Two results from Morita theory
of stable model categories

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TWO RESULTS FROM MORITA THEORY
OF STABLE MODEL CATEGORIES

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ABSTRACT. We prove two results from Morita theory of stable model categories. Both can be regarded as topological versions of recent algebraic theorems. One is on recollements of triangulated categories, which have been studied in the algebraic case by Jørgensen. We give a criterion which answers the following question: When is there a recollement for the derived category of a given symmetric ring spectrum in terms of two other symmetric ring spectra?

The other result is on well generated triangulated categories in the sense of Neeman. Porta characterizes the algebraic well generated categories as localizations of derived categories of DG categories. We prove a topological analogon: a topological triangulated category is well generated if and only if it is triangulated equivalent to a localization of the derived category of a symmetric ring spectrum with several objects. Here 'topological' means triangulated equivalent to the homotopy category of a spectral model category. Moreover, we show that every well generated spectral model category is Quillen equivalent to a Bousfield localization of a category of modules via a single Quillen functor.
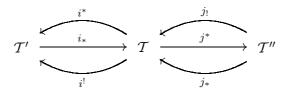
CONTENTS

Introduction

In classical Morita theory [Mor58] questions like these are studied: When are two rings Morita equivalent, that is, when do they have equivalent module categories? When is an abelian category equivalent to the category of modules over some ring? One result is the following. An abelian category $\mathcal{A}$ with (arbitrary) coproducts is equivalent to a category of modules if and only if it has a compact projective generator P [Bas68, Chapter II, Theorem 1.3]. In this case, an equivalence is given by the Hom-functor

$$\operatorname{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \longrightarrow \operatorname{Mod-End}_{\mathcal{A}}(P).$$

A weaker notion than that of classical Morita equivalence is that of derived equivalence first considered by Happel: two rings are derived equivalent if their derived categories are equivalent as triangulated categories. Natural questions are: When are two rings derived equivalent? When is a triangulated category equivalent to the derived category of a ring? Here, ordinary rings can more generally be replaced by differential graded rings (DG rings) or DG algebras over some fixed commutative ring – or 'several objects versions' of such (DG categories). These questions about derived Morita equivalence have been studied among others by Rickard [Ric89] and Keller [Kel94]. As in the classical case, compact generators and certain Hom-functors play an important role.

Using the setting of model categories due to Quillen (cf. [Qui67] or [Hov99]), one can also consider derived categories of other appropriate ring objects (with possibly several objects), such as symmetric ring spectra, and then study similar questions [SS03b].

Recollements. A recollement of triangulated categories is a diagram of triangulated categories

$$
\mathcal{T}' \xrightarrow{\;\;i_*\;\;} \mathcal{T} \xrightarrow{\;\;j^*\;\;} \mathcal{T}''
$$

where (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$, and (j^*, j_*) are adjoint pairs of triangulated functors satisfying some more conditions (see Definition 1.4). This generalizes the notion of triangulated equivalence in so far as a recollement with $\mathcal{T}' = 0$ (resp. $\mathcal{T}'' = 0$) is the same as a triangulated equivalence between $\mathcal{T}$ and $\mathcal{T}''$ (resp. $\mathcal{T}'$). In a recollement, the category $\mathcal{T}$ can be viewed as glued together by $\mathcal{T}'$ and $\mathcal{T}''$. The notion has its origins in the theory of perverse sheaves in algebraic geometry and appeared first in [BBD82], where the authors show among other things that a recollement as above together with t-structures on $\mathcal{T}'$ and $\mathcal{T}''$ induces a t-structure on $\mathcal{T}$.

Jørgensen [Jør06] studies recollements in the case where the involved triangulated categories are derived categories of DG algebras over some fixed commutative ground ring.

He gives a criterion for the existence of DG algebras S and T and a recollement

$$
(*) \qquad \mathsf{D}(S) \; \underset{i_*}{\overset{i^*}{\rightleftarrows}} \; \mathsf{D}(R) \; \underset{j_*}{\overset{j_!}{\rightleftarrows}} \; \mathsf{D}(T)
$$

of derived categories for a given DG algebra R [Jør06, Theorem 3.4].

The derived category of a DG algebra R can be regarded as the homotopy category of the model category of differential graded R-modules. More generally, the homotopy category of every stable model category is a triangulated category in a natural way [Hov99, Chapter 7]. This holds in particular for the category of symmetric spectra in the sense of [HSS00] and for the category of modules over a (symmetric) ring spectrum. For a ring spectrum R let $\mathsf{D}(R)$ denote the homotopy category of modules over R. Given a ring spectrum R we ask, similar to the differential graded case, for a criterion for the existence of ring spectra S and T and a recollement as $(*)$.

One can also study the case where the category of symmetric spectra is more generally replaced by any 'reasonable' monoidal stable model category, including both the case of symmetric spectra and the case of chain complexes ($\mathbb{Z}$-graded and unbounded, over some fixed commutative ground ring) – here a monoid is the same as a DG algebra. The main theorem is Theorem 2.16, which states that a recollement (over a reasonable monoidal stable model category) of the form $(*)$ exists if and only if there are two objects in $\mathsf{D}(R)$ which satisfy certain finiteness and generating conditions. We will proceed in a way similar to Jørgensen's [Jør06]. However, the proofs will sometimes be different and involve the model structure.

Well generated categories. In his book [Nee01b], Neeman introduces the notion of well generated (triangulated) categories, which generalize compactly generated categories. They satisfy, like the compactly generated categories, Brown representability. One advantage over the compactly generated ones is that the class of well generated categories is stable under passing to appropriate localizing subcategories and localizations (cf. Proposition 3.3). A classical example of a compactly generated triangulated category occurring in algebra is the derived category $\mathsf{D}(\mathcal{A})$ of a DG algebra, or more generally, of a DG category $\mathcal{A}$, which is just a 'several objects version' of a DG algebra. By Proposition 3.3, all (appropriate) localizations of $\mathsf{D}(\mathcal{A})$ are well generated again. One could ask whether the converse is also true, that is, whether every well generated triangulated category $\mathcal{T}$ is, up to triangulated equivalence, a localization of the derived category $\mathsf{D}(\mathcal{A})$ for an appropriate DG category $\mathcal{A}$. Porta gives a positive answer if $\mathcal{T}$ is algebraic [Por], [Kel06, Theorem 3.9]. This characterization of algebraic well generated categories can be regarded as a refinement of [Kel94, Theorem 4.3], where Keller characterizes the algebraic compactly generated categories with arbitrary coproducts, up to triangulated equivalence, as the derived categories of DG categories.

A topological version of Keller's theorem has been proved in [SS03b, Theorem 3.9.3(iii)]: the compactly generated topological categories are characterized, up to triangulated equivalence, as the 'derived categories of ring spectra with several objects'. This needs some

explanation. A spectral category is a ring spectrum with several objects, i.e., a small category enriched over the symmetric monoidal model category of symmetric spectra in the sense of [HSS00]. Generalizing the correspondence between ring spectra and DG algebras, spectral categories are the topological versions of DG categories. The derived category of a spectral category $\mathcal{E}$ is the homotopy category of the model category of $\mathcal{E}$-modules. By a topological triangulated category we mean any triangulated category equivalent to the homotopy category of a spectral model category. This is not the same as (but closely related with) a topological triangulated category in the sense of [Sch06], where any triangulated category equivalent to a full triangulated subcategory of the homotopy category of a stable model category is called topological. By [SS03b, Theorem 3.8.2], the homotopy category of any simplicial, cofibrantly generated and proper stable model category is topological.
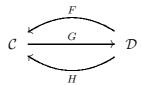
The aim of Part 2 of this paper is to give a characterization of the topological well generated categories. We will prove that every topological well generated triangulated category is triangulated equivalent to a localization of the derived category of a small spectral category such that the acyclics of the localization are generated by a set. On the other hand, the derived category of a small spectral category is compactly generated by the free modules [SS03b, Theorem A.1.1(ii)] and the class of well generated categories is stable under localizations (as long as the acyclics are generated by a set), cf. Proposition 3.3. Hence we get the following characterization (Theorem 4.7): The topological well generated categories are, up to triangulated equivalence, exactly the localizations (with acyclics generated by a set) of derived categories of spectral categories.

Finally, we use Hirschhorn's existence theorem for Bousfield localizations [Hir03, Theorem 4.1.1] to give a lift to the level of model categories in the following sense (Theorem 5.11): Every spectral model category which has a well generated homotopy category admits a Quillen equivalence to a Bousfield localization of a model category of modules (over some endomorphism spectral category). While a rough slogan of a main result in [SS03b] is, 'Compactly generated stable model categories are categories of modules', the corresponding slogan of our result is, 'Well generated stable model categories are localizations of categories of modules'.

Terminology and conventions. Our main reference for triangulated category theory is Neeman's book [Nee01b] and thus we use basically his terminology. One exception concerns the definition of a triangulated category: since we are interested in triangulated categories arising from topology we allow the suspension functor $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$ of a triangulated category $\mathcal{T}$ to be a self-equivalence of $\mathcal{T}$ and do not require it to be an automorphism. In other words, we take the definition of a triangulated category that can be found, for example, in [Mar83, Appendix 2].

Another point of difference is that all our categories are supposed to have Hom-*sets*, not only Hom-*classes*. (In the terminology of [Nee01b], the morphisms between two objects are allowed to form a class. If, between any two objects, they actually form a set, then the category is said to have 'small Hom-sets' in [Nee01b].) Such triangulated 'meta'-categories with Hom-classes arise in the context of Verdier quotients (cf. Remark 1.9(2)). But it turns out that all Verdier quotients we need to consider are in fact 'honest' categories, that is, the morphisms between any two objects form a set.

When we say that a category has (co-)products, we always mean arbitrary set-indexed (co-)products. Adjoint pairs of functors will arise throughout the paper. We use the convention according to which in diagrams the left adjoint functor is drawn above the right adjoint. If we have three functors

$$
\mathcal{C} \underset{H}{\overset{F}{\underset{\longrightarrow}{\rightleftarrows}}} \mathcal{D}
$$

such that (F, G) and (G, H) are adjoint pairs we will call (F, G, H) an adjoint triple.

Part 1. Stable model categories and recollements

We start in Section 1 with a recollection of some notions and lemmas from triangulated category theory which will also be important in Part 2 of this paper. We will then discuss the definition of recollements and some of their properties. Recollements are closely related to localizations and colocalizations. We consider this relation in Section 1.2. An example of a recollement coming from stable homotopy theory is described in Section 1.3.

In Section 2, we introduce 'reasonable' stable model categories, that is, closed symmetric monoidal model categories which are stable and have some other nice properties that allow us to study Morita theory over such categories. Both symmetric spectra and chain complexes are examples of reasonable stable model categories. In [SS03b, Theorem 3.9.3], Schwede and Shipley relate spectral model categories to certain categories of modules via a Quillen pair. We consider a version thereof over reasonable stable model categories in Section 2.3. In Section 2.4, we prove our main result, Theorem 2.16, which gives a criterion for the existence of a recollement for the derived category $\mathsf{D}(R)$, where R is a monoid in a reasonable stable model category.

1. Recollements

1.1. Definition and formal properties. Let us recall some general notions from triangulated category theory.

By a *triangulated subcategory* $\mathcal{U}$ of $\mathcal{T}$ we mean a non-empty full subcategory which is closed under (de-)suspensions and triangles (if two out of three objects in a triangle are in $\mathcal{U}$ then so is the third). Note that $\mathcal{U}$ is then automatically closed under finite coproducts and it contains the whole isomorphism class of an object (i.e., $\mathcal{U}$ is 'replete'). One says $\mathcal{U}$ is *thick* if it is closed under direct summands. If $\mathcal{T}$ has (arbitrary) coproducts, $\mathcal{U}$ is called *localizing* whenever it is closed under coproducts. If $\mathcal{U}$ is localizing it is automatically thick (since in this case, idempotents split in $\mathcal{U}$ [Nee01b, Proposition 1.6.8]).

If $\mathcal{T}$ and $\mathcal{T}'$ are triangulated categories with suspension functors Σ and Σ' a *triangulated* (or *exact*) functor is a functor $F : \mathcal{T} \longrightarrow \mathcal{T}'$ together with a natural isomorphism $F \circ \Sigma \xrightarrow{\cong} \Sigma' \circ F$ such that for every exact triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in $\mathcal{T}$ we get an exact triangle

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow \Sigma' F(X)$$

in $\mathcal{T}'$, whose last arrow involves the natural isomorphism. Unless stated otherwise, by a functor between triangulated categories we always mean a triangulated one. The *kernel* of F is the thick triangulated subcategory of $\mathcal{T}$ containing the objects which are mapped to zero in $\mathcal{T}'$,

$$\ker F = \{X \in \mathcal{T} \mid F(X) \cong 0\}.$$

If $\mathcal{T}$ and $\mathcal{T}'$ have coproducts and F preserves them, then $\ker F$ is localizing. One cannot expect the image of F to be a triangulated subcategory of $\mathcal{T}'$. Even if F is full the image need not be replete. But the *essential image* of F,

$$\operatorname{essim} F = \{X' \in \mathcal{T}' \mid X' \cong F(X) \text{ for some } X \in \mathcal{T}\},$$

is a triangulated subcategory if F is a full (!) triangulated functor. It is localizing if $\mathcal{T}$ and $\mathcal{T}'$ contain coproducts and F preserves them.

If $\mathcal{S}$ is a set of objects of a triangulated category $\mathcal{T}$ with coproducts then $\langle \mathcal{S} \rangle$ denotes the smallest localizing triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}$. (It does exist, it is just the intersection of all localizing triangulated subcategories containing $\mathcal{S}$.)

Example 1.1. If R is a DG algebra, that is, a monoid in the symmetric monoidal model category of chain complexes, then R considered as a module over itself is a generator for $\mathsf{D}(R)$, this is a special case of [Kel94, Section 4.2].

Similarly, if R is a symmetric ring spectrum, that is a monoid in the symmetric monoidal model category of symmetric spectra, then R is a generator for $\mathsf{D}(R)$, which is a special case of [SS03b, Theorem A.1.1(ii)].

For $F : \mathcal{T} \longrightarrow \mathcal{T}'$ let $F(\mathcal{S})$ be the set of all $F(X)$ with $X \in \mathcal{S}$. We have the following (probably well-known)

Lemma 1.2. *Let* $F : \mathcal{T} \longrightarrow \mathcal{T}'$ *be a coproduct preserving triangulated functor between triangulated categories with coproducts and* $\mathcal{S}$ *a set of objects in* $\mathcal{T}$.

(i) *There is an inclusion of (not necessarily triangulated) full subcategories*

$$\mathrm{essim}\left(F|_{\langle \mathcal{S} \rangle}\right) \subset \langle F(\mathcal{S}) \rangle.$$

(ii) *If* F *is full then*

$$\mathrm{essim}\left(F|_{\langle \mathcal{S} \rangle}\right) = \langle F(\mathcal{S}) \rangle$$

as triangulated categories

Proof. Those X in $\mathcal{T}$ for which $F(X)$ is in $\langle F(\mathcal{S}) \rangle$ form a localizing triangulated subcategory containing $\mathcal{S}$ and hence containing $\langle \mathcal{S} \rangle$. So the image (and, as a consequence, the essential image) of $F|_{\langle \mathcal{S} \rangle}$ is contained in $\langle F(\mathcal{S}) \rangle$, as was claimed in (i). For the other inclusion note that since F is full, $\mathrm{essim}\, F|_{\langle \mathcal{S} \rangle}$ is a localizing triangulated subcategory of $\mathcal{T}'$ which contains $F(\mathcal{S})$. This shows (ii). $\qquad\square$

The following lemma is often useful, too.

Lemma 1.3. *Let* $F, G : \mathcal{T} \longrightarrow \mathcal{T}'$ *be coproduct preserving triangulated functors between triangulated categories with coproducts and* $\eta : F \longrightarrow G$ *a natural transformation of triangulated functors. Then those objects* X *for which* η_X *is an isomorphism form a localizing triangulated subcategory of* $\mathcal{T}$. $\qquad\square$

As a definition for recollements we take Jørgensen's [Jør06, Definition 3.1].

Definition 1.4. A *recollement* of triangulated categories is a diagram of triangulated categories

$$\mathcal{T}' \xrightarrow{\quad i_* \quad} \mathcal{T} \xrightarrow{\quad j^* \quad} \mathcal{T}''$$

such that

(i) both $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples, that is, (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$, and (j^*, j_*) are adjoint pairs of triangulated functors,

(ii) $j^* i_* = 0$,

(iii) the functors i_*, $j_!$, and j_* are fully faithful,

(iv) for each object X in $\mathcal{T}$ there are exact triangles
    (a) $j_! j^* X \longrightarrow X \longrightarrow i_* i^* X \longrightarrow \Sigma j_! j^* X$,
    (b) $i_* i^! X \longrightarrow X \longrightarrow j_* j^* X \longrightarrow \Sigma i_* i^! X$,
    where the maps to X are counit maps, the maps out of X are unit maps, and Σ denotes the suspension.

Sometimes we will drop the structure functors i^*, i_*, $i^!$, $j_!$, j^*, and j_* from the notation and simply write $(\mathcal{T}', \mathcal{T}, \mathcal{T}'')$ for a recollement.

Remark 1.5. Here are some formal properties.

(1) Being a left (resp. right) adjoint of $j^* i_* = 0$, the composition of the upper (resp. lower) functors in a recollement is zero:

$$i^* j_! = 0 \quad \text{and} \quad i^! j_* = 0.$$

(2) Provided condition (i) in Definition 1.4 holds, condition (iii) is equivalent to the following. For X' in $\mathcal{T}'$ and X'' in $\mathcal{T}''$ the counit and unit maps

$$i^* i_* X' \longrightarrow X', \quad j^* j_* X'' \longrightarrow X'', \quad X' \longrightarrow i^! i_* X', \quad X'' \longrightarrow j^* j_! X''$$

are natural isomorphisms.

(3) Composing the natural isomorphism $i^* i_* X' \longrightarrow X'$ in (2) with i_* we get that the restriction of $i_* i^*$ to the essential image of i_* is naturally isomorphic to the identity functor.

(4) The third arrow in the exact triangles (a) and (b) of Definition 1.4(iv) is natural in X and uniquely determined. To see the naturality consider a diagram

$$
\begin{array}{ccccccc}
j_! j^* X & \longrightarrow & X & \xrightarrow{\eta_X} & i_* i^* X & \xrightarrow{\psi_X} & j_! j^* \Sigma X \\
{\scriptstyle j_! j^*(f)}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle j_! j^* \Sigma(f)} \\
j_! j^* Y & \longrightarrow & Y & \xrightarrow{\eta_Y} & i_* i^* Y & \xrightarrow{\psi_Y} & j_! j^* \Sigma Y
\end{array}
$$

where the rows are exact triangles as in Definition 1.4(iv)(a) and solid arrows are given such that the left square commutes. The axioms of a triangulated category guarantee the existence of a dotted arrow $\bar{f}$ such that the whole diagram commutes.

We claim that there is only one arrow $\bar{f}$ such that the square in the middle commutes, that is, $\bar{f}\eta_X = \eta_Y f$. It is enough to consider the case $f = 0$ and to show that $\bar{f}$ is necessarily zero, too. But $f = 0$ implies $\bar{f}\eta_X = 0$ and since the representing functor $\mathcal{T}(-, i_* i^* Y)$ is cohomological there exists an arrow $g : j_! j^* \Sigma X \longrightarrow i_* i^* Y$ such that $g\psi_X = \bar{f}$. Now the adjoint map of g with respect to the adjoint pair $(j_!, j^*)$ is a map into $j^* i_* i^* Y$ which is zero by Definition 1.4(ii). Hence g itself is zero and so is $\bar{f}$, proving our claim.

As the unit η is a natural transformation, the map $i_* i^*(f)$ satisfies $i_* i^*(f)\eta_X = \eta_Y f$ and consequently $\bar{f} = i_* i^*(f)$. Since the right square in the diagram is commutative, this shows the naturality of ψ. Taking f to be the identity arrow on X shows the uniqueness of the third arrow ψ_X.

(5) Replacing any of $\mathcal{T}$, $\mathcal{T}'$ or $\mathcal{T}''$ in a recollement by an equivalent triangulated category still gives a recollement.

(6) A recollement with $\mathcal{T}' = 0$ is the same as an equivalence $\mathcal{T} \simeq \mathcal{T}''$ of triangulated categories. Namely $i_* = 0$ implies by Definition 1.4(iv)(b) that $X \cong j_* j^* X$, so j_* is essentially surjective on objects. Since j_* is also fully faithful by Definition 1.4(iii) it is an equivalence of categories with inverses j_* and $j_!$ (which are hence isomorphic). Similarly, a recollement with $\mathcal{T}'' = 0$ is the same as a triangulated equivalence $\mathcal{T}' \simeq \mathcal{T}$.

(7) A map of recollements from $(\mathcal{T}', \mathcal{T}, \mathcal{T}'')$ to $(\mathcal{U}', \mathcal{U}, \mathcal{U}'')$ consists of three triangulated functors $F' : \mathcal{T}' \longrightarrow \mathcal{U}'$, $F : \mathcal{T} \longrightarrow \mathcal{U}$, $F'' : \mathcal{T}'' \longrightarrow \mathcal{U}''$ which commute (up to natural isomorphism) with the structure functors. It is a theorem of Parshall and Scott [PS88, Theorem 2.5] that a map of recollements is determined (up to natural isomorphism) by F' and F (resp. F and F''). Furthermore, if two of F', F and F'' are equivalences then so is the third. This is not true for recollements of abelian categories, see [FP04, Section 2.2].

(8) For every recollement one has

$$\operatorname{essim} i_* = \ker j^*, \quad \operatorname{essim} j_! = \ker i^*, \quad \operatorname{essim} j_* = \ker i^!.$$

Consider, for example, the first equality. The inclusion $\operatorname{essim} i_* \subset \ker j^*$ follows immediately from $j^* i_* = 0$. If, on the other hand, $j^* X = 0$, then the third term in the exact triangle

$$i_* i^! X \longrightarrow X \longrightarrow j_* j^* X \longrightarrow \Sigma i_* i^! X$$

of Definition 1.4(iv)(b) vanishes so that the first map is an isomorphism and thus $X \in \operatorname{essim} i_*$.

Since i_* is fully faithful we have an equivalence of triangulated categories $\mathcal{T}' \simeq \operatorname{essim} i_*$ and hence $\mathcal{T}' \simeq \ker j^*$. Hence, due to Remark 1.5(5), every recollement is 'equivalent' to the recollement

where ι_* is the inclusion with left (resp. right) adjoint ι^* (resp. $\iota^!$).

Example 1.6. The following is the classical example of a recollement arising in algebraic geometry [BBD82, Section 1.4.1]. Let X be a topological space, U an open subspace and F the complement of U in X. Given a sheaf $\mathcal{O}_X$ of commutative rings on X, we denote the restricted sheaves of rings on U, resp. F, by $\mathcal{O}_U$, resp. $\mathcal{O}_F$, and the three categories of sheaves of left modules by $\mathcal{O}_X$-Mod, $\mathcal{O}_U$-Mod, and $\mathcal{O}_F$-Mod. We have six functors

$$
\mathcal{O}_F\text{-Mod} \xrightarrow{\ \ i_*\ \ } \mathcal{O}_X\text{-Mod} \xrightarrow{\ \ j^*\ \ } \mathcal{O}_U\text{-Mod}
$$

where i^* and j^* are restriction functors, i_* and j_* are direct image functors, and $j_!$ is the functor which extends a sheaf on U by 0 outside U to the whole of X, i.e., for every $\mathcal{O}_U$-module $\mathcal{F}$ and every open subset V of X we have $j_!\mathcal{F}(V) = \mathcal{F}(V)$ if $V \subset U$ and $j_!\mathcal{F}(V) = 0$ else. Finally, $i^!$ is defined by

$$
(i^!\mathcal{G})(V \cap F) = \{s \in \mathcal{G}(V) \,|\, \operatorname{supp}(s) \subset F\}
$$

for every $\mathcal{O}_X$-module $\mathcal{G}$ and every open subset V of X.

Let $\mathsf{D}^+(\mathcal{O}_F)$, $\mathsf{D}^+(\mathcal{O}_X)$, and $\mathsf{D}^+(\mathcal{O}_U)$ be the corresponding derived categories of left bounded complexes. The derived functors of i^*, i_*, $i^!$, $j_!$, j^*, and j_* exist and yield a recollement

$$
\mathsf{D}^+(\mathcal{O}_F) \xrightarrow{\ \ i_*\ \ } \mathsf{D}^+(\mathcal{O}_X) \xrightarrow{\ \ j^*\ \ } \mathsf{D}^+(\mathcal{O}_U)\,.
$$

1.2. **Localization and colocalization.** It turns out that the data of a recollement is essentially the same as a triangulated functor j^* which admits both a *localization functor* $j_!$ and a *colocalization functor* j_*. These two notions are defined as follows.

Definition 1.7. If a triangulated functor $F : \mathcal{T} \longrightarrow \mathcal{U}$ admits a fully faithful right adjoint $G : \mathcal{U} \longrightarrow \mathcal{T}$ we call G a *localization functor* and $\mathcal{U}$ a *localization* of $\mathcal{T}$. The objects in the kernel of F are called *(F-)acyclic* and those objects $X \in \mathcal{T}$ for which the unit of the adjunction $X \longrightarrow GF(X)$ is an isomorphism (or, equivalently, which are in the essential image of G) are called *(F-)local*.

Dually, if $F : \mathcal{T} \longrightarrow \mathcal{U}$ admits a fully faithful left adjoint $H : \mathcal{U} \longrightarrow \mathcal{T}$ we call H a *colocalization functor* and $\mathcal{U}$ a *colocalization* of $\mathcal{T}$. The objects in the kernel of F are called *(F-)acyclic* and those objects $X \in \mathcal{T}$ for which the counit of the adjunction $HF(X) \longrightarrow X$ is an isomorphism (or, equivalently, which are in the essential image of H) are called *(F-)colocal*.

Since by [Mar83, Appendix 2, Proposition 11] the adjoint of a triangulated functor is itself triangulated, localization and colocalization functors are always triangulated.

Remark 1.8. If $F : \mathcal{T} \longrightarrow \mathcal{U}$ admits a localization functor $G : \mathcal{U} \longrightarrow \mathcal{T}$, then $\mathcal{U}$ is triangulated equivalent to $\operatorname{essim} G$. The composition $GF : \mathcal{T} \longrightarrow \operatorname{essim} G$ has the inclusion $\operatorname{essim} G \hookrightarrow \mathcal{T}$ as a right adjoint. In other words, the localization $\mathcal{U}$ of $\mathcal{T}$ is equivalent to the triangulated subcategory of local objects, which can then be regarded as a localization of $\mathcal{T}$ with exactly the same acyclics as the original localization of $\mathcal{T}$.

Remark 1.9. Let us compare our definition of localization with others occurring in the literature.

(1) Keller's definition is slightly different from ours, see [Kel06, Section 3.7]: in addition to our definition, the kernel of $F : \mathcal{T} \longrightarrow \mathcal{U}$ is supposed to be generated by a *set* of objects. (The reason for this is that under this additional technical assumption a localization of a *well generated* triangulated category is again well generated, cf. Proposition 3.3.)

(2) The definition given in Neeman's book [Nee01b, Definition 9.1.1] is the following. Given a thick triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$, there always exists a *Verdier quotient* $\mathcal{T}/\mathcal{S}$ together with a universal functor $\mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ with kernel $\mathcal{S}$ [Nee01b, Theorem 2.1.8 and Remark 2.1.10]. In Neeman's terminology, the Hom-'sets' of this triangulated category $\mathcal{T}/\mathcal{S}$ are not necessarily small, that is, they do not form sets but only classes, and hence $\mathcal{T}/\mathcal{S}$ is not an honest category in general. If the Verdier quotient functor $F : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ admits a fully faithful right adjoint $G : \mathcal{T}/\mathcal{S} \longrightarrow \mathcal{T}$ then $\mathcal{T}/\mathcal{S}$ is called a *Bousfield localization* and the functor G is called a *Bousfield localization functor*. It is a consequence of [Nee01b, Theorem 9.1.16] that, if $\mathcal{T}/\mathcal{S}$ is a Bousfield localization, $\mathcal{T}/\mathcal{S}$ is an honest category (i.e., has small Hom-sets).

A Bousfield localization in Neeman's sense is in particular a localization as in Definition 1.7. Namely the right adjoint G, if it exists, is automatically fully faithful. (To see this, it is enough to show that the counit ε of the adjunction (F, G) is an isomorphism. Since F is the identity on objects one has only to check that εF is an isomorphism. But this follows from [Nee01b, Lemma 9.1.7].) On the other hand, by part (iii) of Lemma 1.11(b) below, a localization in our sense is always a Bousfield localization up to triangulated equivalence.

Hence Neeman's notion of Bousfield localization is essentially equivalent to our notion of localization as in Definition 1.7.

(3) In [HPS97] the authors consider stable homotopy categories, i.e., triangulated categories endowed with a closed symmetric monoidal product $\wedge$ and with a certain set of generators – for the complete definition see [HPS97, Definition 1.1.4]. They define a localization functor [HPS97, Definiton 3.1.1] on a stable homotopy category $\mathcal{C}$ to be a pair (L, i), where $L : \mathcal{C} \longrightarrow \mathcal{C}$ is a triangulated functor and $i : \operatorname{id}_{\mathcal{C}} \longrightarrow L$ is a natural transformation such that
  (i) the natural transformation $Li : L \longrightarrow L^2$ is an isomorphism,
  (ii) for all objects X, Y in $\mathcal{C}$ the map $i_X^* : \mathcal{C}(LX, LY) \longrightarrow \mathcal{C}(X, LY)$ given by precomposition with i_X is an isomorphism,
  (iii) if $LX = 0$ then $L(X \wedge Y) = 0$ for all Y.

