Classical solutions for a thin–film equation

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Diese Arbeit widme ich meinen Eltern, die mein Handeln über Jahre unterstützend begleitet haben.

Contents

Chapter 1

Introduction and summary of the results

In this thesis, we address existence, uniqueness and regularity of a nonlinear free boundary problem of fourth order. The underlying equation is

$$\partial_t h + \partial_y (h \; \partial_y^3 h) = 0. \tag{1.1}$$

It is the simplest example of a class of equations, the so called thin–film equations [31]. These equations model the spreading of a thin viscous film with height $h \geq 0$ on a plane, see Figure 1.1. They are valid in the *lubrication approximation regime* which was introduced by Reynolds in 1886 [48]. In this regime, the propagation of the droplet is driven by surface tension and limited by its viscosity. We consider the case of one spatial dimension. Note that equation (1.1) also models the lubrication approximation of a Darcy flow between the two plates of a Hele–Shaw cell [29]. However, the results we present here are more relevant for the spreading of droplets than for the Hele–Shaw flow, since the solutions are allowed to vanish on a set of non–zero measure.

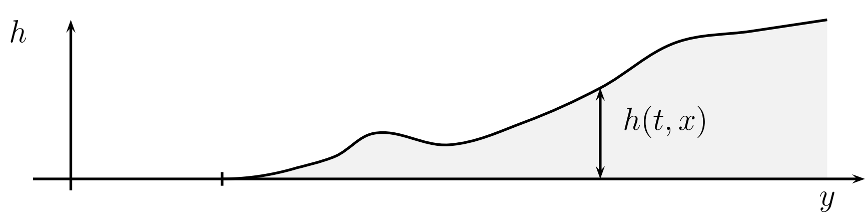

Figure 1.1: Spreading thin film

Equation (1.1) is parabolic in the interior of the *wetted region* $\{h > 0\}$, while the parabolicity degenerates at its boundary. On one hand, this degeneracy in the parabolicity is necessary to keep h non–negative (recall that there is no maximum principle for fourth order parabolic equations). On the other hand, it leads to a hyperbolic behavior of the equation at the moving contact line. As a consequence, solutions of (1.1) show finite speed of propagation of the wetted region [8]. Therefore, (1.1) has to be understood as a free boundary problem: The equation is assumed to hold on the wetted region which itself evolves in time. The mathematical interest lies in the lack of a maximum principle as well as in the degeneracy of the parabolicity. As was first observed by Almgren [1], the evolution is governed by a gradient flow structure. Since the equation does not allow for a maximum principle, our analysis relies on this basic structure.

There exists a well developed theory for weak solutions of (1.1). Long–time existence of weak solutions was shown by Bernis and Friedman [10]. Qualitative properties of solutions also have been investigated, see e.g. [7, 12, 8, 9, 19, 35]. Still, there remain open problems in the framework of weak solutions. One of these questions is the uniqueness of weak solutions. This motivates us to look for a more restricted class of solutions. Indeed, in this work we prove existence and uniqueness for classical solutions of (1.1) if the initial data are a perturbation of the stationary solution. Our classical solutions belong to certain weighted Sobolev spaces: The degeneracy of the parabolicity of (1.1) is reflected by a degeneracy of the weights at the boundary. In fact, these solutions turn out to be smooth for positive times. As a consequence, we obtain smoothness of the free boundary. It should be noted that our existence and uniqueness result only applies for the particular thin–film equation (1.1). Uniqueness for general thin–film equations remains open as well as the case of higher space dimensions. We also consider weighted Hölder spaces. However, our analysis in Hölder spaces is restricted to the analysis of the corresponding linear operator. This is joint work with Giacomelli and Otto, the results have been published as SFB–preprint [28, 27].

The analytical motivation comes from similar work on a related model: The porous media equation is the analogous degenerate parabolic equation of second order. For this equation, there is an existence and regularity theory for classical solutions by Angenent [2, 3], Daskalopoulos and Hamilton [21] and Koch [41].

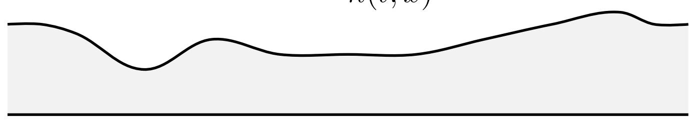

Figure 1.2: Is rupture of the film possible?

In the last chapter of the thesis, we consider the one–dimensional thin–film equation with general mobility h^n where $n \in (0, \infty)$:

$$\partial_t h + \partial_y (h^n \, \partial_y^3 h) = 0. \tag{1.2}$$

The parameter $n \in (0, 3]$ models various slip conditions at the liquid–solid interface [31, 44]. The case $n > 3$ is only of mathematical interest.

We consider a liquid film which initially completely covers the substrate. A natural question to ask is if the model allows for rupture of the liquid film (this is non–trivial since there is no maximum principle for (1.2)). We address a very specific situation: We derive a criterion on the initial data which excludes rupture for nearly flat initial data. The proof is based on energy and entropy estimates for (1.2). It proceeds by a Stampacchia type iterative argument. This is joint work with Giacomelli and already has been published as SFB–preprint [26]. It should be noted that based on a different proof, there exists a similar result by Bertozzi and Pugh [12].

In the following sections of this chapter, we present a summary of our results: In Section 1.1, we formulate (1.1) as a free boundary problem. In Section 1.2 we formally show how (1.2) appears in the lubrication approximation of the Navier–Stokes equations. We briefly review previous results in Section 1.3. In Section 1.4, we formulate a global coordinate transform in order to fix the free boundary of (1.1). Our existence results in Sobolev spaces are stated in Section 1.5. Our results about maximal regularity in Hölder spaces for the corresponding linear operator are presented in Section 1.6. Finally, our non–rupture criterion is presented in Section 1.7.

1.1 Formulation as free boundary problem

In this section, we formulate (1.1) as a free boundary problem. We assume (1.1) to hold on the positivity set:

$$\partial_t h + \partial_y (h\, \partial_y^3 h) = 0 \qquad \text{on } \{h > 0\}. \tag{1.3}$$

At the free boundary $\partial\{h > 0\}$, (1.3) has to be equipped with boundary conditions. For a fourth order parabolic free boundary problem, three conditions are expected for a well–posed problem.

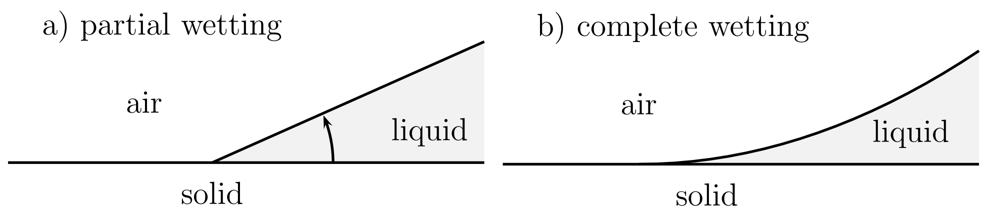

Figure 1.3: Wetting regimes

The first condition is the defining condition for the free boundary:

$$h = 0 \qquad \text{on } \partial\{h > 0\}. \tag{1.4}$$

The second condition models the contact angle of the droplet at the triple point: Since (1.3) describes a quasi–stationary situation, we expect the contact angle to be independent of the speed of propagation. It is described by Young's Law [24, p. 17]: The contact angle is attained by minimization of the surface energy at the triple point between air, liquid and solid. There are two important cases depending on the ratio of the surface tensions between air, liquid and solid (see Figure 1.3): In the *partial wetting regime* the contact angle is non–zero (e.g. water drop on a sheet of plastic). In the *complete wetting regime* the droplet attains a zero contact angle (e.g. water drop on very clean glas). This reflects the fact that the droplet energetically prefers to cover the complete surface. We consider the complete wetting regime:

$$\partial_y h = 0 \qquad \text{on } \partial\{h > 0\}. \tag{1.5}$$

The third boundary condition is induced by the degeneracy of the equation at the free boundary. It is from (1.3) formally clear that the speed V of the free boundary should satisfy

$$V = \partial_y^3 h \qquad \text{on } \partial\{h > 0\}. \tag{1.6}$$

For sufficiently regular solutions, this expression can be justified rigorously (see Lemma 2.12.1). Note that (1.6) allows for spreading of the support as well as for contraction, since $\partial_y^3 h$ may have arbitrary sign at the free boundary.

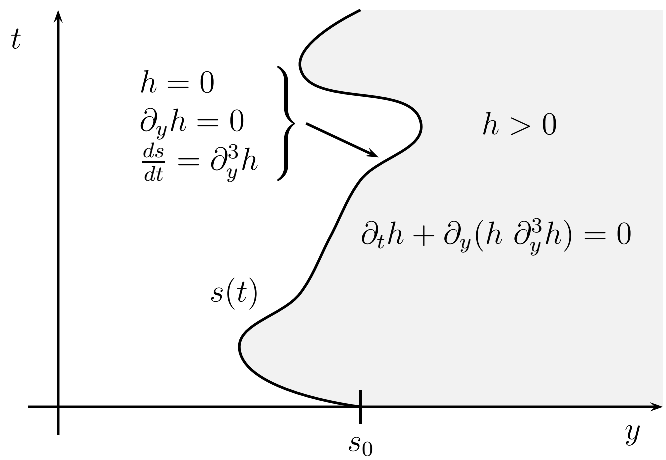

Figure 1.4: Formulation as a free boundary problem

The above considerations result in the following free boundary problem:

$$\begin{cases} \partial_t h + \partial_y (h\, \partial_y^3 h) = 0 & \text{in } \{h > 0\}, \\ h = \partial_y h = 0 & \text{on } \partial\{h > 0\}, \\ V = \partial_y^3 h & \text{on } \partial\{h > 0\}, \\ h = h_0 & \text{on } \{t = 0\}, \end{cases} \tag{1.7}$$

where V describes the speed of propagation, see Figure 1.4.

1.2 Lubrication approximation

In this section, we heuristically show how (1.2) appears as a limit of the Navier–Stokes equations in the regime of lubrication approximation, see [17]. For a rigorous derivation in the special case (1.1) see [29].

Let X (and Z) be the typical horizontal (and vertical) length scale of variations in the physical system, and let T denote the typical time scale. The lubrication approximation is characterized by a separation of length scales in the vertical and horizontal direction, $Z \ll X$, and by a certain time regime, $T \sim (\eta X^4)/(\gamma Z^3)$.

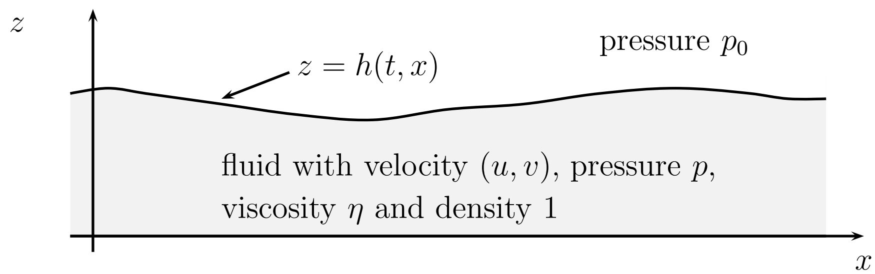

Figure 1.5: The thin film geometry we are looking at

Consider the two–dimensional flow of an incompressible liquid with height $h(t, x)$ and velocity (u, v) on a plate, see Figure 1.5. Horizontal and vertical coordinates are denoted by (x, z). The flow is governed by the 2–d incompressible Navier–Stokes equations

$$\begin{cases} \partial_t u + u\ \partial_x u + v\ \partial_z u = -\partial_x p + \eta\ (\partial_x^2 u + \partial_z^2 u), \\ \partial_t v + u\ \partial_x v + v\ \partial_z v = -\partial_z p + \eta\ (\partial_x^2 v + \partial_z^2 v), \\ \partial_x u + \partial_z v = 0. \end{cases} \tag{1.8}$$

For a capillarity driven flow, the pressure jump at the air–liquid interface is proportional to the mean curvature of the interface: $p_0 - p = \gamma\ \kappa$ (Laplace's Law), see [24, p.7]. In the lubrication regime, this turns into

$$p_0 - p = \gamma\ \partial_x^2 h, \tag{1.9}$$

where p_0 is the constant pressure in the air and γ is the liquid–air surface tension constant.

Taking only the highest order terms into account, the inertia terms in (1.8) can be neglected. To highest order, the pressure is constant in the vertical direction and its variation in the horizontal direction balances the viscous stress generated by $\partial_z^2 u$, i.e. $p = p(t, x)$ and

$$\partial_x p = \eta\ \partial_z^2 u.$$

Hence by (1.9),

$$\eta\ \partial_z^2 u = -\gamma\ \partial_x^3 h \qquad \text{for } 0 < z < h. \tag{1.10}$$

Note that the typical time scale T is chosen to obtain a capillarity driven flow, i.e. both sides of (1.10) have the same scaling.

Equation (1.10) is a second order equation for the tangential velocity u and hence needs to be complemented with two boundary conditions: At the air–liquid interface, it is reasonable to ask for continuity of the tangential component of the shear stress. In the lubrication regime, this turns into

$$\partial_z u = 0 \qquad\qquad \text{at } z = h. \tag{1.11}$$

The boundary condition at the liquid–solid interface is more delicate: For bulk flows it would be standard to impose a no–slip condition at the liquid–solid interface. However, the no–slip condition fails at the moving contact line where it implies infinite energy dissipation [39, 40]. For this reason, various relaxed slip conditions, depending on a parameter $n \in (0,3)$, have been used [31]:

$$u = b^{3-n}\ h^{n-2}\ \partial_z u \qquad\qquad \text{at } z = 0, \tag{1.12}$$

where b is called *slip length*. Different values $n \in (0,3)$ may be seen as simple models to incorporate surface chemistry effects: The case $n = 2$ corresponds to the well–known *Navier slip* condition [22]. The case $n = 1$ has been used as a model for the flow on a porous surface [32, 45]. If $n \neq 2$, then (1.12) depends on the film height, so that the validity of the boundary condition may be questioned. However, molecular dynamics simulations suggest a complex interdependence between slip condition and moving contact line [46, 47].

Integrating (1.10) with respect to the boundary conditions (1.11)–(1.12) yields the profile of the horizontal component u of the velocity. The vertical average $\overline{u} = h^{-1}\int_0^h u\ dz$ of u turns out to be

$$3\eta\ \overline{u} = \gamma\ \partial^3 h \left(h^2 + 3\ b^{3-n} h^{n-1}\right). \tag{1.13}$$

Since the fluid is incompressible we have

$$\partial_t h + \partial_x (h\ \overline{u}) = 0. \tag{1.14}$$

Combining (1.13)–(1.14) we arrive at

$$3\eta\ \partial_t h + \gamma\ \partial_x \left(\left(h^3 + 3b^{3-n} h^n\right) \partial_x^3 h\right) = 0.$$

By scaling in (h, t, x), there are many possibilities for a setting where the constants in the equation turn to 1. Note that the second term in the mobility is dominant for thin films. This motivates to analyze equations of the form (1.2).

1.3 Related previous work

Interestingly, existence theory for classical solutions of (1.1) has not been addressed before. However, there is an existence theory for the analogous second order equation, the porous media equation. In this section we review previous results about weak solutions of (1.1) as well as results on classical solutions for the porous media equation.

Weak solutions for the thin–film equation. There exists a well–developed theory for long–time existence of weak solutions thin–film equations with zero contact angle. These results are not restricted to (1.1) but apply to the more general equation (1.2).

The analysis is essentially based on two a priori estimates. The first describes the decrease of surface energy, in our regime given by the Dirichlet integral,

$$\int (\partial_y h)^2 - \int (\partial_y h_0)^2 = -\iint h^n (\partial_y^3 h)^2.$$

In fact, the flow is completely determined by the decrease of the surface energy: For a suitable choice of inner product, the evolution can be understood as a gradient flow with respect to this energy [1].

Secondly, there is a family of decreasing integrals, the so called entropies: For all $\alpha \in (\frac{1}{2} - n, 2 - n) \setminus \{0, -1\}$ there exists a positive constant $C_{\alpha,n}$ such that

$$\int \frac{h^{\alpha+1}}{\alpha(\alpha+1)} - \int \frac{h_0^{\alpha+1}}{\alpha(\alpha+1)} \ \le\ -\, C_{\alpha,n}^{-1} \iint h^{\alpha+n-3}\, (\partial_y h)^4,$$

see [7, 12].

Long–time existence and regularity properties of non–negative weak solutions in one space dimension have been established in [10, 7, 12]. Existence in higher dimensions has been shown in [14, 34]. These solutions are obtained as a limit of positive solutions of approximating nondegenerate problems. They inherit the regularity of the above two a priori estimates (see Section 4.1 for the exact definition). It is remarkable that these solutions remain nonnegative for nonnegative initial data: This does not hold true for nondegenerate equations of fourth order.

The qualitative behavior of solutions was e.g. analyzed in [7], [12]. In particular, there it is shown that the support of a solution to (1.2) does not grow for $n \geq 7/2$. Indeed, it is conjecture that the sharp lower bound is $n = 3$. Bernis has shown finite speed of propagation for weak solutions of the thin–film equation [8]. If the initial data are sufficiently flat, then the solution starts to move only after a waiting time as was shown in [19, 35].

It should be remarked that in the definition of weak solutions, (1.2) is not explicitly treated as a free boundary problem [10]. In particular, it is not possible to include (1.6) into the definition of weak solutions, since weak solutions do not have sufficiently regularity to control $\|\partial^3 h\|_{C^0}$. The zero contact angle boundary condition (1.5) appears as a natural boundary condition.

Classical solutions for the porous media equation. The analogous parabolic equation of second order is the porous medium equation:

$$\partial_t h - \partial_y (h^n \ \partial_y h) = 0. \tag{1.15}$$

Like (1.1), it is a degenerate parabolic equation, but as a second order equation, it obeys a comparison principle, which excludes both the formation of singularities and the contraction of the support [16]. Long–time existence, uniqueness and regularity of solutions of the corresponding free boundary problem have been shown by Angenent [2, 3, 4] in space dimension $d = 1$. Analogous short–time results were later obtained by Daskalopulos and Hamilton [21] for $d = 2$ and by Koch [41] for arbitrary d. The above results hold for a generic initial datum rather than for perturbations of special solutions (stationary or self–similar).

The general strategy for short–time existence for nonlinear evolution problems is to linearize around the initial data. The general strategy for short–time existence of a free boundary problem like (1.15) or (1.1) is to first transform onto a fixed domain and then to linearize around the initial data. For (1.15), the above authors apply two different coordinate transforms to fix the free boundary: A time–dependent transformation of the spatial variable y [2, 3, 4] or a (localized) hodograph transform [41, 20]. The latter transformation amounts to (locally) interchange dependent and independent variables (near the free boundary).

The next step is to prove maximal regularity for the linearized operator. In order to do so, the above papers apply quite different techniques. The

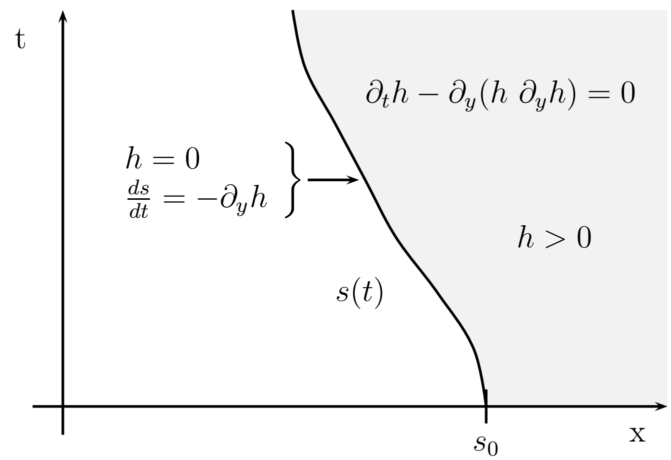

Figure 1.6: The free boundary problem for the porous media equation with $n = 1$

analysis in [2, 3] relies on semi–group theory: the equation is written as an abstract parabolic boundary value problem to which the theory of Da Prato and Grisvard [18] is applied. The main tool in [21] are Schauder estimates in weighted Hölder spaces, obtained by an elaboration of the method of Safonov [49]: Our analysis on Hölder spaces is based on this method. Finally, in [41] the analysis applies the theory of singular integral operators and Gaussian estimates of the fundamental solution of the linearized parabolic equation.

1.4 Global transformation onto a fixed domain

In this section, we consider the situation near a free boundary, say a *left* free boundary (see Figure 1.7). There, we generically expect that any solution looks like (the right wing of) a parabola, i.e. $(y_+)^2/2$. Note that such a parabola itself represents a *stationary solution* for (1.1) with solely one free boundary on the left. This motivates to consider solutions which initially are near this stationary solution:

$$h_0(y) \approx \frac{1}{2}(y_+)^2. \tag{1.16}$$

Let us denote by $s(t)$ the *single free boundary.* In order to transform the

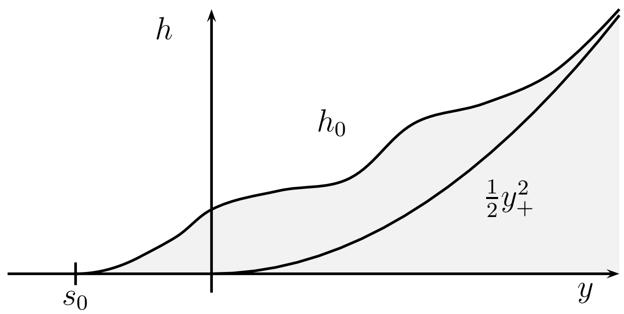

Figure 1.7: Initial data approximate the stationary solution $(x_+)^2$

problem onto a fixed spatial domain, we introduce travelling coordinates by $x = y - s(t)$. We also introduce the new function F by

$$F(t,x) := h(t,x) - \frac{1}{2}x^2 - xs(t). \tag{1.17}$$

Then (1.7) turns into:

$$\begin{cases} \partial_t F + \partial_x(\frac{1}{2}x^2\ \partial_x^3 F) = -\partial_x \left((F - x\ \partial_x F_{|x=0})(\partial_x^3 F - \partial_x^3 F_{|x=0})\right) & \text{in } \mathbb{R}_+^2, \\ F = 0 & \text{on } x = 0, \\ F = F_0 & \text{on } t = 0, \end{cases} \tag{1.18}$$

where we denote $\mathbb{R}_+ = [0, \infty)$. In this transformed problem, the boundary is fixed at the cost of a nonlocal term. Due to the term $xs(t)$ in the right–hand side of (1.17), at least the linear part in (1.18) keeps local.

Since we consider perturbations of the stationary solution, the linear part represents the main part of the operator which we denote by L_0

$$L_0 F := \partial_t F + A_0 F = \partial_t F + \frac{1}{2}\ \partial_x(x^2\ \partial_x^3 F).$$

Note that also L_0 has a gradient flow structure: In fact, it is the gradient flow of $\int \frac{1}{2}x^2(\partial_x^3 F)^2\ dx$ with respect to the inner product $\int (\partial_x F)^2\ dx$. This symmetric structure is important for our subsequent analysis.

1.5 Smooth solutions around the steady state

In Chapter 2, we address long–time existence results in Sobolev spaces for initial data which are a perturbation of the stationary solution $(y_+)^2/2$. This is joint work with Giacomelli and Otto. The results have been published as SFB–preprint [28].

It turns out that the linear part L_0 of the operator motivates to introduce the following semi–norms

$$[F]_{H_m} := \langle F, F\rangle_{H_m}^{1/2},$$
$$\langle F, G\rangle_{H_m} := \int_0^\infty x^{m-1}\ \partial_x^m F\ \partial_x^m G\ dx.$$

In fact, as we will see in Chapter 2, the semi–norms $[F]_{H_m}$ are Lyapunov functionals for the linear operator.

It is natural to ask for the relationship between these norms and the nonlinear part of the operator: Indeed, (1.18) is invariant under the transformation

$$x \mapsto \lambda x, \qquad t \mapsto \lambda^2 t, \qquad F \mapsto \lambda^2 F. \tag{1.19}$$

This scaling invariance selects one single semi–norm out of the sequence of semi–norms $[F]_{H_m}$: The only invariant semi–norm under (1.19) is given by $[F]_{H_4}$. It would be desirable to get a *minimal setting* where we assume just to control the initial datum by the single semi–norm $[F_0]_{H_4}$. However, this seems not to be possible for the following reason: The setting is determined by the goal to control the nonlinear part of the operator in (1.18) by the linear one. In view of (1.18) it is then clear that one needs $\sup_t |F - x\ \partial_x F(t,0)| \ll x^2$. But unfortunately, the estimate one could hope for in terms of scaling barely fails:

$$\|x^{-2}(F - x\ \partial_x F_{|x=0})\|_{C^0} \not\lesssim C\ [F]_{H_4}. \tag{1.20}$$

This motivates to use interpolation semi–norms. They have the same scaling as the corresponding semi–norms but are slightly stronger. For any $m \geq 2$, we define

$$[F]_{H_m^*} \ := \ \int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}[F_-]_{H_{m-1}}^2 + s[F_+]_{H_{m+1}}^2\right)^{1/2}\ \frac{ds}{s},$$

where $F_\pm \in H_{m\pm 1}$. The exact definition of the spaces H_m, H_m^* can be found in Chapter 2.

A comparison between the weighted Sobolev semi–norms and their interpolation counterparts is given in the next lemma:

Lemma 2.1.1. *For any $m \geq 2$ and any function $F \in H_{m+1}$ we have*

$$\frac{1}{C}\ [F]_{H_m} \leq [F]_{H_m^*} \leq C\ [F]_{H_{m-1}}^{1/2}[F]_{H_{m+1}}^{1/2}.$$

The interpolation semi–norms are indeed strong enough to control supremum norms:

Lemma 2.1.2. *For any even $m \geq 2$ and any function $F \in H_m^*$ we have*

$$\|\partial_x^{m/2} F\|_{C^0} \ \leq\ C\ [F]_{H_m^*}.$$

Note that by Lemma 2.1.2 and for $F|_{x=0} = 0$, the statement corresponding to (1.20) holds true for the interpolation semi–norms:

$$\|x^{-2}(F - x\ \partial_x F|_{x=0})\|_{C^0} \ \leq\ C\ \|\partial_x^2 F\|_{C^0} \leq\ C\ [F]_{H_4^*}. \tag{1.21}$$

The main result of Chapter 2 is global existence and uniqueness for perturbations of the stationary solution in weighted Sobolev spaces:

Theorem 2.1.4 (Existence and uniqueness)**.** *There exists an $\epsilon > 0$ such that if $F_0 \in H_4^*$ satisfies*

$$[F_0]_{H_4^*} < \epsilon, \tag{1.22}$$

then there exists a unique solution $F \in X_6^*$ of (1.18)*, and furthermore*

$$[\partial_t F]_{L^2(H_2)^*} + [F]_{C^0(H_4)^*} + [F]_{L^2(H_6)^*} \leq C\ [F_0]_{H_4^*}. \tag{1.23}$$

The space X_6^* is the one induced by (1.23), for the exact definition see (2.19) in Chapter 2. We also obtain estimates on higher derivatives. These estimates eventually yield smoothness of solutions up to the free boundary and of the free boundary itself:

Theorem 2.1.5 (Smoothness)**.** *There exists an $\epsilon > 0$ such that if $F_0 \in H_4^*$ satisfies* (1.22)*, then the solution $F \in X_6^*$ of* (1.18) *belongs to $C^\infty((0,\infty) \times [0,\infty))$; in particular, the free boundary $s(t)$ belongs to $C^\infty((0,\infty))$.*

1.6 Maximal regularity in weighted Hölder spaces

In Chapter 3, we again consider initial configurations that perturb the stationary solution. In contrary to Chapter 2, we measure the distance to the stationary solution in terms of weighted Hölder norms. The main result is maximal regularity for the linear part L_0 of (1.18) in these Hölder norms. It turns out that the nonlinear part of (1.18) is unbounded in the corresponding norms. This is joint work with Giacomelli and Otto. The result has been published as SFB–preprint [27].

The first step is to find an appropriate Hölder norm which is suited to the operator L_0. It is motivated in the following way: Recall that for the standard parabolic operator $\partial_t f + \partial_x^4 f$, the appropriate parabolic metric is given by

$$|t_1 - t_2|^{1/4} + |x_1 - x_2|. \tag{1.24}$$

For $x \approx 1$, the operator L_0 behaves like the standard parabolic operator. Hence the appropriate parabolic metric for L_0 should look like (1.24) for $x \approx 1$ and at the same time satisfy the scaling invariance of L_0 under the transformation:

$$x \mapsto \lambda x, \qquad t \mapsto \lambda^2 t.$$

The only metric which satisfies these two conditions is the Carnot–Caratheodory metric

$$s(z_1, z_2) \ := \ |t_1 - t_2|^{1/4} + |\sqrt{x_1} - \sqrt{x_2}|,$$

where $z_i = (t_i, x_i)$. Corresponding Hölder norms are defined accordingly: For every $\beta \in (0,1)$ and $m \geq 0$, we define

$$[g]_{C_s^\beta} \ := \ \sup_{z_1, z_2 \in \mathbb{R} \times \mathbb{R}_+} \frac{|g(z_1) - g(z_2)|}{s(z_1, z_2)^\beta}.$$

Our main result in Chapter 3 is maximal regularity of the linear operator in weighted Hölder spaces:

Theorem 3.1.1 (Maximal regularity)**.** *Assume that f is smooth and bounded in $\mathbb{R} \times \mathbb{R}_+$. Then for all $m \geq 1$*

$$[\partial_t \partial_x^m f]_{C_s^\beta} + [x^2 \ \partial_x^{m+4} f]_{C_s^\beta} + [x \ \partial_x^{m+3} f]_{C_s^\beta} + [\partial_x^{m+2} f]_{C_s^\beta} \leq C_{\beta,m} \ [\partial_x^m L_0 f]_{C_s^\beta}.$$

This theorem states that each of the single terms, $\partial_t f$, $x^2\ \partial_x^4 f$, $x\ \partial_x^3 f$, $\partial_x^2 f$, has the same regularity as the sum $L_0 f$.

The unboundedness of the nonlinear part of the operator in (1.18) in corresponding Hölder spaces prevents us from developing an existence theory for perturbations of the stationary solution (see Section 3.2). However, Theorem 3.1.1 still could be used as a basis to prove short–time existence for solutions of (1.7) with compactly supported initial data.

1.7 A non–rupture criterion

In Chapter 4, we derive a criterion on flatness of the initial data which excludes rupture of solutions for (1.2):

$$\partial_t h + \partial_y (h^n\ \partial_y^3 h) = 0. \tag{1.25}$$

In contrary to the other parts of the thesis, we work in the framework of weak solutions. This is joint work with Giacomelli. The result has been published as SFB–preprint [26].

Let us fix the setting. We assume (1.25) to hold on a fixed domain $\Omega = (-a, a)$. We choose boundary conditions that ensure conservation of mass. These are either Neumann boundary conditions,

$$\partial_y h(\pm a) = \partial_y^3 h(\pm a)\ =\ 0 \qquad\qquad \text{on } \partial\Omega, \tag{1.26}$$

or periodic boundary conditions,

$$\partial_y^j h(-a) = \partial_y^j h(a) \quad \text{for } j = 0, \ldots, 3 \qquad\qquad \text{on } \partial\Omega. \tag{1.27}$$

We furthermore assume that the initial data is strictly positive, i.e. $h_0 > 0$.

The possibility of rupture depends on the exponent n: The tendency of solutions to stay positive increases for large n. For solutions to (1.25)–(1.26) it first was proved in [10] that rupture cannot happen for $n \geq 4$. This result has been later improved to $n \geq 7/2$ in [7] and [12]. There, it is also shown that touch–down of weak solutions can only occur at isolated points for $n \geq 3/2$. On the other hand, numerical evidence of the formation of dead–cores for small values of n is given in [11, 13, 36].

We would like to point out that there already exists a similar result about positivity for flat initial data. Bertozzi and Pugh have shown that zero contact angle solutions converge uniformly to their mean value [12],

$$\|h(\cdot,t) - \overline{h}\|_{C^0} \leq A\ e^{-ct}.$$

Note that if A is small enough, then global positivity follows. By following the calculation of Bertozzi and Pugh one finds that A is small given a bound on $\|\partial_y h_0\|_{L^2}$ and if $\|h_0 - \overline{h}\|_{L^2}$ is small.

Although our result is qualitatively not new, it might be interesting for its compact structure and its different proof:

Theorem 4.1.2 (Non–rupture criterion)**.** *Let $n > 0$. A positive constant C exists such that for all $h_0 \in H^1(\Omega)$ with*

$$\inf_\Omega h_0 \geq C\ \|\partial_y h_0\|_{L^2(\Omega)}^{1/2}\ \|h_0\|_{L^2(\Omega)}^{1/2} \qquad \text{for } 0 < n \leq 1/2,$$
$$\inf_\Omega h_0 \geq C\ \|\partial_y h_0\|_{L^2(\Omega)}^{2/3}\ \|h_0\|_{L^1(\Omega)}^{1/3} \qquad \text{for } 1/2 < n,$$

there exists a unique solution h of problem (1.25)–(1.26) [resp. (1.25)–(1.27)] with initial datum h_0. Furthermore

$$\inf_{(0,\infty)\times\Omega} h \ \ \geq\ \ \frac{1}{2}\inf_\Omega h_0,$$

and if in addition h_0 belongs to $C^{4+\alpha}(\overline{\Omega})$ and satisfies (1.26) [resp. (1.27)], then h is a classical solution (i.e. $h \in C^{1+\frac{\alpha}{4},4+\alpha}([0,\infty)\times\overline{\Omega})$).

Our proof is based on introducing a new class of test functions, which might find further applications in the analysis of (1.2). The proof combines an interpolation inequality with an application of the classical Stampacchia Lemma [51].

Chapter 2

Smooth solutions around the steady state

In this chapter we consider the situation when the initial data are a perturbation of the stationary solution $(y_+)^2/2$ (see Section 1.4). We identify "minimal conditions" on the initial data under which a unique global solution exists. In fact, this solution turns out to be smooth for positive times and to converge to the stationary solution for large times. As a consequence, we obtain smoothness and large time behavior of the free boundary.

