary for this process to work, that the gauge couplings coincide at a certain energy scale. The renormalization group running of the gauge couplings determines the GUT scale at which this unification is supposed to take place to lie around 10^{16} GeV. In the Standard Model, however, the gauge couplings come only close to each other at these high energies, but they fail to meet in a point. (The three graphs cut out an area which is called the GUT triangle.) An answer to this problem might be provided by supersymmetry.

Supersymmetry is a symmetry that relates bosons and fermions. In four dimensions it has in its simplest version ($\mathcal{N} = 1$) four supercharges such that it associates to every particle one supersymmetric partner. These particles together form a supersymmetry multiplet which is conveniently described in terms of a superfield, because this notation makes cancellations explicit that appear when both particles run around in a loop. In this way a spin-0 boson and a spin-1/2 fermion are described as a chiral superfield and a spin-1 vector boson and a spin-1/2 fermion as a vector superfield. The analog of the Standard Model that is enlarged by this minimal supersymmetry is called the Minimal Supersymmetric Standard Model (MSSM). The spectrum of the MSSM thus consists of the usual Standard Model particles plus their partners: quarks and leptons are accompanied by squarks and sleptons which are spin-0 bosons, and the gauge bosons by the gauginos which are spin-1/2 fermions. While the Standard Model involves only one Higgs field, the MSSM needs two Higgs fields which give mass to the up-type and down-type quarks separately. Their two supersymmetric partners, the higgsinos, are also required because of anomaly cancellation. This setup has remarkable advantages: Quadratic divergences in the loop corrections to the Higgs mass are absent and only a logarithmic divergence appears due to the fact that loops involving particles are cancelled by loops involving their supersymmetric partners. Therefore, supersymmetry solves the aspect of the hierarchy problem that the scales should not mix with each other. In addition, the gauge coupling running shows the beautiful feature that the gauge couplings unify in a point which motivates the consideration of supersymmetric GUTs. Another motivation for supersymmetry comes from string theory (see below), which, when one wants to regard it as the fundamental theory of nature, only makes sense in its supersymmetric version. In a theory where supersymmetry is intact, a particle and its partner possess the same mass. However, no supersymmetric particles have been observed experimentally so far. This means that supersymmetry, if it exists, must be broken somewhere above the presently accessible energy scales. It is the hope of many particle physicists to find indications of supersymmetry at the LHC. For textbook introductions to supersymmetry consider [1–5], the standard review articles are [6–8].

Another approach to resolve the hierarchy between the Planck scale and the weak scale is the introduction of (large) extra dimensions [9, 10]. In this setup the four dimensional Planck mass is given by a product of two factors, one involving a higher dimensional fundamental mass scale and the other the volume of the extra space. By considering large extra dimensions of the order TeV^{-1} the fundamental scale can be chosen of the order of the weak scale and still reproduce the correct four dimensional Planck scale. Hence, there is only *one* fundamental scale (the weak scale) and no hierarchy at all. In order to stabilize this construction one can consider supersymmetric theories in extra dimensions. For overviews on extra dimensions see e. g. [11, 12].

From the higher dimensional supersymmetric theories that exist those with a minimal amount of supersymmetry in five, six and ten dimensions will be relevant for our work. We briefly discuss the supersymmetry multiplets that appear and indicate how they can be described in terms of four dimensional superfields: In five and six dimensions the minimal amount of supercharges is eight and (besides the supergravity and tensor multiplet that are irrelevant for our discussion) the possible supersymmetry multiplets are hyper and gauge multiplets. The hyper multiplet is described in the four dimensional language by two chiral superfields and the gauge multiplet by one four dimensional gauge superfield and a chiral superfield that transforms in the adjoint of the gauge group. In the ten dimensional theory with the minimal amount of sixteen supercharges there are no hyper multiplets and hence the only multiplet that is important for our discussion is the vector multiplet which contains the ten dimensional gauge fields and gauginos. The four dimensional description consists of one four dimensional gauge superfield and three chiral superfields in the adjoint of the gauge group. The four dimensional language for higher dimensional superfields was introduced in [13] for ten dimensions, before it was developed for the vector multiplet in five dimensions [14] and, finally, for the vector and matter multiplets in five, six and ten dimensions [15]. A gauge covariant formalism for five dimensions was given in [16, 17].

The most promising candidate for a theory that could unify the three forces of the Standard Model with general relativity seems to be string theory. In fact, string theory combines all the ideas of grand unification, supersymmetry and extra dimensions. The theory had originally been developed to describe the confinement of quarks due to the strong force by a string that stretches between them. The existence of a massless spin-2 particle in its spectrum which is not realized in the hadronic world was troublesome in this respect, but could eventually be turned into a success when it was realized that this particle could be reinterpreted as the mediator particle of the gravitational force, the graviton, thus giving rise to a theory that could possibly contain both the Standard Model and general relativity [18]. String theory gives up the concept of QFT that particles are essentially point-like objects and proposes instead that the fundamental building blocks of our world are one dimensional strings. The different elementary particles that are observed in nature are then supposed to be described by the different vibrational modes of the string. The string is a mapping from two coordinates (world-sheet coordinates) to a D-dimensional target space. While a string moves in the target space it sweeps out a world sheet just as a point particle sweeps out a world line. Strings can be open and closed and during their propagation in spacetime they can split and merge. For textbook introductions to string theory consider [19–23].

Simple bosonic string theory can only be formulated consistently in 26 spacetime dimensions. It contains a tachyon as the lowest lying state which signals an instability of the theory, and so string theory needs to include fermions. This is solved by the implementation of supersymmetry, and hence a consistent string theory has to be supersymmetric. There are five different supersymmetric string theories (superstring theories): type I, type IIA and IIB and heterotic $E_8 \times E_8$ and $SO(32)$. All of them can only be formulated consistently in ten dimensions. In our work we have applications in the context of the heterotic theories in mind, so we will describe these in a bit more detail. Heterotic string theory was constructed by Gross, Harvey, Martinec and Rohm [24–26]. The heterotic theories involve only closed oriented strings and are hybrid constructions of right-movers from a ten dimensional type II superstring [27, 28] and left-movers from a 26 dimensional bosonic string. Although the bosonic string is involved, the complete theory is free of tachyons. This is achieved, because a tachyon on the supersymmetric side is projected out by the so-called GSO projection [29, 30] such that the tachyon on the bosonic side does not have a counterpart on the superstring side and is removed by level-matching. Sixteen of the left-mover dimensions are compactified on a sixteen dimensional torus giving rise to the gauge group. The possible gauge groups that are consistent with the requirement of anomaly cancellation are $E_8 \times E_8$ and $SO(32)$ [31]. The heterotic string theories have $\mathcal{N} = 1$ supersymmetry in ten dimensions corresponding to sixteen supercharges. Their low-energy limits at weak coupling are given by $\mathcal{N} = 1$ supergravity coupled to $E_8 \times E_8$ and $SO(32)$ super Yang-Mills theory (SYM), respectively.

Although it is far beyond the scope of this thesis, let us also say a few words about the question how string theory can be the fundamental theory of nature when there are actually five different theories. In fact, it was found out that all five string theories are related among each other by a web of duality transformations. The two heterotic theories for example are T-dual to each other which means that if one compactifies the theories on a circle, they can be mapped to each other when one replaces the radius by its inverse [32, 33]. Furthermore, the low energy limit of heterotic $E_8 \times E_8$ theory at strong coupling is given by eleven dimensional supergravity [34, 35] on an interval where each of the E_8 factors is restricted to a ten dimensional boundary [36, 37]. It has been conjectured that a theory called M-theory might exist which would be the underlying unified theory for the five string theories as well as eleven dimensional supergravity.

To make contact to the world around us string theory should provide us with the correct four dimensional physics. The large symmetry groups should be broken to the Standard Model gauge group and the correct particle spectrum should be obtained. A solution might be provided by compactification of the six extra dimensions on a compact manifold whose volume is so small that the extra space is not observable. Hence the quest for the correct description of nature is translated into the the task to choose the right compactification manifold that reproduces the correct low energy physics. Let us consider heterotic string theory for concreteness. Compactification on a six-dimensional torus leaves all sixteen supersymmetries unbroken such that the resulting four-dimensional theory corresponds to $\mathcal{N} = 4$ supersymmetry in four dimensions [38]. We would, however, rather have $\mathcal{N} = 1$ in four dimensions with the remaining supersymmetry broken at energies lower than the compactification scale by some field theoretic formalism. Then one possibility for a class of manifolds that one can choose are the so-called Calabi-Yau manifolds [39]. These manifolds are very hard to describe mathematically. But Dixon, Harvey, Vafa and Witten [40, 41] found out that one can compactify strings on orbifolds which represent a certain limit of Calabi-Yaus. An orbifold is a modification of a torus where certain points of the torus are identified under a symmetry of the torus lattice and a number of fixed points on the torus is invariant under this symmetry. It is not a manifold, because of singularities arising at these fixed points. Nevertheless, string theory can be described consistently on orbifolds. The existence of fixed points is reflected in the spectrum of the string theory. Besides those states that correspond to the dimensionally reduced theory and that are free to propagate through the whole orbifold bulk (untwisted states) there are additional states whose center of mass is localized at the fixed points (twisted states). The existence of theses states is enforced by a string symmetry called modular invariance.

After they were introduced in the context of string theory, orbifold compactifications became popular within higher dimensional field theories to adress questions like supersymmetry and gauge symmetry breaking [14, 42, 43] and the doublet-triplet splitting problem that arises in GUTs [44, 45]. While field theory orbifolds do not have modular invariance at their disposal some information about states that are localized at the fixed points can be obtained from the requirement of local anomaly cancellation [46]: In order for the theory to be consistently defined the anomalies must cancel in the bulk *and* at each of the fixed points separately. For this purpose one has to place fields at the fixed points by hand which are then regarded as analogs of the twisted states in string theory. String theory orbifolds were already known to be globally anomaly-free, and the fact that the twisted states that string theory proposes to live at the orbifold fixed points also meet the stronger requirement of local anomaly cancellation could be shown for prime orbifolds in [47] and for non-prime orbifolds in [48]. The local picture has also become important for string theory phenomenology. By choosing the compactification parameters appropriately, one can ensure that certain GUT gauge groups live locally at the different fixed points of the orbifold. In the limit where the compactification radius goes to zero the global four dimensional gauge group (hopefully that of the Standard Model) arises as the intersection of the local gauge groups. Therefore, string models can be interpreted as orbifold GUTs where GUT gauge groups are localized at lower dimensional subspaces of the string theory orbifold [49–51]. An important issue in string theory model building is to develop good search strategies in order to find realistic models [52–56]. It is especially in the context of the local GUT picture of string theory orbifolds where the results of this thesis about the local gauge coupling running can be applied.

Higher dimensional operators and localization of quantum corrections at orbifold fixed points

We have tried to sketch the picture of high-energy physics phenomenology today. Now we want to introduce some concepts that are relevant in particular for this thesis. The starting point may be seen in the papers by Dienes, Dudas and Gherghetta (DDG) who began to make the extra dimensions scenario fruitful for Grand Unification [57,58]. They proposed to place the MSSM on a higher dimensional spacetime, where the extra space is compactified on an orbifold. Taking only Kaluza-Klein modes up to a certain energy threshold into account, DDG found that the gauge coupling running is modified by a term which goes with a power-law in the cutoff scale (in addition to the usual MSSM gauge coupling running). It was argued that this term would help to realize a low-scale gauge coupling unification. However, it has been observed that neglecting the Kaluza-Klein modes above a certain energy threshold bears difficulties with respect to the consistency of the analysis, because the running is strongly sensitive to the precise choice of the cutoff [59–61]. Later it could be shown that within the framework of spontaneously broken GUT models power-like corrections to the differences of the gauge couplings can be calculated consistently [62–64]. In this setup a discrete symmetry that is realized at high energy scales ensures that the gauge coulings are related to each other such that the sensitivity to the choice of the cutoff vanishes in their difference.

During the investigation of the consistency of the regularization procedure in the DDG model it was realized that taking into account the full tower of Kaluza-Klein modes can lead to the appearance of higher dimensional operators in the renormalization process [65]. This effect will be important for our discussion. In [66] Ghilencea considered a toy model of QED in five and six dimensions with the extra dimensions compactified on an orbifold. In this setup the self-energy of the four dimensional zero-mode photon due to a fermion and its associated Kaluza-Klein tower in the loop was calculated for off-shell external momentum $q^2 \neq 0$. It turned out that in the case of two compact dimensions the quantum effects involve divergent terms that are proportional to q^2. As the external momentum corresponds to a derivative operator in position space, the one-loop counterterms that are necessary to cancel the divergences involve higher dimensional (derivative) operators. These counterterms are not present if the Kaluza-Klein towers are truncated to any large number of modes (as it had been the case for example in the DDG analysis) and obviously could not have been observed in earlier on-shell analysis. It was shown later that higher derivative operators can also be generated in five dimensional models [67]. Here they appear as one-loop counterterms to the mass of a zero-mode scalar field due to the existence of a superpotential that is localized on the ($\mathbb{Z}_2$) fixed points of an $S^1/(\mathbb{Z}_2 \times \mathbb{Z}_2')$ orbifold and couples the scalar field to Kaluza-Klein towers of hyper multiplets in the bulk. Other setups showed the generation of higher derivative operators as counterterms to the mass of a zero-mode scalar field from a localized superpotential and also from bulk gauge interactions on the orbifold $S^1/\mathbb{Z}_2$ in [68] and on $T^2/\mathbb{Z}_N$ in [69]. (For gauge coupling renormalization in warped geometry see [60].)

Another effect that is important for our discussion is the fact that the renormalization of a theory in a (higher-dimensional) space induces effects that are localized on lower-dimensional subspaces. As the authors of [70] found out, loop corrections to a bulk theory that is compactified on a orbifold can give rise to infinities that must be renormalized by terms that are localized on a brane. They considered a simple model of fermions and scalars interacting on a five dimensional space with the fifth coordinate compactified on an $S^1/\mathbb{Z}_2$ orbifold and concluded that even when one chooses a brane localized action to be absent in the first place, the loop corrections of the five-dimensional theory induce nontrivial physics on the fixed points and are responsible for a running of the four dimensional couplings that belong to a theory located at the fixed points.

Local gauge coupling running in supersymmetric theories on orbifolds

Based on the publications [71, 72] this thesis investigates the renormalization of supersymmetric theories in orbifold spacetimes in five, six and ten dimensions. Publication [71] considered for the first time the renormalization of the gauge kinetic term in a theory of bulk hypermultiplets that are coupled to Abelian gauge multiplets in five and six dimensions where the extra dimensions are compactified on an orbifold. Publication [72] extended the same analysis to the situation where the gauge group is non-Abelian. The investigation of the ten dimensional case with the extra six dimensions compactified on an orbifold and a ten dimensional gauge multiplet in the bulk is presented in this thesis for the first time. In order to perform the calculations of supersymmetric Feynman graphs on orbifolds in a straightforward way, in [71, 72] a general method was developed that conveniently extends the path integral calculus of Feynman to fields that respect the orbifold boundary conditions. As a consequence, this allows for a straightforward calculation of loop effects in orbifold field theories. The renormalization of the gauge kinetic term that we pursue here is to be seen as just one application of this method whose advantage over the usual approach lies primarily in the fact that the necessary Kaluza-Klein expansion of the fields is performed at a very late stage of the calculation. This is very helpful, because cancellations within the loop effects can be seen at a very early stage and no unnecessary expansions have to be performed. With the help of this formalism we determine in each of the above cases the renormalization of the kinetic term of the gauge field, which means that we can in particular derive the gauge coupling running.

As explained above, besides the renormalization of the bulk kinetic term of the gauge field, the quantum corrections can give rise to a renormalization of kinetic terms of the gauge field that are localized on the fixed points/fixed planes of the orbifold. In addition, higher dimensional operators can be generated. We investigate these effects.

The thesis is organized as follows:

Chapter 2 contains an introduction to the basic notions of supersymmetric field theory: superspace, superfields and supersymmetric actions. Starting from the supersymmetry algebra we present the well-known central technical identities that are used throughout the thesis.

Chapter 3 provides standard material on the renormalization of the gauge coupling in supersymmetric Abelian and non-Abelian gauge theories in four dimensions. Besides the reproduction of the standard results we exemplify the method of renormalizing in the generating functional of the theory as it is used later in more complicated setups.

Chapter 4 presents the five dimensional calculations from publications [71,72]. We consider a five dimensional spacetime with the fifth dimension compactified on $S^1/\mathbb{Z}_2$ and a hyper and a gauge multiplet in the bulk. The method of calculating Feynman graphs is used to calculate the self-energy graphs directly on the orbifold. We find that while the bulk hyper multiplets generate a linearly divergent loop correction in the bulk, they do not give rise to a renormalization of the four dimensional gauge coupling at the fixed points. This is in particular true for the Abelian theory, where the complete result vanishes, whereas in the non-Abelian case only those terms of the higher dimensional theory are renormalized that do not contribute to the four dimensional gauge coupling. The non-Abelian gauge multiplet, however, induces renormalizations of both bulk and brane gauge couplings. Its loop effects are such that at the fixed points *only* the four dimensional gauge superfield receives a loop correction. A higher dimensional operator is not generated in the five dimensional setup.

Chapter 5 presents the six dimensional calculations from [71,72]. The analysis of the preceeding chapter is generalized to six dimensions with the extra two dimensions compactified on a $T^2/\mathbb{Z}_N$ orbifold and a hyper and a gauge multiplet in the bulk. The cancellation of the loop corrections due to a hyper multiplet that we encountered at the fixed points of $S^1/\mathbb{Z}_2$ is translated into the fact that the four dimensional gauge coupling does not receive loop corrections from a bulk hyper multiplet at the $\mathbb{Z}_2$ fixed points of an even ordered orbifold. Again we observe that in the Abelian case the complete self-energy vanishes, while in the non-Abelian theory only those terms are renormalized that do not contribute to the four dimensional gauge coupling. However, the bulk hyper multiplet induces quantum corrections at those fixed points of $T^2/\mathbb{Z}_N$ that do not belong to a $\mathbb{Z}_2$ subset. The gauge mutiplet induces quantum corrections at all fixed points. At the $\mathbb{Z}_2$ fixed points, however, only the kinetic operator of the four dimensional gauge superfield is renormalized. Another issue encountered in six dimensions is the generation of higher derivative operators at one loop. The operators are generated from loops that involve the hyper multiplet both in the Abelian and in the non-Abelian theory and the gauge superfield in the non-Abelian theory. As our formalism works in position space, we have a resolution where the quantum effects are located: We can see that the higher dimension operators are generated in the bulk, while the fixed points contribute just the usual four dimensional logarithmic running.

Chapter 6 contains unpublished work on gauge coupling renormalization in a ten dimensional spacetime with the six extra coordinates compactified on a $T^6/\mathbb{Z}_N$ orbifold and a gauge multiplet in the bulk. Here we compute the renormalization of the kinetic term of the four dimensional gauge multiplet due to the self-interactions of the ten dimensional bulk gauge multiplet. We find that the renormalization in the bulk vanishes identically. This is a good cross-check, because we reproduce the well-known result that due to the high amount of supersymmetry there is no gauge coupling running in ten dimensions. But the implications of our result are even stronger, because the calculation reveals the fact that no higher dimensional operators are generated as loop counterterms in the bulk. Then we present for the first time the analysis how the theory renormalizes on the lower dimensional subspaces of the orbifold. As the amount of supersymmetry on the lower dimensional subspaces is reduced, certain operators in the action of the ten dimensional vector multiplet are renormalized while others are projected out. Higher dimensional operators are also generated, but this time they are not localized in the bulk of the orbifold, but on its six dimensional subspaces.

Chapter 7 concludes this thesis with a summary and a brief outlook. The technicalities are organized in the appendices.

Chapter 2

Supersymmetric field theory

In this chapter we recapitulate $\mathcal{N} = 1$ supersymmetry in four dimensions. In the context of this thesis, this is important because the higher dimensional supersymmetric multiplets which are considered later can be expressed in a four dimensional language. The emphasis in this chapter is on the technical side, since it provides the notions and identities that will be used in the rest of the thesis. We start with the supersymmetry algebra and its linear representation on superfields living on superspace. The supercovariant derivatives are given together with helpful identities. The chiral and vector superfields form irreducible representations of the supersymmetry algebra. Their restrictions to component fields are discussed. The F-term of a chiral superfield and the D-term of a vector superfield are used to construct invariant actions. It is shown how the fundamental restrictions of the superfields are used to rewrite a superfield action into its component field form. All this material is standard. Textbook introductions to four dimensional supersymmetry can be found in [1–5].

2.1 Supersymmetry algebra and superspace

Our starting point is the $\mathcal{N} = 1$ supersymmetry algebra, which is given by the following commutators and anticommutators of the supersymmetry generator Q and the four momentum P

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2\sigma_{\alpha\dot{\alpha}}{}^m P_m, \\ \{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \\ [P_m, Q_\alpha] &= [P_m, \bar{Q}_{\dot{\alpha}}] = 0. \end{aligned} \tag{2.1}$$

The notion $\mathcal{N} = 1$ refers to the fact that we have only one supersymmetry generator. The Greek indices $\alpha, \beta, \dot{\alpha}, \dot{\beta}$ run from one to two and label the two components of Weyl spinors, while the Latin indices m, n run from one to four and refer to Lorentz four vectors. With the help of parameters ξ^α, $\bar{\xi}_{\dot{\alpha}}$ that anticommute with other spinors and commute with bosonic quantities

$$\begin{aligned}
\{\xi^\alpha, \xi^\beta\} = \{\bar{\xi}_{\dot{\alpha}}, \bar{\xi}_{\dot{\beta}}\} = \{\xi^\alpha, \bar{\xi}_{\dot{\beta}}\} = 0, \\
[\xi^\alpha, a^m] = [\bar{\xi}_{\dot{\alpha}}, a^m] = 0, \\
\{\xi^\alpha, Q_\beta\} = \{\xi^\alpha, \bar{Q}_{\dot{\beta}}\} = [\xi^\alpha, P_m] = 0, \\
\{\bar{\xi}_{\dot{\alpha}}, Q_\beta\} = \{\bar{\xi}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = [\bar{\xi}_{\dot{\alpha}}, P_m] = 0.
\end{aligned} \tag{2.2}$$

the supersymmetry algebra is represented in terms of commutators only

$$\begin{aligned}
\left[\, \xi Q, \bar{\xi}\bar{Q} \,\right] = 2\, \xi \sigma^m \bar{\xi} P_m, \\
\left[\, \xi Q, \xi Q \,\right] = \left[\, \bar{\xi}\bar{Q}, \bar{\xi}\bar{Q} \,\right] = 0, \\
\left[\, P^m, \xi Q \,\right] = \left[\, P^m, \bar{\xi}\bar{Q} \,\right] = 0,
\end{aligned} \tag{2.3}$$

where the following summation convention for spinorial quantities has been employed: Undotted indices are summed from upper left to lower right such that $\xi Q := \xi^\alpha Q_\alpha$, whereas dotted indices are summed from lower left to upper right $\bar{\xi}\bar{Q} := \bar{\xi}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}$. Indices are raised and lowered with the help of the antisymmetric ϵ-tensor with $\epsilon^{12} = \epsilon_{21} = 1$ as

$$\xi^\alpha = \epsilon^{\alpha\beta} \xi_\beta, \qquad \bar{\xi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta}}, \tag{2.4}$$

and two ϵ-tensors are contracted to a Kronecker symbol as $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha_\gamma$ and $\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}_{\dot{\gamma}}$. Summing over the indices in the "wrong" direction brings in a minus sign: $\xi^\alpha \eta_\alpha = -\xi_\alpha \eta^\alpha$ and $\bar{\xi}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}} = -\bar{\xi}^{\dot{\alpha}} \bar{\eta}_{\dot{\alpha}}$. The Pauli matrices σ^m have the fixed index structure $\sigma_{\alpha\dot{\alpha}}{}^m$ such that, for example, the product with spinors in the first line of (2.3) reads $\xi \sigma^m \bar{\xi} = \xi^\alpha \sigma_{\alpha\dot{\alpha}}{}^m \bar{\xi}^{\dot{\alpha}}$.

Representation of the supersymmetry algebra on superfields

In the form (2.3) the supersymmetry algebra may be viewed as a Lie algebra with anticommuting parameters. Hence a group element is defined in the usual way

$$G\big(x^m, \theta, \bar{\theta}\big) = \exp\big(-ix^m P_m + i\theta Q + i\bar{\theta}\bar{Q} \big). \tag{2.5}$$

The representation space for the supersymmetry transformation is therefore an 8-dimensional manifold parameterized by the coordinates x^m, θ_α and $\bar{\theta}_{\dot{\alpha}}$. This manifold is called superspace. Functions which live on superspace are called superfields. A unitary representation of the supersymmetry algebra on superfields is constructed in the following. The multiplication of two group elements gives

$$G\big(a^m,\xi,\bar\xi\big)\,G\big(x^m,\theta,\bar\theta\big) = G\big(a^m + x^m + i\theta\sigma^m\bar\xi - i\xi\sigma^m\bar\theta,\ \theta+\xi,\ \bar\theta+\bar\xi\big). \quad (2.6)$$

where Hausdorff's formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$ has been used. The group multiplication law (2.6) induces a motion in the parameter space

$$\big(x^m,\theta,\bar\theta\big) \to \big(a^m + x^m + i\theta\sigma^m\bar\xi - i\xi\sigma^m\bar\theta,\ \theta+\xi,\ \bar\theta+\bar\xi\big). \quad (2.7)$$

Hence even a pure supersymmetry transformation with $a^m = 0$ induces a motion in the Minkowski coordinates. A superfield $\mathscr{F}$ transforms under the motion (2.7) as

$$\begin{aligned}
\mathscr{F}\big(x^m,\theta,\bar\theta\big) &\to \mathscr{F}\big(x^m + a^m + i\theta\sigma^m\bar\xi - i\xi\sigma^m\bar\theta,\ \theta+\xi,\ \bar\theta+\xi\big) \\
&= \mathscr{F}\big(x^m,\theta,\bar\theta\big) + \big(a^m + i\theta\sigma^m\bar\xi - i\xi\sigma^m\theta\big)\,\frac{\partial\mathscr{F}}{\partial x^m} + \xi^a\frac{\partial\mathcal{F}}{\partial\theta^\alpha} + \bar\xi_{\dot\alpha}\frac{\partial\mathscr{F}}{\partial\bar\theta_{\dot\alpha}} \\
&\equiv U\mathscr{F}\big(x^m,\theta,\bar\theta\big),
\end{aligned} \quad (2.8)$$

where U is the unitary representation that we are looking for. The ansatz

$$U = \exp\big(\,ia^m P_m + i\xi Q + i\bar\xi\bar Q\,\big) \quad (2.9)$$

allows to determine by identification the realization of the supersymmetry algebra on superfields

$$\begin{aligned}
P_m &= -i\partial_m \\
iQ_\alpha &= \ \ \partial_\alpha - i\sigma_{\alpha\dot\alpha}{}^m\bar\theta^{\dot\alpha}\partial_m \\
i\bar Q_{\dot\alpha} &= -\bar\partial_{\dot\alpha} + i\theta^\alpha\sigma_{\alpha\dot\alpha}{}^m\partial_m .
\end{aligned} \quad (2.10)$$

One checks explicitly that P, Q and $\bar Q$ fulfill the supersymmetry algebra (2.1). The derivatives w.r.t. Minkowski coordinates and the dotted and undotted spinorial coordinates, respectively, in (2.10) are defined as

$$\partial_m = \frac{\partial}{\partial x^m}, \qquad \partial_\alpha = \frac{\partial}{\partial\theta^\alpha}, \qquad \bar\partial^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}}. \quad (2.11)$$

Differentiation of spinorial coordinates is defined as

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta, \qquad \bar\partial^{\dot\beta}\bar\theta_{\dot\alpha} = \delta_{\dot\alpha}^{\dot\beta} \quad (2.12)$$

and the squares of the derivatives $\partial^2 = \epsilon^{\alpha\beta}\,\partial_\beta\partial_\alpha$ and $\bar\partial^2 = \epsilon_{\dot\alpha\dot\beta}\,\bar\partial^{\dot\beta}\bar\partial^{\dot\alpha}$ fulfill

$$\partial^2\,\theta^2 = \bar\partial^2\,\bar\theta^2 = -4, \quad (2.13)$$

which will allow us to establish a connection between differentiation and integration w.r.t. Weyl spinors in (2.44).

2.2 Supercovariant derivatives

Supercovariant derivatives D_α and $\bar{D}_{\dot\alpha}$ which anticommute with the supersymmetry generators

$$\begin{aligned}\{D_\alpha, Q_\beta\} &= \{D_\alpha, \bar{Q}_{\dot\beta}\} = 0, \\ \{\bar{D}_{\dot\alpha}, Q_\beta\} &= \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0,\end{aligned} \tag{2.14}$$

and hence commute with an infinitesimal supersymmetry transformation are then defined as

$$\begin{aligned}D_\alpha &= \quad \partial_\alpha + i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m, \\ \bar{D}_{\dot\alpha} &= -\bar\partial_{\dot\alpha} - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m.\end{aligned} \tag{2.15}$$

The supercovariant derivatives fullfil the following anticommutation relations among themselves

$$\begin{aligned}\{D_\alpha, \bar{D}_{\dot\alpha}\} &= -2i\sigma^m_{\alpha\dot\alpha}\partial_m, \\ \{D_\alpha, D_\beta\} &= \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0.\end{aligned} \tag{2.16}$$

It is useful to write out the products D^2 and $\bar{D}^2$ explicitly

$$\begin{aligned}D^2 &= \epsilon^{\alpha\beta}\big(\partial_\beta\partial_\alpha - 2i\,\sigma_{\alpha\dot\alpha}{}^m\bar\theta^{\dot\alpha}\partial_\beta\partial_m + \sigma_{\alpha\dot\alpha}{}^m\bar\theta^{\dot\alpha}\,\sigma_{\beta\dot\beta}{}^n\bar\theta^{\dot\beta}\,\partial_m\partial_n\big), \\ \bar{D}^2 &= \epsilon^{\dot\alpha\dot\beta}\big(\bar\partial_{\dot\alpha}\bar\partial_{\dot\beta} + 2i\,\theta^\alpha\sigma_{\alpha\dot\alpha}{}^m\,\bar\partial_{\dot\beta}\partial_m - \theta^\alpha\sigma_{\alpha\dot\alpha}{}^m\theta^\beta\sigma_{\beta\dot\beta}{}^n\,\partial_m\partial_n\big),\end{aligned} \tag{2.17}$$

in order to note that D^2 ($\bar{D}^2$) is equal to ∂^2 ($\bar\partial^2$) up to terms that contain a total spacetime derivative ∂_m. This means that we can replace ordinary against covariant derivatives as long as we work under a spacetime integral

$$\int d^4x\, D^2 = \int d^4x\,\partial^2\,, \qquad \int d^4x\,\bar{D}^2 = \int d^4x\,\bar\partial^2. \tag{2.18}$$

From (2.17) one reads off that the replacement of ordinary against covariant derivatives is also possible when an expression is restricted to $\bar\theta = 0$ for D^2 or to $\theta = 0$ in the case of $\bar{D}^2$. Straightforward application of the D-algebra (2.16) leads to the following helpful identities

$$\begin{aligned}\big[D_\alpha, \bar{D}^2\big] &= -4i\epsilon^{\dot\alpha\dot\beta}\sigma_{\alpha\dot\alpha}{}^m\bar{D}_{\dot\beta}\partial_m, & D_\alpha D_\beta D_\gamma &= 0, \\ \big[\bar{D}_{\dot\alpha}, D^2\big] &= \quad 4i\epsilon^{\alpha\beta}\sigma_{\alpha\dot\alpha}{}^m D_\beta\partial_m, & \bar{D}_{\dot\alpha}\bar{D}_{\dot\beta}\bar{D}_{\dot\gamma} &= 0, \\ \big[D^2, \bar{D}^2\big] &= -4i\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma_{\alpha\dot\alpha}{}^m\partial_m\big[D_\beta, \bar{D}_{\dot\beta}\big], & D^\alpha\bar{D}^2D_\alpha &= \bar{D}_{\dot\alpha}D^2\bar{D}^{\dot\alpha}, \\ \big\{D^2, \bar{D}^2\big\} &= \quad 2D^\alpha\bar{D}^2D_\alpha + 16\Box, & &\end{aligned} \tag{2.19}$$

where the box in the last line is the d'Alembert operator $\Box = \partial_m\partial^m$.

2.3 Superfields

A power series expansion of a superfield $\mathscr{F} = \mathscr{F}(x, \theta, \bar{\theta})$ in the variables θ_α and $\bar{\theta}_{\dot{\alpha}}$ yields, due to the anticommuting nature of these variables, only a finite number of terms

$$\begin{aligned}\mathscr{F}(x,\theta,\bar{\theta}) \;&=\; f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + \theta^2 m(x) + \bar{\theta}^2 n(x) + \theta\sigma^m\bar{\theta}A_m(x) \\ &\quad + \theta^2\bar{\theta}\bar{\lambda}'(x) + \bar{\theta}^2\theta\psi'(x) + \theta^2\bar{\theta}^2 d(x).\end{aligned} \tag{2.20}$$

As θ and $\bar{\theta}$ can appear at most twice in each of the terms, a superfield is a collection of nine x-dependent component fields. One can now read off that a superfield accommodates four complex scalars f, m, n and d with two bosonic degrees of freedom each, four complex Weyl spinors ϕ_α, $\bar{\chi}_{\dot{\alpha}}$, $\bar{\lambda}'_{\dot{\alpha}}$ and ψ'_α with four fermionic degrees of freedom each and a complex vector A_m with eight bosonic degrees of freedom, such that we have both sixteen bosonic and fermionic degrees of freedom altogether.

While (2.20) is a space saving notation for the relation between a superfield and its component fields, it turns out that for many calculations a different notation is more convenient. For this notation we act with a certain number of covariant derivatives on the superfield $\mathscr{F}$ and subsequently take the restriction to $\theta = \bar{\theta} = 0$. (This will be denoted by a vertical line.) From the expansion (2.20) we calculate the following expressions for $\mathscr{F}$

$$\begin{gathered}\mathscr{F}\,\Big| = f(x), \\ D_\alpha\mathscr{F}\,\Big| = \phi_\alpha(x), \qquad \bar{D}_{\dot{\alpha}}\mathscr{F}\,\Big| = \bar{\chi}_{\dot{\alpha}}(x), \\ \frac{D^2}{-4}\mathscr{F}\,\Bigg| = m(x), \qquad -\frac{1}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}]\,\mathscr{F}\,\Bigg| = \sigma^m{}_{\alpha\dot{\alpha}}A_m(x), \qquad \frac{\bar{D}^2}{-4}\mathscr{F}\,\Bigg| = n(x), \\ -\frac{1}{4}D^2\bar{D}_{\dot{\alpha}}\,\mathscr{F}\,\Bigg| = \bar{\lambda}_{\dot{\alpha}}(x), \qquad -\frac{1}{4}\bar{D}^2 D_\alpha\,\mathscr{F}\,\Bigg| = \psi_\alpha(x) \\ \frac{1}{8}D^\alpha\bar{D}^2 D_\alpha\,\mathscr{F}\,\Bigg| = D(x),\end{gathered} \tag{2.21}$$

where we have performed the field redefinitions

$$\begin{aligned}\bar{\lambda}_{\dot{\alpha}}(x) &= \bar{\lambda}'_{\dot{\alpha}}(x) + \frac{i}{2}\,\sigma^m{}_{\alpha\dot{\alpha}}\,\partial_m\phi^\alpha(x), \\ \psi_\alpha(x) &= \psi'_\alpha(x) - \frac{i}{2}\,\sigma^m{}_{\alpha\dot{\alpha}}\,\partial_m\bar{\chi}^{\dot{\alpha}}(x), \\ D(x) &= 2d(x) - \frac{1}{2}\,\Box f(x).\end{aligned} \tag{2.22}$$

We will refer to the expressions (2.21) as the fundamental restrictions of $\mathscr{F}$. Other restrictions with different combinations of covariant derivatives either vanish or are not independent of the restrictions (2.21) and in this sense not fundamental. It is clear that the definition of the component fields is to a certain degree arbitrary. For example, the overall normalization of the component fields can be adjusted for convenience. And the admixture of component fields from lower restrictions in the definition of component fields from higher restrictions (i. e. the admixture of $\phi(x)$, $\bar{\chi}(x)$ and $f(x)$ in the definitions of $\bar{\lambda}(x)$, $\psi(x)$ and $D(x)$) are artifacts which can be absorbed in a redefinition of the fields as in (2.22). In this thesis we will use this freedom to achieve agreement with the conventions of [1]. We also note here that a superfield does not necessarily have to be a Lorentz scalar as the notation $\mathscr{F}(x, \theta, \bar{\theta})$ may seem to suggest. In fact, for the supersymmetric generalization of the field strength a superfield will be employed that transforms as a Weyl spinor under Lorentz transformations.

The supersymmetry variations of the component fields are now easily obtained from the supersymmetry variation of the superfield: Using (2.9) a pure supersymmetry variation and its infinitesimal version are

$$\mathscr{F} \;\rightarrow\; \exp\left(i\xi Q + i\bar{\xi}\bar{Q}\right)\mathscr{F} \;\simeq\; \mathscr{F} + i\big(\xi Q + \bar{\xi}\bar{Q}\big)\,\mathscr{F} \;\equiv\; \mathscr{F} + \delta_\xi \mathscr{F}. \tag{2.23}$$

From this prescription one can identify the supersymmetry transformation of the component fields using the fundamental restrictions (2.21). The calculation is simplified by the observation that, as far as the restriction is concerned, the supersymmetry generator is proportional to the covariant derivative

$$\delta_\xi \mathscr{F}\,\Big| = i\big(\xi Q + \bar{\xi}\bar{Q}\big)\mathscr{F}\,\Big| = \big(\xi D + \bar{\xi}\bar{D}\big)\mathscr{F}\,\Big|\,, \tag{2.24}$$

which is true for any restriction and not only for $\mathscr{F}\,\Big|$.

Chiral superfields

Superfields form a representation of the supersymmetry algebra, but they are not irreducible representations. To find irreducible representations, we impose constraints on superfields. One possible constraint is to require that the superfield vanishes upon application of a supercovariant derivative operator

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{2.25}$$

A superfield which fulfills this condition is called a chiral superfield. For a chiral superfield there are only three fundamental restrictions

$$\Phi\,\Big| = A(x), \qquad D_\alpha\Phi\,\Big| = \sqrt{2}\,\psi_\alpha(x), \qquad \frac{D^2}{-4}\,\Phi\,\Bigg| = F(x). \tag{2.26}$$

Hence this supermultiplet accomodates a Weyl fermion ψ_α together with its bosonic superpartner A plus a (bosonic) auxilliary field F. The two complex scalars A and F both carry two bosonic degrees of freedom, while ψ_α contributes four fermionic degrees of freedom. Thus, bosonic and fermionic degrees of freedom match within the multiplet. With respect to (2.26) we remark that all other restrictions vanish under the condition (2.25) except for the commutator. But the commutator restriction is for a chiral field not independent of the other restrictions and hence need not be mentioned separately. We see this, because for a chiral superfield the commutator can be replaced by the anticommutator

$$-\frac{1}{2}\left[D_\alpha, \bar{D}_{\dot{\alpha}}\right]\Phi\Big| = \frac{1}{2}\left\{D_\alpha, \bar{D}_{\dot{\alpha}}\right\}\Phi\Big| = -i\sigma_{\alpha\dot{\alpha}}{}^m\partial_m\Phi\Big| \tag{2.27}$$

and thus, upon making use of the algebra for the covariant derivatives, can be traced back to the restriction $\Phi\big|$ in (2.26). The supersymmetry variations of the component fields are calculated by applying the operator (2.24) on the component fields, using the fundamental restrictions (2.26) and the relations for the covariant derivatives (2.19), and we find

$$\begin{aligned}
\delta_\xi A(x) &= \sqrt{2}\,\xi\psi(x),\\
\delta_\xi\psi_\alpha(x) &= i\sqrt{2}\,\sigma_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_m A(x) + \sqrt{2}\,\xi_\alpha F(x),\\
\delta_\xi F(x) &= i\sqrt{2}\,\bar{\xi}\bar{\sigma}^m\partial_m\psi(x).
\end{aligned} \tag{2.28}$$

We observe that a supersymmetry transformation always maps fields on linear combinations of fields in the multiplet and hence the multiplet forms a representation of the supersymmetry algebra. We also note that the highest component F of the chiral multiplet is mapped to a total derivative by the supersymmetry variation which is important for the construction of invariant actions.

Antichiral superfields

In the same way one defines an antichiral superfield which fulfills

$$D_\alpha\bar{\Phi} = 0. \tag{2.29}$$

Its fundamental restrictions are consequently given by

$$\bar{\Phi}\Big| = A^*(x), \qquad \bar{D}_{\dot{\alpha}}\bar{\Phi}\Big| = \sqrt{2}\,\bar{\psi}_{\dot{\alpha}}(x), \qquad \frac{\bar{D}^2}{-4}\,\bar{\Phi}\Big| = F^*(x). \tag{2.30}$$

Again all other restrictions vanish except for the commutator of two covariant derivatives which is proportional to the restriction $\bar{\Phi}\big|$. All expressions for the antichiral superfield are given by the hermitean conjugate of the corresponding expressions of the chiral superfield.

Vector superfields

A vector superfield $V = V(x, \theta, \bar{\theta})$ is defined by the constraint that the field be real

$$V^* = V. \tag{2.31}$$

This requirement relates also the component fields, so that the fundamental restrictions of the vector superfield are defined for convenience as

$$\begin{gathered}
V\Big| = C(x), \\
D_\alpha V\Big| = i\chi_\alpha(x), \qquad \bar{D}_{\dot{\alpha}} V\Big| = -i\bar{\chi}_{\dot{\alpha}}(x), \\
\frac{D^2}{-4} V\Bigg| = m(x), \qquad -\frac{1}{2}\big[D_\alpha, \bar{D}_{\dot{\alpha}}\big]\, V\Bigg| = \sigma_{\alpha\dot{\alpha}}{}^m A_m(x), \qquad \frac{\bar{D}^2}{-4} V\Bigg| = m^*(x), \\
-\frac{1}{4} D^2 \bar{D}_{\dot{\alpha}}\, V\Bigg| = i\bar{\lambda}_{\dot{\alpha}}(x), \qquad -\frac{1}{4}\bar{D}^2 D_\alpha\, V\Bigg| = -i\lambda_\alpha(x) \\
\frac{1}{8} D^\alpha \bar{D}^2 D_\alpha\, V\Bigg| = D(x),
\end{gathered} \tag{2.32}$$

where the complex field is split into real and imaginary parts $m(x) = \frac{i}{2}\big[M(x) + iN(x)\big]$ in the conventions of [1]. With respect to a general superfield, the number of degrees of freedom of a vector superfield is cut in half by the reality condition, i.e. there are eight bosonic and fermionic degrees of freedom: χ_α and ψ_α are complex Weyl spinors with four fermionic degrees of freedom each and C, M, N and D are real scalars with a single bosonic degree of freedom each. A_m is a real vector with four bosonic degrees of freedom and it will be argued in the following that one can identify $A_m(x)$ with a gauge field. Thus, we may also speak of a gauge multiplet. Usually, a gauge field transforms as

$$A_m(x) \to A_m(x) + \partial_m \omega(x), \tag{2.33}$$

where $\omega(x)$ is another real field. This concept is generalized to a supersymmetric gauge transformation by the definition that a vector superfield transforms as

$$V \to V + \Lambda + \bar{\Lambda}, \tag{2.34}$$

where Λ is a chiral and $\bar{\Lambda}$ an antichiral superfield. The combination $\Omega\,(x, \theta, \bar{\theta}) := \big(\Lambda + \bar{\Lambda}\big)(x, \theta, \bar{\theta})$ is real and hence a vector superfield. Its restrictions are calculated to be

$$
\begin{gathered}
\Omega\Big| = A(x) + A^*(x), \\
D_\alpha\Omega\Big| = \sqrt{2}\,\psi_\alpha(x), \qquad \bar{D}_{\dot\alpha}\Omega\Big| = \sqrt{2}\,\bar\psi_{\dot\alpha}(x), \\
\frac{D^2}{-4}\Omega\Bigg| = F(x), \quad -\frac{1}{2}\big[D_\alpha, \bar{D}_{\dot\alpha}\big]\Omega\Bigg| = \sigma_{\alpha\dot\alpha}{}^m\,\partial_m\,\omega(x), \quad \frac{\bar{D}^2}{-4}\Omega\Bigg| = F^*(x), \\
-\frac{1}{4}D^2\bar{D}_{\dot\alpha}\,\Omega\Bigg| = 0, \qquad -\frac{1}{4}\bar{D}^2 D_\alpha\,\Omega\Bigg| = 0 \\
\frac{1}{8}D^\alpha\bar{D}^2 D_\alpha\,\Omega\Bigg| = 0,
\end{gathered}
\tag{2.35}
$$

where $\omega(x) = 2\,\mathrm{Im}\,A(x)$ in the vector field component such that (2.33) is fulfilled under the gauge transformation (2.34). The other component fields transform under (2.34) as

$$
\begin{gathered}
C(x) \to C(x) + \mathrm{Re}\ A(x), \\
\chi_\alpha(x) \to \chi_\alpha(x) - i\sqrt{2}\,\psi_\alpha(x), \\
M(x) + iN(x) \to M(x) + iN(x) - 2iF(x),
\end{gathered}
\tag{2.36}
$$

while the component fields $\lambda_\alpha(x)$ and $D(x)$ are invariant under the generalized gauge transformation. With the help of these gauge transformations, we can fix the Wess-Zumino (WZ) gauge where the components fields $C(x)$, $\chi(x)$, $M(x)$ and $N(x)$ are all identical to zero. For easier reference, we take down again the restrictions of the vector superfield in WZ gauge

$$
\begin{gathered}
-\frac{1}{2}\big[D_\alpha, \bar{D}_{\dot\alpha}\big]\,V\Bigg| = \sigma_{\alpha\dot\alpha}{}^m A_m(x), \\
-\frac{1}{4}D^2\bar{D}_{\dot\alpha}\,V\Bigg| = i\bar\lambda_{\dot\alpha}(x), \qquad -\frac{1}{4}\bar{D}^2 D_\alpha\,V\Bigg| = -i\lambda_\alpha(x), \\
\frac{1}{8}D^\alpha\bar{D}^2 D_\alpha\,V\Bigg| = D(x),
\end{gathered}
\tag{2.37}
$$

while all other restrictions vanish. In WZ gauge the only non-vanishing powers of the vector superfield are V and V^2. This observation will be useful for Taylor expanding the exponential of V, where the series breaks down after the quadratic term. We determine the supersymmetry variation of the component fields by application of the supersymmetry transformation (2.24) on the component fields

$$\begin{aligned}\delta_\xi A^m(x) &= i\xi\sigma^m\bar\lambda(x) + i\bar\xi\bar\sigma^m\lambda(x) + \xi\partial^m\chi(x) - \bar\xi\partial^m\bar\chi(x),\\ \delta_\xi\lambda_\alpha(x) &= i\xi D(x) + \sigma^{mn}\xi F_{mn}(x),\\ \delta_\xi D(x) &= -\xi\sigma^m\partial_m\bar\lambda(x) + \bar\xi\bar\sigma^m\partial_m\lambda(x),\end{aligned} \tag{2.38}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$ is the field strength tensor and we have defined $\sigma^{mn}{}_\alpha{}^\beta = \frac{1}{4}\big(\sigma_{\alpha\dot\alpha}{}^m\bar\sigma^{n\,\dot\alpha\beta} - \sigma_{\alpha\dot\alpha}{}^n\bar\sigma^{m\,\dot\alpha\beta}\big)$. From the first line we can also determine the supersymmetry variation of F^{mn}

$$\delta_\xi F^{mn}(x) = i\partial^m\big(\xi\sigma^n\bar\lambda(x) + \bar\xi\bar\sigma^n\lambda(x)\big) - i\partial^n\big(\xi\sigma^m\bar\lambda(x) + \bar\xi\bar\sigma^m\lambda(x)\big). \tag{2.39}$$

Thus we have checked explicitly that the fields A_m (or F_{mn}), λ_α and D form a representation of the supersymmetry algebra by themselves. We also note that the supersymmetry variation of the highest component D is a total divergence which will help us to establish invariant actions in section 2.4.