The L-local objects in $\mathcal{C}$ are (by definition in [HPS97]) the objects Y for which i_Y is an isomorphism or, equivalently, which are isomorphic to some LX. If $\mathcal{C}_L$ is the full subcategory of L-local objects then $L : \mathcal{C} \longrightarrow \mathcal{C}_L$ is left adjoint to the inclusion $\mathcal{C}_L \hookrightarrow \mathcal{C}$. In other words: we have a localization of triangulated categories as in Definition 1.7, and the two notions of L-local objects (ours and that of [HPS97]) coincide. Note that we did not use condition (iii), which involves the monoidal structure.

On the other hand, if we are given a functor $F : \mathcal{T} \longrightarrow \mathcal{U}$ admitting a localization functor $G : \mathcal{U} \longrightarrow \mathcal{T}$, the composite GF together with the unit of the adjunction $\mathrm{id}_{\mathcal{T}} \longrightarrow GF$ satisfies the first and the second of the above conditions. In so far, if we ignore the monoidal structure, our definition and the one in [HPS97] are equivalent.

Dualizing this definition of localization leads to the notion of colocalization of stable homotopy categories, see [HPS97, Definition 3.1.1]. Each localization L on $\mathcal{C}$ determines a colocalization C on $\mathcal{C}$ and vice versa [HPS97, Lemma 3.1.6]. Two such correspond if and only if there is an exact triangle

$$CX \longrightarrow X \longrightarrow LX \longrightarrow \Sigma(CX)$$

where first map comes from the natural transformation of the colocalization C and the second from natural transformation of the localization L. For each such localization-colocalization pair (L, C) we have $\mathrm{essim}\, L = \ker C$ and $\mathrm{essim}\, C = \ker L$. Hence the L-local objects are exactly the C-acyclics and the C-colocals are exactly the L-acyclics.

Definition 1.10. For a class $\mathcal{A}$ of objects in a triangulated category $\mathcal{T}$, the category $\mathcal{A}^\perp$ is defined as the full subcategory of $\mathcal{T}$ containing those objects which do not receive non-zero graded maps from $\mathcal{A}$, that is,

$$\mathcal{A}^\perp = \{X \in \mathcal{T} \mid \mathcal{T}(\Sigma^n A, X) \cong 0 \text{ for each } n \in \mathbb{Z} \text{ and each } A \in \mathcal{A}\}.$$

In the case where $\mathcal{A}$ consists only of one object A, we simply write $A^\perp$ for $\mathcal{A}^\perp$. Dually, we define

$$^\perp\mathcal{A} = \{X \in \mathcal{T} \mid \mathcal{T}(X, \Sigma^n A) \cong 0 \text{ for each } n \in \mathbb{Z} \text{ and each } A \in \mathcal{A}\}.$$

Note that $\mathcal{A}^\perp$ is a thick triangulated subcategory of $\mathcal{T}$, which is colocalizing (i.e., closed under products) if $\mathcal{T}$ has products. It is localizing if $\mathcal{T}$ has coproducts and all objects in $\mathcal{A}$ are compact, whereas $^\perp\mathcal{A}$ is always a localizing triangulated subcategory if $\mathcal{T}$ has coproducts.

The reader should be warned that there is not a standardized use of $\mathcal{A}^\perp$ and $^\perp\mathcal{A}$ in the literature. Neeman [Nee01b, Definitions 9.1.10 and 9.1.11] writes $\mathcal{A}^\perp$ where Jørgensen [Jør06, Section 3], for example, uses $^\perp\mathcal{A}$ and vice versa. Our definition is the same as Jørgensen's.

In the next lemma, some facts on colocalizations and localizations are summarized. I expect them to be well-known but I do not know a reference for the lemma in the form that will be needed. Hence a complete proof will be given.

Lemma 1.11. *Let* $j^* : \mathcal{T} \longrightarrow \mathcal{T}''$ *be a triangulated functor and* $\mathcal{T}' = \ker j^*$.

(a) *Suppose* j^* *admits a colocalization functor, i.e., a fully faithful left adjoint* $j_!$,

$$\mathcal{T} \xleftarrow[\quad j^* \quad]{\quad j_! \quad} \mathcal{T}''.$$

Then the following statements hold.
(i) *The inclusion* $i_* : \mathcal{T}' \longrightarrow \mathcal{T}$ *has also a left adjoint* i^*.
(ii) *For* X *in* $\mathcal{T}$ *there are natural exact triangles*

$$j_! j^* X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta'_X} i_* i^* X \longrightarrow \Sigma j_! j^* X$$

where ε *is the counit of* $(j_!, j^*)$ *and* η' *is the unit of* (i^*, i_*).
(iii) *The Verdier quotient* $\mathcal{T}/\mathcal{T}'$ *is triangulated equivalent to* $\mathcal{T}''$. *In particular,* $\mathcal{T}/\mathcal{T}'$ *is an honest category (i.e., the Hom-'sets' form actual sets).*
(iv) *For the subcategory of colocal objects, one has*

$$\operatorname{essim} j_! = \ker i^* = {}^{\perp}(\ker j^*).$$

(b) *Dually, suppose* j^* *admits a localization functor, i.e., a fully faithful right adjoint* j_*,

$$\mathcal{T} \xrightarrow[\quad j_* \quad]{\quad j^* \quad} \mathcal{T}''.$$

Then the following statements hold.
(i) *The inclusion* $i_* : \mathcal{T}' \longrightarrow \mathcal{T}$ *has also a right adjoint* $i^!$.
(ii) *For* X *in* $\mathcal{T}$ *there are natural exact triangles*

$$i_* i^! X \xrightarrow{\varepsilon'_X} X \xrightarrow{\eta_X} j_* j^* X \longrightarrow \Sigma i_* i^! X$$

where ε' *is the counit of* $(i_*, i^!)$ *and* η *is the unit of* (j^*, j_*).
(iii) *The Verdier quotient* $\mathcal{T}/\mathcal{T}'$ *is triangulated equivalent to* $\mathcal{T}''$. *In particular,* $\mathcal{T}/\mathcal{T}'$ *is an honest category (i.e., the Hom-'sets' form actual sets).*
(iv) *For the subcategory of local objects, one has*

$$\operatorname{essim} j_* = \ker i^! = (\ker j^*)^{\perp}.$$

Proof. Part (b) follows from (a) by considering opposite categories (then left adjoints become right adjoints and vice versa).

Let us consider part (a) and first prove the statements (i) and (ii) together. Since i_* is triangulated its left adjoint i^* will automatically be triangulated. Let us first define i^* on objects. Statement (ii) tells us what to do. For X in $\mathcal{T}$ take the counit of the adjunction $(j_!, j^*)$ and complete this to an exact triangle

$$(1.12) \qquad\qquad j_! j^* X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta'_X} \varphi X \longrightarrow j_! j^* \Sigma X$$

in $\mathcal{T}$. By assumption, $j_!$ is fully faithful, hence the unit η of the adjunction $(j_!, j^*)$ is an isomorphism. As for all adjoint pairs, the diagram

$$
\begin{array}{ccc}
j^*X & \xrightarrow{\ \eta_{j^*X}\ } & j^*j_!j^*X \\
& {}_{=}\searrow & \Big\downarrow{}^{j^*(\varepsilon_X)} \cong \\
& & j^*X
\end{array}
$$

commutes [ML98, Chapter IV.1, Theorem 1], so that $j^*(\varepsilon_X)$ is an isomorphism. Applying j^* to the triangle (1.12) shows that $\varphi X \in \ker j^*$. Therefore we can define i^*X by $i_* i^* X = \varphi X$.

Given a map $f : X \longrightarrow Y$ in $\mathcal{T}$, the axioms of a triangulated category guarantee the existence of a map $\bar{f}$ such that we get a map of exact triangles

$$
(1.13) \qquad
\begin{array}{ccccccc}
j_!j^*X & \xrightarrow{\varepsilon_X} & X & \xrightarrow{\eta'_X} & i_*i^*X & \longrightarrow & j_!j^*\Sigma X \\
{\scriptstyle j_!j^*(f)}\Big\downarrow & & {\scriptstyle f}\Big\downarrow & & {\scriptstyle \bar{f}}\Big\downarrow & & \Big\downarrow{\scriptstyle j_!j^*\Sigma f} \\
j_!j^*Y & \xrightarrow{\varepsilon_Y} & Y & \xrightarrow{\eta'_Y} & i_*i^*Y & \longrightarrow & j_!j^*\Sigma Y.
\end{array}
$$

Using exactly the same arguments as in Remark 1.5(4) one can show that there is only one map $\bar{f}$ such that the square in the middle commutes, i.e., $\bar{f}\eta'_X = \eta'_Y f$. Consequently, the assignment $f \mapsto \bar{f}$ is additive and compatible with identities and composition. Since i_* is fully faithful we get a functor $i^* : \mathcal{T} \longrightarrow \ker j_*$. To see that (i^*, i_*) is an adjunction it suffices to have a natural transformation (which is then the unit of the adjunction) $X \longrightarrow i_*i^*X$ for X in $\mathcal{T}$ which is universal from X to the functor $i_* : \ker j_* \longrightarrow \mathcal{T}$. Our candidate is the map η'_X defined by the triangle (1.12). It is natural by (1.13). To check that η'_X is universal from X to $i_* : \ker j_* \longrightarrow \mathcal{T}$ let $X' \in \mathcal{T}$ and a map $X \longrightarrow i_*X'$ be given. The composition $j_!j^*X \longrightarrow X \longrightarrow i_*X'$ has zero as an adjoint map with respect to the adjoint pair $(j_!, j^*)$ so it is itself zero. This gives us a commutative diagram (of solid arrows)

$$
\begin{array}{ccccccc}
j_!j^*X & \longrightarrow & X & \xrightarrow{\eta'_X} & i_*i^*X & \longrightarrow & j_!j^*\Sigma X \\
\Big\downarrow & & \Big\downarrow & & {\scriptstyle i_*(h)}\Big\downarrow & & \Big\downarrow \\
0 & \longrightarrow & i_*X' & \xrightarrow{\ =\ } & i_*X' & \longrightarrow & 0
\end{array}
$$

which can be completed into a map of exact triangles via a map $i_*(h)$ for some map $h : i^*X \longrightarrow X'$. As above it follows that h is unique. This shows that η' is in fact the unit of an adjoint pair (i^*, i_*). The exactness of the triangle (1.12) ensures that the statement in (ii) is satisfied.

For part (iii) let $F : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{T}'$ be the the canonical functor into the Verdier quotient and $\varphi = Fj_!$. By the universal property of F there exists a functor ψ such that $\psi F = j^*$.

$$\mathcal{T}' \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{T} \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathcal{T}''$$

$$\mathcal{T}/\mathcal{T}'$$

(with F, φ, ψ labeling the maps to $\mathcal{T}/\mathcal{T}'$)

As $j_!$ is fully faithful the unit of the adjoint pair $(j_!, j^*)$ is an isomorphism and we can conclude that ψ is a left inverse of φ:

$$\psi\varphi = \psi F j_! = j^* j_! \cong \mathrm{id}_{\mathcal{T}''}$$

Let us now apply F to the exact triangle in statement (ii) of part (a) of the lemma so that we get an exact triangle

$$Fj_! j^* X \longrightarrow FX \longrightarrow Fi_* i^* X \longrightarrow F\Sigma j_! j^* X.$$

Since $Fi_* i^* X \cong 0$ we have isomorphisms

$$F \cong F j_! j^* = \varphi j^* = \varphi \psi F$$

and thus by the universal property of F an isomorphism $\mathrm{id}_{\mathcal{T}/\mathcal{T}'} \cong \varphi\psi$. This shows that φ and ψ are inverse triangulated equivalences.

For part (iv) note that $\ker i^* = \mathrm{essim}\, j_!$ can be proved in exactly the same way as in Remark 1.5(8). To see $\mathrm{essim}\, j_! \subset {}^\perp(\ker j^*)$ note that a map $j_! X \longrightarrow Y$ with $Y \in \ker j^*$ corresponds via the adjunction $(j_!, j^*)$ to a map $X \longrightarrow j^* Y = 0$, which has to be the zero map. Hence the map $j_! X \longrightarrow Y$ is itself zero.

It now suffices to prove ${}^\perp(\ker j^*) \subset \ker i^*$. For $X \in {}^\perp(\ker j^*)$ the unit $\eta_X : X \longrightarrow i_* i^* X$ is zero because $i_* i^* X$ is in the essential image of i_*, which is the same as the kernel of j^* (this is again proved as in Remark 1.5(8)). Consider the commutative diagram

$$i^* X$$

$$i^*(\eta_X) \downarrow \qquad \searrow^{=}$$

$$i^* i_* i^* X \xrightarrow[\varepsilon_{i^* X}]{} i^* X$$

involving the unit and counit of the adjunction (i^*, i_*). As we have just seen, $i^*(\eta_X)$ is zero. Since the right adjoint i_* is fully faithful, the counit $\varepsilon_{i^* X}$ is an isomorphism. This implies $i^* X \cong 0$. $\qquad\square$

The next proposition helps us to construct recollements when 'the right part' of a recollement is already given. Together with Remark 1.5(8) it implies that, up to equivalence of triangulated categories, the data of a recollement as in Definition 1.4 is equivalent to the data of Proposition 1.14.
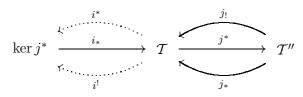
Proposition 1.14. *Let there be given a diagram*

$$\mathcal{T} \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\hspace{1.5cm} j^* \hspace{1.5cm}}}}}} \mathcal{T}''$$

of triangulated categories such that

(i) $(j_!, j^*, j_*)$ *is an adjoint triple of triangulated functors,*

(ii) *at least one of the functors $j_!$ and j_* is fully faithful,*

and let $i_* : \ker j^* \longrightarrow \mathcal{T}$ denote the full inclusion. Then the diagram can be completed into a recollement

$$\ker j^* \xrightarrow{\;\;i_*\;\;} \mathcal{T} \xrightarrow{\;\;j^*\;\;} \mathcal{T}''$$

by functors i^* and $i^!$ which are unique up to isomorphism.

Proof. As left resp. right adjoints of i_*, the functors i^* and $i^!$ have to be unique, and we clearly have $j^* i_* = 0$. Let us assume $j_!$ is fully faithful (in case j_* is fully faithful we could consider opposite categories). By Lemma 1.11(a), parts (i) and (ii), we get the upper half of the recollement,

$$\ker j^* \xrightarrow{\;\;i_*\;\;} \mathcal{T} \xrightarrow{\;\;j^*\;\;} \mathcal{T}''.$$

Then, by part (iii) of the same Lemma, $\mathcal{T}''$ is triangulated equivalent to $\mathcal{T}/\ker j^*$ and hence, by Remark 1.9(2), j_* is automatically fully faithful. Now Lemma 1.11(b), parts (i) and (ii), applies and gives us also the lower part of the recollement,

$$\ker j^* \xrightarrow{\;\;i_*\;\;} \mathcal{T} \xrightarrow{\;\;j^*\;\;} \mathcal{T}''.$$

$\square$

1.3. An example. We will now give an example of a recollement arising from finite localization in stable homotopy theory:

Example 1.15. Throughout this example we will use the notions of stable homotopy category and localization as in [HPS97], see also Remark 1.9(3). Let $\mathcal{C}$ be a stable homotopy category with smash product $\wedge$, internal Hom-functor Hom, and unit $\mathbb{S}$. Recall that a generating set $\mathcal{G}$ is part of the data of $\mathcal{C}$. Suppose that $\mathcal{A}$ is an essentially small $\mathcal{G}$-ideal in $\mathcal{C}$, that is, $\mathcal{A}$ is a thick subcategory such that $G \wedge A \in \mathcal{A}$ whenever $G \in \mathcal{G}$ and

$A \in \mathcal{A}$. Let $\mathcal{D}$ denote the localizing ideal (i.e., localizing subcategory with $C \wedge D \in \mathcal{D}$ whenever $C \in \mathcal{C}$ and $D \in \mathcal{D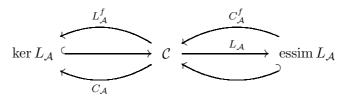}$) generated by $\mathcal{A}$. If all objects of $\mathcal{A}$ are compact, then there exists a localization functor $L_{\mathcal{A}}^f$ on $\mathcal{C}$ whose acyclics are precisely the objects of $\mathcal{D}$ [HPS97, Theorem 3.3.3]. This functor $L_{\mathcal{A}}^f$ is referred to as *finite localization away from* $\mathcal{A}$. Theorem 3.3.3 in [HPS97] also tells us that finite localization is always smashing, that is, the natural transformation $L_{\mathcal{A}}^f \mathbb{S} \wedge - \longrightarrow L_{\mathcal{A}}^f$ (which exists for every localization, cf. [HPS97, Lemma 3.3.1]) is an isomorphism. For the complementary colocalization $C_{\mathcal{A}}^f$ one then has an isomorphism $C_{\mathcal{A}}^f \cong C_{\mathcal{A}}^f \mathbb{S} \wedge -$. In particular, $L_{\mathcal{A}}^f$, resp. $C_{\mathcal{A}}^f$, has a right adjoint $C_{\mathcal{A}} = \operatorname{Hom}(C_{\mathcal{A}}^f \mathbb{S}, -)$, resp. $L_A = \operatorname{Hom}(C_{\mathcal{A}}^f \mathbb{S}, -)$.

Now suppose, in addition, all objects of $\mathcal{A}$ are strongly dualizable. This means, the natural map $\operatorname{Hom}(A, \mathbb{S}) \wedge C \longrightarrow \operatorname{Hom}(A, C)$ is an isomorphism for all $A \in \mathcal{A}$ and all $C \in \mathcal{C}$. Roughly speaking, an object A is strongly dualizable if mapping out of A is the same as smashing with the (Spanier-Whitehead) dual of A. Under these assumptions, by [HPS97, Theorem 3.3.5], the right adjoint functors $L_{\mathcal{A}}$ and $C_{\mathcal{A}}$ form also a localization-colocalization pair such that

$$\ker L_{\mathcal{A}} = \operatorname{essim} C_{\mathcal{A}} = \ker C_{\mathcal{A}}^f = \operatorname{essim} L_{\mathcal{A}}^f = \mathcal{A}^{\perp},$$

(Note that the notation, which we have adopted from [HPS97], might be misleading: the acyclics of $L_{\mathcal{A}}$ are not the objects of $\mathcal{A}$ but those of $\mathcal{A}^{\perp}$.) We hence get a diagram

$$\ker L_{\mathcal{A}} \xhookrightarrow{\quad\quad} \mathcal{C} \xrightarrow{\quad L_{\mathcal{A}} \quad} \operatorname{essim} L_{\mathcal{A}}$$

with $L_{\mathcal{A}}^f$, $C_{\mathcal{A}}$ on the left and $C_{\mathcal{A}}^f$, $L_{\mathcal{A}}$ on the right.

consisting of two adjoint triples because a localization functor can be regarded as a left adjoint for the inclusion of the locals whereas a colocalization can be regarded as a right adjoint for the inclusion of the colocals. Using Proposition 1.14 we can conclude that this diagram is in fact a recollement.

2. Recollements of stable model categories

In this section, we will use some facts on model categories of modules. These are summarized in Section A.1 of the Appendix.

2.1. **Reasonable stable model categories.** Every pointed model category $\mathcal{C}$ supports a suspension functor $\Sigma : \operatorname{Ho}\mathcal{C} \longrightarrow \operatorname{Ho}\mathcal{C}$. This can, for example, be defined on objects by choosing a cofibrant replacement X^{cof} for X in $\mathcal{C}$ and a cone of X^{cof}, that is, a factorization

$$
\begin{array}{ccc}
X^{\operatorname{cof}} & \longrightarrow & * \quad . \\
\searrow & & \nearrow \sim \\
& C_{X^{\operatorname{cof}}} &
\end{array}
$$

The suspension ΣX is then defined as the cofiber of the cofibration $X^{\operatorname{cof}} \rightarrowtail C_{X^{\operatorname{cof}}}$, that is, the coequalizer of the diagram

$$X^{\operatorname{cof}} \underset{0}{\overset{}{\rightrightarrows}} C_{X^{\operatorname{cof}}} \quad,$$

where 0 denotes the unique map factoring through the zero object $*$. On the level of homotopy categories, this construction becomes a well-defined functor. Also note that ΣX is cofibrant. This is because the diagram

$$
\begin{array}{ccc}
X^{\operatorname{cof}} & \rightarrowtail & C_{X^{\operatorname{cof}}} \\
\downarrow & & \downarrow \\
* & \longrightarrow & \Sigma X
\end{array}
$$

is a pushout and cofibrations are preserved by pushouts. The model category is called *stable* if Σ is an equivalence. In this case, $\operatorname{Ho}\mathcal{C}$ is a triangulated category with coproducts where the suspension functor is just Σ. Instead of $\operatorname{Ho}\mathcal{C}(X, Y)$, we will usually write $[X, Y]^{\operatorname{Ho}\mathcal{C}}$ or simply $[X, Y]$ for the abelian group of all morphisms from X to Y. A Quillen functor between stable model categories induces a triangulated functor [Hov99, Proposition 6.4.1] on the level of homotopy categories, which is a triangulated equivalence if the Quillen functor is a Quillen equivalence.

Recall the following definition, see [Jør06, Definition 1.2].

Definition 2.1. An object X of a triangulated category $\mathcal{T}$ is *compact* if

$$\mathcal{T}(X, -) : \mathcal{T} \longrightarrow \operatorname{Ab}$$

preserves coproducts and *self-compact* if the restricted functor $\mathcal{T}(X, -)\mid_{\langle X \rangle}$ preserves coproducts.

Examples 2.2.
(1) Using a result of Neeman [Nee92, Lemma 2.2], one can show that for a ring R, the compact objects in $\mathsf{D}(R)$ are the perfect complexes, that is, the chain complexes which are quasi-isomorphic to a bounded complex of finitely generated projective R-modules [Sch04, Theorem 3.8].
(2) Let $F : \mathcal{T} \longrightarrow \mathcal{T}'$ be a functor between triangulated categories with coproducts. Suppose F preserves coproducts and is fully faithful. If C is a compact object in $\mathcal{T}$ then $F(C)$ is self-compact in $\mathcal{T}'$. To see this note that by Lemma 1.2(ii) any family $(X_i)_{i \in I}$ in $\langle F(C) \rangle$ is up to isomorphism of the form $(F(Y_i))_{i \in I}$ for $Y_i \in \langle C \rangle$. This helps to construct self-compact objects which are not necessarily compact. For example $\mathbb{Z}[\frac{1}{2}]$, the integers with 2 inverted, viewed as an object in $\mathsf{D}(\mathbb{Z})$ is self-compact but not compact [Jør06, Example 1.8].

In the following, we will consider 'reasonable' stable model categories.

Definition 2.3. By a *reasonable* stable model category we mean a stable closed symmetric monoidal model category $(\mathcal{C}, \wedge, \mathbb{S})$ which satisfies the following conditions:

(i) As a model category, $\mathcal{C}$ is cofibrantly generated [Hov99, Definition 2.1.17].
(ii) All objects of $\mathcal{C}$ are small in the sense of [SS00], that is, every object is κ-small with respect to some cardinal κ.
(iii) The monoid axiom holds for $(\mathcal{C}, \wedge, \mathbb{S})$ [SS00, Definition 3.3].
(iv) The unit $\mathbb{S}$ is cofibrant in $\mathcal{C}$ and a compact generator for $\operatorname{Ho}\mathcal{C}$.
(v) The smashing condition holds [SS00, Section 4], that is, for every monoid R in $\mathcal{C}$ and every cofibrant R-module X the functor $- \wedge_R X : \operatorname{Mod-}R \longrightarrow \mathcal{C}$ preserves weak equivalences.

In particular, all statements from Section A.1 (in the Appendix) hold for the case of reasonable stable model categories.
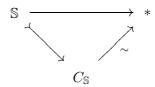
Examples 2.4. We are mainly interested in symmetric spectra and chain complexes. Both form reasonable stable model categories:

(1) Hovey, Shipley and Smith [HSS00] have shown that the category $\operatorname{Sp}^{\Sigma}$ of symmetric spectra with the stable model structure has a smash product $\wedge$ with unit the sphere spectrum $\mathbb{S}$ such that $(\operatorname{Sp}^{\Sigma}, \wedge, \mathbb{S})$ is a closed symmetric monoidal model category which is cofibrantly generated, has only small objects and satisfies the monoid axiom and the smashing condition. The sphere spectrum $\mathbb{S}$ is cofibrant and a compact generator. Hence symmetric spectra form a reasonable stable model category. Monoids in $(\operatorname{Sp}^{\Sigma}, \wedge, \mathbb{S})$ are called *symmetric ring spectra*.
(2) The category $\operatorname{Ch}(k)$ of unbounded chain complexes over some commutative ground ring k form a model category with weak equivalences the quasi-isomorphisms and fibrations the level-wise surjections [Hov99, Section 2.3]. Together with the tensor product and the chain complex $k[0]$ which is k concentrated in dimension 0 this is a reasonable stable model category $(\operatorname{Ch}(k), \otimes, k[0])$.

2.2. **Model categories enriched over a reasonable stable model category.** Let from now on $(\mathcal{C}, \wedge, \mathbb{S})$ be a fixed reasonable stable model category. The goal of this section is to prove Theorem 2.16, which gives a necessary and sufficient criterion for the existence of a recollement with middle term $\mathsf{D}(R)$, where R is a given monoid in $\mathcal{C}$. A $\mathcal{C}$-*model category* in the sense of [Hov99, Definition 4.2.18] is a model category $\mathcal{M}$ together with a Quillen bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ which is associative and unital up to natural and coherent isomorphism (to be precise, the natural coherent isomorphisms are part of the data of the $\mathcal{C}$-model category). In other words, $\mathcal{M}$ is enriched, tensored, and cotensored over $\mathcal{C}$ such that the tensor functor $\otimes$ satisfies the pushout product axiom [Hov99, Definition 4.2.1]. We will denote the enriched Hom-functor by $\operatorname{Hom}_{\mathcal{M}}$. Since $\mathcal{C}$ is stable, the tensor functor is usually denoted by $\wedge$. But to distinguish it from the monoidal product $\wedge$ in $\mathcal{C}$, we will here use $\otimes$. A $\operatorname{Sp}^{\Sigma}$-model category is usually called *spectral* model category.
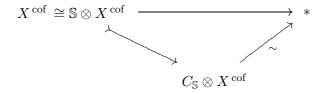
Lemma 2.5. *Every $\mathcal{C}$-model category is stable.*

Proof. Let $\mathcal{M}$ be a $\mathcal{C}$-model category. Note first that $\mathcal{M}$ is pointed since $\mathcal{C}$ is pointed. Namely if 0 denotes the initial and 1 the terminal object of $\mathcal{M}$ apply the left adjoint $- \otimes 1 : \mathcal{C} \longrightarrow \mathcal{M}$ to the map $\mathbb{S} \longrightarrow *$ in $\mathcal{C}$ and get a map $1 \longrightarrow 0$ in $\mathcal{M}$ which has to be an isomorphism.

We define the 1-sphere in $\mathcal{C}$ by $S^1 = \Sigma \mathbb{S}$ and claim that the suspension functor $\Sigma : \operatorname{Ho}\mathcal{M} \longrightarrow \operatorname{Ho}\mathcal{M}$ is isomorphic to $S^1 \otimes^L - : \operatorname{Ho}\mathcal{M} \longrightarrow \operatorname{Ho}\mathcal{M}$. (This left derived functor exists since $S^1 = \Sigma \mathbb{S}$ is cofibrant.) Consider the diagram

$$
\begin{array}{ccc}
\mathbb{S} & \longrightarrow & * \\
& \searrow \quad \nearrow_{\sim} & \\
& C_{\mathbb{S}} &
\end{array}
$$

in $\mathcal{C}$ and apply $- \otimes X^{\operatorname{cof}}$, where $X \in \mathcal{M}$. This is a left Quillen functor, so it preserves the cofibration $\mathbb{S} \rightarrowtail C_{\mathbb{S}}$ and the weak equivalence between the cofibrant objects $C_{\mathbb{S}}$ and $*$. Hence we get a diagram

$$
\begin{array}{ccc}
X^{\operatorname{cof}} \cong \mathbb{S} \otimes X^{\operatorname{cof}} & \longrightarrow & * \\
& \searrow \qquad \nearrow_{\sim} & \\
& C_{\mathbb{S}} \otimes X^{\operatorname{cof}} &
\end{array}
$$

from which we deduce that $\Sigma X \cong \operatorname{cofiber}(\mathbb{S} \otimes X^{\operatorname{cof}} \rightarrowtail C_{\mathbb{S}} \otimes X^{\operatorname{cof}})$ in $\operatorname{Ho}\mathcal{M}$. Now $- \otimes X^{\operatorname{cof}}$ preserves cofibers. Thus we have natural isomorphisms

$$
\Sigma X \cong \operatorname{cofiber}(\mathbb{S} \rightarrowtail C_{\mathbb{S}}) \otimes X^{\operatorname{cof}} \cong \Sigma \mathbb{S} \otimes X^{\operatorname{cof}} \cong S^1 \otimes^L X
$$

in $\operatorname{Ho}\mathcal{M}$ proving our claim. Since $\mathcal{C}$ is stable we can choose a cofibrant object S^{-1} in $\mathcal{C}$ such that $S^1 \wedge S^{-1} \cong \mathbb{S}$ in $\operatorname{Ho}\mathcal{C}$. Then $S^{-1} \otimes^L - : \operatorname{Ho}\mathcal{M} \longrightarrow \operatorname{Ho}\mathcal{M}$ is a quasi-inverse for $\Sigma \cong S^1 \otimes^L - : \operatorname{Ho}\mathcal{M} \longrightarrow \operatorname{Ho}\mathcal{M}$. $\qquad\square$

As in Section A.1 of this paper we consider for a monoid R in a reasonable stable model category the model structure on R-Mod where the fibrations, resp. weak equivalences, are exactly the fibrations, resp. weak equivalences, of the underlying objects in $\mathcal{C}$, [SS00, Section 4]. We denote the homotopy category of R-Mod by $\mathsf{D}(R)$ and call it the *derived* category of R. Just as in the category $\mathcal{C}$ itself, one has the notion of modules in a $\mathcal{C}$-category $\mathcal{M}$ over a monoid R in $\mathcal{C}$.