2.1 Setting and results

We recall the setting which already has been shortly introduced in Section 1.5. Our starting point is the transformed problem (1.18), where the free boundary is already fixed,

$$\begin{cases} L_0 F + \mathcal{N}(F,F) = 0 & \text{in } [0,\infty)\times[0,\infty), \\ F = 0 & \text{on } x = 0, \\ F = F_0 & \text{on } t = 0, \end{cases} \tag{2.1}$$

where L_0 denotes the linear part of (2.1),

$$L_0 F = \partial_t F = \partial_t F + \frac{1}{2}\,\partial_x(x^2\ \partial_x^3 F). \tag{2.2}$$

We define

$$A_0 F := \frac{1}{2}\,\partial_x(x^2\ \partial_x^3 F). \tag{2.3}$$

The nonlinear part of (2.1) corresponds to the bilinear form

$$\mathcal{N}(F,G) := -\partial_x \left((F - x\ \partial_x F_{|x=0})(\partial_x^3 G - \partial_x^3 G_{|x=0}) \right). \tag{2.4}$$

As a consequence of the transformation (1.17), the equation for the position of the free boundary has been separated. It is given by

$$s_0 = -\partial_x F_0(0), \qquad\qquad s(t) = -\partial_x F(t,0).$$

The speed of propagation satisfies the formula

$$\dot{s}(t) = \partial_x^3 F(t,0). \tag{2.5}$$

These two equations can be easily seen to be compatible by formally differentiating the equation in (2.1) with respect to x and using (1.5). Therefore, provided (1.5) holds, (2.1) is equivalent to (1.7) through (1.17) .

We define the following semi–norms

$$[F]_{H_k} := \langle F, F \rangle_{H_k}^{1/2}, \tag{2.6}$$

where

$$\langle F, G \rangle_{H_k} := \int_0^\infty x^{k-1}\ \partial_x^k F\ \partial_x^k G\ dx.$$

For $m \geq 1$, the corresponding norm and space are given by:

$$\|F\|_{H_m}^2 := \sum_{k=1}^m [F]_{H_k}^2,$$

$$H_m := \begin{cases} \text{Completion of } \{F \in C_c^\infty([0,\infty)) : F(0) = 0\} \\ \text{with respect to } \|\cdot\|_{H_m}. \end{cases} \tag{2.7}$$

As we will see later on, these semi–norms $[F]_{H_m}$ are Lyapunov functionals for L_0. Note that the first semi–norm, $k = 1$, is just the Dirichlet integral, which corresponds to the lubrication approximation of the capillary energy. Its boundedness ensures that the boundary condition is preserved under completion. Accordingly, we define the semi–norms

$$[F]_{L^2(H_m)} := \left(\int_0^\infty [F]_{H_m}^2\ dt \right)^{\frac{1}{2}}, \qquad [F]_{C^0(H_m)} := \sup_{t \in (0,\infty)} [F]_{H_m}. \tag{2.8}$$

As motivated in Section 1.5, we introduce interpolation semi–norms: For any $m \geq 2$, we define

$$[F]_{H_m^*} \;:=\; \int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}[F_-]_{H_{m-1}}^2 + s[F_+]_{H_{m+1}}^2\right)^{1/2} \,\frac{ds}{s}, \tag{2.9}$$

where $F_\pm \in H_{m\pm 1}$. The corresponding norm and space are given by

$$\|F\|_{H_m^*} := \sum_{k=2}^{m} [F]_{H_k^*}, \tag{2.10}$$

$$H_m^* := \begin{cases} \text{Completion of } \{F \in C_c^\infty([0,\infty)) : F(0) = 0\} \\ \text{with respect to } \|\cdot\|_{H_m^*}. \end{cases} \tag{2.11}$$

It seems to be crucial to define H_m^* by completion rather than by finiteness of $\|\cdot\|_{H_m^*}$ (the two definitions disagree with each other, see Lemma 2.11.7). A comparison between the weighted Sobolev semi–norms and their interpolation counterparts is given in the next lemma:

Lemma 2.1.1. *For any $m \geq 2$ and any function $F \in H_{m+1}$ we have*

$$\frac{1}{C}\,[F]_{H_m} \leq [F]_{H_m^*} \leq C\,[F]_{H_{m-1}}^{1/2}[F]_{H_{m+1}}^{1/2}. \tag{2.12}$$

The interpolation semi–norms are indeed strong enough to control supremum norms:

Lemma 2.1.2. *For any even $m \geq 2$ and any function $F \in H_m^*$ we have $\partial_x^{m/2} F \in C^0$ and*

$$\|\partial_x^{m/2} F\|_{C^0} \;\leq\; C\,[F]_{H_m^*} \tag{2.13}$$

In particular, we have

$$\|x^{-1}F\|_{C^0} \leq \|\partial_x F\|_{C^0} \leq [F]_{H_2^*}, \tag{2.14}$$

so that the boundary condition in (2.11) is preserved under completion. The proof of the Lemmas 2.1.1–2.1.2 can be found in the appendix.

Finally, we introduce the corresponding parabolic semi–norms. For $m \geq 2$, let

$$\begin{aligned} [F]_{L^2(H_m)^*} &:= \int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}[F_-]_{L^2(H_{m-1})}^2 + s[F_+]_{L^2(H_{m+1})}^2\right)^{1/2} \,\frac{ds}{s}, \\ [F]_{C^0(H_m)^*} &:= \int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}[F_-]_{C^0(H_{m-1})}^2 + s[F_+]_{C^0(H_{m+1})}^2\right)^{1/2} \,\frac{ds}{s}. \end{aligned} \tag{2.15}$$

It seems to be crucial to interpolate the space–time semi–norms rather than taking the temporal norm of the spatial interpolation, as for

$$[F]_{L^2(H_m^*)} := \left(\int_0^\infty [F]_{H_m^*}^2 \, dt\right)^{\frac{1}{2}}, \quad [F]_{C^0(H_m^*)} := \sup_{t\in(0,\infty)} [F]_{H_m^*}. \tag{2.16}$$

Indeed, these semi–norms do not coincide with (2.15) in general. However, there is a lower bound in general and a two-sided bound for tensor products:

Lemma 2.1.3.

i) For any even $m \geq 2$ and any function $F \in C_c([0,\infty)^2)$ we have

$$\begin{aligned} [F]_{L^2(H_m)^*} &\geq [F]_{L^2(H_m^*)}, \\ [F]_{C^0(H_m)^*} &\geq [F]_{C^0(H_m^*)}. \end{aligned} \tag{2.17}$$

ii) For any even $m \geq 2$ and any pair of functions $\alpha, \zeta \in C_c([0,\infty))$ we have

$$\begin{aligned} [\alpha \otimes \zeta]_{L^2(H_m)^*} &\leq \|\alpha\|_{L^2} \, [\zeta]_{H_m^*}, \\ [\alpha \otimes \zeta]_{C^0(H_m)^*} &\leq \|\alpha\|_{C^0} \, [\zeta]_{H_m^*}, \end{aligned} \tag{2.18}$$

where $(\alpha \otimes \zeta)(t,x) = \alpha(t)\, \zeta(x)$.

The proof of Lemma 2.1.3 can be found in Section 2.11.

The semi–norms corresponding to (2.15) and (2.16) on a finite time interval $(0,T)$ are denoted by $[\cdot]_{L^2((0,T);H_m)^*}$, $[\cdot]_{C^0((0,T);H_m)^*}$, $[\cdot]_{L^2((0,T);H_m^*)}$ and $[\cdot]_{C^0((0,T);H_m^*)}$, respectively. The interpolation norms $\|\cdot\|_{L^2(H_m)^*}$ and spaces $L^2(H_m)^*$ are defined by completion exactly as in (2.10)–(2.11).

The semi–norms defined in (2.15) provide an appropriate ambient space for existence and uniqueness of solutions to (2.1) under minimal (in the sense of (1.21) versus (1.20)) assumptions on the initial data. It is given, for $m \geq 4$, by

$$\begin{aligned} \|F\|_{X_{m+2}^*} &:= \sum_{k=4}^{m} \left([\partial_t F]_{L^2(H_{k-2})^*} + [F]_{C^0(H_k)^*} + [F]_{L^2(H_{k+2})^*}\right), \\ X_m^* &:= \begin{cases} \text{Completion of } \{F \in C_c^\infty([0,\infty)^2) : F_{|x=0} = 0\} \\ \text{with respect to } \|\cdot\|_{X_m^*}. \end{cases} \end{aligned} \tag{2.19}$$

It is straightforward to check that the trace of $F \in X_m^*$ at $t = 0$ is well defined (see Lemma 2.11.8).

Here and after, universal constants are denoted by C, and C_k stands for a constant which is universal for fixed k. Our first main result is:

Theorem 2.1.4 (Existence and uniqueness)**.** *There exists an $\epsilon > 0$ such that if $F_0 \in H_4^*$ satisfies*

$$[F_0]_{H_4^*} < \epsilon, \tag{2.20}$$

then there exists a unique solution $F \in X_6^*$ of (2.1)*, and furthermore*

$$[\partial_t F]_{L^2(H_2)^*} + [F]_{C^0(H_4)^*} + [F]_{L^2(H_6)^*} \leq C\ [F_0]_{H_4^*}. \tag{2.21}$$

Equation (2.21) contains enough information to keep track of both the shape of the solution, as follows from

$$\|\partial_x^2 h - 1\|_{C^0} = \|\partial_x^2 F\|_{C^0} \overset{(2.13)}{\leq} C\ [F]_{C^0(H_4^*)} \overset{(2.17)}{\leq} C\ [F]_{C^0(H_4)^*}, \tag{2.22}$$

and the position of the free boundary, as follows from

$$\|\partial_t s\|_{L^2} \overset{(2.5)}{=} \|\partial_x^3 F_{|x=0}\|_{L^2} \overset{(2.13)}{\leq} C\ [F]_{L^2(H_6^*)} \overset{(2.17)}{\leq} C\ [F]_{L^2(H_6)^*}.$$

In particular, (2.22) implies that $h > 0$ for $\epsilon \ll 1$.

The minimal assumption (2.20) turns out to be sufficiently robust to keep all derivatives of the perturbation under control. This yields smoothness of the solution and of the free boundary for positive times:

Theorem 2.1.5 (Smoothness)**.** *There exists an $\epsilon > 0$ such that if $F_0 \in H_4^*$ satisfies* (2.20)*, then the solution $F \in X_6^*$ of* (2.1) *belongs to $C^\infty((0,\infty) \times [0,\infty))$; in particular, the free boundary $s(t)$ belongs to $C^\infty((0,\infty))$.*

It turns out that smoothness may be quantified in terms of $[F_0]_{H_4^*}$ via estimates of the decay of high derivatives of F. Not to overload the elaboration, we shall state and prove them with respect to the semi–norms (2.16):

Theorem 2.1.6 (Decay of high derivatives)**.** *There exists an $\epsilon > 0$ such that if $F_0 \in H_4^*$ satisfies* (2.20)*, then the solution $F \in X_6^*$ of* (2.1) *satisfies*

$$[t^{\frac{k-4}{4}} F]_{C^0(H_k^*)} + [t^{\frac{k-4}{4}} F]_{L^2(H_{k+2}^*)} \leq C_k\ [F_0]_{H_4^*} \qquad \text{for all } k \geq 4. \tag{2.23}$$

The temporal weights are such that estimate (2.23) is scale–invariant with respect to the transformation (1.19).

The bounds so far are oblivious of the value of $[F_0]_{H_1}$. In this sense, they are unable to single out the steady state $\tilde{F} = 0$ among the two–parameter family of stationary solutions to (2.1), $\tilde{F} = (Ax^2 - Bx)$. It is therefore not surprising that $[F_0]_{H_1}$ becomes quantitatively relevant when looking at the long–time behavior of the solution and of the free boundary.

Theorem 2.1.7 (Convergence to the steady state)**.** *There exists an $\epsilon > 0$ such that if $F_0 \in H_4^*$ satisfies* (2.20)*, then the solution $F \in X_6^*$ satisfies*

$$[t^{\frac{k-1}{4}} F]_{C^0(H_k)} + [t^{\frac{k-1}{4}} F]_{L^2(H_{k+2})} \leq C_k \; ([F_0]_{H_1} + [F_0]_{H_4^*}) \quad \text{for all } k \geq 1.$$

In particular, the free boundary converges to zero as $t \to \infty$:

$$|s(t)| \leq C \; (1+t)^{-\frac{1}{4}} ([F_0]_{H_1} + [F_0]_{H_4^*}).$$

In terms of h, $[F_0]_{H_1} < \infty$ implies that the initial data h_0 is close to the reference steady state $(y_+)^2/2$ for $y \gg 1$. In other words, it is the behavior at infinity (not the behavior near $y = 0$) which selects the steady state to which the solution converges for large times. This explains why a general initial position of the free boundary, $s_0 \neq 0$, relaxes to zero in our setting.

In particular, Theorem 2.1.7 implies that solutions to the thin–film equation with $n = 1$ may shrink (take $s_0 < 0$). Such behavior was commonly believed to be possible in view of the existence of local travelling wave profiles of the form $h(t, y) = -\frac{1}{6}(y + t)^3 + A(y + t)_+^2$, $A > 0$, $(y + t)_+ \ll A$. However, to our knowledge it hadn't been rigorously observed so far.

2.2 Overview of the proof

In this section we describe the main ingredients of our method. This will lead us already to the proof of the uniqueness part of Theorem 2.1.4. The proof of the existence part of Theorem 2.1.4, as well as that of the other main results, will be outlined immediately afterwards, together with the plan of the paper. Universal constants are denoted by C. We write $f \lesssim g$, whenever a constant C exists such that $f \leq C \; g$. We write $f \ll g$, whenever $f \leq C^{-1} \; g$ holds for a given sufficiently large constant C.

The bases of our argument are the symmetry and composition properties enjoyed by A_0, as defined in (2.3), which induce the choice of the semi–norms $[\cdot]_{H_k}$, namely (see Lemmas 2.3.2 and 2.3.3)

$$2\langle A_0 F, G\rangle_{H_k} = \langle F, G\rangle_{H_{k+2}}, \qquad 2[A_0 F]_{H_k} = [F]_{H_{k+4}}. \tag{2.24}$$

As can be easily checked at a formal level, they imply the existence of a sequence of Lyapunov functionals for A_0 (see Lemma 2.9.1):

$$\partial_t [F]^2_{H_k} + [F]^2_{H_{k+2}} = 4\langle A_0 F, \partial_t F + A_0 F\rangle_{H_{k-2}}, \qquad k \geq 3.$$

This yields the following existence and uniqueness result for the parabolic linear equation associated to (2.1):

Proposition 2.2.1. *Let $m \geq 4$. For any $F_0 \in H^*_m$ and any $G \in L^2(H_{m-2})^*$ there exists a unique $F \in X^*_{m+2}$ such that*

$$\begin{cases} \partial_t F + A_0 F = G, \\ F_{|t=0} = F_0. \end{cases} \tag{2.25}$$

It satisfies for all $4 \leq k \leq m$ and all $0 < T \leq \infty$ the estimate

$$\begin{aligned} C^{-1}\ [\partial_t F]_{L^2((0,T);H_{k-2})^*} + [F]_{C^0((0,T);H_k)^*} + C^{-1}\ [F]_{L^2((0,T);H_{k+2})^*} \\ \leq [F_0]_{H^*_k} + C\ [G]_{L^2((0,T);H_{k-2})^*}. \end{aligned} \tag{2.26}$$

The main ingredient in the proof of Theorem 2.1.4 is the following a–priori estimate for the nonlinear part of the operator:

Proposition 2.2.2. *For any given $F, G \in X^*_6$ and for any $0 < T \leq \infty$ we have $\mathcal{N}(F,G) \in L^2((0,T), H_2)^*$ and*

$$[\mathcal{N}(F,G)]_{L^2((0,T);H_2)^*} \lesssim [F]_{C^0((0,T);H_4)^*}\ [G]_{L^2((0,T);H_6)^*}. \tag{2.27}$$

The combination of (2.26) and (2.27) yields the a–priori estimate (2.21) which is at the core of Theorem 2.1.4:

Proposition 2.2.3. *There exists an $\epsilon > 0$ such that if $F \in X^*_6$ is a solution of (2.1) with $F_0 \in H^*_4$ such that*

$$[F_0]_{H^*_4} < \epsilon,$$

then

$$[F]_{C^0(H_4)^*} + [F]_{L^2(H_6)^*} \lesssim [F_0]_{H^*_4}. \tag{2.28}$$

Proof. It follows from (2.26) and (2.27) that

$$\begin{aligned}\varphi(T) &:= C^{-1}\ [\partial_t F]_{L^2((0,T);H_2)^*} + [F]_{C^0((0,T);H_4)^*} + C^{-1}\ [F]_{L^2((0,T);H_6)^*} \\ &\overset{(2.26)}{\leq} [F_0]_{H_4^*} + C\ [\partial_t F + A_0 F]_{L^2((0,T);H_2)^*} \\ &\overset{(2.1)}{=} [F_0]_{H_4^*} + C\ [\mathcal{N}(F,F)]_{L^2((0,T);H_2)^*} \\ &\overset{(2.27)}{\leq} [F_0]_{H_4^*} + C\ [F]_{C^0((0,T);H_4)^*}[F]_{L^2((0,T);H_6)^*}. \end{aligned} \tag{2.29}$$

It is easy to check (see Lemma 2.11.8) that $\varphi \in C([0,\infty))$ with $\varphi(0) = [F_0]_{H_4^*}$. Hence (2.29) reads as

$$\varphi(t) \overset{(2.20)}{\leq} \varphi(0) + C\ \varphi^2(t) \quad \text{for all } t > 0. \tag{2.30}$$

If $\varphi(0) < 1/(4C) =: \epsilon$, then

$$C\ \varphi^2 - \varphi + \varphi(0) \geq 0 \iff \begin{cases} \varphi \leq \varphi_1 = \frac{1-\sqrt{1-4C\varphi(0)}}{2C} \\ \qquad\text{or} \\ \varphi \geq \varphi_2 = \frac{1+\sqrt{1-4C\varphi(0)}}{2C} \end{cases}.$$

Since $\varphi(0) \leq \varphi_1$ and φ is continuous, (2.30) implies that

$$\varphi(t) \leq \varphi_1 \lesssim \varphi(0) \qquad\qquad \text{for all } t > 0,$$

and the proof is complete. □

We are now ready to prove the uniqueness of solutions.

Proof of Theorem 2.1.4 – uniqueness. Let $F_1, F_2 \in X_6^*$ be two solutions of (2.1) with initial data $F_0 \in H_4^*$, and let $F = F_1 - F_2$. By (2.26), we have that

$$\begin{aligned}\varphi &:= C^{-1}\ [\partial_t F]_{L^2(H_2)^*} + [F]_{C^0(H_4)^*} + C^{-1}\ [F]_{L^2(H_6)^*} \\ &\lesssim [\mathcal{N}(F_1,F_1) - \mathcal{N}(F_2,F_2)]_{L^2(H_2)^*}. \end{aligned} \tag{2.31}$$

It follows from the definition (2.4) of $\mathcal{N}$ that

$$\mathcal{N}(F_1,F_1) - \mathcal{N}(F_2,F_2) = \mathcal{N}(F_1,F) + \mathcal{N}(F,F_2).$$

Therefore, provided ϵ is sufficiently small for Proposition 2.2.3 to hold, we have that

$$\begin{aligned}
&[\mathcal{N}(F_1, F_1) - \mathcal{N}(F_2, F_2)]_{L^2(H_2)^*} \\
&\quad \lesssim [\mathcal{N}(F_1, F)]_{L^2(H_2)^*} + [\mathcal{N}(F, F_2)]_{L^2(H_2)^*} \\
&\quad \overset{(2.27)}{\lesssim} [F_1]_{C^0(H_4)^*}[F]_{L^2(H_6)^*} + [F]_{C^0(H_4)^*}[F_2]_{L^2(H_6)^*} \\
&\quad \overset{(2.28)}{\lesssim} \epsilon \left([F]_{C^0(H_4)^*} + [F]_{L^2(H_6)^*}\right).
\end{aligned} \tag{2.32}$$

Inserting (2.32) into (2.31), we see that

$$\varphi \lesssim \epsilon\, \varphi,$$

and therefore $\varphi = 0$ if ϵ is sufficiently small. In particular, it follows from Lemma 2.1.3 that $[\partial_t F(t)]_{H_2^*} = 0$ for a.e. t, which by (2.14) implies that $\partial_t F = 0$, that is $F(t, x) = F(0, x) = 0$. □

Of course, the a–priori estimate given by Proposition 2.2.3 is also at the core of the existence part of Theorem 2.1.4, whose proof will be given in section 2.7 using a fixed–point argument. In section 2.4 (see Proposition 2.4.1) we prove well–posedness for the resolvent equation associated to A_0,

$$F + A_0 F = G,$$

which we use in section 2.5 to prove Proposition 2.2.1 via a time–discretization argument.

The results concerned with higher regularity rely on the fact that the symmetric structure of the linear part of the operator is preserved upon differentiation. This yields estimates similar to (2.27) for higher semi–norms of the nonlinear operator. We prove in Proposition 2.8.1 that

$$[\mathcal{N}(F, F)]_{H_k} \;\le\; C\; [F]_{H_4^*}[F]_{H_{k+4}} + C_k\; [F]_{H_6^*}[F]_{H_{k+2}}.$$

Based on this a priori estimate, we prove Theorem 2.1.5 in Section 2.8, Theorem 2.1.6 in Section 2.9 and Theorem 2.1.7 in Section 2.10.

2.3 Basic properties of the linear operator

We first notice:

Lemma 2.3.1. *For all $F \in C^\infty((0,\infty))$ and all $k \geq 0$, we have:*

$$\partial_x^k A_0 F = \frac{1}{2x^{k-1}}\ \partial_x^2 (x^{k+1}\ \partial_x^{k+2} F). \tag{2.33}$$

Proof. Easily obtained by induction. □

Identity (2.33) gives rise to symmetry with respect to each weighted semi–norm:

Lemma 2.3.2. *Let $k \geq 1$. For all $F \in H_{k+4}$ and all $G \in H_{k+2}$, we have:*

$$\langle A_0 F, G\rangle_{H_k} = \frac{1}{2}\langle F, G\rangle_{H_{k+2}}. \tag{2.34}$$

Proof. By (2.7), it suffices to consider $F, G \in C_c^\infty([0,\infty))$ (in fact, the boundary condition $F(0) = 0$ is not needed here). We have

$$\begin{aligned}
\langle A_0 F, G\rangle_{H_k} &= \int_0^\infty x^{k-1}\ \partial_x^k (A_0 F)\ \partial_x^k G\ dx \\
&\overset{(2.33)}{=} \frac{1}{2}\int_0^\infty \partial_x^2 (x^{k+1}\ \partial_x^{k+2} F)\ \partial_x^k G\ dx \\
&= \frac{1}{2}\int_0^\infty x^{k+1}\ \partial_x^{k+2} F\ \partial_x^{k+2} G\ dx.
\end{aligned}$$

Note that all boundary terms at $x = 0$ vanish since $k \geq 1$. □

Applied twice, Lemma 2.3.2 yields:

Lemma 2.3.3. *Let $k \geq 1$. For all $F \in H_{k+4}$, we have:*

$$[F]_{H_{k+4}} = 2\ [A_0 F]_{H_k}. \tag{2.35}$$

The connection between the operator A_0 and the spaces H_m^* can be seen even better in terms of the positive square root of A_0:

$$A_0^{1/2} F = -\frac{1}{\sqrt{2}}\ x\ \partial_x^2 F.$$

Indeed,

$$A_0 F = \frac{1}{2}\ (x^2\ \partial_x^3 F) \overset{(2.33)}{=} \frac{1}{\sqrt{2}}\ x\ \partial_x^2 (\frac{1}{\sqrt{2}}\ x\ \partial_x^2 F).$$

The operator $A_0^{1/2}$ is symmetric and positive definite with respect to all semi–norms $[\cdot]_{H_m}$. More precisely, we have for all $k \geq 1$ and for all $F, G \in H_{k+1}$,

$$\langle A_0^{1/2} F, G \rangle_{H_k} = \frac{1}{\sqrt{2}}\ \langle F, G \rangle_{H_{k+1}}.$$

The last statement follows from an analogous identity to (2.33),

$$\partial_x^k (A_0^{1/2} F) = -\frac{1}{\sqrt{2}\ x^{k-1}}\ \partial_x (x^k\ \partial_x^{k+1} F).$$

This leads to the following representation for the semi–norms $[\cdot]_{H_m}$:

Lemma 2.3.4. *For all $k \geq 1$ and $F \in C_c^\infty([0, \infty))$ with $F(0) = 0$,*

$$[F]_{H_k} = 2^{\frac{k-1}{4}}\ \langle F, A_0^{(k-1)/2} F \rangle_{H_1}^{1/2}. \tag{2.36}$$

We conclude that the complete sequence of semi–norms $[\cdot]_{H_m}$, $m \geq 1$, is generated by $A_0^{1/2}$ and $\langle \cdot, \cdot \rangle_{H_1}$.

2.4 The linear elliptic equation

In this section we prove:

Proposition 2.4.1 (The resolvent equation)**.** *For all $G \in H_1$ there exists a unique solution $F \in H_3$ of*

$$\int_0^\infty \partial_x F\ \partial_x \varphi\ dx + \frac{1}{2} \int_0^\infty x^2\ \partial_x^3 F\ \partial_x^3 \varphi\ dx = \int_0^\infty \partial_x G\ \partial_x \varphi\ dx \qquad \text{for all } \varphi \in H_3. \tag{2.37}$$

Furthermore $F \in H_5$,

$$F + A_0 F = G, \tag{2.38}$$

and a positive constant C (independent of m) exists such that if $G \in H_m$, then $F \in H_{m+4}$ with

$$\sum_{j=0}^{4} [F]_{H_{m+j}} \lesssim\ [G]_{H_m}.$$

The proof of Proposition 2.4.1 proceeds as follows. We first prove existence and uniqueness of weak solutions by the Riesz representation theorem:

Lemma 2.4.2 (Weak solution)**.** *For all $G \in H_1$ there exists a unique solution $F \in H_3$ of (2.37), and*

$$\|F\|_{H_3} \lesssim [G]_{H_1}. \tag{2.39}$$

We then prove that (2.38) holds:

Lemma 2.4.3 (Strong solution)**.** *For all $G \in H_1$, the solution F of* (2.37) *is such that $F \in H_5$,* (2.38) *holds, and*

$$\|F\|_{H_5} \lesssim [G]_{H_1}. \tag{2.40}$$

Finally, higher regularity follows by iterating the argument for Lemma 2.4.3. The rest of the section is concerned with the proofs of Lemma 2.4.2, Lemma 2.4.3 and Proposition 2.4.1.

Proof of Lemma 2.4.2. Clearly, H_3 is a Hilbert space equipped with the inner product

$$(F, G) = \sum_{k=1}^{3} \int_0^\infty x^{k-1}\ \partial_x^k F\ \partial_x^k G\ dx.$$

The form

$$b(F, G) = \int_0^\infty \partial_x F\ \partial_x G\ dx + \frac{1}{2} \int_0^\infty x^2\ \partial_x^3 F\ \partial_x^3 G\ dx$$

is bilinear and symmetric on H_3. By interpolation (see Lemma 2.11.4)

$$\int_0^\infty x\ (\partial_x^2 F)^2\ dx \lesssim \int_0^\infty (\partial_x F)^2\ dx + \int_0^\infty x^2\ (\partial_x^3 F)^2\ dx, \tag{2.41}$$

hence b is also coercive. Existence and uniqueness now follow from the Riesz representation theorem upon the embedding i of H_1 into the dual space of H_3

$$i : H_1 \to \text{dual space of } H_3,$$

$$\langle i(G), \varphi \rangle \;=\; \int_0^\infty \partial_x G\ \partial_x \varphi\ dx$$

and (2.39) follows from (2.41) and from

$$\begin{aligned}\int_0^\infty (\partial_x F)^2\ dx + \frac{1}{2}\int_0^\infty x^2(\partial_x^3 F)^2\ dx &\overset{(2.37)}{=} \int_0^\infty \partial_x F\ \partial_x G\ dx \\ &\le \frac{1}{2}\int_0^\infty (\partial_x F)^2\ dx + \frac{1}{2}\int_0^\infty (\partial_x G)^2\ dx.\end{aligned}$$

□

Proof of Lemma 2.4.3. We set

$$H = x^2\ \partial_x^3 F.$$

Using only $\varphi \in C_c^\infty((0,\infty))$ in (2.37), we gather

$$\partial_x(\partial_x F + \frac{1}{2}\ \partial_x^2 H - \partial_x G) = 0$$

in a distributional sense. Since $[F]_{H_1}+[G]_{H_1} < \infty$ this yields $H \in H^2_{loc}([0,\infty))$ and

$$\frac{1}{2}\ \partial_x^2 H = \partial_x G - \partial_x F + C$$

almost everywhere. In particular, the traces $H(0)$ and $\partial_x H(0)$ exist. Since

$$\int_0^\infty \frac{H^2}{x^2}\ dx + \int_0^\infty (\partial_x G)^2\ dx + \int_0^\infty (\partial_x F)^2\ dx < \infty,$$

the constant vanishes:

$$\frac{1}{2}\ \partial_x^2 H = \partial_x G - \partial_x F. \tag{2.42}$$

We now use $\varphi \in C_c^\infty([0,\infty))$ with $\varphi(0) = 0$ in (2.37),

$$\begin{aligned}0 &= \int_0^\infty \partial_x F\ \partial_x \varphi\ dx + \frac{1}{2}\int_0^\infty H\ \partial_x^3 \varphi\ dx - \int_0^\infty \partial_x G\ \partial_x \varphi\ dx \\ &= \frac{1}{2}(H(0)\ \partial_x^2\varphi(0) - \partial_x H(0)\ \partial_x \varphi(0)) + \int_0^\infty (\partial_x F + \partial_x^2 H - \partial_x G)\ \partial_x \varphi\ dx \\ &\overset{(2.42)}{=} \frac{1}{2}(-H(0)\ \partial_x^2\varphi(0) + \partial_x H(0)\ \partial_x\varphi(0)),\end{aligned}$$

to derive the Neumann boundary conditions

$$H(0) = \partial_x H(0) = 0. \tag{2.43}$$

In view of (2.43), Hardy's inequality in the form of Lemma 2.11.1 yields

$$\begin{aligned}\int_0^\infty \frac{H^2}{x^4}\,dx + \int_0^\infty \frac{(\partial_x H)^2}{x^2}\,dx &\lesssim \int_0^\infty (\partial_x^2 H)^2\,dx \\ &\overset{(2.42)}{\lesssim} [G-F]_{H_1}^2 \\ &\overset{(2.39)}{\lesssim} [G]_{H_1}^2. \end{aligned} \tag{2.44}$$

Next, observe that $\partial_x^5 F = \partial_x^2(H/x^2)$, hence

$$\begin{aligned}\int_0^\infty x^4(\partial_x^5 F)^2\,dx &= \int_0^\infty x^4(\partial_x^2(\frac{H}{x^2}))^2\,dx \\ &\lesssim \int_0^\infty \left((\partial_x^2 H)^2 + \frac{(\partial_x H)^2}{x^2} + \frac{H^2}{x^4}\right)\,dx \\ &\overset{(2.44)}{\lesssim} [G]_{H_1}^2. \end{aligned} \tag{2.45}$$

By interpolation (see Lemma 2.11.4)

$$[F]_{H_4}^2 \lesssim [F]_{H_3}^2 + [F]_{H_5}^2 \overset{(2.39),(2.45)}{\lesssim} [G]_{H_1}^2.$$

Therefore $F \in H_5$ and (2.40) holds. Since $F(0) = G(0) = \partial_x H(0) = 0$ (cf. (2.43)), we obtain from (2.42)

$$\partial_x H = 2(G-F),$$

which in view of the definition of H turns into (2.38), i.e.

$$F + \frac{1}{2}\,\partial_x(x^2\,\partial_x^3 F) = G.$$

□

Proof of Proposition 2.4.1. By induction on m we show that

$$G \in H_m \implies F \in H_{m+4} \text{ with } \sum_{j=0}^{4}[F]_{H_{m+j}} \lesssim [G]_{H_m}. \tag{2.46}$$

By Lemma 2.4.3, the claim is true for $m = 1$. If $G \in H_m$, $m \geq 2$, then by induction $F \in H_{m+3}$ and

$$\|F\|_{H_{m+3}} \lesssim \|G\|_{H_{m-1}} \leq \|G\|_{H_m}. \tag{2.47}$$

In a first step we argue qualitatively that $F \in H_{m+4}$. In a second step we will show the estimate in (2.46). Only in the second step we have to take care that constants do not depend on m.