Projection operators

With the help of the supercovariant derivatives (2.15) one defines the following projection operators on superfields

$$P_0 = \frac{D^\alpha\bar D^2 D_\alpha}{-8\Box}, \qquad P_+ = \frac{\bar D^2 D^2}{16\Box}, \qquad P_- = \frac{D^2\bar D^2}{16\Box}, \tag{2.40}$$

which are idempotent $P_0^2 = P_0$, $P_+^2 = P_+$, $P_-^2 = P_-$ and which fulfill the completeness relation $P_0 + P_+ + P_- = \mathbb{1}$. The projector P_0 is referred to as the transversal projector, P_+ and P_- are referred to as the chiral and antichiral projector, respectively. Their action on chiral and antichiral superfields is

$$\begin{aligned}&P_0\,\Phi = 0, \qquad &&P_+\,\Phi = \Phi, \qquad &&P_-\,\Phi = 0,\\ &P_0\,\bar\Phi = 0, \qquad &&P_+\,\bar\Phi = 0, \qquad &&P_-\,\bar\Phi = \bar\Phi.\end{aligned} \tag{2.41}$$

2.4 Supersymmetric actions

Supersymmetric actions are most conveniently formulated by an integral over superspace. Therefore, we have to introduce integration over the Grassmann variables θ_α and $\bar\theta_{\dot\alpha}$. We define anticommuting volume elements

$$\big\{d\theta^\alpha, d\theta^\beta\big\} = \big\{d\theta^\alpha, \theta^\beta\big\} = 0, \qquad \big\{d\bar\theta_{\dot\alpha}, d\bar\theta_{\dot\beta}\big\} = \big\{d\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\big\} = 0.$$

Integration with respect to the components of the Weyl spinors is defined as in the usual way for Grassmann variables

$$\int d\theta^\alpha = 0, \qquad \int d\theta^\alpha\,\theta^\alpha = 1, \qquad \int d\bar\theta_{\dot\alpha} = 0, \qquad \int d\bar\theta_{\dot\alpha}\,\bar\theta_{\dot\alpha} = 1, \tag{2.42}$$

where *no* summation over α, $\dot{\alpha}$ is implied. This defines a translation invariant integral over Grassmann variables. In the literature one often refrains from raising or lowering the indices of the volume elements (2.4). Instead, one defines the two-dimensional volume elements $d^2\theta$ and $d^2\bar{\theta}$ explicitly such that one can easily include a nontrivial normalization factor of $-\frac{1}{4}$ and the four-dimensional volume element is defined as the product of the former two

$$d^2\theta = -\frac{1}{4}\,\epsilon_{\alpha\beta}\,d\theta^\alpha d\theta^\beta, \qquad d^2\bar{\theta} = -\frac{1}{4}\,\epsilon^{\dot{\alpha}\dot{\beta}} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}}\,, \qquad d^4\theta = d^2\theta\, d^2\bar{\theta}.$$

As we have not defined how to raise or lower the indices of the volume elements, we explicitly write out the summation over the spinor indices $d^2\theta = \frac{1}{2}\,d\theta^1 d\theta^2$ and $\theta^\alpha\theta_\alpha = -2\,\theta^1\theta^2$ for the undotted indices and $d^2\bar{\theta} = -\frac{1}{2}\,d\bar{\theta}_{\dot{1}}\,d\bar{\theta}_{\dot{2}}$ and $\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = 2\,\bar{\theta}_{\dot{1}}\bar{\theta}_{\dot{2}}$ for the dotted indices in order to calculate

$$\int d^2\theta\;\theta^2 = \int d^2\bar{\theta}\;\bar{\theta}^2 = 1. \tag{2.43}$$

In this sense, integration and differentiation w.r.t. Grassmann variables are equivalent, because comparing (2.43) with (2.13) one identifies.

$$\int d^2\theta \;= \frac{\partial^2}{-4}, \qquad \int d^2\bar{\theta} \;= \frac{\bar{\partial}^2}{-4}. \tag{2.44}$$

As we noted already below (2.17), we are allowed to replace ordinary derivatives with covariant derivatives as long as we work under a spacetime integral and so we write down the technically very important equations

$$\begin{aligned}
\int d^4x\, d^2\theta \;&= \int d^4x\, \frac{D^2}{-4}\,, \\
\int d^4x\, d^2\bar{\theta} \;&= \int d^4x\, \frac{\bar{D}^2}{-4}\,, \\
\int d^4x\, d^4\theta \;&= \int d^4x\, \frac{D^2\bar{D}^2}{16}.
\end{aligned} \tag{2.45}$$

These equations allow us to manipulate integrands comfortably by employing the identities (2.17) for the covariant derivatives. They represent the link between a supersymmetric action and the restrictions of the superfields as we will see below. Note that under the spacetime integral D^2 and $\bar{D}^2$ in the last equality commute.

Invariant actions

With the help of Grassmann integration we can now define actions invariant under supersymmetry by integrating a functional of superfields $\mathscr{P}\left[\Phi, \bar{\Phi}, V\right]$ over superspace. As $\mathscr{P}$ is constructed from chiral and vector superfields, it is itself a superfield which might also be chiral or vectorial. In (2.46) and (2.49) we will first give the actions for the two cases and then show that the so-constructed actions are indeed supersymmetry invariants. The action for a chiral functional $\bar{D}_{\dot{\alpha}}\, \mathscr{P}_{\text{chiral}} = 0$ is given by

$$\mathscr{S}_{\text{chiral}} = \int d^4x\, d^2\theta\ \mathscr{P}_{\text{chiral}} + \text{h.c.}. \tag{2.46}$$

One easily verifies that the action does not depend on the spinorial coordinates

$$\partial_\alpha\, \mathscr{S}_{\text{chiral}} = \int d^4x\ D_\alpha\, \frac{D^2}{-4}\, \mathscr{P}_{\text{chiral}} + \text{h.c.} = 0, \tag{2.47}$$

$$\bar{\partial}_{\dot{\alpha}}\, \mathscr{S}_{\text{chiral}} = \int d^4x\ \frac{D^2}{-4}\, \bar{D}_{\dot{\alpha}}\, \mathscr{P}_{\text{chiral}} + \text{h.c.} = 0,$$

where we have written the ordinary derivative as a covariant derivative under the integral sign and used (2.45). In the second line we used the chirality of $\mathscr{P}_{\text{chiral}}$. Hence we can also evaluate the action at $\theta = \bar{\theta} = 0$ without changing anything

$$\mathscr{S}_{\text{chiral}} = \int d^4x\ \frac{D^2}{-4}\, \mathscr{P}_{\text{chiral}}\bigg| + \text{h.c.}\ . \tag{2.48}$$

When we compare the integrand to the last equation of (2.26), we see that we have exactly projected out the F-term of the chiral superfield $\mathscr{P}_{\text{chiral}}$. As we know from (2.28) that the F-term transforms with a total derivative under supersymmetry transformations, the action as defined in (2.46) is invariant under supersymmetry transformations. Expression (2.48) is also the first step in order to expand the superfield action into the component field action. This is simply done by applying the covariant derivatives on a given function $\mathscr{P}$ by using the product rule for anticommunting derivatives and then identifying the fundamental restrictions. An example for this will be given in (2.52).

The action for a vectorial functional $\mathscr{P}^\dagger_{\text{vector}} = \mathscr{P}_{\text{vector}}$ is given by its integral over the whole superspace

$$\mathscr{S}_{\text{vector}} = \int d^4x\, d^4\theta\ \mathscr{P}_{\text{vector}}. \tag{2.49}$$

Obviously, the action is independent of θ and $\bar{\theta}$ such that we can take the restriction to $\theta = \bar{\theta} = 0$ and not change anything. Under the spacetime integral, D^2 and $\bar{D}^2$ commute and so we can use the last line of (2.19) in order to find that

$$\mathscr{S}_{\text{vector}} = \frac{1}{2} \int d^4x\ \frac{D^\alpha \bar{D}^2 D_\alpha}{8}\, \mathscr{P}_{\text{vector}}\bigg|. \tag{2.50}$$

With the help of the fundamental restrictions (2.32), we identify the integrand as the highest component (the D term) of the vector superfield $\mathscr{P}_{\text{vector}}$. We have already seen in (2.38) that D transforms into a total derivative under supersymmetry transformations, therefore $\mathscr{S}_{\text{vector}}$ is invariant.

Component field action

As we remarked above, the identities (2.45) are the starting point for the extraction of the action in component fields from an action formulated in superfields. Let us consider a standard example in all detail: Take $\mathscr{P}\big[\Phi, \bar{\Phi}, V\big] = \bar{\Phi}\,\Phi$. Here $\mathscr{P}$ is a vector superfield, so the action is

$$\mathscr{S} = \int d^4x\, d^4\theta\; \bar{\Phi}\,\Phi. \tag{2.51}$$

We rewrite the superspace integration with the help of covariant derivatives and take the restriction as in (2.50). Then we distribute the covariant derivatives according to the product rule

$$\begin{aligned}
\mathscr{S} &= \frac{1}{16}\int d^4x\, \bar{D}^2 D^2 \big(\bar{\Phi}\,\Phi\big)\Big| = \frac{1}{16}\int d^4x\, \bar{D}^2\big(\bar{\Phi} D^2\Phi\big)\Big| \\
&= \frac{1}{16}\int d^4x\,\Big(\bar{D}^2\bar{\Phi}\, D^2\Phi + 2\epsilon^{\dot{\alpha}\dot{\gamma}}\bar{D}_{\dot{\alpha}}\bar{\Phi}\,\bar{D}_{\dot{\gamma}}D^2\Phi + \bar{\Phi}\,\bar{D}^2 D^2\Phi\Big)\Big| \\
&= \frac{1}{16}\int d^4x\,\Big(\bar{D}^2\bar{\Phi}\, D^2\Phi + 2\epsilon^{\dot{\alpha}\dot{\gamma}}\bar{D}_{\dot{\alpha}}\bar{\Phi}\,\big[\bar{D}_{\dot{\gamma}}, D^2\big]\Phi + \bar{\Phi}\,\big\{\bar{D}^2, D^2\big\}\Phi\Big)\Big| \\
&= \frac{1}{16}\int d^4x\,\Big(\bar{D}^2\bar{\Phi}\, D^2\Phi + 8i\,\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\gamma}}\sigma_{\alpha\dot{\gamma}}{}^{m}\,\bar{D}_{\dot{\alpha}}\bar{\Phi}\;\partial_m D_\beta\Phi + 16\,\bar{\Phi}\,\Box\Phi\Big)\Big| \\
&= \int d^4x\,\Big(F^*(x)F(x) + i\,\partial_m\bar{\psi}_{\dot{\alpha}}(x)\,\bar{\sigma}^{m\,\dot{\alpha}\beta}\,\psi_\beta(x) + A^*(x)\Box A(x)\Big).
\end{aligned} \tag{2.52}$$

In lines one and three we made use of the chirality of the superfield. In the last but one line we took advantage of identities (2.19). Then we used the fundamental restrictions for chiral and antichiral fields in (2.26) and (2.30). Thus, we find that the term $\bar{\Phi}\Phi$ comprises the kinetic terms of the fields in the chiral multiplet.

Delta functions

We conclude this section with a few identities that are extremely useful for Feynman graph calculations in superspace. These identities concern certain combinations of covariant derivatives with delta functions of the Grassmann coordinates. From (2.43) we read off that the two dimensional superspace delta functions are given by

$$\delta^2(\theta) = \theta^2, \qquad \delta^2(\bar\theta) = \bar\theta^2, \qquad \delta^4(\theta) = \delta^2(\theta)\,\delta^2(\bar\theta) \tag{2.53}$$

and the four dimensional delta function is the product of the former two. In Feynman graph calculations these four dimensional delta functions arise from functional differentiation w.r.t. the source terms of superfields as we will see in the next chapter. In the calculation one often encounters the situation that a number of covariant derivatives is sandwiched between two four dimensional delta functions. Then the following identities are extremely useful

$$\begin{gathered}
\delta^\theta_{21}\,\delta^\theta_{21} \;=\; \delta^\theta_{21}\,D_\alpha\,\delta^\theta_{21} \;=\; \delta^\theta_{21}\,D_{\dot\alpha}\,\delta^\theta_{21} \;=\; \delta^\theta_{21}\,D^2\,\delta^\theta_{21} \;=\; \delta^\theta_{21}\,\bar D^2\,\delta^\theta_{21} \;=\; 0, \\
\delta^\theta_{21}\,D_\alpha \bar D^2\,\delta^\theta_{21} \;=\; \delta^\theta_{21}\,D_{\dot\alpha} D^2\,\delta^\theta_{21} \;=\; 0, \\
\delta^\theta_{21}\,\frac{D^2\bar D^2}{16}\,\delta^\theta_{21} \;=\; \delta^\theta_{21}\,\frac{D^\alpha\bar D^2 D_\alpha}{16}\,\delta^\theta_{21} \;=\; \delta^\theta_{21}\,\frac{\bar D^2 D^2}{16}\,\delta^\theta_{21} \;=\; \delta^\theta_{21}
\end{gathered} \tag{2.54}$$

where we have used the short hand $\delta^\theta_{21} = \delta^4(\theta_2 - \theta_1)$ for the four dimensional delta function. The general rule is that such expressions vanish unless the number of D's and $\bar D$'s between the delta functions is equal and there are at least two D's and two $\bar D$'s between the two delta functions. In the next chapter these techniques are applied to Feynman graph calculations in superspace.

Chapter 3

Supersymmetric theory in four dimensions

In this chapter we demonstrate how loop calculations are performed in supersymmetric gauge theories. To this end we consider standard $\mathcal{N} = 1$ supersymmetry in four dimensions that was reviewed in the preceeding chapter. We discuss the well-known actions of a chiral multiplet that is coupled to an Abelian and a non-Abelian gauge multiplet, respectively, both in the superfield and in the component field representation. Then we quantize the theory using the path integral approach. We determine the one-loop corrections to the gauge kinetic term and describe in detail the method of renormalizing 'within the generating functional' as it will be used later on in the thesis in more complicated situations. The difference between the Abelian and the non-Abelian calculation lies in the fact that in the Abelian calculation only the chiral multiplet contributes in the loops whereas in the non-Abelian calculation contributions from the gauge and ghost multiplets have to be added. In both cases we infer the appropriate counterterms and reproduce the standard results for the gauge coupling running.

3.1 Chiral multiplet coupled to an Abelian gauge multiplet

In this section we consider a chiral multiplet that is coupled to an Abelian gauge multiplet in four dimensions. We present the classical action first, then we quantize the theory. In the gauge sector the quantization requires the introduction of Faddeev-Popov ghosts which have a kinetic term but no interaction with the rest of the fields and therefore decouple from the theory. We calculate the loop corrections to the gauge kinetic term due to the hyper multiplet and give a detailed account of how the renormalization 'in the generating functional' is performed. After regularization we infer the counterterm and determine the gauge coupling running in terms of the beta function.

3.1.1 Classical action

First we consider the action for a massless chiral multiplet Φ that is coupled to an Abelian gauge multiplet V on the classical level. The superfield action for these multiplets is given by

$$\mathscr{S}^{\text{4D}}_{\text{Abelian}}\big(\Phi, V\big) = \mathscr{S}_\Phi\big(\Phi, V\big) + \mathscr{S}_V\big(V\big), \tag{3.1}$$

where $\mathscr{S}_\Phi\big(\Phi, V\big)$ is the kinetic action of a chiral multiplet of charge q with its coupling to the gauge multiplet

$$\mathscr{S}^{\text{4D}}_\Phi = \int d^4x\, d^4\theta\; \bar{\Phi}\, e^{2qV}\, \Phi \tag{3.2}$$

and $\mathscr{S}_V\big(V\big)$ is the kinetic action for the gauge multiplet

$$\mathscr{S}^{\text{4D}}_V = \frac{1}{4g^2}\int d^4x \left\{ \int \mathrm{d}^2\theta\; W^\alpha W_\alpha + \int \mathrm{d}^2\bar{\theta}\; \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right\}. \tag{3.3}$$

This action is invariant under the following super-gauge transformation

$$\Phi \to e^{-2q\Lambda}\,\Phi, \qquad \bar{\Phi} \to e^{-2q\bar{\Lambda}}\,\bar{\Phi}, \qquad V \to V + \Lambda + \bar{\Lambda}, \tag{3.4}$$

where Λ is a chiral superfield $\bar{D}\Lambda = 0$ and $\bar{\Lambda}$ its chiral conjugate. As Λ is a superfield that depends on a point in superspace, the transformation is local. W_α and $\bar{W}_{\dot{\alpha}}$ are the gauge invariant supersymmetric field strengths that are constructed from the gauge multiplet V

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V, \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4} D^2 \bar{D}_{\dot{\alpha}} V. \tag{3.5}$$

Because the application of three covariant derivatives of the same type in a row vanishes identically, W_α and $\bar{W}_{\dot{\alpha}}$ are chiral and antichiral superfields, respectively.

Component fields

The fundamental restrictions of the supersymmetric field strength are found by exploiting the fundamental restrictions for the gauge multiplet (2.32). The field strength contains the following component fields

$$\begin{aligned} W_\alpha\Big| &= -i\lambda_\alpha(x), \\ D_\beta W_\alpha\Big| &= -\epsilon_{\beta\alpha} D(x) - i\sigma^{mn}{}_\beta{}^\delta \epsilon_{\delta\alpha}\, F_{mn}(x), \\ \frac{D^2}{-4} W_\alpha\Big| &= \sigma_{\alpha\dot{\alpha}}{}^m \partial_m \bar{\lambda}(x). \end{aligned} \tag{3.6}$$

In contrast to a scalar chiral field in (2.26) the fermionic terms reside in the lowest and highest restriction, while the restriction with one covariant derivative contains the bosonic terms. The component field action is calculated as demonstrated in Section 2.4 with the help of the fundamental restrictions (3.6). One obtains for the kinetic terms of the chiral multiplet

$$\begin{aligned}\mathscr{S}_\Phi = \int d^4x \Big(&- \big(\mathcal{D}_m A(x)\big)^* \big(\mathcal{D}^m A(x)\big) - i\bar{\psi}(x)\bar{\sigma}^m \mathcal{D}_m \psi(x) + F^*(x)F(x) \\ &+ q\, D(x)A^*(x)A(x) - i\sqrt{2}\, q \big(A(x)\bar{\psi}(x)\bar{\lambda}(x) - A^*(x)\lambda(x)\psi(x)\big)\Big) \end{aligned} \tag{3.7}$$

and for the gauge kinetic action

$$\mathscr{S}_V = \frac{1}{g^2} \int d^4x \left(-\frac{1}{4} F_{mn}(x)F^{mn}(x) - i\bar{\lambda}(x)\bar{\sigma}^m \partial_m \bar{\lambda}(x) + \frac{1}{2}\, D^2(x) \right), \tag{3.8}$$

where the covariant derivatives are defined as

$$\begin{aligned} \mathcal{D}_m A(x) &= \big(\partial_m + iqA_m(x)\big)\, A(x), \\ \mathcal{D}_m \psi(x) &= \big(\partial_m + iqA_m(x)\big)\, \psi(x). \end{aligned} \tag{3.9}$$

The gauge field $A_m(x)$ and the gaugino $\lambda(x)$ have the standard kinetic terms. Supersymmetry generates a coupling between the scalar $A(x)$, the fermion $\psi(x)$ and the gaugino $\lambda(x)$. As this action is off-shell, we have a couple of terms that describe the auxilliary fields $F(x)$ and $D(x)$. These can be integrated out via their purely algebraic equations of motion

$$F(x) = 0, \qquad \frac{1}{g^2} D(x) + q\, A^*(x)A(x) = 0 \tag{3.10}$$

which leaves us with an additional four-point self-coupling of the scalar $A(x)$

$$\begin{aligned} &\int d^4x \left(F^*(x)F(x) + \frac{1}{2g^2}\, D^2(x) + q\, D(x)A^*(x)A(x) \right) \\ &\qquad\qquad = -\frac{q^2 g^2}{2} \int d^4x\, \big(A^*(x)A(x)\big)^2. \end{aligned} \tag{3.11}$$

This completes the classical discussion of the four dimensional Abelian action both in the superfield and in the component field language.

3.1.2 Quantization of the action

We want to calculate loop corrections to the gauge kinetic term (3.3), so we have to quantize the action. For our purpose Feynman's path integral approach will prove to be most convenient, so the action (3.1) is quantized by inserting it into the path integral thus defining the generating functional Z of the theory

$$Z\big[J, J_V\big] = \int \mathscr{D}\Phi\, \mathscr{D}V \exp\bigg\{i \int d^4x \,\Big(\mathscr{L}^{\text{4D}}_{\text{Abelian}}\big(\Phi, V\big) + \\ + \int d^4\theta\; J_V V + \int d^2\theta\; J\Phi + \int d^2\bar\theta\; \bar J\bar\Phi\Big)\bigg\}, \qquad (3.12)$$

which is a functional of the source terms in the second line. The sources are superfields: J_V is a vector superfield and J and $\bar J$ are chiral and antichiral superfields, respectively. Note that for notational simplicity we do not indicate the dependence of the action and the generating functional on the antichiral superfields explicitly and that the functional integration runs also over $\mathscr{D}\bar\Phi$. Moreover, the generating functional is normalized to unity in the limit of vanishing sources which we do not indicate explicitly. The source terms of the chiral and antichiral fields in (3.12) can be rewritten such that they also fit under the $\int d^4\theta$ integral

$$\int d^2\theta\; J\Phi + \int d^2\bar\theta\; \bar J\bar\Phi = \int d^4\theta\, \left(\frac{D^2}{-4\Box}\, J\,\Phi + \bar\Phi\, \frac{\bar D^2}{-4\Box}\, \bar J\right), \qquad (3.13)$$

where first the chiral projection operators (2.40) have been used to insert the identity in the form $P_+\, J = J$ and $P_-\, \bar J = \bar J$ and then equations (2.45) have been used under the $\int d^4x$ integral to pull the covariant derivatives into the Grassmann integration.

The action (3.1) is split into free and interacting parts leading to the propagators and vertices, respectively

$$\mathscr{S}^{\text{4D}}_{\text{Abelian}}\big(\Phi, V\big) = \mathscr{S}_{\Phi 2}\big(\Phi\big) + \mathscr{S}_{V2}\big(V\big) + \mathscr{S}_{\Phi\,\text{int}}\big(\Phi, V\big). \qquad (3.14)$$

In the Abelian theory the gauge field action (3.3) is quadratic in the gauge fields and therefore indentical to $\mathscr{S}_{V2}\big(V\big)$. (In the non-Abelian theory we will have an additional term due to the self-interactions of the gauge field.) The free field part of the action (3.1) is given by

$$\mathscr{S}_{\Phi 2}\big(\Phi\big) + \mathscr{S}_{V2}\big(V\big) = \int d^4x\, d^4\theta\, \left(\bar\Phi\Phi + \frac{1}{8}\, V D^\alpha \bar D^2 D_\alpha V\right), \qquad (3.15)$$

where the terms involving the gauge multiplet were rewritten with the help of (2.45) and a subsequent integration by parts.

Figure 1: The four dimensional Abelian theory contains one propagator for the chiral superfield and one for the gauge superfield. There are in principle also two propagators for the ghost superfields, but these fields decouple from the theory.

Chiral field propagator

To find the propagator of the chiral field we consider its free field generating functional

$$Z_0\big[J\big] = \int \mathscr{D}\Phi \ \exp\left\{ i \int d^4x\, d^4\theta \left(\bar{\Phi}\Phi + \frac{D^2}{-4\Box}\, J\,\Phi + \bar{\Phi}\,\frac{\bar{D}^2}{-4\Box}\,\bar{J} \right)\right\}. \tag{3.16}$$

For later use we denote the integrand under the path integral by $K_0\big[\Phi, J\big]$. Under the shift in the integration variable

$$\Phi \to \Phi - \frac{\bar{D}^2}{-4\Box}\bar{J}, \tag{3.17}$$

which has a trivial Jacobian we obtain the result

$$Z_0\big[J\big] = \exp\left\{ -\, i \int d^4x\, d^4\theta \ \bar{J}\,\frac{1}{\Box}\, J \right\}, \tag{3.18}$$

where we have absorbed the constant factor from the Gauss integral in the normalization of the generating functional $Z_0\big[J\big]$.

Gauge superfield propagator

Next we consider the free field generating functional of the gauge superfield and determine the gauge superfield propagator

$$Z_0\big[J_V\big] = \int \mathscr{D}V \exp\left\{ i \int d^4x\, d^4\theta \left(\frac{1}{8}\, V D^\alpha \bar{D}^2 D_\alpha V + J_V V \right)\right\}. \tag{3.19}$$

For later use we denote the integrand under the path integral by $K_0\big[V, J_V\big]$. We would like to perform a shift and a Gauss integration as we have done in the case of the chiral field, but a complication arises, because the quadratic operator $\frac{1}{8}\, D^\alpha \bar{D}^2 D_\alpha$ is not invertible. That this is the case can be seen by showing that the quadratic operator has zero modes, namely the chiral and antichiral fields which it annihilates

$$D^\alpha \bar{D}^2 D_\alpha \Lambda = 0, \qquad D^\alpha \bar{D}^2 D_\alpha \bar{\Lambda} = 0. \tag{3.20}$$

Therefore, one has to perform a gauge fixing. For gauge theories the method of choice to employ a gauge fixing is the Faddeev-Popov method. One defines the Faddeev-Popov determinant by employing a chiral superfield F

$$\Delta[V] = \int \mathscr{D}\bar{\Lambda}\, \mathscr{D}\Lambda\, \delta\big(\Theta^{(\Lambda,\bar{\Lambda})}[V] - F\big)\, \delta\big(\bar{\Theta}^{(\Lambda,\bar{\Lambda})}[V] - \bar{F}\big). \tag{3.21}$$

We choose for the gauge fixing functional the chiral superfield $\Theta[V]$

$$\Theta[V] = \frac{\bar{D}^2}{-4}\sqrt{2}\, V. \tag{3.22}$$

Its highest component is

$$\frac{D^2}{4}\,\Theta[V]\Big| = \frac{1}{\sqrt{2}}\big(\Box C(x) + D(x) - i\,\partial_m A^m(x)\big) \tag{3.23}$$

and thus the imaginary part is chosen such that the gauge fixing implements the Lorentz gauge $\partial_\mu A^\mu = 0$. We insert the identity in the form $\Delta^{-1}[V]\,\Delta[V]$ into the path integral

$$\begin{aligned} Z_0\big[J_V\big] &= \int \mathscr{D}V\, \mathscr{D}\bar{\Lambda}\, \mathscr{D}\Lambda\, \Delta^{-1}[V]\, \delta\big(\Theta^{(\Lambda,\bar{\Lambda})}[V] - F\big)\, \delta\big(\bar{\Theta}^{(\Lambda,\bar{\Lambda})}[V] - \bar{F}\big)\, e^{i\mathscr{S}[V]} \\ &= \int \mathscr{D}V\, \mathscr{D}\bar{\Lambda}\, \mathscr{D}\Lambda\, \Delta^{-1}[V]\, \delta\big(\Theta[V] - F\big)\, \delta\big(\bar{\Theta}[V] - \bar{F}\big)\, e^{i\mathscr{S}[V]} \\ &= \Big(\int \mathscr{D}\bar{\Lambda}\, \mathscr{D}\Lambda\Big) \int \mathscr{D}V\, \Delta^{-1}[V]\, \delta\big(\Theta[V] - F\big)\, \delta\big(\bar{\Theta}[V] - \bar{F}\big)\, e^{i\mathscr{S}[V]} \\ &= \int \mathscr{D}V\, \Delta^{-1}[V]\, \delta\big(\Theta[V] - F\big)\, \delta\big(\bar{\Theta}[V] - \bar{F}\big)\, e^{i\mathscr{S}[V]}. \end{aligned} \tag{3.24}$$

In the second line we have performed a gauge transformation (which leaves $\mathscr{D}\Lambda$, $\mathscr{D}V$ and $\mathscr{S}[V]$ invariant). Then the integrand does no longer depend on $\Lambda, \bar{\Lambda}$, such that we can factor out the integration with respect to those fields. The integration just gives an infinite factor which is absorbed in normalization of the path integral. Next, we integrate with a Gaussian weighting factor $\int \mathscr{D}F\, \mathscr{D}\bar{F}\ \exp\big(-i\,\xi \int d^4x\, d^4\theta\ \bar{F}F\big)$ over $Z_0\big[J_V\big]$

$$\begin{aligned} Z_0\big[J_V\big] &= \int \mathscr{D}F\, \mathscr{D}\bar{F}\, \mathscr{D}V\ \Delta^{-1}[V]\ \delta\big(\Theta[V] - F\big)\, \delta\big(\bar{\Theta}[V] - \bar{F}\big) \\ &\qquad \times \exp\Big\{ - i\,\xi \int d^4x\, d^4\theta\ \bar{F}F \Big\}\, e^{i\mathscr{S}[V]} \end{aligned} \tag{3.25}$$

$$= \int \mathscr{D}V\, \Delta^{-1}[V]\ \exp\Big\{ - i\,\xi \int d^4x\, d^4\theta\ \Theta[V]\bar{\Theta}[V] \Big\}\, e^{i\mathscr{S}[V]}, \tag{3.26}$$

where ξ is the gauge fixing parameter. When we insert the gauge fixing fuctional (3.22) we are able to identify the gauge fixing action as

$$Z_0\big[J_V\big] = \int \mathscr{D}V\, \Delta^{-1}[V]\, e^{i\mathscr{S}[V]+i\mathscr{S}_{gf}}, \qquad \mathscr{S}_{gf} = -\frac{\xi}{8}\int d^4x\, d^4\theta\; \bar{D}^2 V D^2 V. \tag{3.27}$$

We write the gauge fixed generating functional as

$$Z_0\big[J_V\big] = \int \mathscr{D}V\; \exp\left\{ i\int d^4x\, d^4\theta\, \Big(V A_\xi V + J_V V\Big)\right\}. \tag{3.28}$$

Here we have dropped the factor $\Delta^{-1}[V]$, because (as will be shown below) it turns out not to be dependent on V and can thus be drawn out of the path integral and absorbed in the normalization of the generating functional. The quadratic operator A_ξ is invertible

$$\begin{aligned} A_\xi &= \frac{1}{8}\, D^\alpha \bar{D}^2 D_\alpha - \frac{\xi}{16}\,\big(D^2\bar{D}^2 + \bar{D}^2 D^2\big) \\ &= -\Box P_0 - \xi\Box\big(P_+ + P_-\big), \end{aligned} \tag{3.29}$$

which is most easily seen when A_ξ is expressed in terms of projection operators as in the second line of (3.29). So it is easy to show that its inverse is given by

$$\begin{aligned} A_\xi^{-1} &= -\frac{1}{\Box}P_0 - \frac{1}{\xi\Box}\,\big(P_+ + P_-\big) \\ &= \frac{1}{8\Box^2}\,\Big(D^\alpha\bar{D}^2 D_\alpha - \frac{1}{2\xi}\,\big(D^2\bar{D}^2 + \bar{D}^2 D^2\big)\Big). \end{aligned} \tag{3.30}$$

We perform in (3.28) the shift $V \to V - \frac{1}{2}A_\xi^{-1}J_V$ that has a trivial Jacobian and absorb the constant factor in the normalization of the path integral. Taking $\xi = 1$ gives a particularly simple choice of the propagator

$$Z_0\big[J_V\big] = \exp\left\{ i\int d^4x\, d^4\theta\, \left(\frac{1}{4}\, J_V\, \frac{1}{\Box}\, J_V\right)\right\}. \tag{3.31}$$

By choosing a definite gauge, however, we lose the information on the gauge invariance of the result. If one wants to have this feature, one would have to perform the calculation with the gauge parameter left undetermined. Then gauge invariance could be checked, because in the matrix elements any dependence on the artifact ξ must cancel.

Ghost propagator

The inverse of the Faddeev-Popov determinant can be represented in terms of an integral over anticommuting chiral superfields C and C' (ghost superfields)

$$\Delta^{-1}[V] = \int \mathscr{D}C\, \mathscr{D}C' \; \exp\left\{\frac{i}{\sqrt{2}} \int \mathrm{d}^4x \left(\int d^2\theta\, C'\delta_C\Theta + \int d^2\bar{\theta}\, \bar{C}'\delta_C\bar{\Theta}\right)\right\}, \tag{3.32}$$

where the integration over $\mathscr{D}\bar{C}$ and $\mathscr{D}\bar{C}'$ is understood and where the ghost superfields C and C' are independent of each other. In order to make these fields dynamical, the following source terms are introduced

$$\mathscr{S}_{\text{source}} = \int d^4x \Big(\int d^2\theta \left(J_C C + J_{C'} C'\right) + \int d^2\bar{\theta} \left(J_{\bar{C}}\bar{C} + J_{\bar{C}'}\bar{C}'\right)\Big). \tag{3.33}$$

The infinitesimal version of the Abelian gauge transformation of the vector multiplet (3.4) and the analog in Grassmann fields are

$$\delta_\Lambda V = \Lambda + \bar{\Lambda} \qquad \rightarrow \qquad \delta_C V = C + \bar{C}. \tag{3.34}$$

The infinitesimal gauge variation δ_C of the gauge fixing functional is

$$\delta_C\Theta = \sqrt{2}\,\frac{\bar{D}^2}{-4}\left(C + \bar{C}\right), \qquad \delta_C\bar{\Theta} = \sqrt{2}\,\frac{D^2}{-4}\left(C + \bar{C}\right). \tag{3.35}$$

The complete generating functional for the ghost superfields including the source terms can then be represented as

$$Z_0\big[J_C, J_{C'}\big] = \int \mathscr{D}C\, \mathscr{D}C' \; \exp\Big\{ i \int d^4x\, d^4\theta \Big(C'\bar{C} + \bar{C}'C + \\ + \frac{D^2}{-4\Box}\, J_C\, C + \frac{D^2}{-4\Box}\, J_{C'}\, C' - \bar{C}\,\frac{\bar{D}^2}{-4\Box}\,\bar{J}_C - \bar{C}'\frac{\bar{D}^2}{-4\Box}\,\bar{J}_{C'}\Big)\Big\}, \tag{3.36}$$

where purely chiral terms vanish under the full superspace integral. The source terms (3.33) have been rewritten to fit under the $\int d^4\theta$ integral. We perform the shifts

$$C \rightarrow C + \frac{\bar{D}^2}{-4\Box}\bar{J}_{C'}, \qquad C' \rightarrow C' - \frac{\bar{D}^2}{-4\Box}J_C \tag{3.37}$$

in (3.36) and absorb the constant factor in the normalization of the path integral. Hence we obtain the propagators for the ghost superfields

$$Z_0\big[J_C, J_{C'}\big] = \exp\Big\{ i \int d^4x\, d^4\theta \Big(-\bar{J}_{C'}\,\frac{1}{\Box}\,J_C - J_{C'}\,\frac{1}{\Box}\,\bar{J}_C\Big)\Big\}. \tag{3.38}$$

Figure 2: In the four dimensional Abelian theory there are only interactions between the chiral superfield and the vector superfield. Up to fourth order in the fields we have one three point coupling and one four point coupling.

Hence, in the Abelian case, the Faddeev-Popov Lagrangian leads to propagators for the ghost fields, but (3.38) contains no vertices that would connect the ghosts with the rest of the theory. This means that the ghosts are decoupled. In particular, Δ^{-1} is independent of V such that $Z_0\big[J_C, J_{C'}\big]$ can be absorbed in the normalization of the path integral as mentioned below (3.28).

Interaction vertices

Next we determine the interaction vertices. Since our aim is the renormalization of the gauge coupling at one loop, we need vertices up to fourth order in the fields. Then the relevant part of the action (3.14) is

$$\mathscr{S}_{\Phi\,\mathrm{int}}\big(\Phi, V\big) \supset \int d^4x\, d^4\theta\; \bar{\Phi}\left(2qV + 2q^2V^2\right)\Phi. \tag{3.39}$$

This gives a three point coupling that involves the vector superfield and two chiral superfields and a four point coupling that connects two gauge superfields and two chiral superfields. The interactions are depicted in Fig. 2. As we already mentioned above, there are no couplings that would involve ghost superfields.

3.1.3 Renormalization of the gauge kinetic term due to the chiral multiplet

In this section we determine the renormalization of the gauge kinetic term (3.3). In particular, we derive the counterterm which enables us to infer the gauge coupling running. We show in detail how one renormalizes 'in the generating functional' without being forced to calculate really the two-point function to the end. We start with the generating functional of the interacting theory (3.12) where we split the action according to (3.14). Factorizing the different exponentials it is cast into the form

$$Z\big[J, J_V\big] = \int \mathscr{D}\Phi\, \mathscr{D}V \exp\left\{ i \int d^4x\; \mathscr{L}_{\Phi\,\mathrm{int}}\big(\Phi, V\big)\right\} K_0\big[\Phi, J\big]\, K_0\big[V, J_V\big]. \tag{3.40}$$

Here $K_0\big[\Phi, J\big]$ and $K_0\big[V, J_V\big]$ are the integrands under the path integral in the free field generating functionals for the chiral superfield (3.16) and the gauge

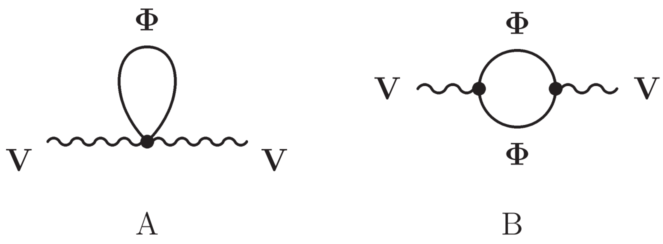


Figure 3: In the Abelian theory, the renormalization of the gauge kinetic term receives contributions only from the chiral superfield in the loop: The tadpole contribution is labelled 3.A and the genuine self-energy graph is denoted as 3.B.

superfield (3.19), respectively, that contain the exponentials of the free field actions and the source terms. Next we insert the expansion of the interaction part (3.39) into (3.40) and expand the exponential that contains this part up to terms that are quadratic in the gauge field V

$$Z\big[J, J_V\big] = \int \mathscr{D}\Phi\, \mathscr{D}V \Bigg(1 + i \int d^4x\, d^4\theta\, \big(2q\, \bar{\Phi}\, V \Phi + 2q^2\, \bar{\Phi}\, V^2 \Phi\big) + \\ - \frac{1}{2} \int \big(d^4x\, d^4\theta\big)_1\, 2q\, \bar{\Phi}_1 V_1 \Phi_1 \int \big(d^4x\, d^4\theta\big)_2\, 2q\, \bar{\Phi}_2 V_2 \Phi_2 + \dots \Bigg) \\ \times K_0\big[\Phi, J\big]\, K_0\big[V, J_V\big], \qquad (3.41)$$

where an index i at the superfields denotes the dependence on the coordinates $(x_i, \theta_i, \bar{\theta}_i)$. The first term in the first line of (3.41) is just the identity. When only this term is present in the expansion, the generating functional corresponds to the free theory for which we know the generating functionals already. The first term under the spacetime integral in the first line $\bar{\Phi}\, V \Phi$ is only the three-point coupling itself which is not interesting for us here and which we neglect. The remaining two terms give rise to self-energy loop corrections to the gauge multiplet that renormalize the gauge kinetic term depicted in Fig. 3. The former of these two terms involves a four-point coupling and corresponds to the tadpole graph 3.A and the latter involves two three-point couplings and corresponds to the genuine self-energy graph 3.B. In order to calculate these graphs, we replace the superfields travelling in the loop by the corresponding functional derivatives w.r.t. the source terms

$$\Phi \rightarrow \frac{\delta}{\delta\big(iJ\big)}, \qquad \bar{\Phi} \rightarrow \frac{\delta}{\delta\big(i\bar{J}\big)}. \qquad (3.42)$$

For the superfields on the external lines such a replacement is tacitly assumed, but their corresponding functional derivatives will not be performed in the calculation and in the end they will be replaced back against the fields. Therefore, in order to distinguish external from internal lines, we leave the gauge fields in the expression, thinking of them for now as replaced by functional derivatives. Having said this, the expression in the big round brackets in (3.41) does not depend on the fields anymore and we are allowed to pull the functional integration through the round brackets right in front of the K_0 factors. Here the integrals over the fields recombine with the respective K_0 factor to give us back our free-field generating functionals Z_0

$$
\begin{aligned}
Z\big[J, J_V\big] = \bigg(1 + 2iq^2 \int d^4x\, d^4\theta\, V^2\, \frac{\delta}{\delta\big(i\bar{J}\big)}\, \frac{\delta}{\delta\big(iJ\big)} + \\
- 2q^2 \int \big(d^4x\, d^4\theta\big)_{12}\, V_1 V_2\, \frac{\delta}{\delta\big(i\bar{J}_1\big)}\, \frac{\delta}{\delta\big(iJ_1\big)}\, \frac{\delta}{\delta\big(i\bar{J}_2\big)}\, \frac{\delta}{\delta\big(iJ_2\big)} + \ldots\bigg) \\
\times Z_0\big[J\big]\, Z_0\big[J_V\big].
\end{aligned}
\tag{3.43}
$$

These derivatives w.r.t. the chiral sources act on the free-field generating functional of the chiral superfield. Performing the functional derivatives leads to the appearance of the chiral delta functions

$$
\frac{\delta J_2}{\delta J_1} = \frac{\bar{D}^2}{-4}\, \delta_{21}, \qquad \frac{\delta \bar{J}_2}{\delta \bar{J}_1} = \frac{D^2}{-4}\, \delta_{21}, \tag{3.44}
$$

where we have defined a short hand for the superspace delta function $\delta_{21} = \delta^4(x_2 - x_1)\, \delta^4(\theta_2 - \theta_1)$. The covariant derivatives that are associated to the delta functions ensure that chirality is preserved. The tadpole diagram 3.A is calculated to be

$$
\begin{aligned}
3.\mathrm{A} &= +2iq^2 \int d^4x\, d^4\theta\, V^2\, \frac{\delta}{\delta\big(i\bar{J}\big)}\, \frac{\delta}{\delta\big(iJ\big)} \\
&= -2q^2 \int \big(d^4x\, d^4\theta\big)_{12}\, V_1^2\, \delta_{21}\, \frac{1}{\Box_2}\, \frac{D_2^2 \bar{D}_2^2}{16}\, \delta_{21} \\
&= -2q^2 \int \big(d^4x\big)_{12}\, d^4\theta\, V_1^2\, \delta_{21}^{(4)}\, \frac{1}{\Box_2}\, \delta_{21}^{(4)},
\end{aligned}
\tag{3.45}
$$

where the derivatives in the first line act on the $Z_0\big[J\big]$ outside the big round round bracket in (3.43). In the second line we are only interested in keeping terms that do not contain sources anymore as usual. We use (2.54) and integrate out the remaining Grassmann delta function such that in the last line only the four dimensional delta function appears $\delta_{21}^{(4)} = \delta^4(x_2 - x_1)$. The final expression is local in the Grassmann coordinates and therefore an index i of a field refers at this stage of the calculation only to the dependence on the Minkowski part of the coordinate $(x_i, \theta, \bar{\theta})$. In the same way, the genuine self-energy graph 3.B is calculated to be

$$
\begin{aligned}
3.\mathrm{B} = &-2q^2 \int \big(d^4x\, d^4\theta\big)_{12}\, V_1 V_2\, \frac{\delta}{\delta(i\bar{J}_1)}\, \frac{\delta}{\delta(iJ_1)}\, \frac{\delta}{\delta(i\bar{J}_2)}\, \frac{\delta}{\delta(iJ_2)} \\
&= 2q^2 \int \big(d^4x\, d^4\theta\big)_{1234}\, V_1 V_2\, \delta_{32}\, \frac{1}{\Box_3} \frac{\bar{D}_3^2 D_3^2}{16}\, \delta_{31}\, \delta_{41}\, \frac{1}{\Box_4} \frac{\bar{D}_4^2 D_4^2}{16}\, \delta_{42} \qquad (3.46) \\
&= 2q^2 \int \big(d^4x\, d^4\theta\big)_{12}\, V_1 V_2\, \frac{1}{\Box_2} \frac{\bar{D}_2^2 D_2^2}{16}\, \delta_{21}\, \frac{1}{\Box_2} \frac{\bar{D}_2^2 D_2^2}{16}\, \delta_{21},
\end{aligned}
$$

where we have integrated out the leading delta functions in the second line. The final step is to take advantage of the identities (2.54) and to integrate out the dependence on one of the θ-coordinates

$$
\begin{aligned}
3.\mathrm{B} = &q^2 \int \big(d^4x\big)_{12}\, d^4\theta\, \big(-V_1 \Box P_0 V_2\big)\, \frac{1}{\Box_2}\, \delta_{21}^{(4)}\, \frac{1}{\Box_2}\, \delta_{21}^{(4)} \qquad (3.47) \\
&+ 2q^2 \int \big(d^4x\big)_{12}\, d^4\theta\, V_1^2\, \frac{\Box_2}{\Box_2}\, \delta_{21}^{(4)}\, \frac{1}{\Box_2}\, \delta_{21}^{(4)}.
\end{aligned}
$$

The first term in (3.47) is (up to the fact that the superfields depend on different coordinates) proportional to the kinetic action of the gauge field. The second term is cancelled by the tadpole contribution (3.45) such that the total self-energy (the sum of the graphs) of the vector multiplet due to graphs with the chiral multiplet in the loop is given by

$$
\Sigma_{\mathrm{VV}}^{\mathrm{chiral}} = 3.\mathrm{A} + 3.\mathrm{B} = q^2 \int \big(d^4x\big)_{12}\, d^4\theta\, \big(-V_1 \Box P_0 V_2\big)\, \frac{1}{\Box_2}\, \delta_{21}^{(4)}\, \frac{1}{\Box_2}\, \delta_{21}^{(4)}. \qquad (3.48)
$$

Note that the self-energy is not local, because the external line gauge fields depend on the different coordinates x_1 and x_2. Having determined the self-energy of the gauge multiplet due to the chiral multiplet in the loop we bring back the functional integration to the front in expression (3.43) and we can finally think of the gauge superfields V on the external lines in the self-energy (3.48) as fields again (and not as replaced by derivatives). So the loop correction to (3.40) due to the chiral multiplet in the loop is given by

$$
Z\big[J, J_V\big] = \int \mathscr{D}\Phi\, \mathscr{D}V \Big(1 + \Sigma_{\mathrm{VV}}^{\mathrm{chiral}} + \dots\Big)\, K_0\big[\Phi, J\big]\, K_0\big[V, J_V\big]. \qquad (3.49)
$$

Finally, we expand the exponentials in K_0 and retain the terms in (3.49) that are quadratic in the gauge field. These are the gauge kinetic term and its loop correction

$$
Z\big[J, J_V\big] \;=\; \int \mathscr{D}\Phi\, \mathscr{D}V \Big(i\, \mu^{d-4} \int d^dx\, d^4\theta\; \frac{1}{8g^2}\, V D^\alpha \bar{D}^2 D_\alpha V + \Sigma_{\mathrm{VV}}^{\mathrm{chiral}} + \dots\Big), \qquad (3.50)
$$

where we have extended the spacetime integration to $d = 4 - 2\epsilon$ dimensions and introduced the factor μ^{d-4} that keeps the mass dimensions of the fields and couplings as they were in four dimensions. This is a necessary step that appears also in the regularization of the divergent self-energy (3.48) with the help of App. D.3. There we extract the divergence and find

$$\Sigma^{\text{chiral}}_{\text{VV}}\Big|_{\text{div}} = i\,\mu^{d-4}\,\frac{q^2}{(4\pi)^2}\left(\frac{1}{\bar{\epsilon}} + \ln\frac{\mu^2}{m^2}\right)\int d^dx\, d^4\theta\,\big(-V\Box P_0 V\big), \tag{3.51}$$

where $\frac{1}{\bar{\epsilon}} = \frac{1}{\epsilon} - \gamma + \ln 4\pi$. The divergent part of the self-energy is local. We require the renormalized generating functional to be finite, hence we have to introduce a counterterm that cancels the divergence. We choose to work in the MS scheme, where the counterterm cancels exactly the pole part $\frac{1}{\epsilon}$ of the self-energy. From (3.12) and (3.50) we can read off that the counterterm which enters as a part of the action into the generating functional is given by the pole piece of the self-energy times a factor of i

$$\Delta\mathscr{S}^{\text{chiral}}_{\text{VV}} = -\mu^{d-4}\frac{q^2}{(4\pi)^2\epsilon}\int d^dx\, d^4\theta\,\big(-V\Box P_0 V\big). \tag{3.52}$$

Bare action, renormalized action and counterterm are related by

$$\mathscr{S}_B = \mathscr{S} + \Delta\mathscr{S}. \tag{3.53}$$

Hence, for the corresponding couplings, we infer the relation

$$\frac{1}{g_B^2} = \left(\frac{1}{g^2} - \frac{q^2}{(4\pi)^2\epsilon}\right)\mu^{-2\epsilon}. \tag{3.54}$$

Taking the derivative w.r.t. μ on both sides and then the limit $\epsilon \to 0$ gives the beta function of the inverse coupling squared

$$\beta_{1/g^2} \equiv \mu\,\frac{\partial}{\partial\mu}\left(\frac{1}{g^2}\right) = -\frac{2q^2}{(4\pi)^2}. \tag{3.55}$$

Here we have used that the bare coupling does not depend on μ. This equation can be integrated and yields the gauge coupling renormalization due to corrections that involve the chiral superfield

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{q^2}{(4\pi)^2}\ln\frac{\mu^2}{\mu_0^2}. \tag{3.56}$$

The reproduction of this standard result closes this section where we have demonstrated the renormalization procedure that will be used throughout this thesis with a concrete example.

