Galois cohomology of Fontaine rings

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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Ottawa

Bonn 2007

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn.

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Tag der Promotion: 5. Juni 2007

Diese Dissertation ist auf dem Hochschulschriftenserver der ULB Bonn http://hss.ulb.uni-bonn.de/diss_online elektronisch publiziert.

Erscheinungsjahr: 2007

Abstract

Let V be a complete discrete valuation ring of mixed characteristic. We express the crystalline cohomology of the special fibre of certain smooth affine V-schemes $X = \mathrm{Spec}(R)$ tensored with an appropriate ring of p-adic periods as the Galois cohomology of the fundamental group of the geometric generic fibre $\pi_1(X_{\bar{V}[1/p]})$ with coefficients in a Fontaine ring constructed from R. This is based on Faltings' approach to p-adic Hodge theory (the theory of almost étale extensions). Using this we deduce maps from p-adic étale cohomology to crystalline cohomology of smooth V-schemes. The results are more general, as the semi-stable case is also considered. In the end we derive an alternative proof of the theorem of Tsuji (the semi-stable conjecture of Fontaine-Jannsen).

Contents

Introduction

Let p be an odd prime number. The purpose of this thesis is to make precise the relationship between crystalline cohomology and Galois cohomology of certain Fontaine rings occuring in Faltings' approach to p-adic Hodge theory ([6], [7]). Let us very briefly recall this approach. Let K^+ be a complete discrete valuation ring of fraction field K of characteristic zero and perfect residue field k of characteristic p. Let X be a proper K^+-scheme with good reduction (to simplify). One constructs a site, usually denoted $\mathscr{X}$, whose cohomology formalizes the idea of glueing $\pi_1(X_{\bar{K}})$-cohomology locally on X. One sheafifies a construction of Fontaine to obtain a sheaf of rings $\mathscr{A}_{\text{crys},n}$ on $\mathscr{X}$ together with transformations

$$H^*(\mathscr{X}, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}_p} A_{\text{cris}} \to H^*(\mathscr{X}, \mathscr{A}_{\text{crys},n}) \leftarrow H^*_{\text{crys}}(X_k|W_n(k), \mathscr{O}) \otimes_{W(k)} A_{\text{cris}}$$

where A_{cris} is the ring of p-adic periods constructed by Fontaine [8], and the group on the right denotes the crystalline cohomology of the special fibre X_k. Then one uses Faltings' theory of almost étale extensions to show that the intermediate cohomology theory $H^*(\mathscr{X}, \mathscr{A}_{\text{crys},n})$ satisfies Poincaré duality and Künneth formula, hence by standard arguments is isomorphic to crystalline cohomology. If X has good reduction then the group on the left is canonically isomorphic to étale cohomology of $X_{\bar{K}}$ tensored with A_{cris} and again standard arguments based on compatibility of characteristic classes, Poincaré duality, and Künneth formula allow one to conclude that the maps are isomorphisms (up to some torsion).

In this thesis we study closely the map

$$H^*_{\text{crys}}(X_k|W_n(k), \mathscr{O}) \otimes_{W(k)} A_{\text{cris}} \to H^*(\mathscr{X}, \mathscr{A}_{\text{crys},n})$$

locally on X. Our main result is that this map is an almost isomorphism up to t-torsion, where $t \in A_{\text{cris}}$ is an element which plays a role analogous to that of $2\pi i$ in the transcendental theory of periods. Here the term 'almost' is used in the sense of almost mathematics ([7], [9]). Using this, one can compare crystalline and p-adic étale cohomology of X without an intermediate cohomology theory, thereby simplifying the approach to p-adic Hodge theory via the theory of almost étale extensions.

Overview

§1: We begin by reviewing the various crystalline sites which we will use in this thesis. This is mainly to fix notation. Afterwards, we review the construction by Fontaine [8] of the final object of the crystalline site of a ring of characteristic p with surjective (absolute) Frobenius. Such final objects are called *Fontaine rings*. We give the proof for the more general log-crystalline site. Then we give some examples of Fontaine rings due to Fontaine and Kato and we recall some of their basic properties.

§2: We first recall the notions of almost ring theory which we will use, following [9]. Then we recall the key input which we shall need, namely Faltings' Almost Purity Theorem

[7]. Afterwards we apply this theorem to certain Fontaine rings, constructed as follows. To simplify assume that $K^+ = W(k)$. Let $\mathrm{Spec}(R)$ be a smooth integral K^+-scheme. Let $Q(R)$ be its field of rational functions and consider the maximal extension $Q(\bar{R})$ of $Q(R)$ such that the normalization $\bar{R}$ of R in $Q(\bar{R})$ is the inductive limit of finite normal integral R-algebras which are étale after inverting p. Then via the theory of almost étale extensions one can show that the ring $\bar{R}/p\bar{R}$ has surjective Frobenius, hence by Fontaine's theorem recalled in §1, we may construct the Fontaine ring

$$A^+ := \lim_n H^0_{\mathrm{crys}}(\bar{R}/p\bar{R}|W_{n+1}(k), \mathscr{O}).$$

Also we can construct another Fontaine ring A^+_∞ as follows. Up to localizing on $\mathrm{Spec}(R)$ we may assume that it has étale local coordinates $T_1, ..., T_d$ which are units. In this case, one says that R is *small*. Let $\bar{K}$ denote the algebraic closure of K and $\bar{K}^+$ its valuation ring, i.e. the normalization of K^+ in $\bar{K}$. Let $\tilde{R} = R \otimes_{K^+} \bar{K}^+$ and let $\tilde{R}_\infty$ denote the ring obtained from $\tilde{R}$ by adding all p-power roots of the the T_i. Define

$$A^+_\infty := \lim_n H^0_{\mathrm{crys}}(\tilde{R}_\infty/p\tilde{R}_\infty|W_{n+1}(k), \mathscr{O}).$$

Then the theory of almost étale extensions applied to these Fontaine rings implies that the canonical homomorphism

$$A^+_\infty/p^n A^+_\infty \to A^+/p^n A^+$$

is the filtering inductive limit of almost Galois coverings and there are canonical almost isomorphisms for each i

$$H^i(\Delta_\infty, A^+_\infty/p^n A^+_\infty) \xrightarrow{\approx} H^i(\Delta, A^+/p^n A^+)$$

where $\Delta := \mathrm{Gal}(\bar{R}[1/p]/\tilde{R}[1/p])$ with quotient $\Delta_\infty := \mathrm{Gal}(\tilde{R}_\infty[1/p]/\tilde{R}[1/p]) \cong \mathbb{Z}_p(1)^d$ (see Corollaries 2.4, 2.5). This also applies to p-adic divided power bases other than $W(k)$ and there is a logarithmic version for schemes with semi-stable singularities (in fact also for schemes with toroidal singularities, *cf.* [7], but we have opted to restrict to the semi-stable case in this thesis).

§3: We construct, via the formalism of crystalline cohomology, a canonical de Rham resolution of the ring A^+. We then (almost) compute the Δ-cohomology of the components of this resolution, by reducing to the case of A^+_∞. The result is the following (*cf.* Theorems 3.1, 3.2)

Theorem 0.1. *There is a canonical morphism of complexes in the derived category*

$$A_{\mathrm{cris}}/p^n A_{\mathrm{cris}} \otimes_{W(k)} \Omega^\bullet_{R/W(k)} \to C^*(\Delta, A^+/p^n A^+)$$

which is an almost quasi-isomorphism up to t^d-torsion.

Here

$$A_{\mathrm{cris}} = \lim_n H^0_{\mathrm{crys}}(\bar{K}^+/p\bar{K}^+|W_{n+1}(k), \mathscr{O})$$

is a ring of p-adic periods constructed by Fontaine and t is the element alluded to above. Over an arbitrary finite extension of $W(k)$ a similar result holds and we allow $\mathrm{Spec}(R)$ to have semi-stable singularities.

§4: We globalize our previous results. One first defines a site $X_{\mathfrak{F}}$ whose cohomology is locally the Galois cohomology of the fundamental group (denoted $\mathscr{X}$ above). The rings A^+/p^nA^+ define a sheaf on this site, denoted $\mathscr{A}_n$. Then one shows that one can almost compute the $X_{\mathfrak{F}}$-cohomology of $\mathscr{A}_n$ on a suitable syntomic site. The point here is that the extension $\tilde{R} \subset \tilde{R}_\infty$ is the inductive limit of syntomic coverings. Then one uses the syntomic construction of crystalline cohomology of Fontaine-Messing, or rather Breuil's more general logarithmic version [4].

§5: We compare the log-crystalline cohomology of the special fibre of a proper semi-stable K^+-scheme to the p-adic étale cohomology of its geometric generic fibre. For this we must generalize certain Artin-Schreier exact sequences to A^+, due to Fontaine in the case of A_{cris}. Then we use a theorem of Faltings from [7] which gives information on the cohomology groups $H^*(X_{\mathfrak{F}}, \mathbb{Z}/p^n\mathbb{Z})$. In the smooth case we do not need this, as we know that the latter is canonically isomorphic to p-adic étale cohomology. Tsuji's theorem for proper semi-stable K^+-schemes (the semi-stable conjecture of Fontaine-Jannsen) then follows from these considerations.

Remarks on notation

- The letter p always denotes an odd prime number.
- $\mathbb{N} = 0, 1, 2, 3, ...$ denotes the set of natural numbers, and for any subset $R \subset \mathbb{R}$ we write R_+ (resp. $R_{>0}$) for the set of elements of R which are greater than or equal to zero (resp. greater than zero).
- By *ring* we mean a commutative ring with unity.
- All monoids considered will be assumed to be commutative.
- If A is a ring and M is an A-module, then we denote by $\Gamma_A(M)$ the divided power polynomial A-algebra defined by M (see [1] or [2] for a construction of this algebra). If $I \subset A$ is an ideal then we denote by $D_A(I)$ the divided power hull of A for the ideal I (*loc.cit.*). If A is a ring and $I \subset A$ is an ideal with a divided power structure $(\gamma_n : I \to A)_{n\in\mathbb{N}}$, then we will often write $x^{[n]} := \gamma_n(x)$ when it is clear which divided power structure is meant. Finally, if $X_1, ..., X_d$ denotes indeterminates, then we write $A\langle X_1, ..., X_d\rangle$ for the divided power polynomial A-algebra in the variables $X_1, ..., X_d$. It is the divided power hull of $A[X_1, ..., X_d]$ for the ideal generated by $X_1, ..., X_d$.

Acknowledgements

First and foremost, I thank my advisor Gerd Faltings for suggesting that I try to compute the Galois cohomology of Fontaine rings via a de Rham complex. His specific suggestion that one should use the logarithm of p-power roots of local coordinates to trivialize constant cocycles was the key to finding the right de Rham complex. I thank him very much for discussions and for his critical comments. This work is also indebted to Lorenzo

Ramero for his reading of drafts as well as numerous comments and suggestions which led to significant improvement. I thank him very much for showing interest in this work. This thesis was written under the auspices of the International Max-Planck Research School on Moduli Spaces (IMPRS) of the Max-Planck-Institut für Mathematik Bonn. I thank the institute for financial support and the staff for their hospitality. I would also like to thank my parents Françoise and Bimal for their support throughout my studies. I would finally like to address a very special thanks to Tülin for her constant encouragement, understanding, and much more.

1 Crystalline cohomology of rings with surjective Frobenius

In this section we recall Fontaine's construction of the final object of the crystalline site of a ring of characteristic $p > 0$ with surjective Frobenius endomorphism. This will play the role of substitute for Poincaré's lemma in our approach to comparison of crystalline and étale cohomology.

1.1 Reminder on crystalline sites

1.1.1. Let Z be a scheme and let $Z_0 \hookrightarrow Z$ be a closed immersion, such that the ideal defining the image of Z_0 in Z is nilpotent and has a divided power structure (we say that $Z_0 \hookrightarrow Z$ is a *divided power thickening*). We usually use the abbreviation DP for "divided power". Let X be a Z_0-scheme. Recall the definition of the crystalline site of a scheme X over the DP-base Z. Its underlying category has for objects DP-thickenings

$$U \hookrightarrow T$$

where U is an open subscheme of X and T is a Z-scheme, such that the canonical morphism $T \to Z$ is a DP-morphism. Morphisms of this category are given by commutative diagrams

$$\begin{array}{ccc} U' & \longrightarrow & T' \\ \downarrow & & \downarrow \\ U & \longrightarrow & T \end{array}$$

where the map $U' \to U$ is an open immersion and $T' \to T$ is a Z-DP-morphism. We define a pretopology on this category by defining coverings to be families of morphisms

$$(U_\alpha \hookrightarrow T_\alpha)_\alpha \to (U \hookrightarrow T)$$

such that $(U_\alpha)_\alpha \to U$ and $(T_\alpha)_\alpha \to T$ are Zariski open coverings. This defines the *crystalline site of X over the DP-base Z*, denoted $(X|Z)_{\mathrm{crys}}$.

1.1.2. To give a sheaf $\mathscr{F}$ on $(X|Z)_{\mathrm{crys}}$ is the same as giving for all $(U \hookrightarrow T) \in \mathrm{ob}(X|Z)_{\mathrm{crys}}$ a sheaf $\mathscr{F}_T$ on the Zariski site of T, together with a morphism

$$g^*_{\mathscr{F}} : g^{-1}\mathscr{F}_T \to \mathscr{F}_{T'}$$

for any morphism $g : (U' \hookrightarrow T') \to (U \hookrightarrow T)$, such that the natural transitivity condition holds for morphisms $(U'' \hookrightarrow T'') \to (U' \hookrightarrow T') \to (U \hookrightarrow T)$ and moreover $g^*_{\mathscr{F}}$ is an isomorphism if $g : T' \to T$ is an open immersion and the square defined by g is cartesian.

In this way we see that the presheaf defined

$$\mathscr{O}(U \hookrightarrow T) := \mathscr{O}_T(T)$$

is in fact a sheaf, called the *structure sheaf* of $(X|Z)_{\mathrm{crys}}$.

1.1.3. Exactly the same construction can be done by replacing the Zariski topology by the étale topology.

1.1.4. Let $(X, M) \to (Z, N)$ be a morphism of log-schemes (see [15]). If $f : X \to (Z, N)$ is a morphism, then we write f^*N for the inverse image log-structure, in contrast with the inverse image sheaf $f^{-1}N$. If M is a pre-log-structure on X then we denote by M^a the associated log-structure. If X is a log-scheme and $U \to X$ is an étale morphism, then the restriction of the log-structure of X to U defines a natural log-structure on U and we will always consider U as a log-scheme for this log-structure. We will assume that the log-structure N on Z is fine and the log-structure M on X is integral. The category of schemes with integral log-structures has finite inverse limits (*cf.* [15] 1.6, 2.8), and in particular fibre products exist (though these are in general different from the fibre products of the category of schemes with log-structures). In this paper we will only consider the category of log-schemes with integral log-structures.

1.1.5. Let $(Z_0, N_0) \hookrightarrow (Z, N)$ be an exact closed immersion, such that $Z_0 \hookrightarrow Z$ is a DP-thickening. The *logarithmic crystalline site* of (X, M) over the DP-log-base (Z, N) is the site whose underlying category has for objects DP-thickenings $U \hookrightarrow T$, where U is an étale X-scheme, T is a log-Z-scheme such that the canonical morphism $T \to Z$ is a DP-morphism, and the closed immersion $U \hookrightarrow T$ is exact. A morphism of this category is a commutative diagram of log-schemes

$$\begin{array}{ccc} U' & \longrightarrow & T' \\ \big\downarrow & & \big\downarrow \\ U & \longrightarrow & T \end{array}$$

where the morphism on the left is étale and the morphism on the right is a DP-morphism. The pretopology on this category is given by defining covering families to be families of morphisms

$$(U_\alpha \hookrightarrow T_\alpha)_\alpha \to (U \hookrightarrow T)$$

such that $(U_\alpha)_\alpha \to U$ and $(T_\alpha)_\alpha \to T$ are coverings for the étale topology, and such that the squares

$$\begin{array}{ccc} U_\alpha & \longrightarrow & T_\alpha \\ \big\downarrow & & \big\downarrow \\ U & \longrightarrow & T \end{array}$$

are cartesian. Given a covering, for any morphism $(U' \hookrightarrow T') \to (U \hookrightarrow T)$, note that the diagram

$$\begin{array}{ccc} U' \times_U U_\alpha & \longrightarrow & T' \times_T T_\alpha \\ \big\downarrow & & \big\downarrow \\ U' & \longrightarrow & T' \end{array}$$

is cartesian, hence $(U' \times_U U_\alpha \hookrightarrow T' \times_T T_\alpha)$ is an object of the logarithmic crystalline site. This defines the *log-crystalline site of X over the DP-base Z*, denoted $((X, M)|(Z, N))_{\mathrm{crys}}$ or simply by $(X|Z)_{\text{log-crys}}$ when it is clear which log-structures are meant for X and Z.

1.1.6. If $X = \mathrm{Spec}(R)$ and $Z = \mathrm{Spec}(S)$ are affine schemes, then we will usually write $(R|S)_{\text{log-crys}}$ instead of $(X|Z)_{\text{log-crys}}$.

1.1.7. Assume in the sequel that Z is annihilated by a power of p. By [16] 2.4.2, if $f : (X, M) \to (X', M')$ is a morphism of log-(Z, N)-schemes with M integral and M' fine, then f induces a morphism of log-crystalline topoi over the DP log-base (Z, N).

1.2 Fontaine's theorem

1.2.1. We begin with a lemma, implicit in [15].

Lemma 1.1. *Let $i : (X_0, M_0) \hookrightarrow (X, M)$ be a nilpotent closed immersion of log-schemes of ideal $\mathscr{I}$. Then*

$$i^*M = i^{-1}M/(1 + \mathscr{I}).$$

In particular, i is exact if and only if the map $i^{-1}M \to M_0$ is surjective and locally for all sections m, m' of $i^{-1}M$ with same image in M_0, there exists $u \in \mathscr{O}_X^*$ such that $m = um'$.

Proof. One first shows easily that $\mathscr{O}_{X_0}^* \cong i^{-1}\mathscr{O}_X^*/(1 + \mathscr{I})$. If $L \to \mathscr{O}_X$ is a log-structure on X together with a morphism of pre-log-structures

$$i^{-1}M \to L$$

then $1 + \mathscr{I} \subset i^{-1}M$ maps to $1 \in L$, so the map factors (necessarily uniquely)

$$i^{-1}M/(1 + \mathscr{I}) \to L.$$

So the claim of the lemma will follow if we can show that $i^{-1}M/(1+\mathscr{I})$ is a log-structure. If $\alpha : i^{-1}M \to i^{-1}\mathscr{O}_X$ is the inverse image by i of the map defining M as a log-structure on X, then α induces an isomorphism $\alpha^{-1}i^{-1}\mathscr{O}_X^* \cong i^{-1}\mathscr{O}_X^*$, whence an isomorphism

$$\alpha^{-1}i^{-1}\mathscr{O}_X^*/(1 + \mathscr{I}) \cong i^{-1}\mathscr{O}_X^*/(1 + \mathscr{I})$$

i.e. $i^{-1}M/(1 + \mathscr{I}) \to i^{-1}\mathscr{O}_X/\mathscr{I} \cong \mathscr{O}_{X_0}$ is a log-structure on X_0. $\square$

1.2.2. Assume S is a ring on which p is nilpotent, and let R be a S/pS-algebra. Let N be an integral monoid defining a log-structure on $\mathrm{Spec}(S)$ and M an integral monoid defining log-structure on $\mathrm{Spec}(R)$ such that we have a commutative square

$$\begin{array}{ccc} N & \longrightarrow & S/pS \\ \downarrow & & \downarrow \\ M & \longrightarrow & R. \end{array}$$

Theorem 1.1. *With the above notation and assumptions, if the (absolute) Frobenius is surjective on R and on M, then the site $(R|S)_{\text{log-crys}}$ has a final object.*

Proof. First note that for any étale map $U \to \operatorname{Spec}(R)$, the (absolute) Frobenius is surjective on $\mathscr{O}_U$. Indeed, since the map is étale, its relative Frobenius is an isomorphism, so this follows from the factorization of the absolute Frobenius of U as the relative Frobenius followed by the pullback of the absolute Frobenius of $\operatorname{Spec}(R)$. We first define the perfection $P(R)$ of R as being the projective limit of the diagram

$$\cdots \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} R$$

where F denotes the (absolute) Frobenius of R. An element of $P(R)$ is given by a sequence $r = (r^{(n)})$ of elements of R indexed by the natural numbers, such that $r^{(n+1)p} = r^{(n)}$ for all n. $P(R)$ is a perfect ring of characteristic p, so its ring of Witt vectors $W(P(R))$ is a flat $\mathbb{Z}_p$-algebra. We write $(r_0, r_1, r_2, ...) \in W(P(R))$ and $r_i = (r_i^{(n)})$ for each $i = 0, 1, 2, ...$ Let $\operatorname{Spec}(R) \hookrightarrow \operatorname{Spec}(A)$ be an object of the site $(R|S)_{\text{log-crys}}$. If $r = (r^{(n)}) \in P(R)$ then define a lift of $r^{(m)}$ to A by choosing lifts $\hat{r}^{(n)} \in A$ of $r^{(n)}$ for all n and setting

$$\tilde{r}^{(m)} := \lim_{n\to\infty} \hat{r}^{(m+n)p^n}.$$

Since p is nilpotent on A and $\operatorname{Ker}(A \to R)$ has a DP-structure, one sees easily that $\tilde{r}^{(m)}$ is a lift of $r^{(m)}$ which is independent of the choices made. Define a map

$$\theta : W(P(R)) \to A$$

by sending $(r_0, ...)$ to $\sum_{i=0}^{\infty} p^i \tilde{r}_i^{(i)}$. Since p is nilpotent on A, these are just the usual Witt polynomials, so the map is indeed a homomorphism of rings. In the case $A = R$, this map is none other than the projection $(r_0, ...) \mapsto r_0^{(0)}$ and in this way we obtain a commutative diagram

$$\begin{array}{ccc} W(P(R)) & \xrightarrow{\theta} & A \\ \downarrow & & \downarrow \\ R & = & R. \end{array}$$

We claim that the map θ is unique for the maps $W(P(R)) \to A$ making the above diagram commute. Indeed, any map $\alpha : W(P(R)) \to A$ is determined by it values on $[r]$, where $[\cdot]$ denotes the Teichmüller lift. If $r = (r^{(n)})$, then write $r(m) := (r^{(m+n)})$ as the sequence "shifted" by m. So for all m we have

$$\alpha([r]) = \alpha([r(m)])^{p^m}$$

and $\alpha([r(m)]) = \theta([r(m)]) + a$ for some $a \in \operatorname{Ker}(A \to R)$. If $p^m A = 0$, then

$$\begin{aligned} \alpha([r]) = \alpha([r(m)])^{p^m} = (\theta([r(m)]) + a)^{p^m} &= \sum_{i=0}^{p^m} \binom{p^m}{i} i! a^{[i]} \theta([r(m)])^{p^m - i} \\ &= \theta([r(m)])^{p^m} = \theta([r]) \end{aligned}$$

thus proving the uniqueness claim.

We may extend this map to a unique homomorphism of S-algebras

$$W(P(R)) \otimes_{\mathbb{Z}} S \to A$$

thereby obtaining a commutative diagram

$$\begin{array}{ccc} W(P(R)) \otimes_{\mathbb{Z}} S & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ R & \overline{\overline{\quad\quad}} & R. \end{array}$$

Define the perfection $P(M)$ of M by

$$P(M) := \left\{ (m^{(n)}) \in M^{\mathbb{N}} \quad : \quad m^{(n+1)p} = m^{(n)} \quad \forall n \in \mathbb{N} \right\}.$$

Via the Teichmüller lift we may consider $P(M)$ as a pre-log-structure on $\mathrm{Spec}(W(P(R)))$. Consider the integral and quasi-coherent pre-log-structure

$$P(M) \oplus N \to W(P(R)).$$

By construction, the natural map $P(M) \oplus N \to M$ induced from the projection $P(M) \to M$ sending $(m^{(n)})$ to $m^{(0)}$, is surjective. Let

$$L := \mathrm{Ker}(P(M)^{\mathrm{gp}} \oplus N^{\mathrm{gp}} \to M^{\mathrm{gp}})$$

where for any integral monoid M we write M^{gp} for the associated group, and define

$$(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log} := W(P(R)) \otimes_{\mathbb{Z}} S \otimes_{\mathbb{Z}} \mathbb{Z}[L].$$

Let us show how to extend the map $W(P(R)) \otimes_{\mathbb{Z}} S \to A$ to a unique map

$$(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log} \to A.$$

Let M_A denote the log-structure of $\mathrm{Spec}(A)$. By the last lemma we have

$$M^a = M_A/(1+I)$$

where $I = \mathrm{Ker}(A \to R)$. Define a map $P(M)^a = P(M^a) \to M_A$ by sending $(m^{(n)})$ to $\hat{m}^{(n)p^n}$, where $\hat{m}^{(n)}$ is a section of M_A lifting $m^{(n)}$ and n is any integer such that $p^n A = 0$. Because of the divided power structure of I it is easy to check that this section is independent of all choices. Since the Frobenius is surjective on M, we check easily as above that this map is unique for the maps defining a commutative square of logarithmic algebras

$$\begin{array}{ccc} W(P(R)) & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ R & \overline{\overline{\quad\quad}} & R. \end{array}$$

The map $P(M) \to P(M)^a \to M_A$ extends uniquely to a map $\lambda_A : P(M) \oplus N \to M_A$. Now if $l \in L$, consider $\lambda_A(l) \in M_A^{\mathrm{gp}}$. Since the image of $\lambda_A(l)$ in M^{gp} is the identity element, by exactness we deduce that $\lambda_A(l) \in A^*$. This defines the map

$$(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log} \to A$$

and hence we obtain a commutative diagram

$$\begin{array}{ccc} (W(P(R)) \otimes_{\mathbb{Z}} S)_{\log} & \longrightarrow & A \\ \downarrow & & \downarrow \\ R & = & R. \end{array}$$

Since the map on the right-hand side is an S-DP-morphism, it follows by the universal property of divided power hulls that $(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log} \to A$ factors over a unique homomorphism

$$(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log}^{\mathrm{DP}} \to A$$

where $(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log}^{\mathrm{DP}}$ is the divided power hull of $(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log}$ for the kernel of the projection onto R sending L to 1. We give

$$\mathrm{Spec}\,(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log}^{\mathrm{DP}}$$

the log-structure associated to the pre-log-structure $P(M) \oplus N$, and claim that this makes

$$\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}\,(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log}^{\mathrm{DP}}$$

into an object of $(R|S)_{\text{log-crys}}$. It suffices to check that it is an exact closed immersion. This follows by construction since the elements $l \in L$ are units.

Now, if $U \to \mathrm{Spec}(R)$ is étale, then for any affine open $\mathrm{Spec}(R') \subset U$ we may construct $(W(P(R')) \otimes_{\mathbb{Z}} S)_{\log}^{\mathrm{DP}}$ by replacing R by R' in the construction above. Then if $\mathrm{Spec}(R') \hookrightarrow \mathrm{Spec}(A')$ is an S-DP-morphism we have unique maps

$$(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log}^{\mathrm{DP}} \to \big(W(P(R')) \otimes_{\mathbb{Z}} S\big)_{\log}^{\mathrm{DP}} \to A'$$

and hence for any object $U \hookrightarrow T$ and any covering $(U_\alpha \hookrightarrow T_\alpha)_\alpha \to (U \hookrightarrow T)$ we may define unique maps

$$\begin{array}{ccc} U_\alpha & \longrightarrow & T_\alpha \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & \mathrm{Spec}\,(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log}^{\mathrm{DP}} \end{array}$$

and since the uniqueness allows us to glue, we are done. □

1.2.3. Remarks.

(a) The proof is inspired by that of Breuil in the case $S = W_n$ in [4]. If we give all schemes the trivial log-structure, then we recover Fontaine's original construction in [8]. We advise the reader unfamiliar with Fontaine's argument first to read the above proof assuming the log-structures to be trivial.

(b) By the uniformity of the construction and the fact that taking the divided power hull commutes with base-change ([2] 3.20, Remark 8), we see that for any n the final object of the crystalline site $(R|S/p^nS)_{\text{log-crys}}$ is given by the S/p^nS-DP-thickening

$$\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}\left(W(P(R)) \otimes_{\mathbb{Z}} S\right)^{\mathrm{DP}}_{\log} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}.$$

In particular, if S is a p-adically complete ring (we say that S is *p-adic base*), then as in [2] one may define a crystalline site $(R|S)_{\text{log-crys}}$ whose cohomology automatically computes the derived projective limit of the cohomology of each $(R|S/p^nS)_{\text{log-crys}}$. Then the proof given in the theorem also works for such S and shows that

$$\mathrm{Spec}(R) \hookrightarrow \mathrm{Spf}\left(\left(W(P(R)) \otimes_{\mathbb{Z}} S\right)^{\mathrm{DP}}_{\log}\right)^{\widehat{}}$$

is the final object of the site $(R|S)_{\text{log-crys}}$, where $(\cdot)^{\widehat{}}$ denotes the p-adic completion.

(c) It is sometimes convenient to give a slightly different construction of the final object, as follows. Suppose $p^mS = 0$. For any S-DP-thickening $\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}(A)$, define a map

$$\theta : W_m(R) \to A$$

by sending $(r_0, ..., r_{m-1})$ to $\sum_{i=0}^{m-1} p^i \hat{r}_i^{p^{m-i}}$, where $\hat{r}_i$ denotes an arbitrary lift of $r_i \in R$ to A. Because of the divided power structure of $\mathrm{Ker}(A \to R)$, this is a well-defined homomorphism of rings. In this way we obtain a commutative square, and to check the uniqueness of θ for such commutative squares, we first reduce to checking it for Teichmüller lifts and then use the fact that the Frobenius is surjective on R. Let $L = \mathrm{Ker}(M^{\mathrm{gp}} \oplus N^{\mathrm{gp}} \to M^{\mathrm{gp}})$, where the map $M^{\mathrm{gp}} \to M^{\mathrm{gp}}$ is given by raising to the p^mth-power. Taking the divided power hull $\left(W_m(R) \otimes_{\mathbb{Z}} S \otimes_{\mathbb{Z}} \mathbb{Z}[L]\right)^{\mathrm{DP}}$ for the kernel of the surjection onto R sending L to 1, we obtain the final object

$$\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}\left(W_m(R) \otimes_{\mathbb{Z}} S \otimes_{\mathbb{Z}} \mathbb{Z}[L]\right)^{\mathrm{DP}}$$

of the site $(R|S)_{\text{log-crys}}$. Details are left to the interested reader (or compare with [4], §4-5).

The advantage of this construction is that it does not use the perfection $P(R)$ of R, but on the other hand it depends on the integer m such that $p^mS = 0$ and is therefore not uniform and it is not true that the previous remark holds for this object.

(d) The existence of the final object of the site $(R|S)_{\text{log-crys}}$ implies that the cohomology of any sheaf of $\mathscr{O}$-modules is canonically isomorphic to the cohomology of its restriction to the étale site of the final object. In particular, the crystalline cohomology of any quasi-coherent $\mathscr{O}$-module vanishes in non-zero degree and we have

$$H^0_{\text{log-crys}}(R|S, \mathscr{O}) = (W(P(R)) \otimes_{\mathbb{Z}} S)^{\text{DP}}_{\log}.$$

In the case S is a p-adic base, using a previous remark we see that

$$\lim_n H^0_{\text{log-crys}}(R|S/p^{n+1}S, \mathscr{O}) = \left((W(P(R)) \otimes_{\mathbb{Z}} S)^{\text{DP}}_{\log}\right)\hat{}.$$

1.2.4. Definition. Let S be a p-adic base and let R be as in the statement of Theorem 1.1 and let

$$A := \lim_n H^0_{\text{log-crys}}(R|S/p^n S, \mathscr{O}).$$

We define the *DP-filtration* $F_p^\bullet$ on A by defining $F_p^i A$ to be the ith divided power of the ideal $F_p^1 A := \operatorname{Ker}(A \to R)$. If $R = \mathcal{R}/p\mathcal{R}$ for a p-adically complete ring $\mathcal{R}$, then we define the *canonical filtration* $F^\bullet$ on A by defining F^i to be the ith divided power of the ideal $F^1 A := \operatorname{Ker}(A \to \mathcal{R})$. The canonical filtration differs from the DP-filtration by the fact that $p \notin F^1$ in general, whereas $p \in F_p^1$.