Let $H = x^{m+1}\ \partial_x^{m+2} F$. We have

$$\begin{aligned} \partial_x^2 H &= \partial_x^2 (x^{m+1}\ \partial_x^{m+2} F) \\ &\overset{(2.33)}{=} 2\ x^{m-1}\ \partial_x^m A_0 F \\ &\overset{(2.38)}{=} 2\ x^{m-1}\ \partial_x^m (G - F) \in L^2_{loc}([0,\infty)). \end{aligned} \tag{2.48}$$

We claim that

$$\exists\ x_n \to 0 :\ H(x_n) \to 0, \tag{2.49}$$

$$\exists\ y_n \to 0 :\ \partial_x H(y_n) \to 0. \tag{2.50}$$

Claim (2.49) follows immediately from

$$[F]_{H_{m+2}} < \infty \implies \exists\ x_n \to 0 :\ x^{\frac{m+2}{2}}\ \partial_x^{m+2} F \to 0.$$

For (2.50), assume by contradiction that

$$\liminf_{x \to 0} |\partial_x H| > 0.$$

Since by (2.47) and (2.48) $H \in C^{\frac{3}{2}}([0,1])$, we may assume without loss of generality that $C > 0$ and $x_0 \in (0,1)$ exist such that $\partial_x H \geq C$ for $x \in (0, x_0)$. Then, using (2.49), $H \geq Cx$, that is $\partial_x^{m+2} F \geq C x^{-m}$. But then, since $m \geq 2$,

$$\int_0^\infty x^{m+1} (\partial_x^{m+2} F)^2\ dx \geq \int_0^{x_0} x^{1-m}\ dx = \infty,$$

a contradiction. Hence (2.50) holds. In view of (2.49) and (2.50), we have by Lemma 2.11.1:

$$\begin{aligned} \int_0^\infty \frac{H^2}{x^{m+3}}\ dx + \int_0^\infty \frac{(\partial_x H)^2}{x^{m+1}}\ dx &\leq C_m \int_0^\infty \frac{(\partial_x^2 H)^2}{x^{m-1}}\ dx \\ &\overset{(2.48)}{\leq} C_m\ [G - F]^2_{H_m} \overset{(2.47)}{\leq} C_m\ \|G\|^2_{H_m}. \end{aligned} \tag{2.51}$$

Therefore

$$\begin{aligned}\int_0^\infty x^{m+3}(\partial_x^{m+4}F)^2\ dx &= \int_0^\infty x^{m+3}(\partial_x^2(\frac{H}{x^{m+1}}))^2\ dx\\ &\le C_m\int_0^\infty \frac{H^2}{x^{m+3}}\ dx + C_m\int_0^\infty \frac{(\partial_x H)^2}{x^{m+1}}\ dx\\ &\quad + \int_0^\infty \frac{(\partial_x^2 H)^2}{x^{m-1}}\ dx\\ &\overset{(2.51)}{\le} C_m\ [G-F]_{H_m}^2.\end{aligned}$$

Hence $F \in H_{m+4}$ in view of Lemma 2.11.6.

We now turn to the quantitative estimate in (2.46). In order to complete the proof, by the interpolation estimates in Lemma 2.11.4 it suffices to show that

$$[F]_{H_m} + [F]_{H_{m+2}} \lesssim [G]_{H_m} \tag{2.52}$$

and

$$[F]_{H_{m+2}} + [F]_{H_{m+4}} \lesssim [G]_{H_m}. \tag{2.53}$$

For the first one, we differentiate (2.38) m times and test it with $x^{m-1}\ \partial_x^m F$:

$$\begin{aligned}[F]_{H_m}^2 + \langle A_0F, F\rangle_{H_m} \overset{(2.34)}{=} [F]_{H_m}^2 + \frac{1}{2}[F]_{H_{m+2}}^2 &= \langle G, F\rangle_{H_m}\\ &\le \frac{1}{2}\left([F]_{H_m}^2 + [G]_{H_m}^2\right),\end{aligned}$$

whence (2.52). For the second one, we differentiate (2.38) m times and test it with $x^{m-1}\ \partial_x^m A_0F$:

$$\begin{aligned}\langle A_0F, F\rangle_{H_m} + [A_0F]_{H_m}^2 &\overset{(2.34)}{=} \frac{1}{2}[F]_{H_{m+2}}^2 + [A_0F]_{H_m}^2\\ &= \langle A_0F, G\rangle_{H_m}\\ &\le \frac{1}{2}\left([A_0F]_{H_m}^2 + [G]_{H_m}^2\right).\end{aligned}$$

Hence

$$\frac{1}{2}\left([F]_{H_{m+2}}^2 + [A_0F]_{H_m}^2\right) \le [G]_{H_m}^2,$$

and (2.53) follows from (2.35). □

2.5 The linear parabolic equation

In this section we prove Proposition 2.2.1. In fact, Proposition 2.2.1 follows from an equivalent statement in terms of the weighted spaces X_m, which are defined by

$$\|F\|_{X_{m+2}} := \sum_{k=3}^{m} \left([\partial_t F]_{L^2(H_{k-2})} + [F]_{C^0(H_k)} + [F]_{L^2(H_{k+2})}\right), \tag{2.54}$$

$$X_m := \begin{cases} \text{Completion of } \{F \in C_c^\infty([0,\infty)^2) : F_{|x=0} = 0\} \\ \text{with respect to } \|\cdot\|_{X_m}. \end{cases} \tag{2.55}$$

It reads as follows:

Proposition 2.5.1. *Let $m \geq 3$. For given $F_0 \in H_m$ and $G \in L^2(H_{m-2})$, there exists a unique solution $F \in X_{m+2}$ of*

$$\begin{cases} \partial_t F + A_0 F = G \\ F_{|t=0} = F_0. \end{cases} \tag{2.56}$$

It satisfies for all $3 \leq k \leq m$ and all $0 < T \leq \infty$ the estimate

$$\begin{aligned} C^{-1}\ [\partial_t F]_{L^2((0,T);H_{k-2})} + [F]_{C^0((0,T);H_k)} + C^{-1}\ [F]_{L^2((0,T);H_{k+2})} \\ \leq [F_0]_{H_k} + C[G]_{L^2((0,T);H_{k-2})}. \end{aligned} \tag{2.57}$$

Before proving 2.5.1 by a time discretization argument, we first show how Proposition 2.2.1 follows from Proposition 2.5.1.

Proof of Proposition 2.2.1. We first prove (2.26) for $F_0 \in C^\infty([0,\infty))$ with $F(0) = 0$ and $G \in C_c^\infty([0,\infty)^2)$ with $G|_{x=0} = 0$. Let F be the corresponding solution of (2.56) as given by Proposition 2.5.1. For any smooth and compactly supported decomposition $F_0 = F_{0-} + F_{0+}$ and $G = G_- + G_+$, let $F_\pm$ be the corresponding solution of (2.56) with data $F_{0\pm}$, $G_\pm$. Due to the linearity of A_0 we have

$$F = F_+ + F_-. \tag{2.58}$$

By Proposition 2.5.1, for every $s > 0$ and every $k \geq 3$ it holds:

$$\begin{aligned} &C^{-1}\left(s^{-1}[\partial_t F_-]_{L^2(H_{k-3})} + s[\partial_t F_+]_{L^2(H_{k-1})}\right) + s^{-1}[F_-]_{C^0(H_{k-1})} + s[F_+]_{C^0(H_{k+1})} \\ &+ C^{-1}\left(s^{-1}[F_-]_{L^2(H_{k+1})} + s[F_+]_{L^2(H_{k+3})}\right) \\ &\qquad \leq s^{-1}[F_{0-}]_{H_{k-1}} + s[F_{0+}]_{H_{k+1}} + C\left(s^{-1}[G_-]_{L^2(H_{k-3})} + s[G_+]_{L^2(H_{k-1})}\right). \end{aligned} \tag{2.59}$$

Equation (2.59) is preserved when taking the infimum over all decompositions $F_{0\pm}$ and $G_\pm$ of F_0 and G on both sides of the equation (cf. Lemma 2.11.6). On the other hand, due to (2.58), the corresponding solutions $F_\pm$ are a subset of arbitrary decompositions of F. Therefore, we arrive at:

$$\begin{aligned} &\inf_{F=F_-+F_+}\left(s^{-1}[\partial_t F_-]_{L^2(H_{k-3})} + s[\partial_t F_+]_{L^2(H_{k-1})}\right) \\ &\quad + \inf_{F=F_-+F_+}\left(s^{-1}[F_-]_{C^0(H_{k-1})} + s[F_+]_{C^0(H_{k+1})}\right) \\ &\quad + \inf_{F=F_-+F_+}\left(s^{-1}[F_-]_{L^2(H_{k+1})} + s[F_+]_{L^2(H_{k+3})}\right) \\ &\lesssim \inf_{F_0=F_{0-}+F_{0+}}\left(s^{-1}[F_{0-}]_{H_{k-1}} + s[F_{0+}]_{H_{k+1}}\right) \\ &\quad + \inf_{G=G_-+G_+}\left(s^{-1}[G_-]_{L^2(H_{k-3})} + s[G_+]_{L^2(H_{k-1})}\right). \end{aligned}$$

Integrating in ds/s yields (2.26) for all $k \geq 4$. Since in particular $F_0 \in H_{m+1}$ and $G \in L^2(H_{m-1})$, we have due to (2.57) that $[F]_{X_{m+3}} < \infty$. With the help of (2.12), one then easily checks that $F \in X^*_{m+2}$.

Arbitrary data $F_0 \in H^*_m$ and $G \in L^2(H_m)^*$ can be approximated by smooth and compactly supported functions $F_{0\nu} \in H_m$ and $G_\nu \in L^2(H_m)$,

$$F_{0\nu} \to F_0 \text{ in } H^*_m, \qquad G_\nu \to G \text{ in } L^2(H_m)^*.$$

By (2.26), the corresponding solutions $F_\nu \in X^*_{m+2}$ converge to a solution $F \in X^*_{m+2}$ of (2.25) with data F_0 and G. Furthermore, F satisfies (2.26) for all $4 \leq k \leq m$. □

In order to prove Proposition 2.5.1, we begin with a discrete counterpart based on Proposition 2.4.1.

Lemma 2.5.2. *Let $m \geq 3$, $h > 0$. For all $F_0 \in H_m$ and all $G \in H_{m-2}$ there exists a solution $F \in H_{m+2}$ of*

$$F + h\ A_0 F = F_0 + h\ G. \tag{2.60}$$

Furthermore, a positive constant C exists such that

$$C^{-1}h\ [\frac{F-F_0}{h}]^2_{H_{k-2}} + [F]^2_{H_k} + C^{-1}h\ [F]^2_{H_{k+2}} \le [F_0]^2_{H_k} + Ch\ [G]^2_{H_{k-2}} \quad (2.61)$$

for all $3 \le k \le m$.

Proof. The existence of a solution $F \in H_{m+2}$ which satisfies (2.60) follows by Proposition 2.4.1 and scaling in x: a solution of (2.38) with right–hand side $F_0(\hat{x}) + h\ G(\hat{x})$ turns into a solution of (2.60) by the change of variables $x = \sqrt{h}\ \hat{x}$.

By (2.60), we have for $3 \le k \le m$

$$\partial_x^{k-2} F + h\ \partial_x^{k-2} A_0 F = \partial_x^{k-2} F_0 + h\ \partial_x^{k-2} G. \quad (2.62)$$

Testing (2.62) by $2x^{k-3}\ \partial_x^{k-2} A_0 F$ and integrating, we see that

$$\begin{aligned}
[F]^2_{H_k} + 2h[A_0F]^2_{H_{k-2}} &\overset{(2.34)}{=} \langle F, A_0 F\rangle_{H_{k-2}} + 2h\ [A_0F]^2_{H_{k-2}} \\
&\overset{(2.62)}{=} 2\ \langle A_0F, F_0 + h\ G\rangle_{H_{k-2}} \\
&\overset{(2.34)}{=} \langle F, F_0\rangle_{H_k} + 2h\ \langle A_0F, G\rangle_{H_{k-2}} \\
&\le \frac{1}{2}\ [F]^2_{H_k} + \frac{1}{2}\ [F_0]^2_{H_k} + h\ [A_0F]^2_{H_{k-2}} + h\ [G]^2_{H_{k-2}}.
\end{aligned}$$

Therefore, by (2.35),

$$[F]^2_{H_k} + C^{-1}h\ [F]^2_{H_{k+2}} \le [F_0]^2_{H_k} + h\ [G]^2_{H_{k-2}}. \quad (2.63)$$

It follows from (2.60) that

$$(F - F_0)/h = G - A_0 F.$$

Hence

$$\begin{aligned}
h\ [\frac{F-F_0}{h}]^2_{H_{k-2}} &\le 2h\ [G]^2_{H_{k-2}} + 2h\ [A_0F]^2_{H_{k-2}} \\
&\overset{(2.35)}{\le} 2h\ [G]^2_{H_{k-2}} + Ch\ [F]^2_{H_{k+2}}. \quad (2.64)
\end{aligned}$$

Equation (2.61) now follows from (2.63) and (2.64). □

We turn to the proof of Proposition 2.5.1:

Proof of Proposition 2.5.1. We only consider the case $T = \infty$, since the proof directly transfers to arbitrary T. Uniqueness is straightforward: the difference F of two solutions solves (2.56) with $G = 0$ and $F_{|x=0} = 0$, and $\partial_x F \in H^1_{loc}([0,\infty)); L^2((0,\infty))$. Therefore, differentiating (2.56) once, testing it by $\partial_x F$ and integrating, we obtain

$$[F(t)]^2_{H_1} + \int_0^t [F]^2_{H_3}\, dt = 0,$$

whence $F = 0$. The rest of the proof is thus concerned with existence.

For a fixed $h > 0$ and $j \in \mathbb{N}_0$, we let $t^h_j := hj$ and

$$\begin{aligned} G^h_j &:= \frac{1}{h}\int_{t^h_j}^{t^h_{j+1}} G(\hat{t},\cdot)\, d\hat{t}, \\ F^h_0 &:= F_0, \\ F^h_{j+1} &:= \begin{cases} \text{solution of (2.60) with data} \\ F^h_j, G^h_j \text{ as given by Lemma 2.5.2.} \end{cases} \end{aligned}$$

By (2.61) we have for $3 \le k \le m$

$$C^{-1}h[\frac{F^h_{j+1} - F^h_j}{h}]^2_{H_{k-2}} + [F^h_{j+1}]^2_{H_k} + C^{-1}h[F^h_{j+1}]^2_{H_{k+2}} \le [F^h_j]^2_{H_k} + Ch[G^h_j]^2_{H_{k-2}}.$$

Summing over j, we obtain

$$\begin{aligned} C^{-1}h\sum_{j=0}^{\infty}\left([\frac{F^h_{j+1} - F^h_j}{h}]^2_{H_{k-2}} + [F^h_{j+1}]^2_{H_{k+2}}\right) + \sup_j [F^h_{j+1}]^2_{H_k} \\ \le [F_0]^2_{H_k} + C[G]^2_{L^2(H_{k-2})}. \end{aligned} \tag{2.65}$$

Define F_h by

$$F_h = \sum_{j=0}^{\infty}\left(\frac{t^h_{j+1} - t}{h}F^h_j + \frac{t - t^h_j}{h}F^h_{j+1}\right)\chi_{[t^h_j, t^h_{j+1})}$$

and G_h by

$$G_h = \sum_{j=0}^{\infty} G^h_j\ \chi_{[t^h_j, t^h_{j+1})}.$$

Clearly

$$G_h \to G \qquad\qquad \text{in } L^2(H_{m-2}).$$

We also have

$$F_{ht} = \frac{F^h_{j+1} - F^h_j}{h} \qquad\qquad \text{on } [t^h_j, t^h_{j+1}) \tag{2.66}$$

and

$$[F_h]_{H_k} \le \max\left\{[F^h_j]_{H_k}, [F^h_{j+1}]_{H_k}\right\} \qquad\qquad \text{on } [t^h_j, t^h_{j+1}).$$

Therefore, by (2.65), we have that

$$C^{-1}[F_{ht}]^2_{L^2(H_{k-2})} + [F_h]^2_{C^0(H_k)} + C^{-1}[F_h]^2_{L^2(H_{k+2})} \le [F_0]^2_{H_k} + C[G]^2_{L^2(H_{k-2})} \tag{2.67}$$

for all $3 \le k \le m$. With help of the time derivative of F and the initial data, locally we can also control low semi–norms. For $T > 1$ and $M > 1$, we have:

$$\sup_{t\in(0,T)} [F_h]^2_{H_1} \lesssim [F_0]^2_{H_1} + T\int_0^T [\partial_t F_h]^2_{H_1}\, dt \overset{(2.67)}{\lesssim} T\left(\|F_0\|^2_{H_m} + \|G\|^2_{L^2(H_{m-2})}\right); \tag{2.68}$$

using Lemma 2.11.4,

$$\sup_{t\in(0,T)} [F_h]^2_{H_2} \lesssim \sup_{t\in(0,T)} \left([F_h]^2_{H_1} + [F_h]^2_{H_3}\right) \tag{2.69}$$

$$\overset{(2.67),(2.68)}{\lesssim} T\left(\|F_0\|^2_{H_m} + \|G\|^2_{L^2(H_{m-2})}\right); \tag{2.70}$$

using that $F(t,0) = 0$,

$$\sup_{t\in(0,T)} \int_0^M F_h^2\, dx \overset{(2.68)}{\le} M^2 \sup_{t\in(0,T)} [F_h]^2_{H_1} \overset{(2.68)}{\le} M^2 T\left(\|F_0\|^2_{H_m} + \|G\|^2_{L^2(H_{m-2})}\right) \tag{2.71}$$

and

$$\int_0^\infty\int_0^M \partial_t F_h^2\, dx\, dt \le M^2\int_0^\infty [\partial_t F_h]^2_{H_1}\, dt \overset{(2.67)}{\le} M^2\left(\|F_0\|^2_{H_m} + \|G\|^2_{L^2(H_{m-2})}\right). \tag{2.72}$$

Collecting (2.67)–(2.72) we conclude that a subsequence exists such that

$$F_h \rightharpoonup F \text{ in } H^1_{loc}([0,\infty)); H^1_{loc}([0,\infty)) \cap L^2_{loc}([0,\infty); H^{m+2}_{loc}([0,\infty)))$$

and (2.57) holds. Furthermore, (2.67)–(2.71) imply that $F \in X_{m+2}$ (note that $F_h \in X_{m+2}$), and the compact embedding $H^1 \subset\subset C$ implies that $F_{|x=0} = F_0$. To prove (2.56), we note that F_h satisfies the approximate equation

$$\partial_t F_h + A_0 \tilde{F}_h = G_h, \tag{2.73}$$

where

$$\tilde{F}_h := F^h_{j+1} \qquad \text{on } [t^h_j, t^h_{j+1}). \tag{2.74}$$

Also $\tilde{F}_h$ is uniformly bounded: by (2.65) and (2.74), we have for all $3 \le k \le m$

$$[\tilde{F}_h]^2_{C^0(H_k)} + [\tilde{F}_h]^2_{L^2(H_{k+2})} \lesssim [F_0]^2_{H_k} + [G]^2_{L^2(H_{k-2})}.$$

By (2.68)–(2.71), locally in space–time we also have uniform control on low semi–norms of $\tilde{F}_h$ since $\tilde{F}_h(t) = F_h(t^h_{j+1})$ for $t \in [t^h_j, t^h_{j+1})$. Therefore

$$\tilde{F}_h \rightharpoonup \tilde{F} \text{ in } L^2_{loc}([0,\infty)); H^1_{loc}([0,\infty)) \cap L^2_{loc}([0,\infty); H^{m+2}_{loc}(0,\infty)),$$

and passing to the limit in (2.73) we obtain that

$$\partial_t F + A_0 \tilde{F} = G.$$

In order to see that $\tilde{F} = F$, it suffices to notice that for any $M > 0$

$$\begin{aligned}\int_0^\infty \int_0^M |\tilde{F}_h - F_h|^2 \, dx \, dt &= \sum_{j=0}^\infty \int_{t^h_j}^{t^h_{j+1}} \int_0^M (t^h_{j+1} - t)^2 \frac{(F^h_{j+1} - F^h_j)^2}{h^2} \, dx \, dt \\ &\overset{(2.72)}{\le} h^2 \int_0^\infty \int_0^M (\partial_t F_h)^2 \, dx \, dt \\ &\overset{(2.66)}{\to} 0 \qquad \text{as } h \to 0.\end{aligned}$$

□

2.6 The main estimate for the nonlinear operator

In this section we prove Proposition 2.2.2. We restrict ourselves to the case $T = \infty$; the proof directly extends to arbitrary T. The result will be achieved by splitting G into $\tilde{G}$ which additionally satisfies $\partial_x^3 \tilde{G}_{|x=0} = 0$ and a remainder of the form $\partial_x^3 G_{|x=0}(t)\ \eta(x)$, and by splitting F into $\tilde{F}$ which additionally satisfies $\partial_x^2 \tilde{F}_{|x=0} = 0$ and a remainder of the form $\partial_x^2 F_{|x=0}(t)\ \xi(x)$. This way, Proposition 2.2.2 will follow from the following two lemmata.

Lemma 2.6.1. *For any given $F, G \in C_c([0,\infty)^2)$ with $F_{|x=0} = 0$ and $\partial_x^3 G_{|x=0} = 0$ we have*

$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right)]_{L^2(H_2)^*} \ \lesssim\ [F]_{C^0(H_4)^*}\ [G]_{L^2(H_6)^*}.$$

Lemma 2.6.2. *For any given $F, G \in C_c([0,\infty)^2)$ with $F_{|x=0} = 0$ and $\partial_x^2 F_{|x=0} = 0$ we have*

$$\begin{aligned} &[\partial_x \left((F - x\ \partial_x F_{|x=0})\ (\partial_x^3 G - \partial_x^3 G_{|x=0}) \right)]_{L^2(H_2)^*} \\ &\quad \lesssim\ [F]_{C^0(H_4)^*}\ \|x^4\ \partial_x^8 G\|_{L^2(L^1)}. \end{aligned}$$

We will need Lemma 2.1.2 in the following form:

Corollary 2.6.3. *Let $F \in [F]_{H_4^*}$. Then*

$$\begin{aligned} &\|x^{-2}\ (F - x\ \partial_x F(0))\|_{C^0} + \|x^{-1}\ (\partial_x F - \partial_x F(0))\|_{C^0} \\ &\quad + \|\partial_x^2 F\|_{C^0} + \|x\ \partial_x^3 F\|_{C^0} \lesssim [F]_{H_4^*}. \end{aligned}$$

Proof. By Lemma 2.1.2, we have that $\|\partial_x^2 F\|_{C^0} \lesssim [F]_{H_4^*}$. Integrating yields

$$\sup_{\hat{x}\in(0,x)} |\partial_x F - \partial_x F(0)| \le \int_0^x |\partial_x^2 F|\ d\hat{x} \le x\ \|\partial_x^2 F\|_{C^0}, \tag{2.75}$$

$$\sup_{\hat{x}\in(0,x)} |F - x\ \partial_x F(0)| \le \int_0^x |\partial_x F - \partial_x F(0)|\ d\hat{x} \overset{(2.75)}{\lesssim} x^2\ \|\partial_x^2 F\|_{C^0}.$$

Finally, by application of Lemma 2.11.3 and Lemma 2.1.1,

$$\|x\ \partial_x^3 F\|_{C^0} \lesssim [F]_{H_4} \lesssim [F]_{H_4^*}.$$

□

Proof of Lemma 2.6.1. We fix F and consider the linear map

$$G \quad \mapsto \quad \partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right).$$

Since according to Lemma 2.1.3, $[F]_{C^0(H_4^*)} \leq [F]_{C^0(H_4)^*}$, by the definition of the semi–norm $L^2(H_2)^*$ it is enough to show

$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right)]_{L^2(H_1)} \lesssim [F]_{C^0(H_4^*)}\ [G]_{L^2(H_5)},$$
$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right)]_{L^2(H_3)} \lesssim [F]_{C^0(H_4^*)}\ [G]_{L^2(H_7)}.$$

These two estimates can be "disintegrated" in time:

$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right)]_{H_1} \lesssim [F]_{H_4^*}\ [G]_{H_5}, \tag{2.76}$$
$$[\partial_x \left((F - x\ \partial_x F_{|x=0})\ \partial_x^3 G \right)]_{H_3} \lesssim [F]_{H_4^*}\ [G]_{H_7}, \tag{2.77}$$

where now we think of F and G as functions of x only.

We start with (2.76):

$$\begin{aligned}
&[\partial_x \left((F - x\ \partial_x F(0))\ \partial_x^3 G \right)]_{H_1} \\
&\quad = \|\partial_x^2 \left((F - x\ \partial_x F(0))\ \partial_x^3 G \right)\|_{L^2} \\
&\quad \lesssim \|\partial_x^2 F\ \partial_x^3 G\|_{L^2} + \|(\partial_x F - \partial_x F(0))\ \partial_x^4 G|\|_{L^2} \\
&\qquad + \|(F - x\ \partial_x F(0))\ \partial_x^5 G\|_{L^2} \\
&\quad \leq \|\partial_x^2 F\|_{C^0}\ \|\partial_x^3 G\|_{L^2} + \|x^{-1}\ (\partial_x F - \partial_x F(0))\|_{C^0}\ \|x\ \partial_x^4 G\|_{L^2} \\
&\qquad + \|x^{-2}\ (F - x\ \partial_x F(0))\|_{C^0}\ \|x^2\ \partial_x^5 G\|_{L^2}.
\end{aligned}$$

It remains to appeal to Corollary 2.6.3 and to Hardy's inequality (cf. Lemma 2.11.1)

$$\|\partial_x^3 G\|_{L^2} \ \lesssim\ \|x\ \partial_x^4 G\|_{L^2} \ \lesssim\ \|x^2\ \partial_x^5 G\|_{L^2} \ =\ [G]_{H_5}.$$

We now turn to (2.77):

$$\begin{aligned}
&[\partial_x \left((F - x\ \partial_x F(0))\ \partial_x^3 G \right)]_{H_3} \\
&\quad = \|x\ \partial_x^4 \left((F - x\ \partial_x F(0))\ \partial_x^3 G \right)\|_{L^2} \\
&\quad \lesssim \|x\ \partial_x^4 F\ \partial_x^3 G\|_{L^2} + \|x\ \partial_x^3 F\ \partial_x^4 G\|_{L^2} \\
&\qquad + \|x\ \partial_x^2 F\ \partial_x^5 G\|_{L^2} + \|x\ (\partial_x F - \partial_x F(0))\ \partial_x^6 G\|_{L^2} \\
&\qquad + \|x\ (F - x\ \partial_x F(0))\ \partial_x^7 G\|_{L^2} \\
&\quad \leq \|x^{3/2}\ \partial_x^4 F\|_{L^2}\ \|x^{-1/2}\ \partial_x^3 G\|_{C^0} + \|x\ \partial_x^3 F\|_{C^0}\ \|\partial_x^4 G\|_{L^2} \\
&\qquad + \|\partial_x^2 F\|_{C^0}\ \|x\ \partial_x^5 G\|_{L^2} + \|x^{-1}\ (\partial_x F - \partial_x F(0))\|_{C^0}\ \|x^2\ \partial_x^6 G\|_{L^2} \\
&\qquad + \|x^{-2}\ (F - x\ \partial_x F(0))\|_{C^0}\ \|x^3\ \partial_x^7 G\|_{L^2}.
\end{aligned} \tag{2.78}$$

The last four terms in (2.78) can be treated as for (2.76), i. e. with Corollary 2.6.3 and Hardy's inequality

$$\|\partial_x^4 G\|_{L^2} \;\lesssim\; \|x\,\partial_x^5 G\|_{L^2} \;\lesssim\; \|x^2\,\partial_x^6 G\|_{L^2} \;\lesssim\; \|x^3\,\partial_x^7 G\|_{L^2} \;=\; [G]_{H_7}. \quad (2.79)$$

The first term in (2.78) requires a different argument: According to Lemma 2.1.1 we have

$$\|x^{3/2}\,\partial_x^4 F\|_{L^2} \;=\; [F]_{H_4} \;\lesssim\; [F]_{H_4^*}.$$

Finally, because of our assumption $\partial_x^3 G_{|x=0} = 0$ we have

$$\|x^{-1/2}\,\partial_x^3 G\|_{C^0} \;\le\; \sup_x x^{-1/2} \int_0^x |\partial_x^4 G| \;\lesssim\; \|\partial_x^4 G\|_{L^2} \;\stackrel{(2.79)}{\lesssim}\; [G]_{H_7}.$$

□

Proof of Lemma 2.6.2. We fix G and consider the linear map

$$F \quad\mapsto\quad \partial_x\big((F - x\,\partial_x F_{|x=0})\,(\partial_x^3 G - \partial_x^3 G_{|x=0})\big)\,.$$

By definition of the semi–norm $L^2(H_m)^*$, it is enough to show:

$$[\partial_x\big((F - x\,\partial_x F_{|x=0})\,(\partial_x^3 G - \partial_x^3 G_{|x=0})\big)]_{L^2(H_1)} \;\lesssim [F]_{C^0(H_3)}\;\|x^4\,\partial_x^8 G\|_{L^2(L^1)},$$
$$[\partial_x\big((F - x\,\partial_x F_{|x=0})\,(\partial_x^3 G - \partial_x^3 G_{|x=0})\big)]_{L^2(H_3)} \;\lesssim [F]_{C^0(H_5)}\;\|x^4\,\partial_x^8 G\|_{L^2(L^1)}.$$

These two estimates follow from the corresponding pointwise (in time) ones:

$$[\partial_x\big((F - x\,\partial_x F_{|x=0})\,(\partial_x^3 G - \partial_x^3 G_{|x=0})\big)]_{H_1} \lesssim [F]_{H_3}\;\|x^4\,\partial_x^8 G\|_{L^1}, \quad (2.80)$$
$$[\partial_x\big((F - x\,\partial_x F_{|x=0})\,(\partial_x^3 G - \partial_x^3 G_{|x=0})\big)]_{H_3} \lesssim [F]_{H_5}\;\|x^4\,\partial_x^8 G\|_{L^1}, \quad (2.81)$$

where now we think of F and G as functions of x only.

Before proving (2.80) and (2.81), we remark that we have

$$\|\partial_x^3 G\|_{C^0} + \|x\,\partial_x^4 G\|_{C^0} + \|x^2\,\partial_x^5 G\|_{C^0} + \|x^3\,\partial_x^6 G\|_{C^0} + \|x^4\,\partial_x^7 G\|_{C^0} \;\lesssim\; \|x^4\,\partial_x^8 G\|_{L^1}. \quad (2.82)$$

Indeed, if $f = \partial_x^3 G$ and $k \in \{0,1,2,3,4\}$, we have

$$x^k\,\partial_x^k f(x) \;=\; (-1)^{k+1}\,x^k \int_x^\infty \partial_x^5 f(x')\,\frac{1}{(4-k)!}(x-x')^{4-k}\,dx',$$

so that

$$
\begin{aligned}
|x^k\ \partial_x^k f(x)| &\lesssim \int_x^\infty |\partial_x^5 f(x')|\ x^k\ |x-x'|^{4-k}\ dx' \\
&\le \int_x^\infty |\partial_x^5 f(x')|\ x'^4\ dx'\ \le\ \|x^4\ \partial_x^5 f\|_{L^1}.
\end{aligned}
$$

We now turn to (2.80). We have:

$$
\begin{aligned}
&[\partial_x \big((F - x\ \partial_x F(0))\ (\partial_x^3 G - \partial_x^3 G(0))\big)]_{H_1} \\
&\quad = \|\partial_x^2 \big((F - x\ \partial_x F(0))\ (\partial_x^3 G - \partial_x^3 G(0))\big)\|_{L^2} \\
&\quad \lesssim \|\partial_x^2 F\ (\partial_x^3 G - \partial_x^3 G(0))\|_{L^2} + \|(\partial_x F - \partial_x F(0))\ \partial_x^4 G\|_{L^2} \\
&\qquad + \|(F - x\ \partial_x F(0))\ \partial_x^5 G\|_{L^2} \\
&\quad \le 2\ \|\partial_x^2 F\|_{L^2}\ \|\partial_x^3 G\|_{C^0} + \|x^{-1}\ (\partial_x F - \partial_x F(0))\|_{L^2}\ \|x\ \partial_x^4 G\|_{C^0} \\
&\qquad + \|x^{-2}\ (F - x\ \partial_x F(0))\|_{L^2}\ \|x^2\ \partial_x^5 G\|_{C^0}.
\end{aligned}
$$

This estimate implies (2.80) because of (2.82) and of Corollary 2.6.3.

We finally address (2.81).

$$
\begin{aligned}
&[\partial_x \big((F - x\ \partial_x F(0))\ (\partial_x^3 G - \partial_x^3 G(0))\big)]_{H_3} \\
&\quad = \|x\ \partial_x^4 \big((F - x\ \partial_x F(0))\ (\partial_x^3 G - \partial_x^3 G(0))\big)\|_{L^2} \\
&\quad \lesssim \|x\ \partial_x^4 F\ (\partial_x^3 G - \partial_x^3 G(0))\|_{L^2} + \|x\ \partial_x^3 F\ \partial_x^4 G\|_{L^2} + \|x\ \partial_x^2 F\ \partial_x^5 G\|_{L^2} \\
&\qquad + \|x\ (\partial_x F - \partial_x F(0))\ \partial_x^6 G\|_{L^2} + \|x\ (F - x\ \partial_x F(0))\ \partial_x^7 G\|_{L^2} \\
&\quad \le 2\ \|x\ \partial_x^4 F\|_{L^2}\ \|\partial_x^3 G\|_{C^0} + \|\partial_x^3 F\|_{L^2}\ \|x\ \partial_x^4 G\|_{L^2} \\
&\qquad + \|x^{-1}\ \partial_x^2 F\|_{L^2}\ \|x^2\ \partial_x^5 G\|_{L^2} + \|x^{-2}\ (\partial_x F - \partial_x F(0))\|_{L^2}\ \|x^3\ \partial_x^6 G\|_{C^0} \\
&\qquad + \|x^{-3}\ (F - x\ \partial_x F(0))\|_{L^2}\ \|x^4\ \partial_x^7 G\|_{C^0} \\
&\quad \overset{(2.82)}{\lesssim} \|x^4 \partial_x^8 G\|_{L^1}\ \big(\|x\ \partial_x^4 F\|_{L^2} + \|\partial_x^3 F\|_{L^2} + \|x^{-1}\ \partial_x^2 F\|_{L^2} \\
&\qquad + \|x^{-2}\ (\partial_x F - \partial_x F(0))\|_{L^2} + \|x^{-3}\ (F - x\ \partial_x F(0))\|_{L^2}\big).
\end{aligned}
$$

Because of $F(0) = \partial_x^2 F(0) = 0$, $F - x\ \partial_x F(0)$ vanishes to second order in $x = 0$; hence, by Hardy's inequality (cf. Corollary 2.11.2) we have

$$
\begin{aligned}
\|x^{-3}\ (F - x\ \partial_x F(0))\|_{L^2}\ &\lesssim\ \|x^{-2}\ (\partial_x F - \partial_x F(0))\|_{L^2}\ \lesssim\ \|x^{-1}\ \partial_x^2 F\|_{L^2} \\
&\lesssim\ \|\partial_x^3 F\|_{L^2}\ \lesssim\ \|x^2\ \partial_x^5 F\|_{L^2}\ =\ [F]_{H_5}.
\end{aligned}
$$

and (2.81) follows. □

We are now ready to prove Proposition 2.2.2.