Example 2.6. If T is a monoid in $\mathcal{C}$ then the category of right T-modules (in $\mathcal{C}$) is a $\mathcal{C}$-model category (and hence stable). The Quillen bifunctor is given by the three functors in (A.1) with $R = S = \mathbb{S}$ (the first functor is tensor, the second cotensor, and the third enrichment). Replacing T by T^{op} shows that left T-modules also form a $\mathcal{C}$-model category. A left module in the $\mathcal{C}$-model category Mod-T over another monoid R is the same as an R-T-bimodule.

Lemma 2.7. *Let R be a monoid in a reasonable stable model category $(\mathcal{C}, \wedge, \mathbb{S})$. Then R is a compact generator for $\mathsf{D}(R)$ and for $\mathsf{D}(R^{\mathrm{op}})$.*

Proof. It suffices to consider the case of $\mathsf{D}(R)$ since R and R^{op} are the same as modules. As $\mathbb{S}$ is cofibrant in $\mathcal{C}$ we have an isomorphism $R \wedge^L \mathbb{S} \cong R$ in $\mathsf{D}(R)$. The Quillen pair induced by extension and restriction of scalars gives us then an isomorphism

$$(2.8) \qquad\qquad [R, X]^{\mathsf{D}(R)} \cong [\mathbb{S}, X]^{\mathrm{Ho}\,\mathcal{C}},$$

which is natural in $X \in \mathsf{D}(R)$. Using this we get

$$\bigoplus_{i \in I} [R, X_i]^{\mathsf{D}(R)} \cong \bigoplus_{i \in I} [\mathbb{S}, X_i]^{\mathrm{Ho}\,\mathcal{C}} \cong \Big[\mathbb{S}, \coprod_{i \in I} X_i\Big]^{\mathrm{Ho}\,\mathcal{C}} \cong \Big[R, \coprod_{i \in I} X_i\Big]^{\mathsf{D}(R)}$$

for any family $(X_i)_{i \in I}$ of objects in $\mathsf{D}(R)$, which shows the compactness of R.

For compact objects, one has the following characterization of being a generator (see [SS03b, Lemma 2.2.1]). A compact object P is a generator for a triangulated category $\mathcal{T}$ with coproducts if and only if P detects if objects are trivial, that is, $X \cong 0$ in $\mathcal{T}$ if and only if $\mathcal{T}(P, \Sigma^n X) = 0$ for all $n \in \mathbb{Z}$. Let $[R, \Sigma^n X]^{\mathsf{D}(R)} = 0$ for all $n \in \mathbb{Z}$. Using again the isomorphism (2.8),

$$0 = [R, \Sigma^n X]^{\mathsf{D}(R)} \cong [\mathbb{S}, \Sigma^n X]^{\mathrm{Ho}\,\mathcal{C}}$$

which implies $X \cong 0$ because $\mathbb{S}$ is a generator for $\mathcal{C}$. Thus R is a generator. $\qquad\square$

2.3. A Quillen pair. Let $\mathcal{M}$ be a $\mathcal{C}$-model category and B a cofibrant and fibrant object in $\mathcal{M}$. Then $E := \mathrm{Hom}_{\mathcal{M}}(B, B)$ is a monoid in $\mathcal{C}$ and there is an action $E \otimes B \longrightarrow B$ of E on B given by the adjoint map of the identity $E \xrightarrow{=} \mathrm{Hom}_{\mathcal{M}}(B, B)$ giving B a left E-module structure.

Theorem 2.9. *Suppose that $\mathcal{M}$ is a $\mathcal{C}$-model category and B a cofibrant and fibrant object in $\mathcal{M}$.*

(i) *There is a Quillen pair*

$$\mathrm{Mod}\text{-}E \; \underset{\mathrm{Hom}_{\mathcal{M}}(B,-)}{\overset{-\otimes_E B}{\rightleftarrows}} \; \mathcal{M}.$$

(ii) *If B is self-compact in $\mathrm{Ho}\,\mathcal{M}$ the restriction $i^!|_{\langle B \rangle}$ of the triangulated functor*

$$i^! = \mathrm{RHom}_{\mathcal{M}}(B, -) : \mathrm{Ho}\,\mathcal{M} \longrightarrow \mathsf{D}(E^{\mathrm{op}})$$

preserves coproducts.

(iii) *If B is self-compact in $\mathrm{Ho}\,\mathcal{M}$ the triangulated functor*

$$i_* = - \otimes_E^L B : \mathsf{D}(E^{\mathrm{op}}) \longrightarrow \mathrm{Ho}\,\mathcal{M}$$

is fully faithful and has essential image

$$\mathrm{essim}\, i_* = \langle B \rangle.$$

Proof. This theorem is a variant of [SS03b, Theorem 3.9.3], in which spectral categories, i.e., Sp^Σ-categories are considered, but in the proof only those properties of Sp^Σ are required which every reasonable stable model category possesses. Moreover, self-compact objects have not been considered in [SS03b]. That is why we have to modify the proof, especially for part (iii).

Part (i) is simply a 'one object version' of [SS03b, Theorem 3.9.3(i)]. For $A \in \mathcal{M}$, the object $\mathrm{Hom}_{\mathcal{M}}(B, A)$ of $\mathcal{C}$ has a canonical right action of $E = \mathrm{Hom}_{\mathcal{M}}(B, B)$ so that the functor $\mathrm{Hom}_{\mathcal{M}}(-, B)$ takes values in Mod-E. If X is a right E-module then $X \otimes_E B$ is defined as the coequalizer in $\mathcal{M}$ of

$$(X \wedge E) \otimes B \;\rightrightarrows\; X \otimes B,$$

where one map is induced by the right action of E on X and the other by the associativity isomorphism $(X \wedge E) \otimes B \cong X \otimes (E \otimes B)$ and the left action of E on B.

For part (ii) one has to check that for any family $(A_j)_{j \in J}$ of objects in $\langle B \rangle$ the canonical map $\coprod_j \mathrm{RHom}_{\mathcal{M}}(B, A_j) \longrightarrow \mathrm{RHom}_{\mathcal{M}}(B, \coprod_j A_j)$ is an isomorphism, or equivalently, the induced map

$$(2.10) \qquad \left[X, \coprod_{j \in J} \mathrm{RHom}_{\mathcal{M}}(B, A_j) \right]^{\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})} \longrightarrow \left[X, \mathrm{RHom}_{\mathcal{M}}(B, \coprod_{j \in J} A_j) \right]^{\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})}$$

is a natural isomorphism for every X in $\mathsf{D}(E^{\,\mathrm{op}})$. But those X for which the map (2.10) is an isomorphism for all (!) families $(A_j)_{j \in J}$ of objects in $\langle B \rangle$ form a localizing triangulated subcategory of $\mathsf{D}(E^{\,\mathrm{op}})$. The right E-module E is contained in this subcategory – to see this, use the compactness of E in $\mathsf{D}(E^{\,\mathrm{op}})$ (Lemma 2.7), the derived adjunction of the Quillen pair from part (i), and the self-compactness of B in $\mathrm{Ho}\,\mathcal{M}$ Since E is a generator for $\mathsf{D}(E^{\,\mathrm{op}})$ it now follows that the map (2.10) is always an isomorphism.

For part (iii) the proof of [SS03b, Theorem 3.9.3(ii)] must be rearranged. The point is that our B is only self-compact, not necessarily compact. We will give the details of the proof, the order is as follows.

(a) $\mathrm{essim}\, i_* \subset \langle B \rangle$
(b) i_* is fully faithful.
(c) $\mathrm{essim}\, i_* \supset \langle B \rangle$

Note that i_*, as a left adjoint, preserves coproducts. Part (a) follows from Lemma 1.2(i) since E is a generator for $\mathsf{D}(E^{\,\mathrm{op}})$ and $E \otimes_E^L B \cong B$ in $\mathrm{Ho}\,\mathcal{M}$. Part (a) implies that i_* maps coproducts in $\mathsf{D}(E^{\,\mathrm{op}})$ to coproducts in $\langle B \rangle$. By part (ii) of this theorem, $i^! |_{\langle B \rangle}$ preserves coproducts as well and so does the composition $i^! i_*$. A left adjoint is fully faithful if and only if the unit of the adjunction is an isomorphism. Hence consider for X in Mod-E the unit

$$(2.11) \qquad\qquad\qquad\qquad X \longrightarrow i^! i_* X,$$

which is a natural transformation between coproduct preserving triangulated functors. By Lemma 1.3 the unit will be an isomorphism if it is so for $X = E$. Since B is fibrant by assumption the unit is in this case the isomorphism

$$E = \mathrm{Hom}_{\mathcal{M}}(B, B) \cong \mathrm{RHom}_{\mathcal{M}}(B, B) \cong \mathrm{RHom}_{\mathcal{M}}(B, E \otimes_E^L B) = i^! i_* E.$$

This shows (b). Now part (c) follows from Lemma 1.2 (ii) because i_* is full by part (b). $\square$

Let R be a monoid in $\mathcal{C}$ and C a cofibrant left R-module which is compact in $\mathsf{D}(R)$. Set $F = \mathrm{Hom}_R(C, C)$ so that C is a left $(F \wedge R)$-module. By Lemma A.4 we get a Quillen pair

$$R\text{-Mod} \;\xleftrightarrows[\mathrm{Hom}_R(C,-)]{-\wedge_F C}\; \text{Mod-}F,$$

whose derived adjunction we will denote by

$$\mathsf{D}(R) \;\xleftrightarrows[j^*]{j_!}\; \mathsf{D}(F^{\mathrm{op}}).$$

Our goal is now to show that the right adjoint j^* has itself a right adjoint j_*. The idea is to imitate [Jør06, Setup 2.1] where a 'dual' module C^* of C is defined such that j^*, the right derived of mapping out of C, is the same as the left derived of tensoring with the dual C^*. To carry this out we must make the technical assumption that R is cofibrant in $\mathcal{C}$. Note that C being a cofibrant R-module also implies that

$$R\text{-Mod-}R \;\xleftrightarrows[\mathrm{Hom}_R(C,-)]{-\wedge_F C}\; \text{Mod-}(F \wedge R),$$

is a Quillen pair (Lemma A.4). Hence we can define

$$C^* = \mathrm{Hom}_R(C, R^{\mathrm{fib}})^{\mathrm{cof}} \quad \text{in} \quad \text{Mod-}(F \wedge R).$$

Here R^{fib} denotes the fibrant replacement of R in R-Mod-R which comes from the functorial factorization, in particular, the weak equivalence $R \xrightarrow{\sim} R^{\mathrm{fib}}$ is also a cofibration (we will need this in Proposition 2.12). Similarly, $\mathrm{Hom}_R(C, R^{\mathrm{fib}})^{\mathrm{cof}}$ is the cofibrant replacement of $\mathrm{Hom}_R(C, R^{\mathrm{fib}})$ in Mod-$(F \wedge R)$.

Proposition 2.12. *Suppose that R is cofibrant and a monoid in $\mathcal{C}$, C is a cofibrant and fibrant left R-module which is compact in $\mathsf{D}(R)$, and let F and C^* be defined as above. Then there are triangulated functors*

$$\mathsf{D}(R) \;\xrightarrow{\ j^*\ }\; \mathsf{D}(F^{\mathrm{op}})$$

given by $j_! = - \wedge_F^L C$, $j^* = \mathrm{RHom}_R(C, -)$, $j_* = \mathrm{RHom}_F(C^*, -)$ such that
  (i) $j^* \cong C^* \wedge_R^L -$,
  (ii) $(j_!, j^*, j_*)$ *is an adjoint triple,*
  (iii) $j_!$ *and j_* are fully faithful.*

Proof. Note first that we have a Quillen pair

$$R\text{-Mod} \;\xleftrightarrows[\mathrm{Hom}_F(C^*,-)]{C^*\wedge_R -}\; \text{Mod-}F$$

since C^* is cofibrant as a right $(F \wedge R)$-module.

Ad (i). The compactness of C implies that $j^* = \mathrm{RHom}_R(C, -)$ preserves coproducts. (This can be proved in the same manner as Theorem 2.9(ii).) Hence it suffices to give a natural transformation between the functors j^* and $C^* \wedge_R^L -$ which is an isomorphism in $\mathsf{D}(F^{\mathrm{op}})$ for the generator R (Lemma 1.3). For X in R-Mod a map

$$(2.13) \qquad\qquad C^* \wedge_R^L X \longrightarrow \mathrm{RHom}_R(C, X)$$

in $\mathsf{D}(F^{\mathrm{op}})$ corresponds via adjunction to a map

$$(2.14) \qquad\qquad (C^* \wedge_R^L X) \wedge_F^L C \longrightarrow X$$

in $\mathsf{D}(R)$. For X^{cof} the cofibrant replacement of X in R-Mod we have isomorphisms

$$(C^* \wedge_R^L X) \wedge_F^L C \cong (C^* \wedge_R X^{\mathrm{cof}}) \wedge_F C \cong (C^* \wedge_F C) \wedge_R X^{\mathrm{cof}}$$

in $\mathsf{D}(R)$. Using the bimodule map $C^* = \mathrm{Hom}_R(C, R^{\mathrm{fib}})^{\mathrm{cof}} \longrightarrow \mathrm{Hom}_R(C, R^{\mathrm{fib}})$ (which is a weak equivalence, but we do not need that here) and the evaluation

$$\mathrm{Hom}_R(C, R^{\mathrm{fib}}) \wedge_F C \longrightarrow R^{\mathrm{fib}}$$

we get a map

$$(C^* \wedge_F C) \wedge_R X^{\mathrm{cof}} \longrightarrow R^{\mathrm{fib}} \wedge_R X^{\mathrm{cof}}.$$

The functor $- \wedge_R X^{\mathrm{cof}} : R\text{-Mod-}R \longrightarrow R\text{-Mod}$ is left Quillen because R being cofibrant in $\mathcal{C}$ implies that X^{cof} is cofibrant in $\mathcal{C}$ by Corollary A.6. Hence smashing with X^{cof} preserves the trivial cofibration $R \longrightarrow R^{\mathrm{fib}}$ and we get

$$R^{\mathrm{fib}} \wedge_R X^{\mathrm{cof}} \cong R \wedge_R X^{\mathrm{cof}} \cong X^{\mathrm{cof}} \cong X$$

in $\mathsf{D}(R)$. Altogether this defines a natural map (2.14) and via adjunction the desired map (2.13). Setting $X = R$ gives us the isomorphism $C^* \wedge_R^L R \cong C^* \cong \mathrm{RHom}_R(C, R)$ in $\mathsf{D}(F^{\mathrm{op}})$.

Ad (ii). We know already that $(j_!, j^*)$ and $(C^* \wedge_R^L -, j_*)$ are adjoint pairs. Hence it follows from part (i) of this proposition that $(j_!, j^*, j_*)$ is an adjoint triple.

Ad (iii). In the homotopy category, C is compact and in particular self-compact. Furthermore, C is cofibrant and fibrant by assumption. Hence $j_!$ is fully faithful by Theorem 2.9. The fact that the right adjoint j_* of j^* is also fully faithful can be deduced formally from Proposition 1.14. Alternatively, we check that the counit of the adjoint pair (j^*, j_*) is an isomorphism. Let X be a right F-module. We show that X and $j^* j_* X = \mathrm{RHom}_R(C, \mathrm{RHom}_F(C^*, X))$ represent the same contravariant functor on $\mathsf{D}(F^{\mathrm{op}})$. For $Z \in \mathrm{Mod}\text{-}F$ we have natural isomorphisms

$$
\begin{aligned}
[Z, \mathrm{RHom}_R(C, \mathrm{RHom}_F(C^*, X))]^{\mathrm{Ho}\, F^{\mathrm{op}}} &\cong [Z \wedge_F^L C, \mathrm{RHom}_F(C^*, X)]^{\mathsf{D}(R)} \\
&\cong [(C^* \wedge_R^L (Z \wedge_F^L C), X]^{\mathsf{D}(F^{\mathrm{op}})} \\
&\cong [\mathrm{RHom}_R(C, Z \wedge_F^L C), X]^{\mathsf{D}(F^{\mathrm{op}})} \\
&\cong [Z, X]^{\mathsf{D}(F^{\mathrm{op}})}.
\end{aligned}
$$

The first and second are adjunction isomorphisms, the third uses part (i) of this proposition and the last is induced by the unit $Z \longrightarrow \mathrm{RHom}_R(C, Z \wedge_F^L C)$ of the adjoint pair $(j_!, j^*)$, which is an isomorphism because $j_!$ is faithful. This gives an isomorphism $j^* j_* X \xrightarrow{\cong} X$ which is indeed the counit. $\qquad\square$

Lemma 2.15. *Suppose we are in the situation of Proposition 2.12. Then*

$$C^\perp = \ker j^*.$$

Proof. Let $X \in C^\perp$. We want to show $X \in \ker j^*$, that is, $\mathrm{RHom}_R(C, X) \cong 0$ in $\mathsf{D}(F^{\mathrm{op}})$. Let us again use the characterization for generators of triangulated categories with coproducts given in [SS03b, Lemma 2.2.1]. Since F is a generator for $\mathsf{D}(F^{\mathrm{op}})$ it suffices to show that $\mathrm{RHom}_R(C, X) \in F^\perp$. One has isomorphisms

$$
\begin{aligned}
\left[\Sigma^n F, \mathrm{RHom}_R(C, X)\right]^{\mathsf{D}(F^{\mathrm{op}})} &\cong \left[\Sigma^n F \wedge^L_F C, X\right]^{\mathsf{D}(R)} \\
&\cong \left[\Sigma^n(F \wedge^L_F C), X\right]^{\mathsf{D}(R)} \\
&\cong \left[\Sigma^n C, X\right]^{\mathsf{D}(R)} \\
&\cong 0,
\end{aligned}
$$

where the last one is because of our assumption, $X \in C^\perp$. This shows $C^\perp \subset \ker j^*$.

If, on the other hand, $X \in \ker j^*$ we have

$$
\begin{aligned}
\left[\Sigma^n C, X\right]^{\mathsf{D}(R)} &\cong \left[C, \Sigma^{-n} X\right]^{\mathsf{D}(R)} \\
&\cong \left[F \wedge^L_F C, \Sigma^{-n} X\right]^{\mathsf{D}(R)} \\
&\cong \left[F, \mathrm{RHom}_R(C, \Sigma^{-n} X)\right]^{\mathsf{D}(F^{\mathrm{op}})} \\
&\cong \left[F, \Sigma^{-n} \mathrm{RHom}_R(C, X)\right]^{\mathsf{D}(F^{\mathrm{op}})} \\
&\cong 0,
\end{aligned}
$$

where the last isomorphism is induced by $\mathrm{RHom}_R(C, X) \cong 0$ in $\mathsf{D}(F^{\mathrm{op}})$. Hence $X \in C^\perp$ and the proof is complete. □

2.4. **The main theorem.** We are now able to prove the main theorem which gives a necessary and sufficient criterion for the existence of a certain recollement.

Theorem 2.16. *Let R be a monoid in the reasonable stable model category $\mathcal{C}$ and let B and C be left R-modules. Then the following are equivalent.*

  (i) *There is a recollement*

  where S and T are monoids in $\mathcal{C}$ such that $i_*(S) \cong B$ and $j_!(T) \cong C$.
  (ii) *In the derived category $\mathsf{D}(R)$, the module B is self-compact, C is compact, $B^\perp \cap C^\perp = 0$, and $B \in C^\perp$.*

Proof. (i) $\Rightarrow$ (ii) For this implication, the proof of [Jør06, Theorem 3.4, (i) $\Rightarrow$ (ii)] can be translated literally. The details are as follows. By definition, the triangulated functor i_*

is fully faithful and, as a left adjoint, preserves coproducts. Since S is compact in $\mathsf{D}(S)$ by Lemma 2.7, we can conclude that $B \cong i_*(S)$ is self-compact in $\mathsf{D}(R)$ (cf. Example 2.2(2)). To see the compactness of $C \cong j_!(T)$ use the adjunction isomorphism

$$[j_!(T), -]^{\mathsf{D}(R)} \cong [T, j^*(-)]^{\mathsf{D}(T)},$$

the fact that j^* preserves coproducts (as a left adjoint), and the compactness of T in $\mathsf{D}(T)$.

For $X \in B^\perp \cap C^\perp$ we have

$$0 \cong [\Sigma^n B, X]^{\mathsf{D}(R)} \cong [\Sigma^n i_*(S), X]^{\mathsf{D}(R)} \cong [\Sigma^n S, i^! X]^{\mathsf{D}(S)}$$

and

$$0 \cong [\Sigma^n C, X]^{\mathsf{D}(R)} \cong [\Sigma^n j_!(T), X]^{\mathsf{D}(R)} \cong [\Sigma^n T, j^* X]^{\mathsf{D}(T)}$$

for each $n \in \mathbb{Z}$, and this implies by [SS03b, Lemma 2.2.1] $i^! X = 0$ and $j^* X = 0$. Using the exact triangle in Definition 1.4(iv)(b) we can conclude that $X = 0$ and thus we get $B^\perp \cap C^\perp = 0$.

It remains to show that $B \in C^\perp$. For each $n \in \mathbb{Z}$ we have

$$[\Sigma^n C, B]^{\mathsf{D}(R)} \cong [\Sigma^n j_!(T), i_*(S)]^{\mathsf{D}(R)} \cong [\Sigma^n T, j^* i_*(S)]^{\mathsf{D}(T)} = 0 \,,$$

where the last equation holds because of $j^* i_* = 0$.

(ii) $\Rightarrow$ (i) We want to use Proposition 2.12 and therefore need a monoid in $\mathcal{C}$ which is cofibrant in $\mathcal{C}$. We can take a cofibrant replacement R^{cof} of our monoid R in the model category of monoids in $\mathcal{C}$ using the model structure of [SS00, Theorem 4.1(3)]. The same theorem tells us that R^{cof} is cofibrant in $\mathcal{C}$. Schwede and Shipley have shown that whenever the smashing condition is satisfied (see Definition 2.3; we need the smashing condition only for this application) a weak equivalence between monoids (in our case the weak equivalence $R^{\mathrm{cof}} \xrightarrow{\sim} R$) induces a Quillen equivalence between the module categories and in particular a triangulated equivalence between the homotopy categories [SS00, Theorem 4.3]. The properties 'compactness' and 'self-compactness' as well as the 'generating condition' in part (ii) of our theorem are of course preserved by triangulated equivalences. Hence we can without loss of generality assume that R is cofibrant in $\mathcal{C}$ (Remark 1.5(5)). We can also assume that C is cofibrant and fibrant in R-Mod (otherwise we take a cofibrant and fibrant replacement).

Now the proof of [Jør06, Theorem 3.4, (ii) $\Rightarrow$ (i)] can be imitated. Let $E = \mathrm{Hom}_R(B, B)$ and $F = \mathrm{Hom}_R(C, C)$. We apply Proposition 2.12 and get the right part

$$(2.17) \qquad \mathsf{D}(R) \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\hspace{1.2cm} j^* \hspace{1.2cm}}}}}} \mathsf{D}(F^{\mathrm{op}})$$

of a recollement. Let ι denote the inclusion of $C^\perp$ in $\mathsf{D}(R)$. By Lemma 2.15, $C^\perp$ is just the kernel of j^*. Hence we can use Proposition 1.14 to complete the diagram (2.17) into

a recollement

$$(2.18) \qquad C^{\perp} \xrightarrow{\quad \iota \quad} \mathsf{D}(R) \xrightarrow{\quad j^* \quad} \mathsf{D}(F^{\mathrm{op}}) \quad .$$

Our next claim is

$$(2.19) \qquad\qquad\qquad C^{\perp} = \langle B \rangle.$$

Since $B \in C^{\perp}$ by assumption, $\langle B \rangle$ is contained in the localizing subcategory $C^{\perp}$. Now let X be in $C^{\perp}$. We apply Theorem 2.9 to the case $\mathcal{M} = R$-Mod and consider the counit $\varepsilon_X : i_* i^! X \longrightarrow X$ of the adjunction

$$\mathsf{D}(E^{\mathrm{op}}) \xleftarrow[\;\; i^! \;\;]{\;\; i_* \;\;} \mathsf{D}(R) \quad .$$

As for any adjunction, we have a commutative diagram

$$i^! X \xrightarrow[\cong]{\eta_{i^! X}} i^! i_* i^! X$$

$$\searrow{\scriptstyle =} \qquad \downarrow{\scriptstyle i^!(\varepsilon_X)}$$

$$i^! X \, .$$

Here the unit $\eta_{i^! X}$ is an isomorphism since the left adjoint i_* is fully faithful by Theorem 2.9(iii). Hence $i^!(\varepsilon_X)$ is also an isomorphism. If we extend ε_X to an exact triangle

$$i_* i^! X \xrightarrow{\;\varepsilon_X\;} X \longrightarrow Y \longrightarrow \Sigma i_* i^! X$$

and apply the triangulated functor $i^!$ we can conclude that $\mathrm{RHom}_R(B, Y) = i^! Y = 0$. This implies $Y \in B^{\perp}$ because

$$[\Sigma^n B, Y]^{\mathsf{D}(R)} \cong [E \otimes_E^L B, \Sigma^{-n} Y]^{\mathsf{D}(R)} \cong [E, \Sigma^{-n} \, \mathrm{RHom}_R(B, Y)]^{\mathsf{D}(E^{\mathrm{op}})} = 0.$$

Moreover, the first term $i_* i^! X$ of the exact triangle is in the essential image of i_*, which is by Theorem 2.9(iii) equal to $\langle B \rangle$. But we have already shown that $\langle B \rangle \subset C^{\perp}$, so $i_* i^! X$ is in $C^{\perp}$, as the middle term X of the exact triangle is by assumption. It follows that Y is also in $C^{\perp}$ and hence in $C^{\perp} \cap B^{\perp} = 0$. But $Y = 0$ implies that $\varepsilon_X : i_* i^! X \longrightarrow X$ is an isomorphism. Thus X is in $\mathrm{essim}\, i_*$, which is the same as $\langle B \rangle$ by Theorem 2.9(iii). This proves equation (2.19).

Since $i_* : \mathsf{D}(E^{\mathrm{op}}) \longrightarrow \mathsf{D}(R)$ is fully faithful it restricts to an equivalence of triangulated categories

$$i_* : \mathsf{D}(E^{\mathrm{op}}) \xrightarrow{\;\simeq\;} \mathrm{essim}\, i_* = C^{\perp}.$$

Composing this equivalence with the recollement (2.18) yields a recollement

$$
\mathsf{D}(E^{\,\mathrm{op}}) \xrightarrow{\ i_* \ } \mathsf{D}(R) \xrightarrow{\ j^* \ } \mathsf{D}(F^{\,\mathrm{op}})
$$

with $i_*(E) = E \otimes_E^L B \cong B$ and $j^!(F) = F \otimes_F^L C \cong C$. This completes the proof. $\qquad\square$

Example 2.20. Let $\mathcal{C}$ be the model category of chain complexes over $\mathbb{Z}$, $R = \mathbb{Z}$, $B = \mathbb{Z}[\frac{1}{2}]$ (i.e., the integers with 2 inverted, cf. Examples 2.2(2)), and $C = \mathbb{Z}/2$. Then it is verified in [Jør06, Example 3.5] that B is self-compact, C is compact, $B^{\perp} \cap C^{\perp} = 0$, and $B \in C^{\perp}$, so that Theorem 2.16 applies and yields a recollement for $\mathsf{D}(\mathbb{Z})$.

Remark 2.21. Let us consider two special cases.
(1) If $\mathcal{C}$ is the model category of chain complexes (over some commutative ground ring) Theorem 2.16 is just the same as Jørgensen's [Jør06, Theorem 3.4].
(2) Assume $B = 0$ in the theorem. Then statement (i) reads as follows: There is a triangulated equivalence

$$
\mathsf{D}(R) \underset{j^*}{\overset{j_!}{\longleftrightarrow}} \mathsf{D}(T)
$$

where T is a monoid in $\mathcal{C}$ such that $j_!(T) \cong C$. Namely, $i_*(S) \cong 0$ means we are in the degenerate case where S is zero, which implies that $\mathsf{D}(S)$ is zero. Then, by Remark 1.5(6), the recollement in (i) is the same as a triangulated equivalence j^* with inverse $j_!$.

Consider the second statement of Theorem 2.16. Of course, the trivial module is self-compact and $0^{\perp}$ is the whole category $\mathsf{D}(R)$. Since a compact C in $\mathsf{D}(R)$ is a generator if and only if $C^{\perp} = 0$, statement (ii) reads as: C is a compact generator for the derived category $\mathsf{D}(R)$.

