Nonlinear and Stochastic Dynamical Systems Modeling Price Dynamics

Aspects of Financial Economics in Oil Markets

INAUGURAL-DISSERTATION

zur Erlangung des Grades eines Doktors
der Wirtschafts- und Gesellschaftswissenschaften
durch die
Rechts- und Staatswissenschaftliche Fakultät
der Rheinischen Friedrich-Wilhelms-Universität
Bonn

vorgelegt von Simon Jäger
aus Heidelberg

Dekan:	Prof. Dr. Gerhard Wagner
Erstreferent:	Prof. Dr. Klaus Sandmann
Zweitreferent:	Prof. Dr. Frank Riedel
Tag der mündlichen Prüfung:	28.02.2008
Erscheinungsjahr:	2008

Diese Dissertation ist auf dem Hochschulschriftenserver der ULB Bonn
$http://hss.ulb.uni-bonn.de/diss\_online$ elektronisch publiziert.

Abstract

This thesis has as main aims:

- analyzing the effects of nonlinearities and stochasticity on price dynamics of assets, in general,
- modeling the price dynamics of oil, as a special commodity,
- determining option prices and optimal hedging strategies for commodities like oil.

In many economic and financial processes mathematical modeling leads to nonlinear and stochastic dynamical systems. The interplay of stochastic and nonlinear effects is important under many aspects. Whereas the dynamic behavior of deterministic dynamical system may be characterized by the attractors of its trajectories, stochastic "perturbations" will lead to a even more complex behavior e.g. to transitions, even to jumps between attractors like equilibria. This fact gives an explanation of the observed multi-modality of distributions for prices of assets.

In the first section of this thesis we consider as a typical case the dynamic of exchange rates. A simple nonlinear stochastic differential equation containing economic data in its coefficients is used to model the dollar/pound exchange rates generalizing the standard linear approach. Due to the simplicity of the model, it is easier in this case to investigate the influence of the nonlinearities. We present a numerical method to solve the corresponding inverse problem, determining the nonlinearities arising in the drift. The simulations of the model system achieved result showing very convincing agreement with real data for evolution of the exchange rates as well as for the distribution of the rates.

In its main sections the thesis analyzes the interplay of supply, demand and prices of oil, one of the most important commodities, crucial for the economic, but also for human development, in general. Reducing the underlying complex network to a model system as simple as possible and combining nonlinear and stochastic effects, we are able to describe the complex dynamical behavior, the dynamics of the price, supply and demand observed in real data. We derive, discuss and validate the reduced model including essential factors like the strategy of OPEC and the contribution by the Non-OPEC producers. The resulting reduced model system consists of several nonlinear stochastic differential equations leading to a higher dimensional forward Kolmogorov equation for the oil price distribution. So far, the numerical methods able to handle direct or inverse problems arising for this system are not available.

Due to the fact that commodity markets in general, the oil markets in special are incomplete, determining option prices leads additional difficulties. Using the concept of indifference pricing we evaluate price and hedging strategies of financial contracts. An exponential utility function is chosen to determine optimal sell respectively buy prices. Using a duality principle, we are led to optimization problem for martingale measures with respect to the wealth process. Here the relative entropy is naturally involved. The arising stochastic optimization problems are directly connected with Hamilton-Jacobi-Bellman equations. The option prices are determined by the solution of this nonlinear, in applications high dimensional system of partial differential equations. Due to this numerical complexity, we restricted our simulation to the case where the price of the commodity can be described by a single nonlinear stochastic differential equation with a volatility, depending on one stochastic variable modeled by an other stochastic differential equation.

This special case is studied in numerical simulations to the full extend. The (bid and ask) option prices are computed and their dependence on nonlinearities and parameters is studied. The optimal hedging strategies are determined. The numerical algorithms used to solve the Hamilton-Jacobi-Bellman-system and to compute the option price are tested choosing as test the system corresponding to the Black-Scholes case with explicit solutions.

Zusammenfassung

Die wesentlichen Ziele dieser Arbeit sind:

- die Analyse nichtlinearer stochastischer Effekte auf die Preisdynamik von Vermögenswerten (Assets), im Allgemeinen,
- die Modellierung der Preisdynamik von Öl, als einem zentralen Rohstoff,
- die Bestimmung von Optionspreisen und optimalen Anlagestrategien für Rohstoffe wie Öl.

Bei vielen Finanz- und volkswirtschaftlichen Prozessen führt die mathematische Modellierung zu nichtlinearen stochastischen dynamischen Systemen. Das Zusammenspiel von Stochastik und nichtlinearen Effekten ist unter vielen Gesichtspunkten wichtig. Während das dynamische Verhalten deterministischer Systeme durch die Attraktoren der Trajektorien charakterisiert werden kann, führen stochastische "Störungen" zu einem noch weit komplexeren Verhalten, so zu Übergängen, selbst zu Sprüngen zwischen Attraktoren, z.B. zwischen Gleichgewichten. Diese Tatsache liefert eine Erklärung für beobachtbare multi-modale Preisverteilungen für Vermögenswerte.

Im ersten Abschnitt dieser Arbeit betrachten wir als einen typischen Fall die Dynamik von Wechselkursen. Zur Modellierung des Dollar/Pfund Wechselkurs verwenden wir eine einzelne stochastische Differentialgleichung, deren Koeffizienten durch ökonomische Variable bestimmt werden. Wegen der einfachen Struktur des Modells, ist es in diesem Fall leichter den Einfluss von Nichtlinearitäten zu untersuchen. Wir erstellen numerische Methoden, um das dazugehörige "Inverse Problem" zu lösen, Nichtlinearitäten zu bestimmen, die im Driftterm auftreten. Die Simulationen der Modelgleichungen liefern Ergebnisse, die überraschend gut mit den realen Daten für die Entwicklung des Wechselkurses und dessen Verteilung übereinstimmen.

Im zentralen Teil der Arbeit analysieren wir das Wechselspiel von Angebot, Nachfrage und Preis von Öl, eines der wichtigsten Rohstoffe, wichtig nicht nur für die Wirtschaft, sondern auch allgemein für die Entwicklung humaner Aktivitäten. Das dem Markt zugrunde liegende, komplexe Netzwerk wird auf ein möglichst einfaches Modelsystem reduziert, wobei das Wechselspiel nichtlinearer und stochastischer Effekte eine wichtige Rolle spielt. So kann das komplexe dynamische Verhalten von Preis, Angebot und Nachfrage besser beschrieben werden. Wir erstellen, diskutieren und validieren ein vereinfachtes Modell, das wichtige Faktoren einbezieht, wie die OPEC-Strategie und den Beitrag der Nicht-OPEC Produzenten. Dieses reduzierte Modelsystem besteht aus mehreren nichtlinearen stochastischen Differentialgleichungen und führt somit zu einer hochdimensionalen Forward-Kolmogorov Gleichung für die Verteilung des Ölpreises. Numerische Methoden, die in der Lage wären, die direkten und inversen Probleme der entwickelten Modellgleichungen zu lösen, existieren bisher noch nicht.

Rohstoffmärkte, insbesondere der Ölmarkt, sind unvollständig. Diese Tatsache führt zu zusätzlichen Schwierigkeiten bei der Bestimmung von Optionspreisen und Absicherungsstrategien. Wir ermitteln die Preise und Strategien zur Absicherung von Finanzverträgen mit Konzepten des "Indifference Pricing". Um eine optimale Kauf- und Verkaufspreise festzulegen, wird eine exponentielle Nutzenfunktion gewählt. Unter Anwendung des Dualitätsprinzips, ergibt sich ein Optimierungsproblem für Martingalmaße mit Bezug auf den Vermögensprozess. Hier kommt ganz natürlich die relative Entropie ins Spiel. Das zugehörige stochastische Optimierungsproblem ist direkt verbunden mit Hamilton-Jacobi-Bellman Gleichungen. Die Optionspreise werden durch die Lösung dieses Systems nichtlinearer in den Anwendungen hochdimensionaler partieller Differentialgleichungen ermittelt. Wegen der numerischen Komplexität, beschränken wir unsere Simulation jedoch auf den Fall, in dem der Rohstoffpreis als eine einzelne nichtlineare stochastische Differentialgleichung beschrieben werden kann. Dabei hängt die auftretende Volatilität von einer stochastischen Variablen ab, die ihrerseits durch eine weitere stochastische Differentialgleichung modelliert wird.

Dieser Spezialfall wird in den numerischen Simulationen im vollen Umfang untersucht. Die (Kauf- und Verkauf-) Optionspreise werden berechnet und in Abhängigkeit von den Nichtlinearitäten und den Parametern analysiert. Die optimalen Anlagestrategien werden bestimmt. Die verwendeten numerischen Algorithmen zur Lösung des Hamilton-Jacobi-Bellman-Systems und zur Bestimmung des Optionspreises werden getestet, indem als Testfall das Black-Scholes System mit den entsprechenden expliziten Lösungen herangezogen wird.

To my Parents

Acknowledgement

This dissertation combines the main results of my doctoral studies at the Bonn Graduate School of Economics (BGSE), University of Bonn. I am taking the opportunity to thank all who have supported me in this research project. It is a pleasure for me to express my gratitude to my advisor Prof. Dr. Klaus Sandmann for enduring support and encouragement as well as for many valuable suggestions concerning my work. He offered me many possibilities to improve my scientific horizon, to enter the exciting area of commodity markets and to get into contact with the right people. I would like to thank Prof. Dr. Frank Riedel for his critical review of my thesis.

I enjoyed the opportunity the close cooperation with Prof. Dr. Ekaterina Kostina, University of Marburg. I have benefited from her great experience in numerical optimizations. I am grateful to Dr. Christoph Reisinger, Oxford Center for Industrial and Applied Mathematics, for starting a cooperation analyzing and developing numerical methods for the derived multidimensional pricing PDE arising for commodity derivatives. The results are not presented in this thesis. However, the numerous discussions and the invitation to Oxford were extremely helpful. I am very thankful to Dr. Malte Braack, University of Heidelberg for the possibility to use the software GASCOIGNE. Thanks also to Sanae Rujivan, University of Heidelberg, for the exchange of ideas concerning commodity price modeling and for introducing me to Thai culture.

I am expressing my gratitude to Prof. Dr. Kristian Miltersen for giving me the opportunity to study for one year in his team at the Norwegian School of Economics and Business Administration (NHH). The research environment in Norway was highly inspiring. I would like to thank Prof. Dr. Rögnvaldur Hannesson for introducing me to petroleum economics, for many interesting discussions and the contacts e.g. to STATOIL. I would like to thank my colleagues in Norway especial Tao Lin for plenty of useful discussions on various energy issues.

Let me express my appreciation of the substantial support by the Bonn Graduate School of Economics, the German Research Foundation (DFG) and the German Academic Exchange Service (DAAD). Prof. Dr. Urs Schweizer, Chairman of the Graduate School, deserves special credit for his efficient support.

I would like to thank the members of the Institute of Banking and Finance, University of Bonn, who made time in Bonn personally and scientifically very enjoyable: Sven Balder, Michael Brandl, An Chen, Haishi Huang, Birgit Koos, Dr. Eva Lütkebohmert, Dr. Antje Mahayni, Anne Ruston, Xia Su, Jens Wannenwetsch and Manuel Wittke. I appreciate very much the time we spent together, the seminars and the intensive discussions afterwards at the "Treppchen". Sven Balder deserves special thanks for integrating this active group.

Last, but not least at all I am grateful to my parents, my sisters and my grandmother for their never-ending support during these years. I dedicate this dissertation my family.

Contents

List of Figures

Chapter 1

Introduction

Capturing the price behavior of assets and forecasting future developments is essential in financial asset management and international policy. Many activities, such as trading and risk management, are directly dependent on the quality of pricing financial and physical assets in order to evaluate derivatives, devise hedging strategies or estimate the financial risk of a firm's portfolio position. Whereas statistical methods are traditionally important tools in the quantitative approach to economic and financial processes mathematical modeling and simulation of the systems are playing still not the role they deserve. In order to deal with the specifics of individual markets and their particular behavior, extensions for traditional stochastic models, for instance the Black-Scholes model, are in demand.

Traditional approaches in modeling price processes such as foreign exchange rates and commodity prices in particular crude oil prices are working with the hypothesis of a single long run price equilibrium (see e.g. Taylor, Peel, and Sarno (2001), Geman (2005) and Cartea and Figueroa (2005)). Deviations from this reversion level are expected to be temporary, thus dynamics of oil prices are mainly driven by one attracting equilibrium. Recent developments on oil markets e.g. the transition to significant higher price levels clearly demonstrate the weakness of this approach. In contrast to the popular mean reversion explanation, we will show that the history of oil price movements is rather characterized by multiple attractors which are changing in time but on a slow time scale. Simultaneously, we face various attractors, creating natural zones of instabilities. Small random perturbations may push balanced markets from one equilibrium into another, producing serious price movements all of a sudden.

Mathematical modeling of economic and financial processes leads in general to nonlinear deterministic and stochastic dynamical systems. Model based statistics has to be developed in order to improve integration and exploitation of economic knowledge. Stochastic nonlinear dynamical systems, describing the arising processes more adequately, have to be investigated with the aim to obtain better qualitative and more precise quantitative answers. The interplay of nonlinearities in the dynamics and the stochastic influences in the system is highly important, and not enough taken into account (see e.g. Krugman (2000) and Borovkova, Dehling, Renkema, and Tulleken (2003)). It is known from mathematical modeling of population dynamics that this interaction may lead to effects which cannot be explained otherwise: e.g. multi-modality can be traced back to multiple steady states in the dynamical systems, observed jumps and strong oscillations in the data can be described by stochastic changes of attractors (compare e.g. Skorokhod, Hoppensteadt, and Salehi (2002)). Especially now, where there seems to be a rather uncontrolled development of oil prices, a rational analysis of the interactions responsible for the observed effects is absolutely necessary.

Quite a few market participants hold the increased activity of non-commercials in the futures markets, such as hedge funds responsible for large price movements particularly during phases of transitions[1]. As soon as the dynamics has leveled out at a new equilibrium these arguments are not used anymore. In opposite to this ad hoc explanation, which cannot easily be quantified and validated, we follow a new concept of prevailing market forces. In the deterministic model, any trajectory starting in a sufficiently small neighborhood (domain attraction) of a stable equilibrium point will converge to it. However in a randomly perturbed nonlinear system, small perturbation can cause a crossing of these domains, pushing the balanced market from one equilibrium into another. As a consequence, both regime switches and rare events will arise. The attractors or quasi-steady states are evolving in time according to market determinants, e.g. global supply and demand. This investigation is focussed mainly to the dynamics of the oil market. However, we will also include the exchange market as another case where multi-modal distributions are arising.

The dynamics of financial derivatives is more than a stochastic process depending on other stochastic background processes. Available information e.g. about the oil industry in our case should be used to specify the mathematical description. The oil market leads to a complex system of interactions not accessible as complete network. The complexity has to be reduced by phenomenological or mathematical arguments. It is an effective strategy to start with a more detailed and therefore complex mathematical model and to reduce it to a submodel describing essential features of the system. One of our aims is to include the model for oil price dynamics into a rational pricing of options on oil. Therefore we did not deal with a detailed derivation of the reduced model system using mathematical tools for systems reduction like sensitivity analysis or multi-scale techniques. The final model system we obtain is still large enough to cause problems in parameter estimation and numerical simulation. To our opinion, it is necessary to combine in a better way statistics and modeling based on economic facts. The model developed for the oil market tries to capture especially the market strategies of OPEC using switches in oil production in order to influence the oil prices. We formulate mathematically these switching rules, relating the price dynamics to oil supply and demand. We are going to give a more precise survey on the methods used and the results achieved in this thesis.

[1]Compare e.g. 134th (Extraordinary) Meeting of the OPEC Conference

Methods

Modeling Price Dynamics

We are trying to improve the understanding of the complex dynamics on oil markets by a more detailed analysis of the interacting main factors. The main state variables supply, demand and oil price satisfy stochastic differential equations. For the oil price drift and volatility are depending nonlinearly on the supply, demand, and the price. The nonlinearity is chosen such that canceling the noise term leads to a deterministic dynamics which is characterized by the evolution of so-called quasi-steady states. These are varying in time and playing the role of steady states in an autonomous system. They are called “stable” or “unstable” states if they are locally in time attractive or repulsive. It is important that time intervals may arise where there exist several quasi-steady states. In these intervals stochasticity can lead to transitions between these time-varying states, to jumps and a behavior similar to instabilities. We take the simplest nonlinearity allowing such a behavior: a polynomial of order three. Thus the attractors depend essentially on a few key factors of the oil market e.g. increased demand for oil, production constraints and different strategic supply behavior by OPEC and Non-OPEC countries. We assume that the quasi-steady states are evolving on a slower time scale, changing their location and possibly their stability.

We model the subsystem of supply and demand taking into account basic characteristics of oil markets structure. In recent years, OPEC has tried to control the market using supply as control variable, since it is determined just by the producer and seems to be most effective. It is a rational strategy to keep the oil price or more general an objective functional in a prescribed interval, which is optimal for its own interest. To obtain a dynamic, which is relatively robust, a switching rule between different supply policies is proposed in analogy to switching rule in a heating system: The supply is increased (decreased), if the value of the objective functional becomes larger (smaller) than the upper (lower) threshold. The Non-OPEC group is mainly producing and selling oil to an extend which is determined essentially by its supply potentials. Here we rely on predictions obtained by models including more detailed information about oil fields, infrastructure and technology.

Remark: The methods developed for the dynamics of the crude oil price are applicable also to other areas in economy. Important areas are the pricing of natural resources (soybeans or copper), energy prices (electricity, gas) or foreign exchange rates. Of course, each of these field will have its own special properties. However, they also share common methodical features from a view point of modeling, simulation, and validation. Therefore, there is opening up a huge area of application of the concepts and methods. The crude oil price can be considered as special asset price and therefore it is natural to try to apply the developed methods to assets in general. As an example we will present a foreign exchange rate model where the equilibria are determined by money supply and demand, in particular interest rates home and abroad.

Parameter Estimation

In general the estimation of model parameters is a crucial problem. The parameters have to be estimated such that the observed behavior of the economic process is reproduced by the simulation of model equations up to a small error, acceptable in the considered specific problem. To quantify the deviation, least-square error functional is used. Developing methods and algorithms for parameter estimation for systems as they are arising in finance is a problem by itself. Here we developed an estimation algorithm in case of a single nonlinear stochastic differential equation using the corresponding Fokker-Planck equation (see e.g. Jäger and Kostina (2005) and Jäger and Kostina (2006)). This non stochastic partial differential equations contains a diffusion and a drift term, the unknown parameters of which have to be determined solving the corresponding inverse problem. A least square functional subject to the forward Kolmogorov equation measuring the deviation from the data is minimized. We apply a generalized Gauss-Newton algorithm constrained by the forward Kolmogorov equation and some initial and boundary conditions. Our algorithms can cover the problem to calibrate the model equation describing the evolution of exchange rates. However, since the algorithm is based on multiple shooting techniques we are restricted in dimension of the system. However, in reality we are forced to deal with systems of higher dimension, where algorithms of similar quality are still in demand.

Derivative Pricing

The suggested model for the oil price dynamic was derived with a focus on aspects of financial decisions. Here, one essential aim is to provide more detailed, accessible information to be used e.g. for rational pricing of contracts on oil. Its advantages are a rather simple structure reflecting the essential features, its interfaces to more refined modeling if necessary, the combination of nonlinear and stochastic effects neglected too often in existing theories.

We derive pricing and hedging strategy for the oil market, which in its nature is incomplete. Whereas in a complete market derivative contracts can be priced uniquely by construction of replicating portfolios and application of the no-arbitrage principle, the situation for incomplete markets is more complicated. Many different option prices are consistent with no-arbitrage, each corresponding to different martingale measure. There is no longer a unique price, that means there exist no unique-preference independent martingale measure. Incompleteness arises whenever the number of sources of risk exceed the number of traded assets. As a consequence, issuing a contingent claim incorporates some unavoidable risk and therefore the issuer's valuation and hedging strategy have to take into account implicit or explicit assumptions on the agent's attitudes towards them.

Here we have chosen an exponential utility function to measure the gain in wealth and an optimization method to determine a fair price. Due to a duality theorem, this is equivalent to selecting a martingale measure by an optimization. This includes the relative entropy functional quantifying the "distance" of measures. In formulating the stochastic optimization problem the concept of indifference pricing is used taking into account the following fact. An agent is indifferent with respect to expected utility of alternative investment strategies, to be more precise of investing in a pure Merton portfolio or issuing a contingent claim. The later differ by receiving a premium p initially and accepting the liability associated with the claim. The indifference price results from equating both alternatives

Using this approach, we have investigate the Hamilton-Jacobi-Bellman equations, a system of nonlinear partial differential equations connected with the optimization problem. Here an end-value-boundary-value problem has to be solved, where the boundary values have to be chosen properly. In this thesis we are mainly interested in the numerical solutions, which determine the proper martingale measure,the option pricing and the hedging strategies. Whereas theory does not limit the system size, the numerical algorithms for larger systems are not available to the extend needed.

Each state variable in the model, described by a stochastic differential equation, corresponds one space variable in the diffusion system (Hamilton-Jacobi-Bellman-equation). Therefore, numerical algorithms for partial differential equations in high dimensions are urgently needed. According to the state of the art, we have to restrict ourself to small size systems.

Using our model for oil price dynamics including supply and demand, we are lead to more variables than we can tread numerically at this moment. In cooperation with Reisinger (Oxford) we are working on simulations on our full model using algorithms based on thin-grid and multi-grid methods. In order to test our approach, we restrict ourself in the chapter on pricing, as far as the numerics is concerned just on the case where supply and demand are not modeled explicitly, however include an additional variable modeling the stochastic volatility. Thus, the Hamilton-Jacobi-Bellman equations have to be solved in 2 dimensions. We are able to use the software developed for nonlinear evolution equations.

The reduced system is of interest in itself. Taking into account the existence of multiple quasi-steady states and randomly perturbed volatility, we can explain the distinctive features of observed price distributions. The presented nonlinear system generalizes the Pilipovic model which is widely used to describe the price dynamics on energy markets.

Results

Applying concepts and methods of nonlinear, stochastic dynamical systems we achieved the following results:

Nonlinear Price Dynamics

- Observed price movements can be traced back to transitions between multiple quasi-steady states leading to multi-modal distributions.
- We present a weighted least squares method subject to the distributional dynamics described by the Focker-Planck equation. The numerical algorithm has proven to be efficient.

Oil Market Dynamics

- Derivation of a system of stochastic differential equations representing a reduced model for supply, demand and price dynamics of oil, including global economic factors like GDP, development of resources and technologies. Including economic factors in form of data or by modeling them in more in detail, a deeper understanding of price dynamics could be achieved and the quantitative description of the processes improved.
- Modeling and simulation of controls as exercised by OPEC. The hypothesis claiming that OPEC uses as tool a price range with barriers changing slowly in time is formulated in a model with switching rules and the model is validated using real data.
- Simulation of oil price dynamics and comparison with real data. The simulations showed that main features of the dynamics are reflected by the model system. Though the system is kept as simple as possible, systematic parameter identification based on accessible data remains a problem to be attacked in future.

Pricing Contracts on Oil

- Given a market model, an answer to the question how to price derivative contracts on crude oil is provided. We present a nonlinear pricing PDE and the corresponding hedging strategy for a complex system in an incomplete market setting.
- The presented pricing method is applied to a nonlinear stochastic volatility model which represents an extension of the Pilipovic Model widely used in modeling price dynamics.
- A numerical methods for solving the end value - boundary value problem for the Hamilton-Jacobi-Bellman is presented and tested. The numerical approximation error to the Hamilton-Jacobi-Bellman System could be approved as very small. The differences between complete and incomplete markets, linear and nonlinear mean reversion processes are demonstrated.

Overview of the content

In **Chapter 2** we identify and demonstrate *nonlinear phenomena* like multi-modal distributions, steady states, attractors in various markets especially in commodity and foreign exchange rate markets (see e.g. Jäger (2006)). We develop a simple estimation technique which enables us to locate these attractors. In order to integrate these observations into a more general view, we give a very brief survey of pricing methods in economics and finance. We review some characteristics of nonlinear price systems in particular we illustrate the idea of transitions between the minima of a potential function. Based on this approach we introduce the concept of *"generalized mean reversion processes"* and define the notion of so called quasi-steady states.

In **Chapter 3** we summarize the fundamental link between *nonlinear diffusions and the distributional dynamics*. Especially, a single stochastic differential equation with nonlinear drift is analyzed. Here, the nonlinearity in the drift is playing a crucial role. Efficient algorithms are required in order to calibrate the models and to identify the system parameters. We choose an estimation technique based on the idea of minimizing a weighted least square functional subject to a the Fokker-Planck equation. Taking into account nonlinear effects in volatility and drift and dependence on economic data, one obtains equations where the standard numerical methods are not sufficient. The coefficients are rapidly oscillatory due to the fact that real data are included, and strong instabilities may arise caused by the nonlinearities in the drift.

The presented numerical methods are part of a joint work with E. Kostina (compare Jäger and Kostina (2006), manuscript in revision) and used to simulate the dynamics of the dollar/pound exchange rate.

Chapter 4 begins with a short overview of the *world oil market*, addressing the issue of international trade, nature of crude oil, distribution of supply and proven reserves, global demand and inventory movements. The oil market is depending on many factors interacting in a complex network which seems almost not accessible by modeling. However, following the common ansatz of systems theory, we reduce a complex system to a simplified one modeling just the components, which we think are dominant. By presenting and discussing a flow diagram illustrating the more complex situation, the implicit assumptions are made clear and possibilities of refining our simplified model are indicated. Models of subsystems including more specific details are investigated by teams of analysts e.g. in oil companies. The suggested model is derived with a focus on aspects of financial decisions discussed in chapter 5.

We formulate model equations for price, supply and demand. Here, the following aspects are considered: strategies and prevailing restrictions in production and selling, distribution of proven oil reserves, exploration and development prospects by OPEC and Non-OPEC countries, composition of the global energy demand, the evolution of driving forces such as increasing wealth, booming car sales, technological and economic development, in particular for emerging countries.

The *total supply* is obtained as the sum of supply by *OPEC* and *Non-OPEC*. In section 4.3.1 we present a model for the OPEC “price band strategy” and validate it with data from 2000 – 2005. The control of supply by the member states of OPEC depends essentially on the choice of adequate thresholds, switching and policy rules, as well as capacity limits. In contrast Non-OPEC countries are assumed to follow no policy coordination. The model presented in section 4.3.2 propose a crude oil supply of Non-OPEC countries which is essentially determined by accessible resources, and supply potentials. Analysis and predicting these benchmarks is a complex problem by itself. We use as an input into our model determine a rough guideline for Non-OPEC prospects. Projects from 2004 – 2009 are quite well observable. Thus in short term large deviations are not to be expected.

Recent developments on oil markets reveal the importance of a careful analysis of *oil demand*. In section 4.4 we introduce the concepts of potential demand, budget conditioned demand and actual demand which takes into account economic and technological evolution of different regions and countries, budget restrictions as well as strategic behavior of consumers. To keep the system tractable we will treat two groups: Developed and emerging countries. They differ mainly through their consumption level and growth rates.

Taking into account these information the quasi-steady states are driven by global oil supply and demand perspectives. The market simulations are presented in section 4.5.

Chapter 5 deals with the problem of *optimal investments* in commodity markets which are assumed to be incomplete. Furthermore, the underlying price process is allowed to follow a nonlinear stochastic differential equation and the concept of convenience yield is embedded. Following Hodges and Neuberger (1989), we define in section 5.1.4 the notion of *indifference price* of a contingent claim which is based on the idea of comparing two optimal investment strategies with and without involving the contingent claim.

Instead of solving the resulting variational problem directly, dual problems can be formulated which leads to an optimization over martingale measures and turns out to be simpler than the original problem. We review the key duality results of Delbaen, Grandits, Rheinländer, Samperi, Schweizer, and Stricker (2002) and will show that the price of a contingent claim can be interpreted as the Lagrange formulation of the following optimization problem: Maximize the expected payoff as functional on a set of measures with finite entropy relative to the minimal entropy measure.

Applying results of stochastic optimization we offer in section 5.1.5 a detailed derivation including necessary and sufficient conditions of the pricing PDE. We deduce expressions for the optimal trading strategies corresponding to the Merton investment, the investment including the contingent claim. In addition the indifference hedging strategy is defined considering the relation between these different strategies.

Section 5.3 provides a numerical approximation of the derived pricing PDE and the corresponding hedging strategies. In order to be able to use the software GASCOIGNE we reformulate the nonlinear price system as an initial-boundary value problem the integrated weak form basic for the used finite element method.

The last section 5.4 analyzes the pricing and hedging strategies in different cases: We test our numerical method in the standard Black-Scholes situation, a special case, where the explicit solutions is known. We illustrate numerically the effect of introducing stochastic volatility to the oil price model. In particular, we study the asymptotic behavior and monotonicity with respect to the risk aversion parameter as well as with respect to the volume scaling. Finally we compare the linear and the nonlinear mean reversion processes.

Chapter 2

Nonlinear Price Dynamics

Contents

At the beginning we motivate the use of nonlinear stochastic systems in order to capture the price dynamics of energy commodities, in particular of crude oil. We give a very short introduction to pricing methods applied in petroleum economics and finance. Here, we focus on a multiple equilibrium approach by Cremer and Salehi-Isfahani (1989) which explains large price movements as a consequence of multiple equilibria in a competitive oil market and sketch established concepts in finance. In general, one observes that nonlinear dynamics has not received enough attention.

Here we propose to model the processes by nonlinear stochastic dynamical systems. Due to the stochastic effects the trajectories may jump between regions attracted by different states, e.g. by steady states of a deterministic nonlinear process (equilibria in the sense of dynamical systems). A mathematical theory of such processes is presented by Skorokhod, Hoppensteadt, and Salehi (2002). In reality, these attracting states are changing in time on a large time scale. Since their short time effect is similar to that of steady states, we are using the term "quasi-steady" states to indicate their time dependence. It is important to remark that these "switches" between regions of attraction may occur also on the small time scale since they are excited by stochastic "perturbations". Based on this concept, we consider a generalization of the commonly used mean reversion process to the case of multiple equilibria. The traditional mean reversion model used to describe price dynamics in Finance is included as a special case. However, it seams to be necessary to include nonlinear effects also for modeling financial processes. Furthermore, assuming that quasi-steady states are depending on a set of fundamental key economic variables, economic information can be taken into account.

In this thesis we give two examples: a model for foreign exchange rates and crude oil price. For the later we are going to develop a coupled system describing crude oil supply and demand, whereas in case of foreign exchange rates we return to the structural monetary exchange rate theory. As far as modeling and simulation of processes is concerned, the two disciplines finance and economics are mainly coexisting and not linked to the extend necessary for better understanding and predicting. Many important common territories are waiting to be better explored and developed. Here, we hope to give a small contribution to link these fields.

2.1 Nonlinear Phenomena in Real Data

This section delivers some basic descriptive statistics for selected commodity prices and foreign exchange rates and illustrates their frequency distributions. It turns out that many economic and financial time series exhibit a clustering which can be explained e.g. by a range of possible outcomes during a crisis, like the Persian Gulf crisis in 1990. This clustering behavior is getting more and more into the focus of mathematical modeling of financial price processes. Recent analysis of spot and futures prices (e.g. by Borovkova, Dehling, Renkema, and Tulleken (2003)) detect a similar behavior in case of agricultural and energy commodities. In order to detect the underlying attractors a simple estimation technique is offered and applied to the oil market.

The oil price data for our investigation is taken from Energy Information Administration[1] whereas the remainder data is taken from the International Monetary Fund's International Financial Statistics database. The later is available over a longer time horizon however with a lower frequency (usually monthly). *West Texas Intermediate Cushing* (WTI Spot Price Free On Board) is taken as a proxy for the global oil price. Here, daily realizations are available from the end of the eighties. To demonstrate the crude oil price development over the last thirty years, the trajectory of the *Refiner Acquisition Cost of Imported Crude Oil* (IRAC) is displayed in addition.

Table 2.1 provides the data sources, the sample frequencies and the different time frames. Furthermore, the first four moments of the underlying price process are computed. Most statistical texts describe kurtosis as a measure of the "peakedness" of a distribution. However, Darlington (1970) shows that a far better term for describing kurtosis is "bimodality". In the following, the standardized kurtosis is used, i.e. data from a normal distribution will have a kurtosis value of zero. The kurtosis of a perfectly symmetrical bimodal distribution is a constant -2 (see e.g. Chissom (1970)). The histograms for different asset markets in figure 2.1 and the small kurtosis indicate bi-modality in the data samples. This is particular true for crude oil.

[1] http://www.eia.doe.gov/oil_gas/petroleum/info_glance/petroleum.html

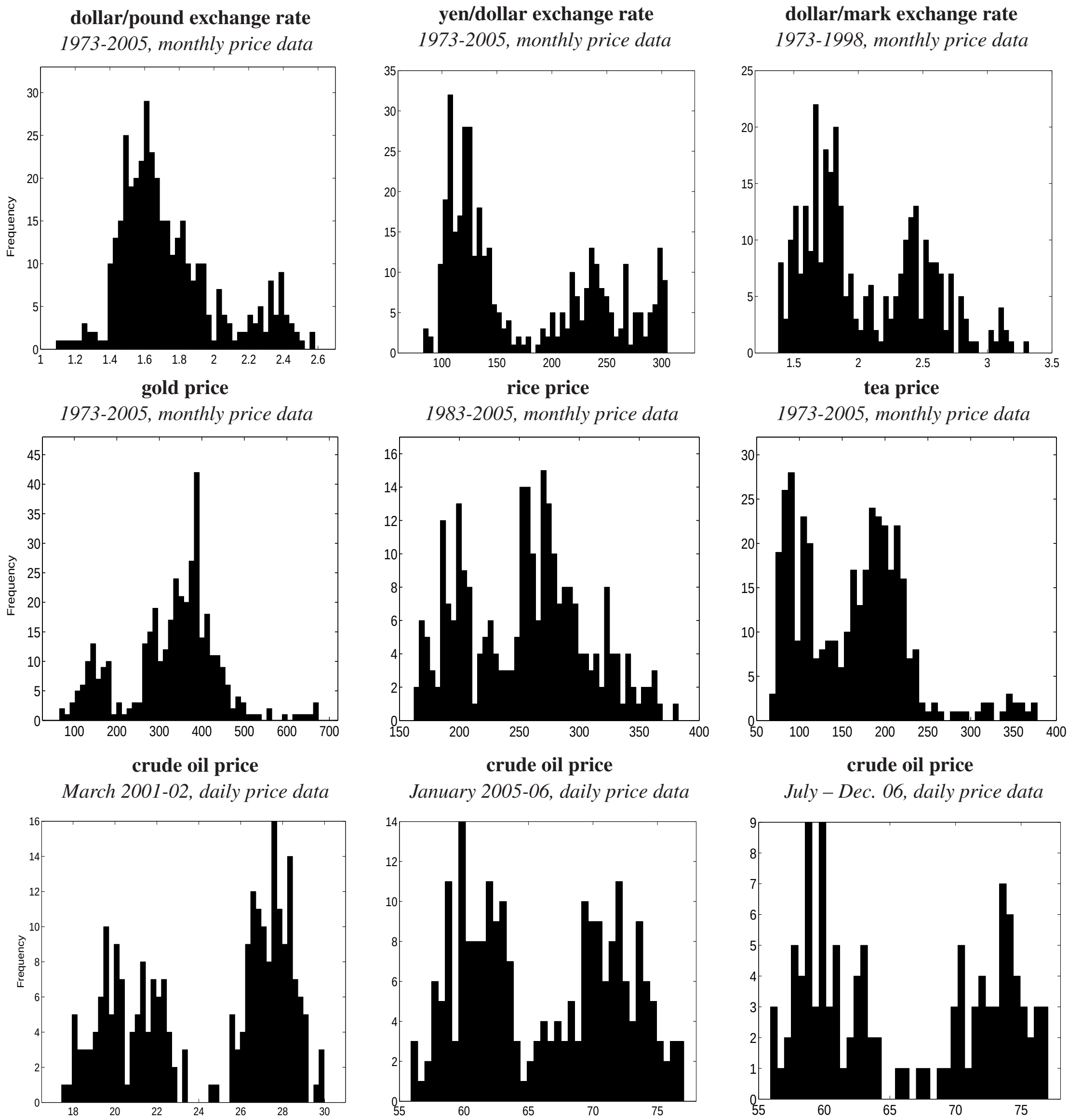


Figure 2.1: Clustering in Asset Prices.

Asset	Frequency	Source	Period	Mean	S.D.	Skewness	Kurtosis	LF
Exchange Rates								
$/£	monthly	IMF	1985 - 2005	1.74	0.29	0.90	-0.30	1
Yen/$	monthly	IMF	1973 - 2005	174.08	67.53	0.56	-1.22	1
$/DM	monthly	IMF	1973 - 1998	2.05	0.45	0.56	-0.74	1
Commodities								
Gold	monthly	IMF	1973 - 2005	332.26	124.60	-0.21	-0.87	1
Rice	monthly	IMF	1988 - 2005	254.39	52.03	0.17	-0.84	1
Tea	monthly	IMF	1988 - 2005	139.12	50.77	0.26	-1.31	1
Crude Oil								
WTI	daily	EIA	March 2001-02	24.39	3.67	-0.30	-1.49	1
	daily	EIA	January 2005-06	66.13	5.83	0.10	-1.38	1
	daily	EIA	July - Dec. 2006	65.45	6.59	0.35	-1.45	1

Table 2.1: Data Sources and Basic Statistics

In order to evaluate the hypothesis whether the price process X has a normal distribution or not the Jarque-Bera test is generally used. However, to take into account for small samples, we perform the Lilliefors test for normality which exhibits more precisely the kurtosis and fat tails (see e.g. Judge, Hill, Griffiths, Lutkepohl, and Lee (1998)). It compares the empirical distribution of X with a normal distribution having the same mean and variance as X. The result *LF* is 1 if we can reject the hypothesis that X has a normal distribution, or 0 if we cannot reject that hypothesis. We reject the hypothesis if the test is significant at the 5% level. The selected distributions are far from being normally distributed over the specific time periods. The Lilliefors tests rejects for all data sets the normal hypothesis. The same is true for the logarithm of the prices.

To give an additional motivation to investigate nonlinear price diffusion processes, the evolution of the price density is computed in figure 2.2 estimated using a kernel smoothing method. Thereby, we detect a bi-modality in the realized oil price distribution indicating two major centers of concentration during this bumpy period of time.

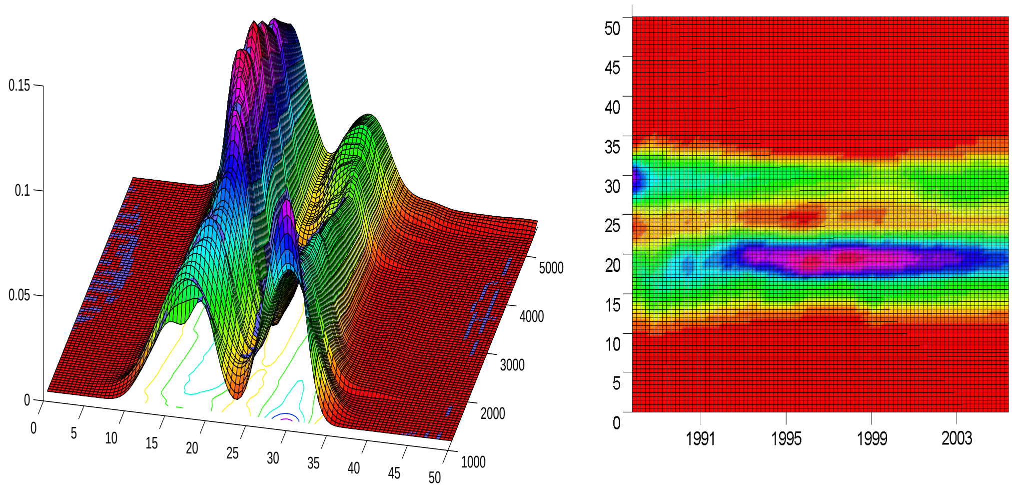


Figure 2.2: Dynamics of Attracting Domains (1988–2005). *The left side shows the distributional dynamics of historic oil price data from 1988 to 2005. The picture on the right hand side represents the corresponding contour plot emphasizing once more the attracting regions around* $X_1 = 20$ *and* $X_3 = 30$ *dollar per barrel.*

To avoid misunderstandings, we have to be careful in interpreting these results. The shape of the density function is of course sensitive to the chosen time periods. Neglecting the turbulent oil market in the mid eighties, the bi-modality is less distinctive. In spite of this, there is a demand for more flexible modeling of the underlying process.

The time series in the figure above indicate multi-modality that means change of attractors in time. This feature can be explained by assuming that the underlying processes is nonlinear and multiple attractors are present. Restricting oneself to a single attractor, one has to assume that it is changing in time, and one has to model its dynamical change in a rational way. Whereas the ansatz with multiple attractors has less difficulties, the single attractor ansatz is no clear alternative. The ”stability” of the attractors influences the timescales of the switches.

2.2 Identifying Constant Attractors

Obviously, the past thirty years of oil price history is characterized by distinctive price fluctuations. Figure 2.4 shows monthly and daily oil prices over different time periods. Applying an estimation technique which is going to be developed at the end of this section, we locate two constant price attractors or quasi-steady states. Naturally, we do not expect the attractors to be time invariant. Their dynamics will essentially depend on the economic environment, in our case on conditions of the global oil market, like supply and demand. Especially, in the latest months of 2006, there was a significant pressure primarily due the growing demand for crude oil from Asia driving the oil price to heights not reached ever before. It seems that the attractors of its dynamics will be in higher regions.

Since all quasi-steady states are supposed to fluctuate, each single equilibrium may change its properties being either attracting or repelling. The interval between the attracting levels can be considered as a natural zone of instability. We consider the case where prices high above or below this region are forced back. However, within the natural zone prices may strongly oscillate by crossing different domains of attractions several times mainly due to random perturbations (e.g. political or economic instabilities, or environmental conditions).

In order to locate the attracting equilibria $X^+ = c_A$ and $X^- = c_B$, we present a simple least square estimation technique. The obtained values give a first intuition of long run equilibrium states. However, if these steady states depend on economic data such as global oil supply and demand, more sophisticated methods have to be applied in order to detect the price attractors. Here, we refer to chapter 3.