3.2 Chiral multiplet coupled to a non-Abelian gauge multiplet

In this section we consider a chiral multiplet that is coupled to a non-Abelian gauge multiplet in four dimensions. We present the classical action first, then we quantize the theory and calculate loop corrections to the gauge kinetic term. There are two major differences compared to the Abelian discussion in the last section. First, the non-Abelian action contains gauge multiplet self-interactions which give rise to additional loop corrections to the gauge kinetic term. Second, the Faddeev-Popov ghosts couple to the gauge multiplet and also contribute to loops. We calculate their contributions to the loop correction of the gauge kinetic term in addition to the non-Abelian version of the hyper multiplet corrections. The latter require only minimal modifications w.r.t. the Abelian calculation in the previous section. We discuss briefly the renormalization process and its differences compared to the preceeding section. After regularization we infer the counterterms and determine the gauge coupling running in terms of the beta functions due to the different contributions.

3.2.1 Classical action

We consider a chiral multiplet that is coupled to a non-Abelian gauge multiplet. The gauge multiplet is represented by a Lie-algebra valued superfield, i.e. the gauge superfield is contracted with the Hermitean generators of the gauge group $V = V^i T_i$, and it transforms in the adoint representation. The algebra of the generators $[T_i, T_j] = f_{ij}{}^k T_k$ defines the purely imaginary structure coefficients. The Killing metric, denoted by η_{ij}, is used to raise and lower adjoint indices, for example $f_{ijk} = f_{ij}{}^\ell \eta_{\ell k}$. We denote the trace in the representation of the chiral multiplet by tr and the trace in the adjoint representation by $\mathrm{tr}_{\mathbf{Ad}}$. The latter is given by $\mathrm{tr}_{\mathbf{Ad}}(XY) = -f_{ijk} f_{\ell mn} \eta^{jm} \eta^{kn} X^i Y^\ell$, where the matrices X and Y are defined in the adjoint: $(X)_{jk} = X^i (T_i)_{jk} = X^i f_{ijk}$, etc. Then the action that describes this theory is given by

$$\mathscr{S}^{\text{4D}}_{\text{non-Abelian}}\big(\Phi, V\big) = \mathscr{S}_\Phi\big(\Phi, V\big) + \mathscr{S}_V\big(V\big), \tag{3.57}$$

where $\mathscr{S}_\Phi\big(\Phi, V\big)$ is the kinetic action of the chiral multiplet with its coupling to the gauge multiplet

$$\mathscr{S}_\Phi = \int d^4x\, d^4\theta\; \bar{\Phi}\, e^{2qV}\, \Phi \tag{3.58}$$

and $\mathscr{S}_V$ is the kinetic action of the gauge multiplet

$$\mathscr{S}_V = \frac{1}{4g^2} \int d^4x\; \mathrm{tr}\Big[\int \mathrm{d}^2\theta\; W^\alpha W_\alpha + \int \mathrm{d}^2\bar{\theta}\; \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \Big]. \tag{3.59}$$

Note that Φ is a vector and $\bar{\Phi}$ a transposed vector with respect to the gauge group. The action is invariant under the following supergauge transformation

$$\Phi \to e^{-2\Lambda}\,\Phi, \qquad \bar{\Phi} \to \bar{\Phi}\,e^{-2\bar{\Lambda}}, \qquad e^{2V} \to e^{2\bar{\Lambda}}\,e^{2V}\,e^{\Lambda}, \tag{3.60}$$

where $\Lambda = \Lambda^i T_i$ is a Lie algebra-valued chiral superfield and $\bar{\Lambda}$ its chiral conjugate. The transformation law for e^{2V} implies in particular that its inverse transforms as $e^{-2V} \to e^{-2\Lambda}e^{-2V}e^{-2\bar{\Lambda}}$ such that the product $e^{2V}e^{-2V} = 1$ is invariant under the gauge transformation. The non-Abelian supersymmetric field strengths are defined as

$$W_\alpha = -\frac{1}{8}\bar{D}^2\Big(e^{-2V}\,D_\alpha e^{2V}\Big), \qquad \bar{W}_{\dot\alpha} = -\frac{1}{8}D^2\Big(e^{-2V}\,\bar{D}_\alpha e^{2V}\Big) \tag{3.61}$$

and transform covariantly under a gauge transformation

$$W_\alpha \to e^{-2\Lambda}\,W_\alpha\,e^{2\Lambda}, \qquad \bar{W}_{\dot\alpha} \to e^{2\bar{\Lambda}}\,\bar{W}_{\dot\alpha}\,e^{-2\bar{\Lambda}}. \tag{3.62}$$

W_α and $\bar{W}_{\dot\alpha}$ are chiral and antichiral superfields, respectively.

Component fields

The fundamental restrictions of the chiral spinor field have the same form as their Abelian counterparts in (3.6)

$$\begin{aligned} W_\alpha\Big| &= -i\lambda_\alpha(x), \\ D_\beta W_\alpha\Big| &= -\epsilon_{\beta\alpha}D(x) - i\sigma^{mn}{}_\beta{}^\delta\epsilon_{\delta\alpha}\,F_{mn}(x), \\ \frac{D^2}{-4}\,W_\alpha\Bigg| &= \sigma_{\alpha\dot\alpha}{}^m\mathcal{D}_m\bar{\lambda}^{\dot\alpha}(x), \end{aligned} \tag{3.63}$$

with the difference that they contain the non-Abelian component field strength and the gauge covariant derivative of the gaugino $\bar{\lambda}$ defined as

$$\begin{aligned} \mathcal{D}_m\bar{\lambda}^{\dot\alpha}(x) &= \partial_m\bar{\lambda}^{\dot\alpha}(x) + i\left[A_m(x), \bar{\lambda}^{\dot\alpha}(x)\right], \\ F_{mn}(x) &= \partial_m A_n(x) - \partial_n A_m(x) + i\left[A_m(x), A_n(x)\right]. \end{aligned} \tag{3.64}$$

The restriction reveals the component field action for the kinetic term of the chiral multiplet

$$\begin{aligned} \mathscr{S}_\Phi = \int d^4x\,\Big(&-\big(\mathcal{D}_m A(x)\big)^\dagger\big(\mathcal{D}^m A(x)\big) - i\bar{\psi}(x)\bar{\sigma}^m\mathcal{D}_m\psi(x) + F^\dagger(x)F(x) \\ &+ A^\dagger(x)D(x)A(x) - i\sqrt{2}\left(\bar{\psi}(x)\bar{\lambda}(x)A(x) - A^\dagger(x)\lambda(x)\psi(x)\right)\Big) \end{aligned} \tag{3.65}$$

and for the gauge multiplet

$$\mathscr{S}_V = \frac{1}{g^2}\int d^4x \,\mathrm{tr}\Big[-\frac{1}{4}\,F_{mn}(x)F^{mn}(x) - i\,\bar\lambda(x)\bar\sigma^m\mathcal{D}_m\bar\lambda(x) + \frac{1}{2}\,D^2(x)\Big]. \quad (3.66)$$

The covariant derivatives for $A(x)$ and $\psi(x)$ are defined as

$$\begin{aligned}\mathcal{D}_mA(x) &= \big(\partial_m + iA_m(x)\big)A(x),\\ \mathcal{D}_m\psi(x) &= \big(\partial_m + iA_m(x)\big)\psi(x)\end{aligned} \quad (3.67)$$

and the non-Abelian field strength tensor $F_{mn}(x)$ and gauge covariant derivative of the gaugino $\mathcal{D}_m\lambda$ were defined in (3.64).

3.2.2 Quantization of the action

Many things are just as in the Abelian case, so we restrict ourselves to present the results where everything is the same or present the deviations. The action (3.57) is quantized by inserting it into the path integral thus defining the generating functional of the theory

$$Z\big[J,J_V\big] = \int \mathscr{D}\Phi\,\mathscr{D}V\,\exp\Big\{i\int d^4x\,\Big(\mathscr{L}^{4\mathrm{D}}_{\text{non-Abelian}}\big(\Phi,V\big) + \\ + \int d^4\theta\,\mathrm{tr}\big[J_VV\big] + \int d^2\theta\,J\Phi + \int d^2\bar\theta\,\bar J\bar\Phi\Big)\Big\}, \quad (3.68)$$

where the source terms J_V, J and $\bar J$ represent a matrix, a transposed vector and a vector w.r.t. the gauge group. The action (3.57) is split into free and interacting parts leading to the propagators and vertices, respectively

$$\mathscr{S}^{4\mathrm{D}}_{\text{non-Abelian}}\big(\Phi,V\big) = \mathscr{S}_{\Phi 2}\big(\Phi\big) + \mathscr{S}_{V2}\big(V\big) + \mathscr{S}_{\Phi\,\mathrm{int}}\big(\Phi,V\big) + \mathscr{S}_{V\mathrm{int}}\big(V\big). \quad (3.69)$$

The difference to the Abelian theory is that the non-Abelian action (3.69) contains a self-interaction part $\mathscr{S}_{V\mathrm{int}}\big(V\big)$ of the gauge multiplet. The quadratic part of the action is given by

$$\mathscr{S}_{\Phi 2}\big(\Phi\big) + \mathscr{S}_{V2}\big(V\big) = \int d^4x\,d^4\theta\,\Big(\bar\Phi\Phi + \mathrm{tr}\Big[\frac{1}{8g^2}\,VD^\alpha\bar D^2D_\alpha V\Big]\Big). \quad (3.70)$$

Chiral field propagator

The chiral superfield propagator has the same form as in the Abelian case (3.18)

$$Z_0[J,\bar J] = \exp\Big\{i\int d^4x\,d^4\theta\,\Big(-\bar J\,\frac{1}{\Box}\,J\Big)\Big\}. \quad (3.71)$$

Gauge superfield propagator

As in the Abelian case, also here the gauge fixing has to be performed to obtain an invertible quadratic operator for the gauge superfield. The gauge fixing functional is the same,

$$\Theta[V] = \frac{\bar{D}^2}{-4}\sqrt{2}\,V, \tag{3.72}$$

only that it is now Lie algebra valued $\Theta = \Theta^i\,T_i$ and so is the field $F = F^i\,T_i$ that is used for the integration with a Gaussian weighting factor in the form $\int \mathscr{D}F\,\mathscr{D}\bar{F}\,\exp\big\{-i\,\xi\int d^4x\,d^4\theta\,\mathrm{tr}\big[\bar{F}F\big]\big\}$ over $Z_0\big[J_V\big]$ with the result

$$Z_0[J_V] = \int \mathscr{D}V\,\Delta^{-1}[V]\,\exp\left\{-\,i\,\xi\int d^4x\,d^4\theta\;\mathrm{tr}\left[\Theta[V]\,\bar{\Theta}[V]\right]\right\}\,e^{i\mathscr{S}[V]}. \tag{3.73}$$

Then the gauge fixing action is the non-Abelian version of (3.27)

$$\mathscr{S}_{gf} = -\frac{\xi}{8}\int d^4x\,d^4\theta\;\mathrm{tr}\left[\bar{D}^2 V D^2 V\right]. \tag{3.74}$$

In the non-Abelian case the factor $\Delta^{-1}[V]$ depends on V and the ghost superfields. However, it does not contain a term purely quadratic in V. Therefore, it does not contribute to the gauge superfield propagator which turns out to be the non-Abelian version of (3.31),

$$Z_0[J_V] = \exp\left\{i\int d^4x\,d^4\theta\;\mathrm{tr}\left[\frac{1}{4}\,J_V\,\frac{1}{\Box}\,J_V\right]\right\}. \tag{3.75}$$

Ghost propagator

The inverse of the Faddeev-Popov determinant can be represented in terms of an integral over anticommuting Lie algebra valued chiral superfields $C = C^i\,T_i$ and $C' = C'^{\,i}\,T_i$

$$\Delta^{-1}[V] = \int \mathscr{D}C\,\mathscr{D}C'\;\exp\left\{\frac{i}{\sqrt{2}}\,\mathrm{tr}\int \mathrm{d}^4x\left[\int d^2\theta\,C'\delta_C\Theta + \int d^2\bar{\theta}\,\bar{C}'\delta_C\bar{\Theta}\right]\right\}, \tag{3.76}$$

where the integration over $\mathscr{D}\bar{C}$ and $\mathscr{D}\bar{C}'$ is understood. A difference to the Abelian calculation comes about, because the non-Abelian gauge variation of the gauge multiplet (3.60) enters in (3.76) which is different from the Abelian gauge variation (3.4). The infinitesimal non-Abelian gauge variation reads

$$\delta_\Lambda V = L_V(\Lambda - \bar{\Lambda}) + \coth\left(L_V(L_V(\Lambda + \bar{\Lambda}))\right), \tag{3.77}$$

Figure 4: Besides the interactions of the vector superfield with the chiral superfield that are also present in the Abelian theory and were depicted in Fig. 2, there are self-couplings of the vector superfield in the non-Abelian case, too.

where $L_V(X) = [V, X]$ is the Lie derivative and the hyperbolic cotangent is defined via its power series

$$\coth\left(L_V(L_V(X))\right) = X + \frac{1}{3}\left[V,[V,X]\right] + \dots \,. \tag{3.78}$$

Taking a look at the power series expansion we note that the leading term of the non-Abelian gauge variation (3.77) contains its Abelian counterpart $\Lambda + \bar{\Lambda}$ from (3.34). The infinitesimal gauge variation $\delta_C V$ is defined with Λ ($\bar{\Lambda}$) replaced by C ($\bar{C}$) in (3.77). With the infinitesimal gauge variation δ_C of the gauge fixing functional

$$\delta_C \Theta = \sqrt{2}\,\frac{\bar{D}^2}{-4}\Big(L_V(C - \bar{C}) + \coth\left(L_V(L_V(C + \bar{C}))\right)\Big) \tag{3.79}$$

we rewrite the inverse of the Faddeev-Popov determinant as

$$\Delta^{-1}[V] = \int \mathscr{D}C\, \mathscr{D}C' \;\exp\Big\{i\, \mathscr{S}_{gh}\big(C, C', V)\big]\Big\}, \tag{3.80}$$

where $\mathscr{S}_{gh}$ is the ghost action that involves the ghost and gauge multiplets

$$\mathscr{S}_{gh}\big(C, C', V\big)$$

$$= \int d^4x\, d^4\theta\; \mathrm{tr}\Big[(C' + \bar{C}')\Big[L_V(C - \bar{C}) + \coth\left(L_V(L_V(C + \bar{C}))\right)\Big]\Big]. \tag{3.81}$$

We separate the ghost action into quadratic and interaction terms

$$\mathscr{S}_{gh}\big(C, C', V\big) = \mathscr{S}_{gh2}\big(C, C'\big) + \mathscr{S}_{gh\,\mathrm{int}}\big(C, C', V\big). \tag{3.82}$$

Inserting the definition of the Lie derivative and the expansion of the hyperbolic cotangent in (3.80) we see that the quadratic part of the non-Abelian generating functional for the ghost fields agrees with the Abelian version. Thus, after adding source terms (which are Lie algebra valued) nothing changes for the propagators except for the obligatory trace

$$Z[J_C] = \exp\Big\{i \int d^4x\, d^4\theta\; \mathrm{tr}\Big[-\bar{J}_{C'}\,\frac{1}{\Box}\,J_C - J_{C'}\,\frac{1}{\Box}\,\bar{J}_C\Big]\Big\}. \tag{3.83}$$

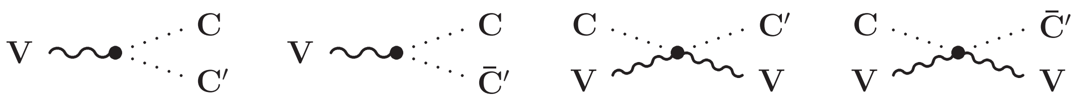


Figure 5: In the non-Abelian theory the ghost superfields couple to the vector superfield. There are two interactions involving two ghost superfields and one vector superfield and two interactions involving two ghost superfields and two vector superfields.

Interaction vertices

The couplings of the chiral superfield to the vector superfield up to fourth order in the fields that stem from the kinetic term of the chiral superfield $\mathscr{S}_{\Phi\,\mathrm{int}}$ in (3.58) are the same as in the Abelian case without the charge q

$$\mathscr{S}_{\Phi\,\mathrm{int}}(\Phi, V) \supset \int d^4x\, d^4\theta\; \bar{\Phi}\left(2V + 2V^2\right)\Phi. \tag{3.84}$$

The vector superfield interaction $\mathscr{S}_{V\mathrm{int}}$ is expanded to fourth order in the fields

$$\mathscr{S}_{V\mathrm{int}}(V) \supset \int d^4x\, d^4\theta\; \mathrm{tr}\Bigg[\frac{1}{4}\left[V, D^\alpha V\right]\bar{D}^2 D_\alpha V - \frac{1}{8}\left[V, D^\alpha V\right]\bar{D}^2\left[V, D_\alpha V\right] - \frac{1}{6}\left[V,\left[V, D^\alpha V\right]\right]\bar{D}^2 D_\alpha V\Bigg]. \tag{3.85}$$

These self-interactions have been depicted in Fig. 4. In the non-Abelian theory there are also ghost superfield interactions. The expansion of (3.81) with the help of (3.78) to fourth order in the fields yields the following interaction part of ghost superfields with the gauge superfield

$$\mathscr{S}_{gh\,\mathrm{int}}(V, C, C') \supset \int d^4x\, d^4\theta\; \mathrm{tr}\left[(C' + \bar{C}')[V, C - \bar{C}] + \frac{1}{3}(C' + \bar{C}')\left[V, [V, C + \bar{C}]\right]\right]. \tag{3.86}$$

This means that we have a three point coupling between two ghosts and one gauge superfield and a four point coupling between two ghosts and two gauge superfields. Via these interactions that are depicted in Fig. 5 the ghosts will affect graphs in which gauge superfields appear.

3.2.3 Renormalization of the gauge kinetic term due to the chiral multiplet

Here we discuss the renormalization of the gauge kinetic term due to the chiral multiplet. The interactions are contained in $\mathscr{S}_{\Phi\,\mathrm{int}}(\Phi, V)$ in analogy to the

Abelian calculation. The graphs that arise from these interactions are the same as in the Abelian case and were depicted in Fig. 3. The calculation proceeds along the same lines as in Section 3.1.3. The source terms of the chiral superfield are now vectors w.r.t. the gauge group. Hence the chiral superfields have to be replaced against derivatives w.r.t. source terms in the following way

$$\Phi^a \to \frac{\delta}{\delta\big(iJ_a\big)}, \qquad \bar{\Phi}_a \to \frac{\delta}{\delta\big(i\bar{J}^a\big)}. \tag{3.87}$$

where a is now an index in the representation of the chiral multiplet. The functional derivatives w.r.t. the sources that lead to the chiral delta functions are defined as

$$\frac{\delta J_{2b}}{\delta J_{1a}} = \frac{\bar{D}^2}{-4}\,\delta_{21}\,\delta^a_b, \qquad \frac{\delta \bar{J}^b_2}{\delta \bar{J}^a_1} = \frac{D^2}{-4}\,\delta_{21}\,\delta^b_a. \tag{3.88}$$

The results for the graphs from the calculation in Section 3.1.3 can be taken over when the square of the charge of the chiral superfield q^2 is formally replaced against the trace in the representation of the chiral multiplet 'tr'. Hence, the self-energy due to the chiral multiplet in the loop is given by

$$\Sigma^{\text{chiral}}_{\text{VV}} = \int \big(d^4x\big)_{12}\, d^4\theta\ \text{tr}\Big[-V_1\Box P_0 V_2\Big]\,\frac{1}{\Box_2}\,\delta^{(4)}_{21}\,\frac{1}{\Box_2}\,\delta^{(4)}_{21}. \tag{3.89}$$

The path from the self-energy to the counterterm is the same as in 3.1.3 with the exception that we have only obtained the quadratic part of the full gauge kinetic term. Hence, in order to write down a well-defined counterterm, we promote the operator $\text{tr}\big[-V_1\Box P_0 V_2\big]$ to the full kinetic term for the gauge superfield

$$\Delta\mathscr{S}^{\text{chiral}}_{\text{VV}} = -\mu^{d-4}\frac{1}{(4\pi)^2\epsilon}\ \text{tr}\int d^dx\,\left[\frac{1}{4}\int \mathrm{d}^2\theta\ W^\alpha W_\alpha + \frac{1}{4}\int \mathrm{d}^2\bar{\theta}\ \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right]. \tag{3.90}$$

The gauge coupling running is found in the same way as in Section 3.1.3 to be

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{1}{(4\pi)^2}\ \ln\frac{\mu^2}{\mu_0^2}. \tag{3.91}$$

This result finishes the discussion of the loop corrections to the gauge kinetic term due to the chiral multiplet in the non-Abelian theory.

3.2.4 Renormalization of the gauge kinetic term due to the gauge multiplet

Now we turn to the renormalization of the gauge kinetic term due to interactions in the gauge sector, namely gauge multiplet self-interactions and interactions of the gauge multiplet with the ghosts. The interactions are contained in $\mathscr{S}_{V\text{int}}\,(V)$

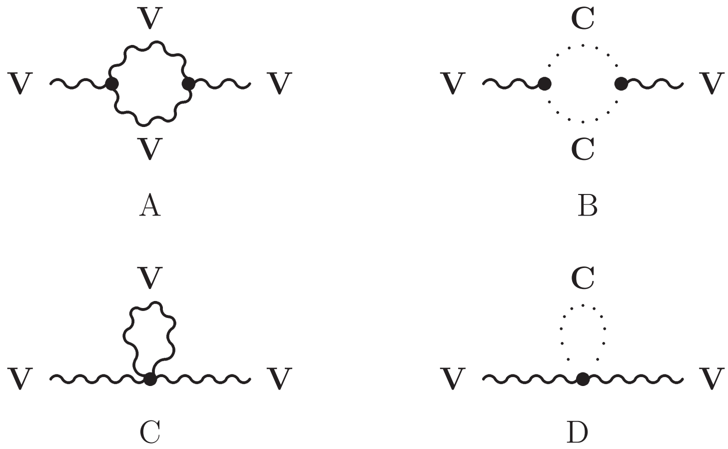


Figure 6: In the non-Abelian case the vector superfield self-energy receives contributions also from loop graphs that involve both the vector superfield itself and ghost superfields. In each case there is one genuine self-energy graph and one tadpole contribution. The graphs are labelled 6.A to 6.D.

and in $\mathscr{S}_{gh\,\mathrm{int}}\,(V, C, C')$, respectively. Four graphs that contribute to the renormalization can be constructed. Two of them contain the gauge multiplet in the loop and the other two involve the ghosts. In each case there is one genuine self-energy graph and one tadpole. The graphs are depicted in Fig. 6. The calculation is straightforward. After expanding the exponential that contains the interactions in the generating functional as described in (3.41) and picking out the terms that correspond to the graphs, one selects two of the gauge multiplets as the external line. The rest of the fields that propagate in the loop are replaced against the derivatives w.r.t. the corresponding source terms

$$V^i \to \frac{\delta}{\delta\big(iJ_V\big)_i}, \qquad C^i \to \frac{\delta}{\delta\big(iJ_C\big)_i}, \qquad C'^i \to \frac{\delta}{\delta\big(iJ'_C\big)_i}. \tag{3.92}$$

where i is an adjoint index. For the antichiral ghost fields the rules involve a hermitean conjugation. The functional derivatives are applied to the propagators that contain the source terms. They lead to the following delta functions

$$\frac{\delta J_{V2}{}^i}{\delta J_{V1}{}^j} = \delta_{21}\,\delta^i_j, \qquad \frac{\delta J_{C2}{}^i}{\delta J_{C1}{}^j} = \frac{\bar{D}^2}{-4}\,\delta_{21}\,\delta^i_j, \qquad \frac{\delta J_{C'2}{}^i}{\delta J_{C'1}{}^j} = \frac{\bar{D}^2}{-4}\,\delta_{21}\,\delta^i_j. \tag{3.93}$$

The genuine self-energy graph with the gauge multiplet in the loop gives the result

$$\begin{aligned} 6.\mathrm{A} = \frac{1}{2}\, f_{ijk} f_{\ell mn}\, \eta^{kn}\eta^{mj} \int \big(d^4x\big)_{12}\, d^4\theta\; V_1^i\, \Box_2 \Big(\big(P_+ + P_-\big) - 5P_0\Big)\, V_2^\ell \\ \times \frac{1}{\Box_2}\,\delta_{21}\,\frac{1}{\Box_2}\,\delta_{21}. \end{aligned} \tag{3.94}$$

The genuine self-energy graph with the ghost in the loop amounts to

$$\begin{aligned} 6.\mathrm{B} = f_{ijk} f_{\ell mn}\, \eta^{kn} \eta^{mj} \int \big(d^4x\big)_{12}\, d^4\theta\, \Big[V_1^i\, V_2^\ell\, \frac{\Box_2}{\Box_2}\, \delta_{21}\, \frac{1}{\Box_2}\, \delta_{21} \\ - \frac{1}{2}\, V_1^i\, \Box_2 \Big(P_+ + P_- + P_0 \Big)\, V_2^\ell\, \frac{1}{\Box_2}\, \delta_{21}\, \frac{1}{\Box_2}\, \delta_{21} \Big]. \qquad (3.95) \end{aligned}$$

The tadpole graph which involves the gauge multiplet gives

$$6.\mathrm{C} \quad = \quad -\frac{1}{3}\, f_{ijk} f_{\ell mn}\, \eta^{kn} \eta^{mj} \int \big(d^4x\big)_{12}\, d^4\theta\; V_1^i\, V_2^\ell\, \delta_{21}\, \frac{1}{\Box_2}\, \delta_{21} \qquad (3.96)$$

and the tadpole graph with the ghost in the loop leads to

$$6.\mathrm{D} \quad = \quad -\frac{2}{3}\, f_{ijk} f_{\ell mn}\, \eta^{kn} \eta^{mj} \int \big(d^4x\big)_{12}\, d^4\theta\; V_1^i\, V_2^\ell\, \delta_{21}\, \frac{1}{\Box_2}\, \delta_{21}. \qquad (3.97)$$

The sum of the two tadpole graphs cancels against the first line of graph 6.B. In the sum of the remaining terms the dependence on the transversal directions cancels. Only the longitudinal contribution is left and therefore the sum of the graphs corresponds to the following self-energy due to the vector superfield V and the ghost superfields C

$$\Sigma_{\mathrm{VV}}^{(\mathrm{V,C})} \quad = \quad -3 \int \big(d^4x\big)_{12}\, d^4\theta\; \mathrm{tr}_{\mathbf{Ad}} \Big[- V_1 \Box_2 P_0 V_2 \Big]\, \frac{1}{\Box_2}\, \delta_{21}^{(4)}\, \frac{1}{\Box_2}\, \delta_{21}^{(4)}. \qquad (3.98)$$

For the reproduction of the standard results assume that the chiral multiplets are in the fundamental representation. Then we use that $\mathrm{tr}_{\mathbf{Ad}}\big[T_a T_b\big] = C(A)\,\mathrm{tr}\big[T_a T_b\big]$ to rewrite the trace in the adjoint into the trace in the fundamental where $C(A)$ is the coefficient in $\mathrm{tr}_{\mathbf{Ad}}\big[T_a T_b\big] = C(A)\,\eta_{ab}$. Then we obtain the following divergence in position space

$$\Sigma_{\mathrm{VV}}^{(\mathrm{V,C})}\Big|_{\mathrm{div}} = -i\, \frac{3\, C(A)\, \mu^{d-4}}{(4\pi)^2} \left(\frac{1}{\bar{\epsilon}} + \ln \frac{\mu^2}{m^2} \right) \int d^dx\, d^4\theta\; \mathrm{tr}\Big[- V_1 \Box_2 P_0 V_2 \Big]. \qquad (3.99)$$

The corresponding counterterm is obtained after the quadratic operator has been promoted to the full gauge kinetic term as explained above (3.90)

$$\Delta \mathscr{S}_{\mathrm{VV}}^{(\mathrm{V,C})} = \mu^{d-4}\, \frac{3\, C(A)}{(4\pi)^2 \epsilon}\, \mathrm{tr} \int d^dx \left[\frac{1}{4} \int \mathrm{d}^2\theta\; W^\alpha W_\alpha + \frac{1}{4} \int \mathrm{d}^2\bar{\theta}\; \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha} \right]. \qquad (3.100)$$

Then the gauge coupling running is given by

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{C(A)}{(4\pi)^2}\, \ln \frac{\mu^2}{\mu_0^2}. \qquad (3.101)$$

This result concludes our discussion of the renormalization of the gauge kinetic term in four dimensional supersymmetric theories.

Chapter 4

Supersymmetric theory in five dimensions

In this chapter we consider a five dimensional spacetime with the fifth dimension compactified on the orbifold $S^1/\mathbb{Z}_2$. The field content under consideration consists of a hyper multiplet and a gauge multiplet in the bulk. As in the four dimensional analysis of the preceeding chapter, we aim at the renormalization of the gauge kinetic term. The orbifold bulk can be considered as a flat five dimensional Minkowski space and the determination of the five dimensional renormalization is straightforward. But due to the existence of the orbifold fixed points the spacetime does not have a trivial structure as in the four dimensional calculation. This gives rise to the possibility that the bulk gauge fields may induce quantum corrections that are localized at the fixed points. As we deal with a higher dimensional theory there is in principle the chance that higher dimensional operators are generated in the renormalization process. We will see that in five dimensions this is not the case. As has become apparent from the discussion in the last chapter, in the Abelian case it is only the hyper multiplet that leads to loop corrections of the gauge kinetic term, while in the non-Abelian case also the particles from the gauge sector contribute to the loop effects. We discuss the Abelian case in the first and the non-Abelian case in the second section. This chapter introduces our method of calculating Feynman graphs directly on the orbifold. The method is discussed in most detail in Section 4.1.3 for the simple orbifold $S^1/\mathbb{Z}_2$ and will be applied to more complicated orbifolds in the following chapters.

4.1 Hyper multiplet coupled to an Abelian gauge multiplet

In this section we consider a hyper multiplet coupled to an Abelian gauge multiplet in five dimensions with the fifth dimension compactified on the orbifold

$S^1/\mathbb{Z}_2$. We begin our discussion with a review of the five dimensional multiplets and how they are described in a four dimensional superfield language. In the Abelian theory only the hyper mutiplet contributes to the renormalization of the gauge kinetic term, so we quantize only the hyper multiplet part of the action. We present our method how the theory on flat space can be extended to the five dimensional orbifold $S^1/\mathbb{Z}_2$. To this end we introduce orbifold compatible delta functions that arise from functional differentiation. We show that the renormalization of the gauge kinetic term due to the hyper multiplet vanishes both in the bulk and at the fixed points but for different reasons.

4.1.1 Classical action in five dimensional Minkowski space

We consider a classical supersymmetric theory of a hyper multiplet that is coupled to an Abelian gauge multiplet in five dimensional Minkowski space in the language of four dimensional superfields [13, 15, 16, 73]. The degrees of freedom of the five dimensional hyper multiplet are described by two four dimensional chiral multiplets Φ_+ and Φ_- and the degrees of freedom of the five dimensional gauge multiplet are contained in one four dimensional vector multiplet V and one four dimensional chiral multiplet S. The superfield action for these multiplets is given by

$$\mathscr{S}^{5\mathrm{D}}_{\mathrm{Abelian}}\big(\Phi_+,\Phi_-,V,S\big)=\mathscr{S}_H\big(\Phi_+,\Phi_-,V,S\big)+\mathscr{S}_V\big(V,S\big), \tag{4.1}$$

where $\mathscr{S}_H$ is the kinetic action of the hyper multiplet with its coupling to the five dimensional gauge multiplet

$$\begin{aligned}\mathscr{S}_H=\int d^5x\bigg\{&\int d^4\theta\,\big(\bar{\Phi}_+e^{2qV}\Phi_++\bar{\Phi}_-e^{-2qV}\Phi_-\big)\\&+\int d^2\theta\;\Phi_-\big(\partial_5+\sqrt{2}\,qS\big)\Phi_++\int d^2\bar{\theta}\;\bar{\Phi}_+\big(-\partial_5+\sqrt{2}\,q\bar{S}\big)\bar{\Phi}_-\bigg\}.\end{aligned} \tag{4.2}$$

The five dimensions are described by the coordinates (x^m,y), the integration runs over the five dimensions $\int d^5x=\int d^4x\,dy$ and the derivative into the fifth direction has been denoted by $\partial_5=\frac{\partial}{\partial y}$. This action is invariant under the following supergauge transformation

$$\begin{aligned}\Phi_+&\to e^{-2q\Lambda}\Phi_+, &\qquad V&\to V+\Lambda+\bar{\Lambda},\\ \Phi_-&\to e^{+2q\Lambda}\Phi_-, &\qquad S&\to S+\sqrt{2}\,\partial_5\Lambda,\end{aligned} \tag{4.3}$$

where Λ is a chiral superfield and $\bar{\Lambda}$ its conjugate. The kinetic action $\mathscr{S}_V$ for the five dimensional gauge multiplet in a four dimensional superfield language comprises the standard terms for the four dimensional gauge field V and one extra term for the four dimensional chiral multiplet S

$$\mathscr{S}_V = \frac{1}{g^2} \int d^5x \left\{ \frac{1}{4} \int d^2\theta \, W^\alpha W_\alpha + \frac{1}{4} \int d^2\bar{\theta} \, \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} + \int d^4\theta \, V_5^2 \right\}, \tag{4.4}$$

where W_α, $\bar{W}_{\dot{\alpha}}$ are the gauge invariant Abelian field strengths as defined in four dimensions (3.5) and V_5 is a gauge invariant combination of the superfields V and S

$$V_5 = \frac{1}{\sqrt{2}} \big(S + \bar{S} \big) - \partial_5 V. \tag{4.5}$$

It is straightforward to rewrite the gauge kinetic action in the following form

$$\mathscr{S}_V = \frac{1}{g^2} \int d^5x \, d^4\theta \left\{ \frac{1}{8} V D^\alpha \bar{D}^2 D_\alpha V + \partial_5 V \, \partial_5 V - \sqrt{2} \, \partial_5 V \big(S + \bar{S} \big) + \bar{S} S \right\}. \tag{4.6}$$

This is the gauge kinetic term for which we want to determine the renormalization. Expression (4.6) is obtained from (4.4) by using the identities (2.45) and some partial integrations. There is a mixing between the four dimensional vector multiplet V and the chiral multiplet S. This mixing is of importance when one wants to obtain the propagators for V and S, but as we do not have loop corrections from the gauge sector in the Abelian theory, we will refrain from doing this here. We come back to this issue when we consider the non-Abelian theory in Section 4.2.

This description clearly does not bear manifest five dimensional Lorentz invariance. Lorentz invariance is recovered after the auxiliary fields are eliminated by their equations of motion. Therefore, this description is not an off-shell formulation of the five dimensional supersymmetric theory. However, for us the main advantage of this approach is that perturbation theory is greatly simplified over a component approach and all kinds of cancellations due to $\mathcal{N} = 1$ supersymmetry are built in.

4.1.2 Quantization of the action

After this strictly classical discussion of the five dimensional hyper and vector multiplets we now turn towards the quantization of the theory using path integral methods. To this end we insert the action into the path integral thus defining

Figure 7: Here we depict the drawing conventions for the propagators of the two chiral multplets that compose the five dimensional hyper multiplet. There are two chiral multiplet propagators: The first one corresponds to the diagonal terms in (4.8), the second one refers to the off–diagonal parts.

the generating functional of the theory

$$
\begin{aligned}
&Z\big[J_+, J_-, J_V, J_S\big] = \\
&\qquad \int \mathscr{D}\Phi_+\, \mathscr{D}\Phi_-\, \mathscr{D}V\, \mathscr{D}S\ \exp\Big\{ i \int d^5x\ \Big(\mathscr{L}^{5\mathrm{D}}_{\text{Abelian}}\big(\Phi_+, \Phi_-, V, S\big) + \\
&\qquad + \int d^2\theta\ J_+\Phi_+ + \int d^2\bar{\theta}\ \bar{J}_+\bar{\Phi}_+ + \int d^2\theta\ J_-\Phi_- + \int d^2\bar{\theta}\ \bar{J}_-\bar{\Phi}_- + \\
&\qquad\qquad + \int d^4\theta\ J_V V + \int d^2\theta\ J_S S + \int d^2\bar{\theta}\ J_{\bar{S}}\bar{S} \Big) \Big\}, \qquad (4.7)
\end{aligned}
$$

that involves the sources J_+, J_-, J_V and J_S. In the following we determine only the propagators of the four dimensional chiral superfields Φ_+ and Φ_-. Then the interactions involving these fields can be obtained by functional differentiation with respect to the chiral sources J_+ and J_-. As there are no graphs involving the vector superfield that would lead to a renormalization of the gauge kinetic term, we do not discuss the vector superfield propagators here.

Hyper multiplet propagator

By considering the quadratic part of the hyper multiplet action (4.2) and using some standard superspace identities, we obtain

$$
\begin{aligned}
&Z_0\big[J_+, J_-\big] = \\
&\int \mathscr{D}\Phi_+ \mathscr{D}\Phi_-\, \exp\Big\{ i \int d^5x\, \mathrm{d}^4\theta\, \big(J_+ \bar{J}_-\big)\, \frac{-1}{\Box + \partial_5^2} \begin{pmatrix} 1 & \partial_5 \frac{D^2}{-4\Box} \\ -\partial_5 \frac{\bar{D}^2}{-4\Box} & 1 \end{pmatrix} \begin{pmatrix} \bar{J}_+ \\ J_- \end{pmatrix} \Big\}. \qquad (4.8)
\end{aligned}
$$

Hence as for massive chiral multiplets in four dimensions we have both non-chiral propagators between $\bar{J}_\pm$ and $J_\pm$, as well as chiral propagators between J_+ and J_- and their conjugates. In Fig. 7 we depict our drawing conventions of these chiral propagators: The first propagator in this picture gives the correlation between the sources $\bar{J}_\pm$ and $J_\pm$, and the second one between J_+ and J_-. Obviously, there is also the conjugate propagator between $\bar{J}_+$ and $\bar{J}_-$.

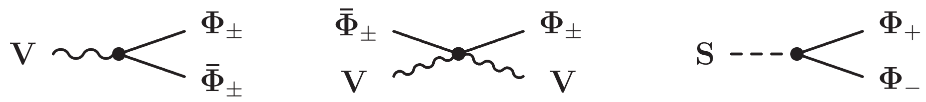


Figure 8: These interaction vertices involve the coupling of the gauge superfields V and S to the hyper multiplet chiral superfields Φ_+ and Φ_-.

Interactions

The only interactions in the Abelian theory are given by vertices that involve the coupling of the five dimensional hyper multiplet to the five dimensional gauge multiplet, i.e. the coupling of the chiral superfields Φ_+ and Φ_- to the vector superfield V and the chiral superfield S. The expansion of the hyper multiplet action (4.2) up to fourth order in the fields gives rise to the interactions

$$\begin{aligned}\mathscr{S}_{\mathrm{H}} \supset \int d^5x \Big\{ \int \mathrm{d}^4\theta\, \bar{\Phi}_\pm(\pm 2qV + 2q^2V^2)\Phi_\pm \\ + \int d^2\theta\, \sqrt{2}\, q\Phi_- S\Phi_+ + \int d^2\bar{\theta}\, \sqrt{2}\, q\bar{\Phi}_+\bar{S}\bar{\Phi}_- \Big\},\end{aligned} \tag{4.9}$$

which have been depicted in Fig. 8. This finishes the discussion of propagators and vertices of the five dimensional Abelian theory.

4.1.3 Orbifold compatible calculus for $S^1/\mathbb{Z}_2$

Up to now, the treatment was completely standard with no deviation to a treatment in uncompactified five dimensional Minkowski space and the orbifold has not been mentioned yet. Starting from the action (4.1) where the integration $\int dy$ runs over the non-compact fifth dimension and where general superfields $\Phi_+(x,y)$, $\Phi_-(x,y)$, $V(x,y)$ and $S(x,y)$ are allowed in the Hilbert space (we drop the dependence on the Grassmann coordinates for a while) we will now render the action compatible with the orbifold $M^4 \times S^1/\mathbb{Z}_2$. Details on the geometry of $S^1/\mathbb{Z}_2$ and its eigenfunctions can be found in App. A.1. The first step is to put the action on a circle of radius R. This means that now the integration in the extra dimension runs over the circle $\int_0^{2\pi R} dy$. From the Hilbert space of fields that live on the uncompactified space only those fields form the Hilbert space on the circle that fulfill the periodic boundary condition

$$\begin{aligned}\Phi_+^{\text{circle}}(x, y + 2\pi) &= \Phi_+^{\text{circle}}(x, y), \qquad & V^{\text{circle}}(x, y + 2\pi) &= V^{\text{circle}}(x, y), \\ \Phi_-^{\text{circle}}(x, y + 2\pi) &= \Phi_-^{\text{circle}}(x, y), \qquad & S^{\text{circle}}(x, y + 2\pi) &= S^{\text{circle}}(x, y),\end{aligned} \tag{4.10}$$

while all non-periodic fields are projected out. The usual explicit mode expansion of the periodic fields involves infinite sums over cosine and sine modes. In our formalism we will not perform an explicit mode expansion of the fields at this stage. We will simply restrict the fields in the action (4.1) to the fields on the circle which respect equations (4.10).