1.2.5. Let R and S be as in Theorem 1.1 and let $h : A \to R$ be a homomorphism of logarithmic S/pS-algebras, where A has a fine log-structure. Denote also by h the associated morphism of log-crystalline sites

$$h : (R|S)_{\text{log-crys}} \to (A|S)_{\text{log-crys}}.$$

Then for any sheaf of abelian groups $\mathscr{F}$ on $(R|S)_{\text{log-crys}}$, the ith direct image sheaf $R^i h_* \mathscr{F}$ is the sheaf on $(A|S)_{\text{log-crys}}$ associated to the presheaf

$$(U \hookrightarrow T) \rightsquigarrow H^i_{\text{log-crys}}(\operatorname{Spec}(R) \times_{\operatorname{Spec}(A)} U|T, \mathscr{F}).$$

Assume that the Frobenius is surjective on R and on the integral monoid M defining the log-structure of R. If $\mathscr{E}$ is a quasi-coherent sheaf of $\mathscr{O}$-modules on $(R|S)_{\text{log-crys}}$ then it follows immediately from Theorem 1.1 that for $i \neq 0$

$$R^i h_* \mathscr{E} = 0$$

and hence for all i

$$H^i_{\text{log-crys}}(R|S, \mathscr{E}) \cong H^i_{\text{log-crys}}(A|S, h_* \mathscr{E})$$

which is again zero for $i \neq 0$.

Proposition 1.1. *With the above notation and assumptions, $h_* \mathscr{O}$ is a quasi-coherent crystal of $\mathscr{O}$-modules on $(A|S)_{\text{log-crys}}$.*

Proof. Assume $p^n S = 0$. Let $\mathcal{P}$ be the presheaf on $(A|S)_{\text{log-crys}}$ defined

$$\mathcal{P}(U \hookrightarrow T) := H^0_{\text{log-crys}}(\operatorname{Spec}(R) \times_{\operatorname{Spec}(A)} U|T, \mathscr{O}).$$

Consider a morphism $g : (U' \hookrightarrow T') \to (U \hookrightarrow T)$ of $(A|S)_{\text{log-crys}}$. Then $g : T' \to T$ is an open map, and hence

$$\left(g^{-1}\mathcal{P}|_T \otimes_{g^{-1}\mathscr{O}_T} \mathscr{O}_{T'}\right)(T') = \mathcal{P}(g(T')) \otimes_{\mathscr{O}_T(g(T))} \mathscr{O}_{T'}(T').$$

We claim that, up to localizing on U', we have

$$g^{-1}\mathcal{P}|_T \otimes_{g^{-1}\mathscr{O}_T} \mathscr{O}_{T'} \cong \mathcal{P}|_{T'}.$$

We have a commutative square

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & g(U') \arrow[d] \\ T' \arrow[r] & g(T') \end{tikzcd}$$

Let C be the unique $g(T')$-scheme making the following square cartesian

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & g(U') \arrow[d] \\ C \arrow[r] & g(T'). \end{tikzcd}$$

Then $C \to g(T')$ is étale, and we have a commutative square

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & C \arrow[d] \\ T' \arrow[r] & g(T'). \end{tikzcd}$$

Since the left vertical arrow is a nilpotent thickening, there exists a unique morphism $T' \to C$ making the resulting diagram commute. Hence we have a commutative diagram

$$\begin{tikzcd} U' \arrow[r, equal] \arrow[d] & U' \arrow[r] \arrow[d] & g(U') \arrow[d] \\ T' \arrow[r] & C \arrow[r] & g(T') \end{tikzcd}$$

and so we reduce to proving the claim in the following two cases

I. $g : T' \to g(T')$ is étale

II. $g : U' \to g(U')$ is an isomorphism.

In case I we will use the construction of the final object given in 1.2.3 Remark (c) and in case II we will used the construction of the final object given in the proof of Theorem 1.1. We ask the reader to refer to these parts for notation.

Let us prove the claim in case I. Let Λ be a finitely generated integral monoid defining the log-structure on A and define

$$L := \mathrm{Ker}(M^{\mathrm{gp}} \oplus \Lambda^{\mathrm{gp}} \to M^{\mathrm{gp}})$$

where the map $\Lambda \to M$ is the canonical one and $M \to M$ is given by raising to the p^nth power. The square

$$\begin{array}{ccc} W_n(R \otimes_A g^{-1}\mathscr{O}_U) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} g^{-1}\mathscr{O}_T & \longrightarrow & W_n(R \otimes_A \mathscr{O}_{U'}) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} \mathscr{O}_{T'} \\ \downarrow & & \downarrow \\ R \otimes_A g^{-1}\mathscr{O}_U & \longrightarrow & R \otimes_A \mathscr{O}_{U'}. \end{array}$$

is cocartesian, where the map

$$W_n(R \otimes_A g^{-1}\mathscr{O}_U) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} g^{-1}\mathscr{O}_T \to R \otimes_A g^{-1}\mathscr{O}_U$$

(resp.

$$W_n(R \otimes_A \mathscr{O}_{U'}) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} \mathscr{O}_{T'} \to R \otimes_A \mathscr{O}_{U'})$$

is the $g^{-1}\mathscr{O}_T$-linear map (resp. $\mathscr{O}_{T'}$-linear map) deduced from the map sending $(r_0, ..., r_{n-1}) \in W_n(R \otimes_A g^{-1}\mathscr{O}_U)$ (resp. $\in W_n(R \otimes_A \mathscr{O}_{U'})$) to $r_0^{p^n}$ and L to 1. Taking divided power hulls for the kernels of these maps and using the flatness of the map

$$W_n(R \otimes_A g^{-1}\mathscr{O}_U) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} g^{-1}\mathscr{O}_T \to W_n(R \otimes_A \mathscr{O}_{U'}) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} \mathscr{O}_{T'}$$

we obtain

$$g^{-1}\mathcal{P}|_T \otimes_{g^{-1}\mathscr{O}_T} \mathscr{O}_{T'} \cong \mathcal{P}|_{T'}.$$

In case II, we can assume U' is affine (and hence so are T' and $g(T')$). Define

$$L := \mathrm{Ker}(P(M)^{\mathrm{gp}} \oplus \Lambda^{\mathrm{gp}} \to M^{\mathrm{gp}}).$$

Let $\mathscr{I}_{T'}$ (resp. $\mathscr{I}_{g(T')}$) denote the ideal sheaf of U' in T' (resp. U' in $g(T')$). Define the pairs

$$\begin{aligned} (B, I) &= \big(W(P(R \otimes_A \mathscr{O}_{U'})) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} \mathscr{O}_{g(T')}, W(P(R \otimes_A \mathscr{O}_{U'})) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} \mathscr{I}_{g(T')}\big) \\ (B', I') &= (W(P(R \otimes_A \mathscr{O}_{U'})) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} \mathscr{O}_{T'}, W(P(R \otimes_A \mathscr{O}_{U'})) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} \mathscr{I}_{T'}) \end{aligned}$$

Note that since the ring of Witt vectors of a perfect ring of characteristic p is $\mathbb{Z}$-torsion free, $I \subset B$ (resp. $I' \subset B'$) is an ideal. We define a divided power structure on I by setting $(x \otimes y)^{[i]} := x^i \otimes y^{[i]}$ for all $x \in W(P(R \otimes_A \mathscr{O}_{U'})) \otimes_{\mathbb{Z}} \mathbb{Z}[L]$ and $y \in \mathscr{I}_{g(T')}$, where the box exponent denotes the divided power structure on $\mathscr{I}_{g(T')}$. We extend to sums via the binomial formula. This is a well-defined DP-structure. Similarly we define a

DP-structure on I'. Since $\mathscr{O}_{g(T')} \to \mathscr{O}_{T'}$ is a DP-morphism, one sees immediately that the canonical map $B \to B'$ is a DP-morphism. Moreover we have

$$B/I \cong B'/I'.$$

Define

$$\begin{aligned} J &= \operatorname{Ker}(B \to R \otimes_A \mathscr{O}_{U'}) \\ J' &= \operatorname{Ker}(B' \to R \otimes_A \mathscr{O}_{U'}) \end{aligned}$$

for the canonical maps. Then $I \subset J$ and $I' \subset J'$, and via the map $B \to B'$, J maps to J', and by definition we have $B/J \cong B'/J'$. So by [1] Ch. I, Prop. 2.8.2, we have a canonical isomorphism

$$D_B(J) \otimes_B B' \cong D_{B'}(J')$$

where $D_B(J)$ denotes the divided power hull of B for the ideal J. This is precisely

$$g^{-1}\mathcal{P}|_T \otimes_{g^{-1}\mathscr{O}_T} \mathscr{O}_{T'} \cong \mathcal{P}|_{T'}$$

and hence we have shown the claim.

Sheafifying we see that

$$h_*\mathscr{O}|_{T'} \cong g^* \left(h_*\mathscr{O}|_T\right)$$

so $h_*\mathscr{O}$ is a crystal of $\mathscr{O}$-modules. Quasi-coherence is a special case of the proof of case I above. □

1.2.6. Let $\mathscr{T}$ be a topos and $\mathscr{A}$ a sheaf of rings on $\mathscr{T}$ such that we have a ringed topos $(\mathscr{T}, \mathscr{A})$. Denote by $\mathfrak{Qcoh}(\mathscr{T}, \mathscr{A})$ the category of quasi-coherent sheaves of $\mathscr{A}$-modules. If A is a ring, then denote by $\mathfrak{Mod}(A)$ the category of A-modules. Another useful fact is the following

Proposition 1.2. *Let R and S be as before. The global sections functor defines an equivalence of categories*

$$\mathfrak{Qcoh}\left((R|S)_{\text{log-crys}}, \mathscr{O}\right) \leftrightsquigarrow \mathfrak{Mod}\left(H^0_{\text{log-crys}}(R|S, \mathscr{O})\right).$$

Proof. The inverse functor is defined by sending an $A := H^0_{\text{log-crys}}(R|S, \mathscr{O})$-module M to the log-crystalline sheaf

$$(U \hookrightarrow T) \rightsquigarrow M \otimes_A \mathscr{O}_T.$$

Now the result follows by definition of a quasi-coherent crystal. □

1.3 Some Fontaine rings

Let us show that one can give an explicit description of the crystalline cohomology of certain rings with surjective Frobenius, following the proof of Fontaine's theorem.

1.3.1. The first motivating example is Fontaine's ring A_{cris}, constructed as follows. Let K be a complete discrete valued field of characteristic zero and of perfect residue field k of characteristic $p > 0$. Denote by K^+ its valuation ring (i.e. its subring of elements of valuation at least zero). Let $\bar{K}$ be the algebraic closure of K and let $\bar{K}^+$ be its valuation ring. Let us construct the crystalline cohomology of the structure sheaf on the site $(\bar{K}^+/p\bar{K}^+|W)_{\mathrm{crys}}$, where $W := W(k)$ is a p-adic base. Define

$$A_{\mathrm{inf}}(K^+) := W(P(\bar{K}^+/p\bar{K}^+)).$$

As in the proof of Fontaine's theorem, there is a canonical map

$$\theta : A_{\mathrm{inf}}(K^+) \to \hat{\bar{K}}^+$$

defined by choosing lifts, where $\hat{\bar{K}}^+$ is the p-adic completion of $\bar{K}^+$. First note that if $(r^{(n)}) \in A_{\mathrm{inf}}(K^+)/pA_{\mathrm{inf}}(K^+) = P(\bar{K}^+/p\bar{K}^+)$, then, as in the proof of Fontaine's theorem, we may define $\tilde{r}^{(n)} \in \hat{\bar{K}}^+$ by the formula

$$\tilde{r}^{(n)} := \lim_{m\to\infty} \hat{r}^{(n+m)p^m}$$

where $\hat{r}^{(n+m)}$ denotes an arbitrary lift of $r^{(n+m)}$ to $\hat{\bar{K}}^+$. One sees easily that the association sending $(r^{(n)})$ to $\tilde{r}^{(n)}$ is a bijection between $A_{\mathrm{inf}}(K^+)/pA_{\mathrm{inf}}(K^+)$ and the set of sequences (x_n) of elements of $\hat{\bar{K}}^+$ such that $x_{n+1}^p = x_n$ for all n. This enables us to define a valuation on $A_{\mathrm{inf}}(K^+)/pA_{\mathrm{inf}}(K^+)$ by the rule

$$v(r^{(n)}) := v_p(\tilde{r}^{(0)})$$

where v_p is the valuation of $\hat{\bar{K}} = \hat{\bar{K}}^+[1/p]$, normalized by $v_p(p) = 1$. Since $A_{\mathrm{inf}}(K^+)/pA_{\mathrm{inf}}(K^+)$ is a valuation ring, it is an integral domain. Hence, if an element of its fraction field has valuation greater than or equal to zero, it lies in $A_{\mathrm{inf}}(K^+)/pA_{\mathrm{inf}}(K^+)$, because this is true for $\hat{\bar{K}}^+ \subset \hat{\bar{K}}$. In particular, elements of valuation zero are units.

Consider the elements

$$\underline{p} := (p, p^{1/p}, p^{1/p^2}, ...) \in A_{\mathrm{inf}}(K^+)/pA_{\mathrm{inf}}(K^+)$$

and

$$\underline{1} := (1, \zeta_p, \zeta_{p^2}, ...) \in A_{\mathrm{inf}}(K^+)/pA_{\mathrm{inf}}(K^+)$$

where ζ_{p^n} denotes a p^nth root of unity. We claim that $\mathrm{Ker}(\theta)$ is a principal ideal, of which $\xi := [\underline{p}] - p$ and $\frac{[\underline{1}]-1}{[\underline{1}]^{1/p}-1}$ are generators. First notice that $\mathrm{Ker}(\theta \mod p)$ is the set of elements of valuation one, so since ξ clearly has valuation one, it generates this ideal. Also, in $P(\bar{K}^+/p\bar{K}^+)$ we have

$$v\left(\frac{\underline{1}-1}{\underline{1}^{1/p}-1}\right) = v\left(\frac{(\underline{1}^{1/p}-1)^p}{\underline{1}^{1/p}-1}\right) = (p-1)v(\underline{1}^{1/p}-1) = (p-1)\frac{1}{p-1}$$

so $\frac{[\underline{1}]-1}{[\underline{1}]^{1/p}-1}$ is also a generator of $\mathrm{Ker}(\theta \mod p)$. This proves the claim modulo p, hence in general since $A_{\mathrm{inf}}(K^+)$ is p-adically complete.

Taking the divided power hull of $A_{\mathrm{inf}}(K^+)$ for $\mathrm{Ker}(\theta)$ and then the p-adic completion we obtain a "ring of p-adic periods" denoted A_{cris} or $A_{\mathrm{cris}}(K^+)$. Thus elements of $A_{\mathrm{cris}}(K^+)$ are represented by sequences $\sum_n x_n \xi^{[n]}$, with $x_n \in A_{\mathrm{inf}}(K^+)$ tending to zero p-adically with n. By construction, we have

$$A_{\mathrm{cris}}(K^+) = \lim_n H^0_{\mathrm{crys}}(\bar{K}^+/p\bar{K}^+|W_n(k), \mathscr{O}).$$

As a Fontaine ring, $A_{\mathrm{cris}}(K^+)$ is endowed with a canonical filtration (Def. 1.2.4)

$$F^i A_{\mathrm{cris}}(K^+) = (\mathrm{Ker}(\theta))^{[i]}$$

(ith divided power of the ideal $\mathrm{Ker}(\theta)$). Define an element of $F^1 A_{\mathrm{cris}}$ by

$$t := \log([\underline{1}]) = -\sum_{n>0}(n-1)!(1-[\underline{1}])^{[n]}.$$

This element plays a very important role in p-adic Hodge theory. By functoriality of crystalline cohomology, A_{cris} has a Frobenius endomorphism Φ. One usually writes

$$B^+_{\mathrm{cris}} := A_{\mathrm{cris}}(K^+)[1/p]$$

and

$$B_{\mathrm{cris}} := B^+_{\mathrm{cris}}[1/t].$$

The canonical filtration induces a filtration on B^+_{cris} and we define the filtration of B_{cris} by

$$F^i B_{\mathrm{cris}} := \cup_j t^{-j} F^{i+j} B^+_{\mathrm{cris}}.$$

Since $\Phi(t) = pt$, it follows that the action of Φ extends to B_{cris} in a natural way.

1.3.2. The schemes we will consider will not generally have smooth lifts to $W_n(k)$, but they will have smooth lifts to a formal divided power lifting Σ of K^+/pK^+, defined as follows. Firstly, making a choice of uniformizer π of K^+ determines a presentation $K^+ = W(k)[u]/(E(u))$, where $E(u)$ is the minimal equation of π over $W(k)$, i.e. $E(u)$ is an Eisenstein equation of degree e, where e is the ramification index of K^+ over $W(k)$. So $W_n(k)[u]$ is a canonical smooth $W_n(k)$-lift of $K^+/pK^+ = k[u]/(u^e)$. Taking the divided power hull for the kernel of the surjection $W_n(k)[u] \to K^+/pK^+$, we obtain $\Sigma_n = W_n(k)[u]\langle u^e\rangle$. It has a lifting of the absolute Frobenius of K^+/pK^+ defined as the unique homomorphism sending u to u^p and restricting to the canonical Frobenius on $W_n(k)$. Finally we define

$$\Sigma := \lim_n \Sigma_{n+1}.$$

1.3.3. Let us compute the crystalline cohomology of $\bar{K}^+/p\bar{K}^+$ over the p-adic base Σ. This reduces to computing the kernel of the canonical map

$$A_{\mathrm{inf}}(K^+) \otimes_{\mathbb{Z}} \Sigma \to \bar{K}^+/p\bar{K}^+.$$

We claim that it is generated by ξ and $[\underline{\pi}] \otimes 1 - 1 \otimes u$, where

$$\underline{\pi} = (\pi, \pi^{1/p}, \pi^{1/p^2}, ...) \in A_{\mathrm{inf}}(K^+)/pA_{\mathrm{inf}}(K^+).$$

To see this, assume $\sum_i x_i \otimes u^i$ lies in the kernel, and write

$$\begin{aligned}
\sum_i x_i \otimes u^i &= \sum_i (x_i \otimes 1)(1 \otimes u - [\underline{\pi}] \otimes 1 + [\underline{\pi}] \otimes 1)^i \\
&= \sum_i \sum_{j=0}^{i} \binom{i}{j} (x_i \cdot [\underline{\pi}]^{i-j} \otimes 1)(1 \otimes u - [\underline{\pi}] \otimes 1)^j \\
&= \sum_i x_i \cdot [\underline{\pi}]^i \otimes 1 + \sum_i \sum_{j=1}^{i} \binom{i}{j} (x_i \cdot [\underline{\pi}]^{i-j} \otimes 1)(1 \otimes u - [\underline{\pi}] \otimes 1)^j.
\end{aligned}$$

Since $\sum_i \sum_{j=1}^i \binom{i}{j} (x_i \cdot [\underline{\pi}]^{i-j} \otimes 1)(1 \otimes u - [\underline{\pi}] \otimes 1)^j$ obviously maps to zero, we see that $\sum_i x_i \cdot [\underline{\pi}]^i \otimes 1$ must also map to zero. From the commutative diagram

$$\begin{array}{ccc}
A_{\mathrm{inf}}(K^+) & \longrightarrow & A_{\mathrm{inf}}(K^+) \otimes_{\mathbb{Z}} \Sigma \\
\downarrow & & \downarrow \\
\hat{\bar{K}}^+ & = & \hat{\bar{K}}^+
\end{array}$$

we deduce that $\sum_i x_i \cdot [\underline{\pi}]^i \in \xi \cdot A_{\mathrm{inf}}(K^+)$, thereby proving the claim.

We define B^+ to be the p-adic completion of the divided power hull of $A_{\mathrm{inf}}(K^+) \otimes_{\mathbb{Z}} \Sigma$ for the ideal generated by ξ and $[\underline{\pi}] - u$. We have

$$B^+ = \lim_n H^0_{\mathrm{crys}}(\bar{K}^+/p\bar{K}^+ | \Sigma_n, \mathscr{O}).$$

B^+ is endowed with the canonical filtration F defined by the divided powers of the ideal $\mathrm{Ker}(B^+ \to \hat{\bar{K}}^+)$. Its interpretation in terms of crystalline cohomology also implies that it is endowed with a Frobenius endomorphism.

1.3.4. Warning. Do not confuse B^+ with B^+_{cris}!

1.3.5. Let us turn to the logarithmic case. Define the *canonical log-structure* on Σ_n to be the fine log-structure associated to the pre-log-structure

$$\mathcal{L}(u) : \mathbb{N} \to \Sigma_n : 1 \mapsto u.$$

Composing this with the canonical map $\Sigma_n \to K^+/p^n K^+$ defines a pre-log-structure on the latter, making it into a log-Σ_n-scheme. The associated log-structure is also the inverse image of the canonical log-structure on K^+ defined

$$K^+ - \{0\} \to K^+.$$

Similarly, the canonical log-structure on $\bar{K}^+$ is given by

$$\bar{K}^+ - \{0\} \to \bar{K}^+$$

and we endow $\bar{K}^+/p\bar{K}^+$ with the inverse image log-structure. If we fix roots of π, then it is the log-structure associated to the pre-log-structure

$$M : \mathbb{Q}_+ \to \bar{K}^+/p\bar{K}^+ : \alpha \mapsto \pi^\alpha.$$

Note that although this pre-log-structure depends on choices of roots of π, its associated log-structure does not, as the quotient of two choices of roots of π will be a unit of $\bar{K}^+/p\bar{K}^+$. Let

$$L := \mathrm{Ker}(P(M)^{\mathrm{gp}} \oplus \mathcal{L}(u)^{\mathrm{gp}} \to M^{\mathrm{gp}}).$$

Since $P(M)$ consists of sequences (x_n) of non-negative rational numbers such that $p \cdot x_{n+1} = x_n$ for all n, we see that $P(M)^{\mathrm{gp}}$ consists of sequences (x_n) of rational numbers such that $p \cdot x_{n+1} = x_n$ for all n, i.e. $P(M)^{\mathrm{gp}} \cong \mathbb{Q}$. So we see that L is the kernel of the map

$$\mathbb{Q} \oplus \mathbb{Z} \to \mathbb{Q} : (\alpha, m) \mapsto \alpha + m$$

i.e. L consists of pairs $(m, -m) \in \mathbb{Z}^2$. Note that under the map to $A_{\mathrm{inf}}(K^+) \otimes_{\mathbb{Z}} \Sigma$, $(m \cdot p^{-n})_n \in P(M)$ maps to $[\underline{\pi}]^m \otimes 1$ and $m \in \mathcal{L}(u)$ maps to $1 \otimes u^m$, so one should think of $(m, -m)$ as $[\underline{\pi}]^m \otimes u^{-m}$.

Define

$$B^+_{\mathrm{log}} := \lim_n H^0_{\text{log-crys}}(\bar{K}^+/p\bar{K}^+|\Sigma_n, \mathscr{O}).$$

By construction, B^+_{log} is the p-adic completion of the divided power hull of $A_{\mathrm{inf}}(K^+) \otimes_{\mathbb{Z}} \Sigma \otimes_{\mathbb{Z}} \mathbb{Z}[L]$ for the kernel of the canonical surjection onto $\hat{\bar{K}}^+$. Suppose $\sum_i x_i \otimes u^i \otimes l_i$ lies in the kernel of the canonical map

$$A_{\mathrm{inf}}(K^+) \otimes_{\mathbb{Z}} \Sigma \otimes_{\mathbb{Z}} \mathbb{Z}[L] \to \hat{\bar{K}}^+.$$

Then we have

$$\sum_i x_i \otimes u^i \otimes l_i = \sum_i x_i \otimes u^i \otimes 1 + \sum_i x_i \otimes u^i \otimes (l_i - 1)$$

so $\sum_i x_i \otimes u^i \in (\xi, [\underline{\pi}] - u) \cdot A_{\mathrm{inf}}(K^+) \otimes_{\mathbb{Z}} \Sigma$. Hence we deduce that $\sum_i x_i \otimes u^i \otimes l_i$ lies in the ideal generated by ξ, $[\underline{\pi}] - u$ and $l - 1$ as l ranges over the elements of L.

Proposition 1.3 (Kato)**.** *Every choice of sequence of roots of π determines an isomorphism*

$$B^+_{\mathrm{log}}/p^n B^+_{\mathrm{log}} \simeq A^+_{\mathrm{cris}}/p^n A^+_{\mathrm{cris}} \langle X \rangle$$

where $A^+_{\mathrm{cris}}/p^n A^+_{\mathrm{cris}} \langle X \rangle$ is a divided power polynomial ring in one indeterminate X over the ring $A^+_{\mathrm{cris}}/p^n A^+_{\mathrm{cris}}$.

Proof. It suffices to show that $A^+_{\text{cris}}/p^n A^+_{\text{cris}}\langle X\rangle$ has the universal property. Make

$$C := A_{\text{inf}}(K^+)/p^n A_{\text{inf}}(K^+)\left[X, \frac{1}{1+X}\right]$$

in to a $W_n[u]$-algebra by sending u to $[\underline{\pi}]\cdot(1+X)^{-1}$. If $(U \hookrightarrow T)$ is an affine object of $(\bar{K}^+/p\bar{K}^+|\Sigma_n)_{\text{log-crys}}$, then define a $W_n[u]$-algebra map

$$C \to \mathscr{O}_T$$

extending the canonical map $\theta_T : A_{\text{inf}}(K^+)/p^n A_{\text{inf}}(K^+) \to \mathscr{O}_T$ by sending X to $\theta_T([\underline{\pi}])\cdot u^{-1}-1$ (this element exists in $\mathscr{O}_T$ by Lemma 1.1). If we are given another map

$$\alpha : C \to \mathscr{O}_T$$

then by $W_n[u]$-linearity we must have $\alpha([\underline{\pi}]\cdot(1+X)^{-1}) = u$. But

$$u = \alpha([\underline{\pi}]\cdot(1+X)^{-1}) = \alpha([\underline{\pi}])\cdot(1+\alpha(X))^{-1} = \theta_T([\underline{\pi}])\cdot(1+\alpha(X))^{-1}$$

so $\alpha(X) = \theta_T([\underline{\pi}])\cdot u^{-1}-1$, hence the map is unique. Taking the divided power hull for the kernel of the map in the case $\mathscr{O}_T = \bar{K}^+/p\bar{K}^+$ it is clear that we obtain the ring

$$A^+_{\text{cris}}/p^n A^+_{\text{cris}}\langle X\rangle\left[\frac{1}{1+X}\right] \cong A^+_{\text{cris}}/p^n A^+_{\text{cris}}\langle X\rangle .$$

Let us endow this ring with the pre-log-structure

$$P(M)\oplus\mathcal{L}(u) \to A^+_{\text{cris}}/p^n A^+_{\text{cris}}\langle X\rangle : \Big((m^{(i)})_{i\in\mathbb{N}}, n\Big) \mapsto [\underline{\pi}]^{m(0)}\otimes u^n$$

where we write u for its image $[\underline{\pi}]\cdot(1+X)^{-1}$ in $A^+_{\text{cris}}/p^n A^+_{\text{cris}}\langle X\rangle$, and M is the log-structure of $\bar{K}^+/p\bar{K}^+$ defined above. We need to check that it belongs to the log-crystalline site, i.e. the closed immersion associated to the surjective homomorphism

$$A^+_{\text{cris}}/p^n A^+_{\text{cris}}\langle X\rangle \to \bar{K}^+/p\bar{K}^+$$

is exact. For this it suffices to check that the elements $l \in L := \operatorname{Ker}(P(M)^{\text{gp}}\oplus\mathcal{L}(u)^{\text{gp}} \to M^{\text{gp}})$ are units. But from the computation of L preceding the statement of the proposition, we know that $l = (1+X)^m$ for some integer m, hence everything follows. □

So one should think of X as $[\underline{\pi}]\otimes u^{-1}-1$. $B^+_{\log}$ is also endowed with the canonical filtration F defined as the divided power filtration of the ideal $F^1B^+_{\log} := \operatorname{Ker}(B^+_{\log} \to \hat{\bar{K}}^+)$ and a Frobenius endomorphism. Define the *monodromy operator* N on $B^+_{\log}$ as the unique $A_{\text{cris}}(K^+)$-linear derivation satisfying

$$N(X) = 1+X.$$

1.3.6. There is another ring B_{dR}^+ constructed by Fontaine, defined

$$B_{\text{dR}}^+ := \lim_n \left(A_{\text{cris}}^+(K^+) \otimes_{W(k)} K\right)/I^{n+1}$$

where $I := \text{Ker}(A_{\text{cris}}^+(K^+) \otimes_{W(k)} K \to \hat{\bar{K}})$. It is know [8] that B_{dR}^+ is a discrete valuation ring of uniformizer t and residue field $\hat{\bar{K}}$, and that the natural map $A_{\text{cris}}^+(K^+) \otimes_{W(k)} K \to B_{\text{dR}}^+$ is injective. We have $\text{gr}_I^i B_{\text{dR}}^+ \cong \hat{\bar{K}}(i)$ (Tate twist), where we use the logarithm $\log : \mathbb{Z}_p(1) \to A_{\text{cris}}^+(K^+)$ to identify $\hat{\bar{K}}t^i$ with $\hat{\bar{K}}(i)$.

Proposition 1.4 (Fontaine)**.** *Let $\alpha \in \mathbb{Q}$. If $p > 2$, then*

(i) $t^{p-1} \in p \cdot F^1 A_{\text{cris}}^+(K^+)$

(ii) $\frac{t \cdot p^{\max(v_p(\alpha),0)}}{[\underline{1}]^\alpha - 1} \in A_{\text{cris}}^+(K^+)$.

Proof. For (i), first recall that $1 - [\underline{1}] = u \cdot \xi \cdot (1 - [\underline{1}]^{1/p})$ for some unit $u \in A_{\text{inf}}(K^+)$. Since $(1 - \underline{1}^{1/p})^{p-1} \in A_{\text{inf}}(K^+)/pA_{\text{inf}}(K^+)$ is an element of valuation one, it follows that it lies in the ideal generated by ξ. Hence $(1 - \underline{1})^{p-1}$ lies in the ideal generated by ξ^p. But we have $\xi^p = p!\xi^{[p]}$ in $A_{\text{cris}}^+(K^+)$, so we deduce that $(1 - [\underline{1}])^{p-1} \in p \cdot A_{\text{cris}}^+(K^+)$. Now, we have

$$t = -\sum_{n=1}^{p}(n-1)!(1-[\underline{1}])^{[n]} - \sum_{n=p+1}^{\infty}(n-1)!(1-[\underline{1}])^{[n]}$$

and the second sum is divisible by p, so it suffices to consider

$$\left(\sum_{n=p+1}^{\infty}(n-1)!(1-[\underline{1}])^{[n]}\right)^{p-1} = \sum_{j_1+\ldots+j_p=p-1} \frac{(p-1)!(1-[\underline{1}])^{\sum i \cdot j_i}}{j_1! \cdots j_p! \prod_{i=1}^{p} i^{j_i}}.$$

Here each summand has p-adic valuation j_p in the denominator and at least $\left[\frac{\sum i \cdot j_i}{p-1}\right] \geq j_p + 1$ in the numerator. This proves (i).

For (ii), we separate in two cases:

I. $v_p(\alpha) \geq 0$

II. $v_p(\alpha) < 0$.

In case I, first assume that $\alpha \in \mathbb{Z}$. Then $[\underline{1}]^\alpha - 1 \in \text{Ker}(A_{\text{cris}}(K^+) \to \hat{\bar{K}}^+)$, so the divided power series $\log([\underline{1}]^\alpha)$ exists and converges to $\alpha \cdot t$. We have

$$[\underline{1}]^\alpha - 1 = \exp(\alpha \cdot t) - 1 = \alpha \cdot t \cdot \sum_{n>0} \frac{(\alpha \cdot t)^{n-1}}{n!}$$

and

$$\frac{t^{n-1}}{n!} = \frac{p^{q_n}}{n!} q_n!(t^{p-1}/p)^{[q_n]} t^{r_n}$$

where $n-1 = q_n(p-1)+r_n$ and $q_n = \left[\dfrac{n-1}{p-1}\right] \geq v_p(n!)$. So $u_\alpha := \sum_{n>0} \dfrac{(\alpha \cdot t)^{n-1}}{n!}$ converges to a unit of $A^+_{\mathrm{cris}}(K^+)$ for all $\alpha \in \mathbb{Z}$. Hence assertion (ii) in this case. If $\alpha = \frac{x}{y}$ with x, y integers and $v_p(y) = 0$, then $\frac{[\underline{1}]^x-1}{[\underline{1}]^\alpha-1} \in A^+_{\mathrm{cris}}(K^+)$, so assertion (ii) holds in this case.