Proof of Proposition 2.2.2. We first assume that $F, G \in C_c^\infty([0,\infty))$ with $F|_{x=0} = G|_{x=0} = 0$. We fix two functions $\xi, \eta \in C_c^\infty([0,\infty))$ of the spatial variable only with

$$\begin{array}{rclclcl} \xi(0) &=& \partial_x\xi(0) &=& 0, & & \partial_x^2\xi(0) = 1, \\ \eta(0) &=& \partial_x\eta(0) &=& \partial_x^2\eta(0) = 0, & & \partial_x^3\eta(0) = 1. \end{array} \tag{2.83}$$

We use these function to split G and F into

$$F = \tilde{F} + \partial_x^2 F_{|x=0} \otimes \xi, \quad G = \tilde{G} + \partial_x^3 G_{|x=0} \otimes \eta.$$

Because of (2.83), we have $\partial_x^3 \tilde{G}_{|x=0} = 0$, so that we may apply Lemma 2.6.1 to the couple $(F, \tilde{G})$. Likewise, we have $\partial_x^2 \tilde{F}_{|x=0} = 0$, so that we may apply Lemma 2.6.2 to the couple $(\tilde{F}, \partial_x^3 G_{|x=0} \otimes \eta)$. This yields

$$\begin{aligned}
&[\mathcal{N}(F,G)]_{L^2(H_2)^*} \\
&\quad \lesssim [\mathcal{N}(F,\tilde{G})]_{L^2(H_2)^*} + [\mathcal{N}(\tilde{F}, \partial_x^3 G_{|x=0} \otimes \eta)]_{L^2(H_2)^*} \\
&\qquad + [\mathcal{N}(\partial_x^2 F_{|x=0} \otimes \xi, \partial_x^3 G_{|x=0} \otimes \eta)]_{L^2(H_2)^*} \\
&\quad \lesssim [F]_{C^0(H_4)^*} [\tilde{G}]_{L^2(H_6)^*} + [\tilde{F}]_{C^0(H_4)^*} \|x^4 \partial_x^8 (\partial_x^3 G_{|x=0} \otimes \eta)\|_{L^2(L^1)} \\
&\qquad + [(\partial_x^2 F_{|x=0} \partial_x^3 G_{|x=0}) \otimes \mathcal{N}(\xi,\eta)]_{L^2(H_2)^*} \\
&\quad \lesssim [F]_{C^0(H_4)^*} \big([G]_{L^2(H_6)^*} + [\partial_x^3 G_{|x=0} \otimes \eta]_{L^2(H_6)^*}\big) \\
&\qquad + \big([F]_{C^0(H_4)^*} + [\partial_x^2 F_{|x=0} \otimes \xi]_{C^0(H_4)^*}\big) \|\partial_x^3 G_{|x=0} \otimes (x^4 \partial_x^8 \eta)\|_{L^2(L^1)} \\
&\qquad + [(\partial_x^2 F_{|x=0} \partial_x^3 G_{|x=0}) \otimes \mathcal{N}(\xi,\eta)]_{L^2(H_2)^*}.
\end{aligned}$$

We now appeal to part ii) of Lemma 2.1.3:

$$\begin{aligned}
&[\mathcal{N}(F,G)]_{L^2(H_2)^*} \\
&\quad \lesssim [F]_{C^0(H_4)^*} \big([G]_{L^2(H_6)^*} + \|\partial_x^3 G_{|x=0}\|_{L^2} [\eta]_{H_6^*}\big) \\
&\qquad + \big([F]_{C^0(H_4)^*} + \|\partial_x^2 F_{|x=0}\|_{C^0} [\xi]_{H_4^*}\big) \|\partial_x^3 G_{|x=0}\|_{L^2} \|x^4 \partial_x^8 \eta\|_{L^1} \\
&\qquad + \|\partial_x^2 F_{|x=0}\|_{C^0} \|\partial_x^3 G_{|x=0}\|_{L^2} [\mathcal{N}(\xi,\eta)]_{H_2^*} \\
&\quad \lesssim [F]_{C^0(H_4)^*} \big([G]_{L^2(H_6)^*} + \|\partial_x^3 G_{|x=0}\|_{L^2}\big) \\
&\qquad + \big([F]_{C^0(H_4)^*} + \|\partial_x^2 F_{|x=0}\|_{C^0}\big) \|\partial_x^3 G_{|x=0}\|_{L^2} \\
&\qquad + \|\partial_x^2 F_{|x=0}\|_{C^0} \|\partial_x^3 G_{|x=0}\|_{L^2}.
\end{aligned}$$

We now evoke Lemma 2.1.2:

$$\begin{aligned}
&[\mathcal{N}(F,G)]_{L^2(H_2)^*} \\
&\quad \lesssim [F]_{C^0(H_4)^*}\ \big([G]_{L^2(H_6)^*} + [G]_{L^2(H_6^*)}\big) \\
&\qquad + \big([F]_{C^0(H_4)^*} + [F]_{C^0(H_4^*)}\big)\ [G]_{L^2(H_6^*)} + [F]_{C^0(H_4^*)}\ [G]_{L^2(H_6^*)}.
\end{aligned}$$

We conclude the proof of estimate (2.27) for smooth F and G using part i) of Lemma 2.1.3. By density, (2.27) holds for all $F, G \in X_6^*$. Finally, since $\mathcal{N}$ is an operator which maps $C_c^\infty([0,\infty))^2 \to C_c^\infty([0,\infty))$, we obtain that $\mathcal{N}$ maps $X_6^* \to L^2(H_2)^*$, and therefore $\mathcal{N}(F,G) \in L^2(H_2)^*$. ☐

2.7 Existence

In this section we complete the proof of Theorem 2.1.4. Uniqueness and (2.21) have already been shown in Section 2.2. Hence, we are left to prove:

Proposition 2.7.1. *There exists an $\epsilon > 0$ s.t. for all $F_0 \in H_6^*$ satisfying* (2.20) *there exists $F \in X_6^*$ which solves* (2.1).

Proof. For $\delta > 0$ to be chosen later, let

$$X = \{\ F \in X_6^* : \|F\|_{X_6^*} \leq \delta\ \} \tag{2.84}$$

and define

$$S(F) := L^*\mathcal{N}(F,F), \tag{2.85}$$

where L^*G is the unique solution $F \in X_6^*$ of (2.25) with initial data F_0 as given by Proposition 2.2.1:

$$\partial_t S(F) + A_0(S(F)) = \mathcal{N}(F,F).$$

Hence, in order to prove Proposition 2.7.1 it suffices to show that S has a fixed point in X. In fact, we shall prove that S is a contraction in X. By Proposition 2.2.1,

$$\|S(F)\|_{X_6^*} \lesssim [F_0]_{H_4^*} + [\mathcal{N}(F,F)]_{L^2(H_2)^*}. \tag{2.86}$$

Furthermore, since the difference of two functions $F, \tilde{F} \in X$ satisfies

$$\begin{gathered}
\partial_t(S(F) - S(\tilde{F})) + A_0(S(F) - S(\tilde{F})) = \mathcal{N}(F,F) - \mathcal{N}(\tilde{F},\tilde{F}), \\
(F - \tilde{F})(0,x) = 0,
\end{gathered}$$

again by Proposition 2.2.1 and the definition (2.4) of $\mathcal{N}$ we see that

$$\begin{aligned}\|S(F) - S(\tilde{F})\|_{X_6^*} &\leq [\mathcal{N}(F,F) - \mathcal{N}(\tilde{F},\tilde{F})]_{L^2(H_2)^*} \\ &\lesssim [\mathcal{N}(F, F-\tilde{F})]_{L^2(H_2)^*} + [\mathcal{N}(F-\tilde{F},\tilde{F})]_{L^2(H_2)^*}. \end{aligned} \tag{2.87}$$

We now argue as in the proof of Proposition 2.2.3: by Proposition 2.2.2, we have $\mathcal{N}(F,G) \in L^2(H_2)^*$ and

$$[\mathcal{N}(F,F)]_{L^2(H_2)^*} \lesssim \|F\|^2_{X_6^*}.$$

Therefore for $F \in X$

$$\|SF\|_{X_6^*} \overset{(2.86)}{\lesssim} \epsilon + \delta^2. \tag{2.88}$$

Note that, as a consequence of (2.88), $S(0) \in X$ for $\epsilon \ll \delta$, hence X is non–empty. In view of (2.87) and Proposition 2.10.1 we get

$$\begin{aligned}\|SF - S\tilde{F}\|_{X_6^*} &\lesssim \ (\|F\|_{X_6^*} + \|\tilde{F}\|_{X_6^*})\ \|F-\tilde{F}\|_{X_6^*}. \\ &\lesssim \ \delta\ \|F-\tilde{F}\|_{X_6^*}. \end{aligned} \tag{2.89}$$

Choosing $\delta = \sqrt{\epsilon}$ and ϵ sufficiently small, (2.88) and (2.89) turn into

$$\|SF\|_{X_6^*} \leq \delta,$$

$$\|SF - S\tilde{F}\|_{X_6^*} \leq \frac{1}{2}\|F-\tilde{F}\|_{X_6^*},$$

and the proof is complete. □

2.8 Regularity

In this section we prove Theorem 2.1.5. We begin with higher–order estimates for the nonlinear operator $\mathcal{N}$, defined in (2.4). The key point of the next proposition is, that the constant in front of the highest order term $[F]_{H_{k+2}}$ does not depend on k:

Proposition 2.8.1. *Let $k \geq 5$ and $F \in H_{k+2}$. Then*

$$[\mathcal{N}(F,F)]_{H_{k-2}} \leq \ C\ [F]_{H_4^*}[F]_{H_{k+2}} + C'_k\ [F]_{H_6^*}[F]_{H_k}. \tag{2.90}$$

Proof. We have

$$
\begin{aligned}
[\mathcal{N}(F,F)]_{H_{k-2}} &= \|x^{\frac{k-3}{2}}\ \partial_x^{k-1}\left((F - x\ \partial_x F(0))(\partial_x^3 F - \partial_x^3 F(0)\right)\|_{L^2}. \\
&\le \|x^{\frac{k-3}{2}}\ \partial_x^{k-1} F\ (\partial_x^3 F - \partial_x^3 F(0))\|_{L^2} \\
&\quad + \sum_{j=0}^{k-2} \binom{k-1}{j} \|x^{\frac{k-3}{2}}\ \left(\partial_x^j (F - x\ \partial_x F(0))\ \partial_x^{k+2-j} F\right)\|_{L^2}.
\end{aligned}
$$

We have by $\mathcal{N}$ by Lemma 2.1.2 and Corollary 2.11.2:

$$
\|x^{\frac{k-3}{2}}\ \partial_x^{k-1} F\ (\partial_x^3 F - \partial_x^3 F(0))\|_{L^2} \lesssim [F]_{H_6^*}\ \|x^{\frac{k-3}{2}}\ \partial_x^{k-1} F\|_{L^2} \lesssim [F]_{H_6^*}[F]_{H_k}.
$$

Therefore it remains to estimate

$$
\binom{k-1}{j} \|x^{\frac{k-3}{2}}\ \partial_x^j (F - x\ \partial_x F(0))\ \partial_x^{k+2-j} F\|_{L^2}, \qquad j = 0, \dots, k-2.
$$

For $0 \le j \le 3$ we apply Corollary 2.6.3 and Corollary 2.11.2 to get

$$
\begin{aligned}
&\binom{k-1}{j} \|x^{\frac{k-3}{2}}\ \partial_x^j (F - x\ \partial_x F(0))\ \partial_x^{k+2-j} F\|_{L^2} \\
&\lesssim k^j\ \|x^{j-2}\ \partial_x^j (F - x\ \partial_x F(0))\|_{C^0}\ \|x^{\frac{k+1-2j}{2}}\ \partial_x^{k+2-j} F\|_{L^2} \\
&\lesssim\ [F]_{H_4^*} \|x^{\frac{k+1}{2}}\ \partial_x^{k+2} F\|_{L^2} \\
&= [F]_{H_4^*}[F]_{H_{k+2}}
\end{aligned}
$$

(note that the constant in Hardy's inequality ensures that the estimate is independent of k). This already proves (2.90) for $k = 5$. For $k \ge 6$ and $4 \le j \le k-2$, we estimate, using Lemma 2.11.1, Lemma 2.11.3 and Corollary 2.11.5,

$$
\begin{aligned}
&\binom{k-1}{j} \|x^{\frac{k-3}{2}}\ \partial_x^j (F - x\ \partial_x F(0))\ \partial_x^{k+2-j} F\|_{L^2} \\
&\le C_k\ \|x^{\frac{j-2}{2}}\ \partial_x^j F\|_{C^0}\ \|x^{\frac{k-1-j}{2}}\ \partial_x^{k+2-j} F\|_{L^2} \\
&\le C_k\ \|x^{\frac{j-1}{2}}\ \partial_x^{j+1} F\|_{L^2} \|x^{\frac{k-1-j}{2}}\ \partial_x^{k+2-j} F\|_{L^2} \\
&\le C_k\ [F]_{H_{j+2}}[F]_{H_{k+4-j}} \\
&\le C_k\ [F]_{H_6}[F]_{H_k} \\
&\overset{(2.12)}{\le}\ C_k\ [F]_{H_6^*}[F]_{H_k}.
\end{aligned}
$$

□

To the proof of Theorem 2.1.5 we premise the following intermediate result:

Proposition 2.8.2. *There exists an* $\epsilon > 0$ *such that if* $F_0 \in H_m$, $m \geq 5$, *satisfies* (2.20)*, then the solution* F *of* (2.1) *given by Theorem 2.1.4 is such that* $F \in X_{m+2}$ *(see* (2.55)*), and furthermore*

$$[F]_{C^0(H_k)} + [F]_{L^2(H_{k+2})} \leq C_k \, \|F_0\|_{H_k} \qquad \text{for all } 5 \leq k \leq m.$$

Proof. Since F has been obtained in Theorem 2.1.4 as the unique fixed point of the map S (see (2.85)) on X (see (2.84)), by Proposition 2.5.1 the sequence $F^{(n)}$ defined as the unique solution of

$$\begin{cases} \partial_t F^{(n+1)} + A_0 F^{(n+1)} = \mathcal{N}(F^{(n)}, F^{(n)}) \\ F^{(n)}_{|x=0} = F_0 \end{cases}$$

(with, say, $F^{(0)} = 0$) satisfies $F \in X_{m+2}$ and converges to F in X. Hence, Proposition 2.8.2 follows immediately by dominated convergence once we have shown that

$$[\partial_t F^{(n+1)}]_{L^2(H_{k-2})} + [F^{(n+1)}]_{C^0(H_k)} + [F^{(n+1)}]_{L^2(H_{k+2})} \leq C_k \, \|F_0\|_{H_k} \tag{2.91}$$

for all $5 \leq k \leq m$ and all n sufficiently large (cf. Lemma 2.11.6). To see (2.91), we write:

$$\begin{aligned} &[\partial_t F^{(n+1)}]_{L^2(H_{k-2})} + [F^{(n+1)}]_{C^0(H_k)} + [F^{(n+1)}]_{L^2(H_{k+2})} \\ &\overset{(2.57)}{\lesssim} [F_0]_{H_k} + [\mathcal{N}(F^{(n)}, F^{(n)})]_{L^2(H_{k-2})} \\ &\overset{(2.90)}{\lesssim} [F_0]_{H_k} + [F^{(n)}]_{C^0(H_4^*)} [F^{(n)}]_{L^2(H_{k+2})} + C_k' \, \|[F^{(n)}]_{H_6^*} [F^{(n)}]_{H_k}\|_{L^2}. \end{aligned} \tag{2.92}$$

We first assume that $k \leq 6$, hence in this case the constant C_k' is universal. We use Cauchy–Schwarz in the form

$$\|[F^{(n)}]_{H_6^*} [F^{(n)}]_{H_k}\|_{L^2} \leq [F^{(n)}]_{L^2(H_6^*)} [F^{(n)}]_{C^0(H_k)}.$$

Since $\|F^{(n)} - F\|_X \to 0$, using (2.20) and (2.21) we may absorb the last two terms on the right hand side for n sufficiently large and ϵ sufficiently small, thus getting (2.91) for $5 \leq k \leq 6$:

$$[\partial_t F^{(n+1)}]_{L^2(H_{k-2})} + [F^{(n+1)}]_{C^0(H_k)} + [F^{(n+1)}]_{L^2(H_{k+2})} \lesssim [F_0]_{H_k}, \quad 5 \leq k \leq 6. \tag{2.93}$$

For $k \geq 6$ we only absorb the first term on the right hand side of (2.92), whereas for the second one we use Cauchy–Schwarz in the form

$$\|[F^{(n)}]_{H_6^*}[F^{(n)}]_{H_k}\|_{L^2} \leq [F^{(n)}]_{C^0(H_6^*)}[F^{(n)}]_{L^2(H_k)}$$

and arrive for n sufficiently large at

$$\begin{aligned}
&[\partial_t F^{(n+1)}]_{L^2(H_{k-2})} + [F^{(n+1)}]_{C^0(H_k)} + [F^{(n+1)}]_{L^2(H_{k+2})} \\
&\overset{(2.90)}{\lesssim} [F_0]_{H_k} + C_k' \, [F^{(n)}]_{C^0(H_6^*)}[F^{(n)}]_{L^2(H_k)}. \\
&\overset{(2.93)}{\lesssim} [F_0]_{H_k} + C_k' \, [F^{(n)}]_{L^2(H_k)}.
\end{aligned}$$

A straightforward induction on k starting from (2.93) concludes the proof. □

We are now ready to prove Theorem 2.1.5.

Proof of Theorem 2.1.5. Fix $\eta > 0$, and let $\tau_n \uparrow \eta$. In view of (2.21), (2.17) and (2.12), there exists $t_1 \in (0, \tau_1)$ such that $[F(t_1)]_{H_6} < \infty$. Choosing $F(t_1)$ as an initial data in Proposition 2.8.2, we obtain in particular that

$$\int_{t_1}^{\infty} [F]_{H_8}^2 \, dt < \infty.$$

Hence, there exists $t_2 \in (\tau_1, \tau_2)$ such that $[F]_{H_8}^2 < \infty$. Iterating this argument, we conclude that $F(\cdot + \eta, \cdot) \in X_{m+2}$ for all m. A reiterated application of Corollary 2.11.2 then implies that $\partial_x F(\cdot + \eta, \cdot) \in C^0([0,\infty), H^s([0,\infty))$ for all s. Regularity in time then follows by differentiating the equation, and the arbitrariness of η completes the proof. □

2.9 Decay of high derivatives

In this section we prove Theorem 2.1.6. We shall use the following:

Lemma 2.9.1. *Let $k \geq 3$. If $F \in X_{k+2}$, then $[F]_{H_k}^2 \in W_{loc}^{1,1}([0,\infty))$ and*

$$\partial_t [F]_{H_k}^2 + [F]_{H_{k+2}}^2 = 4\langle A_0 F, \partial_t F + A_0 F\rangle_{H_{k-2}}. \tag{2.94}$$

Proof. The lemma is not trivial since a priori there is no control on $\partial_x^k \partial_t F$. However its proof is standard, and we sketch it for completeness. For $T > 0$ fixed, let $\varphi \in C_c^\infty([0,\infty))$ such that $\varphi = 1$ on $(0,T)$ and let

$$\tilde{F} = \begin{cases} \varphi(t)F(t) & t \geq 0 \\ \varphi(-t)F(-t) & t < 0. \end{cases}$$

Let η be a mollifier, $\eta_\epsilon(t) = \epsilon^{-1}\eta(t\epsilon^{-1})$, and $\tilde{F}_\epsilon = \eta_\epsilon * \tilde{F}$. As is well known,

$$\begin{cases} [\tilde{F}_\epsilon]_{H_j} & \to & [\tilde{F}]_{H_j} & \text{in } L^2(\mathbb{R}) & \text{for } 1 \leq j \leq k+2, \\ [\partial_t \tilde{F}_\epsilon]_{H_j} & \to & [\partial_t \tilde{F}]_{H_j} & \text{in } L^2(\mathbb{R}) & \text{for } 1 \leq j \leq k-2. \end{cases} \tag{2.95}$$

Since each $\tilde{F}_\epsilon$ is smooth in time, we have:

$$[\tilde{F}_\epsilon(t_2)]_{H_k}^2 - [\tilde{F}_\epsilon(t_1)]_{H_k}^2 = 2\int_{t_1}^{t_2} \langle \tilde{F}_\epsilon, \partial_t \tilde{F}_\epsilon \rangle_{H_k} \, dt \overset{(2.34)}{=} 4\int_{t_1}^{t_2} \langle A_0 \tilde{F}_\epsilon, \partial_t \tilde{F}_\epsilon \rangle_{H_{k-2}} \, dt. \tag{2.96}$$

Let $\epsilon, \delta > 0$. Choosing $t = t_2$ and $t_1 = -\infty$ in (2.96), we see that

$$\begin{aligned} &[\tilde{F}_\epsilon(t)]_{H_k}^2 - [\tilde{F}_\delta(t)]_{H_k}^2 \\ &\quad = 4\int_{-\infty}^{t} \Big(\langle A_0 \tilde{F}_\epsilon, \partial_t \tilde{F}_\epsilon - \partial_t \tilde{F}_\delta \rangle_{H_{k-2}} + \langle A_0 \tilde{F}_\epsilon - A_0 \tilde{F}_\delta, \partial_t \tilde{F}_\delta \rangle_{H_{k-2}} \Big) \, dt \\ &\quad \overset{(2.35)}{\leq} \left(\int_{-\infty}^{\infty} \Big([\tilde{F}_\epsilon^2 - \tilde{F}_\delta]_{H_{k+2}} + [\partial_t \tilde{F}_\epsilon^2 - \partial_t \tilde{F}_\delta]_{H_{k-2}} \Big) \, dt \right)^{\frac{1}{2}} \\ &\qquad\quad \left(\int_{-\infty}^{\infty} \Big([\tilde{F}_\epsilon]_{H_{k+2}}^2 + [\partial_t \tilde{F}_\delta]_{H_{k-2}}^2 \Big) \, dt \right)^{\frac{1}{2}} \\ &\quad \overset{(2.95)}{=} o_{\delta,\epsilon}(1). \end{aligned}$$

Hence, $[\tilde{F}_\epsilon]_{H_k}^2$ is a Cauchy sequence in $C([0,T])$, and therefore $[\tilde{F}_\epsilon]_{H_k} \to [\tilde{F}]_{H_k}$ in $C([0,T])$ (the identification of the limit follows from (2.95)). Since $\tilde{F} = F$ in $[0,T]$ and T is arbitrary, this proves the continuity of $[F]_{H_k}^2$. Passing to the limit in (2.96) we see that for $0 \leq t < t_2$

$$\frac{[F(t_2)]_{H_k}^2 - [F(t_1)]_{H_k}{}^2}{t_2 - t_1} = \frac{4}{t_2 - t_1} \int_{t_1}^{t_2} \langle A_0 F, \partial_t F \rangle_{H_{k-2}} \, dt.$$

Passing to the limit as $t_2 \to t$ and using (2.35) we complete the proof. ∎

We are now ready to prove Theorem 2.1.6.

Proof of Theorem 2.1.6. The starting point is:

$$\begin{aligned}\partial_t[F]^2_{H_k} + [F]^2_{H_{k+2}} &\overset{(2.94)}{=} 4\langle A_0 F, \mathcal{N}(F,F)\rangle_{H_{k-2}} \\ &\overset{(2.35)}{\leq} \frac{1}{2}[F]^2_{H_{k+2}} + 2\,[\mathcal{N}(F,F)]^2_{H_{k-2}} \\ &\overset{(2.90)}{\leq} \frac{1}{2}[F]^2_{H_{k+2}} + C\,[F]^2_{H_4^*}[F]^2_{H_{k+2}} + C_k'\,[F]^2_{H_6^*}[F]^2_{H_k},\end{aligned}$$

which holds for all $t > 0$ and all $k \geq 5$. Therefore

$$\partial_t[F]^2_{H_k} + [F]^2_{H_{k+2}} \leq C\,[F]^2_{H_4^*}[F]^2_{H_{k+2}} + C_k'\,[F]^2_{H_6^*}[F]^2_{H_k} \tag{2.97}$$

Note that the constant in the first term on the right-hand side of (2.97) does not depend on k, and that by Theorem 2.1.4 and (2.17), $\sup_t [F]_{H_4^*} \lesssim \epsilon$. Therefore, for ϵ sufficiently small we arrive at

$$\partial_t[F]^2_{H_k} + [F]^2_{H_{k+2}} \leq C_k'\,[F]^2_{H_6^*}[F]^2_{H_k}. \tag{2.98}$$

Bringing this together with the weights in time, as in the statement of the theorem, we get:

$$\begin{aligned}\partial_t(t^{\frac{k-4}{2}}[F]^2_{H_k}) + t^{\frac{k-4}{2}}[F]^2_{H_{k+2}} &= t^{\frac{k-4}{2}}(\partial_t[F]^2_{H_k} + [F]^2_{H_{k+2}}) + C_k\,t^{\frac{k-6}{2}}[F]^2_{H_k} \\ &\overset{(2.98)}{\lesssim} C_k'\,t^{\frac{k-4}{2}}[F]^2_{H_6^*}[F]^2_{H_k} + C_k\,t^{\frac{k-6}{2}}[F]^2_{H_k}. \end{aligned} \tag{2.99}$$

In the case $k = 6$ this turns by interpolation (cf. Corollary 2.11.5) into

$$\partial_t(t[F]^2_{H_6}) + t[F]^2_{H_8} \lesssim t[F]^2_{H_6^*}[F]^2_{H_6} + [F]^2_{H_6} \lesssim t[F]^2_{H_4}[F]^2_{H_8} + [F]^2_{H_6}.$$

For $\epsilon \ll 1$ we absorb on the left hand side and integrate in time. This yields

$$\sup_t\ t[F]^2_{H_6} + \int_0^\infty t[F]^2_{H_8}\,dt \lesssim \int_0^\infty [F]^2_{H_6}. \overset{(2.21)}{\lesssim} [F_0]^2_{H_4^*}. \tag{2.100}$$

The same argument, using also (2.100), yields the analogous statement for $k = 8$:

$$\sup_t\ t^2[F]^2_{H_8} + \int_0^\infty t^2[F]^2_{H_{10}}\,dt \lesssim [F_0]^2_{H_4^*}. \tag{2.101}$$

Interpolation between (2.21) and (2.101), using Lemma 2.1.1, yields

$$\sup_t \; t[F]^2_{H_6^*} + \int_0^\infty t[F]^2_{H_8^*}\; dt \lesssim [F_0]^2_{H_4^*}. \tag{2.102}$$

For arbitrary k, this argument would break down due to the k dependence of the constants. We instead integrate (2.99) directly to obtain

$$\begin{aligned} &\sup_t \; (t^{\frac{k-4}{2}}[F]^2_{H_k}) + \int_0^\infty t^{\frac{k-4}{2}}[F]^2_{H_{k+2}}\; dt \\ &\quad \leq C_k' \int_0^\infty t^{\frac{k-4}{2}}[F]^2_{H_6^*}[F]^2_{H_k}\; dt + C_k \int_0^\infty t^{\frac{k-6}{2}}[F]^2_{H_k} \\ &\quad \overset{(2.102)}{\leq} C_k \int_0^\infty t^{\frac{k-6}{2}}[F]^2_{H_k}\; dt. \end{aligned}$$

The last equation is the basis for an induction argument, starting from $k = 6$, which yields for all even $k \geq 4$

$$\sup_t \; (t^{\frac{k-4}{2}}[F]^2_{H_k}) + \int_0^\infty t^{\frac{k-4}{2}}[F]^2_{H_{k+2}}\; dt \leq C_k \; [F_0]^2_{H_4^*}. \tag{2.103}$$

Interpolation between the case $k = 4$ (cf. (2.21)) and (2.103) yields (2.23) for all interpolation norms in between, and completes the proof of Theorem 2.1.6. □

2.10 Convergence to the steady state

In this section we prove Theorem 2.1.7. We premise another short lemma about the nonlinear part of the operator:

Proposition 2.10.1. *Let $F \in H_6^*$. Then*

$$[\mathcal{N}(F, F)]_{H_1} \lesssim [F]_{H_3}[F]_{H_6^*}. \tag{2.104}$$

Proof. We write using Lemma 2.1.2, Corollary 2.6.3 and Corollary 2.11.2:

$$\begin{aligned}
[\mathcal{N}(F,F)]_{H_1} &= \|\partial_x^2\left((F - x\,\partial_x F(0))(\partial_x^3 F - \partial_x^3 F(0))\right)\|_{L^2} \\
&\lesssim \|\partial_x^2 F\,(\partial_x^3 F - \partial_x^3 F(0))\|_{L^2} + \|(\partial_x F - \partial_x F(0))\,\partial_x^4 F\|_{L^2} \\
&\qquad + \|(F - x\,\partial_x F(0))\,\partial_x^5 F\|_{L^2} \\
&\leq \|\partial_x^2 F\|_{L^2}\|\partial_x^3 F - \partial_x^3 F(0)\|_{C^0} \\
&\qquad + \|x^{-1}\,(\partial_x F - \partial_x F(0))\|_{C^0}\,\|x\,\partial_x^4 F\|_{L^2} \\
&\qquad + \|x^{-2}\,(F - x\,\partial_x F(0))\|_{C^0}\|x^2\,\partial_x^5 F\|_{L^2} \\
&\leq [F]_{H_3}[F]_{H_6^*} + [F]_{H_4^*}[F]_{H_5} \\
&\leq [F]_{H_3}[F]_{H_6^*}.
\end{aligned}$$

□

Proof of Theorem 2.1.7. We first show that

$$\sup_t\,[F]_{H_1}^2 + \int_0^\infty [F]_{H_3}^2\,dt \lesssim [F_0]_{H_1}^2. \tag{2.105}$$

We write using Cauchy–Schwarz

$$\begin{aligned}
\partial_t [F]_{H_1}^2 &\overset{(2.94)}{=} 2\langle F, \partial_t F\rangle_{H_1} \\
&= 2\langle F, -A_0 F + \mathcal{N}(F,F)\rangle_{H_1} \\
&\overset{(2.34)}{\leq} -[F]_{H_3}^2 + 2[F]_{H_1}[\mathcal{N}(F,F)]_{H_1} \\
&\overset{(2.104)}{\leq} -[F]_{H_3}^2 + C\,[F]_{H_1}[F]_{H_3}[F]_{H_6^*} \\
&\leq -\frac{1}{2}[F]_{H_3}^2 + C\,[F]_{H_1}^2[F]_{H_6^*}^2.
\end{aligned} \tag{2.106}$$

Therefore

$$\begin{aligned}
\log\left(\frac{[F(t)]_{H_1}^2}{[F_0]_{H_1}^2}\right) &\lesssim \int_0^t [F]_{H_6^*}^2\,dt \\
&\overset{(2.21)}{\lesssim} [F_0]_{H_4^*}^2,
\end{aligned}$$

which implies that $[F]_{H_1} \lesssim [F_0]_{H_1}$. Inserting this information into (2.106) we obtain (2.105). By an analogous argument as in the proof of Theorem 2.1.6

this leads to

$$\begin{aligned}
&\sup_t \left(t^{\frac{k-1}{2}} [F]^2_{H_k} \right) + \int_0^\infty t^{\frac{k-1}{2}} [F]^2_{H_{k+2}} \, dt \\
&\lesssim C_k' \int_0^\infty t^{\frac{k-1}{2}} [F]^2_{H_6^*} [F]^2_{H_k} \, dt + C_k \int_0^\infty t^{\frac{k-3}{2}} \, [F]^2_{H_k} \, dt \\
&\overset{(2.102)}{\lesssim} C_k \int_0^\infty t^{\frac{k-3}{2}} [F]^2_{H_k} \, dt
\end{aligned}$$

for every $k \geq 1$. An induction argument as in the proof of Theorem 2.1.6 completes the proof. □

2.11 Weighted Sobolev spaces

The basic tool for the weighted Sobolev spaces is the Hardy inequality, introduced in [38] (see also the detailed survey for inequalities in weighted spaces in [43]). Let us mention that similar weighted spaces and tools are also used in [50]. For the convenience of the reader we derive the form of Hardy inequality as we need it:

Lemma 2.11.1 (Hardy inequality)**.** *Let $k \neq -1$. Assume that $F \in H^1_{loc}((0, \infty))$ is such that*

$$\|x^{(k+2)/2} \, \partial_x F\|_{L^2} < \infty$$

and

$$\begin{aligned}
\exists \alpha_n \downarrow 0: \ \ F(\alpha_n) \to 0 \qquad\qquad & \textit{if } k < -1, \\
\exists \beta_n \uparrow \infty: \ \ F(\beta_n) \to 0 \qquad\qquad & \textit{if } k > -1.
\end{aligned}$$

Then

$$\|x^{k/2} \, F\|_{L^2} \leq \frac{2}{k+1} \, \|x^{(k+2)/2} \, \partial_x F\|_{L^2}. \tag{2.107}$$

Proof. First we observe that if $k < -1$, then

$$\begin{aligned}
F(x) &= F(\alpha_n) + \int_{\alpha_n}^x \partial_x F \, dx \\
&\leq o_n(1) + \left(\int_{\alpha_n}^x x^{k+2} (\partial_x F)^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\alpha_n}^x x^{-k-2} \, dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Hence, passing to the limit as $n \uparrow \infty$,

$$F(x) = x^{-(k+1)/2}\ o(1) \qquad \text{as } x \to 0.$$

If $k > -1$, the same argument applied to $F(1/x)$ yields that

$$F(x) = x^{-(k+1)/2}\ o(1) \qquad \text{as } x \to \infty.$$

Let now $0 < \alpha < \beta$. Taking also into account the sign of one of the boundary terms, we have:

$$\begin{aligned}\int_\alpha^\beta x^k F^2\ dx &= \left[\frac{1}{k+1} x^{k+1} F^2\right]_\alpha^\beta - \frac{2}{k+1}\int_\alpha^\beta x^{k+1} F\ \partial_x F\ dx \\ &\leq o(1) + \frac{1}{2}\int_\alpha^\beta x^k F^2\ dx + \frac{2}{(k+1)^2}\ \int_0^\infty x^{k+2}(\partial_x F)^2\ dx \\ &\qquad \text{as } \alpha, \frac{1}{\beta} \to 0,\end{aligned}$$

and Lemma 2.11.1 follows by monotone convergence. □

It follows immediately from Lemma 2.11.1 that:

Corollary 2.11.2. *Let $k \neq -1$. Assume that $F \in H^1_{loc}((0,\infty))$ is such that*

$$\|x^{(k+2)/2}\ \partial_x F\|_{L^2} < \infty$$

and

$$\begin{aligned}\|x^{-1/2}F\|_{L^2((0,1))} &< \infty \qquad &\text{if } k < -1,\\ \|x^{-1/2}F\|_{L^2((1,\infty))} &< \infty \qquad &\text{if } k > -1.\end{aligned}$$

Then (2.107) *holds.*

The Hardy inequality implies the following supremum estimates:

Lemma 2.11.3. *Let $k \geq 1$. Assume $F \in H^1_{loc}((0,\infty))$ is such that*

$$\|x^{(k-1)/2}\ F\|_{L^2} + \|x^{(k+1)/2}\ \partial_x F\|_{L^2} < \infty.$$

Then

$$\|x^{k/2}F\|_{C^0} \lesssim \|x^{(k+1)/2}\ \partial_x F\|_{L^2}.$$

Proof. The integrability of F at $x = 0$ implies that a sequence $\alpha_n \to 0$ exists such that $\alpha_n^k F^2(\alpha_n) \to 0$. Therefore

$$\begin{aligned}
\sup_{x\in(\alpha_n,\infty)} x^k F^2 &\leq o_n(1) + k\int_0^\infty x^{k-1}F^2\ dx + 2\int_0^\infty x^k F\ \partial_x F\ dx \\
&\leq o_n(1) + (k+1)\int_0^\infty x^{k-1}F^2\ dx + \int_0^\infty x^{k+1}(\partial_x F)^2\ dx \\
&\overset{(2.107)}{\lesssim} o_n(1) + \int_0^\infty x^{k+1}(\partial_x F)^2\ dx.
\end{aligned}$$

Passing to the limit as $n \to \infty$ completes the proof. □

We derive the interpolation inequalities in two different forms. In the first lemma we use less assumptions on the integrability of the estimated function. We also use general weights. Secondly, we give the proof of Lemma 2.1.1.