The second step is to place the theory on the orbifold. The fundamental domain of the $S^1/\mathbb{Z}_2$ is the interval from 0 to πR and thus a calculation on the orbifold would require half the integration range of the circle. However, we choose to keep working on the circle (the 'covering space' of the orbifold). This means in particular that we have to describe fields living on the orbifold in terms of fields living on the circle as we will see now. Placing fields on the orbifold amounts to a second projection. First the orbifold action has to be implemented such that the action $\mathscr{S}$ is left invariant. We implement the $\mathbb{Z}_2$ orbifold twist on the fields by requiring that the fields transform as

$$\begin{aligned} \Phi_+^{\text{orbifold}} &\to \Phi_+^{\text{orbifold}}, \qquad & V^{\text{orbifold}} &\to V^{\text{orbifold}}, \\ \Phi_-^{\text{orbifold}} &\to -\Phi_-^{\text{orbifold}}, \qquad & S^{\text{orbifold}} &\to -S^{\text{orbifold}}, \end{aligned} \tag{4.11}$$

under the $\mathbb{Z}_2$ orbifold twist $y \to -y$. (Please keep in mind that the subscripts at the fields Φ_+ and Φ_- refer to the sign of the charge in the action (4.2) and have nothing to do with the parity under the orbifold twist.) The Hilbert space on the orbifold only contains fields from the Hilbert space on the circle that fulfill the requirement (4.11), i.e. that are orbifold compatible, all other fields from the Hilbert space of the circle are projected out. In the conventional mode expansion the orbifold even fields Φ_+ and V are described only by the cosine modes (they have a zero mode with a KK tower on top) and the orbifold odd fields Φ_- and S only consist of the sine modes (they only have a KK tower, but no zero mode). In order to be able to work on the covering space the fields on the orbifold are expressed as linear combinations of the fields living on the circle. This is achieved as follows

$$\begin{aligned} \Phi_+^{\text{orbifold}}(x,y) &= \frac{1}{2}\big(\Phi_+^{\text{circle}}(x,y) + \Phi_+^{\text{circle}}(x,-y)\big), \\ \Phi_-^{\text{orbifold}}(x,y) &= \frac{1}{2}\big(\Phi_-^{\text{circle}}(x,y) - \Phi_-^{\text{circle}}(x,-y)\big), \\ V^{\text{orbifold}}(x,y) &= \frac{1}{2}\big(V^{\text{circle}}(x,y) + V^{\text{circle}}(x,-y)\big), \\ S^{\text{orbifold}}(x,y) &= \frac{1}{2}\big(S^{\text{circle}}(x,y) - S^{\text{circle}}(x,-y)\big), \end{aligned} \tag{4.12}$$

Clearly, these combinations of fields are orbifold compatible, i.e. they obey (4.11). This construction generalizes straightforwardly to more complicated situations. Next we read off the required orbifold transformation behaviour of the sources from corresponding terms in (4.7). The sources have to transform as

$$J_+^{\text{orbifold}} \to J_+^{\text{orbifold}}, \qquad J_-^{\text{orbifold}} \to -J_-^{\text{orbifold}}, \tag{4.13}$$

under the orbifold twist $y \to -y$. Therefore, also the orbifold compatible sources can be constructed from the sources living on the circle as

$$
\begin{aligned}
J_+^{\text{orbifold}}(x,y) &= \frac{1}{2}\big(J_+^{\text{circle}}(x,y) + J_+^{\text{circle}}(x,-y)\big),\\
J_-^{\text{orbifold}}(x,y) &= \frac{1}{2}\big(J_-^{\text{circle}}(x,y) - J_-^{\text{circle}}(x,-y)\big).
\end{aligned}
\tag{4.14}
$$

With this knowledge we come to the central point, namely the definition of the functional derivatives w.r.t. these sources

$$
\frac{\delta J_{+2}^{\text{orbifold}}}{\delta J_{+1}^{\text{orbifold}}} = \frac{\bar{D}^2}{-4}\,\tilde{\delta}_{21}^{(+)}, \qquad \frac{\delta J_{-2}^{\text{orbifold}}}{\delta J_{-1}^{\text{orbifold}}} = \frac{\bar{D}^2}{-4}\,\tilde{\delta}_{21}^{(-)}. \tag{4.15}
$$

Here the appropriate covariant derivatives have been included in the definition to maintain chirality. The twiddle on the delta functions indicates the fact that these are orbifold compatible delta functions, i.e. they are in accordance with the transformation behaviour of the source terms on the l.h.s. of equations (4.15) as it was specified in (4.13). The orbifold compatible delta functions for the Abelian theory on the $S^1/\mathbb{Z}_2$ look as follows

$$
\begin{aligned}
\tilde{\delta}_{21}^{(+)} &= \frac{1}{2}\Big(\delta(y_2 - y_1) + \delta(y_2 + y_1)\Big)\delta^4(x_2 - x_1)\,\delta^4(\theta_2 - \theta_1),\\
\tilde{\delta}_{21}^{(-)} &= \frac{1}{2}\Big(\delta(y_2 - y_1) - \delta(y_2 + y_1)\Big)\delta^4(x_2 - x_1)\,\delta^4(\theta_2 - \theta_1).
\end{aligned}
\tag{4.16}
$$

These delta functions are the key elements of our formalism of calculating Feynman graphs directly on the orbifold, since they contain all the geometric information about the orbifold compatible superfields. Therefore, it is important to develop some of their properties: We first check that the delta functions transform in the same way as the corresponding sources. Under an orbifold transformation $y_2 \to -y_2$ of the second coordinate the delta functions transform as the sources in the numerators in (4.15)

$$
\tilde{\delta}_{21}^{(+)} \to \tilde{\delta}_{21}^{(+)}, \qquad \tilde{\delta}_{21}^{(-)} \to -\tilde{\delta}_{21}^{(-)}. \tag{4.17}
$$

Under an orbifold transformation $y_1 \to -y_1$ of the first coordinate, the orbifold compatible delta functions transform inversely, corresponding to the sources in the denominators of (4.15). In this simple situation of a $\mathbb{Z}_2$ symmetry the inverse transformation is the same as (4.17).

Then we want to mention also a helpful technical detail: In the calculation of a graph on an orbifold one frequently encounters derivative operators that act on orbifold compatible delta functions. Then it is often helpful to switch the variable w.r.t. which the derivative acts. For derivatives w.r.t. the uncompactified coordinates this gives just the ordinary minus sign, because of

$\frac{\partial}{\partial x_2}\,\delta^4(x_2 - x_1) = -\frac{\partial}{\partial x_1}\,\delta^4(x_2 - x_1)$. However, for derivatives w.r.t. the fifth coordinate changing a spacetime index of the y-derivative also changes the type of orbifold compatible delta function

$$\big(\partial_5\big)_2\,\tilde{\delta}_{21}^{(+)} = -\big(\partial_5\big)_1\,\tilde{\delta}_{21}^{(-)}. \tag{4.18}$$

Identities analogous to (4.18) will appear in the non-Abelian and higher dimensional cases in more complicated forms.

Going through the construction again, we notice that the important objects for a Feynman graph calculation are the orbifold compatible delta functions. Their transformation properties were derived from the transformation properties of the source terms which in turn were derived from the transformation properties of the fields. The explicit definition of orbifold compatible fields in (4.12) and orbifold compatible sources in (4.14), however, is not important for the calculation at all. We will therefore not construct orbifold compatible fields and orbifold compatible sources anymore, but instead only infer the orbifold compatible delta functions from the required transformation behaviour of the fields and source terms. This remark completes the discussion of the Abelian supersymmetric field theory on the orbifold $S^1/\mathbb{Z}_2$. The method is extended to cover the non-Abelian case in Section 4.2.3.

4.1.4 Renormalization of the gauge kinetic term due to the hyper multiplet

In this section we calculate the hyper multiplet contributions to the gauge kinetic term of the Abelian supersymmetric theory in five dimensions with the fifth dimension compactified on the orbifold $S^1/\mathbb{Z}_2$. The gauge kinetic term for the five dimensional gauge multiplet (4.6) contains four different parts in the four dimensional superfield language. Therefore, we have to determine four different self-energies. By Σ_{VV} we denote the self-energy which contains the loop corrections for the first two terms in (4.6) that involve only the four dimensional gauge multiplet V. The self-energies that contain the loop corrections to the terms which mix the four dimensional gauge field with the chiral adjoint field S and its conjugate $\bar{S}$ are denoted by Σ_{VS} and $\Sigma_{V\bar{S}}$, respectively. The last term in (4.6) involves only the chiral adjoint field and is loop corrected by the self-energy $\Sigma_{S\bar{S}}$. For each part the loop corrections are determined separately.

From now on we will assume that the theory has been placed on the orbifold as described in the preceeding Section 4.1.3. All fields are now considered to be orbifold compatible and we forget about the superscript at the fields that indicated this explicitly. In the following calculation of Feynman graphs the only place where the orbifold compatibility appears is in the fact that we have to use the orbifold compatible delta functions (4.16) instead of the usual delta functions.

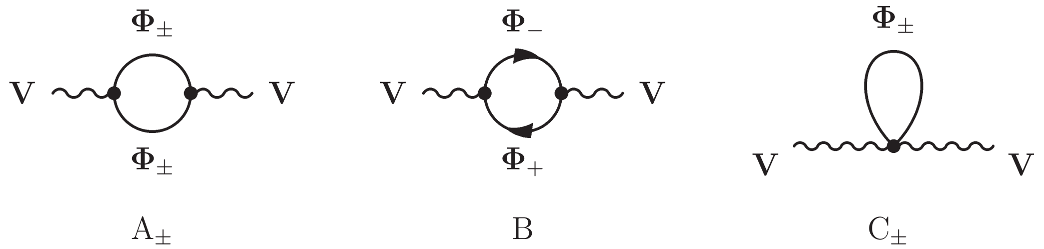


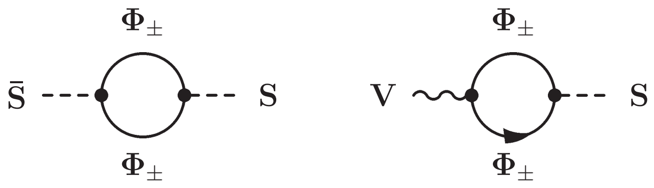
Figure 9: The five dimensional gauge multiplet receives VV self-energy corrections from the hyper multiplet. The proper self-energy graphs are labeled 9.A$_\pm$ and 9.B. The tadpole graph is denoted by 9.C$_\pm$.

The first step of the calculation proceeds along the same lines as it was worked out in detail in Section 3.1.3 for an analogous calculation in four dimensional Minkowski space. The interactions that involve the coupling of the hyper multiplet to the gauge multiplet in (4.9) are used to construct the relevant terms in the expansion of the generating functional which correspond to the graphs with the hyper multiplet in the loop. Those graphs that contribute to the Σ_{VV} self-energy have been depicted in Fig. 9. Graph 9.A$_\pm$ contains the propagators that connect the chiral sources $J_\pm$ with the anti-chiral sources $\bar{J}_\pm$, while diagram 9.B involves the chiral sources J_+ and J_-. This diagram also has a hermitean conjugate partner, which we refer to as $\overline{9.\mathrm{B}}$. The tadpole graphs are given as diagrams 9.C$_\pm$. Those graphs which contribute to the $\Sigma_{V\bar{S}}$ and $\Sigma_{S\bar{S}}$ self-energies are depicted in Fig. 10. The Σ_{VS} self-energy is given by the complex conjugate of $\Sigma_{V\bar{S}}$. In order to calculate these graphs the chiral fields on the internal lines are replaced by the functional derivatives w.r.t. the chiral source terms

$$\Phi_+ \to \frac{\delta}{\delta\big(iJ_+\big)}, \qquad \Phi_- \to \frac{\delta}{\delta\big(iJ_-\big)}. \tag{4.19}$$

These functional derivatives act on the exponential of the propagators as described in Section 3.1.3. The only difference to a calculation in Minkowski space is that the functional differentiation w.r.t these orbifold compatible sources produces orbifold compatible delta functions (4.16) according to (4.15) and not the usual delta functions.

The second step of the calculation is concerned with reducing the number of orbifold compatible delta functions in the amplitude (9.A$_\pm$ and 9.B for example contain four of them) and replacing them by ordinary delta functions. This step is plausible when one regards the twiddle on the delta function as a projector that projects the amplitude from uncompactified space to the orbifold. (In fact, formally taking away all twiddles obviously just reproduces the amplitude in Minkowski space.) Because the square of a projector gives the projector again, it is possible to 'remove' superfluous twiddles from the amplitude leaving only usual delta functions. A worked out example how the reduction of the orbifold compatible delta functions to ordinary delta functions is performed in a loop am-



A$_\pm$ B$_\pm$

Figure 10: The five dimensional gauge multiplet receives $\bar{S}S$ self-energy corrections from the hyper multiplet as is depicted in figure 10.A$_\pm$. In addition the hyper multiplet gives rise to mixing between the four dimensional superfields V and S in Fig. 10.B$_\pm$.

plitude can be found in App. B. The reduction of all but two orbifold compatible delta functions is always unambiguous and therefore we present all amplitudes at the level of two orbifold compatible delta functions. The amplitudes which correspond to the graphs with the hyper multiplet in the loop in Figs. 9 and 10 are provided in App. C.1. There we employ a six dimensional non-Abelian notation which has a straightforward reduction to the five dimensional Abelian notation as is explained in the appendix. The sum of the amplitudes due to the graphs in Fig. 9 is the VV self-energy which takes the form

$$\begin{aligned}\Sigma_{VV} = q^2 \int \big(d^5x\big)_{12} d^4\theta \Big\{ & - V_1 \,\Box_2 P_0\, V_2 \,\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\delta}_{21}^{5(+)} \,\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\tilde{\delta}}_{21}^{5(+)} \\ & - V_1 \,\Box_2 P_0\, V_2 \,\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\tilde{\delta}}_{21}^{5(-)} \,\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\delta}_{21}^{5(-)} \\ & + 2\, \partial_5 V_1\, \partial_5 V_2 \,\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\delta}_{21}^{5(+)} \,\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\delta}_{21}^{5(-)} \Big\}\end{aligned} \tag{4.20}$$

and the graphs in Fig. 10 constitute the $V\bar{S}$ and $S\bar{S}$ self-energies

$$\Sigma_{V\bar{S}} = -\,2\sqrt{2}\, q^2 \int \big(d^5x\big)_{12} d^4\theta\; \partial_5 V_1\, \bar{S}_2 \,\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\delta}_{21}^{5(+)} \,\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\delta}_{21}^{5(-)}, \tag{4.21}$$

$$\Sigma_{S\bar{S}} = 2\, q^2 \int \big(d^5x\big)_{12} d^4\theta\; S_1 \bar{S}_2 \,\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\delta}_{21}^{5(+)} \,\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\delta}_{21}^{5(-)}. \tag{4.22}$$

The Σ_{VS} self-energy is given by the complex conjugate of (4.21).

The third step involves the remaining two orbifold compatible delta functions and requires a little more care. We see that the expressions in the first two lines of (4.20) employ the same orbifold compatible delta function twice. In such a situation we can simply replace one of the two orbifold delta functions against an ordinary delta function as in the second step. This last orbifold delta function cannot be replaced anymore and it gives the information how the amplitude is distributed on the orbifold. In the rest of the terms of the self-energies, however, two *different* types of delta functions appear and in this situation cancellations can appear. There are two ways to proceed: The first one is to simply insert the definitions of the orbifold delta functions (4.16) and to multiply the terms out. Then all cancellations are explicit. For this simple case of a $\mathbb{Z}_2$ orbifold this is a viable method. The second way is equivalent and more convenient especially in the case of more general $\mathbb{Z}_N$ symmetries where the orbifold compatible delta functions are sums of N terms. In this way one takes one half times the sum of the linear combination of both possibilities to replace an orbifold compatible delta function against an ordinary delta function. Then, for example, the $S\bar{S}$ self-energy reads

$$\Sigma_{S\bar{S}} = q^2 \int \left(d^5x\right)_{12} d^4\theta \; S_1\bar{S}_2 \left(\frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\delta}_{21}^{5(+)}\, \frac{1}{(\Box+\partial_5^2)_2}\, \delta_{21}^{(5)} + \right.$$

$$\left. + \frac{1}{(\Box+\partial_5^2)_2}\, \delta_{21}^{(5)}\, \frac{1}{(\Box+\partial_5^2)_2}\, \tilde{\delta}_{21}^{5(-)} \right), \qquad (4.23)$$

where the second and the first orbifold delta functions have been replaced against an ordinary delta function in the first and the second lines, respectively. Here $\delta_{21}^{(5)} = \delta^4(x_2-x_1)\,\delta(y_2-y_1)$ is the ordinary five dimensional delta function. It is clear from (4.16) that in the $S\bar{S}$ self-energy the dependence on $\delta(y_2+y_1)$ cancels

$$\Sigma_{S\bar{S}} = 2q^2 \int \left(d^5x\right)_{12} d^4\theta \; S_1\bar{S}_2\, \frac{1}{(\Box+\partial_5^2)_2}\, \delta_{21}^{(5)}\, \frac{1}{(\Box+\partial_5^2)_2}\, \delta_{21}^{(5)}. \qquad (4.24)$$

So for these terms we are left with a pure bulk amplitude that has no fixed point contribution.

Bulk renormalization

In the first two lines of (4.20) the reduction to an expression with just one orbifold delta function is straightforward as described in the last paragraph. Expanding the last orbifold delta function with (4.16) and picking out the terms with two ordinary delta function corresponds to the bulk amplitude. For the rest of the terms of the self-energies we have just discussed that they give a pure bulk contribution. Then we add all terms to find the correction to the gauge kinetic term

$$\Sigma_{\text{bulk}}^{\text{hyper}} = q^2 \int \left(d^5x\right)_{12} d^4\theta \left(-V_1\,\Box_2 P_0\, V_2 + \partial_5 V_1\,\partial_5 V_2 - \sqrt{2}\,\partial_5 V_1\left(S_2+\bar{S}_2\right) + S_1\bar{S}_2 \right)$$

$$\times \frac{1}{(\Box+\partial_5^2)_2}\, \delta(y_2-y_1)\, \frac{1}{(\Box+\partial_5^2)_2}\, \delta(y_2-y_1). \qquad (4.25)$$

Note that all terms appear with the correct coefficients to recombine to an expression which is proportional to the gauge kinetic term. The complete self-energy is divergent and we can extract the leading terms with the help of the identities in App. D.1. It is a well-known fact that dimensional regularization in five dimensions hides a linear divergence and does not produce a divergent pole piece. Hence, the leading term of the self-energy reads

$$\Sigma^{\text{hyper}}_{\text{bulk}} = -\frac{i|m|q^2\mu^{-2\epsilon}}{(4\pi)^2}\int d^dx\, d^4\theta\, \left(-V\,\Box P_0\,V + \partial_5 V\partial_5 V - \sqrt{2}\partial_5 V(S+\bar{S}\,) + S\bar{S}\right), \tag{4.26}$$

where $d = 5-2\epsilon$ and m is an infrared regulator mass which is introduced in the appendix. In the MS scheme only a pole part leads to a counterterm. Hence, there is no renormalization of the gauge kinetic term and no influence on the gauge coupling running in the five dimensional bulk.

Fixed point renormalization

In order to find the amplitude at the fixed points, we pick out the terms with one delta function $\delta(y_2-y_1)$ and one delta function $\delta(y_2+y_1)$ in the first two lines of (4.20). We find that the fixed point contributions of the first and the second line cancel each other. For the rest of the terms it was shown that they are pure bulk amplitudes. Therefore, the self-energies (4.20) - (4.22) vanish at the fixed points and there is no renormalization of the gauge coupling at the fixed points of $S^1/\mathbb{Z}_2$. This is a different statement than that for the bulk result, where no counterterm is obtained because of the properties of the regularization, but the self-energy (4.25) is non-zero. In the next chapter this result will be generalized to the statement that for an Abelian gauge theory in the bulk there is no gauge coupling renormalization at the $\mathbb{Z}_2$ fixed points of a $T^2/\mathbb{Z}_N$ orbifold in six dimensions.

4.2 Hyper multiplet coupled to a non-Abelian gauge multiplet

In this section we consider a hyper multiplet that is coupled to a non-Abelian gauge multiplet in five dimensions with the fifth dimension compactified on the orbifold $S^1/\mathbb{Z}_2$. We discuss the five dimensional action in terms of four dimensional superfields and quantize the theory. As in the four dimensional setup the quantization of the gauge sector of the non-Abelian theory requires gauge fixing and leads to the introduction of ghost superfields. We build our presentation on the argumentation and notions of the four dimensional discussion that is provided in Section 3.2. We place the theory on the orbifold $S^1/\mathbb{Z}_2$ with the method developed in the preceeding Section 4.1 and calculate the loop corrections to the gauge kinetic term due to graphs that involve the hyper, gauge and ghost multiplets.

4.2.1 Classical action in five dimensional Minkowski space

In this section we consider the classical supersymmetric theory of a hyper multiplet that is coupled to a non-Abelian gauge multiplet. The description in terms of four dimensional superfields [13, 15, 16, 73] involves the same superfields Φ_+, Φ_-, V and S as in the Abelian case of the preceeding Section 4.1. The chiral fields Φ_+ and Φ_- transform in a given representation (for example the fundamental or adjoint representation) of the gauge group. The four dimensional gauge multiplet $V = V^i T_i$ and the four dimensional chiral multiplet $S = S^i T_i$ both transform in the adjoint representation of the gauge group. The notation for the algebra, the Killing metric and the different traces has been introduced in Section 3.2. The superfield action for the various multiplets is given by

$$\mathscr{S}^{5\text{D}}_{\text{non-Abelian}}\big(\Phi_+, \Phi_-, V, S\big) = \mathscr{S}_H\big(\Phi_+, \Phi_-, V, S\big) + \mathscr{S}_V\big(V, S\big), \tag{4.27}$$

where $\mathscr{S}_H$ is the kinetic action of the hyper multiplet with its coupling to the five dimensional gauge multiplet

$$\begin{aligned}\mathscr{S}_H = \int d^5x \, \bigg\{ \int d^4\theta \, \big(\bar{\Phi}_+ e^{2V} \Phi_+ + \Phi_- e^{-2V} \bar{\Phi}_-\big) + \\ + \int d^2\theta \, \Phi_-\big(\partial_5 + \sqrt{2}\, S\big)\Phi_+ + \int d^2\bar{\theta} \, \bar{\Phi}_+\big(- \partial_5 + \sqrt{2}\, \bar{S}\big)\bar{\Phi}_- \bigg\}.\end{aligned} \tag{4.28}$$

This action is invariant under the following super-gauge transformation

$$\begin{aligned} &\Phi_+ \to e^{-2\Lambda}\, \Phi_+, \qquad && e^{2V} \to e^{2\bar{\Lambda}}\, e^{2V}\, e^{2\Lambda}, \\ &\Phi_- \to \Phi_-\, e^{2\Lambda}, \qquad && S \to e^{-2\Lambda}\left(S + \tfrac{1}{\sqrt{2}}\,\partial_5\right) e^{2\Lambda}, \end{aligned} \tag{4.29}$$

where Λ is a chiral superfield and $\bar{\Lambda}$ its conjugate. The kinetic action $\mathscr{S}_V$ for the five dimensional gauge multiplet in a four dimensional superfield language comprises the standard terms for the four dimensional gauge field V and one extra term for the four dimensional chiral multiplet S

$$\mathscr{S}_V = \frac{1}{g^2}\int d^5x \,\mathrm{tr}\Bigg[\frac{1}{4}\int d^2\theta \; W^\alpha W_\alpha + \frac{1}{4}\int d^2\bar{\theta}\; \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} + \\ + \frac{1}{4}\int d^4\theta \; e^{2V_5}e^{-2V}e^{2V_5}e^{-2V}\Bigg], \quad (4.30)$$

where the non-Abelian field strengths W_α, $\bar{W}_{\dot{\alpha}}$ are defined as in the four dimensional case (3.61) with the same gauge transformation properties (3.62). The superfield V_5 is a combination of the superfields V and S such that e^{2V_5} also transforms covariantly

$$e^{2V_5} = \partial_5\, e^{2V} - \sqrt{2}\, e^{2V} S - \sqrt{2}\,\bar{S}e^{2V}, \qquad e^{2V_5} \rightarrow e^{2\bar{\Lambda}}\, e^{2V_5}\, e^{2\Lambda}, \quad (4.31)$$

so that the vector multiplet action is gauge invariant. The reduction to the Abelian case in Section 4.1 is trivial, where we found in particular that the super field strengths W_α and V_5 are gauge invariant. When we compute the renormalization of the vector multiplet at one loop, we perform a direct computation rather than a background field method. Therefore, we will be able to recover only the quadratic part of the vector multiplet action

$$\mathscr{S}_{V2} = \frac{1}{g^2}\int d^5x\, d^4\theta\, \mathrm{tr}\left[\frac{1}{8} V D^\alpha \bar{D}^2 D_\alpha V + (\partial_5 V)^2 - \sqrt{2}\partial_5 V\left(S+\bar{S}\right) + \bar{S}S\right]. \quad (4.32)$$

In the non-Abelian discussion we quantize also the gauge sector. Therefore, we have to care about the mixing between the four dimensional gauge multiplet V and the chiral adjoint multiplet S that we already observed in the Abelian quadratic action in Section 4.2. The presence of this mixing would complicate the calculations, but luckily, it can be removed by a suitable choice of gauge fixing, as we discuss below. For the non-Abelian action the same comments about five dimensional Lorentz invariance apply as for the Abelian action (4.27).

4.2.2 Quantization of the action

Now we turn to the quantization of the theory using the path integral approach. The generating functional reads

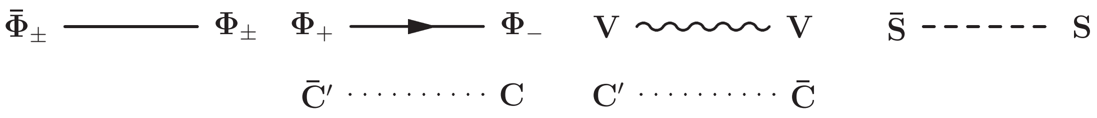


Figure 11: Here we depict our drawing conventions for the propagators in the non-Abelian theory. The chiral multiplet propagators have the same form as in the Abelian case in (4.8). The first chiral multiplet propagator corresponds to the diagonal terms in (4.8), while the second refers to the off–diagonal parts. In the gauge sector we have propagators for the V and S multiplets presented in (4.41) and the ghost multiplets (4.45).

$$
\begin{aligned}
&Z\big[J_+, J_-, J_V, J_S\big] = \\
&\quad \int \mathscr{D}\Phi_+ \, \mathscr{D}\Phi_- \, \mathscr{D}V \, \mathscr{D}S \, \exp\bigg\{ i \int d^5x \, \Big(\mathscr{L}^{\text{5D}}_{\text{non-Abelian}}\big(\Phi_+, \Phi_-, V, S\big) + \\
&\qquad + \int d^2\theta \; J_+ \Phi_+ + \int d^2\bar\theta \; \bar{J}_+ \bar\Phi_+ + \int d^2\theta \; J_- \Phi_- + \int d^2\bar\theta \; \bar{J}_- \bar\Phi_- + \\
&\qquad + \operatorname{tr}\Big[\int d^4\theta \; J_V V + \int d^2\theta \; J_S S + \int d^2\bar\theta \; J_{\bar S} \bar S \Big] \Big) \bigg\}. \qquad (4.33)
\end{aligned}
$$

As usual the interactions can be obtained by functional differentiation with respect to the sources, after the original superfields are integrated out using their corresponding quadratic actions.

Hyper multiplet propagator

By considering the quadratic part of the hyper multiplet action (4.28) we find the same hyper multiplet propagator as in the Abelian case (4.8).

Gauge multiplet propagator

For the five dimensional gauge multiplet we need to do more work because of gauge invariance. The problem of resulting zero modes can be made manifest by representing the quadratic action (4.32) in the following matrix form

$$
\begin{aligned}
Z_0\big[J_V, J_S\big] = \int \mathscr{D}V \exp\bigg\{ i \int d^5x \, \operatorname{tr}\Big[\int d^4\theta \; \frac{1}{g^2} \, \bar{\mathbf{v}} \, \mathbf{A} \, \mathbf{v} + \int d^4\theta \; J_V V \\
+ \int d^2\theta \; J_S S + \int d^2\bar\theta \; \bar{J}_S \bar{S} \Big] \bigg\}, \qquad (4.34)
\end{aligned}
$$

where the vector $\mathbf{v}$ and the hermitean matrix $\mathbf{A}$ are given by

$$\mathbf{v} = \begin{pmatrix} V \\ S \\ \bar{S} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} -\Box P_0 - \partial_5^2 & \frac{1}{\sqrt{2}} P_+ \partial_5 & \frac{1}{\sqrt{2}} P_- \partial_5 \\ -\frac{1}{\sqrt{2}} P_+ \partial_5 & \frac{1}{2} P_+ & 0 \\ -\frac{1}{\sqrt{2}} P_- \partial_5 & 0 & \frac{1}{2} P_- \end{pmatrix}. \tag{4.35}$$

Here P_0 is the transversal projector and P_+ and P_- are its chiral counterparts defined in (2.40). The operator $\mathbf{A}$ has chiral zero modes corresponding to the gauge directions $\mathbf{x}$. Indeed, we see that

$$\mathbf{A}\mathbf{x} = 0 \qquad \text{for} \qquad \mathbf{x} = \delta_\Lambda \mathbf{v} = \begin{pmatrix} \Lambda + \bar{\Lambda} \\ \sqrt{2}\, \partial_5 \Lambda \\ \sqrt{2}\, \partial_5 \bar{\Lambda} \end{pmatrix}. \tag{4.36}$$

This shows explicitly that also in five dimensions in order to define the propagator of the vector multiplet, we need to perform a gauge fixing to modify the quadratic form $\mathbf{A}$ so that it becomes invertible.

The procedure to determine the gauge fixed action follows the conventional four dimensional superfield methods for gauge multiplets as they were reviewed in Chapter 3. In generalization of (3.72) we choose the gauge fixing functional [17, 74]

$$\Theta = \frac{\bar{D}^2}{-4} \left(\sqrt{2}\, V + \frac{1}{\Box}\, \partial_5 \bar{S} \right). \tag{4.37}$$

To motivate this choice, we observe that taking the imaginary part of the highest restriction

$$\frac{D^2}{-4}\, \Theta \bigg| = \frac{1}{\sqrt{2}} \left(\Box C + D + \partial_5 \varphi - \mathrm{i} \partial_M A^M \right) \tag{4.38}$$

reveals that the gauge fixing functional Θ incorporates the five dimensional Lorentz invariant gauge fixing $\partial_M A^M = 0$. The gauge fixing condition $\Theta = F$, with F an arbitrary chiral superfield, is implemented into the path integral via the standard procedure as the argument of a delta function together with a compensating Fadeev-Popov determinant $\Delta(\Theta)$. We include the Gaussian weighting factor $\exp i \int d^5x d^4\theta\, \mathrm{tr} \bar{F} F$ and perform the functional integration over F. Because of the delta functions, that implement the gauge fixing, this Gaussian integration is trivial and results in the gauge fixing action

$$\mathscr{S}_{\mathrm{gf}} = -\frac{1}{g^2} \int d^5x\, d^4\theta\, \mathrm{tr} \left[V \left(\Box + \frac{1}{8} D^\alpha \bar{D}^2 D_\alpha \right) V + \sqrt{2} V \partial_5 \left(S + \bar{S} \right) + \partial_5 \bar{S}\, \frac{1}{\Box}\, \partial_5 S \right] \tag{4.39}$$

in extension of (3.74). Combining this gauge fixing action with (4.34) into the gauge fixed action gives rise to invertible quadratic operators

$$\mathscr{S}_{\mathrm{V2}} + \mathscr{S}_{\mathrm{gf}} = \int d^5x\, d^4\theta \;\mathrm{tr}\Big[-V\big(\Box + \partial_5^2\big)V + \bar{S}\Big(1 + \frac{\partial_5^2}{\Box}\Big)S\Big]. \tag{4.40}$$

Here we see a further motivation for the gauge fixing functional (4.37): The mixing between the V and the S and $\bar{S}$ fields, which was present in (4.34), has been removed. Consequently, the propagators for V and S are decoupled

$$Z_0\big[J_V, J_S\big] = \exp\Big\{ i \int d^5x\, d^4\theta \;\mathrm{tr}\Big[\frac{1}{4}\, J_V \,\frac{1}{\Box + \partial_5^2}\, J_V + \bar{J}_S \,\frac{-1}{\Box + \partial_5^2}\, J_S\Big]\Big\}. \tag{4.41}$$

This decoupling amounts to a major simplification of the supergraph computations performed in Section 4.2.5. Notice that the nonlocal term in (4.40) has given rise to a perfectly regular propagator for the S superfield. The propagators are depicted in Fig. 11.

Ghost propagator

To finish the description of the gauge fixing procedure, we rewrite the Faddeev-Popov determinant $\Delta(\Theta)$ using anti-commuting Lie algebra valued ghost superfields C and C' as demonstrated in Chapter 3. The central point for determining their action are the infinitesimal versions of the gauge variations (4.29) of the fields V and S

$$\delta_\Lambda S = \sqrt{2}\,\partial_5\Lambda + 2\big[S, \Lambda\big], \tag{4.42}$$

$$\delta_\Lambda V = L_V(\Lambda - \bar{\Lambda}) + \coth(L_V) L_V(\Lambda + \bar{\Lambda}),$$

that are present in the gauge fixing functional (4.37). The variation of the four dimensional vector multiplet is of course the same as in the four dimensional procedure, cf. (3.77). Then the infinitesimal gauge variation δ_C of the gauge fixing functional reads

$$\delta_C\Theta = \sqrt{2}\,\frac{\bar{D}^2}{-4}\Big(L_V(C - \bar{C}) + \coth\big(L_V(L_V(C + \bar{C}))\big) + \\ + \frac{\partial_5}{\Box}\big(\sqrt{2}\,\partial_5\bar{C} - 2\big[\bar{S}, \bar{C}\big]\big)\Big) \tag{4.43}$$

and the ghost action $\mathscr{S}_{gh}\big(C, C', V\big)$ is determined straightforwardly in analogy to (3.81).

$$\mathscr{S}_{\mathrm{gh}} = \frac{1}{\sqrt{2}} \int d^5x\, d^4\theta \;\mathrm{tr}\Big[\sqrt{2}\,(C' + \bar{C}')\Big(L_V(C - \bar{C}) + \coth\big(L_V(L_V(C + \bar{C}))\big)\Big) \\ + C'\,\frac{\partial_5}{\Box}\Big(\sqrt{2}\,\partial_5\bar{C} - 2\big[\bar{S}, \bar{C}\big]\Big) + \bar{C}'\,\frac{\partial_5}{\Box}\Big(\sqrt{2}\,\partial_5 C + 2\big[S, C\big]\Big)\Big]. \tag{4.44}$$

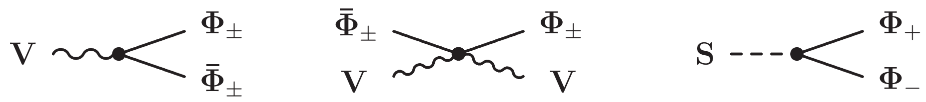


Figure 12: These interaction vertices involve the coupling of the gauge superfields V and S to the hyper multiplet chiral superfields Φ_+ and Φ_-.

From this action the ghost propagators are determined easily

$$Z_0\big[J_C, J_C'\big] = \exp\left\{ i \int d^5x\, d^4\theta\ \mathrm{tr}\Big[-\bar{J}_C' \frac{1}{\Box + \partial_5^2}\, J_C - J_C' \frac{1}{\Box + \partial_5^2}\, \bar{J}_C \Big]\right\}. \quad (4.45)$$

Again, even though the (quadratic) action (4.44) appears to include non-local terms, the ghosts have perfectly normal 5D propagators. These propagators are given in Fig. 11. Even though there are two types of propagators, we use only one notation for both of them, because the two propagators are the same.

This completes our description of the quantum field theory of hyper and gauge multiplets in five dimensions. The vertices can be obtained straightforwardly by expanding the various actions. In the following we will only give those interaction terms that will be relevant for the computations of the gauge kinetic action at one loop.

Interactions

For the hyper multiplet action (4.28) the expansion to fourth order gives rise to the interactions

$$\mathscr{S}_{H\,\mathrm{int}} \supset \int d^5x\, \mathrm{tr}\Big[\int \mathrm{d}^4\theta\ \bar{\Phi}_\pm (\pm 2V + 2V^2)\Phi_\pm + \int d^2\theta \sqrt{2}\Phi_- S \Phi_+ + \int d^2\bar{\theta}\sqrt{2}\bar{\Phi}_+ \bar{S} \bar{\Phi}_- \Big]. \quad (4.46)$$

We have depicted the corresponding vertices in Fig. 12. Performing the expansion to fourth order in the gauge sector (4.30) leads to the following interactions

$$\begin{aligned}\mathscr{S}_{V\mathrm{int}} \supset \int d^5x\, d^4\theta\ \mathrm{tr}\Big[&\frac{1}{4}\big[V, D^\alpha V\big]\bar{D}^2 D_\alpha V - \frac{1}{8}\big[V, D^\alpha V\big]\bar{D}^2\big[V, D_\alpha V\big] \\ &- \frac{1}{6}\big[V,\big[V, D^\alpha V\big]\big]\bar{D}^2 D_\alpha V + \sqrt{2}\,\partial_5 V\big[V, \bar{S} - S\big] - 2\,S\big[V, \bar{S}\big] \\ &+ \frac{1}{3}\,\partial_5 V\big[V,\big[V,\partial_5 V\big]\big] - \frac{2\sqrt{2}}{3}\,\partial_5 V\big[V,\big[V, S + \bar{S}\big]\big] + 2\,S\big[V,\big[V,\bar{S}\big]\big]\Big]. \end{aligned} \quad (4.47)$$

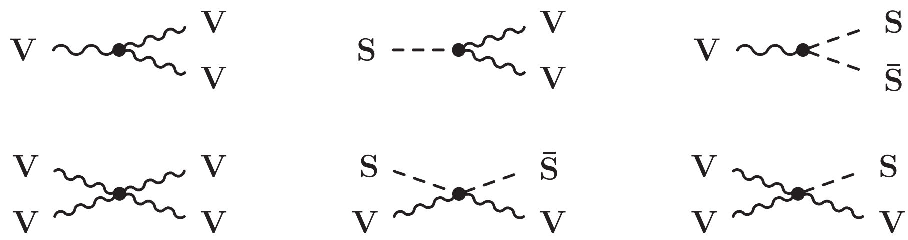


Figure 13: These vertices encode the self-interactions of the gauge multiplet involving the vector superfield V and the chiral superfield S.

To indicate that we display interaction terms of the expansion up to fourth order we use the notation "$\supset$". In deriving (4.47) from (4.30) we have rewritten the (anti-)chiral superspace integrals into full superspace integration in the standard way. We use the convention that the derivative operator ∂_5 only acts on the field it is immediately adjacent to. The interaction vertices have been collected in Fig. 13.

In the ghost sector we obtain the following interactions from the expansion of (4.44)

$$\mathscr{S}_{gh\,\mathrm{int}} \supset \int d^5x\, d^4\theta\, \mathrm{tr}\Big[\big(C' + \bar{C}'\big)\big[V, C - \bar{C}\big] + \frac{1}{3}\big(C' + \bar{C}'\big)\big[V, \big[V, C + \bar{C}\big]\big] + \sqrt{2}\,\frac{\partial_5}{\Box}\,C'\big[\bar{S}, \bar{C}\big] - \sqrt{2}\,\frac{\partial_5}{\Box}\,\bar{C}'\big[S, C\big]\Big]. \qquad (4.48)$$

These vertices are depicted in Fig. 14. One might worry about a possible non-locality of the interaction of a V field with two ghosts C' and $\bar{C}$ in (4.48), because the term contains a four dimensional d'Alembertian operator $\Box$ in the denominator. But such terms do not necessarily pose a problem, because physical amplitudes may also contain a bunch of supercovariant derivatives, which give rise to additional $\Box$ operators in the numerator so that cancellations can take place. In our calculation this issue does not arise at all, because it is impossible to construct one loop corrections to the $\bar{S}S$ self-energy with ghosts in the loop. The only graphs that could be constructed would be one loop contributions that are purely chiral, such that they vanish upon superspace integration.

4.2.3 Orbifold compatible calculus for $S^1/\mathbb{Z}_2$

In this section we extend the method of rendering a theory orbifold compatible as it was presented in detail in Section 4.1.3 to cover the non-Abelian case. The fields of the non-Abelian theory are allowed to transform with a rotation in the gauge degrees of freedom under the orbifold twist. The matrix that represents

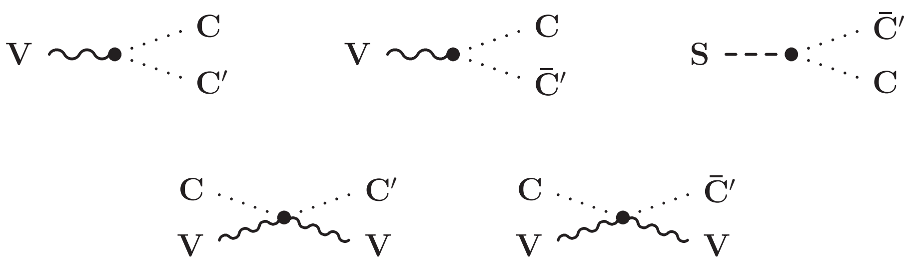


Figure 14: The ghosts C and C' only interact with the vector multiplet superfields V and S.

this rotation enters in the orbifold compatible delta functions. We start with the transformation behaviour of the fields under the orbifold twist. They have to be orbifold compatible such that their action is invariant under the orbifold symmetry. This means that they must transform covariantly under the orbifold action

$$\begin{aligned} \Phi_+ &\to Z\Phi_+, & V &\to ZVZ, \\ \Phi_- &\to -\Phi_- Z, & S &\to -ZSZ, \end{aligned} \tag{4.49}$$

where Z is the rotation in the gauge degrees of freedom. As we indicated in Section 4.1.3 such orbifold compatible fields and sources can always be constructed by taking suitable linear combinations of the fields defined on the covering space and their $\mathbb{Z}_2$ reflections. Invariance of the action implies that the transformation of the hyper and vector multiplets are encoded in a single unitary matrix Z. Because this is a $\mathbb{Z}_2$ action, the matrix Z fulfills $Z^2 = \mathbb{1}$. Hence Z is a real symmetric matrix with the eigenvalues ± 1. As it is often convenient to make the adjoint indices on V and S explicit, we introduce the matrix $Q^i{}_j$ to write the transformation rules for the V and S superfields as

$$V^i \to Q^i{}_j V^j, \qquad S^i \to -Q^i{}_j S^j, \qquad Q^i{}_j = \mathrm{tr}\left[T^i Z T_j Z\right]. \tag{4.50}$$

The invariance of the action requires that the matrix Q fulfills

$$Q^i{}_{i'}\, Q^j{}_{j'}\, \eta_{ij} = \eta_{i'j'}, \qquad f_{ijk}\, Q^i{}_{i'}\, Q^j{}_{j'}\, Q^k{}_{k'} = f_{i'j'k'}\ , \tag{4.51}$$

such that it is orthogonal with respect to the Killing metric η_{ij}. We infer that all matrix elements $Q^i{}_j$ are real. And due to the $\mathbb{Z}_2$ symmetry we know that $Q^2 = \mathbb{1}$ and hence Q is a real symmetric matrix. In the computation of the one loop self-energies we will be making frequent use of the properties of the matrices Z and Q. The $\mathbb{Z}_2$ properties of orbifold compatible fields imply that orbifold compatible source terms have to transform as

$$\begin{aligned} J_+ &\to J_+ Z, & J_V{}^i &\to Q^i{}_j\, J_V{}^j, \\ J_- &\to -Z J_-, & J_S{}^i &\to -Q^i{}_j\, J_S{}^j, \end{aligned} \tag{4.52}$$

where we have used the orthogonality of Q in (4.51). For Feynman supergraph computations that employ the path integral formalism it is important to know the orbifold compatible delta functions obtained by functional differentiation w.r.t. orbifold compatible sources:

$$\begin{aligned} \frac{\delta J_{+2b}}{\delta J_{+1a}} &= \frac{\bar{D}^2}{-4}\,\tilde{\delta}_{21}^{(+)a}{}_b, & \frac{\delta J_{V2}{}^i}{\delta J_{V1}{}^j} &= \tilde{\delta}_{21}^{(V)i}{}_j, \\ \frac{\delta J_{-2}{}^b}{\delta J_{-1}{}^a} &= \frac{\bar{D}^2}{-4}\,\tilde{\delta}_{21}^{(-)b}{}_a, & \frac{\delta J_{S2}{}^i}{\delta J_{S1}{}^j} &= \frac{\bar{D}^2}{-4}\,\tilde{\delta}_{21}^{(S)i}{}_j. \end{aligned} \tag{4.53}$$

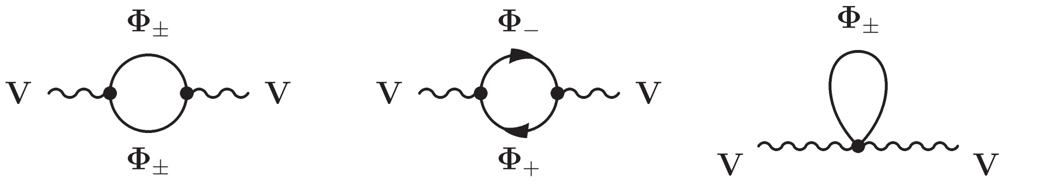
Because (except for J_V) all these sources are chiral, the functional differentiation w.r.t. them leads to chiral delta functions in superspace: $-\frac{1}{4}\bar{D}^2\delta^4(\theta_2-\theta_1)$. For later convenience we have defined the superspace orbifold compatible delta functions, indicated as $\tilde{\delta}$, containing full Grassmann delta functions $\delta^4(\theta_2-\theta_1)$. As a consequence, the factor $-\frac{1}{4}\bar{D}^2$ appears explicitly for the chiral sources in (4.53). From the transformation properties of the sources we infer that the orbifold compatible delta functions are given by

$$\begin{aligned} \tilde{\delta}_{21}^{(+)a}{}_b &= \frac{1}{2}\Big(\delta^a{}_b\,\delta(y_2-y_1) + Z^a{}_b\,\delta(y_2+y_1)\Big)\ \delta^4(x_2-x_1)\delta^4(\theta_2-\theta_1), \\ \tilde{\delta}_{21}^{(-)b}{}_a &= \frac{1}{2}\Big(\delta^b{}_a\,\delta(y_2-y_1) - Z^b{}_a\,\delta(y_2+y_1)\Big)\ \delta^4(x_2-x_1)\delta^4(\theta_2-\theta_1), \\ \tilde{\delta}_{21}^{(V)i}{}_j &= \frac{1}{2}\Big(\delta^i{}_j\,\delta(y_2-y_1) + Q^i{}_j\,\delta(y_2+y_1)\Big)\delta^4(x_2-x_1)\ \delta^4(\theta_2-\theta_1), \\ \tilde{\delta}_{21}^{(S)i}{}_j &= \frac{1}{2}\Big(\delta^i{}_j\,\delta(y_2-y_1) - Q^i{}_j\,\delta(y_2+y_1)\Big)\delta^4(x_2-x_1)\ \delta^4(\theta_2-\theta_1). \end{aligned} \tag{4.54}$$

Note that, since the matrix elements of Q are real, the orbifold delta function for $\bar{S}$ is the same as the orbifold delta function for S in five dimensions. These delta functions are the key elements of our formalism for calculating Feynman graphs directly on the orbifold, since they contain all the geometric information about the orbifold compatible superfields. Therefore, it is important to develop some of their properties: All delta functions are symmetric in their spacetime and gauge indices, while under a reflection of either y_1 or y_2 the delta functions transform as

$$\begin{aligned} \tilde{\delta}_{21}^{(+)a}{}_b &\to +Z^a{}_{a'}\,\tilde{\delta}_{21}^{(+)a'}{}_b, & \tilde{\delta}_{21}^{(V)i}{}_j &\to +Q^i{}_{i'}\,\tilde{\delta}_{21}^{(V)i'}{}_j, \\ \tilde{\delta}_{21}^{(-)a}{}_b &\to -Z^a{}_{a'}\,\tilde{\delta}_{21}^{(-)a'}{}_b, & \tilde{\delta}_{21}^{(S)i}{}_j &\to -Q^i{}_{i'}\,\tilde{\delta}_{21}^{(S)i'}{}_j. \end{aligned} \tag{4.55}$$

In calculating amplitudes one often makes use of partial integration. But as the delta function is a function of two coordinates (x_2, y_2) and (x_1, y_1), one sometimes needs to change the coordinate w.r.t. which a derivative ∂_5 acts before one can perform the partial integration. When this ∂_5 acts on the delta function, the



A± B C±

Figure 15: The gauge multiplet receives VV self-energy corrections from the hyper multiplet. The proper self-energy graphs are labeled 15.A$_\pm$ and 15.B. The tadpole graph is denoted by 15.C$_\pm$.

change of the coordinate may not only bring in a minus sign as one expects, but may also switch between the types of delta functions:

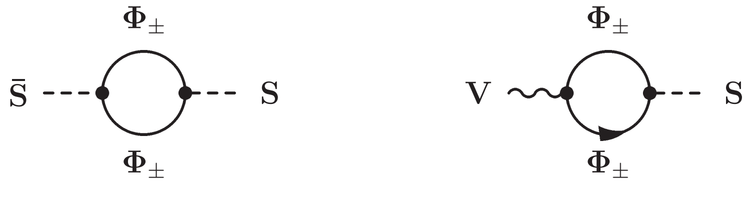
$$
\begin{aligned}
(\partial_5)_2\, \tilde{\delta}_{21}^{(+)a}{}_b &= -(\partial_5)_1\, \tilde{\delta}_{21}^{(-)a}{}_b\,, & (\partial_5)_2\, \tilde{\delta}_{21}^{(V)i}{}_j &= -(\partial_5)_1\, \tilde{\delta}_{21}^{(S)i}{}_j\,, \\
(\partial_5)_2\, \tilde{\delta}_{21}^{(-)a}{}_b &= -(\partial_5)_1\, \tilde{\delta}_{21}^{(+)a}{}_b\,, & (\partial_5)_2\, \tilde{\delta}_{21}^{(S)i}{}_j &= -(\partial_5)_1\, \tilde{\delta}_{21}^{(V)i}{}_j.
\end{aligned}
\tag{4.56}
$$

With this technology we are ready to perform supergraph calculations in the non-Abelian theory in five dimensions with the extra dimension compactified on the orbifold $S^1/\mathbb{Z}_2$.