Part 2. **Topological well generated categories**

We will characterize the topological well generated triangulated categories. Our main result, Theorem 4.7, states that a topological triangulated category is well generated if and only if it is a localization of the derived category of a spectral category (alias ring spectrum with several objects) such that the acyclics are generated by a set. This is a topological version of Porta's characterization of the algebraic well generated categories [Por], [Kel06, Theorem 3.9], as appropriate localizations of the derived categories of DG categories, that is, DG algebras with several objects.

In Section 3 we will recall some definitions from triangulated category theory (such as α-small, α-perfect, α-compact, well generated), which can all be found in [Nee01b], although sometimes stated in a different way. We make frequent use of the results in Neeman's book and we have decided to use Neeman's original definition of well generated categories although it is not as easily stated as Krause's characterization [Kra01]. It would be interesting to think about a proof of the classification theorem using Krause's characterization (which, as far as I know, Porta takes as a definition when he classifies the algebraic well generated categories). We will write down a proof for the fact that the class of well generated triangulated categories is stable under forming subcategories that are generated by a set of objects and localizations whose acyclics are generated by a set of objects, cf. Proposition 3.3.

Section 4 starts with giving the definitions of (symmetric) ring spectra with several objects ('spectral categories') and module spectra over such. The derived category of a spectral category is the homotopy category of its modules. We describe a Quillen pair defined by Schwede and Shipley in [SS03b] between a spectral model category $\mathcal{K}$ and the category of modules over some 'endomorphism' spectral category $\mathcal{E}$ (depending on the choice of a set $\mathcal{G}$ of certain objects in $\mathcal{K}$). The induced triangulated adjoint functors (Lemma 4.1) form the basis for the proof of the characterization theorem. We will define an appropriate homology functor from the derived category of $\mathcal{E}$ into the abelian category of $\mathcal{G}$-modules which reflects isomorphisms, i.e., the isomorphisms in the derived category of $\mathcal{E}$ are exactly the quasi-isomorphisms with respect to this homology functor. In Section 4.2, we give a brief sketch of the proof for the characterization theorem. The details are presented in Section 4.3.

The last section gives a lift of one implication of the main theorem from Section 4 to the level of model categories. Using Hirschhorn's existence theorem for Bousfield localizations [Hir03, Theorem 4.1.1], we show that a spectral model category which has a well generated homotopy category is Quillen equivalent (via a single Quillen functor) to a Bousfield localization of a model category of modules over some spectral category (Theorem 5.11).

3. Well generated categories

3.1. **Terminology.** Throughout this section we let $\mathcal{T}$ be a triangulated category with (arbitrary set-indexed) coproducts. Let α be an infinite cardinal. An α-*localizing* subcategory of $\mathcal{T}$ is a triangulated subcategory which is closed under α-coproducts, i.e., coproducts of strictly less than α objects. A triangulated subcategory is *localizing* if it is closed under all coproducts. For $\alpha \geq \aleph_1$, α-localizing subcategories are always *thick*, i.e., closed under direct summands [Nee01b, Remark 3.2.7]. We call a set $\mathcal{S}$ of objects in $\mathcal{T}$ a *weak generating* set for $\mathcal{T}$ if it is, up to isomorphism, closed under (de-)suspensions and any

object $T \in \mathcal{T}$ is zero if and only if $\mathcal{T}(S, T) = 0$ for all $S \in \mathcal{S}$. If $\mathcal{S}$ is a weak generating set for $\mathcal{T}$, then a map $X \longrightarrow Y$ in $\mathcal{T}$ is an isomorphism if and only if the induced map $\mathcal{T}(S, X) \longrightarrow \mathcal{T}(S, Y)$ is an isomorphism for all $S \in \mathcal{S}$. To see this, consider the cofiber Z of $X \longrightarrow Y$, which is zero if and only if $X \longrightarrow Y$ is an isomorphism, and use that $\mathcal{T}(S, -) : \mathcal{T} \longrightarrow \text{Ab}$ is homological, i.e., it maps the triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ to a long exact sequence of abelian groups. By $\langle \mathcal{S} \rangle$, resp. $\alpha\text{-loc}\langle \mathcal{S} \rangle$, we denote the smallest localizing, resp. α-localizing, subcategory of $\mathcal{T}$ which contains a given set of objects $\mathcal{S}$. If $\mathcal{T} = \langle \mathcal{S} \rangle$ then $\mathcal{S}$ is called a *generating* set for $\mathcal{T}$. A generating set closed under (de-)suspensions is also a weak generating set for $\mathcal{T}$. (The converse will hold if $\mathcal{S}$ is $\aleph_1$-perfect [Nee01b, Proposition 8.4.1]. For example, any set of compact generators is $\aleph_1$-perfect. We will give the definition of α-perfect below.)

Let α be an infinite cardinal. An object $T \in \mathcal{T}$ is α-*small* if any map $T \longrightarrow \coprod_{i \in I} X_i$ into an arbitrary coproduct in $\mathcal{T}$ factors through some sub-coproduct

$$T \longrightarrow \coprod_{i \in I'} X_i \hookrightarrow \coprod_{i \in I} X_i$$

with $|I'| < \alpha$, i.e., the cardinality of I' is strictly smaller than α.

A class $\mathcal{S}$ of objects in $\mathcal{T}$ is called α-*perfect* (for an infinite cardinal α) if it satisfies the following.

(i) $0 \in \mathcal{S}$

(ii) Any map $S \longrightarrow \coprod_{i \in I} T_i$ in $\mathcal{T}$ with $S \in \mathcal{S}$ and $|I| < \alpha$ factors as

$$(*) \qquad\qquad S \longrightarrow \coprod_{i \in I} S_i \xrightarrow{\coprod f_i} \coprod_{i \in I} T_i$$

  with $S_i \in \mathcal{S}$ and maps $f_i : S_i \longrightarrow T_i$ in $\mathcal{T}$.

(iii) If a composite such as $(*)$ vanishes, every map f_i can be factored as $S_i \xrightarrow{g_i} S_i' \xrightarrow{h_i} T_i$ with $S_i' \in \mathcal{S}$ such that the composite $S \longrightarrow \coprod_{i \in I} S_i \longrightarrow \coprod_{i \in I} S_i'$ already vanishes.

We will now give the definitions of 'α-compactly generated' and 'well generated'. The original definition is from Neeman's book [Nee01b, Definition 8.1.6 and Remark 8.1.7].

Definition 3.1. Let α be an infinite regular cardinal. A set of objects in a triangulated category with coproducts is called an α-*compact generating set* if it is a weak generating set which is α-perfect and contains only α-small objects.

An α-*compactly generated category* is a triangulated category with coproducts which admits an α-compact generating set. A triangulated category which is β-compactly generated for some infinite regular cardinal β is called *well generated*.

Let us recall some basic statements concerning smallness and compactness. The proofs can be found in [Nee01b, Chapters 3 and 4]. By $\mathcal{T}^{(\alpha)}$ we denote the triangulated subcategory of α-small objects in $\mathcal{T}$. It is a thick triangulated subcategory, which is α-localizing if α is regular (i.e., α cannot be written as the sum of less than α cardinals, all strictly smaller than α). For $\alpha \leq \beta$, we have $\mathcal{T}^{(\alpha)} \subset \mathcal{T}^{(\beta)}$. An object T is $\aleph_0$-small if and only if it is compact, that is, if and only if the covariant Hom-functor $\mathcal{T}(T, -) : \mathcal{T} \longrightarrow \text{Ab}$ commutes with coproducts.

Every triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ contains a unique maximal α-perfect class, denoted by $\mathcal{S}_\alpha$, which is a thick triangulated subcategory. We set $\mathcal{T}^\alpha = (\mathcal{T}^{(\alpha)})_\alpha$ and call the objects of this thick triangulated subcategory α-*compact*. Hence α-compact objects are in particular α-small. If α is regular, the α-compact objects form an α-localizing subcategory. For $\alpha \leq \beta$, we have $\mathcal{T}^\alpha \subset \mathcal{T}^\beta$. Note that any class of objects in a triangulated category with coproducts is $\aleph_0$-perfect (use that finite coproducts are also products). Hence $\aleph_0$-compact is the same as $\aleph_0$-small, which is the same as compact.

Remark 3.2. All objects of an α-compact generating set $\mathcal{S}$ are α-compact. Such an $\mathcal{S}$ is not only a weak generating set but also a generating set in the sense that $\mathcal{T} = \langle \mathcal{S} \rangle$. This is because $\mathcal{S}$ is in particular $\aleph_1$-perfect by [Nee01b, Lemma 4.2.1]. Note also that $\mathcal{T}$ is α-compactly generated if and only if the subcategory $\mathcal{T}^\alpha$ of α-compact objects has a small skeleton which is a weak generating set [Nee01b, Remark 8.4.3]. Such a skeleton is then an α-compact generating set for $\mathcal{T}$. If $\mathcal{T}$ is well generated then $\mathcal{T}^\beta$ is essentially small for all infinite β.

The following characterization of well generated triangulated categories, which is due to Krause [Kra01], is easier stated than Neeman's original definition. A triangulated category with coproducts is well generated if and only if there is a weak generating set $\mathcal{S}$ consisting of α-small objects for some cardinal α such that the following holds: given any set-indexed family of maps $X_i \longrightarrow Y_i$, $i \in I$, with the induced maps $\mathcal{T}(S, X_i) \longrightarrow \mathcal{T}(S, Y_i)$ being surjective for all $S \in \mathcal{S}$, then the induced map $\mathcal{T}(S, \coprod_{i \in I} X_i) \longrightarrow \mathcal{T}(S, \coprod_{i \in I} Y_i)$ is also surjective. Note that α can be chosen to be regular by enlarging it if necessary. One important property of well generated categories $\mathcal{T}$ is that they satisfy Brown representability (cf. [Nee01b, Proposition 8.4.2]). This means, every homological functor $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$ which maps coproducts to products is naturally isomorphic to $\mathcal{T}(-, X)$ for some X.

3.2. **Subcategories and localizations of well generated categories.** Of course, all compactly generated ($= \aleph_0$-compactly generated) triangulated categories are well generated. An example of a well generated category which is not compactly generated is the derived category of sheaves on a non-compact, connected manifold of dimension bigger than 1. This is discussed in [Nee01a]. A whole class of examples comes from the following proposition. For our definition of localization and the comparison to Neeman's definition see Definition 1.7 and Remark 1.9(2).

Proposition 3.3. *Let $\mathcal{T}$ be a well generated triangulated category and $\mathcal{T}'$ a localizing subcategory which is generated by a set of objects. Then $\mathcal{T}'$ is well generated.*

Furthermore, the quotient $\mathcal{T}/\mathcal{T}'$ is a localization of $\mathcal{T}$ (and has in particular honest Hom-sets), and $\mathcal{T}/\mathcal{T}'$ is also well generated.

This proposition does not appear in Neeman's book in this general form, so we will give a proof for it. Using the tools given in [Nee01b], this will not be hard although somewhat technical. Recall from Section 1 that the essential image essim F of a triangulated functor $F : \mathcal{T} \longrightarrow \mathcal{T}'$ is the full subcategory consisting of all objects which are, up to isomorphism, in the image of F. It is a triangulated subcategory if F is full, and a localizing subcategory

if, in addition, F is a coproduct preserving functor between triangulated categories with coproducts.

Proof of Proposition 3.3. Let $\mathcal{S}'$ be a set with $\mathcal{T}' = \langle \mathcal{S}' \rangle$. Since $\mathcal{T}$ is well generated, we have $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$, where α runs through all (infinite) regular cardinals [Nee01b, Proposition 8.4.2]. Hence

$$\coprod_{S' \in \mathcal{S}'} S' \in \mathcal{T}^{\alpha_1}$$

for some regular α_1. But $\mathcal{T}^{\alpha_1}$ is thick, so we have $\mathcal{S}' \subset \mathcal{T}^{\alpha_1}$. Since $\mathcal{T}$ is well generated, there exists an α_2-compact generating set $\mathcal{S}$ for some regular α_2. If we put $\alpha = \max(\aleph_1, \alpha_1, \alpha_2)$, we have

$$\mathcal{S} \subset \mathcal{T}^\alpha, \quad \mathcal{S}' \subset \mathcal{T}' \cap \mathcal{T}^\alpha, \quad \mathcal{T}' = \langle \mathcal{S}' \rangle, \quad \text{and} \quad \mathcal{T} = \langle \mathcal{S} \rangle.$$

That means we are in the situation assumed in [Nee01b, Theorem 4.4.9]. This theorem tells us that for all regular $\beta \geq \alpha$,

$$\beta\text{-loc}\langle \mathcal{S}' \rangle = (\mathcal{T}')^\beta, \quad \beta\text{-loc}\langle \mathcal{S} \rangle = \mathcal{T}^\beta,$$

and the canonical functor $\mathcal{T}^\beta / (\mathcal{T}')^\beta \longrightarrow \mathcal{T}/\mathcal{T}'$ factors over an equivalence

$$\mathcal{T}^\beta / (\mathcal{T}')^\beta \overset{\simeq}{\longrightarrow} (\mathcal{T}/\mathcal{T}')^\beta.$$

(We need $\alpha \geq \aleph_1$ only for this last equivalence.)

By [Nee01b, Proposition 3.2.5], $\beta\text{-loc}\langle \mathcal{S}' \rangle$ is essentially small for all infinite β. Consequently, $(\mathcal{T}')^\beta$ is also essentially small for all infinite β: namely, for all regular $\beta \geq \alpha$ we have $(\mathcal{T}')^\beta = \beta\text{-loc}\langle \mathcal{S}' \rangle$ and for an arbitrary infinite β there exists a regular $\beta' \geq \alpha$ with $\beta \leq \beta'$ and hence $(\mathcal{T}')^\beta \subset (\mathcal{T}')^{\beta'}$. Let $\text{Sk}(\mathcal{T}')^\beta$ denote a skeleton of $(\mathcal{T}')^\beta$. From $\mathcal{S}' \subset \beta\text{-loc}\langle \mathcal{S}' \rangle = (\mathcal{T}')^\beta$ for $\beta \geq \alpha$ we can deduce $\mathcal{T}' = \langle \mathcal{S}' \rangle \subset \langle \text{Sk}(\mathcal{T}')^\beta \rangle$, hence $\mathcal{T}' = \langle \text{Sk}(\mathcal{T}')^\beta \rangle$. This shows $\mathcal{T}'$ is β-compactly generated for $\beta \geq \alpha$ (cf. Remark 3.2) and thus well generated.

In particular, $\mathcal{T}'$ satisfies Brown representability. By [Nee01b, Proposition 9.1.19], this implies that $\mathcal{T}/\mathcal{T}'$ is a localization of $\mathcal{T}$ (and hence has honest Hom-sets [Nee01b, Remark 9.1.17]).

By Remark 3.2, since $\mathcal{T}$ is well generated, $\mathcal{T}^\beta$ is essentially small – and so is the quotient $(\mathcal{T}/\mathcal{T}')^\beta \simeq \mathcal{T}^\beta / (\mathcal{T}')^\beta$ for any regular $\beta \geq \alpha$ and hence for all infinite β. We now want to show that $\text{Sk}(\mathcal{T}/\mathcal{T}')^\beta$ generates $\mathcal{T}/\mathcal{T}'$ for $\beta \geq \alpha$. Let $q : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{T}'$ denote the canonical triangulated functor into the quotient. Since the functor $\mathcal{T}^\beta / (\mathcal{T}')^\beta \longrightarrow (\mathcal{T}/\mathcal{T}')^\beta$ is an equivalence, every object of $\text{Sk}(\mathcal{T}/\mathcal{T}')^\beta$ is, up to isomorphism, of the form $q(X)$ for some $X \in \text{Sk}\,\mathcal{T}^\beta$, so that we have $\langle \text{Sk}(\mathcal{T}/\mathcal{T}')^\beta \rangle = \langle q(\text{Sk}\,\mathcal{T}^\beta) \rangle$. Moreover, since $\mathcal{T}/\mathcal{T}'$ is a localization, the functor q preserves coproducts and we can apply Lemma 1.2. Hence we have $\langle q(\text{Sk}\,\mathcal{T}^\beta) \rangle \supset \text{essim}\, q \mid_{\langle \text{Sk}\,\mathcal{T}^\beta \rangle}$. Furthermore, $\langle \text{Sk}\,\mathcal{T}^\beta \rangle = \mathcal{T}$ because $\mathcal{T}$ is in particular β-compactly generated, and hence $\text{essim}\, q \mid_{\langle \text{Sk}\,\mathcal{T}^\beta \rangle} = \text{essim}\, q \mid_{\mathcal{T}} = \mathcal{T}/\mathcal{T}'$. Altogether, what we get is $\langle \text{Sk}(\mathcal{T}/\mathcal{T}')^\beta \rangle \supset \mathcal{T}/\mathcal{T}'$ and thus $\langle \text{Sk}(\mathcal{T}/\mathcal{T}')^\beta \rangle = \mathcal{T}/\mathcal{T}'$. This shows that $\mathcal{T}/\mathcal{T}'$ is well generated. $\square$

As an immediate consequence of Proposition 3.3, appropriate localizations of the homotopy category of (symmetric) spectra, which is compactly generated, are well generated and hence satisfy Brown representability. An example of a triangulated category which is not well generated is the opposite category of a compactly generated category. This

(and other 'non-examples') can be found in [Nee01b, Appendix E]. The older result of
Boardman [Boa70] which says that the stable homotopy category is not self-dual can be
regarded as a consequence of this fact.

    The following lemma will be useful in Section 4.

Lemma 3.4. *Let $\mathcal{T}$ be a well generated triangulated category and $\mathcal{W}$ a set of maps in $\mathcal{T}$
which is, up to isomorphism, closed under (de-)suspensions, let $\mathcal{S}$ be a set consisting of
one cofiber for each map in $\mathcal{W}$, and let $\mathcal{W}$-loc consist of all $X \in \mathcal{T}$ for which the induced
map $f^* : \mathcal{T}(B, X) \longrightarrow \mathcal{T}(A, X)$ is an isomorphism for all $f : A \longrightarrow B$ in $\mathcal{W}$.*

    Then there exists a localization of $\mathcal{T}$ with acyclics $\langle \mathcal{S} \rangle$ and locals $\mathcal{W}$-loc.

Proof. By Proposition 3.3, the quotient $\mathcal{T}/\langle \mathcal{S} \rangle$ is a localization and it has $\langle \mathcal{S} \rangle$ as subcat-
egory of acyclics. It remains to show that $\mathcal{W}$-loc is the subcategory of local objects. By
Lemma 1.11(b)(iv) it suffices to check that $\langle \mathcal{S} \rangle^\perp = \mathcal{W}$-loc. Actually, we will show that

(3.5) $$\mathcal{S}^\perp = \langle \mathcal{S} \rangle^\perp = \mathcal{W}\text{-loc}.$$

The inclusion $\mathcal{S}^\perp \supset \langle \mathcal{S} \rangle^\perp$ follows immediately from $\mathcal{S} \subset \langle \mathcal{S} \rangle$. Let us show $\langle \mathcal{S} \rangle^\perp \supset \mathcal{W}$-loc:
Given $X \in \mathcal{W}$-loc and $C \in \mathcal{S}$, there exists an exact triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow \Sigma A$$

with $f \in \mathcal{W}$. We apply $\mathcal{T}(-, X)$ to the triangle and get a long exact sequence

$$\cdots \longleftarrow \mathcal{T}(A, X) \xleftarrow{f^*} \mathcal{T}(B, X) \longleftarrow \mathcal{T}(C, X) \longleftarrow \mathcal{T}(\Sigma A, X) \xleftarrow{(\Sigma f)^*} \mathcal{T}(\Sigma B, X) \longleftarrow \cdots .$$

Since X is in $\mathcal{W}$-loc, the maps $(\Sigma^n f)^*$ are isomorphisms for all integers n. Hence
$\mathcal{T}(\Sigma^n C, X) = 0$ and thus $\mathcal{S} \subset {}^\perp(\mathcal{W}\text{-loc})$. Since ${}^\perp(\mathcal{W}\text{-loc})$ is a localizing subcategory,
it follows that $\langle \mathcal{S} \rangle \subset {}^\perp(\mathcal{W}\text{-loc})$, which is equivalent to $\langle \mathcal{S} \rangle^\perp \supset \mathcal{W}$-loc.

    It remains to check $\mathcal{W}$-loc $\supset \mathcal{S}^\perp$. Given $X \in \mathcal{S}^\perp$ and $f \in \mathcal{W}$, we let C be the cofiber of
$f : A \longrightarrow B$ so that we obtain a long exact sequence

$$\cdots \longleftarrow \mathcal{T}(\Sigma^{-1}C, X) \longleftarrow \mathcal{T}(A, X) \xleftarrow{f^*} \mathcal{T}(B, X) \longleftarrow \mathcal{T}(C, X) \longleftarrow \cdots .$$

Now C and $\Sigma^{-1}C$ are in $\mathcal{S}$ and X is in $\mathcal{S}^\perp$, hence we have $\mathcal{T}(C, X) = \mathcal{T}(\Sigma^{-1}C, X) = 0$.
This implies f^* is an isomorphism, which shows that X is in $\mathcal{W}$-loc. $\square$

4. Classification of topological well generated categories

4.1. **Spectral model categories versus model categories of modules.** In this sec-
tion, we will consider 'spectral categories' (or 'symmetric ring spectra with several objects')
and modules over such. From now on, when we say '(ring) spectrum' we will always mean
'symmetric (ring) spectrum'. Spectral categories are $\mathcal{C}$-categories as discussed in Sec-
tion A.2, where $\mathcal{C}$ is now the closed symmetric monoidal model category $(\mathrm{Sp}^\Sigma, \wedge, \mathbb{S})$ of
symmetric spectra. This means, a spectral category $\mathcal{R}$ consists of a set of objects and
for any two objects R and R' in $\mathcal{R}$ there is a Hom-spectrum $\mathcal{R}(R, R')$ together with an
identity 'element' $\mathbb{S} \longrightarrow \mathcal{R}(R, R)$ for each R in $\mathcal{R}$ and composition morphisms

$$\mathcal{R}(R', R'') \wedge \mathcal{R}(R, R') \longrightarrow \mathcal{R}(R, R'')$$

which are associative and unital with respect to the identity elements [SS03b, Section 3.3].

A *right module* over a spectral category is a spectral functor

$$X : \mathcal{R}^{\mathrm{op}} \longrightarrow \mathrm{Sp}^{\Sigma}.$$

This means, X is a family of spectra $X(R)$, $R \in \mathcal{R}$, together with maps of spectra

$$\mathcal{R}(R, R') \longrightarrow \mathrm{Hom}_{\mathrm{Sp}^{\Sigma}}\left(X(R'), X(R)\right)$$

which are compatible with composition and identities. By adjunction, these maps correspond to a right action of $\mathcal{R}$ on X, i.e., maps of spectra

$$X(R') \wedge \mathcal{R}(R, R') \longrightarrow X(R)$$

which are associative and unital. The category Mod-$\mathcal{R}$ over a spectral category $\mathcal{R}$ has as objects right $\mathcal{R}$-modules, and a morphism $X \longrightarrow Y$ of $\mathcal{R}$-modules is family of maps of spectra $X(R) \longrightarrow Y(R)$ which are compatible with the action of $\mathcal{R}$. Note that a spectral category which consists only of one object is the same as a ring spectrum and the modules are just ordinary modules as considered in Part 1 of this paper.

As in the 'one object version', the category Mod-$\mathcal{R}$ is a spectral (and then by Lemma 2.5 also a stable) model category where maps are weak equivalences (resp. fibrations) if and only if they are objectwise weak equivalences (resp. fibrations) of symmetric spectra in the stable model structure [SS03b, Theorem A.1.1]. The homotopy category of Mod-$\mathcal{R}$ will be denoted by $\mathsf{D}(\mathcal{R}^{\mathrm{op}})$ and we call it the *derived category* of $\mathcal{R}$ because it is the topological analog of the derived category of a DG category. The *free* modules $F_R = \mathcal{R}(-, R) : \mathcal{R}^{\mathrm{op}} \longrightarrow \mathrm{Sp}^{\Sigma}$, for $R \in \mathcal{R}$, form a set of compact generators for $\mathsf{D}(\mathcal{R}^{\mathrm{op}})$ [SS03b, Theorem A.1.1].

Let $\mathcal{K}$ be a spectral model category, that means the model category $\mathcal{K}$ is enriched, tensored and cotensored over Sp^{Σ} and the tensor functor satisfies the pushout product axiom [Hov99, Definition 4.2.1]. Recall that we considered $\mathcal{C}$-model categories for a more general $\mathcal{C}$ in Section 2.2. Let $\mathcal{G}$ be a set of cofibrant and fibrant objects in $\mathcal{K}$ and let $\mathcal{E}$ be the full spectral subcategory of $\mathcal{K}$ with objects $\mathcal{G}$. In [SS03b, Section 3.9] Schwede and Shipley define a spectral Quillen adjunction

$$\mathcal{K} \xrightarrow[\mathrm{Hom}(\mathcal{G}, -)]{\overset{- \wedge_{\mathcal{E}} \mathcal{G}}{\xleftarrow{\hspace{2cm}}}} \mathrm{Mod}\text{-}\mathcal{E},$$

where for $A \in \mathcal{K}$, the value of the right adjoint is given by $\mathrm{Hom}(\mathcal{G}, A) : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathrm{Sp}^{\Sigma}$, $G \mapsto \mathrm{Hom}_{\mathcal{K}}(G, A)$. The left adjoint is defined by an appropriate coequalizer [SS03b, Theorem 3.9.3]. (We considered a one object version of this Quillen pair in Section 2.3 over a more general category $\mathcal{C}$.) We have $\mathrm{Hom}(\mathcal{G}, G) = F_G$ for $G \in \mathcal{G}$, which follows immediately from the definition of $\mathrm{Hom}(\mathcal{G}, -)$. The counit $\epsilon_G : \mathrm{Hom}(\mathcal{G}, G) \wedge_{\mathcal{E}} \mathcal{G} \longrightarrow G$ is an isomorphism for each $G \in \mathcal{G}$. The reason is, that for $A \in \mathcal{K}$ the induced map of spectra

$$\epsilon_G^* : \mathrm{Hom}_{\mathcal{K}}(G, A) \longrightarrow \mathrm{Hom}_{\mathcal{K}}\left(\mathrm{Hom}(\mathcal{G}, G) \wedge_{\mathcal{E}} \mathcal{G}, A\right)$$

is the composition of the map

$$\mathrm{Hom}_{\mathcal{K}}(G, A) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}\text{-}\mathcal{E}}\left(\mathrm{Hom}(\mathcal{G}, G), \mathrm{Hom}(\mathcal{G}, A)\right)$$
$$= \mathrm{Hom}_{\mathrm{Mod}\text{-}\mathcal{E}}\left(\mathrm{Hom}_{\mathcal{K}}(-, G), \mathrm{Hom}_{\mathcal{K}}(-, A)\right),$$

which is an isomorphism by the enriched Yoneda lemma, with the adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}\text{-}\mathcal{E}}\left(\mathrm{Hom}(\mathcal{G}, G), \mathrm{Hom}(\mathcal{G}, A)\right) \cong \mathrm{Hom}_{\mathcal{K}}\left(\mathrm{Hom}(\mathcal{G}, G) \wedge_{\mathcal{E}} \mathcal{G}, A\right).$$

This shows that ϵ_G is an isomorphism. In particular, we obtain an isomorphism $F_G \wedge_{\mathcal{E}} \mathcal{G} \cong G$.

Let us denote the derived adjunction by

$$\operatorname{Ho}\mathcal{K} \xleftrightarrow[F]{J} \mathsf{D}(\mathcal{E}^{\,\mathrm{op}}).$$

Since both $\mathcal{K}$ and Mod-$\mathcal{E}$ are stable, $\operatorname{Ho}\mathcal{K}$ and $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ are triangulated categories with coproducts and J and F are triangulated functors. Since $G \in \mathcal{G}$ is cofibrant we have $F(G) \cong F_G$ on the homotopy level.

We want to see that the counit $\varepsilon : JF(G) \longrightarrow G$ is also an isomorphism for objects G of $\mathcal{G}$. The following is in general true for Quillen pairs

$$\mathcal{C} \xleftrightarrow[U]{V} \mathcal{D}.$$

If C is a fibrant object of $\mathcal{C}$ such that $U(C)$ is cofibrant in $\mathcal{D}$ and the counit

$$\epsilon_C : VU(C) \longrightarrow C$$

is an isomorphism in $\mathcal{C}$, then the counit $\varepsilon_C : (LV)(RU)(C) \longrightarrow C$ of the derived adjunction is an isomorphism in $\operatorname{Ho}\mathcal{C}$. In our case, G is fibrant in $\mathcal{K}$ by assumption. Thus, in order show that the counit $\varepsilon : JF(G) \longrightarrow G$ is an isomorphism, it suffices to prove that $F(G) \cong F_G$ is cofibrant in Mod-$\mathcal{E}$. This can be seen by analyzing the proof of [SS03b, Theorem A.1.1]. Alternatively, if we accept that Mod-$\mathcal{E}$ has a model structure with weak equivalences and fibrations objectwise in $\operatorname{Sp}^{\Sigma}$, we can consider the Quillen pair

$$\text{Mod-}\mathcal{E} \xleftrightarrow[\mathrm{ev}_G]{-\wedge F_G} \text{Mod-}\mathbb{S} = \operatorname{Sp}^{\Sigma}$$

where $\mathrm{ev}_G(X) = X(G)$ and $Y \wedge F_G$ is given by $(Y \wedge F_G)(G') = X \wedge F_G(G') = X \wedge \mathcal{E}(G', G)$ with the obvious right action of $\mathcal{E}$. Applying the left Quillen functor $- \wedge F_G$ to the (cofibrant) sphere spectrum $\mathbb{S}$ shows that F_G is a cofibrant module.