In case II, write $\alpha = \frac{x}{yp^n}$ with x, y coprime integers and $v_p(x/y) = 0$. By case I we have $([\underline{1}]^x-1)^{-1} \in t^{-1} \cdot A^+_{\mathrm{cris}}(K^+)$, and $\frac{[\underline{1}]^x-1}{[\underline{1}]^\alpha-1} \in A^+_{\mathrm{cris}}(K^+)$, hence assertion (ii) in this case. □

Corollary 1.1. $\mathrm{gr}^i B^+_{\mathrm{cris}} \cong \hat{\bar{K}}(i)$

Proof. We know that $\mathrm{gr}^i B^+_{\mathrm{cris}}$ is generated by $\left(\frac{[\underline{1}]-1}{[\underline{1}]^{1/p}-1}\right)^i$ as a $\mathrm{gr}^0 B^+_{\mathrm{cris}} = \hat{\bar{K}}$-module. Since $(\underline{1}]^{1/p}-1)$ maps to $p^{\frac{1}{p-1}}x$ in $\hat{\bar{K}}^+$ for some unit x, we deduce that $\mathrm{gr}^i B^+_{\mathrm{cris}}$ is generated by $([\underline{1}]-1)^i$. In the proof of the last proposition, we have seen that $t = v \cdot ([\underline{1}]-1)$ for some unit $v \in 1 + t \cdot A^+_{\mathrm{cris}}(K^+)$, so we deduce that $\mathrm{gr}^i B^+_{\mathrm{cris}}$ is generated by t^i as a $\hat{\bar{K}}$-module. Finally the logarithm $\log : \mathbb{Z}_p(1) \to A^+_{\mathrm{cris}}(K^+)$ identifies $\mathbb{Z}_p(i)$ with the $\mathbb{Z}_p$-submodule of $A^+_{\mathrm{cris}}(K^+)$ generated by t^i. □

Proposition 1.5. *Every choice of uniformizer $\pi \in K^+$ determines a $\mathrm{Gal}(\bar{K}/K)$-equivariant isomorphism of $\hat{\bar{K}}$-modules for all n*

$$\mathrm{gr}^n B^+_{\log} \cong \mathrm{gr}^n B^+[1/p] \cong \oplus_{i+j=n} E^i \cdot \hat{\bar{K}}(j)$$

where E is an Eisenstein equation for π and $i, j \in \mathbb{N}$.

Proof. The proof for B^+ works for $B^+_{\log}$, so consider the former. Making a choice of uniformizer π determines an Eisenstein equation $E \in \Sigma$ and defines a surjection $\Sigma \to K^+$. Consider the ring

$$B^+ \hat{\otimes}_\Sigma K^+.$$

It is a filtered B^+ module via the canonical map, and clearly

$$\mathrm{gr}^0(B^+ \hat{\otimes}_\Sigma K^+) = \hat{\bar{K}}^+.$$

Hence for all n there is a canonical surjection

$$\mathrm{gr}^n B^+[1/p] \cong (\mathrm{gr}^n B^+[1/p]) \otimes_{\hat{\bar{K}}^+} \mathrm{gr}^0(B^+ \hat{\otimes}_\Sigma K^+) \to \mathrm{gr}^n(B^+ \hat{\otimes}_\Sigma K^+)[1/p].$$

Since the completion of $(B^+ \hat{\otimes}_\Sigma K^+)[1/p]$ for its filtration is clearly B^+_{dR}, they have the same graded, i.e.

$$\mathrm{gr}^n(B^+ \hat{\otimes}_\Sigma K^+)[1/p] \cong \hat{\bar{K}}(n).$$

So the surjection $\mathrm{gr}^n B^+[1/p] \to \mathrm{gr}^n(B^+ \hat{\otimes}_\Sigma K^+)[1/p]$ admits a section, namely the canonical map

$$\hat{\bar{K}}(n) \cong \mathrm{gr}^n B^+_{\mathrm{cris}} \to \mathrm{gr}^n B^+[1/p].$$

Now, if E is an Eisenstein equation for π, then we have an exact sequence

$$0 \longrightarrow (E \cdot \Sigma_n)^{\mathrm{DP}} \longrightarrow \Sigma_n \longrightarrow K^+/p^n K^+ \longrightarrow 0$$

which, tensored with B^+ yields and exact sequence

$$0 \longrightarrow (E \cdot B^+/p^n B^+)^{\mathrm{DP}} \longrightarrow B^+/p^n B^+ \longrightarrow B^+ \otimes_{\Sigma_n} K^+/p^n K^+ \longrightarrow 0$$

Now an easy induction on r shows that we have split short exact sequences for all r

$$0 \to \oplus_{i+j=r, i>0} E^i \cdot \mathrm{gr}^j (B^+ \hat{\otimes}_\Sigma K^+)[1/p] \to \mathrm{gr}^r B^+[1/p] \to \mathrm{gr}^r (B^+ \hat{\otimes}_\Sigma K^+)[1/p] \to 0$$

as claimed. $\square$

2 Almost ring theory

The basic ideas of almost ring theory were discovered by Faltings and developed in his papers on p-adic Hodge theory. More recently, a book devoted to the subject has been written by O. Gabber and L. Ramero [9], thereby laying comprehensive foundations for Faltings' approach to p-adic Hodge theory. We begin by recalling the necessary definitions and theorems and then apply them to the study of Fontaine rings.

2.1 Reminder on almost ring theory

2.1.1. The basic object of study in almost ring theory is the category of modules on rings Λ (commutative with unity) together with an ideal $\mathfrak{m}$ satisfying $\mathfrak{m}^2 = \mathfrak{m}$. We assume further that $\mathfrak{m}$ is the countable union of principal ideals (x^α), indexed by an infinite additive monoid $\Gamma \subset \mathbb{Q}_{>0}$, such that x^α is not a zero divisor for all $\alpha \in \Gamma$ and furthermore for all $\alpha, \beta \in \Gamma$ we have

$$x^\alpha \cdot x^\beta = u \cdot x^{\alpha+\beta}$$

for some unit $u \in \Lambda$.

By localizing the category of Λ-modules at the subcategory of elements annihilated by $\mathfrak{m}$, one obtains the category of almost Λ-modules. We'll say that two Λ-modules M, N are *almost isomorphic* if they are isomorphic as almost Λ-modules, and we write this as $M \approx N$. A morphism $M \to N$ of the category of Λ-modules is represented by a diagram of real maps of Λ-modules

$$\begin{tikzcd} M & N \arrow[d, "\approx"'] \\ M' \arrow[u, "\approx"] \arrow[r] & N' \end{tikzcd}$$

where the vertical maps are descend to almost isomorphisms.

We can also define the category of almost Λ-algebras (commutative with unity). This is done by formally inverting the homomorphisms of Λ-algebras $A \to B$ whose kernel and cokernel are annihilated by $\mathfrak{m}$. In this category morphisms admit a similar description as the category of almost Λ-modules above, except of course that we require all maps to be homomorphisms of Λ-algebras.

2.1.2. The motivating example is the case $\Lambda = \bar{K}^+$, $\Gamma = \mathbb{Q}_{>0}$, and $x^\varepsilon = p^\varepsilon$ for all $\varepsilon \in \mathbb{Q}_{>0}$. An example of almost isomorphism of $\bar{K}^+$-modules is given by the canonical homomorphism of differentials

$$\Omega_{K_\infty^+/K^+} \otimes_{K_\infty^+} \bar{K}^+ \to \Omega_{\bar{K}^+/K^+}$$

where K_∞^+ denotes the normalization of K^+ in the extension of K generated by all p-power roots of unity.

2.1.3. We will always refer to morphisms in these almost categories with the adjective "almost", e.g. "almost homomorphism", etc. So if we do not use the adjective "almost" we will be referring to usual real morphisms.

2.1.4. Let A be a Λ-algebra. An A-module M is

- *almost flat* if the functor $(-) \otimes_A M$ is almost exact
- *almost faithfully flat* if the functor $(-) \otimes_A M$ is almost exact and faithful
- *almost projective* if $\mathrm{Ext}^1_A(M, N) \approx 0$ for any A-module N
- *almost finitely generated* (resp. *almost finitely presented*) if for all $\alpha \in \Gamma$ there exists a finitely generated (resp. finitely presented) A-module N, together with maps of A-modules $f_\alpha : M \to N$ and $g_\alpha : N \to M$ such that $f_\alpha \circ g_\alpha = g_\alpha \circ f_\alpha = x^\alpha$
- an *almost projective module of finite rank* if its is almost projective, almost finitely generated, and there is an integer r such that $\wedge_A^{r+1} M \approx 0$
- an *almost projective module of rank r* if it is an almost projective module of finite rank with $\wedge_A^{r+1} M \approx 0$ and $L := \wedge_A^r M$ is an *almost invertible* A-module, i.e. the canonical map $L \otimes_A L^* \to A$ is an almost isomorphism, where $N^* := \mathrm{Hom}_A(N, A)$ for any A-module N.

By [9] Lemma 3.12 almost flatness, almost faithful flatness, and almost projectivity are all preserved by arbitrary base change. By [9] Prop. 2.4.18, every almost finitely generated projective A-module is almost finitely presented and every almost finitely presented flat A-module is almost projective. Also, by [9] Prop. 4.3.27, for every almost projective A-module M of finite rank, say $\wedge_A^{r+1} M \approx 0$, there exists a decomposition

$$A \approx \prod_{i=0}^{r} A_i$$

such that $M \otimes_A A_i$ is an almost projective A_i-module of rank i.

2.1.5. A homomorphism of Λ-algebras $A \to B$ is

- *almost unramified* if it makes B into an almost projective $B \otimes_A B$-module
- *almost étale* if it makes B into an almost flat and almost unramified A-algebra

- an *almost étale covering* if it is almost étale and makes B into an almost projective A-module of finite rank.

As in the classical case, almost étale coverings form a category (see [9], §3.1).

Lemma 2.1. *(i) Almost étale coverings are stable by base change.*

(ii) If $A \to B \to C$ are homomorphisms such that $A \to B$ and the composition $B \to C$ are almost étale coverings, then so is $B \to C$.

(iii) An almost projective module of finite rank which is everywhere of non-zero rank is almost faithfully flat. In particular, an almost étale covering which is everywhere of non-zero rank is almost faithfully flat.

Proof. By [9] Lemma 3.1.2, assertions (i) and (ii) are true for almost étale homomorphisms. For (i), note that the notions of almost flatness, almost finite generation, and almost finite presentation are all stable by base change. Hence almost finitely generated projectivity is also stable by base change. Finally, using that taking exterior powers commutes with base change we obtain that almost projectivity of finite rank is stable by base change, proving (i).

For (ii), we need to check that C is an almost projective B-module of finite rank. Since B is almost unramified over A, by [9] Cor. 3.1.9, the canonical exact sequence

$$0 \longrightarrow I \longrightarrow B \otimes_A B \longrightarrow B$$

splits to define an almost isomorphism of $B \otimes_A B$-modules $B \oplus I \approx B \otimes_A B$. So I is almost finitely generated as a $B \otimes_A B$-module, and by transitivity of almost finite generation ([9] Lemma 2.4.7) I is also almost finitely generated as a B-module. Tensoring over B with the almost flat B-module C we obtain an almost isomorphism

$$C \oplus C \otimes_B I \approx C \otimes_A B$$

hence $C \approx C \otimes_A B / C \otimes_B I$. Note that $C \otimes_A B$ is an almost finitely presented B-module because $A \to C$ is an almost étale covering. Also, $C \otimes_B I$ is almost finitely generated as a $C \otimes_A B = C \otimes_B (B \otimes_A B)$-module, so it is almost finitely generated as a B-module. By [9] Lemma 2.3.18, it follows that C is almost finitely presented as a B-module. Since C is also an almost étale (hence almost flat) B-algebra, we deduce that it is an almost projective B-module. Finally, for all i the canonical map $\wedge^i_A C \to \wedge^i_B C$ is obviously surjective, hence C is an almost projective B-module of finite rank, as required.

For (iii), it suffices to show that an almost projective A-module P of non-zero constant rank is faithfully flat. By [9] Prop. 2.4.28, we know that the almost faithful flatness of an A-module M is equivalent to the almost surjectivity of the canonical map $M \otimes_A M^* \to A$. Write

$$E_{P/A} := \mathrm{Im}(P \otimes_A P^* \to A).$$

It is an ideal. We claim that $E_{\wedge^i_A P/A} \to E_{P/A}$ is almost injective for all $i > 0$. To see this, let $B = A/E_{P/A}$. By [9] Prop. 2.4.28, we have $E_{P \otimes_A B/A} = E_{P/A} \cdot B = 0$, so $P \otimes_A B \approx 0$

because P is almost projective. Hence $E_{\wedge_A^i P/A} \cdot B = E_{\wedge_B^i (P \otimes_A B)/B} \approx 0$ and the claim follows. Now, since $\wedge_A^i P$ is almost invertible for some $i > 0$, we have $E_{\wedge_A^i P} \approx A$ and this completes the proof. $\square$

2.1.6. The behaviour of almost étale morphisms of $\mathbb{F}_p$-algebras under the Frobenius endomorphism is studied in some detail in [9]. We will need the following

Theorem 2.1 (Gabber-Ramero)**.** *Let $A \to B$ be an almost étale homomorphism of $\mathbb{F}_p$-algebras. Then the commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle\Phi}\big\downarrow & & {\scriptstyle\Phi}\big\downarrow \\ A & \longrightarrow & B \end{array}$$

is almost cocartesian, where Φ denotes the (absolute) Frobenius endomorphism. In particular, the relative Frobenius of B over A is an almost isomorphism.

This is Theorem 3.5.13 of [9]. We will need the following corollary.

Corollary 2.1. *If $A \to B$ is an almost étale homomorphism of $\mathbb{F}_p$-algebras, then the induced homomorphism of Witt vectors*

$$W_n(A) \to W_n(B)$$

is almost étale.

Proof. The proof is the same as the classical case [14]. Let us reproduce it here. By [9] Cor. 3.2.11 (iii) an almost flat lift of an almost étale homomorphism is unique, hence it suffices to prove that the above homomorphism is almost flat. For this, we use the filtration of the Witt vectors given by the powers of the Verschiebung V (*cf.* [14] 0., Prop. 1.5.8). For any ring A, define $V^i W_n(A)$ to be the image of $W_n(A)$ under the i-fold iteration of the Verschiebung V. This defines a filtration of $W_n(A)$ and set $\mathrm{gr}_V W_n(A) := \oplus_{i \geq 0} V^i W_n(A)/V^{i+1} W_n(A)$. If A is a ring of characteristic p, then $VW_n(A)$ is an ideal with a canonical divided power structure ([14], 0., 1.4.3) and in particular is nilpotent, so this filtration is finite.

By [14] 0., 1.3.15, for any ring A of characteristic p we have canonical isomorphisms

$$\mathrm{gr}_V W_n(A) \cong \oplus_{m<n} \Phi_*^m(A)$$

where Φ^m denotes the m-fold iteration of Φ and $\Phi_*^m(A)$ denotes A considered as a module over itself via the map Φ^m. From the theorem it follows that we have a canonical almost isomorphism

$$B \otimes_A \mathrm{gr}_V W_n(A) \approx \mathrm{gr}_V W_n(B).$$

By the almost version of the standard flatness criterion ([3] Ch. 3, §5, no. 2, Thm. 1), the almost flatness of $W_n(A) \to W_n(B)$ follows. $\square$

2.1.7. If R is a ring and X is a finite set, then we write $R^{\times X}$ for the product $\prod_{x\in X} R_x$, where $R_x = R$ for all $x \in X$. We say that an almost étale homomorphism $A \to B$ of Λ-algebras is a *formal almost Galois covering* of group G if G is a finite group acting by A-algebra automorphisms on B such that the canonical map

$$B \otimes_A B \to B^{\times G}$$

induced by the maps $B \to B^{\times G} : b \mapsto (b, b, ..., b)$ and $B \to B^{\times G} : b \mapsto (g(b))_{g\in G}$, is an almost isomorphism.

We say that an almost étale homomorphism $A \to B$ of Λ-algebras is an *almost Galois covering* if it is a formal almost Galois covering and in addition is almost faithfully flat.

Note that a (formal) almost Galois covering is preserved under arbitrary base change.

Proposition 2.1. *Let $A \to B$ be a formal almost Galois covering of group G, and M a B-module with semi-linear G-action. Then*

(i) $M^G \otimes_A B \approx M$

(ii) $H^i(G, M) \approx 0$ *for all* $i \neq 0$.

Proof. Because B is an almost projective $B \otimes_A B$-module, there exists a $B \otimes_A B$-linear homomorphism $B \to B \otimes_A B$ which is a section of the canonical multiplication map. This gives a real map $\mathfrak{m} \otimes_\Lambda B \to B \otimes_A B$ or equivalently $B \to \mathrm{Hom}_\Lambda(\mathfrak{m}, B \otimes_A B)$. Then the image e of 1 in $\mathrm{Hom}_\Lambda(\mathfrak{m}, B \otimes_A B)$ is an idempotent. Moreover, for all $\varepsilon \in \Gamma$ we may consider $x^\varepsilon e$ as an element of $B \otimes_A B$, say $x^\varepsilon e = \sum_i x_i \otimes y_i$. Since $A \to B$ is an almost Galois covering, it follows that $B \otimes_A B$ is almost isomorphic to $\prod_{g\in G} g(e) \cdot B$ with G acting transitively on the $g(e)$. This implies that

$$\mathrm{Tr}(x^\varepsilon e) := \sum_{g\in G} g(x^\varepsilon e) = x^\varepsilon$$

and if $g \neq \mathrm{id}$ then $g(x^\varepsilon e)$ maps to zero in B. This allows us to construct a map

$$\begin{array}{rcl} s : M & \to & B \otimes_A M^G \\ m & \mapsto & \displaystyle\sum_i x_i \otimes \mathrm{Tr}(y_i m). \end{array}$$

We claim that composition either way with the natural map $B \otimes_A M^G \to M$ is multiplication by x^ε. One way we have

$$\begin{aligned} s(m) = \sum_i \sum_{g\in G} x_i \otimes g(y_i m) &= \sum_i x_i \otimes y_i m + \sum_{g\neq \mathrm{id}} \left(\sum_i x_i \otimes g(y_i) \right) \cdot (1 \otimes g(m)) \\ &= (x^\varepsilon e)(1 \otimes m) + \sum_{g\neq id} g(x^\varepsilon e) \cdot (1 \otimes g(m)). \end{aligned}$$

So $s(m)$ maps to zero in x^ε in B since $g(x^\varepsilon e)$ maps to zero in B for all $g \neq \mathrm{id}$. Going the other way, let $z \otimes m \in B \otimes_A M^G$. Then

$$s(z \cdot m) = \sum_i x_i \otimes \mathrm{Tr}(y_i z) = \sum_i x_i \,\mathrm{Tr}(y_i z) \otimes m = \sum_{g \in G} \left(\sum_i x_i \cdot g(y_i) \right) \cdot (g(z) \otimes m)$$

and since $\sum_i x_i \cdot g(y_i) = 0$ for $g \neq \mathrm{id}$, it follows that $s(z \cdot m) = \sum_i x_i y_i z \otimes m = x^\varepsilon m$, as required. This proves the claim and therefore also (i).

For (ii), it suffices to show that for all $\varepsilon \in \Gamma$ we have $x^\varepsilon = \mathrm{Tr}(b)$ for some $b \in B$. Consider the A-ideal

$$I := \{\mathrm{Tr}(b) | b \in B\}\,.$$

By the above, we have $x^\varepsilon = \sum_i x_i \,\mathrm{Tr}(y_i) \in I \cdot B$. Taking norms we find

$$\mathrm{N}_{B/A}(\mathfrak{m} \cdot B) \subset \mathrm{N}_{B/A}(I \cdot B) \subset I$$

and since $\mathfrak{m}^n = \mathfrak{m}$ for all $n > 0$ we deduce that $\mathfrak{m} \subset I$. □

2.1.8. Let A be a Λ-algebra and $I \subset A$ an ideal such that $I^2 = 0$.

Theorem 2.2 (Faltings)**.** *The category of almost étale coverings of A/I is equivalent to the category of almost étale coverings of A.*

For the proof, see [7] 3. Theorem. More generally, we have ([9] Thms. 3.2.18, 3.2.28)

Theorem 2.3 (Gabber-Ramero)**.** *The category of almost étale homomorphisms of A/I is equivalent to the category of almost étale homomorphisms of A.*

Proposition 2.2. *The equivalence in Theorem 2.3 preserves almost faithful flatness.*

Proof. Following [9], one defines a functor from almost Λ-modules to real Λ-modules by

$$M \rightsquigarrow M_! := \mathfrak{m} \otimes_\Lambda M.$$

Since $\mathfrak{m}$ is a flat Λ-module, this functor is exact and $M_! \approx M$. Similarly, one defines a functor from almost Λ-algebras to real Λ-algebras by the exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow \Lambda \oplus B_! \longrightarrow B_{!!} \longrightarrow 0$$

where the map $\mathfrak{m} \to \Lambda \oplus B_!$ is given by $x \mapsto (x, -x \otimes 1)$, and $\Lambda \oplus B_!$ is given the ring structure $(x, a) \cdot (y, b) := (x \cdot y, x \cdot b + y \cdot a + a \cdot b)$. Then $B_{!!} \approx B$. Moreover, $A \to B$ is almost faithfully flat if and only if $A_{!!} \to B_{!!}$ is almost faithfully flat ([9] Remark 3.1.3 (ii)). Finally note that if $I \subset A$ is an ideal, then $I_! \subset A_{!!}$ is an ideal and $A_{!!}/I_! \cong (A/I)_{!!}$. Since in our case $A_{!!} \to B_{!!}$ is a nilpotent thickening of a faithfully flat homomorphism, it suffices to show that $A_{!!} \to B_{!!}$ is flat. By the usual criterion for flatness, it suffices to show that $\mathrm{Tor}_1^{A_{!!}}(B_{!!}, (A/I)_{!!}) = 0$. But this follows from [9], Prop. 2.5.34. □

Corollary 2.2. *If $A \to B$ is an almost étale covering, then it is a (formal) almost Galois covering of group G if and only if $A/I \to B/I \cdot B$ is a (formal) almost Galois covering of group G.*

Proof. Since almost faithful flatness is preserved under the equivalence, it suffices to check that formal almost Galois coverings are preserved. Since the functor $\otimes_B(B/I \cdot B)$ is an equivalence of the category of almost étale coverings of B with the category of almost étale coverings of $B/I \cdot B$, it follows that the canonical map $B \otimes_A B \to B^{\times G}$ is an almost isomorphism if and only the same is true after tensor product $\otimes_B(B/I \cdot B)$, hence the claim. $\square$

We refer to [9] for an in-depth study of almost ring theory.

2.2 Almost purity

2.2.1. Fix $c \in \{1, \pi\} \subset K^+$, and define

$$O(c) := K^+[T_1, ..., T_r, T_{r+1}^{\pm 1}, ..., T_{d+1}^{\pm 1}]/(T_1 \cdots T_r - c).$$

Following Faltings, we say that a K^+-algebra R is *small* if there is an étale map

$$O(c) \to R.$$

Every smooth (resp. semi-stable) K^+-scheme has a Zariski open covering by small affines with $c = 1$ (resp. $c = \pi$). Note that with these choices of c, R is a regular ring; in particular, it is a finite product of integrally closed domains, and we shall assume that R is an integrally closed domain. There is a natural fine log-structure on R associated to the pre-log-structure

$$\mathbb{N}^r \to R : (n_1, ..., n_r) \mapsto \prod_{i=1}^{r} T_i^{n_i}.$$

This makes $\mathrm{Spec}(R) \to \mathrm{Spec}(K^+)$ into a a log-smooth morphism, where $\mathrm{Spec}(K^+)$ is given the trivial (resp. canonical) log-structure for $c = 1$ (resp. $c = \pi$).

2.2.2. Fix $c \in \{1, \pi\}$. For any $n \in \mathbb{N}$, let $K_n = K[X]/(X^{p^n} - c)$. Let K_n^+ denote the normalization of K^+ in K_n. If R is a small K^+-algebra, then we define

$$R_n := R \otimes_{O(c)} K_n^+[T_1^{1/p^n}, ..., T_{d+1}^{1/p^n}].$$

Note that the normalization of K_n^+ in R_n is unramified over K_n^+ because R is small. If $K \subset L$ is a finite extension, then write $L_n = L \cdot K_n$ and let L_n^+ be the normalization of K^+ in L_n. Define

$$O(c)_{n,L} := O(c) \otimes_{K^+} L_n^+[X_1, ..., X_{d+1}]/(X_1^{p^n} - T_1, ..., X_{d+1}^{p^n} - T_{d+1}).$$

Lemma 2.2. *$O(c) \otimes_{K^+} L_n^+$ and $O(c)_{n,L}$ are integrally closed domains.*

Proof. Firstly note that $O := O(c)$ is a regular integral domain, in particular it is a Krull domain. Note that if the ring

$$O_n := O(c)[X_1, ..., X_{d+1}]/(X_1^{p^n} - T_1, ..., X_{d+1}^{p^n} - T_{d+1})$$

is normal, then it contains $O(c)_{n,K}$ because of the relation $\prod_{i=1}^{r} T_i = c$. So it suffices to show that O_n is normal, because it is clear that it is integral. Since O is a Krull ring and O_n is a finite free O-module, by [3] Ch. 7, §4, no. 2, Thm. 2, it follows that

$$O_n = \cap_{\text{height}(\mathfrak{p})=1}(O_n)_{\mathfrak{p}}$$

where the intersection is taken over the localizations at all prime ideals $\mathfrak{p} \subset O$ of height one. So by [3] Ch. 5, §1, no. 2, Prop. 8, it suffices to show that each $(O_n)_{\mathfrak{p}}$ is integrally closed. Hence we reduce to the case O is a discrete valuation ring. If O is of equal characteristic zero, then O_n is étale over O, hence normal. If O is of mixed characteristic, then π is a uniformizer for O and if $c = \pi$ then we must have $\pi = u \cdot T_i$ for some unit u and $1 \leq i \leq r$. It suffices to show that

$$O' = O[X]/(X^{p^n} - T_j)$$

is normal for all j. If $j = i$ and $c = \pi$, then the equation $X^{p^n} - T_i = 0$ is an Eisenstein equation, hence O' is a discrete valuation ring ([17] Ch. I, §6, Prop. 17) and in particular is normal. If $j \neq i$ or $c = 1$, then T_j is a unit of O. Let $F(X) = X^{p^n} - T_j$, and let A be the normalization of O in $Q(O')$ (the fraction field of O'). Let $\mathfrak{c}$ denote the annihilator of A/O'. Then we have the formula ([17] Ch. III, §6, Cor. 1):

$$\mathfrak{c} = (F'(T_j^{1/p^n})) \cdot \mathfrak{D}^{-1}$$

where

$$\mathfrak{D}^{-1} = \{y \in Q(O') : \mathrm{Tr}(xy) \in O \quad \forall x \in A\}$$

is the codifferent. Now $(F'(T_j^{1/p^n})) = (p^n)$ so it suffices to show that $\mathfrak{D} \subset (p^n)$. For this we compute traces, so consider the conjugates of T_j^{k/p^n} in a suitably large Galois extension of $Q(O)$. These are given by multiples $\zeta \cdot T_j^{k/p^n}$, where ζ is some p^nth root of unity. Hence for all $0 < k < p^n$ we have

$$\mathrm{Tr}(T_j^{k/p^n}) = T_j^{k/p^n} \sum_{l=0}^{p^n-1} \zeta^l = 0.$$

So if $\sum_{k=0}^{p^n-1} a_k T_j^{k/p^n}$ denotes a typical element of $Q(O')$ with $a_k \in Q(O)$ then

$$\mathrm{Tr}\left(\sum_{k=0}^{p^n-1} a_k T_j^{k/p^n}\right) = p^n a_0$$

so $\mathfrak{D} = (p^n)$. This proves that O' is integrally closed, hence so is O_n.

Now, since $O(c)_{n,L}$ is an integral domain, it suffices to show that it is normal. Since L_n is a finite extension of K_n and normality is stable by étale localization, up to making a finite unramified extension of K_n we may assume that L_n is a totally ramified extension of K_n, in particular $L_n^+ \cong K_n^+[X]/(f)$, where f is an Eisenstein polynomial ([17] Ch. I, §6, Prop. 18). Since K_n^+ is a principal ideal domain, L_n^+ is a free K_n^+-module, and hence $O(c)_{n,L}$ is a finite free O_n-module. Since O_n is a Krull ring, by the same reasoning as above we reduce to the case O_n is a discrete valuation ring. If O_n is of equal characteristic zero, then $O(c)_{n,L}$ is étale over O_n, hence normal. If O_n is of mixed characteristic, then π^{1/p^n} is a uniformizer for O_n and $O(c)_{n,L} \cong O_n[X]/(f)$. Since f is an Eisenstein equation, by [17] Ch. I, §6, Prop. 17, $O(c)_{n,L}$ is a discrete valuation ring, in particular a normal ring. □

For R small, define $R_{n,L} := R \otimes_{O(c)} O(c)_{n,L}$.

Proposition 2.3. *Assume $K \cap R = K^+$. If R is an integral domain and K^+ is integrally closed in R, then $R_{n,L}$ is an integrally closed domain for all n and L.*

Proof. (Ramero) Since $R_{n,L}$ is étale over $O(c)_{n,L}$ it follows that R is a normal ring. So it suffices to show that $R_{n,L}$ is an integral domain.

We first show that $S := R \otimes_{K^+} K_n^+$ is an integral domain. Note that since K^+ is integrally closed in R we have $Q(R) \cap K_n = K$. Now, the kernel of the canonical map

$$S \to R \cdot K_n^+$$

is contained in the kernel of

$$Q(R) \otimes_{K^+} K_n^+ \to Q(R) \cdot K_n$$

and $Q(R) \otimes_K K_n$ is a finite dimensional $Q(R)$ vector space of dimension $[K_n : K]$. Since $Q(R) \cap K_n = K$, the same is true of $Q(R) \cdot K_n$, hence the map is an isomorphism. This proves that S is an integral domain.

We now show that $R_n = R_{n,K}$ is an integral domain. It suffices to show that its spectrum is connected. Since R_n is finite flat over S the image of a connected component of R_n under the morphism $f : \mathrm{Spec}(R_n) \to \mathrm{Spec}(S)$ is both open and closed, hence equal to S because f is generically finite étale (in particular every connected component of R_n dominates S). So it suffices to show that f has a single connected fibre. Let $\mathfrak{q}$ be a generic point of $\mathrm{Spec}(S/\pi S)$, considered as a prime ideal of S, and let $\mathfrak{p} = \mathfrak{q} \cap O$, where $O := O(c) \otimes_{K^+} K_n^+$. Let $S_{\mathfrak{q}}^h$ resp. $O_{\mathfrak{p}}^h$ denote the henselization of S at $\mathfrak{q}$ resp. O at $\mathfrak{p}$. Since the prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ have height one, these are discrete valuation rings. Let $O_n := O(c)_n$. The extension

$$S_{\mathfrak{q}}^h \to O_{\mathfrak{p}}^h$$

is finite étale of degree f (say) and the extension

$$O_{\mathfrak{p}}^h \to O_{n,\mathfrak{p}}^h$$

is totally ramified of degree e (say). So the composite $S^h_{\mathfrak{q}} \cdot O^h_{n,\mathfrak{p}}$ is an extension of $S^h_{\mathfrak{q}}$ of degree $n = e \cdot f$. On the other hand, the canonical map

$$S^h_{\mathfrak{q}} \otimes_O O^h_{n,\mathfrak{p}} \to S^h_{\mathfrak{q}} \cdot O^h_{n,\mathfrak{p}}$$

is a surjective map of free $S^h_{\mathfrak{q}}$-modules of same rank, hence it is an isomorphism. Thus, $S^h_{\mathfrak{q}} \otimes_O O^h_{n,\mathfrak{p}} \cong S^h_{\mathfrak{q}} \cdot O^h_{n,\mathfrak{p}}$ is connected, and so (since $S^h_{\mathfrak{q}}$ is a henselian noetherian local ring) has connected special fibre

$$(S^h_{\mathfrak{q}} \otimes_O O^h_{n,\mathfrak{p}})/\mathfrak{q} \cdot (S^h_{\mathfrak{q}} \otimes_O O^h_{n,\mathfrak{p}}) \cong (S_{\mathfrak{q}}/\mathfrak{q}) \otimes_O O_n.$$

This implies that the fibre of $R_n = S \otimes_O O_n$ over $\mathfrak{q}$ is connected, hence so is R_n.