We estimate the attracting equilibria by minimizing the following least square functional:

$$\Phi \;\; = \textstyle\int_A (f - c_A)^2 + \int_B (f - c_B)^2 \quad \rightarrow \quad \min \tag{2.1}$$

where $B = I - A$. The basic idea is illustrated in the following figure.

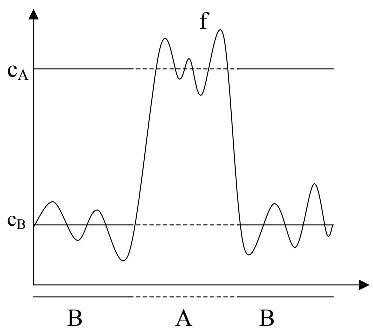
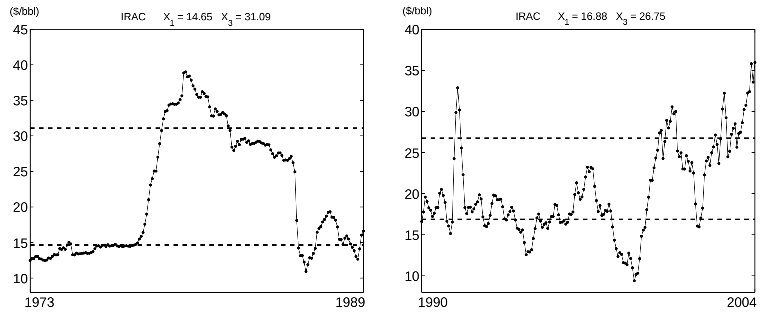
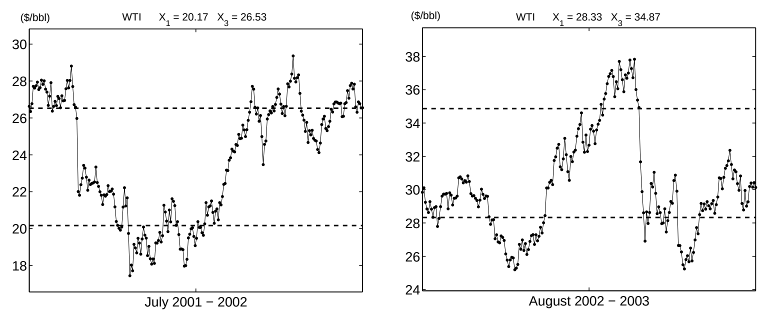


Figure 2.3: Identifying Constant Attractors

Long Term Horizon (monthly price data)



Mid Term Horizon (one year, daily price data)



Short Term Horizon (six months, daily price data)

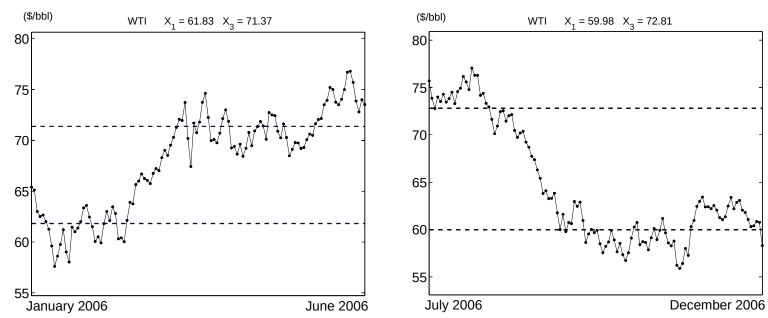


Figure 2.4: Oil Price Movements (1973 – 2005). *The figure illustrates monthly and daily price fluctuations across different time frames. The dashed lines represent estimated constant steady states. However, we assume in the following that dynamics of these attractors is essentially driven by oil market fundamentals such as global production capacities or global economic activity.*

Assuming that there exist a solution, A fixed:

$$\int_A (f - c_A) = 0 \qquad \Leftrightarrow \qquad c_A = \frac{1}{|A|}\int_A f \quad \text{and} \quad c_B = \frac{1}{|B|}\int_B f.$$

As a consequence

$$\int_A (f^2 - 2fc_A + c_A^2) = \int_A f^2 - c_A^2|A|.$$

Therefore, minimizing (2.1) is equivalent to

$$\begin{aligned} & \int_A f^2 - \int_A fc_A + \int_B f^2 - \int_B fc_B \to \min \\ \Leftrightarrow \qquad & \int_A fc_A + \int_B fc_B \to \max \\ \Leftrightarrow \quad & \frac{1}{|A|}\left(\int_A f\right)^2 + \frac{1}{|B|}\left(\int_B f\right)^2 \to \max \end{aligned}$$

Let $A_\alpha = \{x|f(x) = \alpha\}$, we compute the optimal α, finding the maximum value of Φ

$$\begin{aligned} \Phi(\alpha) &= \frac{1}{|A_\alpha|}\left(\int_{A_\alpha} f\right)^2 + \frac{1}{|B_\alpha|}\left(\int_{B_\alpha} f\right)^2 \\ &= \frac{(\int_{A_\alpha} f)^2}{|A_\alpha|} + \frac{(\int_I f - \int_{A_\alpha} f)^2}{|B_\alpha|}\ , \end{aligned}$$

where

$$\int_{A_\alpha} f = \int_I f\text{sign}(f-\alpha)^+ \qquad \text{and} \qquad |A_\alpha| = \int_I \text{sign}(f-\alpha)^+.$$

This algorithm is very easy to implement and provides a first tool in order to get a picture of the underlying time series.

The following question arises immediately: ***Why do we find such a pattern?*** In the following two sections we present a selection of attempts to explain these phenomena with a focus on the crude oil market. We want to establish a general perspective leading to a nonlinear mean reversion approach, a system of stochastic differential equations with multiple quasi-steady states.

2.3 Pricing Methods in Economics

Price jumps on the oil market between different levels have already been detected during the eighties and created a numerous theoretical and empirical studies about the structure of the world oil market and the role of OPEC as a cartel. Sometimes OPEC is divided into different blocks addressing in particular the role of Saudi-Arabia or dividing into groups according to their financial needs, absorptive capacity, cost of extraction, and size of reserves (compare e.g Adelman (1982), Griffin and Teece (1982) and Mabro (1991)). In contrast to that, a few approaches are working with the hypothesis of a competitive market where changes in oil prices are determined by a mismatch of supply and demand rather than a cartel behavior (see e.g. MacAvoy (1982)). An extensive survey with different classifications of models on OPEC behavior and as well as a brief comment on diverse empirical studies is offered by Cremer and Salehi-Isfahani (1991).

Multiple Equilibrium Story

We want to take up the idea of multiple equilibria and try to reveal fundamental characteristics of the crude oil market using the properties of nonlinear interacting systems. Associated with a backward bending supply curve and a relative inelastic demand curve, the idea of multiple equilibria in oil markets was first mentioned by Cremer and Salehi-Isfahani (1989). Surprisingly, this approach is mostly ignored in the oil market analysis. The importance of multiple equilibrium approaches in economics and the potential instability of markets is recently addressed by Krugman (2000) revisiting the energy crisis and pointing out the pioneering work of Cremer and Salehi-Isfahani. To quote Krugman:

> *"Aside from the evident weakness of OPEC viewed as a cartel, the history of the rise and fall of oil prices is very suggestive of some sort of multiple equilibrium story. The original surge in oil prices came suddenly and unexpectedly, with a long-term effect from a short-term restriction of supply - not what you would expect from a cartel gradually learning about its market power, but very much what you would expect if events "tipped" the market from one equilibrium into another. Why didn't the multiple-equilibrium view gain more acceptance? This is something of a puzzle. Perhaps it seemed too exotic at the time, especially applied in such a down-to-earth (below-the-earth?) industry as oil. What is particularly odd is that fancy, multiple-equilibrium stories are now very fashionable when applied to high-tech sectors. But everyone has lost interest in the old energy issue."*

According to Cremer and Salehi-Isfahani oil differs from other commodity markets like agricultural commodities (rice, tea, or soybeans) not in the existence of a cartel, but in three other facts: it is an

- exhaustible resource,
- production is controlled by national governments, and
- oil is the dominant source of national income for major exporters.

The property that oil is an exhaustible resource means that not extracting is a form of investment. Thus the producing country has to decide at the margin whether oil in the ground is more valuable than extracted oil. When oil revenues become very large relative to Gross National Product (GNP), structural limitations on their useful disposal force the country to limit or even reduce oil exports. If a natural resource producing country does not want to spend all of the massive cash flow generated by a sudden price increase on consumption, it must pursue one of three strategies:

- engage in real investments at home,
- invest abroad or
- "invest" by cutting oil extraction, and hence reducing supply.

However, the optimal investment strategy of oil exporting countries is bounded by limited *absorptive capacity* and *imperfections in the international capital* markets, a circumstance well known in development economics. The concept of limited absorptive capacity of a country pays attention to the fact that the rate at which investment expenditure can be turned into productive capacity declines with the total size of total investment. Imperfect capital markets may arise due to e.g. political risks or uncertainty regarding property rights. Large foreign investments in physical capital assets (real estates, industrial concerns, etc.) are vulnerable by seizures. This is particular true for countries like Iran or Libya with strained diplomatic relations to the western hemisphere. In addition the policy makers might fear of appearing to busy accumulating foreign wealth at the expense of domestic investment or consumption. The quoted authors prove that these conditions lead to a *backward bending supply* curve which is depicted on the left in figure 2.5.

A hint to the existence of a backward bending supply curve can be given by the following simple argument: Suppose there exist a revenue $\bar{H}_t$ at time t such that income above $\bar{H}_t$ has zero marginal utility, then for a given price of oil X_t the output Q_t will satisfy

$$X_t \cdot Q_t \leq \bar{H}_t, \text{ or } Q_t \leq \bar{H}_t / X_t.$$

This implies that for high price the supply becomes very small, and hence the supply curve must at some point start declining. Obviously, the assumption of a fixed target revenue is too restrictive. Cremer and Salehi-Isfahani replace this concept of a target revenue with a more realistic specification of development strategy. Using a dynamic economic growth model for oil they prove that the constraints of absorptive capacity and imperfect capital markets are sufficient to produce a backward bending supply curve for oil. The basic ideas of the underlying model are sketched below.

Underlying Model

According to Cremer and Salehi-Isfahani oil producers maximize their discounted sum of utility from consumption (C_t), extraction $(S_t - S_{t+1})$, and investment $(K_{t+1} - K_t)$:

$$\max_{C_t, K_{t+1}, S_{t+1}} U_t(C_t) + V_t(K_{t+1}, S_{t+1}) \frac{1}{1+\rho}.$$

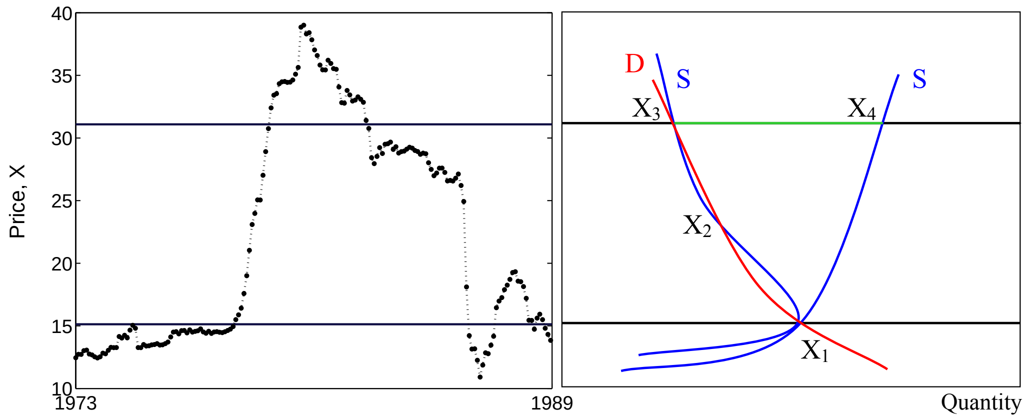


Figure 2.5: Backward Bending Supply Curve *On the left side we depict the time period as analyzed by Cremer and Salehi-Isfahani (1973-1989) and the estimated constant equilibria* $X_1 = 15.06$ *and* $X_3 = 31.19$. *On the right we depict the supply (blue) and demand (red) curves. In case of a backward bending supply curve we obtain three equilibria;* X_1 *and* X_3 *are stable, whereas* X_2 *is unstable.*

Here $V_t(K_t, S_t)$ represents the maximum utility which can be derived over time from holding the amount of capital K_t and oil S_t. The constraints defining the feasible set are:

$$\begin{aligned} C_t + I_t &= Y_t + X_t R_t && \textit{national accounts}, \\ K_{t+1} - K_t &= g_t(I_t) && \textit{capital accumulation}, \\ S_{t+1} &= S_t - R_t && \textit{depletion part}, \\ Y_t &= f_t(K_t) && \textit{production function}, \end{aligned}$$

where ρ measures the time preference of the country. I_t is the investment, R_t the quantity of oil extracted, X_t the price of oil, and Y_t the GNP (Gross National Product). In this model the value of extra income declines because of sharply diminishing marginal utility of consumption and constraints on absorptive capacity.

Consequences of a Backward Bending Supply Curve

Given a standard demand function D, there arise three intersections X_1, X_2, X_3 with the backward bending supply curve S. The intersection X_2 is unstable. Whereas, the low and high price equilibria X_1 and X_3 are stable. Due to external shocks, the price may jump from X_1 to X_3. Under standard assumptions the high price level would not last over a long period of time. In order to maintain this price level OPEC would be forced to absorb somehow the resulted excess supply $\overline{X_3X_4}$ (green line). However, in reality, the higher oil price level survived without production rationing by OPEC. The competitive model of Cremer and Salehi-Isfahani provides an answer to this puzzle. Since both X_1 and X_3 are seen as competitive equilibria, there exists no access supply to absorb.

To sum up the findings of Cremer and Salehi-Isfahani: Already moderate exogenous shocks can create discontinuous jumps between the different equilibrium levels X_1 and X_3. Thus, large price movements can be explained by multiple equilibria in a competitive market environment.

2.4 Pricing Methods in Finance

A first intuitive model for the dynamics of crude oil prices is a system of ordinary differential equations (deterministic approach):

$$dX = \mu(t, X, Y)dt. \tag{2.2}$$

$Y(t)$ describes external, e.g. economic or political effects. The oil price $X(t)$ is a solution of the ordinary differential equations (2.2). The function μ could be a polynomial or a rational function in X with coefficients depending on t and $X(t)$.

Actually, real processes are in general random processes that means we have to consider randomly perturbed dynamical systems. This class of dynamical systems were first studied by Bogoliubov and Krylov using methods from ergodic theory and Markov processes. Skorokhod, Hoppensteadt, and Salehi (2002) are providing a comprehensive survey of the mathematical theory and methods for randomly perturbed differential equations and their applications. Modeling processes in economics and finance leads naturally to such systems, denoted by the equation

$$dX = \mu(t, X, Y, Z)dt, \tag{2.3}$$

where Z describes the random perturbation. However, the results of mathematical research on nonlinear differential equations with stochastic perturbed coefficients have to the best of our knowledge only rarely been used in modeling economic or financial processes. For instance, the information obtained about the transitions of the trajectories of the perturbed system between the stable steady states of the unperturbed one should be taken into account in analyzing real time series. We will pursue this remark in a forthcoming investigation and consider here a perturbation caused by adding to equation (2.2) a stochastic process:

$$dX = \mu(t, X, Y)dt + \sigma(t, X, Y)dW. \tag{2.4}$$

where W denotes a Wiener process (Brownian motion). The functions, $\mu(\cdot)$ and $\sigma^2(\cdot)$ are called the drift and the diffusion functions of the price process. First of all, the functions μ and σ are unknown and need to be determined by modeling and by means of data. Economic working hypothesis are necessary to provide the functional dependence of economic variables. To discover market forces responsible for these price variations we introduce in chapter 4 a coupled interacting system which describes the dynamics of crude oil price, supply, and demand. In this section we concentrate on a selection of financial models. Their drawbacks in capturing recent developments in the oil market lead to the introduction of generalized mean reversion process.

The idea of the following section is to consider some ad hoc specifications for the drift and volatility function which are widely used in the literature of energy markets (compare e.g. Pilipovic (1997) and Geman (2005)) and to discuss their short-comings.

Geometric Brownian Motion

Based on the price model of Samuelson, Black and Scholes (1973) and Merton (1973) approached the problem of pricing financial instruments, such as futures and options. Since than, the geometric Brownian motion is a basic building block of modern finance. An enormous selection of pricing tools have been developed on the basis of this model. At first glance, it seems to be appealing to take over these concepts and techniques to the oil industry. Accordingly, the crude oil spot price dynamics is assumed to follow basically a Geometric Brownian motion (see e.g. Brennan and Schwartz (1985), Gibson and Schwartz (1990) and Brennan (1991)). The dynamics is given by

$$dX = \mu X dt + \sigma X dW, \tag{2.5}$$

where both μ and σ are constant. For any arbitrary initial value $X_0 = x_0$, the analytic solution is given as:

$$X(t) = \exp\left((\mu - \sigma^2/2)t + \sigma W(t)\right) x_0. \tag{2.6}$$

In many cases this approach yields to explicit solutions which we usually do not obtain for a more complex dynamics. As a consequence, the Black-Scholes framework serves as reference, however, adjustments taking into account the different nature of commodities are necessary. Using a model of a competitive spot market for an exhaustible resource under demand uncertainty, Lund (1993) shows that the geometric Brownian motion can hardly be an equilibrium price process under reasonable assumptions. Furthermore, empirical studies have shown that energy and commodities prices experience significant deviations from log-normality (compare e.g. Bessembinder, Coughenour, Seguin, and Smoller (1995)).

Mean Reversion Processes

As a consequence, the mean reversion process is probably the most commonly price model used by oil market practitioners. According to this modeling approach, the price dynamics of commodities exhibit a linear mean reverting drift, where the price moves in the direction of its long-run equilibrium. This effect is assured by setting the drift term:

$$\mu(t, X, Y) \quad = \quad \kappa_0(X_1 - X). \tag{2.7}$$

Here, κ_0 is a positive constant measuring the speed of adjustment towards X_1.

To understand the price dynamics, it is worthwhile to analyze the ordinary differential equation, i.e. focusing on the drift term. If X is higher (lower) than the long run equilibrium X_1, the sign is negative (positive). Thus the price is always reverting to its attracting level. Similar to an oscillating pendulum or spring is always pulled back towards its rest position. If the price is far away from X_1 the mean reversion term becomes larger, forcing X back to its attraction level.

Mean reversion can be motivated e.g. by global supply and demand responses to the prevailing price level. When crude oil prices are relatively high, existing producers will increase their production rate and new producers will enter the market, whereas consumers will at first replenish their stocks. Thereby creating a downward pressure on prices. As long as the price is higher than a the equilibrium price, this downward pressure is expected to last. There again, when prices are relatively low supply will decrease since, for instance, some of the high cost producers will exit the market and demand increases enforcing an upward pressure on prices (compare e.g. Hasset and Metcalf (1995)).

In case of a constant volatility $\sigma(t, X, Y) = \sigma_0$ we obtain the *Ornstein-Uhlenbeck or Vasicek model*. It was introduced in 1977 by Vasicek and is applied since then to a number of prices processes in financial markets such as interest rates, commodity prices and foreign exchange rates. The solution of the Ornstein-Uhlenbeck process is given by:

$$X(t) = e^{-\kappa_0 t} x_0 + \sigma_0 e^{-\kappa t} \int_0^t \exp(\kappa_0 s) dW(s) \tag{2.8}$$

Most modifications are concerning the modeling of the volatility function. A major approach in energy markets is the Pilipovic model. Pilipovic (1997) proposed a linear volatility function of the form: $\sigma(t, X, Y) = \sigma_0 X$. In addition he allows the long-term equilibrium price X_1 to be driven by a secondary stochastic differential equation leading to the following two-factor model:

$$\begin{aligned} dX_1 &= \kappa_1 (X_2 - X_1) dt + \sigma_1 X_1 dW_1, \\ dX_2 &= \kappa_2 X_2 dt + \sigma_2 X_2 dW_2 \end{aligned} \tag{2.9}$$

where $\kappa_i, \sigma_i, i = 0, 1$ are positive constants, and dZ determines the stochastic perturbation in the equilibrium price. (2.9) can be written in a vector form:

$$dX = M dt X + \Sigma dW X \tag{2.10}$$

where M, Σ and W are the following matrices

$$M = \begin{pmatrix} -\kappa_1 & \kappa_1 \\ 0 & -\kappa_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad W = \begin{pmatrix} -W_1 & 0 \\ 0 & -W_2 \end{pmatrix}.$$

M and Σ are constant matrices. Using the calculus for matrices one obtains similar to 2.6 for the solution of the equation with initial conditions $X(t_0) = X_0$ the following representation

$$X(t) = \exp(M - \Sigma' \Sigma / 2) t + \Sigma W(t)) X_0. \tag{2.11}$$

In this context, we would like to make the following remarks:

- Mean reversion can be either in the prices, or the natural log of prices. The later approach is suggested e.g. by Schwartz (1997). Accordingly, the mean reversion is applied to the log of the price $x = \ln(X)$ rather than to the price X itself:

  $$dx = \kappa(x_1 - x) dt + \sigma dW,$$

  where $x_1 = X_1 - (\sigma^2 / 2\kappa)$.

- Both, a pure Geometric Brownian motion as well as a simple mean reversion model are not in part able to capture fundamental phenomena of energy and commodity markets: Price distributions look very different from what we observe in the figures 2.1 - 2.4. Sudden large price movements due to extreme events and regime switching can not be addressed.

  This difference is particularly important for evaluating derivative contracts and managing risks in e.g. the petroleum, natural gas or electricity industries. To overcome this problem, financial mathematics offers are a variety of extensions e.g. introducing additional jump components or taking into account nonlinear effects.

- To our opinion, the auxiliary process for the long run equilibrium X_1 is to naive. We expect the long run equilibrium to depend on a set of economic fundamentals. It is not just a new log-normally distributed variable. In Chapter 4 (3) we introduce a dynamics of the quasi-steady states in case of the oil (foreign exchange) market.

Jump Diffusion Models

In order to consider these characteristic phenomena, an additional random jump component is added to equation (2.4) (see e.g. Øksendal (1998) and Geman (2005)):

$$dX = \mu(t, X, Y)dt + \sigma(t, X, Y)dW + JXdQ(\lambda). \tag{2.12}$$

Its value depends on the probability of occurrence of a jump, the expected size, and their expected standard deviation. Again it is possible to work with different assumptions on drift and volatility functions. In this context of oil markets it seems to be reasonable to consider a mean reversion process. Accordingly, the oil price evolves with mean reverting drift and two random terms: a diffusion and a Poisson process embodying a random jump. The arrival of jumps is governed by a Poisson process dQ with arrival frequency parameter λ. The proportional jump size J may be a constant or drawn from a probability distribution.

It is assumed that abnormal news (political as well as economic) generate these discrete jumps of random size. However, we believe that this convenient extension provides only a partial explanation of the occurrence of these large price movements. In many energy markets, temporary price spikes are often the result of supply shocks. This is particularly true for electricity prices where abnormal discontinuous price jumps are usually a result of outages, transmission constraints, etc. Barlow (2002) offers an alternative to this “common adding” of a jump process. Inspired by Föllmer and Schweizer (1993) a simple microeconomic model, the author derives a nonlinear Ornstein-Uhlenbeck which accounts for sudden price jumps, whenever there is a mismatch in supply and demand. Thus, taking into account the characteristics of the electricity market, in particular missing effective storage, a localized market, capacity constraints on transmission lines, it is possible to derive a prices process that exhibits widely price jumps. In the following we are going to sketch the principle idea of Barlow (2002).

Nonlinear Ornstein-Uhlenbeck

Barlow (2002) considers simple functional forms of electricity supply and demand. The supply $S_t(X)$ is increasing in price X whereas the demand is decreasing $D_t(X)$. As a consequence there is a (unique) equilibrium spot price X_t for each point in time t

$$S_t(X_t) = D_t(X_t). \tag{2.13}$$

The supply is supposed to be limited, non-random and independent of time

$$S_t(X) \;=\; \alpha_0 - \alpha_1 X^{\alpha_2}, \tag{2.14}$$

where α_0, α_1 are positive and $\alpha_2 < 0$. The energy demand is assumed to be inelastic

$$D_t(X) = D_t. \tag{2.15}$$

If the demand exceeds the maximum supply α_0, then X_t is cut off at some maximum price $\overline{X}$. Given that S is invertible, the market equilibrium condition (2.13) implies a model for the spot prices:

$$X_t \;=\; \begin{cases} S^{-1}(D_t) = \left(\dfrac{\alpha_0 - D_t}{\alpha_1}\right)^{1/\alpha_2}, & D_t < \alpha_0 - \varepsilon\alpha_1 \\ \overline{X} = \varepsilon^{1/\alpha_2}, & D_t \geq \alpha_0 - \varepsilon\alpha_1. \end{cases} \tag{2.16}$$

Taking into account random effects on the demand side

$$D_t \;=\; \beta - \sigma_1 Y_t, \tag{2.17}$$

$$dY_t \;=\; -\lambda dY_t dt + dW_t, \tag{2.18}$$

we obtain for $D_t < \alpha_0 - \varepsilon\alpha_1$

$$\begin{aligned} X_t \;&=\; \left(\frac{\alpha_0 - \beta}{\alpha_1} + \frac{\sigma_1 Y}{\alpha_1}\right)^{1/\alpha_2} = (1 + \alpha_2 Z)^{1/\alpha_2} \\ \text{with} \qquad Z_t \;&=\; \frac{\alpha_0 - \alpha_1 - \beta}{\alpha_1\alpha_2} + \frac{\sigma}{\alpha_1\alpha_2} Y_t. \end{aligned}$$

As a result, the nonlinear Ornstein-Uhlenbeck process is given as

$$\begin{aligned} X_t \;&=\; \begin{cases} f_{\alpha_2}(Z_t), & 1 + \alpha_2 Z_t > \varepsilon, \\ \varepsilon^{1/\alpha_2}, & 1 + \alpha_2 Z_t \leq \varepsilon, \end{cases} \\ dZ_t \;&=\; \lambda(\alpha_0 - Z_t)dt + \sigma dW_t, \end{aligned} \tag{2.19}$$

where $f_{\alpha_2}(z) = (1 + \alpha_2 z)^{1/\alpha_2}$ and $f_0(z) = e^z$. The nonlinear model captures both mean-reverting behavior and price spikes observed in electricity markets. In addition, classical spot price models are included as special cases. Figure 2.6 shows a simulation of a nonlinear Ornstein-Uhlenbeck process.

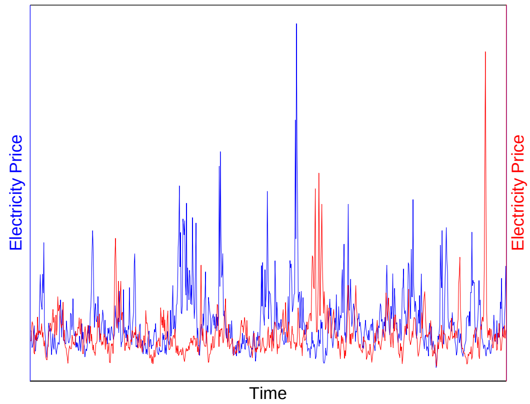


Figure 2.6: Nonlinear Ornstein-Uhlenbeck Process

In conclusion, we want to make some comments on Barlow's approach:

- In this model it is assumed that there is exactly one steady state, given as intersection point of the supply and demand curves. The model is stationary with respect to supply and demand.
- It does not provide a satisfactory explanation of the relation between spot and future prices. To obtain this, it seems necessary to look at multi-factor models. Similar extensions would also be needed to use this model for option pricing.
- The electricity price arises from a model for each supply and demand curves. It is easy in principle to incorporate additional factors to account for long-term effects, or changes in market structure. According to the author, such an extension of the model would be essential for handling options and future prices in a proper way.

Convenience Yield Models

Schwartz (1997) and Miltersen and Schwartz (1998) developed convenience yield models to price commodity futures and options with stochastic convenience yields and interest rates. The introduction of an additional state variable allows to take into account richer shapes of curves than one-factor models (especially for long term maturities) and richer volatility structures. The convenience yield is an unobserved quantity related to the physical ownership of the asset. According to Brennan (1958) and Brennan and Schwartz (1985) the convenience yield represents the overall benefit minus the cost that a holder of a commodity receives by holding commodity. It can be compared with a dividend yield for a stock.

Commodity pricing models are obtained via various assumptions on the behavior of C. A review of the literature of spot convenience yield models is offered by e.g. Ludkovski and Carmona (2003). Here, we present the basic spot model for convenience yield introduced by Schwartz (1997) and Gibson and Schwartz (1990) which is probably the most famous model of commodity prices. It is a 2-factor model with a stochastic mean reverting convenience yield C driving the geometric Brownian motion commodity spot price X_t

$$\begin{aligned} dX &= (\mu - C)X dt + \sigma X dW_1, \\ dC &= \kappa_c(C_1 - C)dt + \sigma_c dW_2. \end{aligned} \tag{2.20}$$

The increments to standard Brownian motion are correlated with: $dW_1 dW_2 = \rho dt$. The Ornstein Uhlenbeck process relies on the hypothesis that there is a level of stocks, which satisfies the needs of industry under normal conditions. In order to evaluate commodity contracts, we come back to this issue in chapter 5. The essential part of the following sections deals with modeling of the price process itself.

2.5 Nonlinear Price Systems

2.5.1 Analysis of Multi-stable Systems

In the following we make some comments on the basic idea of multiple steady state systems perturbed by random noise. Consider a deterministic dynamical system of the form

$$\dot{X} = g(X, \theta),$$

where g depends on the state variable X and a parameter θ.

Definition 1 (Multiple Steady State System)**.**
An autonomous dynamical system is called a multiple steady state system if and only if there exist parameter values θ such that $g(X, \theta) = 0$ has more than one solutions.

Let us assume that g is the X-gradient of a potential G. That means,

$$g(X, \theta) = \nabla_X G(X, \theta).$$

In our case the price dynamics is a scalar equation. A primitive G to the drift function is a potential. The coefficients of the polynomial are the parameters θ. G is a polynomial of fourth order.

Critical Points of the Deterministic System

Such gradient systems have been studied extensively in mechanics, chemistry and biology (see e.g. Metzner, Schütte, and Vanden-Eijnden (2006) and Horenko, Dittmer, Fischer, and Schütte (2006)). Steady states of the system are critical points $X_j(\theta)$ of the potential, that means

$$\nabla_X G(X_j(\theta), \theta) = 0$$

Strict local minima are stable steady states of the system. The critical points X_j and the derivative of g in X_j, that means the Hessian of G are crucial for the behavior of the trajectories. The trajectories of the system can be considered as curves on the graph of G. The graph of G depending on the system parameters illustrates the properties of the dynamical system.

Figure 2.7 shows the graph of the potential function on X and an additional parameter, deforming continuously the case of two minima and one maximum into the case of one minimum. The trajectories move on this surface, if there are no perturbations, generically to a stable minimum.

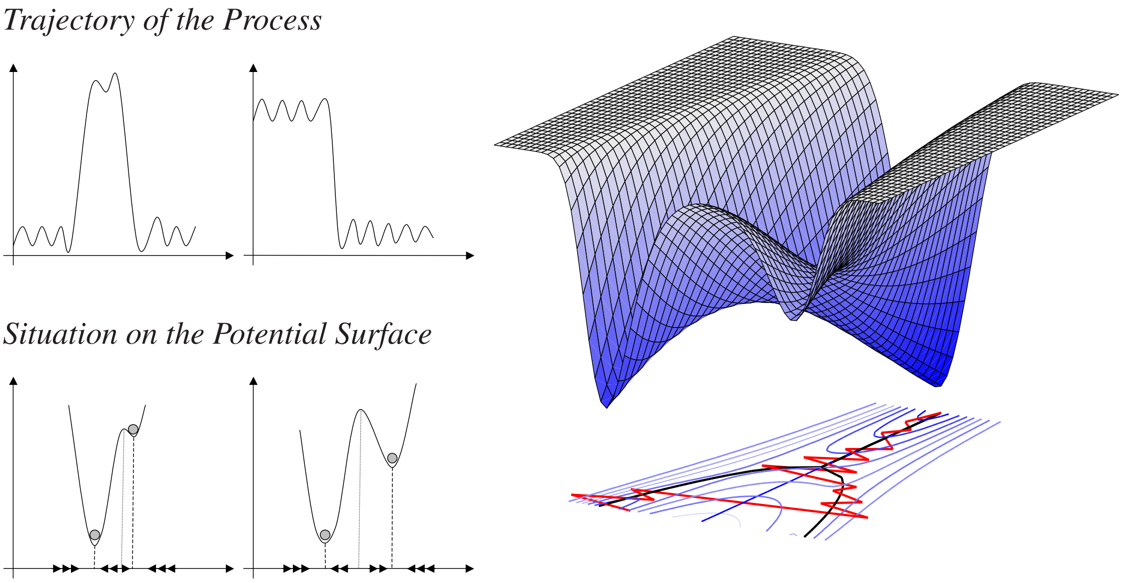


Figure 2.7: Transition between the Minima of the Potential Function

The projection of the critical points in the space of variables (X, q) is a system of bifurcating curves forming a pitchfork (black, full line indicates "stable", dotted "unstable"). If g is stochastically perturbed the critical points may change location and character. Trajectories may be driven from a neighborhood of a steady state to another one. The projection of a trajectory (red) to the stochastic dynamical system is a curve oscillatory around this pitchfork:

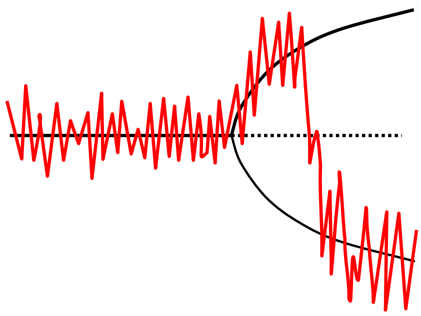


Figure 2.8: Pitchfork

Dynamical systems with multiple steady states and stochastic perturbations are highly important for applications in many areas: mechanics, neuronal networks, molecular dynamics, ..., finally also in economic and financial systems. The common feature of all processes modeled by these systems is the fact that their dynamics can be described as transition between discrete states. Here we refer to Skorokhod, Hoppensteadt, and Salehi (2002) where the dynamics of such stochastic systems is analyzed and probabilities for transitions between the steady states are determined. These investigations provide information also for models in economics as considered here. Multiple steady states may arise for fixed parameters in model equations, but can also be observed in the data of the real systems, for instance in the dynamics of the price of oil. The concept of multiple steady states is basic to our approach in modeling the price dynamics. In the following we represent polynomials as products of their linear factors. Here the zeros may be complex and have to be considered as functions of the coefficients of the polynomial. The drift term, e.g. the zeros of the polynomial, may depend on stochastic processes, the dynamics may be perturbed by a Wiener process with a volatility, which also may depend on stochastic processes.

2.5.2 Generalized Mean Reversion Process

The oil price itself follows a stochastic nonlinear equation with drift and volatility depending on the supply, demand, and the price. We are going to formulate the equations for the coupled dynamics of these quantities. The concept we are using can be described as follows:

The stochastic differential equation for the price will contain a nonlinear drift. The nonlinearity should be such that the deterministic equation, arising by cancelation of the noise term, gives rise to a dynamic which essentially can be described by the evolution of in time varying states, playing the role of steady states in case of an autonomous system. This states will be called quasi-steady states. They may be called "stable" or "unstable" if locally in time attractive or repulsive. The main reason for this approach consist in the possibility to allow time intervals where there exist several quasi-steady states. In these intervals stochasticity can lead to transitions between these time-varying states, to jumps and a behavior similar to instabilities. There exists a "zone" of transition between this states, containing the trajectories of the price. In intervals where there is only one such quasi-steady state, approximation to this state will observed. We take the simplest nonlinearity allowing the described behavior: a polynomial of order 3, which has in the field of complex numbers exactly 3 roots, where at least one has to be real. Hence we allow for both periods of single and multiple equilibria. We start from the "ansatz":

$$dX = \kappa_x(X_1 - X)(X_2 - X)(X_3 - X)dt + \sigma(X)dW \qquad (2.21)$$

Here we may assume that X_1 is a real root, where the others X_2, X_3 have to be complex conjugate if they are not real.

The dynamics can be revealed studying the sign of the drift term. Allowing for time varying steady states, we define the price attractors as

$$X^{+} = \max\{X_1, X_2, X_3\} \quad \text{and} \quad X^{-} = \min\{X_1, X_2, X_3\}$$

Whenever the actual oil price lies above the steady state $X^{+}(t, X(t), Y(t))$ there is a tendency for the price to fall back down to the price band (negative sign of the drift term), whenever the price is far below $X^{-}(t, X(t), Y(t))$ the price rise back. As a consequence, prices high above or below this zone of instability will not persist over a long period of time.

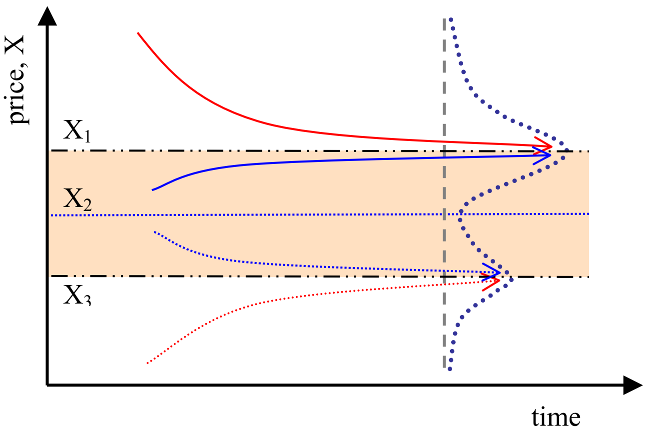


Figure 2.9: Generalized Mean Reversion Process – Root Decomposition. *In case of constant equilibria the quasi-steady state X_2 is repelling, whereas $X^{-} = X_1$ and $X^{+} = X_3$ are attracting. By crossing X_2 due to e.g. small random perturbations, the domain of attraction is changed and a new attracting level might be reached. However, we expect these quasi-steady states X_j to be dependent on key indicators of the oil industry Y. Naturally, these oil market determinants will change over time. Hence the attractors may change both their characteristics and location.*

Simulating a large number of price paths, we will get a multiple peaked terminal distribution. Here, intimated by points. The dynamics of the price distribution is essentially determined by the drift and volatility function. We will address this issue in greater detail in section 4.

2.5.3 Dynamics of the Quasi-Steady States

Future oil price changes given by $dX/dt = \mu(t, X(t), Y(t))$ depends essentially on some economic data $Y(t)$ described by a deterministic or stochastic function. Suppose that the drift term $\mu(t, X, Y) = \mu^{*}(X, Y)$ (autonomous case) and that μ^{*} has discrete zeros $X_j(Y), j = 1, ..., k$. Then $X_j(Y)$ are equilibria of the equation. In economics, there is rather often assumed that the trajectories of the system are tending to an equilibrium for large times. However, in reality this assumption is not valid, also the systems will not be autonomous. Zeros X_j of $\mu(t, X(t), Y(t))$ will depend on t and $Y(t)$. If changes in time are slow, $X_j(t)$ play locally in time the role of equilibria. We call them quasi-steady states.

We are going to explain the dynamics of the oil price steady states in structural terms, that means in terms of evolution of global oil supply and demand, and variables that determine supply and demand (e.g. investment and production capacities, real income, and technological progress). As a consequence, recent high oil prices can be traced back to a structural upwards trend, driving the steady states up to $60 dollars a barrel. In a first step, we describe the dependence of the price attractors on supply and demand and shortly discuss the resulting dynamics of the price distribution. In section 5 we present new concepts and approaches exploring rules in the dynamics of supply and demand.

Chapter 3

Distributional Dynamics and Nonlinear Diffusions

Contents

Diffusion processes are widely used for mathematical modeling in finance e.g. in modeling asset prices, including stock prices, interest rates, commodity prices, or foreign exchange rates. In the previous chapter, we presented a non-linear stochastic continuous-time model that captures the main characteristics of price dynamics. The generalized mean reversion process discloses various features of observed price movements such as multi-modality of the distributions, multiple equilibria, and regime switching. The dynamics of the quasi-steady states depend substantially on the economic environment.

We briefly review this fundamental relationship between nonlinear diffusion and the distributional dynamics, especially in case oil prices. As soon as the drift term and volatility function are not only depending on price but also on market fundamentals such as proven reserves, supply strategies, or the growing demand for oil of ambitious economies, it is possible to give distributional characteristics an economic interpretation. In addition this connection can be used to estimate the unknown system parameters.

This chapter is part of joint works with E. Kostina (2005 and 2006):

- **Parameter Estimation for Forward Kolmogorov Equation with Application to Nonlinear Exchange Rate Dynamics**, *Proceedings in Applied Mathematics*
- **An Inverse Problem for a Nonlinear Stochastic Differential Equation modeling Price Dynamics**, *Preprint Series: Interdisciplinary Center for Scientific Computing, University of Heidelberg* and *in review for Applied Mathematical Finance*.

It is a common strategy in financial econometrics to derive the likelihood function from the transitional probability density. Since explicit solutions are rare, the econometrics literature offers a wide range of approximations. A short overview is offered in this chapter. In this context, it is popular to work with the hypothesis of stationary or invariant probability density functions. Such distributions have been analyzed by e.g. Creedy, Lye, and Martin (1996) in connection with nonlinear price dynamics of foreign exchange rates. However, this assumption is not always justified. As a result, we propose an alternative non-stationary method of estimating diffusion processes taking into account the full dynamics of the transition probability function.

In order to calibrate the models, efficient algorithms identifying the system parameters are in demand. Taking into account nonlinear effects in volatility and drift and dependence on observed economic data, which are not directly modeled, one obtains problems which cannot be treated by standard numerical methods. The coefficients are rapidly oscillatory and strong instabilities may arise. To handle these problems we develop numerical methods, which are used to simulate the nonlinear dynamics of exchange rates depending on economic data. The model reveals a significant connection between exchange rates and its fundamentals. Furthermore, it is consistent with traditional flexible exchange rate models.

3.1 Price Dynamics and the Transitional Distribution

Stochastic differential equations describing price diffusion processes can be linked directly to diffusion equations, to the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\mu_i(x)f(x,t)) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i \partial x_j}(\sigma_{ij}^2(x)f(x,t)). \tag{3.1}$$

This partial differential equation defines the transitional distribution of the oil price at each point in time. In some special cases this equation, also known as the forward Kolmogorov equation, can be explicitly solved (see tabular 3.1). They can be used for testing algorithms. Therefore, we include them in our survey.