The unit $\eta_{F_G} : F_G \longrightarrow FJ(F_G)$ is also an isomorphism for every free module F_G. This follows from $F_G \cong F(G)$ and $F(\varepsilon_G)\eta_{F(G)} = \operatorname{id}_{F(G)}$, which holds in general for adjunctions.

Let us summarize these facts in the following

Lemma 4.1. *Let $\mathcal{K}$ be a spectral model category, $\mathcal{G}$ a set of cofibrant and fibrant objects, and $\mathcal{E}$ the full spectral subcategory of $\mathcal{K}$ with objects $\mathcal{G}$.*

Then the derived category $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ of $\mathcal{E}$ has the free modules F_G, $G \in \mathcal{G}$, as a compact generating set. There is an adjoint pair of triangulated functors

$$\operatorname{Ho}\mathcal{K} \xleftrightarrow[F]{J} \mathsf{D}(\mathcal{E}^{\,\mathrm{op}}),$$

under which the objects in $\mathcal{G}$ correspond to the free modules, that is, $F(G) \cong F_G$ and $J(F_G) \cong G$, such that the counit $\varepsilon_G : JF(G) \longrightarrow G$ and the unit $\eta_{F_G} : F_G \longrightarrow FJ(F_G)$ are isomorphisms for $G \in \mathcal{G}$.

$\square$

Remark 4.2. As the counit $\varepsilon_G : JF(G) \longrightarrow G$ is an isomorphism for $G \in \mathcal{G}$ the functor F is fully faithful when the source is in $\mathcal{G}$, that is, for $G \in \mathcal{G}$ and arbitrary $A \in \text{Ho}\,\mathcal{K}$, the map

$$[G, A]^{\text{Ho}\,\mathcal{K}} \longrightarrow [F(G), F(A)]^{\text{D}(\mathcal{E}^{\text{op}})},$$
$$g \mapsto F(g),$$

is an isomorphism. Namely, if we compose this map with the adjunction isomorphism $[F(G), F(A)]^{\text{D}(\mathcal{E}^{\text{op}})} \longrightarrow [JF(G), A]^{\text{Ho}\,\mathcal{K}}$, we obtain the map

$$\varepsilon_G^* : [G, A]^{\text{Ho}\,\mathcal{K}} \longrightarrow [JF(G), A]^{\text{Ho}\,\mathcal{K}}$$

induced by the colimit ε_G, which is an isomorphism by Lemma 4.1.

Schwede and Shipley have shown that the Quillen pair

$$\mathcal{K} \xleftrightarrow[\text{Hom}(\mathcal{G},-)]{-\wedge_{\mathcal{E}}\mathcal{G}} \text{Mod-}\mathcal{E},$$

is in fact a Quillen equivalence if $\mathcal{G}$ is a set of compact generators for $\text{Ho}\,\mathcal{K}$. In particular, F and J are then inverse equivalences. We will study the case where $\mathcal{G}$ is not necessarily a compact generating set for $\text{Ho}\,\mathcal{K}$ but only an appropriate α-compact generating set. The goal is to show that under this assumption F is fully faithful and hence $\text{Ho}\,\mathcal{K}$ is a localization of $\text{D}(\mathcal{E}^{\text{op}})$.

As a triangulated category, $\text{Ho}\,\mathcal{K}$ is in particular an Ab-category (or a ring with several objects), i.e., enriched over the closed symmetric monoidal category Ab of abelian groups. Regarding the set $\mathcal{G}$ of objects in $\mathcal{K}$ as a full subcategory of $\text{Ho}\,\mathcal{K}$, we get an Ab-category which we denote for simplicity again by $\mathcal{G}$.

Definition 4.3. The category Mod-$\mathcal{G}$ of right $\mathcal{G}$-modules has as objects the Ab-functors $\mathcal{G}^{\text{op}} \longrightarrow \text{Ab}$ and as morphisms natural transformations. We define a functor

$$H_0 : \text{D}(\mathcal{E}^{\text{op}}) \longrightarrow \text{Mod-}\mathcal{G}$$

by $H_0(X) = [F|_{\mathcal{G}}(-), X] : \mathcal{G}^{\text{op}} \longrightarrow \text{Ab}.$

The category Mod-$\mathcal{G}$ is abelian and has objectwise defined coproducts and products. Recall that a functor from a triangulated category to an abelian category is called *homological* if it maps exact triangles to (long) exact sequences [Nee01b, Definition 1.1.7].

Lemma 4.4. *The functor $H_0 : \text{D}(\mathcal{E}^{\text{op}}) \longrightarrow \text{Mod-}\mathcal{G}$ is homological and preserves coproducts and products. If $\mathcal{G}$ is up to isomorphism closed under (de-)suspensions, then H_0 reflects isomorphisms, i.e., $f : X \longrightarrow Y$ is an isomorphism in $\text{D}(\mathcal{E}^{\text{op}})$ if and only if $H_0(f) : H_0(X) \longrightarrow H_0(Y)$ is an isomorphism in Mod-$\mathcal{G}$.*

Proof. Consider an exact triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ in $\text{D}(\mathcal{E}^{\text{op}})$. The induced sequence

$$\cdots \longrightarrow H_0(X) \longrightarrow H_0(Y) \longrightarrow H_0(Z) \longrightarrow H_0(\Sigma X) \longrightarrow \cdots$$

is exact if and only if it is objectwise exact, i.e.,

$$\cdots \longrightarrow [F(G),X] \longrightarrow [F(G),Y] \longrightarrow [F(G),Z] \longrightarrow [F(G),\Sigma X] \longrightarrow \cdots$$

is a long exact sequence for each $G \in \mathcal{G}$. But this is true since $[F(G),-] : \mathsf{D}(\mathcal{E}^{\mathrm{op}}) \longrightarrow \mathrm{Ab}$ is homological.

Since products in Mod-$\mathcal{G}$ are objectwise, it is clear that H_0 preserves them. Now consider a family $(X_i)_{i \in I}$ of objects in $\mathsf{D}(\mathcal{E}^{\mathrm{op}})$. Since $F(G)$ is compact, the canonical map

$$\bigoplus_{i \in I} [F(G),X_i] \longrightarrow \left[F(G),\coprod_{i \in I} X_i\right]$$

is an isomorphism for every $G \in \mathcal{G}$. Hence $\coprod_{i \in I} H_0(X_i) \longrightarrow H_0\left(\coprod_{i \in I} X_i\right)$ is an isomorphism in Mod-$\mathcal{G}$.

Now let $\mathcal{G}$ be closed under (de-)suspensions (up to isomorphism). Then the set

$$F(\mathcal{G}) = \{F(G) \,|\, G \in \mathcal{G}\}$$

is in particular a weak generating set for $\mathsf{D}(\mathcal{E}^{\mathrm{op}})$ (as a consequence of Lemma 4.1). Given a map $f : X \longrightarrow Y$ in $\mathsf{D}(\mathcal{E}^{\mathrm{op}})$ with cofiber Z. We then have the following logical equivalences:

$$
\begin{aligned}
f \text{ is an isomorphism} \quad &\Longleftrightarrow \quad Z \cong 0 \\
&\Longleftrightarrow \quad [F(G),Z] \cong 0 \text{ for all } G \in \mathcal{G} \\
&\Longleftrightarrow \quad [F(G),X] \longrightarrow [F(G),Y] \text{ is an isomorphism for all } G \in \mathcal{G} \\
&\Longleftrightarrow \quad H_0(X) \longrightarrow H_0(Y) \text{ is an isomorphism in Mod-}\mathcal{G},
\end{aligned}
$$

where the third equivalence uses that $[F(G),-]$ is homological. $\qquad\qquad\square$

4.2. **The characterization theorem and the strategy of proof.** Let us state the main result, Theorem 4.7, fix some notation, and sketch the proof before giving the details in Section 4.3. Porta considers those well generated triangulated categories which are algebraic, that is, triangulated equivalent to the stable category of a Frobenius category [Kel06, Section 3.6]. One feature of algebraic categories is that they allow certain 'derived Hom-functors' into derived categories of DG categories. In the topological case, we would like to have such derived Hom-functors into derived categories of spectral categories. Lemma 4.1 provides such a functor F if the triangulated category in question is the homotopy category of a spectral model category. This leads us to the following

Definition 4.5. A triangulated category $\mathcal{T}$ is called *topological* if it is triangulated equivalent to the homotopy category of a spectral model category.

Examples 4.6. In particular, any stable model category Quillen equivalent to a spectral model category has a homotopy category which is topological. Here are three classes of such model categories.

(1) Schwede and Shipley have proved that every simplicial, cofibrantly generated, proper stable model category is Quillen equivalent to a spectral model category [SS03b, Theorem 3.8.2].

(2) By [Hov01, Theorems 9.1 and 8.11], a simplicial, cellular, left proper stable model category for which the domains of the generating cofibrations are cofibrant is Quillen equivalent to a spectral model category.

(3) Another class of examples arises from [Dug06]. Using [Hov01, Theorem 8.11] one can deduce from [Dug06, Propositions 5.5(a) and 5.6(a)] that presentable stable model categories are Quillen equivalent to spectral model categories.

By [SS03b, Theorem A.1.1], the derived category of a spectral category is compactly generated. Hence Proposition 3.3 yields one implication of the following characterization theorem.

Theorem 4.7. *Let $\mathcal{T}$ be a topological triangulated category. Then the following are equivalent.*

   (i) *$\mathcal{T}$ is well generated.*

  (ii) *$\mathcal{T}$ is triangulated equivalent to a localization of the derived category of a spectral category where the acyclics are generated by a set.*

The implication (i) $\Rightarrow$ (ii) is more involved and the proof is given in Section 4.3. Let us from now on and for the rest of this paper use the following

Notation 4.8. We assume that $\mathcal{K}$ is a spectral model category having a well generated homotopy category $\mathrm{Ho}\,\mathcal{K}$. Then there exists a regular infinite cardinal α such that the full subcategory $(\mathrm{Ho}\,\mathcal{K})^\alpha$ of α-compact objects in $\mathrm{Ho}\,\mathcal{K}$ has a small skeleton $\mathcal{G}$ which is an α-compact generating set for $\mathrm{Ho}\,\mathcal{K}$ (see Remark 3.2). We fix such a cardinal α and such a generating set $\mathcal{G}$, which is then, up to isomorphism, closed under (de-)suspensions, triangles, and α-coproducts (that is, coproducts of strictly less than α objects). Moreover, by choosing cofibrant and fibrant replacements, we can assume all objects in $\mathcal{G}$ are cofibrant and fibrant in $\mathcal{K}$. We let $\mathcal{E}$ be the full spectral subcategory of $\mathcal{K}$ with objects $\mathcal{G}$. As above, by slight abuse of notation, we regard $\mathcal{G}$ not only as a set but also as an Ab-category with objects $\mathcal{G}$. Since $\mathcal{G}$ is in particular closed under finite coproducts and contains a zero object, it is also an additive category.

Consider the following diagram.

(4.9)

Here Mod_α-$\mathcal{G}$ is a suitable subcategory of Mod-$\mathcal{G}$ (see Definition 4.10) and $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ is the corresponding subcategory of $\mathsf{D}(\mathcal{E}^{\mathrm{op}})$ of those objects whose homology lies in Mod_α-$\mathcal{G}$ (see Definition 4.11). The pair (J, F) is the adjoint pair from Lemma 4.1. It restricts to an adjoint pair $(\widetilde{J}, \widetilde{F})$.

If we can show that

(1) $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ is a localization of $\mathsf{D}(\mathcal{E}^{\mathrm{op}})$, i.e., there exists a left adjoint L for the inclusion R, and

(2) $\widetilde{J}$ and $\widetilde{F}$ are inverse equivalences of triangulated categories,

then it follows that $\mathrm{Ho}\,\mathcal{K}$ is a localization of $\mathsf{D}(\mathcal{E}^{\mathrm{op}})$.

For the proof of (2) we will consider the unit

$$X \longrightarrow \widetilde{F}\widetilde{J}(X) \quad \text{and the counit} \quad \widetilde{J}\widetilde{F}(A) \longrightarrow A$$

of the adjunction and show that they are isomorphisms for all objects $X \in \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ and $A \in \mathrm{Ho}\,\mathcal{K}$. This is easy to see for the free modules F_G (which lie in fact in the subcategory $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ and form a set of generators for it) and for the generators $G \in \mathcal{G}$ of $\mathrm{Ho}\,\mathcal{K}$. Then it suffices to prove that both $\widetilde{J}$ and $\widetilde{F}$ preserve coproducts. Of course, in the case of the left adjoint $\widetilde{J}$ this is true. The problem is to show that $\widetilde{F}$ also preserves coproducts. In his paper [Kra01], Krause defines a functor $\Phi : \mathrm{Ho}\,\mathcal{K} \longrightarrow \mathrm{Mod}_\alpha$-$\mathcal{G}$ which preserves coproducts and is isomorphic to $\widetilde{H}_0\widetilde{F}$ (where $\widetilde{H}_0$ is the restriction of H_0). This helps us to reduce the question whether $\widetilde{F}$ preserves coproducts to the following: Does $\widetilde{H}_0$ preserve coproducts of objects which are in the essential image of $\widetilde{F}$?

4.3. **Proof of the characterization theorem.** Recall that $\mathcal{K}$, α, $\mathcal{G}$, and $\mathcal{E}$ are as in Notation 4.8. Let us first define the categories Mod_α-$\mathcal{G}$ and $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ occurring in the diagram (4.9). We will then prove a series of lemmas and finally the remaining part of Theorem 4.7.

Definition 4.10. The category Mod_α-$\mathcal{G}$ is defined as the full subcategory of Mod-$\mathcal{G}$ with objects those functors $\mathcal{G}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$ which send α-coproducts in $\mathcal{G}$ to products in Ab. The functor

$$\Phi : \mathrm{Ho}\,\mathcal{K} \longrightarrow \mathrm{Mod}_\alpha\text{-}\mathcal{G}$$

is defined by $A \mapsto [-, A]|_\mathcal{G}$.

Note that Φ indeed takes values in Mod_α-$\mathcal{G}$: the functor $[-, A]$ maps even arbitrary coproducts in $\mathrm{Ho}\,\mathcal{K}$ to products in Mod_α-$\mathcal{G}$. Neeman denotes the category Mod_α-$\mathcal{G}$ by $\mathcal{E}x(\mathcal{G}^{\mathrm{op}}, \mathcal{A}b)$ [Nee01b, Defintion 6.1.3], Krause uses $\mathrm{Prod}_\alpha(\mathcal{G}^{\mathrm{op}}, \mathrm{Ab})$ [Kra01]. It is an abelian subcategory of Mod-$\mathcal{G}$, which is closed under products [Nee01b, Lemma 6.1.4 and Lemma 6.1.5], but it is not closed under coproducts in general. Nevertheless, Mod_α-$\mathcal{G}$ does have coproducts – which cannot be objectwise in general, since they have to be different from those in Mod-$\mathcal{G}$. An explicit description of the coproducts in Mod_α-$\mathcal{G}$ can be found in Neeman's book [Nee01b, Section 6.1] (the definition together with the proof of the universal property takes twelve and a half pages). Krause shows that Mod_α-$\mathcal{G}$ is equivalent to the category of coherent functors $(\mathrm{Add}\,\mathcal{G})^{\mathrm{op}} \longrightarrow \mathrm{Ab}$, where $\mathrm{Add}\,\mathcal{G}$ denotes the closure of $\mathcal{G}$ in $\mathrm{Ho}\,\mathcal{K}$ under all coproducts and direct summands [Kra01, Lemma 2].

In the category of these coherent functors, coproducts have a nicer description [Kra02, Lemma 1]. However, we do not need to know what the coproducts in Mod_α-$\mathcal{G}$ look like – the only thing we will need is the fact that the functor $\Phi : \mathrm{Ho}\,\mathcal{K} \longrightarrow \mathrm{Mod}_\alpha$-$\mathcal{G}$ preserves coproducts [Kra01, Theorem C].

Definition 4.11. The category $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ is the full subcategory of $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ having as objects those X for which $H_0(X)$ is in Mod_α-$\mathcal{G}$. The functor

$$\widetilde{H}_0 : \mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}}) \longrightarrow \mathrm{Mod}_\alpha\text{-}\mathcal{G}$$

is the restriction of H_0.

Remark 4.12. Note that $\widetilde{H}_0$ reflects isomorphisms since H_0 does. Moreover, $\widetilde{H}_0$ is homological, that is, it sends exact triangles to long exact sequences. The reason is the following: $H_0 R$ is homological because R is a triangulated functor and H_0 is homological, so $r\widetilde{H}_0 = H_0 R$ is homological. Since r is exact $\widetilde{H}_0$ has to be homological. But it is not so easy to see that $\widetilde{H}_0$ preserves coproducts – this will be a consequence of Proposition 4.33.

Lemma 4.13. *The category $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ is a triangulated subcategory of $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$. It is colocalizing, i.e., closed under products. If $\alpha = \aleph_0$ then $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}}) = \mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$.*

Proof. Let $(G_i)_{i \in I}$ be a family of objects in $\mathcal{G}$ with $|I| < \alpha$. Assume X is in $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$. Then the canonical map

$$\Big[F\Big(\coprod_{i \in I} G_i\Big), X\Big] \longrightarrow \prod_{i \in I}[F(G_i), X]$$

is an isomorphism. We want to show that ΣX is also in $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$. Since both F and coproducts commute with the desuspension Σ^{-1}, the canonical map

$$\Big[F\Big(\coprod_{i \in I} G_i\Big), \Sigma X\Big] \longrightarrow \prod_{i \in I}[F(G_i), \Sigma X]$$

is isomorphic to

$$\Big[F\Big(\coprod_{i \in I} \Sigma^{-1}(G_i)\Big), X\Big] \longrightarrow \prod_{i \in I}[F\Sigma^{-1}(G_i), X].$$

But this map is an isomorphism because $\mathcal{G}$ is closed under desuspensions and X is in $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$. Similarly, $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ is closed under desuspensions.

Consider a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ in $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ such that X and Y are in $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$. Since H_0 is homological and products of exact sequences in Ab are exact again,

we get a commutative diagram of abelian groups with long exact columns:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
(H_0X)\Big(\coprod_{i\in I} G_i\Big) & \xrightarrow{\cong} & \prod_{i\in I}(H_0X)(G_i) \\
\downarrow & & \downarrow \\
(H_0Y)\Big(\coprod_{i\in I} G_i\Big) & \xrightarrow{\cong} & \prod_{i\in I}(H_0Y)(G_i) \\
\downarrow & & \downarrow \\
(H_0Z)\Big(\coprod_{i\in I} G_i\Big) & \longrightarrow & \prod_{i\in I}(H_0Z)(G_i) \\
\downarrow & & \downarrow \\
(H_0\Sigma X)\Big(\coprod_{i\in I} G_i\Big) & \xrightarrow{\cong} & \prod_{i\in I}(H_0\Sigma X)(G_i) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

We can apply the 5-lemma and get and isomorphism

$$
(H_0Z)\Big(\coprod_{i\in I} G_i\Big) \xrightarrow{\cong} \prod_{i\in I}(H_0Z)(G_i).
$$

This shows Z is in $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ and thus $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ is closed under triangles. Since $\mathrm{Mod}_\alpha\text{-}\mathcal{G}$ is closed under products in $\mathrm{Mod}\text{-}\mathcal{G}$ and H_0 preserves products, $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ is also closed under products.

Now let $\alpha = \aleph_0$. Then $\mathrm{Mod}_\alpha\text{-}\mathcal{G}$ contains all additive functors $\mathcal{G}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$ which map finite coproducts to products. But this is true for all additive functors and thus $\mathrm{Mod}_\alpha\text{-}\mathcal{G} = \mathrm{Mod}\text{-}\mathcal{G}$ and $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}}) = \mathsf{D}(\mathcal{E}^{\mathrm{op}})$. $\square$

Lemma 4.14. *The functor* $F : \mathrm{Ho}\,\mathcal{K} \longrightarrow \mathsf{D}(\mathcal{E}^{\mathrm{op}})$ *factors through* $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$. *Consequently we get an adjoint pair of triangulated functors*

$$
\mathrm{Ho}\,\mathcal{K} \; \underset{\widetilde{F}}{\overset{\widetilde{J}}{\rightleftarrows}} \; \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}}).
$$

Moreover, the composition $\widetilde{H}_0\widetilde{F}$ *is isomorphic to* Φ.

Proof. We have to check that for $A \in \mathrm{Ho}\,\mathcal{K}$ the functor $H_0 F(A)$ sends α-coproducts in $\mathcal{G}$ to products in Ab. Let $(G_i)_{i \in I}$ be a family in $\mathcal{G}$ with $|I| < \alpha$. Using the adjunction (J, F) and the fact that its counit $JF(G) \longrightarrow G$ is an isomorphism for $G \in \mathcal{G}$ (see Lemma 4.1) we can conclude that the canonical map

$$\left[F\Big(\coprod_{i \in I} G_i\Big), F(A)\right]^{\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})} \longrightarrow \prod_{i \in I} [F(G_i), F(A)]^{\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})}$$

is isomorphic to the map $[\coprod_{i \in I} G_i, A]^{\mathrm{Ho}\,\mathcal{K}} \longrightarrow \prod_{i \in I} [G_i, A]^{\mathrm{Ho}\,\mathcal{K}}$, which is an isomorphism by the universal property of the coproduct. Hence $H_0 F(A)$ maps α-coproducts to products. This yields a functor $\widetilde{F} : \mathrm{Ho}\,\mathcal{K} \longrightarrow \mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ with $R\widetilde{F} = F$, which is left adjoint to $\widetilde{J} = JR$.

Using Remark 4.2, we get an isomorphism

$$\widetilde{H}_0 \widetilde{F}(A)(G) = H_0 F(A)(G) = [F(G), F(A)]^{\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})} \cong [G, A]^{\mathrm{Ho}\,\mathcal{K}} = \Phi(A)(G)$$

which is natural in $G \in \mathcal{G}$ and $A \in \mathrm{Ho}\,\mathcal{K}$. $\qquad\square$

Proposition 4.15. *The category* $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ *is a localization of* $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$, *i.e., there exists a left adjoint* L *for the inclusion* R. *Let* $\mathcal{W}$ *denote the set of the canonical maps*

$$\coprod_{i \in I} F(G_i) \longrightarrow F\left(\coprod_{i \in I} G_i\right)$$

where $(G_i)_{i \in I}$ *runs through all families in* $\mathcal{G}$ *with* $|I| < \alpha$. *(Strictly speaking, we allow one and only one set* I *for each cardinality smaller than* α *to ensure all the considered maps really form a set.) Then the acyclics are generated by the set* $\mathcal{S}$ *containing one cofiber for each map in* $\mathcal{W}$. *Moreover, the subcategory* $\mathcal{W}$-loc *(see Lemma 3.4) is equal to* $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$.

Proof. Since $\mathcal{G}$ is closed under (de-)suspensions so is $\mathcal{W}$. We know that $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ is well generated (even compactly generated by the free modules), so we can apply Lemma 3.4 and get a localization of $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ with $\mathcal{W}$-loc as the class of local objects and $\langle \mathcal{S} \rangle$ as the class of acyclics. An object X of $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ is in $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ by definition if and only if $H_0(X) : \mathcal{G}^{\,\mathrm{op}} \longrightarrow$ Ab sends α-coproducts to products, i.e., if and only if the canonical map

$$(4.16) \qquad\qquad H_0(X)\Big(\coprod_{i \in I} G_i\Big) \longrightarrow \prod_{i \in I} H_0(X)(G_i)$$

is an isomorphism. Using the definition of H_0 and the fact that $[-, X] : \mathsf{D}(\mathcal{E}^{\,\mathrm{op}}) \longrightarrow$ Ab maps coproducts to products we see that the map (4.16) is isomorphic to the map

$$\left[F\Big(\coprod_{i \in I} G_i\Big), X\right] \longrightarrow \left[\coprod_{i \in I} F(G_i), X\right],$$

which is an isomorphism if and only if X is in $\mathcal{W}$-loc. This shows $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}}) = \mathcal{W}$-loc. By Remark 1.8, $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}}) = \mathcal{W}$-loc is equivalent to $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})/\langle \mathcal{S} \rangle$ and thus a localization of $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ with acyclics $\langle \mathcal{S} \rangle$. Let L be the composition

$$\mathsf{D}(\mathcal{E}^{\,\mathrm{op}}) \longrightarrow \mathsf{D}(\mathcal{E}^{\,\mathrm{op}})/\langle S \rangle \overset{\simeq}{\longrightarrow} \mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}}).$$

The inclusion $R : \mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}}) \longrightarrow \mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ is then a fully faithful right adjoint for L. $\square$

As an immediate consequence of Proposition 4.15 we get the following

Corollary 4.17. *The category $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ has coproducts, namely $\coprod_{i\in I} X_i = L\left(\coprod_{i\in I} RX_i\right)$ for any indexing set I. The canonical map $X_j \longrightarrow \coprod_i X_i$ is the composition*

$$X_j \overset{\cong}{\longrightarrow} LR(X_j) \longrightarrow L\left(\coprod_{i\in I} RX_i\right),$$

where the first map is the inverse of the counit (which is an isomorphism since the right adjoint R is fully faithful) and the second map is L applied to the canonical map of the coproduct in $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$. $\square$

Remark 4.18. The inclusion of abelian categories $r : \mathrm{Mod}_\alpha\text{-}\mathcal{G} \longrightarrow \mathcal{G}$ has a left adjoint, too. But since this left adjoint (which is discussed in [Nee01b, Section 7.5]) is not exact in general, it will not be useful for us.

Lemma 4.19. *The functor $\widetilde{F} : \mathrm{Ho}\,\mathcal{K} \longrightarrow \mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ is isomorphic to the composition LF. Moreover, $\widetilde{F}$ preserves α-coproducts of objects which lie in $\mathcal{G}$.*

Proof. Since R is fully faithful we have an isomorphism $LR \cong \mathrm{id}_{\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})}$, which is given by the counit of the adjoint pair (L, R). This yields an isomorphism $LF = LR\widetilde{F} \cong \widetilde{F}$.

As a consequence of Proposition 4.15, for any family $(G_i)_{i\in I}$ with $G_i \in \mathcal{G}$ and $|I| < \alpha$, the map

$$L\left(\coprod_{i\in I} F(G_i)\right) \longrightarrow LF\left(\coprod_{i\in I} G_i\right)$$

is an isomorphism. As a left adjoint, L commutes with coproducts. Using the isomorphism $LF \cong \widetilde{F}$, we see that $\widetilde{F}$ preserves α-coproducts of objects in $\mathcal{G}$. $\square$

Lemma 4.20. *The functor $L : \mathsf{D}(\mathcal{E}^{\,\mathrm{op}}) \longrightarrow \mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ preserves α-compact objects.*

Proof. Since $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ is well generated (even compactly generated), every object is β-compact for some β [Nee01b, Proposition 8.4.2]. By [Nee01b, Lemma 4.4.4] it suffices then to show that the acyclics of the localization

$$\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}}) \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$$

have a generating set containing only α-compact objects. Let C be a cofiber of a map

$$\coprod_{i\in I} F(G_i) \longrightarrow F\left(\coprod_{i\in I} G_i\right)$$

where $G_i \in \mathcal{G}$ and $|I| < \alpha$. The acyclics are generated by such cofibers C (Proposition 4.15). Recall that all α-compact objects form an α-localizing triangulated subcategory. As a free module, each $F(G_i)$ is compact (Lemma 4.1) and hence α-compact – and so is the α-coproduct $\coprod_{i \in I} F(G_i)$. Since $\coprod_{i \in I} G_i$ is up to isomorphism in $\mathcal{G}$, the object $F\left(\coprod_{i \in I} G_i\right)$ is also α-compact and so is the cofiber C.

$\square$

Lemma 4.21. *The functor* $\widetilde{F} : \mathrm{Ho}\,\mathcal{K} \longrightarrow \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ *is fully faithful when the source is in* $\mathcal{G}$, *that is, for* $G \in \mathcal{G}$ *and arbitrary* $A \in \mathrm{Ho}\,\mathcal{K}$, *the map*

$$
\begin{aligned}
[G, A]^{\mathrm{Ho}\,\mathcal{K}} &\longrightarrow [\widetilde{F}(G), \widetilde{F}(A)]^{\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})}, \\
g &\mapsto \widetilde{F}(g),
\end{aligned}
$$

is an isomorphism.

Proof. We will again use $\widetilde{F} = LF$ (Lemma 4.19). By Remark 4.2, F is fully faithful when the source is in $\mathcal{G}$. Since R is fully faithful, the left adjoint L is fully faithful on $\mathrm{essim}\,R$ and in particular on $\mathrm{essim}\,F \subset \mathrm{essim}\,R$.

$\square$

Lemma 4.22. *The set* $\widetilde{F}(\mathcal{G}) = \{\widetilde{F}(G) \,|\, G \in \mathcal{G}\}$ *is an* α-*compact generating set (in the sense of Definition* 3.1*) for* $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$.

Proof. First of all, $\widetilde{F}(\mathcal{G})$ is a weak generating set for $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$. Recall what this means: it is closed under (de-)suspensions up to isomorphism, and $[\widetilde{F}(G), X]^{\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})} = 0$ for all $G \in \mathcal{G}$ implies $X = 0$. This holds because $F(\mathcal{G})$ is a weak generating set for $\mathsf{D}(\mathcal{E}^{\mathrm{op}})$. More is true, we have $\langle \widetilde{F}(\mathcal{G}) \rangle = \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$. Namely, using the Lemmas 4.19 and 1.2, we get

$$
\langle \widetilde{F}(\mathcal{G}) \rangle = \langle LF(\mathcal{G}) \rangle \supset \mathrm{essim}\,L|_{\langle F(\mathcal{G}) \rangle} = \mathrm{essim}\,L = \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})
$$

and thus $\langle \widetilde{F}(\mathcal{G}) \rangle = \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$.