4.2.4 Renormalization of the gauge kinetic term due to the hyper multiplet

The calculation of the hyper multiplet contributions to the gauge kinetic term follows closely the computation of the hyper multiplet loop corrections in the Abelian case that were performed in Section 4.1. The supergraphs that are necessary for the calculation are the same as in the Abelian case and are depicted again for easier reference in Figs. 15 and 16. The results for these graphs are provided in App. C.1, where a six dimensional notation is employed that has a straightforward reduction to five dimensions. On the level of two orbifold compatible delta functions the VV self-energy takes the form

$$
\begin{aligned}
\Sigma_{VV} = \int \big(d^5x\big)_{12} d^4\theta\, \mathrm{tr}\Big[&- V_1\, \frac{1}{(\Box + \partial_5^2)_2}\, \tilde{\delta}_{21}^{5(+)}\, \Box_2 P_0\, V_2\, \frac{1}{(\Box + \partial_5^2)_2}\, \tilde{\tilde{\delta}}_{21}^{5(+)} \\
&- V_1\, \frac{1}{(\Box + \partial_5^2)_2}\, \tilde{\tilde{\delta}}_{21}^{5(-)}\, \Box_2 P_0\, V_2\, \frac{1}{(\Box + \partial_5^2)_2}\, \tilde{\delta}_{21}^{5(-)} \\
&+ 2\, \partial_5 V_1\, \frac{1}{(\Box + \partial_5^2)_2}\, \tilde{\delta}_{21}^{5(+)}\, \partial_5 V_2\, \frac{1}{(\Box + \partial_5^2)_2}\, \tilde{\delta}_{21}^{5(-)} \Big].
\end{aligned}
\tag{4.57}
$$



A$_\pm$ B$_\pm$

Figure 16: The gauge multiplet receives $\bar{S}S$ self-energy corrections from the hyper multiplet as is depicted in figure 16.A$_\pm$. In addition the hyper multiplet gives rise to mixing between the four dimensional superfields V and S, see 16.B$_\pm$.

The $V\bar{S}$ and $S\bar{S}$ self-energies are given by

$$\Sigma_{V\bar{S}} = -2\sqrt{2}\int \left(d^5x\right)_{12} d^4\theta \ \mathrm{tr}\Big[\ \partial_5 V_1 \ \frac{1}{(\Box+\partial_5^2)_2}\ \tilde{\delta}_{21}^{5(+)}\ \bar{S}_2 \frac{1}{(\Box+\partial_5^2)_2}\ \tilde{\delta}_{21}^{5(-)}\ \Big], \tag{4.58}$$

$$\Sigma_{S\bar{S}} = 2\int \left(d^5x\right)_{12} d^4\theta \ \mathrm{tr}\Big[\ S_1 \ \frac{1}{(\Box+\partial_5^2)_2}\ \tilde{\delta}_{21}^{5(+)}\ \bar{S}_2 \frac{1}{(\Box+\partial_5^2)_2}\ \tilde{\delta}_{21}^{5(-)}\ \Big] \tag{4.59}$$

and the VS self-energy is just the complex conjugate of the result for $V\bar{S}$. Equipped with the terminology of Section 4.1 we can head directly for the renormalization of the non-Abelian theory in the bulk and at the fixed points.

Bulk renormalization

The bulk amplitude is found by picking out the terms with two ordinary delta functions from the self-energies (4.57)-(4.59). We add all terms to find the correction to the gauge kinetic term in the bulk

$$\Sigma_{\mathrm{bulk}}^{\mathrm{hyper}} = \int \left(d^5x\right)_{12} d^4\theta \ \mathrm{tr}\Big[-V_1\Box_2 P_0 V_2 + \partial_5 V_1 \partial_5 V_2 - \sqrt{2}\partial_5 V_1 (S_2+\bar{S}_2\,) + S_1\bar{S}_2\Big]$$
$$\times \ \frac{1}{(\Box+\partial_5^2)_2}\ \delta(y_2-y_1)\ \frac{1}{(\Box+\partial_5^2)_2}\ \delta(y_2-y_1). \tag{4.60}$$

All terms appear with the correct coefficients to recombine to an expression which is proportional to the gauge kinetic term. The extraction of the leading terms of this divergent self-energy with the help of the identities in App. D.1 leads to the same observation as in the Abelian calculation in Section 4.1.4, namely that dimensional regularization hides the linear divergence. The leading term of the self-energy reads

$$\Sigma_{\mathrm{bulk}}^{\mathrm{hyper}} = -\frac{i|m|\,\mu^{-2\epsilon}}{(4\pi)^2}\int d^dx\, d^4\theta \ \mathrm{tr}\Big[-V\Box P_0 V + \left(\partial_5 V\right)^2 - \sqrt{2}\partial_5 V(S+\bar{S}\,) + S\bar{S}\Big], \tag{4.61}$$

where $d = 5 - 2\epsilon$ and m is an infrared regulator mass. Therefore, in the same way as in the Abelian calculation, we do not obtain a counterterm in the MS scheme. Hence, also here we do not have a renormalization of the gauge kinetic term and there is no effect on the gauge coupling running in the five dimensional bulk.

Fixed point renormalization

In order to find the amplitude at the fixed points, we pick out the terms with one delta function $\delta(y_2 - y_1)$ and one delta function $\delta(y_2 + y_1)$ from the self-energies (4.57) - (4.59). We add all terms to find the self-energy at the fixed points

$$\Sigma^{\text{hyper}}_{\text{fp}} = \frac{1}{2}\int \left(d^5x\right)_{12} d^4\theta \,\text{tr}\Big[[\partial_5 V_1, Z]\partial_5 V_2 - \sqrt{2}\,[\partial_5 V_1, Z](\bar{S}_2 - S_2\,) + [S_1, Z]\bar{S}_2\Big]$$
$$\times \frac{1}{(\Box + \partial_5^2)_2}\,\delta(y_2 - y_1)\,\frac{1}{(\Box + \partial_5^2)_2}\,\delta(y_2 + y_1). \tag{4.62}$$

Note that for the kinetic term of the four dimensional gauge multiplet the cancellation at the fixed points is complete and this term is absent in (4.62). We obtain the remaining terms of the five dimensional gauge kinetic action with the fields enclosed in commutators. This shows that in the case when Z is proportional to the identity and in the Abelian case the amplitude vanishes at the fixed points which is consistent with our previous findings in Section 4.1.4. The divergent part of this amplitude is found to be

$$\Sigma^{\text{hyper}}_{\text{fp}}\Big|_{\text{div}} = \frac{i\mu^{-2\epsilon}}{2(4\pi)^2}\Big(\frac{1}{\bar{\epsilon}} + \ln\frac{\mu^2}{m^2}\Big)\int d^5x\, d^4\theta\, \text{tr}\Big[Z\big[\bar{S} - \sqrt{2}\,\partial_5 V, S - \sqrt{2}\,\partial_5 V\big]\Big]\,\delta(2y). \tag{4.63}$$

As we saw at the level of the amplitude, in the Abelian case the hyper multiplet does not induce a correction at the fixed points. The $(\partial_5 V)^2$ parts of this expression have been obtained before, see [75]. As explained in Section 3.2 in the non-Abelian theory the counterterm is only well defined when we take the nonlinear extension. So we arrive at the expression

$$\Delta\mathscr{S}^{\text{hyper}}_{\text{fp}} = -\frac{\mu^{-2\epsilon}}{2(4\pi)^2\epsilon}\int d^dx\, d^4\theta\, \text{tr}\Big[Z\Big[\big(\bar{S} - \tfrac{1}{\sqrt{2}}\,\partial_5\big)e^{2V}, \big(S + \tfrac{1}{\sqrt{2}}\,\partial_5\big)e^{-2V}\Big]\Big]\,\delta(2y). \tag{4.64}$$

From the counterterm (4.64) we can read off that there is no renormalization of the gauge kinetic term of the four dimensional gauge multiplet at the fixed points. The only renormalization effects involve the other operators from the five dimensional gauge kinetic term. Hence, the counterterm (4.64) does not lead to a running of the four dimensional gauge coupling at the fixed points.

We remark that (4.64) is gauge invariant only w.r.t. the zero mode supergauge group defined by $\partial_5\Lambda = \partial_5\bar{\Lambda} = 0$ and $[Z,\Lambda] = [Z,\bar{\Lambda}] = 0$. The second condition is a consequence of the orbifold projection at the $\mathbb{Z}_2$ fixed points. However, for the full supergauge group at the fixed points $\partial_5\Lambda$ and $\partial_5\bar{\Lambda}$ do not necessarily vanish nor commute with Z. Consequently this expression, as it stands, is not gauge invariant under the full bulk gauge transformations. As we will speculate below (5.79) this might be cured by a Wess-Zumino-Witten-like term.

4.2.5 Renormalization of the gauge kinetic term due to vector multiplet self-interactions

In this section we compute the one loop corrections to the gauge kinetic term due to the particles from the gauge sector. Because of the gauge fixing described in Section 4.2.2 we encounter the superfields V, S and the ghosts C, C' in the loops.

The supergraphs that give the loop corrections to the self-energy Σ_{VV} have been depicted in Fig. 17 and those that give loop corrections to the Σ_{VS} and $\Sigma_{V\bar{S}}$ self-energies are displayed in Fig. 18. In the first two lines of Fig. 17 the genuine self-energy graphs are labeled 17.A to 17.D. Because there are two ghost propagator diagrams, 17.D gives rise to four contributions. We use the notation 17.E to 17.G to indicate the tadpole supergraphs in Fig. 17. The contributions from these tadpole graphs are necessary to cancel non-gauge invariant terms from the total amplitude. The first graph in Fig. 18 is the $S\bar{S}$ self-energy diagram. Finally, diagrams 18.B and 18.C give the self-energy due to the mixing between S and V. The results of these graphs are given on the level of two orbifold compatible delta functions in App. C.2 in a six dimensional notation which has a straightforward reduction to five dimensions as explained in the appendix. The explicit calculation of graph 17.A can be found as a worked out example in App. B to illustrate the main steps that are required for the calculation of such supergraphs on orbifolds. The sum of the respective graphs gives the self-energies which we present at the level of two orbifold compatible delta functions. The VV self-energy is found to be

$$
\begin{aligned}
&\Sigma_{VV} = \\
&f_{ijk} f_{\ell mn} \int \left(d^5x\right)_{12} d^4\theta \Big[-3\, V_1^i \Box_2 P_0 V_2^\ell \frac{1}{(\Box+\partial_5^2)_2} \widetilde{\delta}_{21}^{5(V)mj} \frac{1}{(\Box+\partial_5^2)_2} \widetilde{\delta}_{21}^{5(V)nk} \\
&\qquad + V_1^i \Box_2 P_0 V_2^\ell \frac{1}{(\Box+\partial_5^2)_2} \widetilde{\delta}_{21}^{5(\bar{S})mj} \frac{1}{(\Box+\partial_5^2)_2} \widetilde{\delta}_{21}^{5(S)nk} \qquad (4.65) \\
&\qquad + 2\, \partial_5 V_1^i \partial_5 V_2^\ell \frac{1}{(\Box+\partial_5^2)_2} \widetilde{\delta}_{21}^{5(V)mj} \frac{1}{(\Box+\partial_5^2)_2} \widetilde{\delta}_{21}^{5(\bar{S})nk} \Big].
\end{aligned}
$$

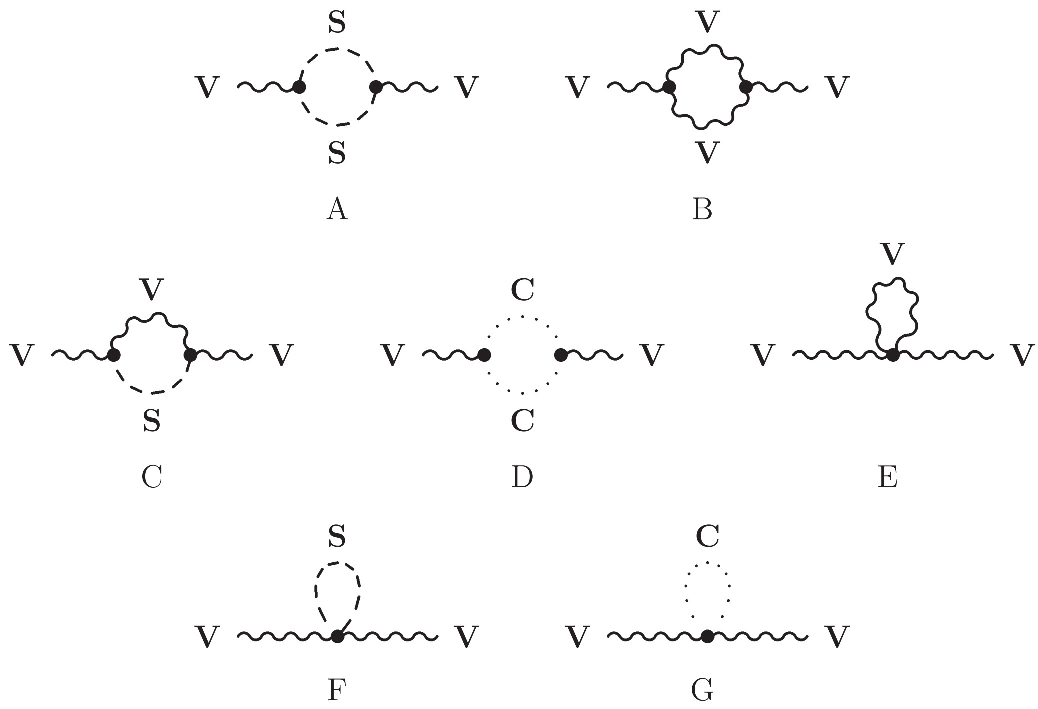


Figure 17: The gauge contributions to the VV part of the gauge multiplet self-energy are due to the V self-coupling , the interactions with the chiral superfield S and the ghost superfields C and C'. The genuine self-energy graphs are labeled 17.A to 17.D. The tadpole graphs are referred to as 17.E to 17.G.

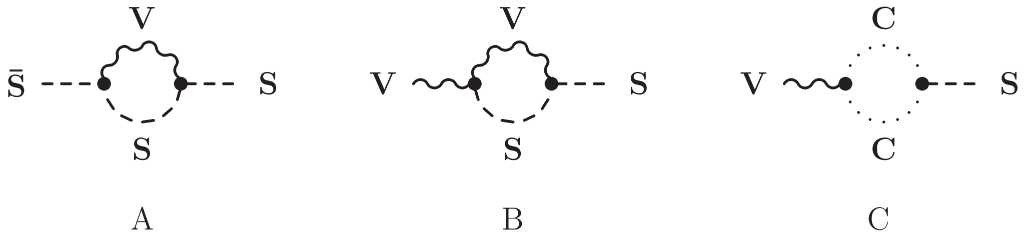


Figure 18: The $S\bar{S}$ self-energy graph is given in figure 18.A. The mixing between the four dimensional superfields V and S corresponding to the third term of (4.32) is renormalized by the diagrams 18.B and 18.C. The last diagram has ghosts in the loop.

As we have explained in Section 4.1.4, the last two orbifold delta functions have to be handled with care. In the first two lines of (4.65) the same orbifold compatible delta function is employed twice, since in five dimensions the orbifold delta functions for S and $\bar{S}$ are equal. Hence, in the first two lines one of these orbifold delta functions can be straightforwardly reduced to an ordinary delta function. But for the last line in (4.65) cancellations appear. As we explained in Section 4.1.4 it is convenient to write the amplitude as half of the sum of both possibilities to reduce one delta function. Then the fixed point contribution of the last term in (4.65) vanishes, leaving only a bulk contribution. For the same reason also the VS, $V\bar{S}$ and $S\bar{S}$ self-energies, given in Fig. 18, only have a bulk contribution, because their two orbifold compatible delta function expressions are given by

$$\Sigma_{V\bar{S}} = -2\sqrt{2}\, f_{ijk} f_{\ell mn} \int \left(d^5x\right)_{12} d^4\theta\; \partial_5 V_1^i \bar{S}_2^\ell \,\frac{1}{(\Box + \partial_5^2)_2}\, \widetilde{\delta}_{21}^{5(V)mj} \frac{1}{(\Box + \partial_5^2)_2}\, \widetilde{\delta}_{21}^{5(\bar{S})nk} \tag{4.66}$$

for $V\bar{S}$, the complex conjugate for VS, and

$$\Sigma_{\bar{S}S} = 2\, f_{ijk} f_{\ell mn} \int \left(d^5x\right)_{12} d^4\theta\; S_1^i \bar{S}_2^\ell \,\frac{1}{(\Box + \partial_5^2)_2}\, \widetilde{\delta}_{21}^{5(V)mj} \frac{1}{(\Box + \partial_5^2)_2}\, \widetilde{\delta}_{21}^{5(\bar{S})nk} \tag{4.67}$$

for the $S\bar{S}$ self-energy. By combining these results and expanding the final orbifold compatible delta functions according to their definitions in (4.54), we can identify the bulk and fixed point contributions.

Bulk renormalization

The bulk amplitude is the sum of all terms with two ordinary delta functions from the self-energies (4.65)-(4.67). We find the following correction to the gauge kinetic term in the bulk

$$\Sigma_{\text{bulk}}^{\text{gauge}} = f_{ijk}f_{\ell mn}\eta^{mj}\eta^{nk}\int\left(d^5x\right)_{12}d^4\theta\,\Big[-V_1^i\,\Box_2 P_0\,V_2^\ell+\partial_5 V_1^i\,\partial_5 V_2^\ell+ \tag{4.68}$$

$$-\sqrt{2}\,\partial_5 V_1^i(S_2^\ell+\bar{S}_2^\ell\,)+S_1^i\bar{S}_2^\ell\Big]\,\frac{1}{(\Box+\partial_5^2)_2}\,\delta(y_2-y_1)\,\frac{1}{(\Box+\partial_5^2)_2}\,\delta(y_2-y_1).$$

Dimensional regularization of this linearly divergent result with the help of the identities in App. D.1 gives the following leading terms in analogy to the other bulk results that we found in five dimensions

$$\Sigma_{\text{bulk}}^{\text{gauge}} = \frac{i|m|\mu^{-2\epsilon}}{(4\pi)^2}\int d^dx\,d^4\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[-V\Box P_0 V+\big(\partial_5 V\big)^2-\sqrt{2}\partial_5 V(S+\bar{S})+S\bar{S}\Big], \tag{4.69}$$

where $d=5-2\epsilon$ and m is an infrared regulator mass. Therefore, in the MS scheme there is no counterterm to the gauge kinetic term in the five dimensional bulk and no running of the gauge coupling due to this contribution.

Fixed points renormalization

The fixed point amplitude is obtained as the sum of all terms with one delta function $\delta(y_2-y_1)$ and one delta function $\delta(y_2+y_1)$ from the self-energies (4.65)-(4.67). We find the following self-energy at the fixed points

$$\begin{aligned}\Sigma_{\text{fp}}^{\text{gauge}} &= 2f_{ijk}f_{\ell mn}\,\eta^{mj}\,Q^{nk}\int\left(d^5x\right)_{12}d^4\theta\,\Big[-V_1^i\,\Box_2 P_0\,V_2^\ell\Big]\\ &\quad\times\frac{1}{(\Box+\partial_5^2)_2}\,\delta(y_2+y_1)\,\frac{1}{(\Box+\partial_5^2)_2}\,\delta(y_2-y_1).\end{aligned} \tag{4.70}$$

As we remarked above, except for the first two lines in (4.65), all self-energy contributions are purely bulk and vanish at the fixed points. Only the kinetic term of the four dimensional gauge superfield is loop corrected by a non-vanishing self-energy. The divergent piece of this self-energy reads

$$\Sigma_{\text{fp}}^{\text{gauge}}\Big|_{\text{div}} = -\frac{2i\mu^{-2\epsilon}}{(4\pi)^2}\Big(\frac{1}{\bar{\epsilon}}+\ln\frac{\mu^2}{m^2}\Big)\int d^5x\,d^4\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[-Q\,V(x,y)\,\Box P_0\,V(x.y)\Big]\,\delta(2y). \tag{4.71}$$

As explained in Section 3.2 in the non-Abelian theory the counterterm is only well defined when we take the nonlinear extension. So we arrive at the counterterm

$$\Delta\mathscr{S}_{\text{fp}}^{\text{gauge}} = \frac{2\mu^{-2\epsilon}}{(4\pi)^2\epsilon}\int d^dx\,d^4\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[Q\,\Big(\frac{1}{4}\int d^2\theta\,W^\alpha W_\alpha+\frac{1}{4}\int d^2\bar{\theta}\,\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\Big)\Big]\,\delta(2y). \tag{4.72}$$

This counterterm involves the gauge kinetic term of the four dimensional gauge multiplet. Hence, there is a running of the four dimensional gauge coupling at the fixed points which is read off from the general expression (4.72) as demonstrated in the final equations of Section 3.1.3 after one has chosen the representation of the chiral multiplet and the matrix Q for a specific orbifold theory.

Chapter 5

Supersymmetric theory in six dimensions

In this chapter the results of the preceeding chapter are generalized to a six dimensional spacetime with the two extra dimensions compactified on the orbifold $T^2/\mathbb{Z}_N$. Again we consider a hyper multiplet and a gauge multiplet in the bulk and aim at the renormalization of the gauge kinetic term. We determine the counterterms in the bulk and at the fixed points. We find that in even ordered $T^2/\mathbb{Z}_N$ orbifolds those fixed points that form a $\mathbb{Z}_2$ subset must be distinguished from the other fixed points, because at these fixed points special cancellations appear. Moreover, we observe the generation of a higher dimensional operator as loop counterterm in the six dimensional bulk. We consider the Abelian theory in which only the hyper multiplet leads to loop corrections in the first section. The non-Abelian theory in which also the particles from the gauge sector contribute to loop effects is discussed in the second section. Our method of calculating Feynman graphs directly on the orbifold is adjusted to cover the case of this more complicated orbifold. For a study of supersymmetric gauge theory on $T^2/\mathbb{Z}_2$ which has been performed in the component field formalism and is thus complementary to our superfield approach we refer the reader to [76].

5.1 Hyper multiplet coupled to an Abelian gauge multiplet

In this section we consider a hyper multiplet coupled to an Abelian gauge multiplet in six dimensions with the extra two dimensions compactified on the orbifold $T^2/\mathbb{Z}_N$. The six dimensional multiplets have the same decomposition into four dimensional multiplets as in five dimensions, but their superfield action is slightly changed. In the Abelian theory only the hyper mutiplet contributes to the renormalization of the gauge kinetic term, so we quantize only the hyper multiplet part of the action. Then we extend our orbifold compatible calculus for Feynman graphs to the case of $T^2/\mathbb{Z}_N$. This involves a generalization of the orbifold compatible delta functions that arise from functional differentiation w.r.t. the preceeding chapter. We determine the renormalization of the gauge kinetic term due to the hyper multiplet in the bulk and at the two classes of fixed points.

5.1.1 Classical action in six dimensional Minkowski space

First we consider the classical action for a hyper multiplet that is coupled to an Abelian gauge multiplet in six dimensions. The action in terms of four dimensional superfield [13, 15] is similar to the five dimensional action and we will indicate the deviations. The hyper multiplet action with its coupling to the gauge multiplet takes the form

$$\mathscr{S}_H = \int d^6x \left\{ \int d^4\theta \, \big(\bar{\Phi}_+ e^{2qV} \Phi_+ + \Phi_- e^{-2qV} \bar{\Phi}_-\big) + \right. \\ \left. + \int d^2\theta \; \Phi_- \big(\partial + \sqrt{2}\, qS\big)\Phi_+ + \int d^2\bar{\theta} \; \bar{\Phi}_+ \big(- \bar{\partial} + \sqrt{2}\, q\bar{S}\big)\bar{\Phi}_- \right\}. \quad (5.1)$$

Here we employ complex coordinates $z = \frac{1}{2}(x_5 - ix_6)$ and $\bar{z} = \frac{1}{2}(x_5 + ix_6)$ such that the six dimensional derivative operators are $\partial = \partial_5 + i\partial_6$ and $\bar{\partial} = \partial_5 - i\partial_6$. Most of the changes w.r.t. the five dimensional case have to do with the question if the five dimensional derivative operator ∂_5 has to be replaced by ∂ or $\bar{\partial}$. So the only terms in the hyper multiplet action that are changed w.r.t. the five dimensional action (4.2) are those that contained the five dimensional derivative operator ∂_5. The action is invariant under the super-gauge transformation

$$\begin{aligned} \Phi_+ &\to e^{-2q\Lambda}\Phi_+, \qquad & V &\to V + \Lambda + \bar{\Lambda}, \\ \Phi_- &\to e^{+2q\Lambda}\Phi_-, \qquad & S &\to S + \sqrt{2}\,\partial\Lambda. \end{aligned} \quad (5.2)$$

Notice that in both expressions (5.1) and (5.2) the holomorphic derivative ∂ appears only in those places where chiral superfields are present. The kinetic action $\mathscr{S}_V$ for the six dimensional gauge multiplet involves the four dimensional gauge multiplet V and the chiral multiplet S as in five dimensions

$$\mathscr{S}_V = \frac{1}{g^2} \int d^6x \left\{ \frac{1}{4} \int d^2\theta \; W^\alpha W_\alpha + \frac{1}{4} \int d^2\bar{\theta} \; \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} + \right. \\ \left. + \int d^4\theta \; \Big[\Big(\sqrt{2}\,\bar{\partial}V - \bar{S}\Big)\Big(\sqrt{2}\,\partial V - S\Big) - \partial V \bar{\partial} V \Big] \right\}, \quad (5.3)$$

but in six dimensions it is not possible to represent this result in terms of a single gauge invariant vector superfield like the superfield V_5 defined in (4.5). It is straightforward to rewrite the gauge kinetic action in the following form

$$\mathscr{S}_V = \frac{1}{g^2}\int d^6x\, d^4\theta\left[\frac{1}{8}V D^\alpha \bar{D}^2 D_\alpha V + \bar{\partial}V\,\partial V - \sqrt{2}\,\bar{\partial}V S - \sqrt{2}\,\partial V\bar{S} + \bar{S}S\right]. \tag{5.4}$$

In this form it is clear that the same self-energies Σ_{VV}, Σ_{VS}, $\Sigma_{V\bar{S}}$ and $\Sigma_{S\bar{S}}$ as in five dimensions have to be calculated in order to determine the renormalization of the gauge kinetic action.

5.1.2 Quantization of the action

The generating functional looks exactly the same as in five dimensions (4.7) with the five dimensional action replaced against the six dimensional action.

Hyper multiplet propagator

The propagators for the two chiral multiplets that form the six dimensional hyper multiplet,

$$Z_0\big[J_+, J_-\big] =$$

$$\int \mathscr{D}\Phi_+\mathscr{D}\Phi_-\,\exp\left\{i\int d^6x\,d^4\theta\,\big(J_+\bar{J}_-\big)\,\frac{-1}{\Box+\partial\bar{\partial}}\begin{pmatrix}1 & \bar{\partial}\frac{D^2}{-4\Box}\\ -\partial\frac{\bar{D}^2}{-4\Box} & 1\end{pmatrix}\begin{pmatrix}\bar{J}_+\\ J_-\end{pmatrix}\right\}, \tag{5.5}$$

are the direct generalizations of their five dimensional counterparts given in (4.8). The six dimensional propagators are depicted in the same way as in five dimensions in Fig. 7. As the multiplets from the gauge sector do not lead to a renormalization of the gauge kinetic term in the Abelian theory, we do not discuss the gauge and ghost superfield propagators here.

Interactions

The interactions of the hyper multiplet with the gauge multiplet do not contain any derivatives in the extra dimensions and are therefore just the six dimensional version of (4.9) depicted in Fig. 8.

5.1.3 Orbifold compatible calculus for $T^2/\mathbb{Z}_N$

The properties of $T^2/\mathbb{Z}_N$ are more complicated than those of $S^1/\mathbb{Z}_2$, they are described in App. A.2. The hyper and gauge multiplets on the orbifold need to be covariant w.r.t. the $\mathbb{Z}_N$ orbifold action. Hence, their transformation behaviour under $z \to e^{-i\varphi}z$ is found to be

$$\Phi_+ \to e^{in_\pm\varphi}\Phi_+, \qquad \Phi_- \to e^{-in_\pm\varphi}\Phi_-, \qquad V\to V \qquad S\to e^{i\varphi}S, \tag{5.6}$$

where $n_\pm$ are integers between 0 and $N-1$ such that

$$e^{i(1+n_+ +n_-)\varphi} = 1. \tag{5.7}$$

is fulfilled. As the two numbers are related by (5.7), we need only one of them in principle. However, it turns out to be convenient to keep using the notation $n_\pm$. The reduction to the $\mathbb{Z}_2$ orbifold group is attained for $n_+ = 0$ and $n_- = 1$. Then many of the properties of the five dimensional case, discussed in Section 4.1, are recovered. To obtain the orbifold compatible delta functions for the chiral superfields, we write down the transformation behaviour of orbifold compatible sources under $z \to e^{-i\varphi} z$

$$J_\pm \to e^{-in_\pm\varphi} J_\pm, \qquad \bar{J}_\pm \to e^{in_\pm\varphi} J_\pm. \tag{5.8}$$

Then we define the functional derivatives w.r.t. the sources with the appropriate factors of D^2 and $\bar{D}^2$ to maintain chirality

$$\frac{\delta J_{\pm 2}}{\delta J_{\pm 1}} = \frac{\bar{D}^2}{-4}\,\tilde{\delta}_{21}^{(\pm)}, \qquad \frac{\delta \bar{J}_{\pm 2}}{\delta \bar{J}_{\pm 1}} = \frac{D^2}{-4}\,\tilde{\tilde{\delta}}_{21}^{(\pm)}. \tag{5.9}$$

These definitions have to be in accordance with the transformation behaviour of the source terms (5.8). The following orbifold compatible delta functions ensure that this is the case

$$\begin{aligned} \tilde{\delta}_{21}^{(\pm)} &= \frac{1}{N}\sum_{b=0}^{N-1} e^{ibn_\pm\varphi}\,\delta^2(z_2 - e^{ib\varphi}z_1)\,\delta^4(x_2-x_1)\,\delta^4(\theta_2-\theta_1), \\ \tilde{\tilde{\delta}}_{21}^{(\pm)} &= \frac{1}{N}\sum_{b=0}^{N-1} e^{-ibn_\pm\varphi}\,\delta^2(z_2 - e^{ib\varphi}z_1)\,\delta^4(x_2-x_1)\,\delta^4(\theta_2-\theta_1). \end{aligned} \tag{5.10}$$

One checks that the so-defined delta functions transform in the same way as the corresponding sources under an orbifold transformation $z_2 \to e^{-i\varphi}z_2$ of the second coordinate

$$\tilde{\delta}_{21}^{(\pm)} \to e^{-in_\pm\varphi}\,\tilde{\delta}_{21}^{(\pm)}, \qquad \tilde{\tilde{\delta}}_{21}^{(\pm)} \to e^{in_\pm\varphi}\,\tilde{\tilde{\delta}}_{21}^{(\pm)} \tag{5.11}$$

and inversely under an orbifold transformation $z_1 \to e^{-i\varphi}z_1$ of the first coordinate

$$\tilde{\delta}_{21}^{(\pm)} \to e^{in_\pm\varphi}\,\tilde{\delta}_{21}^{(\pm)}, \qquad \tilde{\tilde{\delta}}_{21}^{(\pm)} \to e^{-in_\pm\varphi}\,\tilde{\tilde{\delta}}_{21}^{(\pm)}. \tag{5.12}$$

Derivatives with respect to the compactified coordinates always act on the $\delta^2(z_2 - e^{ib\varphi}z_1)$ factor. Therefore, changing a spacetime index of such a derivative also changes the type of delta function and implies a complex conjugation

$$\bar{\partial}_2\,\tilde{\delta}_{21}^{(\pm)} = -\bar{\partial}_1\,\tilde{\tilde{\delta}}_{21}^{(\mp)}. \tag{5.13}$$

This completes the discussion of the supersymmetric Abelian gauge theory on the six dimensional orbifold $T^2/\mathbb{Z}_N$. We have seen, that even though many properties are very similar to the ones encountered for the $S^1/\mathbb{Z}_2$ orbifold discussed in Section 4.1, there are also some important additional complications in the six dimensional case.

5.1.4 Renormalization of the gauge kinetic term due to the hyper multiplet

The calculation of the renormalization of the gauge kinetic term in six dimensions with the two extra dimensions compactified on $T^2/\mathbb{Z}_N$ parallels the procedure described for five dimensions with the extra dimension compactified on $S^1/\mathbb{Z}_2$ carried out in Section 4.1.4. The expansion of the hyper multiplet action to fourth order in the gauge coupling (4.46) remains valid in six dimensions. This means that we have to calculate the same set of graphs as in five dimensions depicted in Figs. 9 and 10. The replacement of fields against the derivatives w.r.t. the corresponding source terms is performed as

$$\Phi_\pm \to \frac{\delta}{\delta\big(iJ_\pm\big)}, \qquad \bar{\Phi}_\pm \to \frac{\delta}{\delta\big(i\bar{J}_\pm\big)}. \tag{5.14}$$

The results for the graphs are provided in App. C.1 in a non-Abelian notation which has a simple reduction to the Abelian calculation performed here. The self energies Σ_{VV}, Σ_{VS}, $\Sigma_{V\bar{S}}$ and $\Sigma_{\bar{S}S}$ are similar to those that we found in five dimensions. The differences arise from the changes of the derivative operators in the quadratic part of the gauge multiplet action and from the fact that the orbifold compatible delta functions are no longer real and we have to distinguish between $\tilde{\delta}$ and $\tilde{\tilde{\delta}}$

$$\begin{aligned}\Sigma_{VV} = q^2 \int \big(d^6x\big)_{12} d^4\theta \ \Big(&-V_1\,\Box_2 P_0\,V_2\Big)\frac{1}{(\Box+\partial\bar{\partial})_2}\,\tilde{\delta}^{6(+)}_{21}\,\frac{1}{(\Box+\partial\bar{\partial})_2}\,\tilde{\tilde{\delta}}^{6(+)}_{21} \\ &+\Big(-V_1\,\Box_2 P_0\,V_2\Big)\frac{1}{(\Box+\partial\bar{\partial})_2}\,\tilde{\tilde{\delta}}^{6(-)}_{21}\,\frac{1}{(\Box+\partial\bar{\partial})_2}\,\tilde{\delta}^{6(-)}_{21} \\ &+2\,\partial V_1\,\bar{\partial}V_2\frac{1}{(\Box+\partial\bar{\partial})_2}\,\tilde{\delta}^{6(+)}_{21}\,\frac{1}{(\Box+\partial\bar{\partial})_2}\,\tilde{\delta}^{6(-)}_{21}\ ,\end{aligned} \tag{5.15}$$

$$\Sigma_{V\bar{S}} = -2\sqrt{2}\,q^2 \int \big(d^6x\big)_{12} d^4\theta\ \partial V_1\,\bar{S}_2\,\frac{1}{(\Box+\partial\bar{\partial})_2}\,\tilde{\delta}^{6(+)}_{21}\,\frac{1}{(\Box+\partial\bar{\partial})_2}\,\tilde{\delta}^{6(-)}_{21}\ , \tag{5.16}$$

$$\Sigma_{S\bar{S}} = 2q^2 \int \big(d^6x\big)_{12} d^4\theta\ S_1\bar{S}_2\,\frac{1}{(\Box+\partial\bar{\partial})_2}\,\tilde{\delta}^{6(+)}_{21}\,\bar{S}_2\frac{1}{(\Box+\partial\bar{\partial})_2}\,\tilde{\delta}^{6(-)}_{21}. \tag{5.17}$$

After the reduction of one more orbifold compatible delta function following the three steps described in Section 4.1.4 we split the amplitude into three different contributions. The bulk amplitude has two ordinary delta functions and is recovered for $b=0$ in the expansion of the last orbifold delta function. The amplitude at the $\mathbb{Z}_2$ fixed points of an even ordered orbifold is obtained for $b=N/2$. It is related to the amplitude at the fixed points of $S^1/\mathbb{Z}_2$. The sum in the last delta function that runs over all values of b except for 0 and $N/2$ gives the contributions of the amplitude at those fixed points that are not in a $\mathbb{Z}_2$ subsector of the $\mathbb{Z}_N$ action. These three different contributions are presented separately.

Bulk renormalization

The bulk contribution of the self-energies (5.56)-(5.58) is given by the $b = 0$ contribution from the delta functions. We extract the divergence with the help of App. D.1 and determine the local bulk counterterm in six dimensions. It consists of a sum of two terms. The first term in the sum is the counterterm to the six dimensional gauge kinetic term and the second term is a higher derivative operator that is generated in the renormalization process

$$\Delta\mathscr{S}^{\text{hyper}}_{\text{bulk}} = \Delta\mathscr{S}^{\text{hyper}}_{\text{gkt}} + \Delta\mathscr{S}^{\text{hyper}}_{\text{HDO}}\,. \tag{5.18}$$

The piece of the counterterm that renormalizes the bulk gauge kinetic term reads

$$\begin{aligned}\Delta\mathscr{S}^{\text{hyper}}_{\text{gkt}} &= -\frac{2q^2m^2\mu^{-2\epsilon}}{(4\pi)^3N\epsilon}\int d^dx\left\{\frac{1}{4}\int d^2\theta\; W^\alpha W_\alpha + \text{h.c.} + \right.\\ &\left. + \int d^4\theta\;\Big[\Big(\sqrt{2}\,\bar\partial V - \bar S\Big)\Big(\sqrt{2}\,\partial V - S\Big) - \partial V\bar\partial V\Big]\right\}\end{aligned} \tag{5.19}$$

and the higher derivative operator that is generated as loop counterterm is also localized in the bulk

$$\begin{aligned}\Delta\mathscr{S}^{\text{hyper}}_{\text{HDO}} &= \frac{q^2\mu^{-2\epsilon}}{3(4\pi)^3N\epsilon}\int d^dx\left\{\frac{1}{4}\int d^2\theta\; W^\alpha\big(\Box{+}\partial\bar\partial\big)W_\alpha + \text{h.c.} + \right.\\ &\left. + \int d^4\theta\;\Big[\Big(\sqrt{2}\,\bar\partial V - \bar S\Big)\Big(\Box{+}\partial\bar\partial\Big)\Big(\sqrt{2}\,\partial V - S\Big) - \partial V\big(\Box{+}\partial\bar\partial\big)\bar\partial V\Big]\right\},\end{aligned} \tag{5.20}$$

where $d = 6 - 2\epsilon$. From (5.19) we can read off the gauge coupling running in the six dimensional bulk. The relation between bare coupling and renormalized coupling is given by

$$\frac{1}{g_B^2} = \left(\frac{1}{g^2} - \frac{2\,q^2m^2}{(4\pi)^3N\epsilon}\right)\mu^{-2\epsilon} \tag{5.21}$$

such that the beta function of the inverse coupling squared is found to be

$$\beta_{1/g^2} = -\frac{4\,q^2m^2}{(4\pi)^3N} \tag{5.22}$$

and the running

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{2\,q^2m^2}{(4\pi)^3N}\ln\frac{\mu^2}{\mu_0^2}. \tag{5.23}$$

Next we discuss the conterterms at the two classes of fixed points.

Fixed point self-energy

First we display the divergent self-energy contributions at all fixed points combines. We take one half of both possibilities to reduce one of the two delta functions as described in Section 4.1.4 and extract the four dimensional divergence at the fixed points according to App. D.2. After expansion of the last orbifold compatible delta function with (5.10) and integration over the ordinary delta function $\delta^2(z_2 - z_1)$ the self-energies at the fixed points read

$$\Sigma_{VV,\,\mathrm{fp}}\Big|_{\mathrm{div}} = \frac{i\mu^{-2\epsilon}}{(4\pi)^2 N\epsilon} \sum_{b=1}^{N-1} \int d^d x\, d^4\theta \,\Big\{ -\frac{1}{2}\Big(p(b) + p(-b)\Big) V \,\Box P_0\, V + \\ + p(b)\Big(\bar{\partial} V \partial V - \sqrt{2}\,\bar{\partial} V S - \sqrt{2}\,\partial V \bar{S} + S\bar{S}\Big)\Big\}\, \delta^2\big((1-e^{ib\varphi})z\big), \quad (5.24)$$

where $d = 6 - 2\epsilon$ and $p(b)$ is a sum of phase factors $p(b) = e^{ibn_+\varphi} + e^{ibn_-\varphi}$. Now we separate the contributions at the different fixed points.

$\mathbb{Z}_2$ fixed point renormalization

The contribution of the self-energies located at the $\mathbb{Z}_2$ fixed points of even ordered orbifolds can be read off from (5.24) for $b = N/2$. Due to the fact that the numbers n_+ and n_- are related to each other by (5.7) the quantities $p(b)$ and $p(-b)$ vanish for $b = N/2$ and so does the divergent self-energy at these fixed points (5.24). Therefore, in the Abelian case, there is no renormalization at the $\mathbb{Z}_2$ fixed points due to the hyper multiplet at all. This is the generalization of the result that we found a vanishing self-energy due to the hyper multiplet in the Abelian theory at the fixed points of $S^1/\mathbb{Z}_2$ in Section 4.1.4.

Non-$\mathbb{Z}_2$ fixed point renormalization

Now we consider the contributions at the rest of the fixed points which do not lie in a $\mathbb{Z}_2$ subsector of an even ordered orbifold. Their contribution can be read off from (5.24) for $b \neq N/2$. The condition $e^{ib\varphi}z = z$ that is implemented by the remaining delta function describes the same fixed point set for b and $-b$. Therefore, we sum the contributions to b and $-b$ explicitly in the form

$$\sum_{\substack{b=1\\ b\neq N/2}}^{N-1} A(b)\, \delta^2\big((1-e^{ib\varphi})z\big) = \sum_{b=1}^{[N/2]_*} \big(A(b) + A(-b)\big)\, \delta^2\big((1-e^{ib\varphi})z\big). \quad (5.25)$$

The symbol $[N/2]_*$ is defined via

$$[N/2]_* = \begin{cases} \frac{1}{2}\big(N-2\big) & \text{if N even} \\ \\ \frac{1}{2}\big(N-1\big) & \text{if N odd} \end{cases} \tag{5.26}$$

and we have used that the delta function $\delta^2\big((1-e^{ib\varphi})z\big)$ is symmetric under a reflection of b. Then the local counter term is written as

$$\Delta\mathscr{S}^{\text{hyper}}_{\text{non-}\mathbb{Z}_2} = -\frac{2\,\mu^{-2\epsilon}}{(4\pi)^2 N\epsilon} \sum_{b=1}^{[N/2]_*} c(b) \int d^d x \left\{ \frac{1}{4} \int d^2\theta \; W_\alpha W^\alpha + \text{h.c.} + \right.$$

$$\left. + \int d^4\theta \left[\Big(\sqrt{2}\,\bar{\partial} V - \bar{S}\Big)\Big(\sqrt{2}\,\partial V - S\Big) - \partial V \bar{\partial} V \right] \right\} \delta^2\big((1-e^{ib\varphi})z\big). \tag{5.27}$$

where $c(b)$ is defined as the combination of phase factors

$$c(b) = \cos\big(b\varphi\, n_+\big) + \cos\big(b\varphi\,(n_+ + 1)\big). \tag{5.28}$$

The gauge coupling running of the four dimensional gauge coupling at the fixed points is read off from this general counterterm as demonstrated in Section 3.1.3 after one has chosen the order N of the orbifold and specified the orbifold twist phase n_+. This Abelian result will also serve as a cross check for the non-Abelian result in Section 5.2.4.

5.2 Hyper multiplet coupled to a non-Abelian gauge multiplet

In this section we consider a hyper multiplet which is coupled to a non-Abelian gauge multiplet in six dimensions with the two extra dimensions compactified on $T^2/\mathbb{Z}_N$. In the non-Abelian discussion we quantize also the gauge sector of the theory and extend the six dimensional orbifold compatible calculus that was discussed for the Abelian case in Section 5.2.3 to cover the non-Abelian case. We calculate the loop effects for the gauge kinetic term due to to hyper, gauge and ghost multiplets and distinguish contributions in the bulk and at the two classes of fixed points.

5.2.1 Classical action in six dimensional Minkowski space

The classical action for a hyper multiplet that is coupled to a non-Abelian gauge multiplet in six dimensions in terms of four dimensional superfields [13, 15] has many similarities both with the six dimensional Abelian action and the five dimensional non-Abelian action. The kinetic action of the hyper multiplet with its coupling to the six dimensional gauge multiplet $\mathscr{S}_H$ is given by the same expression as in the six dimensional Abelian theory (5.1) for unit charge. The kinetic action for the six dimensional gauge multiplet is given by

$$\mathscr{S}_V = \frac{1}{g^2}\int d^6x \left\{ \mathrm{tr}\left[\frac{1}{4}\int d^2\theta\, W^\alpha W_\alpha + \frac{1}{4}\int d^2\bar{\theta}\, \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right] + \right. \tag{5.29}$$
$$\left. + \,\mathrm{tr}\int d^4\theta \left[\left(-\tfrac{1}{\sqrt{2}}\,\bar{\partial} + \bar{S}\right) e^{2V} \left(\tfrac{1}{\sqrt{2}}\,\partial + S\right) e^{-2V} + \frac{1}{4}\,\bar{\partial} e^{2V}\partial e^{-2V}\right]\right\}$$

with the notation for the complex coordinates given below (5.3). This means that w.r.t. the five dimensional non-Abelian action (4.30) the second line is modified. The non-Abelian super-gauge transformation for the six dimensional theory reads

$$\begin{aligned} \Phi_+ &\to e^{-2q\Lambda}\Phi_+, \qquad & e^{2V} &\to e^{2\bar{\Lambda}} e^{2V} e^{2\Lambda}, \\ \Phi_- &\to \Phi_- e^{+2q\Lambda}, \qquad & S &\to e^{-2\Lambda}\left(S + \tfrac{1}{\sqrt{2}}\,\partial\right) e^{2\Lambda}. \end{aligned} \tag{5.30}$$

The action (5.29) reproduces the six dimensional non-Abelian SYM theory when its restriction to component fields with V in WZ gauge is taken. To render the action fully gauge invariant, a Wess-Zumino-Witten term has to be added [13]. This term, however, does not have a quadratic piece and therefore does not influence the quadratic part of the gauge multiplet action

$$\mathscr{S}_{V2} = \frac{1}{g^2} \int d^5x\, d^4\theta\; \mathrm{tr}\left[\frac{1}{8} V D^\alpha \bar{D}^2 D_\alpha V + \bar{\partial} V\, \partial V - \sqrt{2}\,\bar{\partial} V S - \sqrt{2}\,\partial V \bar{S} + \bar{S} S\right]. \tag{5.31}$$

This is the straightforward extension of the five dimensional quadratic piece (4.32). Therefore, in order to determine the renormalization of the gauge kinetic term, we have to find the same self-energies as in the five dimensional case.