Finally, we show that $R_{n,L}$ is an integrally closed domain. The surjective map

$$R_{n,L} \cong R_n \otimes_{K_n^+} L_n^+ \to R_n \cdot L^+$$

has kernel contained in the kernel of the map

$$Q(R_n) \otimes_{K_n} L_n \to Q(R_n) \cdot L$$

where $Q(A)$ denotes the fraction field of A for any integral domain A. Since K_n^+ is integrally closed in R_n, it follows that $L_n \cap Q(R_n) = K_n$, hence $Q(R_n) \cdot L$ is a $Q(R_n)$ vector space of dimension $[L_n : K_n]$. Since the same is true for $Q(R_n) \otimes_{K_n} L_n$, it follows that they are isomorphic, and hence $R_n \otimes_{K_n^+} L_n^+ \cong R_n \cdot L^+$ is an integral domain. $\square$

2.2.3. Define

$$K_\infty^+ := \bigcup_{n \in \mathbb{N}} K_n^+.$$

We will now consider the almost ring theory of the pair $(K_\infty^+, \mathfrak{m}_\infty)$, where $\mathfrak{m}_\infty \subset K_\infty^+$ is the maximal ideal. Note that there is a sequence of rational numbers $\varepsilon_n = \frac{1}{p^n(p-1)}$ occuring as p-adic valuations of elements of K_∞^+, and tending to zero with n. Namely $\varepsilon_n = v_p(\zeta_{p^{n+1}} - 1)$, where $\zeta_{p^{n+1}}$ denotes a primitive p^{n+1}th root of unity. Since K_∞^+ is a valuation ring we see that we indeed have $\mathfrak{m}_\infty^2 = \mathfrak{m}_\infty$.

2.2.4. Let S be a finite integral R-algebra. We say that S is *étale in characteristic zero* if $R_K \to S_K$ is étale. For all n, let S_∞ be the normalization of $R_\infty \otimes_R S$, where

$$R_\infty := \bigcup_{n \in \mathbb{N}} R_n.$$

The following striking theorem is the key input we will use. It is usually refered to as the *Almost Purity Theorem*.

Theorem 2.4 (Faltings)**.** *If S is a finite integral normal R-algebra which is étale in characteristic zero, then the canonical homomorphism*

$$R_\infty \to S_\infty$$

is an almost étale covering of K_∞^+-algebras.

The proof of this theorem is given in [7]. If R is assumed to be classically smooth (i.e. $c = 1$), then a slightly more general statement has been proven in [10].

We define

$$\tilde{R} := R \otimes_{K^+} \bar{K}^+$$

and

$$\tilde{R}_\infty := R_\infty \otimes_{K_\infty^+} \bar{K}^+.$$

If K^+ is integrally closed in R and p is not a unit in R, then by Proposition 2.3 these are integrally closed domains. If S is a finite integral normal $\tilde{R}$-algebra which is étale in characteristic zero, then we may write S as the direct limit of finite integral normal $R \otimes_{K^+} L^+$-algebras, as L ranges over the finite field extensions of K. In this way we see that if we define S_∞ to be the normalization of $R_\infty \otimes_R S$, then $R_\infty \to S_\infty$ is the filtering inductive limit of almost étale coverings. Since this homomorphism factors over $\tilde{R}_\infty \to S_\infty$, then by Lemma 2.1 we deduce that $\tilde{R}_\infty \to S_\infty$ is the filtering inductive limit of almost étale coverings.

Corollary 2.3. *If S is a finite integral normal $\tilde{R}$-algebra which is étale in characteristic zero, then the (absolute) Frobenius is surjective on S_∞/pS_∞.*

Proof. (Faltings) If $S = R$, then by étaleness over

$$O(c)_{\infty,\bar{K}} := \operatorname{colim}_{n,L} O(c)_{n,L}$$

it suffices to show that the Frobenius is surjective on the latter. But the latter is a quotient of $\bar{K}^+/p\bar{K}^+[T_i^{p^{-\infty}}]$ and every element of this ring has the form

$$\sum v_N T_1^{n_1/p^{m_1}} \cdots T_e^{n_e/p^{m_e}} = \left(\sum v_N^{1/p} T_1^{n_1/p^{m_1+1}} \cdots T_e^{n_e/p^{m_e+1}}\right)^p$$

so the claim in this case is clear.

Using that the relative Frobenius of an almost étale homomorphism is almost isomorphism ([9] Thm. 3.5.13) we deduce that the Frobenius is almost surjective on S_∞/pS_∞. This implies that for all $x \in S_\infty$, there exists $y, z \in S_\infty$ such that $p^{1/2}x = y^p + pz$. But then $y^p = p^{1/2}(x - p^{1/2})$, so $y = p^{1/2p}w$ for some $w \in S_\infty$ because S_∞ is integrally closed, and hence $x = w^p + p^{1/2}z$. Then same trick shows that $z = u^p + p^{1/2}v$ and so $x \equiv (w + p^{1/2p}u)^p \mod p$. □

2.3 Almost étale coverings of Fontaine rings

By Corollary 2.3 we may, following Fontaine, construct the Fontaine rings

$$\begin{aligned} A_\infty^+ &:= \lim_n H^0_{\text{crys}}(\tilde{R}_\infty/p\tilde{R}_\infty|\Sigma_n, \mathscr{O}) \\ A_\infty^+(S) &:= \lim_n H^0_{\text{crys}}(S_\infty/pS_\infty|\Sigma_n, \mathscr{O}) \end{aligned}$$

where S is a finite integral normal $\tilde{R}$-algebra, étale in characteristic zero. Then the canonical homomorphism

$$A_\infty^+/p^n A_\infty^+ \to A_\infty^+(S)/p^n A_\infty^+(S)$$

is a nilpotent thickening of the almost étale homomorphism $\tilde{R}_\infty/p\tilde{R}_\infty \to S_\infty/pS_\infty$ and we will show that it is an almost étale homomorphism. We first set up the almost ring theory in this context.

2.3.1. We will consider the almost ring theory of the ring $A_{\rm inf}(K^+) = W(P(\hat{K^+}/p\hat{K^+}))$ with respect to the ideal $\mathfrak{a}$, union of the principal ideal $[\underline{p}]^\varepsilon$ for all positive rational exponents $\varepsilon > 0$. Note that $[\underline{p}]^\varepsilon$ is not a zero divisor: this is true modulo p since $P(\bar{K}^+/p\bar{K}^+)$ is a valuation ring (hence an integral domain) and so in general since $A_{\rm inf}(K^+)$ is p-torsion free, being the ring of Witt vectors of a perfect ring of characteristic p.

2.3.2. Let R be a small K^+-algebra and S a finite integral $\tilde{R}$-algebra. We define the canonical log-structure $\mathcal{L}_{\rm can}(S)$ on S by

$$\mathcal{L}_{\rm can}(S) : (S[1/p])^* \cap S \subset S.$$

For $S = \tilde{R}$, $\mathcal{L}_{\rm can}(\tilde{R})$ is the log-structure associated to the pre-log-structure

$$M_\infty : \mathbb{Q}_+ \oplus \mathbb{N}^r \to \tilde{R} : (\alpha, (n_1, .., n_r)) \mapsto \alpha \cdot \prod_{i=1}^{r} T_i^{n_i}.$$

Let $i : \mathrm{Spec}(S/pS) \hookrightarrow \mathrm{Spec}(S)$ be the canonical map. Define a pre-log-structure on S/pS by

$$\mathcal{L}_\infty(S/pS) := \left\{ l \in i^*\mathcal{L}_{\rm can}(S) | \exists n \in \mathbb{N} : l^{p^n} \in \mathrm{Im}\left(\mathcal{L}_{\rm can}(\tilde{R}) \to S/pS\right)\right\}$$

By taking inductive limits, this defines a log-structure on S_∞/pS_∞ denoted $\mathcal{L}_\infty(S_\infty/pS_\infty)$.

Proposition 2.4. *(i) $\mathcal{L}_\infty(\tilde{R}_\infty/p\tilde{R}_\infty)$ is the log-structure associated to the log-structure*

$$\mathbb{Q}_+ \oplus \mathbb{N}[1/p]^{d+1} \to \tilde{R}_\infty/p\tilde{R}_\infty : (\alpha, (n_1, .., n_{d+1})) \mapsto \alpha \cdot \prod_{i=1}^{d+1} T_i^{n_i}.$$

(ii) $\mathcal{L}_\infty(S_\infty/pS_\infty)$ is the inverse image of the log-structure $\mathcal{L}_\infty(\tilde{R}_\infty/p\tilde{R}_\infty)$.

Proof. Since $\mathcal{L}_\infty(\tilde{R}_\infty/p\tilde{R}_\infty)$ is obtained from $i^*\mathcal{L}_{\rm can}(\tilde{R})$ by taking p-power roots of its elements, (i) follows from the description of $i^*\mathcal{L}_{\rm can}(\tilde{R})$ given above and the fact that the Frobenius is surjective on $\tilde{R}_\infty/p\tilde{R}_\infty$.

Since $\mathcal{L}_\infty(S_\infty/pS_\infty)$ is obtained by taking p-power roots of $i^*\mathcal{L}_{\rm can}(\tilde{R})$ and the Frobenius is surjective on $\tilde{R}_\infty/p\tilde{R}_\infty$, (ii) is clear. □

2.3.3. Let R be a small integral K^+-algebra and let T be a p-adically complete logarithmic Σ-algebra such that R/pR is a T-algebra. In practice we will have $T = \Sigma$ or T will be a formal Σ-lift of R/pR. For all $n \geq 1$, write $T_n := T/p^nT$. We assume that the log-structure of T is integral and quasi-coherent and we denote the monoid generating it by $\mathcal{L}(T)$. Let S be a finite integral normal $\tilde{R}$-algebra étale in characteristic zero. Define

$$\begin{aligned} A^+_{\log,\infty,T} &:= \lim_n H^0_{\text{log-crys}}(\tilde{R}_\infty/p\tilde{R}_\infty | T_{n+1}, \mathscr{O}) \\ A^+_{\log,\infty,T}(S) &:= \lim_n H^0_{\text{log-crys}}(S_\infty/pS_\infty | T_{n+1}, \mathscr{O}). \end{aligned}$$

Proposition 2.5. *The canonical homomorphism*

$$A^+_{\log,\infty,T}/p^n A^+_{\log,\infty,T} \to A^+_{\log,\infty,T}(S)/p^n A^+_{\log,\infty,T}(S)$$

is an almost étale homomorphism.

Proof. Firstly, by Corollary 2.1 the canonical homomorphism

$$W_n(\tilde{R}_\infty/p\tilde{R}_\infty) \to W_n(S_\infty/pS_\infty)$$

is almost étale. By definition (*cf.* 1.2.3 Remark (c)) we have

$$(W_n(S_\infty/pS_\infty) \otimes_{\mathbb{Z}} T_n)_{\log} = W_n(S_\infty/pS_\infty) \otimes_{\mathbb{Z}} T_n \otimes_{\mathbb{Z}} \mathbb{Z}[L]$$

where $L := \mathrm{Ker}(M_\infty^{\mathrm{gp}} \oplus \mathcal{L}(T)^{\mathrm{gp}} \to M_\infty^{\mathrm{gp}})$, where the map $M_\infty \to M_\infty$ is raising to the power p^n. This is exactly the same L as for $\tilde{R}_\infty/p\tilde{R}_\infty$, hence we see that there is a canonical almost étale homomorphism

$$\Big(W_n(\tilde{R}_\infty/p\tilde{R}_\infty) \otimes_{\mathbb{Z}} T_n\Big)_{\log} \to (W_n(S_\infty/pS_\infty) \otimes_{\mathbb{Z}} T_n)_{\log}.$$

This implies that we have an almost cocartesian square

$$\begin{array}{ccc} \Big(W_n(\tilde{R}_\infty/p\tilde{R}_\infty) \otimes_{\mathbb{Z}} T_n\Big)_{\log} & \longrightarrow & (W_n(S_\infty/pS_\infty) \otimes_{\mathbb{Z}} T_n)_{\log} \\ \downarrow & & \downarrow \\ \tilde{R}_\infty/p\tilde{R}_\infty & \longrightarrow & S_\infty/pS_\infty \end{array}$$

where the vertical maps are the canonical maps, i.e. the T_n-linear maps induced from $W_n(A) \to A : (a_0, ..., a_{n-1}) \mapsto a_0^{p^n}$ and sending L to 1. Hence by the next lemma we conclude that the canonical map

$$A^+_{\log,\infty,T}/p^n A^+_{\log,\infty,T} \otimes_{\left(W_n(\tilde{R}_\infty/p\tilde{R}_\infty)\otimes_{\mathbb{Z}} T_n\right)_{\log}} (W_n(S_\infty/pS_\infty) \otimes_{\mathbb{Z}} T_n)_{\log} \to A^+_{\log,\infty,T}(S)/p^n A^+_{\log,\infty,T}(S)$$

is an almost isomorphism, thereby completing the proof. $\square$

In the proof we have made use of the following almost analogue of a well-known result on divided power hulls.

Lemma 2.3. *Let A, B be Λ-algebras. Suppose there is a homomorphism $A \to B$ making B into an almost flat A-algebra. Then for any ideal $I \subset A$ the canonical map*

$$D_A(I) \otimes_A B \to D_B(I \cdot B)$$

is an almost isomorphism, where $D_A(I)$ denotes the divided power hull of A for the ideal I.

Proof. For any ring A and any module M let $\Gamma_A(M)$ denote the divided power algebra generated by M (see [2] 3.9). Since B is an almost flat A-algebra, the canonical map $I \otimes_A B \to I \cdot B$ is an almost isomorphism, hence $\Gamma_B(I \otimes_A B) \approx \Gamma_B(I \cdot B)$ by [9] Lemma 8.1.13. But

$$\Gamma_B(I \otimes_A B) \cong \Gamma_A(I) \otimes_A B$$

so $\Gamma_B(I \cdot B)$ is an almost flat $\Gamma_A(I)$-algebra. Hence if $J \subset \Gamma_A(I)$ denotes the ideal defining $D_A(I)$, then we have $J \otimes_A B \cong J \otimes_{\Gamma_A(I)} \Gamma_B(I \otimes_A B) \approx J \cdot \Gamma_B(I \cdot B)$, and therefore

$$D_A(I) \otimes_A B = \frac{\Gamma_A(I) \otimes_A B}{J \otimes_A B} \approx \frac{\Gamma_B(I \cdot B)}{J \cdot \Gamma_B(I \cdot B)} = D_B(I \cdot B).$$

$\square$

2.3.4. Define $\bar{R}$ to be the normalization of R in the maximal extension of $Q(R)$ which is étale in characteristic zero. Then if K^+ is integrally closed in R we have

$$\pi_1(\mathrm{Spec}(\tilde{R}[1/p])) = \mathrm{Gal}(\bar{R}[1/p]/\tilde{R}[1/p])$$

and $\tilde{R}_\infty[1/p] \to \bar{R}[1/p]$ is the inductive limit of Galois coverings of $\tilde{R}_\infty[1/p]$. Define

$$\Delta := \mathrm{Gal}(\bar{R}[1/p]/\tilde{R}[1/p])$$

and

$$\Delta_\infty := \mathrm{Gal}(\tilde{R}_\infty[1/p]/\tilde{R}[1/p]).$$

Then Δ_∞ is a quotient of Δ and let $\Delta_0 := \mathrm{Ker}(\Delta \to \Delta_\infty)$. Define

$$A^+_{\log,T} := \lim_n H^0_{\text{log-crys}}(\bar{R}/p\bar{R}|T_{n+1}, \mathscr{O}).$$

Lemma 2.4. *If K^+ is integrally closed in R, then the homomorphism $\tilde{R}_\infty \to \bar{R}$ is the filtering inductive limit of almost Galois coverings.*

Proof. Let us first show that every non-zero almost étale covering of $\tilde{R}_\infty$ is almost faithfully flat. More generally if A is an integral domain, then any almost étale covering of A is almost faithfully flat. Indeed, if $\varphi \in \mathrm{Hom}_\Lambda(\mathfrak{m}, A)$ is an idempotent, then for all $\varepsilon > 0$ we have

$$x^{\varepsilon/2} \cdot \varphi(x^{\varepsilon/2}) = \varphi(x^\varepsilon) = \varphi(x^{\varepsilon/2}) \cdot \varphi(x^{\varepsilon/2})$$

so $x^{\varepsilon/2} = \varphi(x^{\varepsilon/2})$, whence $\varphi = \mathrm{id}$. So A has no almost idempotents, hence every almost projective A-module of finite rank is either almost zero or everywhere of constant non-zero rank, hence almost faithfully flat.

Now, let S be a finite integral normal R-algebra, such that $R[1/p] \to S[1/p]$ is a Galois covering. Let L^+ be the integral closure of K^+ in S, and write $L = Q(L^+)$. Then by the almost purity theorem, the homomorphism $R_\infty \to S_\infty$ is an almost étale covering and factors over the almost étale covering $R_\infty \to R_{\infty,L}$. By Lemma 2.1 it follows that $R_{\infty,L} \to S_\infty$ is an almost étale covering. We claim that it is an almost Galois covering. First note that $R_{\infty,L}[1/p] \to S_\infty[1/p]$ is a Galois covering, since it is obtained by base change from the Galois covering $R_L[1/p] \to S[1/p]$. If e denotes the idempotent image of 1 under a section of the multiplication map $S_\infty[1/p] \otimes_{R_{\infty,L}[1/p]} S_\infty[1/p] \to S_\infty[1/p]$, then we have a canonical isomorphism

$$S_\infty[1/p] \otimes_{R_{\infty,L}[1/p]} S_\infty[1/p] \cong \prod_{g \in G} g(e) \cdot S_\infty[1/p].$$

By almost étaleness, for all $\varepsilon > 0$ we have $p^\varepsilon e \in S_\infty \otimes_{R_{\infty,L}} S_\infty$, hence

$$\mathrm{Hom}_{K_\infty^+}(\mathfrak{m}_\infty, S_\infty \otimes_{R_{\infty,L}} S_\infty) \cong \mathrm{Hom}_{K_\infty^+}(\mathfrak{m}_\infty, \prod_{g \in G} g(e) \cdot S_\infty)$$

i.e. $S_\infty \otimes_{R_{\infty,L}} S_\infty \approx \prod_{g \in G} g(e) \cdot S_\infty$. This proves the claim. The same argument with S replaced by $S \otimes_{L^+} E^+$ for E^+ the normalization of L^+ in a finite Galois extension$L \subset E$ proves that

$$\tilde{R}_\infty \to \tilde{S}_\infty$$

is the inductive limit of almost Galois coverings, where $\tilde{S} = S \otimes_L \bar{K}^+$.

Now we write $\bar{R}[1/p]$ as the inductive limit of Galois coverings of $R[1/p]$. For each such Galois covering, the normalization of R in it is a finite integral R-algebra S. Up to replacing S_n by an irreducible component (in particular finite étale) we can assume that S_n is integral for n large enough, hence S_∞ is contained in $\bar{R}$. Now we clearly have

$$\mathrm{colim}_S\, S_\infty = \mathrm{colim}_S\, \tilde{S}_\infty = \bar{R}$$

and the claim follows. □

In particular, $\tilde{R}_\infty \to \bar{R}$ is almost faithfully flat.

Corollary 2.4. *The canonical homomorphism*

$$A^+_{\log,\infty,T}/p^n A^+_{\log,\infty,T} \to A^+_{\log,T}/p^n A^+_{\log,T}$$

is the filtering inductive limit of almost Galois coverings.

Proof. First note that since $\tilde{R}_\infty/p\tilde{R}_\infty \to \bar{R}/p\bar{R}$ is the filtering inductive limit of almost Galois coverings, so is

$$W_n(\tilde{R}_\infty/p\tilde{R}_\infty) \to W_n(\bar{R}/p\bar{R}).$$

Looking at the proof of Proposition 2.5 we note two things. Firstly, the canonical homomorphism

$$\Big(W_n(\tilde{R}_\infty/p\tilde{R}_\infty) \otimes_{\mathbb{Z}} T_n\Big)_{\log} \to \big(W_n(\bar{R}/p\bar{R}) \otimes_{\mathbb{Z}} T_n\big)_{\log}$$

is the inductive limit of almost Galois coverings. Secondly, we have an almost cocartesian square

$$\begin{array}{ccc} \Big(W_n(\tilde{R}_\infty/p\tilde{R}_\infty)\otimes_{\mathbb{Z}} T_n\Big)_{\log} & \longrightarrow & \big(W_n(\bar{R}/p\bar{R})\otimes_{\mathbb{Z}} T_n\big)_{\log} \\ \big\downarrow & & \big\downarrow \\ \tilde{R}_\infty/p\tilde{R}_\infty & \longrightarrow & \bar{R}/p\bar{R}. \end{array}$$

So we deduce that the canonical map

$$A^+_{\log,\infty,T}/p^n A^+_{\log,\infty,T} \otimes_{\left(W_n(\tilde{R}_\infty/p\tilde{R}_\infty)\otimes_{\mathbb{Z}} T_n\right)_{\log}} \big(W_n(\bar{R}/p\bar{R})\otimes_{\mathbb{Z}} T_n\big)_{\log} \to A^+_{\log,T}/p^n A^+_{\log,T}$$

is an almost isomorphism, and since tensor product commutes with inductive limits, we are done. □

Corollary 2.5. *(i) The canonical map*

$$A^+_{\log,\infty,T}/p^n A^+_{\log,\infty,T} \to \Big(A^+_{\log,T}/p^n A^+_{\log,T}\Big)^{\Delta_0}$$

is an almost isomorphism.

(ii) $A^+_{\log,T}/p^n A^+_{\log,T}$ *is a discrete* Δ*-module and for all* $i \neq 0$ *we have*

$$H^i(\Delta_0, A^+_{\log,T}/p^n A^+_{\log,T}) \approx 0.$$

Proof. Part (i) follows by a limit argument from Proposition 2.1 (i) and the fact that the homomorphism $A^+_{\log,\infty,T}/p^n A^+_{\log,\infty,T} \to A^+_{\log,T}/p^n A^+_{\log,T}$ is almost faithfully flat. For (ii), note that $A^+_{\log,T}/p^n A^+_{\log,T}$ is the divided power hull of $W_n(\bar{R}/p\bar{R})\otimes_{\mathbb{Z}} T_n \otimes_{\mathbb{Z}} \mathbb{Z}[L]$ and hence is a discrete Δ-module. Again, a limit argument proves (ii) via Proposition 2.1 (ii). □

In particular, via the Hochschild-Serre spectral sequence of Galois cohomology we deduce canonical almost isomorphisms for all i

$$H^i(\Delta_\infty, A^+_{\log,\infty,T}/p^n A^+_{\log,\infty,T}) \approx H^i(\Delta, A^+_{\log,T}/p^n A^+_{\log,T}).$$

This almost isomorphism will enable us to express the right-hand side in terms of crystalline cohomology.

3 Galois cohomology of Fontaine rings

In this section we construct a natural de Rham resolution of the Fontaine rings $A^+_{\log,\Sigma_n}$ and then we compute the Galois cohomology of its components. Assume throughout that R is a small integral K^+-algebra and that K^+ is integrally closed in R.

3.1 Canonical de Rham resolutions of Fontaine rings

3.1.1. If $c = 1$, then let $v = 1$, and if $c = \pi$ then let $v = u$, where $\Sigma_n = W_n(k)[u]\langle u^e \rangle$. Define

$$\Theta(c) := \Sigma[T_1, ..., T_r, T_{r+1}^{\pm 1}, ..., T_{d+1}^{\pm 1}]/(T_1 \cdots T_r - v).$$

Since R/pR is étale over $O(c)/pO(c)$, for all $n \geq 1$ there exists an étale $\Theta(c)/p^n\Theta(c)$-algebra lifting R/pR, unique up to canonical isomorphism. Denote it by $\mathcal{R}_n$, and define $\mathcal{R} := \lim_n \mathcal{R}_{n+1}$. Note that $\mathcal{R}/p^n\mathcal{R} = \mathcal{R}_n$. We endow $\mathcal{R}_n$ with log-structure associated to

$$\mathcal{N} : \mathbb{N}^{r+1} \to \mathcal{R}_n : (n_0, ..., n_r) \mapsto u^{n_0} \cdot \prod_{i=1}^{r} T_i^{n_i}$$

thus making it into a logarithmic Σ_n-algebra.

3.1.2. Consider the canonical map

$$h : \mathrm{Spec}(\bar{R}/p\bar{R}) \to \mathrm{Spec}(R/pR)$$

and the associated morphism of log-crystalline topoi with respect to the DP-base Σ_n. By Proposition 1.1, $h_*\mathscr{O}$ is a quasi-coherent crystal of $\mathscr{O}$-modules on $(\mathcal{R}_n|\Sigma_n)_{\text{log-crys}}$. Since $\mathcal{R}_n$ is a log-smooth lift of R/pR, by [15] Thm. 6.2, there is an integrable quasi-nilpotent logarithmic connection d on $h_*\mathscr{O}(\mathcal{R}_n)$ whose associated de Rham complex computes the log-crystalline cohomology of $\bar{R}/p\bar{R}$ over the DP-base Σ_n. Since this cohomology vanishes in non-zero degree by Theorem 1.1, it follows that the augmentation

$$A^+_{\log,\Sigma}/p^n A^+_{\log,\Sigma} \cong H^0_{\text{log-crys}}(\bar{R}/p\bar{R}|\Sigma_n, \mathscr{O}) \to h_*\mathscr{O}(\mathcal{R}_n) \otimes_{\mathcal{R}_n} \omega^\bullet_{\mathcal{R}_n/\Sigma_n}$$

is a quasi-isomorphism, where $\omega^i_{\mathcal{R}_n/\Sigma_n} := \wedge^i_{\mathcal{R}_n} \omega^1_{\mathcal{R}_n/\Sigma_n}$ and $\omega^1_{\mathcal{R}_n/\Sigma_n}$ is the sheaf of logarithmic differentials ([15] 1.7). This quasi-isomorphism is the analogue of Poincaré's lemma which we will use in our comparison strategy. Our aim is to (almost) compute the Δ-cohomology of the components of this resolution.

From now on we write

$$A^+ := A^+_{\log,\Sigma}$$

and

$$M^+ := \lim_n H^0_{\text{log-crys}}(\bar{R}/p\bar{R}|\mathcal{R}_n, \mathscr{O}).$$

Note that $M^+/p^nM^+ \cong h_*\mathscr{O}(\mathcal{R}_n)$.

3.1.3. Let

$$\tilde{O}(c)_\infty := \bigcup_{n,L} O(c)_{n,L}$$

so that $\tilde{R}_\infty = R \otimes_{O(c)} \tilde{O}(c)_\infty$. The same argument as above can be applied to the morphism

$$h_\infty : \mathrm{Spec}(\tilde{O}(c)_\infty/p\tilde{O}(c)_\infty) \to \mathrm{Spec}(O(c)/pO(c)).$$

By Proposition 1.1 the sheaf $h_{\infty,*}\mathscr{O}$ is a quasi-coherent crystal of $\mathscr{O}$-modules on the site $(O(c)/pO(c)|\Sigma_n)_{\text{log-crys}}$ and hence we have a canonical isomorphism

$$h_{\infty,*}\mathscr{O}(\mathcal{R}_n) \cong h_{\infty,*}\mathscr{O}\left(\Theta(c)/p^n\Theta(c)\right) \otimes_{\Theta(c)/p^n\Theta(c)} \mathcal{R}_n.$$

Note that in the notation of the previous section we have

$$h_{\infty,*}\mathscr{O}(\mathcal{R}_n) = A^+_{\log,\infty,\mathcal{R}}/p^n A^+_{\log,\infty,\mathcal{R}}.$$

Also, since $\mathcal{R}_n$ is a log-smooth Σ_n-lift of R/pR, there is an integrable quasi-nilpotent logarithmic connection d on $h_{\infty,*}\mathscr{O}\left(\mathcal{R}_n\right)$ whose associated de Rham complex is a resolution of

$$A^+_{\log,\infty,\Sigma}/p^n A^+_{\log,\infty,\Sigma}.$$

Define

$$A^+_\infty = A^+_\infty(R) := A^+_{\log,\infty,\Sigma}$$

and

$$M^+_\infty = M^+_\infty(R) := \lim_n h_{\infty,*}\mathscr{O}(\mathcal{R}_n).$$

3.1.4. Note that, in the notation of §2.3, we have

$$\begin{aligned} M^+_\infty &= A^+_{\log,\infty,\mathcal{R}} \\ M^+ &= A^+_{\log,\mathcal{R}} \end{aligned}$$

in particular, the canonical map $M^+_\infty/p^n M^+_\infty \to M^+/p^n M^+$ is the inductive limit of almost Galois coverings, $M^+_\infty/p^n M^+_\infty \approx \left(M^+/p^n M^+\right)^{\Delta_0}$, and we have canonical almost isomorphisms for all i

$$H^i(\Delta_\infty, M^+_\infty/p^n M^+_\infty) \approx H^i(\Delta, M^+/p^n M^+).$$

3.1.5. Define

$$\begin{aligned} A_{\text{cris},\infty}(R) &:= \lim_n H^0_{\text{crys}}(\tilde{R}_\infty/p\tilde{R}_\infty|W_{n+1}(k), \mathscr{O}) \\ A_{\text{cris}}(R) &:= \lim_n H^0_{\text{crys}}(\bar{R}/p\bar{R}|W_{n+1}(k), \mathscr{O}) \end{aligned}$$

This is just the classical crystalline cohomology, i.e. we ignore the log-structures.

Lemma 3.1. *Let A be an integrally closed domain of characteristic zero such that the Frobenius is surjective on A/pA. Assume that A contains all p-power roots of p. Let*

$$\theta : W(P(A/pA)) \to \hat{A}$$

denote the canonical map constructed in the proof of Theorem 1.1. Then $\operatorname{Ker}(\theta)$ *is a principal ideal generated by $\xi := [\underline{p}] - p$, where*

$$\underline{p} := (p, p^{1/p}, p^{1/p^2}, \ldots) \in P(A/pA).$$

Proof. We assume $x \in \mathrm{Ker}(\theta \mod p)$. Write $x = (x^{(n)})_n$ with $x^{(n+1)p} = x^{(n)}$ for all n. Then we have $x^{(1)p} = 0$. Let $\hat{x}^{(1)} \in A$ be a lift of $x^{(1)}$. Then $\hat{x}^{(1)p} = py$ for some $y \in A$ so $\frac{\hat{x}^{(1)}}{p^{1/p}} \in A$ because A is integrally closed. Hence $x^{(1)} \in p^{1/p} \cdot A/pA$. Continuing in this manner we find that $x \in \underline{p} \cdot P(A/pA)$. This proves the claim modulo p, and since $W(P(A/pA))$ is p-adically complete and $\hat{A}$ is p-torsion free the lemma follows. $\square$

Proposition 3.1. *In the case $R = O(c)$, every choice of p-power roots of $\pi, T_1, ..., T_{d+1}$ determines an isomorphism*

$$M^+_\infty/p^n M^+_\infty \simeq \frac{A_{\mathrm{cris},\infty}(R)/p^n A_{\mathrm{cris},\infty}(R)\,\langle X, X_1, ..., X_{d+1}\rangle}{(1 + X(c) - \prod_{i=1}^r (1 + X_i))}$$

where the $X, X_1, ..., X_{d+1}$ are indeterminates and $X(c) = 0$ if $c = 1$, $X(c) = X$ if $c = \pi$.