By Lemma 4.20, each $\widetilde{F}(G) = LF(G)$ is α-compact in $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$. Note that this does not yet imply that $\widetilde{F}(\mathcal{G})$ is an α-compact generating set for $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ – we still have to show it is α-perfect. In fact, we will show that $\widetilde{F}(\mathcal{G})$, regarded as a subcategory of $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$, is equivalent to the category $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})^\alpha$ of all α-compact objects in $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$, which is by definition α-perfect.

The set $\mathcal{G}$ is by assumption (cf. Notation 4.8), up to isomorphism, closed under (de-)suspensions, triangles, and α-coproducts. The same holds for the set $\widetilde{F}(\mathcal{G})$ in $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ because $\widetilde{F}$ is triangulated, fully faithful on $\mathcal{G}$ (Lemma 4.21), and preserves α-coproducts of objects in $\mathcal{G}$ (Lemma 4.19). As a consequence, the full subcategory $\overline{\widetilde{F}(\mathcal{G})}$ of $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ consisting of all objects isomorphic to some object in $\widetilde{F}(\mathcal{G})$ is an α-localizing triangulated subcategory of $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$. This implies that the inclusion of $\widetilde{F}(\mathcal{G})$ in $\overline{\widetilde{F}(\mathcal{G})}$ gives an equivalence of categories

$$
(4.23) \qquad\qquad \widetilde{F}(\mathcal{G}) \simeq \overline{\widetilde{F}(\mathcal{G})} = \alpha\text{-loc}\langle \widetilde{F}(\mathcal{G}) \rangle,
$$

where $\alpha\text{-loc}\langle\widetilde{F}(\mathcal{G})\rangle$ is the smallest α-localizing subcategory containing $\widetilde{F}(\mathcal{G})$. We can now apply [Nee01b, Lemma 4.4.5] to get an equality $\alpha\text{-loc}\langle\widetilde{F}(\mathcal{G})\rangle = \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})^\alpha$. Together with (4.23) this implies that the inclusion $\widetilde{F}(\mathcal{G}) \hookrightarrow \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})^\alpha$ is an equivalence of categories and in particular $\widetilde{F}(\mathcal{G})$ is an α-compact generating set for $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$. $\qquad\square$

Lemma 4.24. *The functor* $\widetilde{F} : \operatorname{Ho}\mathcal{K} \longrightarrow \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ *preserves* α-coproducts.

Proof. For a family $(A_i)_{i\in I}$ of objects in $\operatorname{Ho}\mathcal{K}$ with $|I| < \alpha$ let

$$\gamma : \coprod_{i\in I} \widetilde{F}(A_i) \longrightarrow \widetilde{F}\left(\coprod_{i\in I} A_i\right)$$

denote the canonical map. Since $\widetilde{F}(\mathcal{G})$ is a weak generating set for $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ (see Lemma 4.22), it suffices to show that for each $G \in \mathcal{G}$ the induced map
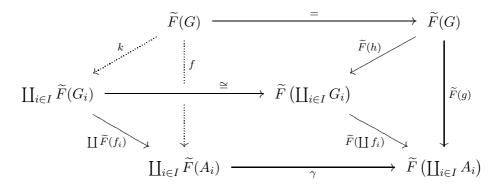
$$\gamma_* : \left[\widetilde{F}(G), \coprod_{i\in I} \widetilde{F}(A_i)\right] \longrightarrow \left[\widetilde{F}(G), \widetilde{F}\left(\coprod_{i\in I} A_i\right)\right]$$

is bijective.

The surjectivity can be seen as follows. Any given morphism $\widetilde{F}(G) \longrightarrow \widetilde{F}\left(\coprod_{i\in I} A_i\right)$ is by Lemma 4.21 of the form $\widetilde{F}(g)$ for some $g : G \longrightarrow \coprod_{i\in I} A_i$. Since $\mathcal{G}$ is an α-compact generating set for $\operatorname{Ho}\mathcal{K}$, this map g factors as

$$G \xrightarrow{h} \coprod_{i\in I} G_i \xrightarrow{\coprod f_i} \coprod_{i\in I} A_i.$$

Consider the following commutative diagram (of solid arrows).

The horizontal arrow in the middle is an isomorphism since we know already by Lemma 4.19 that $\widetilde{F}$ preserves α-coproducts of objects of $\mathcal{G}$. Hence there exists a dotted arrow k such that the whole diagram commutes. If we define f to be the composition $\left(\coprod \widetilde{F}(f_i)\right)k$ then $\gamma_*(f) = \gamma f = \widetilde{F}(g)$ and hence γ_* is surjective.

To prove the injectivity of γ_* consider a morphism $f : \widetilde{F}(G) \longrightarrow \coprod_{i\in I} \widetilde{F}(A_i)$ such that $\gamma f = 0$. By Lemma 4.22, $\widetilde{F}(\mathcal{G})$ is in particular an α-perfect generating set for $\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$.
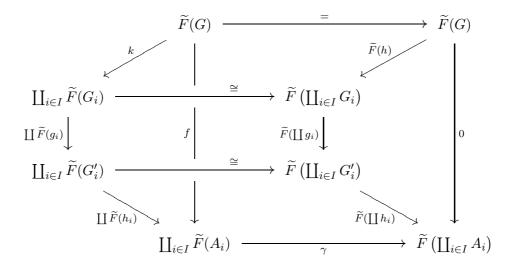
This implies that f can be factored as

$$\widetilde{F}(G) \xrightarrow{k} \coprod_{i\in I} \widetilde{F}(G_i) \xrightarrow{\coprod \widetilde{F}(f_i)} \coprod_{i\in I} \widetilde{F}(A_i),$$

where we used that $\widetilde{F}$ is full for arrows with source in $\mathcal{G}$ (Lemma 4.21). Composing k with the isomorphism $\coprod_{i\in I} \widetilde{F}(G_i) \xrightarrow{\cong} \widetilde{F}\left(\coprod_{i\in I} G_i\right)$ yields a map $\widetilde{F}(G) \longrightarrow \widetilde{F}\left(\coprod_{i\in I} G_i\right)$ which is of the form $\widetilde{F}(h)$ for some $h : G \longrightarrow \coprod_{i\in I} G_i$. We can conclude that $\widetilde{F}\left(\coprod f_i\right)\widetilde{F}(h) = \gamma f = 0$ and, since $\widetilde{F}$ is faithful for morphisms with source in $\mathcal{G}$ (Lemma 4.21), this implies $\left(\coprod f_i\right) h = 0$. But the definition of an α-perfect class allows us then to factor each $f_i : G_i \longrightarrow A_i$ as

$$G_i \xrightarrow{g_i} G_i' \xrightarrow{h_i} A_i$$

with $G_i' \in \mathcal{G}$ such that the composition $G \xrightarrow{h} \coprod G_i \xrightarrow{\coprod g_i} \coprod G_i'$ already vanishes. We finally have a commutative diagram

where the two horizontal arrows in the middle are isomorphisms by Lemma 4.19. We have just seen that the composition $\widetilde{F}(\coprod g_i)\widetilde{F}(h)$ vanishes. As a consequence, the composition $\left(\coprod \widetilde{F}(g_i)\right)k$ also vanishes and so does f. This shows γ_* is injective. $\qquad\square$

Proposition 4.25. *The homological functor $\widetilde{H}_0 : \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}}) \longrightarrow \mathrm{Mod}_\alpha\text{-}\mathcal{G}$ preserves coproducts of objects which are in the essential image of $\widetilde{F} : \mathrm{Ho}\,\mathcal{K} \longrightarrow \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$.*

Proof. Since $r : \mathrm{Mod}_\alpha\text{-}\mathcal{G} \longrightarrow \mathrm{Mod}\text{-}\mathcal{G}$ is fully faithful, it suffices to show that for any set-indexed family $(X_i)_{i\in I}$ of objects in $\mathrm{essim}\,\widetilde{F}$ the map

$$r\left(\coprod_{i\in I} \widetilde{H}_0(X_i)\right) \longrightarrow r\widetilde{H}_0\left(\coprod_{i\in I} X_i\right)$$

is an isomorphism. This map is the composition of the following isomorphisms, each of which will be explained below.

$$(4.26) \qquad r\left(\coprod_{i\in I}\tilde{H}_0(X_i)\right) \cong r\operatorname*{colim}_{I'\subset I,|I'|<\alpha}\left(\coprod_{i\in I'}\tilde{H}_0(X_i)\right)$$

$$(4.27) \qquad\qquad\qquad \cong \operatorname*{colim}_{I'\subset I,|I'|<\alpha} r\left(\coprod_{i\in I'}\tilde{H}_0(X_i)\right)$$

$$(4.28) \qquad\qquad\qquad \cong \operatorname*{colim}_{I'\subset I,|I'|<\alpha} r\tilde{H}_0\left(\coprod_{i\in I'}X_i\right)$$

$$(4.29) \qquad\qquad\qquad = \operatorname*{colim}_{I'\subset I,|I'|<\alpha} H_0R\left(\coprod_{i\in I'}X_i\right)$$

$$(4.30) \qquad\qquad\qquad \cong H_0R\left(\coprod_{i\in I}X_i\right)$$

$$(4.31) \qquad\qquad\qquad = r\tilde{H}_0\left(\coprod_{i\in I}X_i\right)$$

Ad (4.26). It is a general fact from category theory that a coproduct can be expressed as such a colimit.

Ad (4.27). The inclusion $r : \text{Mod}_\alpha\text{-}\mathcal{G} \longrightarrow \text{Mod-}\mathcal{G}$ preserves α-filtered colimits [Nee01b, Lemma A.1.3] and the colimit in question is indeed α-filtered in the sense of [Nee01b, Definition A.1.2]: it is a colimit over the category $\mathcal{I}$ with objects the subsets I' of I which have cardinality (strictly) less than α. Morphisms are the inclusions between two such subsets. Let $\mathcal{J}$ be a subcategory of $\mathcal{I}$ with less than α morphisms. In particular, $\mathcal{J}$ has less than α objects and we can conclude

$$\left|\bigcup_{J\in\mathcal{J}}J\right| \leq \left|\coprod_{J\in\mathcal{J}}J\right| = \sum_{J\in\mathcal{J}}|J| < \alpha$$

where, for the last inequality, we used that α is regular and hence any sum of less than α cardinals, all smaller than α, is itself smaller than α. Then $\bigcup_{J\in\mathcal{J}} J$ is an object of $\mathcal{I}$ admitting an arrow

$$J' \longrightarrow \bigcup_{J\in\mathcal{J}}J,$$

for every J' in $\mathcal{J}$. This shows that $\mathcal{I}$ is an α-filtered category.

Ad (4.28). By Lemma 4.24, $\widetilde{F}$ preserves α-coproducts. Since $\Phi \cong \tilde{H}_0\widetilde{F}$ (Lemma 4.14) and since Φ preserves arbitrary coproducts [Kra01, Theorem C], the functor $\tilde{H}_0$ has to preserve α-coproducts which are in the essential image of $\widetilde{F}$.

Ad (4.29). Here only the the equality $H_0R = r\tilde{H}_0$ is used.

Ad (4.30). Recall that colimits in Mod-$\mathcal{G}$ are formed objectwise. Using the adjunction isomorphism of the pair (L, R) and the isomorphism $LF \cong \widetilde{F}$ (Lemma 4.19), we obtain

for $G \in \mathcal{G}$ a natural isomorphism

$$\operatorname*{colim}_{I' \subset I, |I'| < \alpha} H_0 R \left(\coprod_{i \in I'} X_i \right) (G) \quad = \quad \operatorname*{colim}_{I' \subset I, |I'| < \alpha} \left[F(G), R \left(\coprod_{i \in I'} X_i \right) \right]^{\mathsf{D}(\mathcal{E}^{\mathrm{op}})}$$

$$\cong \quad \operatorname*{colim}_{I' \subset I, |I'| < \alpha} \left[\widetilde{F}(G), \coprod_{i \in I'} X_i \right]^{\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})}.$$

By Lemma 4.22, $\widetilde{F}(G)$ is in particular α-small, that is, any map from $\widetilde{F}(G)$ into a co-product factors through some sub-coproduct of less than α objects. Hence the canonical monomorphism

$$\operatorname*{colim}_{I' \subset I, |I'| < \alpha} \left[\widetilde{F}(G), \coprod_{i \in I'} X_i \right]^{\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})} \longrightarrow \left[\widetilde{F}(G), \coprod_{i \in I} X_i \right]^{\mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})}$$

is an isomorphism. Using again the adjunction isomorphism and $LF \cong \widetilde{F}$ we get an isomorphism $\left[\widetilde{F}(G), \coprod_{i \in I} X_i \right] \cong H_0 R \left(\coprod_{i \in I} X_i \right) (G)$ and hence the isomorphism (4.30).

Ad (4.31). This is again nothing but $H_0 R = r \widetilde{H}_0$. $\qquad \square$

Corollary 4.32. *The functor $\widetilde{F} : \operatorname{Ho} \mathcal{K} \longrightarrow \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ preserves coproducts.*

Proof. Since $\widetilde{H}_0$ reflects isomorphisms (Remark 4.12), it suffices to check that for any family $(A_i)_{i \in I}$ of objects in $\operatorname{Ho} \mathcal{K}$ the map

$$\widetilde{H}_0 \left(\coprod_{i \in I} \widetilde{F}(A_i) \right) \longrightarrow \widetilde{H}_0 \widetilde{F} \left(\coprod_{i \in I} A_i \right)$$

is an isomorphism. But this follows from Proposition 4.25 because $\widetilde{H}_0 \widetilde{F}$ is isomorphic to Φ (Lemma 4.14) and $\Phi : \operatorname{Ho} \mathcal{K} \longrightarrow \operatorname{Mod}_\alpha \text{-} \mathcal{G}$ preserves coproducts by [Kra01, Theorem C]. $\qquad \square$

Proposition 4.33. *The adjoint functors*

$$\operatorname{Ho} \mathcal{K} \overset{\widetilde{J}}{\underset{\widetilde{F}}{\longleftrightarrow}} \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}}).$$

are inverse equivalences of triangulated categories.

Proof. Consider for $X \in \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$ and $A \in \operatorname{Ho} \mathcal{K}$ the unit $\eta_X : X \longrightarrow \widetilde{F} \widetilde{J} X$ and the counit $\varepsilon_A : \widetilde{J} \widetilde{F}(A) \longrightarrow A$ of the adjunction. It suffices to show that both are isomorphisms for all objects $A \in \operatorname{Ho} \mathcal{K}$ and $X \in \mathsf{D}_\alpha(\mathcal{E}^{\mathrm{op}})$. The counit of the restricted adjunction $(\widetilde{J}, \widetilde{F})$ is the same as the counit of the adjunction (J, F), which is an isomorphism for $A \in \mathcal{G}$ by Lemma 4.1. Since both $\widetilde{J}$ and $\widetilde{F}$ preserve coproducts ($\widetilde{J}$ as a left adjoint and $\widetilde{F}$ by

Corollary 4.32) and since $\mathcal{G}$ generates $\operatorname{Ho}\mathcal{K}$, we can apply Lemma 1.3 and conclude that the counit is an isomorphisms for all objects.

If we apply the inclusion functor R to the unit of an object in $\widetilde{F}(\mathcal{G})$ we get a map

$$F(G) = R\widetilde{F}(G) \longrightarrow R\widetilde{F}\widetilde{J}\widetilde{F}(G) = FJF(G),$$

which is (up to isomorphism) the unit of the free module F_G with respect to the adjoint pair (J, F) and hence an isomorphism by Lemma 4.1. Again, since both $\widetilde{J}$ and $\widetilde{F}$ preserve coproducts and $\widetilde{F}(\mathcal{G})$ is a generating set for $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ by Lemma 4.22 we can conclude that the unit of the adjoint pair $(\widetilde{J}, \widetilde{F})$ is an isomorphism for all objects. $\qquad\square$

Note that, in particular, $F = R\widetilde{F}$ is fully faithful. Now we have all what we need to finish the proof of the characterization theorem.

Proof of Theorem 4.7, (i) $\Rightarrow$ (ii). Let $\mathcal{T}$ be a topological triangulated category, i.e., $\mathcal{T}$ is triangulated equivalent to the homotopy category of some spectral model category $\mathcal{K}$. If $\mathcal{T}$ is well generated then so is $\operatorname{Ho}\mathcal{K}$ and we can choose a regular cardinal α and a generating set $\mathcal{G}$ as in Notation 4.8 and let $\mathcal{E}$ (as before) be the full spectral subcategory of $\mathcal{K}$ with objects $\mathcal{G}$. By Proposition 4.33, $\operatorname{Ho}\mathcal{K}$ is equivalent to $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$, which is by Proposition 4.15 a localization of $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ with acyclics being generated by a set.
$\qquad\square$

Remark 4.34. The proof of Theorem 4.7 can also be read as a proof for the characterization of the topological *compactly* generated categories as the derived categories of spectral categories. Namely, compactly generated means we can choose $\alpha = \aleph_0$ and in this case $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}}) = \mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$, see Lemma 4.13.

5. A lift to the model category level

5.1. Bousfield localizations, properness, and cellularity.

Let us recall some notions from Hirschhorn's book [Hir03]. If $\mathcal{M}$ is a model category and $\mathcal{C}$ is a class of morphisms in $\mathcal{M}$, then an object W of $\mathcal{M}$ is called $\mathcal{C}$-*local* if it is fibrant and for every element $f : A \longrightarrow B$ of $\mathcal{C}$ the induced map $f^* : \operatorname{map}(B, W) \longrightarrow \operatorname{map}(A, W)$ is a weak equivalence of simplicial sets [Hir03, Definition 3.1.4]. Here $\operatorname{map}(X, Y)$ denotes a homotopy function complex between X and Y, which is a simplicial set that can in general be obtained by (co-)simplicial resolutions [Hir03, Section 17.4]. Given a cofibrant object X and a fibrant object Y in a simplicial model category, $\operatorname{map}(X, Y)$ can be chosen to be the (fibrant) simplicial set given by the enrichment [Hir03, Example 17.1.4]. Note that model categories of modules over a spectral category are spectral and thus simplicial [SS03b, Theorem A.1.1 and Lemma 3.5.2]). A $\mathcal{C}$-*local equivalence* is a map $g : X \longrightarrow Y$ such that for every $\mathcal{C}$-local object W of $\mathcal{M}$ the induced map $g^* : \operatorname{map}(Y, W) \longrightarrow \operatorname{map}(X, W)$ is a weak equivalence of simplicial sets [Hir03, Definition 3.1.4]. Every weak equivalence in $\mathcal{M}$ is in particular a $\mathcal{C}$-local equivalence [Hir03, Proposition 3.1.5]. For us, a Bousfield localization is what Hirschhorn calls a *left* Bousfield localization [Hir03, Definition 3.3.1]:

Definition 5.1. The *Bousfield localization* of a model category $\mathcal{M}$ with respect to a class of maps $\mathcal{C}$ is a model category $L_{\mathcal{C}}\mathcal{M}$ which has the same underlying category as $\mathcal{M}$ such that

- the weak equivalences in $L_{\mathcal{C}}\mathcal{M}$ are the $\mathcal{C}$-local equivalences,
- the cofibrations in $L_{\mathcal{C}}\mathcal{M}$ are the cofibrations in $\mathcal{M}$,
- and the fibrations in $L_{\mathcal{C}}\mathcal{M}$ are the maps which have the right lifting property with respect to those maps which are both cofibrations and weak equivalences in $L_{\mathcal{C}}\mathcal{M}$.

Remark 5.2. Here are some basic properties for a Bousfield localization $L_{\mathcal{C}}\mathcal{M}$ of $\mathcal{M}$.

(1) Since every weak equivalence in $\mathcal{M}$ is a $\mathcal{C}$-local equivalence and the cofibrations in $\mathcal{M}$ and $L_{\mathcal{C}}\mathcal{M}$ are the same, every fibration in $L_{\mathcal{C}}\mathcal{M}$ is in particular a fibration in $\mathcal{M}$.

(2) There is a Quillen pair

$$L_{\mathcal{C}}\mathcal{M} \; \underset{Q}{\overset{P}{\longleftrightarrow}} \; \mathcal{M}$$

where P and Q are the identity functors on underlying categories. The functor P has the following universal property. The left derived $P^L : \operatorname{Ho}\mathcal{M} \longrightarrow \operatorname{Ho}L_{\mathcal{C}}\mathcal{M}$ of P maps the images in $\operatorname{Ho}\mathcal{M}$ of the elements in $\mathcal{C}$ to isomorphisms in $\operatorname{Ho}L_{\mathcal{C}}\mathcal{M}$ and P is universal with this property, i.e., if $F : \mathcal{M} \longrightarrow \mathcal{N}$ is an arbitrary left Quillen functor whose left derived functor takes the images in $\operatorname{Ho}\mathcal{M}$ of the elements in $\mathcal{C}$ to isomorphisms in $\operatorname{Ho}L_{\mathcal{C}}\mathcal{N}$, then there exists a unique left Quillen functor $\bar{F} : L_{\mathcal{C}}\mathcal{M} \longrightarrow \mathcal{N}$ such that $\bar{F}P = F$ [Hir03, Theorem 3.3.19 and Definition 3.1.1].

(3) The right derived $Q^R : \operatorname{Ho}L_{\mathcal{C}}\mathcal{M} \longrightarrow \operatorname{Ho}\mathcal{M}$ is fully faithful. In particular, if $\mathcal{M}$ and $L_{\mathcal{C}}\mathcal{M}$ are stable, $\operatorname{Ho}L_{\mathcal{C}}\mathcal{M}$ is a localization of $\operatorname{Ho}\mathcal{M}$ in the sense of Definition 1.7. This can be seen as follows. Let X and Y be $L_{\mathcal{C}}$-fibrant objects, so that $Q^R(X) \cong X$ and $Q^R(Y) \cong Y$. We can further assume that X is cofibrant. Then the right derived Q^R is on morphisms the map induced by the identity

$$L_{\mathcal{C}}\mathcal{M}(X,Y)/\sim \; \longrightarrow \; \mathcal{M}(X,Y)/\sim$$

where $\sim$ denotes the equivalence relation given by (left) homotopy. This map is clearly surjective. Assume we have $f \sim g$ in $\mathcal{M}$ via a cylinder object

in $\mathcal{M}$. But this is also a cylinder object in $L_{\mathcal{C}}\mathcal{M}$ and thus $f \sim g$ in $L_{\mathcal{C}}\mathcal{M}$.

We want to apply Hirschhorn's existence theorem for Bousfield localizations [Hir03, Theorem 4.1.1]. It states that for any left proper cellular model category $\mathcal{M}$ and any *set* $\mathcal{C}$ of morphisms, the Bousfield localization $L_{\mathcal{C}}\mathcal{M}$ exists. Recall that a model category is called *left proper* if pushouts along cofibrations preserve weak equivalences [Hir03, Definition 13.1.1].

Definition 5.3. A spectral category $\mathcal{R}$ is called *pointwise cofibrant* if the symmetric spectrum $\mathcal{R}(R, R')$ is cofibrant for all R, R' in $\mathcal{R}$.

Lemma 5.4. *If $\mathcal{R}$ is a spectral category which is pointwise cofibrant (Definition 5.3), then Mod-$\mathcal{R}$ is left proper.*

Proof. Since weak equivalences in Mod-$\mathcal{R}$ are defined objectwise and the stable model category Sp^{Σ} of symmetric spectra is left proper [HSS00, Theorem 5.5.2], this is an immediate consequence of Corollary A.13. □

A *cellular* model category is a certain kind of cofibrantly generated model category that allows sets I, resp. J, of generating cofibrations, resp. trivial cofibrations, such that I and J satisfy some finiteness conditions on the domains and codomains of their elements. The precise definition of cellular is technical and we will not give it here; the only place we will use the very definition is the proof of Proposition 5.6. The reader is referred to Hirschhorn's book [Hir03, Definition 12.1.1]. (To be precise, we need a slightly stronger version of the definition from the book, namely the one Hirschhorn has given in earlier drafts; see the proof of Proposition 5.6.) Examples of cellular model categories are (pointed or unpointed) simplicial sets, topological spaces, and spectra, as the following lemma states.

Lemma 5.5. *The stable model category Sp^{Σ} of symmetric spectra is cellular.*

Proof. Hovey has shown that symmetric spectra with the projective model structure form a cellular model category [Hov01, Theorem A.9] where the set of generating cofibrations can be chosen as

$$I = \{\mathrm{F}_n(\partial\Delta[r]_+) \longrightarrow \mathrm{F}_n(\Delta[r]_+) \mid r, n \geq 0\}.$$

Here, $\partial\Delta[r]_+ \longrightarrow \Delta[r]_+$ denotes the inclusion of the boundary of the simplicial r-simplex into the simplicial r-simplex (plus additional basepoints, respectively); these simplicial maps from a set of generating cofibrations for the model category of pointed simplicial sets. The functor F_n from pointed simplicial sets to symmetric spectra is left adjoint to the n-th evaluation functor. The category of symmetric spectra with the stable model structure can be obtained as a Bousfield localization of the projective model structure [Hov01, Section 8]. Hence, by [Hir03, Theorem 4.1.1], the stable model category of symmetric spectra is also cellular and the set of generating cofibrations can still be chosen to be I. □

Proposition 5.6. *If $\mathcal{R}$ is a spectral category which is pointwise cofibrant (Definition 5.3), then Mod-$\mathcal{R}$ is cellular.*

A one object version of this proposition occurs in a paper by Hovey [Hov98, Proposition 2.9] – but he does not give the proof there because the definition of cellular is somewhat technical. We will give the proof using ideas of an unpublished pre-version of Hirschhorn's book, where enriched diagram categories have been studied.

Proof of Proposition 5.6. Let I be a set of generating cofibrations and J a set of generating trivial cofibrations which make the category of symmetric spectra into a cellular model

category. Recall that I can be chosen as

$$I = \{F_n(\partial\Delta[r]_+) \longrightarrow F_n(\Delta[r]_+) \mid r, n \geq 0\}.$$

By [SS03b, Theorem A.1.1], Mod-$\mathcal{R}$ is a cofibrantly generated model category. A set of generating cofibrations is given by

$$\widetilde{I} = \bigcup_{R \in \mathcal{R}} I \wedge F_R$$

where $F_R = \mathcal{R}(-, R)$ is the free module with respect to R and $I \wedge F_R$ consists of all maps of the form $f \wedge F_R : A \wedge F_R \longrightarrow B \wedge F_R$ for some $f : A \longrightarrow B$ in I. Similarly, a set of generating trivial cofibrations is given by

$$\widetilde{J} = \bigcup_{R \in \mathcal{R}} J \wedge F_R.$$

We have to show that $\widetilde{I}$ and $\widetilde{J}$ satisfy the following three conditions [Hir03, Definition 12.1.1].

(i) For every element $A \longrightarrow B$ of I and every object R of $\mathcal{R}$, the modules $A \wedge F_R$ and $B \wedge F_R$ are compact relative to $\widetilde{I}$ in the sense of [Hir03, Definition 10.8.1].

(ii) For every element $A \longrightarrow B$ of J and every object R of $\mathcal{R}$, the module $A \wedge F_R$ is cofibrant and small relative to $\widetilde{I}$ in the sense of [Hir03, Definition 10.5.12].

(iii) The cofibrations in Mod-$\mathcal{R}$ are effective monomorphisms [Hir03, Definition 10.9.1].

Note that condition (ii) is stronger than in [Hir03, Definition 12.1.1] (where the domains of the generating trivial cofibrations are not required to be cofibrant). It occurs in this stronger form in earlier drafts of Hirschhorn's book, and as Hirschhorn pointed out to me, the earlier definition of cellular is the right one for the existence theorem of Bousfield localizations.

Ad (i). Fix an element $A \longrightarrow B$ of I. Let $X \longrightarrow Y$ be a relative $\widetilde{I}$-cell complex [Hir03, Definition 10.5.8]. This means, $X \longrightarrow Y$ is the composition of a λ-sequence

$$X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_\beta \longrightarrow \cdots \quad (\beta < \lambda)$$

(for some ordinal λ) such that $X_{\beta+1}$ is obtained from X_β by attaching a set of $\widetilde{I}$-cells, i.e., there are pushout diagrams

$$\begin{array}{ccc} \coprod_i A_i \wedge F_{R_i} & \longrightarrow & \coprod_i B_i \wedge F_{R_i} \\ \downarrow & & \downarrow \\ X_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

for elements $A_i \longrightarrow B_i$ of I and objects R_i of $\mathcal{R}$. Such a relative $\widetilde{I}$-cell complex is *presented* if a particular λ-sequence and certain gluing maps for the pushout diagrams are specified so that one can consider subcomplexes of $X \longrightarrow Y$, see [Hir03, Section 10.6] for the details.