Table 3.1: Explicit Solutions for Mean Reverting Processes.

Vasicek (1977)

SDE	$dX = C_0(X_1 - X)dt + \sigma dW$	$M = X_0 e^{-C_0\Delta} + X_1(1 - e^{-C_0\Delta})$
PDF	$f(X,t) = (2\pi V^2)^{-\frac{1}{2}} \exp\left(-\frac{(X-M)^2}{2V^2}\right)$	$V^2 = \frac{\sigma^2}{2C_0}(1 - e^{-2C_0\Delta})$

Cox, Ingersoll, and Ross (1985)

SDE	$dX = C_0(X_1 - X)dt + \sigma\sqrt{X}dW$
PDF	$f(X,t) = \rho \exp(-u - v)\,(v/u)^{\frac{q}{2}}\, I_q(2\sqrt{uv})$

I_q is the modified Bessel function of the first kind of order q,
$\rho = \frac{2C_0}{\sigma^2(1-\exp(-C_0\Delta))}$, $q = \frac{2C_0X_1}{\sigma^2} - 1 \geq 0$, $u = \rho X_0 \exp(-C_0\Delta)$, and $v = \rho X$

If the price dynamics exhibit complex dynamics, it is not possible to derive elementary solutions. Therefore, it is common practise to approximate the solution, e.g. by assuming stationarity. In case of one state variable x, this is done e.g. by Creedy, Lye, and Martin (1996).

3.2 Stationary or Invariant Density

We are going to analyze the stationary case discussing conditions on drift and volatility appropriate for price dynamics. Unlike the transitional density it is possible to derive analytic expression for the stationary density. The stationary density of a nonlinear diffusion process is found be setting $\partial f/\partial t = 0$. This converts the diffusion equation (3.1) into an elliptic equation, for $n = 1$ into an ordinary differential equation for the stationary density:

$$0 = -\frac{d}{dx}(\mu f) + \frac{1}{2}\frac{d^2}{dx^2}\left(\sigma^2 f\right). \tag{3.2}$$

We assume that μ and σ are continuous functions in $\mathbb{R}$, $\mu(x) \geq 0$ and $\sigma(x) > 0$ in $\mathbb{R} \setminus \{0\}$. Applying the product rule, (3.2) can be rewritten as:

$$0 = -\frac{d}{dx}(\mu f) + \frac{1}{2}\frac{d}{dx}\left(f d\sigma^2/dx + \sigma^2 df/dx\right). \tag{3.3}$$

By integration with respect to x we obtain a first-order linear differential equation

$$c = -\mu f + \frac{1}{2}\left(f d\sigma^2/dx + \sigma^2 df/dx\right), \tag{3.4}$$

where c is a constant of integration. Define κ by

$$\begin{aligned} \kappa &= 2\,\frac{\mu - \sigma\frac{d}{dx}\sigma}{\sigma^2} \qquad (3.5)\\ &= \frac{2\mu}{\sigma} - 2\frac{d}{dx}\log\sigma \end{aligned}$$

for $x \neq 0$ and obtain

$$\frac{df}{dx} = \frac{2c}{\sigma^2} + \kappa f. \tag{3.6}$$

This equation has the general solution

$$f(x) = \exp\left(\int_{x_0}^{x}\kappa(s)ds\right)\left\{f(x_0) + 2c\int_{x_0}^{x}\frac{1}{\sigma^2(\xi)}\exp\left(-\int_{x_0}^{\xi}\kappa(\xi)ds\right)d\xi\right\}, \tag{3.7}$$

where x_0 is a positive fixed number and the representation holds at first for $x > 0$, since $\sigma(x)$ may vanish for $x = 0$. We have to find appropriate conditions that f is be a density for a price, which is restricted to $[0, \infty]$ in its values that means f should be a nonnegative function supported in $[0, \infty]$, $f(0) = 0$ and $\int_0^\infty f(x)dx = 1$. Since prices should not get negative and f should be integrable, it is natural to assume that σ vanishes at $x = 0$ fast enough, and the drift term is negative for large values of x. Here, the case $\sigma(x) = \sigma_0 x^\alpha$, $\frac{1}{2} \leq \alpha \leq 1$ and the following drift function are of interest:

$$\mu(x) = \eta(x_1 - x)(x_2 - x)(x_3 - x), \qquad \text{with} \qquad \eta > 0.$$

We assume

$$\int_{x_0}^{x} \kappa(s)ds \quad \text{converges for } \ x \to 0 \quad \text{in } \ \mathbb{R} \cup 0\{-\infty, \infty\}.$$

However, we are mainly interested in the case $\int_{x_0}^x \kappa(s)ds \to -\infty$. In case of a finite limit we obtain that the density does not vanish in $] - \infty, 0[$.

We formulate the following assumptions:

I Behaviour for $0 < x \leq x_1, \ \ x_1$ small:

(a) $\quad \sigma^2\kappa(x) \geq \delta > 0$

$\quad \mu(0) - \sigma \frac{d\sigma}{dx}(0) > 0 \qquad$ (Feller Condition)

(b) $\quad 0 \leq \sigma(x) \leq \gamma_1^0 x^{\frac{1}{2}}$ or

(b') $\quad \sigma(x) \geq \gamma_1^0 x^\alpha, \quad 0 \leq \alpha < \frac{1}{2}$

II Behaviour for $x \geq x_1 > 0, \ \ x_1$ large:

$\quad \gamma_1^\infty x^{\frac{1}{2}} \leq \sigma(x) \leq \gamma_2^\infty x^2$

$\quad \mu(x) \leq -\beta x.$

We show the following lemma.

Lemma

1. I (a), (b) and $\sigma(0) = 0$ imply $\int_{x_0}^x \kappa(s)ds \to -\infty$ for $x \to 0$
2. I (b') and $\sigma(0) = 0$ imply $\int_{x_0}^x \kappa(s)ds \to +\infty$ for $x \to 0$
3. II implies $\exp\left(\int_{x_0}^x \kappa(s)ds\right) \leq \frac{const}{x^\rho}, \quad \rho > 1$. That means this function is integrable on $[0, \infty]$.

Proof: Taking into account the various assumptions, we get

1. $\kappa(s) \geq \dfrac{\delta}{\sigma^2} \geq \dfrac{\delta}{\gamma_1^0 x}$

$$\int_{x_0}^{x} \kappa(s)ds \leq -\int_{x}^{x_1} \frac{\delta^*}{s} ds = \delta^* \log \frac{x}{x_1} \to -\infty, \quad \text{for } x \downarrow 0.$$

2. $\int_{x_0}^{x} \kappa(s)ds = \int_{x_0}^{x} \frac{2\mu(s)}{\sigma^2(s)} - 2\log\frac{\sigma(x)}{\sigma(x_0)} \to +\infty$

   The integrand on the right hand side can be bounded by $\frac{const}{|x|^{2\alpha}}$, with $2\alpha < 1$ since $\sigma(x) \to 0$ for $x \to 0$ we obtain the claim.

   Remark: This fact excludes this case. If $\sigma(0) \neq 0$ one obtains a finite limit for the integral. In this situation one has to expect negative prices.

3. For $x > x_1$ and x large, we obtain

   $$\int_{x_1}^{x} \kappa(s)ds \leq \int_{x_1}^{x} -\frac{\beta s}{\gamma_1^{\infty} s^2} ds + 2\log\frac{\sigma(x_1)}{\sigma(x)} = \log\left(\frac{\sigma^2(x_1)x^{\beta^*}}{\sigma^2(x)x^{\beta}}\right)$$

   This implies together with the estimate of σ from below

   $$\exp\left(\int_{x_0}^{x} \kappa(s)ds\right) \leq const(x_1)\frac{1}{x^{1+\beta^*}}, \quad \text{for } x \to \infty.$$

   Thus, the left hand side is integrable on $[0, \infty]$.

Proposition:
Assume I(a), I(b) and define

$$f(x) = \eta \exp\left(\int_{x_0}^{x} \kappa(s)ds\right), \tag{3.8}$$

where η is given by

$$\eta = \left(\int_{0}^{\infty} \exp\left(\int_{x_0}^{\xi} \kappa(s)ds\right) d\xi\right)^{-1}.$$

Claim: f is the unique density function for the price.

Proof: It remains to be shown that $c = 0$ in the general solution.

Assume $c > 0$:
choose $0 < x_1 \leq x_0$ such that $0 < \kappa(x)$ for all $0 < x \leq x_1$.

$$\begin{aligned} -\int_{x}^{x_0} \frac{2c}{\sigma^2(\xi)} \exp\left(-\int_{x_0}^{\xi} \kappa(s)ds)\right) d\xi &\leq \int_{x_0}^{x_1} \frac{2c}{\sigma^2(\xi)} d\xi \exp\left(-\int_{x_0}^{\xi} \kappa(s)ds)\right) \\ &\leq -\int_{x}^{x_1} \frac{2c}{(\gamma_1^0)^2|x|} \exp\left(-\int_{x_0}^{x_1} \kappa(s)ds\right) \\ &\leq -const(x_1)\log(\frac{x_1}{x}) \to -\infty \quad \text{for } x \to 0. \end{aligned}$$

This implies negative values for f, which are impossible.

Assume $c < 0$:
It suffices to estimate the following integral from below

$$\int_x^{x_1} \frac{1}{\sigma^2\kappa}(\xi)\kappa(\xi)\exp\left(\kappa(s)ds\right)d\xi$$

where x_1 is small enough.

Setting $\rho(x_1) = \inf\left\{\frac{1}{\sigma^2\kappa}(\xi)|0 < \xi \leq x_1\right\}$ and observing $0 < \delta < \sigma^2\kappa$ we continue the inequality

$$\begin{aligned}
&\geq \rho(x_1)\left(\int_x^{x_1}\left(\kappa(\xi)\exp\left(\int_\xi^x \kappa(s)ds\right)\right)\right)\\
&= \rho(x_1)\left(\int_x^{x_1} -\left(\frac{d}{d\xi}\exp\left(\int_\xi^x \kappa(s)ds\right)\right)d\xi\right)\\
&= \rho(x_1)\left\{1-\exp\left(\int_{x_1}^x \kappa(s)ds\right)\right\}\\
&= \rho(x_1) \text{ for } x \to \infty
\end{aligned}$$

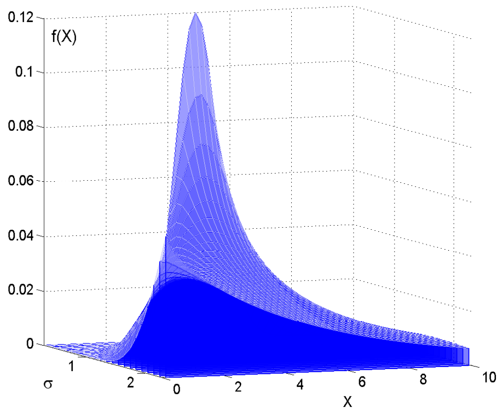
From the formula for the general solution we see that for negative c the boundary condition $f(0) = 0$ cannot be fulfilled.

Formula (3.8) shows that the critical point of the density function at which the density's derivative vanishes (i.e. $f^{'}(x) = 0$) are exactly the roots of $\mu - \sigma\frac{d}{dx}\sigma$. The local maxima and minima of the density are called mode and anti-mode, respectively. Thus, multi-modality is generally a result of multiple steady states in a dynamical system. In cases $\sigma^2(x) = 1$ the equilibria of the deterministic system $dX/dt = \mu(t, X, Y)$ are exactly the modes and anti-modes of the corresponding probability density function. Otherwise, the modes and anti-modes are shifted away from the equilibria of the deterministic system.

Cobb (1978) and Cobb and Zack (1985) study the properties of these multi-modal distributions belonging to the exponential family within the context of sets of critical points and the theory of singularities and catastrophes. The generalized exponential family of distributions provides great flexibility in modeling not only symmetric fat-tailed distributions, but also distributions that are skew and possible even multi-modal. Many common unimodal families e,g. normal, gamma, inverse gamma, and beta densities are included as special cases. Figure 3.1 shows a sequence of stationary probability distributions functions generated from linear and nonlinear stochastic price systems.

Equations (3.8) demonstrates that the dynamics of f and the shape of f^* is entirely determined by the drift and volatility function of the price process. We model the drift term as polynomial of order three with zeros evolving in time following structural changes in the market. A potential problem with these distributions is that they are time independent, This fact is a loss of information: The empirical distribution is a temporally aggregated distribution of a sequence of transitional densities (see Creedy, Lye, and Martin (1996)). Taking into account the important role of economic processes in modeling and simulation of the price distribution contributes helps to achieve a better understanding of market dynamics.

Linear Mean Reversion (CIR)



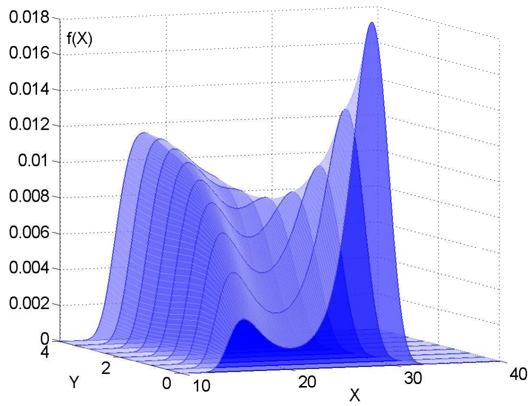
Drift and volatility functions

$\mu = C_0(X_1 - X)$
$\sigma = \sigma\sqrt{X}$

Stationary PDF

$f = \left(\frac{X}{2V^2}\right)^{\frac{X_1}{2V^2}-1} \exp\left(-\frac{X}{2V^2} / \Gamma\left(\frac{X_1}{2V^2}\right)\right)$
$M = X_1, \quad V^2 = \frac{\sigma^2}{2C_0},$

Nonlinear Mean Reversion



Drift and volatility functions

$\mu = C_0(X_1 - X)(X_2 - X)(X_3 - X)$
$\sigma = \sigma\sqrt{X}$

Stationary PDF

$f = \exp\left(-\frac{1}{3}\frac{C_0}{\sigma^2}\xi^3 + \frac{b}{2\sigma^2}\xi^2 - \frac{c}{\sigma^2}\xi|^{X}_{X_0}\right)\left(\frac{X}{X_0}\right)^{\frac{d-\sigma^2}{\sigma^2}}$
$b = C_0(X_1 + X_2 + X_3),$
$c = C_0(X_1X_2 + X_1X_3 + X_2X_3),$
$d = C_0X_1X_2X_3$
here, we choose
$X_1 = 16 + Y, X_2 = 20, X_3 = 28 - 2Y, \sigma = 0.3$

Figure 3.1: Distributional Dynamics. *We show stationary solutions to the Fokker-Planck equations in dependence on σ (on the left) and on parameter Y controlling the nonlinear drift (on the right). The simulations illustrate that nonlinear drift terms may lead to stationary distributions with multiple peaks, as observed in real data.*

3.3 Parameter Estimation

An important issue in finance is the parameter estimation (inverse problem) of stochastic differential equations. Recent approaches for modeling the dynamics of asset prices such as interest rates, commodity prices, or foreign exchange rates are based on diffusion processes with nonlinear drift terms and nonlinear volatility functions (see e.g. Ait-Sahalia (1999)). In order to apply the model to predictions or control of real processes, the model has to be able to reproduce the real process data quantitatively under changing conditions. Values for the unknown parameters and the initial data of the initial value problem have to be estimated, such that the observed behavior of the economic process is reproduced in an optimal way.

Traditionally, given the probability density distribution of the price process, the maximum likelihood method can be applied to estimate the unknown parameters. Except for very simple linear drift and volatility functions, the forward Kolmogorov equation cannot be solved explicitly and the likelihood function cannot be given in an analytic formula. To overcome these difficulties, a large number of techniques has been developed approximating the transition densities e.g. solving the Kolmogorov partial differential equation numerically (see Lo (1988)). Furthermore the transitional distribution is approximated by either assuming stationarity (see Creedy, Lye, and Martin (1996)), applying an Euler discretization (Kloeden and Platen (1992)), deriving an explicit expansion for the transition using Hermite expansion (see Aït-Sahalia (2002)), or computing the transition density using simulation techniques (see Pedersen (1995) and Brandt and Santa-Clara (2002). The later method is closely related to the Bayesian method proposed by Elerian, Chib, and Shephard (2001) and Eraker (2001). For a detailed comparison of approximation techniques for transition densities of diffusion processes we refer to Jensen and Poulsen (1999).

In the following we present the main principles of the stationary approach and provide a general framework to estimate unknown parameters of price diffusion processes used in mathematical finance. In case of one space dimension Creedy, Lye, and Martin (1996) applied the stationary approach to estimate nonlinear exchange rate dynamics. By contrast, this thesis is considering the full time dependent situation and, thus, generalizes results obtained for the quasi-steady state situation. The principle idea is based on minimizing a weighted least squares functional constrained by the forward Kolmogorov equation capturing the dynamics of the price probability density distribution. Beginning with the formulation of the estimation problem we describe a generalized Gauss-Newton algorithm constrained by the forward Kolmogorov equation and some initial and boundary conditions. Therefore, we do not assume the restrictive hypothesis of a time-independent distribution and, thus, generalize the equilibrium approach.

3.3.1 Problem Formulation

In order to solve the inverse problem for stochastic differential equations modeling the relevant processes, this thesis is solving the inverse problem of the corresponding forward Kolmogorov equation, a non-stochastic partial differential containing a diffusion and a drift term (compare 3.1). The parameters of these terms have to be recovered numerically from the available data. Since there exists already an extensive research on solving inverse problems for diffusion-transport equations, it seemed to be rather natural to use the forward Kolmogorov equation to determine the missing parameters. However, it very soon became obvious, that the arising coefficients are such that the available algorithms did not work well enough. Therefore, it was necessary to improve and to adjust the numerical methods for the inverse problem. This thesis is presenting and testing an improved algorithm overcoming these difficulties.

We consider a stochastic process $X_t, t \in [t_0, T]$ with the probability space $(\Omega, \mathcal{F}, P)$ and the distribution $F_t(x) = P(X_t \leq x), t \in [t_0, T]$ and $x \in \mathbb{R}$. It is assumed that the drift term and the volatility function of the stochastic process (2.4) depend on the spatial variable x and the unknown parameter vector p: $\mu = \mu(t, x; p)$ and $\sigma^2 = \sigma^2(t, x; p)$. To estimate the unknown parameters we make use of the forward Kolmogorov equation for the density function $f(t, x) = \frac{dF_t(x)}{dx}$ of the stochastic process X_t. The transitional distribution at each point of time satisfies the partial differential equation:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(\mu f) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 f). \tag{3.9}$$

Ideally, we would like to know the explicit solution which would allow to compute the maximum likelihood estimation. Unfortunately, the exact transition density is only known in few cases (see table 3.1). Thus, several approximations for the transition functions are proposed in the empirical literature. Here, we make some brief comments on the stationary approach before introducing the non-stationary approach.

3.3.2 Stationary Approach

As described in section 3.2, the stationary density f^* of a nonlinear diffusion process is found by setting $\partial f / \partial t = 0$:

$$f^*(x) = \exp\left(-\int_0^x \frac{2\mu(\xi)}{\sigma^2(\xi)} d\xi - 2 \ln \sigma(x)\right) \eta^\star. \tag{3.10}$$

The normalizing constant $\eta^\star$ is chosen such that the integral of f^* over its domain is 1. It is nothing but straightforward to use the stationary distribution (3.10) to estimate the unknown parameter via the well known maximum-likelihood technique. Creedy, Lye, and Martin (1996), for instance, make use of this idea to identify the drift term and volatility function of a nonlinear exchange rate model.

In general, quasi stationarity is assuming that the trajectories of the underlying process are tending very fast to stationary points. To quote Creedy et al. *"if prices are flexible, the speed of convergence to the stationary distribution is fast"*. This assumption may not be justified in real situations. In fact, the existence of simple stationary points by itself cannot be assumed in reality, since the model systems are in general not autonomous.

There will exist states which evolve slowly in time and locally play the role of stationary points. In the following we use the expression stationary states also for such states, despite the fact they may evolve in time slowly. The restriction to stationary distributions in strict sense has to be considered as an approximation, which can be too rigorous. Taking into account real time dependent data, as Creedy, Lye, and Martin (1996) are doing, one should consider the full time-dependence.

3.3.3 Non-stationary Approach

In the following, we drop the assumption of a stationary distribution. Using the fact that the price distribution of the stochastic price process satisfies the forward Kolmogorov equation, we estimate the unknown parameters p, by solving the following optimization problem

Minimize the weighted least squares functional

$$\min_p \quad \sum_{j=1}^{N} \left(\eta_j - \int_0^\infty f(t_j, x, p) x dx \right)^2 / \omega_j^2, \tag{3.11}$$

subject to the forward Kolmogorov equation

$$\frac{\partial f(t,x,p)}{\partial t} = -\frac{\partial(\mu(t,x,p) f(t,x,p))}{\partial x} + \frac{1}{2} \frac{\partial^2(\sigma^2(t,x,p) f(t,x,p))}{\partial x^2}, \quad 0 \le t \le T, \tag{3.12}$$

a state condition

$$\int_0^\infty f(t,x,p) dx = 1, \quad 0 \le t \le T, \tag{3.13}$$

an initial condition

$$f(t,x,p) \mid_{t=0} = f_0(x,p), \tag{3.14}$$

and two boundary conditions

$$\left. \mu(t,x,p) - \frac{1}{2} \frac{\partial(\sigma^2(t,x,p) f(t,x,p))}{\partial x} \right|_{x=x_{min}} = 0, \tag{3.15}$$

$$\left. \mu(t,x,p) - \frac{1}{2} \frac{\partial(\sigma^2(t,x,p) f(t,x,p))}{\partial x} \right|_{x=x_{max}} = 0, \quad 0 \le t \le T \ .$$

Here, the least squares functional (3.11) can be interpreted as a weighted norm of the difference between the real values η_j of the random variable x at time points $t_j, j = 1, ..., N$, and their expected values. The parameters p will be estimated by minimizing this functional subject to the forward Kolmogorov equation for the density function $f(t, x, p)$ (3.12), the state condition (3.13), initial conditions (3.14) and boundary conditions (3.15). In the thesis, we assume that the initial density $f_0(x, p)$ is given by

$$f_0(x, p) := \exp\left(-(x - x_0)^2\right)\eta^\star, \quad \eta^\star \text{ is a normalizing constant,}$$

with an additional parameter x_0 to estimate.

3.4 Numerical Methods

The optimization problem is a parameter estimation problem with partial differential equations as constraints. To solve this problem we apply the so-called boundary value problem (BVP) approach, see Bock (1987). The basic idea consists in parameterizing the dynamic equations like a boundary value problem and then performing simultaneously the minimization of the cost function subject to the constraints given by the discretized boundary value problem. We apply a generalized Gauss-Newton methods with trust region globalization techniques to solve the nonlinear least squares problem using a tailored linear algebra to exploit the special structures arising from the multiple shooting discretization. In the following subsections we illustrate the basic ideas of the applied numerical methods.

Spatial Discretization

In a first step, we reduce the forward Kolmogorov equation (3.12) into a system of ordinary differential equations (ODE) employing the method of lines Schiesser (1991). The initial conditions (3.14) and the integrals appearing in the problem formulation (3.11)-(3.15) are transformed accordingly. The state condition (3.13) and the boundary conditions (3.15) are taken into account in the transformation of the forward Kolmogorov equation (3.12) and are implicitly included into the resulting ODE system. As a consequence, the parameter estimation problem (3.11)-(3.15) results in a nonlinear least squares ODE constrained problem which can be formally written as

$$\begin{aligned} \min_{p} \quad & ||r_1(y, p)||_2^2 := \sum_{j=1}^{m_1} \left(\eta_j - B(t_j, y, p)\right)^2 / \omega_j^2 \qquad (3.16) \\ \text{s.t.} \quad & \dot{y} = \phi(t, y, p), \quad 0 \le t \le T, \text{ and } y(0) = y_0. \end{aligned}$$

For solving problem (3.16) we use the boundary value problem approach according to which the ODEs are parameterized by multiple shooting and are treated as implicit constraints in the minimization problem.

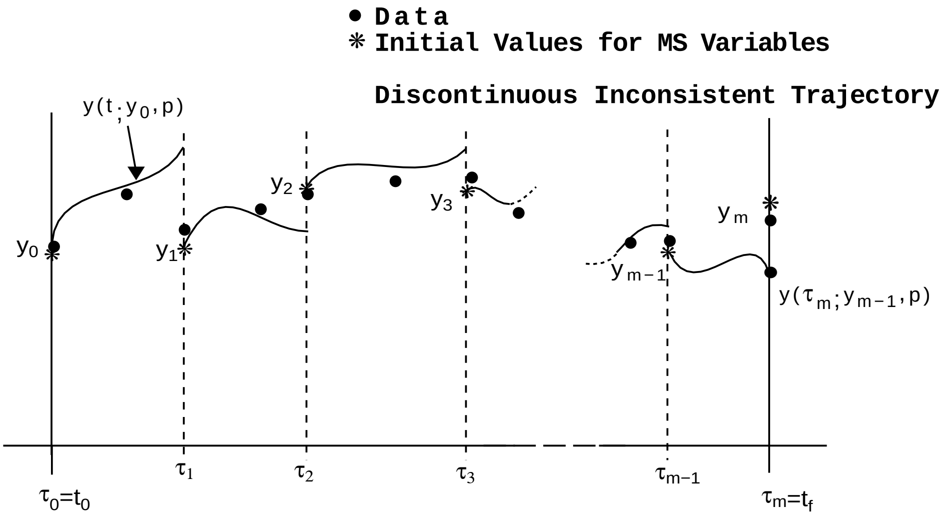


Figure 3.2: Multiple Shooting

Parameterization in Time - Multiple Shooting

We parameterize the semidiscretized parameter estimation problem (3.16) in time using multiple shooting approach. The scheme of the multiple shooting consists in the following. First, one chooses a suitable grid of multiple shooting nodes τ_j

$$0 = \tau_0 < \tau_1 < \ldots < \tau_m = T,$$

covering the interval where measurements are given.

At each grid point the values of the state variables y_j are chosen as additional unknowns and m initial value problems

$$\dot{y} \;=\; \phi(t, y, p), \quad y(\tau_j) = y_j, \tag{3.17}$$

are solved on each subinterval $I_j := [\tau_j, \tau_{j+1}]$ to yield a solution $y(t; y_j, p)$ for $t \in I_j$. The principle of multiple shooting is depicted in the figure below.

Solutions of dynamic systems, generated by this procedure, are usually not continuous at τ_j. This has to be enforced by additional matching conditions

$$h_j(y_j, y_{j+1}, p) := y(\tau_{j+1}; y_j, p) - y_{j+1}, \quad j = 0, ..., m-1.$$

Inserting the computed values $y(t_i, y_j, p)$, $\tau_j \leq t_i \leq \tau_{j+1}$, into problem (3.16) one obtains a constrained problem in the variables $(y, p) := (y_0, \ldots, y_m, p)$:

$$\begin{aligned} \min \quad & ||r_1(y, p)||_2^2 \\ \text{s.t.} \quad & h_j(y_j, y_{j+1}, p) = 0, \quad j = 0, ..., m-1. \end{aligned} \tag{3.18}$$

Multiple shooting possesses several advantages which are discussed to large extent e.g. in Bock (1987).

Generalized Gauss-Newton Method with Trust Region Globalization

For the solution of nonlinear constrained least squares problems of the presented type, Bock (1983) proposed a generalization of the Gauss-Newton Method which was only applicable to unconstrained least squares problems. The numerical method has proven to be stable and efficient for a series of real life parameter estimation problems constrained by ordinary differential equations and differential-algebraic equations.

The parameterization of the dynamic system yields to a finite dimensional, possibly large scale, nonlinear equality constrained approximation problem, which can be formally written as

$$\begin{aligned} \min \quad & ||r_1(s)||_2^2, \\ s.t. \quad & r_2(s) = 0. \end{aligned} \tag{3.19}$$

Here, the variables are parameters and values of the state variables at each multiple shooting node, $s := (y, p)$, $n :=$ dim s, the equalities $r_2(s) = 0$ represent the matching conditions induced by multiple shooting, $r_2(s) = (h_0^T(s), ..., h_{m-1}(s)^T)^T$. We assume that the functions $r_i : D \subset R^n \to R^{m_i}$, $i = 1, 2$, are twice-continuously differentiable.

The basic steps of the generalized Gauss-Newton algorithm with trust region globalization applied to the nonlinear constrained least squares problem are:

1. Start with an initial guess s^0.
2. Improve the solutions iteratively by

$$s^{k+1} = s^k + \Delta s^k, \tag{3.20}$$

where the increment Δs^k is the solution of the linearized problem

$$\min_{\Delta s \in R^n} \quad ||r_1(s) + J_1(s)\Delta s||_2^2, \tag{3.21}$$

subject to possible relaxed constraints

$$r_2(s) + J_2(s)\Delta s = (1 - \alpha)r_2(s), \; 0 < \alpha \leq 1, \tag{3.22}$$

and a trust region constraint

$$||\Delta s||_2^2 \leq \Delta^2. \tag{3.23}$$

Here, $J_i(s) = \dfrac{\partial r_i(s)}{\partial s}$, $i = 1, 2$ are the Jacobians, Δ is the trust region radius at the $k-$th iteration and α a relaxation factor that ensures the feasibility of linear constraints and the trust region constraint in the problem (3.21)-(3.23).

Following theory of nonlinear programming, we may conclude that if the Jacobians $J_1(s)$ and $J_2(s)$ satisfy two regularity assumptions on a domain D

$$\text{rank } J_2(s) = m_2, \tag{3.24}$$

$$\text{rank } J = n, \ J = J(s) = \begin{pmatrix} J_1(s) \\ J_2(s) \end{pmatrix} \tag{3.25}$$

then a linearized trust region problem (3.21)-(3.23) has a unique solution Δs, a unique Lagrange vector $\lambda \in R^{m_2}$, and a unique Levenberg-Marquardt parameter $\lambda_{LM} \geq 0 \in R$ satisfying the following Kuhn-Tucker conditions

$$\begin{aligned} (J_1^T(s)J_1(s) + \lambda_{LM} I)\Delta s \quad &+ J_2^T(s)\lambda &= -J_1^T(s)r_1(s), \\ J_2(s)\Delta s \quad & &= -\alpha r_2(s), \end{aligned} \tag{3.26}$$

and the complementarity condition, namely $\lambda_{LM} = 0$ if $||\Delta s|| \leq \Delta$.

Using (3.26) one can easily show that under the regularity conditions (3.24) and (3.25) Δs can be formally written with the help of a solution operator $\mathcal{L}(s, \lambda_{LM}, \alpha)$:

$$\begin{aligned} \Delta s &= -\mathcal{L}(s, \lambda_{LM}, \alpha) r(s), \quad r(s) = \begin{pmatrix} r_1(s) \\ r_2(s) \end{pmatrix}, \\ \mathcal{L}(s, \lambda_{LM}, \alpha) &= (I \quad 0) \begin{pmatrix} J_1^T(s)J_1(s) + \mu I & J_2^T(s) \\ J_2(s) & 0 \end{pmatrix}^{-1} \begin{pmatrix} J_1^T(s) & 0 \\ 0 & \alpha I \end{pmatrix}. \end{aligned}$$

Note that at the solution $s = s^*$ of the nonlinear problem (3.19) the following relations hold $\lambda_{LM} = 0$ and $\alpha = 1$ and the solution operator $\mathcal{L}(s, 0, 1)$ is a generalized inverse, that satisfies $\mathcal{L}(s, 0, 1)J\mathcal{L}(s, 0, 1) = \mathcal{L}(s, 0, 1)$. The operator $\mathcal{L}(s, 0, 1)$ plays a special role in statistical assessment of parameter estimation.

Evaluation of Functions and Jacobians

In the course of the Gauss-Newton method the entries in the objective function and constraints and their derivatives must be evaluated frequently. The main computational effort in multiple shooting arises in the solution of the initial value problems (3.17) and the computation of the solution derivatives with respect to the unknowns. Efficient error controlled numerical integration methods that also deliver derivatives of the solution are required.

We use the integrator DAESOL Bauer (2001), a Backward Differentiation Formula (BDF) method with variable mesh formulas based on Newton interpolation. It uses true variable mesh error estimates for order and stepsize control, and a nonlinear implicit system treatment which employs strategies developed for continuation problems.

The calculation of derivatives employs "Internal Numerical Differentiation" (IND) procedures which compute derivatives of the internally generated discretization schemes. This procedure is stable in the sense of backward analysis, accurate and allows derivative error control. Moreover, it is less expensive - computing time gains of up to 80% over usual forward differences are achieved. One of the unique features responsible for the fast performance of the multiple shooting method is the adaptive accuracy strategy which keeps integration tolerances below two decimals for the most part. However, it depends strongly on the use of IND. For a detailed discussion the reader is referred to Bock (1987).

Computing a Trust-Region Step

To compute the trust-region step Δs^k at the point s^k we have to solve problem (3.21)-(3.23). It may happen that the linearized constraints $r_2(s^k) + J_2(s^k)\Delta s^k = 0$ and the trust-region constraint $||\Delta s^k||_2^2 \le (\Delta^k)^2$ are inconsistent. To overcome this difficulty we relax the linear constraints and choose the relaxation factor α^k, $0 < \alpha^k \le 1$ such that the constraints

$$\alpha r_2(s^k) + J_2(s^k)\Delta s^k = 0, \quad ||\Delta s^k||_2^2 \le (\Delta^k)^2 \tag{3.27}$$

are feasible. The rules of choosing α^k will be described later.

Consider now the relaxed problem (3.21)-(3.23). Following a composite-step approach we compute the solution Δs^k of problem (3.21)-(3.23), which consists of a tangential and a normal components. This can be efficiently done by employing a block-LQ decomposition.

Block-LQ Decomposition

The Jacobian J in the problem (3.21)-(3.23) has a very specific structure induced by the multiple shooting, which allows very effective recursive block decompositions. We describe here LQ-decomposition which is preferable for computing trust region step because it allows to compute the trust-region step Δs^k exactly. Here, L is a lower triangular and Q is an orthogonal matrix respectively. Not only for the sake of simplicity, but rather for improving stability properties of the decomposition we handle the parameters p as constant state variables (with derivative zero) and include them in the differential variables $s_j := (y_j, p)$. The Jacobian under consideration has the form (for the sake of simplicity we omit the point s^k and the index k):

$$J = \begin{pmatrix} D_1^0 & D_1^1 & & \dots & & D_1^m \\ G^0 & H^0 & & & & \\ & \ddots & \ddots & & 0 & \\ & & \ddots & \ddots & & \\ & 0 & & \ddots & \ddots & \\ & & & & G^{m-1} & H^{m-1} \end{pmatrix}, \quad r = \begin{pmatrix} r_1 \\ h_0 \\ \vdots \\ \vdots \\ \vdots \\ h_{m-1} \end{pmatrix}$$

where

$$D_1^j := \partial r_1/\partial(s_j),\ G^j := \begin{pmatrix} \partial y(\tau_{j+1})/\partial(s_j) \end{pmatrix},\ H^j := \begin{pmatrix} -\mathcal{I} & 0 \end{pmatrix}, \quad j = 0, ..., m.$$

In the first step, we compute LQ-decomposition of the block $[G^0,\ H^0]$:

$$[G^0,\ H^0] = [L^0,\ 0]Q^0,$$

with Q^0 orthogonal and L^0 lower triangular and compute necessary changes in the corresponding blocks:

$$[0,\ G^1] = [T^0,\ \tilde{G}^1]Q^0,\ \begin{bmatrix} D_1^0 & D_1^1 \end{bmatrix} = \begin{bmatrix} \tilde{D}_1^0 & \hat{D}_1^1 \end{bmatrix} Q^0.$$

The next step of decomposition matrix is now given by

$$J^1 = \begin{pmatrix} \tilde{D}_1^0 & \hat{D}_1^1 & & \dots & & D_1^m \\ L^0 & 0 & & \dots & & 0 \\ T^0 & \tilde{G}^1 & H^1 & 0 & \dots & 0 \\ 0 & 0 & G^2 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & & 0 & G^{m-1} & H^{m-1} \end{pmatrix}$$

Now, we compute LQ-decomposition of the block $[\tilde{G}^1,\ H^1]$:

$$[\tilde{G}^1,\ H^1] = [L^1, 0]Q^1$$

and the necessary changes in the corresponding blocks:

$$[0,\ G^2] = [T^1, \tilde{G}^2]Q^1,\ \begin{bmatrix} \hat{D}_1^1 & D_1^2 \end{bmatrix} = \begin{bmatrix} \tilde{D}_1^1 & \hat{D}_1^2 \end{bmatrix} Q^1.$$

We proceed with this procedure until the last multiple shooting block is processed.

$$[\tilde{G}^{m-1},\ H^{m-1}] = [L^{m-1},\ 0]Q^{m-1};\ \begin{bmatrix} \hat{D}_1^{m-1} & D_1^m \end{bmatrix} = \begin{bmatrix} \tilde{D}_1^{m-1} & \tilde{D}_1^m \end{bmatrix} Q^{m-1}.$$

As a result, we get the decomposition $J = J^m Q$ where

$$J^m = \begin{pmatrix} \tilde{D}_1^0 & \tilde{D}_1^1 & & \dots & \dots & \tilde{D}_1^m \\ L^0 & 0 & & \dots & \dots & 0 \\ T^0 & L^1 & 0 & \dots & \dots & 0 \\ 0 & T^1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & T^{m-2} & L^{m-1} & 0 \end{pmatrix}.$$

With $\Delta\tilde{s} = Q\Delta s$, the first m parts of the transformed increments (forming normal component of the trust-region step) can be computed recursively

$$\begin{aligned} \Delta\tilde{s}_0 &= -(L^0)^{-1}\tilde{h}^0; \\ \Delta\tilde{s}_j &= (L^j)^{-1}(-\tilde{h}^j - T^{j-1}\Delta\tilde{s}_{j-1}), \qquad j = 1, ..., m-1. \end{aligned}$$

In order to find the last (tangential) part $\tilde{s}_m$ we solve the so-called condensed problem

$$\min_{\tilde{s}_m} \qquad ||r_1 + \alpha \sum_{i=0}^{m-1} \tilde{D}_1^i \Delta\tilde{s}_i + \tilde{D}_1^m \tilde{s}_m||_2^2, \tag{3.28}$$

$$\text{s.t.} \qquad ||\tilde{s}_m||_2^2 \leq \bar{\Delta}^2 := \Delta^2 - \alpha^2 \sum_{i=0}^{m-1} ||\Delta\tilde{s}_i||_2^2. \tag{3.29}$$

This problem is solved by a classical trust-region algorithm. To recover the original increment, a recursive orthogonal transformation is performed $\Delta s = \Delta s = Q^T \Delta\tilde{s}$.

Reduced Approach

In order to reduce the number of evaluations of derivatives to minimum, we may exploit point conditions, e.g. known initial and multipoint conditions, see Schlöder (1988). This approach is especially preferable for parameter estimation in large-scale ODE, resulting from a semidiscretization of PDEs, with only few degrees of freedom in the initial values, like in case of the problem under investigation in this paper.

Assume for simplicity that part of the equality constraints only depend on variables at one multiple shooting point. This results in entries in the linear system of the form:

$$A^i \Delta s^i = a^i, i = 0, ..., m.$$

In the first step of the reduced approach we evaluate the block A^0 and compute an LQ-decomposition

$$A^0 = [L_A^0,\ 0] Q_A^0,$$

with Q_A^0 orthogonal and L_A^0 lower triangular. Then the solution manifold can be represented as $\Delta s^0 = \Delta s_N^0 + \Delta s_T^0$,
where $\Delta s_N^0 = (Q_A^0)^T \begin{pmatrix} (L_A^0)^{-1} a^0 \\ 0 \end{pmatrix}$ and $\Delta s_T^0 = (Q_A^0)^T \begin{pmatrix} 0 \\ s_T^0 \end{pmatrix} =: \mathcal{N} s_T^0$ with s_T^0 free.

Now, we insert this solution into the first matching condition

$$G^0 \Delta s^0 + H^0 \Delta s^1 = h^0 \tag{3.30}$$

which then can be rewritten as

$$\mathcal{G}^0 s_T^0 - H^0 \Delta s^1 = h^0 - G^0 \Delta s_N^0, \quad \mathcal{G}^0 := G^0 \mathcal{N}. \tag{3.31}$$

We may apply to the matrix $[\mathcal{G}^0, H^0]$ the decomposition procedure described in the previous section, determine the solution manifold and proceed to the next multiple shooting interval.

The advantages of the reduced approach are obvious. To generate the linearized matching conditions in the form (3.31), only the matrix $\mathcal{G}^0$ of the directional derivatives of the initial value problem (IVP) with respect to the columns of $\mathcal{N}$ and one directional derivative $G^0 \Delta s_N^0$ have to be computed. The matrix G^0 itself is not needed. Thus, the effort for the (costly) computation of derivatives of the solution of the ODE is reduced considerably.

Computing the Relaxation Parameter α

The definition of $\bar{\Delta}$ (3.29) motivates the choice of α. If we choose

$$\alpha = \min \left(1, \frac{\sqrt{2}}{2} \frac{\Delta}{\sum_{i=0}^{m-1} ||\Delta \tilde{s}_i||_2^2} \right)$$

then $\bar{\Delta}^2 \geq \frac{1}{2}\Delta^2$, that gives us enough freedom to work on reducing the objective function.

Control of Trust Region Radii

The number Δ is the so-called trust region radius that characterizes the region in which the linearized model (3.21)-(3.23) is considered to be a good approximation to the nonlinear problem. In general, the step Δs is accepted, if it produces sufficient improvement in an appropriate merit function $T(s)$. In trust-region methods, the improvement is evaluated through the ratio of the actual reduction in a merit function to the predicted reduction, that is a prediction of what the reduction in the merit function will be according to the approximation of the original problem. A traditional choice of the merit function is the so-called exact l_1-penalty function

$$T_1(s) \quad = \quad \frac{1}{2}||r_1(s)||_2^2 + \sum_{i=1}^{m_2} \beta_i |r_{2i}(s)| \tag{3.32}$$

Here, $\beta_i > 0$, $i = 1, ..., m_2$, are the penalty parameters that have to be determined in the algorithm to guarantee the global convergence of the method. Different strategies used for updating the penalty parameters and the trust-region radius and corresponding convergence theory based on classical choice of the merit function can be found e.g. in Conn, Gould, and Toint (2000).