Proof. We check the universal property. Given a sequence of p-power roots of T_i, define

$$\underline{T_i} := (T_i, T_i^{1/p}, T_i^{1/p^2}, ...) \in P(\tilde{R}_\infty/p\tilde{R}_\infty).$$

Consider the ring

$$C := \frac{W_n(P(\tilde{R}_\infty/p\tilde{R}_\infty))\left[X, \frac{1}{1+X}, X_1, \frac{1}{1+X_1}, ..., X_{d+1}, \frac{1}{1+X_{d+1}}\right]}{(1 + X(c) - \prod_{i=1}^r (1 + X_i))}.$$

Make C into a $W_n[u][T_1, ..., T_r, T_{r+1}^{\pm 1}, ..., T_{d+1}^{\pm 1}]/(\prod_{i=1}^r T_i - u)$-algebra by sending u to $[\underline{\pi}] \cdot (1 + X)^{-1}$ and T_i to $[\underline{T_i}] \cdot (1 + X_i)^{-1}$. For an affine object $(U \hookrightarrow T)$ of the site $(\tilde{R}_\infty/p\tilde{R}_\infty|\Theta(c)/p^n\Theta(c))_{\text{log-crys}}$ define a map

$$C \to \mathscr{O}_T$$

extending the canonical map $\theta_T : W_n(P(\tilde{R}_\infty/p\tilde{R}_\infty)) \to \mathscr{O}_T$ by sending X to $\theta_T([\underline{\pi}]) \cdot u^{-1} - 1$ and X_i to $\theta_T([\underline{T_i}]) \cdot T_i^{-1} - 1$ (note that by Lemma 1.1 these elements exist in $\mathscr{O}_T$). One checks easily that this is the unique map of $W_n[u][T_1, ..., T_r, T_{r+1}^{\pm 1}, ..., T_{d+1}^{\pm 1}]/(\prod_{i=1}^r T_i - u)$-algebras extending θ_T. It follows from Lemma 3.1 that the kernel of the map to $C \to \tilde{R}_\infty/p\tilde{R}_\infty$ is the ideal generated by ξ and $X, X_1, ..., X_{d+1}$. So the divided power hull C^{DP} of C for this ideal is precisely the ring in the statement of the proposition. By the uniqueness of θ_T we see that it suffices now to show that C^{DP} defines an object of $(\tilde{R}_\infty/p\tilde{R}_\infty|\Theta(c)/p^n\Theta(c))_{\text{log-crys}}$. This reduces to showing that the closed immersion associated to the surjection $C^{\mathrm{DP}} \to \tilde{R}_\infty/p\tilde{R}_\infty$ is exact, where we endow C^{DP} with the log-structure associated to

$$P(\mathcal{L}_\infty(\tilde{R}_\infty/p\tilde{R}_\infty)) \oplus \mathscr{N} \oplus \mathcal{L}(u) \to C$$

induced from the maps $P(\mathcal{L}_\infty(\tilde{R}_\infty/p\tilde{R}_\infty)) \to C : \left(m^{(n)}\right) \mapsto [(m^{(n)})]$ and $\mathscr{N} \oplus \mathcal{L}(u) \to \Theta(c)/p^n\Theta(c) \to C$. By Lemma 1.1, this reduces to showing that

$$L := \mathrm{Ker}\left(P(\mathcal{L}_\infty(\tilde{R}_\infty/p\tilde{R}_\infty))^{\mathrm{gp}} \oplus \mathscr{N}^{\mathrm{gp}} \to \mathcal{L}_\infty(\tilde{R}_\infty/p\tilde{R}_\infty)^{\mathrm{gp}}\right)$$

corresponds to units of C^{DP}. Recall that

$$\begin{array}{rcl} \mathcal{L}_\infty(\tilde{R}_\infty/p\tilde{R}_\infty) : \mathbb{Q}_+ \oplus \mathbb{N}[1/p]^r & \to & \tilde{R}_\infty/p\tilde{R}_\infty \\ (\alpha, (n_1, ..., n_r)) & \mapsto & \alpha \cdot \prod_{i=1}^r T_i^{n_i}. \end{array}$$

So L is the kernel of the map

$$\begin{array}{rcl} P\left(\mathbb{Q}_+ \oplus \mathbb{N}[1/p]^r\right)^{\mathrm{gp}} \oplus \mathbb{Z}^{r+1} & \to & \mathbb{Q} \oplus \mathbb{Z}[1/p]^{r+1} \\ \big((\alpha \cdot p^{-n})_n, (x_1 \cdot p^{-n})_n, ..., (x_r \cdot p^{-n})_n, (n_0, ..., n_r)\big) & \mapsto & \Big(\alpha^{(0)} + n_0, x_1 + n_1, ..., x_r + n_r\Big) \end{array}$$

with $\alpha \in \mathbb{Q}$ and $x_1, ..., x_r, n_0, ..., n_r \in \mathbb{N}$. That is, L consists of the $2(r+1)$-tuples $((n_0 \cdot p^{-n})_n, (n_1 \cdot p^{-n})_n, ..., (n_r \cdot p^{-n})_n, (-n_0, ..., -n_r))$. Note that

$$X, X_1, ..., X_r \in \mathrm{Ker}(C^{\mathrm{DP}} \to \tilde{R}_\infty/p\tilde{R}_\infty)$$

and hence $1 + X, 1 + X_1, ..., 1 + X_r$ are units of C^{DP}. Define a map

$$\begin{array}{rcl} L & \to & C^{\mathrm{DP}} \\ \big((n_0 \cdot p^{-n})_n, (n_1 \cdot p^{-n})_n, ..., (n_r \cdot p^{-n})_n, (-n_0, ..., -n_r)\big) & \mapsto & (1+X)^{n_0} \prod_{i=1}^r (1+X_i)^{n_i}. \end{array}$$

It is easy to see that this map is compatible with the pre-log-structure of C^{DP}, and hence we are done. □

A simple adaptation of the above proof shows the following. We leave the details to the interested reader.

Proposition 3.2. *Every choice of p-power roots of π determines isomorphisms*

$$\begin{array}{rcl} A_\infty^+/p^n A_\infty^+ & \simeq & A_{\mathrm{cris},\infty}(R)/p^n A_{\mathrm{cris},\infty}(R) \langle X \rangle \\ A^+/p^n A^+ & \simeq & A_{\mathrm{cris}}(R)/p^n A_{\mathrm{cris}}(R) \langle X \rangle \end{array}$$

where the X is an indeterminate.

Morally, $X = [\underline{\pi}] \otimes u^{-1} - 1$. We define the *monodromy operator* N on $A^+/p^n A^+$ (resp. $A_\infty^+/p^n A_\infty^+$) as the unique $A_{\mathrm{cris}}(R)$-linear (resp. $A_{\mathrm{cris},\infty}(R)$-linear) derivation satisfying

$$N(X) = 1 + X.$$

3.2 Computations in Galois cohomology

3.2.1. Note that making a choice of p-power roots of the local coordinates T_i defines an element

$$\underline{T_i} = (T_i, T_i^{1/p}, T_i^{1/p^2}, ...) \in P(\tilde{R}_\infty/p\tilde{R}_\infty).$$

Lemma 3.2. *Assume $R = O(c)$ and make a choice of roots of the T_i. Every element of $A_{\mathrm{cris},\infty}(R)/p^n A_{\mathrm{cris},\infty}(R)$ can be written as a finite sum of the form*

$$\sum_{k\geq 0} x_k \xi^{[k]}$$

with $x_k \in W_n(P(\tilde{R}_\infty/p\tilde{R}_\infty))$ of the form

$$\sum_m p^m v_m \prod_{i=1}^{d+1} [\underline{T_i}]^{\alpha_{i,m}}$$

where $\alpha_{i,m} \in \mathbb{N}[1/p]$ for $1 \leq i \leq r$ and $\alpha_{i,m} \in \mathbb{Z}[1/p]$ for $r+1 \leq i \leq d+1$ and $v_m \in A_{\inf}(K^+)$.

Proof. The lemma will follow from Lemma 3.1 once we show that we can get the x_k in the desired form. Since the ring of Witt vectors $W_n(A)$ of a ring A of characteristic p is equal to its subring of elements of the form $\sum_m p^m[a_m]$, where $a_m \in A$ and $[\cdot]$ denotes the Teichmüller lift, it suffices to prove the claim modulo p. By definition, we obtain an element of $P(\tilde{R}_\infty/p\tilde{R}_\infty)$ by taking roots of $r \in \tilde{R}_\infty/p\tilde{R}_\infty$. We may write

$$r = \sum_N v_N \prod_{i=1}^{d+1} T_i^{n_i}$$

where $v_N \in \bar{K}^+/p\bar{K}^+$ and $n_i \in \mathbb{N}[1/p]$. Make a choice of p-power roots of elements of $\bar{K}^+/p\bar{K}^+$ (this will not affect the statement of the lemma). Define

$$\underline{r} := \sum_N \underline{v_N} \prod_{i=1}^{d+1} \underline{T_i}^{n_i}.$$

Taking pth roots of r we get

$$\begin{aligned}
r^{(1)} &= \sum_N v_N^{1/p} \prod_{i=1}^{d+1} T_i^{n_i/p} + p^{a_1/p} r_1 \\
r^{(2)} &= \sum_N v_N^{1/p^2} \prod_{i=1}^{d+1} T_i^{n_i/p^2} + p^{a_1/p^2} r_1^{p/p^2} + p^{a_2 p/p^2} r_2 \\
&\dots \\
(r^{(n)}) &= \underline{r} + \underline{p}^{a_1}\underline{r_1} + \underline{p}^{a_2 p}\underline{r_2} + \dots
\end{aligned}$$

where $a_i \in \mathbb{N}$. Now recall that $\underline{p}$ has divided powers in $A_{\mathrm{cris},\infty}(R)/pA_{\mathrm{cris},\infty}(R)$ so we get

$$(r^{(n)}) = \underline{r} + \underline{p}^{a_1}\underline{r_1}.$$

The lemma follows. $\square$

Note that $\Delta_\infty \cong \mathbb{Z}_p(1)^d$. Let $\sigma_2, ..., \sigma_{d+1}$ be a choice of topological generators of Δ_∞. For each $2 \le i \le d+1$, make a choice of roots of T_i such that $\sigma_i(T_i^{1/p^n}) = \zeta_{p^n} T_i^{1/p^n}$ for a p^nth root of unity ζ_{p^n}. Define

$$\underline{1} := (1, \zeta_p, \zeta_{p^n}, ...) \in P(\bar{K}^+/\bar{K}^+)$$

so that $\sigma(\underline{T_i}) = \underline{1} \cdot \underline{T_i}$. Furthermore, define

$$t := \log([\underline{1}]).$$

Proposition 3.3. *Assume $R = O(c)$. The subring*

$$\left(A_\infty^+/p^n A_\infty^+\right)^{\Delta_\infty} \subset A_\infty^+/p^n A_\infty^+$$

consists of elements which can be written $\sum_{k,j} x_{k,j} \xi^{[k]} X^{[j]}$ with $x_{k,j} \in W_n(P(\tilde{R}_\infty/p\tilde{R}_\infty))$ of the form

$$\sum_m p^m v_m \prod_{i=1}^{d+1} [\underline{T_i}]^{\alpha_{i,m}}$$

where $v_p(\alpha_{i,m}) \ge n - m$ and $v_m \in A_{\text{inf}}(K^+)$. In particular

$$\left(A_\infty^+\right)^{\Delta_\infty} \cong B^+.$$

Proof. We have

$$\left(A_\infty^+/p^n A_\infty^+\right)^{\Delta_\infty} = \bigcap_i \operatorname{Ker}(\sigma_i - 1).$$

From the exact sequences

$$0 \longrightarrow p^{n-1}A_\infty^+/p^n A_\infty^+ \longrightarrow A_\infty^+/p^n A_\infty^+ \longrightarrow A_\infty^+/p^{n-1}A_\infty^+ \longrightarrow 0$$

we see that it suffices to prove the assertion for A_∞^+/pA_∞^+. Note that if $v_p(\alpha) > 0$, then $\alpha \in \mathbb{Z}$ and

$$\underline{1}^\alpha - 1 = (\underline{1} - 1 + 1)^\alpha - 1 = \sum_{r>0} \alpha(\alpha-1)\cdots(\alpha - r + 1)(\underline{1} - 1)^{[r]} = 0$$

so

$$(\sigma_i - 1)\underline{T_i}^\alpha = (\underline{1}^\alpha - 1)\underline{T_i}^\alpha = 0.$$

It follows easily from this that $\bigcap_i \operatorname{Ker}(\sigma_i - 1)$ consists of the elements in the statement of the proposition. $\square$

Corollary 3.1. *The canonical map*

$$\left(A_\infty^+(O(c))/p^n A_\infty^+(O(c))\right)^{\Delta_\infty} \to \left(M_\infty^+(O(c))/p^n M_\infty^+(O(c))\right)^{\Delta_\infty}$$

has image in $B_{\log}^+/p^n B_{\log}^+ \otimes_{\Sigma_n} \Theta(c)/p^n\Theta(c)$.

Proof. Since we have made a choice of roots of T_i we have $X_i = [\underline{T_i}] \otimes T_i^{-1} - 1$. If $\alpha \in \mathbb{Z}[1/p]$ with $v_p(\alpha) \ge n$, then $\alpha \in \mathbb{Z}$ and we have

$$[\underline{T_i}]^\alpha = T_i^\alpha([\underline{T_i}] \otimes T_i^{-1} - 1 + 1)^\alpha = T_i^\alpha + \sum_{r>0} \alpha(\alpha-1)\cdots(\alpha - r + 1)X_i^{[r]} = T_i^\alpha$$

so $[\underline{T_i}]^\alpha \in \Theta(c)/p^n\Theta(c)$ as required. $\square$

3.2.2. Recall that R is a small integral K^+ with K^+ integrally closed in R, and $\mathcal{R}_n$ is the étale $\Theta(c)/p^n\Theta(c)$-algebra lifting R/pR.

Theorem 3.1. *We have*

$$t \cdot \left(F_p^i M_\infty^+(R)/p^n F_p^i M_\infty^+(R)\right)^{\Delta_\infty} = t \cdot F_p^i B_{\log}^+/p^n F_p^i B_{\log}^+ \otimes_{\Sigma_n} \mathcal{R}_n.$$

Proof. By dévissage in p it suffices to prove the statement modulo p. The proof for $i = 0$ will work for all i, so we assume $i = 0$. Recall that by quasi-coherence of $h_{\infty,*}\mathscr{O}$, we have a canonical isomorphism

$$M_\infty^+(R)/p^n M_\infty^+(R) \cong M_\infty^+(O(c))/p^n M_\infty^+(O(c)) \otimes_{\Theta(c)/p^n\Theta(c)} \mathcal{R}_n.$$

Since the invariants under Δ_∞ are the elements in $\bigcap_i \operatorname{Ker}(\sigma_i - 1)$, we see that we may assume that $R = O(c)$. In this case, by Proposition 3.1 we have

$$M_\infty^+/p^n M_\infty^+ \cong \frac{A_\infty^+/p^n A_\infty^+ \langle X_1, ..., X_{d+1}\rangle}{(1 + X(c) - \prod_{i=1}^r (1 + X_i))}.$$

Let $Y_i := \frac{-X_i}{1+X_i}$ for $i = 1, ..., d+1$. Then

$$M_\infty^+/p^n M_\infty^+ \cong \frac{A_\infty^+/p^n A_\infty^+ \langle Y_1, ..., Y_{d+1}\rangle}{((1 + X(c))^{-1} - \prod_{i=1}^r (1 + Y_i))}.$$

Note that the connection d on M_∞^+ acts by $T_i \frac{\partial}{\partial T_i}(Y_i^{[n+1]}) = Y_i^{[n]}$ because $Y_i = [\underline{T_i}]^{-1} \otimes T_i - 1$. For all $2 \leq i \leq d+1$, define an "integration" map

$$\int_i : M_\infty^+ \to M_\infty^+$$

to be the unique morphism of A_∞^+-modules sending $Y_i^{[n]}$ to $([\underline{T_i}] \otimes 1) \cdot Y_i^{[n+1]}$ and fixing the other $Y_j^{[m]}$. It is a one-sided inverse of $\partial_i := \frac{\partial}{\partial T_i}$, hence Δ_∞-equivariant up to adding terms in $\operatorname{Ker}(\partial_i)$. Note that if $\int_i m \in A_\infty^+$, then $m = \partial_i \int_i m = 0$. From the above description, it is clear that every element of M_∞^+/pM_∞^+ can be written as a finite sum of the form

$$m = x_0 + \int_i x_1 + \int_i \int_i x_2 + ... + \int_i^{\times n} x_n$$

with $x_j \in \operatorname{Ker}(\partial_i)$. We will prove by induction on n that if $m \in \operatorname{Ker}(\sigma_i - 1)$, then $t \cdot m \in t\,(\operatorname{Ker}(\partial_i))^{\sigma_i=1}[T_i]$. By $\operatorname{Ker}(\partial_i)^{\sigma_i=1}[T_i]$ we mean the sub-algebra of $M_\infty^+(O(c))/pM_\infty^+(O(c))$ generated by $\operatorname{Ker}(\partial_i)^{\sigma_i=1}$ and T_i (not the polynomial algebra!). This will prove the theorem, as

$$\bigcap_i \operatorname{Ker}(\partial_i)^{\sigma_i=1}[T_i] \subset \left(A_\infty^+(O(c))/pA_\infty^+(O(c))\right)^{\Delta_\infty}[T_1, ..., T_{d+1}] \subset B_{\log}^+/pB_{\log}^+ \otimes_{\Sigma_1} \Theta(c)/p\Theta(c).$$

For $n = 0$ this is trivial, so assume the result known for $n-1$. We have

$$m = \sigma_i(m) = \sigma_i \sum_{j=0}^{n} \int_i^{\times j} x_j = a + \sigma_i(x_0) + \sum_{j=1}^{n} \int_i \sigma_i \int_i^{\times j-1} x_j$$

for some $a \in \mathrm{Ker}(\partial_i)$, and hence

$$0 = a + \sigma_i(x_0) - x_0 + \sum_{j=1}^{n} \int_i (\sigma_i - 1) \int_i^{\times j-1} x_j.$$

Applying ∂_i we get

$$\sum_{j=1}^{n} (\sigma_i - 1) \int_i^{\times j-1} x_j = 0$$

so by induction we deduce that

$$t \sum_{j=1}^{n} \int_i^{\times j-1} x_j \in t \cdot (\mathrm{Ker}(\partial_i))^{\sigma_i = 1} [T_i].$$

Hence we have

$$tm = tx_0 + t \int_i x$$

for some $x \in t \cdot (\mathrm{Ker}(\partial_i))^{\sigma_i=1} [T_i]$. We may write

$$x = \sum_n b_n T_i^n$$

with $b_n \in (\mathrm{Ker}(\partial_i))^{\sigma_i = 1}$. We have

$$T_i^n = \sum_{r=0}^{n} \frac{n!}{r!} Y_i^{[n-r]}$$

so

$$\int_i T_i^n = [\underline{T_i}]^{n+1} \sum_{r=0}^{n} \frac{n!}{r!} Y_i^{[n+1-r]}.$$

Lifting to $M_\infty^+[1/p]$, by substituting the above series expressions for T_i^{n+1} we find

$$\int_i T_i^n = \frac{T_i^{n+1}}{n+1} - \frac{[\underline{T_i}]^{n+1}}{n+1}$$

and hence

$$(\sigma_i - 1) \int_i T_i^n = \frac{1 - [\underline{1}]^{n+1}}{n+1} [\underline{T_i}]^{n+1}.$$

Recall (*cf.* the proof of the Proposition 1.4) that

$$1 - [\underline{1}]^{n+1} = 1 - \exp((n+1)t) = -(n+1)t \sum_{m \geq 0} \frac{((n+1)t)^m}{(m+1)!}$$

where $v_{n+1} := \sum_{m \geq 0} \dfrac{((n+1)t)^m}{(m+1)!}$ is a unit of $A_{\text{cris}}(K^+)$. So we obtain

$$(\sigma_i - 1)x_0 = -\sum_n b_n(\sigma_i - 1)\int_i T_i^n = \sum_n b_n v_{n+1} t \underline{T_i}^{n+1}. \tag{1}$$

On the other hand, we can write $x_0 = \sum_n c_n \underline{T_i}^n$ with $c_n \in (\operatorname{Ker}(\partial_i))^{\sigma_i = 1}$, so

$$(\sigma_i - 1)x_0 = \sum_n c_n(\underline{1}^n - 1)\underline{T_i}^n = \sum_{v_p(n)=0} n c_n v_n t \underline{T_i}^n. \tag{2}$$

Since by Proposition 3.3 we know that b_n and c_n can only have powers of $\underline{T_i}$ divisible by p, on comparing equations 1 and 2 we see that

$$t \sum_{v_p(n+1) \geq 1} b_n \underline{T_i}^{n+1} = \sum_{v_p(n+1) \geq 1} b_n v_{n+1} t \underline{T_i}^{n+1} = 0$$

where in the first equality we have used that $v_{n+1} \equiv 1 \mod p$ if $n+1 \equiv 0 \mod p$. But

$$\begin{aligned} t \sum_{v_p(n+1) \geq 1} b_n \underline{T_i}^{n+1} &= t \sum_{v_p(n+1) \geq 1} b_n T_i^{n+1} \sum_{r=0}^{n+1} \frac{(n+1)!}{(n+1-r)!} X_i^{[r]} \\ &= t \sum_{v_p(n+1) \geq 1} b_n T_i^{n+1} \end{aligned}$$

so

$$t \sum_{v_p(n+1) \geq 1} b_n T_i^{n+1} = 0$$

and we may write

$$t \int_i x = t \sum_{n, v_p(n+1)=0} \frac{b_n}{n+1}(T_i^{n+1} - \underline{T_i}^{n+1})$$

hence

$$\begin{aligned} tm &= t \sum_{n, v_p(n+1)=0} \left(\frac{b_n}{n+1} T_i^{n+1} + \frac{(n+1)c_{n+1} - b_n}{n+1} \underline{T_i}^{n+1} \right) \\ &\quad + t \sum_{n, v_p(n+1) \geq 1} c_{n+1} T_i^{n+1}. \end{aligned}$$

Here we have used that if $v_p(n+1) > 0$ then $T_i^{n+1} = \underline{T_i}^{n+1}$. Note that

$$(\sigma_i - 1) \sum_{n, v_p(n+1)=0} c_{n+1} \underline{T_i}^{n+1} = (\sigma_i - 1)x_0$$

and

$$(\sigma_i - 1) \sum_{n, v_p(n+1)=0} \frac{b_n}{n+1} \underline{T_i}^{n+1} = (\sigma_i - 1) x_0$$

hence

$$(\sigma_i - 1) \sum_{n, v_p(n+1)=0} \frac{(n+1)c_{n+1} - b_n}{n+1} \underline{T_i}^{n+1} = 0.$$

Thus we indeed get that

$$tm \in t \left(\mathrm{Ker}(\partial_i)\right)^{\sigma_i = 1} [T_i]$$

thereby completing the induction and therefore the proof. □

The previous theorem, together with the following one, constitute the core results of this paper.

Theorem 3.2. *For all $i \neq 0$, the B^+-module*

$$H^i(\Delta_\infty, F_p^j M_\infty^+ / p^n F_p^j M_\infty^+)$$

is annihilated by t^d.

Proof. We give the proof for $j = 0$ to simplify, but the same proof works for any j. Also assume that $c = \pi$, since the case $c = 1$ is similar but simpler. Firstly, by Corollary 2.5, $M_\infty^+/p^n M_\infty^+$ is a discrete p-torsion Δ_∞-module, so we have canonical isomorphisms for all i

$$\mathrm{Ext}^i_{\mathbb{Z}_p[[\Delta_\infty]]}(\mathbb{Z}_p, M_\infty^+/p^n M_\infty^+) \cong H^i(\Delta_\infty, M_\infty^+/p^n M_\infty^+).$$

where the Ext-group is taken in the category of topological $\mathbb{Z}_p[[\Delta_\infty]]$-modules. Since $\Delta_\infty \cong \mathbb{Z}_p(1)^d$, we have an isomorphism of rings

$$\mathbb{Z}_p[[\Delta_\infty]] \simeq \mathbb{Z}_p[[\sigma_2 - 1, ..., \sigma_{d+1} - 1]].$$

This implies that the Koszul complex $L := \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} L_i$, where L_i is the complex defined

$$0 \longrightarrow \mathbb{Z}_p[[\Delta_\infty]] \xrightarrow{\sigma_i - 1} \mathbb{Z}_p[[\Delta_\infty]] \longrightarrow 0,$$

is a homological resolution of $\mathbb{Z}_p$ by free compact $\mathbb{Z}_p[[\Delta_\infty]]$-modules. Since this resolution is free compact and $M_\infty^+/p^n M_\infty^+$ is discrete, we clearly have

$$\mathrm{Hom}_{\mathbb{Z}_p[[\Delta_\infty]]}(L, M_\infty^+/p^n M_\infty^+) \cong L \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} M_\infty^+/p^n M_\infty^+.$$

We first show that for all i and all $m \in M_\infty^+/p^n M_\infty^+$, $t \cdot m$ lies in the image of the endomorphism $\sigma_i - 1$, and from this we will deduce the statement of the theorem. Since

$$M_\infty^+(R)/p^n M_\infty^+(R) \cong M_\infty^+(O(c))/p^n M_\infty^+(O(c)) \otimes_{\Theta(c)/p^n \Theta(c)} \mathcal{R}_n$$

we can assume that $R = O(c)$. Fix some $2 \leq i \leq d+1$. Define $Y_i = \frac{-X_i}{1+X_i} = [\underline{T_i}^{-1}] \otimes T_i - 1$. Every element of $M^+_\infty(O(c))/p^n M^+_\infty(O(c))$ is the sum of monomials of the form

$$\mu(c) = x[\underline{cT_i^{-1}}]^\alpha Y_i^{[n]}, \quad \text{or} \quad \mu = x[\underline{T_i}]^\alpha Y_i^{[n]}$$

with x invariant under σ_i and $\alpha \in \mathbb{N}[1/p]$. For $m = \mu(c), \mu$, we will show that $tm = (\sigma_i - 1) f_i(m)$ for some $f_i(m)$ by induction on n. If $n = 0$, then we distinguish three cases: $\alpha = 0$, $v_p(\alpha) \geq 0$, and $v_p(\alpha) < 0$. If $\alpha = 0$ then take

$$f_i(m) = m \log(X_i - 1).$$

Since $\sigma_i \log(X_i - 1) = \log([\underline{1}](X_i - 1)) = t + \log(X_i - 1)$ we obtain the claim in this case. If $v_p(\alpha) \geq 0$ then $\alpha \in \mathbb{N}$ and

$$\begin{aligned} [\underline{\pi T_i^{-1}}]^\alpha &= (uT_i^{-1})^\alpha([\underline{\pi T_i}] \otimes u^{-1}T_i - 1 + 1)^\alpha \\ &= (uT_i^{-1})^\alpha \left(1 + \sum_{r=0}^{\alpha} \frac{\alpha!}{(\alpha - r)!}([\underline{\pi T_i}] \otimes u^{-1}T_i - 1)^{[r]}\right) \end{aligned}$$

so

$$\begin{aligned} ([\underline{1}]^\alpha - 1)[\underline{\pi T_i^{-1}}]^\alpha &= (\sigma_i - 1)[\underline{\pi T_i^{-1}}]^\alpha \\ &= (uT_i^{-1})^\alpha(\sigma_i - 1)\left(\sum_{r=1}^{\alpha} \frac{\alpha!}{(\alpha - r)!}([\underline{\pi T_i}] \otimes u^{-1}T_i - 1)^{[r]}\right) \\ &= (\sigma_i - 1)\left(\alpha \cdot (uT_i^{-1})^\alpha \left(\sum_{r=1}^{\alpha} \frac{(\alpha-1)!}{(\alpha - r)!}([\underline{\pi T_i}] \otimes u^{-1}T_i - 1)^{[r]}\right)\right) \end{aligned}$$

and hence

$$t[\underline{\pi T_i^{-1}}]^\alpha = (\sigma_i - 1)\left(\frac{t\alpha}{[\underline{1}]^\alpha - 1}(uT_i^{-1})^\alpha \left(\sum_{r=1}^{\alpha} \frac{(\alpha-1)!}{(\alpha - r)!}([\underline{\pi T_i}] \otimes u^{-1}T_i - 1)^{[r]}\right)\right)$$

(recall that $\dfrac{t\alpha}{[\underline{1}]^\alpha - 1} \in A_{\mathrm{cris}}(K^+)$ by Proposition 1.4). Similarly we have

$$t[\underline{T_i}]^\alpha = (\sigma_i - 1)\left(\frac{t\alpha}{[\underline{1}]^\alpha - 1}T_i^\alpha \left(\sum_{r=1}^{\alpha} \frac{(\alpha-1)!}{(\alpha - r)!}([\underline{T_i}] \otimes T_i^{-1} - 1)^{[r]}\right)\right).$$

If $v_p(\alpha) < 0$, then

$$t[\underline{\pi T_i}^{-1}]^\alpha = (\sigma_i - 1)\left(\frac{t}{[\underline{1}]^\alpha - 1}[\underline{\pi T_i}^{-1}]^\alpha\right)$$

and

$$t[\underline{T_i}]^\alpha = (\sigma_i - 1)\left(\frac{t}{[\underline{1}]^\alpha - 1}[\underline{T_i}]^\alpha\right).$$

This begins the induction. As in the proof of Theorem 3.1, for each i we define the "integration" map

$$\int_i : M_\infty^+ \to M_\infty^+$$

to be the unique homomorphism of A_∞^+-modules sending $Y_i^{[n]}$ to $[\underline{T_i}]Y_i^{[n+1]}$ and fixing $Y_j^{[m]}$ for $j \neq i$. We have

$$\begin{aligned} \sigma_i \int_i ([\underline{T_i}^{-1}] \otimes T_i - 1)^{[n]} &= [\underline{1}][\underline{T_i}]([\underline{1}]^{-1}[\underline{T_i}^{-1}] \otimes T_i - 1)^{[n+1]} \\ &= \sum_{r=0}^{n+1} [\underline{1}]^{1-(n+1-r)}[\underline{T_i}]([\underline{T_i}^{-1}] \otimes T_i - 1)^{[n+1-r]}([\underline{1}]^{-1} - 1)^{[r]}. \end{aligned}$$

Also

$$\begin{aligned} \int_i \sigma_i([\underline{T_i}^{-1}] \otimes T_i - 1)^{[n]} &= \int_i \sum_{r=0}^{n} [\underline{1}]^{-(n-r)}([\underline{T_i}^{-1}] \otimes T_i - 1)^{[n-r]}([\underline{1}]^{-1} - 1)^{[r]} \\ &= \sum_{r=0}^{n} [\underline{1}]^{-(n-r)}([\underline{T_i}^{-1}] \otimes T_i - 1)^{[n+1-r]}([\underline{1}]^{-1} - 1)^{[r]} \end{aligned}$$

hence

$$\sigma_i \int_i ([\underline{T_i}^{-1}] \otimes T_i - 1)^{[n]} - \int_i \sigma_i([\underline{T_i}^{-1}] \otimes T_i - 1)^{[n]} = [\underline{1}][\underline{T_i}]([\underline{1}]^{-1} - 1)^{[n+1]}.$$

So we find

$$\begin{aligned} \sigma_i \int_i \mu(c) - \int_i \sigma_i \mu(c) &= x\sigma_i([\underline{\pi T_i}^{-1}]^\alpha)\left(\sigma_i \int_i Y_i^{[n]} - \int_i \sigma_i Y_i^{[n]}\right) \\ &= x[\underline{\pi T_i}^{-1}]^\alpha [\underline{1}]^{1-\alpha}[\underline{T_i}]([\underline{1}]^{-1} - 1)^{[n+1]} \end{aligned}$$

and

$$\sigma_i \int_i \mu - \int_i \sigma_i \mu = x\sigma_i([\underline{T_i}]^\alpha)\left(\sigma_i \int_i Y_i^{[n]} - \int_i \sigma_i Y_i^{[n]}\right) = x[\underline{T_i}]^\alpha [\underline{1}]^{1+\alpha}[\underline{T_i}]([\underline{1}]^{-1} - 1)^{[n+1]}.$$

But $([\underline{1}]^{-1} - 1)^{[n+1]} = (tv)^{[n+1]} = t\dfrac{v^{n+1}t^n}{(n+1)!}$ for some unit $v \in A_{\mathrm{cris}}(K^+)$, and $\dfrac{t^n}{(n+1)!} \in A_{\mathrm{cris}}(K^+)$ (see Proposition 1.4 and its proof). So, for $m = \mu(c), \mu$ we have that

$$\sigma_i \int_i tm - \int_i \sigma_i tm = (\sigma_i - 1)m'$$

for some m' and by induction hypothesis $tm = (\sigma_i - 1)f_i(m)$, hence

$$\int_i tm = \int_i (\sigma_i - 1)f_i(m) = (\sigma_i - 1)\int_i f_i(m) - \left(\sigma_i \int_i f_i(m) - \int_i \sigma_i f_i(m)\right)$$

lies in the image of $\sigma_i - 1$. This completes the induction, and proves that for all i

$$H_0(L_i \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} M_\infty^+/p^n M_\infty^+)$$

is annihilated by t.