We have to show that there exists a cardinal γ such that for every presented $\widetilde{I}$-cell complex $X \longrightarrow Y$, every map $A \wedge F_R \longrightarrow Y$ factors through a subcomplex $X \longrightarrow Y'$ of

size at most γ (i.e., the set of cells has cardinality at most γ). Via the (Quillen) adjunction

$$(5.7) \qquad\qquad \text{Mod-}\mathcal{R} \; \xrightleftharpoons[\text{ev}_R]{-\wedge F_R} \; \text{Mod-}\mathbb{S} = \text{Sp}^\Sigma \quad ,$$

a map $A \wedge F_R \longrightarrow Y$ corresponds to a map $A \longrightarrow Y(R)$. Since colimits are preserved by the evaluation functor (Corollary A.13), $X(R) \longrightarrow Y(R)$ is the composition of the presented λ-sequence

$$X(R) = X_0(R) \longrightarrow X_1(R) \longrightarrow X_2(R) \longrightarrow \cdots \longrightarrow X_\beta(R) \longrightarrow \cdots$$

and $X_{\beta+1}(R)$ is obtained from $X_\beta(R)$ by attaching I'-cells, where

$$I' = \bigcup_{R' \in \mathcal{R}} I \wedge F_{R'}(R).$$

But this means that $X(R) \longrightarrow Y(R)$ is a presented I'-cell complex. If we can show that A is compact relative to I', we can conclude that there exists a cardinal γ (which does not depend on $X \longrightarrow Y$ and $A \wedge F_R \longrightarrow Y$) such that $A \longrightarrow Y(R)$ factors through a subcomplex of size at most γ. Via adjunction, this corresponds to a factorization of $A \wedge F_R \longrightarrow Y$ through a subcomplex of the same size.

It remains to prove that A is compact relative to I' in Sp^Σ. For that purpose we want to apply [Hir03, Proposition 11.4.9]. A stable cofibration of symmetric spectra is in particular a level cofibration [HSS00, Corollary 5.1.5] and hence a monomorphism. Since $F_{R'}(R) = \mathcal{R}(R, R')$ was assumed to be cofibrant, all elements in I' are cofibrations in Sp^Σ. The domains of the elements of I', which are of the form $\text{F}_n(\partial\Delta[r]_+) \wedge F_{R'}(R)$ [HSS00, Definition 3.3.2], are cofibrant since $F_{R'}(R)$ is cofibrant by assumption and every spectrum of the form $\text{F}_n(K)$ for a pointed simplicial set K is cofibrant [HSS00, Proposition 3.4.2]. By [Hir03, Corollary 12.3.4], it follows that the domains of the elements of I' are compact (relative to I). Since Sp^Σ is cellular (Lemma 5.5) and A is the domain of an element of I, the object A is compact (relative to I). Now we have all we need to apply [Hir03, Proposition 11.4.9], which tells us that A is also compact relative to I'. Hence we have shown that $A \wedge F_R$ is compact relative to $\widetilde{I}$. The same proof shows that $B \wedge F_R$ is compact relative to $\widetilde{I}$.

Ad (ii). Let $A \longrightarrow B$ be an element of J and R an object of $\mathcal{R}$. Since Sp^Σ together with J as a set of generating trivial cofibrations is cellular, A is cofibrant. Applying the left Quillen functor $- \wedge F_R : \text{Sp}^\Sigma \longrightarrow \text{Mod-}\mathcal{R}$ shows that $A \wedge F_R$ is cofibrant. For the smallness, we need to show that there exists a cardinal κ such that for all $\lambda \geq \kappa$ and all λ-sequences

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_\beta \longrightarrow \cdots$$

in $\text{Mod-}\mathcal{R}$ where the maps $X_\beta \longrightarrow X_{\beta+1}$ are relative $\widetilde{I}$-cell complexes, the map $\text{colim}_{\beta<\lambda} \text{Mod-}\mathcal{R}(A \wedge F_R, X_\beta) \longrightarrow \text{Mod-}\mathcal{R}(A \wedge F_R, \text{colim}_{\beta<\lambda} X_\beta)$ is an isomorphism of sets [Hir03, Definition 10.4.1]. Using again the adjunction (5.7) and the fact that ev_R preserves colimits (Corollary A.13), we can conclude that it suffices to show A is small relative to relative I'-cell complexes in Sp^Σ. Since Sp^Σ is cellular, A (which is the domain of an element of J) is small relative to I and thus, by [Hir03, Proposition 11.2.3], relative to the class of all cofibrations. But the elements of I' are cofibrations because

$F_{R'}(R) = \mathcal{R}(R, R')$ is cofibrant by assumption. Hence all relative I'-cell complexes are cofibrations. This shows that A is small relative to relative I'-cell complexes.

Ad (iii). We have to show that every cofibration $f : A \longrightarrow B$ in Mod-$\mathcal{R}$ is the equalizer of the canonical pair of maps $B \rightrightarrows B \amalg_A B$. We can choose an equalizer E of this pair and get an induced map g.

$$E \longrightarrow B \rightrightarrows B \amalg_A B \quad .$$

$$\begin{array}{c} {}_{g}\Big\uparrow \quad \nearrow {}^{f} \\ A \end{array}$$

By Corollary A.13, $\mathrm{ev}_R(f)$ is a cofibration in Sp^{Σ} and thus an effective monomorphism by Lemma 5.5. Since for every R in $\mathcal{R}$ the map ev_R preserves limits, $\mathrm{ev}_R(g)$ is an isomorphism for each R. Hence g is an isomorphism. $\qquad\square$

5.2. **Well generated stable model categories.** By a well generated stable model category we mean a stable model category whose homotopy category is well generated as a triangulated category (Definition 3.1). One implication of Theorem 4.7 says that a topological well generated triangulated category is equivalent to a localization of the derived category of a spectral category. Theorem 5.11 lifts this result to the level of model categories. Roughly speaking, well generated spectral model categories are localizations of categories of modules. Recall from Examples 4.6 that there are, up to Quillen equivalence, rather large classes of spectral model categories.

Lemma 5.8. *Let $\mathcal{K}$ be a left proper spectral model category and $\mathcal{C}$ a set of maps in $\mathcal{K}$ such that the localization $\mathrm{L}_{\mathcal{C}}\mathcal{K}$ exists. Assume the domains and codomains of the maps in $\mathcal{C}$ are cofibrant and the image $\mathcal{W}$ of $\mathcal{C}$ in $\mathrm{Ho}\,\mathcal{K}$ is, up to isomorphism, closed under (de-)suspensions.*

Then the following are equivalent for an object X in $\mathcal{K}$.

(i) *X is $\mathrm{L}_{\mathcal{C}}$-fibrant (i.e., fibrant in $\mathrm{L}_{\mathcal{C}}\mathcal{K}$).*
(ii) *X is $\mathcal{C}$-local.*
(iii) *X is fibrant (in the original model structure) and, considered as an object in $\mathrm{Ho}\,\mathcal{K}$, it lies in $\mathcal{W}$-loc (cf. Lemma 3.4).*

Proof. (i) $\Leftrightarrow$ (ii) This is [Hir03, Propostion 3.4.1].

(ii) $\Leftrightarrow$ (iii) By definition, X is $\mathcal{C}$-local if and only if it is fibrant (in the original model structure) and for every element $f : A \longrightarrow B$ of $\mathcal{C}$ the induced map $f^* : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$ is a weak equivalence of simplicial sets. Now the simplicial enrichment of $\mathcal{K}$ is induced by the spectral enrichment, that is, $f^* : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$ is the level zero map of a map of spectra, which are fibrant (i.e., Ω-spectra) since A and B are cofibrant and X is fibrant. Hence f^* is a map of loop spaces and thus a weak equivalence if and only if for the distinguished basepoint, $\pi_n(f^*) : \pi_n \mathrm{map}(B, X) \longrightarrow \pi_n \mathrm{map}(A, X)$ is an isomorphism for all $n \geq 0$. Since A and B are cofibrant and X is fibrant, the map $\pi_n(f^*)$ is naturally isomorphic to the map $f^* : [\Sigma^n B, X] \longrightarrow [\Sigma^n A, X]$ of morphism groups in $\mathrm{Ho}\,\mathcal{K}$. Since $\mathcal{W}$ is closed under (de-)suspensions, the $\mathrm{L}_{\mathcal{C}}$-fibrant objects in $\mathcal{K}$

are exactly the fibrant objects X such that the induced map $f^* : [B, X] \longrightarrow [A, X]$ is an isomorphism for all $f : A \longrightarrow B$ in $\mathcal{W}$. In other words, the $\mathrm{L}_\mathcal{C}$-fibrant objects are precisely the fibrant objects which, considered as objects in $\mathrm{Ho}\,\mathcal{K}$, lie in $\mathcal{W}$-loc. $\qquad\square$

Corollary 5.9. *If in the situation of Lemma* 5.8

$$(*) \qquad\qquad \mathrm{Ho}\,\mathrm{L}_\mathcal{C}\mathcal{K} \underset{Q^R}{\overset{P^L}{\rightleftarrows}} \mathrm{Ho}\,\mathcal{K}$$

denotes the derived adjunction, then $\operatorname{essim} Q^R = \mathcal{W}$-loc, *and the model category* $\mathrm{L}_\mathcal{C}\mathcal{K}$ *is stable, such that* $\mathrm{Ho}\,\mathrm{L}_\mathcal{C}\mathcal{K}$ *is a localization of* $\mathrm{Ho}\,\mathcal{K}$ *in the sense of Definition* 1.7 *via* $(*)$.

Proof. The essential image of Q^R contains all objects which are, up to isomorphism in $\mathrm{Ho}\,\mathcal{K}$, $\mathrm{L}_\mathcal{C}$-fibrant. By Lemma 5.8, the $\mathrm{L}_\mathcal{C}$-fibrant objects are the fibrant objects which are in $\mathcal{W}$-loc. Since both $\operatorname{essim} Q^R$ and $\mathcal{W}$-loc are replete subcategories of $\mathrm{Ho}\,\mathcal{K}$ (i.e., they contain the whole isomorphism class of any of their objects), it follows that they coincide.

In particular, $\operatorname{essim} Q^R$ is a (colocalizing) triangulated subcategory of $\mathrm{Ho}\,\mathcal{K}$ and hence closed under (de-)suspensions. This implies that $\mathrm{Ho}\,\mathrm{L}_\mathcal{C}\mathcal{K}$ (which is isomorphic to $\operatorname{essim} Q^R$ since Q^R is fully faithful, see Remark 5.2(3)) is also stable. The details are as follows. Let Σ, resp. Ω, denote the suspension, resp. desuspension, in $\mathrm{Ho}\,\mathcal{K}$, and Σ', resp. Ω', the suspension, resp. desuspension, in $\mathrm{Ho}\,\mathrm{L}_\mathcal{C}\mathcal{M}$. We have to show that Σ' and Ω' are inverse equivalences. As a left derived, P^L commutes with suspension, while Q^R commutes with desuspension [Hov99, Proposition 6.4.1]. Let us first check that Q^R also commutes with suspension. Since $\operatorname{essim} Q^R$ is a triangulated subcategory of $\mathrm{Ho}\,\mathcal{K}$, there is an isomorphism $\Sigma Q^R X \cong Q^R Y$ for some Y in $\mathrm{Ho}\,\mathrm{L}_\mathcal{C}\mathcal{K}$. Using that the counit of the adjunction (P^L, Q^R) is an isomorphism (Q^R is fully faithful) we get an induced isomorphism

$$Y \cong P^L Q^R Y \cong P^L \Sigma Q^R X \cong \Sigma' P^L Q^R X \cong \Sigma' X$$

so that $\Sigma Q^R X$ is naturally isomorphic to $Q^R \Sigma' X$. Now we have isomorphisms

$$\Sigma'\Omega' \cong P^L Q^R \Sigma'\Omega' \cong P^L \Sigma Q^R \Omega' \cong P^L \Sigma\Omega Q^R \cong P^L Q^R \cong \mathrm{id}$$

and

$$\Omega'\Sigma' \cong P^L Q^R \Omega'\Sigma' \cong P^L \Omega Q^R \Sigma' \cong P^L \Omega\Sigma Q^R \cong P^L Q^R \cong \mathrm{id}$$

which show that Ω' and Σ' are inverse equivalences. $\qquad\square$

Lemma 5.10. *Let* $\mathcal{K}$ *be a left proper spectral model category and* $\mathcal{C}$ *a set of maps in* $\mathcal{K}$, *closed under (de-)suspensions in* $\mathrm{Ho}\,\mathcal{K}$ *(up to isomorphism). If the localization* $\mathrm{L}_\mathcal{C}\mathcal{K}$ *exists, it is a left proper spectral model category.*

Proof. Bousfield localizations of left proper model categories are always left proper [Hir03, Proposition 3.4.4]. For the proof of the spectral part, we can without loss of generality assume that the domains and codomains of the maps in $\mathcal{C}$ are cofibrant (otherwise we could cofibrantly replace them, this would, up to isomorphism of model categories, have no effect on $\mathrm{L}_\mathcal{C}\mathcal{K}$.) Recall that the underlying categories of $\mathrm{L}_\mathcal{C}\mathcal{K}$ and $\mathcal{K}$ are the same. Since $\mathcal{K}$ is spectral, it has a tensor, a cotensor, and an enriched Hom-functor. We show that the same functors make $\mathrm{L}_\mathcal{C}\mathcal{K}$ into a spectral category. It suffices to verify the pushout

product axiom [Hov99, Definition 4.2.1]. Let $f : A \rightarrowtail B$ be a cofibration in Sp^{Σ} and $g : X \rightarrowtail Y$ a cofibration in $\mathrm{L}_{\mathcal{C}}\mathcal{K}$. Since $\mathcal{K}$ and $\mathrm{L}_{\mathcal{C}}\mathcal{K}$ have the same cofibrations and since the pushout product axiom holds for $\mathcal{K}$, the map

$$f \,\square\, g : (A \wedge Y) \amalg_{A \wedge X} (B \wedge X) \longrightarrow B \wedge Y$$

is a cofibration, which is trivial if f is. Now assume the cofibration g is trivial in $\mathrm{L}_{\mathcal{C}}\mathcal{K}$. We have to show that $f \,\square\, g$ is also trivial. By Corollary 5.9, $\mathrm{L}_{\mathcal{C}}\mathcal{K}$ is stable. So using the general fact [Hov99, Theorem 1.2.10(iv)] that a map is a weak equivalence if and only if its image in the homotopy category is an isomorphism, it suffices to show that the cofiber of $f \,\square\, g$ is trivial in $\mathrm{Ho}\,\mathrm{L}_{\mathcal{C}}\mathcal{K}$. Let C be the cofiber of f and Z the cofiber of g. As cofibers of cofibrations both C and Z are cofibrant; Z is also trivial (in $\mathrm{Ho}\,\mathrm{L}_{\mathcal{C}}\mathcal{K}$) because g was assumed to be trivial. The cofiber of $f \,\square\, g$ is just $C \wedge Z$. By what we have already shown, the left adjoint $- \wedge Z : \mathrm{Sp}^{\Sigma} \longrightarrow \mathrm{L}_{\mathcal{C}}\mathcal{K}$ preserves (trivial) cofibrations, hence it is a left Quillen functor and the kernel of its left derived $- \wedge^{L} Z : \mathrm{Ho}\,\mathrm{Sp}^{\Sigma} \longrightarrow \mathrm{Ho}\,\mathrm{L}_{\mathcal{C}}\mathcal{K}$ is a localizing triangulated subcategory of $\mathrm{Ho}\,\mathrm{Sp}^{\Sigma}$. It contains the sphere spectrum $\mathbb{S}$ because Z is trivial in $\mathrm{Ho}\,\mathrm{L}_{\mathcal{C}}\mathcal{K}$. Since the sphere spectrum is a generator for $\mathrm{Ho}\,\mathrm{Sp}^{\Sigma}$, the kernel of $- \wedge^{L} Z$ is the whole stable homotopy category $\mathrm{Ho}\,\mathrm{Sp}^{\Sigma}$. In particular, since C is cofibrant, we have isomorphisms $C \wedge Z \cong C \wedge^{L} Z \cong 0$ in $\mathrm{Ho}\,\mathrm{L}_{\mathcal{C}}\mathcal{K}$. This shows that the cofiber of $f \,\square\, g$ is trivial and finishes the proof. $\qquad\square$

Theorem 5.11. *Every well generated spectral model category admits a right Quillen equivalence to a Bousfield localization of the model category of modules over some spectral category.*

Proof. Let $\mathcal{K}$ be a well generated spectral model category, we can thus assume we are in the situation of Notation 4.8, where we fixed a sufficiently nice generating set $\mathcal{G}$ for $\mathrm{Ho}\,\mathcal{K}$ and we defined $\mathcal{E}$ to be the full spectral subcategory of $\mathcal{K}$ having $\mathcal{G}$ as set of objects. Recall that there is a spectral Quillen adjunction [SS03b, Section 3.9]

$$\mathcal{K} \underset{\mathrm{Hom}(\mathcal{G},-)}{\overset{- \wedge_{\mathcal{E}} \mathcal{G}}{\longleftarrow\!\!\!\longrightarrow}} \mathrm{Mod}\text{-}\mathcal{E},$$

which we already considered in Section 4.1. The spectral category $\mathcal{E}$ will not be pointwise cofibrant (Definition 5.3) in general. But by [SS03a, Proposition 6.3], the spectral categories with a fixed set of objects form a model category with pointwise weak equivalences and pointwise fibrations and where cofibrant objects are in particular pointwise cofibrant. So we can chose a cofibrant replacement $\mathcal{E}^{\,\mathrm{cof}}$ of $\mathcal{E}$ with the same set of objects $\mathcal{G}$, such that $\mathcal{E}^{\,\mathrm{cof}}$ is pointwise cofibrant. By [SS03a, Theorem 7.2], the corresponding module categories are related by a Quillen equivalence (given by extension and restriction of scalars), which we denote by

$$\mathrm{Mod}\text{-}\mathcal{E} \underset{U}{\overset{V}{\longleftarrow\!\!\!\longrightarrow}} \mathrm{Mod}\text{-}\mathcal{E}^{\,\mathrm{cof}}.$$

We define a set $\mathcal{C}$ of maps in $\mathrm{Mod}\text{-}\mathcal{E}^{\,\mathrm{cof}}$ by modifying the set $\mathcal{W}$ of the maps

$$\coprod_{i \in I} F(G_i) \longrightarrow F\Big(\coprod_{i \in I} G_i\Big)$$

in $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$, which we used in Proposition 4.15 to get the localization $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ of $\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$. We let $\mathcal{C}$ be the set of maps

$$\left(U\!\left(\left(\coprod_{i\in I}\mathrm{Hom}(\mathcal{G},G_i)^{\,\mathrm{cof}}\right)^{\mathrm{fib}}\right)\right)^{\mathrm{cof}} \;\longrightarrow\; \left(U\!\left(\mathrm{Hom}\!\left(\mathcal{G},\coprod_{i\in I}G_i\right)^{\mathrm{fib}}\right)\right)^{\mathrm{cof}}$$

where $(G_i)_{i\in I}$ runs through all families in $\mathcal{G}$ with $|I|<\alpha$ and we allow one and only one set I for each cardinality smaller than α. The decorations 'cof' and 'fib' denote cofibrant and fibrant replacements. Let $\mathcal{W}'$ denote the image in $\mathsf{D}((\mathcal{E}^{\,\mathrm{cof}})^{\mathrm{op}})$ of the maps in $\mathcal{C}$. Then $\mathcal{W}\subset\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ and $\mathcal{W}'\subset\mathsf{D}((\mathcal{E}^{\,\mathrm{cof}})^{\mathrm{op}})$ correspond via the inverse triangulated equivalences V^L and U^R.

By Lemma 5.4 and Proposition 5.6, Mod-$\mathcal{E}^{\,\mathrm{cof}}$ is left proper cellular and we can apply Hirschhorn's existence theorem for Bousfield localizations [Hir03, Theorem 4.1.1], so that we obtain a Bousfield localization $\mathrm{L}_\mathcal{C}\mathrm{Mod}\text{-}\mathcal{E}^{\,\mathrm{cof}}$ (Definition 5.1) together with a Quillen functor pair

$$\mathrm{L}_\mathcal{C}\mathrm{Mod}\text{-}\mathcal{E}^{\,\mathrm{cof}} \quad\overset{P}{\underset{Q}{\rightleftarrows}}\quad \mathrm{Mod}\text{-}\mathcal{E}^{\,\mathrm{cof}}$$

where P and Q are the identity functors on underlying categories and the right derived Q^R is fully faithful (Remark 5.2(3)). Consider the following two diagrams of solid arrows.

The left one is a diagram of model categories, the right one a diagram of homotopy categories, resp. triangulated categories. The right diagram contains all derived functors from the left one, but there are also functors which are only defined on the triangulated level since $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ has only been defined as a triangulated category (Definition 4.11) and not as the homotopy category of any model category. The adjoint pairs (J,F), (L,R) and $(\widetilde{J},\widetilde{F})$ have been studied in Section 4; $\widetilde{J}$ and $\widetilde{F}$ are inverse triangulated equivalences (Proposition 4.33).

If we apply the composition JV^L of left derived functors to a map in $\mathcal{C}$ we get the map $J(\coprod_{i\in I}F(G_i)) \longrightarrow JF(\coprod_{i\in I}G_i)$. Since $J=\widetilde{J}L$, this map is an isomorphism by Proposition 4.15. Hence the universal property of the Bousfield localization yields a Quillen functor pair $(\overline{-\wedge_\mathcal{E}\mathcal{G}},\overline{\mathrm{Hom}(\mathcal{G},-)})$ such that $(\overline{-\wedge_\mathcal{E}\mathcal{G}})\circ P = (-\wedge_\mathcal{E}\mathcal{G})\circ V$ (and hence $Q\circ(\overline{\mathrm{Hom}(\mathcal{G},-)}) = U\circ\mathrm{Hom}(\mathcal{G},-)$).

Our goal is to show that $\overline{\mathrm{Hom}(\mathcal{G},-)}$ is a Quillen equivalence. It suffices to check that $\bar{J}$ is a triangulated equivalence. By Corollary 5.9, the essential image of Q^R is the same as $\mathcal{W}'$-loc. Since $\mathsf{D}_\alpha(\mathcal{E}^{\,\mathrm{op}})$ is the same as $\mathcal{W}$-loc (Proposition 4.15) and $\mathcal{W}\subset\mathsf{D}(\mathcal{E}^{\,\mathrm{op}})$ corresponds to $\mathcal{W}'\subset\mathsf{D}((\mathcal{E}^{\,\mathrm{cof}})^{\mathrm{op}})$ via the equivalences V^L and U^R, we get an induced

equivalence of categories $H : \mathrm{Ho}(\mathrm{L}_{\mathcal{C}}\mathrm{Mod}\text{-}\mathcal{E}^{\,\mathrm{cof}}) \longrightarrow \mathsf{D}((\mathcal{E}^{\,\mathrm{cof}})^{\,\mathrm{op}})$ such that $RH \cong V^L Q^R$. Recall that the right adjoint Q^R is fully faithful and the counit is hence an isomorphism $P^L Q^R \cong \mathrm{id}$. Moreover, the equality $(\overline{-\wedge_{\mathcal{E}} \mathcal{G}}) \circ P = (-\wedge_{\mathcal{E}} \mathcal{G}) \circ V$ gives us an isomorphism $\bar{J} P^L \cong J V^L$. Using these, we obtain an isomorphism

$$\widetilde{J}H \cong JRH \cong JV^L Q^R \cong \bar{J} P^L Q^R \cong \bar{J}$$

and thus $\bar{J}$ is an equivalence since $\widetilde{J}$ and H are equivalences. This shows that the Quillen pair $(\overline{-\wedge_{\mathcal{E}} \mathcal{G}}, \overline{\mathrm{Hom}(\mathcal{G},-)})$ is indeed a Quillen equivalence.

$\square$

Corollary 5.12. *Let $\mathcal{K}$ be a model category. Then the following are equivalent.*

(i) *$\mathcal{K}$ is Quillen equivalent to a well generated spectral model category.*

(ii) *$\mathcal{K}$ is Quillen equivalent to a Bousfield localization $\mathrm{L}_{\mathcal{C}}\mathrm{Mod}\text{-}\mathcal{R}$ for some pointwise cofibrant spectral category $\mathcal{R}$ and some set $\mathcal{C}$ of morphisms in $\mathrm{Mod}\text{-}\mathcal{R}$ whose image in $\mathsf{D}(\mathcal{R}^{\,\mathrm{op}}) = \mathrm{Ho}\,\mathrm{Mod}\text{-}\mathcal{R}$ is, up to isomorphism, closed under (de-)suspensions.*

Proof. (i) $\Rightarrow$ (ii) This is Theorem 5.11. Its proof shows that the image of $\mathcal{C}$ in the homotopy category is indeed closed, up to isomorphism, under (de-)suspensions.

(ii) $\Rightarrow$ (i) We may assume that the domains and codomains of the elements of $\mathcal{C}$ are cofibrant (otherwise we could replace them cofibrantly, this would not have any effect on the localization). Let $\mathcal{W}$ denote the image of $\mathcal{C}$ in $\mathsf{D}(\mathcal{R})$. By Lemma 5.10, $\mathrm{L}_{\mathcal{C}}\mathrm{Mod}\text{-}\mathcal{R}$ is spectral. Its homotopy category is a localization of the compactly generated triangulated category $\mathsf{D}(R^{\,\mathrm{op}}) = \mathrm{Ho}\,\mathrm{Mod}\text{-}\mathcal{R}$ by Corollary 5.9. Here, the acyclics are generated by the set containing one cofiber for each map in $\mathcal{W}$ (cf. Lemma 3.4). Hence, by Theorem 4.7, the homotopy category of $\mathrm{L}_{\mathcal{C}}\mathrm{Mod}\text{-}\mathcal{R}$ is well generated.

$\square$

APPENDIX A. MODULE CATEGORIES

A.1. **The one object case.** We use the terminology of [Hov99, Chapter 4]. Let $\mathcal{C}$ be a closed symmetric monoidal category with $\wedge$ denoting the monoidal product, $\mathbb{S}$ the unit, and Hom the internal Hom-functor (the *set* of morphisms from X to Y will be denoted by $\mathcal{C}(X,Y)$). Let R, S and T be monoids therein. We can then consider module categories, even bimodule categories like R-Mod-S (which is isomorphic to the category of $R \wedge S^{\mathrm{op}}$-modules). There are bifunctors

$$\wedge_S : R\text{-Mod-}S \ \times \ S\text{-Mod-}T \ \longrightarrow \ R\text{-Mod-}T,$$
(A.1)
$$\mathrm{Hom}_R : (R\text{-Mod-}S)^{\mathrm{op}} \ \times \ R\text{-Mod-}T \ \longrightarrow \ S\text{-Mod-}T,$$
$$\mathrm{Hom}_T : (S\text{-Mod-}T)^{\mathrm{op}} \ \times \ R\text{-Mod-}T \ \longrightarrow \ R\text{-Mod-}S,$$

where the last one should better be denoted by $\mathrm{Hom}_{T^{\mathrm{op}}}$ instead, but let us allow ourselves this slight abuse of notation. The object $X \wedge_S Y$ is defined as the coequalizer in $\mathcal{C}$ of the diagram

$$X \wedge S \wedge Y \ \rightrightarrows \ X \wedge Y$$

where the upper map uses the right action of S on X and the lower the left action of S on Y. This gives an object in $\mathcal{C}$ which has a left action of R via the left action of R on X and a right action of T via the right action of T on Y. Similarly, $\mathrm{Hom}_R(X,Y)$ is defined as the equalizer in $\mathcal{C}$ of the diagram

$$\mathrm{Hom}(X,Y) \ \rightrightarrows \ \mathrm{Hom}(R \wedge X, Y).$$

Here, both maps can be defined via their adjoint maps

$$R \wedge X \wedge \mathrm{Hom}(X,Y) \ \rightrightarrows \ Y,$$

where the upper map is first using the action of R on X and then the evaluation map, and the lower is first evaluation and then the action of R on Y. The left S-action on $\mathrm{Hom}_R(X,Y)$ comes from the right S-action on the contravariant variable X, the right T-action comes from the right T-action on Y. The three bifunctors (A.1) give an adjunction of two variables in the sense of [Hov99, Definition 4.1.12]. Note that the forgetful functor and the functor given by smashing with the free module yield an adjunction

(A.2)
$$R\text{-Mod} \ \underset{\xleftarrow{\hspace{1cm}}}{\overset{R \wedge -}{\xrightarrow{\hspace{1cm}}}} \ \mathcal{C}.$$

From now on let $\mathcal{C}$ be a closed symmetric monoidal *model* category [Hov99, Definition 4.2.6] which is cofibrantly generated [Hov99, Definition 2.1.17], has only small objects (small in the sense of [SS00], i.e., κ-small with respect to the whole category for some cardinal κ), and satisfies the monoid axiom [SS00, Definition 3.3]. There are many examples of such model categories: simplicial sets, symmetric spectra, stable module categories, chain complexes ($\mathbb{Z}$-graded, unbounded, over some commutative ground ring), and others [SS00, Section 5]. We are mainly interested in symmetric spectra and chain complexes. Then both the module category over a fixed monoid in $\mathcal{C}$ and the category of monoids in $\mathcal{C}$ have a cofibrantly generated model structure where fibrations, resp. weak equivalences, are just fibrations, resp. weak equivalences, in the underlying category $\mathcal{C}$ and cofibrations are

determined by the lifting property with respect to trivial fibrations [SS00, Theorem 4.1]. In particular, the adjunction (A.2) is a Quillen functor pair (recall that we use the convention according to which the left adjoint arrow is drawn above the right adjoint). If R is a monoid in $\mathcal{C}$, the homotopy category of R-Mod will be called the *derived* category of R and we denote it by $\mathsf{D}(R)$.