However, it is well known that already in mildly ill-conditioned problems such a trust region control strategy may be very inefficient since it may produce very small radia. Therefore we use the trust region generalization of the "restrictive monotonicity test" (RMT), see Bock, Kostina, and Schlöder (2000), that has proved to be very effective in practical applications. The idea of the RMT for control of trust region is that at s^k we consider a modified nonlinear problem:

$$\min_s \; ||r_1(s)||_2^2 + \lambda_{LM}^k ||s - s^k||_2^2, \;\; r_2(s) = (1 - \alpha^k) r_2(s^k), \tag{3.33}$$

for some values of λ_{LM}^k and α^k, and choose the maximal trust-region radius Δ^k for which the iterates of the simplified Gauss-Newton method, i.e. Gauss-Newton method with keeping Jacobian $J(s^k)$ fixed at all iterations, applied to (3.33) are contracting. This leads to the following *restrictive monotonicity test:*

Compute Δs^k as a solution of (3.21)-(3.23) with given Δ^k

$$\Delta s^k = \mathcal{L}(s^k, \alpha^k, \lambda_{LM}^k) F(s^k).$$

This corresponds to the first iteration of Gauss-Newton method applied to solve (3.33). The second iteration, $\tilde{\Delta} s^k$, of the simplified Gauss Newton applied to (3.33) solves the linearized problem

$$\begin{aligned} \min_{\Delta s \in R^n} \quad & ||r_1(s^k + \Delta s^k) + J_1(s^k)\Delta s||_2^2 + \lambda_{LM}^k ||\Delta s^k + \tilde{\Delta} s^k||_2^2, \qquad (3.34) \\ \text{s.t.} \quad & r_2(s^k + \Delta s^k) + J_2(s^k)\Delta s = (1 - \alpha^k) r_2(s^k), \end{aligned}$$

and can be written as

$$\tilde{\Delta} s^k = \mathcal{L}(s^k, \alpha^k, \lambda_{LM}^k) F(s^k + \Delta s^k). \tag{3.35}$$

We accept the step Δs^k if

$$||\tilde{\Delta s}^k|| \leq \frac{\eta}{2} ||\Delta s^k|| \text{ for some } 0 < \eta < 2.$$

The restrictive monotonicity test has shown very good performance in practice, for the theoretical justification of the test we refer the reader to Bock, Kostina, and Schlöder (2000).

Statistical Sensitivity Analysis for the Estimates

The first results for statistical sensitivity analysis were obtained by Gauss (1805, 1995). The discussion of the statistical sensitivity analysis for the unconstrained case can be found e.g. in Bard (1974). Here we give the results for the *constrained* least squares problems which are presented in Bock (1987) and Bock, Kostina, and Kostyukova (2004). If the experimental data is normally distributed then the estimated solution s^* of the parameter estimation problem is also a random variable which is normally distributed in the first order

$$s^* \sim \mathcal{N}\left(s^{\text{true}}, C\right)$$

with the (unknown) true value s^{true} as expected value and the variance-covariance matrix C given by

$$C = C(s^*) = J^+ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} J^{+T}. \tag{3.36}$$

Here, $J^+ := \mathcal{L}(s^*, 0, 1)$. The variance-covariance matrix describes the confidence ellipsoid which is an approximation of the nonlinear confidence region of the estimated variables. The matrix C can be cheaply computed using the decompositions of the Jacobians that are computed anyway in the Gauss-Newton method.

The $100\beta\%$ confidence ellipsoid ($0 \leq \beta \leq 1$) can be described by

$$G_L(\beta; s^*) = \{s^* + \Delta s \mid \Delta s = -J^+(s^*) \begin{pmatrix} \eta \\ 0 \end{pmatrix}, \ \|\eta\|_2^2 \leq \gamma^2(\beta)\}.$$

Here, the probability factor $\gamma(\beta)$ is given by

$$\gamma^2(\beta) = \chi^2_{n-m_1}(1-\beta)$$

where n is the dimension of s, m_1 is the dimension of the constraints of the parameter estimation problem (3.19), and $\chi^2_{n-m_2}(1-\beta)$ is the quantile of the χ^2 distribution.

The diagonal elements of the covariance matrix play an important role in the statistical assessment of the estimates as well, namely they are used to compute confidence intervals $\theta_i = \sqrt{C_{ii}}\gamma(\beta)$ for each variable s_i, $i = 1, ..., n$, since

$$G_L(\beta, s^*) \subset \bigtimes_{i=1}^{n} [s_i^* - \theta_i, s_i^* + \theta_i].$$

At the solution the statistical average of the residuals, the so called common factor, can be computed by

$$\zeta = \sqrt{||r_1(s^*)||_2^2/(m_1 + m_2 - n_p)}$$

where m_1 is the number of measurements, m_2 is the number of constraints, n_p is the number of parameters. It can be used to check whether the model reproduces the measurements within the expected statistical variation.

Let us note that in multiple shooting statistical information can be computed for all variables including the values at multiple shooting nodes.

Overall Algorithm

Let $\varepsilon > 0$, $\delta > 0$, $0 < \eta_1 < \eta_2 < 2$ and $0 < \gamma_1 < 1 < \gamma_2$ be specified constants. Let s^0 and Δ^0 be given.

For $k = 0, 1, 2, ...$ do until convergence (that is until $||\Delta s^k|| > \varepsilon$)

1. Compute Δs^k, λ^k and λ^k_{LM} as the solution of problem (3.21)-(3.23).
2. Compute $\tilde{\Delta} s^k$ as the solution of problem (3.34).
3. If $||\tilde{\Delta} s^k|| > \eta_2/2||\Delta s^k||$ then do not accept the step, decrease the trust region radius $\Delta^k := \gamma_1 \Delta^k$ and go to 1.
4. Otherwise accept the new point $s^{k+1} = s^k + \Delta s^k$.
5. If $||\tilde{\Delta} s^k|| > \eta_1/2||\Delta s^k||$ then increase the trust region radius $\Delta^{k+1} = \gamma_2 \Delta^k$.

3.5 Application to Foreign Exchange Rates

Mathematical modeling and simulation have become important tools for the analysis of data and the prediction of economic and financial processes. For a long time, mainly stationary systems and stationary fixpoints of model equations were considered, not taking properly into account the importance of dynamical effects. Therefore, after thirty years of flexible price movements, modeling the development of foreign exchange rates remains a challenge. Beginning with the seminal empirical work of Meese and Rogoff (1983) it still seems questionable whether any structural exchange rate model would be of systematic value. Usually, a pure random walk process outperforms all classical exchange rate approaches based on monetary fundamentals. Hence one is inclined to conclude that flexible exchange rates are ordinary stochastic processes in which the different states of the economic environment are of secondary importance.

Standard approaches in modeling price dynamics e.g. foreign exchange rates and commodity prices are working with the hypothesis of a single long run price equilibrium (see e.g. Creedy, Lye, and Martin (1996) and Geman (2005)). Deviations from this reversion level are expected to be temporary, thus the dynamics is mainly driven by one attracting equilibrium. However, the interplay of nonlinearities in the dynamics and the stochastic influences in the system are highly important, but not enough taken into account. These interactions may lead to effects which cannot be explained otherwise: e.g. multi-modal distributions can be traced back to multiple states in the dynamical system, observed jumps and strong oscillation in the historical data can be explained by stochastic changes of attractors. Small random perturbations may push a balanced market from one equilibrium into another, reflecting both regime switches and rare events.

Another defect is getting more and more obvious in the analysis of economic data sets: There is a lack of combining stochastics and nonlinear dynamical systems in methodology, pure statistics and modeling based on economic facts in theory. Model based statistics has to be developed in order to integrate and exploit economic knowledge better. Stochastic nonlinear dynamical systems, describing the arising processes more adequate, have to be investigated with the aim to get better qualitative and more precise quantitative answers.

Models of Exchange Rate Determinants

In order to determine the dynamics of our steady-states, we refer to the structural monetary exchange rate theory which goes back to Frenkel (1976), Mussa (1976) and Bilson (1978a and 1978b). Starting point of the monetary exchange rate model is the Purchasing Power Parity (PPP). With X denoting the home currency price of foreign exchange, P and $\bar{P}$ denoting the prices home and abroad, the purchasing power parity implies that the exchange rate between domestic and foreign countries equals to the ratio between domestic and foreign prices

$$X = P/\bar{P}. \tag{3.37}$$

Here, and in the following, a bar indicates the foreign country. To get a general insight into alternative interpretations and a doctrinal perspective of the Purchasing Power Parity the interested reader is referred to the work of Frenkel (1978).

Monetary theory suggest that the evolution of the exchange rate is determined by the relative price of two moneys (see Dornbusch (1980)). As a result of this, the exchange rate behavior reflects the dynamics of the relative demands for two moneys. A common specification of the money demand function M^D is

$$M^D = PY^\eta e^{-\lambda i}. \tag{3.38}$$

Here, P denotes the prices and Y the real income. The parameters η and λ represent the income elasticity and the interest rate semi-elasticity of demand for money respectively. For simplicity, it is a rather common practise to expect identical coefficients for all countries home and abroad. That's why the foreign money demand function is given by

$$\bar{M}^D = \bar{P}\bar{Y}^\eta e^{-\lambda \bar{i}}. \tag{3.39}$$

Assuming market-clearing (money demand equals money supply $M^D = M^S = M$) in both countries and rearranging these term in order to isolate the price level, we obtain

$$\begin{aligned} P &= MY^{-\eta} e^{\lambda i}, \\ \bar{P} &= \bar{M}\bar{Y}^{-\eta} e^{\lambda \bar{i}}. \end{aligned} \tag{3.40}$$

Putting (3.40) into the Purchasing Power Parity (3.37) yields immediate to the monetary exchange rate equation

$$X = \frac{P}{\bar{P}} = \frac{M}{\bar{M}} \frac{Y^{-\eta}}{\bar{Y}^{-\eta}} \frac{e^{\lambda i}}{e^{\lambda \bar{i}}}. \tag{3.41}$$

Accordingly, the relative change in money supply, interest rates and real income affect the dynamics of the exchange rate. A rise in domestic relative income induces appreciation whereas an increase in domestic interest rates induces depreciation.

The log-linear representation of the fundamental monetary price equation is

$$x = \alpha_1 m + \alpha_2 \bar{m} + \alpha_3 y + \alpha_4 \bar{y} + \alpha_5 i + \alpha_6 \bar{i}, \tag{3.42}$$

where m is the logarithm of the money supply, y is the logarithm of real income, i is the nominal interest rate, and x is the logarithm of the exchange rate.

The main characteristics of these initial exchange rate models is the idea that exchange rates are determined by the relative behavior of a set of underlying economic variables (home versus foreign variables). It is widely accepted, that these initial exchange rate models have some validity when considered as a long-run equilibrium (see e.g. MacDonald and Taylor (1993)). In the literature there is a controversial discussion about which economic variables should be included and even the direction of influence is ambiguous. In consequence, the traditional flexible exchange rate approach only serve as a common reference point. A selective literature survey on the economics of exchange rates over the last decades is offered e.g. by Taylor (1995).

Modeling the Dynamics of Attractors

By modeling the dynamics of the quasi-steady states as a product of N economic key variables (e.g. money supply M, real output Y and the exponential of nominal interest rates i)

$$X_j(Z) = C_j Z_1^{\alpha_{j1}} \cdots Z_N^{\alpha_{jN}}, \quad j = 1, 2, 3, \tag{3.43}$$

the standard linear monetary exchange rate model is embedded into the nonlinear model (compare equation (3.42)). Hence, the dynamics of the attractors and with that of the exchange rates are driven by well known economic relationships. As first step, we choose a constant volatility. Further investigation could be done by taking into account the influence of economic fundamentals and lagged effects on the volatility. However, the estimation results seem to justify the concentration at the time being on modeling the drift term.

In this section we apply the methods described earlier to analyze the behavior of the dollar/pound exchange rate during the post-Bretton Woods period. In order to illustrate the qualitative improvements of the nonlinear model, we take the standard mean reversion model as a benchmark

$$\begin{array}{rrcl} \textit{Linear model} & \mu(t, X, Z) & = & C_0(X_1 - X), \\ \textit{Nonlinear model} & \mu(t, X, Z) & = & C_0(X_1 - X)(X_2 - X)(X_3 - X). \end{array}$$

The data for our investigation is taken from the International Monetary Fund's *International Financial Statistics* database, and run from March 1973 through July 2005. In particular, the dollar/pound exchange rate X is given in line "ag" (expressed as home currency per unit of foreign currency). The dynamics of the exchange rate attractors are determined by the relative behavior of the interest rates i^{US} and i^{UK}

$$X_j(i^{US}, i^{UK}) = C_j \left(\exp\left\{i^{US}\right\}\right)^{\alpha_{j1}} \left(\exp\left\{i^{UK}\right\}\right)^{\alpha_{j2}}.$$

Here, we restrict ourselves to short term interest rates. We use the 90 days treasury bill rates for Britain and the United States. The data is taken from the IFS CD 2005 and is given in line "$60c$". Adding further exchange rate fundamentals such as monetary aggregates or income measures does not improve substantially the explanatory power of the model.

In a first step, the volatility is chosen as it very common in modeling financial processes $\sigma(X) = \sigma_0 \sqrt{X}$. Further investigation could be done taking into account the influence of economic fundamentals and lagged effects on the volatility. However, our results seem to justify the concentration at the time being on modeling the drift term.

Applying the presented numerical methods to the dynamics of dollar/pound exchange rates and comparing both linear and nonlinear approaches, we achieve the following results:

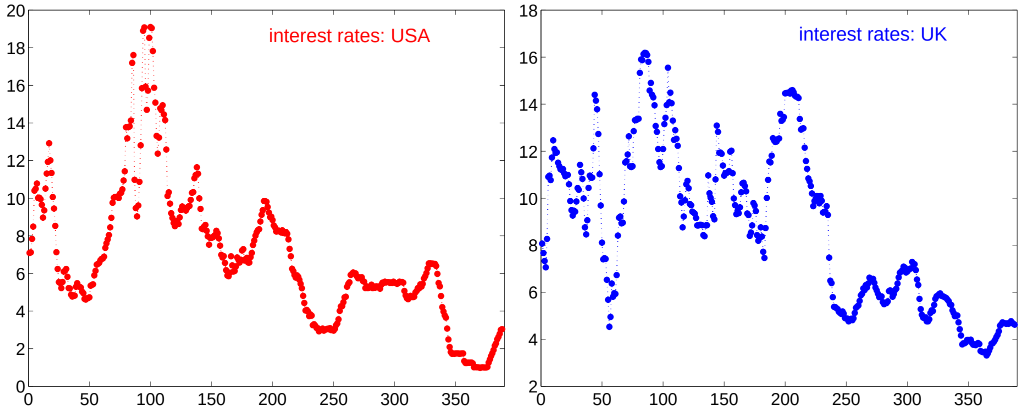
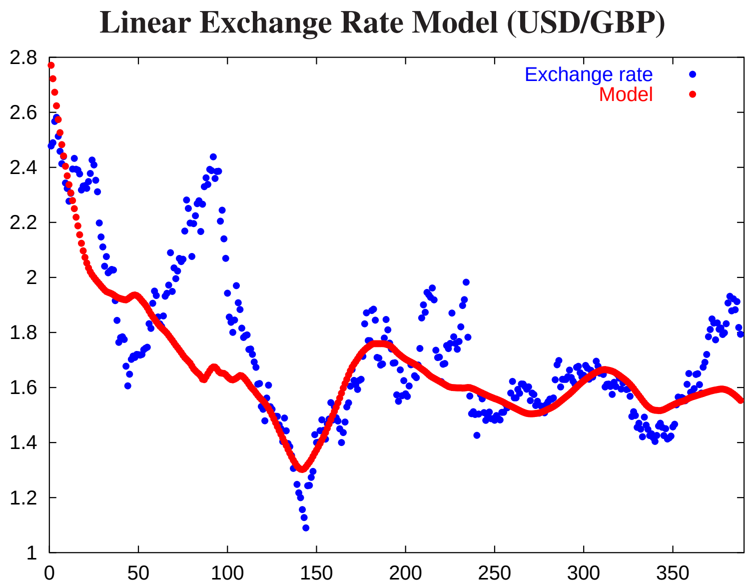


Figure 3.3: Economic Data: Interest Rates

Table 3.2: Estimates of the dollar/pound exchange rate

Linear: $\mu(t, X, Z) = C_0(X_1 - X)$
Nonlinear: $\mu(t, X, Z) = C_0(X_1 - X)(X_2 - X)(X_3 - X)$
Attractors: $X_j(Z) = C_j \exp(i^{US})^{\alpha_{j1}} \exp(i^{UK})^{\alpha_{j2}}$

Linear model			**Nonlinear model**		
parameter	estimated value	± standard deviations	parameter	estimated value	± standard deviations
x_0	2.829	± 0.082	x_0	2.623	± 0.021
C_0	0.047	± 0.001	C_0	0.102	± 0.001
C_1	1.824	± 0.013	C_1	1.052	± 0.006
α_{11}	-0.034	± 0.001	α_{11}	0.025	± 0.002
α_{12}	-0.020	± 0.001	α_{12}	0.016	± 0.001
σ	0.020	± 0.001	C_2	2.180	± 0.030
			α_{21}	- 0.002	± 0.002
			α_{22}	-0.013	± 0.002
			C_3	1.722	± 0.060
			α_{31}	-0.008	± 0.004
			α_{32}	-0.027	± 0.005
			σ	0.017	± 0.001



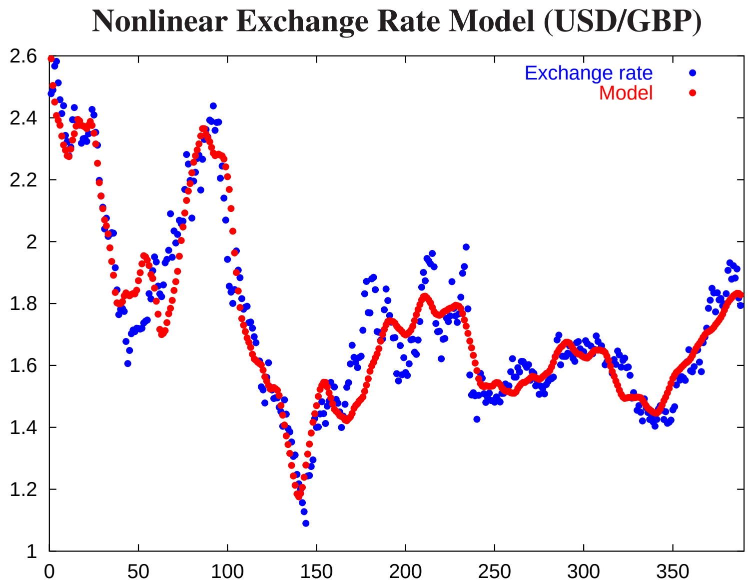


Figure 3.4: Simulation Results. *The figure shows the simulation results of the dollar/pound exchange rate from 1973 to 2005 (dotted in red) and the real data (dotted in blue). In both linear (above) and nonlinear (below) approaches, the quasi-steady states depend substantially on the relative change of nominal interest rates. It can be observed that by taking into account the interplay of nonlinearity and stochastic perturbations improves the quality of pricing substantially.*

- By considering multiple steady states we are able to capture the historic price dynamics and distribution characteristics for the dollar/pound exchange rate. Contrary to the linear model, the generalized mean reversion process detects main turning points over a period of thirty years. Timing and direction of changes are caught surprisingly well.

- To compare the quality of the different models, we compute the root-mean-squared-error (RMSE) and the mean-average-percentage-error (MAPE) over different time periods.

Table 3.3: Diagnostics of the dollar/pound exchange rate

	Linear model	**Nonlinear model**
RMSE[1]	0.101	0.056
MAPE[2]	7.85	4.49

[1]*Root Mean Squared Error,* [2]*Mean Average Percentage Error*

These figures demonstrate a substantial improvement of quality of pricing.

- Beside the crucial role of nonlinearities interacting with stochastic disturbances, the results highlight the importance which a stronger involvement of economic key variables has for the development of the foreign exchange rates. The set of economic data is given and not modeled. The latter is important for predictions. However, here we are mainly interested in investigating the influence of the economic variables and the effect of the nonlinearities. Therefore, the reduction to this simpler case is justified at this state of research.

- We estimate the parameters generalizing results obtained for the quasi-steady state situation. Least squares problems constrained by partial and ordinary differential equations are already solved for real life problems such as chemical reaction systems (see e.g. Bock et al (2000)). The applied numerical method is characterized by both high accuracy and efficiency.

- In order to make forecasts, a modeling of the underlying explanatory variables is in demand. As a consequence, we get a higher dimensional system of nonlinear differential equation. In this case, the application of the presented forward Kolmogorov method depends on numerical algorithms for high dimensional problems. Recently, several different numerical methods have been developed for direct simulation for the differential equation in high dimensions, e.g. using thin grid techniques. Parameter identification in high dimension is still one of the challenges not yet overcome.

Conclusion

This thesis treats the challenging inverse problem of the identification of the dollar/pound exchange rate mechanism; numerical results are discussed. Large price movements and multi-modal distributions can be explained by the transition between different quasi-steady states which generalizes the linear mean reversion process. The attractors depend on functions of highly oscillating market fundamentals, e.g. nominal interest rates. Modeling these determinants would lead to a system of stochastic differential equations and therefore to high dimensional Kolmogorov equations. This extension will be considered in future work.

Chapter 4

Modeling Oil Price Dynamics

Contents

The dynamics of supply, demand and prices for energy, oil and gas in particular, is crucial not only for the economic development, but also for the development of human activities in general. A rational analysis for their interactions and their dependence on various factors, based on mathematical modeling and simulation, is a key tool for better understanding, predicting and controlling the energy market. This thesis analyzes such dynamical systems, with the aim to provide an oil market model with a rather simple structure to be used e.g. for rational pricing of contracts on oil. Businesses operating in the petroleum, natural gas, or electricity industries are particularly vulnerable to market risk or more specifically, price risk as a consequence of the extreme volatility of energy commodity prices. Thus, commodity price risk plays a dominant role in the energy industries, and the use of derivatives has become a common means of helping energy firms, investors, and customers manage risks that arise from the high volatility. The starting point in financial mathematics is the Black-Scholes model, see Chapter 2. This standard model is being extended in many directions in order to deal with the specifics of individual markets and their very particular behavior. This is true e.g. for fixed income products or electricity markets. The following investigation is focussed on the dynamics of oil prices, an important territory but to our surprise rarely explored.

Basic ideas of this chapter are published in the *Proceedings of the 29th IAEE International Conference*, compare Jäger (2006).

In general, price dynamics of commodities are attracting more and more interest.

Strong growth in the demand for oil worldwide, particular in China and other developing countries, is generally accepted as a driving force behind the sharp price increases seen over the past three years. Other factors contributing to the upward trend include a tight supply situation, concerns about economic and political situations in e.g. the Middle East, supply disruptions caused by weather events (e.g. hurricane Katrina in August 2005). The future path of prices is key uncertainty facing the world oil market. At the beginning we will describe the crucial building blocks of the market, in order to be able to model and simulate them. Thus, we describe trade patterns and the major oil market drivers: crude prices (grade differentials), supply (distribution, production and global reserves), inventory movements, demand (global demand trends) and in particular the strategic behavior of OPEC. Main steps are the reduction of the underlying complex network to a model system as simple as possible and the combination of nonlinear and stochastic effects describing the complex dynamical behavior observed in real data.

4.1 World Oil Market: A Survey

Over the past thirty years crude oil has become the biggest commodity market in the world. Today, there is more trade internationally in oil than in anything else. This is true whether one measures trade by volume, by its value, or by the carrying capacity needed to move the goods. The volatility of the oil price (see e.g. Figure 2.4) and the resulting hedging needs triggered the development of a sophisticated financial market. Thus, for example the monthly volume of energy related futures contracts on the NYMEX has grown from approximately 1.700,000 contracts per month in January 1982 to 7 million contracts per month in January 2000. In addition many market participants entered into OTC contracts to manage price risks. Crude oil is a key primary energy source. At the price levels which have usually ruled in the past no other fuels can compete in terms of price and convenience in usage. Despite a possible future reduction of its share in the global energy market, crude oil remains an important commodity, and even if the western hemisphere would approach a saturation in near future, the oil market continues to play a significant role in shaping global economic and political developments.

Oil has distinct advantages as a carrier of energy. It has a high content of energy per weight unit which minimizes transportation costs. Compared to alternative sources of energy like natural gas or electricity, crude oil is rather easily handled being fluid and storable without much costs. The cost of gas transportation over large distances yields to limited trade between regional markets, whereas the transportation costs of oil across the world is comparatively small leading to a global market while gas is not. Thus transportation and storage play a critical role. They are not just the physical link between producers, refiners and consumers; their associated costs are a primary factor in determining the pattern of world trade. Altogether, it is a common hypothesis to consider oil as a commodity, which is traded on a single global market at major exchanges like New York Mercantile Exchange and International Petroleum Exchange.

4.1.1 Economic Network

The factors determining the oil markets are nodes in a large and complex network, which cannot be fully integrated and visualized in a concise flow diagram. However, despite its shortcomings we present a network illustrating the dependence of the oil price on the different factors. Here, we emphasized the importance of the factors "supply" and "demand" which are quite often treated to superficially in modeling.

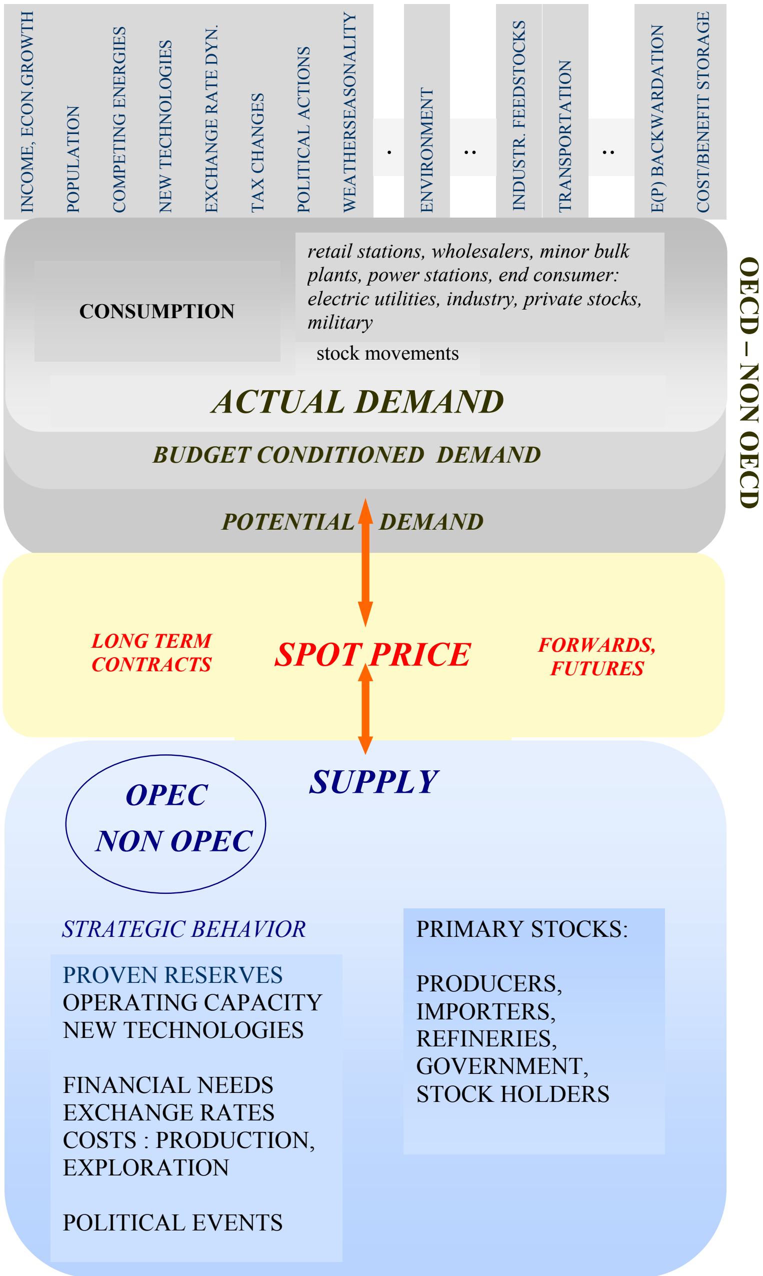


Figure 4.1: Flow Diagram

Also this investigation cannot tread these factors to an extend as they should be covered. Several academic and commercial research teams are very active in modeling and simulating the global energy market in general, and the oil-gas-market especially, including detailed information also on production respectively on exploration and development of new resources and technologies (compare e.g. "Petroleum Market Model" and "Oil and Gas Supply Module" of the National Energy Modeling System, U.S. Department of Energy/Energy Information Administration (2005)) However, approaches similar to the concepts pursued here in a very reduced system could not be found. Also we were seriously confronted with the problem getting real data basic for a more detailed modeling.

4.1.2 The Price and Nature of Oil

In general, crude oil is a naturally occurring substance which is found in widely differing amounts in various countries throughout the world. Oil is not perfectly homogenous but is a mixture of complex hydrocarbons together with certain trace elements. Oil is not used directly for any important purpose; rather it is refined in order to split it into different products (e.g. gasoline, diesel, heating oil) which are either used directly for final consumption or are in turn further processed.

The price and price differentials between crude oils reflect the relative ease of refining. Most simply, crude oils are classified by their density and sulfur content. Less dense (or "lighter") crudes generally have a higher share of light hydrocarbons – higher value products – that can be recovered with simple distillation. The denser ("heavier") crude oils produce a greater share of lower-valued products with simple distillation and require additional processing to produce the desired range of products. Some crude oils also have a higher sulfur content, an undesirable characteristic with respect to both processing and product quality.

For pricing purposes, crude oils of similar quality are often compared to a single representative crude oil, a "benchmark," of the quality class. A 'representative' crude is taken as the focal point for explanation. If necessary, the price of other crudes are derived by applying a series of adjustment factors to take into account the differences in the properties of the various crudes. Over the longer term the prices of all crudes tend to move sufficiently closely together to make the use of benchmark crude an attractive simplification for the attempts to understand the workings of the world oil market. Figure 4.1.2 demonstrates the fact that the price of different types of crude oil move strongly together. Thus, it is reasonable to concentrate on a single crude, where data exist over longer periods of time. In our analysis West Texas Intermediate (WTI) serves as the reference point.

The outstanding position of crude oil as the major commodity of international trade can be explained by the fact that there is a substantial discrepancy between production and consumption. Oil deposits tend to be in countries and areas that are not major consumers of oil. The Middle East is the most important producing area for oil, producing almost one-third of the world total in 2005, however, consumes less than 20 percent of its own

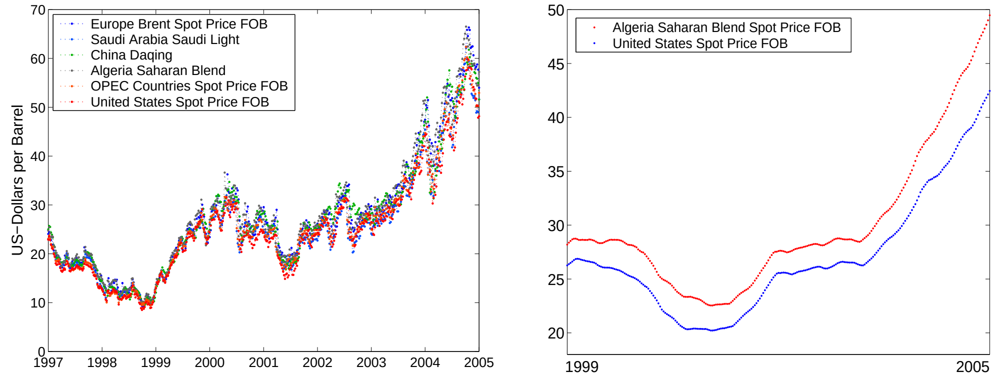


Figure 4.2: Crude Oil Price Differentials 1997-2005
This figure shows the historical price data of different crude oil brands (left side) and the moving average of two selected crude oil prices (on the right).

production. In Africa more than two-thirds of what is produced is exported out of the region. The deficit areas are North America, Europe, and Asia Pacific. Thus, e.g. Japan is the main consumer of oil in Asia, but produces no oil of its own. This mismatch between the distribution of oil deposits and the location of major demand areas is essentially to understand recent history of oil price movements. The establishment of OPEC and its role as a major player on the oil market goes essentially back to this world wide imbalance. In case of low inventory levels, mismatches of supply and demand, e.g. by cutbacks of OPEC production yields to strong price movements. Figure 4.3 illustrates some examples of historical events indicating the large influence of supply and demand changes over the last thirty years.

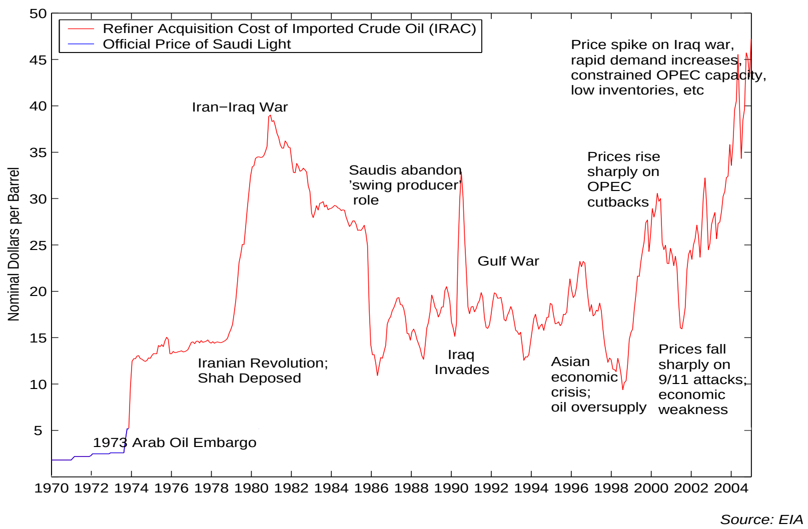


Figure 4.3: Major Events and World Oil Prices (1970-2005). *This figure shows the oil price data from 1970 to 2005. Since than the global oil price exhibits several jumps up and down linked to abnormal shocks either on supply or demand side.*

In the following we want to explain the main determinants of crude oil supply and demand. Studying the specific supply and demand conditions is an essential step to understand the world oil market in general and the vulnerability of oil prices in particular.

4.1.3 Crude Oil Supply and Proven Reserves

The supply of oil is provided by several groups of producers differing in their production policy. There is the politically best organized OPEC (Organization of Petroleum Exporting Countries) group established in Baghdad, Iraq, in September 1960. Original OPEC members include Iran, Iraq, Kuwait, Saudi-Arabia, and Venezuela. Between 1960 and 1975, the organization expanded to include Qatar (1961), Indonesia (1962), Libya (1962), the United Arab Emirates (1967), Algeria (1969), and Nigeria (1971).

OPEC own the largest part of "proven reserves" of low cost oil with a today's estimated quantity of 814 billion barrels in 2004 (79% of the global deposit). Whereas their share in oil supply is about 42% in 2004, it is estimated by the International Energy Agency (IEA) to rise over fifty percent in 2025[1]. This increase reflects the fact that the reserves elsewhere in the world will be depleted more and more, whereas the oil reserves in the Middle East will stay dominant. While global demand continues to rise, so will global oil dependence on oil supply from OPEC. Figures 4.4 and 4.5 show the distributions of global oil supply in 2005 and proven oil reservers. The member states of OPEC are marked blue, whereas the non-OPEC countries are depicted in green. The brightness indicates the influence of each country. The extraordinary position of OPEC can be explained by the size of proven reserves and the comparable simple access to oil. This leads to lower extraction costs and the possibility to react more quickly to changing market conditions, e.g. by switching production "immediately" on (off). This is in particular true for Saudi-Arabia which is often considered as a dominant player within the group of OPEC.

Due to this windfall OPEC is able to exhibit a 'proactive' supply policy. In 1982 OPEC started to assign explicit crude oil production quotas to each member country. Previously, the OPEC members had coordinated the offer prices they posted for their crude oil. The actual production policy is assigned to common objectives which are not always in line with pure profit maximization. Joint actions require of course national oil companies, which is the case for OPEC members. In general, the member states of OPEC are highly dependent on current oil revenues and their future perspectives. Accordingly, OPEC pursues a long-term strategy: stabilizing the oil market by monitoring recent crude oil price movements. To quote from 134th (Extraordinary) Meeting of the OPEC Conference (press release No 2 (2005)):

> *"In a tight environment, too high oil price levels may affect the prospects for economic growth, especially in developing countries, and therefore threaten future oil demand growth. On the other hand, low oil price levels would*

[1]IEA, World Energy Outlook 2005.

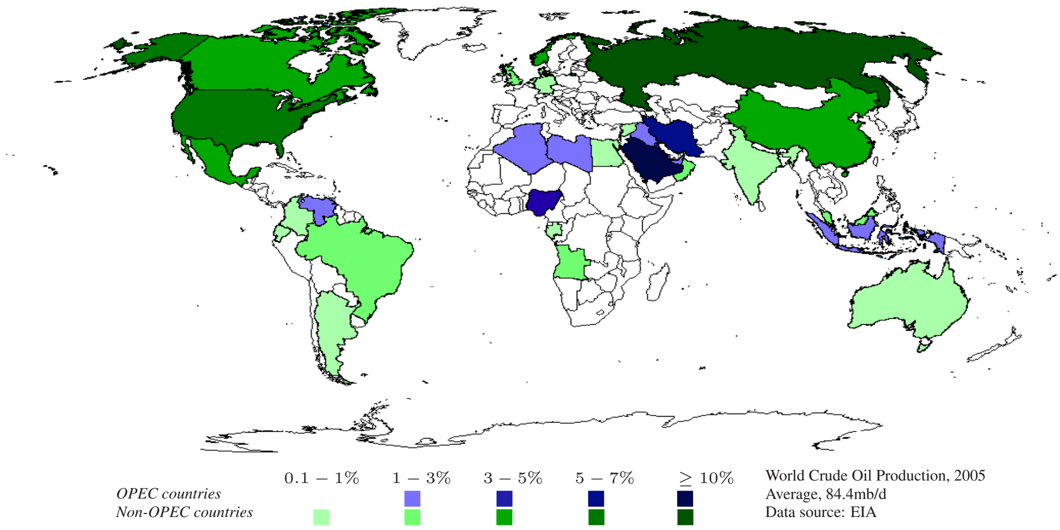


Figure 4.4: Global Crude Oil Supply

> *place strains upon the aspirations of OPEC Member Country populations for their economic development and social progress."*

In section 4.3.1 we offer a more detailed discussion about OPEC's objectives and formulate a switching rule between different supply policies as examined by OPEC.

Non-OPEC countries (countries that are not members of OPEC) produced 58% of the world's oil in 2004. This group includes large oil producers like Russia (8.9 mb/d), United States (5.4 mb/d), China (3.4 mb/d), Mexico (3.3 mb/d), Norway (2.9 mb/d), Canada (2.3 mb/d), and United Kingdom (1.8 mb/d). Their production as share of world total oil production varied from 48% in 1973 to 71% in 1985, with an average over the last thirty years of 60%. In contrast to OPEC they do not possess considerable spare capacities and the proven oil reserves are relative to OPEC limited. We refer the reader again to the figures 4.4 and 4.5 illustrating the worldwide supply and reserve distributions. It is generally assumed in oil industry that an expansion of non-OPEC production will be constrained by a declining potential of many traditional fields and relative to OPEC by limited and high cost-reserves, e.g. deepwater production.

The rising cost structure is due to increased geological maturity, which can be observed e.g. in the North Sea or the US onshore production. Even if an increase in production in the Gulf of Mexico and non-conventional output in Canada can offset the decline in onshore 'lower 48' US production and Canadian conventional oil output, Non-OPEC countries are faced with increasing extraction costs. Due to capacity additions in Africa, the Commonwealth of Independent States (CIS)[2] and Latin America there is still some room to grow in the mid term. However, in the long term a gradual decline in aggregate production level is to be expected.

[2]The Commonwealth of Independent States (CIS) is the international organization, or alliance, consisting of 11 former Soviet Republics: Armenia, Azerbaijan, Belarus, Georgia, Kazakhstan, Kyrgyzstan, Moldova, Russia, Tajikistan, Ukraine, and Uzbekistan.

Most major non-OPEC countries have private oil sectors. Usually, their governments have little control over production levels. So far there has been hardly any explicit cooperation with OPEC to restrain output in recent history. There is one exception worthwhile to mention: Mexico. Similar to OPEC member countries, Mexico's oil sector is in public hands, with PEMEX as the only oil company in Mexico. The existence of an independent private industry has direct consequences on the non-OPEC supply policy. Private companies do not hold back profitable production, and maintain little spare production capacity. Accordingly, in case of any disruption, OPEC would be the primary source of additional oil to displace the loss other than stocks.

Proven Oil Reserves

Oil is a non-renewable resource. Therefore, proven oil reserves are crucial for the future price dynamics. It is worthwhile to get an idea of the definition of the underlying concept. An excellent introduction and critical examination is given by Mitchell (2004). Oil is formed underground from remains of organic material such as debris from marine organisms and on-time forests. This is an ongoing process but the rate of formation of new oil is too slow; what we are using up now is immensely greater than whatever new oil is being formed. Over time we will run out of oil, however a proper rating strongly depends on the definition of "proven" reserves.