Now, consider the Koszul complex $L = \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} L_i$. For any complex K of $\mathbb{Z}_p[[\Delta_\infty]]$-modules we have short exact sequences

$$0 \to H_0(L_i \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} H_p(K)) \to H_p(L_i \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} K) \to H_1(L_i \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} H_{p-1}(K)) \to 0.$$

Applying this inductively to $K = L_{\leq e} \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} M_\infty^+/p^n M_\infty^+$, with $L_{\leq e} := \otimes_{i \leq e} L_i$ the theorem follows. $\square$

3.2.3. Finally, we would like the resolution

$$A^+/p^n A^+ \to M^+/p^n M^+ \otimes_{\mathcal{R}_n} \omega^\bullet_{\mathcal{R}_n/\Sigma_n}$$

to be filtered. This probably follows from a general filtered crystalline Poincaré lemma but we can prove this directly.

Proposition 3.4. *For all i we have $H^i F_p^r(M^+/p^n M^+ \otimes_{\mathcal{R}_n} \omega^\bullet_{\mathcal{R}_n/\Sigma_n}) = 0$.*

Proof. We claim that every element of $F_p^r M^+/p F_p^r M^+$ can be written as a $W_n(P(\bar{R}/p\bar{R}))$-linear combination of monomials of the form

$$\mu := x_0 \cdot \prod_i (1 \otimes x_i - [\underline{\bar{x}_i}] \otimes 1)^{[m_i]}$$

with $x_0 \in F_p^{r-m} A_{\mathrm{cris}}(K^+)/p^n F_p^{r-m} A_{\mathrm{cris}}(K^+)$ where $m = \sum_i m_i$, and $x_i \in \mathcal{R}_n$, $\bar{x}_i$ the image of x_i in R/pR and $\underline{\bar{x}_i} \in P(\bar{R}/p\bar{R})$ a sequence of p-power roots of $\bar{x}_i$. To see this, consider the canonical map

$$\theta_{\mathcal{R}} : W_n(P(\bar{R}/p\bar{R})) \otimes_{\mathbb{Z}} \mathcal{R}_n \to \bar{R}/p\bar{R}.$$

If $\theta_{\mathcal{R}}(\sum_i a_i \otimes x_i) = 0$, then write

$$\sum_i a_i \otimes x_i = \sum_i (a_i \otimes 1)(1 \otimes x_i - [\underline{\bar{x}_i}] \otimes 1) + \sum_i a_i [\underline{\bar{x}_i}] \otimes 1.$$

Since we must have $\sum_i a_i [\underline{\bar{x}_i}] \in \mathrm{Ker}(\theta) = (\xi, p)$, where θ is the canonical map

$$\theta : W_n(P(\bar{R}/p\bar{R})) \to \bar{R}/p\bar{R}$$

this proves the claim.

Now, if $T_2, ..., T_{d+1}$ denote étale local coordinates on $\mathcal{R}_n$, then we have

$$\frac{\partial \mu}{\partial T_k} = x_0 \sum_i (1 \otimes \frac{\partial x_i}{\partial T_k})(1 \otimes x_i - [\underline{\bar{x}_i}] \otimes 1)^{[m_i - 1]} \cdot \prod_{j \neq i} (1 \otimes x_j - [\underline{\bar{x}_j}] \otimes 1)^{[m_j]}$$

so $\dfrac{\partial}{\partial T_k}(F_p^r M^+/p^n F_p^r M^+) \subset F_p^{r-1}M^+/p^n F^{r-1}M^+$ (Griffiths transversality). Define operators on differential forms

$$h_k(xdT_k) := \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\partial^n x}{\partial T_k^n} (1 \otimes T_k - \underline{[T_k]} \otimes 1)^{[n+1]}.$$

Note that this is a finite sum by quasi-nilpotence of the connection d. One checks easily that

$$h_k : F_p^r M^+/p^n F_p^r M^+ \cdot dT_k \to F_p^{r+1} M^+/p^n F_p^{r+1} M^+$$

is a homomorphism of abelian groups for all k. We extend this to a all differential forms of order at least 1 by

$$H_k(xdT_{i_1} \wedge \cdots \wedge dT_{i_n}) := (-1)^{j-1} h_k(xdT_k) dT_{i_1} \wedge \cdots \wedge \widehat{dT_{i_j}} \wedge \cdots \wedge dT_{i_n}$$

if $k = i_j$ for some $1 \leq j \leq n$ and otherwise we define $H_k(xdT_{i_1} \wedge \cdots \wedge dT_{i_n}) = 0$. Define another operator on differential forms by

$$\frac{\partial}{\partial T_k}(xdT_{i_1} \wedge \cdots \wedge dT_{i_n}) \wedge dT_k := \frac{\partial x}{\partial T_k} dT_k \wedge dT_{i_1} \wedge \cdots \wedge dT_{i_n}$$

so that for any differential form η we have

$$\sum_k \frac{\partial \eta}{\partial T_k} \wedge dT_k = d\eta.$$

Note that if $k \neq l$, then $H_k\left(\dfrac{\partial \eta}{\partial T_l} \wedge T_l\right) = \dfrac{\partial H_k(\eta)}{\partial T_l} \wedge T_l$. Let $\eta = xdT_{i_1} \wedge \cdots \wedge dT_{i_n}$. If $k = i_j$ for some j we have

$$\begin{aligned} dH_k(\eta) &= \eta + \sum_{l \notin \{i_1, \ldots, i_n\}} \frac{\partial H_k(\eta)}{\partial T_l} \wedge T_l \\ &= \eta - \sum_{l \notin \{i_1, \ldots, i_n\}} H_k\left(\frac{\partial \eta}{\partial T_l} \wedge T_l\right) \end{aligned}$$

so that

$$dH_k(\eta) + H_k(d\eta) = \eta.$$

In general, write $\eta = \sum_I x_I dT_I$, where $I \subset \{2, ..., d+1\}$, $\sharp I = n$ for some fixed n, and $dT_I := \wedge_{i \in I} dT_i$. This is well-defined up to sign, which will not matter in the sequel. Moreover, we may and will assume that no two I's are equal. For all such η, choose a smallest possible subset $J_\eta \subset \cup_I I$ such that for all I there exists some $j_I \in J_\eta \cap I$. Since J_η is smallest possible we see that for all I there is a single $\{j_I\} = J_\eta \cap I$. With these choices, we define

$$H_{J_\eta}(\eta) := \sum_{j \in J_\eta} H_j(\eta).$$

Since we have

$$d(x_I dT_I) = \sum_{l \notin I} \frac{\partial x_I}{dT_l} dT_l \wedge dT_I$$

is it clear that $J_{d\eta} \subset J_\eta$. Then

$$H_{J_\eta}(\eta) = \sum_I \sum_{j \in J_\eta \cap I} H_j(x_I dT_I) = \sum_I H_{j_I}(x_I dT_I)$$

hence

$$dH_{J_\eta}(\eta) = \sum_I dH_{j_I}(x_I dT_I) = \eta - \sum_I H_{j_I} d(x_I dT_I) = \eta - H_{J_\eta} d(\eta).$$

So if in particular η is closed, then $\eta = dH_{J_\eta}(\eta)$. □

3.3 Application to Hodge-Tate cohomology

If we replace Σ everywhere in the above by K^+, then we obtain similar results which can be applied to recover Galois cohomology computations of Faltings.

3.3.1. The case where Σ is replaced with K^+ is a de Rham theory, rather than a crystalline one.

Lemma 3.3. *We have canonical isomorphisms*

$$\begin{array}{rcl} A^+/p^n A^+ \otimes_{\Sigma_n} K^+/p^n K^+ & \cong & H^0_{\text{log-crys}}(\bar{R}/p\bar{R}|K^+/p^n K^+, \mathscr{O}) \\ M^+/p^n M^+ \otimes_{\Sigma_n} K^+/p^n K^+ & \cong & H^0_{\text{log-crys}}(\bar{R}/p\bar{R}|R/p^n R, \mathscr{O}) \end{array}$$

and similarly when we replace $\bar{R}$ by $\tilde{R}_\infty$.

Proof. Let T_n be one of Σ_n or $\mathcal{R}_n$. In the latter case note that $\mathcal{R}_n \otimes_{\Sigma_n} K^+/p^n K^+ \cong R/p^n R$ by very definition of $\mathcal{R}_n$. Let S be one of $\bar{R}/p\bar{R}$ or $\tilde{R}_\infty/p\tilde{R}_\infty$. Making a choice of uniformizer π of K^+ defines a surjection $\Sigma \to K^+$. We have a short exact sequence

$$0 \longrightarrow E \longrightarrow T_n \xrightarrow{u \mapsto \pi} T_n \otimes_{\Sigma_n} K^+/p^n K^+ \longrightarrow 0$$

and tensoring with the flat abelian group $W(P(S))$ we obtain a short exact sequence

$$0 \to W(P(S)) \otimes_{\mathbb{Z}} E \to W(P(S)) \otimes_{\mathbb{Z}} T_n \to W(P(S)) \otimes_{\mathbb{Z}} T_n \otimes_{\Sigma_n} K^+/p^n K^+ \to 0$$

so $I := W(P(S)) \otimes_{\mathbb{Z}} E \subset W(P(S)) \otimes_{\mathbb{Z}} T_n$ is an ideal. It has a DP-structure defined $(x \otimes y)^{[r]} = x^r \otimes y^{[r]}$ for all $x \in W(P(S))$ and all $y \in E$, where the box exponent denotes the DP-structure on I, and then extended to sums via the binomial formula. Let M be an integral monoid defining the log-structure on S, and define

$$L := \mathrm{Ker}(P(M)^{\mathrm{gp}} \oplus N \to M^{\mathrm{gp}})$$

where $N \to T_n$ is a fine monoid defining the pre-log-structure. By definition N also induces the pre-log-structure on $T_n \otimes_{\Sigma_n} K^+/p^n K^+$. Define

$$\begin{aligned} B &:= W(P(S)) \otimes_{\mathbb{Z}} T_n \otimes_{\mathbb{Z}} \mathbb{Z}[L] \\ B' &:= W(P(S)) \otimes_{\mathbb{Z}} T_n \otimes_{\Sigma_n} K^+/p^n K^+ \otimes_{\mathbb{Z}} \mathbb{Z}[L]. \end{aligned}$$

Then the homomorphism $B \to B'$ is a DP-morphism, where the ideal $I' := 0 \subset B'$ is given the canonical DP-structure, and $B/I = B'/I'$. Define $J := \mathrm{Ker}(B \to S)$, $J' := \mathrm{Ker}(B' \to S)$. Then $I \subset J$ and $I' \subset J'$ and we have $B/J = B'/J'$. So by [1] Ch. I Prop. 2.8.2, there is a canonical isomorphism of divided power hulls

$$D_B(J) \otimes_B B' \cong D_{B'}(J')$$

which is precisely what we wanted to show. □

3.3.2. Now the previous results hold with Σ_n replaced by $K^+/p^n K^+$: the augmentation

$$A^+/p^n A^+ \otimes_{\Sigma_n} K^+/p^n K^+ \to M^+/p^n M^+ \otimes_{\Sigma_n} K^+/p^n K^+ \otimes_R \omega^\bullet_{R/K^+}$$

is a filtered quasi-isomorphism and for $i \neq 0$ we have

$$t^d H^i(\Delta, F_p^r M^+/p^n F_p^r M^+ \otimes_{\Sigma_n} K^+/p^n K^+) \approx 0$$

and moreover

$$tH^0(\Delta, F_p^r M^+/p^n F_p^r M^+ \otimes_{\Sigma_n} K^+/p^n K^+) \approx F_p^r B^+/p^n F_p^r B^+ \otimes_{\Sigma_n} R/p^n R.$$

3.3.3. Let us compute the graded of $A^+ \hat{\otimes}_{\Sigma} K^+[1/p]$ for the canonical filtration.

Lemma 3.4. *Let $R_1 \to R_2$ be a flat homomorphism of $\mathbb{F}_p$-algebras. If the Frobenius is surjective on R_i for $i = 1, 2$, then the homomorphism*

$$W_n(P(R_1)) \to W_n(P(R_2))$$

is flat.

Proof. Let I be a finite index set, and let $\left((x_i^{(n)})_n\right)_{i \in I}$ (resp. $\left((y_i^{(n)})_n\right)_{i \in I}$) be a family of elements of $P(R_1)$ (resp. $P(R_2)$) indexed by I. Assume that

$$\sum_i (x_i^{(n)})_n \cdot (y_i^{(n)})_n = 0$$

Then we must have $\sum_i x_i^{(n)} \cdot y_i^{(n)} = 0$ for all n, hence $\sum_i x_i^{(n)} \otimes y_i^{(n)} = 0$ in $\left\langle x_i^{(n)} \right\rangle \otimes_{R_1} R_2$, where $\left\langle x_i^{(n)} \right\rangle$ denotes the ideal of R_2 generated by the $x_i^{(n)}$. This implies that $\sum_i (x_i^{(n)})_n \otimes (y_i^{(n)})_n = 0$ in $\left\langle (x_i^{(n)})_n \right\rangle \otimes_{P(R_1)} P(R_2)$, which proves the flatness of $P(R_1) \to P(R_2)$.

Now, since $P_1 := P(R_1)$ and $P_2 := P(R_2)$ are perfect rings of characteristic p, the $W_n := W_n(\mathbb{F}_p)$-modules $W_n(P_1)$ and $W_n(P_2)$ are flat. In particular, since $(p) = p \cdot W_n$ is a flat W_n-module, $p \cdot W_n(P_1) \cong (p) \otimes_{W_n} W_n(P_1)$ is a flat $W_n(P_1)$-module and hence we have a resolution of P_1 by flat $W_n(P_1)$-modules

$$0 \to (p) \otimes_{W_n} W_n(P_1) \to W_n(P_1) \to P_1 \to 0$$

Tensoring with $W_n(P_2)$ we obtain an exact sequence

$$0 \to \mathrm{Tor}_1^{W_n(P_1)}(W_n(P_2), P_1) \to (p) \otimes_{W_n} W_n(P_2) \to W_n(P_2)$$

hence $\mathrm{Tor}_1^{W_n(P_1)}(W_n(P_2), P_1) = 0$. By the standard flatness criterion, this implies that

$$W_n(P_1) \to W_n(P_2)$$

is flat. $\square$

Now, since the map $W(P(\bar{K}^+/p\bar{K}^+)) \to W(P(\bar{R}/p\bar{R}))$ is flat it follows that

$$A^+ \otimes_\Sigma K^+/p^n K^+ \cong W(P(\bar{R}/p\bar{R})) \otimes_{W(P(\bar{K}^+/p\bar{K}^+))} B^+_{\log} \otimes_\Sigma K^+/p^n K^+$$

so $A^+ \otimes_\Sigma K^+/p^n K^+$ is a flat $B^+_{\log} \otimes_\Sigma K^+/p^n K^+$-algebra and hence

$$\mathrm{gr}^i(A^+ \otimes_\Sigma K^+/p^n K^+) \cong \bar{R}/p^n \bar{R} \otimes_{\bar{K}^+/p^n \bar{K}^+} \mathrm{gr}^i(B^+_{\log} \otimes_\Sigma K^+/p^n K^+).$$

Since $B^+_{\log} \hat{\otimes}_\Sigma K^+ \subset B^+_{\mathrm{dR}}$ it follows that

$$\mathrm{gr}^i(B^+_{\log} \hat{\otimes}_\Sigma K^+)[1/p] \cong \hat{\bar{K}}(i)$$

and hence

$$\mathrm{gr}^i(A^+ \hat{\otimes}_\Sigma K^+)[1/p] \cong \hat{\bar{R}}(i)[1/p].$$

3.3.4. Let G be a filtered $A_{\mathrm{cris}}(K^+)$-module. We endow

$$G \otimes_{A_{\mathrm{cris}}(K^+)} B_{\mathrm{cris}}$$

with the tensor product filtration. Assume in addition that G is p-adically complete and $G/p^n G$ is a discrete Δ-module for all $n \geq 1$ and that the Δ-action is $A_{\mathrm{cris}}(K^+)$-linear. Then we define

$$H^*(\Delta, F^r(G \otimes_{A_{\mathrm{cris}}(K^+)} B_{\mathrm{cris}})) := \mathrm{colim}_s\, H^*(\Delta, F^{r+s} G) \otimes_{A_{\mathrm{cris}}(K^+)} F^{-s} B_{\mathrm{cris}}.$$

This is a cohomological functor because $A_{\mathrm{cris}}(K^+) \to B_{\mathrm{cris}}$ is flat. Define

$$\begin{aligned} B_{\log} \hat{\otimes}_\Sigma K^+ &:= B^+_{\log} \hat{\otimes}_\Sigma K^+ \otimes_{A_{\mathrm{cris}}(K^+)} B_{\mathrm{cris}} \\ A \hat{\otimes}_\Sigma K^+ &:= A^+ \hat{\otimes}_\Sigma K^+ \otimes_{A_{\mathrm{cris}}(K^+)} B_{\mathrm{cris}} \\ M \hat{\otimes}_\Sigma K^+ &:= M^+ \hat{\otimes}_\Sigma K^+ \otimes_{A_{\mathrm{cris}}(K^+)} B_{\mathrm{cris}}. \end{aligned}$$

It follows from our computations that for any $r \in \mathbb{Z}$ and all $i \neq 0$ we have

$$H^i(\Delta, F^r M \hat{\otimes}_\Sigma K^+) \approx 0$$

and $H^0(\Delta, F^r M \hat{\otimes}_\Sigma K^+) \approx F^r B_{\log} \hat{\otimes}_\Sigma K^+ \hat{\otimes}_{K^+} \hat{R}$, so we deduce that $H^*(\Delta, \mathrm{gr}^r A \hat{\otimes}_\Sigma K^+)$ is almost represented by the de Rham complex

$$\cdots \to \omega^i_{R/K^+} \hat{\otimes}_R \hat{\bar{R}}(r-i)[1/p] \to \omega^{i+1}_{R/K^+} \hat{\otimes}_R \hat{\bar{R}}(r-i-1)[1/p] \to \cdots$$

Since by a theorem of Tate [18] there are no $\mathrm{Gal}(\bar{K}/K)$-equivariant homomorphisms $\hat{\bar{K}}(i) \to \hat{\bar{K}}(j)$ for $i \neq j$ we see that this is a complex with differential zero, hence

$$H^i(\Delta, \mathrm{gr}^r A \hat{\otimes}_\Sigma K^+) \cong \omega^i_{R/K^+} \hat{\otimes}_R \hat{\bar{R}}(r-i)[1/p].$$

Noting that by definition we have

$$\begin{aligned} H^i(\Delta, \mathrm{gr}^0 A \hat{\otimes}_\Sigma K^+) &= H^i(\Delta, \hat{\bar{R}}) \otimes_{A_{\mathrm{cris}}(K^+)} B_{\mathrm{cris}} \\ &= H^i(\Delta, \hat{\bar{R}}) \otimes_{A_{\mathrm{cris}}(K^+)} \mathrm{gr}^0 B_{\mathrm{cris}} = H^i(\Delta, \hat{\bar{R}})[1/p] \end{aligned}$$

we recover the Galois cohomology computations of Faltings [7], albeit without any information on the p-torsion.

3.3.5. We can recover the p-torsion information as follows. Using the logarithm $\log : \mathbb{Z}_p(1) \to B^+_{\log}$ we identify $\mathbb{Z}_p(i)$ with $\mathbb{Z}_p t^i \subset B^+_{\log}$ and this defines an injective map

$$\bar{K}^+/p^n\bar{K}^+(i) \to \mathrm{gr}^i B^+_{\log} \otimes_\Sigma K^+/p^n K^+$$

whose cokernel it annihilated by $i! \cdot ([\underline{1}]^{1/p} - 1)^i$ because up to units we have

$$t^i = ([\underline{1}]^{1/p} - 1)^i i! \xi^{[i]}.$$

Recall that by Tate there are almost no $\mathrm{Gal}(\bar{K}/K)$-equivariant homomorphisms

$$\bar{K}^+/p^n\bar{K}^+(i) \to \bar{K}^+/p^n\bar{K}^+(j)$$

unless $i = j$. There is a commutative square of complexes

$$\begin{array}{ccc} F^{d+1}\left(B^+_{\log}/p^n B^+_{\log} \otimes_\Sigma \omega^\bullet_{R/K^+}\right) & \longrightarrow & F^d\left(B^+_{\log}/p^n B^+_{\log} \otimes_\Sigma \omega^\bullet_{R/K^+}\right) \\ \downarrow & & \downarrow \\ C^*(\Delta, F^{d+1}A^+ \otimes_\Sigma K^+/p^n K^+) & \longrightarrow & C^*(\Delta, F^d A^+ \otimes_\Sigma K^+/p^n K^+) \end{array}$$

hence a morphism on cokernels

$$\mathrm{gr}^d\left(B^+_{\log}/p^n B^+_{\log} \otimes_\Sigma \omega^\bullet_{R/K^+}\right) \to C^*(\Delta, \mathrm{gr}^d A^+ \otimes_\Sigma K^+/p^n K^+).$$

Up to $d! \cdot ([\underline{1}]^{1/p} - 1)^d$-torsion the de Rham complex on the left-hand side has zero differential, hence up to torsion for all i we get a map

$$\omega^i_{R/K^+} \otimes_{K^+} \bar{K}^+/p^n\bar{K}^+(d-i) \to H^i(\Delta, \bar{R}/p^n\bar{R}(d)).$$

This map is an almost isomorphism up to $d! \cdot ([\underline{1}]^{1/p} - 1)^d$-torsion, as can be checked by computing the right-hand side.

4 Globalization

In this section we globalize our previous results. We begin by the defining a site which will enable us to glue Galois cohomology, following Faltings. Then we review the construction by Fontaine-Messing of the syntomic crystalline site and its logarithmic generalization by Breuil. Finally we link the two constructions to express crystalline cohomology as the cohomology of a sheaf on the Faltings site.

4.1 Faltings cohomology

4.1.1. Let X be a scheme of finite type over a field k. Denote by $X_{\text{ét}}$ the étale site of X and by $X_{\text{Fét}}$ the site whose underlying category consists of finite étale morphisms. There is a canonical continuous morphism of sites

$$\rho : X_{\text{ét}} \to X_{\text{Fét}}.$$

Recall that a scheme of finite type is said to be a $K(\pi, 1)$ if the adjunction map

$$\mathbb{L} \to R\rho_* \rho^* \mathbb{L}$$

is a quasi-isomorphism for all locally constant constructive sheaves $\mathbb{L}$ of order prime to the characteristic of k. M. Artin has proved in SGA 4, Exp. XI, that every smooth scheme over an algebraically closed field is locally (for the Zariski topology) a $K(\pi, 1)$.

Theorem 4.1 (Faltings)**.** *If X is a smooth K^+-scheme, then every point of X has a neighbourhood U such that $U_{\bar{K}}$ is a $K(\pi, 1)$.*

See [5] 2.1 Lemma for the proof.

4.1.2. Let X be a normal K^+-scheme of finite type. Define a site $X_{\mathfrak{F}}$ as follows. The category underlying $X_{\mathfrak{F}}$ has for objects the pairs (U, V), where $U \to X$ is étale and $V \to U_{\bar{K}}$ is finite étale. A morphism $(U', V') \to (U, V)$ is given by a commutative diagram of étale morphisms

$$\begin{array}{ccc} U' & \longrightarrow & V' \\ \downarrow & & \downarrow \\ U & \longrightarrow & V. \end{array}$$

Coverings are given by families of morphisms $(U_\alpha, V_\alpha)_\alpha \to (U, V)$ such that $\coprod_\alpha U_\alpha \to U$ and $\coprod_\alpha V_\alpha \to V$ are surjective.

Let $\bar{x}$ be a geometric point of X, $X_{\bar{x}} \otimes \bar{K} := \operatorname{Spec}(\mathscr{O}_{X,\bar{x}} \otimes_{K^+} \bar{K})$, and let $\bar{y}$ be a geometric point of $X_{\bar{x}} \otimes \bar{K}$. The points of the topos $X_{\mathfrak{F}}$ are given by the fibre functors

$$\mathscr{F}_{(\bar{x},\bar{y})} := \operatorname{colim}_{Y \to X_{\bar{x}} \otimes \bar{K}} \mathscr{F}(Y)$$

where the inductive limit is taken over all finite étale neighbourhoods of $\bar{y}$. By a finite étale neighbourhood of a geometric point $\bar{x} \to X$ we mean a finite étale morphism

$Y \to X$ and a morphism $\bar{x} \to Y$ such the composition $\bar{x} \to Y \to X$ is the original map. We will usually refer to $(\bar{x}, \bar{y})$ as the point of the topos $X_{\mathfrak{F}}$. This topos has enough points and the cohomology of quasi-compact X commutes with filtering inductive limits ([7], 3.).

4.1.3. Examples of sheaves on $X_{\mathfrak{F}}$ include the constant sheaf $\mathbb{Z}/p^n\mathbb{Z}$ and the sheaf $\bar{\mathscr{O}}_n$ defined

$$\bar{\mathscr{O}}_n(U, V) := \mathscr{O}(V^\nu/p^n V^\nu)$$

where V^ν denotes the integral closure of U in the quasi-coherent $\mathscr{R}(U)$-algebra $\mathscr{R}(V)$ (for any scheme X we denote by $\mathscr{R}(X)$ its sheaf of algebras of rational functions, i.e. the product of its local rings at its generic points, see [12] II, 6.3, for details about $\mathscr{R}(X)$).

Define a sheaf $\mathscr{A}_n^+$ as the sheaf associated to the presheaf

$$(U, V) \rightsquigarrow H^0_{\text{log-crys}}(V^\nu \otimes_{\mathbb{Z}} \mathbb{F}_p | \Sigma_n, \mathscr{O}).$$

4.1.4. Let us show how to locally compute the cohomology of $X_{\mathfrak{F}}$. There is a canonical continuous morphism of sites

$$\beta : X_{\mathfrak{F}} \to X_{\bar{K},\text{Fét}}$$

induced by the functor sending a finite étale cover $Y \to X_{\bar{K}}$ to the object (X, Y) of $X_{\mathfrak{F}}$.

Proposition 4.1. *Suppose X is a normal strictly local K^+-scheme, $\bar{x} \to X$ its closed point, and let $X_{\bar{K}} = \coprod Y_i$ be its decomposition into irreducible components.*

- *(i) β_* is an exact and faithful functor.*
- *(ii) For any choice of geometric points $\bar{y}_i \to Y_i$, any sheaf of abelian groups $\mathscr{F}$ on $X_{\mathfrak{F}}$, and any n we have*

$$H^n(X_{\mathfrak{F}}, \mathscr{F}) \cong \prod_i H^n(\pi_1(Y_i, \bar{y}_i), \mathscr{F}_{(\bar{x},\bar{y}_i)}).$$

Proof. For (i), let $\bar{y} \to X_{\bar{K}}$ be a geometric point. By definition we see that

$$(\beta_*\mathscr{F})_{\bar{y}} = \operatorname{colim}_Y (\beta_*\mathscr{F})(Y) = \operatorname{colim}_Y \mathscr{F}(Y) = \mathscr{F}_{(\bar{x},\bar{y})}.$$

This proves that β_* is exact and faithful.

(ii) follows from (i) because by [11] we have

$$H^n(X_{\bar{K},\text{Fét}}, \beta_*\mathscr{F}) \cong \prod_i H^n(\pi_1(Y_i, \bar{y}_i), \mathscr{F}_{(\bar{x},\bar{y}_i)}).$$

□

The natural functor $X_{\mathfrak{F}} \to X_{\bar{K},\text{ét}}$ sending (U, V) to V induces a continuous morphism of sites denoted

$$\alpha : X_{\bar{K},\text{ét}} \to X_{\mathfrak{F}}.$$

Proposition 4.2. *Suppose X is a smooth K^+-scheme. Then for any locally constant constructible sheaf $\mathbb{L}$ on $X_{\bar{K},\mathrm{Fét}}$, the adjunction map*

$$\beta^*\mathbb{L} \to R\alpha_*\alpha^*\beta^*\mathbb{L}$$

is a quasi-isomorphism.

Proof. The claim is local on X, so we can assume that X is strictly local and $X_{\bar{K}}$ is a $K(\pi, 1)$. Let $\bar{x}$ be the closed point of X. First note that if $\mathbb{L}$ is a locally constant constructible sheaf then it is representable by a finite étale covering Z of $X_{\bar{K}}$, hence for any finite étale covering Y of $X_{\bar{K}}$ we have $(\beta^*\mathbb{L})(Y) = \mathrm{Hom}_{X_{\bar{K},\mathrm{Fét}}}(Y, Z) = \mathbb{L}(Y)$. In particular, for any geometric point $\bar{y} \to X_{\bar{K}}$ we have

$$(\beta_*\beta^*\mathbb{L})_{\bar{y}} = (\beta^*\mathbb{L})_{(\bar{x},\bar{y})} = \mathrm{colim}_Y(\beta^*\mathbb{L})(Y) = \mathrm{colim}_Y\,\mathbb{L}(Y) = \mathbb{L}_{\bar{y}}$$

so $\mathbb{L} \cong \beta_*\beta^*\mathbb{L}$. Now, we have a commutative diagram of sites

$$\begin{tikzcd} X_{\bar{K},\mathrm{ét}} \arrow[r, "\alpha"] \arrow[d, "\rho"'] & X_{\mathfrak{F}} \arrow[d, "\beta"'] \\ X_{\bar{K},\mathrm{Fét}} \arrow[r, equal] & X_{\bar{K},\mathrm{Fét}} \end{tikzcd}$$

so

$$R\alpha_*\alpha^*\beta^*\mathbb{L} = R\alpha_*\rho^*\mathbb{L}$$

and

$$\mathbb{L} \cong \beta_*\beta^*\mathbb{L} \to \beta_*R\alpha_*\rho^*\mathbb{L} \cong R\beta_*R\alpha_*\rho^*\mathbb{L} \cong R\rho_*\rho^*\mathbb{L} \cong \mathbb{L}.$$

Since β_* is exact and faithful this completes the proof. $\square$

Corollary 4.1. *If X is a smooth K^+-scheme and $\mathbb{L}$ a locally constant constructible sheaf on $X_{\bar{K},\mathrm{Fét}}$, then for all i we have canonical isomorphisms*

$$H^i_{\mathrm{ét}}(X_{\bar{K}}, \mathbb{L}) \cong H^i(X_{\mathfrak{F}}, \beta^*\mathbb{L}).$$

4.1.5. Assume X is a proper semi-stable K^+-scheme. Let us compute the cohomology of the sheaf $\hat{\bar{\mathscr{O}}} := \lim_n \bar{\mathscr{O}}_{n+1}$ on $X_{\mathfrak{F}}$ up to p-torsion. For this we use the Leray spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{ét}}(X, R^ju_*\bar{\mathscr{O}}_{n+1}) \Longrightarrow H^{i+j}(X_{\mathfrak{F}}, \bar{\mathscr{O}}_{n+1})$$

where $u : X_{\mathfrak{F}} \to X_{\mathrm{ét}}$ is the projection onto the étale site. Recall (*cf.* §3.3) that

$$R^ju_*\bar{\mathscr{O}}_{n+1} \cong \omega^j_{R/K^+} \otimes \bar{K}^+/p^n\bar{K}^+(-j) \oplus (d!(\zeta-1)^d\text{-torsion})$$

where ζ denotes a p-th root of unity. Hence, taking the projective limit of the spectral sequence and inverting p we obtain a spectral sequence

$$E_2^{i,j} = H^i(X, \omega^j_{X/K^+}) \otimes_{K^+} \hat{\bar{K}}(-j) \Longrightarrow H^{i+j}(X_{\mathfrak{F}}, \hat{\bar{\mathscr{O}}})[1/p]$$

which degenerates at E_2 because of the Tate twists, i.e.

$$H^{i+j}(X_{\mathfrak{F}}, \hat{\bar{\mathscr{O}}})[1/p] \cong \oplus_{i,j} H^i(X, \omega^j_{X/K^+}) \otimes_{K^+} \hat{\bar{K}}(-j).$$

4.2 Syntomic cohomology

4.2.1. Recall that a morphism of schemes $X \to Y$ is *syntomic* if locally $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ with

$$B = A[X_1, ..., X_n]/(f_1, ..., f_m)$$

for a regular sequence $(f_1, ..., f_m)$ in $A[X_1, ..., X_n]$ and $A[X_1, ..., X_n]/(f_1, ..., f_i)$ is flat as an A-module for all i. Locally here can mean either for the Zariski or étale topologies.