Lemma 2.11.4 (Interpolation inequality)**.** *Let $k \geq 0$. A universal constant C exists such that for all $F \in H^2_{loc}((0,\infty))$ with*

$$\int_0^\infty x^k F^2\ dx + \int_0^\infty x^{k+2}(\partial_x^2 F)^2\ dx < \infty,$$

it holds that:

$$\int_0^\infty x^{k+1}(\partial_x F)^2\ dx \leq C\left(\int_0^\infty x^k F^2\ dx\right)^{\frac{1}{2}}\left(\int_0^\infty x^{k+2}(\partial_x^2 F)^2\ dx\right)^{\frac{1}{2}}.$$

Proof. We claim that

$$\exists \beta_n \to \infty : \partial_x F(\beta_n) \to 0. \tag{2.108}$$

If not, then we would have $|\partial_x F(x)| \geq C$ for $x \in (x_0,\infty)$ for some $C > 0$ and $x_0 > 0$. By the continuity of $\partial_x F$, without loss of generality we would have $\partial_x F \geq C$, and therefore $F(x) \geq F(x_0) + Cx$, in (x_0,∞), in contradiction with the integrability assumption of F. Hence (2.108) holds, and by Lemma 2.11.1 we obtain that

$$\int_0^\infty x^k(\partial_x F)^2\ dx \lesssim (k+1)^{-2}\int_0^\infty x^{k+2}(\partial_x^2 F)^2\ dx. \tag{2.109}$$

We have for $0 < \alpha < \beta$:

$$\begin{aligned}\int_\alpha^\beta x^{k+1}(\partial_x F)^2\ dx = &-\int_\alpha^\beta x^{k+1}\ F\ \partial_x^2 F\ dx \\ &-(k+1)\int_\alpha^\beta x^k F\ \partial_x F\ dx + [x^{k+1}F\ \partial_x F]_\alpha^\beta \\ \overset{(2.109)}{\lesssim} &\left(\int_\alpha^\beta x^k F^2\ dx\right)^{\frac{1}{2}}\left(\int_\alpha^\beta x^{k+2}(\partial_x^2 F)^2\ dx\right)^{\frac{1}{2}} \\ &+[x^{k+1}F\ \partial_x F]_\alpha^\beta.\end{aligned}$$

It remains to show that sequences $\alpha_n \to 0, \beta_n \to \infty$ exist such that

$$[x^{k+1}F\ \partial_x F]_{\alpha_n}^{\beta_n} = o_n(1).$$

For β_n, assume on the contrary that C and x_0 exist such that

$$C \le x^{k+1}|F\ \partial_x F| \le x^{k+1}F^2 + x^{k+1}(\partial_x F)^2 \qquad \text{for all } x > x_0.$$

Then

$$\infty = \int_{x_0}^\infty \frac{C}{x}\ dx \le \int_{x_0}^\infty x^k F^2 + x^k(\partial_x F)^2\ dx,$$

in contradiction with (2.109). The argument for α_n is identical. □

We have also been using the following corollary of Lemma 2.11.4:

Corollary 2.11.5. *Let $1 \le l < k < m$. There exists a constant C_{lm} such that for all $F \in H_m$,*

$$[F]_{H_k} \le C_{lm}\ [F]_{H_l}^{\frac{m-k}{m-l}}[F]_{H_m}^{\frac{k-l}{m-l}} \tag{2.110}$$

and

$$[F]_{H_k^*} \le C_{lm}\ [F]_{H_l}^{\frac{m-k}{m-l}}[F]_{H_m}^{\frac{k-l}{m-l}}. \tag{2.111}$$

Proof. (2.110) follows by repeated application of Lemma 2.11.4, and (2.111) follows from (2.110) and Lemma 2.1.1. □

We now prove Lemma 2.1.1.

Proof of Lemma 2.1.1. The proof of

$$[F]_{H_m^*} \lesssim [F]_{H_{m-1}}^{1/2} \, [F]_{H_{m+1}}^{1/2}. \tag{2.112}$$

is straightforward. Decompose

$$F = F \, \chi_{(0,s^*)} + F \, \chi_{(s^*,\infty)} =: F_+ + F_-.$$

Using this decomposition in (2.9) and optimizing in s^* yields (2.112).

We turn to the proof of

$$[F]_{H_m} \lesssim [F]_{H_m^*}. \tag{2.113}$$

The first step is to argue that

$$[F]_{H_m^*} \gtrsim \left(\int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1} \, [F_-]_{H_{m-1}}^2 + s \, [F_+]_{H_{m+1}}^2 \right) \, \frac{ds}{s} \right)^{1/2}. \tag{2.114}$$

Hence in terms of

$$K^2(s) := \inf_{F=F_-+F_+} \left(s^{-1} \, [F_-]_{H_{m-1}}^2 + s \, [F_+]_{H_{m+1}}^2 \right),$$

we have to show

$$\left(\int_0^\infty K^2(s) \, \frac{ds}{s} \right)^{1/2} \lesssim \int_0^\infty K(s) \, \frac{ds}{s}.$$

This follows from the stronger statement that

$$\sup_{s \in (0,\infty)} K(s) \lesssim \int_0^\infty K(s) \, \frac{ds}{s},$$

which in turn follows from

$$K(s) \lesssim K(s') \qquad \text{for } \frac{s}{2} \le s' \le s, \tag{2.115}$$

since

$$K(s) = \frac{2}{s} \int_{s/2}^s K(s) \, ds' \overset{(2.115)}{\lesssim} \frac{1}{s} \int_{s/2}^s K(s') \, ds' \lesssim \int_{s/2}^s K(s') \, \frac{ds'}{s'}.$$

Inequality (2.115) can be seen as follows:

$$\begin{aligned} K(s) &= \inf_{F=F_-+F_+} \left(s^{-1}\ [F_-]^2_{H_{m-1}} + s\ [F_+]^2_{H_{m+1}}\right) \\ &\le \inf_{F=F_-+F_+} \left(s'^{-1}\ [F_-]^2_{H_{m-1}} + 2s'\ [F_+]^2_{H_{m+1}}\right) \\ &\le 2\ K(s') \quad \text{for} \quad s' \ \le\ s\ \le\ 2s'. \end{aligned}$$

The second step is to argue that

$$\frac{\pi}{2}\ \langle F, A_0^{\frac{m-1}{2}}\ F\rangle_{H_1} \ = \left(\int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}\ [F_-]^2_{H_{m-1}} + s\ [F_+]^2_{H_{m+1}}\right)\ \frac{ds}{s}\right)^{1/2}. \tag{2.116}$$

By density, we may assume that $F \in C_c^\infty([0,\infty))$ with $F(0) = 0$. In view of (2.36), we have

$$\begin{aligned} &\inf_{F=F_-+F_+} \left(s^{-1}\ [F_-]^2_{H_{m-1}} + s\ [F_+]^2_{H_{m+1}}\right) \\ &\qquad = \inf_{F=F_-+F_+} \left(s^{-1}\ \langle F_-, A_0^{\frac{m-2}{2}}\ F_-\rangle_{H_1} + s\ \langle F_+, A_0^{\frac{m}{2}}\ F_+\rangle_{H_1}\right). \end{aligned}$$

The minimization can now be carried out explicitly. The minimizers are given by

$$F_- = s^2 A_0\ (\mathrm{id} + s^2 A_0)^{-1}\ F, \qquad\qquad F_+ = (\mathrm{id} + s^2 A_0)^{-1}\ F$$

(the invertibility of $I + s^2 A_0$ follows from Proposition 2.4.1), so that

$$\begin{aligned} &\inf_{F=F_-+F_+} \left(s^{-1}\ \langle F_-, A_0^{\frac{m-2}{2}}\ F_-\rangle_{H_1} + s\ \langle F_+, A_0^{\frac{m}{2}}\ F_+\rangle_{H_1}\right) \\ &\qquad = s\ \langle F, A_0^{\frac{m}{2}}\ (\mathrm{id} + s^2\ A_0)^{-1}\ F\rangle_{H_1}. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_0^\infty \inf_{F=F_-+F_+} \left(s^{-1}\ [F_-]^2_{H_{m-1}} + s\ [F_+]^2_{H_{m+1}}\right)\ \frac{ds}{s} \\ &\qquad = \langle F, A_0^{\frac{m-2}{2}}\ \int_0^\infty (\mathrm{id} + s^2\ A_0)^{-1}\ A_0 F\ ds\rangle_{H_1} \\ &\qquad = \frac{\pi}{2}\langle F, A_0^{\frac{m-2}{2}}\ A_0^{\frac{1}{2}}\ F\rangle_{H_1}. \end{aligned}$$

We have used the following representation formula for $A_0^{\frac{1}{2}}$:

$$A_0^{\frac{1}{2}} F = \int_0^\infty (\mathrm{id} + s^2\ A_0)^{-1}\ A_0 F\ ds, \tag{2.117}$$

which holds in view of Proposition 2.4.1 (see [6]). Equation (2.113) now follows from (2.36):

$$[F]^2_{H_m} \stackrel{(2.36)}{=} C\ \langle F, A_0^{\frac{m-1}{2}} F\rangle_{H_1} \quad \stackrel{(2.114),(2.116)}{\lesssim} \quad [F]^2_{H^*_m}.$$

□

Proof of Lemma 2.1.2. We first appeal to Hardy's inequality

$$\begin{aligned} [F]_{H_{m-1}} &= \|x^{\frac{m-2}{2}}\ \partial_x^{m-1} F\|_{L^2} \gtrsim \|\partial_x^{\frac{m}{2}} F\|_{L^2}, \\ [F]_{H_{m+1}} &= \|x^{\frac{m}{2}}\ \partial_x^{m+1} F\|_{L^2} \gtrsim \|\partial_x^{\frac{m}{2}+1} F\|_{L^2}, \end{aligned}$$

to obtain

$$[F]_{H^*_m} \gtrsim \int_0^\infty \left(\inf_{F=F_-+F_+} \left(s^{-1}\ \|\partial_x^{\frac{m}{2}} F_-\|^2_{L^2} + s\ \|\partial_x^{\frac{m}{2}+1} F_+\|^2_{L^2} \right) \right)^{1/2}\ \frac{ds}{s}.$$

It is convenient to introduce $f = \partial_x^{\frac{m}{2}} F$. Because of the above, it is enough to show

$$\|f\|_{C^0} \lesssim \int_0^\infty \left(\inf_{f=f_-+f_+} \left(s^{-1}\ \|f_-\|^2_{L^2} + s\ \|\partial_x f_+\|^2_{L^2} \right) \right)^{1/2}\ \frac{ds}{s}. \tag{2.118}$$

By even reflection, we may prove (2.118) for functions f on the real line instead of the half–line. This allows as to use the Fourier transform $\hat{f}$. Because of $\sup_x |f| \lesssim \int_{-\infty}^\infty |\hat{f}|\ dk$, it is enough to show

$$\begin{aligned} &\int_{-\infty}^\infty |\hat{f}|\ dk \\ \lesssim &\int_0^\infty \left(\inf_{\hat{f}=\hat{f}_-+\hat{f}_+} \left(s^{-1} \int_{-\infty}^\infty |\hat{f}_-|^2\ dk + s \int_{-\infty}^\infty k^2\ |\hat{f}_+|^2\ dk \right) \right)^{1/2}\ \frac{ds}{s}. \end{aligned} \tag{2.119}$$

The minimizer on the right-hand side can be explicitly computed to be $\hat{f}_+ = (1+s^2k^2)^{-1}\hat{f}$, so that

$$
\begin{aligned}
&\inf_{\hat{f}=\hat{f}_-+\hat{f}_+}\left(s^{-1}\int_{-\infty}^{\infty}|\hat{f}_-|^2\,dk + s\int_{-\infty}^{\infty}k^2\,|\hat{f}_+|^2\,dk\right)\\
&= \int_{-\infty}^{\infty}\inf_{\hat{f}=\hat{f}_-+\hat{f}_+}\left(s^{-1}\,|\hat{f}_-|^2 \;+ s\,k^2\,|\hat{f}_+|^2\right)\,dk\\
&= \int_{-\infty}^{\infty}\frac{s\,k^2}{1+s^2\,k^2}\,|\hat{f}|^2\,dk.
\end{aligned}
$$

Hence we obtain in particular

$$
\begin{aligned}
&\int_0^{\infty}\left(\inf_{\hat{f}=\hat{f}_-+\hat{f}_+}\left(s^{-1}\int_{-\infty}^{\infty}|\hat{f}_-|^2\,dk + s\int_{-\infty}^{\infty}k^2\,|\hat{f}_+|^2\,dk\right)\right)^{1/2}\frac{ds}{s}\\
&\qquad\gtrsim \int_0^{\infty}\left(s^{-1}\int_{s^{-1}}^{2\,s^{-1}}|\hat{f}|^2\,dk\right)^{1/2}\frac{ds}{s}.
\end{aligned}
\tag{2.120}
$$

On the other hand, we have by Cauchy–Schwarz

$$
\begin{aligned}
\int_{-\infty}^{\infty}|\hat{f}|\,dk \;&\sim\; \int_0^{\infty}\int_{s^{-1}}^{2\,s^{-1}}|\hat{f}|\,dk\,\frac{ds}{s}\\
&\lesssim\; \int_0^{\infty}\left(s^{-1}\int_{s^{-1}}^{2\,s^{-1}}|\hat{f}|^2\,dk\right)^{1/2}\frac{ds}{s}.
\end{aligned}
\tag{2.121}
$$

Hence (2.119) follows from (2.120) and (2.121). Continuity of $\partial_x^{\frac{m}{2}}F$ follows by an approximation argument, since our spaces are defined by completion of smooth functions. □

We turn to the proof of Lemma 2.1.3:

Proof of Lemma 2.1.3. We start with part i). It is convenient to introduce the abbreviations

$$
\begin{aligned}
K(s,t) &:= \inf_{F(t)=F_-+F_+} \left(s^{-1}\ [F_-]^2_{H_{m-1}} + s\ [F_+]^2_{H_{m+1}}\right)^{1/2}, \\
K_2(s) &:= \inf_{F=F_-+F_+} \left(s^{-1}\ [F_-]^2_{L^2(H_{m-1})} + s\ [F_+]^2_{L^2(H_{m+1})}\right)^{1/2} \\
&= \inf_{F=F_-+F_+} \| \left(s^{-1}\ [F_-(\cdot)]^2_{H_{m-1}} + s\ [F_+(\cdot)]^2_{H_{m+1}}\right)^{1/2} \|_{L^2} \\
&= \| \inf_{F(\cdot)=F_-+F_+} \left(s^{-1}\ [F_-]^2_{H_{m-1}} + s\ [F_+]^2_{H_{m+1}}\right)^{1/2} \|_{L^2} \\
&= \|K(s,\cdot)\|_{L^2}, \qquad (2.122) \\
K_\infty(s) &:= \inf_{F=F_-+F_+} \left(s^{-1}\ [F_-]^2_{C^0(H_{m-1})} + s\ [F_+]^2_{C^0(H_{m+1})}\right)^{1/2} \\
&\geq \inf_{F=F_-+F_+} \| \left(s^{-1}\ [F_-(\cdot)]^2_{H_{m-1}} + s\ [F_+(\cdot)]^2_{H_{m+1}}\right)^{1/2} \|_{C^0} \\
&= \| \inf_{F(\cdot)=F_-+F_+} \left(s^{-1}\ [F_-]^2_{H_{m-1}} + s\ [F_+]^2_{H_{m+1}}\right)^{1/2} \|_{C^0} \\
&= \|K(s,\cdot)\|_{C^0}. \qquad (2.123)
\end{aligned}
$$

We now obtain by the triangle inequality in L^2 and L^∞ respectively:

$$
\begin{aligned}
[F]_{L^2(H_m)^*} &= \int_0^\infty K_2(s)\ \frac{ds}{s} \\
&\overset{(2.122)}{=} \int_0^\infty \|K(s,\cdot)\|_{L^2}\ \frac{ds}{s} \\
&\geq \| \int_0^\infty K(s,\cdot)\ \frac{ds}{s} \|_{L^2} \\
&= \ [F]_{L^2(H_m^*)},
\end{aligned}
$$

$$
\begin{aligned}
[F]_{C^0(H_m)^*} &= \int_0^\infty K_\infty(s)\ \frac{ds}{s} \\
&\overset{(2.123)}{\geq} \int_0^\infty \|K(s,\cdot)\|_{C^0}\ \frac{ds}{s} \\
&\geq \| \int_0^\infty K(s,\cdot)\ \frac{ds}{s} \|_{C^0} \\
&= [F]_{C^0(H_m^*)}.
\end{aligned}
$$

We turn to part ii) and fix $\alpha(t)$. We consider the linear operator $\zeta \mapsto \alpha \otimes \zeta$. Then the inequalities follow from interpolating the standard estimates

$$[\alpha \otimes \zeta]_{L^2(H_{m-1})} \leq \|\alpha\|_{L^2} \, [\zeta]_{H_{m-1}},$$
$$[\alpha \otimes \zeta]_{L^2(H_{m+1})} \leq \|\alpha\|_{L^2} \, [\zeta]_{H_{m+1}}$$

and

$$[\alpha \otimes \zeta]_{C^0(H_{m-1})} \leq \|\alpha\|_{C^0} \, [\zeta]_{H_{m-1}},$$
$$[\alpha \otimes \zeta]_{C^0(H_{m+1})} \leq \|\alpha\|_{C^0} \, [\zeta]_{H_{m+1}}$$

respectively. □

In (2.7), we have defined H_m as completion of

$$\mathcal{D} = \{C_c^\infty([0,\infty)) : F(0) = 0\}$$

with respect to $\|\cdot\|_{H_m}$. Similarly, in (2.11), we have defined H_m^* as completion of $\mathcal{D}$ with respect to $\|\cdot\|_{H_m^*}$. For the convenience of the reader we show in the next two lemmata that

$$H_m = \{F \in H_{loc}^m : F(0) = 0, \|F\|_{H_m} < \infty\}, \tag{2.124}$$

but

$$H_m^* \subsetneq \{F \in H_{loc}^m : F(0) = 0, \|F\|_{H_m^*} < \infty\}. \tag{2.125}$$

For (2.124), we have to prove:

Lemma 2.11.6. *For all $m \geq 1$, $\mathcal{D}$ is dense in*

$$W_m := \{F \in H_{loc}^m : F(0) = 0, \|F\|_{H_m} < \infty\}$$

with respect to $\|\cdot\|_{H_m}$.

Proof. Locally, density for standard Sobolev spaces translates directly to weighted norms, i.e.

$$\mathcal{D}_0 := C^\infty((0,\infty)) \cap W_m \qquad \text{is dense in } W_m \text{ with respect to } \|\cdot\|_{H_m}.$$

Therefore it suffices to consider $F \in \mathcal{D}_0$. We first show that

$$\mathcal{D}_1 := C^\infty([0,\infty)) \cap W_m \quad \text{is dense in } \mathcal{D}_0 \text{ with respect to } \|\cdot\|_{H_m}. \tag{2.126}$$

Define for all $\delta > 0$

$$F_\delta(x) := \int_0^x \partial_{\hat{x}} F(\hat{x} + \delta)\ d\hat{x}.$$

Of course $F_\delta \in \mathcal{D}_1$ and for all $1 \leq k \leq m$

$$\lim_{\delta \to 0} \int_a^{\frac{1}{a}} x^{k-1}\ (\partial_x^k F_\delta - \partial_x^k F)^2\ dx = 0. \tag{2.127}$$

for any $a > 0$. On the other hand

$$\limsup_{\delta \to 0}\ \int_0^a x^{k-1} (\partial_x^k F_\delta - \partial_x^k F)^2\ dx \leq 2 \limsup_{\delta \to 0}\ \int_0^{a+\delta} x^{k-1} (\partial_x^k F)^2\ dx = o_a(1) \tag{2.128}$$

and

$$\limsup_{\delta \to 0}\ \int_{\frac{1}{a}}^\infty x^{k-1} (\partial_x^k F_\delta - \partial_x^k F)^2\ dx \leq 2 \limsup_{\delta \to 0}\ \int_{\frac{1}{a}}^\infty x^{k-1} (\partial_x^k F)^2\ dx = o_a(1). \tag{2.129}$$

Convergence of $F_\delta \to F$ in W_m follows from (2.127), (2.128), (2.129). Hence (2.126) holds, and it remains to show that

$$\mathcal{D} \text{ is dense in } \mathcal{D}_1 \text{ with respect to } \|\cdot\|_{H_m}.$$

Note that, since $F(0) = 0$,

$$\int_0^\infty \frac{F^2}{x^2}\ dx \lesssim \int_0^\infty (\partial_x F)^2\ dx \qquad \text{(by Lemma 2.11.1).} \tag{2.130}$$

Let η be a cut–off function s.t. $\eta = 0$ on $(0, 1)$ and $\eta = 1$ on $(2, \infty)$, and $\eta_R(x) = \eta(x/R)$. Letting $F_R := (1 - \eta_R)F$, for every $R > 1$ and every $1 \leq k \leq m$ we have:

$$\begin{aligned}
[F - F_R]_{H_k}^2 &= \int_0^\infty x^{k-1} (\partial_x^{k-1} (\eta_R \partial_x F))^2\ dx \\
&\leq \int_0^\infty x^{k-1} \eta_R^2\ (\partial_x^k F)^2\ dx + C_k \sum_{j=1}^{k-1} \int_0^\infty x^{k-1} (\partial_x^{k-j} \eta_R)^2 (\partial_x^j F)^2\ dx \\
&\quad + \int_0^\infty x^{k-1} (\partial_x^k \eta_R)^2 F^2\ dx
\end{aligned}$$

Since $\partial_x^{k-j}\eta_R \lesssim R^{j-k}\text{supp}(\partial_x^{k-j}\eta_R)$, it follows that

$$
\begin{aligned}
[F-F_R]_{H_k}^2 &\le \int_R^\infty x^{k-1}(\partial_x^k F)^2\,dx + C_k\sum_{j=1}^{k-1}\int_R^{2R} x^{k-1}R^{2j-2k}(\partial_x^j F)^2\,dx \\
&\quad + \int_R^{2R} x^{k-1}R^{-2k}F^2\,dx \\
&\overset{R>1}{\le} C_k\sum_{j=1}^{k}\int_R^\infty x^{j-1}(\partial_x^j F)^2\,dx + \int_R^{2R} x^{-2}F^2\,dx \\
&\overset{(2.130)}{\le} C_k\sum_{j=1}^{k}\int_R^\infty x^{j-1}(\partial_x^j F)^2\,dx \\
&\to 0 \qquad\qquad \text{for } R\to\infty.
\end{aligned}
$$

This concludes the proof. □

The statement (2.125) is a consequence of

Lemma 2.11.7. *For all even $m \ge 2$ we have*

$$\|x\|_{H_m^*} < \infty, \qquad \textit{but} \qquad x \notin H_m^*.$$

Proof. It follows directly from Lemma 2.1.2 that $x \notin H_m^*$, since

$$1 = \|\partial_x x\|_{C^0} \overset{(2.13)}{\lesssim} [F]_{H_2^*},$$

and therefore x cannot be approximated in H_2^* by functions with compact support.

In order to prove the first claim we note that

$$[x]_{H_1} = \infty, \qquad [x]_{H_k} = 0 \text{ for all } k \ge 2.$$

Therefore $[x]_{H_k^*} = 0$ for all $k \ge 3$ and it remains to prove that $[x]_{H_2^*} < \infty$. By (2.11), it is enough to find a decomposition $x = F_- + F_+$ which ensures finiteness of $[x]_{H_2^*}$. Let η be a cut–off function s.t. $\eta = 0$ on $(0,1)$ and $\eta = 1$ on $(2,\infty)$. We decompose

$$x = \frac{x\ \eta(x/s)}{1+\ln^2 s} + x\ (1 - \frac{\eta(x/s)}{1+\ln^2 s}) =: F_- + F_+,$$

A straightforward calculation and using (2.11) yields $[x]_{H_2^*} < \infty$. □

We conclude by pointing out simple properties of X_6^*.

Lemma 2.11.8. *$X_6^* \subset C([0,\infty); H_4^*)$. In particular, for any $F \in X_6^*$ the trace $F|_{t=0}$ is well defined in H_4^*. In addition, for any $F \in X_6^*$ the function*

$$\varphi(T) = [\partial_t F]_{L^2((0,T);H_2)^*} + [F]_{C^0((0,T);H_4)^*} + [F]_{L^2((0,T);H_6)^*}$$

is continuous in $[0,\infty)$ with $\varphi(0) = [F|_{t=0}]_{H_4^*}$.

Proof. By translation invariance, it is enough to show continuity of $t \mapsto F(t)$ in H_4^* at $t = 0$. Let $F_\nu \in C_c^\infty([0,\infty)^2)$ such that $\|F - F_\nu\|_{X_6^*} \to 0$. Then, for a given $\epsilon > 0$ there exists $\nu^* \in \mathbb{N}_0$ such that $\sup_t [F(t) - F_{\nu^*}(t)]_{H_4^*} < \epsilon/4$, and since F_{ν^*} is smooth, there exists a $\delta > 0$ s.t. $[F_{\nu^*}(t) - F_{\nu^*}(s)]_{H_4^*} < \epsilon/2$ for all $0 < s < t < \delta$. Hence,

$$[F(t) - F(s)]_{H_4^*} < \frac{\epsilon}{2} + [F_{\nu^*}(t) - F_{\nu^*}(s)]_{H_4^*} < \epsilon \quad \text{for all } 0 < s < t < \delta.$$

The completeness of H_4^* now implies that F is continuous in this space at $t = 0$. The second statement follows by the same argument, noting that

$$[F]_{L^2((0,T);H_k)^*} \lesssim [F]^{1/2}_{L^2((0,T);H_{k-1})}[F]^{1/2}_{L^2((0,T);H_{k+1})} \tag{2.131}$$

for all $k \geq 2$ ((2.131) is an easy generalization of (2.112) which we leave to the reader). ☐

2.12 Speed of propagation

In this section, we show that condition (1.6) does not have to be included into the formulation of the free boundary problem (1.7). It can be recovered for any solution h having only the regularity that corresponds to our minimal setting.

We consider (1.3), (only) equipped with the boundary conditions (1.4)–(1.5). We assume that the free boundary propagates with arbitrary speed $V \in L^2([0,\infty))$. Applying the coordinate transform $x = y - \int_0^t V\, dt$, we get the following problem on a fixed domain

$$\begin{cases} \partial_t h - \partial h\, V + \partial(h\, \partial^3 h) = 0 & \text{in } (0,\infty)^2, \\ h = \partial h = 0 & \text{for } x = 0, \\ h = h_0 & \text{for } t = 0. \end{cases} \tag{2.132}$$

We consider h to be a solution (2.132) with the regularity of Theorem 2.1.4, i.e. $h \in X_6^*$ and furthermore we assume that

$$\|\partial^2 h_{|x=0} - 1\|_{C^0} \leq C < 1. \tag{2.133}$$

This is reasonable since every solution $F \in X_6^*$ of Theorem 2.1.4 satisfies (by Lemmas 2.1.2–2.1.3)

$$\|\partial^2 h - 1\|_{C^0} \overset{(1.17)}{=} \|\partial^2 F\|_{C^0} \lesssim [F]_{C^0(H_4)^*} \lesssim [F_0]_{H_4^*} < \epsilon.$$

We have the following Lemma:

Lemma 2.12.1. *Let* $V \in L^2(\mathbb{R}_+)$ *and let* $h \in X_6^*$ *be a solution of* (2.132) *such that* (2.133) *holds. Then*

$$V = \partial_y^3 h_{|x=0}. \tag{2.134}$$

Proof. Our solution is sufficiently regular such that (2.132) can be differentiated once in space (this is well–defined in L^1_{loc} away from the boundary). We get:

$$\begin{aligned} 0 &= \partial_{xt}^2 h - \partial^2 h V + \partial^2 (h\ \partial^3 h) \\ &= \partial_{xt}^2 h - \partial^2 h V \\ &\quad + (\partial^2 h\ \partial^3 h) + 2\ (\partial h\ \partial^4 h) + (h\ \partial^5 h). \end{aligned} \tag{2.135}$$

In the following, we argue that even the trace of this expression at $x = 0$ is well defined in $L^2([0, \infty))$. For this, it is enough to show that

$$\begin{array}{ll} \partial_{xt}^2 h,\ \ \partial^3 h,\ \ x\ \partial^4 h,\ \ x^2\ \partial^5 h & \in L^2(C^0), \\ \partial^2 h,\ \ x^{-1}\ \partial h,\ \ x^{-2}\ h & \in C^0(C^0). \end{array} \tag{2.136}$$

Indeed, by Lemmas 2.1.1–2.1.2–2.1.3–2.11.3, we have

$$\begin{aligned} &\|\partial_{t,x}^2 h\|_{L^2(C^0)} + \|\partial^2 h\|_{C^0(C^0)} + \|\partial^3 h\|_{L^2(C^0)} \\ &\quad \lesssim [\partial_t h]_{L^2(H_2)^*} + [h]_{C^0(H_4)^*} + [h]_{L^2(H_6)^*} \\ &\quad \leq \|h\|_{X_6^*} \end{aligned} \tag{2.137}$$

and

$$\|x\ \partial^4 h\|_{L^2(C^0)} \lesssim \|x^2\ \partial^5 h\|_{L^2(C^0)} \lesssim [h]_{L^2(H_6)} \leq \|h\|_{X_6^*}. \tag{2.138}$$

By the boundary conditions $h_{|x=0} = \partial h_{|x=0}$ we have

$$\|x^{-2}\ h\|_{C^0(C^0)} \leq \|x^{-1}\ \partial h\|_{C^0(C^0)} \leq \|\partial^2 h\|_{C^0(C^0)}. \tag{2.139}$$

The inequalities (2.137)–(2.139) together yield (2.136). Since by definition of X_6^*, $C_c^\infty(\mathbb{R}_+^2)$ is a dense subspace of X_6^* it follows that

$$(x\ \partial^4 h)_{|x=0} = (x^2\ \partial^5 h)_{|x=0} = 0.$$

Finally, in view of (2.132) we have $\partial_{xt}^2 h_{|x=0} = 0$. Hence (2.135) simplifies to

$$\partial^2 h_{|x=0}\ (V - \partial^3 h_{|x=0}) = 0.$$

Using (2.133), this yields (2.134). □

Chapter 3

Maximal regularity for L_0 in Hölder spaces

This chapter yields a first step toward a short–time regularity theory for (1.1) in Hölder spaces. Starting point is (1.18), i.e. we consider perturbations of the stationary solution as in Chapter 2. It turns out that the nonlinear part of (1.18) is unbounded in our Hölder norms. For this reason, the analysis is restricted to the linear part L_0. The main result is maximal regularity for L_0 in weighted Hölder spaces.

3.1 Setting and result

In this chapter, we consider the linear equation

$$L_0 f = g, \tag{3.1}$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}_+$. We denote $\mathbb{R}_+ := [0, \infty)$. Recall that

$$L_0 f = \partial_t f + A_0 f = \partial_t f + \frac{1}{2}\, \partial(x^2\, \partial^3 f).$$

The first step consists in finding the appropriate metric to analyze (3.1): As we have seen in Section 1.6, it is the Carnot–Caratheodory metric

$$s(z_1, z_2) \ := \ |t_1 - t_2|^{1/4} + |\sqrt{x_1} - \sqrt{x_2}|, \tag{3.2}$$

for $z_i = (t_i, x_i) \in \mathbb{R}_\times\mathbb{R}_+$. It is invariant under the scaling of the standard second order parabolic equation, i. e.

$$x \mapsto \lambda x, \qquad t \mapsto \lambda^2 t \tag{3.3}$$

and it is, near $x = 1$, equivalent to the appropriate parabolic metric for the standard fourth order linear operator $|t_1 - t_2|^{1/4} + |x_1 - x_2|$.