5.2.2 Quantization of the action

The theory is quantized by inserting it into the path integral. The generating functional is the same as in five dimensions (4.33) with the five dimensional non-Abelian action replaced against the six dimensional non-Abelian action.

Hyper multiplet propagator

The propagator for the hyper multiplet in the six dimensional non-Abelian theory is the same as the propagator in the six dimensional Abelian theory (5.5) and we can use the drawing conventions of the five dimensional non-Abelian theory in Fig. 11.

Gauge multiplet propagator

The quadratic part of the vector multiplet action can be represented as

$$\begin{aligned} Z_0\big[J_V, J_S\big] = \int \mathscr{D}V \exp\Big\{ i \int d^6x\; \mathrm{tr}\Big[\int d^4\theta\; \frac{1}{g^2}\; \bar{\mathbf{v}}\, \mathbf{A}\, \mathbf{v} + \int d^4\theta\; J_V V \\ + \int d^2\theta\; J_S S + \int d^2\bar{\theta}\; \bar{J}_S \bar{S} \Big]\Big\}, \end{aligned} \tag{5.32}$$

where the vector $\mathbf{v}$ and the hermitean matrix $\mathbf{A}$ are given by

$$\mathbf{v} = \begin{pmatrix} V \\ S \\ \bar{S} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} -\Box P_0 - \partial\bar{\partial} & \frac{1}{\sqrt{2}} P_+ \bar{\partial} & \frac{1}{\sqrt{2}} P_- \partial \\ -\frac{1}{\sqrt{2}} P_+ \partial & \frac{1}{2} P_+ & 0 \\ -\frac{1}{\sqrt{2}} P_- \bar{\partial} & 0 & \frac{1}{2} P_- \end{pmatrix}. \tag{5.33}$$

The operator $\mathbf{A}$ has chiral zero modes corresponding to the gauge directions $\mathbf{x}$

$$\mathbf{A}\mathbf{x} = 0 \qquad \text{for} \qquad \mathbf{x} = \delta_\Lambda \mathbf{v} = \begin{pmatrix} \Lambda + \bar{\Lambda} \\ \sqrt{2}\,\partial\Lambda \\ \sqrt{2}\,\partial\bar{\Lambda} \end{pmatrix}. \tag{5.34}$$

This shows explicitly that also in six dimensions in order to define the propagator of the vector multiplet, we need to perform a gauge fixing to modify the quadratic form $\mathbf{A}$ so that it becomes invertible. In six dimensions, the gauge fixing functional (4.37) is generalized to

$$\Theta = \frac{\bar{D}^2}{-4}\left(\sqrt{2}V + \frac{1}{\Box}\,\partial\bar{S}\right) \tag{5.35}$$

and the restriction to the highest component now yields

$$\frac{D^2}{-4}\,\Theta\,\Bigg| = \frac{1}{\sqrt{2}}\left(\Box C + D + \partial_6 A_5 - \partial_5 A_6 + \mathrm{i}\partial_M A^M\right). \tag{5.36}$$

Hence the imaginary part gives rise to a six dimensional Lorentz invariant gauge fixing for the vector field A_M. The gauge fixing action is

$$\mathscr{S}_{\mathrm{gf}} = -\frac{1}{g^2}\int d^6x\, d^4\theta\ \mathrm{tr}\Big[V\Big(\Box + \frac{1}{8}D^\alpha\bar{D}^2D_\alpha\Big)V + \sqrt{2}V\Big(\partial\bar{S} + \bar{\partial}S\Big) + \partial\bar{S}\,\frac{1}{\Box}\,\bar{\partial}S\Big], \tag{5.37}$$

such that the gauge fixed action finally involves an invertible quadratic operator

$$\mathscr{S}_{\mathrm{V2}} + \mathscr{S}_{\mathrm{gf}} = \int d^6x\, d^4\theta\,\mathrm{tr}\Big[-V\big(\Box + \partial\bar{\partial}\big)V + \bar{S}\,\frac{\Box + \partial\bar{\partial}}{\Box}\,S\Big]. \tag{5.38}$$

The propagators for the four dimensional superfields V and S are decoupled

$$Z_0\big[J_V, J_S\big] = \exp\Big\{i\int d^6x\, d^4\theta\ \mathrm{tr}\Big[\frac{1}{4}\,J_V\,\frac{1}{\Box + \partial\bar{\partial}}\,J_V + \bar{J}_S\,\frac{-1}{\Box + \partial\bar{\partial}}\,J_S\Big]\Big\}. \tag{5.39}$$

Since the five dimensional propagators (4.41) contain only ∂_5^2, this six dimensional result is precisely as expected.

Ghost propagator

Finally, in order to determine the ghost propagators in six dimensions we have to take into account the following modifications: The infinitesimal version of the six dimensional transformation law (5.30) for the superfield S reads

$$\delta_\Lambda S = \sqrt{2}\,\partial\Lambda + 2\big[S,\Lambda\big], \tag{5.40}$$

This leads to the infinitesimal gauge variation δ_C of the gauge fixing functional

$$\delta_C\Theta = \sqrt{2}\,\frac{\bar{D}^2}{-4}\Big(L_V(C-\bar{C}) + \coth\left(L_V(L_V(C+\bar{C}))\right) + \\ + \frac{\partial}{\Box}\left(\sqrt{2}\,\bar{\partial}\bar{C} - 2\big[\bar{S},\bar{C}\big]\right)\Big). \tag{5.41}$$

The ghost action is determined straightforwardly in analogy to (4.44)

$$\mathscr{S}_{\text{gh}} = \frac{1}{\sqrt{2}} \int d^6x\, d^4\theta\ \text{tr}\Big[\sqrt{2}\,(C' + \bar{C}')\Big(L_V(C - \bar{C}) + \coth\big(L_V(L_V(C + \bar{C}))\big)\Big)$$
$$+\, C'\,\frac{\partial}{\Box}\Big(\sqrt{2}\,\bar{\partial}\bar{C} - 2\big[\bar{S}, \bar{C}\big]\Big) + \bar{C}'\,\frac{\bar{\partial}}{\Box}\Big(\sqrt{2}\,\partial C + 2\big[S, C\big]\Big)\Big], \quad (5.42)$$

which means that w.r.t. the five dimensional version (4.44) only the last two terms are modified. The ghost propagators are therefore the straightforward generalization of the five dimensional version (4.45)

$$Z_0\big[J_C, J_C'\big] = \exp\Big\{ i \int d^6x\, d^4\theta\ \text{tr}\Big[-\bar{J}_C' \frac{1}{\Box + \partial\bar{\partial}} J_C - J_C' \frac{1}{\Box + \partial\bar{\partial}} \bar{J}_C \Big]\Big\}. \quad (5.43)$$

Thus, we see that the propagators in six dimensions are to a large extent simple generalizations of the five dimensional propagators given in Section 4.2.2. Therefore, we use the same conventions to draw the six dimensional propagators as given in Fig. 11.

Interactions

The interactions that involve the hyper multiplet are the same as in five dimensions. They can be found in (4.46) and are depicted in Fig. 12. The following interactions from $\mathscr{S}_V$ involve the gauge multiplet and the chiral adjoint multiplet S

$$\mathscr{S}_{V\text{int}} \supset \int d^6x\, d^4\theta\ \text{tr}\Big[\frac{1}{4}\,\big[V, D^\alpha V\big]\bar{D}^2 D_\alpha V - \frac{1}{8}\,\big[V, D^\alpha V\big]\bar{D}^2\big[V, D_\alpha V\big] +$$
$$-\,\frac{1}{6}\,\big[V, \big[V, D^\alpha V\big]\big]\bar{D}^2 D_\alpha V + \sqrt{2}\,\partial V\,\big[V, \bar{S}\big] - \sqrt{2}\,\bar{\partial} V\,\big[V, S\big] +$$
$$-\,2\,S\big[V, \bar{S}\big] + \frac{1}{3}\,\bar{\partial} V\,\big[V, \big[V, \partial V\big]\big] - \frac{2\sqrt{2}}{3}\,\partial V\,\big[V, \big[V, \bar{S}\big]\big] +$$
$$-\,\frac{2\sqrt{2}}{3}\,\bar{\partial} V\,\big[V, \big[V, S\big]\big] + 2\,S\big[V, \big[V, \bar{S}\big]\big]\Big]. \quad (5.44)$$

For these interactions we use the drawing conventions of Fig. 13 also in six dimensions. In the ghost sector the following interactions which are depicted in Fig. 14 are obtained from the expansion of the ghost action (5.42)

$$\mathscr{S}_{gh\,\text{int}} \supset \int d^6x\, d^4\theta\ \text{tr}\Big[(C' + \bar{C}')\,\big[V, C - \bar{C}\big] + \frac{1}{3}\,(C' + \bar{C}')\,\big[V, \big[V, C + \bar{C}\big]\big]$$
$$+\,\sqrt{2}\,\frac{\partial}{\Box}\,C'\big[\bar{S}, \bar{C}\big] - \sqrt{2}\,\frac{\bar{\partial}}{\Box}\,\bar{C}'\big[S, C\big]\Big]. \quad (5.45)$$

We already mentioned above that in six dimensions gauge invariance requires the presence of an additional WZW term for the gauge multiplet V [13]. This term leads in principle to a three point gauge field self-interaction. However, it turns out that all graphs that can be constructed with this additional interaction add up to zero because of the symmetry of the structure constants. Thus, for our calculation, we do not need these interactions and are left with the same set of relevant graphs in six dimensions as in the five dimensional situation.

5.2.3 Orbifold compatible calculus for $T^2/\mathbb{Z}_N$

We will now describe which boundary conditions the fields have to fulfill in order to live on the orbifold $T^2/\mathbb{Z}_N$ instead of uncompactified Minkowski space. The hyper and gauge multiplets on the orbifold need to be covariant w.r.t. the $\mathbb{Z}_N$ orbifold action. Hence, their transformation behaviour under $z \to e^{-i\varphi}z$ is required to be

$$\begin{aligned} \Phi_+ &\to Z_+\Phi_+, & V &\to Z_+ V \bar{Z}_+, \\ \Phi_- &\to \Phi_- Z_-, & S &\to \bar{Z}_- S \bar{Z}_+, \end{aligned} \tag{5.46}$$

with the properties $Z_+^N = Z_-^N = 1$, because the transformations are $\mathbb{Z}_N$ actions. Invariance of the action requires in addition that the matrices Z_+ and Z_- be unitary and that they are related to each other via

$$Z_+ Z_- e^{i\varphi} = \mathbb{1}. \tag{5.47}$$

Therefore, we only need the matrix Z_+ in principle. However, it turns out to be convenient to keep using the notation $Z_\pm$. The transformation rules for the V and S superfields with the adjoint indices made explicit are given by

$$V^i \to Q^i{}_j\, V^j, \qquad S^i \to e^{+i\varphi}\, Q^i{}_j\, S^j, \qquad Q^i{}_j = \mathrm{tr}[T^i Z_+ T_j \bar{Z}_+]. \tag{5.48}$$

This implies that all matrix elements $Q^i{}_j$ are real. Invariance of the action requires Q to have the properties (4.51) and $Q^N = \mathbb{1}$ as it defines a $\mathbb{Z}_N$ action. The reduction to the $\mathbb{Z}_2$ orbifold group with $\varphi = \pi$ and $Z_+ = -Z_- = Z$ is interesting, because then many of the properties of the five dimensional case, discussed in Section 4.2.3, are recovered.

To obtain the orbifold compatible delta functions for the various superfields, we write down the transformation behaviour of orbifold compatible sources under $z \to e^{-i\varphi}z$

$$\begin{aligned} J_+ &\to J_+ Z_+^{-1}, & J_V{}^i &\to Q^i{}_j\, J_V{}^j, \\ J_- &\to Z_-^{-1} J_-, & J_S{}^i &\to e^{-i\varphi}\, Q^i{}_j\, J_S{}^j, \end{aligned} \tag{5.49}$$

where the orthogonality property of Q in (4.51) has been used. This is also reflected in the orbifold compatible delta functions for the $T^2/\mathbb{Z}_N$ orbifold

$$\begin{aligned}
\tilde{\delta}_{21}^{(+)a}{}_b &= \frac{1}{N}\sum_{b=0}^{N-1}\left[Z_+^b\right]{}^a{}_b\,\delta^2(z_2-e^{ib\varphi}z_1)\,\delta^4(x_2-x_1)\,\delta^4(\theta_2-\theta_1),\\
\tilde{\delta}_{21}^{(-)b}{}_a &= \frac{1}{N}\sum_{b=0}^{N-1}\left[Z_-^b\right]{}^b{}_a\,\delta^2(z_2-e^{ib\varphi}z_1)\,\delta^4(x_2-x_1)\,\delta^4(\theta_2-\theta_1),\\
\tilde{\delta}_{21}^{(V)i}{}_j &= \frac{1}{N}\sum_{b=0}^{N-1}\left[Q^{-b}\right]{}^i{}_j\,\delta^2(z_2-e^{ib\varphi}z_1)\,\delta^4(x_2-x_1)\,\delta^4(\theta_2-\theta_1),\\
\tilde{\delta}_{21}^{(S)i}{}_j &= \frac{1}{N}\sum_{b=0}^{N-1}e^{ib\varphi}\left[Q^{-b}\right]{}^i{}_j\,\delta^2(z_2-e^{ib\varphi}z_1)\,\delta^4(x_2-x_1)\,\delta^4(\theta_2-\theta_1).
\end{aligned}\tag{5.50}$$

We check the orbifold compatibility: Under $z_2 \longrightarrow e^{-i\varphi}z_2$ these delta functions transform in the same way as the corresponding sources

$$\begin{aligned}
\tilde{\delta}_{21}^{(+)a}{}_b &\longrightarrow \left[Z_+^{-1}\right]{}^a{}_{a'}\,\tilde{\delta}_{21}^{(+)a'}{}_b, &\qquad \tilde{\delta}_{21}^{(V)i}{}_j &\longrightarrow Q^i{}_{i'}\,\tilde{\delta}_{21}^{(V)i'}{}_j,\\
\tilde{\delta}_{21}^{(-)a}{}_b &\longrightarrow \left[Z_-^{-1}\right]{}^a{}_{a'}\,\tilde{\delta}_{21}^{(-)a'}{}_b, &\qquad \tilde{\delta}_{21}^{(S)i}{}_j &\longrightarrow e^{-i\varphi}\,Q^i{}_{i'}\,\tilde{\delta}_{21}^{(S)i'}{}_j.
\end{aligned}\tag{5.51}$$

Under $z_1 \longrightarrow e^{-i\varphi}z_1$ they transform as the inverse of the corresponding sources, so that these delta functions are indeed orbifold compatible

$$\begin{aligned}
\tilde{\delta}_{21}^{(+)a}{}_b &\longrightarrow \left[Z_+\right]{}^a{}_{a'}\,\tilde{\delta}_{21}^{(+)a'}{}_b, &\qquad \tilde{\delta}_{21}^{(V)i}{}_j &\longrightarrow \left[Q^{-1}\right]{}^i{}_{i'}\,\tilde{\delta}_{21}^{(V)i'}{}_j,\\
\tilde{\delta}_{21}^{(-)a}{}_b &\longrightarrow \left[Z_-\right]{}^a{}_{a'}\,\tilde{\delta}_{21}^{(-)a'}{}_b, &\qquad \tilde{\delta}_{21}^{(S)i}{}_j &\longrightarrow e^{i\varphi}\left[Q^{-1}\right]{}^i{}_{i'}\,\tilde{\delta}_{21}^{(S)i'}{}_j.
\end{aligned}\tag{5.52}$$

In contrast to the orbifold compatible delta functions (4.54) in five dimensions, these delta functions are no longer symmetric in their indices: The exchange of the spacetime labels results in

$$\begin{aligned}
\tilde{\delta}_{12}^{(+)a}{}_b &= \tilde{\tilde{\delta}}_{21}^{(+)}{}^a{}_b, &\qquad \tilde{\delta}_{12}^{(V)i}{}_j &= \tilde{\delta}_{21}^{(V)}{}_j{}^i,\\
\tilde{\delta}_{12}^{(-)a}{}_b &= \tilde{\tilde{\delta}}_{21}^{(-)}{}^a{}_b, &\qquad \tilde{\delta}_{12}^{(S)i}{}_j &= \tilde{\delta}_{21}^{(\bar{S})}{}_j{}^i,
\end{aligned}\tag{5.53}$$

because Z_+ and Z_- are unitary and Q is orthonormal. Derivatives with respect to the compactified coordinates always act on the $\delta^2(z_2 - e^{ib\varphi}z_1)$ factor. Therefore, changing a spacetime index of such a derivative also changes the type of delta function as

$$\begin{aligned}
\bar{\partial}_2\,\tilde{\delta}_{21}^{(+)a}{}_b &= -\bar{\partial}_1\,\tilde{\tilde{\delta}}_{21}^{(-)}{}^a{}_b, &\qquad \partial_2\,\tilde{\delta}_{21}^{(V)i}{}_j &= -\partial_1\,\tilde{\delta}_{21}^{(\bar{S})i}{}_j,\\
\bar{\partial}_2\,\tilde{\delta}_{21}^{(-)a}{}_b &= -\bar{\partial}_1\,\tilde{\tilde{\delta}}_{21}^{(+)}{}^a{}_b, &\qquad \partial_2\,\tilde{\delta}_{21}^{(S)i}{}_j &= -\partial_1\,\tilde{\delta}_{21}^{(V)i}{}_j.
\end{aligned}\tag{5.54}$$

Notice that for the hyper multiplet delta functions a complex conjugation is involved in (5.53) and (5.54). We are now ready to perform loop calculations also in the non-Abelian theory in six dimensions with the extra two dimensions compactified on the orbifold $T^2/\mathbb{Z}_N$.

5.2.4 Renormalization of the gauge kinetic term due to the hyper multiplet

In this section we determine the renormalization of the gauge kinetic term in the non-Abelian theory in six dimensions where the extra dimensions are compactified on the orbifold $T^2/\mathbb{Z}_N$. There are many similarities to the calculation in the five dimensional non-Abelian theory on the orbifold $S^1/\mathbb{Z}_2$ which we described in Section 4.2.4. Therefore, it will suffice for us to indicate the points that deviate from our treatment in this section. The expansion of the hyper multiplet action to fourth order in the gauge coupling (4.46) remains valid in six dimensions. Hence, we have to calculate the same set of graphs as in five dimensions that was depicted in Figs. 15 and 16. The results for these graphs are provided in App. C.1. The replacement of the fields on the external lines against the derivatives w.r.t. the corresponding source terms is performed as

$$\Phi_+^a \to \frac{\delta}{\delta\big(iJ_{+a}\big)}, \quad \bar{\Phi}_{+a} \to \frac{\delta}{\delta\big(i\bar{J}_+^a\big)}, \quad \Phi_{-a} \to \frac{\delta}{\delta\big(iJ_-^a\big)}, \quad \bar{\Phi}_-^a \to \frac{\delta}{\delta\big(i\bar{J}_{-a}\big)}. \tag{5.55}$$

Then the self-energies Σ_{VV}, Σ_{VS}, $\Sigma_{V\bar{S}}$ and $\Sigma_{\bar{S}S}$ are calculated as

$$\begin{aligned}\Sigma_{VV}^{\text{hyper}} = \int \big(d^6x\big)_{12} d^4\theta \, \text{tr}\Big[&- V_1 \frac{1}{(\Box + \partial\bar{\partial})_2} \tilde{\delta}_{21}^{6(+)} \, \Box_2 P_0 \, V_2 \frac{1}{(\Box + \partial\bar{\partial})_2} \tilde{\tilde{\delta}}_{21}^{6(+)} \\ &- V_1 \frac{1}{(\Box + \partial\bar{\partial})_2} \tilde{\tilde{\delta}}_{21}^{6(-)} \, \Box_2 P_0 \, V_2 \frac{1}{(\Box + \partial\bar{\partial})_2} \tilde{\delta}_{21}^{6(-)} \\ &+ 2\, \partial V_1 \frac{1}{(\Box + \partial\bar{\partial})_2} \tilde{\delta}_{21}^{6(+)} \, \bar{\partial} V_2 \frac{1}{(\Box + \partial\bar{\partial})_2} \tilde{\delta}_{21}^{6(-)} \Big],\end{aligned} \tag{5.56}$$

$$\Sigma_{V\bar{S}}^{\text{hyper}} = -2\sqrt{2} \int \big(d^6x\big)_{12} d^4\theta \, \text{tr}\Big[\partial V_1 \frac{1}{(\Box + \partial\bar{\partial})_2} \tilde{\delta}_{21}^{6(+)} \, \bar{S}_2 \frac{1}{(\Box + \partial\bar{\partial})_2} \tilde{\delta}_{21}^{6(-)} \Big], \tag{5.57}$$

$$\Sigma_{S\bar{S}}^{\text{hyper}} = 2 \int \big(d^6x\big)_{12} d^4\theta \;\; \text{tr}\Big[S_1 \frac{1}{(\Box + \partial\bar{\partial})_2} \tilde{\delta}_{21}^{6(+)} \, \bar{S}_2 \frac{1}{(\Box + \partial\bar{\partial})_2} \tilde{\delta}_{21}^{6(-)} \Big]. \tag{5.58}$$

We reduce one more orbifold compatible delta function in the by now familiar three steps of Section 4.1.4. In the same way as in the Abelian calculation in

Section 5.1.4 we distinguish self-energy contritutions in the bulk, at $\mathbb{Z}_2$ fixed points and non-$\mathbb{Z}_2$ fixed points.

Bulk renormalization

The bulk contribution of the self-energies is extracted for the term with $b = 0$ in the expansion of the last orbifold delta function. We extract the divergence with the help of App. D.1 and determine the local bulk counterterm in six dimensions. In the same way as in the Abelian calculation the counterterm splits into a piece that renormalizes the bulk gauge kinetic term and a higher derivative operator

$$\Delta\mathscr{S}^{\text{hyper}}_{\text{bulk}} = \Delta\mathscr{S}^{\text{hyper}}_{\text{gkt}} + \Delta\mathscr{S}^{\text{hyper}}_{\text{HDO}}. \tag{5.59}$$

The first term which renormalizes the bulk gauge kinetic term reads

$$\begin{aligned}\Delta\mathscr{S}^{\text{hyper}}_{\text{gkt}} = \frac{-2m^2\mu^{-2\epsilon}}{(4\pi)^3 N\epsilon}\int d^dx \,\bigg\{ \int d^2\theta \;\text{tr}\left[\frac{1}{4}\,W^\alpha W_\alpha\right] + \text{h.c.} + \\ + \int d^4\theta\;\text{tr}\left[\left(-\tfrac{1}{\sqrt{2}}\,\bar\partial + \bar S\right)e^{2V}\left(\tfrac{1}{\sqrt{2}}\,\partial + S\right)e^{-2V} + \frac{1}{4}\,\partial e^{-2V}\bar\partial e^{2V}\right]\bigg\}\end{aligned} \tag{5.60}$$

and the second term which is the higher derivative operator is given by

$$\begin{aligned}\Delta\mathscr{S}^{\text{hyper}}_{\text{HDO}} = \frac{\mu^{-2\epsilon}}{3(4\pi)^3 N\epsilon}\int d^dx\,\bigg\{ \int d^2\theta\;\text{tr}\left[\frac{1}{4}\,W^\alpha(\Box + \partial\bar\partial)W_\alpha\right] + \text{h.c.} + \\ + \int d^4\theta\;\text{tr}\bigg[\left(-\tfrac{1}{\sqrt{2}}\,\bar\partial + \bar S\right)e^{2V}\left(\Box + \partial\bar\partial\right)\left(\tfrac{1}{\sqrt{2}}\,\partial + S\right)e^{-2V} + \\ + \frac{1}{4}\,\partial e^{-2V}\left(\Box + \partial\bar\partial\right)\bar\partial e^{2V}\bigg]\bigg\},\end{aligned} \tag{5.61}$$

where $d = 6 - 2\epsilon$. The counterterm (5.60) to the bulk gauge kinetic term influences the six dimensional gauge coupling running. The six dimensional bare gauge coupling is related to the renormalized gauge coupling and the counterterm by

$$\frac{1}{g_B^2} = \left(\frac{1}{g^2} - \frac{2m^2}{(4\pi)^3 N\epsilon}\right)\mu^{-2\epsilon}. \tag{5.62}$$

Then we find that the beta function reads

$$\beta_{1/g^2} = -\frac{4m^2}{(4\pi)^3 N}. \tag{5.63}$$

Integration between the energy scales μ_0 and μ determines the running

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{2m^2}{(4\pi)^3 N} \ln \mu^2. \tag{5.64}$$

This concludes the discussion of the bulk results.

Fixed point self-energy

The fixed point self-energy is the sum over all values of b without the bulk contribution at $b = 0$. After we integrate out the ordinary delta function $\delta^2(z_2 - z_1)$, we obtain

$$\Sigma_{\rm fp}\Big|_{\rm div} = \frac{i\mu^{-2\epsilon}}{(4\pi)^2 N\epsilon} \sum_{b=1}^{N-1} \int d^d x\, d^4\theta\, {\rm tr}\Big[-\frac{1}{2} V \big(Z_+^b + \bar{Z}_-^b + Z_-^b + \bar{Z}_+^b\big) \Box P_0 V + \\ + \partial V Z_+^b \bar{\partial} V + Z_-^b \partial V \bar{\partial} V - \sqrt{2}\, \partial V Z_+^b \bar{S} - \sqrt{2}\, Z_-^b \partial V \bar{S} - \sqrt{2}\, \bar{Z}_+^b \bar{\partial} V S + \\ - \sqrt{2}\, \bar{\partial} V \bar{Z}_-^b S + + S Z_+^b \bar{S} + Z_-^b S \bar{S} \Big]\, \delta^2\big((1 - e^{ib\phi}) z\big), \tag{5.65}$$

where $d = 6 - 2\epsilon$. Now we distinguish the contributions at the two classes of fixed points.

$\mathbb{Z}_2$ fixed point renormalization

For even ordered orbifolds we have $\mathbb{Z}_2$ fixed points. The contribution of the self-energies located at these fixed points is given for $b = N/2$. When the matrices Z_+ and Z_- are taken to this power, they have special properties

$$Z_+^{N/2} = -Z_-^{N/2}, \qquad \bar{Z}_+^{N/2} = Z_+^{-N/2} = Z_+^{N/2}, \qquad \bar{Z}_-^{N/2} = Z_-^{-N/2} = Z_+^{N/2}, \tag{5.66}$$

which are a consequence of the relation between the matrices Z_+ and Z_- (5.47) and the unitarity of Z_+ and Z_-. This means that the matrix $Z_+^{N/2}$ has the same properties as the matrix Z in the five dimensional case that we considered in Section 4.2.3. Hence, we observe special cancellations in the self-energies at the $\mathbb{Z}_2$ fixed points of an even ordered orbifold

$$\Delta \mathscr{S}_{\mathbb{Z}_2}^{\rm hyper} = -\frac{\mu^{-2\epsilon}}{(4\pi)^2 N\epsilon} \int d^d x\, d^4\theta\, {\rm tr}\Big[Z_+^{N/2} \Big[\big(\bar{S} - \tfrac{1}{\sqrt{2}} \bar{\partial}\big) e^{2V}, \big(S + \tfrac{1}{\sqrt{2}} \partial\big) e^{-2V} \Big] + \\ + \frac{1}{4} Z_+^{N/2} \Big[\bar{\partial} e^{2V}, \partial e^{-2V} \Big] \Big]\, \delta^2\big(2z\big). \tag{5.67}$$

In particular, the part of the self-energy proportional to the kinetic term of the four dimensional gauge multiplet vanishes. This means that there is no renormalization of the four dimensional gauge coupling at the $\mathbb{Z}_2$ fixed points of an even ordered orbifold.

We note that we recover the rest of the operators in the gauge kinetic term (5.29) enclosed in commutators. In the same way as discussed below (4.64) the counterterm is invariant under the zero mode gauge group, but it is not gauge invariant under the full bulk gauge transformation.

Non-$\mathbb{Z}_2$ fixed point renormalization

Now we regard the terms of the divergent self-energies with $b \neq N/2$. As in the Abelian calculation we sum the contributions to b and $-b$ in the form (5.25). We introduce the algebra element A_+ that corresponds to the unitary matrix Z_+ via $Z_+ \equiv e^{iA_+}$ to write the local counterterm as

$$\Delta\mathscr{S}^{\text{hyper}}_{\text{non-}\mathbb{Z}_2} = \frac{-2\,\mu^{-2\epsilon}}{(4\pi)^2 N\epsilon} \sum_{b=1}^{[N/2]_*} \int d^d x \left\{ \frac{1}{4} \int d^2\theta \,\mathrm{tr}\Big[P_1(b)\, W_\alpha W^\alpha \Big] + \text{h.c.} + \right.$$
$$+ \int d^4\theta\, \mathrm{tr}\left[P_2(b) \left(\left(-\tfrac{1}{\sqrt{2}}\,\bar\partial + \bar S \right) e^{2V} \left(\tfrac{1}{\sqrt{2}}\,\partial + S \right) e^{-2V} + \frac{1}{4}\,\bar\partial e^{2V} \partial e^{-2V} \right) \right] +$$
$$\left. + \int d^4\theta\, \mathrm{tr}\left[P_3(b) \left(\left(\tfrac{1}{\sqrt{2}}\,\partial + S \right) e^{-2V} \left(-\tfrac{1}{\sqrt{2}}\,\bar\partial + \bar S \right) e^{2V} + \frac{1}{4}\,\partial e^{-2V} \bar\partial e^{2V} \right) \right] \right\} \times$$
$$\times\, \delta^2\big((1 - e^{ib\varphi}) z \big). \qquad (5.68)$$

Here $P_i(b)$ are the following combinations that involve the algebra element A_+

$$P_1(b) = P_2(b) + P_3(b), \quad P_2(b) = \cos\big(bA\big)_+, \quad P_3(b) = \cos\Big(b\,(A_+ + \varphi) \Big) \qquad (5.69)$$

and the counterterm (5.68) is the non-linear extension of the quadratic piece. We remark that by a formal replacement of the matrix A_+ by the product of a scalar and the orbifold phase $a_+\varphi$ and of the trace tr by the square of the charge q^2 one obtains the Abelian result found in Section 5.1. In order to read off the gauge coupling running of the four dimensional gauge coupling at the non-$\mathbb{Z}_2$ fixed points from the first line of (5.68), one has to specify the order of the orbifold N and the matrix A_+ (or equivalently Z_+).

5.2.5 Renormalization of the gauge kinetic term due to the gauge multiplet

Here we determine the renormalization of the gauge kinetic term due to the loop corrections that involve the gauge and ghost multiplets. The relevant interactions are contained in $\mathscr{S}_{V\text{int}}$ and $\mathscr{S}_{gh\,\text{int}}$. Because the graphs that involve interactions from the WZW term vanish identically, as remarked in Section 5.2.2, we are left with the same set of graphs as in the five dimensional non-Abelian theory. The graphs are depicted in Figs. 17 and 18. The results for these graphs are presented in App. C.2. Here we display the self-energies at the level of two orbifold compatible delta functions. The VV self-energy reads

$$\begin{aligned}
\Sigma_{VV}^{\text{gauge}} = & \\
& f_{ijk}f_{\ell mn}\int (d^6x)_{12}\, d^4\theta \Big[-3\, V_1^i \Box_2 P_0 V_2^\ell \frac{1}{(\Box+\partial\bar{\partial})_2}\widetilde{\delta}_{21}^{6(V)mj}\frac{1}{(\Box+\partial\bar{\partial})_2}\widetilde{\delta}_{21}^{6(V)nk} \\
& + V_1^i \Box_2 P_0 V_2^\ell \frac{1}{(\Box+\partial\bar{\partial})_2}\widetilde{\delta}_{21}^{6(\bar{S})mj}\frac{1}{(\Box+\partial\bar{\partial})_2}\widetilde{\delta}_{21}^{6(S)nk} \\
& + 2\,\partial V_1^i \bar{\partial} V_2^\ell \frac{1}{(\Box+\partial\bar{\partial})_2}\widetilde{\delta}_{21}^{6(V)mj}\frac{1}{(\Box+\partial\bar{\partial})_2}\widetilde{\delta}_{21}^{6(\bar{S})nk}\Big] \qquad (5.70)
\end{aligned}$$

and the $V\bar{S}$ self-energy is given by

$$\begin{aligned}
\Sigma_{V\bar{S}}^{\text{gauge}} = & \\
& -2\sqrt{2}\, f_{ijk}f_{\ell mn}\int (d^6x)_{12}\; d^4\theta\, \partial V_1^i \bar{S}_2^\ell \frac{1}{(\Box+\partial\bar{\partial})_2}\widetilde{\delta}_{21}^{6(V)mj}\frac{1}{(\Box+\partial\bar{\partial})_2}\widetilde{\delta}_{21}^{6(\bar{S})nk}, \qquad (5.71)
\end{aligned}$$

with the complex conjugate result for the VS self-energy. Finally,

$$\Sigma_{\bar{S}S}^{\text{gauge}} = 2\, f_{ijk}f_{\ell mn}\int \big(d^6x\big)_{12} d^4\theta\, S_1^i \bar{S}_2^\ell \frac{1}{(\Box+\partial\bar{\partial})_2}\widetilde{\delta}_{21}^{6(V)mj}\frac{1}{(\Box+\partial\bar{\partial})_2}\widetilde{\delta}_{21}^{6(\bar{S})nk} \qquad (5.72)$$

is the result for the $S\bar{S}$ self-energy. We reduce one more orbifold compatible delta function and distinuish the contributions in the bulk and at the two classes of fixed points

Bulk renormalization

The bulk counterterm for $b = 0$ in the expansion of the remaining orbifold compatible delta function is again a sum a of counterterm to the gauge kinetic term and a higher derivative operator

$$\Delta\mathscr{S}_{\text{bulk}}^{\text{gauge}} = \Delta\mathscr{S}_{\text{gkt}}^{\text{gauge}} + \Delta\mathscr{S}_{\text{HDO}}^{\text{gauge}}\,. \qquad (5.73)$$

The counterterm to the gauge kinetic term is given by

$$\Delta\mathscr{S}^{\text{gauge}}_{\text{gkt}} = \frac{2m^2\mu^{-2\epsilon}}{(4\pi)^3 N\epsilon}\int d^dx\,\Bigg\{\int d^2\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[\frac{1}{4}\,W^\alpha W_\alpha\Big] + \text{h.c.} + \\ + \int d^4\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[\Big(-\tfrac{1}{\sqrt{2}}\,\bar\partial + \bar S\Big)\,e^{2V}\,\Big(\tfrac{1}{\sqrt{2}}\,\partial + S\Big)\,e^{-2V} + \frac{1}{4}\,\partial e^{-2V}\,\bar\partial e^{2V}\Big]\Bigg\}. \quad (5.74)$$

And the higher dimensional operator that is generated as a loop counterterm in the bulk reads

$$\Delta\mathscr{S}^{\text{gauge}}_{\text{HDO}} = -\frac{\mu^{-2\epsilon}}{3(4\pi)^3 N\epsilon}\int d^dx\,\Bigg\{\int d^2\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[\frac{1}{4}\,W^\alpha\big(\Box + \partial\bar\partial\big)W_\alpha\Big] + \text{h.c.} + \\ + \int d^4\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[\Big(-\tfrac{1}{\sqrt{2}}\,\bar\partial + \bar S\Big)\,e^{2V}\big(\Box + \partial\bar\partial\big)\Big(\tfrac{1}{\sqrt{2}}\,\partial + S\Big)\,e^{-2V} + \\ + \frac{1}{4}\,\partial\,e^{-2V}\big(\Box + \partial\bar\partial\big)\,\bar\partial\,e^{2V}\Big]\Bigg\}, \quad (5.75)$$

where $d = 6-2\epsilon$. The gauge coupling running in the six dimensional bulk can be read off from the counterterm to the gauge kinetic term in (5.74). To this end the trace in the adjoint representation of the gauge group has to be rewritten into the trace in the representation of the hyper multiplet which appears in the action. Then the running of the six dimensional bulk gauge coupling is obtained in the same way as in the preceeding Section 5.2.5 for the hyper multiplet calculation.

By gauge invariance we can infer some additional effects. As we remarked above, the action is not gauge invariant unless also a Wess-Zumino-Witten term is added [13]. Therefore, to preserve supergauge invariance, also this Wess-Zumino-Witten term has to be renormalized. Moreover, because a higher derivative operator is generated, also a higher derivative analogue of the Wess-Zumino-Witten term must exist and renormalize. We have not performed an explicit calculation to confirm the renormalization. However, we can say that the Wess-Zumino-Witten term and its higher derivative counterpart will have to renormalize with the same multiplicative coefficients as the corresponding terms in the quadratic part of the action in order for the theory to be gauge invariant at the one-loop level.

Fixed point self-energy

The self-energy at the fixed points is obtained for the terms with $b \neq 0$ in the expansion of the last orbifold compatible delta function. After the extraction of the four dimensional divergence and the integration over the ordinary delta function we obtain

$$\Sigma^{\text{gauge}}_{\text{fp}}\Big|_{\text{div}} = -\frac{i\mu^{-2\epsilon}}{(4\pi)^2 N\epsilon}\sum_{b=1}^{N-1}\int d^dx\, d^4\theta\, \Big\{ p_1(b)\,\mathrm{tr}_{\mathbf{Ad}}\Big[-\frac{1}{2}\,Q^{-b}V\Box P_0 V\Big] +$$
$$+\, p_2(b)\,\mathrm{tr}_{\mathbf{Ad}}\Big[Q^{-b}\Big(\partial V\bar\partial V - \sqrt{2}\,\partial V\bar S - \sqrt{2}\,S\bar\partial V + S\bar S\Big)\Big]\Big\}\,\delta^2\big((1-e^{ib\varphi})z\big), \tag{5.76}$$

where the quantities $p_1(b)$ and $p_2(b)$ involve sums of phases

$$p_1(b) = 6 - e^{ib\varphi} - e^{-ib\varphi}, \quad p_2(b) = 1 + e^{-ib\varphi} \tag{5.77}$$

and $d = 6 - 2\epsilon$. Now we distinguish the contributions at the different fixed points.

$\mathbb{Z}_2$ fixed point renormalization

Also in the loop correction from the gauge sector we observe special cancellations at the $\mathbb{Z}_2$ fixed points of an even ordered orbifold that are not present at the other fixed points. This is because at the $\mathbb{Z}_2$ fixed points that are given for $b = \frac{N}{2}$ all combinations proportional to $p_2(b)$ vanish and only the counterterm to the gauge kinetic term of the four dimensional gauge field survives

$$\Delta\mathscr{S}^{\text{gauge}}_{\mathbb{Z}_2} = \frac{4\,\mu^{-2\epsilon}}{(4\pi)^2 N\,\epsilon}\int d^dx\,\Big\{\frac{1}{4}\int d^2\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[Q^{N/2}W^\alpha W_\alpha\Big] + \text{h.c.}\Big\}\,\delta^2\big(2z\big). \tag{5.78}$$

As always, after one has specified the representation of the hyper multiplet, the order of the orbifold and the transformation matrix Q, one can read off the running of the four dimensional gauge coupling at the $\mathbb{Z}_2$ fixed points from the general counterterm (5.78).

Non-$\mathbb{Z}_2$ fixed point renormalization

As usual the contributions of the last orbifold compatible delta function with $b \neq 0$ and $b \neq \frac{1}{2}$ give the counterterm at the non-$\mathbb{Z}_2$ fixed points. We sum the contributions to b and $-b$ in the form (5.25) and obtain the result

$$\Delta\mathscr{S}^{\text{gauge}}_{\text{non-}\mathbb{Z}_2} = \frac{\mu^{-2\epsilon}}{(4\pi)^2 N\epsilon}\sum_{b=1}^{[N/2]_*}\int d^dx\,\Big\{\frac{1}{4}\int d^2\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[P_1(b)\,W^\alpha W_\alpha\Big] + \text{h.c.} +$$
$$+\int d^4\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[P_2(b)\Big(\Big(\tfrac{1}{\sqrt{2}}\,\partial + S\Big)e^{-2V}\Big(-\tfrac{1}{\sqrt{2}}\,\bar\partial + \bar S\Big)e^{2V} + \frac{1}{4}\,\partial e^{-2V}\bar\partial e^{2V}\Big)\Big]\Big\}$$
$$\times\,\delta^2\big((1-e^{ib\varphi})z\big), \tag{5.79}$$

where $d = 6 - 2\epsilon$ and the expressions $P_1(b)$ and $P_2(b)$ are defined as

$$P_1(b) = 6\cos bH - \cos b(H+\varphi) - \cos b(H-\varphi),$$
$$P_2(b) = \cos bH + \cos b(H+\varphi). \tag{5.80}$$

Here we have introduced the hermitean matrix H via $Q = e^{iH}$. To arrive at this expression we have used that the matrices $\cos bH$, etc. are symmetric, which is a consequence of the fact that Q is orthogonal. The symbol $[N/2]_*$ has been defined below (5.25). The gauge coupling running of the four dimensional gauge coupling at the non-$\mathbb{Z}_2$ fixed points can be read off from the first line of (5.79) after the orbifold has been specified.

The counterterm (5.79) is gauge invariant under the zero mode supergauge group, as defined below (4.64). However, as we discussed there also this term is not gauge invariant under the full supergauge transformations. This is not surprising when one takes into account that (5.29) is also not gauge invariant by itself: One needs to add a Wess-Zumino-Witten term to make the theory gauge invariant. Therefore we expect that also the expression above can be made gauge invariant by adding a suitable extension of a Wess-Zumino-Witten interaction.

Chapter 6

Supersymmetric theory in ten dimensions

In this chapter we consider a gauge multiplet in a ten dimensional spacetime where the extra six dimensions are compactified on the orbifold $T^6/\mathbb{Z}_N$. For completeness we start by presenting briefly the Abelian action where it is immediately clear that no renormalization of the gauge kinetic term takes place. Then we focus on the non-Abelian case and calculate loop corrections to the gauge kinetic term due to the gauge multiplet self-interactions and ghost multiplets in the loop. We check the well-known fact that there is no renormalization of the gauge coupling in the ten dimensional bulk because of the high amount of supersymmetry. In addition, we find that no higher dimensional operator is required as loop counterterm in the ten dimensional bulk. We derive a general expression for the divergent amplitude at the fixed points/planes of the orbifold. Then we specify to the case of a $T^6/\mathbb{Z}_4$ orbifold in order to obtain a definite result for the counterterms. We find that on the six dimensional fixed torus a six dimensional higher derivative operator has to be introduced as loop counterterm in addition to an ordinary six dimensional counterterm. The four dimensional fixed points of the orbifold support a counterterm for the kinetic term of the four dimensional gauge multiplet.

6.1 Abelian gauge multiplet

The action for an Abelian gauge multiplet in ten dimensions represented in four dimensional superfields has been derived in [13, 15]. The degrees of freedom of the ten dimensional gauge multiplet are contained in one four dimensional gauge multiplet V and three chiral fields S_I, where I is the family index. The action is then given by

$$\mathscr{S}_V = \frac{1}{g^2}\int d^{10}x \left\{ \int d^2\theta \left(\frac{1}{4}W^\alpha W_\alpha + \frac{1}{2}\epsilon^{IJK}S_I\partial_J S_K\right) + \text{h.c.} + \right.$$
$$\left. + \int d^4\theta \left[\left(\sqrt{2}\,\bar\partial_I V - \bar S_I\right)\left(\sqrt{2}\,\partial^I V - S^I\right) - \bar\partial_I V \partial^I V\right] \right\}. \tag{6.1}$$

Here we employ the following notation for the complex coordinates in the extra dimensions

$$z_I = \frac{1}{2}\left(x_{3+2I} - \mathrm{i}\,x_{4+2I}\right), \qquad \bar z_I = \frac{1}{2}\left(x_{3+2I} + \mathrm{i}\,x_{4+2I}\right), \tag{6.2}$$

where the index I runs from 1 to 3 and is the same index as the family index, because the derivative operators and the chiral fields S form a covariant derivative structure. The holomorphic and antiholomorphic derivatives are

$$\partial_I = \partial_{3+2I} + \mathrm{i}\,\partial_{4+2I}, \qquad \bar\partial_I = \partial_{3+2I} - \mathrm{i}\,\partial_{4+2I}. \tag{6.3}$$

The ten dimensional action for the Abelian gauge multiplet (6.1) is closely related to the six dimensional Abelian action (5.3). In the second line it is only the family index that has to be added. A new piece that involves the three chiral superfields is the term in the superpotential. This term is necessary such that the action in Lorentz invariant in ten dimensions. The action (6.1) is invariant under the following super-gauge transformation

$$S_I \to S_I + \sqrt{2}\,\partial_I\Lambda, \quad V \to V + \Lambda + \bar\Lambda. \tag{6.4}$$

The action (6.1) involves only kinetic terms for the four dimensional gauge multiplet V and the three chiral multiplets S_I. Hence in the Abelian theory no renormalization takes place and we proceed directly to the non-Abelian theory in the next section.

6.2 Non-Abelian gauge multiplet

In this section we consider a non-Abelian gauge multiplet in ten dimensions with the extra six dimensions compactified on the orbifold $T^6/\mathbb{Z}_N$. We discuss the ten dimensional action in terms of four dimensional superfields and quantize the theory. Then we focus on the renormalization of the gauge kinetic term, where in this discussion we renormalize only the operators that involve the four dimensional gauge multiplet due to loop corrections from gauge and ghost multiplets. This will suffice for us to check that the renormalization in the bulk vanishes identically and to obtain a general formula for the gauge coupling running at the fixed points/planes of the orbifold. In the last section we specify to the orbifold $T^6/\mathbb{Z}_4$ and infer explicit counterterms.