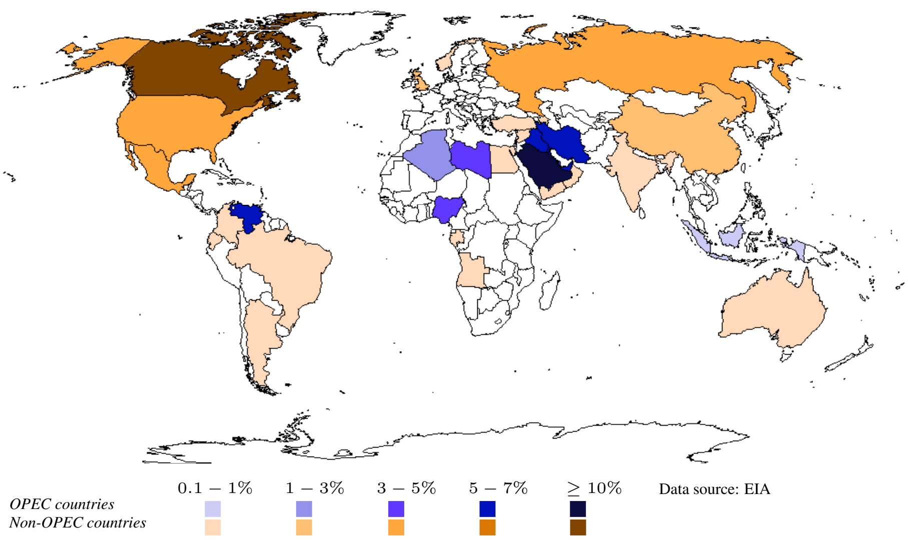


Figure 4.5: Proven Oil Reserves

Estimates of "proven" reserves assume current economics, feasible technology and geology. Thus, proven reserves can be augmented through exploration and development of new discoveries through technological improvements, as well as through the existence of more favorable economic conditions. Historically, estimates of world oil reserves have generally trended upward (see Figure 4.6). As at the end of, 2005, proven world oil reserves, as reported by BP's Statistical Review of World Energy (2006)[3] were estimated at 1200.7 thousand million barrels i.e. 430.3 thousand million barrels (about 56 percent) higher than the estimate for 1985. A very popular approach to measure whether the supply of oil is keeping up with demand is to track the size of world oil reserves and compare it to the rate of production. Figure 4.7 shows the reserve to production ratios (measured in years) for alternate years from 1981 to 2005. While reserve-to-production (R/P) ratio is lower in 2005 than in 2003 it is actually higher than in 1981. Although oil is a limited resource, at the rate of production in 2005, global oil reserves are sufficient to last more than 40 years.

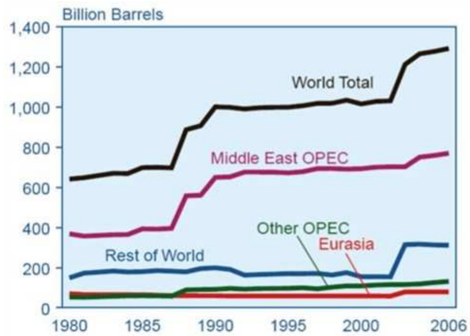


Figure 4.6: Crude Oil Reserves
Source: International Energy Outlook 2006, Energy Information Administration.

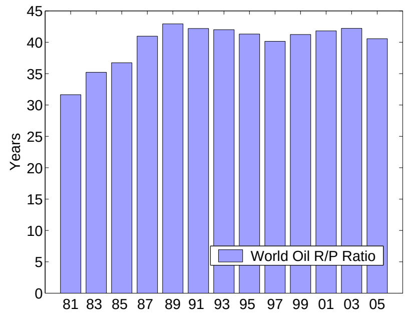


Figure 4.7: World Oil R/P Ratios
Source: The BP statistical Review of World Energy, June 2005.

As a result, there appears to be little cause for concern on the world level that the oil is physically running out in the near future. Even though we assume that global oil is exhaustible, we do not take the pessimistic view of Colin Campbell and others of his school (compare e.g. Campbell (1997)) which predict that a peak of crude oil production will soon cause a catastrophic world wide economic depression[4].

4.1.4 Demand

The demand for oil is the result of our demand for energy. Oil is needed e.g. for industrial production, electric power generation, and transportation. The main consumers of oil continue to be the advanced economies: the United States (24.64%), OECD-Europe (18.46%) and total OECD[5] (62%) in 2005. However, the economic development in Asia is a major new force in the world, and its oil consumption accounts for most of the increase during the last years. Of the 4.8 million barrel increased in daily world oil consumption from 2001 to 2004, 70% came from non-OECD countries and 50% came from non-OECD countries in Asia.

[3]http://www.bp.com/http://www.bp.com/productlanding.do?categoryId=6842&contentId=7021390

[4]see www.oilcrisis.com

[5]The Organization for Economic Cooperation and Development consists of "OECD Europe," Canada, Japan, South Korea, the United States, and "Other OECD."

The new demand has been coming primarily from China and India. China is emerging as a major energy consumer of oil. With 1.3 billion people and a rapidly growing economy, China is forgoing deals with the world major energy players and has now overtaken Japan as the second largest oil consumer in the world. It has been increasingly looking overseas for oil needed to maintain its booming economy. They even subsidize the use of oil domestically to mitigate the adverse impact of high oil prices on their economy.

The demand for oil is essentially determined by the general level of income or economic activity. The current rush for oil is mainly caused from developing countries where income per capita has reached a level at which vehicle ownership rises quickly. The demand for transport accounts for over 60% of the total oil consumption increase. Since Asia is expected to experience the highest growth rates in the world of 5.5% per year, this development is going to continue (compare e.g. International Energy Outlook 2005 from the Energy Information Administration). E.g. the vehicle ownership in China is predicted to jumped from 21 vehicles per 1000 people in 2003 to 387 vehicles in 2030.[6]

Furthermore the demand is affected by the price of oil and the relative price to competing forms of energy like coal or gas. As a consequence of the first oil shock in 1973 - 74, there has been a significant decline in non-transport oil demand, which has not been reversed by the oil price collapse of the mid 1980s. This imperfect price reversibility of oil demand is studied by Gately (1992) and Dargay and Gately (1995). This effect results from energy conservation and fuel switching to other energy sources such as natural gas and electricity. E.g. within electricity generation, the OECD has switched away from oil, back to coal or nuclear power like in France and Japan. There is a number of explanations for this irreversibility. One is the technological development in energy efficiency, e.g. in heating systems, or industrial processes. Building insulation will not be removed when prices fall. The asymmetric price response deliver an explanation why OPEC is concerned about market stability. However, not all products are easily replaced. Oil remains the only type of fuel used for aviation, and the alternatives for automobile are still not worked out to the last.

4.1.5 Inventory Movements

The movement in stocks of any commodity can play a major role in explaining short-run price variations. However, by their nature, substantial stock accumulations or draw-downs are unlikely to persist over longer periods of time. In models primarily intended for long-run forecasting, stocks have been ignored by imposing the assumption that there are no changes in stock levels. We do not come up with a submodel for changes in oil inventories. Submodules can be implemented in future work. We work with the hypothesis that changes in stocks are captured within the demand for crude oil.

[6]Source: International Energy Agency; United Nations Yearbook.

4.2 Modeling the Dynamics of the Attractors

Having analyzed some facts about the crude oil market, we are now going to formulate the complete oil market model. The price dynamics is assumed to follow the generalized mean reversion process. For convenience, we recall the stochastic differential equation:

$$dX \;=\; \kappa(X_1 - X)(X_2 - X)(X_3 - X)dt + \sigma\sqrt{X}dW.$$

The interaction of price X, supply S and actual demand D is taken care of by modeling the dynamics of the roots $X_i, i = 1, 2, 3$. Thus, the influence on the price is characterized via the quasi-states. Taking into account the qualitative behavior of the real price dynamics, one expects the time derivative of the price will satisfy the following qualitative properties

$$\begin{aligned} \frac{dX}{dt}(t) < 0 \quad &\text{if} \quad D - S < 0 \;\text{ and} \\ \frac{dX}{dt}(t) \geq 0 \quad &\text{if} \quad D - S \geq 0. \end{aligned}$$

In order to quantify these qualitative relations, we assume the following relation:

$$\begin{aligned} dX_i \;&=\; \eta(\delta)X_i dt\,, \qquad \text{where} \quad i = 1, 2, 3 \quad \text{and} \\ \delta_t \;&:=\; D(t) - S(t). \end{aligned} \tag{4.1}$$

Thus, the factor η depends on the difference δ between actual demand and supply. It is the same for all i. For negative δ the value of η is set strongly negative, assuming that the price is falling. However, if the supply is less than the actual demand, η is set smaller positive, since the price is expected to rise, however with a smaller speed, since the actual demand is not directly observable in real data. Only in special cases, it is easier to anticipate the demand (e.g. caused by seasonal effects like gasoline consumption in holiday traffic). Here prescribing their influence on the price, we choose as an example a piecewise linear function of the difference between supply and demand (see figure 4.8):

$$\eta(\delta) = -\varphi\left(-\delta/\delta_1\right)\eta_1 + \varphi\left(\delta/\delta_2\right)\eta_2 \tag{4.2}$$

where we define the auxiliary function

$$\varphi(t) = \min\left([t]^+, 1\right),$$

in order to smooth a transition between to states.

Thus, the quasi-steady states of the price process are pushed by the structural terms supply and demand. Price movements can be explained by jumps between the equilibria and changes in their own dynamics according to (4.1) which might evolve on a different time scale.

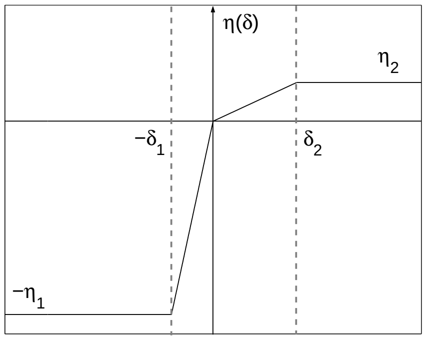


Figure 4.8: Price Impact of Supply and Demand. *The dynamics of the price attractors is substantially driven by the development of crude oil supply and demand. In the case of excess supply (demand) the price will fall (rise). To take into account that worldwide crude oil demand is far more uncertain, we allow for asymmetric reaction behavior.*

It can be shown that the evolution of the price distribution and the shape of stationary density function is entirely determined by the drift and volatility function of the price process. Including the role of economic key factors in the determination of the drift and volatility function and taking into account nonlinearity improves the quantitative and qualitative understanding of the shape and the dynamics of future price distributions.

4.3 Modeling Oil Supply

Modeling of Supply and Demand as Subsystems

Modeling price dynamics has to include models for supply and demand. Up to now, there exist approaches and applications to various commodities such as electricity, gas, gold, or even rice prices. Most mathematical models in financial applications are not enough taking into account knowledge about individual market characteristics. To our opinion it is very important for risk management as well as for pricing derivative contracts to include specific risks. Thus, the supply of oil is strongly affected by OPEC and Non-OPEC supply policies, the distribution of proven oil reserves and as well as by the different stages of development in e.g. technology and income. It is the aim of our investigation to include major determinants of the oil market. However, we also have to keep the system manageable, e.g. for the pricing contingent claims written on oil. In the subsequent, we describe the development of crude oil supply and demand and their interaction with the oil price.

Supply of Oil and Development of World Oil Production

The total supply S is decomposed into the sum of the supply by OPEC, denoted by S_o, and the supply S_n delivered by Non-OPEC increasingly by Russia and Frontier States:

$$S = S_o + S_n.$$

The supply by OPEC is modelled more in detail, since it is following a strategy different from "produce and sell" politics. OPEC member countries are intervening by direct action in the market, changing the supply in coordinated actions with the aim to keep the prices in a regime which seems to be most favorable. We are going to design a model equation for S_o in the following section 4.3.1. The Non-OPEC states are covered in section 4.3.2.

4.3.1 Modeling Oil Supply by the Member States of OPEC

In order to control the market, the oil producers can use two instrumental variables: price and supply. The control variable "supply" seems to be more efficient, since it is determined just by the producer. In fact, OPEC frequently used the supply, to control prices and the profit. It is a rational strategy to try to keep the oil price X or more general an objective functional $\phi(X)$ in an prescribed interval $[\phi^-(t), \phi^+(t)]$, which is optimal for its interest. We define the following switching rule, which leads to rather robust dynamics, despite rapid transitions: The supply is increased, if the value of the objective functional is moving beyond $\phi^+(t)$, it is decreased if it is moving below $\phi^-(t)$. This strategy is similar to the switches used to control heating systems. The burner is switched on if the temperature falls below a lower threshold temperature T^-, and it is switch off, if the temperature becomes larger than an upper threshold temperature T^+.

To illustrate the processes even better, we mention the analogy to the heating system. There the control is carried out by a thermostat, e.g. realized by a device containing a piece of bi-metal, bending due to temperature changes. On this metal is sitting a drop of mercury, which may role from one side to the other if the friction is overcome by gravity. Thus, electric contact is opened or closed. This is a nonlinear dynamical process which is not modelled in detail, but roughly by simpler switching rules. Also in the case of oil supply there are background processes which are very roughly modelled by switching rules. Again similar to the temperature thresholds, the thresholds for the price or more general for the value of a control functional of the price will depend on time and might be randomly perturbed.

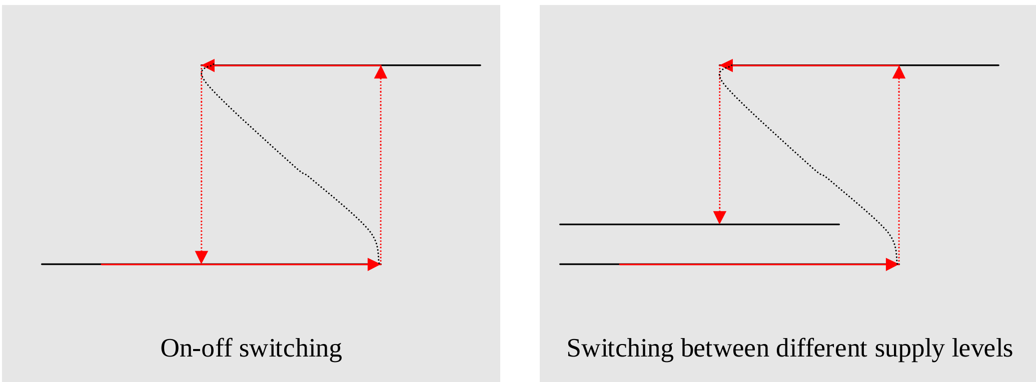


Figure 4.9: Switching Behavior. *Coming from below (above) OPEC keeps the prevailing policy. In case of overshooting the upper ϕ^+ (lower ϕ^-) threshold the member states are assumed to change their supply behavior. Within ϕ^- and ϕ^+, there is a zone of multiple states. If we want to know which policy OPEC is exercising one has to consider the history. The figure on the ride hand side illustrates the switching between different supply levels. In the following we assume that OPEC reacts according to prevailing oil market conditions. Hence supply changes by OPEC members are state dependent.*

There are not just jumps between zero and full heating in case of the burner, jumps between a smaller and higher oil supply in the case of oil, but transitions between the states. If we formulate this transition dynamics with help of differential equations, switching between two "policies", a reduction p_1 and an increase p_2, are introduced. These are functions of the supply, adapted to oil production and transportation facilities.

Supply Control and Switching Functions

Following this concept we define a function switching w with values between zero and one. It is used to control the supply in the following way

$$dS_o/dt = wp_1 + (1-w)p_2, \tag{4.3}$$

where p_1 and p_2 are functions of the supply. p_1 is negative and p_2 positive. The oil producers can choose between two different policies. These supply policies adapt to the economic environment, e.g. recent price fluctuations by changing from one state to another with asymmetric thresholds, $\phi^+(t)$ and $\phi^-(t)$. Here, the switching function w determines the weight of every single policy.

To derive an equation for w we interpret $w(t)$ as the fraction of the facilities with reduced oil delivery and $v(t)$ the fraction with full oil delivery. By definition we have: $w(t) + v(t) = 1$. The dynamics for the fractions is given by

$$\begin{aligned} \frac{dw}{dt} &= -\alpha(\phi)w + \beta(\phi)v \\ \frac{dv}{dt} &= +\alpha(\phi)w - \beta(\phi)v, \end{aligned}$$

where α is the specific rate of change from reduced to full delivery of oil, and β is the specific rate of change from full ones to reduced ones. Both coefficients are depending on a simple objective (decision) function, ϕ. Alternatively, we could reduce this system to a single equation

$$\frac{dw}{dt} = -\left(\alpha(\phi) + \beta(\phi)\right) w + \beta(\phi). \tag{4.4}$$

Each protagonist analyzes the oil market by comparing its decision function $\phi(X)$ with predefined upper and lower thresholds, $\phi^+(t)$ and $\phi^-(t)$. Accordingly, crossing these asymmetric thresholds triggers to change their supply policy. If the objective function $\phi(X)$ is smaller than $\phi^-(t)$ the OPEC countries are assumed to switch from full to the reduced state in order to stabilize the market.

The rate of change between these different states is modelled by

$$\alpha = \lambda\varphi\left(\frac{\phi - \phi^-}{\phi^+ - \phi^-}\right) \quad \text{and} \quad \beta = \lambda\varphi\left(\frac{\phi^+ - \phi}{\phi^+ - \phi^-}\right) \tag{4.5}$$

where λ determines the speed of transition between these two states. Correspondingly, a large λ would generate a fast switching. The dynamics of the coefficients $\alpha(\phi)$ and $\beta(\phi)$ is depicted below.

Policy Functions

We suppose that OPEC controls the economic 'temperature' of the oil market by switching between different supply policies p_1 (decreasing, 'off') and p_2 (increasing, 'on'),

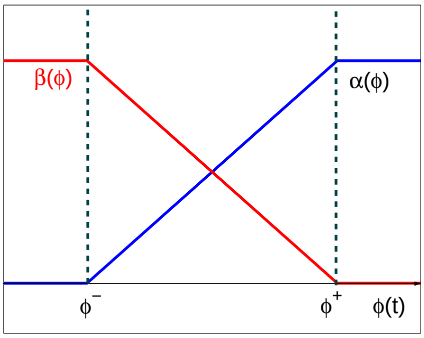


Figure 4.10: Rate of Changes: $\alpha(\phi)$ **and** $\beta(\phi)$
As long as the OPEC's objective $\phi(X)$ *is below* $\phi^-(t)$ *the rate of change* $\alpha(\phi) = 0$ *whereas* $\beta(\phi) = \lambda$. *As a result the weight* w *is increasing and* $v := (1 - w)$ *is decreasing by the same amount* ($\frac{dw}{dt} = -\frac{dv}{dt} = \lambda v$). *Thus to stabilize the market OPEC is going to enforce policy* p_1, *reducing supply. The speed of transition is primarily determined by the size of* λ.

using a strategy similar to the mechanism of a thermostat:

$$\begin{aligned} p_1 &= -\kappa_1(L_o^- - S_o)^2 S_o \quad \text{and} \\ p_2 &= \quad \kappa_2(L_o^+ - S_o)^2 S_o. \end{aligned} \tag{4.6}$$

Here κ_1 and κ_2 express the speed of adjustment towards the upper and lower supply limits L_o^- and L_o^+. In a first step, we assume that both are constant over the considered time horizon. However, the real data show that we are running simultaneously into a period of an increased demand for oil and maturing Non-OPEC oil fields. In order to maintain its ability to affect the oil market, OPEC needs to make a gradual adjustment of its upper supply limit. It is reasonable to assume that OPEC aligns these limits with the growing residual demand of oil, which is measured as the difference between expected global oil demand and potential Non-OPEC supply.

Objective Function and Thresholds

It remains to formulate a rule which defines when OPEC is going to switch between different supply strategies. Up to 2005, the OPEC group pursued an official price band mechanism which is considered to be in the interest of both, the profit of oil producers and the global economy. According to that mechanism OPEC committed itself to raise its crude oil production by 0.5 mb/d on a pro-rata basis if the reference basket price stays above the upper threshold $\phi^+ = \$28$ for twenty days in a row and to cut output by a similar volume if price falls below $\phi^- = \$22$. If this fails to move the price into the desired range, another 0.5mb/d production adjustment would be made 20 days later. Each such intervention takes time to be fully enforced. Therefore we may a continuous transition, instead of discrete jumps.

At the recent 134th (Extraordinary) Meeting of the Conference in 2005, OPEC noted that they failed to calm the market and that a *"new realism"* is required[7]. The reference price have remained outside the official price band for over a year. Furthermore, is contrary to common expectations, the world economy seems to be able to cope fairly well with the current price level. This is supported by the following statements of Alan Greenspan from end of 2004[8]:

> *"The impact of the current surge in oil prices, though noticeable, is likely to prove less consequential to economic growth and inflation than in the 1970s."*

As a consequence, OPEC took the decision to suspend its official price band temporarily[9]. However, since OPEC is a part of a global interacting economic system, the Conference stressed that

> *"the Organization remains firm in its commitment to maintaining a stable market with prices at reasonable levels conducive to expansion of production capacity and supply growth to meet rising demand, as well as to ensuring that there is enough oil to fuel global economic growth in the 21st Century".*

On its 45^{th} Anniversary OPEC adopted a long-term strategy, on the occasion of its Ministerial Conference Meeting in Vienna, 20 September 2005. The strategy re-emphasizes OPEC's commitment to support oil market stability.

> *"... extreme price levels, either too high or too low, are damaging for both producers and consumers, and points the necessity of being proactive under all market conditions.*

These statements shows that OPEC is decided to keep the dynamics of market as far as possible under control. It is obvious that a price band mechanism makes only sense in case market conditions change to a degree which urges to reassess the objectives and the switching rules e.g. upper and lower thresholds. In recent months the oil price seems to have leveled out at a new equilibrium around \$60 per barrel (see estimated upper price attractor X_3 in figure 2.4). To balance market stability and their short term profit, OPEC has to adjust and to reassess a new target range. As a consequence, the actions of OPEC are becoming more predictable and reliable, a fact that is very important for market.

[7] OPEC commentary, "The new realism", *http://www.opec.org/opecna/commentaries/2005/com020205.htm*

[8] The Federal Reserve Board: Speech, Greenspan: *http://www.federalreserve.gov/boarddocs/speeces2004/20041015*

[9] See 134th (Extraordinary) Meeting of the OPEC Conference, press release No 2 (2005)

Simulation of OPEC Supply

In the following we are testing the model for the OPEC strategy using historical price and supply data from 2000 to 2005. Figure 4.11 shows the results of a simulation using the following assumptions:

1. OPEC aligns its policy decision with the OPEC reference basket price, an average of various light sweet and heavier sour crude oils. Since daily data were not available from the Energy Information Administration[10], we used daily West Texas Intermediate (WTI) as a proxy.
2. We choose upper and lower switching thresholds $\phi^- = 25$ and $\phi^+ = 31$ dollars a barrel. In order to take into account the different qualities of the crude, the target thresholds are higher than the official price band. Since OPEC is a rather heterogenous group of countries with different interests, there has been a permanent discussion about reasonable thresholds. Accordingly, the target was always subject to small adjustments. In our simulation we allow for a slight increase of the thresholds by assuming a drift term $0.15 \cdot \phi$ and the volatility function $0.2\sqrt{\phi}$.
3. We set upper and lower supply limits $L_o^- = 27$ and $L_o^+ = 35$ mb/d, $\kappa_1 = \kappa_2 = 0.15$ and $\lambda = 1000$ leading to a fast transition to the new supply.

The following figures show

- the oil price data and the price threshold,
- the weights w respectively $(1 - w)$ for the policy 1 respectively policy 2,
- and the real data and the results of simulation of the supply (md/b) by OPEC.

[10]The data is taken from the Energy Information Administration: *http://tonto.eia.doe.gov/dnav/pet/pet_sum_top.asp*

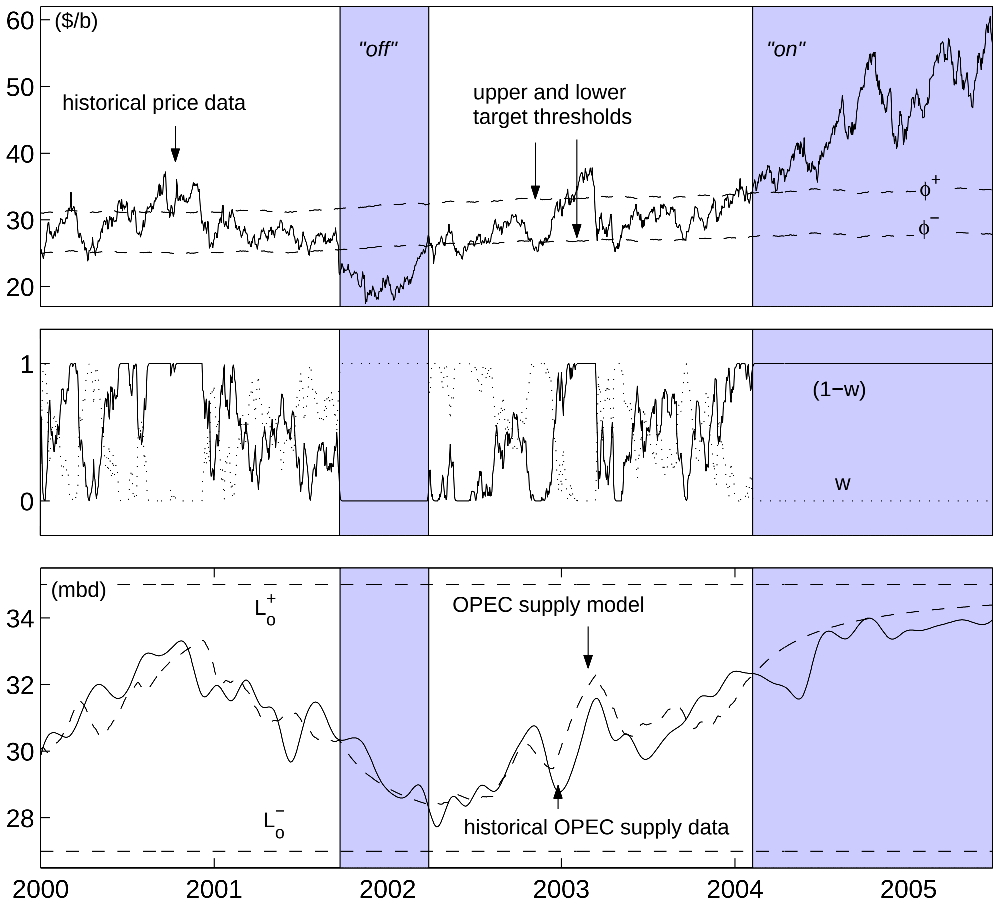


Figure 4.11: Simulation Results of OPEC Supply. *This figure illustrates the underlying idea of switching between different supply policies and compares real data with the OPEC supply obtained by simulating equations (4.3) – (4.6). The upper figure shows the historic price data from 2000 to 2005 as well as the simulated thresholds ϕ^+ and ϕ^-. As soon as the real price data crosses these thresholds OPEC adjusts it policy. The mid figure reveals the underlying idea of OPEC's switching behavior by presenting the corresponding switching functions w and $(1-w)$ which determine the weights for the policies p_1 and p_2. In order to illustrates two different extreme cases $w = 1$ and $w = 0$, we picked out two sample periods in which OPEC pursued either a policy of decreasing or policy of increasing supply of oil. The expressions 'off' and 'on' express the analogy to the control of a heating system. At the bottom we show the real and simulated data, where the dashed curve represent the simulated supply dynamics. Note that the OPEC supply model detects main turning points in this period. Timing and direction of changes are caught surprisingly well.*

Remark: The results of the simulation will surely improve, if system parameters are determined by more sophisticated methods of parameter identification. Since the results are already quite satisfactory we did not apply these techniques. As a result we summarize: The developed switching rules seem to describing the supply strategy of OPEC quite well.

4.3.2 Modeling Oil Supply by Non-OPEC States

Now we are concerned with countries outside the OPEC organization. They are responsible for a significant share of 58% world oil trade. Non-OPEC production is expected to rise in coming years, with the greatest increases in the former Soviet Union (FSU), including states bordering the Caspian Sea. However, mature provinces such as the North Sea (including first of all Norway and United Kingdom), are already suffering from higher than forecasted rates of depletion. Exploration and development of new reserves are not able to compensate for the the growing importance of depletion.

In modeling the supply by states outside of OPEC, we assume that their actions are following a simpler pattern:

Most of these states are not pursuing an actively controlled, coordinated policy. They are producing and selling oil to an extend which is determined mainly by their production limits. There is a critical price below which the costs of production or storage cannot be covered anymore. However, this price level seems to not have been touched during the recent history. Therefore it is irrelevant to discuss possible strategies in this case.

Global crude oil resources are assumed to be exhaustible. This is particularly true for Non-OPEC countries holding around 20% of the total proven oil reserves in 2004. As a consequence, there exists a naturally given upper supply limit R_n. In general, these accessible non-OPEC resources will not be fully supplied to the consumer. Quite often, political instabilities create harmful environments for investments. As a consequence, there is a multitude of factors, e.g. undeveloped export systems, preventing oil companies to make use of the total accessible resources $L_n \leq R_n$. In order to explain the difference between accessible resources R_n and potential supply L_n, a detailed analysis of each Non-OPEC country is essential.

Analyzing and predicting the 'potential' of oil production is a complex problem by itself, and in fact cannot be handled totally isolated, but involves a more detailed modeling of the oil market. Here, we choose an iterative approach used in modeling complex systems. We refer to an approximation to the future oil supply of Non-OPEC countries. In their study "The Oil Supply and Demand Context for Security of Oil Supply to the EU from the GCC Countries (2005)", R. Skinner and R. Arnott offer a detailed description of country and company policies. We are using such predictions as input into our model to determine a rough guideline for the Non-OPEC supply available for the consumers. They have investigated Non-OPEC supply prospects by breaking it down into regions and capacities of different categories.

Figure 4.12 illustrates the estimated potential future supply up to 2020 obtained by this analysis. Obviously, these figures are subjected to uncertainty, which is according to the authors less for the time period up to 2010. The projection from 2004-2009 is based on actual projects that are currently running and quite well observable. In the short term large deviations from the predictions are not to be expected.

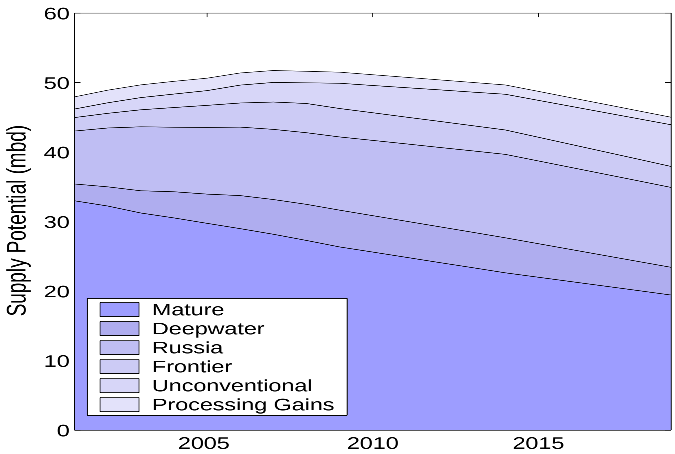


Figure 4.12: Non-OPEC Supply 'Potential' (2002-2020). *Based on the analysis of Skinner and Arnott (2005) the oil production of Non-OPEC countries is expected to peak within the next decade at around 53 mb/d, despite the expected successful new explorations. As a consequence of this, OPEC needs to make strong efforts in order to meet increasingly global demand for oil.*

The following stochastic differential equation for Non-OPEC supply reflects the described strategy:

$$dS_n \;=\; \mu_n(L_n - S_n)S_n\left(1 - \chi_{[\psi^-,\psi^+]}\right)dt + \sigma_n(S_n)dW_n, \qquad (4.7)$$

where χ is the characteristic function. We assume that Non-OPEC countries produce crude oil on average close to their given potentials. In the neighborhood of L_n, we allow for a simple strategic behavior. In order to assure a steady income stream, we assume that Non-OPEC countries are carefully increasing their supply, if the price is below ψ^-. A further acceleration can be observed if prices are higher than ψ^+.

Simulation of Non-OPEC Supply

Similar to the OPEC supply, we are testing the Non-OPEC approach using historical data from 2000 to 2005. We illustrate the simulated deterministic and stochastic model of Non-OPEC supply. Their dynamics is essentially determined by the potential Non-OPEC supply.

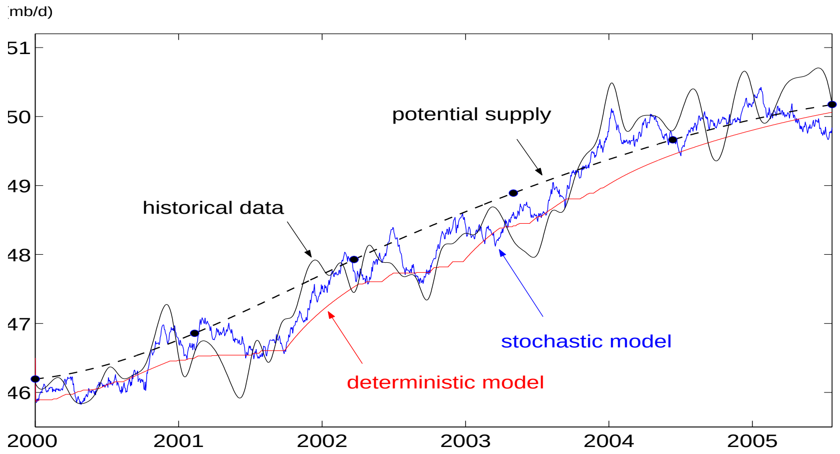


Figure 4.13: Non-OPEC Supply (2000-2005)

The suggested model is able to capture the essential part of the supply dynamics.

4.4 Modeling Oil Demand

It is generally accepted that one possible explanation why oil prices have moved higher is the rapidly increasing demand for oil over the past few years, e.g. due to higher personal wealth and booming car sales especially in emerging countries such as China and India. Surprisingly, the assumption made up to now that too high oil prices will cause demand to break down has turned out to be wrong. In fact, income effects seem to dominate price effects (see e.g. Paul Horsnell (2004)).

In modeling the dynamics of price, supply and demand, the term demand has to be analyzed more carefully[11]. It is not sufficient to consider the amount of oil bought on the market. This quantity at a given time t is less or equal to the supply. The oil price dynamics is influenced by the difference of this amount and the supply. If it is negative, the price will decrease due to this surplus of supply, on a fast time scale. In order to understand the factors of the price dynamics in the periods where there is no surplus on the market, we have to have a refined view of demand, which we are structuring in the following way:

- *Potential demand* (D^*)
  is the total amount of oil, which at a fixed time independently of the actual price could be used in all fields e.g. in industry, traffic, or heating. The potential demand is measured in quantities like barrel, but not in price. For the time evolution of D^* we set up a simple submodel including the dynamics e.g. of the economic and technological evolution.

[11] Observe: Here one should start from a microscopic modeling and by limiting pass to a macroscopic description. However, this is a task which has not been carried out up to now and which is a difficult problem by itself (compare e.g. Mc Cauley 2004).

- *Budget conditioned demand* (D^B)
  According to the budget situation and following a rating of expenses to be made, only a certain amount of financial resources will be available. The amount of oil which can be bought at time t and a price $X(t)$ is called budget conditioned demand $D^B(t)$.

- *Actual demand* (D)
  Since the costumers will follow a strategy of buying, the actual demand will depend on the price X. The purchase of oil will be reduced or partially postponed depending on assessment of the current price level. Determining the actual demand from D^B and X requires a more careful analysis of the strategies chosen. Here, we chose a rather crude modeling in order to reduce the complexity of the global model. In case, a more detailed model is available, the model equations could be just connected with the system as more detailed description.

4.4.1 Modeling of Actual Demand

We are suggesting here a simple model which reflects the strategic approach of mainly the costumers. We assume the following relation between the actual and the budget conditioned demand:

$$D(t) = D^0(t) + \gamma(X(t))[D^B(t) - D^0(t)]^+, \tag{4.8}$$

here $D^0(t) \leq D^B(t)$ is a minimal demand which is necessary to maintain e.g. military or health care. The second term models the response of the consumers to the price level. The factor γ can be chosen as a piecewise linear function. Thus, the fact that there will be an amount of oil absolutely necessary for the vital requirements is taken into account. On the other hand a strategic behavior is possible due to the second term in (4.8). It seems to be stringent to set value of γ equal to 1 if the price is not higher than a critical price X^*

$$\gamma(X) = \varphi\left(1 + \vartheta(X^* - X)\right).$$

This choice has the following effect: In case of prices below this threshold value X^*, price increases will be still accepted by the market, if the actual demand is higher than the supply.

4.4.2 Modeling of Budget Conditioned Demand

The budget conditioned demand $D^B(t)$ is determined by the gross domestic product $Z_1(t)$, the actual oil price $X(t)$ and the potential demand $D^*(t)$:

$$D^B(t) = \min\left\{\varepsilon(X(t))Z_1(t), D^*(t)\right\}. \tag{4.9}$$

The factor ε describes the effect of the potential demand on the demand conditioned by the budget. Its value is between zero and a maximal factor less than 1.

$$\varepsilon(X) = \xi_0 + \frac{\xi_1}{\xi_2 + \xi_3 X}.$$

4.4.3 Modeling of Potential Demand

The potential demand and its increase are depending strongly on the economic and technological level of the regions and countries. Whereas in developed areas the energy demand in general, the oil demand in particulary, are slowly changing and to a certain extend have almost reached saturation, an explosion in energy demand in regions with fast industrial and economic growth and huge still growing populations, e.g. in China and India, is setting the market under high pressure. The most important motors of the potential oil demand and thus of the oil prices are gross domestic product, population size, number of vehicles technological change, and seasonality in weather, climate.

Modeling of the potential oil demand, we cannot get involved with a detailed treatment of the forces driving the oil prices and vice versa strongly influenced in their development by oil. We restrict ourself to a very reduced modeling of the evolution of the potential demand. We consider two groups of consumers, one representing the developed regions, the other one the emerging regions. The quantity we are interested in is just the sum of the potential demand of these two groups. According to the standard approach we use a log-linear demand specifications:

$$D^*(t) = c \prod_{i=1}^{n} Z_i(t)^{\theta_i}, \tag{4.10}$$

where c is a constant, $Z = (Z_1, ..., Z_n)$ a vector of variables determining the potential demand and $\theta = (\theta_1, ..., \theta_n)$ is the according vector of elasticities of the potential demand.

The demand for oil, which is used in virtually almost all lines of production, is strongly affected by the general level of income or activity in an economy. In our case the economic activity is expressed via the gross domestic product Z_1. Here, we assume an exponential growing with different growth rates μ_m and μ_e where m denotes "matured" regions and e represents "emerging" regions. Besides these principal long run driving forces, one might consider further impact factors like the growing number of car sales as well as improvements of oil efficiency. Thus, Huntington (1993) includes for instance $Z_2 = e^{gt}$, where g measures the rate of autonomous improvements in oil efficiency. A more detailed analysis goes beyond the scope of this thesis. However, it is possible to customize the model e.g. by introducing lagged variables, taking at least roughly into account relevant, but too complex and therefore not explicitly modeled subsystems. Including additional determinants and testing different possible scenarios the model may be refined.

4.5 Market Simulations

We model the oil market using the state variables: price X, supply S, and actual demand D. Both quantities supply and demand are divided into two groups: OPEC and Non-OPEC, and developing and developed regions respectively.

$$\begin{aligned} dX &= \kappa_x(c_1X^* - X)(c_2X^* - X)(c_3X^* - X)dt + \sigma\sqrt{X}dW \\ dX^* &= \eta(\delta)X^*dt, \qquad \text{where } \delta = D - S \\ dS_n &= \mu_n(L_n - S_n)S_n\,\chi^*(\psi(t))dt \\ dS_o &= (wp_1 + (1-w)p_2)dt \\ dw &= (-(\alpha+\beta)w + \beta)dt \end{aligned}$$

In order to smooth a transition between to states, we choose the auxiliary function

$$\varphi(t) = \min\left([t]_+, 1\right).$$

The dynamics of the quasi-steady states is determined by the factor η. Here, we chose as an example a piecewise linear function of the difference between supply and demand:

$$\eta(\delta) = \frac{(\eta_2 + \eta_1)}{2\delta}\varphi\left(\frac{t + \eta_1}{2\delta}\right) - \eta_1.$$

The strategic supply behaviour of Non-OPEC and OPEC countries is reflected in:

NON-OPEC:

$$\begin{aligned} \chi^* &= 1 - \chi_{[\psi_-,\psi_+]}, \qquad \text{where} \quad \chi \text{ is the characteristic function} \\ \psi &= X, \end{aligned}$$

OPEC:

$$\begin{aligned} p_1 &= -\kappa_1(L_o^- - S_o)^2 S_o \qquad \text{and} \qquad p_2 = \kappa_2(L_o^+ - S_o)^2 S_o \\ \alpha &= \upsilon\varphi\left(\frac{\phi - \phi^-}{\phi^+ - \phi^-}\right) \qquad \text{and} \qquad \beta = \upsilon\varphi\left(\frac{\phi^+ - \phi}{\phi^+ - \phi^-}\right). \\ \phi &= X. \end{aligned}$$

To take into account different development perspectives, we consider two groups differing mainly in their growth dynamics. In each group the actual demand is given by:

$$\begin{aligned} D &= D^0 + \gamma\left[D^B - D^0\right]^+, \\ D^B &= \min\left\{\varepsilon Z_1, D^*\right\}, \\ D^* &= \beta\, Z^\theta, \end{aligned}$$

where β is a constant, $Z = (Z_1, ..., Z_n)$ a vector of variables determining the potential demand D^*, Z_1 is the gross domestic product, and $\theta = (\theta_1, ..., \theta_n)$ is the according vector of elasticities of potential demand. In a first step, we assume an exponential growing domestic product Z_1 in both developed and developing regions

$$\begin{aligned} \gamma &= \min\{1, [1 + \vartheta(X^* - X)]_+\}, \\ \varepsilon &= \xi_0 + \frac{\xi_1}{\xi_2 + \xi_3 X}. \end{aligned}$$

The simulation results of the oil market model are shown in figure 4.14 and figure 4.15. The initial and parameter values (compare Table 03) are chosen such that the model captures the major developments on the oil market from 2000 to 2005.

Table 03: Initial and parameter values

initial values (2000)		**parameter values**				
X	\$31 /bbl	κ_x	0.5	σ_x	0.33	*price*
X_1	\$23 /bbl	δ_i	8	η_i	1	
X_2	\$29 /bbl	λ	100	κ_i	0.015	*OPEC supply*
X_3	\$34 /bbl	L_o^-	26 mbbl/d	L_o^+	35 mbbl/d	
S_0	30 mbbl/d	ϕ^-	\$24 /bbl	ϕ^+	\$30 /bbl	
S_n	46 mbbl/d	ψ^-	\$26 /bbl	ψ^+	\$30 /bbl	*non-OPEC supply*
Z_1^m	\$58 bn/d	μ_n	10%			
Z_1^e	\$29 bn/d	μ_m	2.5%	μ_e	5%	*demand*
		ξ_0	0.0008	ξ_1	1	
		ξ_2	2	ξ_3	1000	
		θ	.36	γ	1	

The first figure compares the historic price data with a sample path of the simulated stochastic oil market model. In addition, we plotted the evolvement of the multiple equilibria X_j during this period of time. Simulating the interacting system of OPEC and Non-OPEC supply, global demand developments in matured and emerging regions, and considering nonlinear price dynamics, we can explain observed phenomena in the oil market. Large price movements can be interpreted as resulting from multiple equilibria, tending towards a level of \$60 a barrel in 2005. In particular the growing demand of emerging countries put a pressure on the attractors. Crude oil prices are fluctuating between these gradually increasing quasi-steady states. Prices far above or below this zone of instability are driven back and instabilities arise within this attracting region.