There is another way to motivate the metric (3.2): Under the transformation $\hat{x} = 2\sqrt{x}$, (2.2) turns into the standard parabolic operator plus terms which have the same scaling as $\partial^4 f$, but which have less derivatives:

$$L_0 f \ = \partial_t f + \partial_{\hat{x}}^4 f + \dots \ .$$

This means that the Carnot–Caratheodory metric corresponds to the standard parabolic metric (1.24) in the $\hat{x}$ variable.

Having a metric, we can define for any $\beta \in (0, 1)$ corresponding homogeneous Hölder norms. We define for every $m \in \mathbb{N}_0$,

$$[g]_{C_s^{m,\beta}(\Omega)} \ := \ \sup_{z_1, z_2 \in \Omega} \frac{|\partial^m g(z_1) - \partial^m g(z_2)|}{s(z_1, z_2)^\beta}. \tag{3.4}$$

The domain of integration is omitted if it coincides with $\Omega = \mathbb{R} \times \mathbb{R}_+$.

Which regularity can we expect from equation (3.1)? Maximal regularity means that each single term on the left hand side of the equation is estimated by their sum. Hence, having a bound on $[\partial^m L_0 f]_{C_s^\beta}$, this means to have an estimate of f in terms of

$$[\partial_t \partial^m f]_{C_s^\beta} + [x^2 \ \partial^{m+4} f]_{C_s^\beta} + [x \ \partial^{m+3} f]_{C_s^\beta} + [\partial^{m+2} f]_{C_s^\beta}.$$

The main result of this chapter is indeed maximal regularity:

Theorem 3.1.1. *Assume that f is smooth and bounded in $\mathbb{R} \times \mathbb{R}_+$. Then for all $m \geq 1$*

$$[\partial_t \partial^m f]_{C_s^\beta} + [x^2 \ \partial^{m+4} f]_{C_s^\beta} + [x \ \partial^{m+3} f]_{C_s^\beta} + [\partial^{m+2} f]_{C_s^\beta} \leq C_{\beta,m} \ [\partial^m L_0 f]_{C_s^\beta}.$$

Note that existence of solutions in these Hölder spaces follows by combining the existence results in Chapter 2 with Theorem 3.1.1. For this reason, we only address maximal regularity, but not existence issues for L_0.

Instead of proving Theorem (3.1.1) directly, we fix m and prove the corresponding homogeneous estimate for the m–th derivative of f. We use the notation

$$F := \partial^m f, \qquad\qquad L_m F := \partial^m L_0 F. \tag{3.5}$$

From Chapter 2, (2.33) we infer that

$$L_m F = \partial_t F + \frac{1}{2x^{m-1}}\ \partial^2(x^{m+1}\ \partial^2 F).$$

For later reference, we note that (2.33) can be rewritten in terms of F in the following form: For all $k \in \mathbb{N}_0$,

$$\partial^k L_m F = \partial^t \partial^k F + \frac{1}{2x^{m+k-1}}\ \partial^2(x^{m+k+1}\ \partial^{k+2} F). \tag{3.6}$$

Theorem 3.1.1 is a consequence of the following statement in terms of F:

Theorem 3.1.2. *Assume that F is smooth and bounded in $\mathbb{R} \times \mathbb{R}_+$. Then we have for any $m \geq 1$*

$$[\partial_t F]_{C^\beta_s} + [x^2\ \partial^4 F]_{C^\beta_s} + [x\ \partial^3 F]_{C^\beta_s} + [\partial^2 F]_{C^\beta_s} \leq C_{\beta,m}\ [L_m F]_{C^\beta_s}.$$

Let us compare Theorem 3.1.1 with the corresponding result from Proposition 2.56 in Chapter 2: The Sobolev norms in Chapter 2 include a weight that depends on m. The reason for this is that each operator L_m has a different set of Lyapunov functionals (see Lemma 3.3.1). In contrary, the Carnot–Caratheodory metric does not depend on m, it is the appropriate metric for all L_m. This comes from the fact that it is only induced by the scaling invariance (3.3). However, the constant in the estimate of Theorem 3.1.2 depends on m, while it is universal in Proposition 2.56.

The operators L_m embed into the following more general class of operators

$$L_{A,B} = \partial_t F + x^2\ \partial^4 F + A\ x\ \partial^3 F + B\ \partial^2 F,$$

where $(A, B) \in \mathbb{R}^2$. Theorem 3.1.2 ensures maximal regularity in the case $A_m = 2(m+1)$, $B_m = m(m+1)$ and for integers $m \geq 1$. It does not seem to be possible to extend our method to every $(A, B) \in \mathbb{R}^2$. However, it seems to be that by optimizing the estimates, our argument should cover the case (A_m, B_m) for real $m \geq 1$. It even should be possible to extend the results to all (A_m, B), where $B \leq B_m$: This is because our argument is based on the decrease of certain Lyapunov functionals. They still decrease for smaller values of B, in fact the operator gets "better" for smaller B. For $m < 1$, our argument might fail since in this case our Lyapunov functionals have weights with negative power.

To prove Theorem 3.1.2, we follow the method of Safonov [49]. It does not rely on the fundamental solution. Instead, it is based on the following two ingredients:

- local L^∞–estimates on derivatives of a local solution for the homogeneous equation ("inner estimates"),
- and a global L^∞–bound on a solution of the inhomogeneous equation with localized right hand side ("L^∞–bound").

We denote (square–shaped) neighborhoods of $(0,0)$ in the metric space $(\mathbb{R} \times \mathbb{R}_+, s)$ by

$$P_r \;:=\; \{z : 0 \leq x \leq r, -r^2 \leq t \leq 0\} \;=\; I_r \times B_r, \tag{3.7}$$

where

$$I_r \;=\; [-r^2, 0], \quad B_r \;=\; [0, r].$$

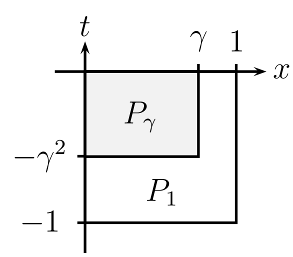

Figure 3.1: Two neighborhoods P_γ, P_1 of $(0,0)$

Our inner estimates as well as the L^∞–bound are stated in the next two propositions:

Proposition 3.1.3 (Inner estimates)**.** *For any $k, l \in \mathbb{N}_0$, $m \geq 1$ and $\gamma \in (0,1)$, there exists a positive constant C such that*

$$\|\partial_t^l \partial^k F\|_{C^0(P_{\gamma R})} \;\leq\; C\; R^{-2l-k-(m-1)/2}\; \|x^{(m-1)/2} F\|_{C^0(P_R)}$$

for any $R > 0$ and any smooth function F in P_R which satisfies

$$L_m F = 0. \tag{3.8}$$

A smooth function f in $I \times \mathbb{R}_+$ is said to have exponential decay if

$$\lim_{x\to\infty} x^{-m}\partial^k f(t,x) = 0$$

for all $m, k \in \mathbb{N}_0$ and all $t \in I$. With this understanding, we have:

Proposition 3.1.4 (L^∞–bound)**.** *Let F be smooth and have exponential decay, Assume that F satisfies*

$$\begin{aligned} \operatorname{supp} L_m F &\subseteq P_s, \\ F(-s^2,\cdot) &= 0. \end{aligned} \tag{3.9}$$

for some $s > 0$. Then for any $m \geq 1$, there exists a constant C such that

$$\|x^{(m-1)/2}F\|_{C^0(\mathbb{R}\times\mathbb{R}_+)} \leq C\ s^2\ \|x^{(m-1)/2}L_m F\|_{C^0(P_s)}.$$

In the next section, we discuss why the estimate for the nonlinear part of the operator (1.18) fails for the Hölder norms. In Section 3.3 we outline the proof of Propositions 3.1.3 and 3.1.4, and the main steps in going from these to Theorem 3.1.2. Proofs of the various steps involved are given in Sections 3.4 to 3.8. Before all that, let us summarize the subsequent notations.

Notation and conventions. In what follows, $\alpha \in \left(0, \frac{1}{2}\right)$ and $\beta = 2\alpha$ denote Hölder exponents. The positive constant $\gamma < 1$, serves as scaling factor between an inner square $P_{\gamma r}$ and an outer square P_r. The integer constants k, l, m will be related to the order of differentiation and to powers of x. We always assume $m \geq 1$. *In this chapter*, we denote by C any constant depending on α, β, γ, k, l, m. We furthermore write

$$A \lesssim B$$

whenever a positive constant C, depending (at most) on the above mentioned parameters α, β, γ, k, l, m, exists such that $A \leq C\,B$. Differentiation of a function f with respect to space, resp. time, is denoted by ∂f, resp. by $\partial_t f$.

3.2 Incompatibility of our norms with (1.18)

Incompatibility. In order to get existence of a solution for the nonlinear equation (1.7), it would be desirable to have a theory for both the linear part L_0 and the nonlinear part $\mathcal{N}$ of (1.18). However, Theorem 3.1.1 is only concerned with the linear part L_0. In this section, we comment on the compatibility of our Hölder norms with (1.18). Recall that

$$\mathcal{N}(f,g) \;=\; -\partial\left((f - x\,\partial f_{|x=0})(\partial^3 g - \partial^3 g_{|x=0})\right).$$

Our aim would be to prove existence for (1.18) by a fix–point argument as in Section 2.7: One then needs maximal regularity for L_0 and boundedness for $\mathcal{N}$.

We first consider the *minimal setting* (compare Section 1.5): The scaling invariance

$$x \mapsto \lambda x, \qquad t \mapsto \lambda^2 t, \qquad F \mapsto \lambda^2 F \tag{3.10}$$

of (1.18) singles out one of the semi–norms in (3.4): The invariant homogeneous norm corresponds to $m = 0$, $\beta = 0$. This would mean to impose control on the initial data in terms of

$$\|\partial^2 f_0\|_{C^0} + \|x\;\partial^3 f_0\|_{C^0} + \|x^2\;\partial^4 f_0\|_{C^0}. \tag{3.11}$$

However, the linear estimate is not compatible with $\beta = 0$. This could be fixed by imposing more control on the initial data: For instance for any $k \geq 0$ and any $\beta \in (0,1)$, one could use the norm:

$$\begin{aligned} &\|\partial f_0\|_{C^\beta_s} + \|x\;\partial^2 f_0\|_{C^\beta_s} + \|x^2\;\partial^3 f_0\|_{C^\beta_s} \\ +&\|\partial^{k+2} f_0\|_{C^\beta_s} + \|x\;\partial^{k+3} f_0\|_{C^\beta_s} + \|x^2\;\partial^{k+4} f_0\|_{C^\beta_s}, \end{aligned} \tag{3.12}$$

since norms of type (3.12) control the norm (3.11). In order to apply our linear theory to the norms in the first line of (3.12), it would be necessary to integrate (1.18) once in space. This is possible since (1.18) has divergence form.

Can we get boundedness of $\mathcal{N}$ in the above norms? We first consider the minimal setting: In order to get existence, one would need boundedness of $\|\mathcal{N}(f,g)\|_{C^0}$. In view of $\mathcal{N}(f,f)$, this requires an estimate for $\|\partial^3 f_{|x=0}\|_{C^0}$. However, this expression cannot be controlled in the setting (3.11) (recall that $\partial^3 f_{|x=0}$ denotes the speed of propagation for the free boundary of (1.7)). By doing the computations, it turns out that the nonlinear operator remains to be unbounded in the corresponding spaces even in the stronger setting (3.12).

In this case, the unboundedness of $\mathcal{N}(f,g)$ comes from the norm with the lowest order in scaling (i.e. the first line of (3.12)).

The above considerations indicate that the coordinate transform (1.17) together with the fix–point argument of Section 2.7 does not yield an existence result for (1.7).

It should be noted that the above mentioned difficulties do not apply in a localized setting. Therefore, the linear theory in this chapter is suited as a basis of short–time existence for compactly supported initial data.

Hodograph transform. A natural idea to regain boundedness of the nonlinear operator in the global setting would be to use a different coordinate transform to fix the free boundary of (1.18): The nonlocal term $\partial^3 f_{|x=0}$ seems to be intimately related to the specific transform (1.17). Indeed, Daskalopoulos and Hamilton [21] and Koch [41] have applied a different coordinate transform to fix the free boundary for the porous media equation: a *hodograph transform*. This transformation consists in interchanging dependent and independent variables: In view of (1.18), it means to introduce a new function $Y(t,x)$ by

$$h(t, Y(t,x)) \;=\; \frac{1}{2}x^2.$$

In the new coordinates, the free boundary is fixed at $x = 0$. In fact, this transformation does not create a nonlocal term. One easily gets that the linear operator in the new variables is in leading order given by

$$L_2 f = \partial_t f + \frac{1}{2x}\,\partial^2(x^3\,\partial^2 f).$$

Interestingly, this operator equals the second spatial derivative of L_0. Therefore we could in principally apply the linear theory in this chapter also to this operator. However, the invariant norm in this setting is given by

$$\|\partial f_0\|_{C^0} + \|x\,\partial^2 f_0\|_{C^0} + \|x^2\,\partial^3 f_0\|_{C^0}.$$

In order to control this norm with the linear theory, it would be necessary to integrate the transformed nonlinear equation at least once in space. However, it turns out that the transformed nonlinear equation is not (easily) integrable. This casts a doubt on the hope that the Hodograph transformation helps to get an existence theory in the global setting, i.e. for (1.18).

3.3 Outline of the proof

Inner estimates and L^∞–bound. Because of the lack of a maximum principle, we base our analysis on energy methods. Observe that the spatial part of L_m is symmetric and positive definite with respect to the inner product

$$(F, F) = \int_{\mathbb{R}_+} x^{m-1}\, F^2. \tag{3.13}$$

This gives us a countable number of energy estimates — the time derivatives of (3.13):

Lemma 3.3.1 (Global energy estimates)**.** *Let F be a smooth, exponentially decaying solution of $L_m F = 0$ in $\mathbb{R} \times \mathbb{R}_+$. Then for any $k \in \mathbb{N}_0$*

$$\frac{d}{dt} \int_{\mathbb{R}_+} x^{m+k-1} \left(\partial^k F\right)^2 = - \int_{\mathbb{R}_+} x^{m+k+1} \left(\partial^{k+2} F\right)^2. \tag{3.14}$$

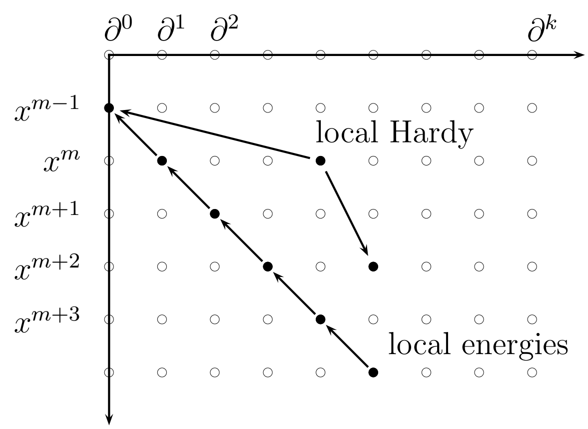

Figure 3.2: Lemma 3.3.2 and Lemma 3.3.3 can be visualized as "journey" in the above graph. Every knot represents an integral of form $\int x^l (\partial^k f)^2$.

Note that Lemma 3.3.1 is a parabolic reformulation of Lemma 2.3.2. By a localization argument, (3.14) can be transformed into the following local estimates at the boundary points $x = 0$:

Lemma 3.3.2 (Local energy estimates)**.**
Under the assumptions of Proposition 3.1.3 we have

$$\iint_{P_{\gamma R}} x^{m+k+1} \left(\partial^{k+2} F\right)^2 \lesssim R^{-2} \iint_{P_R} x^{m+k-1} \left(\partial^k F\right)^2$$

for any $R > 0$.

To convert these weighted integrals into norms, we need the following localized form of the Hardy inequality:

Lemma 3.3.3 (Local Hardy inequality)**.** *For all* $l \in \mathbb{N}_0$ *and* $k \in \mathbb{N}_0$ *we have*

$$\int_{B_\gamma} x^l \left(\partial^k \varphi\right)^2 \lesssim \int_{B_\gamma} x^{m-1}\varphi^2 + \int_{B_\gamma} x^{l+2} \left(\partial^{k+1}\varphi\right)^2 \tag{3.15}$$

for any $\gamma > 0$ *and any smooth* φ *in* B_γ.

The combination of Lemmas 3.3.2 and 3.3.3 gives as intermediate step:

Lemma 3.3.4. *Under the assumptions of Proposition 3.1.3, we have*

$$\iint_{P_{\gamma R}} \left(\partial_t^l \partial^k F\right)^2 \lesssim R^{-4l-2k-m+1} \iint_{P_R} x^{m-1} F^2$$

for every $R > 0$.

A standard interpolation — $\sup_t |\varphi| \lesssim \int (\varphi^2 + (\partial_t \varphi)^2)$ — converts it into a pointwise bound in time. By interpolating in space, the L^2–estimates can then be transformed into L^∞–estimates:

Lemma 3.3.5. *Under the assumptions of Proposition 3.1.3, we have*

$$\sup_{(t,x)\in P_{\gamma R}} \left|\partial_t^l \partial^k F\right|^2 \lesssim R^{-4l-2k-m-2} \iint_{P_R} x^{m-1} F^2$$

for every $R > 0$.

This obviously implies Proposition 3.1.3. The proof of Lemmas 3.3.1-3.3.5 is given in Section 3.4.

For the proof of Proposition 3.1.4, we have to consider the inhomogeneous equation with zero "initial" data and compactly supported right-hand side, i.e.

$$\begin{aligned} \operatorname{supp}(L_m f) &\subseteq P_s, \\ f(-s^2, \cdot) &= 0. \end{aligned}$$

We prove the L^∞-bound by using control over the first and the third energy, combined with the following standard interpolation:

Lemma 3.3.6. *For any* $m \geq 1$ *we have*

$$\sup_{x\in\mathbb{R}_+} x^{m-1} F^2 \lesssim \int_{\mathbb{R}_+} x^{m-1} F^2 + \int_{\mathbb{R}_+} x^{m+1} (\partial^2 F)^2. \tag{3.16}$$

The proofs of Lemma 3.3.6 and Proposition 3.1.4 are given in Section 3.5.

The remainder estimates. We will derive Schauder estimates along the lines of the polynomial approximation method introduced by Safonov [49]. This approach was used by Daskalopoulos and Hamilton in [20] to analyze the second order equation (1.15) and its variants in two space dimensions. It is based on estimating the remainder between a function and its Taylor polynomial.

The Taylor polynomials which we need are (mostly) of degree $n = 2$ respectively $n = 4$ in space, and 1 in time. We use the following notation

$$T_n^{z_0} F := \sum_{i=0}^{n} \frac{1}{i!}\, \partial^i F(z_0)(x - x_0)^i + \partial_t F(z_0)(t - t_0), \quad z_0 = (t_0, x_0). \tag{3.17}$$

The remainder is denoted by

$$R_n^{z_0} F \ := \ F - T_n^{z_0} F.$$

We use the abbreviations

$$\mu := (m - 1)/2,$$
$$\alpha := \beta/2.$$

The remainder estimates are divided into two parts. First we obtain remainder estimates at the boundary:

Lemma 3.3.7 (Remainder estimate at the boundary)**.** *Let F be smooth in P_1. Then*

$$\|x^\mu R_2^0 F\|_{C^0(P_r)} \ \lesssim \ r^{\mu+2+\alpha} \left(\|F\|_{C^0(P_1)} + [L_m F]_{C_s^\beta(P_1)} \right)$$

for all $0 < r \leq 1$.

We then appeal to standard Schauder estimates for uniformly parabolic equations. Rewritten in terms of our metric s, these hold on parabolic squares away from the boundary

$$Q_r \ := \ \left\{ z \ : \ r/4 \leq x \leq r, -r^2 \leq t \leq 0 \right\}.$$

Lemma 3.3.8 (Scaled remainder estimate in the interior)**.** *Let F be smooth in P_1. Then*

$$\|R_4^{(0,r/2)} F\|_{C^0(Q_r)} \ \lesssim \ \|F\|_{C^0(Q_r)} + r^{2+\alpha}\, [L_m F]_{C_s^\beta(Q_r)}$$

for all $0 < r \leq 1$.

The combination of Lemmas 3.3.7 and 3.3.8 yields

Lemma 3.3.9 (Remainder estimate near the boundary)**.** *Let F be smooth in P_1. Then*

$$\|R_4^{(0,r/2)} F\|_{C^0(Q_r)} \lesssim \ r^{2+\alpha} \left(\|F\|_{C^0(P_1)} + [L_m F]_{C_s^\beta(P_1)} \right)$$

for all $0 < r \leq 1$.

The proof of Lemma 3.3.7 can be found in Section 3.6. Lemmas 3.3.8–3.3.9 are proved in Section 3.7.

Schauder estimates. The first step in the Schauder estimate is to combine the remainder estimates from Lemma 3.3.7 and 3.3.8 to obtain Hölder continuity in space at the boundary in the following sense:

Lemma 3.3.10 (Hölder continuity in space at the boundary)**.**
Let F be smooth in P_1. Then

$$\begin{aligned} &|x^2 \ \partial^4 F(t,x)| + |x \ \partial^3 F(t,x)| \\ &+ |\partial^2 F(t,x) - \partial^2 F(t,0)| + |\partial_t F(t,x) - \partial_t F(t,0)| \\ &\lesssim \ s((t,x),(t,0))^\beta \ (\|F\|_{C^0(P_1)} + [L_m F]_{C_s^\beta(P_1)}) \end{aligned}$$

for all $(t,x) \in P_{1/2}$.

We combine standard Schauder estimates for fourth order parabolic operator with Lemma 3.3.9 to obtain

Lemma 3.3.11 (Hölder continuity in the interior)**.** *Let F be smooth in P_1. Then*

$$\begin{aligned} &[\partial_t F]_{C_s^\beta(Q_r)} + [\partial^2 F]_{C_s^\beta(Q_r)} + [x \ \partial^3 F]_{C_s^\beta(Q_r)} + [x^2 \ \partial^4 F]_{C_s^\beta(Q_r)} \\ &\qquad \lesssim \ \|F\|_{C^0(P_1)} + [L_0 F]_{C_s^\beta(P_1)}. \end{aligned}$$

for all $0 < r \leq 1/2$.

We combine Lemmas 3.3.10–3.3.11 to obtain Hölder continuity in time and space at the boundary:

Lemma 3.3.12 (Hölder continuity at the boundary)**.**
Let F be smooth in P_1. Then

$$\begin{aligned} &|F_t(t_1,x_1) - F_t(t_2,0)| + |\partial^2 F(t_1,x_1) - \partial^2 F(t_2,0)| \\ &+|x_1\ \partial^3 F(t_1,x_1)| + |x_1^2\ \partial^4 F(t_1,x_1)| \\ &\lesssim\ s((t_1,x_1),(t_2,0))^\beta\ (\|F\|_{C^0(P_1)} + [L_0 F]_{C_s^\beta(P_1)}) \end{aligned}$$

for all $(t_1,x_1),(t_2,0) \in P_{1/4}$.

Lemmas 3.3.11 and 3.3.12 combine to

Lemma 3.3.13 (Local Hölder continuity)**.** *Let F be smooth in P_2. Then*

$$\begin{aligned} &[\partial_t F]_{C_s^\beta(P_{1/4})} + [\partial^2 F]_{C_s^\beta(P_{1/4})} + [x\ \partial^3 F]_{C_s^\beta(P_{1/4})} + [x^2\ \partial^4 F]_{C_s^\beta(P_{1/4})} \\ &\lesssim\ \|F\|_{C^0(P_2)} + [L_0 F]_{C_s^\beta(P_2)}. \end{aligned}$$

Theorem 3.1.2 is an easy corollary of Lemma 3.3.13. The proofs of Lemma 3.3.13 and Theorem 3.1.2 can be found in Section 3.8.

3.4 Inner estimates

In this section we give the proof of Proposition 3.1.3, following the outline in the previous section. The assumptions of Proposition 3.1.3 are assumed to hold throughout the section.

Proof of Lemma 3.3.1. We need to show that

$$\frac{d}{dt}\int_{\mathbb{R}_+} x^{m+k-1}\left(\partial^k F\right)^2 \ =\ -\int_{\mathbb{R}_+} x^{m+k+1}\left(\partial^{k+2} F\right)^2. \qquad (3.18)$$

Indeed, we have by (3.6)

$$\partial^k L_m F \overset{(3.6)}{=}\ \partial_t \partial^k F + \frac{1}{2x^{m+k-1}}\ \partial^2(x^{m+k+1}\ \partial^{k+2} F) = 0. \qquad (3.19)$$

Testing (3.19) with $x^{m+k-1}\ \partial^k F$ yields

$$\partial_t \int_{\mathbb{R}_+} x^{m+k-1}(\partial^k F)^2 + \int_{\mathbb{R}_+} \partial^2(x^{m+k+1}\ \partial^{k+2} F)\ \partial^k F = 0$$

and the statement follows from integration by parts. □

Proof of Lemma 3.3.2. By the scale invariance (3.3) and the definition (3.7) of P_R, it suffices to prove the statement for $R = 1$, that is,

$$\iint_{P_\gamma} x^{m+k+1}\left(\partial^{k+2}F\right)^2 \lesssim \iint_{P_1} x^{m+k-1}\left(\partial^k F\right)^2.$$

Choose a smooth non–negative cut–off function ζ such that $\zeta = 1$ on P_γ, $\zeta = 0$ outside of P_1. Using a product Ansatz, we can assume that in addition

$$\partial\zeta \;=\; 0 \quad \text{for } x \in B_\gamma. \tag{3.20}$$

From $L_m F = 0$, it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}_+} \zeta^4\; x^{m+k-1}(\partial^k F)^2 \\
&\quad= \int_{\mathbb{R}_+} \partial_t(\zeta^4)\; x^{m+k-1}(\partial^k F)^2 + 2\int_{\mathbb{R}_+} \zeta^4\; x^{m+k-1}(\partial^k F)\; \partial_t\partial^k F \\
&\quad\overset{(3.6)}{=} \int_{\mathbb{R}_+} \partial_t(\zeta^4)\; x^{m+k-1}(\partial^k F)^2 - \int_{\mathbb{R}_+} \zeta^4\; (\partial^k F)\; \partial^2(x^{m+k+1}\partial^{k+2}F) \\
&\quad= -\int_{\mathbb{R}_+} \zeta^4\; x^{m+k+1}(\partial^{k+2}F)^2 + R,
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
R :=& \int_{\mathbb{R}_+} \partial_t(\zeta^4)\; x^{m+k-1}(\partial^k F)^2 - \int_{\mathbb{R}_+} \partial^2(\zeta^4)\; x^{m+k+1}(\partial^k F)\; (\partial^{k+2}F) \\
&+ 2\int_{\mathbb{R}_+} \partial(\zeta^4)\; x^{m+k+1}(\partial^{k+1}F)\; (\partial^{k+2}F).
\end{aligned}$$

We claim that

$$R \;\le\; \frac{1}{2}\int_{B_1} \zeta^4\; x^{m+k+1}(\partial^{k+2}F)^2 + C\int_{B_1} x^{m+k-1}(\partial^k F)^2. \tag{3.22}$$

If (3.22) holds, then an integration of (3.21) in time over $(-\infty, 0)$ yields

$$0 \;\le\; -\frac{1}{2}\iint_{P_1} \zeta^4\; x^{m+k+1}(\partial^{k+2}F)^2 + C\iint_{P_1} x^{m+k-1}(\partial^k F)^2$$

such that

$$\iint_{P_\gamma} x^{m+k+1}(\partial^{k+2}F)^2 \lesssim \iint_{P_1} x^{m+k-1}(\partial^k F)^2$$

and the statement of the lemma follows. It remains to prove (3.22). The estimate for the first term in R is immediate:

$$\int_{\mathbb{R}_+} \partial_t(\zeta^4)\ x^{m+k-1}(\partial^k F)^2 \lesssim \int_{B_1} x^{m+k-1}(\partial^k F)^2.$$

The second term can be estimated by Young's inequality,

$$\begin{aligned}
&\left|\int_{\mathbb{R}_+} \partial^2(\zeta^4)x^{m+k+1}(\partial^k F)\ (\partial^{k+2}F)\right| \\
&\quad= \left|\int_{\mathbb{R}_+} \left(12\zeta^2(\partial\zeta)^2 + 4\zeta^3\ \partial^2\zeta\right)\ x^{m+k+1}(\partial^k F)\ (\partial^{k+2}F)\right| \\
&\quad\le \frac{1}{6}\int_{\mathbb{R}_+} \zeta^4 x^{m+k+1}(\partial^{k+2}F)^2 \\
&\qquad + C\int_{\mathbb{R}_+} \left((\partial\zeta)^4 + \zeta^2\ (\partial^2\zeta)^2\right) x^2\ x^{m+k-1}(\partial^k F)^2 \\
&\quad\le \frac{1}{6}\int_{B_1} x^{m+k+1}(\partial^{k+2}F)^2 + C\int_{B_1} x^{m+k-1}(\partial^k F)^2.
\end{aligned}$$

For the third term we use the equality

$$(\partial^{k+1}F)^2 \ = \ -\partial^k F\ \partial^{k+2}F + \partial^2(\frac{1}{2}(\partial^k F)^2). \tag{3.23}$$

Hence,

$$\begin{aligned}
&\left|\int_{\mathbb{R}_+} \partial(\zeta^4)x^{m+k+1}(\partial^{k+1}F)\ (\partial^{k+2}F)\right| \\
&\quad= 2\left|\int_{\mathbb{R}_+} \left(\zeta^3\partial\zeta x^{m+k+1}\right)\partial(\partial^{k+1}F)^2\right| \\
&\quad= 2\left|\int_{\mathbb{R}_+} \partial\left(\zeta^3\partial\zeta x^{m+k+1}\right)(\partial^{k+1}F)^2\right| \\
&\quad\overset{(3.23)}{\lesssim} \left|\int_{\mathbb{R}_+} \partial\left(\zeta^3\partial\zeta x^{m+k+1}\right)(\partial^k F)(\partial^{k+2}F)\right| \\
&\qquad + \left|\int_{\mathbb{R}_+} \partial^3\left(\zeta^3\partial\zeta x^{m+k+1}\right)(\partial^k F)^2\right|.
\end{aligned} \tag{3.24}$$

The integral in in line (3.24) can be estimated similarly to the calculation above. For the integral in line (3.24) we recall (3.20) which ensures that

$|\partial^3(\zeta^3\partial\zeta x^{m+k+1})| \lesssim x^{k-1}$. One then gets using Cauchy–Schwarz,

$$\left|\int_{\mathbb{R}_+} \partial(\zeta^4)x^{m+k+1}(\partial^{k+1}F)\ (\partial^{k+2}F)\right| \\ \leq \frac{1}{6}\int_{B_1}\zeta^4 x^{m+k+1}(\partial^{k+2}F)^2 + C\int_{B_1} x^{m+k-1}(\partial^k F)^2,$$

which concludes the proof. □

The (global) Hardy inequality is stated in Lemma 2.11.1 in Chapter 2. The proof of its localized version which we give here is based on a similar reasoning:

Proof of Lemma 3.3.3. An integration by parts yields

$$\int_0^\gamma x^l(\partial^k\varphi)^2 \;=\; -\frac{2}{l+1}\int_0^\gamma x^{l+1}(\partial^k\varphi)(\partial^{k+1}\varphi) + \frac{1}{l+1}\gamma^{l+1}(\partial^k\varphi)(\gamma)^2$$

(the boundary term at $x=0$ vanishes). The first term can be estimated by Cauchy–Schwarz and Young's inequality:

$$\frac{2}{l+1}\int_0^\gamma x^{l+1}(\partial^k\varphi)(\partial^{k+1}\varphi) \;\leq\; \frac{1}{2}\int_0^\gamma x^l(\partial^k\varphi)^2 + C\int_0^\gamma x^{l+2}(\partial^{k+1}\varphi)^2.$$

For the second term, we have by standard interpolation

$$\partial^k\varphi(\gamma)^2 \;\lesssim\; \int_{\gamma/2}^\gamma \varphi^2 + \int_{\gamma/2}^\gamma (\partial^{k+1}\varphi)^2.$$

Hence,

$$\partial^k\varphi(\gamma)^2 \;\lesssim\; \int_0^\gamma x^{m-1}\varphi^2 + \int_0^\gamma x^{l+2}(\partial^{k+1}\varphi)^2.$$

Together, these estimates conclude the proof of Lemma 3.3.3. □

In the following, we prove Lemmas 3.3.4 – 3.3.5, thus completing the proof of Proposition 3.1.3.