6.2.1 Classical action in ten dimensional Minkowski space

Here we introduce the classical action of a non-Abelian gauge multiplet in ten dimensions. The decomposition into four dimensional multiplets is the same as in the Abelian case. The action for the ten dimensional gauge multiplet in terms of these four dimensional superfields reads [13, 15]

$$\begin{aligned}
\mathscr{S}_V = \\
&\frac{1}{g^2}\int d^{10}x \left\{ \int d^2\theta \; \mathrm{tr}\Big[\frac{1}{4}\, W^\alpha W_\alpha + \frac{1}{2}\,\epsilon^{IJK} S_I\Big(\partial_J S_K + \frac{\sqrt{2}}{3}\,[S_J, S_K]\Big)\Big] + \mathrm{h.c.} \right.\\
&\left. + \int d^4\theta \; \mathrm{tr}\left[\Big(-\tfrac{1}{\sqrt{2}}\,\bar{\partial}_I + \bar{S}_I\Big) e^{2V}\Big(\tfrac{1}{\sqrt{2}}\,\partial^I + S^I\Big) e^{-2V} + \frac{1}{4}\,\bar{\partial}_I e^{2V}\,\partial^I e^{-2V}\right]\right\}, \qquad (6.5)
\end{aligned}$$

where we employ the same notation for the (anti-)holomorphic derivatives as in the preceeding Section 6.1. The last line is just the generalization of the corresponding part in the six dimensional non-Abelian action (6.5). The additional superpotential term that in the non-Abelian theory also involves a self-coupling of the chiral adjoint multiplets S_I appears only in ten dimensions. The ten dimensional super-gauge transformation is given by

$$S_I \to e^{-2\Lambda}\left(S_I + \frac{1}{\sqrt{2}}\,\partial_I\right) e^{2\Lambda}, \quad e^{2V} \to e^{2\bar{\Lambda}} e^{2V} e^{2\Lambda}. \qquad (6.6)$$

The action (6.5) in its component field form reproduces the ten dimensional SYM theory when the restrictions of the four dimensional gauge multiplet V are chosen in WZ gauge. In order for the ten dimensional action to be fully super-gauge invariant under (6.6) a WZW term has to be added [13] like in the six dimensional non-Abelian theory discussed in Section 5.2. In the same way as in six dimensions, this term does not have a quadratic piece such that the quadratic action of the gauge multiplet consists only of the quadratic terms in (6.5)

$$\mathscr{S}_{V2} = \frac{1}{g^2}\int d^{10}x\, d^4\theta\ \mathrm{tr}\Big[\frac{1}{8}\,V D^\alpha \bar{D}^2 D_\alpha V + \bar{\partial}_I V\,\partial^I V - \sqrt{2}\,\bar{\partial}_I V S^I +$$

$$-\sqrt{2}\,\partial_I V S^I + \bar{S}_I S^I + \frac{1}{2}\,\epsilon^{IJK}\Big(S_I\partial_J\frac{D^2}{-4\Box}S_K + \bar{S}_I\bar{\partial}_J\frac{\bar{D}^2}{-4\Box}\bar{S}_K\Big)\Big]. \quad (6.7)$$

We note that the new superpotential term gives rise to the last two terms such that in principle two more self-energy calculations are required to obtain the full renormalization of the ten dimensional gauge kinetic term. In the following we will restrict ourselves to determine the loop corrections to the first two operators in (6.7) that involve only the four dimensional gauge multiplet V. In our notation this corresponds to the calculation of the self-energy Σ_{VV}.

6.2.2 Quantization of the action

To this end we quantize the action. The generating functional of the theory reads

$$Z\big[J_V, (J_S)_I\big] = \int \mathscr{D}V \prod_{I=1}^{3}\mathscr{D}S_I\ \exp\Big\{i\int d^{10}x\ \Big(\mathscr{L}^{10\mathrm{D}}_{\text{non-Abelian}}(V, S_I) +$$

$$+\,\mathrm{tr}\Big[\int d^4\theta\ J_V V + \int d^2\theta\ (J_S)_I S^I + \int d^2\bar{\theta}\ (\bar{J}_S)_I \bar{S}^I\Big]\Big)\Big\}. \quad (6.8)$$

The quadratic action (6.7) determines the free-field generating functional from which we obtain the propagators. As in the preceeding chapters this will require a gauge fixing and the introduction of ghosts.

Vector superfield propagators

The free-field generating functional that involves the quadratic action (6.7) can be represented as

$$Z_0\big[J_V, (J_S)_I\big]$$

$$= \int \mathscr{D}V \prod_{I=1}^{3}\mathscr{D}S_I\ \exp\Big\{i\int d^{10}x\ \mathrm{tr}\Big[\int d^4\theta\ \frac{1}{g^2}\,\bar{\mathbf{v}}_{(I)}\,\mathbf{A}^{(IJ)}\,\mathbf{v}_{(J)} + \int d^4\theta\ J_V V +$$

$$+\int d^2\theta\ (J_S)_I S^I + \int d^2\bar{\theta}\ (\bar{J}_S)_I \bar{S}^I\Big]\Big\}, \quad (6.9)$$

where the vector $\mathbf{v}$ and the hermitean matrix $\mathbf{A}$ are given by

$$\mathbf{v}_{(I)} = \begin{pmatrix} V \\ S_I \\ \bar{S}_I \end{pmatrix}, \quad \mathbf{A}^{(IJ)} = \begin{pmatrix} -\Box P_0 - \partial_I \bar{\partial}^I & \frac{1}{\sqrt{2}}\, P_+ \bar{\partial}_I\, \eta^{IJ} & \frac{1}{\sqrt{2}}\, P_- \partial_I\, \eta^{IJ} \\ -\frac{1}{\sqrt{2}}\, P_+ \eta^{IJ} \partial_J & \frac{1}{2}\, P_+ \eta^{IJ} & -\frac{1}{2}\, \epsilon^{IJK} \bar{\partial}_K \frac{\bar{D}^2}{-4\Box} \\ -\frac{1}{\sqrt{2}}\, P_- \eta^{IJ} \bar{\partial}_J & -\frac{1}{2}\, \epsilon^{IJK} \partial_K \frac{D^2}{-4\Box} & \frac{1}{2}\, P_- \eta^{IJ} \end{pmatrix}. \tag{6.10}$$

The operator $\mathbf{A}$ has chiral zero modes corresponding to the gauge directions $\mathbf{x}$

$$\mathbf{A}^{(IJ)}\, \mathbf{x}_{(J)} = 0 \qquad \text{for} \qquad \mathbf{x}_{(J)} = \delta_\Lambda \mathbf{v}_{(J)} = \begin{pmatrix} \Lambda + \bar{\Lambda} \\ \sqrt{2}\, \partial_J \Lambda \\ \sqrt{2}\, \partial_J \bar{\Lambda} \end{pmatrix}. \tag{6.11}$$

Therefore, we have to perform a gauge fixing in order to modify the quadratic form $\mathbf{A}$ so that it becomes invertible. The gauge fixing procedure follows closely the gauge fixing in six dimensions described in Section 5.2.2. The gauge fixing functional (5.35) is generalized to

$$\Theta = \frac{\bar{D}^2}{-4} \left(\sqrt{2} V + \frac{1}{\Box}\, \partial_I \bar{S}^I \right) \tag{6.12}$$

and the restriction to the highest component of the gauge fixing functional yields

$$\frac{D^2}{-4}\, \Theta \bigg| = \frac{1}{\sqrt{2}} \Big(\Box C + D + \partial_6 A_5 - \partial_5 A_6 + \partial_8 A_7 - \partial_7 A_8 + \\ + \partial_{10} A_9 - \partial_9 A_{10} - \mathrm{i} \partial_M A^M \Big). \tag{6.13}$$

Hence the imaginary part gives rise to a ten dimensional Lorentz invariant gauge fixing for the vector field A_M. In extension of the six dimensional result (5.37) the gauge fixing action is found to be

$$\mathscr{S}_{\text{gf}} = -\frac{1}{g^2} \int d^{10}x\, d^4\theta\, \mathrm{tr} \Big[V \Big(\Box + \frac{1}{8} D^\alpha \bar{D}^2 D_\alpha \Big) V + \\ + \sqrt{2} V \Big(\partial_I \bar{S}^I + \bar{\partial}_I S^I \Big) + \partial_I \bar{S}^I\, \frac{1}{\Box}\, \bar{\partial}_J S^J \Big) \Big] \tag{6.14}$$

and the gauge fixed action involves an invertible quadratic operator

$$\mathscr{S}_{\text{V2}} + \mathscr{S}_{\text{gf}} = \int d^{10}x\, d^4\theta\, \mathrm{tr} \Big[-V \big(\Box + \partial_I \bar{\partial}^I \big) V + \bar{S}_I\, \frac{\Box\, \delta^{IJ} + \partial^I \bar{\partial}^J}{\Box}\, S_J + \\ + \frac{1}{2}\, \epsilon^{IJK} \Big(S_I \partial_J\, \frac{D^2}{-4\Box}\, S_K + \bar{S}_I \bar{\partial}_J \frac{\bar{D}^2}{-4\Box}\, \bar{S}_K \Big) \Big]. \tag{6.15}$$

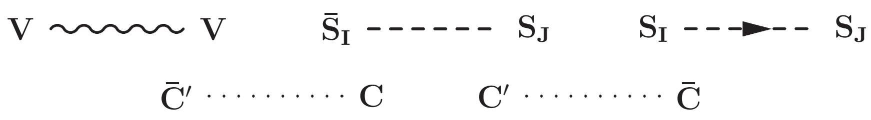


Figure 19: Only in ten dimensions there is a propagator that connects two chiral adjoint superfields S corresponding to off-diagonal elements in the quadratic form $\mathbf{A}$.

Hence the propagator of the four dimensional gauge multiplet V is decoupled from the propagators of the three chiral adjoint fields S_I

$$\begin{aligned} & Z_0\big[J_V, \big(J_S\big)_I\big] \\ & \quad = \exp\left\{ i \int d^{10}x\, d^4\theta\, \mathrm{tr}\left[\, \frac{1}{4}\, J_V \, \frac{1}{\Box + \partial_L \bar{\partial}^L}\, J_V - \big(\bar{J}_S\big)_I \, \frac{\delta^{IJ}}{\Box + \partial_L \bar{\partial}^L} \big(J_S\big)_J + \right.\right. \\ & \qquad \left.\left. - \big(J_S\big)_I \, \frac{\frac{1}{2}\,\epsilon^{IJK} \bar{\partial}_K}{\Box + \partial_L \bar{\partial}^L}\, \frac{D^2}{-4\Box} \, \big(J_S\big)_J - \big(\bar{J}_S\big)_I \, \frac{\frac{1}{2}\,\epsilon^{IJK} \partial_K}{\Box + \partial_L \bar{\partial}^L}\, \frac{\bar{D}^2}{-4\Box} \, \big(\bar{J}_S\big)_J \right]\right\}. \end{aligned} \tag{6.16}$$

The first line is the direct generalization of the propagators in the six dimensional non-Abelian theory (5.39). But the second line which is due to ten dimensional Lorentz invariance contains new propagators that connect chiral adjoint fields S_I with the same chirality which have no analog in six dimensions (and obviously vanish when not all three S_I fields are present). The propagators are depicted in Fig. 19.

Ghost superfield propagators

What remains is the determination of the ten dimensional ghost propagators. To this end it is necessary to consider the infinitesimal version of the ten dimensional transformation law (6.6) for the three superfields S_I

$$\delta_\Lambda S_I = \sqrt{2}\, \partial_I \Lambda + 2\big[S_I, \Lambda\big]. \tag{6.17}$$

This determines the infinitesimal gauge variation δ_C of the gauge fixing functional

$$\begin{aligned} \delta_C \Theta = \sqrt{2}\, \frac{\bar{D}^2}{-4} \Big(& L_V(C - \bar{C}) + \coth\big(L_V(L_V(C + \bar{C}))\big) + \\ & + \frac{1}{\Box}\, \partial_I \big(\sqrt{2}\, \bar{\partial}^I \bar{C} - 2\big[\bar{S}^I, \bar{C}\big]\big) \Big), \end{aligned} \tag{6.18}$$

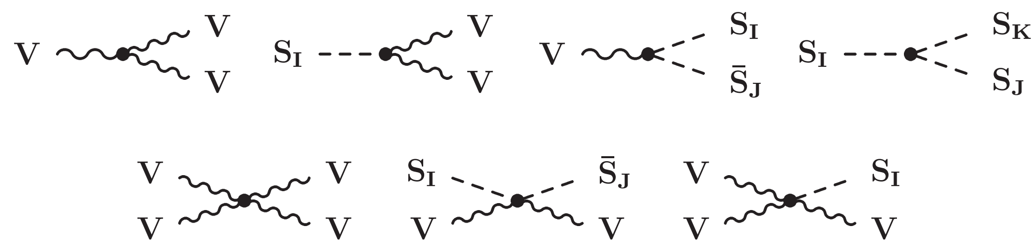


Figure 20: The ten dimensional theory contains a vertex involving three chiral adjoint superfields S.

which enters in the definition of the ghost action. The ten dimensional ghost action is a straightforward generalization of its six dimensional counterpart (5.42)

$$\mathscr{S}_{gh} = \frac{1}{\sqrt{2}} \int d^{10}x\, d^4\theta\, \mathrm{tr}\Big[\sqrt{2}\,(C' + \bar{C}')\Big(L_V(C - \bar{C}) + \coth\big(L_V(L_V(C + \bar{C}))\big)\Big)$$
$$+ C'\,\frac{\partial_I}{\Box}\Big(\sqrt{2}\,\bar{\partial}^I \bar{C} - 2\big[\bar{S}^I, \bar{C}\big]\Big) + \bar{C}'\,\frac{\bar{\partial}_I}{\Box}\Big(\sqrt{2}\,\partial^I C + 2\big[S^I, C\big]\Big)\Big]. \quad (6.19)$$

Therefore, the ten dimensional ghost propagators are as simple as in six dimensions (5.43)

$$Z_0\big[J_C, J_C'\big] = \exp\left\{ i \int d^{10}x\, d^4\theta\, \mathrm{tr}\Big[-\bar{J}_C'\,\frac{1}{\Box + \partial_I \bar{\partial}^I}\, J_C - J_C'\,\frac{1}{\Box + \partial_I \bar{\partial}^I}\, \bar{J}_C \Big] \right\}. \quad (6.20)$$

The drawing conventions for the ghost propagators are given in Fig. 19.

Interactions

We expand the gauge multiplet action (6.5) to the fourth order in the fields to obtain the interactions between the four dimensional gauge multiplet V and the three chiral adjoint multiplets S_I

$$\begin{aligned}\mathscr{S}_{V\mathrm{int}} \supset \int d^{10}x\, d^4\theta\, \mathrm{tr}\Big[&\frac{1}{4}\,\big[V,D^\alpha V\big]\bar{D}^2 D_\alpha V - \frac{1}{8}\,\big[V,D^\alpha V\big]\bar{D}^2\big[V,D_\alpha V\big] + \\ &-\frac{1}{6}\,\big[V,\big[V,D^\alpha V\big]\big]\bar{D}^2 D_\alpha V + \sqrt{2}\,\partial_I V\big[V,\bar{S}^I\big] - \sqrt{2}\,\bar{\partial}_I V\big[V,S^I\big] + \\ &-2\,S_I\big[V,\bar{S}^I\big] + \frac{1}{3}\,\bar{\partial}_I V\big[V,\big[V,\partial^I V\big]\big] + \\ &-\frac{2\sqrt{2}}{3}\,\partial_I V\big[V,\big[V,\bar{S}^I\big]\big] - \frac{2\sqrt{2}}{3}\,\bar{\partial}_I V\big[V,\big[V,S^I\big]\big] + 2\,S_I\big[V,\big[V,\bar{S}^I\big]\big] + \\ &+\frac{1}{3\sqrt{2}}\,\epsilon^{IJK}\,\frac{D^2}{-4}\,S_I\big[S_J,S_K\big] - \frac{1}{3\sqrt{2}}\,\epsilon^{IJK}\,\frac{\bar{D}^2}{-4}\,\bar{S}_I\big[\bar{S}_J,\bar{S}_K\big]\Big]. \end{aligned} \tag{6.21}$$

Most of the interactions are generalizations of the six dimensional case. But only in ten dimensions there is a three point vertex that connects three chiral adjoint fields S_I of the same chirality (and its hermitean conjugate) in the last line of (6.21). The interactions are depicted in Fig. 20.

The ghost action (6.19) is expanded in order to find the interactions that connect the four dimensional gauge multiplet V and the three chiral adjoint multiplets S_I with the ghost multiplets C and C' which we depict in Fig. 21

$$\begin{aligned}\mathscr{S}_{gh\,\mathrm{int}} \supset \int d^{10}x\, d^4\theta\, \mathrm{tr}\Big[&\big(C'+\bar{C}'\big)\big[V,C-\bar{C}\big] + \frac{1}{3}\,\big(C'+\bar{C}'\big)\big[V,\big[V,C+\bar{C}\big]\big] \\ &+\sqrt{2}\,\frac{\partial_I}{\Box}\,C'\big[\bar{S}^I,\bar{C}\big] - \sqrt{2}\,\frac{\bar{\partial}^I}{\Box}\,\bar{C}'\big[S,C\big]\Big]. \end{aligned} \tag{6.22}$$

In principle there is also a three point self-interaction of the gauge multiplet from the WZW term which we mentioned above. But in the same way as in the six dimensional calculation all graphs that can be constructed from this interaction vanish due to the symmetry of the structure constants. Therefore, we do not consider this interaction here.

6.2.3 Orbifold compatible calculus for $T^6/\mathbb{Z}_N$

Now we extend our orbifold compatible calculus such that it covers the ten dimensional non-Abelian theory. The four dimensional gauge multiplet V and the three chiral adjoint fields S_I have to be covariant w.r.t. the $\mathbb{Z}_N$ orbifold action. Hence, their transformation behaviour under the orbifold twist $z_I \to e^{-i\varphi_I} z_I$ is found to be

$$V \to Z\,V\,\bar{Z}, \qquad S_I \to e^{i\varphi_I} Z S_I Z^{-1}, \qquad \bar{S}_I \to e^{-i\varphi_I}\bar{Z}^{-1}\bar{S}_I\bar{Z} \tag{6.23}$$

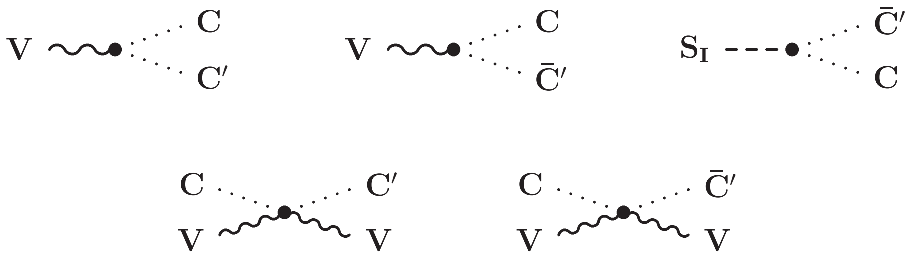


Figure 21: Those vertices that connect the four dimensional gauge multiplet V with the ghost multiplets are of the same type as in lower dimensions. A change occurs in the upper right vertex which connects the ghosts to the family of three chiral adjoint multiplets S_I.

with the properties $Z^N = 1$, because the transformations are $\mathbb{Z}_N$ actions. Invariance of the action requires in addition that the matrix Z be unitary and that the sum of the phases fulfills

$$\sum_I \varphi_I = 0 \bmod 2\pi. \tag{6.24}$$

The transformation rules for the V and S superfields with the adjoint indices made explicit read

$$V^i \to Q^i{}_j\, V^j, \qquad S_I^i \to e^{i\varphi_I}\, Q^i{}_j\, S_I^j, \qquad Q^i{}_j = \mathrm{tr}\big[T^i Z T_j \bar{Z}\big]. \tag{6.25}$$

This implies that the matrix Q has the same properties as in the six dimensional non-Abelian theory considered in Section 5.2.3. To obtain the orbifold compatible delta functions for the various superfields, we write down the transformation behaviour of orbifold compatible sources under $z_I \to e^{-i\varphi_I} z_I$

$$J_V{}^i \to Q^i{}_j\, J_V{}^j\,, \qquad \big(J_S\big)_I^i \to e^{-i\varphi_I}\, Q^i{}_j\, \big(J_S\big)_I^j, \tag{6.26}$$

where the orthogonality property of Q in (4.51) has been used. The orbifold compatible delta functions arise from functional differentiation w.r.t. the source terms as

$$\frac{\delta\big(J_V\big)_2{}^i}{\delta\big(J_V\big)_1{}^j} = \tilde{\delta}_{21}^{(V)i}{}_j\,, \quad \frac{\delta\big(J_S\big)_2{}^{iI}}{\delta\big(J_S\big)_1{}^{jJ}} = \frac{\bar{D}^2}{-4}\,\tilde{\delta}_{21}^{(S_I)i}{}_j\,\delta^I_J, \quad \frac{\delta\big(\bar{J}_S\big)_2{}^{iI}}{\delta\big(\bar{J}_S\big)_1{}^{jJ}} = \frac{D^2}{-4}\,\tilde{\delta}_{21}^{(\bar{S}_I)i}{}_j\,\delta^I_J, \tag{6.27}$$

where the orbifold compatible delta functions in (6.27) are defined for the $T^6/\mathbb{Z}_N$ orbifold as

$$\tilde{\delta}_{21}^{(V)i}{}_j = \frac{1}{N}\sum_{b=0}^{N-1}\big[Q^{-b}\big]^i{}_j \prod_{J=1}^{3}\delta^2\big(z_{2J} - e^{ib\varphi_J}z_{1J}\big)\,\delta^4\big(x_2 - x_1\big)\,\delta^4\big(\theta_2 - \theta_1\big),$$
$$\tilde{\delta}_{21}^{(S_I)i}{}_j = \frac{1}{N}\sum_{b=0}^{N-1} e^{ib\varphi_I}\big[Q^{-b}\big]^i{}_j \prod_{J=1}^{3}\delta^2\big(z_{2J} - e^{ib\varphi_J}z_{1J}\big)\,\delta^4\big(x_2 - x_1\big)\,\delta^4\big(\theta_2 - \theta_1\big),$$
$$\tilde{\delta}_{21}^{(\bar{S}_I)i}{}_j = \frac{1}{N}\sum_{b=0}^{N-1} e^{-ib\varphi_I}\big[Q^{-b}\big]^i{}_j \prod_{J=1}^{3}\delta^2\big(z_{2J} - e^{ib\varphi_J}z_{1J}\big)\,\delta^4\big(x_2 - x_1\big)\,\delta^4\big(\theta_2 - \theta_1\big).$$
$$(6.28)$$

We check the orbifold compatibility: Under $z_{2J} \to e^{-i\varphi_J}z_{2J}$ these delta functions transform in the same way as the corresponding sources in (6.27)

$$\tilde{\delta}_{21}^{(V)i}{}_j \to Q^i{}_{i'}\,\tilde{\delta}_{21}^{(V)i'}{}_j\,,$$
$$\tilde{\delta}_{21}^{(S_I)i}{}_j \to e^{-i\varphi_I}\,Q^i{}_{i'}\,\tilde{\delta}_{21}^{(S)i'}{}_j\,, \qquad \tilde{\delta}_{21}^{(\bar{S}_I)i}{}_j \to e^{i\varphi_I}\,Q^i{}_{i'}\,\tilde{\delta}_{21}^{(S)i'}{}_j \tag{6.29}$$

and under $z_{1J} \to e^{-i\varphi}z_{1J}$ they transform as the inverse of the corresponding sources in (6.27), such that these delta functions are indeed orbifold compatible

$$\tilde{\delta}_{21}^{(V)i}{}_j \to \big[Q^{-1}\big]^i{}_{i'}\,\tilde{\delta}_{21}^{(V)i'}{}_j\,,$$
$$\tilde{\delta}_{21}^{(S_I)i}{}_j \to e^{i\varphi_I}\big[Q^{-1}\big]^i{}_{i'}\,\tilde{\delta}_{21}^{(S_I)i'}{}_j, \qquad \tilde{\delta}_{21}^{(\bar{S}_I)i}{}_j \to e^{-i\varphi_I}\big[Q^{-1}\big]^i{}_{i'}\,\tilde{\delta}_{21}^{(S_I)i'}{}_j. \tag{6.30}$$

As the properties of the Q matrix are the same as in six dimensions the exchange of the spacetime labels results in

$$\tilde{\delta}_{12}^{(V)i}{}_j = \tilde{\delta}_{21}^{(V)}{}_j{}^i\,, \qquad \tilde{\delta}_{12}^{(S_I)i}{}_j = \tilde{\delta}_{21}^{(\bar{S}_I)}{}_j{}^i, \tag{6.31}$$

because Z is unitary and Q is orthonormal. Derivatives with respect to the compactified coordinates always act on the $\delta^2\big(z_{2J} - e^{ib\varphi_J}z_{1J}\big)$ factor. Therefore, changing a spacetime index of such a derivative also changes the type of delta function as

$$\partial_{2I}\,\tilde{\delta}_{21}^{(S_I)i}{}_j = -\partial_{1I}\,\tilde{\delta}_{21}^{(V)i}{}_j\,, \qquad \partial_{2I}\,\tilde{\delta}_{21}^{(V)i}{}_j = -\partial_{1I}\,\tilde{\delta}_{21}^{(\bar{S}_I)i}{}_j. \tag{6.32}$$

This completes the discussion of the supersymmetric non-Abelian field theory in ten dimensions with the extra six dimensions compactified on the orbifold $T^2/\mathbb{Z}_N$. We are now ready to perform loop calculations in this theory.

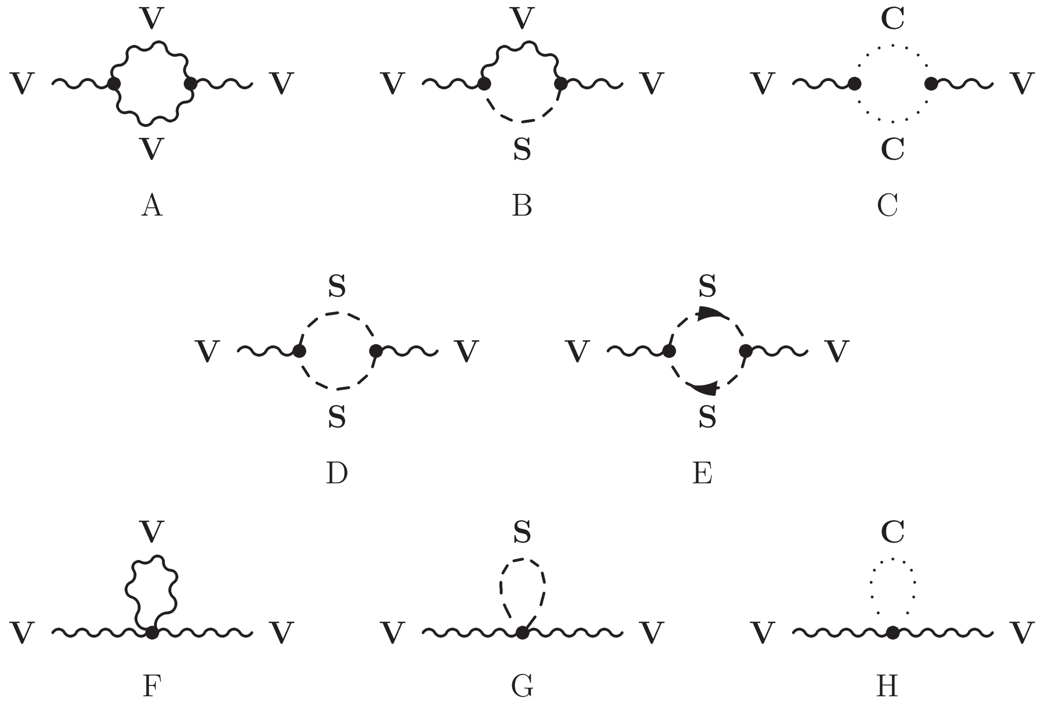


Figure 22: Graphs that contribute to the Σ_{VV} self-energy in ten dimensions. Most of the graphs are straightforward extensions of the graphs that were calculated in lower dimensions. A graph that can only be constructed in the ten dimensional theory, however, is the graph labelled 22.E.

6.2.4 Renormalization of the gauge kinetic term due to the gauge multiplet

In this section we calculate loop corrections to the gauge kinetic term in the non-Abelian theory in ten dimensions where the extra six dimensions are compactified on the orbifold $T^6/\mathbb{Z}_N$. As we already remarked below (6.5) we will not perform the complete renormalization for all operators in the gauge kinetic term, but instead restrict ourselves to calculate the Σ_{VV} self-energy which renormalizes the first two operators in (6.5) that contain only the four dimensional gauge multiplet V. The set of graphs that is relevant for this caclulation can be found in Fig. 22. Many of these graphs have counterparts that we considered in the analysis in lower dimensions in Sections 4.2.5 and 5.2.5 and their generalization to ten dimensions is straightforward. One graph, however, only exists in the ten dimensional theory. This is the graph 22.E that involves the new propagators which connect chiral adjoint fields of the same chirality in (6.16). The results for all graphs from Fig. 22 are provided in App. C.3 on the level of two orbifold compatible delta functions. The sum of the graphs defines the Σ_{VV} self-energy

$$\Sigma_{VV} = 3\, f_{ijk} f_{\ell mn} \int \left(d^{10}x\right)_{12} d^4\theta \Big(-V_1^i \Box_2 P_0 V_2^\ell \Big) \frac{1}{\boldsymbol{\Box}_2} \widetilde{\delta}_{21}^{10(V)mj} \frac{1}{\boldsymbol{\Box}_2} \widetilde{\delta}_{21}^{10(V)nk}$$

$$- f_{ijk} f_{\ell mn} \sum_I \int \left(d^{10}x\right)_{12} d^4\theta \Big(-V_1^i \Box_2 P_0 V_2^\ell \Big) \frac{1}{\boldsymbol{\Box}_2} \widetilde{\delta}_{21}^{10(\bar{S}_I)mj} \frac{1}{\boldsymbol{\Box}_2} \widetilde{\delta}_{21}^{10(S_I)nk}$$

$$+ 2\, f_{ijk} f_{\ell mn} \int \left(d^{10}x\right)_{12} d^4\theta\, \partial_I V_1^i\, \bar{\partial}^I V_2^\ell \frac{1}{\boldsymbol{\Box}_2} \widetilde{\delta}_{21}^{10(V)mj} \frac{1}{\boldsymbol{\Box}_2} \widetilde{\delta}_{21}^{10(\bar{S}_I)nk}$$

$$- f_{ijk} f_{\ell mn} \sum_{IKCF} \epsilon^{IKC} \epsilon^{IKF} \int \left(d^{10}x\right)_{12} d^4\theta\, \partial_C V_1^i\, \bar{\partial}^F V_2^\ell \frac{1}{\boldsymbol{\Box}_2} \widetilde{\delta}_{21}^{10(S_I)mj} \frac{1}{\boldsymbol{\Box}_2} \widetilde{\delta}_{21}^{10(S_K)nk}, \tag{6.33}$$

where we use the bold box to denote the ten dimensional d'Alembert operator $\boldsymbol{\Box} = \Box + \partial_M \bar{\partial}^M$. Now we reduce one more orbifold compatible delta function to an ordinary delta function and obtain the results in the bulk and at the fixed points.

Bulk renormalization

The bulk result is obtained for the $b = 0$ contribution in the expansion of the last orbifold compatible delta function (6.28). The bulk contribution of the self-energy vanishes identically and there is no renormalization at all. This result was to be expected, since it is known that there is no gauge coupling renormalization in an $\mathcal{N} = 1$ supersymmetric theory in ten dimensions. Clearly, the presence of the new graph 22.E that is constructed from the new propagators specific to ten dimensions is indespensable for this result to hold. Moreover, we note that no higher dimensional operator is generated in the bulk as loop counterterm.

Fixed point self-energy

The other terms with $b \neq 0$ in the expansion of the last orbifold compatible delta function constitute the self-energy contributions at the fixed points and planes of the $T^6/\mathbb{Z}_N$ orbifold

$$\Sigma_{VV,\,\mathrm{fp}} =$$

$$-\sum_{b=0}^{N-1} P_1(b) \int \left(d^{10}x\right)_{12} d^4\theta\, \mathrm{tr}_{\mathbf{Ad}} \Big[-Q^{-b}\, V_1 \Box_2 P_0 V_2 \Big] \frac{1}{\boldsymbol{\Box}_2} \delta_{21} \frac{1}{\boldsymbol{\Box}_2} \prod_{J=1}^{3} \delta^2\big(z_{2J} - e^{ib\varphi_J} z_{1J}\big)$$

$$-\sum_{b=0}^{N-1} \sum_{I=1}^{3} P_2(b) \int \left(d^{10}x\right)_{12} d^4\theta\, \mathrm{tr}_{\mathbf{Ad}} \Big[Q^{-b}\, \partial_I V_1 \bar{\partial}^I V_2 \Big] \frac{1}{\boldsymbol{\Box}_2} \delta_{21} \frac{1}{\boldsymbol{\Box}_2} \prod_{J=1}^{3} \delta^2\big(z_{2J} - e^{ib\varphi_J} z_{1J}\big), \tag{6.34}$$

where we have only indicated the delta functions in the compact dimensions for notational simplicity. The quantities that involve the phase factors are defined as

$$P_1(b) = \frac{1}{N}\left(3 - \frac{1}{2}\sum_{J=1}^{3}\left(e^{ib\varphi_I} + e^{-ib\varphi_I}\right)\right), \quad P_2(b) = \frac{1}{N}\left(1 + e^{-ib\varphi_I} - \sum_{J\neq I} e^{ib\varphi_J}\right). \tag{6.35}$$

Because a $T^6/\mathbb{Z}_N$ orbifold has a quite complicated geometry with fixed points and fixed planes, the exact divergence structure of the self-energy depends on the respective orbifold under consideration, represented by the orbifold twist. Therefore, we will now consider a specific example with a definite twist.

6.2.5 Renormalization of the gauge kinetic term due to the gauge multiplet on $T^6/\mathbb{Z}_4$

As an example we specify to the orbifold $T^6/\mathbb{Z}_4$. For details on the geometry of this orbifold we refer the reader to [48, 77]. For this orbifold the twist is given by

$$\varphi = 2\pi\left(\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}\right), \tag{6.36}$$

thus fulfilling the constraint on the phases (6.24). This means that in the second twisted sector there is effectively no orbifold action on the first complex torus and the twist corresponds to a $\mathbb{Z}_2$ orbifold action in the remaining two tori

$$2\varphi = 2\pi\left(0, -\frac{1}{2}, -\frac{1}{2}\right). \tag{6.37}$$

Hence, in the $T^6/\mathbb{Z}_4$ orbifold we have to distinguish six dimensional contributions which are localized at the fixed torus in the first complex plane for $b = 2$ in (6.34) and contributions which are localized at the four dimensional fixed points in the $b = 1, 3$ sector from (6.34). The six dimensional fixed torus supports a six dimensional amplitude whose divergence is the same as that of the bulk contribution in the six dimensional calcuation discussed in Chapter 5. The amplitude at the fixed points gives rise to the usual four dimensional divergence.

Fixed torus renormalization

The counterterm that is localized at the six dimensional fixed torus in the first complex plane for $b = 2$ is a sum of a counterterm to a gauge kinetic term of a six dimensional gauge multiplet and a higher derivative operator

$$\Delta\mathscr{S}^{\text{gauge}}_{\text{torus}} = \Delta\mathscr{S}^{\text{gauge}}_{\text{gkt}} + \Delta\mathscr{S}^{\text{gauge}}_{\text{HDO}}. \tag{6.38}$$

The counterterm to the gauge kinetic term of the six dimensional gauge multiplet is given by

$$\Delta\mathscr{S}^{\text{gauge}}_{\text{gkt}} = \frac{m^2\mu^{-2\epsilon}}{(4\pi)^3\epsilon}\int d^dx\,\Big\{\int d^2\theta\,\text{tr}_{\mathbf{Ad}}\Big[\frac{1}{4}\,Q^{-2}W^\alpha W_\alpha\Big] + \text{h.c.} +$$
$$+\int d^4\theta\,\text{tr}_{\mathbf{Ad}}\Big[Q^{-2}\Big(\Big(-\tfrac{1}{\sqrt{2}}\,\partial_1 + S_1\Big)\,e^{2V}\Big(\tfrac{1}{\sqrt{2}}\,\bar\partial_1 + \bar S_1\Big)\,e^{-2V} + \frac{1}{4}\,\partial_1 e^{-2V}\bar\partial_1 e^{2V}\Big)\Big]\Big\}$$
$$\times\,\delta^2\big((1-e^{i2\varphi_2})z_2\big)\,\delta^2\big((1-e^{i2\varphi_3})z_3\big). \qquad (6.39)$$

Here we have displayed the full counterterm that is proportional to the action of a six dimensional gauge multiplet that lives on the first torus. Notice that we have only calculated explicitly those terms that involve only the four dimensional gauge multiplet and its derivatives. In order for the theory to reproduce the complete action for the six dimensional gauge multiplet, also the terms that involve one of the three chiral adjoint multiplets S and its chiral conjugate $\bar S$ have to be present (the one with the same index as the derivative operator). This observation is strengthened by our experience from the last chapters where we saw that the $\partial V\bar\partial V$ operator always renormalizes in the same way as the terms that involved the associated chiral adjoint field S and its chiral conjugate. In addition, a higher derivative operator is localized at the fixed torus

$$\Delta\mathscr{S}^{\text{gauge}}_{\text{HDO}} = -\frac{\mu^{-2\epsilon}}{6(4\pi)^3\epsilon}\int d^dx\,\Big\{\int d^2\theta\,\text{tr}_{\mathbf{Ad}}\Big[\frac{1}{4}\,Q^{-2}W^\alpha(\Box + \partial_1\bar\partial_1)W_\alpha\Big] + \text{h.c.} +$$
$$+\int d^4\theta\,\text{tr}_{\mathbf{Ad}}\Big[Q^{-2}\Big(\Big(-\tfrac{1}{\sqrt{2}}\,\partial_1 + S_1\Big)\,e^{2V}\big(\Box + \partial_1\bar\partial_1\big)\Big(\tfrac{1}{\sqrt{2}}\,\bar\partial_1 + \bar S_1\Big)\,e^{-2V} +$$
$$+\frac{1}{4}\,\partial_1\,e^{-2V}\big(\Box + \partial_1\bar\partial_1\big)\,\bar\partial_1\,e^{2V}\Big)\Big]\Big\}\;\delta^2\big((1-e^{i2\varphi_2})z_2\big)\,\delta^2\big((1-e^{i2\varphi_3})z_3\big), \quad (6.40)$$

where $d = 10 - 2\epsilon$. The gauge coupling running of the six dimensional gauge coupling of the theory that is localized at the fixed torus can be read off from (6.39) after the matrix Q has been specified.

Fixed points renormalization

The delta functions $\prod_{J=1}^3 \delta^2\big((1-e^{i\varphi_J})z_J\big)$ and $\prod_{J=1}^3 \delta^2\big((1-e^{-i\varphi_J})z_J\big)$ describe the same fixed point set, so we add the contributions to $b = 1$ and $b = 3$ to find the counterterm at the four dimensional fixed points

$$\Delta\mathscr{S}_{\mathrm{fp}}^{\mathrm{gauge}} = \frac{\mu^{-2\epsilon}}{(4\pi)^2\epsilon}\int d^dx \left\{ \int d^2\theta \, \mathrm{tr}_{\mathbf{Ad}}\left[\frac{1}{4}\,Q^{-1}W^\alpha W_\alpha\right] + \mathrm{h.c.}\right\} \tag{6.41}$$

$$\prod_{J=1}^{3}\delta^2\big((1-e^{i\varphi_J})z_J\big). \tag{6.42}$$

Here we give only the counterterm to the four dimensional gauge multiplet which is important for the running of the four dimensional gauge coupling of the theory that lives at the fixed points. The divergent self-energy piece that involves the derivatives of the V field has a more complicated structure as we can see from (6.34). The determination of this self-energy piece is straightforward, but in order to combine it to a well-defined counterterm, we would have to know the contributions from the chiral adjoint multiplets S_I, too. In any case the running of the four dimensional gauge coupling of the theory that lives at the fixed points can be read off from the counterterm (6.41) once the matrix Q has been specified.

Chapter 7

Summary and Outlook

This thesis focused on the renormalization of supersymmetric gauge theories in higher dimensions where the extra space is compactified on an orbifold. Feynman's path integral calculus was extended to fields that respect the orbifold boundary conditions in order to provide a convenient general framework for loop calculations in field theories on orbifolds. This method was centered around the notion of orbifold compatible delta functions which arise from functional differentiation and in which the information on the geometry of the orbifold is encoded. The orbifold compatible calculus was presented in Sections 4.1.3 and 4.2.3 for the Abelian and non-Abelian theory of a hyper and a gauge multiplet in five dimensions with the extra dimension compactified on $S^1/\mathbb{Z}_2$, in Sections 5.1.3 and 5.2.3 for the Abelian and non-Abelian theory with the same field content in six dimensions with the extra two dimensions compactified on $T^2/\mathbb{Z}_N$ and in Section 6.2.3 for the non-Abelian theory of a gauge multiplet in ten dimensions with the extra six dimensions compactified on $T^6/\mathbb{Z}_N$.

The orbifold compatible calculus supported an analysis that focused on the renormalization of the gauge kinetic term. We showed that the renormalization of a theory in a higher dimensional non-trivial spacetime exhibits a rich structure with three principal effects: Besides the fact that loop corrections lead to the ordinary renormalization of the bulk gauge kinetic term, the introduction of both localized gauge kinetic terms at the fixed points or planes of the orbifold and higher dimensional operators may be required as loop counterterms.

In Section 4.1.4 it was shown that the loop corrections to the gauge kinetic term due to a bulk hyper multiplet in the Abelian five dimensional theory with the fifth dimension compactified on $S^1/\mathbb{Z}_2$ only give rise to a linearly divergent counterterm in the bulk. No localized gauge kinetic terms have to be introduced as counterterms and no higher dimensional operator is generated. For the non-Abelian theory on the same orbifold the calculation in Section 4.2.4 showed that the hyper multiplet induces the same linear divergence in the bulk as in the Abelian theory. At the fixed points some operators from the action of the five dimensional gauge multiplet are renormalized, but not the gauge kinetic term of the four dimensional gauge multiplet. According to our calculation in Section 4.2.5 the gauge and ghost multiplets also induce a linearly divergent counterterm in the bulk. It has the opposite sign as compared to the hyper multiplet contribution. The loop corrections of the gauge and ghost multiplets at the fixed points are such that from the action of the five dimensional gauge multiplet only the gauge kinetic term of the four dimensional gauge multiplet is renormalized. No higher dimensional operators are generated in the five dimensional non-Abelian theory.

In Section 5.1.4 it was demonstrated that the loop corrections to the gauge kinetic term due to a bulk hyper multiplet in the Abelian six dimensional theory with the extra two dimensions compactified on $T^2/\mathbb{Z}_N$ lead to a higher dimensional operator in the bulk. This higher dimensional operator has to be introduced as a loop counterterm in addition to the ordinary six dimensional bulk counterterm to the gauge kinetic term. At the fixed points which form a $\mathbb{Z}_2$ subset of an even ordered orbifold no localized renormalization effects appear in generalization of the five dimensional result. At the non-$\mathbb{Z}_2$ fixed points we find an ordinary four dimensional counterterm. Also in the non-Abelian calculation on the same orbifold performed in Section 5.2.4 the loop corrections due to the hyper multiplet generate a higher dimensional operator in the bulk in addition to the ordinary six dimensional counterterm. At the $\mathbb{Z}_2$ fixed points some of the operators of the six dimensional gauge kinetic term are renormalized, but not the gauge kinetic term of the four dimensional gauge multiplet. At the non-$\mathbb{Z}_2$ fixed points we obtain the usual four dimensional counterterm. As we describe in Section 5.2.5 the gauge and ghosts multiplets induce a higher dimensional operator in the bulk as loop counterterm in addition to the counterm to the six dimensional gauge kinetic term. At the $\mathbb{Z}_2$ fixed points only the gauge kinetic term of the four dimensional gauge multiplet is renormalized. The non-$\mathbb{Z}_2$ fixed points support an ordinary four dimensional counterterm.

In Section 6.2.4 we calculated the loop corrections due to the self-interactions of a non-Abelian gauge multiplet in the ten dimensional theory where the extra six dimensions are compactified on $T^6/\mathbb{Z}_N$. We checked with our calculation the fact that the loop corrections to the bulk gauge kinetic term vanish. Moreover, our analysis revealed that no higher derivative operator is generated in the ten dimensional bulk and we derived the divergent self-energies at the fixed points and planes in general. To calculate the divergences and counterterms explicitly, we specified to the situation that the extra six dimensions are compactified on $T^6/\mathbb{Z}_4$ in Section 6.2.5. We showed that on the six dimensional fixed torus a six dimensional higher derivative operator has to be introduced as loop counterterm in addition to an ordinary six dimensional counterterm. The four dimensional fixed points of the orbifold support a counterterm for the kinetic term of the four dimensional gauge multiplet.

The general results presented in this thesis are ready for application once a concrete orbifold is specified and the local spectra on the fixed points/planes are known. In particular, they can be applied to obtain the local gauge coupling running in orbifolds whose spectra are generated from a heterotic string orbifold construction without Wilson lines like the $T^6/\mathbb{Z}_4$ spectra considered in [48]. Realistic 'local GUT' string theory models, however, associate Wilson lines to the fixed points in order to provide different phenomenologically promising GUT gauge groups at different fixed points. In order to address the renormalization of these 'local GUT' string models our work can be straightforwardly extended to include Wilson lines. There are two possible paths one can follow: Wilson lines correspond to non-trivial periodicity conditions of the fields under torus lattice shifts and so the periodicity conditions of the fields and their sources have to be modified. This would be reflected in a change of the orbifold compatible delta functions. The second possibility is to take advantage of the fact that a field that transforms non-trivially under a torus lattice shift can be expressed as a periodic field that is minimally coupled to a constant background gauge field. In this picture the inclusion of Wilson lines requires a minimal modification of the field content to include these background gauge fields.

Another point is that in this investigation we took only part of the renormalization into account. The fact that higher dimensional operators and localized gauge kinetic terms are generated as loop counterterms means that they cannot be set to zero at the tree level and should be considered right from the start in a fully consistent analysis [78]. The full renormalization of the theory should include a discussion of all higher dimensional operators. For example, the operators that can be constructed from a gauge field strength F in six dimensions involve besides FF and $F\Box F$ also terms like F^3 where the indices are contracted in all possible ways. In ten dimensions there are even more complicated structures, because higher dimensional operators like $F^2\Box F$ and $F\Box^k F$ with k up to the value three are allowed. Only the analysis of the full renormalization could provide an answer to the question how many free parameters the theory actually contains.

In principle, string theory predicts these higher dimensional operators as α'-corrections at the tree level. However, explicit loop calculations that are required to infer the renormalization of the tree level coupling constants are very difficult. Calculations might be performed at the one-loop level in string theory, but two loop calculations are technically very hard to manage. Therefore, for higher loop calculations one has to resort to field theory methods. The orbifold compatible calculus presented in this thesis represents a fully consistent framework for calculations at any loop order and seems to be an appropriate tool to address these questions.

Appendix A

Orbifolds

In this appendix we give a brief definition of orbifolds and discuss the orbifolds $S^1/\mathbb{Z}_2$ and $T^2/\mathbb{Z}_N$ in a little more detail. For the geometry of $T^6/\mathbb{Z}_4$ we refer the reader to [48, 77]. An orbifold O is defined as a quotient space of a manifold M and a discrete symmetry group P

$$O = M/P, \tag{A.1}$$

where P is called the point group. We are interested in the situation that the manifold is an n dimensional torus T^n which can be obtained from n dimensional flat space by dividing it by a lattice Λ

$$T^n = \mathbb{R}^n/\Lambda \tag{A.2}$$

Then we can write the orbifold in our case as

$$O = \mathbb{R}^n/(P \ltimes \Lambda) = \mathbb{R}^n/S \tag{A.3}$$

where the space group S is the semi-direct product of the discrete symmetry group and the torus lattice shifts. For more details we refer the reader to the original literature [40, 41].