For example, the homomorphism $R \to R_n$ is syntomic, because the equations $X_1^{p^n} - T_1, ..., X_{d+1}^{p^n} - T_{d+1}$ have no common roots. Hence $\tilde{R} \to \tilde{R}_\infty$ is the inductive limit of syntomic homomorphisms. It is well-known that syntomic morphisms are stable by base change and composition. Thus we may speak of the syntomic topology.

Following Breuil, we want to consider a crystalline site in which we localise for the syntomic topology rather than the étale topology. We also need a logarithmic version. For this we must restrict to logarithmic syntomic morphisms which lift locally, the so-called *log-syntomic morphisms* of [4].

4.2.2. If M is an integral monoid and $m \in M^{\mathrm{gp}}$, one say that m is *simplifiable* in M if there exists $n, n' \in M$ such that $m = n/n'$ and for all $p, p' \in M$ such that $m = p/p'$ there exists $m_0 \in M$ such that $p = n_0 n$ and $p' = n_0 n'$; one then says that $m = n/n'$ is a *simplified expression* for m ("écriture simplifiée" in French). One says that m is *regular in* M if m is simplifiable in M and of infinite order in M^{gp}. Let $m_1, ..., m_r \in M^{\mathrm{gp}}$ and let $M_i = \langle m_1, ..., m_i \rangle$ be the subgroup of M^{gp} generated by $m_1, ..., m_i$, $M_0 = 0$. We say that $(m_1, ..., m_r)$ is a *regular sequence* in M if m_i is regular in $M/M_{i-1} := \mathrm{Im}(M \to M^{\mathrm{gp}}/M_{i-1})$ for $1 \leq i \leq r$. The sequence $(m_1, ..., m_r)$ is regular if and only if $(n_1 - n_1', ..., n_r - n_r')$ is a regular sequence in $\mathbb{Z}[M]$, where $m_i = n_i/n_i'$ is a simplified expression for m_i for all i, in which case the canonical map

$$\mathbb{Z}[M]/(n_1 - n_1', ..., n_i - n_i') \to \mathbb{Z}[M/M_i]$$

is an isomorphism for all i ([4], Prop. 2.1.8).

Finally, one defines a morphism of integral monoids $M \to N$ to be *syntomic* if it is injective and if we can write $N = (M \oplus \mathbb{N}X_1 \oplus \cdots \oplus \mathbb{N}X_r)/\langle x_1, ..., x_s \rangle$ for some regular sequence $(x_1, ..., x_s)$ in $M \oplus \mathbb{N}X_1 \oplus \cdots \oplus \mathbb{N}X_r$ and moreover the morphism $M \to N$ is *universally integral*. This last condition means that for any morphism integral monoids $M \to M'$ the push out of $N \leftarrow M \to M'$ is integral. By [15] Prop. 4.1, $M \to N$ is universally integral if and only if $\mathbb{Z}[M] \to \mathbb{Z}[N]$ is flat. This condition ensures that if $M \to N$ is syntomic then so is $\mathbb{Z}[M] \to \mathbb{Z}[N]$.

4.2.3. Define a morphism of fine log-schemes $f : (X, M) \to (Y, N)$ to be *log-syntomic* if it is flat and locally for the étale topology it is *standard log-syntomic*, i.e. we have $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, $M_A = P^a$ and $N_B = Q^a$, and a commutative diagram

$$\begin{array}{ccc} Q & \longrightarrow & B \\ \downarrow & & \downarrow \\ P & \longrightarrow & A \end{array}$$

such that $Q \to P$ is syntomic and $B \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P] \to A$ is syntomic. Log-syntomic morphisms are stable under base change by a morphism of fine log-schemes and the composition of two log-syntomic morphisms is again log-syntomic. This enables us, for every fine log-scheme X, to define the *log-syntomic site* X_{syn} whose objects are log-syntomic morphisms $U \to X$ and whose coverings are given by families of log-syntomic morphisms $U_\alpha \to U$ such that $\coprod U_\alpha \to U$ is surjective.

Let X be a fine Z_0-scheme. Define the *log-syntomic crystalline site* of X, denoted $(X_{\mathrm{syn}}|Z_n)_{\text{log-crys}}$, by taking the category whose objects are log-DP-Z_n-thickenings $(U \hookrightarrow T)$ with $U \to X$ log-syntomic. Morphisms are given by the usual commutative diagrams of DP-morphisms. Coverings are given by families of morphisms

$$((U_\alpha \hookrightarrow T_\alpha) \to (U \hookrightarrow T))_\alpha$$

such that $(T_\alpha \to T)_\alpha$ is a log-syntomic covering and the commutative squares

$$\begin{array}{ccc} U_\alpha & \longrightarrow & T_\alpha \\ \downarrow & & \downarrow \\ U & \longrightarrow & T \end{array}$$

are cartesian.

4.2.4. The reason for introducing log-syntomic morphisms is the fact that they locally lift over nilpotent thickenings, see [4] Lemme 3.2.2. This makes the log-syntomic crystalline site particularly useful. We thank C. Breuil for indicating to us the idea of proof of the following proposition, and we take the opportunity to give a complete proof which we have not seen in the literature but is well-known to the experts (*cf.* [4] Cor. 3.2.3).

Proposition 4.3. *For any sheaf $\mathscr{F}$ on $(X_{\mathrm{syn}}|Z_n)_{\text{log-crys}}$, the presheaf $\mathscr{F}^{\text{log-crys}}$ on X_{syn} defined*

$$\mathscr{F}^{\text{log-crys}}(U) := H^0_{\text{log-crys}}(U|Z_n, \mathscr{F})$$

is a sheaf.

Proof. Let $(U_\alpha \to U)_\alpha$ be a log-syntomic covering. We claim that for every object $(U_\alpha \hookrightarrow T_\alpha)$ of the log-crystalline site of U_α over the DP-log-base Z_n there exists a cartesian square

$$\begin{array}{ccc} U_\alpha & \longrightarrow & T_\alpha \\ \downarrow & & \downarrow \\ U & \longrightarrow & T \end{array}$$

with $(U \hookrightarrow T)$ an object of the log-crystalline site of U over Z_n and $T_\alpha \to T$ a logarithmic DP-Z_n-morphism (not necessarily syntomic). Up to localizing for the étale topology we may assume that each $U_\alpha \to U$ is standard log-syntomic given by the square

$$\begin{array}{ccc} Q & \longrightarrow & \mathscr{O}_U \\ \downarrow & & \downarrow \\ P & \longrightarrow & \mathscr{O}_{U_\alpha} \end{array}$$

with $Q \to P$ and $\mathscr{O}_U \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P] \to \mathscr{O}_{U_\alpha}$ syntomic. Up to further localization (*cf.* [4] Lemme 2.2.1.2) we may assume that the log-structure of T_α is given by a fine monoid $\hat{P}$ and we have a commutative diagram

$$\begin{array}{ccc} \hat{P} & \longrightarrow & \mathscr{O}_{T_\alpha} \\ \downarrow & & \downarrow \\ P & \longrightarrow & \mathscr{O}_{U_\alpha}. \end{array}$$

Define

$$\mathscr{O}_T := \mathscr{O}_{T_\alpha} \times_{\mathscr{O}_{U_\alpha}} \mathscr{O}_U.$$

It is naturally endowed with the pre-log-structure given by the monoid $\hat{Q} := \hat{P} \times_P Q$. We claim that $T := \mathrm{Spec}(\mathscr{O}_T)$ makes the obvious commutative square cartesian, i.e. we claim that the canonical map

$$\mathscr{O}_{T_\alpha} \otimes_{\mathscr{O}_T} \mathscr{O}_U \to \mathscr{O}_{U_\alpha}$$

is an isomorphism. It is clearly surjective, hence it suffices to prove that it is injective. Suppose $\sum_i x_i \otimes y_i$ lies in its kernel. Let $\hat{y}_i$ be a lift to $\mathscr{O}_{T_\alpha}$ of the image of y_i in $\mathscr{O}_{U_\alpha}$ for all i. Then

$$1 \otimes y_i = \hat{y}_i \otimes 1$$

by definition of $\mathscr{O}_T$. Then $\sum_i x_i \otimes y_i = (\sum_i x_i \hat{y}_i) \otimes 1$ and $z := \sum_i x_i \hat{y}_i \in \mathrm{Ker}(\mathscr{O}_{T_\alpha} \to \mathscr{O}_{U_\alpha})$. But then we must have $z \otimes 1 = 1 \otimes 0 = 0$, again by the very definition of $\mathscr{O}_T$. This proves the claim. Moreover, it follows from classical arguments (*cf.* [2] 5.11 Lemma) that $(U \hookrightarrow T)$ is an object of the log-crystalline site of U over Z_n and that $T_\alpha \to T$ is a log-DP-morphism, thereby proving the claim.

Now, the natural map $\hat{Q} \to \hat{P}$ may not be syntomic, but we may modify $\hat{P}$ in order to make it syntomic, as follows. Since $Q \to P$ is syntomic we may write

$$P = \frac{Q \oplus \mathbb{N}X_1 \oplus \cdots \oplus \mathbb{N}X_r}{\langle g_1, ..., g_s \rangle}.$$

Let $z_1, .., z_r$ denote a choice of lifts to $\hat{P}$ of $X_1, ..., X_r \in P$. This defines a map

$$\hat{Q} \oplus \mathbb{N}Z_1 \oplus \cdots \oplus \mathbb{N}Z_r \to \hat{P} : Z_i \mapsto z_i.$$

Clearly, the kernel of this map consists of lifts of elements of $\langle g_1, ..., g_s \rangle$. Let $\hat{g}_i$ be a choice of lift of g_i lying in this kernel. Then the map factors over

$$\hat{P}' := \frac{\hat{Q} \oplus \mathbb{N}Z_1 \oplus \cdots \oplus \mathbb{N}Z_r}{\langle \hat{g}_1, ..., \hat{g}_s \rangle} \to \hat{P}$$

and the natural homomorphism $\hat{Q} \to \hat{P}'$ is syntomic by construction.

Now, write $\mathscr{O}_{U_\alpha} = \mathscr{O}_U \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P][X_i]/(f_j)$ for a regular sequence (f_j). Let x_i denote the image of X_i in $\mathscr{O}_{U_\alpha}$ for all i and let $\hat{x}_i$ denote a choice of lifts to $\mathscr{O}_{T_\alpha}$. Define an $\mathscr{O}_T$-algebra homomorphism

$$\mathscr{O}_T \otimes_{\mathbb{Z}[\hat{Q}]} \mathbb{Z}[\hat{P}'][X_i] \to \mathscr{O}_{T_\alpha} : X_i \mapsto \hat{x}_i.$$

Note that this map is surjective, i.e. $\mathscr{O}_{T_\alpha}$ is generated by the $\hat{x}_i$ as an $\mathscr{O}_T \otimes_{\mathbb{Z}[\hat{Q}]} \mathbb{Z}[\hat{P}']$-algebra. This is true modulo a nilpotent ideal generated by elements in the image of the canonical map $\mathscr{O}_T \otimes_{\mathbb{Z}[\hat{Q}]} \mathbb{Z}[\hat{P}'] \to \mathscr{O}_{T_\alpha}$, hence must be true in general. Moreover, it is clear that the kernel I of the map consists of lifts of elements of the elements of the ideal (f_j). For all j, choose a lift $\hat{f}_j \in \mathscr{O}_T \otimes_{\mathbb{Z}[\hat{Q}]} \mathbb{Z}[\hat{P}'][X_i]$ of f_j lying in I, and define

$$\begin{aligned} \mathscr{O}_{T'_\alpha} &:= \frac{\mathscr{O}_T \otimes_{\mathbb{Z}[\hat{Q}]} \mathbb{Z}[\hat{P}'][X_i]}{(\hat{f}_j)} \\ T'_\alpha &:= \operatorname{Spec}(\mathscr{O}_{T'_\alpha}). \end{aligned}$$

Then we have a commutative diagram of DP-Z_n-morphisms

$$\begin{tikzcd} U_\alpha \arrow[r] \arrow[d, equal] & T_\alpha \arrow[d] \\ U_\alpha \arrow[r] \arrow[d] & T'_\alpha \arrow[d] \\ U \arrow[r] & T \end{tikzcd}$$

in which the lower square is cartesian and the map $T'_\alpha \to T$ is log-syntomic by construction. In other words, we have shown that for any $(U_\alpha \hookrightarrow T_\alpha)$ as above, there exists $(U \hookrightarrow T)$ and a lifting $T'_\alpha \to T$ of the log-syntomic morphism $U_\alpha \to U$ together with a morphism of the log-crystalline site of U_α over Z_n

$$\begin{tikzcd} U_\alpha \arrow[r] \arrow[d, equal] & T_\alpha \arrow[d] \\ U_\alpha \arrow[r] & T'_\alpha. \end{tikzcd}$$

This implies that the projective limit

$$\mathscr{F}^{\text{log-crys}}(U_\alpha) = \lim_{(U_\alpha \hookrightarrow T_\alpha)} \mathscr{F}(T_\alpha)$$

can be computed over the subcategory of the log-crystalline site of U_α consisting of $(U_\alpha \hookrightarrow T_\alpha)$ which are lifts of $(U \hookrightarrow T)$ for some T. In other words, $\mathscr{F}^{\text{log-crys}}(U)$ is the equalizer of the double arrow

$$\prod_\alpha \mathscr{F}^{\text{log-crys}}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathscr{F}^{\text{log-crys}}(U_\alpha \times_U U_\beta)$$

as required. $\square$

4.2.5. Let X be a fine log-K^+-scheme and $K \subset L$ a finite field extension. Define a site $X_{\mathfrak{F}-\mathrm{syn},L}$ as follows. Objects are given by pairs (U,V), where

- $U \to X$ is étale
- $V \to U_{L^+}$ is a log-syntomic morphism such that the pair $(U, V \times_{\mathrm{Spec}(L^+)} \mathrm{Spec}(\bar{K}))$ belongs to $X_{\mathfrak{F}}$.

A morphism $(U', V') \to (U, V)$ is given by a commutative diagram

$$\begin{array}{ccc} V' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \end{array}$$

where $U' \to U$ is étale and $V' \to V$ is and log-syntomic. Coverings are given by families of morphisms $(U_\alpha, V_\alpha) \to (U, V)$ such that $(U_\alpha \to U)_\alpha$ is an étale covering, and $(V_\alpha \to V)_\alpha$ is a surjective family of log-syntomic morphisms.

The natural functor sending (U, V) to $(U, V \otimes_{L^+} \bar{K})$ defines a continuous morphism of sites

$$s_L : X_{\mathfrak{F}} \to X_{\mathfrak{F}-\mathrm{syn},L}.$$

Lemma 4.1. *For all $i \neq 0$ and all r we have $R^i s_{L,*} F_p^r \mathscr{A}_n \approx 0$.*

Proof. Clearly, $R^i s_{L,*} F_p^j \mathscr{A}_n$ is associated to the presheaf

$$(U, V) \rightsquigarrow H^i(X_{\mathfrak{F}(U,V)}, F_p^j \mathscr{A}_n)$$

where $X_{\mathfrak{F}(U,V)}$ is the site consisting of pairs (U', V') of $X_{\mathfrak{F}}$ with a morphism $(U', V') \to (U, V)$. By Proposition 4.1 this is the same as the sheaf associated to the presheaf

$$(U, V) \rightsquigarrow H^i(V_{\text{Fét}}, F_p^j \mathscr{A}_n).$$

If we take $U = \mathrm{Spec}(R)$ where R is a small integral K^+-algebra with K^+ integrally closed in R, and $V = \mathrm{Spec}(\tilde{R}_\infty)$, then $V \to U$ is the filtering inductive limit of log-syntomic coverings and $V \to \mathrm{Spec}(\bar{R})$ is the filtering inductive limit of almost Galois coverings, hence by Corollary 2.4 and Proposition 2.1 for $i \neq 0$ we have

$$H^i(V_{\text{Fét}}, F_p^j \mathscr{A}_n) \cong H^i(\mathrm{Gal}(\bar{R}[1/p]/\tilde{R}_\infty[1/p]), F_p^j \mathscr{A}_n(\bar{R}[1/p])) \approx 0.$$

$\square$

4.2.6. For any fine log-Z_0-scheme Y, let Y_{syn} denote the log-syntomic site and $(Y_{\mathrm{syn}}|Z_n)_{\text{log-crys}}$ the syntomic log-crystalline site. Define a sheaf of ideals $\mathscr{I}$ on $(Y_{\mathrm{syn}}|Z_n)_{\text{log-crys}}$ by

$$\mathscr{I}(U \hookrightarrow T) := \mathrm{Ker}(\mathscr{O}_T \to \mathscr{O}_U).$$

It has a canonical divided power structure and we will write $\mathscr{I}^{[r]}$ for its rth divided power.

By Proposition 4.3, the presheaf $\mathscr{I}^{[r],\text{log-crys}}$ on Y_{syn} defined

$$\mathscr{I}^{[r],\text{log-crys}}(U) = H^0_{\text{log-crys}}(Y|Z_n, \mathscr{I}^{[r]})$$

is a sheaf and, by arguments of [4] §3.3, there are canonical isomorphisms for all i

$$H^i(Y_{\text{syn}}, \mathscr{I}^{[r],\text{log-crys}}) \cong H^i_{\text{log-crys}}(Y|Z_n, \mathscr{I}^{[r]}).$$

Now, the functor sending (U,V) to $(V/pV \to U_{L^+/pL^+})$ defines a continuous morphism of sites

$$a_L : X_{L^+/pL^+,\text{syn}} \to X_{\mathfrak{F}-\text{syn},L}$$

Lemma 4.2. *For all $i \neq 0$ we have $R^i a_{L,*}\mathscr{I}^{[r],\text{log-crys}} = 0$.*

Proof. Clearly, $R^i a_{L,*}\mathscr{I}^{[r],\text{log-crys}}$ is the sheaf associated to the presheaf

$$(U,V) \rightsquigarrow H^i((V/pV)_{\text{syn}}, \mathscr{I}^{[r],\text{log-crys}}) \cong H^i_{\text{log-crys}}(V/pV|Z_n, \mathscr{I}^{[r]})$$

and if $U = \operatorname{Spec}(R)$ with R small integral and $V = \operatorname{Spec}(S)$ then the ind-log-syntomic covering $V_\infty := \operatorname{Spec}(\tilde{S}_\infty) \to V$ has no log-crystalline cohomology in non-zero degree by Proposition 2.3 and Theorem 1.1. □

Clearly, there is a canonical morphism of sheaves on $X_{\mathfrak{F}-\text{syn},L}$

$$a_{L,*}\mathscr{I}^{[r],\text{log-crys}} \to s_{L,*}F^r_p\mathscr{A}_n.$$

Let

$$u_L : X_{\mathfrak{F}-\text{syn},L} \to X_{\text{ét}}$$

be the projection onto the étale site of X. Our Galois cohomology computations from the last section can be reformulated as follows.

Theorem 4.2. *For all i the canonical morphisms*

$$\operatorname{colim}_L R^i u_{L,*}a_{L,*}\mathscr{I}^{[r],\text{log-crys}} \to \operatorname{colim}_L R^i u_{L,*}s_{L,*}F^r_p\mathscr{A}_n$$

have kernel and cokernel almost annihilated by t^d.

Proof. This is a reformulation of our Theorems 3.1 and 3.2. □

Corollary 4.2. *For all i there is a canonical homomorphism*

$$H^i_{\text{log-crys}}(X_{K^+/pK^+}|\Sigma_n, \mathscr{O}) \otimes_{\Sigma_n} B^+_{\log}/p^n B^+_{\log} \to H^i(X_{\mathfrak{F}}, \mathscr{A}_n)$$

which is a filtered almost isomorphism up to t^d-torsion.

Proof. This follows from Theorem 4.2, using the flatness of $B^+_{\log}/p^n B^+_{\log}$ over Σ_n ([16] Prop. 3.3). □

5 Comparison of étale and crystalline cohomology

In this section we compare étale and crystalline cohomology theories. We denote by X a semi-stable K^+-scheme throughout.

5.1 Chern classes

The key ingredient in the comparison of cohomology theories is Poincaré duality. For this we must compare characteristic classes, which on regular schemes reduces to comparing Chern classes.

5.1.1. Let us begin by defining Chern classes in Faltings cohomology. If $\mathscr{L}$ is a line bundle on X, then it defines a class in $[\mathscr{L}] \in H^1(X, \mathscr{O}_X^*)$ which maps to $H^1(X_{\mathfrak{F}}, \bar{\mathscr{O}}_n^*)$, where $\bar{\mathscr{O}}_n^*$ denotes the sheaf of units of the sheaf of rings $\bar{\mathscr{O}}_n$. Via the Kummer sequence

$$0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z}(1) \longrightarrow \bar{\mathscr{O}}_n^* \xrightarrow{p^n} \bar{\mathscr{O}}_n^* \longrightarrow 0$$

we define the first Chern class $c_1(\mathscr{L}) \in H^2(X_{\mathfrak{F}}, \mathbb{Z}/p^n\mathbb{Z}(1))$ as the image of $[\mathscr{L}]$ under the composition $H^1(X, \mathscr{O}_X^*) \to H^1(X_{\mathfrak{F}}, \bar{\mathscr{O}}_n^*) \to H^2(X_{\mathfrak{F}}, \mathbb{Z}/p^n\mathbb{Z}(1))$. Note that by construction it is clear that under the canonical map to p-adic étale cohomology $H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z}(1))$, $c_1(\mathscr{L})$ maps to $c_1^{\text{ét}}(\alpha^*\mathscr{L})$, where

$$c_1^{\text{ét}} : H^1(X, \mathscr{O}_X^*) \to H^2_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z}(1))$$

is the cycle class map. We also have a crystalline cycle class map

$$c_1^{\text{crys}} : H^1(X, \mathscr{O}_X^*) \to H^2_{\text{log-crys}}(X_{K^+/pK^+}|\Sigma_n, \mathscr{O}).$$

5.1.2. Let H be a cohomology theory with a theory of Chern classes satisfying the projective bundle formula. Recall that the splitting principle for H means that we can find a proper smooth surjective morphism $f : X' \to X$ such that $f^*\mathscr{E}$ has a filtration with line bundle subquotients and $f^* : H(X) \to H(X')$ injective. It is trivial for rank 1 vector bundles and is constructed by induction on the rank r of $\mathscr{E}$. Consider the canonical morphism $\pi : \mathbb{P}(\mathscr{E}) \to X$. By the projective bundle formula, we know that π^* is injective on cohomology and the vector bundle $\pi^*\mathscr{E}/\mathscr{O}(1)$ has rank $r-1$, so by induction there exists a morphism $g : X' \to \mathbb{P}(\mathscr{E})$ with the required properties. Then the composition

$$f : X' \xrightarrow{g} \mathbb{P}(\mathscr{E}) \xrightarrow{\pi} X$$

does the job. Moreover, if X is a proper semi-stable (resp. smooth) K^+-scheme, then so is X'. If α is a Chern root of $\mathscr{E}$, then $f^*(\alpha)$ is the Chern class of a line bundle on X'. This reduces the checking of properties of Chern classes of vector bundles to that of line bundles.

5.1.3. Using the logarithm $\log : \mathbb{Z}/p^n\mathbb{Z}(1) \to A_{\mathrm{cris}}(K^+)/p^n A_{\mathrm{cris}}(K^+)$ we define maps for all $i, j \in \mathbb{Z}$

$$H^i(X_{\mathfrak{F}}, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}(j) \to H^i(X_{\mathfrak{F}}, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} A_{\mathrm{cris}}(K^+)/p^n A_{\mathrm{cris}}(K^+).$$

Recall that by Corollary 4.2 we have defined a canonical transformation

$$H^i_{\text{log-crys}}(X_{K^+/pK^+}|\Sigma_n, \mathscr{O}) \to H^i(X_{\mathfrak{F}}, \mathscr{A}_n).$$

Proposition 5.1. *Let $\mathscr{L}$ be a vector bundle on X. The transformation*

$$H^i(X_{\mathfrak{F}}, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} A_{\mathrm{cris}}(K^+)/p^n A_{\mathrm{cris}}(K^+) \to H^i(X_{\mathfrak{F}}, \mathscr{A}_n)$$

sends the Chern class $c_1(\mathscr{L})$ to $c_1^{\mathrm{crys}}(\mathscr{L}^{\otimes -1}) = -c_1^{\mathrm{crys}}(\mathscr{L})$.

Proof. Let $\mathfrak{U} = \cup_i U_i$ be an affine open covering of X trivializing $\mathscr{L}$. Up to further localization we may assume that $U_i = \mathrm{Spec}(R_i)$ with R_i small integral K^+-algebra. Let $U_{i,j} = U_i \cap U_j$ and let $f_{i,j} \in \mathscr{O}^*_{U_{i,j}}$ be a set of trivializing functions for $\mathscr{L}$. For all i let Z_i be the étale $\Theta(c)/p^n\Theta(c)$-algebra lifting $R_i \otimes \mathbb{F}_p$ and let $Z_{i,j} = Z_i \otimes_{\Theta(c)} Z_j$. Let $\hat{f}_{i,j}$ be a lift of $f_{i,j} \mod p$ to $Z_{i,j}$. Let D_i and $D_{i,j}$ be the divided power hulls for the respective closed immersions. Note that by étaleness over $\Theta(c)/p^n\Theta(c)$, these closed immersions are exact, hence the logarithmic and classical divided power hulls coincide. The crystalline Chern class $c_1^{\mathrm{crys}}(\mathscr{L})$ is represented by the 1-cocycle

$$\mathrm{dlog}(\hat{f}_{i,j}) \in \check{C}^1(\mathfrak{U}, D_{i,j} \otimes_{Z_{i,j}} \omega^1_{Z_{i,j}/\Sigma}).$$

On the other hand, we defined $c_1(\mathscr{L})$ to be the image of the class of $\mathscr{L}$ in the Picard group under the coboundary map arising from the Kummer exact sequence, i.e. it is represented by the 1-cocycle

$$f_{i,j}^{p^{-n}} \in \check{C}^1(\mathfrak{U}, C^1(\Delta, \mathbb{Z}/p^n\mathbb{Z}(1)))$$

and under the logarithm $\log : \mathbb{Z}/p^n\mathbb{Z}(1) \to \mathscr{A}_n$ it maps to

$$\log\left(f_{i,j}^{p^{-n}}\right) \in \check{C}^1(\mathfrak{U}, C^1(\Delta, A^+/p^n A^+)).$$

So it suffices to show that for all i, j, $\mathrm{dlog}(\hat{f}_{i,j})$ maps to $\log\left(f_{i,j}^{p^{-n}}\right)$ up to coboundaries, so we can restrict to a single $U_{i,j} = \mathrm{Spec}(R)$. We have morphisms of complexes

$$C^*(\Delta, A^+/p^n A^+) \xrightarrow{\sim} C^*(\Delta, M^+/p^n M^+ \otimes_{\mathcal{R}_n} \omega^\bullet_{\mathcal{R}_n/\Sigma_n}) \longleftarrow B^+_{\log}/p^n B^+_{\log} \otimes_{\Sigma_n} \omega^\bullet_{\mathcal{R}_n/\Sigma_n}$$

and $\mathrm{dlog}(\hat{f}_{i,j})$ is the image under d of $\log\left(\left[f_{i,j}^{p^{-n}}\right]^{-1} \otimes \hat{f}_{i,j}\right) \in C^0(\Delta, M^+/p^n M^+)$ and the latter has image $\log\left(f_{i,j}^{-p^{-n}}\right) = -\log\left(f_{i,j}^{p^{-n}}\right) \in C^1(\Delta, M^+/p^n M^+)$. $\square$

5.2 Artin-Schreier theory for Fontaine rings

In this section we work locally on the special fibre of X, so we my assume that R is a local ring of X at a point of the special fibre X_k. In particular p lies in the Jacobson radical of R. As usual, define

$$A^+ := \lim_n H^0_{\text{log-crys}}(\bar{R}/p\bar{R}|\Sigma_{n+1}, \mathscr{O}).$$

Let Φ denote the Frobenius endomorphism of A^+ lifting the absolute Frobenius of A^+/pA^+.

Lemma 5.1. *For all n we have an exact sequence*

$$0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow A^+/p^nA^+ \xrightarrow{\Phi-1} A^+/p^nA^+ \longrightarrow 0.$$

Proof. For all $x \in A^+/p^nA^+$, the equation $\Phi(X) - X = x$ defines an étale covering of A^+/p^nA^+, so it suffices to show that it induces the trivial covering of $\bar{R}/p\bar{R}$. This reduces to showing the statement with A^+/p^nA^+ replaced by $\bar{R}/p\bar{R}$. It suffices to show that for all $x \in \bar{R}$ the equation $X^p - X = x$ is soluble in X. We can assume that $x \in S$ for S a finite normal domain such that $R \to S$ is étale in characteristic zero. Then S is a semi-local ring whose Jacobson radical contains p. Let $f(X) = X^p - X - x$. We claim that $f(X) = 0$ defines a finite étale cover of S. Indeed, $f'(X) = pX^{p-1} - 1$ is a unit of $S[X]/(f(X))$ since p lies in the Jacobson radical. So $f(X) = 0$ is finite étale over S, in particular has a root in $\bar{R}$. $\square$

Define

$$\begin{aligned} A_{\text{inf}}(R) &:= W(P(\bar{R}/p\bar{R})) \\ A_{\text{cris}}(R) &:= \lim_n H^0_{\text{crys}}(\bar{R}/p\bar{R}|W_{n+1}(k), \mathscr{O}). \end{aligned}$$

Let Φ denote the canonical Frobenius-lifts of $A_{\text{inf}}(R)$ and $A_{\text{cris}}(R)$. Recall that $t := \log([\underline{1}])$ and by Proposition 1.4 we have $t^{p-1} \in p \cdot F^1 A_{\text{cris}}(K^+)$, so for all n we have

$$\frac{t^n}{(n+1)!} = \frac{p^q q!}{(n+1)!} \left(\frac{t^{p-1}}{p}\right)^{[q]} t^r$$

where $n = q(p-1) + r$ and $q = \left[\dfrac{n}{p-1}\right] \geq v_p((n+1)!)$. Hence we may define for all n

$$t^{\{n\}} := \frac{t^n}{p^q \cdot r! \cdot q!} \in A_{\text{cris}}(K^+).$$

The following theorem is a generalization of Thm. 5.37 of [8]. Its proof is based on one given in a course by Faltings in the case $R = K^+$.

Theorem 5.1. *(i) For $x \in A_{\text{inf}}(R)$, $x \in ([\underline{1}] - 1) \cdot A_{\text{inf}}(R)$ if and only if $\Phi^n(x) \in \xi \cdot A_{\text{inf}}(R)$ for all n.*

(ii) If $x \in F^m A_{\text{cris}}(R)$ and $\Phi(x) = p^m tx$, then $x \in \mathbb{Z}_p t^{\{m\}}$.