Simulating a large number of price trajectories, we compute a sequence of density estimates using a kernel smoothing method. Figure 4.15 emphasizes the capacity of nonlinear processes to capture multi-modal price distributions. Thus multi-modality is the result of multiple equilibria in nonlinear stochastic dynamical system. Thereby, the modes correspond to attracting equilibria while antimodes correspond to repelling equilibria. These quasi-steady states are driven by economic variables like crude oil supply and demand.

Although there is an increase in supply of oil and therefore the prices should decrease, the growing demand offsets this effect driving the attracting domain to new heights. According to our model assumptions, OPEC is trying to stabilize the oil market by increasing their supply as soon as the upper threshold is hit. As a consequence of recent high oil prices, OPEC has only a small range left to react by a substantial increase of supply. In order to preserve its instruments for significant control of the market, OPEC has to invest in new production capacities and regain the instruments for price control.

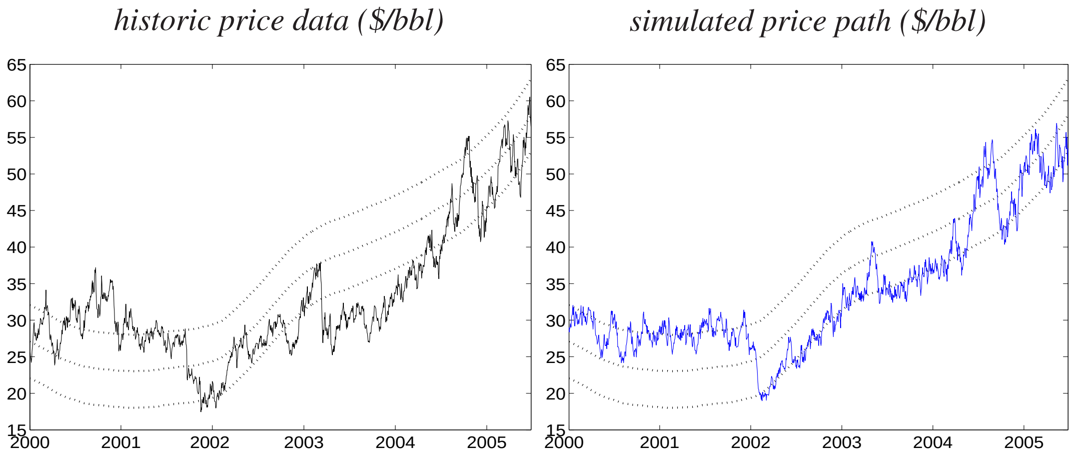


Figure 4.14: Price Trajectory (2000-2005). *This figure illustrates the trajectories of the oil price as function of time (left: historic daily data 2000-05, right: simulation of the model with estimated parameters and a selected realization of the stochastic process, dotted: trajectories of the 3 quasi-steady states).*

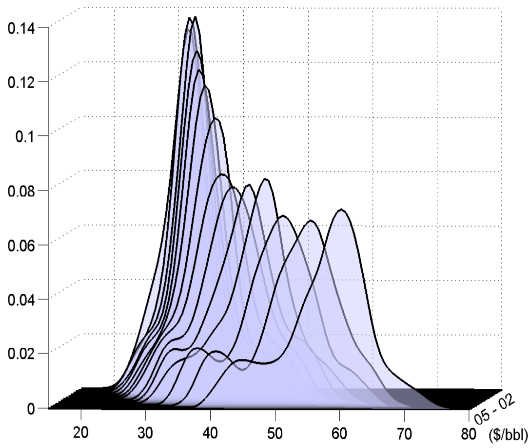


Figure 4.15: Simulated Distributional Dynamics. *The figure shows the price distribution at different time points, obtained by averaging over a large number of realizations. The distributions shows two distinguished maxima, that means, two price levels are expressed at the same time.*

Conclusion

One essential aim is to provide more detailed, accessible information to be used e.g. for rational pricing of contracts on oil. Analyzing the complex network of interactions in the oil market and applying concepts and methods of nonlinear, stochastic dynamical systems we achieved the following results:

- Derivation of a system of stochastic differential equations representing reduced model for supply, demand and price dynamics of oil, including global economic factors like gross domestic product, development of natural resources and technologies.
- Modeling and simulation of controls as exercised by OPEC.
- Simulation of oil price dynamics and comparison with real data.

The simulations showed that main features of the dynamics are reflected by the model system. Its advantages are a rather simple structure reflecting essential structures, its extensibility to more refined modeling if necessary, the combination of nonlinear and stochastic effects neglected to often in existing theories. Though the system is kept as simple as possible, systematic parameter identification based on accessible data remains a problem to be attacked in future.

Chapter 5

Pricing and Hedging Contracts on Oil

Contents

For more than 20 years, businesses in the petroleum and natural gas industry have used derivatives to reduce their exposure to volatile prices. Derivatives are financial instruments which derive their value from that of an underlying asset. The asset that underlies a derivative can be physical commodity (e.g. crude oil or wheat), foreign or domestic currencies, treasury bonds, company stock indices representing the value of groups of securities or commodities, or even an intangible commodity such as a weather-related index (e.g., rainfall, heating degree days). The price movements of the underlying asset is very sensitive to the market structure, e.g. market integration, price deregulations and the price volatility is caused by e.g. shifts in the supply and demand. Modeling oil prices, we entered a rarely explored, but important territory. In general, price dynamics of commodities like oil are attracting more and more interest. Besides of the increased complexity and size of the relevant systems, an additional difficulty has to be overcome: incompleteness of the markets. We derived models as simple as possible, but including essential factors. In this chapter we present concepts and methods leading to a "fair" pricing for derivative contracts on oil. In principle, the suggested method can be applied to a large class of system modeling the price dynamics. However, the resulting systems of equations are getting even in our model to large for existing numerical algorithms. Nonlinearities are still increasing the problems. Therefore, at this moment we are restricted to limits in dimension as far as simulation is concerned. We restrict ourselves in this chapter to a price dynamics modeled by just one stochastic differential equation with a stochastic volatility, controlled by a stochastic process, also modeled by a stochastic differential equation. We remark that the theoretical results hold true also for more general systems modeling the price dynamics, especially the equation derived in Chapter 4 modeling the oil price.

System Dynamics and Numerical Challenges

Considering oil price dynamics including a detailed modeling of demand and supply as suggested in this thesis we end up with a system of nonlinear stochastic differential equations. Since we are confronted with an incomplete market, we have to chose a new approach in deriving equations for an option price. Here we decided for the method of indifference pricing. Using optimization methods to remove the uncertainty and fixing a fair price, we end up with Hamilton-Jacobi-Bellman equations, a system of strongly nonlinear partial differential equations. Whereas theory is not restricted by the size of the systems, dimension poses challenges to the numerical simulations. The problem of high dimensions arises also in the case of complete markets and its solution is of independent interest, but cannot be attacked in the frame of this investigation. Solving linear and nonlinear partial differential equations in general, Fokker-Planck equations or Hamilton-Jacobi-Bellman equations especially, in higher dimensions is a central topic of algorithmic research. Using e.g. sparce grid methods or main component methods, the curse of dimensionality can be at essentially eased. Whereas for linear systems efficient algorithms could be developed recently for high dimensions, nonlinearities are still restricting the dimension. To compute the option prices using the model for the oil price, developed in the first chapters of this thesis, we are even in the simplest case confronted with at least 5 respectively 7 spatial variables. Developing reliable algorithms for the Hamilton-Jacobi-Bellman is the main aim of an ongoing project. In order to be able to use available, well tested numerical software, we consider in this chapter a reduced model leading to 4 spatial variables. Up to now, very often Monte-Carlo-Methods are used to simulate huge systems or to compute high dimensional integrals. Despite the fact, that they offered chance to attack high dimensional problems, they show disadvantages e.g. large computing times. According to the state of the art, we expect decisive progress in using direct methods solving nonlinear partial differential equations in large numbers of variables.

In order to illustrate the approach and to study possible effects e.g. of the nonlinearities we analyze a smaller system, modeling just a stochastic differential equation for the oil price, with a nonlinear drift and a stochastic volatility controlled by a scalar process. That means that the resulting Hamilton-Jacobi-Bellman describes the time evolution of the price in two independent "space" variables. Thus the dimension gets a size treatable by algorithms developed for nonlinear diffusion transport problem. We are going to compute the price and hedging strategies for oil contracts numerically using these algorithms.

Linear Drift versus Nonlinear Drift

We have already commented the effect of nonlinearities on the dynamics of prices for commodities. However, their influence on pricing options has to be discussed. In the case of nonlinear drift terms mathematical difficulties may arise in changing to a new reference measure by a Girsanov transformation. Whereas in the in complete market situation the drift terms may be eliminated, this does not hold for incomplete markets as treated in this chapter. The optimal valuation and hedging of contingent claims in incomplete markets has been studied by a number of authors e.g. Föllmer and Sondermann (1986), Föllmer and Schweizer (1991), Schweizer (1996) and Frittelli (2000). The standing assumption in the existing literature is that asset prices follow a geometric Brownian motion. This assumption facilitates the analysis considerably. Zariphopoulou is one of the few authors studying the nonlinear case. In 1999 he treated a generalization of the Merton's problem of optimal consumption and portfolio choice for nonlinear stock dynamics.

Complete versus Incomplete Market

As realized in the pioneering work of Black, Scholes, Merton and others, financial assets in complete markets can be priced uniquely by construction of replicating portfolios and application of the no arbitrage principle. In recent years, however, financial applications are increasingly led to incomplete market models, since volatilities are driven by stochastic processes, quite a few assets are not traded or the market cannot react properly to dynamics of the underlying processes. In this chapter we assume that the model supports a single traded asset with price process X and a second auxiliary process Y, which may correspond to a related but non-traded stock respectively commodity or a diffusion process which drives the dynamics of X. In the following we assume that Y governs the volatility of the price process. Stochastic volatility models address many of the short-comings of popular option pricing models like Black-Scholes and Cox-Ingersoll-Ross and become increasingly the focus of attention in pricing and hedging commodity derivatives (see e.g. Richter and Sørensen (2002)). Empirical studies strongly suggest that volatility is not constant, but has a random component. ARCH/GARCH models, whose continuous-time diffusion limits are stochastic volatility models, provide much better descriptions of the data (see e.g. Duffie, Gray, and Hoang (2002)). In addition stochastic volatility models capture non-flat implied volatility surfaces. Here, the stochastic volatility environment serves as a simple example of an incomplete market. In this context, there is no unique pricing measure and the volatility risk premium which plays a crucial role in derivative pricing and hedging is not determined. Very often this unknown process is taken to be zero, a constant or a deterministic function of present volatility. The following figure gives a survey on the approach to option pricing we have chosen. We make use of the so called duality between optimization problems of utility functionals and functional like the entropy functional on martingale measures.

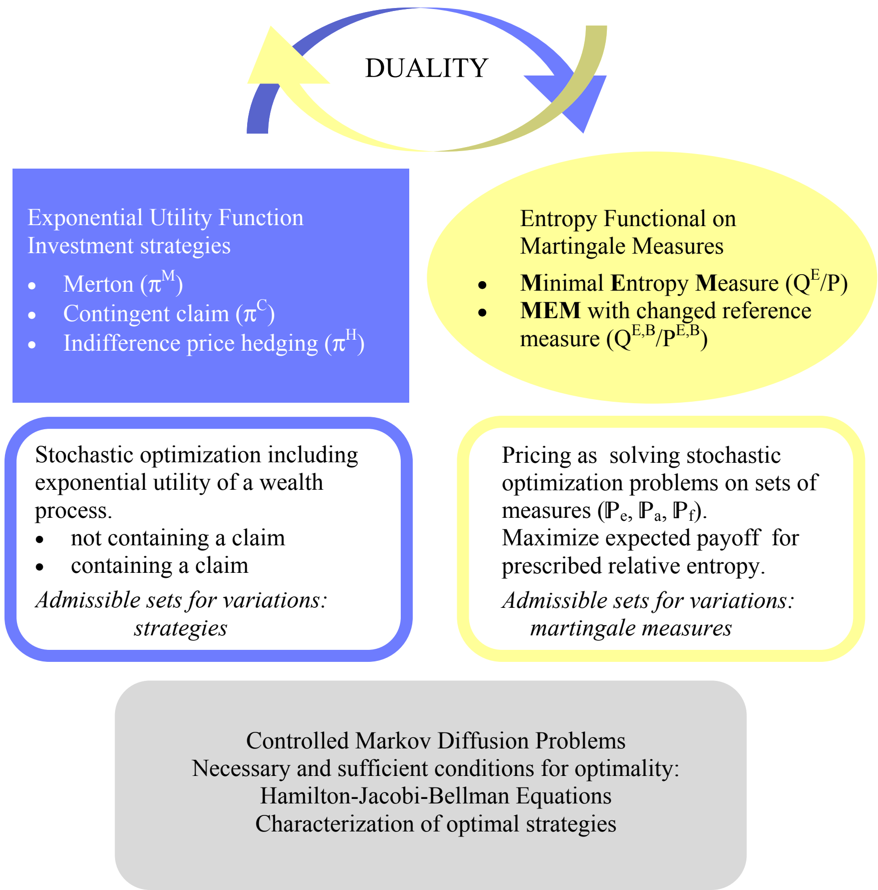


Figure 5.1: Duality Approach

5.1 Pricing in Incomplete Markets

The randomly perturbed volatility leads to an incomplete market, where the second source of randomness cannot be perfectly hedged by trading the underlying stocks (e.g. commodities) and bonds. As a consequence there are no unique preference independent options prices. Instead it is necessary to choose a pricing measure from the family of equivalent martingale measures, thus making implicit or explicit assumptions about utility and preferences. Here, we decided for the method of indifference pricing with exponential utility following in the concepts of Ilhan, Jonsson, and Sircar (2004) and Ilhan and Sircar (2006) including more general, nonlinear dynamics. We outline the main features and concentrate on the steps specific for our market situation. For a more detailed description we refer to the literature.[1]

5.1.1 Market Dynamics

In preparation for the specific problem of pricing options on oil, we start with a general system modeling the price dynamics of an asset:

$$\begin{aligned} dX_t &= \mu(t, X_t, Y_t)dt + \sigma(t, X_t, Y_t)dW_t^1, \quad X_0 = x \\ dY_t &= b(t, Y_t)dt + a(t, Y_t)\left(\rho dW_t^1 + \sqrt{(1-\rho^2)}dW_t^2\right), \quad Y_0 = y. \end{aligned} \tag{5.1}$$

Both W^1 and W^2 are independent Brownian motions with respect to the measure P. The correlation between the price X and the auxiliary process Y is measured by ρ, which is assumed to be constant and takes a value in $[-1, 1]$. Below, we will replace $\sqrt{1-\rho^2}$ by $\hat{\rho}$. The assumption the coefficients have to satisfy will be formulated later. In principle, X and Y could be vector valued quantities. For simplicity of notation we will restrict ourself to demonstrate the scalar case.

5.1.2 Pricing with Equivalent Martingale Measures

Suppose there is an equivalent martingale measure P^* under which the discounted price process $\widetilde{X} = e^{-rt}X$ is a martingale. Throughout this thesis we assume constant deterministic interest rates r. With respect to P^* the price of any contingent claim at time t with payoff B at time T is given by

$$C_t(t, X_t, Y_t) = E^{P^*}[e^{-r(T-t)}B(X_T, Y_T)]. \tag{5.2}$$

In order to construct an equivalent martingale measure one makes the following Ansatz transforming the Brownian motions

$$dW_t^{\lambda,1} = dW_t^1 + \frac{\mu(t, X_t, Y_t) - rX_t}{\sigma(t, X_t, Y_t)}dt, \tag{5.3}$$

$$dW_t^{\lambda,2} = dW_t^2 + \lambda_t dt. \tag{5.4}$$

[1] We would like to thank Aytaç Ilhan for very stimulating discussions on the subject.

In parallel the measure P has to be transformed to new measure P^λ such that $W^{\lambda,1}$ and $W^{\lambda,2}$ are two independent Brownian motions on $(\Omega, \mathcal{F}, P^\lambda)$, using the well-known theorem of Girsanov. As a result we obtain a set of measures parameterized by a process $\lambda_t = \lambda(t, X_t, Y_t)$. The Radon-Nikodym derivatives of these measures P^λ are given by the following formula under assumptions on the coefficients of the stochastic differential equations (5.1)

Theorem 1 (Girsanov)**.**

$$\begin{aligned} \frac{dP^\lambda}{dP} &= \exp\left\{ -\int_0^T \theta_s^{(1)} dW_s^1 - \int_0^T \theta_s^{(2)} dW_s^2 - \frac{1}{2}\int_0^T \left((\theta_s^{(1)})^2 + (\theta_s^{(2)})^2\right) ds \right\} \\ &=: \mathcal{E}(-\theta^{(1)}.W^1 - \theta^{(2)}.W^2) \qquad \textit{(Doléan-Dade exponential)} \end{aligned}$$

where

$$\begin{aligned} \theta^{(1)} &= \frac{\mu(t,X,Y) - rX}{\sigma(t,X,Y)} =: \gamma(t,X,Y), \\ \theta^{(2)} &= \lambda(t,X,Y). \end{aligned}$$

We assume that the functions μ, σ, b, a are bounded and satisfy the usual Lipschitz conditions. We shall make an assumption on the pair $(\theta^{(1)}, \theta^{(2)})$ such that P^λ is well-defined as a probability measure. γ_t and λ_t are adapted to $\mathcal{F}_t$ and satisfies the integrability conditions $\int_0^T \gamma_s^2 ds < \infty$ and $\int_0^T \lambda_s^2 ds < \infty$. This is particularly true if $\sigma(t,X,Y)$ is bounded off from zero and λ_t is bounded.

The term γ may be interpreted as the market price per unit of oil risk and is sometimes called Sharpe ratio, whereas λ represents the market risk of the auxiliary process e.g. stochastic volatility. Any admissible choice of λ leads to an equivalent martingale measure P^λ and thus to a parameterized set of no arbitrage derivative prices

$$C_t(t, X_t, Y_t) = E^{P^\lambda}[e^{-r(T-t)} B(X_T, Y_T)]. \tag{5.5}$$

Under P^λ the discounted price process $\widetilde{X}$ is a martingale and accordingly the stochastic differential equation (5.1) changes to

$$\begin{aligned} dX &= rX dt + \sigma(t,X,Y) dW^{\lambda,1} \\ dY &= [b(t,Y) - \rho a(t,Y)\gamma(t,X,Y) - \hat{\rho} a(t,Y)\lambda(t,X,Y)]\, dt \\ &\quad + a(t,Y)[\rho dW^{\lambda,1} + \hat{\rho} dW^{\lambda,2}]. \end{aligned} \tag{5.6}$$

In order to apply the available mathematical theory for existence, uniqueness and positivity of solutions of differential equations, or in order to carry out the transformations according to Girsanov, one has to make assumptions on the nonlinearities in the model systems. One has very often to assume that nonlinearities are global Lipschitz continuous, or that they are bounded or bounded off from zero. We agree on the convention, that in general this nonlinearities are cut off properly close to zero respectively close to infinity. Such assumptions are mainly needed for analytical reasons, whereas in our numerical tests the dynamics did not drive the solutions into the regions where we performed a cut-off.

5.1.3 Selecting an Optimal Martingale Measure

We want to determine a strategy which optimizes properly chosen utility functionals. The following definitions concerning measures are basic for formulating and solving the variational problems.

Definition 2 (Martingale Measures, Relative Entropy)**.**

1. *The sets of absolute continuous and equivalent (local) martingale measures for $\widetilde{X}$ are denoted by $\mathbb{P}_a$ and $\mathbb{P}_e$.*
2. *L^s_P denotes the set of P-measurable functions f such that $|f|^s$ has finite integral with respect to P. For $s \geq 1$, $\mathcal{L}^s_P$ is the set of measures Q contained in $\mathbb{P}_a$ such that the Radon-Nikodym derivative $\frac{dQ}{dP} \in L^s_P$. Furthermore, denote by $\mathbb{P}_f$ the set of Q contained in $\mathbb{P}_a$ such that*

$$H(Q|P) = \mathbb{E}^P \left[\frac{dQ}{dP} \log \frac{dQ}{dP} \right]$$

is finite. $H(Q|P)$ is called relative entropy of Q with respect to P.

In this context, we want to make the following two remarks:

1. Here we are confronted with the problem to select a proper martingale measure. To this end one solves an optimization for a properly chosen functional of the Radon-Nikodym derivative. For $s > 1$ one chooses the s-moment, that means for $s = 2$ one prefers the *minimal variance measure*. In the limit case $s = 1$, one chooses the relative entropy as functional and obtains the *minimal relative entropy measure*. The relative entropy $H(Q|P)$ of Q with respect to P provides an other possibility to quantify the 'distance' between two probability measures Q and P (even though it is not a metric). According to the analysis of the q-measures its range of admissible applications is larger than that for $q > 1$. For a detailed discussion we refer to Hobson (2004).
2. The choice of the functional consists mainly in selecting a utility function, evaluation the evolution of the wealth of the agent. It is known that the minimal variance is connected with a quadratic utility function, whereas the relative entropy corresponds to an exponential utility function. Mathematically the corresponding optimization problems can be considered as dual (see e.g. Delbaen, Grandits, Rheinländer, Samperi, Schweizer, and Stricker (2002) and Kramkov and Schachermayer (1999)). In the approach here, we decided for an exponential utility and thus for the entropy concept, similar to the investigations of Ilhan, Jonsson, and Sircar (2004) and Ilhan and Sircar (2006). In practice the concept of utility functions is not yet fully accepted as a tool in pricing due to the fact that evaluations seem to escape all attempts of quantitative modeling. However, in incomplete markets we are forced to select a measure optimal with respect to a functional. That means in the dual formulation: We have to select a utility function and strategy which lead to an optimal utility with respect to the wealth.

In the following section we agree on the general assumption

$$\mathbb{P}_e \cap \mathbb{P}_f \neq \emptyset. \tag{5.7}$$

5.1.4 Indifference Pricing

Fundamental concepts of the indifference pricing mechanism are already going back to Bernoulli (1738) and can be explained using the idea of a *certainty equivalent*, i.e. a certain amount of money that makes the agent indifferent - with respect to expected utility - between the return from a random payoff and this amount. Analyzing the optimal replication of contingent claims under transaction costs, Hodges and Neuberger (1989) were the first to adopt this concept of static certainty equivalence to a dynamic one. The pricing mechanism is based on comparison of maximal expected utilities corresponding to investment opportunities with and without involving the contingent claim. Starting with an initial wealth $v > 0$, the agent is assumed to maximize his expected utilities by trading in a portfolio of a risky asset X and a bank account A. The agent's attitude towards risk is modeled via an exponential utility function $U(v) = -e^{-\alpha v}$, where the *risk aversion parameter* α is a positive constant.

Wealth consists of various segments which have to be evaluated in a common unit. In case of commodities like oil one has beside of the actual market price to take into account the cost and benefits resulting e.g. from storing the goods. In case of storable commodities the concept of *convenience yield* is widely used. In the following we include a convenience yield C, which is defined as the benefit associated with holding the underlying physical good rather than the derivative contract (see e.g. Brennan (1991)). This premium arises e.g. from the value of the flexibility of being able to use the physical commodity in a production process at short notice thus avoiding to shut down production facilities or breaching contracts. It is commonly treated like dividend yields on a standard equity if assumed to be constant. To keep the complexity controllable, we assume that it evolves according to

$$dC = cXdt. \tag{5.8}$$

Extensions to stochastic mean reversion models as suggested by e.g. Gibson and Schwartz (1990) and Miltersen and Schwartz (1998) are not considered here.

Definition 3 (Discounted Gain Process)**.**

The total gain generated by holding a commodity is the total gain of capital and dividend (convenience)

$$dG = dX + cXdt. \tag{5.9}$$

In order to price contracts on oil, the discounted gain process $\widetilde{G} := Ge^{-rt}$, is going to play the role of the price process in the previous part. Taking into account this change we obtain the following system:

Assume that P^λ is chosen with respect to $\widetilde{G}$ according to theorem of Girsanov. Then $dW^{\lambda,1}$ and $W^{\lambda,2}$ defined by

$$dW_t^{\lambda,1} = dW_t^1 + \gamma_t dt \qquad \text{where} \qquad \gamma_t = \frac{\mu(t, X_t, Y_t) + (c-r)X_t}{\sigma(t, X_t, Y_t)} \tag{5.10}$$

$$dW_t^{\lambda,2} = dW_t^2 + \lambda_t dt \qquad \lambda_t = \lambda(t, X_t, Y_t) \tag{5.11}$$

and the discounted gain process is a martingale

$$d\widetilde{G} = e^{-rt}\sigma(t, X, Y)dW^{\lambda,1}. \tag{5.12}$$

The price dynamics of the underlying commodity follows

$$\begin{aligned} dX &= (r-c)X dt + \sigma(t, X, Y)dW^{\lambda,1} \\ dY &= [b(t, Y) - \rho a(t, Y)\gamma(t, X, Y) - \hat{\rho} a(t, Y)\lambda(t, X, Y)]\, dt \\ &\quad + a(t, Y)[\rho dW^{\lambda,1} + \hat{\rho} dW^{\lambda,2}]. \end{aligned} \tag{5.13}$$

5.1.4.1 Optimal investment strategy

Definition 4 (Wealth Process)**.**

1. *The wealth process at time t under a strategy (π, η) is given by*

$$V_t = \pi_t G_t + \eta_t A_t$$

where π_t, η_t denotes the number of units held at time t. The dynamics of the bank account A is determined by the deterministic evolution:

$$dA = rAdt.$$

2. *A wealth process is said to be self-financing, iff*

$$dV_t = \pi_t dG_t + \eta_t dA_t.$$

Accordingly, the dynamics of the discounted wealth process is given by:

$$\begin{aligned} d\widetilde{V} &= \pi e^{-rt}\left[(\mu(t, X, Y) + (c-r)X)dt + \sigma(t, X, Y)dW^1\right] \\ &= \pi d\widetilde{G}. \end{aligned} \tag{5.14}$$

Definition 5 (Admissible Trading Strategies)**.**

The set of admissible trading strategies is defined as

$$\Theta := \left\{ \pi \in L_{\widetilde{G}} : \int_0^t \pi \, \mathrm{d}\widetilde{G} \quad \text{is a } Q\text{-martingale for all } Q \in \mathbb{P}_f \right\}.$$

$L_{\widetilde{G}}$ represents the set of $\widetilde{G}$-integrable processes. We remark that alternative choices of Θ are used, however, the results do not depend on the specific choice (see Delbaen, Grandits, Rheinländer, Samperi, Schweizer, and Stricker (2002)).

Given an initial wealth, an agent can choose the following strategies to achieve an optimal investment:

Strategy 1: **Merton investment**

The investor aims to maximize the expected utility of the discounted terminal wealth resulting from trading in stocks and bank account

S1: $$u(v,0) := \sup_{\pi\in\Theta} \mathbb{E}\left\{U\left(\widetilde{V}_T(v,\pi)\right)\right\}.$$

Strategy 2: **Issuing contingent claims**

The agent sells n financial contract today, maximizing the expected utility from net final wealth received from an initial capital, the compensation p^s at time t and the obligation to pay out B at expiration date T

S2: $$u(v+p^s(n\widetilde{B},\alpha),-n) := \sup_{\pi\in\Theta} \mathbb{E}\left\{U\left(\widetilde{V}_T\left(v+p^s(n\widetilde{B},\alpha),\pi\right)-n\widetilde{B}\right)\right\}.$$

Here, the constant parameter α measures the agent's risk aversion.

5.1.4.2 Indifference Price

Following Hodges and Neuberger (1989), the indifference price results from setting the utilities of the discounted net terminal equal.

Definition 6 (Indifference Price)**.**

If the corresponding equation has a unique solution the selling price of a contingent claim is defined as a function p^s *such that the agent is indifferent towards the scenarios* $S1$ *and* $S2$

$$u(v,0) \quad = \quad u(v+p^s(n\widetilde{B},\alpha),-n). \tag{5.15}$$

In analogy, the buyer's indifference price is given by $p^b(n\widetilde{B},\alpha) = -p^s(-n\widetilde{B},\alpha)$.

With regard to the definition of the indifference price, we want to mention further interpretations and observations:

1. In the literature, p^b and p^s are also referred to as the *reservation buying (bid) price* and *reservation selling (ask) price*. Note that this is subjective valuation from the point of view of the agent and does not reflect a price at which trading occurs.
2. In general, the method leads to two different prices, a lower and an upper bound for the price seller and buyer agree on in a real trade. Using rational arguments we obtain an option price interval instead of a single option price. In a complete market bid and ask price coincide defining the Black-Scholes price. For the exponential

utility function chosen here, these price bands are narrow if $n \cdot \alpha$ is small. The bounds will depend on the second derivative of the utility function and certainly on the risk aversion.

3. These price intervals are used in arguing against indifference pricing, since fixing a price is postponed to a deal between seller and buyer. However, choosing a martingale measure e.g. by setting $\lambda = 0$ is rather arbitrary. As an additional counter argument it is quite often pointed out that the numerical calculations of indifference option prices are more time-consuming than computing the diffusion equation for ordinary option prices. This reasoning neglects the fact that high dimensions are crucial for the complexity also in linear diffusion equations.

4. It has to be expected that the pricing depends nonlinear on the volume of the option. However, it is often assumed that $p(n) = n \cdot p$. Results on volume-scaling are going to be discussed in section 5.4.2.

In the following we are considering the selling price of a contingent claim. Therefore, we skip the index s for simplicity.

The investment problem $(S1)$ dates back to Merton. Merton used dynamic programming to solve an investor's optimal portfolio in a complete market where asset prices follow Markovian diffusions. The solution u satisfies a Hamilton-Jacobi-Bellman equation. In a number of well known special cases Merton was able to solve such partial differential equations analytically. The second problem $(S2)$ can be addressed in the same way.

Instead of solving the variational problems directly, dual problems can be formulated and solved, see e. g. Becherer (2004). This reformulation leads to an optimization over martingale measures and turns out to be simpler than the original problem. Furthermore, we remark that in this approach the problems are no longer restricted to the Markovian case. The following theorem was proven by Delbaen, Grandits, Rheinländer, Samperi, Schweizer, and Stricker (2002) and Becherer (2004).

Theorem 2 (Duality Result on Exponential Utility Optimization)**.**

Assumptions:

1. *There exist equivalent local martingale measures with finite relative entropy, that means* $\mathbb{P}_f \cap \mathbb{P}_e \neq \emptyset$.

2. *The integrability assumption* $\mathbb{E}\{e^{(\alpha+\varepsilon)\widetilde{B}}\} < \infty$ *and* $\mathbb{E}\{e^{-\varepsilon\widetilde{B}}\} < \infty$ *holds for some* $\varepsilon > 0$. *This implies that* $\widetilde{B}$ *is in* $L^1(Q)$ *for all* $Q \in \mathbb{P}_f$.

Then

$$\sup_{\pi\in\Theta} \mathbb{E}^P\Big\{-\exp\Big(-\alpha\Big(\widetilde{V}_T(v,\pi) - \widetilde{B}\Big)\Big)\Big\} = -\exp\left(\alpha \sup_{Q\in\mathbb{P}_f(P)}\left(\mathbb{E}^Q\{\widetilde{B}\} - \frac{1}{\alpha}H(Q|P) - v\right)\right)$$

and there exist a unique $Q^{\widetilde{B}}$ *maximizing the right hand side .*

In order to guarantee integrability condition, we have to cut off the corresponding payoff function for values at a finite barrier. We make this assumption for all part of this thesis.

Applying the duality results to the corresponding optimization problems $(S1)$ and $(S2)$ we obtain the indifference price for a contingent claim.

Theorem 3 (Exponential Pricing)**.**

Let the assumptions of theorem 2 be satisfied, then the following relation holds

$$p(\alpha, \widetilde{B}) = \sup_{Q \in \mathbb{P}_f(P)} \left(\mathbb{E}^Q\{\widetilde{B}\} - \frac{1}{\alpha} H(Q|P) \right) - \sup_{Q \in \mathbb{P}_f(P)} \left(-\frac{1}{\alpha} H(Q|P) \right) \tag{5.16}$$

Proof:

Applying the duality result to the definition of the indifference price,

$$u(v + p^s, -\widetilde{B}) = u(v, 0),$$

we obtain (5.16) by simple calculations

$$\begin{aligned}
\sup_{\pi \in \Theta} \mathbb{E}^P \Big\{ -\exp\left(-\alpha \left(\widetilde{V}_T(v+p,\pi) - \widetilde{B}\right)\right) \Big\} &= \sup_{\pi \in \Theta} \mathbb{E}^P \Big\{ -\exp\left(-\alpha \widetilde{V}_T(v,\pi)\right) \Big\} \\
-\exp\left(\alpha \sup_{Q \in \mathbb{P}_f} \left(\mathbb{E}^Q\{\widetilde{B}\} - \frac{1}{\alpha} H(Q|P) - v - p \right)\right) &= -\exp\left(\alpha \sup_{Q \in \mathbb{P}_f} \left(-\frac{1}{\alpha} H(Q|P) - v \right)\right) \\
\sup_{Q \in \mathbb{P}_f} \left(\mathbb{E}^Q\{\widetilde{B}\} - \frac{1}{\alpha} H(Q|P) - p \right) &= \sup_{Q \in \mathbb{P}_f} \left(-\frac{1}{\alpha} H(Q|P) \right).
\end{aligned}$$

The representation (5.16) of the option price suggests changing from the real life measure to the minimal entropy martingale measure as reference. We are going to profit from this measure transformation in computing the option price.

Theorem 4 (Minimal Entropy Martingale Measure)**.**

Under the assumption (5.7) there exists a unique solution Q^E in $\mathbb{P}_f(P) \cap \mathbb{P}_e(P)$ minimizing the relative entropy in $\mathbb{P}_f(P)$:

$$Q^E = \arg \min_{Q \in \mathbb{P}_f(P)} H(Q|P). \tag{5.17}$$

Its density has the form

$$\frac{dQ^E}{dP} = c^E \exp\left\{ -\alpha \widetilde{V}_T(\pi^E) \right\} , \tag{5.18}$$

where

$$\pi^E = \arg\max_{\pi\in\Theta(P)} \mathbb{E}\left\{-\exp\{-\alpha V_T(\pi)\}\right\} \tag{5.19}$$

and

$$\log c^E = H(Q^E|P) < \infty. \tag{5.20}$$

For the proof see Frittelli (2000) and Kabanov and Stricker (2002).

The following theorem represents the option price as supremum of one functional, defined with respect to minimal entropy measure as reference measure, and thus simplifies formula (5.16).

Theorem 5 (Alternative Price Representation)**.**

Suppose assumption (5.7) and $Q^E \in \mathcal{L}^2_P$. Than the option price is characterized by

$$p(\alpha, \widetilde{B}) = \sup_{Q\in\mathbb{P}_f(Q^E)} \left(\mathbb{E}^Q\{\widetilde{B}\} - \frac{1}{\alpha} H(Q|Q^E)\right). \tag{5.21}$$

Proof:

The proof of theorem 5 is based on a change of measures which leads to the following relationship:

$$H(Q|P) \;=\; H(Q|Q^E) + H(Q^E|P). \tag{5.22}$$

At first glance this relation is not obvious. Therefore, it is necessary to make some further comments.

$$\begin{aligned}
H(Q|P) &= \mathbb{E}^P\left\{\frac{dQ}{dP}\log\frac{dQ}{dP}\right\} \\
&= \mathbb{E}^Q\left\{\log\frac{dQ}{dP}\right\} \\
&= \mathbb{E}^Q\left\{\log\left(\frac{dQ}{dQ^E}\frac{dQ^E}{dP}\right)\right\} \\
&= \mathbb{E}^{Q^E}\left\{\frac{dQ}{dQ^E}\log\frac{dQ}{dQ^E}\right\} + \mathbb{E}^Q\left\{\log\frac{dQ^E}{dP}\right\} \\
&= \mathbb{E}^{Q^E}\left\{\frac{dQ}{dQ^E}\log\frac{dQ}{dQ^E}\right\} + \mathbb{E}^Q\left\{\log\left(c^E e^{-\alpha\widetilde{V}_T(\pi^E)}\right)\right\} \\
&= H(Q|Q^E) + H(Q^E|P) - \alpha\mathbb{E}^Q\left\{\widetilde{V}_T(\pi^E)\right\}.
\end{aligned}$$

The last term on the right hand side is zero since the discounted wealth process $\widetilde{V}$ is a martingale under Q.

We know from 5.16, that

$$
\begin{aligned}
p(\alpha, \widetilde{B}) &= \sup_{Q \in \mathbb{P}_f(P)} \left(\mathbb{E}^Q\{\widetilde{B}\} - \frac{1}{\alpha} H(Q|P) \right) + \inf_{Q \in \mathbb{P}_f(P)} \left(\frac{1}{\alpha} H(Q|P) \right) \\
&= \sup_{Q \in \mathbb{P}_f(P)} \left(\mathbb{E}^Q\{\widetilde{B}\} - \frac{1}{\alpha} H(Q|P) \right) + \frac{1}{\alpha} H(Q^E|P) \\
&= \sup_{Q \in \mathbb{P}_f(P)} \left(\mathbb{E}^Q\{\widetilde{B}\} - \frac{1}{\alpha} \Big(H(Q|P) - H(Q^E|P) \Big) \right) \\
&= \sup_{Q \in \mathbb{P}_f(P)} \left(\mathbb{E}^Q\{\widetilde{B}\} - \frac{1}{\alpha} H(Q|Q^E) \right)
\end{aligned}
$$

Given the representation (5.21) for the indifference price, it is worthwhile to note that

1. in case of an exponential utility function the indifference price does not depend on the initial wealth level v.
2. the expression (5.21) can be interpreted as the Lagrange formulation of the following optimization: Maximize the expected payoff as functional on a set of measures in $\mathbb{P}_f$ with fixed entropy relative to the minimal entropy measure. $\frac{1}{\alpha}$ is the Lagrange factor.

5.1.5 Pricing PDE

5.1.5.1 Hamilton-Jacobi-Bellman Equation

The relative entropy of P^λ with respect to P is given by

$$
H(P^\lambda|P) = \mathbb{E}^{P^\lambda} \left\{ \frac{1}{2} \int_0^T \big(\gamma_s^2(t, X, Y) + \lambda_s^2(t, X, Y) \big) \, ds \right\}. \qquad (5.23)
$$

In order to compute minimal entropy using the technique of stochastic optimization, we consider processes (X, Y) starting at time t in (x, y) and derive an equation for

Definition 7.

$$
\psi(t, x, y) := \sup_\lambda \mathbb{E}^{P^\lambda} \left\{ -\frac{1}{2} \int_t^T \big(\gamma_s^2 + \lambda_s^2 \big) \, ds \mid X_t = x, Y_t = y \right\}. \qquad (5.24)
$$

In order to derive the option value, we have to solve the stochastic control problem

$$
p(\widetilde{B}) = \sup_\lambda \mathbb{E}^{P^\lambda} \left\{ \widetilde{B} - \frac{1}{2\alpha} \int_0^T \lambda_s^2 \, ds \right\}. \qquad (5.25)
$$

and introduce the auxiliary function

Definition 8.

$$\phi(t,x,y) \quad := \quad \sup_{\lambda} \mathbb{E}^{P^{\lambda}} \left\{ \widetilde{B} - \frac{1}{2\alpha} \int_t^T \lambda_s^2 \, ds \mid X_t = x, Y_t = y \right\}. \tag{5.26}$$

We can apply results of the stochastic optimization (compare e.g. Fleming and Soner (2006)) and derive the Hamilton-Jacobi-Bellman equations for ψ and ϕ, a system of nonlinear partial differential equation with respect to (t, x, y) as variables.

Theorem 6 (Nonlinear Pricing System)**.**

The Hamilton-Jacobi-Bellman equations associated with this stochastic control problem are

$$\frac{\partial \psi}{\partial t} + \mathcal{L}_{xy}\psi + \frac{1}{2}\hat{\rho}^2 a^2(t,y) \left(\frac{\partial \psi}{\partial y} \right)^2 = \frac{1}{2}\gamma^2(t,x,y) \; , \tag{5.27}$$

$$\frac{\partial \phi}{\partial t} + \mathcal{L}_{xy}\phi + \hat{\rho}^2 a^2(t,y) \frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial y} + \frac{1}{2}\alpha\hat{\rho}^2 a^2(t,y) \left(\frac{\partial \phi}{\partial y} \right)^2 = 0 \; , \tag{5.28}$$

where

$$\mathcal{L}_{yx} \quad = \quad \mathcal{L}_y + \mathcal{L}_x + \rho a(t,y)\sigma(t,x,y) \frac{\partial^2}{\partial x \partial y} \; , \tag{5.29}$$

$$\mathcal{L}_x \quad = \quad (r-c)x\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x,y)\frac{\partial^2}{\partial x^2} \; , \tag{5.30}$$

$$\mathcal{L}_y \quad = \quad \left[b(t,y) - \rho a(t,y)\gamma(t,x,y) \right] \frac{\partial}{\partial y} + \frac{1}{2}a^2(t,y)\frac{\partial^2}{\partial y^2} \tag{5.31}$$

and γ is defined in (5.10). $t \in [0,T]$ and $x, y \in R^2$.

The terminal conditions are

$$\psi(T,x,y) \quad = \quad 0 \qquad \textit{and} \tag{5.32}$$

$$\phi(T,x,y) \quad = \quad \widetilde{B}(X_T, Y_T). \tag{5.33}$$

This statement holds under the following: μ and σ satisfy the conditions formulated in definition 9. Furthermore, λ and γ need to be continuous functions.

A sketch of the proof is given in the next section.

Here, the following remarks should be taken into account.

1. The solution ϕ describes the value of the contract, given at time 0 by

   $$p(n\widetilde{B}, \alpha) = \phi(0, x_0, y_0).$$

   The information on the minimal entropy measure is contained in ψ.

2. Sofar, only terminal conditions are prescribed. The problems of the asymptotic behavior at infinity in space, uniqueness and existence of solutions, of approximation by solutions on bounded domains will discussed in the next section.