A.1 One extra dimension compactified on the orbifold $S^1/\mathbb{Z}_2$

To describe the orbifold $S^1/\mathbb{Z}_2$, we begin by defining the circle S^1 by the identifications

$$y \sim y + \Lambda_W, \qquad \Lambda_W = 2\pi R\,\mathbb{Z}, \tag{A.4}$$

where Λ_W is the winding mode lattice. The length of the circle (the "volume" of a fundamental region of the lattice Λ_W) is equal to $\mathrm{Vol}_W = 2\pi R$. We denote the delta function on the torus by $\delta(y) = \delta_{\mathbb{R}}(y + \Lambda_W)$. The momentum in the fifth direction p^5 is quantized and takes values in the Kaluza-Klein lattice such that the five dimensional integral is defined as

$$\int \frac{d^5p}{(2\pi)^5} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\pi R} \sum_{p^5 \in \Lambda_K}, \qquad \Lambda_K = \mathbb{Z}/R. \tag{A.5}$$

The volume of the Kaluza-Klein lattice is given by $\mathrm{Vol}_K = \frac{1}{R}$.

To construct the orbifold $S^1/\mathbb{Z}_2$, we need to divide out a $\mathbb{Z}_2$ point group. We implement the $\mathbb{Z}_2$ action as a reflection $y \to -y$. This implies that the derivative in the extra dimension transforms as $\partial_5 \to -\partial_5$. The fundamental domain of the $S^1/\mathbb{Z}_2$ orbifold is the interval $[0, \pi R]$. It has two fixed points located at $y = 0$ and $y = \pi R$. The delta function that peaks at these two fixed points is given by $\delta(2y)$ and can be expanded into

$$\delta(2y) = \frac{1}{2} \left(\delta(y) + \delta(y - \pi R) \right). \tag{A.6}$$

The normalization using the number of fixed points, 2 for $S^1/\mathbb{Z}_2$, ensures that the integral of this delta function over the circle is unity.

A.2 Two extra dimensions compactified on the orbifold $T^2/\mathbb{Z}_N$

Next we consider compactification of two dimensions on the orbifold $T^2/\mathbb{Z}_N$. Because the torus T^2 is compact, the only possible values for the orbifold order N are $2, 3, 4, 6$, but we will keep our discussion general here. The torus T^2 is defined by the identifications

$$z \sim z + \Lambda_W, \qquad \Lambda_W = \pi \Big(R_1 \mathbb{Z} + R_2 e^{i\theta} \mathbb{Z} \Big). \tag{A.7}$$

Here Λ_W denotes the winding mode lattice of the torus with the volume $\mathrm{Vol}_W = (2\pi)^2 R_1 R_2 \sin\theta$, where R_1 and R_2 are the radii of the torus and θ defines its angle, i.e. $\theta = \pi/2$ gives the square torus. Inspired by the string literature, we can introduce the complex structure modulus U and the Kähler modulus T of the torus

$$\Lambda_W = \pi \sqrt{\frac{\mathrm{Im}\ (T)}{\mathrm{Im}\ (U)}} \left(\mathbb{Z} + U\mathbb{Z} \right), \quad U = \frac{R_2}{R_1} e^{i\theta}, \quad T = i R_1 R_2 \sin\theta. \tag{A.8}$$

In terms of these variables the volume of the torus reads $\mathrm{Vol}_W = (2\pi)^2\, \mathrm{Im}\ (T)$. The momenta p and $\bar{p}$ of the torus mode functions $\psi_p(z, \bar{z}) = e^{i(pz + \bar{p}\bar{z})}$ are quantized: p lies on the Kaluza-Klein lattice Λ_K (and $\bar{p}$ on the complex conjugate lattice). The six dimensional momentum integral is defined as

$$\int \frac{d^6p}{(2\pi)^6} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(2\pi)^2 \mathrm{Im}\ (T)} \sum_{p \in \Lambda_K}, \qquad \Lambda_K = \frac{i}{\sqrt{\mathrm{Im}\ (T) \mathrm{Im}\ (U)}} \left(\mathbb{Z}\bar{U} + \mathbb{Z} \right). \tag{A.9}$$

The volume of the Kaluza-Klein lattice is given by $\mathrm{Vol}_K = 1/\mathrm{Im}\ (T)$.

To define the orbifold $T^2/\mathbb{Z}_N$, we implement the $\mathbb{Z}_N$ action of the orbifold group as $z \to e^{-i\varphi} z$, with $\varphi = \frac{2\pi}{N}$. Consequently, the holomorphic derivative ∂ transforms as $\partial \to e^{i\varphi}\partial$. The delta function, that peaks at the orbifold fixed points z_f, is given by

$$\delta^2\left((1 - e^{i\varphi}) z \right) = \frac{1}{4|\sin\frac{1}{2}\varphi|^2} \sum_f \delta^2(z - z_f), \tag{A.10}$$

in terms of the torus delta function $\delta^2(z)$. The factor $4|\sin\frac{1}{2}\varphi|^2$ equals the number of fixed points of the $T^2/\mathbb{Z}_N$ orbifold.

Appendix B

Example of a supergraph calculation on an orbifold

This appendix contains a worked out example that illustrates the self-energy calculations on orbifolds that we have performed. We have chosen the VV self energy contribution due to the chiral superfield S in the five dimensional non-Abelian theory. The graph is depicted in Fig. 17.A. As a supergraph it is quite simple and therefore we can focus on the special issues of computing diagrams in the five dimensional space where the extra dimension is compactified on $S^1/\mathbb{Z}_2$. The techniques that we present here can be extended to the higher dimensional calculations.

From the graph 17.A we see that the corresponding term in the expansion of the generating functional is the quadratic term that contains the $SV\bar{S}$ interaction term twice. The interaction term is given in (4.47). To calculate the self-energy graph the S and $\bar{S}$ superfields on the external lines are replaced by the corresponding sources that act on the exponential of the propagators (4.41) as we showed in detail in Section 3.1.3. After functional differentiations we obtain the orbifold compatible delta functions for the non-Abelian theory in five dimensions (4.54). The expression for the supergraph 17.A on the orbifold reads

$$
\begin{aligned}
17.\mathrm{A} = 2\, f_{ijk} f_{\ell mn} \int \big(d^5x\, d^4\theta\big)_{1234}\, V_1^i V_2^\ell\, \tilde{\delta}^{(S)}_{31}{}_{p}{}^{j} \frac{\eta^{pp'}}{(\Box + \partial_5^2)_2} \frac{\bar{D}_2^2 D_2^2}{16} \tilde{\delta}^{(S)}_{32}{}_{p'}{}^{n} \times \\
\times\; \tilde{\delta}^{(S)}_{42}{}_{q}{}^{m} \frac{\eta^{qq'}}{(\Box + \partial_5^2)_1} \frac{\bar{D}_1^2 D_1^2}{16}\, \tilde{\delta}^{(S)}_{41}{}_{q'}{}^{k}. \qquad \text{(B.1)}
\end{aligned}
$$

Here we have used that in five dimensions there is no distinction between the orbifold delta functions for S and $\bar{S}$ as we remark below (4.54). Now we try to replace as many orbifold compatible delta functions by ordinary delta functions as possible. The strategy to replace an orbifold delta function by an ordinary one is always the same: One expands the orbifold delta function into a sum and performs a substitution such that all the summands are equal. For example, we can replace the first orbifold delta function in the final factor in the expression (B.1) for diagram 17.A. We begin by expanding the first delta function by inserting its definition

$$17.\mathrm{A} = 2\, f_{ijk} f_{\ell mn} \int \left(d^5x\, d^4\theta\right)_{1234} V_1^i V_2^\ell\, \tilde{\delta}^{(S)}_{31\ \ p}{}^{j} \frac{\eta^{pp'}}{(\Box + \partial_5^2)_2} \frac{\bar{D}_2^2 D_2^2}{16}\, \tilde{\delta}^{(S)}_{32\ \ p'}{}^{n} \times$$
$$\times\, \frac{1}{2}\Big(\delta_q^m \delta(y_4 - y_2) - Q_q{}^m \delta(y_4 + y_2)\Big) \delta^4(x_4 - x_2)\delta^4(\theta_4 - \theta_2)\, \frac{\eta^{qq'}}{(\Box + \partial_5^2)_1} \frac{\bar{D}_1^2 D_1^2}{16}\, \tilde{\delta}^{(S)}_{41\ \ q'}{}^{k}. \tag{B.2}$$

We perform the reflection $y_4 \to -y_4$ to show that

$$-Q_q{}^m \int dy_4\ \ \delta(y_4 + y_2)\, \eta^{qq'}\, \tilde{\delta}^{(S)}_{41\ \ q'}{}^{k} \ = \ \delta_q^m \int dy_4\ \ \delta(y_4 - y_2)\, \eta^{qq'}\, \tilde{\delta}^{(S)}_{41\ \ q'}{}^{k}, \tag{B.3}$$

where we have used the transformation properties (4.55) of $\tilde{\delta}^{(S)}_{41\ \ q'}{}^{k}$ and the orthogonality of Q in (4.51). Here we have not copied the propagators because they contain the operator ∂_5^2 which is invariant under this reflection. Substituting this back into the original expression, we obtain

$$17.\mathrm{A} = 2\, f_{ijk} f_{\ell mn} \int \left(d^5x\, d^4\theta\right)_{1234} V_1^i V_2^\ell \tilde{\delta}^{(S)}_{31\ \ p}{}^{j} \frac{\eta^{pp'}}{(\Box + \partial_5^2)_2} \frac{\bar{D}_2^2 D_2^2}{16}\, \tilde{\delta}^{(S)}_{32\ \ p'}{}^{n} \times$$
$$\times\ \delta(y_4 - y_2)\delta^4(x_4 - x_2)\delta^4(\theta_4 - \theta_2) \frac{1}{(\Box + \partial_5^2)_1} \frac{\bar{D}_1^2 D_1^2}{16}\, \tilde{\delta}^{(S)mk}_{41}. \tag{B.4}$$

Hence we have removed the orbifold projection on the first delta function. In the same fashion we can reduce one of the orbifold delta functions in the first factor. We choose to make the replacement

$$\tilde{\delta}^{(S)}_{31\ \ p}{}^{j} \to \delta_p^j\, \delta(y_3 - y_1)\delta^4(x_3 - x_1)\delta^4(\theta_3 - \theta_1). \tag{B.5}$$

Now we integrate over $(x,\theta)_3$ and $(x,\theta)_4$ and are left with

$$17.\mathrm{A} = 2\, f_{ijk} f_{\ell mn} \int \left(d^5x\, d^4\theta\right)_{12} V_1^i V_2^\ell\ \times$$
$$\times\ \frac{1}{(\Box + \partial_5^2)_2} \frac{\bar{D}_2^2 D_2^2}{16}\, \tilde{\delta}^{(S)nj}_{21}\ \frac{1}{(\Box + \partial_5^2)_2} \frac{D_2^2 \bar{D}_2^2}{16}\, \tilde{\delta}^{(S)mk}_{21}. \tag{B.6}$$

We have now arrived at the level of two orbifold compatible delta functions. In general one must be careful to reduce one more orbifold compatible delta function as described in Section 4.1.4. But as the remaining two orbifold delta functions are of the same type, we are on the safe side and choose to expand the second delta function

$$17.\mathrm{A} = 2\, f_{ijk} f_{\ell mn} \int \left(d^5x\, d^4\theta\right)_{12} V_1^i V_2^\ell \,\frac{1}{(\Box + \partial_5^2)_2} \frac{\bar{D}_2^2 D_2^2}{16}\, \tilde{\delta}_{21}^{(S)nj} \times$$

$$\times\, \frac{1}{(\Box + \partial_5^2)_2} \frac{D_2^2 \bar{D}_2^2}{16} \frac{1}{2} \Big(\eta^{mk} \delta(y_2 - y_1) - Q^{mk} \delta(y_2 + y_1)\Big) \delta^4(x_2 - x_1) \delta^4(\theta_2 - \theta_1). \tag{B.7}$$

Performing the transformation $y_1 \to -y_1$ one shows that

$$-f_{ijk}\, Q^{mk} \int dy_1\, V_1^i\, \delta(y_2{+}y_1)\, \tilde{\delta}_{21}^{(S)nj} = f_{ijk}\, \eta^{mk} \int dy_1\, V_1^i\, \delta(y_2{-}y_1)\, \tilde{\delta}_{21}^{(S)nj}. \tag{B.8}$$

Here we used that both the transformation of V in (4.50) and of the orbifold compatible delta function in (4.55) bring in a matrix Q. Then we applied the orthogonality property of Q in (4.51) in order to place the indices of all three Q's alike. Subsequently, we took advantage of the fact that three Q's contracted with the structure constants leave the structure constants invariant, as found in (4.51). Thus, we find

$$17.\mathrm{A} = 2\, f_{ijk} f_{\ell mn} \eta^{mk} \int \left(d^5x\, d^4\theta\right)_{12} V_1^i V_2^\ell \times$$

$$\times\, \frac{1}{(\Box + \partial_5^2)_2} \frac{\bar{D}_2^2 D_2^2}{16}\, \delta_{21}^{(S)nj}\, \frac{1}{(\Box + \partial_5^2)_2} \frac{D_2^2 \bar{D}_2^2}{16}\, \delta_{21}. \tag{B.9}$$

Hence we see that in this diagram we have been able to replace all but one orbifold compatible delta functions by ordinary delta functions. The final step in the evaluation of this diagram in the coordinate space representation is to make the expression local in the Grassmann variables. Making use of standard identities for the covariant supersymmetric derivatives, we perform the integration over θ_2

$$17.\mathrm{A} = f_{ijk} f_{\ell mn} \eta^{mk} \int (d^5x)_{12}\, d^4\theta\, \Big[- V_1^i \Box P_0 V_2^\ell\, \frac{1}{(\Box + \partial_5^2)_2}\, \widetilde{\delta}_{21}^{5(S)nj} \frac{1}{(\Box + \partial_5^2)_2}\, \delta_{21}^5$$

$$+\, 2\, V_1^i V_2^\ell\, \frac{\Box_2}{(\Box + \partial_5^2)_2}\, \widetilde{\delta}_{21}^{5(S)nj} \frac{1}{(\Box + \partial_5^2)_2}\, \delta_{21}^5 \Big]. \tag{B.10}$$

Since the expression only contains θ_1, it is local in θ_1 and we simply dropped the subscript "1" on θ. The structure of the calculation is the same in higher dimensions except for the fact that the orbifold compatible delta functions involve N summands on a $\mathbb{Z}_N$ orbifold instead of two. The result for the six dimensional counterpart to the graph from this example calculation can be found in (C.6) in App. C. One observes that the reduction of the six dimensional result to five dimensions straightforward by making use of the fact that $\tilde{\delta}^{(\bar{S})} = \tilde{\delta}^{(S)}$. Hence we refer to App. C for the expressions for the other diagrams in Fig. 17.

Appendix C

Explicit results for loop graphs on orbifolds

This appendix provides the amplitudes for the loop graphs that have been calculated. The amplitudes are presented on the level of two orbifold compatible delta functions as far as the spacetime part of the delta functions is concerned. As the Grassmann factor of the orbifold compatible delta functions does not feel the orbifold we have already integrated out the dependence on one of the Grassmann variables thus rendering the expressions local in θ, i.e. the integrands only depend on one Grassmann variable θ and all indices at the fields refer only to the spacetime variables, cf. the last line in the example given in (3.45). App. C.1 contains the results for the five and six dimensional loop graphs which involve the hyper multiplet. The five and six dimensional results for the gauge and ghost multiplet loops are presented in App. C.2. Finally, App. C.3 provides the results for the ten dimensional loop graphs due to the ten dimensional gauge and ghost multiplets.

C.1 Loops in five and six dimensions involving the hyper multiplet

Here we give the amplitudes for the loop graphs that involve the hyper multiplet in five and six dimensions. The amplitudes as they stand are taken from the non-Abelian calculation performed in six dimensions with the two extra dimensions compactified on $T^2/\mathbb{Z}_N$. In particular, this means that the expressions correspond directly to the six dimensional version of the graphs in Figs. 15 and 16 and make use of the orbifold compatible delta functions (5.50). They give rise to the self-energies (5.56)-(5.58). The Abelian result in six dimensions is obtained by formally replacing the trace 'tr' against the square of the charge q^2 and interpreting the orbifold delta funtcions as those from the Abelian calculation defined in (5.10). Then the amplitudes correspond to the six dimensional version of the graphs in Figs. 9 and 10 and give rise to the self-energies (5.15)-(5.17).

The amplitudes reduce to those in the five dimensional non-Abelian calculation when one replaces $z = y$, neglects any dependence on $\bar{z}$, changes the derivatives ∂ and $\bar{\partial}$ to ∂_5 and uses the orbifold compatible delta functions (4.54). Then the amplitudes correspond to the graphs in Figs. 15 and 16 and give rise to the self-energies (4.57)-(4.59). The five dimensional Abelian case is obtained from the five dimensional non-Abelian case by formally replacing the trace 'tr' against the square of the charge q^2 and interpreting the delta functions as those in (4.16). Then the amplitudes correspond to the graphs depicted in Figs. 9 and 10 and they give rise to the self-energies (4.57)-(4.59).

The graphs which contribute to the Σ_{VV} self-energy are

$$15.\mathrm{A}_{\pm} = -2\int\left(d^6x\right)_{12}d^4\theta \;\mathrm{tr}\Big[V_1\frac{1}{(\Box+\partial\bar{\partial})_2}\tilde{\delta}_{21}^{6(+)}\Big(\frac{1}{2}\Box P_0V_2 - V_2\Box_2\Big)\frac{1}{(\Box+\partial\bar{\partial})_2}\tilde{\tilde{\delta}}_{21}^{6(+)}$$

$$+ V_1\frac{1}{(\Box+\partial\bar{\partial})_2}\tilde{\tilde{\delta}}_{21}^{6(-)}\Big(\frac{1}{2}\Box P_0V_2 - V_2\Box_2\Big)\frac{1}{(\Box+\partial\bar{\partial})_2}\tilde{\delta}_{21}^{6(-)}\Big], \tag{C.1}$$

$$15.\mathrm{B} = 4\int\left(d^6x\right)_{12}d^4\theta \;\mathrm{tr}\Big[V_1\frac{\partial_1}{(\Box+\partial\bar{\partial})_2}\tilde{\tilde{\delta}}_{21}^{6(+)}V_2\frac{\bar{\partial}_2}{(\Box+\partial\bar{\partial})_2}\tilde{\delta}_{21}^{6(-)}\Big], \tag{C.2}$$

$$15.\mathrm{C}_{\pm} = -2\int\left(d^6x\right)_{12}d^4\theta \;\mathrm{tr}\Big[V_1V_2\,\tilde{\delta}_{21}^{6(+)}\frac{1}{(\Box+\partial\bar{\partial})_2}\tilde{\tilde{\delta}}_{21}^{6(+)}$$

$$+ V_1V_2\,\tilde{\delta}_{21}^{6(-)}\frac{1}{(\Box+\partial\bar{\partial})_2}\tilde{\tilde{\delta}}_{21}^{6(-)}\Big]. \tag{C.3}$$

The graphs which contribute to the $\Sigma_{S\bar{S}}$ and $\Sigma_{V\bar{S}}$ self-energies are

$$16.\mathrm{A}_{\pm} = 2\int\left(d^6x\right)_{12}d^4\theta \;\mathrm{tr}\Big[S_1\frac{1}{(\Box+\partial\bar{\partial})_2}\tilde{\delta}_{21}^{6(+)}\bar{S}_2\frac{1}{(\Box+\partial\bar{\partial})_2}\tilde{\delta}_{21}^{6(-)}\Big], \tag{C.4}$$

$$16.\mathrm{B}_{\pm} = -2\sqrt{2}\int\left(d^6x\right)_{12}d^4\theta \;\mathrm{tr}\Big[\partial V_1\frac{1}{(\Box+\partial\bar{\partial})_2}\tilde{\delta}_{21}^{6(+)}\bar{S}_2\frac{1}{(\Box+\partial\bar{\partial})_2}\tilde{\delta}_{21}^{6(-)}\Big]. \tag{C.5}$$

C.2 Loops in five and six dimensions involving the gauge multiplet

Here we provide the amplitudes for the loop graphs from Figs. 17 and 18 that involve the gauge and ghost multiplets in five and six dimensions. These diagrams appear only in the non-Abelian theories. As they are presented here, they are obtained in the calculation performed in six dimensions where the extra two dimensions are compactified on $T^2/\mathbb{Z}_N$. This means that the delta functions are taken to be those in (5.50) and the corresponding self-energies are given in (5.70)-(5.72).

The result for the five dimensional calculation with the extra dimension compactified on $S^1/\mathbb{Z}_2$ is obtained by replacing $z = y$, neglecting any dependence on $\bar{z}$, changing the derivatives ∂ and $\bar{\partial}$ to ∂_5 and interpreting the delta functions as those in (5.50). Then the amplitudes give rise to the self-energies (4.65)-(4.67).

The graphs which contribute to the Σ_{VV} self-energy are

$$\begin{aligned} 17.\mathrm{A} = f_{ijk} f_{\ell mn} \int (d^6x)_{12}\, d^4\theta \, \Big[V_1^i \Box P_0 V_2^\ell \, \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(\bar{S})mj} \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(S)nk} \\ - 2\, V_1^i V_2^\ell \, \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(\bar{S})mj} \frac{\Box_2}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(S)nk} \Big], \quad (C.6) \end{aligned}$$

$$\begin{aligned} 17.\mathrm{B} = \frac{1}{2} \, f_{ijk} f_{\ell mn} \int (d^6x)_{12}\, d^4\theta \; V_1^i \Box \left(P_+ + P_- - 5P_0 \right) V_2^\ell \, \times \\ \times \, \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(V)mj} \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(V)nk}, \quad (C.7) \end{aligned}$$

$$\begin{aligned} 17.\mathrm{C} = f_{ijk} f_{\ell mn} \int (d^6x)_{12}\, d^4\theta \, \Big[\partial V_1^i \bar{\partial} V_2^\ell \, \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(V)mj} \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(\bar{S})nk} \\ - \partial V_1^i V_2^\ell \, \frac{\bar{\partial}_2}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(V)mj} \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(\bar{S})nk} \\ - V_1^i \bar{\partial} V_2^\ell \, \frac{\partial_1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(V)mj} \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(\bar{S})nk} \\ + V_1^i V_2^\ell \, \frac{\partial_1 \bar{\partial}_2}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(V)mj} \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(\bar{S})nk} \Big], \quad (C.8) \end{aligned}$$

$$\begin{aligned} 17.\mathrm{D} = f_{ijk} f_{\ell mn} \int (d^6x)_{12}\, d^4\theta \, \Big[2 V_1^i V_2^\ell \, \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(V)mj} \frac{\Box_2}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(V)nk} \\ - \frac{1}{2} \, V_1^i \Box \left(P_+ + P_- + P_0 \right) V_2^\ell \, \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(V)mj} \frac{1}{(\Box + \partial\bar{\partial})_2} \, \widetilde{\delta}_{21}^{6(V)nk} \Big], \quad (C.9) \end{aligned}$$

$$17.\mathrm{E} \; = \; -\frac{1}{3} \, f_{ijk} f_{\ell mn} \, \eta^{nk} \int (d^6x)_{12}\, d^4\theta \;\; V_1^i V_2^\ell \; \widetilde{\delta}_{21}^{6(V)}{}_a{}^j \frac{\eta^{ab}}{(\Box + \partial\bar{\partial})_2} \; \widetilde{\delta}_{21}^{6(V)}{}_b{}^m, \quad (C.10)$$

$$17.\mathrm{F} = 2\, f_{ijk} f_{\ell mn}\, \eta^{nk} \int (d^6x)_{12}\, d^4\theta\; V_1^i V_2^\ell\; \widetilde{\delta}_{21}^{6(\bar{S})}{}_a{}^m\; \frac{\eta^{ab}}{(\Box + \partial\bar{\partial})_2}\; \widetilde{\delta}_{21}^{6(S)}{}_b{}^j, \tag{C.11}$$

$$17.\mathrm{G} = -\frac{2}{3}\, f_{ijk} f_{\ell mn}\, \eta^{nk} \int (d^6x)_{12}\, d^4\theta\; V_1^i V_2^\ell\; \widetilde{\delta}_{21}^{6(V)}{}_a{}^j\; \frac{\eta^{ab}}{(\Box + \partial\bar{\partial})_2}\; \widetilde{\delta}_{21}^{6(V)}{}_b{}^m. \tag{C.12}$$

The graphs which contribute to the $\Sigma_{S\bar{S}}$ and $\Sigma_{V\bar{S}}$ self-energies are

$$18.\mathrm{A} = 2\, f_{ijk} f_{\ell mn} \int (d^6x)_{12}\, d^4\theta\; S_1^i \bar{S}_2^\ell\; \frac{1}{(\Box + \partial\bar{\partial})_2}\; \widetilde{\delta}_{21}^{6(V)mj} \frac{1}{(\Box + \partial\bar{\partial})_2}\; \widetilde{\delta}_{21}^{6(\bar{S})nk}, \tag{C.13}$$

$$\begin{aligned} 18.\mathrm{B} = -\sqrt{2}\, f_{ijk} f_{\ell mn} \int (d^6x)_{12}\, d^4\theta \Big[& 2\, \partial V_1^i \bar{S}_2^\ell\; \frac{1}{(\Box + \partial\bar{\partial})_2}\; \widetilde{\delta}_{21}^{6(V)mj} \frac{1}{(\Box + \partial\bar{\partial})_2}\; \widetilde{\delta}_{21}^{6(\bar{S})nk} \\ & + V_1^i \bar{S}_2^\ell\; \frac{1}{(\Box + \partial\bar{\partial})_2}\; \widetilde{\delta}_{21}^{6(V)mj} \frac{\partial_1}{(\Box + \partial\bar{\partial})_2}\; \widetilde{\delta}_{21}^{6(\bar{S})nk} \Big], \end{aligned} \tag{C.14}$$

$$18.\mathrm{C} = \sqrt{2}\, f_{ijk} f_{\ell mn} \int (d^6x)_{12}\, d^4\theta\; V_1^i \bar{S}_2^\ell\; \frac{1}{(\Box + \partial\bar{\partial})_2}\; \widetilde{\delta}_{21}^{6(V)mj} \frac{\partial_1}{(\Box + \partial\bar{\partial})_2}\; \widetilde{\delta}_{21}^{6(\bar{S})nk}. \tag{C.15}$$

C.3 Loops in ten dimensions involving the gauge multiplet

Here we provide the amplitudes corresponding to the loop graphs in Fig. 22 which involve the gauge and ghost multiplets. These results are obtained in the calculation that is performed in ten dimensions where the extra six dimensions are compactified on the orbifold $T^6/\mathbb{Z}_N$. The delta functions are given in (6.28) and the amplitudes give rise to the Σ_{VV} self-energy in (6.33). The expressions for the graphs are calculated to be

$$\begin{aligned} 22.\mathrm{A} = f_{ijk} f_{\ell mn} \sum_I \int \big(d^{10}x\big)_{12}\, d^4\theta \Big[& V_1^i \Box P_0 V_2^\ell\; \frac{1}{\blacksquare_2}\; \widetilde{\delta}_{21}^{10(\bar{S}_I)mj} \frac{1}{\blacksquare_2}\; \widetilde{\delta}_{21}^{10(S_I)nk} \\ & - 2\, V_1^i V_2^\ell\; \frac{1}{\blacksquare_2}\; \widetilde{\delta}_{21}^{10(\bar{S}_I)mj} \frac{\Box_2}{\blacksquare_2}\; \widetilde{\delta}_{21}^{10(S_I)nk} \Big], \end{aligned} \tag{C.16}$$

$$\begin{aligned} 22.\mathrm{B} = \frac{1}{2}\, f_{ijk} f_{\ell mn} \int \big(d^{10}x\big)_{12}\, d^4\theta\; & V_1^i \Box \left(P_+ + P_- - 5\, P_0\right) V_2^\ell\; \times \\ & \times\; \frac{1}{\blacksquare_2}\; \widetilde{\delta}_{21}^{10(V)mj} \frac{1}{\blacksquare_2}\; \widetilde{\delta}_{21}^{10(V)nk}, \end{aligned} \tag{C.17}$$

$$22.\mathrm{C} = f_{ijk} f_{\ell mn} \int (d^{10}x)_{12}\, d^4\theta \, \Big[\, \partial_I V_1^i \bar{\partial}_I V_2^\ell \, \frac{1}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(V)mj} \frac{1}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(\bar{S}_I)nk}$$

$$- \partial_I V_1^i V_2^\ell \, \frac{\bar{\partial}_2^I}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(V)mj} \frac{1}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(\bar{S}_I)nk}$$

$$- V_1^i \bar{\partial}_I V_2^\ell \, \frac{\partial_1^I}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(V)mj} \frac{1}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(\bar{S}_I)nk}$$

$$+ \, V_1^i V_2^\ell \, \frac{\partial_{1I} \, \bar{\partial}_2^I}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(V)mj} \frac{1}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(\bar{S}_I)nk} \Big], \quad \text{(C.18)}$$

$$22.\mathrm{D} = f_{ijk} f_{\ell mn} \int (d^{10}x)_{12} \, d^4\theta \, \Big[\, 2 \, V_1^i V_2^\ell \, \frac{1}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(V)mj} \frac{\square_2}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(V)nk}$$

$$- \frac{1}{2} \, V_1^i \square \big(P_+ + P_- + P_0\big) V_2^\ell \, \frac{1}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(V)mj} \frac{1}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(V)nk} \Big], \quad \text{(C.19)}$$

$$22.\mathrm{E} = 2 \, f_{ijk} f_{\ell mn} \sum_{IKCF} \epsilon^{IKC} \epsilon^{IKF} \int (d^{10}x)_{12} \, d^4\theta \; V_1^i V_2^\ell \, \times$$

$$\times \, \frac{\partial_{1C}}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(S_K)mj} \, \frac{\bar{\partial}_{1F}}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(\bar{S}_K)nk}, \quad \text{(C.20)}$$

$$22.\mathrm{F} = -\frac{1}{3} \, f_{ijk} f_{\ell mn} \, \eta^{nk} \int (d^{10}x)_{12} \, d^4\theta \; V_1^i V_2^\ell \, \widetilde{\delta}_{21}^{10(V)}{}_a{}^j \, \frac{\eta^{ab}}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(V)}{}_b{}^m, \quad \text{(C.21)}$$

$$22.\mathrm{G} = 2 \, f_{ijk} f_{\ell mn} \, \eta^{nk} \sum_I \int (d^{10}x)_{12} \, d^4\theta \; V_1^i V_2^\ell \, \widetilde{\delta}_{21}^{10(\bar{S}_I)}{}_a{}^m \, \frac{\eta^{ab}}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(S_I)}{}_b{}^j, \quad \text{(C.22)}$$

$$22.\mathrm{H} = -\frac{2}{3} \, f_{ijk} f_{\ell mn} \, \eta^{nk} \int (d^{10}x)_{12} \, d^4\theta \; V_1^i V_2^\ell \, \widetilde{\delta}_{21}^{10(V)}{}_a{}^j \, \frac{\eta^{ab}}{\boldsymbol{\square}_2} \, \widetilde{\delta}_{21}^{10(V)}{}_b{}^m. \quad \text{(C.23)}$$

The bold box denotes the ten dimensional d'Alembert operator $\boldsymbol{\square} = \square + \partial_M \bar{\partial}^M$.

Appendix D

Regularization of the divergent self-energy integrals

In this appendix we provide the regularization of the divergent self-energy integrals that appear in the main text of the thesis.

D.1 Regularization of a divergence in the five and six dimensional bulk

Here we demonstrate the regularization of the divergent integral in the five and six dimensional bulk. We want to discuss both five and six dimensions at the same time and employ a six dimensional notation which has a straightforward reduction to five dimensions. The divergent bulk self-energies have the structure

$$\mathcal{I}_D = \int \left(d^D x\right)_{12} \; A(x_1)B(x_2) \, \frac{1}{(\Box + \partial\bar{\partial} - m^2)_2} \, \delta_{21} \, \frac{1}{(\Box + \partial\bar{\partial} - m^2)_2} \, \delta_{21}, \quad \text{(D.1)}$$

where we introduced an infrared regulator mass m. $A(x_1)$ and $B(x_2)$ represent the operators from the action that depend on the first and the second spacetime coordinate, respectively, and are local in the Grassmann variables. The delta functions in (D.1) denote the delta functions on the circle or the torus in the five and six dimensional cases, respectively. For an application to the five dimensional bulk self-energies one replaces $\partial\bar{\partial} \to \partial_5^2$ and uses $z = y$, $\bar{z} = 0$. We insert a Fourier transformation (D.21) to represent this integral in momentum space as

$$\mathcal{I}_D = \frac{1}{2^2} \int \frac{d^d k}{(2\pi\mu^2)^d} \sum_{\ell \in \Lambda_K} \mathrm{Vol}_W \, A(k,l) \, B(-k,-l) \, I_D, \quad \text{(D.2)}$$

where Vol_W is the volume of the circle or the torus in the five and six dimensional cases, respectively. The μ dependence is a result of our Fourier transformation conventions D.4. Here k is the continuous external momentum in four dimensions and n the discrete Kaluza-Klein momentum in the extra dimensions. In order to find the counterterms, we need to calculate the divergent part of

$$I_D = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\mathrm{Vol}_W} \sum_{n \in \Lambda_K} \frac{1}{p^2 + |n|^2 + m^2} \frac{1}{(p-k)^2 + |n-l|^2 + m^2}. \tag{D.3}$$

This has been done in App. D.3: We extend the four dimensional momentum integral to $d = 4 - 2\epsilon$ dimensions. As notation we keep $D - d$ to be either 1 or 2, so that also the total number of dimensions D becomes ϵ dependent. The divergent part takes the form

$$I_D\Big|_{div} = i\alpha_1 + i\alpha_2(k^2 + |l|^2). \tag{D.4}$$

In five dimensions dimensional regularization hides a linear divergence. The leading term for α_1 is found to be

$$\alpha_1 = -\frac{1}{(4\pi)^2}|m|, \qquad \alpha_2 = 0 \tag{D.5}$$

and the second term α_2 is not present in five dimensions. In six dimensions we obtain the coefficients

$$\alpha_1 = \frac{m^2}{(4\pi)^3}\left(\frac{1}{\bar{\epsilon}} + \ln\frac{\mu^2}{m^2}\right), \qquad \alpha_2 = \frac{1}{6(4\pi)^3}\left(\frac{1}{\bar{\epsilon}} + \ln\frac{\mu^2}{m^2}\right), \tag{D.6}$$

where $\frac{1}{\bar{\epsilon}} = \frac{1}{\epsilon} - \gamma + \ln 4\pi$. In six dimensions $\alpha_2 \neq 0$ and the second term in (D.4) requires the introduction of a higher dimensional operator as loop counterterm in the action. Transforming back into position space we obtain the local terms

$$\mathcal{I}_D\Big|_{div} = i\int d^D x \left[\alpha_1\, A(x)B(x) - \alpha_2\, A(x)\big(\Box + \partial\bar{\partial}\big)B(x)\right]. \tag{D.7}$$

D.2 Regularization of a divergence at the four dimensional fixed points

The structure of a divergent self-energy contribution at a class of four dimensional fixed points which fulfill $z = e^{ik\varphi}z$ is represented as

$$\mathcal{J}_D = \int (d^D x)_{12}\, A(x_1)\, B(x_2) \times$$
$$\times \frac{1}{(\Box + \partial\bar{\partial} - m^2)_2}\, \delta\big(z_2 - e^{ik\varphi}z_1\big)\, \frac{1}{(\Box + \partial\bar{\partial} - m^2)_2}\, \delta\big(z_2 - z_1\big) \tag{D.8}$$

with obvious reduction to five dimensions. In the delta function only the compact dimensions have been indicated for notational simplicity. In momentum space

$$\mathcal{J}_D = \frac{1}{2^2} \int \frac{d^d k}{(2\pi\mu)^{2d}} \sum_{\ell_1,\ell_2} (2\pi)^d A(k, e^{ik\varphi}\ell_1 + \ell_2) B(-k, -\ell_1 - \ell_2)\, J_0. \tag{D.9}$$

The divergence is due to the four dimensional integral

$$J_0 = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + |\ell_1|^2 + m^2} \frac{1}{(p-k)^2 + |\ell_2|^2 + m^2}, \tag{D.10}$$

which is calculated in (D.20). One obtains after the transformation into position space

$$\mathcal{J}_D\big|_{div}$$
$$= \frac{i}{(4\pi)^2} \Big(\frac{1}{\bar\epsilon} + \ln \frac{\mu^2}{m^2} \Big) \int d^d x \left(d^2 z\right)_{12} \; A(x, z_1) B(x, z_2)\; \delta^2(z_2 - e^{ik\varphi} z_1) \delta^2(z_2 - z_1). \tag{D.11}$$

This expression is local in the uncompactified four dimensional directions. In the compactified dimensions, it is localized at the fixed points, because of the two delta functions with the two different arguments.

D.3 Regularization of the momentum integral

In order to determine the counterterms, we need to calculate the divergent part of the integral

$$I_D = i \int \frac{d^d p_E}{(2\pi)^d} \frac{1}{\mathrm{Vol}_W} \sum_n \frac{1}{p^2 + |n|^2 + m^2} \frac{1}{(p-k)^2 + |n-l|^2 + m^2}, \tag{D.12}$$

which is obtained after a Wick rotation. We can replace the integration over the volume of the continuous momenta by the integration over the radius

$$\int \frac{d^d p_E}{(2\pi)^d} = \frac{2(\mu)^{4-d}}{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}\right)} \int_0^\infty dp\, p^{d-1}. \tag{D.13}$$

The non-compact four dimensional integral is extended to $d = 4 - 2\epsilon$ dimensions using the standard procedure of dimensional regularization of scalar integrals. Furthermore we use the identity

$$\frac{1}{M_i^2} = \frac{1}{\mu^2} \int_0^\infty dt\; e^{-tM_i^2/\mu^2}, \tag{D.14}$$

where M_i are the momentum-dependent denominators of (D.12). With the help of a Feynman parameter s the integral (D.12) can be written as

$$I_D = i\frac{1}{(4\pi)^{d/2}}\frac{1}{(2\pi)^{D-d}\,\mathrm{Vol}}\int_0^1 ds\int_0^\infty \frac{dt}{t^{d/2-1}}e^{-\frac{t}{\mu^2}\left[s(1-s)(k^2+|l|^2)+m^2\right]}\theta_K\left[\begin{array}{c} sl \\ 0 \end{array}\right]\left(0|\tfrac{2it}{\mu^2}\right), \tag{D.15}$$

where for $D-d=1,\,2$ we take for θ_K the θ function of the circle or the torus, defined in (D.25) and (D.32), respectively. After application of the Poisson resummation formula (D.38) which is valid both on the circle and on the torus, we obtain

$$I_D = i\frac{\mu^{D-d}}{(4\pi)^{D/2}}\int_0^1 ds\int_0^\infty \frac{dt}{t^{D/2-1}}e^{-t\left[s(1-s)(k^2+|l|^2)+m^2\right]/\mu^2}\theta_W\left[\begin{array}{c} 0 \\ -sl \end{array}\right]\left(0|\tfrac{i\mu^2}{2t}\right). \tag{D.16}$$

Because $\theta_W - 1$ cannot lead to UV divergencies, we can put $\theta_W = 1$ in order to determine the divergent part. We find

$$I_D\big|_{\mathrm{div}} = i\frac{1}{\mu^d}\left(\frac{\mu^2}{m^2}\right)^2\left(\frac{m^2}{4\pi}\right)^{\frac{D}{2}}\sum_{n'\geq 0}(-)^{n'}\frac{\Gamma\left(n'+2-\frac{D}{2}\right)\Gamma(n'+1)}{\Gamma(2n'+2)}\left(\frac{K^2}{m^2}\right)^{n'}. \tag{D.17}$$

In the six dimensional case we obtain

$$I_{6-2\epsilon}\big|_{\mathrm{div}} = \frac{i}{(4\pi)^3}\Big[\Big(\frac{1}{\bar\epsilon}+\ln\frac{\mu^2}{m^2}\Big)\Big(m^2+\frac{1}{6}(k^2+|l|^2)\Big)+m^2\Big], \tag{D.18}$$

where we have defined $\frac{1}{\bar\epsilon} = \frac{1}{\epsilon} - \gamma + \ln 4\pi$. Here only the terms with $n' \in \{0,1\}$ contribute to the divergent part and we have neglected terms with higher n'. In the five dimensional case the expression reads

$$I_{5-2\epsilon}\big|_{\mathrm{div}} = -i\frac{1}{(4\pi)^2}|m|, \tag{D.19}$$

where only the $n'=0$ term has been taken into account. The four dimensional case can also be traced back when one neglects the summation $\frac{1}{\mathrm{Vol}_W}\sum_{n'}$. This results in

$$I_{4-2\epsilon}\big|_{\mathrm{div}} = \frac{i}{(4\pi)^2}\Big(\frac{1}{\epsilon}-\gamma+\ln 4\pi+\ln\frac{\mu^2}{m^2}\Big). \tag{D.20}$$

D.4 Fourier transformation conventions

In the evaluation of the divergent self-energies we need to perform Fourier transformations between coordinate and momentum space. We describe our conventions for the six dimensional situation compactified on the torus T^2. Because of the notation in five and six dimensions introduced in App. A.1 and App. A.2, respectively, the reduction to the five dimensional integrals on the circle S^1 is straightforward. We define the Fourier transformation as

$$A(x,z) = \int \frac{d^d p}{(2\pi\,\mu)^d} \sum_{n\in\Lambda_K} A(p,n) e^{i(px+nz+\bar{n}\bar{z})} \tag{D.21}$$

and

$$A(p,n) = \frac{2\mu^d}{\mathrm{Vol}_W} \int d^d x d^2 z A(x,z) e^{-i(px+nz+\bar{n}\bar{z})}. \tag{D.22}$$

We have introduced the regularization scale μ such that the coordinate and momentum Fourier transforms have the same mass dimension. The coordinate space delta function is given by

$$\delta^d(x_2-x_1)\delta^2(z_2-z_1) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\mathrm{Vol}_W} \sum_{n\in\Lambda_K} e^{i(p(x_2-x_1)+n(z_2-z_1)+\bar{n}(\bar{z}_2-\bar{z}_1))}. \tag{D.23}$$

The delta function in momentum space can be expanded as

$$\delta^d(p_2-p_1)\delta^2(n_2-n_1) = 2\int \frac{d^d x d^2 z}{(2\pi)^d \mathrm{Vol}_W} e^{i(p_2-p_1)x+(n_2-n_1)z+(\bar{n}_2-\bar{n}_1)\bar{z})}. \tag{D.24}$$

D.5 Genus one theta functions

The genus one theta function on the Kaluza Klein lattice is defined as

$$\theta_K \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\sigma|\tau) = \sum_{n\in\mathbb{Z}/R} e^{i\frac{\tau}{2}(n-\alpha)^2 + i[(\sigma-\beta)(n-\alpha)]}. \tag{D.25}$$

The theta function is translation invariant under a shift of α by an element of the Kaluza Klein lattice or a shift of β by an element of the winding mode lattice

$$\theta_K \begin{bmatrix} \alpha+n \\ \beta \end{bmatrix} (\sigma|\tau) = \theta_K \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\sigma|\tau), \qquad n \in \Lambda_K, \tag{D.26}$$

$$\theta_K \begin{bmatrix} \alpha \\ \beta+w \end{bmatrix} (\sigma|\tau) = \theta_K \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\sigma-w|\tau), \qquad w \in \Lambda_W. \tag{D.27}$$

The genus one theta function on the winding mode lattice is defined as

$$\theta_W \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\sigma|\tau) = \sum_{w\in 2\pi R} e^{i\frac{\tau}{2}(w-\alpha)^2 + i(w-\alpha)(-\beta)}. \tag{D.28}$$

The relation between θ_K and θ_W is

$$\theta_K \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\sigma|\tau) = R\sqrt{\frac{2\pi}{-i\tau}}\, e^{-\frac{i}{2\tau}\tau^2 + i\alpha\beta}\, \theta_W \begin{bmatrix} \beta \\ -\alpha \end{bmatrix} \left(\frac{-\sigma}{\tau}\middle|\frac{-1}{\tau}\right). \tag{D.29}$$

This can be obtained by using Poisson resummation, which allows us to rewrite a complex exponential function that is summed over the Kaluza-Klein lattice Λ_K into a delta function that is summed over the winding mode lattice and vice versa. Concretely, we have

$$\frac{1}{2\pi R}\sum_{n\in\Lambda_K} e^{iny} = \sum_{w\in\Lambda_W}\delta(y-w), \qquad y\in\mathbb{R} \tag{D.30}$$

and

$$R\sum_{w\in\Lambda_W} e^{iwp} = \sum_{n\in\Lambda_K}\delta(p-n), \qquad p\in\mathbb{R}. \tag{D.31}$$

D.6 Genus two theta functions

The genus two theta function on the Kaluza-Klein lattice Λ_K is defined by

$$\theta_K\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](\sigma|\tau) = \sum_{n\in\Lambda_K} e^{i\frac{\tau}{2}|n-\alpha|^2+i\left[(\bar\sigma-\bar\beta)(\bar n-\bar\alpha)+(\sigma-\beta)(n-\alpha)\right]}. \tag{D.32}$$

Also the genus two theta function fulfills the translation invariance properties (D.26) and (D.27). The genus two theta function on the winding mode lattice is defined by

$$\theta_W\left[\begin{array}{c}\beta\\ \alpha\end{array}\right](\sigma|\tau) = \sum_{w\in\Lambda_W} e^{2i\tau|w-\beta|^2+i(\bar w-\bar\beta)(2\sigma-\bar\alpha)+(w-\beta)(2\bar\sigma-\alpha)}. \tag{D.33}$$

The relation between θ_K and θ_W is given by

$$\theta_K\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](\sigma|\tau) = \mathrm{Vol}\,\frac{2\pi}{-i\tau}\,e^{i\left(-\frac{2}{\tau}|\sigma|^2+\alpha\beta+\bar\alpha\bar\beta\right)}\,\theta_W\left[\begin{array}{c}\beta\\ -\alpha\end{array}\right]\left(\frac{-\sigma}{\tau}\middle|\frac{-1}{\tau}\right). \tag{D.34}$$

This is obtained by Poisson resummation on the torus

$$\frac{1}{\mathrm{Vol}_W}\sum_{n\in\Lambda_K} e^{i(nz+\bar n\bar z)} = \sum_{w\in\Lambda_W}\frac{1}{2}\,\delta^2(z-w), \qquad z\in\mathbb{C}, \tag{D.35}$$

$$\frac{1}{\mathrm{Vol}_K}\sum_{w\in\Lambda_w} e^{i(wp+\bar w\bar p)} = \sum_{n\in\Lambda_K} 2\,\delta^2(p-n), \qquad p\in\mathbb{C}, \tag{D.36}$$

where

$$\delta^2(p) = \frac{1}{2}\,\delta(p_5)\delta(p_6), \qquad \delta^2(z) = 2\,\delta(x_5)\delta(x_6). \tag{D.37}$$

Therefore, in the case $\tau=\frac{2it}{\mu^2}$, $\alpha = sl$, and $\sigma=\beta=0$

$$\theta_K\left[\begin{array}{c}sl\\ 0\end{array}\right]\left(0|\tfrac{2it}{\mu^2}\right) = \mathrm{Vol}\left(\frac{\pi\mu^2}{t}\right)^{\frac{D-d}{2}}\theta_W\left[\begin{array}{c}0\\ -sl\end{array}\right]\left(0|\tfrac{i\mu^2}{2t}\right) \tag{D.38}$$

holds both for the theta functions on the circle and on the torus.

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