Lemma A.3. *If in* (A.1) S *is cofibrant in* $\mathcal{C}$ *then* $\wedge_S$ *together with* Hom_R *and* Hom_T *is a Quillen bifunctor (in the sense of* [Hov99, Definition 4.2.1]*) and hence induces an adjunction of two variables on the level of homotopy categories.*

Proof. We have to check the *pushout product axiom* [Hov99, Definition 4.2.1]. It suffices to do this for the generating cofibrations and the generating trivial cofibrations. Such a generating cofibration in R-Mod-S is of the form $R \wedge A \wedge S \rightarrowtail R \wedge B \wedge S$ with $A \rightarrowtail B$ a cofibration in $\mathcal{C}$, similarly for a generating trivial cofibration. If $S \wedge X \wedge T \rightarrowtail S \wedge Y \wedge T$ is a generating cofibration in S-Mod-T, the pushout product map is isomorphic to the map

$$R \wedge (A \wedge S \wedge Y \amalg_{A \wedge S \wedge X} B \wedge S \wedge X) \wedge T \longrightarrow R \wedge (B \wedge S \wedge Y) \wedge T.$$

This is a cofibration in R-Mod-T since S is cofibrant in $\mathcal{C}$ (hence smashing with S preserves cofibrations), the pushout product axiom holds in $\mathcal{C}$, and $R \wedge - \wedge T \cong$ preserves cofibrations (as a left Quillen functor). If one of the above cofibrations is a trivial one the same proof shows that the pushout product map is a trivial cofibration. $\square$

In particular, if S is cofibrant, smashing over S with a cofibrant bimodule gives a left Quillen functor between bimodule categories. If we do not assume S to be cofibrant, the functor $\wedge_S$ in (A.1) is not a Quillen bifunctor in general. For example, if the unit $\mathbb{S}$ is cofibrant in $\mathcal{C}$, the monoid S is not cofibrant in $\mathcal{C}$, and $R = T = \mathbb{S}$, then S is cofibrant as a right resp. left S-module (take $R = \mathbb{S}$ in the Quillen adjunction $(S \wedge_R -, f^*)$ in Lemma A.5) whereas $S \wedge_S S \cong S$ is not cofibrant in $\mathcal{C}$ by assumption. That is, $\wedge_S$ is not a Quillen bifunctor in this case.

However, in the case where S is not necessarily assumed to be cofibrant, smashing over S with an R-S-bimodule can be a left Quillen functor:

Lemma A.4. *Suppose that the unit* $\mathbb{S}$ *in* $\mathcal{C}$ *is cofibrant. If moreover* $X \in R$-Mod-S *is cofibrant in* R-Mod *then the adjoint pair*

$$S\text{-Mod-}T \xrightleftharpoons[\mathrm{Hom}_R(X,-)]{X \wedge_S -} R\text{-Mod-}T$$

is a Quillen pair.

Recall that we use the convention according to which the left adjoint arrow is drawn above the right adjoint. The derived adjoint pair will be denoted by

$$\mathsf{D}(S \wedge T^{\mathrm{op}}) \xrightleftharpoons[\mathrm{RHom}_R(X,-)]{X \wedge_S^L -} \mathsf{D}(R \wedge T^{\mathrm{op}}) \quad .$$

Proof. We show that $\mathrm{Hom}_R(X, -)$ is a right Quillen functor. In the diagram

$$
\begin{array}{ccc}
S\text{-Mod-}T & \xleftarrow{\ \mathrm{Hom}_R(X,-)\ } & R\text{-Mod-}T \\
\downarrow & & \downarrow \\
\mathcal{C} & \xleftarrow{\ \mathrm{Hom}_R(X,-)\ } & R\text{-Mod}
\end{array}
$$

the vertical functors are the forgetful functors. They preserve and reflect fibrations and trivial fibrations by the definition of the model structure on module categories. As a consequence of Lemma A.3, $\mathrm{Hom}_R(X, -) : R\text{-Mod} \longrightarrow \mathcal{C}$ is a right Quillen functor, so it preserves both fibrations and trivial fibrations. Now it follows that

$$
\mathrm{Hom}_R(X, -) : R\text{-Mod-}T \longrightarrow S\text{-Mod-}T
$$

also preserves them. $\square$

Lemma A.5. *Let $f : R \longrightarrow S$ be a map of monoids in $\mathcal{C}$. The induced functor (restriction of scalars)*

$$
f^* : S\text{-Mod} \longrightarrow R\text{-Mod}
$$

has both a left adjoint $S \wedge_R -$ and a right adjoint $\mathrm{Hom}_R(S, -)$. Moreover, $(S \wedge_R -, f^*)$ is always a Quillen pair, and $(f^*, \mathrm{Hom}_R(S, -))$ is a Quillen pair whenever the unit $\mathbb{S}$ is cofibrant in $\mathcal{C}$ and S is cofibrant in R-Mod.

Proof. For the definition of the left adjoint (extension of scalars) $S \wedge_R - : R\text{-Mod} \longrightarrow S\text{-Mod}$, the S-S-bimodule S is considered as an S-R-bimodule via restriction of scalars along the map $\mathrm{id} \wedge f^{\,\mathrm{op}} : S \wedge R^{\,\mathrm{op}} \longrightarrow S \wedge S^{\,\mathrm{op}}$. Since (trivial) fibrations are just (trivial) fibrations in $\mathcal{C}$, they are preserved by the restriction of scalars functor f^*.

For the definition of the right adjoint $\mathrm{Hom}_R(S, -) : R\text{-Mod} \longrightarrow S\text{-Mod}$, we consider S as an R-S-bimodule. Now f^* is the same as the functor $S \wedge_S - : S\text{-Mod} \longrightarrow R\text{-Mod}$, which has $\mathrm{Hom}_R(S, -)$ as a right adjoint. If $\mathbb{S}$ is cofibrant in $\mathcal{C}$ and S is cofibrant as an R-module, we can apply Lemma A.4 to deduce that $(f^*, \mathrm{Hom}_R(S, -))$ is a Quillen pair. $\square$

Consider the special case of the map of monoids $\iota : R \longrightarrow R \wedge S^{\,\mathrm{op}}$. The right adjoint of the restriction functor is $\mathrm{Hom}_R(R \wedge S, -) \cong \mathrm{Hom}_{\mathcal{C}}(S, -)$.

Corollary A.6. *Suppose that $\mathbb{S}$ is cofibrant in $\mathcal{C}$. Let R and S be monoids such that S is cofibrant in $\mathcal{C}$. Then we have a Quillen pair*

$$
R\text{-Mod-}S \; \underset{\mathrm{Hom}(S,-)}{\overset{\iota^*}{\rightleftarrows}} \; R\text{-Mod}.
$$

In particular, a cofibrant R-S-bimodule is then also cofibrant as an R-module. $\square$

Many monoidal model categories have the property that smashing with a cofibrant module X over some monoid R preserves weak equivalences. The functor $- \wedge_R X$ induces then a functor between the homotopy categories without being a Quillen functor in general.

Symmetric spectra have this smashing property [HSS00, Lemma 5.4.4] and there are many other examples [SS00, Section 5] including chain complexes.

A.2. **The several objects case.** Let us now consider the case of monoids with several objects, which we need for Part 2 of this paper. We use results from [SS03a, Sections 6 and 7]; results for the case of spectral categories can also be found in [SS03b, Appendix A]. As in Section A.1, we fix a closed symmetric monoidal model category $(\mathcal{C}, \wedge, \mathbb{S})$ which is cofibrantly generated, has only small objects, and satisfies the monoid axiom. The reader is encouraged to think of $\mathcal{C}$ as the category of symmetric spectra (cf. Example 2.4(1)).

A $\mathcal{C}$-*category* or a *monoid in $\mathcal{C}$ with several objects* [Bor94, Definition 6.2.1] is a small category $\mathcal{R}$ that is enriched over $\mathcal{C}$. This means, for any two objects R and R' of $\mathcal{R}$ there is a Hom-object $\mathcal{R}(R, R')$ in $\mathcal{C}$ together with an identity 'element' $\mathbb{S} \longrightarrow \mathcal{R}(R, R)$ for each R in $\mathcal{R}$ and composition morphisms

$$\mathcal{R}(R', R'') \wedge \mathcal{R}(R, R') \longrightarrow \mathcal{R}(R, R'')$$

which are associative and unital with respect to the identity elements. When $\mathcal{R}$ has only one object, it is the same as a monoid in $\mathcal{C}$. A *morphism* of $\mathcal{C}$-categories is a $\mathcal{C}$-functor [Bor94, Definition 6.2.3]. If $\mathcal{C}$ is the category of symmetric spectra, $\mathcal{C}$-categories are called spectral categories; if $\mathcal{C}$ is the category of chain complexes (Example 2.4(2)), then $\mathcal{C}$-categories are called DG categories. We have required the smallness condition (i.e., the objects of a $\mathcal{C}$-category form a set) since we want to consider module categories over $\mathcal{C}$-categories.

A *left module* over a $\mathcal{C}$-category $\mathcal{R}$ is a $\mathcal{C}$-functor $X : \mathcal{R} \longrightarrow \mathcal{C}$, i.e., an object $X(R)$ in $\mathcal{C}$ for every $R \in \mathcal{R}$ and morphisms in $\mathcal{C}$

$$\mathcal{R}(R, R') \longrightarrow \mathrm{Hom}_{\mathcal{C}}\big(X(R), X(R')\big)$$

which are compatible with composition and identities. By adjunction, these maps correspond to a left action of $\mathcal{R}$ on X, i.e., maps in $\mathcal{C}$

$$\mathcal{R}(R, R') \wedge X(R) \longrightarrow X(R')$$

which are associative and unital. A *right module* over $\mathcal{R}$ is a left $\mathcal{R}^{\,\mathrm{op}}$-module. A morphism $X \longrightarrow Y$ of $\mathcal{R}$-modules is a family $X(R) \longrightarrow Y(R)$ of maps in $\mathcal{C}$ compatible with the action of $\mathcal{R}$. An important point which distinguishes the several objects case from the one object case is, that there is not just one free $\mathcal{R}$-module but *one for each object* of $\mathcal{R}$, namely the *free module* $F_R^{\mathcal{R}}$ with respect to R. It is defined by $F_R^{\mathcal{R}}(R') = \mathcal{R}(R, R')$ (hence we have $F_R^{\mathcal{R}^{\,\mathrm{op}}}(R') = \mathcal{R}(R', R)$ for the free right modules). We will sometimes omit the upper index and just write F_R. Note that the enriched Yoneda lemma yields an adjoint pair

$$(\mathrm{A.7}) \qquad \qquad \mathcal{R}\text{-Mod} \; \underset{\mathrm{ev}_R}{\overset{F_R \wedge -}{\rightleftarrows}} \; \mathcal{C}$$

where ev_R is the evaluation functor with $\mathrm{ev}_R(X) = X(R)$ and $F_R \wedge Y$ is given by

$$(F_R \wedge Y)(R') = F_R(R') \wedge Y = \mathcal{R}(R, R') \wedge Y$$

with the obvious left action of $\mathcal{R}$.

Schwede and Shipley [SS03a, Theorem 6.1] have shown that the category $\mathcal{R}$-Mod of $\mathcal{R}$-modules is a cofibrantly generated model category with weak equivalences and fibrations

defined objectwise. As usual, the cofibrations are determined by the lifting property. A set of generating (trivial) cofibrations is given by all maps of the form

$$F_R \wedge A \longrightarrow F_R \wedge B$$

for $A \longrightarrow B$ a generating (trivial) cofibration in $\mathcal{C}$. Note that the adjunction (A.7) is indeed a Quillen functor pair. The homotopy category of $\mathcal{R}$-Mod is called the *derived category* of $\mathcal{R}$ and we denote it by $\mathsf{D}(\mathcal{R})$.

The *smash product* $\mathcal{R} \wedge \mathcal{S}$ of two $\mathcal{C}$-categories $\mathcal{R}$ and $\mathcal{S}$ [SS03b, Section A.2] has as set of objects the product of the sets of objects of $\mathcal{R}$ and $\mathcal{S}$. The morphism objects are given by

$$\mathcal{R} \wedge \mathcal{S}\left((R,S),(R',S')\right) = \mathcal{R}(R,R') \wedge \mathcal{S}(S,S').$$

This allows us to consider bimodule categories as $\mathcal{R}$-Mod-$\mathcal{S}$ with objects the $\mathcal{R} \wedge \mathcal{S}^{\,\mathrm{op}}$-modules. Note that a spectral category $\mathcal{R}$ can itself be regarded as an $\mathcal{R} \wedge \mathcal{R}^{\,\mathrm{op}}$-module in a natural way. For $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ spectral categories, there are, as in the one object case, bifunctors

$$\wedge_{\mathcal{S}} : \mathcal{R}\text{-Mod-}\mathcal{S} \ \times \ \mathcal{S}\text{-Mod-}\mathcal{T} \longrightarrow \mathcal{R}\text{-Mod-}\mathcal{T},$$

(A.8) $$\operatorname{Hom}_{\mathcal{R}} : (\mathcal{R}\text{-Mod-}\mathcal{S})^{\,\mathrm{op}} \ \times \ \mathcal{R}\text{-Mod-}\mathcal{T} \longrightarrow \mathcal{S}\text{-Mod-}\mathcal{T},$$

$$\operatorname{Hom}_{\mathcal{T}} : (\mathcal{S}\text{-Mod-}\mathcal{T})^{\,\mathrm{op}} \ \times \ \mathcal{R}\text{-Mod-}\mathcal{T} \longrightarrow \mathcal{R}\text{-Mod-}\mathcal{S},$$

which form an adjunction of two variables [Hov99, Definition 4.1.12]. For example, if $X \in \mathcal{R}$-Mod-$\mathcal{S}$ and $Y \in \mathcal{S}$-Mod-$\mathcal{T}$, then $(X \wedge_{\mathcal{S}} Y)(R,T)$ is the coequalizer of the diagram

$$\coprod_{S,S'\in\mathcal{S}} X(R,S) \wedge \mathcal{S}(S',S) \wedge Y(S',T) \implies \coprod_{S''\in\mathcal{S}} X(R,S'') \wedge Y(S'',T)$$

where the upper map uses the right action of $\mathcal{S}$ on X and the lower the left action of $\mathcal{S}$ on Y. The left action of $\mathcal{R}$ on X and the right action of $\mathcal{T}$ on Y yield the $\mathcal{R} \wedge \mathcal{T}^{\,\mathrm{op}}$-module structure. Similarly, $\operatorname{Hom}_{\mathcal{R}}$ and $\operatorname{Hom}_{\mathcal{T}}$ (which should be more precisely denoted by $\operatorname{Hom}_{\mathcal{T}^{\,\mathrm{op}}}$) are given by certain equalizers.

To state the next lemma, which is an analog to Lemma A.3, we need the following

Definition A.9. A $\mathcal{C}$-category is *pointwise cofibrant* if $\mathcal{R}(R,R')$ is cofibrant in $\mathcal{C}$ for all $R, R' \in \mathcal{R}$.

Lemma A.10. *If in* (A.8) $\mathcal{S}$ *is pointwise cofibrant, then* $\wedge_{\mathcal{S}}$ *together with* $\operatorname{Hom}_{\mathcal{R}}$ *and* $\operatorname{Hom}_{\mathcal{T}}$ *is a Quillen bifunctor (in the sense of* [Hov99, Definition 4.2.1]*) and hence induces an adjunction of two variables on the level of homotopy categories.*

Proof. The proof is a generalization of the proof of Lemma A.3. We have to verify the pushout product axiom [Hov99, Definition 4.2.1] for generating (trivial) cofibrations. Let $A \rightarrowtail B$ and $X \rightarrowtail Y$ be cofibrations in $\mathcal{C}$. We have to consider the pushout product of the maps

$$F_R^{\mathcal{R}} \wedge A \wedge F_S^{\mathcal{S}\,\mathrm{op}} \longrightarrow F_R^{\mathcal{R}} \wedge B \wedge F_S^{\mathcal{S}\,\mathrm{op}} \quad \text{and} \quad F_{S'}^{\mathcal{S}} \wedge X \wedge F_T^{\mathcal{T}\,\mathrm{op}} \longrightarrow F_{S'}^{\mathcal{S}} \wedge Y \wedge F_T^{\mathcal{T}\,\mathrm{op}}$$

for $R \in \mathcal{R}$, $S, S' \in \mathcal{S}$ and $T \in \mathcal{T}$. By definition, $F_S^{\mathcal{S}\,\mathrm{op}} \wedge_{\mathcal{S}} F_{S'}^{\mathcal{S}}$ is the coequalizer of the diagram

$$\coprod_{S_1, S_2 \in \mathcal{S}} \mathcal{S}(S_1, S) \wedge \mathcal{S}(S_2, S_1) \wedge \mathcal{S}(S', S_2) \; \rightrightarrows \; \coprod_{S_3 \in \mathcal{S}} \mathcal{S}(S_3, S) \wedge \mathcal{S}(S', S_3)$$

which is just $\mathcal{S}(S', S)$. Moreover, as a left adjoint, $F_R^{\mathcal{R}} \wedge - \wedge F_T^{\mathcal{T}\,\mathrm{op}} \cong F_{(R,T)}^{\mathcal{R} \wedge \mathcal{T}\,\mathrm{op}}$ preserves colimits. Thus the pushout product map is isomorphic to

$$F_R^{\mathcal{R}} \wedge \left(A \wedge \mathcal{S}(S', S) \wedge Y \coprod_{A \wedge \mathcal{S}(S', S) \wedge X} B \wedge \mathcal{S}(S', S) \wedge Y \right) \wedge F_T^{\mathcal{T}\,\mathrm{op}}$$
$$\longrightarrow F_R^{\mathcal{R}} \wedge \left(B \wedge \mathcal{S}(S', S) \wedge Y \right) \wedge F_T^{\mathcal{T}\,\mathrm{op}}$$

But since $\mathcal{S}(S', S)$ is cofibrant by assumption and the pushout product axiom holds in $\mathcal{C}$, this map is just the image of a cofibration in $\mathcal{C}$ under the left Quillen functor

$$F_R^{\mathcal{R}} \wedge - \wedge F_T^{\mathcal{T}\,\mathrm{op}} : \mathcal{C} \longrightarrow \mathcal{R}\text{-Mod-}\mathcal{T}$$

and hence a cofibration. The same arguments show that the pushout product map is a trivial cofibration whenever one of the cofibrations $A \rightarrowtail B$ and $X \rightarrowtail Y$ is trivial. $\square$

Consequently, if $\mathcal{S}$ is pointwise cofibrant, smashing over $\mathcal{S}$ with a cofibrant bimodule gives a left Quillen functor.

We now want to prove an analog of Lemma A.4. For this purpose we need a further notion. If $\mathcal{R}$ is $\mathcal{C}$-category, then $\mathbb{S}_{\mathcal{R}}$ denotes the 'discrete' $\mathcal{C}$-category associated to $\mathcal{R}$, i.e., $\mathbb{S}_{\mathcal{R}}$ has the same objects as $\mathcal{R}$, and $\mathbb{S}_{\mathcal{R}}(R, R')$ is the unit $\mathbb{S}$ if $R = R'$ and the trivial object $*$ otherwise [SS03b, Section A.1]. An $\mathbb{S}_{\mathcal{R}}$-module is simply a family of objects in $\mathcal{C}$ indexed by the objects of $\mathcal{R}$, and Mod-$\mathbb{S}_{\mathcal{R}}$ carries the product model structure, that is, weak equivalences, cofibrations and fibrations are defined objectwise. There is a canonical morphism $f : \mathbb{S}_{\mathcal{R}} \longrightarrow \mathcal{R}$ of $\mathcal{C}$-functors induced by the unit maps. It induces the restriction of scalars functor

$$f^* : \mathcal{R}\text{-Mod} \longrightarrow \mathbb{S}_{\mathcal{R}}\text{-Mod}, \quad X \mapsto X \circ f.$$
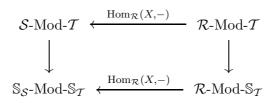
Lemma A.11. *Suppose that the unit $\mathbb{S}$ in $\mathcal{C}$ is cofibrant. If moreover $X \in \mathcal{R}$-Mod-$\mathcal{S}$ is cofibrant in $\mathcal{R}$-Mod-$\mathbb{S}_{\mathcal{S}}$ then the adjoint pair*

$$\mathcal{S}\text{-Mod-}\mathcal{T} \; \underset{\mathrm{Hom}_{\mathcal{R}}(X, -)}{\overset{X \wedge_{\mathcal{S}} -}{\rightleftarrows}} \; \mathcal{R}\text{-Mod-}\mathcal{T}$$

is a Quillen pair.

Proof. Note that via restriction of scalars along $\mathcal{R} \wedge \mathbb{S}_{\mathcal{S}} \longrightarrow \mathcal{R} \wedge \mathcal{S}$, the module X can indeed also be considered as an object of $\mathcal{R}$-Mod-$\mathbb{S}_{\mathcal{S}}$. We show that $\mathrm{Hom}_{\mathcal{R}}(X, -)$ is a right

Quillen functor. In the diagram

$$\begin{array}{ccc} \mathcal{S}\text{-Mod-}\mathcal{T} & \xleftarrow{\ \operatorname{Hom}_{\mathcal{R}}(X,-)\ } & \mathcal{R}\text{-Mod-}\mathcal{T} \\ \downarrow & & \downarrow \\ \mathbb{S}_{\mathcal{S}}\text{-Mod-}\mathbb{S}_{\mathcal{T}} & \xleftarrow{\ \operatorname{Hom}_{\mathcal{R}}(X,-)\ } & \mathcal{R}\text{-Mod-}\mathbb{S}_{\mathcal{T}} \end{array}$$

the vertical functors are induced by restriction of scalars. Hence they preserve and reflect fibrations and trivial fibrations. Since $\mathbb{S}$ was assumed to be cofibrant, the discrete $\mathcal{C}$-category $\mathbb{S}_{\mathcal{S}}$ is pointwise cofibrant. Hence we can apply Lemma A.10, which implies that $\operatorname{Hom}_{\mathcal{R}}(X,-) : \mathcal{R}\text{-Mod-}\mathbb{S}_{\mathcal{T}} \longrightarrow \mathbb{S}_{\mathcal{S}}\text{-Mod-}\mathbb{S}_{\mathcal{T}}$ is a right Quillen functor since X is cofibrant in $\mathcal{R}\text{-Mod-}\mathbb{S}_{\mathcal{S}}$, so it preserves fibrations and trivial fibrations. Consequently, $\operatorname{Hom}_{\mathcal{R}}(X,-) : \mathcal{R}\text{-Mod-}\mathcal{T} \longrightarrow \mathcal{S}\text{-Mod-}\mathcal{T}$ preserves them as well. $\qquad\square$

Lemma A.12. *Let $f : \mathcal{R} \longrightarrow \mathcal{S}$ be a morphism of $\mathcal{C}$-categories. Then the induced functor (restriction of scalars)*

$$f^* : \mathcal{S}\text{-Mod} \longrightarrow \mathcal{R}\text{-Mod}, \quad X \mapsto X \circ f,$$

has both a left adjoint $\mathcal{S} \wedge_{\mathcal{R}} -$ and a right adjoint $\operatorname{Hom}_{\mathcal{R}}(\mathcal{S}, -)$. Moreover, $(\mathcal{S} \wedge_{\mathcal{R}} -, f^*)$ is always a Quillen pair, and $(f^*, \operatorname{Hom}_{\mathcal{R}}(\mathcal{S}, -))$ is a Quillen pair whenever the unit $\mathbb{S}$ is cofibrant in $\mathcal{C}$ and $\mathcal{S}$ is cofibrant in $\mathcal{R}\text{-Mod-}\mathbb{S}_{\mathcal{S}}$.

Proof. Note that for the definition of the left adjoint (extension of scalars) $\mathcal{S} \wedge_{\mathcal{R}} - : \mathcal{R}\text{-Mod} \longrightarrow \mathcal{S}\text{-Mod}$, the $\mathcal{S} \wedge \mathcal{S}^{\operatorname{op}}$-module $\mathcal{S}$ is considered as an $\mathcal{S} \wedge \mathcal{R}^{\operatorname{op}}$-module via restriction of scalars along the map $\operatorname{id} \wedge f^{\operatorname{op}} : \mathcal{S} \wedge \mathcal{R}^{\operatorname{op}} \longrightarrow \mathcal{S} \wedge \mathcal{S}^{\operatorname{op}}$. Clearly, f^* preserves fibrations and trivial fibrations and $(\mathcal{S} \wedge_{\mathcal{R}} -, f^*)$ is thus a Quillen pair.

For the definition of the right adjoint $\operatorname{Hom}_{\mathcal{R}}(\mathcal{S}, -)$, we consider $\mathcal{S}$ as an $\mathcal{R} \wedge \mathcal{S}^{\operatorname{op}}$-module. Restriction of scalars is the same as the functor $\mathcal{S} \wedge_{\mathcal{S}} - : \mathcal{S}\text{-Mod} \longrightarrow \mathcal{R}\text{-Mod}$. This functor has $\operatorname{Hom}_{\mathcal{R}}(\mathcal{S}, -) : \mathcal{R}\text{-Mod} \longrightarrow \mathcal{S}\text{-Mod}$ as a right adjoint, which is by Lemma A.11 a right Quillen functor, whenever $\mathbb{S}$ is cofibrant in $\mathcal{C}$ and $\mathcal{S}$ is cofibrant in $\mathcal{R}\text{-Mod-}\mathbb{S}_{\mathcal{S}}$. $\qquad\square$

The following corollary is used in Section 5.2. Note that if $\mathcal{C}$ is the category of symmetric spectra, the unit $\mathbb{S}$ is indeed cofibrant.

Corollary A.13. *For each object R in a $\mathcal{C}$-category $\mathcal{R}$, the evaluation functor*

$$\operatorname{ev}_R : \mathcal{R}\text{-Mod} \longrightarrow \mathcal{C}$$

with $\operatorname{ev}_R(X) = X(R)$ preserves fibrations, weak equivalences, limits and colimits. Moreover, if the unit $\mathbb{S}$ is cofibrant in $\mathcal{C}$ and $\mathcal{R}$ is pointwise cofibrant (Definition A.9), ev_R preserves cofibrations and trivial cofibrations.

Proof. The assumption that $\mathcal{R}$ is pointwise cofibrant implies that $\mathcal{R}$, considered as an bimodule, is cofibrant in $\mathbb{S}_{\mathcal{R}}\text{-Mod-}\mathbb{S}_{\mathcal{R}}$ (in fact, both conditions are equivalent). Hence we can apply Lemma A.12 to the canonical morphism of $\mathcal{C}$-categories $f : \mathbb{S}_{\mathcal{R}} \longrightarrow \mathcal{R}$ and use

the fact that in a module category over a discrete $\mathcal{C}$-category everything (limits, colimits, fibrations, cofibrations, and weak equivalences) is defined objectwise. $\qquad\square$

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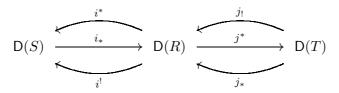
Summary

Morita theory is the study of equivalences of module categories and related questions. There are different settings for Morita theory:

- the classical case of ordinary rings ('monoids in the monoidal category of abelian groups'), modules, and equivalences of abelian categories (Morita, since the 1950s),
- the differential graded (DG) case, where DG algebras ('monoids in the monoidal category category of chain complexes'), DG modules over such and derived equivalences of triangulated categories are considered (Happel, Rickard, Keller, since the 1980s),
- the context of stable model categories, with symmetric ring spectra ('monoids in the monoidal category of symmetric spectra'), module spectra, and equivalences of their homotopy categories or even Quillen equivalences (Schwede, Shipley, since the 2000s).

In this paper, we prove two results from Morita theory of stable model categories. They can be regarded as topological counterparts of Morita theorems which have recently been proved in the DG context. One is on recollements of triangulated categories, the other on well generated categories.

Recollements. If R is a symmetric ring spectrum, Schwede's and Shipley's work on Quillen model structures of module categories allow us to define the derived category $\mathsf{D}(R)$ as the homotopy category of R-modules. Motivated by a similar result of Jørgensen for the DG case, we give a criterion of when the triangulated category $\mathsf{D}(R)$ can be obtained as 'glued together' from two other such derived categories. More precisely, we give a criterion for the existence of symmetric ring spectra S and T and a recollement of the derived categories, that is, a diagram of triangulated categories and functors

$$
\mathsf{D}(S) \xrightarrow{\ i_* \ } \mathsf{D}(R) \xrightarrow{\ j^* \ } \mathsf{D}(T)
$$

satisfying among others the condition that each arrow drawn above another in converse direction is left adjoint to the latter. Our result states, that such a recollement exists if and only if there are two R-modules B and C satisfying certain finiteness and generating conditions. If such modules exist, the (opposites of) the endomorphism ring spectra of B and C will serve as S and T. In fact, we do not work over the category of symmetric spectra but over a more general one including chain complexes so that we get a generalization of the result in the DG context.

Well generated categories. The well generated triangulated categories (due to Neeman) generalize the compactly generated triangulated categories and share with them the important property of Brown representability. But unlike the class of compactly generated ones, the class of well generated categories is stable under passing to appropriate localizing subcategories and localizations. While Porta characterizes the well generated *algebraic* triangulated categories as localizations of the derived categories of DG algebras with several objects (alias DG categories), we use the setting of Quillen model categories to prove that a *topological* triangulated category is well generated if and only if it is a

localization of the derived category of a symmetric ring spectrum with several objects (alias spectral categories). Here, a triangulated category is called topological if it is equivalent to the homotopy category of a spectral model category. We also give a lift to the level of model categories, that is, we show that any spectral model category which has a well generated homotopy category is Quillen equivalent (via a single Quillen functor) to a Bousfield localization of the model category of modules over some spectral category. While a result of Schwede and Shipley can be summarized by the slogan 'Compactly generated stable model categories are categories of modules', the corresponding slogan of our result is, 'Well generated stable model categories are localizations of categories of modules'.