3. Note that the drift term of the process for the oil price enters through γ. Thus, the multiple equilibria of the price dynamics enter in both partial differential equations.

4. In a few cases like Geometric Brownian motion

$$\gamma^{BS}(t, Y) = \frac{\mu + c - r}{\sigma(t, Y)}$$

does not depend on X. Thus, $\psi(t, X, Y) = \psi(t, Y)$ and accordingly, the Hamilton-Jacobi-Bellman equation simplifies essentially. $\mathcal{L}_{xy}$ reduces to $\mathcal{L}_y$.

5.1.5.2 Derivation of the Pricing PDE

We are going to apply the following theorem linking the solution of a stochastic optimization problem of the following class with a corresponding partial differential equation (see Fleming and Soner (2006)).

Definition 9 (Controlled Markov Diffusion Problem)**.**

Consider the $\mathbb{R}^n$-valued process x(s) governed by a system of stochastic differential equations of the form

$$dx = \mu(s, x(s), u(s))ds + \sigma(s, x(s), u(s))dw(s),$$

where $u(s) \in U$ is the control applied at time s and $w(s)$ is a Brownian motion of dimension d. We formulate the following notations and assumptions: $[t_0, t_1[\times \mathbf{R}^n =: Q_0$, $\bar{Q}_0$ is a closure of Q_0, U is a closed subset of $\mathbf{R}^m$, the control is a measurable function on $[t_0, t_1[$ with values in U. μ and σ are continuous in $\bar{Q}_0 \times U$ and continuously differentiable with respect to Q_0 variables. Furthermore we assume that μ and σ are Lipschitz

$$|\mu(s, y, u) - \mu(t, x, u)| \leq \textit{const}(|s - t| + |y - x|)$$
$$|\sigma(s, y, u) - \sigma(t, x, u)| \leq \textit{const}(|s - t| + |y - x|)$$

We assume for the drift and volatility $\mu, \sigma : \bar{Q}_0 \times \bar{U} \to \mathbf{R}^n$ respectively $\mathbf{R}^{n,n}$ are continuous and continuously differentiable in the first variable.

Let f and g be the "running" respectively "terminal cost" function. $f : Q_0 \times U \to \mathbf{R}$ and $g : Q_0 \to \mathbf{R}$ are continuous. Consider a functional of the process and the control of the form

$$J(t, x; u) = \mathbb{E}_{tx} \left\{ \int_t^\tau f(s, x(s), u(s))ds + g(\tau, x(\tau)) \right\}.$$

The index indicates that the expectation is taken at time t with x(t) equals to x. The problem to maximize $J(t, x; u)$ over all $u \in \mathcal{A}$

$$F(t, x) = \sup_{\mathcal{A}} J(t, x; u)$$

is called a controlled Markov diffusion problem (CMD).

Definition 10 (Hamilton Function)**.**

Assume $p \in \mathbb{R}^n, A = (A_{ij}), i, j = 1, ..., n$, $a = \sigma\sigma'$ and tr abbreviates trace, that means tr $aA = \sum_{i,j=1}^n a_{ij} A_{ij}$. Define the Hamilton function

$$\mathcal{H}(t, x, p, A) \quad := \quad \sup_v \left[\mu(t, x, v) \cdot p + \frac{1}{2}\, tra(t, x, v)A + f(t, x, v) \right].$$

Using the techniques of control theory one derives the following necessary condition formulated with help of the Hamilton function, known as Hamilton-Jacobi-Bellman equation (compare Fleming and Soner (2006)).

Theorem 7 (HJB Equation associated to a CMD Problem)**.**

Assume, that the infimum F of a CMD problem is smooth. Then the following differential equation holds:

$$\frac{\partial F}{\partial t} + \mathcal{H}(t, x, D_x F, D_x^2 F) = 0 \tag{5.34}$$

where $D_x F$ denotes the gradient of F in x and $D_{xx} F = (F_{x_i x_j}), i, j = 1, ..., n$. This equation is called the Hamilton-Jacobi-Bellman (HJB) partial differential equation associated with the optimal stochastic control problem.

Here, some technical remarks are necessary:

1. The assumption, that F is smooth, is in general not satisfied. The concept of classical solutions of a differential equation has to be generalized. Here we refer to the mathematical literature on weak solutions, for the Hamilton-Jacobi-Bellman system especially on so called viscosity solutions.

2. The Hamilton-Jacobi-Bellman equation formulate a necessary, however not a sufficient condition that F is the infimum of J. However, we will see in Theorem that in case of smooth enough solutions F that it is the wanted J-minimum. For sufficient conditions see Fleming and Soner (2006).

In the following two steps, we show that applying the Hamilton-Jacobi-Bellman theorem leads directly to the partial differential equations (5.27) and (5.28).

5.1.5.3 Derivation of the Necessary Condition

The two functions ψ and ϕ are characterized by the optimization problems 5.24 respectively 5.26.

Optimization Problem (5.24):

Under P^λ the discounted gain process is a martingale and the dynamics of the underlying processes X and Y evolve according to (5.13). Using (5.34) the Hamilton-Jacobi-Bellman partial differential equation is given by

$$\frac{\partial \psi}{\partial t} + [b(y) - \rho a(y)\gamma(x,y)]\frac{\partial \psi}{\partial y} + (r-c)x\frac{\partial \psi}{\partial x} + \frac{1}{2}a^2(y)\frac{\partial^2 \psi}{\partial y^2} + \frac{1}{2}\sigma^2(x,y)\frac{\partial^2 \psi}{\partial x^2}$$
$$+ \rho a(y)\sigma(x,y)\frac{\partial \psi}{\partial x \partial y} + \sup_\lambda \left\{ -\hat{\rho} a(y)\lambda\frac{\partial \psi}{\partial y} - \frac{1}{2}\lambda^2 \right\} = \frac{1}{2}\gamma^2(x,y).$$

The terminal condition are the same as in (5.32). The term in the bracket is maximal for

$$\lambda_t^E(t,X,Y) = -\hat{\rho} a(t,Y)\frac{\partial \psi}{\partial y}(t,X,Y). \tag{5.35}$$

We use the index E in order to indicate that the variational problem is related to the entropy. Substituting of the optimal λ yields to

$$\frac{\partial \psi}{\partial t} + \mathcal{L}_{xy}\psi + \frac{1}{2}\hat{\rho}^2 a^2(y)\left(\frac{\partial \psi}{\partial y}\right)^2 = \frac{1}{2}\gamma^2(x,y)$$

where $\mathcal{L}_{xy}$ is defined in (5.29).

Optimization Problem (5.26)

We start with the measure change from P to the minimal entropy measure P^{λ^E} and obtain for the stochastic processes including the transformed Brownian motions with respect to the minimal entropy measure:

$$\begin{aligned} dX_t &= (r-c)X_t dt + \sigma(t, Y_t, X_t) dW^{\lambda^E,1}, \\ dY_t &= \left(b(Y_t) - \rho a(Y_t)\gamma(t, X, Y) + \hat{\rho}^2 a^2(Y_t) \frac{\partial \psi}{\partial y} \right) dt + a(Y_t) \left(\rho dW_t^{\lambda^E,1} + \hat{\rho} dW_t^{\lambda^E,2} \right). \end{aligned}$$

Now we consider the optimization problem (5.26) defining ϕ. We have to change from the minimal entropy measure to measures P^λ with Radon-Nikodym derivative

$$\frac{dP^\lambda}{dP^{\lambda^E}} = \exp\left(-\int_0^T \lambda_s dW_s^{\lambda^E,2} - \frac{1}{2}\int_0^T \lambda_s^2 ds \right). \tag{5.36}$$

These measures are the martingale measures equivalent to the minimal entropy martingale measure. We have to represent the processes X and Y using P^λ as reference measure. The following transformation concerning the Brownian motions are required:

$$dW_t^{\lambda,1} = dW_t^{\lambda^E,1}, \tag{5.37}$$

$$dW_t^{\lambda,2} = dW_t^{\lambda^E,2} + \lambda_t dt. \tag{5.38}$$

$W^{\lambda,1}$ and $W^{\lambda,2}$ are two independent Brownian motions on $(\Omega, \mathcal{F}, P^\lambda)$. With respect to the new measure X_t and Y_t satisfy

$$\begin{aligned} dX_t &= (r-c)X_t dt + \sigma(t, Y_t, X_t) dW^{\lambda,1}, \\ dY_t &= \left[b(Y_t) - \rho a(Y_t)\gamma(t, X, Y) + \hat{\rho}^2 a^2(Y_t) \frac{\partial \psi}{\partial y} - \hat{\rho} a(Y_t)\lambda(t, X, Y) \right] dt \\ &\quad + a(Y_t)[\rho dW_t^{\lambda,1} + \hat{\rho} dW_t^{\lambda,2}]. \end{aligned}$$

Having performed these transformations the optimization problem defining ϕ gets the structure of a CMD problem and we can apply theorem 7. Thus, ϕ satisfies the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned} &\frac{\partial \phi}{\partial t} + (r-c)x\frac{\partial \phi}{\partial x} + \left[b(y) - \rho a(y)\gamma + \hat{\rho}^2 a^2(y)\frac{\partial \psi}{\partial y} \right] \frac{\partial \phi}{\partial y} \\ &\quad + \frac{1}{2}a^2(y)\frac{\partial^2 \phi}{\partial y^2} + \frac{1}{2}\sigma^2(x,y)\frac{\partial^2 \phi}{\partial x^2} + \rho a(y)\sigma(x,y)\frac{\partial \phi}{\partial x \partial y} \\ &\quad + \sup_\lambda \left\{ -\hat{\rho} a(y)\lambda \frac{\partial \phi}{\partial y} - \frac{1}{2\alpha}\lambda^2 \right\} = 0 \end{aligned}$$

with the terminal condition

$$\phi(T, x, y) = \widetilde{B}(X_T, Y_T).$$

The term in the bracket is maximal for

$$\lambda_t^p(t, X, Y) = -\alpha\hat{\rho}a(t, Y)\frac{\partial\phi}{\partial y}(t, X, Y). \tag{5.39}$$

The indicator p is related to the variational problem for option price. We obtain

$$\frac{\partial\phi}{\partial t} + \mathcal{L}_{xy}\phi + \hat{\rho}^2 a^2(y)\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial y} + \frac{1}{2}\alpha\hat{\rho}^2 a^2(y)\left(\frac{\partial\phi}{\partial y}\right)^2 \quad = \quad 0. \tag{5.40}$$

5.1.5.2 Sufficient Condition

We show that for λ_t^E respectively λ_t^p determined in (5.35) and (5.39) the corresponding functionals attain a maximum. We are going to use the following characterization of the minimal relative entropy measure. For a proof see Grandits and Rheinländer (2002).

Theorem 8 (Alternative Characterization of Q^E)**.**

Assume there exists $\bar{Q} \in \mathbb{P}_e(P) \cap \mathbb{P}_f(P)$. Then $\bar{Q} = Q^E$ if and only if the following holds:

(i) $\displaystyle\frac{d\bar{Q}}{dP} = \exp\left\{c + \int_0^T \nu_t d\widetilde{G}\right\}$ *for a constant c and $\widetilde{G}-$integrable ν,*

(ii) $\displaystyle\mathbb{E}^Q\left\{\int_0^T \nu_t d\widetilde{G}_t\right\} = 0$ *for $Q = \bar{Q}, Q^E$.*

We are going to consider the two situations separately.

Case 1: Optimization Problem (5.24)

Consider the measure P^{λ^E} defined by

$$\frac{dP^{\lambda^E}}{dP} := \exp\left\{-\int_0^T \gamma_t dW_t^1 - \int_0^T \lambda_t^E dW_t^2 - \frac{1}{2}\int_0^T \left(\gamma_t^2 + \left(\lambda_t^E\right)^2\right) dt\right\}, \tag{5.41}$$

where $\quad \gamma(t, X, Y) = \frac{\mu(t,X,Y)+(c-r)X}{\sigma(t,X,Y)} \quad$ and $\quad \lambda^E(t, X, Y) = -\hat{\rho}a(t, Y)\frac{\partial\psi}{\partial y}(t, X, Y).$

We are going to show that $\frac{dP^{\lambda^E}}{dP}$ can be represented in form (i) in theorem 8 and that condition *(ii)* is fulfilled. We have to rewrite the integrals as integrals with respect to

$$d\widetilde{G} = e^{-rt}\sigma(t, X, Y)(dW_t^1 + \gamma_t dt) = e^{-rt}\sigma(t, X, Y)dW^{\lambda^E,1}. \tag{5.42}$$

The main step is to substitute $-\frac{1}{2}\int_0^T(\gamma_t^2 + (\lambda_t^E)^2)dt$ in (5.41). We start with the Ito-formula for $\psi(t, X(t), Y(t))$:

$$\begin{aligned}
0 &= \psi(0, X(0), Y(0)) - \psi(T, X(T), Y(T)) \\
&+ \int_0^T \left\{ \frac{\partial \psi}{\partial t} + (r-c)X\frac{\partial \psi}{\partial x} + \left(b(y) - \gamma\rho a(y) + a^2(y)\hat{\rho}^2\frac{\partial \psi}{\partial y}\right)\frac{\partial \psi}{\partial y} \right. \\
&+ \left. \frac{1}{2}\sigma^2(t,x,y)\frac{\partial^2 \psi}{\partial x^2} + \rho a(y)\sigma(t,x,y)\frac{\partial^2 \psi}{\partial x \partial y} + \frac{1}{2}a^2(y)\frac{\partial^2 \psi}{\partial y^2} \right\} dt \\
&+ \int_0^T \left(\sigma(t,x,y)\frac{\partial \psi}{\partial x} + a(y)\rho\frac{\partial \psi}{\partial y}\right) dW^{\lambda^E,1} + \int_0^T a(y)\hat{\rho}\frac{\partial \psi}{\partial y} dW^{\lambda^E,2}.
\end{aligned}$$

Taking into account the Hamilton-Jacobi-Bellman equation (5.27)

$$\frac{\partial \psi}{\partial t} + \mathcal{L}_{xy}\psi + \frac{1}{2}\hat{\rho}^2 a^2(t,y)\left(\frac{\partial \psi}{\partial y}\right)^2 = \frac{1}{2}\gamma^2(t,x,y)$$

and the transformations (5.10), (5.11) and (5.42), we obtain

$$\begin{aligned}
-\frac{1}{2}\int_0^T \left(\gamma_t^2 + \left(\lambda_t^E\right)^2\right) dt &= \psi(T, X(T), Y(T)) - \psi(0, X(0), Y(0)) \\
&- \int_0^T \left(\frac{\partial \psi}{\partial x} + \frac{a(y)\rho}{\sigma(t,x,y)}\frac{\partial \psi}{\partial y}\right) d\widetilde{G} + \int_0^T \lambda_t^E dW_t^2 - \int_0^T \gamma^2 dt.
\end{aligned}$$

Thus, (5.41) can be expressed as

$$\frac{dP^{\lambda^E}}{dP} = \exp\left\{c + \int_0^T \nu_t d\widetilde{G}\right\}, \tag{5.43}$$

where $c = -\psi(0, x, y)$ and

$$\nu = -\frac{1}{e^{-rt}\sigma(t,x,y)}\left(\gamma(t,x,y) + \sigma(t,x,y)\frac{\partial \psi}{\partial x} + \rho a(t,y)\frac{\partial \psi}{\partial y}\right). \tag{5.44}$$

For ν given in (5.44) condition (ii) in theorem 8 is fulfilled

$$\begin{aligned}
&\mathbb{E}^{P^{\lambda^E}}\left\{\int_0^T \nu_t d\widetilde{G}_t\right\} = \\
-\ &\mathbb{E}^{P^{\lambda^E}}\left\{\int_0^T \frac{e^{rt}}{\sigma(t,x,y)}\left(\gamma(t,x,y) + \sigma(t,x,y)\frac{\partial \psi}{\partial x} + \rho a(t,y)\frac{\partial \psi}{\partial y}\right) dW_t^{\lambda^E}\right\} = 0.
\end{aligned}$$

This follows from the fact that $W_t^{\lambda^E}$ is Brownian motion for P^{λ^E}. Since Q^E is the minimal relative entropy measure for the discounted gains process, the condition is also fulfilled for Q^E. Therefore, the functional to be optimized assumes its minimum in

λ^E defined in (5.35) and the corresponding measure is the minimal entropy martingale measure, $Q^E = P^{\lambda^E}$.

Case 2: Optimization Problem (5.26):

We proceed similar to case 1, however, showing that λ^p defined in (5.39) and the corresponding measure are solutions to an optimization problem equivalent to the problem (5.26), formulated in the following theorem.

Theorem 9 (Indifference Price as Minimal Entropy)**.**

Under assumption 5.7 and $Q^E \in \mathcal{L}_P^2$

$$p(\alpha, \widetilde{B}) = \frac{1}{\alpha}\left(\sup_{Q\in\mathbb{P}_f(P^{E,\widetilde{B}})} -H(Q|P^{E,\widetilde{B}}) - \log c^{E,\widetilde{B}}\right), \tag{5.45}$$

where

$$\frac{dP^{E,\widetilde{B}}}{dQ^E} = c^{E,\widetilde{B}} e^{\alpha\widetilde{B}}, \qquad \textit{with} \qquad (c^{E,\widetilde{B}})^{-1} = \mathbb{E}^{Q^E}\{e^{\alpha\widetilde{B}}\}. \tag{5.46}$$

Proof: By a change of measure obtain

$$\begin{aligned} H(Q|Q^E) &= \mathbb{E}^{Q^E}\left\{\frac{dQ}{dQ^E}\log\frac{dQ}{dQ^E}\right\} \\ &= \mathbb{E}^Q\left\{\log\left(\frac{dQ}{dP^{E,\widetilde{B}}}\frac{dP^{E,\widetilde{B}}}{dQ^E}\right)\right\} \\ &= \mathbb{E}^Q\left\{\log\frac{dQ}{dP^{E,\widetilde{B}}} + \log\left(c^{E,\widetilde{B}}e^{\alpha\widetilde{B}}\right)\right\} \\ &= \mathbb{E}^{P^{E,\widetilde{B}}}\left\{\frac{dQ}{dP^{E,\widetilde{B}}}\log\frac{dQ}{dP^{E,\widetilde{B}}}\right\} + \log c^{E,\widetilde{B}} + \alpha\mathbb{E}^Q\{\widetilde{B}\} \\ &= H(Q|P^{E,\widetilde{B}}) + \log c^{E,\widetilde{B}} + \alpha\mathbb{E}^Q\{\widetilde{B}\}. \end{aligned} \tag{5.47}$$

Set the result (5.47) in (5.21) and obtain (5.45).

Since the optimization problem determining the option price can be formulated as an minimization of the entropy with respect to the reference measure $P^{E,\widetilde{B}}$, we can proceed as in the first case. We know there exists a unique solution $Q^{E,\widetilde{B}}$. Let P^{λ^p} be the measure associated with λ^p defined 5.39. Substitute in

$$\frac{dP^{\lambda^p}}{dQ^E} = \exp\left\{-\int_0^T \lambda^p dW^{\lambda^E,2} - \frac{1}{2}\int_0^T (\lambda^p)^2 dt\right\} \tag{5.48}$$

the following equation

$$\begin{aligned} -\frac{1}{2}\int_0^T (\lambda^p)^2 dt &= \alpha\Big(\phi(T,X(T),Y(T)) - \phi(0,X(0),Y(0))\Big) \\ &- \alpha\int_0^T \left(\frac{\partial\phi}{\partial x} + \frac{a(y)\rho}{\sigma(t,x,y)}\frac{\partial\phi}{\partial y}\right) d\widetilde{G} + \int_0^T \lambda^p dW^{\lambda^E,2}. \end{aligned}$$

Here we used the Ito-formula for $\phi(t, X(t), Y(t))$ and the representation (5.40). Hence, (5.48) can be written in the form of (i) in theorem 8, that means

$$\frac{dP^{\lambda^p}}{dQ^E} = \exp\left\{c + \int_0^T \nu d\widetilde{G}\right\}, \tag{5.49}$$

where $c = \alpha(\phi(T,x,y) - \phi(0,x,y))$ and

$$\nu = -\alpha e^{rt}\left(\frac{\partial\phi}{\partial x} + \frac{\rho a(y)}{\sigma(t,x,y)}\frac{\partial\phi}{\partial y}\right). \tag{5.50}$$

In addition, we change the measure to the reference measure $P^{E,\widetilde{B}}$. Thus,

$$\frac{dP^{\lambda^p}}{dP^{E,\widetilde{B}}} = \exp\left\{c^* + \int_0^T \nu d\widetilde{G}\right\},$$

where ν is given in (5.50) and $c^* = -\log\left(\mathbb{E}^{Q^E}\Big\{e^{\alpha\widetilde{B}}\Big\}\right) - \alpha\phi(0,x,y)$.

As a consequence, we obtain

$$\mathbb{E}^{P^{\lambda^p}}\left\{\int_0^T \nu_t d\widetilde{G}_t\right\} = -\alpha\mathbb{E}^{P^{\lambda^p}}\left\{\int_0^T e^{rt}\left(\frac{\partial\phi}{\partial x}d\widetilde{G} + \rho a(y)\frac{\partial\phi}{\partial y}dW^1\right)\right\} = 0.$$

The condition holds true for $Q^{E,\widetilde{B}}$ the minimal entropy measure with respect to the reference measure $P^{E,\widetilde{B}}$. As a consequence, the measures must be identical according to the lemma, $Q^{E,\widetilde{B}} = P^{E,\widetilde{B}}$. The functional to be optimized assumes a minimum in λ^p defined in (5.39). Condition (ii) in theorem 8 is satisfied and we conclude that the minimum is achieved in λ^p.

5.2 Optimal Trading Strategies

This section analyzes hedging strategies in more detail. Equations for the optimal strategies will be derived using the solution of the Hamilton-Jacobi-Bellman equations associated with the optimizing problem. At first we define the following hedging strategies:

Definition 11 (Optimal Strategies)**.**

Merton Portfolio Strategy

$$\pi_M := \arg\sup_{\pi\in\Theta} \mathbb{E}^P \left\{\exp\left(-\alpha\widetilde{V}_T(v,\pi)\right)\right\} \tag{5.51}$$

Contingent Claim Strategy

$$\pi_C := \arg\sup_{\pi\in\Theta} \mathbb{E}^P \left\{\exp\left(-\alpha\left(\widetilde{V}_T(v+p^s,\pi) - n\widetilde{B}\right)\right)\right\} \tag{5.52}$$

Hedging Strategy

$$\pi_H := \arg\sup_{\pi\in\Theta} E^{Q^E}\{-\exp(-\alpha(V(\pi) - \widetilde{B}))\} \tag{5.53}$$

One derives immediately the following relation between these strategies.

Lemma 1 (Relations of the Strategies)**.**

$$\pi_C = \pi_M + \pi_H \tag{5.54}$$

The strategy π_c is very intuitive. The first term represents the optimal investment strategy in absence of a claim π_M. The second term is the adjustment to this strategy caused by the introduction of the claim. π_H is also called the *utility indifference hedging* strategy.

For the proof change from reference measure Q^E to P in the definition of the hedging strategy and apply the following theorem of Frittelli (2000) and Kabanov and Stricker (2002).

Theorem 10 (Fritelli, Kabanov and Stricker)**.**

Under assumption 5.7, Q^E exists, is unique, is in $\mathbb{P}_f(P) \cap \mathbb{P}_e(P)$ and its density has the form

$$\frac{dQ^E}{dP} = c^M \exp\left(-\alpha\widetilde{V}_T(\pi^M)\right), \tag{5.55}$$

where

$$\log c^M = H(Q^E|P) < \infty.$$

Now we are able to compute the optimal strategies using the solution of the Hamilton-Jacobi-Bellman equations.

Theorem 11 (Formulas for Optimal Trading Strategies)**.**

$$\begin{aligned}\widetilde{\pi}_M &= \overbrace{\frac{\mu(t,X)+(c-r)X}{\alpha\sigma^2(t,x,y)}}^{a} + \overbrace{\frac{1}{\alpha}\left(\frac{\partial\psi}{\partial x}+\frac{\rho a(y)}{\sigma(t,x,y)}\frac{\partial\psi}{\partial y}\right)}^{b},\\ \widetilde{\pi}_H &= \underbrace{\frac{\partial\phi}{\partial x}}_{c} + \underbrace{\frac{\rho a(y)}{\sigma(t,x,y)}\frac{\partial\phi}{\partial y}}_{d},\end{aligned}$$

where the tilde marks the discounted values, i.e. $\widetilde{\pi}^i := e^{-rt}\pi^i$*. The remaining strategy* $\widetilde{\pi}_C$ *can be obtained via Lemma 1.*

The individual terms can be interpreted, respectively, as follows

1. (a) the Merton ratio,
   (b) the volatility hedging term for the Merton portfolio,
   (c) the delta hedging for the claim and
   (d) the corresponding volatility hedging term.
2. The nonlinear dependence on the price dynamics leads to an additional term $\frac{\partial\psi}{\partial x}$ in (b). If γ and λ do not depend on X, e.g. in case of a geometric Brownian motion, this term disappears.

Derivation of the formula for π_M

We compare two alternative representations of the Radon-Nikodym derivative of Q^E in (5.43) and theorem (10):

$$\begin{aligned}\text{[5.43] :} \qquad \frac{dQ^E}{dP} &= \exp\left(c+\int_0^T \nu d\widetilde{G}\right)\\ \text{[Theorem 10]:} \qquad \frac{dQ^E}{dP} &= c^M\exp\left(-\alpha\int_0^T \pi^M d\widetilde{G}\right).\end{aligned}$$

The uniqueness of the minimal entropy martingale measure implies $\nu = -\alpha\pi^M$ and finally

$$\widetilde{\pi}^M = \frac{\mu(t,X)+(c-r)X}{\alpha\sigma^2(t,x,y)} + \frac{1}{\alpha}\left(\frac{\partial\psi}{\partial x}+\frac{\rho a(y)}{\sigma(t,x,y)}\frac{\partial\psi}{\partial y}\right).$$

Derivation of formula for π_H

We compare two alternative representations of the derivative of P^{λ^p} with respect to Q^E: At first we recall

$$\frac{dP^{\lambda^p}}{dQ^E} = \exp\left(c+\int_0^T \nu d\widetilde{G}\right)$$

is obtained in (5.49). P^{λ^p} can be determined as minimal entropy measure with respect to a measure $Q^{\widetilde{B},E}$ defined through its derivative with respect to Q^E in (5.46). Due to this fact we can apply formula (5.18) to P^{λ^p} and the reference measure $Q^{\widetilde{B},E}$, remarking that on the right hand side in (5.18) the optimal strategy in this case is just π_H. We obtain

$$\frac{dP^{\lambda^p}}{dQ^E} = \frac{dP^{\lambda^p}}{dP}\frac{dP}{dQ^E} = c^H \exp\left\{-\alpha\left(\widetilde{V}_T(\pi^H) - \widetilde{B}\right)\right\}.$$

As in the no claim case ν in (5.50) is equal to $-\alpha\pi^H$ and therefore

$$\widetilde{\pi}^H = \frac{\partial\phi}{\partial x} + \frac{\rho a(y)}{\sigma(t,x,y)}\frac{\partial\phi}{\partial y}.$$

5.3 Numerical Approximation to the Hamilton-Jacobi-Bellman System

In the last section we showed that the option price and the hedging strategy can be computed using the solution of the Hamilton-Jacobi-Bellman system. Here, we cannot deal with the analytic problems concerning existence, uniqueness and smoothness of solutions and refer to Fleming and Soner (2006) and the recent literature quoted there. The system of partial differential equations poses several obstacles to overcome.

- It is an end-value problem in time valid in whole space. Changing time direction transfers the problem to an initial value problem. In order to reduce the problem in a bounded domain in space, one has to put properly chosen boundary conditions. A detailed discussion of this topic is offered by Fleming and Soner (2006).
- The equations contain diffusion and transport terms interacting with each other. Diffusion may degenerate, transport terms may get dominant. This fact is creating additional problems for computing. The numerical algorithms have to be adjusted to the case that transport is getting dominant.
- The system of PDE contains terms with quadratic growth in the gradients. In general, such nonlinearities may lead to singularities, and therefore, adequate analytical and numerical methods have to be used. We did not run into difficulties of this type in our simulations, whereas the first to topics came up.

We are going to apply a software tool for nonlinear evolution equations (GASCOIGNE), developed by Rannacher and his coworkers (IWR, University of Heidelberg, Version 2006). It based on multigrid methods and on discretization using stabilized finite elements. It includes error control and adaptive mesh refinement. It is proven to be a flexible tool for simulations applications to optimal control problem e.g. for fluid flow, particle transport in fluids, multi-component diffusion-reaction systems e.g. in combustion problems, as well as for parameter identification of partial differential equations. Basic information is available at *http://gascoigne.uni-hd.de/*.

In order to use this toolkit we have to reformulate the PDE changing the time direction, introducing a computational domain in space and properly chosen boundary condition on its boundary, and finally writing the equations in an integrated form with test functions. This form is basic for the finite element method applied in the program.

At first we replace the end-value problem by an initial value problem transforming the time t to $\tau = T - t$. This leads to a change of signs. Furthermore, in order to adjust the representation of the differential operator to the finite element method, we rewrite the equations such that the leading part of the differential operator is given in divergence form and obtain altogether:

$$
\begin{aligned}
\frac{\partial \psi}{\partial \tau} - \beta \nabla \psi - div(D \nabla \psi) - c_\psi \left(\frac{\partial \psi}{\partial y} \right)^2 &= -\frac{1}{2}\gamma^2, \\
\frac{\partial \phi}{\partial \tau} - \beta \nabla \phi - div(D \nabla \phi) - c_\phi \left(\frac{\partial \phi}{\partial y} \right)^2 - k \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial y} &= 0,
\end{aligned}
$$

where the coefficients are defined as follows

$$
\begin{aligned}
\beta &= \begin{pmatrix} (r-c)x \\ b(y) - \rho a(y) \gamma(x,y) \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \frac{\partial \sigma^2}{\partial x}(x,y) + \frac{\partial (\rho a \sigma)}{\partial y}(x,y) \\ \frac{\partial a^2}{\partial y}(y) + \frac{\partial (\rho a \sigma)}{\partial x}(x,y) \end{pmatrix}, \\
D &= \frac{1}{2} \begin{pmatrix} \sigma^2(x,y) & \rho a(y) \sigma(x,y) \\ \rho a(y) \sigma(x,y) & a^2(y) \end{pmatrix}, \\
c_\psi &= \frac{1}{2} \hat{\rho}^2 a^2(y), \qquad c_\phi = \frac{1}{2} \alpha \hat{\rho}^2 a^2(y), \\
k &= \hat{\rho}^2 a^2(y), \qquad \gamma = \frac{\mu(x,y) + (c-r)x}{\sigma(x,y)}.
\end{aligned}
$$

The second term of β arises due to a correction necessary to balance the interchanging differentiation with multiplication. Thus we have to assume that both diffusion terms σ and a are differentiable with respect to x and y.

Formulation of the Initial and Boundary Conditions

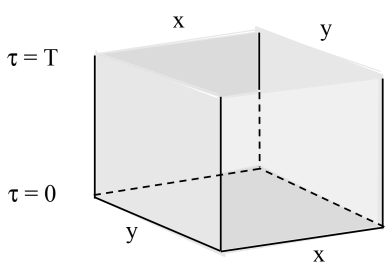
The following rectangle in time and space is chosen as computational domain

$\Omega := [0,T] \times [x_{min}, x_{max}] \times [y_{min}, y_{max}]$ specified here as $[0,1] \times [0,200] \times [-10,10]$.

The conditions on the parabolic boundary $\partial^* \Omega$ of Ω are

$\psi(0,x,y) = 0,$
$\frac{\partial \psi}{\partial x}(\tau,x,y) = 0,$
$\frac{\partial \psi}{\partial y}(\tau,x,y) = 0$ for all boundaries;

$\phi(0,x,y) = \max(x-S,0),$
$\phi(\tau,x_{min},y) = 0,$
$\phi(\tau,x_{max},y) = e^{-r\tau}(x-S),$
$\frac{\partial \phi}{\partial y}(\tau,x,y) = 0$ for both boundaries.



The mathematical theory guarantees the existence of a unique solution to this initial-boundary-value problem. However, it also has to be shown, that the solution can be considered as an approximation to the solution in the whole space. According to the theory (see Fleming and Soner (2006), chapter IX) we know that the solutions in Ω approximate the global solution in each compact set if x_{min} and y_{min} tend to $-\infty$ and x_{max} and y_{max} tend to $+\infty$. Since there is no error estimate available, we have to control the dependence of the "approximative" solutions testing the dependence of solutions on the domain size.

Weak Formulation

In order to solve the problem numerically, let us now derive the weak formulation of the differential equations, that means let us express them in an integrated form. Multiply the equations with test functions η_1 and η_2 and integrate on Ω.

The test functions η_1, η_2 are chosen as a smooth functions where η_2 vanishes on $\partial^*\Omega$ if $x = x_{max}$. We obtain

$$\int_\Omega \left(\frac{\partial\psi}{\partial\tau}\right)\eta_1 - \int_\Omega (\beta\nabla\psi)\,\eta_1 + \int_\Omega (D\nabla\psi)\,\nabla\eta_1 - \int_\Omega \left(c_\psi\left(\frac{\partial\psi}{\partial y}\right)^2\right)\eta_1 = -\gamma$$

$$\int_\Omega \left(\frac{\partial\phi}{\partial\tau}\right)\eta_2 - \int_\Omega (\beta\nabla\phi)\,\eta_2 + \int_\Omega (D\nabla\phi)\,\nabla\eta_2 - \int_\Omega \left(c_\phi\left(\frac{\partial\phi}{\partial y}\right)^2\right)\eta_2 - \int_\Omega \left(k\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial y}\right)\eta_2 = 0$$

This formulation is used in numerical algorithms by choosing the test functions out of a finite element basis. The input for the program uses the data of the equations in this form. Since we are not analyzing the underlying numerical algorithms in this thesis, we just use this information here, otherwise the weak system will be no longer used explicitly.

5.4 Test Problems and Numerical Results

In the following we are going to examine the indifference pricing in case of linear and nonlinear mean reversion processes with stochastic volatility. Before, however, we are testing our approach in the special case where the price dynamics is a geometric Brownian Motion with constant volatility. That means for a Black-Scholes model, where explicit solutions for the considered options are known. Our numerical methods designed for the more general situation should lead in this simple case to a good approximation of the known explicit solutions. Confirmed by this simple test we are going to compare the pricing results in complete and incomplete market setting. At first we introduce a simple one factor model which is widely used to describe the spot price dynamics on energy markets, in particular for electricity prices.

5.4.1 Testing the Algorithm in the Black-Scholes Case

In the linear case the general market dynamics is given by

M1:
$$\begin{aligned} dX &= \mu X dt + f(Y) X dW \\ dY &= bY dt + a\left(\rho dW^1 + \sqrt{(1-\rho^2)} dW^2\right). \end{aligned} \tag{5.56}$$

Thus, the price dynamics is driven by a stochastic volatility process $\sigma(X, Y) = f(Y)X$ where f is a positive function. It is reasonable to assume that f is monotone increasing and bounded. We choose for our tests as example a function, which is often used:

$$f(Y) = \arctan(Y-1)/2\pi + 0.3.$$

An excellent overview about different specifications of stochastic volatility models is given by Fouque, Papanicolaou, and Sircar (2000).

In the simple case that a and b are zero, we are getting the fundamental Black-Scholes partial differential equation for option prices for commodities

$$\frac{\partial P}{\partial t} + \frac{1}{2} f^2(Y_0) X^2 \frac{\partial^2 P}{\partial X^2} + (r-c) X \frac{\partial P}{\partial X} - rP = 0. \tag{5.57}$$

In order relate this equation to the Hamilton-Jacobi-Bellman equation we set $\phi = e^{rt} P$ and observe that ϕ satisfies the Hamilton-Jacobi-Bellman equation associated with the given problem. This is in agreement with the fact that the option price at time t_0 is $e^{-rt_0}\phi(t_0)$ as a consequence of our definitions. ψ does not enter in the equation for ϕ in this situation.

In addition we have to take into account the boundary conditions prescribed by the particular derivative. In the following we are going to analyze a European Call option with strike price S and maturity date T. The appropriate boundary condition is

$$P(T, X_T) = \max(X_T - S, 0).$$

Changing the variable X and by "variation of constants" the equation can be transformed to the *heat equation*

$$\frac{\partial z}{\partial t} + \eta \frac{\partial^2 z}{\partial x^2} = 0,$$

where explicit solutions are known.

The following function is an explicit solution to the extended Black-Scholes equation:

$$\begin{aligned} P_t(t, X, Y_0) &= X e^{-c(T-t)} N(d_1) - S e^{-r(T-t)} N(d_2) \\ d_1 &= \frac{\log(X/S) + (r - c + \frac{1}{2} f^2(Y_0))(T-t)}{f(Y_0)\sqrt{T-t}} \\ d_2 &= = d_1 - f(Y_0)(T-t) \end{aligned} \tag{5.58}$$

where $N(x)$ is the cumulative function of a normal distribution $\mathcal{N}(0,1)$

$$N(x) \;=\; \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

The explicit formula of the call price is expressed in terms of the current price X, the strike price S, the volatility $f(Y_0)$, the constant interest rates r, the dividend yield c and the time to maturity $T-t$. Note that in case of complete markets, the no-arbitrage price of the derivative does not depend on μ.

We can compute the solution numerically in a bounded computational domain using the software GASCOIGNE and compare the result with the explicit global solution. The difference is of such a small size that its visualization in the following figures needs an interpretation. Color blue represents the graph of the explicit solution, red the graph of the numerical approximation. If in discretisation pixels cannot be discriminated the color of the pixel closed to the observer is visualized. We also plotted the relative error in the region of practical interest.

Furthermore, we remark that increasing the volatility the error increases if we keep the computational domain fixed. Scaling the domain properly will reduce the approximation error.

Setting the coefficients a and b in (5.56) equal to zero enables us to test the quality of the applied algorithm in the complete market setting. In addition, we suppose the following parameter values: $\mu = 0.1, r = 0.05, c = 0.02$ and the strike price $S = 75$.

Figures 5.2 and 5.3 illustrate the evolution of the option price on the left in dependence of the auxiliary process Y at $\tau = 1$ and on the right in dependence of the time to maturity τ at a fixed Y for a range of values of oil prices $X \in [40, 100]$. For a better illustration of the quality of the numerical approximation we plot a smaller part of the computational domain $\Omega = [0,1] \times [0,200] \times [-10,10]$. Figure 5.3 zooms into the neighborhood of the strike price S for different Ys and τs. The according relative approximation errors are demonstrated in Figure 5.4. In the domain, where the price is close to zero, the error is high. However in the region of interest it is reasonable low. The mean absolute relative approximation error for the chosen examples is 0.0020 on the left and 0.0017 on the right.

Summary 1.

The numerical methods suggested to solve the pricing problem lead in the special case of the Black-Scholes situation to very good agreement of the obtained approximations with explicit solutions available for this special case. Increasing the volatility by a factor d *may require enlarging the computational domain in space by a factor d, to get the same quality of approximation.*

Having positively tested the quality of the numerical methods in a special case, we are prepared to study the influence of incompleteness on the option prices.

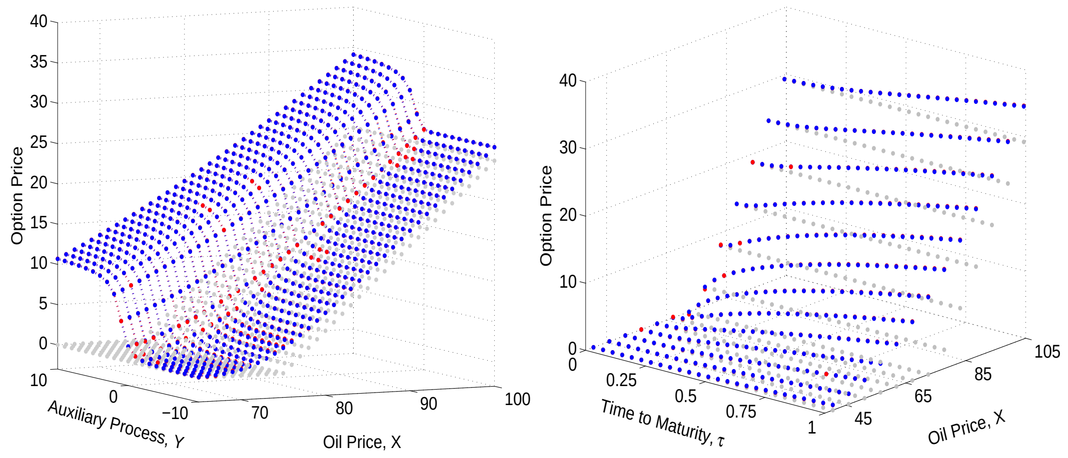


Figure 5.2: Explicit Solution (blue) versus Numerical Approximation (red) for the Black-Scholes Case. *We consider a European Call option at $\tau = 1$ on the left and $Y = 10$ on the right.*

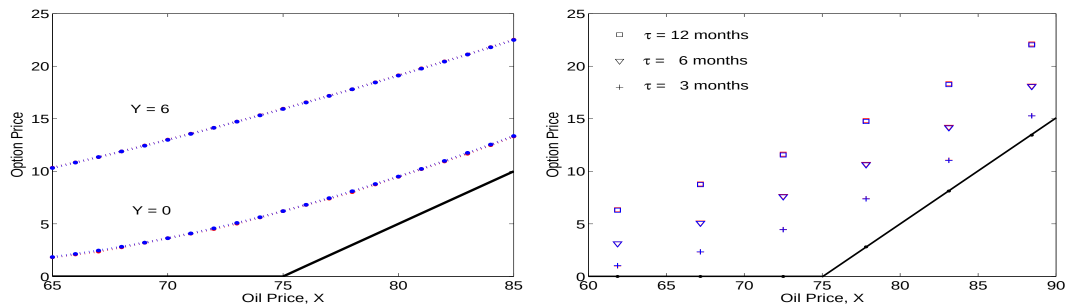


Figure 5.3: Zooming of Figure 5.2. *This figure illustrates 5.2 in the neighborhood of the strike price. This zooming is done for a better assess of the quality of the numerical approximation.*

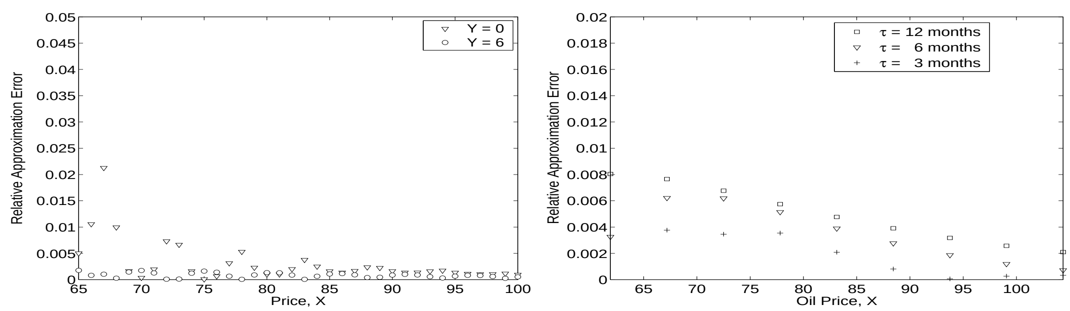


Figure 5.4: Relative Approximation Errors

5.4.2 Complete vs Incomplete

Incompleteness results from random perturbations of the auxiliary process Y which can not be traded. The correlation between the price of a commodity X and the auxiliary process Y is measured by ρ. In general, the correlation may depend on time $\rho(t) \in [-1, 1]$, but we will assume that ρ is constant, which is the case quite often in practice. For equity markets a lower stock price decreases the value of equity relative to the debt, thereby increasing the leverage of the firm and accordingly the risk of holding the stock. As a result, volatilities in the stock market go up, when prices go down. This relationship is called “leverage effect” and goes back to Black (1976).