Proof of Lemma 3.3.4. By the scale invariance (3.3) and the definition (3.7) of P_R, it suffices to prove the statement for $R=1$, that is,

$$\iint_{P_\gamma}\left(\partial_t^l\partial^k F\right)^2 \;\lesssim\; \iint_{P_1} x^{m-1}F^2. \tag{3.25}$$

Let us first consider the case $l = 0$: We apply the localized Hardy estimate $m + k - 1$ times. This yields

$$\iint_{P_\gamma} (\partial^k F)^2 \overset{(3.15)}{\lesssim} \iint_{P_\gamma} x^{m-1} F^2 + \iint_{P_\gamma} x^{2m+2k-2} (\partial^{m+2k-1} F)^2. \tag{3.26}$$

It remains to argue that for any $k \in \mathbb{N}_0$

$$\iint_{P_\gamma} x^{2m+2k-2} (\partial^{m+2k-1} F)^2 \lesssim \iint_{P_1} x^{m-1} F^2. \tag{3.27}$$

Indeed, fix k and select a sequence $\{\gamma_j\}$ strictly decreasing from 1 to γ. The local energy estimates (Lemma 3.3.2) yield

$$\iint_{P_\gamma} x^{2m+2k-2} (\partial^{m+2k-1} F)^2 \lesssim \ldots \lesssim \iint_{P_{\gamma_2}} x^m (\partial F)^2 \lesssim \iint_{P_1} x^{m-1} F^2. \tag{3.28}$$

In the case $l = 1$, choose $\gamma < \gamma_1 < 1$. We use the fact that $\partial_t F$ is again a solution of (3.8). Hence we obtain by the above

$$\iint_{P_\gamma} (\partial_t \partial^k F)^2 \lesssim \iint_{P_{\gamma_1}} x^{m-1} (\partial_t F)^2. \tag{3.29}$$

Using the equation (3.8), i.e.

$$\partial_t F = -x^2 \, \partial^4 F - 2(m+1) \, x \, \partial^3 F - \frac{1}{2} m(m+1) \, \partial^2 F,$$

this turns into

$$\begin{aligned}
\iint_{P_\gamma} (\partial_t \partial^k F)^2 &\overset{(3.29)}{\lesssim} \iint_{P_{\gamma_1}} x^{m+3} (\partial^4 F)^2 + \iint_{P_{\gamma_1}} x^{m+1} (\partial^3 F)^2 \\
&\quad + \iint_{P_{\gamma_1}} x^{m-1} (\partial^2 F)^2 \\
&\overset{(3.15)}{\lesssim} \iint_{P_{\gamma_1}} x^{m+3} \, (\partial^4 F)^2 + \iint_{P_{\gamma_1}} x^{m-1} F^2 \\
&\overset{(3.28)}{\lesssim} \iint_{P_1} x^{m-1} F^2.
\end{aligned}$$

The case of general l follows by iteration of the above argument. □

By standard interpolation the estimate can be converted into a pointwise bound in time and space which proves Lemma 3.3.5:

Proof of Lemma 3.3.5. By scaling, it suffices to prove the statement for $R = 1$, i.e.

$$\sup_{(t,x)\in P_\gamma} \left|\partial_t^l \partial^k F\right|^2 \lesssim \iint_{P_1} x^{m-1} F^2. \tag{3.30}$$

We first prove that

$$\sup_{t\in I_\gamma} \int_{B_\gamma} \left(\partial_t^l \partial^k F\right)^2 \lesssim \iint_{P_1} x^{m-1} F^2. \tag{3.31}$$

This follows at once from the straightforward inequality

$$\sup_{t\in I_\gamma} |\varphi|^2 \lesssim \int_{I_\gamma} \varphi^2 + \int_{I_\gamma} (\partial_t \varphi)^2.$$

Indeed, using Cauchy–Schwarz

$$\begin{aligned} \sup_{t\in I_\gamma} \int_{B_\gamma} \left(\partial_t^l \partial^k F\right)^2 &\lesssim \iint_{P_\gamma} \left(\partial_t^l \partial^k F\right)^2 + \iint_{P_\gamma} \left(\partial_t^{l+1} \partial^k F\right)^2 \\ &\overset{(3.25)}{\lesssim} \iint_{P_1} x^{m-1} F^2. \end{aligned}$$

Equation (3.30) follows using the same interpolation:

$$\begin{aligned} \sup_{(t,x)\in P_\gamma} \left|\partial_t^l \partial^k F\right|^2 &= \sup_{t\in I_\gamma} \sup_{x\in B_\gamma} \left|\partial_t^l \partial^k F\right|^2 \\ &\lesssim \sup_{t\in I_\gamma} \left\{ \int_{B_\gamma} \left(\partial_t^l \partial^k F\right)^2 + \int_{B_\gamma} \left(\partial_t^l \partial^{k+1} F\right)^2 \right\} \\ &\overset{(3.31)}{\lesssim} \iint_{P_1} x^{m-1} F^2. \end{aligned}$$

□

3.5 The L^∞–bound

In this section, we give the proof of Proposition 3.1.4 following the outline in Section 3.3. We start by giving the proof of Lemma 3.3.6:

Proof of Lemma 3.3.6. We first consider $m \geq 2$. In this case, we claim that

$$\sup_{x\in\mathbb{R}_+} x^{m-1}F^2 \lesssim \int_{\mathbb{R}_+} x^m\ (\partial F)^2. \tag{3.32}$$

By the interpolation Lemma 2.11.4, (3.16) follows from (3.32). In order to prove (3.32), we note that by Hardy's inequality

$$\int_{\mathbb{R}_+} x^{m-2}\ F^2 \ \lesssim \ \int_{\mathbb{R}_+} x^m\ (\partial F)^2. \tag{3.33}$$

By integration by parts and using (3.33), one can estimate

$$\begin{aligned}
\sup_{x\in\mathbb{R}_+} x^{m-1}F^2 &\lesssim \int_0^\infty \left|\partial\left(x^{m-1}F^2\right)\right| \\
&\lesssim \left(\int_0^\infty x^{m-2}F^2 + \int_0^\infty x^{m-1}\left|F\partial F\right|\right) \\
&\lesssim \int_0^\infty x^{m-1}\left|F\ \partial F\right| \\
&\lesssim \int_0^\infty x^{m-2}F^2 + \int_0^\infty x^m(\partial F)^2 \\
&\overset{(3.33)}{\lesssim} \int_0^\infty x^m(\partial F)^2,
\end{aligned}$$

which yields (3.32).

It remains to consider the case $m = 1$: Indeed, (3.32) does not hold true for $m = 1$: A counterexample is given by $F = \ln|\ln x|$ for $x \ll 1$. However, by using the Hardy inequality, (3.16) can be directly obtained through

$$\sup_{x\in\mathbb{R}_+} F^2 \lesssim \int_{\mathbb{R}_+} F^2 + \int_{\mathbb{R}_+} (\partial F)^2 \lesssim \int_{\mathbb{R}_+} F^2 + \int_{\mathbb{R}_+} x^2\ (\partial^2 F)^2,$$

which concludes the proof. □

The proof of Proposition 3.1.4 is based on Lemma 3.3.6 together with control over the first and the third Lyapunov functional of L_m.

Proof of Proposition 3.1.4. Recall the inhomogeneous equation

$$\partial_t F + \frac{1}{2x^{m-1}}\ \partial^2(x^{m+1}\ \partial^2 F) \ = \ G, \tag{3.34}$$

i.e. we denote $G = L_m F$. We assume without loss of generality that $s = 1$, that is,

$$\begin{aligned} \operatorname{supp} G \ &\subseteq \ P_1, \\ F(-1,\cdot) \ &= \ 0. \end{aligned} \tag{3.35}$$

By multiplying (3.34) with $x^{m-1}F$ and integrating in time, one gets using (3.35)

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \frac{1}{2}\int x^{m-1}F^2 + \iint \frac{1}{2}x^{m+1}(\partial^2 F)^2 \\ &\quad = \iint_{P_1} x^{m-1}F\ G \\ &\quad \overset{(3.35)}{\leq} \frac{1}{4}\iint_{P_1} x^{m-1}F^2 + \iint_{P_1} x^{m-1}G^2 \\ &\quad \leq \frac{1}{4}\sup_{t\in\mathbb{R}}\int x^{m-1}F^2 + \iint_{P_1} x^{m-1}G^2. \end{aligned}$$

Hence by absorbing on the left hand side,

$$\sup_{t\in\mathbb{R}}\int x^{m-1}F^2 \ \lesssim \ \iint_{P_1} x^{m-1}G^2 \leq \sup_{(t,x)\in P_1} x^{m-1}G^2. \tag{3.36}$$

Derivating (3.34) twice yields

$$\partial_t\partial^2 F + \frac{1}{2x^{m+1}}\ \partial^2(x^{m+3}\ \partial^4 F) \overset{(3.6)}{=} \partial^2 G,$$

By multiplying this equation with $x^{m+1}\partial^2 F$ and integrating in time, one gets

$$\begin{aligned}\sup_{t\in\mathbb{R}}\int \frac{1}{2}x^{m+1}(\partial^2 F)^2 + \iint \frac{1}{2}x^{m+3}(\partial^4 F)^2 \\ &= \iint_{P_1} x^{m+1}\partial^2 F\ \partial^2 G \\ &= \iint_{P_1} \partial^2\left(x^{m+1}\partial^2 F\right)\ G \\ &= m(m+1)\iint_{P_1} x^{m-1}\partial^2 F\ G + 2(m+1)\iint_{P_1} x^m \partial^3 F\ G \\ &\quad + \iint_{P_1} x^{m+1}\partial^4 F\ G.\end{aligned}$$

By Cauchy–Schwarz and Hardy's inequality one easily arrives at

$$\begin{aligned}\sup_{t\in\mathbb{R}}\int x^{m+1}(\partial^2 F)^2 + \iint x^{m+3}(\partial^4 F)^2 \\ \le \frac{1}{4}\iint_{P_1} x^{m+3}(\partial^4 F)^2 + C\iint_{P_1} x^{m-1}G^2,\end{aligned}$$

Absorbing on the left hand side yields

$$\sup_{t\in\mathbb{R}}\int x^{m+1}(\partial^2 F)^2 \lesssim \iint_{P_1} x^{m-1}G^2 \lesssim \sup_{(t,x)\in P_1} x^{m-1}G^2. \tag{3.37}$$

In view of (3.36), (3.37), the proof is concluded by Lemma 3.3.6. □

3.6 Remainder estimates at the boundary

In this section, we give a proof of the remainder estimates at the boundary following the outline in Section 3.3. The first lemma is at the core of Safonov's approach to Hölder estimates, see also [42]. Here we see how the inner estimates and L^∞-bound come into the polynomial approximation argument.

Lemma 3.6.1. *Assume that F is smooth in P_s. Then there exists a polynomial p of degree 2 in x and degree 1 in t such that*

$$\|x^\mu(F-p)\|_{C^0(P_r)} \lesssim \frac{r^{\mu+3}}{s^{\mu+3}}\ \|x^\mu F\|_{C^0(P_s)} + s^{\mu+2}\ \|L_m F\|_{C^0(P_s)}$$

for all $0 < r \le s$.

Proof. By the scale invariance (3.3), we may assume without loss of generality that $s = 1$. The idea is to split F into a near–field part h and a far–field part k:

$$F \;=\; h + k. \tag{3.38}$$

To this purpose, we choose a cut–off function ζ such that $\zeta = 1$ in $P_{1/2}$ and $\zeta = 0$ outside of P_1. The near–field part is defined as the solution of

$$L_m h \;=\; \zeta\; L_m F, \tag{3.39}$$

with vanishing "initial" data $h(-1,\cdot) \equiv 0$.

Note that the right hand side of (3.39) is smooth and compactly supported. In Proposition 2.5.1, we have given a proof for existence of a solution that is smooth up to the boundary in the case $m = 0$. With slight changes, the proof can be adapted to the case $m \geq 1$.

The L^∞–bound in Proposition 3.1.4 implies that

$$\|x^\mu h\|_{C^0(P_1)} \;\lesssim\; \|x^\mu L_m F\|_{C^0(P_1)} \;\leq\; \|L_m F\|_{C^0(P_1)}. \tag{3.40}$$

The far field part is defined via $k := F - h$ so that (3.38) holds.

We choose p to be the Taylor polynomial in zero of the far–field part, i. e.

$$p(t,x) \;=\; (T^0_2 k)(t,x) \;=\; k(0,0) + \partial k(0,0)x + \frac{1}{2}\,\partial^2 k(0,0)x^2 + \partial_t k(0,0)t.$$

Since we have by construction

$$L_m k \;=\; L_m F - \zeta\; L_m F \;=\; 0 \qquad\qquad \text{in } P_{1/2},$$

the inner estimate in Proposition 3.1.3 yields

$$\|k\|_{C^3(P_{1/4})} \;\lesssim\; \|x^\mu k\|_{C^0(P_{1/2})} \leq \|x^\mu k\|_{C^0(P_1)}. \tag{3.41}$$

Since p approximates k as a Taylor polynomial of second order, we thus obtain for $r \leq 1/4$

$$\begin{aligned}
\|x^\mu (k-p)\|_{C^0(P_r)} \;&\lesssim\; r^\mu\; \|k-p\|_{C^0(P_r)} \\
&\lesssim\; r^\mu\; \sup_{P_r}(x^3, |t|x, t^2)\; \|k\|_{C^3(P_{1/4})} \\
&\overset{(3.41)}{\lesssim}\; r^{\mu+3}\; \|x^\mu k\|_{C^0(P_1)}.
\end{aligned}$$

In case of $r \geq 1/4$ we have $r \sim 1$ and hence

$$\begin{aligned}
\|x^\mu(k-p)\|_{C^0(P_r)} &\leq \|x^\mu k\|_{C^0(P_1)} + \|x^\mu p\|_{C^0(P_1)} \\
&\lesssim \|x^\mu k\|_{C^0(P_1)} \\
&\quad + \max\{|k(0)|, |\partial k(0)|, |\partial^2 k(0)|, |\partial_t k(0)|\} \\
&\overset{(3.41)}{\lesssim} \|x^\mu k\|_{C^0(P_1)} \\
&\lesssim r^{\mu+3}\ \|x^\mu k\|_{C^0(P_1)}.
\end{aligned}$$

We thus have for any $r \leq 1$

$$\|x^\mu(k-p)\|_{C^0(P_r)} \lesssim r^{\mu+3}\ \|x^\mu k\|_{C^0(P_1)}. \tag{3.42}$$

We are now ready to complete the proof:

$$\begin{aligned}
\|x^\mu(F-p)\|_{C^0(P_r)} &\overset{(3.38)}{\leq} \|x^\mu(k-p)\|_{C^0(P_r)} + \|x^\mu h\|_{C^0(P_r)} \\
&\overset{(3.42)}{\lesssim} r^{\mu+3}\ \|x^\mu k\|_{C^0(P_1)} + \|x^\mu h\|_{C^0(P_1)} \\
&\overset{(3.38)}{\leq} r^{\mu+3}\ \|x^\mu F\|_{C^0(P_1)} + 2\ \|x^\mu h\|_{C^0(P_1)} \\
&\overset{(3.40)}{\lesssim} r^{\mu+3}\ \|x^\mu F\|_{C^0(P_1)} + \|L_m F\|_{C^0(P_1)}.
\end{aligned}$$

□

The next next lemma connects polynomial approximation to Hölder norms.

Lemma 3.6.2. *For every smooth function F in P_1 with $T_2^0 F = 0$ we have*

$$\sup_{0\leq\rho\leq 1} \frac{\|x^\mu F\|_{C^0(P_\rho)}}{\rho^{\mu+2+\alpha}} \lesssim \|x^\mu F\|_{C^0(P_1)} + \sup_{0\leq\rho\leq 1} \frac{\|L_m F\|_{C^0(P_\rho)}}{\rho^\alpha}. \tag{3.43}$$

Proof. Let the numbers q, r, s satisfy

$$0 \leq q \leq r \leq s \leq 1.$$

Let p be any polynomial of degree 2 in x and degree 1 in t. For such polynomials, there is the inverse estimate

$$\|x^\mu p\|_{C^0(P_r)} \lesssim \frac{r^{\mu+2}}{q^{\mu+2}} \|x^\mu p\|_{C^0(P_q)}. \tag{3.44}$$

By scaling invariance, it suffices to prove (3.44) for $r \geq 1$ and $q = 1$: Let $p(t,x) = a_0 + a_1 x + a_2 x^2 + a_3 t$. Since these polynomials form a finite dimensional vector space on P_1, all norms for this space are equivalent. Therefore

$$\sum_{i=0}^{3} |a_i| \lesssim \|x^\mu p\|_{C^0(P_1)}.$$

Hence, one gets

$$\|x^\mu p\|_{C^0(P_r)} \lesssim r^{\mu+2} \sum_{i=0}^{3} |a_i| \lesssim r^{\mu+2} \|x^\mu p\|_{C^0(P_1)}.$$

This proves (3.44). We conclude that

$$\begin{aligned}
\|x^\mu F\|_{C^0(P_r)} &\leq \|x^\mu (F-p)\|_{C^0(P_r)} + \|x^\mu p\|_{C^0(P_r)} \\
&\overset{(3.44)}{\lesssim} \|x^\mu (F-p)\|_{C^0(P_r)} + \frac{r^{\mu+2}}{q^{\mu+2}} \|x^\mu p\|_{C^0(P_q)} \\
&\leq \|x^\mu (F-p)\|_{C^0(P_r)} + \frac{r^{\mu+2}}{q^{\mu+2}} \|x^\mu (F-p)\|_{C^0(P_q)} \\
&\quad + \frac{r^{\mu+2}}{q^{\mu+2}} \|x^\mu F\|_{C^0(P_q)} \\
&\overset{q\leq r}{\leq} 2\, \frac{r^{\mu+2}}{q^{\mu+2}} \|x^\mu (F-p)\|_{C^0(P_r)} + \frac{r^{\mu+2}}{q^{\mu+2}} \|x^\mu F\|_{C^0(P_q)}.
\end{aligned}$$

According to Lemma 3.6.1 we thus obtain

$$\begin{aligned}
&\|x^\mu F\|_{C^0(P_r)} \\
&\quad\lesssim \frac{r^{\mu+2}}{q^{\mu+2}} \left(\frac{r^{\mu+3}}{s^{\mu+3}} \|x^\mu F\|_{C^0(P_s)} + s^{\mu+2} \|L_m F\|_{C^0(P_s)} \right) + \frac{r^{\mu+2}}{q^{\mu+2}} \|x^\mu F\|_{C^0(P_q)} \\
&\quad\leq r^{\mu+2} \left(\frac{r^{\mu+3}}{q^{\mu+2} s^{\mu+3}} \|x^\mu F\|_{C^0(P_s)} + \frac{1}{q^{\mu+2}} \|x^\mu F\|_{C^0(P_q)} + \frac{s^{\mu+2}}{q^{\mu+2}} \|L_m F\|_{C^0(P_s)} \right).
\end{aligned}$$

We rewrite this as

$$\begin{aligned}
\frac{\|x^\mu F\|_{C^0(P_r)}}{r^{\mu+2+\alpha}} &\lesssim \frac{r^{\mu+3-\alpha}}{q^{\mu+2} s^{1-\alpha}} \frac{\|x^\mu F\|_{C^0(P_s)}}{s^{\mu+2+\alpha}} + \frac{q^\alpha}{r^\alpha} \frac{\|x^\mu F\|_{C^0(P_q)}}{q^{\mu+2+\alpha}} \\
&\quad + \frac{s^{\mu+2+\alpha}}{q^{\mu+2}\, r^\alpha} \frac{\|L_m F\|_{C^0(P_s)}}{s^\alpha}.
\end{aligned}$$

Hence,

$$\frac{\|x^\mu F\|_{C^0(P_r)}}{r^{\mu+2+\alpha}} \lesssim \left(\frac{r^{\mu+3-\alpha}}{q^{\mu+2}s^{1-\alpha}} + \frac{q^\alpha}{r^\alpha}\right) \sup_{0<\rho\leq 1} \frac{\|x^\mu F\|_{C^0(P_\rho)}}{\rho^{\mu+2+\alpha}} + \frac{s^{\mu+2+\alpha}}{q^{\mu+2}\ r^\alpha} \sup_{0<\rho\leq 1} \frac{\|L_m F\|_{C^0(P_\rho)}}{\rho^\alpha}.$$

Since $\alpha \in (0,1)$ we may first choose $q \leq r$ sufficiently small and then $s \geq r$ sufficiently large so that the term in the brackets becomes smaller than, say, $1/2$. Hence there exists $C < \infty$ such that

$$\sup_{0\leq r\leq \frac{1}{C}} \frac{\|x^\mu F\|_{C^0(P_r)}}{r^{\mu+2+\alpha}} \leq \frac{1}{2} \sup_{0<\rho\leq 1} \frac{\|x^\mu F\|_{C^0(P_\rho)}}{\rho^{\mu+2+\alpha}} + C \sup_{0<\rho\leq 1} \frac{\|L_m F\|_{C^0(P_\rho)}}{\rho^\alpha}. \quad (3.45)$$

On the other hand, we trivially have

$$\sup_{\frac{1}{C}\leq r\leq 1} \frac{\|x^\mu F\|_{C^0(P_r)}}{r^{\mu+2+\alpha}} \lesssim \|x^\mu F\|_{C^0(P_1)}. \quad (3.46)$$

The lemma follows from combining (3.45) with (3.46) since

$$\sup_{0<\rho\leq 1} \frac{\|x^\mu F\|_{C^0(P_\rho)}}{\rho^{\mu+2+\alpha}} < \infty$$

by our assumption that $T_2^0 F = 0$. □

We are now ready to prove the remainder estimates at the boundary:

Proof of Lemma 3.3.7. Recall that we have to show

$$\sup_{0<\rho\leq 1} \frac{\|x^\mu R_2^0 F\|_{C^0(P_\rho)}}{\rho^{\mu+2+\alpha}} \lesssim \|F\|_{C^0(P_1)} + [L_m F]_{C_s^\beta(P_1)}. \quad (3.47)$$

We apply (3.43) to $R_2^0 F$ and obtain

$$\sup_{0<\rho\leq 1} \frac{\|x^\mu R_2^0 F\|_{C^0(P_\rho)}}{\rho^{\mu+2+\alpha}} \lesssim \|x^\mu R_2^0 F\|_{C^0(P_1)} + \sup_{0\leq\rho\leq 1} \frac{\|L_m R_2^0 F\|_{C^0(P_\rho)}}{\rho^\alpha}. \quad (3.48)$$

We start by remarking that $L_m T_2^0 F = (L_m F)(0)$ so that

$$L_m R_2^0 F \ = \ L_m F - (L_m F)(0).$$

Therefore by definition of P_ρ

$$\begin{aligned}\|L_m R_2^0 F\|_{C^0(P_\rho)} &= \|L_m F - (L_m F)(0)\|_{C^0(P_\rho)} \\ &\leq \sup_{z\in P_\rho} s^\beta(z,0)\ [L_m F]_{C_s^\beta(P_1)} \\ &\lesssim \rho^\alpha\ [L_m F]_{C_s^\beta(P_1)}\end{aligned}$$

with $\beta = 2\alpha$. Hence (3.48) turns into

$$\sup_{0<\rho\leq 1} \frac{\|x^\mu R_2^0 F\|_{C^0(P_\rho)}}{\rho^{\mu+2+\alpha}} \lesssim \|x^\mu R_2^0 F\|_{C^0(P_1)} + [L_m F]_{C_s^\beta(P_1)}. \tag{3.49}$$

It remains to show

$$\|x^\mu R_2^0 F\|_{C^0(P_1)} \lesssim \|F\|_{C^0(P_1)} + [L_m F]_{C_s^\beta(P_1)}. \tag{3.50}$$

By polynomial scaling, i. e. (3.44), we have for every $\rho \leq 1$:

$$\begin{aligned}&\|x^\mu T_2^0 F\|_{C^0(P_1)} \\ &\quad\lesssim \frac{1}{\rho^{\mu+2}}\|x^\mu T_2^0 F\|_{C^0(P_\rho)} \\ &\quad\leq \frac{1}{\rho^{\mu+2}}\|x^\mu R_2^0 F\|_{C^0(P_\rho)} + \frac{1}{\rho^{\mu+2}}\|x^\mu F\|_{C^0(P_\rho)} \\ &\quad\overset{(3.49)}{\lesssim} \rho^\alpha \left(\|x^\mu R_2^0 F\|_{C^0(P_1)} + [L_m F]_{C_s^\beta(P_1)}\right) + \frac{1}{\rho^2}\|F\|_{C^0(P_\rho)} \\ &\quad\leq \rho^\alpha \|x^\mu F\|_{C^0(P_1)} + \rho^\alpha \|x^\mu T_2^0 F\|_{C^0(P_1)} \\ &\qquad\quad +\rho^\alpha [L_m F]_{C_s^\beta(P_1)} + \frac{1}{\rho^2}\|F\|_{C^0(P_\rho)} \\ &\quad\leq \left(\rho^\alpha + \frac{1}{\rho^2}\right)\|F\|_{C^0(P_1)} + \rho^\alpha \|x^\mu T_2^0 F\|_{C^0(P_1)} \\ &\qquad\quad + \rho^\alpha\ [L_m F]_{C_s^\beta(P_1)}.\end{aligned}$$

By choosing a sufficiently small $\rho > 0$, the above turns into

$$\|x^\mu T_2^0 F\|_{C^0(P_1)} \lesssim \|F\|_{C^0(P_1)} + [L_m F]_{C_s^\beta(P_1)}. \tag{3.51}$$

This implies (3.50):

$$\begin{aligned}\|x^\mu R_2^0 F\|_{C^0(P_1)} &\leq \|F\|_{C^0(P_1)} + \|x^\mu T_2^0 F\|_{C^0(P_1)} \\ &\overset{(3.51)}{\lesssim} \|F\|_{C^0(P_1)} + [L_m F]_{C_s^\beta(P_1)}.\end{aligned}$$

□

3.7 Remainder estimates near the boundary

We first prove parabolic estimates in the interior

Proof of Lemma 3.3.8. Since L_m is uniformly fourth–order parabolic in the cube $(-1,0)\times(1/4,1)$, we have by standard local Schauder estimates [23]

$$\begin{aligned}&\|R_4^{(0,1/2)}F\|_{C^0((-1/2,0)\times(1/3,2/3))}\\&\qquad\lesssim\ \|F\|_{C^0((-1,0)\times(1/4,1))}+[L_mF]_{C^\beta((-1,0)\times(1/4,1))},\end{aligned}$$

where $[\cdot]_{C^\beta}$ is the Hölder semi–norm w. r. t. the standard metric for fourth–order parabolic operators, i. e. $|t_1-t_2|^{1/4}+|x_1-x_2|$. Since this standard metric is equivalent to our metric $|t_1-t_2|^{1/4}+|\sqrt{x_1}-\sqrt{x_2}|$ on $(-1,0)\times(1/4,1)$, the above estimate turns into

$$\begin{aligned}&\|R_4^{(0,1/2)}F\|_{C^0((-1/2,0)\times(1/3,2/3))}\\&\qquad\lesssim\ \|F\|_{C^0((-1,0)\times(1/4,1))}+[L_mF]_{C^\beta_s((-1,0)\times(1/4,1))}.\end{aligned}$$

Since the operator is invariant under the rescaling $(t,x)\rightsquigarrow(r^2t,rx)$, the last estimate entails

$$\|R_4^{(0,r/2)}F\|_{C^0((-r^2/2,0)\times(r/3,2r/3))}\ \lesssim\ \|F\|_{C^0(Q_r)}+r^{2+\alpha}[L_mF]_{C^\beta_s(Q_r)}. \tag{3.52}$$

In the last step, we appeal to an inverse estimate for polynomials of order at most four (as in (3.44)):

$$\begin{aligned}\|R_4^{(0,r/2)}F\|_{C^0(Q_r)}&\le\|T_4^{(0,r/2)}F\|_{C^0(Q_r)}+\|F\|_{C^0(Q_r)}\\&\lesssim\|T_4^{(0,r/2)}F\|_{C^0((-r^2/2,0)\times(r/3,2r/3))}+\|F\|_{C^0(Q_r)}\\&\le\|R_4^{(0,r/2)}F\|_{C^0((-r^2/2,0)\times(r/3,2r/3))}+2\|F\|_{C^0(Q_r)}.\end{aligned}\tag{3.53}$$

The combination of (3.52) and (3.53) yields as desired

$$\|R_4^{(0,r/2)}F\|_{C^0(Q_r)}\ \lesssim\ \|F\|_{C^0(Q_r)}+r^{2+\alpha}[L_mF]_{C^\beta_s(Q_r)}. \tag{3.54}$$

□

We are now ready to prove Lemma 3.3.9:

Proof of Lemma 3.3.9. Recall that we want to prove for $r \leq 1$:

$$\|R_4^{(0,r/2)} F\|_{C^0(Q_r)} \lesssim r^{2+\alpha} \left(\|F\|_{C^0(P_1)} + [L_m F]_{C_s^\beta(P_1)} \right). \tag{3.55}$$

We notice that $R_4^{(0,r/2)} T_2^0 F = 0$ which yields

$$R_4^{(0,r/2)} F \; = \; R_4^{(0,r/2)} R_2^0 F.$$

This allows us to make use Lemma 3.3.8 (applied to $R_2^0 F$) and then of Lemma 3.3.7 (applied to F):

$$\begin{aligned}
&\|R_4^{(0,r/2)} F\|_{C^0(Q_r)} \\
&\quad = \|R_4^{(0,r/2)} R_2^0 F\|_{C^0(Q_r)} \\
&\quad \overset{(3.54)}{\lesssim} \; \|R_2^0 F\|_{C^0(Q_r)} + r^{2+\alpha} [L_m R_2^0 F]_{C_s^\beta(Q_r)} \\
&\quad \lesssim \frac{1}{r^\mu} \, \|x^\mu R_2^0 F\|_{C^0(Q_r)} + r^{2+\alpha} [L_m R_2^0 F]_{C_s^\beta(Q_r)} \\
&\quad \leq \; \frac{1}{r^\mu} \, \|x^\mu R_2^0 F\|_{C^0(P_r)} + r^{2+\alpha} [L_m R_2^0 F]_{C_s^\beta(P_1)} \\
&\quad \overset{(3.47)}{\lesssim} \; r^{2+\alpha} \|F\|_{C^0(P_1)} + r^{2+\alpha} [L_m F]_{C_s^\beta(P_1)} + r^{2+\alpha} [L_m R_2^0 F]_{C_s^\beta(P_1)}.
\end{aligned} \tag{3.56}$$

We now appeal to

$$L_m R_2^0 F - L_m F \; = \; -L_m T_2^0 F \; = \; const,$$

so that

$$[L_m R_2^0 F]_{C_s^\beta(P_1)} \; = \; [L_m F]_{C_s^\beta(P_1)}. \tag{3.57}$$

Equation (3.55) now follows from (3.56) and (3.57). □

3.8 Schauder estimates

In this section we give the prove of the Schauder estimates, hence completing the proof of Theorem 3.1.2. We follow the outline of the proof of the Schauder estimates in Section 3.3.

Proof of Lemma 3.3.10. Recall that we want to show for $(t,x) \in P_{1/2}$:

$$\begin{aligned} &|x^2\ \partial^4 F(t,x)| + |x\ \partial^3 F(t,x)| \\ &+|\partial^2 F(t,x) - \partial^2 F(t,0)| + |\partial_t F(t,x) - \partial_t F(t,0)| \\ &\lesssim\ s((t,x),(t,0))^\beta\ (\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)}). \end{aligned}$$

By the translation in time invariance, it suffices to show that for $0 \le x \le \frac{1}{2}$:

$$\begin{aligned} &|x^2\ \partial^4 F(0,x)| + |x\ \partial^3 F(0,x)| \\ &+|\partial^2 F(0,x) - \partial^2 F(0,0)| + |\partial_t F(0,x) - \partial_t F(0,0)| \\ &\lesssim\ x^\alpha(\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)}), \end{aligned} \tag{3.58}$$

where $\beta = 2\alpha$. In order to establish this, we will argue that

$$\begin{aligned} &|x^2\ \partial^4 F(0,x)| + |x\ \partial^3 F(0,x)| \\ &+|\partial^2 F(0,x) - \partial^2 F(0,0)| + |\partial_t F(0,x) - \partial_t F(0,0)| \\ &\lesssim\ \frac{1}{x^2}\|R_4^{(0,x)} F\|_{C^0(Q_{2x})} + \frac{1}{x^{\mu+2}}\|x^\mu R_2^0 F\|_{C^0(P_{2x})}. \end{aligned} \tag{3.59}$$

Indeed, the remainder estimates (3.47) and (3.55) allow us to conclude (3.58) from (3.59).