(iii) The following sequence is exact for all m

$$0 \longrightarrow \mathbb{Z}_p t^{\{m\}} \longrightarrow F^m A_{\text{cris}}(R) \xrightarrow{p^{-m}\Phi - 1} \tfrac{1}{p^m} A_{\text{cris}}(R)$$

and $\operatorname{Coker}(F^m A_{\text{cris}}(R) \to \dfrac{1}{p^m} A_{\text{cris}}(R))$ *is annihilated by p^{2m}.*

Proof. (i): Let $\theta : A_{\text{inf}}(R) \to \hat{\bar{R}}$ be the canonical map. Write $x = \xi x_0$. We have

$$\theta(\Phi(\xi)) = \theta([\underline{p}]^p - p) = p(p^{p-1} - 1) \neq 0$$

hence $\theta(\Phi(x_0)) = 0$ since $\hat{\bar{R}}$ is p-torsion free. So we can write $\Phi(x_0) = \xi x_1$ and $x_0 = \Phi^{-1}(\xi)\Phi^{-1}(x_1)$. The same argument for all $n \geq 1$ implies that x is divisible by $\xi\Phi^{-1}(\xi)\Phi^{-2}(\xi) \cdots \Phi^{-n}(\xi)$ for all n. Reducing modulo p, we have $\xi = \underline{p}$ so we see that x is divisible by

$$\underline{p}^{\sum_{n \geq 0} p^{-n}} = \underline{p}^{\frac{p}{p-1}}.$$

Now recall that $P(\bar{K}^+/p\bar{K}^+)$ is a valuation ring and $v(\underline{1} - 1) = \frac{p}{p-1} = v\left(\underline{p}^{\frac{p}{p-1}}\right)$ so $\underline{p}^{\frac{p}{p-1}} = u \cdot (\underline{1} - 1)$ for some unit $u \in P(\bar{K}^+/p\bar{K}^+)$. This proves (i) modulo p, hence in general since $A_{\text{inf}}(R)$ is p-adically complete and $\hat{\bar{R}}$ is p-torsion free.

(ii): We first claim that every element of $A_{\text{cris}}(R)$ can be written as a sum

$$\sum_{n=0}^{\infty} a_n t^{\{n\}}$$

where $a_n \in A_{\text{inf}}(R)$ converges p-adically. It suffices to prove this modulo p, where $\frac{t^{p-1}}{p} = u \cdot \frac{\xi^p}{p}$. Now write $n = ap + b$ with $0 \leq b < p$. Note that $v_p(n!) \leq a + v_p(a!)$, so we can write

$$\xi^{[n]} = \frac{(\xi^p/p)^a}{a!} \xi^b \frac{p^a a!}{n!} = t^{\{a(p-1)\}} \xi^b \frac{p^a a!}{n!}$$

and the claim follows. Now suppose that $x \in F^m A_{\text{cris}}(R)$, $\Phi(x) = p^m x$. By Lemma 5.1 the claim is true for $m = 0$, so assume that $m \geq 1$. Write $x = \sum a_n t^{\{n\}}$. We claim that we may assume that $a_n = 0$ for all $n < m$. Since $\Phi^n(a_0) \in \xi \cdot A_{\text{inf}}(R)$ for all n, hence $a_0 \in ([\underline{1}] - 1) \cdot A_{\text{cris}}(R) = t \cdot A_{\text{cris}}(R)$. So we may assume that $a_0 = 0$. If $m = 1$ we are done, so assume $m > 1$. Then $\Phi^n(a_1 t) \in F^2 A_{\text{cris}}(R)$. Since $t = ([\underline{1}] - 1) \cdot v$ for some unit v we have $\Phi^n(a_1 t) = p^n t \Phi^n(a_1) = v p^n([\underline{1}] - 1)\Phi^n(a_1) \in F^2 A_{\text{cris}}(R)$, so $p^n([\underline{1}] - 1)\Phi^n(a_1) \in F^2 A_{\text{cris}}(R) \cap A_{\text{inf}}(R) = F^2 A_{\text{inf}}(R)$, hence we must have $\Phi^n(a_1) \in F^1 A_{\text{inf}}(R)$ for all n, whence we can assume that $a_1 = 0$. Continuing in this manner we see that we may assume that $x = \sum_{n \geq m} a_n t^{\{n\}}$. Since $\Phi^n(x) = p^{mn} x$ for all n we have

$$x = \frac{\Phi^n(x)}{p^{mn}} = \sum_{n \geq m} \Phi^n(a_n) p^{n-m} t^{\{n\}} \equiv \Phi^m(a_m) t^{\{m\}} \mod p^n.$$

So $x = x_0 t^{\{m\}}$, where $x_0 = \Phi^m(a_m)$. But then $\Phi(x_0) = x_0$, hence $x_0 \in \mathbb{Z}_p$, thereby proving (ii).

(iii): First observe that if $a' = \sum_{n>m} a_n t^{\{m\}}$, then

$$a' \equiv a' - p^{-m}\Phi(a') \mod p$$

so it suffices to show that for all $a \in A_{\inf}(R)$ and $n_0 \leq m$ there exists $x \in p^m F^m A_{\mathrm{cris}}(R)$ and $b = \sum_{n>n_0} b_n t^{\{n\}}$ with $b_n \in A_{\inf}(R)$ such that

$$p^{-m}\Phi(x) - x = at^{\{n_0\}} + b.$$

Recall that $\zeta := \dfrac{[\underline{1}] - 1}{[\underline{1}]^{1/p} - 1}$ is a generator of the principal ideal $F^1 A_{\inf}(R)$. Set $x = X\zeta^{m-n_0} t^{\{n_0\}}$ for X an indeterminate. We have

$$p^{-m}\Phi(x) - x = p^{-m}\Phi(X)\Phi(\zeta)^{m-n_0} p^{n_0} t^{\{n_0\}} - X\zeta^{m-n_0} t^{\{n_0\}}.$$

Recall that $[\underline{1}] - 1 = v \cdot t$, where $v = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}$ is a unit of $A_{\mathrm{cris}}(K^+)$ (*cf.* the proof of Proposition 1.4). Hence we have

$$\Phi(\zeta) = \frac{pt\Phi(v)}{tv} = p\frac{\Phi(v)}{v}$$

and $w := \frac{\Phi(v)}{v} \in 1 + t \cdot A_{\mathrm{cris}}(K^+)$ is a unit. Substituting, we find

$$p^{-m}\Phi(x) - x = w^{m-n_0}\Phi(X)t^{\{n_0\}} - X\zeta^{m-n_0} t^{\{n_0\}} = t^{\{n_0\}}(\Phi(X) - X\zeta^{m-n_0}) + b$$

where b is of the form $b = \sum_{n>n_0} b_n t^{\{n\}}$ with $b_n \in A_{\inf}(R)$. So it suffices find a solution in $A_{\inf}(R)$ to the equation

$$\Phi(X) - X\zeta^{m-n_0} = a.$$

Since $A_{\inf}(R)$ is p-adically complete, it suffices to find a solution modulo p, i.e. it suffices, for all n, to find a solution in $\bar{R}/p\bar{R}$ to the equation

$$X^p - u_n p^{\frac{m-n_0}{p^n}} X = a_n$$

where $(a_n)_n = a \mod p$ and $(u_n)_n = \big(\zeta/\underline{p}\big)^{m-n_0}$ is a unit of $P(\bar{K}^+/p\bar{K}^+)$. Let $m_0 = m - n_0$. Note that if $m_0 \geq p^n$, then this equation reads $X^p = a$ and has a solution since the Frobenius is surjective on $\bar{R}/p\bar{R}$. So we may assume that $m_0 < p^n$. Let $\hat{u}_n$ and $\hat{a}_n$ be lifts of u_n and a_n respectively to $\bar{R}$. Note that $\hat{u}_n$ is a unit because p lies in the Jacobson radical of $\bar{R}$. We claim that the equation

$$X^p - \hat{u}_n p^{\frac{m_0}{p^n}} X = \hat{a}_n$$

has a solution in $\bar{R}$. Equivalently, if S denotes a finite integral normal R-algebra containing $\hat{u}_n$,$\hat{a}_n$ and $p^{p^{-n}}$, then we claim that the equation $F(X) := X^p - \hat{u}_n p^{\frac{m_0}{p^n}} X = \hat{a}_n = 0$ defines a finite étale covering of $S[1/p]$. Indeed, we have

$$F'(X) = pX^{p-1} - \hat{u}_n p^{\frac{m_0}{p^n}} = \hat{u}_n p^{\frac{m_0}{p^n}} (\hat{u}_n^{-1} p^{1-\frac{m_0}{p^n}} X^{p-1} - 1)$$

and $\hat{u}_n^{-1}p^{1-\frac{m_0}{p^n}}X^{p-1}-1$ is a unit of $S[X]/(F(X))$ because $\left(p^{1-\frac{m_0}{p^n}}\right)^{p^n} = p^{p^n-m_0} \in pS$ lies in the Jacobson radical of S, hence so does $p^{1-\frac{m_0}{p^n}}$. So $\frac{S[X]}{(F(X))}[1/p]$ is a finite étale covering of $S[1/p]$, and in particular the equation $F(X)=0$ has a solution in $\bar{R}$. $\square$

Let $T := \log(X+1) \in B_{\log}^+$, where X is as in Proposition 1.3.

Corollary 5.1. *For any $r \geq 0$, the sequence*

$$0 \longrightarrow \oplus_{a\in\mathbb{N}}\mathbb{Z}_p t^a T^{r-a} \longrightarrow F^r A^+ \xrightarrow{p^{-r}\Phi-1} \frac{1}{p^r}A^+ \longrightarrow 0$$

is exact up to torsion annihilated by a fixed p-power depending only on r.

Proof. We have already shown the claim for $r=0$, and we will show it by induction on r. We have just shown the analogous claim for $A_{\mathrm{cris}}(R)$. We claim to have exact sequences for all $r, n \geq 1$

$$0 \longrightarrow F^r A_{\mathrm{cris}}(R)/p^n F^r A_{\mathrm{cris}}(R) \longrightarrow F^r A^+/p^n F^r A^+ \xrightarrow{N} F^{r-1}A^+/p^n F^{r-1}A^+ \longrightarrow 0$$

where N is the monodromy operator. This is essentially shown in [4] Prop. 6.2.1. Let us give the argument. Recall that A^+/p^nA^+ is DP-polynomial ring in one variable X over $A_{\mathrm{cris}}(R)/p^nA_{\mathrm{cris}}(R)$ (Prop. 3.2). Let $x=\sum_i x_i X^{[i]}$ with $x_i \in A_{\mathrm{cris}}(R)$. Then

$$N(x) = \left(\sum_{i\neq 0} x_i X^{[i]}\right)(1+X).$$

If $N(x)=0$, then since $1+X$ is a unit we have $\sum_{i\neq 0} x_i X^{[i]} = 0$, whence $x_i = 0$ for all $i \neq 0$, so $x \in A_{\mathrm{cris}}(R)$. It remains to see that N is surjective. It suffices to show that each $X^{[i]}$ lies in the image of N. If $i=0$, then $1 = N(\log(X+1))$. We have $N(X^{[i+1]}) = X^{[i]} + (i+1)X^{[i+1]}$, so it suffices to show that $(i+1)X^{[i+1]}$ lies in the image of N. Repeating this we see that it suffices to show that $(i+1)(i+2)\cdots(i+j)X^{[i+j]}$ lies in the image of N for some $j \in \mathbb{N}$. But for j large enough this is zero.

Since N satisfies $N\Phi = p\Phi N$, for all $r \geq 1$ we obtain commutative diagrams with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^r A_{\mathrm{cris}}(R)[1/p] & \longrightarrow & F^r A^+[1/p] & \xrightarrow{N} & F^{r-1}A^+[1/p] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p^{-r}\Phi-1} & & \downarrow{\scriptstyle p^{-r}\Phi-1} & & \downarrow{\scriptstyle p^{-(r-1)}\Phi-1} & & \\
0 & \longrightarrow & A_{\mathrm{cris}}(R)[1/p] & \longrightarrow & A^+[1/p] & \xrightarrow{N} & A^+[1/p] & \longrightarrow & 0.
\end{array}$$

By induction on r, we may assume that the right vertical arrow is surjective. Since the left vertical arrow is surjective, it follows that the middle vertical arrow is surjective. So by the snake lemma it remains to prove the exactness of the sequence

$$0 \longrightarrow \mathbb{Q}_p(r) \longrightarrow \oplus_{a\in\mathbb{N}}\mathbb{Q}_p t^a T^{r-a} \xrightarrow{N} \oplus_{a\in\mathbb{N}}\mathbb{Q}_p t^a T^{r-1-a} \longrightarrow 0$$

but $N(t^aT^b) = bt^aT^{b-1}$ so this is clear. $\square$

5.3 Finiteness of Faltings cohomology

Assume in this section that X is a proper semi-stable K^+-scheme.

5.3.1. Recall that we have defined a sheaf of rings $\bar{\mathscr{O}}_1$ on the topos $X_{\mathfrak{F}}$. The absolute Frobenius F is surjective on $\bar{\mathscr{O}}_1$ and so we may consider the projective limit $P(\bar{\mathscr{O}}_1)$ of

$$\cdots \xrightarrow{F} \bar{\mathscr{O}}_1 \xrightarrow{F} \bar{\mathscr{O}}_1 \xrightarrow{F} \bar{\mathscr{O}}_1.$$

Define

$$\mathscr{A}_{\mathrm{inf},n} := W_n(P(\bar{\mathscr{O}}_1)).$$

The following theorem of Faltings is the key to obtaining information about the cohomology groups $H^i(X_{\mathfrak{F}}, \mathbb{Z}/p^n\mathbb{Z})$.

Theorem 5.2 (Faltings). *For all i and any $n \geq 1$ we have canonical almost isomorphisms*

$$H^i(X_{\mathfrak{F}}, \mathscr{A}_{\mathrm{inf},n}) \approx H^i(X_{\mathfrak{F}}, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}} A_{\mathrm{inf}}(K^+)/p^n A_{\mathrm{inf}}(K^+).$$

Let us recall the main ideas of the proof ([7], p. 223-227). Firstly, recall that the kernel of the canonical map

$$\mathscr{A}_{\mathrm{inf},1} \to \bar{\mathscr{O}}_1$$

is generated by $\xi = [\underline{p}] - p$. Using duality, one shows that the cohomology groups $H^i(X_{\mathfrak{F}}, \bar{\mathscr{O}}_1)$ – essentially Hodge cohomology – are almost finitely presented $\bar{K}^+/p\bar{K}^+$-modules. Then, by dévissage in ξ, one derives that $H^i := H^i(X_{\mathfrak{F}}, \mathscr{A}_{\mathrm{inf},1})$ are almost finitely presented $A_{\mathrm{inf}}(K^+)/pA_{\mathrm{inf}}(K^+)$-modules, in fact almost projective of constant rank r (*loc.cit.* p. 223-226). This is the crucial step in showing that

$$\mathrm{Hom}_{\Phi, A_{\mathrm{inf}}(K^+)}(H^i, A_{\mathrm{inf}}(K^+)/pA_{\mathrm{inf}}(K^+))$$

is an $\mathbb{F}_p$-vector space of dimension r. Here the homomorphisms are Frobenius-linear almost maps of $A_{\mathrm{inf}}(K^+)$-modules. Then one chooses $0 < \varepsilon < 1$, $a := \underline{p}^{\varepsilon}$, and one considers the short exact sequences (on the special fibre of X)

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathscr{A}_{\mathrm{inf},1} \xrightarrow{\Phi - a} \mathscr{A}_{\mathrm{inf},1} \longrightarrow 0.$$

Using that this sequence is exact for variable a enables one to show that almost Frobenius invariants of H^i coincide with Frobenius invariants, so that H^i is almost generated by real Frobenius invariants, thereby proving the theorem modulo p.

For all i, consider the $\mathbb{Q}_p$-vector space

$$H^i(X_{\mathfrak{F}}, \mathbb{Q}_p) := \Big(\lim_n H^i(X_{\mathfrak{F}}, \mathbb{Z}/p^{n+1}\mathbb{Z})\Big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Corollary 5.2. *For all i we have* $\dim_{\mathbb{Q}_p} H^i(X_{\mathfrak{F}}, \mathbb{Q}_p) = \dim_K H^i_{\mathrm{dR}}(X_K/K)$.

Proof. If we let $\mathscr{A}_{\mathrm{inf}} := \lim_n \mathscr{A}_{\mathrm{inf},n+1}$ and $\hat{\bar{\mathscr{O}}} := \lim_n \bar{\mathscr{O}}_{n+1}$ then

$$H^i(X_{\mathfrak{F}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \hat{\bar{K}} \cong H^i(X_{\mathfrak{F}}, \mathscr{A}_{\mathrm{inf}}/\xi\mathscr{A}_{\mathrm{inf}})[1/p] = H^i(X_{\mathfrak{F}}, \hat{\bar{\mathscr{O}}})[1/p]$$

is the Hodge cohomology of $X \otimes_{K^+} \hat{\bar{K}}$ so the result follows from the degenerescence of the Hodge spectral sequence by descent. $\square$

5.3.2. Note that if X is a smooth K^+-scheme, then by Corollary 4.1 we already know that

$$H^i(X_{\mathfrak{F}}, \mathbb{Z}/p^n\mathbb{Z}) \cong H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})$$

so that in this case we do not have to resort to these difficult computations.

5.4 Comparison

In this section we assume that X is a proper semi-stable K^+-scheme. We will put all of the previous theory together to show that the log-crystalline cohomology of the special fibre of X is an admissible filtered (φ, N)-module in the sense of Fontaine. From this we will deduce a comparison map to p-adic étale cohomology of the geometric generic fibre of X.

5.4.1. Recall that we have defined a sheaf of rings $\mathscr{A}_n$ on $X_{\mathfrak{F}}$. As a sheaf of Fontaine rings, it is endowed with both the DP-filtration, denoted $F_p^\bullet$, and the canonical filtration denoted $F^\bullet$ (Def. 1.2.4). Define

$$H^i(X_{\mathfrak{F}}, \mathscr{A}^+) := \lim_n H^i(X_{\mathfrak{F}}, \mathscr{A}_{n+1}).$$

It has a filtration defined

$$F^r H^i(X_{\mathfrak{F}}, \mathscr{A}_n) := \operatorname{Im}(H^i(X_{\mathfrak{F}}, F^r \mathscr{A}_n) \to H^i(X_{\mathfrak{F}}, \mathscr{A}_n)).$$

We define

$$H^i(X_{\mathfrak{F}}, \mathscr{A}) := H^i(X_{\mathfrak{F}}, \mathscr{A}^+) \otimes_{A_{\text{cris}}(K^+)} B_{\text{cris}}$$

and endow it with the tensor product filtration.

5.4.2. Define a complex of sheaves $\mathscr{L}_n(r)$ on $X_{\mathfrak{F}}$ by the distinguished triangle

$$\mathscr{L}_n(r) \longrightarrow F^r \mathscr{A}_n \xrightarrow{\Phi - p^r} \mathscr{A}_n.$$

Note that by Corollary 5.1 there is a canonical morphism of complexes

$$L_n(r) := \oplus_{a \in \mathbb{N}} (\mathbb{Z}/p^n\mathbb{Z}) t^a T^{r-a} \to \mathscr{L}_n(r).$$

Setting $L^+(r) := \lim_n L_{n+1}(r)$ and $\mathscr{L}^+(r) := \lim_n \mathscr{L}_{n+1}(r)$ we deduce a canonical isomorphism

$$H^i(X_{\mathfrak{F}}, L^+(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^i(X_{\mathfrak{F}}, \mathscr{L}^+(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

and of course we have $H^i(X_{\mathfrak{F}}, L^+(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^i(X_{\mathfrak{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Z}_p} L^+(r)$.

5.4.3. Let $B_{\text{st}}^+ := B_{\text{cris}}^+[T]$ with $T := \log(1 + X)$ as before. Note that by Proposition 1.3 we know that $B_{\text{st}}^+ = B_{\log}^+[1/p]^{N-nil}$ is the subring of monodromy nilpotent elements of $B_{\log}^+[1/p]$. As usual, write $B_{\text{st}} = B_{\text{st}}^+[1/t]$.

5.4.4. Let $\mathcal{L}(u)$ be the pre-log-structure on Σ_n. Its inverse image to Spec(k) defines a pre-log-structure

$$\mathcal{L}(\pi) : \mathbb{N} \to k$$

and composing with the Teichmüller lift $[\cdot] : k \to W_n(k)$ this defines a pre-log-structure on $W_n(k)$, denoted $\mathcal{L}(\pi)$. Consider the log-crystalline site of X_k over $W_n(k)$ with the latter endowed with the log-structure associated to $\mathcal{L}(\pi)$. Define

$$\begin{aligned} H^i_{\text{log-crys}}(X_k|W(k)) &:= \lim_n H^i_{\text{log-crys}}(X_k|W_{n+1}(k), \mathscr{O}) \\ H^i_{\text{log-crys}}(X_{K^+/pK^+}|\Sigma) &:= \lim_n H^i_{\text{log-crys}}(X_{K^+/pK^+}|\Sigma_{n+1}, \mathscr{O}). \end{aligned}$$

Recall that by [13] Lemma 5.2, for all i we have canonical isomorphisms

$$H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} \Sigma[1/p] \cong H^i_{\text{log-crys}}(X_{K^+/pK^+}|\Sigma)[1/p].$$

The Frobenius Φ on the finite $W(k)$-module $H^i_{\text{log-crys}}(X_k|W(k))$ is related to the monodromy N by the relation $N\Phi = p\Phi N$. Since Φ is an isogeny on $H^i_{\text{log-crys}}(X_k|W(k))$, it follows that N is nilpotent. Hence, the set of monodromy nilpotent elements of $H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} B_{\log}$ is precisely $H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} B_{\text{st}}$.

5.4.5. It follows from Corollary 4.2 that there are canonical almost isomorphisms for all i

$$H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} B_{\log} \approx H^i(X_{\mathfrak{F}}, \mathscr{A}).$$

Since $B_{\text{st}} \subset B_{\text{dR}}$ and the latter is a field, it follows that B_{st} is an integral domain and hence the map

$$H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} B_{\text{st}} \to H^i(X_{\mathfrak{F}}, \mathscr{A})$$

is injective. So we can define a filtration

$$F^r(H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} B_{\text{st}}) := H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} B_{\text{st}} \cap F^r H^i(X_{\mathfrak{F}}, \mathscr{A}).$$

This filtration is clearly exhaustive.

Lemma 5.2. *This filtration on $H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} B_{\text{st}}$ induces the same filtration on $H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} B_{\text{st}} \otimes_{W(k)} K^+$ as the filtration $F^\bullet_H$ induced by the canonical filtration on B_{st} and the Hodge filtration on $H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} K$.*

Proof. Note that the canonical map $B_{\text{st}} \otimes_{W(k)} K^+ \to B_{\log} \hat{\otimes}_\Sigma K$ is filtered for the canonical filtration on both sides and induces an isomorphism on gradeds, so it suffices to prove that the Hodge filtration coincides with the canonical filtration on

$$H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} B_{\log} \hat{\otimes}_\Sigma K \approx H^i(X_{\mathfrak{F}}, \mathscr{A} \hat{\otimes}_\Sigma K)$$

induced from the canonical filtration on the sheaf of Fontaine rings

$$\mathscr{A} \hat{\otimes}_\Sigma K := \mathscr{A}^+ \hat{\otimes}_\Sigma K^+ \otimes_{A_{\text{cris}}(K^+)} B_{\text{cris}}.$$

We claim that the spectral sequence

$$E_1^{i,j} = H^{i+j}(X_{\mathfrak{F}}, \mathrm{gr}^j \mathscr{A}^+ \otimes_\Sigma K^+/p^n K^+) \Longrightarrow \mathrm{gr}^j H^{i+j}(X_{\mathfrak{F}}, \mathscr{A}^+ \otimes_\Sigma K^+/p^n K^+)$$

degenerates up to p-torsion independent of n. Indeed, up to p-torsion independent of n we have

$$E_1^{i,j} = H^{i+j}(X_{\mathfrak{F}}, \bar{\mathscr{O}}_n(j)) = \oplus_m H^{i+j-m}(X, \omega^m_{X/K^+} \otimes_{K^+} \bar{K}^+/p^n \bar{K}^+)(j-m)$$

and by looking at the Tate twists one sees easily that the differentials of the spectral sequence must be zero up to p-torsion independent of n. This proves that

$$\mathrm{gr}^j H^{i+j}(X_{\mathfrak{F}}, \mathscr{A} \hat{\otimes}_\Sigma K) \cong \oplus_m H^{i+j-m}(X, \omega^m_{X/K^+}) \otimes_{K^+} \hat{\bar{K}}(j-m)$$

which is visibly the same as $\mathrm{gr}^j_H(H^{i+j}_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)} B_{\log} \hat{\otimes}_\Sigma K)$. $\square$

5.4.6. Let $D := H^i_{\text{log-crys}}(X_k|W(k))[1/p]$. Define $L(r) := \oplus_{a+b=r, b\geq 0} \mathbb{Q}_p t^a T^b$. Let

$$\tilde{H}^i(X_{\mathfrak{F}}, \mathbb{Z}_p) := \mathrm{Im}\left(H^i(X_{\mathfrak{F}}, \mathbb{Z}_p) \to H^i(X_{\mathfrak{F}}, \mathscr{A}^+)\right).$$

Then it follows from the above that

$$\tilde{H}^i(X_{\mathfrak{F}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} L^+(r)[1/p] = F^r H^i(X_{\mathfrak{F}}, \mathscr{A}^+)[1/p]^{\Phi = p^r}$$

hence, inverting t we deduce

$$\tilde{H}^i(X_{\mathfrak{F}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} L(r) = F^r H^i(X_{\mathfrak{F}}, \mathscr{A})^{\Phi = p^r}.$$

Intersecting with $D \otimes_{W(k)[1/p]} B_{\mathrm{st}}$ we get

$$\tilde{H}^i(X_{\mathfrak{F}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} L(r) \cap D \otimes_{W(k)[1/p]} B_{\mathrm{st}} = F^r(D \otimes_{W(k)[1/p]} B_{\mathrm{st}})^{\Phi = p^r}.$$

Recall that Fontaine's functor $V_{\mathrm{st}}(-)$ on filtered (φ, N)-modules E is defined

$$V_{\mathrm{st}}(E) := (E \otimes_{W(k)[1/p]} B_{\mathrm{st}})^{\Phi=1, N=0} \cap F^0(E \otimes_{W(k)[1/p]} B_{\mathrm{st}} \otimes_{W(k)} K^+).$$

Using the fact that $L(0)^{N=0} = \mathbb{Q}_p$, from Lemma 5.2 we deduce that

$$V_{\mathrm{st}}(D) = \tilde{H}^i(X_{\mathfrak{F}}, \mathbb{Q}_p) \cap D \otimes_{W(k)[1/p]} B_{\mathrm{st}}$$

and hence

$$\dim_{\mathbb{Q}_p} V_{\mathrm{st}}(D) \leq \dim_{\mathbb{Q}_p} H^i(X_{\mathfrak{F}}, \mathbb{Q}_p) = \dim_{W(k)[1/p]} D.$$

Let us prove the reverse inequality. We have a commutative diagram

$$\begin{tikzcd}
H^i(X_{\mathfrak{F}}, \mathbb{Q}_p) \arrow[r] \arrow[d, equal] & H^i(X_{\mathfrak{F}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\inf}(V)[1/p] \arrow[r] \arrow[d] & F^0 H^i(X_{\mathfrak{F}}, \mathscr{A} \hat{\otimes}_\Sigma K) \arrow[d] \\
H^i(X_{\mathfrak{F}}, \mathbb{Q}_p) \arrow[r] & H^i(X_{\mathfrak{F}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \hat{\bar{K}} \arrow[r, "\cong"] & \mathrm{gr}^0 H^i(X_{\mathfrak{F}}, \mathscr{A} \hat{\otimes}_\Sigma K).
\end{tikzcd}$$

from which we see that $H^i(X_{\mathfrak{F}}, \mathbb{Q}_p)$ injects into the Frobenius invariants of $F^0 H^i(X_{\mathfrak{F}}, \mathscr{A}\hat{\otimes}_\Sigma K)$ and injects also into $\mathrm{gr}^0\, H^i(X_{\mathfrak{F}}, \mathscr{A}\hat{\otimes}_\Sigma K) = \mathrm{gr}^0(D \otimes_{W_n(k)[1/p]} B_{\mathrm{st}} \otimes_{W_n(k)} K^+)$. Hence we deduce that it must inject into $D \otimes_{W_n(k)[1/p]} B_{\mathrm{st}}$, i.e. we have the inequality

$$\dim_{\mathbb{Q}_p} V_{\mathrm{st}}(D) \geq \dim_{\mathbb{Q}_p} H^i(X_{\mathfrak{F}}, \mathbb{Q}_p)$$

thereby proving the following

Theorem 5.3. *$D := H^i_{\text{log-crys}}(X_k|W(k))[1/p]$ is an admissible filtered (φ, N)-module and $V_{\mathrm{st}}(D) = H^i(X_{\mathfrak{F}}, \mathbb{Q}_p)$.*

5.4.7. By Theorem 5.3 there is a canonical map

$$D \otimes_{W(k)[1/p]} B_{\mathrm{st}} \cong H^i(X_{\mathfrak{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} \to H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}.$$

We call it the *comparison transformation.* It is compatible with $\mathrm{Gal}(\bar{K}/K)$-action, cup products, Frobenius, monodromy, and filtrations after tensor product $\otimes_{W(k)[1/p]} K$.

Proposition 5.2. *The comparison transformation is compatible with cycle class maps up to sign.*

Proof. Firstly, since X is regular, every coherent sheaf on X has a finite resolution by vector bundles, hence it suffices to check that the comparison tranformation is compatible with Chern classes up to sign. Since both log-crystalline cohomology and the p-adic étale cohomology satisfy the projective bundle formula, by the splitting principle this reduces to showing that the comparison transformation is compatible with the first Chern class of a line bundle up to sign, which has already been checked in Proposition 5.1. $\square$

Theorem 5.4. *The comparison transformation is an isomorphism.*

Proof. Write $H^i_{\text{log-crys}} := H^i_{\text{log-crys}}(X_k|W(k))$ and $H^i_{\text{ét}} = H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$. By Zariski's connectedness theorem and [11] Exp. X, Prop. 1.2, up to replacing K^+ by a finite étale covering, we may assume that $X \to \mathrm{Spec}(K^+)$ has geometrically irreducible fibres of dimension d. Since the comparison transformation is compatible with cycle class maps up to sign, we obtain a diagram which is commutative up to sign

$$\begin{array}{ccccc} H^i_{\text{log-crys}} \otimes_{W(k)} H^{2d-i}_{\text{log-crys}} \otimes_{W(k)[1/p]} B_{\mathrm{st}} & \xrightarrow{\cup} & H^{2d}_{\text{log-crys}} \otimes_{W(k)[1/p]} B_{\mathrm{st}} & \xrightarrow{\sim} & B_{\mathrm{st}} \\ \downarrow & & \downarrow & & \| \\ H^i_{\text{ét}} \otimes_{\mathbb{Q}_p} H^{2d-i}_{\text{ét}} \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} & \xrightarrow{\cup} & H^{2d}_{\text{ét}} \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} & \xrightarrow{\sim} & B_{\mathrm{st}} \end{array}$$

so by Poincaré duality we obtain a one-sided inverse to the comparison transformation

$$H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} \to H^i_{\text{log-crys}}(X_k|W(k)) \otimes_{W(k)[1/p]} B_{\mathrm{st}}.$$

If this transformation commutes with cup products then the same argument gives it a one-sided inverse and hence the theorem will follow. Let

$$\delta : X \hookrightarrow X \times X$$

be the diagonal immersion. Cup products in both cohomology theories are defined by the composition

$$H^i(X) \otimes H^j(X) \longrightarrow H^{i+j}(X \times X) \xrightarrow{\delta^*} H^{i+j}(X)$$

where the first map is induced by the Künneth decomposition. Since the transformation is compatible with Künneth decompositions, by duality we see that the dual transformation commutes with cup products if and only if the comparison transformation commutes with δ_*. But δ_* is characterized by $\mathrm{Tr}_{X\times X}(\delta_*(x) \cup y) = \mathrm{Tr}_X(x \cup \delta^*(y))$, where Tr denote the trace maps, so compatibility with δ_* follows from the compatibility with cup products and characteristic classes. $\square$

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