In contrast to equity markets, we assume that the correlation coefficient of the oil price and its volatility is positive. This reflects that the volatility is tending to high values when spot prices are high. In this case supplies and inventories are tending to be scarce. Thus, the arrival of new supply or demand information may have significant effect on prices. This *“inverse leverage effect”* is found in empirical studies for a large number of commodities such as oil, gas and soybeans (see Richter and Sørensen (2002) and Geman (2005)).

We want to illustrate the effect of incompleteness introducing stochastic volatility in the linear case, e.g. allowing for $a \neq 0$. We compare this new situation with the former one where $a = 0$ holds. Since Y_t is a stochastic process, $f(Y)$ is fluctuating between the lower and upper boundaries, chosen here as 0.05 and 0.55. In case of the complete market we select as volatility $C_0 \cdot X$. Comparing both approaches, we expect that for $C_0 > f(0)$ the price p_2 for a Call option of a stochastic volatility model is lower than the price p_1 for the complete case , and for $C_0 < f(0)$ the price p_2 is larger than p_1. A numerical example is given in figure 5.5.

We have chosen the following parameter:

Model Parameter for M1.

μ	0.1	ρ	0.25	r	0.05
a	0.5	b	-5	c	0.02

Summary 2.

Numerical tests support the conjecture: $(p_1 - p_2)(C_0 - f(0))$ *is nonnegative.*

Risk Aversion

In the following we want to give a numerical example of the dependence on the risk aversion parameter α. The following limit results for the risk aversion are proven by

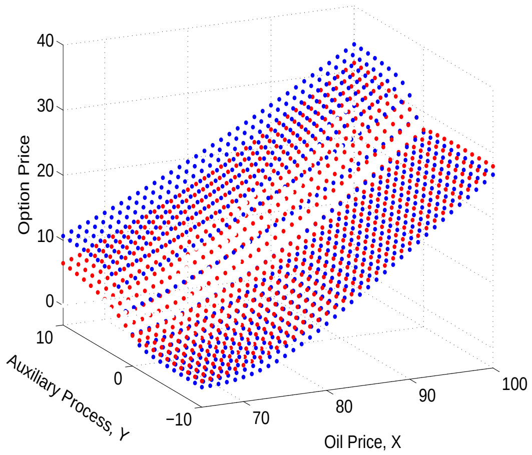


Figure 5.5: Constant versus Stochastic Volatility.

Becherer (2001) and Delbaen, Grandits, Rheinländer, Samperi, Schweizer, and Stricker (2002).

Theorem 12 (Risk Aversion Asymptotics and Monotonicity in α)**.**

- $\lim_{\alpha \uparrow \infty} p(\alpha, B) = \sup_{Q \in \mathbb{P}_e} \mathbb{E}^Q\{B\}$,

- $\lim_{\alpha \downarrow 0} p(\alpha, B) = \mathbb{E}^{Q^E}\{B\}$,

- *Since $H(Q|P) \geq H(Q^E|P)$ for any $Q \in \mathbb{P}_f$ by definition of Q^E, $p(\alpha, B)$ is increasing in α.*

In figure 5.6 we plot the dependence of the option price on the risk parameter α. We choose α in an interval 0.01 and 150. In the literature, however, α is generally chosen between zero and one (see e.g. Davis and Zariphopoulou (1993), Clewlow and Hodges (1997) and Monoyios (2004)). Users may feel unable to specify the required risk aversion coefficient, another reason why they may not be satisfied with the concept of utility functions. However, as seen in our simulations, the result for the indifference price is rather insensitive with respect to the risk aversion factor, as long as it is not to large. Moreover, quantifying risks can not be avoided in general.

Summary 3.

The numerical results illustrate the monotonicity of the indifference price with respect to the risk parameter α as claimed in the theorem 12. The following observation can be made: the changes of the price are rather small for α not too large (up to approximately 10 in our test cases and for a single unit).

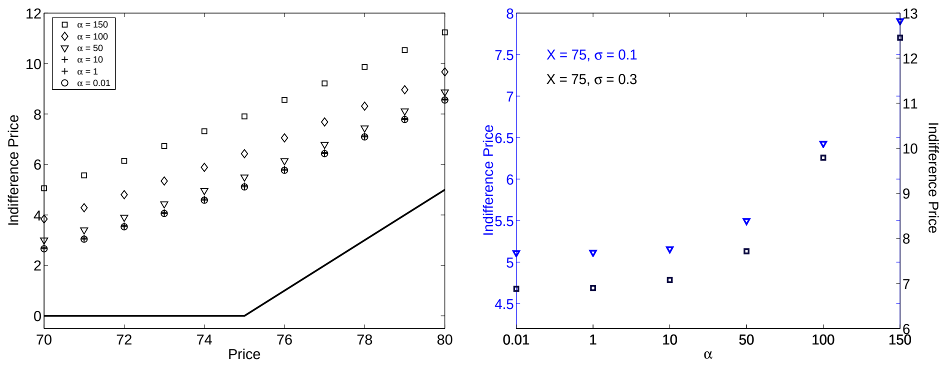


Figure 5.6: Risk Aversion Asymptotics and Monotonicity in α. *These figures demonstrate the dependence of the indifference price on the risk aversion parameter α for a fixed $Y = -6$ (left hand) and fixed $Y = \{-6, 1\}$ and $X = 75$ (right hand).*

Volume-Scaling

Due to the fact that the Hamilton-Jacobi-Bellman system contains nonlinearities we expect nonlinear dependence of the option price on the volume of the option. The following scaling formula is an easy consequence of the representation

$$p(\widetilde{B}) = \sup_{Q \in \mathbb{P}_f(P)} \left(\mathbb{E}^Q\{\widetilde{B}\} - \frac{1}{\alpha} H(Q|P) \right) - \sup_{Q \in \mathbb{P}_f(P)} \left(-\frac{1}{\alpha} H(Q|P) \right)$$

obtained in 5.16.

Lemma 2 (Volume-Scaling)**.**

$$p(nB, \alpha) = np(B, n\alpha) \geq np(B, \alpha)$$

for $n \in (0, 1]$ and, if B is bounded, the previous equation holds for $n \in (0, \infty)$.

Summary 3 and this scaling law suggest the conjecture

$$p(B, n\alpha) \approx p(B, \alpha)$$

if $n\alpha$ is smaller than approximately 10 in our situation.

These facts suggest the following definition.

Definition 12 (Volume-Risk Aversion)**.**

$$VRA := n \cdot \alpha$$

where n is the number of units of stocks treated in the contract and α measures the risk aversion per unit.

Summary 4.

If the VRA is small, the indifference price is linear in the volume. This fails for large VRA . The (seller) price is getting super additive with respect to volume.

According to our agreement the price p represents the sellers price p^s. Recalling the relation to the buyers price p^b:

$$p^b(nB,\alpha) = -p^s(-nB,\alpha)$$

and using the formula of the lemma 2 we obtain:

$$\begin{aligned} p^b(nB,\alpha) &= -p^s(-nB,\alpha) \\ &= -np^s(-B,n\alpha) \leq -np^s(-B,\alpha) \\ &= np^b(B,\alpha). \end{aligned}$$

The *super-additivity of the sellers price* transforms to *sub-additivity of the buyers price*. This is in agreement with the fact: A seller requires more than twice the price for taking on twice the risk. The investor is not willing to pay twice as much for twice as many options, but requires a reduction in this price to take on the additional risk. The figures 5.7 - 5.9 show the influence of a growing volume. They are results of simulations for the Hamilton-Jacobi-Bellman system associated to the model equations M1. The previously chosen parameters are used with exception of the risk aversion parameter which is taken now significantly higher setting $\alpha = 10$.

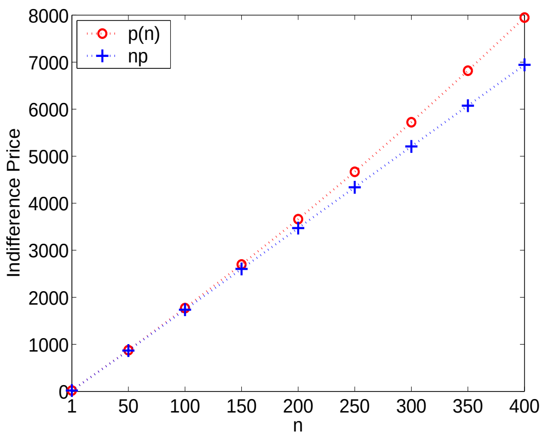


Figure 5.7: Volume-Scaling. *This figure illustrates the effect of volume-scaling comparing $np(B,\alpha)$ dotted in blue versus $p(nB,\alpha)$ marked in red.*

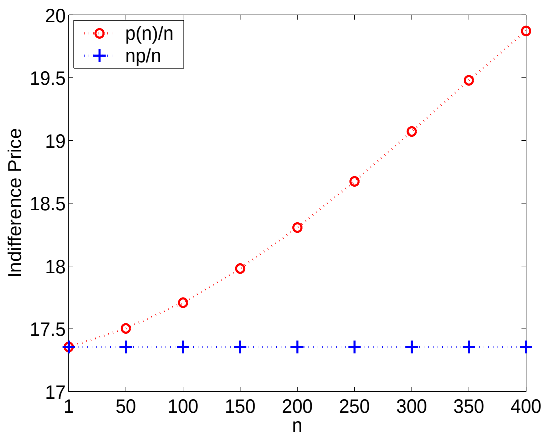


Figure 5.8: Volume-Scaling per unit, *that means we plot $p(B,\alpha)$ and $p(nB,\alpha)/n$ in blue and red respectively.*

Figure 5.7 illustrates that the price is approximately linear in the volume for small volume-risk aversion. Also for small volume-risk aversion, the bid and ask prices are approximately the same as shown in figure 5.10. In figure 5.8 we observe, the average seller price for n volumes in one options is strictly increasing with the number of volumes. The size of the option price interval increases with the volatility. Figure 5.9 demonstrates this effects for bid and ask prices by visualizing the results for different values of Y, controlling the volatility.

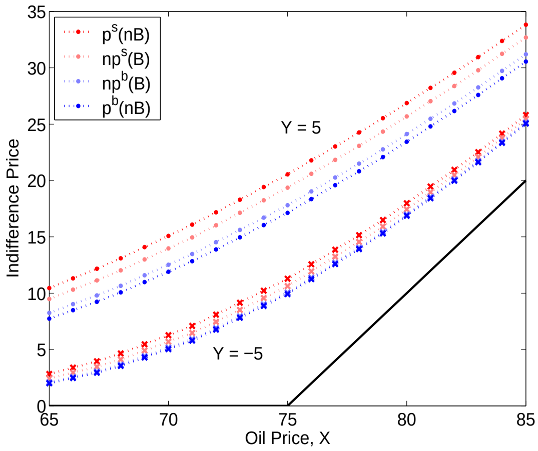


Figure 5.9: Bid-Ask Spread, Volume Scaling. *This figure describes the buy and sell price computed for a one year European Call option.*

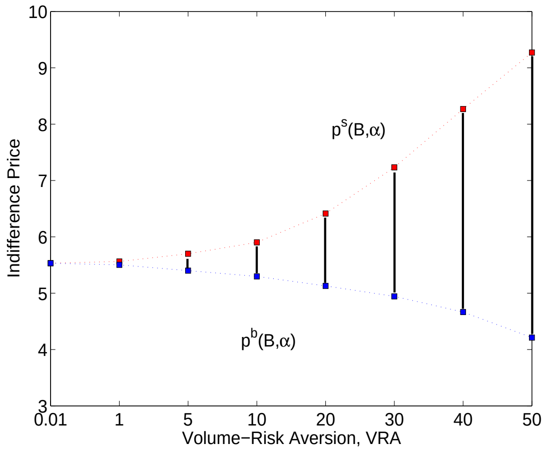


Figure 5.10: Bid-Ask Spread, Volume-Risk Aversion. *This figure illustrates the bid-ask spread for $X = 75$ and $Y = 0$ in dependence of the volume-risk aversion VRA $= n\alpha$.*

5.4.3 Linear vs Nonlinear Mean Reversion

It is widely assumed that commodity prices exhibit a mean reversion. Bessembinder, Coughenour, Seguin, and Smoller (1995) present empirical evidence in case of agricultural commodities, crude oil or various metals for this behavior. In 1997 Pilipovic proposed a linear mean reverting process which is popular in the energy industry. For a more detailed description and comments we refer to chapter 2. Here, we are going to analyze the following two modifications of the Pilipovic model:

- **Stochastic volatility**

$$\begin{aligned} \mathbf{M2}: \qquad dX &= \kappa(X)(X_1 - X)dt + f(Y)XdW^1 \\ f(Y) &= \frac{1}{2\pi}\arctan(Y-1) + 0.3 \\ dY &= bYdt + a(\rho dW^1 + \hat{\rho} dW^2), \end{aligned}$$

- **Multiple equilibria**

$$\mathbf{M3}: \qquad dX = \kappa(X)(X_1 - X)(X_2 - X)(X_3 - X)dt + f(Y)XdW^1.$$

Here we are especially interested in nonlinear drift terms, particulary in nonlinearities like polynomials of degree 3 with 1 or 3 real zeros, such that the corresponding deterministic dynamics do not explode or become zero. However we have to know that under the influence of a stochastic perturbation the situation is preserved.

Applying Girsanov transformation $dW^{\lambda,1} = dW^1 + \gamma dt$, the discounted gain process is a martingale, the drift term of dG under Q^E is zero. In contrast to the previous case of a geometric Brownian motion, X does not cancels out. Thus, in the neighborhood of $X = 0$, we run into troubles. In order to avoid this problem, we introduce a mean reversion rate

$$\kappa(X) = \begin{cases} \frac{k}{\varepsilon} X & X < \varepsilon \\ k & X \geq \varepsilon. \end{cases} \tag{5.59}$$

Taken into account that in case of the crude oil markets, X_0 is far above zero and the attractive forces of the long run equilibrium level, this assumption (5.59) is rather a technical condition.

In order to compare the linear and nonlinear approach, we assume that the speed of adjustment towards a long run equilibrium is in both cases the same. This speed is given by derivative of μ in the equilibrium, in the special case considered here, we obtain

$$\kappa_{Linear} \approx \kappa(X_{j+1} - X_j)(X_{j+2} - X_j), \text{ where } X_4 = X_1. \tag{5.60}$$

In the nonlinear case, the relative position of the steady states influences their stability. Thus, we have to adjust the size of κ in the context of multiple equilibria. Taking into account this remarks, we are setting $\kappa_{Linear} = 0.8$ and $\kappa = 0.008$. The remaining parameters are left unchanged.

It is a standard technique to approximate nonlinear systems locally by linearization. In the nonlinear approach we are faced with two stable attractors, if $X_1 < X_2 < X_3$ and constant X_1 and X_3. We assume that prices are fluctuating between these two equilibria. However, their attraction may keep the price in the neighborhood, thus avoiding a jump to an other equilibrium. In this situation, the system could be locally approximated by a linear equation. In order to illustrate the effect of multiple equilibria we have avoid this situation by the proper choice of parameters. Using the parameters of the following table, we compute the indifference price and hedging strategies for the linear and nonlinear price dynamics of commodities (see figure 5.11-5.13).

Parameter for M2 and M3: Linear versus Nonlinear Drift.

	Linear	Nonlinear		Linear	Nonlinear
κ	0.8	0.008	a	0.2	0.2
X_1	75	65	b	-5	-5
X_2	–	75	ρ	0.25	0.25
X_3	–	85	α	0.01	0.01
r	0.05	0.05	c	0.03	0.03

Figure 5.11 and 5.12 illustrate the price of a European Call option with time to maturity of one year. Figure 5.12 plots the price difference resulting from assuming different underlying price processes. The price dynamics leads to an option price which is plotted in the linear (blue) and in the nonlinear case (red). Following theorem 11, the dynamic hedging strategy is simply computed from the derivatives of ϕ with respect to x and y. This is done numerically. The result is plotted in figure 5.13.

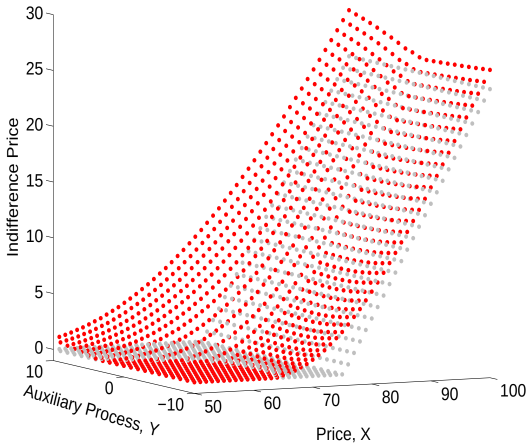


Figure 5.11: Nonlinear Oil Price Dynamics. *This figure shows the price of a European Call option with one year time to maturity. The underlying dynamics is described by M3, where the nonlinear drift term is crucial for the price dynamics.*

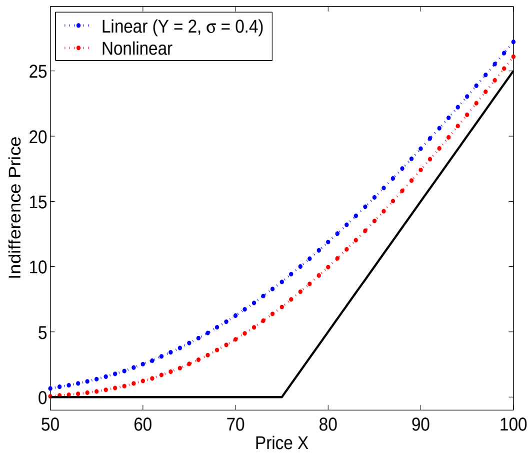


Figure 5.12: Indifference Price for Linear/Nonlinear Price Dynamics. *This figure gives a numerical example taking into account the effect of the nonlinearity on the option price.*

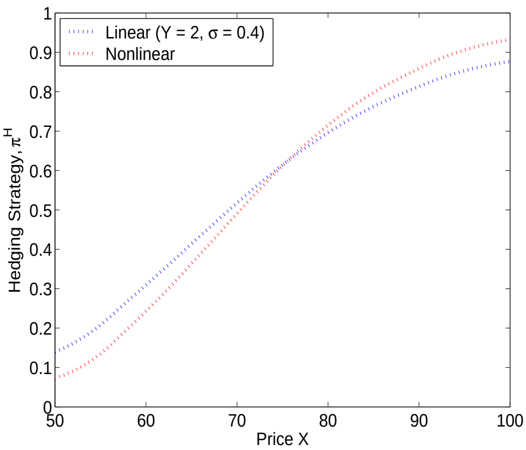


Figure 5.13: Hedging Strategy π^M *for the linear and nonlinear case at time $\tau = 1$.*

Summary 5.

We are able to evaluate derivative contracts on commodities like oil or gas where the underlying price dynamics exhibit a generalized mean reversion process taking into account multiple quasi-steady states and stochastic volatility.

The shape of the option price is essentially determined by the terminal and boundary conditions, in particular the random payoff structure of the underlying contract. Our numerical example demonstrates that the nonlinearities have essential effects on quantitative relations. The price for a European Call option based on the suggested nonlinear drift term is consistently lower than the price for a linear approach. E.g. the mean absolute percentage deviation is around 10 percent. This price difference leads immediately to different hedging strategies.

These simulation results are obtained assuming that the long run equilibria are constant. However, in general we will have quasi-steady states, slowly changing in time, but strongly influencing the dynamics. We are very well aware of the fact that our results might dependent on this simplifications. Locally in time, the approximation will be justified. The most important contribution given by this analysis is the better understanding of the effects to be expected.

Chapter 6

Conclusion

André Kostolany (1906 – 1999), world-renowned stock market expert and speculator, successful due to an enormous practical experience accumulated over decades, expresses a general attitude towards a more rational approach to real markets:

"At the stock exchange the feeling often says to us, what for we make, and the understanding, what we are to avoid".

"An der Börse sagt uns oft das Gefühl, was mir machen, und der Verstand, was wir vermeiden sollen".

Mathematical modeling and simulation is a rational approach to a better understanding of real processes, and in general an approximation to reality, which can help to analyze consequences of our assumptions on reality. For instance, by helping to identify decisions and actions with high risk. Computational economics and finance is providing tools not just for scientific analysis but also for decision making and planning in real markets.

In this thesis we used nonlinear, stochastic dynamical system as a main tool for modeling and simulating markets of assets and commodities. We followed as guideline the idea that the interaction of nonlinear and stochastic effects is responsible for the evolution and structure formation also in economic and financial markets. Choosing the price dynamics of the oil market as test case we were forced to reduce the model to a very simplified one, which still could describe the main features of the system. A balance has to be achieved between a more detailed modeling of all factors, especially the economic ones, and size and complexity of the model system, where all necessary data will be accessible and which is computable with available computational tools in an acceptable time. We developed a setting such that the model system can be enlarged and specified if the situation requires including more and more detailed information. Despite the simplifications, which also could have proven to be an oversimplification, the models lead to simulations showing very satisfying agreement with real data. Whereas the modeling could have been done in more detail, the main limitations is caused by the lack of data and proper tools for estimate the systems parameters, the size and the scales of the systems leading to computational challenges, which up to now only partially could be mastered. The following central mathematical problems are to develop

- numerical methods in solving nonlinear partial differential in higher dimensions,
- statistical methods to estimate the systems parameters, here especially estimating volatilities,
- methods to determine from real market data quantitatively validation, necessary to tread incomplete markets and to achieve a rational pricing,
- methods to reduce complex systems, e.g. modeling the demand and supply.

Solutions to these problems will allow treading more realistic models. So far, in computing option prices we were limited to a model simpler as the one developed in chapter 4. In an ongoing investigation jointly with Reisinger (Oxford), we are developing algorithms to solve the Hamilton-Jacobi-Bellman system in higher dimension including the model for price, supply and demand, developed in this thesis. Analyzing markets for commodities in general, one is forced to include more economic, technological and political information. E.g. it is necessary not just to consider a general stochastic "background" process influencing a price dynamics, but to try to get to a better understanding of the real processes involved. Here, we tried to give contributions in this direction: Whereas in the case of exchange rates we included directly economic data in the model equation, we modeled main factors for the price dynamics in case of the oil market. We consider our investigation as an example for research necessary in general for other important commodities. It is clear that we are in general confronted with incomplete markets leading to the difficulties we experienced here. We believe that especially research in this field and the transfer of theoretical achievements to real life applications has to be promoted.

Emphasizing the importance of interplay between stochastic and nonlinear effects, we tried to show that the effects of dynamics of real processes can be understood which otherwise could not be explained. E.g. the phenomena of rare events can at least partially be explained as rapid switching between (quasi-) stationary states, rather distant in phase space and different in stability. Here the mathematical theory mentioned in section 2.4 and presented in the monograph Skorokhod, Hoppensteadt, and Salehi (2002) in an other context becomes important. It provides a proper basis for more general investigations of economic and financial processes, which from a modeling point of view could be considered as a huge nonlinear and stochastic dynamical system. Concepts of statistical physics and molecular modeling, developed for huge networks of interactions, could help analyzing and simulating these model systems. We hope that our investigations could help to achieve a better understanding of the oil price dynamics in particular, and due to the portability of the methods also of commodity markets in general. The following topics

- extensions of the reduced model for the oil model, more detailed modeling of the demand and supply, especially by analyzing the technological development, by determining and quantifying indicators for factors for the market,

- analysis of development of new oil fields, of new alternative energy resources,
- modeling of gas markets,
- derivation of reduced models starting from multi-agent models for complex markets,
- optimization of strategies e.g. like the strategy of OPEC in producing and selling oil,
- optimization of strategies to invest in commodities in general, in oil especially.

If real processes can be approached by rational concepts at all, if they follow some rules, then it should be possible, to formulate their behavior in basic model equation, to describe and simulate their dynamics with mathematical and computational tools. However, the belief in the feasibility of this concept has to be complemented by clear understanding of its limitations.

Bibliography

ADELMAN, M. (1982): "OPEC as a Cartel," in *OPEC Behaviour and World Oil Prices*, ed. by J. Griffin, and M. Teece, pp. 37 – 63, London. George Allen and Unwin.

AIT-SAHALIA, Y. (1999): "Transition Densities for Interest Rate and Other Nonlinear Diffusions," *Journal of Finance*, 54, 1311 – 1395.

AÏT-SAHALIA, Y. (2002): "Maximum-Likelihood Estimation of Discretely-Sampled Diffusions: A Closed-Form Approximation Approach," *Econometrica*, 70, 223 – 262.

BARD, Y. (1974): *Nonlinear Parameter Estimation*. Academic Press, NY.

BARLOW, M. (2002): "A Diffusion Model for Electricity Prices," *Mathematical Finance*, 12, 287 – 298.

BAUER, I. (2001): "Numerische Verfahren Zur Lösung Von Anfangswertaufgaben und Zur Generierung Von Ersten und Zweiten Ableitungen mit Anwendungen in Chemie und Verfahrenstechnik," Ph.D. thesis, Universität Heidelberg.

BECHERER, D. (2001): "Rational Hedging and Valuation with Utility-Based Preferences," Ph.D. thesis, Technical University of Berlin, e-published: edocs.tu-berlin.de/diss/2001/becherer-dirk.htm.

——— (2004): "Utility Indifference Hedging and Valuation Via Reaction Diffusion Systems," *Proceedings of the Royal Society*, 460, 27 – 51.

BERNOULLI, D. (1738): "Specimen Theoriae Novae de Mensura Sortis," in *Commentarii Academiae Sientiarum Imperialis Petropolitanae*, pp. 175–192. Translated by L. Sommer (1954): Econometrica 22, 23-36.

BESSEMBINDER, H., J. F. COUGHENOUR, P. J. SEGUIN, AND M. M. SMOLLER (1995): "Mean Reversion in Equilibrium Asset Prices: Evidence from the Futures Term Structure," *Journal of Finance*, 1, 361 – 375.

BILSON, J. F. (1978a): "Rational Expactations and the Exchange Rate," in *The Economics of Exchange Rates*, ed. by J. Frankel, and H. Johnson, pp. 75 – 96. Addison Wesley.

——— (1978b): "The Monetary Approach to the Exchange Rate - Some Empirical Evidence," *IMF Staff Papers*, 25, 48 – 75.

BLACK, F. (1976): "Studies of Stock Market Volatility Changes," *Proceedings of the American Statistical Association, Business and Economic Statistics*, pp. 177–181.

BLACK, F., AND M. SCHOLES (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637 – 654.

BOCK, H. (1983): "Recent Advances in Parameter Identification Techniques for O. D. E.," in *Numerical Treatment of Inverse Problems, Progress in Scientific Computing*, ed. by P. Deuflhard, and E. Hairer, pp. 95 –121. , Birkhäuser: Boston.

——— (1987): "Randwertproblemmethoden Zur Parameteridentifizierung in Systemen Nichtlinearer Differentialgleichungen," *Bonner Mathematische Schriften*, 183(183).

BOCK, H., E. KOSTINA, AND O. I. KOSTYUKOVA (2004): "Covariance Matrices for Constrained Parameter Estimation Problems," *SIAM Journal on Matrix Analysis and Applications*.

BOCK, H., E. KOSTINA, AND J. SCHLÖDER (2000): "On the Role of Natural Level Functions to Achieve Global Convergence for Damped Newton Methods," in *System Modelling and Optimization: Methods, Theory and Applications*, ed. by Powell, and Scholtes. Kluwer.

BOROVKOVA, S., H. DEHLING, J. RENKEMA, AND H. TULLEKEN (2003): "A Potential Function Approach to Commodity Prices Modelling," *Computational Economics*, 22, 139 –161.

BRANDT, M., AND P. SANTA-CLARA (2002): "Simulated Likelihood Estimation of Diffusions with an Application to Exchange Rate Dynamics in Incomplete Markets," *Journal of Financial Econometrics*, 63, 161 – 210.

BRENNAN, M. (1958): "The Supply of Storage," *American Economic Review*, 47, 50 – 72.

BRENNAN, M., AND E. SCHWARTZ (1985): "Evaluating Natural Resource Investments," *The Journal of Business*, 58, 135–157.

BRENNAN, M. J. (1991): "The Price of Convenience and the Valuation of Commodity Contingent Claims," in *Stochastic Models and Option Values*, ed. by D. Lund, and B. Øksendal. North-Holland, Amsterdam.

CAMPBELL, C. (1997): *The Coming Oil Crisis*. Multi-Science Publishing Company and Petroconcusltants.

CARTEA, A., AND M. FIGUEROA (2005): "Pricing in Electricity Markets: A Mean Reverting Jump Diffusion Model with Seasonality," *Applied Mathematical Finance*, 12, 313 – 335.

CHISSOM, B. (1970): "Interpretation of the Kurtosis Statistic," *The American Statistician*, pp. 19 – 23.

CLEWLOW, L., AND S. HODGES (1997): "Optimal Delta-Hedgin under Transaction Costs," *Journal of Economic Dynamics and Control*, 21, 1353 – 1376.

COBB, L. (1978): "Stochastic Catastrophe Models and Multimodal Distriubtions," *Behavioural Science*, 23, 360 – 374.

COBB, L., AND S. ZACK (1985): "Applications of Catastrophe Theory for Statistical Modelling in the Biosciences," *Journal of the American Statistical Association*, 80, 793 – 802.

CONN, A., N. GOULD, AND P. TOINT (2000): "Trust Region Methods," *SIAM*.

COX, J., J. INGERSOLL, AND S. ROSS (1985): "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385 – 407.

CREEDY, J., J. LYE, AND V. L. MARTIN (1996): "A Non-Linear Model of the Real US/UK Exchange Rate," *Journal of Applied Econometrics*, 11, 669 – 686.

CREMER, J., AND D. SALEHI-ISFAHANI (1989): "The Rise and Fall of Oil Prices: A Competitive View," *Annales d'economie et de statistique*, (15/16), 427 – 454.

CREMER, J., AND D. SALEHI-ISFAHANI (1991): *Models of the Oil Market*. Hardwood Academic Publishers.

DARGAY, J., AND D. GATELY (1995): "The Imperfect Price Reversibility of Non-Transport Oil Demand in the OECD," *Energy Economics*, 17, 59 – 71.

DARLINGTON, R. (1970): "Is Kurtosis Really Peakedness?," *The American Statistician*, pp. 19 – 22.

DAVIS, M.H.A., P. V., AND T. ZARIPHOPOULOU (1993): "European Option Pricing with Transaction Costs," *SIAM Journal of Control and Optimization*, 31, 470 – 493.

DELBAEN, F., P. GRANDITS, T. RHEINLÄNDER, D. SAMPERI, M. SCHWEIZER, AND C. STRICKER (2002): "Exponential Hedging and Entropic Penalties," *Mathematical Finance*, 12, 99 –123.

DORNBUSCH, R. (1980): "Exchange Rate Economics: Where Do We Stand?," *Brookings Papers on Economic Activity*, 1, 143 – 185.

DUFFIE, D., S. GRAY, AND P. HOANG (2002): *Volatility in Energy Prices*chap. 2, pp. 38 – 55. Managing Energy Price Risk, 2nd Ed., Robert Jameson ed., Risk Publications, London.

ELERIAN, O., S. CHIB, AND N. SHEPHARD (2001): "Likelihood Inference for Discretly Observed Non-Linear Diffusion," *Econometrica*, 69, 959 – 993.

ERAKER, B. (2001): "MCMC Analysis of Diffusion Models with Application to Finance," *Journal of Business and Economic Statistics*, 19, 177 – 191.

FLEMING, W., AND H. SONER (2006): *Controlled Markov Processes and Viscosity Solutions, Second Edition*. Springer-Verlag, New York, Heidelberg, Berlin.

FÖLLMER, H., AND M. SCHWEIZER (1991): "Hedging of Contingent Claims under Incomplete Information," in *Applied Stochastic Analysis. Stochastics Monographs*, ed. by M. H. A. Davis, and R. J. Elliott, pp. 389 – 414. Gordon and Breach.

——— (1993): "A Microeconomic Appraoch to Diffusion Models for Stock Prices," *Mathematical Finance*, 3, 1 – 23.

FÖLLMER, H., AND D. SONDERMANN (1986): "Hedging of Non-Redundant Contingent Claims," in *Contributions to Mathematical Economics*, ed. by W. Hildenbrand, and A. Mas-Colell, pp. 205 –223. North-Holland.

FOUQUE, J., G. PAPANICOLAOU, AND K. SIRCAR (2000): *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press.

FRENKEL, J. A. (1976): "A Monetary Approach to the Exchange Rate: Doctrinal Aspects and Empirical Evidence," *Scandinavian Journal of Economics*, 78, 200 – 224.

——— (1978): "Purchasing Power Parity: A Doctrinal Perspective and Evidence from 1920s.," *Journal of International Economics*, 8, 169 – 191.

FRITTELLI, M. (2000): "The Minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets," *Mathematical Finance*, 10, 39 – 52.

GATELY, D. (1992): "Imperfec Price-Reversibility of Oil Demand: Asymmetric Response of US Gasoline Consumption to Price Increases and Declines," *The Energy Journal*, 13, 179 – 207.

GAUSS, C. F. (1805): *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium (Orignial with Translation)*. Hamburg.

——— (1995): "Theory of Combinations of Observations Least Subject to Errors, Original with Translation," *SIAM*, Philadelphia.

GEMAN, H. (2005): *Commodities and Commodity Derivatives: Modeling and Pricing for Agriculturals, Metals, and Energy*. John Wiley & Sons Ltd.

GIBSON, R., AND E. SCHWARTZ (1990): "Stochastic Convenience Yield and the Pricing of Oil Contingent Claims," *Journal of Finance*, 45, 959–976.

GRANDITS, P., AND T. RHEINLÄNDER (2002): "On the Minimal Entropy Martingale Measure," *Annals of Probability*, 30, 1003 – 1038.

GRIFFIN, J., AND M. TEECE (1982): *OPEC Behaviour and World Oil Prices*. Geoge Allen and Unwin, London.

HASSET, K., AND G. METCALF (1995): "Investment under Alternative Return Assumptions: Comparing Random Walks and Mean-Reversion," *Journal of Economic Dynamics and Control*, 19, 1471 – 1488.

HOBSON, D. (2004): "Stochastic Volatility Models, Correlation, and the Q-Optimal Measure," *Mathematical Finance*, 4, 537 –556.

HODGES, S., AND A. NEUBERGER (1989): "Optimal Replication of Contingent Claims Under Transaction Costs," *Rev. Futures Markets*, 8, 222 – 239.

HORENKO, I., E. DITTMER, A. FISCHER, AND C. SCHÜTTE (2006): "Automated

Model Reduction for Complex Systems Exhibiting Metastability," *Multiscale Modeling and Simulation*, 5, 802 – 827.

ILHAN, A., M. JONSSON, AND R. SIRCAR (2004): "Portfolio Optimization with Derivatives and Indifference Pricing," in *Forthcoming in Volume on Indifference Pricing*, ed. by R. Carmona. Princeton University Press.

ILHAN, A., AND R. SIRCAR (2006): "Optimal Static-Dynamic Hedges for Barrier Options," *Mathematical Finance*, 16, 359–385.

JÄGER, S. (2006): "Modelling and Simulating Oil Market Dynamics," in *Proceedings of the 29th IAEE International Conference: Securing Energy in Insecure Times*.

JÄGER, S., AND E. KOSTINA (2005): "Parameter Estimation for the Forward Kolmogorov Equation with Application to Non-Linear Exchange Rate Dynamics," *Proceedings in Applied Mathematics and Mechanics.*, 3(4), 123 – 143.

——— (2006): "An Inverse Problem for a Nonlinear Stochastic Differential Equation Modelling Price Dynamics," *Preprint Series Interdisciplinary Center for Scientific Computing and in review for Applied Mathematical Finance*.

JENSEN, B., AND R. POULSEN (1999): "A Comparison of Approximization Techniques for Transition Densities of Diffusion Processes," Discussion paper, Aarhus University.

JUDGE, G., R. HILL, W. GRIFFITHS, H. LUTKEPOHL, AND T. LEE (1998): *Introduction to the Theory and Practice of Econometrics*. New York, Wiley.

KABANOV, Y., AND C. STRICKER (2002): "On the Optimal Portfolio for the Exponential Utility Maximization: Remarks to the Six-Author Paper," *Mathematical Finance*, 12, 125 – 134.

KLOEDEN, P. E., AND E. PLATEN (1992): *The Numerical Solution of Stochastic Differential Equations*. Springer-Verlag.

KRAMKOV, D. O., AND W. SCHACHERMAYER (1999): "The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets," *Annals of Applied Probability*, 9, 904 – 950.

KRUGMAN, P. (2000): "The Energy Crisis Revisited," *Internet: http://web.mit.edu/krugman/www/opec.html*.

LO, A. (1988): "Maximum Likelihood Estimation of Generalized Ito Processes With Discretely Sampled Data," *Econometric Theory*, 4, 231 – 247.

LUND, D. (1993): "The Lognormal Diffusion Is Hardly an Equilibrium Price Process for Exhaustible Resources," *Journal of Environmental Economics and Management*, 25, 235 – 241.

MABRO, R. (1991): "OPEC and the Price of Oil," *The Energy Journal*, 13, 1 – 17.

MACAVOY, P. (1982): *Crude Oil Prices as Determined by OPEC and Market Fundamentals*. Mass: Ballinger, Cambridge.

MACDONALD, R., AND M. P. TAYLOR (1993): "The Monetary Approach to the Exchange Rate," *International Monetary Fund*, 40(1), 89 – 106.

MEESE, R. A., AND K. ROGOFF (1983): "Empirical Exchange Rate Models of the Seventies: Do They Fit Out of Sample?," *Journal of International Economics*, 14, 3 – 24.

MERTON, R. (1973): "Theory of Rational Option Pricing," *Bell Journal of Economics, The RAND Corporation.*, 4, 141 – 183.

METZNER, P., C. SCHÜTTE, AND E. VANDEN-EIJNDEN (2006): "Illustration of Transition Path Theory on a Collection of Simple Examples," *The Journal of Chemical Physics*, 125, 1 – 17.

MILTERSEN, K., AND E. SCHWARTZ (1998): "Pricing of Options on Commodity Futures with Stochastic Term Structure of Convenience Yields and Interest Rates," *Journal of Financial and Quantitative Analysis*, 33, 33 – 59.

MITCHELL, J. (2004): "Petroleum Reserves in Question," in *Sustainable Development Programme, SDB BP 04/03*, http://www.chathamhouse.org.uk/pdf/research/sdp/Mitchell%20SDP.pdf. Chatham House, Oxford Institute for Energy Studies.

MONOYIOS, M. (2004): "Performance of Utility-Based Strategies for Hedging Basis Risk," *Quantitative Finance*, 4, 245 – 255.

MUSSA, M. L. (1976): "The Exchange Rate, the Balance of Payments and Monetary and Fiscal Policy Under a Regime of Controlled Floating," *Scandinavian Journal of Economics*, 78, 229 – 248.

ØKSENDAL, B. (1998): *Stochastic Differential Equations: An Introduction with Applications*. Springer, New York.

PEDERSEN, A. R. (1995): "A New Approach to Maximum-Likelihood Estimation for Stochastic Differential Equations Based on Discrete Observations," *Scandinavian Journal of Statistics*, 22, 55 – 71.

PILIPOVIC, D. (1997): *Energy Risk: Valuing and Managing Energy Derivatives*. McGraw-Hill.

RICHTER, M., AND C. SØRENSEN (2002): "Stochastic Volatility and Seasonality in Commodity Futures and Options: The Case of Soybeans," WP 20024, ISBAN 87-90705-61-0.

SCHIESSER, W. (1991): *The Numerical Method of Lines: Integration of Partial Differential Equations*. Academic, San Diego.

SCHLÖDER, J. (1988): "Numerische Methoden Zur Behandlung Hochdimensionaler Aufgaben der Parameteridentifizierung," *Bonner Mathematische Schriften*, 197.

SCHWARTZ, E. (1997): "The Stochastic Behavior of Commodity Pices: Implications for Valuation and Hedging," *The Journal of Finance*, 52, 923 – 973.

SCHWEIZER, M. (1996): "Approximation Pricing and the Variance-Optimal Martingale Measure," *Annals of Probability*, 24, 206 – 236.

SKOROKHOD, A., F. HOPPENSTEADT, AND H. SALEHI (2002): *Random Perturbation Methods with Applications in Science and Engineering*. Springer, New York, Heidelberg, Berlin.

TAYLOR, M. (1995): "The Economics of Exchange Rates," *Journal of Economic Literature*, 33, 13 – 47.

TAYLOR, M. P., D. A. PEEL, AND L. SARNO (2001): "Nonlinear Mean-Reversion In Real Exchange Rates: Toward a Solution to the Purchasing Power Parity Puzzles," *International Economic Review*, 42(4), 1015 –1042.

VASICEK, O. (1977): "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177 – 188.

ZARIPHOPOULOU, T. (1999): "Optimal Investment and Consumption Models with Nonlinear Stock Dynamics," *Mathematical Methods of Operation Research*, 50, 271–296.