We now address (3.59). We start with $x^2\ \partial^4 F(0,x)$. By definition of $T_4^{(0,x)}$, we have

$$\partial^4 F(0,x)\ =\ \partial^4 (T_4^{(0,x)} F)(0,x).$$

Since $\partial^4(T_2^0 F) = 0$, this can be reformulated as

$$\partial^4 F(0,x)\ =\ \partial^4 (T_4^{(0,x)} F - T_2^0 F)(0,x).$$

We now appeal to the following inverse estimate for polynomials p of order at most 4:

$$\begin{aligned} &x^4\ |\partial^4 p(0,x)| + x^3\ |\partial^3 p(0,x)| + x^2\ |\partial^2 p(0,x)| + x^2\ |\partial_t p(0,x)| \\ &\lesssim\ \|p\|_{C^0(Q_{2x})}. \end{aligned} \tag{3.60}$$

Applied to $T_4^{(0,x)}F - T_2^0 F$, (3.60) yields

$$\begin{aligned}
|x^2\ \partial^4 F(0,x)| &\lesssim \frac{1}{x^2}\|T_4^{(0,x)}F - T_2^0 F\|_{C^0(Q_{2x})} \\
&= \frac{1}{x^2}\|R_4^{(0,x)}F - R_2^0 F\|_{C^0(Q_{2x})} \\
&\lesssim \frac{1}{x^2}\|R_4^{(0,x)}F\|_{C^0(Q_{2x})} + \frac{1}{x^{\mu+2}}\|x^\mu R_2^0 F\|_{C^0(Q_{2x})} \\
&\leq \frac{1}{x^2}\|R_4^{(0,x)}F\|_{C^0(Q_{2x})} + \frac{1}{x^{\mu+2}}\|x^\mu R_2^0 F\|_{C^0(P_{2x})}.
\end{aligned}$$

Also the next term in (3.59) can be written in a similar way

$$\partial^3 F(0,x) \ = \ \partial^3(T_4^{(0,x)}F - T_2^0 F)(0,x)$$

so that likewise

$$\begin{aligned}
|x\ \partial^3 F(0,x)| &\lesssim \frac{1}{x^2}\|T_4^{(0,x)}F - T_2^0 F\|_{C^0(Q_{2x})} \\
&\leq \frac{1}{x^2}\|R_4^{(0,x)}F\|_{C^0(Q_{2x})} + \frac{1}{x^{\mu+2}}\|x^\mu R_2^0 F\|_{C^0(P_{2x})}.
\end{aligned}$$

For the third term in (3.59) we notice that

$$\partial^2 F(0,x) - \partial^2 F(0,0) \ = \ \partial^2(T_4^{(0,x)}F - T_2^0 F)(0,x)$$

so that as above

$$\begin{aligned}
|\partial^2 F(0,x) - \partial^2 F(0,0)| &\lesssim \frac{1}{x^2}\|T_4^{(0,x)}F - T_2^0 F\|_{C^0(Q_{2x})} \\
&\leq \frac{1}{x^2}\|R_4^{(0,x)}F\|_{C^0(Q_{2x})} + \frac{1}{x^{\mu+2}}\|x^\mu R_2^0 F\|_{C^0(P_{2x})}.
\end{aligned}$$

Finally, the same argument applies to the fourth term in (3.59): Since $\partial_t(T_2^0 F) = const$, we have

$$\partial_t F(0,x) - \partial_t F(0,0) \ = \ \partial_t(T_4^{(0,x)}F - T_2^0 F)(0,x),$$

so that also here, using (3.60),

$$|\partial_t F(0,x) - \partial_t F(0,0)| \lesssim \frac{1}{x^2}\|R_4^{(0,x)}F\|_{C^0(Q_{2x})} + \frac{1}{x^{\mu+2}}\|x^\mu R_2^0 F\|_{C^0(P_{2x})}.$$

This establishes (3.59). □

Proof of Lemma 3.3.11. Since L_m is uniformly fourth–order parabolic in the cube $(-2,0)\times(1/8,2)$, we have by standard local Schauder estimates

$$\begin{aligned}&[x^2\ \partial^4 F]_{C^\beta((-1,0)\times(1/4,1))}+[x\ \partial^3 F]_{C^\beta((-1,0)\times(1/4,1))}\\&+[\partial^2 F]_{C^\beta((-1,0)\times(1/4,1))}+[\partial_t F]_{C^\beta((-1,0)\times(1/4,1))}\\&\lesssim\ \|F\|_{C^0((-2,0)\times(1/8,2))}+[L_m F]_{C^\beta((-2,0)\times(1/8,2))}.\end{aligned}$$

Since the standard metric for fourth–order parabolic operators is equivalent to our metric on $(-2,0)\times(1/8,2)$, the above estimate turns into

$$\begin{aligned}&[x^2\ \partial^4 F]_{C^\beta_s((-1,0)\times(1/4,1))}+[x\ \partial^3 F]_{C^\beta_s((-1,0)\times(1/4,1))}\\&+[\partial^2 F]_{C^\beta_s((-1,0)\times(1/4,1))}+[\partial_t F]_{C^\beta_s((-1,0)\times(1/4,1))}\\&\lesssim\ \|F\|_{C^0((-2,0)\times(1/8,2))}+[L_m F]_{C^\beta_s((-2,0)\times(1/8,2))}.\end{aligned}$$

Since the operator L_m and its components, i. e. $x^2\ \partial^4 F$, $\ldots$, $\partial_t F$, scale the same under $(t,x)\rightsquigarrow(r^2t,rx)$, the last estimate entails

$$\begin{aligned}&[x^2\ \partial^4 F]_{C^\beta_s(Q_r)}+[x\ \partial^3 F]_{C^\beta_s(Q_r)}+[\partial^2 F]_{C^\beta_s(Q_r)}+[\partial_t F]_{C^\beta_s(Q_r)}\\&\lesssim\ \frac{1}{r^{2+\alpha}}\|F\|_{C^0((-2r^2,0)\times(r/8,2r))}+[L_m F]_{C^\beta_s((-2r^2,0)\times(r/8,2r))}.\end{aligned}\tag{3.61}$$

We now apply (3.61) to R^0_2F. Since $\partial^4(R^0_2F)=\partial^4F$, $\partial^3(R^0_2F)=\partial^3F$, $\partial^2(R^0_2F)-\partial^2F=const$, $\partial_t(R^0_2F)-\partial_tF=const$ and thus $L_mR^0_2F-L_mF=const$, (3.61) yields with F replaced by R^0_2F

$$\begin{aligned}&[x^2\ \partial^4 F]_{C^\beta_s(Q_r)}+[x\ \partial^3 F]_{C^\beta_s(Q_r)}+[\partial^2 F]_{C^\beta_s(Q_r)}+[\partial_t F]_{C^\beta_s(Q_r)}\\&\quad\lesssim\ \frac{1}{r^{2+\alpha}}\|R^0_2F\|_{C^0((-2r^2,0)\times(r/8,2r))}+[L_m F]_{C^\beta_s((-2r^2,0)\times(r/8,2r))}\\&\quad\lesssim\ \frac{1}{r^{\mu+2+\alpha}}\|x^\mu R^0_2F\|_{C^0((-2r^2,0)\times(r/8,2r))}+[L_m F]_{C^\beta_s((-2r^2,0)\times(r/8,2r))}.\end{aligned}\tag{3.62}$$

We now evoke Lemma 3.3.7: Since $(-2r^2,0)\times(r/8,2r)\subset P_1$ for $0\le r\le\frac{1}{2}$ we have

$$\frac{1}{r^{2+\alpha}}\|x^\mu R^0_2F\|_{C^0((-2r^2,0)\times(r/8,2r))}\ \lesssim\ \|F\|_{C^0(P_1)}+[L_mF]_{C^\beta_s(P_1)}.\tag{3.63}$$

The lemma follows from the combination of estimates (3.62) and (3.63). □

Proof of Lemma 3.3.12. In view of Lemma 3.3.10, it remains to show for $(t_1, 0)$, $(t_2, 0) \in P_{1/4}$:

$$\begin{aligned} &|\partial^2 F(t_1,0) - \partial^2 F(t_2,0)| + |\partial_t F(t_1,0) - \partial_t F(t_2,0)| \\ &\quad \lesssim\ s((t_1,0),(t_2,0))^\beta \left(\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)}\right). \end{aligned}$$

By translation in time invariance, it suffices to show for $-1/16 \leq t \leq 0$

$$\begin{aligned} &|\partial^2 F(t,0) - \partial^2 F(0,0)| + |\partial_t F(t,0) - \partial_t F(0,0)| \\ &\quad \lesssim\ (-t)^{\beta/4} \left(\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)}\right). \end{aligned} \tag{3.64}$$

We introduce $x := (-t)^{1/2} \leq 1/2$ and write

$$\begin{aligned} &|\partial^2 F(t,0) - \partial^2 F(0,0)| + |\partial_t F(t,0) - \partial_t F(0,0)| \\ &\quad \leq\ |\partial^2 F(t,0) - \partial^2 F(t,x)| + |\partial_t F(t,0) - \partial_t F(t,x)| \\ &\quad\quad + |\partial^2 F(t,x) - \partial^2 F(0,x)| + |\partial_t F(t,x) - \partial_t F(0,x)| \\ &\quad\quad + |\partial^2 F(0,x) - \partial^2 F(0,0)| + |\partial_t F(0,x) - \partial_t F(0,0)|. \end{aligned} \tag{3.65}$$

Since $(t,x),(0,x) \in P_{1/2}$, Lemma 3.3.10 yields

$$\begin{aligned} &|\partial^2 F(t,0) - \partial^2 F(t,x)| + |\partial_t F(t,0) - \partial_t F(t,x)| \\ &\quad + |\partial^2 F(0,x) - \partial^2 F(0,0)| + |\partial_t F(0,x) - \partial_t F(0,0)| \\ &\quad \lesssim\ \sqrt{x}^\beta \left(\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)}\right) \\ &\quad =\ (-t)^{\beta/4} \left(\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)}\right). \end{aligned} \tag{3.66}$$

Since $(t,x),(0,x) \in Q_x$, Lemma 3.3.11 yields

$$\begin{aligned} &|\partial^2 F(t,x) - \partial^2 F(0,x)| + |\partial_t F(t,x) - \partial_t F(0,x)| \\ &\quad \lesssim\ (-t)^{\beta/4} \left(\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)}\right). \end{aligned} \tag{3.67}$$

Inserting (3.66) and (3.67) into (3.65) yields (3.64).

□

Proof of Lemma 3.3.13. Let g denote any of the four functions $x^2\,\partial^4 F$, $x\,\partial^3 F$, $\partial^2 F$, $\partial_t F$. Recall that we want to show for any $z_1, z_2 \in P_{1/4}$

$$|g(z_1) - g(z_2)|\ \lesssim\ s(z_1,z_2)^\beta \left(\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)}\right).$$

Without loss of generality we may assume $x_2 \le x_1$. We distinguish two cases:

$$\left\{ \begin{array}{ll} \text{Case I:} & \sqrt{x_1} \ge 2s(z_1,z_2) \\ \text{Case II:} & \sqrt{x_1} \le 2s(z_1,z_2) \end{array} \right\}. \tag{3.68}$$

In case I we have

$$x_2 \;\ge\; \frac{x_1}{4} \quad \text{and} \quad |t_1 - t_2| \;\le\; \frac{x_1^2}{16}.$$

This implies that there exists a $t \in (-1/16, 0)$ such that

$$z_1,\; z_2 \;\in\; (t,0) + Q_{x_1}.$$

By translation invariance in time, we may appeal to Lemma 3.3.11 which yields as desired

$$\begin{aligned} |g(z_1) - g(z_2)| \;&\lesssim\; s(z_1,z_2)^\beta \left(\|F\|_{C^0((t,0)+P_1)} + [L_m F]_{C^\beta_s((t,0)+P_1)} \right) \\ &\le\; s(z_1,z_2)^\beta \left(\|F\|_{C^0(P_2)} + [L_m F]_{C^\beta_s(P_2)} \right). \end{aligned}$$

In case II we write

$$|g(z_1) - g(z_2)| \;\le\; |g(z_1) - g(t_1,0)| + |g(t_1,0) - g(t_2,0)| + |g(t_2,0) - g(z_2)|$$

and evoke Lemma 3.3.12 which yields

$$\begin{aligned} |g(z_1) - g(z_2)| &\lesssim (s(z_1,(t_1,0)) + s((t_1,0),(t_2,0)) + s((t_2,0),z_2))^\beta \\ &\quad \times \left(\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)} \right) \\ &= \left(\sqrt{x_1} + |t_1 - t_2|^{1/4} + \sqrt{x_2} \right)^\beta \left(\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)} \right) \\ &\overset{(3.68)}{\le} (5s(z_1,z_2))^\beta \left(\|F\|_{C^0(P_1)} + [L_m F]_{C^\beta_s(P_1)} \right). \end{aligned}$$

□

Theorem 3.1.2 is an easy corollary of Lemma 3.3.13:

Proof of Theorem 3.1.2. According to Lemma 3.3.13 we have

$$\begin{aligned} &[x^2\, \partial^4 F]_{C^\beta_s(P_{1/4})} + [x\, \partial^3 F]_{C^\beta_s(P_{1/4})} + [\partial^2 F]_{C^\beta_s(P_{1/4})} + [\partial_t F]_{C^\beta_s(P_{1/4})} \\ &\qquad \lesssim\; \|F\|_{C^0(P_2)} + [L_m F]_{C^\beta_s(P_2)} \end{aligned} \tag{3.69}$$

for all F smooth in P_2. Let F be smooth and bounded in $\mathbb{R} \times \mathbb{R}_+$. Consider

$$F_R(t,x) := F(R^2(t-1/8), Rx).$$

Then

$$\begin{aligned}
&[x^2\ \partial^4 F_R]_{C_s^\beta(P_{1/4})} + [x\ \partial^3 F_R]_{C_s^\beta(P_{1/4})} + [\partial^2 F_R]_{C_s^\beta(P_{1/4})} + [\partial_t F]_{C_s^\beta(P_{1/4})} \\
&= R^{2+\frac{\beta}{2}}\ \Big([x^2\ \partial^4 F_R]_{C_s^\beta(A_R)} + [x\ \partial^3 F_R]_{C_s^\beta(A_R)} + [\partial^2 F_R]_{C_s^\beta(A_R)} + [\partial_t F]_{C_s^\beta(A_R)}\Big)
\end{aligned}$$

where $A_R := (-R^2/8, R^2/8) \times (0, R/4)$. Furthermore

$$\begin{aligned}
\|F_R\|_{C^0(P_2)} &\leq \|F\|_{C^0(\mathbb{R}\times\mathbb{R}_+)}, \\
[L_m F_R]_{C_s^\beta(P_2)} &\leq R^{2+\frac{\beta}{2}}\ [L_m F]_{C_s^\beta(\mathbb{R}\times\mathbb{R}_+)}.
\end{aligned}$$

Therefore (3.69) turns into

$$\begin{aligned}
&[x^2\ \partial^4 F]_{C_s^\beta(A_R)} + [x\ \partial^3 F]_{C_s^\beta(A_R)} + [\partial^2 F]_{C_s^\beta(A_R)} + [\partial_t F]_{C_s^\beta(A_R)} \\
&\qquad \lesssim\ R^{-2-\frac{\beta}{2}}\ \|F\|_{C^0(\mathbb{R}\times\mathbb{R}_+)} + [L_m F]_{C_s^\beta(\mathbb{R}\times\mathbb{R}_+)}.
\end{aligned}$$

Since $\|F\|_{C^0(\mathbb{R}\times\mathbb{R}_+)} < \infty$, the theorem follows as $R \to \infty$. □

Chapter 4

A non–rupture criterion

In this chapter, we address the more general thin–film equation (1.2). By assuming flat and positive initial data, we give a positive lower bound for the solution, i.e. we give a compact flatness–criterion on the initial data which ensures global positivity of the solution of the thin–film equation. For the proof we use a new class of test functions in combination with the application of Stampacchias Lemma.

4.1 Setting and result

In this chapter we consider the thin–film equation with general exponent of mobility $n \in (0, \infty)$

$$\partial_t h + \partial_y (h^n \ \partial_y^3 h) = 0 \tag{4.1}$$

on the fixed domain $\Omega = (-a, a)$. We assume either Neumann boundary conditions,

$$\partial_y h(\pm a) = \partial_y^3 h(\pm a) \ = \ 0, \tag{4.2}$$

or periodic boundary conditions,

$$\partial_y^j h(-a) = \partial_y^j h(a) \quad \text{for } j = 0, \ldots, 3. \tag{4.3}$$

In this chapter, we it is convenient to work in the framework of weak solutions: In one space dimension, weak solutions to (4.1)–(4.2) [resp. (4.1)–(4.3)] may be defined as follows:

Definition 4.1.1. *Let $h_0 \in H^1(\Omega)$, $h_0 \geq 0$. A weak solution of* (4.1)–(4.2) *[resp.* (4.1)–(4.3)*] with initial datum h_0 is a non–negative function $h \in C([0,\infty) \times \overline{\Omega})$ such that:*

(i) $h_t \in L^2((0,\infty);(H^1(\Omega))')$;

(ii) $h_{xxx} \in L^2_{loc}(\{h>0\})$ *and* $h^{\frac{n}{2}} h_{xxx} \in L^2(\{h>0\})$;

(iii) for all [$|\Omega|$–periodic] $\varphi \in L^2((0,\infty); H^1(\Omega))$

$$\int_0^\infty < h_t, \varphi > dt = \int_0^\infty \int_\Omega \chi_{\{h>0\}} h^n \, h_{xxx} \, \varphi_x \, ;$$

(iv) $h(0,x) = h_0(x)$;

(v) $h_x = 0$ *on* $\partial\Omega \cap \{h>0\}$ *for a.e.* $t>0$ *[resp.* $h(t,\cdot)$ $|\Omega|$*–periodic].*

Long–time existence of weak solutions and their regularity properties have been established in [10, 7, 12]. Weak solutions are unique as long as they are positive (see e.g. [37, Theorem 7.1]).

We have the following result:

Theorem 4.1.2. *Let $n>0$. A positive constant C exists such that for all $h_0 \in H^1(\Omega)$ with*

$$\inf_\Omega h_0 \geq C \, \|\partial_y h_0\|^{1/2}_{L^2(\Omega)} \, \|h_0\|^{1/2}_{L^2(\Omega)} \qquad \text{for } 0<n\leq 1/2, \tag{4.4}$$

$$\inf_\Omega h_0 \geq C \, \|\partial_y h_0\|^{2/3}_{L^2(\Omega)} \, \|h_0\|^{1/3}_{L^1(\Omega)} \qquad \text{for } 1/2<n, \tag{4.5}$$

there exists a unique solution h of problem (4.1)–(4.2) [resp. (4.1)–(4.3)] with initial datum h_0. Furthermore

$$\inf_{(0,\infty)\times\Omega} h \geq \frac{1}{2} \inf_\Omega h_0, \tag{4.6}$$

and if in addition h_0 belongs to $C^{4+\alpha}(\overline{\Omega})$ and satisfies (4.2) [resp. (4.3)], then h is a classical solution (i.e. $h \in C^{1+\frac{\alpha}{4},4+\alpha}([0,\infty)\times\overline{\Omega})$).

The core of the result is of course the lower bound (4.6). The main ingredient for its proof is a new class of test functions for the thin–film equation. It is given by

$$\varphi_\gamma(s) = \int_s^\infty \int_u^\infty \frac{(\gamma - v)_+^2}{v^n}\, dv\, du. \tag{4.7}$$

Testing (4.1) with (4.7), we show that as long as it is positive, the solution h satisfies the following inequality for $0 < \delta < \gamma < \gamma_0 = \inf_\Omega h_0$:

$$\gamma_0^{-n} \sup_{(0,T)} \int_\Omega (\delta - h)_+^2 + \int_0^T \int_\Omega \chi_{\{h<\delta\}} (\partial_y^2 h)^2 \lesssim (\gamma - \delta)^{-2} \int_0^T \int_\Omega \chi_{\{h<\gamma\}} (\partial_y h)^4. \tag{4.8}$$

We use the following interpolation to close the estimate:

Lemma 4.1.3. *A positive constant C exists such that*

$$\int_\Omega \chi_{\{h<\delta\}} (\partial_y h)^4 \le C \left(\int_\Omega \chi_{\{h<\delta\}} (\partial_y h)^2 \right)^{\frac{1}{2}} \left(\int_\Omega (\delta - h)_+^2 \right)^{\frac{1}{2}} \times \left(\int_\Omega \chi_{\{h<\delta\}} (\partial_y^2 h)^2 \right) \tag{4.9}$$

for all $\delta \in \mathbb{R}$ and all $h \in C^2(\overline{\Omega})$ such that either $\partial_y h = 0$ on $\partial\Omega$ or h is $|\Omega|$–periodic.

Then (4.8) turns into an inequality for the integral quantity

$$G_T(\xi) = \int_0^T \int_\Omega \chi_{\{h<\gamma_0-\xi\}} (\partial_y h)^4 \tag{4.10}$$

to which the following calculus tool may be applied provided (4.4)–(4.5) hold:

Lemma 4.1.4 (Stampacchia)**.** *Assume that a given non–negative, non–increasing function $G : (0, \rho_0) \to \mathbb{R}$ satisfies:*

$$G(\xi) \le \frac{c_0}{(\xi - \eta)^\alpha} (G(\eta))^\beta \qquad \text{for all } 0 \le \eta < \xi \le \rho_0 \tag{4.11}$$

for some constants $c_0 > 0$, $\alpha > 0$ and $\beta > 1$. Assume further that

$$\rho_0^\alpha \ \ge \ 2^{\frac{\alpha\beta}{\beta-1}}\, c_0\ (G(0))^{\beta-1}. \tag{4.12}$$

Then $G(\rho_0) = 0$.

This classical Lemma (see [51, Lemma 4.1]), or suitable extensions of it, have been successfully applied to study finite speed of propagation and waiting time phenomena for thin–film and other degenerate parabolic equations and systems [19, 33, 35, 5, 15, 30, 25].

The chapter is organized as follows: In Section 4.2 we prove Lemma 4.1.3, and in Section 4.3 we prove Theorem 4.1.2.

Notation and conventions. In what follows, we denote by C positive constants only depending on n. We write $f \lesssim g$, whenever a constant C exists such that $f \le C\ g$. We write $f \ll g$, whenever $f \le C^{-1}\ g$ holds for a given sufficiently large constant C. The domain of integration is omitted whenever it coincides with $(-a, a)$ (for Neumann boundary conditions) or with a period (for periodic boundary conditions).

4.2 Proof of Lemma 4.1.3

If h is $|\Omega|$–periodic, we translate the domain of integration in such a way that $\partial_y h = 0$ on its boundary. Hence, we only need to prove (4.9) for this case. If $h \ge \delta$ everywhere, there is nothing to prove. Else, since h is continuous, $\{h < \delta\}$ splits into countably many connected components (b_i, c_i):

$$\{h < \delta\} = \bigcup_{i=1}^{\infty} (b_i, c_i). \tag{4.13}$$

On each connected component (b, c) (we omit the index for notational convenience), we have:

$$\int_b^c (\partial_y h)^2 = [-(\delta - h)\ \partial_y h]_b^c + \int_b^c (\delta - h)\ \partial_y^2 h.$$

At both b and c, it either holds that $h = \delta$ (if $b \ne -a$ or $c \ne a$) or $\partial_y h = 0$ (otherwise). Hence the boundary term vanishes, and by Hölder we obtain:

$$\int_b^c (\partial_y h)^2 \le \left(\int_b^c (\delta - h)^2 \right)^{\frac{1}{2}} \left(\int_b^c (\partial_y^2 h)^2 \right)^{\frac{1}{2}}. \tag{4.14}$$

Note also that there always exists $y_0 \in [b,c]$ such that $\partial_y h(y_0) = 0$: Indeed, this is true by the hypothesis if $b = -a$ or $c = a$, and follows immediately from $h(b) = h(c) = \delta$ otherwise. Therefore

$$\sup_{(b,c)}(\partial_y h)^2 \lesssim \int_b^c |\partial_y h\ \partial_y^2 h| \le \left(\int_b^c (\partial_y h)^2\right)^{\frac{1}{2}} \left(\int_b^c (\partial_y^2 h)^2\right)^{\frac{1}{2}}. \tag{4.15}$$

We obtain from (4.14) and (4.15) that

$$\begin{aligned}
\int_b^c (\partial_y h)^4 \ &\le \ \left(\sup_{(b,c)}(\partial_y h)^2\right) \int_b^c (\partial_y h)^2 \\
&\overset{(4.15)}{\lesssim} \left(\int_b^c (\partial_y h)^2\right)^{\frac{1}{2}} \left(\int_b^c (\partial_y h)^2\right) \left(\int_b^c (\partial_y^2 h)^2\right)^{\frac{1}{2}} \\
&\overset{(4.14)}{\lesssim} \left(\int_b^c (\partial_y h)^2\right)^{\frac{1}{2}} \left(\int_b^c (\delta - h)^2\right)^{\frac{1}{2}} \left(\int_b^c (\partial_y^2 h)^2\right).
\end{aligned} \tag{4.16}$$

Adding over i we conclude that

$$\begin{aligned}
\int \chi_{\{h<\delta\}} (\partial_y h)^4 &\overset{(4.13)}{=} \sum_{i=1}^{\infty} \int_{b_i}^{c_i} (\partial_y h)^4 \\
&\overset{(4.16)}{\lesssim} \sum_{i=1}^{\infty} \left(\int_{b_i}^{c_i} (\partial_y h)^2\right)^{\frac{1}{2}} \left(\int_{b_i}^{c_i} (\delta - h)^2\right)^{\frac{1}{2}} \left(\int_{b_i}^{c_i} (\partial_y^2 h)^2\right) \\
&\le \left(\int \chi_{\{h<\delta\}} (\partial_y h)^2\right)^{\frac{1}{2}} \left(\int \chi_{\{h<\delta\}} (\delta - h)^2\right)^{\frac{1}{2}} \\
&\quad \times \sum_{i=1}^{\infty} \int_{b_i}^{c_i} (\partial_y^2 h)^2,
\end{aligned}$$

and the proof of Lemma 4.1.3 is complete.

4.3 Proof of Theorem 1

Unless stated otherwise, in what follows h will denote indifferently a solution of either (4.1)–(4.2) or of (4.1)–(4.3). Global existence follows from (minor modifications of, in case of periodic boundary conditions) the works of Bernis and Friedman [10, Thm. 3.1], Beretta, Bertsch and Dal Passo [7, Prop. 1.1] and Bertozzi and Pugh [12, Thm. 2.2].

We preliminarily point out some well–known facts. Testing the equation with $\varphi = 1$, we immediately see that

$$\int h(t) \;=\; \int h_0 \quad \text{for all } t > 0. \tag{4.17}$$

Let $\gamma_0 = \inf_{\Omega} h_0 > 0$, and let

$$T \;<\; T_0 \;=\; \sup\{\tau : \inf_{(0,\tau)\times\Omega} h > \gamma_0/2\} \tag{4.18}$$

($T_0 > 0$ since h is continuous). By (ii) in Definition 4.1.1, we have that

$$h \in L^2((0,T); H^3(\Omega)). \tag{4.19}$$

In particular,

$$h(t) \in C^2(\overline{\Omega}) \quad \text{for a.e. } t \in (0,T). \tag{4.20}$$

In case of Neumann boundary conditions, (4.19) and (v) in Definition 4.1.1 imply that $\partial_y h \in L^2((0,T); (H^1_0(\Omega) \cap H^2(\Omega)))$. In addition, (iii) in Definition 4.1.1 yields $h_{yt} \in L^2((0,T); (H^1_0(\Omega) \cap H^2(\Omega))')$. Therefore $\partial_y h \in C([0,T]; L^2(\Omega))$, and

$$\begin{aligned} &\int (\partial_y h)^2(t_2) - \int (\partial_y h)^2(t_1) \\ &\quad = -\int_{t_1}^{t_2} \int < h_t, \partial_y^2 h > \\ &\quad = -\int_{t_1}^{t_2} \int h^n\, (\partial_y^3 h)^2 \quad \text{for all } 0 \le t_1 < t_2 \le T. \end{aligned} \tag{4.21}$$

A similar argument yields (4.21) in the case of periodic boundary conditions. In particular

$$\int (\partial_y h)^2(t) \;\le\; \int (\partial_y h_0)^2 \quad \text{for all } t \in [0,T]. \tag{4.22}$$

As first observed in [7, 12], testing the equation by h^α one sees that for all $\alpha \in (\frac{1}{2} - n, 2 - n) \setminus \{0, -1\}$ there exists a constant $C_{\alpha,n} \ge 1$ such that

$$\int \frac{h^{\alpha+1}(t)}{\alpha(\alpha+1)} + C_{\alpha,n}^{-1} \int_0^t \int h^{\alpha+n-3}\, (\partial_y h)^4 \;\le\; \int \frac{h_0^{\alpha+1}}{\alpha(\alpha+1)} \tag{4.23}$$

for all $t \in [0,T]$.

We are now ready to prove the lower bound. We test the equation with $\varphi'_\gamma(h)$, with φ_γ given by (4.7). Since

$$\varphi''_\gamma(s) \quad = \quad s^{-n}\ (\gamma - s)_+^2, \tag{4.24}$$

integrating by parts we obtain

$$\begin{aligned}
\int_0^t \int \varphi'_\gamma(h)\ h_t \quad &= \quad \int_0^t \int \varphi''_\gamma(h)\ \partial_y h\ h^n\ \partial_y^3 h \\
&\overset{(4.24)}{=} \quad \int_0^t \int (\gamma - h)_+^2\ \partial_y h\ \partial_y^3 h \\
&= \quad -\int_0^t \int (\gamma - h)_+^2\ (\partial_y^2 h)^2 + \frac{2}{3} \int_0^t \int \chi_{\{h<\gamma\}} (\partial_y h)^4
\end{aligned}$$

(note that, by (4.20), all integrations by parts are admissible, and the boundary terms vanish because of (v) in Definition 4.1.1). Therefore, for $\gamma < \gamma_0$ we obtain

$$\sup_{t\in(0,T)} \int \varphi_\gamma(h(t)) + \int_0^T \int (\gamma - h)_+^2\ (\partial_y^2 h)^2 \le \frac{2}{3} \int_0^T \int \chi_{\{h<\gamma\}} (\partial_y h)^4. \tag{4.25}$$

We note that

$$\begin{aligned}
\varphi_\gamma(s) \quad &\geq \quad \int_s^\delta \int_u^\delta \frac{(\gamma - v)_+^2}{v^n}\ dv\ du \\
&\gtrsim \quad \frac{(\gamma-\delta)^2}{\delta^n} (\delta - s)_+^2 \quad \text{for all } 0 < \delta < \gamma.
\end{aligned} \tag{4.26}$$

Hence, for $0 < \delta < \gamma < \gamma_0$, we estimate the left–hand side of (4.25) from below as follows:

$$\begin{aligned}
\int \varphi_\gamma(h) \quad &\overset{(4.26)}{\gtrsim} \quad \gamma_0^{-n}\ (\gamma-\delta)^2 \int (\delta - h)_+^2, \\
\int_0^T \int (\gamma - h)_+^2\ (\partial_y^2 h)^2 \quad &\geq \quad (\gamma - \delta)^2 \int_0^T \int \chi_{\{h<\delta\}} (\partial_y^2 h)^2.
\end{aligned}$$

Therefore (4.25) becomes

$$\gamma_0^{-n} \sup_{(0,T)} \int (\delta - h)_+^2 + \int_0^T \int \chi_{\{h<\delta\}} (\partial_y^2 h)^2 \lesssim (\gamma - \delta)^{-2} \int_0^T \int \chi_{\{h<\gamma\}} (\partial_y h)^4. \tag{4.27}$$

In view of (4.20) and (v) in Definition 4.1.1, we may use Lemma 4.1.3 to close the estimate:

$$
\begin{aligned}
&\int_0^T \int \chi_{\{h<\delta\}} (\partial_y h)^4 \\
&\overset{(4.1.3)}{\lesssim} \int_0^T \left(\int \chi_{\{h<\delta\}} (\partial_y h)^2 \right)^{\frac{1}{2}} \left(\int (\delta - h)_+^2 \right)^{\frac{1}{2}} \left(\int \chi_{\{h<\delta\}} (\partial_y^2 h)^2 \right) \\
&\overset{(4.22)}{\leq} \left(\int (\partial_y h_0)^2 \right)^{\frac{1}{2}} \left(\sup_{t\in(0,T)} \int (\delta - h(t))_+^2 \right)^{\frac{1}{2}} \int_0^T \int \chi_{\{h<\delta\}} (\partial_y^2 h)^2 \\
&\overset{(4.27)}{\lesssim} \left(\int (\partial_y h_0)^2 \right)^{\frac{1}{2}} \frac{\gamma_0^{\frac{n}{2}}}{(\gamma - \delta)^3} \left(\int_0^T \int \chi_{\{h<\gamma\}} (\partial_y h)^4 \right)^{\frac{3}{2}}. \qquad (4.28)
\end{aligned}
$$

Let $G_T(\xi)$ be defined by (4.10). Then (4.28) may be rewritten as follows:

$$
G_T(\xi) \lesssim \frac{\|\partial_y h_0\|_{L^2}\, \gamma_0^{\frac{n}{2}}}{(\xi - \eta)^3} \, (G_T(\eta))^{\frac{3}{2}} \quad \text{for all } 0 < \eta < \xi < \gamma_0. \qquad (4.29)
$$

Assume for a moment that a $\rho_0 > 0$ exists such that

$$
\|\partial_y h_0\|_{L^2}^2 \, \gamma_0^n \, G_T(0) \ll \rho_0^6 < \left(\frac{\gamma_0}{3} \right)^6. \qquad (4.30)
$$

Then, Lemma 4.1.4 implies that $G_T(\rho_0) = 0$, that is $\partial_y h = 0$ on $\{h < \gamma_0 - \rho_0\}$, that is $\inf_{(0,T)\times\Omega} h \geq \gamma_0 - \rho_0 > 2\gamma_0/3$. Since $T < T_0$ is arbitrary, this means that $T_0 = \infty$ and proves (4.6). In order to find such ρ_0, we note that

$$
\begin{aligned}
G_T(0) &= \int_0^T \int \chi_{\{h<\gamma_0\}} (\partial_y h)^4 \\
&\leq \gamma_0^{3-\alpha-n} \int_0^T \int h^{\alpha+n-3} (\partial_y h)^4,
\end{aligned}
$$

and therefore a ρ_0 satisfying (4.30) exists in particular if

$$
\|\partial_y h_0\|_{L^2}^2 \int_0^T \int h^{\alpha+n-3} (\partial_y h)^4 \ll \gamma_0^{3+\alpha}. \qquad (4.31)
$$

We now distinguish two cases:

$\underline{n > \frac{1}{2}}$. If $n > \frac{3}{2}$, the set $(-\infty, -1) \cap (\frac{1}{2} - n, 2 - n)$ is not empty, and therefore

$$\int_0^T \int h^{\alpha+n-3} (\partial_y h)^4 \overset{(4.23)}{\lesssim} \int h_0^{\alpha+1} \leq \gamma_0^\alpha \int h_0.$$

If $\frac{1}{2} < n \leq \frac{3}{2}$, the set $(-1, 0) \cap (\frac{1}{2} - n, 2 - n)$ is not empty, and therefore

$$\int_0^T \int h^{\alpha+n-3} (\partial_y h)^4 \overset{(4.23)}{\lesssim} \int h(T)^{\alpha+1} \overset{(4.18)}{\lesssim} \gamma_0^\alpha \int h(T) \overset{(4.17)}{\leq} \gamma_0^\alpha \int h_0.$$

Altogether, for $n > \frac{1}{2}$ we have that (4.31) holds in particular if

$$\|\partial_y h_0\|_{L^2}^2 \int h_0 \ll \gamma_0^3,$$

which is (4.4).

$\underline{n \leq \frac{1}{2}}$. We choose $\alpha = 1 \in (\frac{1}{2} - n, 2 - n)$ in (4.23), so that

$$\int_0^T \int h^{\alpha+n-3} (\partial_y h)^4 \lesssim \int h_0^2.$$

Therefore (4.31) holds in particular if

$$\left(\int (\partial_y h_0)^2\right) \left(\int h_0^2\right) \ll \gamma_0^4,$$

which is (4.5).

Hence, we have shown that under the assumption of Theorem 4.1.2, any solution of (4.1)–(4.2) [resp. (4.1)–(4.3)] with initial datum h_0 is uniformly bounded below by a positive constant. Uniqueness and regularity then follows from standard parabolic theory, see e.g. [23, Thm. 6.3].

Chapter 5

Notation

$C(k), C_k$	constants depending on k
C	$\begin{cases} \text{Chapter 1, 2:} & \text{universal constant} \\ \text{Chapter 3:} & \text{constant depending on } \alpha, \beta, \gamma, k, l, m \\ \text{Chapter 4:} & \text{constant depending on } n \end{cases}$
$f \lesssim g$	$f \leq g$ for some constant C
$f \gtrsim g$	$g \leq C\ f$ for some constant C
$f \sim g$	$f \lesssim g \lesssim f$
$f \ll g$	$f \leq C^{-1}\ g$ for a given sufficiently large constant C
$\mathbb{N}_0$	$\{0, 1, \dots\}$
$\mathbb{R}_+$	$[0, \infty)$

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