Type III von Neumann algebras in the Theory of Infinite-dimensional Groups

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Summary

In this dissertation, we study two von Neumann algebras generated by *regular representations* of infinite-dimensional groups $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$. Regular representations for general infinite-dimensional groups were defined in 1985 by Alexander Kosyak, in his Ph.D. dissertation ([**Kos85**]). $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$ are the groups of *finite*, but of infinite order upper triangular matrices with units on the diagonal. Both groups are direct limits of finite-dimensional nilpotent groups of upper triangular matrices, corresponding to two different embeddings. In [**Kos92**] and [**Kos01**], Alexander Kosyak studied right and left regular representations for these groups. These representations act on the space $L^2(B^{\mathbb{N}}, d\mu_b)$ (resp. $L^2(B^{\mathbb{Z}}, d\mu_b)$), where $B^{\mathbb{N}}$ (resp. $B^{\mathbb{Z}}$) is the space of *arbitrary* upper triangular matrices and μ_b is a quasi-invariant Gaussian measure, depending on a set of weights b. Kosyak found sufficient (and in the first case necessary) conditions on the measure μ_b for the right representations to be irreducible. Moreover, examples of measures that give rise to both reducible and irreducible regular representations for the two groups were given.

Next, we consider von Neumann algebras $\mathfrak{A}^{R,b}$ (and $\mathfrak{A}^{L,b}$), generated by the right (and left) regular representation. If the right representation is reducible, the corresponding von Neumann algebra $\mathfrak{A}^{R,b}$ is a non-type I algebra (according to the Murray-von Neumann classification). Kosyak also studied the conditions on the measure, when $\mathfrak{A}^{L,b}$ is the commutant of $\mathfrak{A}^{R,b}$. We prove that, in this case, the constant function 1 is cyclic and separating for these algebras. The corresponding modular operator and conjugation are well defined, similarly as in the case of locally compact groups. Our main theorem says that if the condition for the right von Neumann algebra to be the commutant of the left one holds, both von Neumann algebras are type III_1 factors, according to the classification of Alain Connes. In the case of $B_0^{\mathbb{N}}$, we show this by proving the triviality of the fixed point algebra of $\mathfrak{A}^{R,b}$ w.r.t. the modular evolution.

To prove the type III_1 factor property for the von Neumann algebra generated by the regular representations of the group $B_0^{\mathbb{Z}}$, we consider the crossed product (denoted by $\mathcal{N}$), of $\mathfrak{A}^{R,b}$ with $\mathbb{R}$, w.r.t. the modular group σ. The latter crossed product is an invariant of type III factors, called the *non-commutative flow of weights* (its center is called the *flow of weights*) and was defined by Connes and Takesaki. Moreover, a theorem of the same authors implies that, if $\mathcal{N}$ is a factor, then the algebra $\mathfrak{A}^{R,b}$ (and hence also its commutant $\mathfrak{A}^{L,b}$) is a type III_1 factor. In the last chapter of this dissertation we prove that the center of $\mathcal{N}$ is trivial and hence the type III_1 factor property of $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$.

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Contents

CHAPTER 1

Preliminaries

1. Introduction

This dissertation deals with certain analogues of group von Neumann algebras for two infinite-dimensional groups. A von Neumann group algebra of a locally compact group is generated by regular representations of this group. Regular representations play an important role in the representation theory of locally compact groups. The decomposition of a regular representation into irreducible ones contains all the irreducible representations for finite and compact groups and many irreducible representations of locally compact Lie groups. In the case of locally compact groups a regular representation is always reducible, since along with a right regular representation there exists a left regular representation, commuting with it. Moreover, the commutant of a right regular representation is generated by operators of the left regular representation ([**Dix69b**]). Since the beginning of the sixties, regular representation for infinite-dimensional groups have been studied (see e.g. [**AHKTV83, AHK78, Kos92, Kos94, Kos01**]). For general infinite-dimensional groups they were defined in 1985 in the Ph.D. dissertation of Alexander Kosyak. For a review of regular representations of infinite-dimensional groups we refer to e.g. [**Kos94**]. In this case, the situation can be completely different from the locally compact case. The right regular representations can for example be irreducible.

In this dissertation we shall consider two infinite dimensional groups. They are the groups of finite, but of infinite order upper triangular matrices with units on the diagonal. In other words, inductive limits of nilpotent groups of upper triangular matrices, where by a nilpotent Lie group we mean a Lie group with a nilpotent Lie algebra. The first limit, denoted $B_0^{\mathbb{N}}$, is obtained by considering the embedding which extends the matrix with a row and a column in one direction, where as the second limit, $B_0^{\mathbb{Z}}$, uses the two-sided embedding, and thus the matrices can also have negative indices. In [**Kos92**] and [**Kos01**], Alexander Kosyak studied analogues of right and left regular representations for these groups. They depend on a quasi-invariant Gaussian measure. Kosyak found sufficient (and in the first case necessary) conditions on the measure for the right representations to be irreducible. Moreover, examples of measures that give rise to both reducible and irreducible regular representations for the two groups were given.

Von Neumann group algebras were already studied in the famous papers, on rings of operators, by Murray and von Neumann ([**MvN36, MvN37, vN40, MvN43**]). There, the authors introduced the example of a von Neumann algebra of a discrete countable i.c.c. group (see later), they proved that this was a type II_1 factor, according to their classification. A von Neumann algebra is called a *factor* when its center is trivial. The latter subdivided the class of factors into five types, type I_n, type I_∞, type II_1, type II_∞ and type III. The type III case was the most mysterious at that time and it was not before the early seventies that Alain Connes found a finer classification of type III factors ([**Con73**]). In the next section of this chapter we shall discuss some basics of von Neumann algebra theory and classification of factors.

In general, a theorem Connes ([**Con76**]) says that a von Neumann group algebra of a locally compact connected separable group can be at most of type II_∞, and hence cannot be of type III. It is known that for infinite-dimensional groups von Neumann algebras generated by regular representations can be of type III. The first example of such a type III von Neumann algebra was studied in [**AHKTV83**]. It is the factor generated by the energy representations of the (infinite-dimensional) group of smooth mapping from $\mathbb{R}$ into $SU(2)$. In this dissertation, we provide other examples of type III factors generated by regular representations.

After defining the regular representations for $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$, we consider von Neumann algebras, generated by them. Since the regular representations depend on the measure, so do the von Neumann algebras. In case of an irreducible representation, the corresponding von Neumann algebra is a factor of type I_∞. However, it is certainly not clear which type of von Neumann algebra is generated by a reducible regular representation and when it is a factor. In this dissertation we give an answer to these questions. Kosyak found a condition on the measure ($E(b) < \infty$[1]) for the left von Neumann algebra to be the commutant of the right. Moreover A. Kosyak and R. Zekri ([**KZ00, KZ01**]) found sufficient conditions for the algebras to be factors. In the current work we prove that if $E(b) < \infty$, then the algebras are factors. Hence, the conditions of Kosyak and Zekri are not necessary. Furthermore, these factors are hyperfinite (i.e. generated by a family of matrix algebras) and of type III_1 according to the classification of A. Connes ([**Con73**]). This also means that they are all mutually isomorphic (due to a theorem of Haagerup, [**Haa87**]).

The structure of this work is as follows. In the first chapter, we introduce some background. We start by reviewing basic theory of von Neumann algebras, in particular classification of factors. Then, we discuss some basics of Gaussian measures on infinite dimensional spaces.

[1] see Chapters 1 and 2 for the definition

In the second chapter we consider the group $B_0^{\mathbb{N}}$ together with its regular representations, and prove the above results. A parallel study of this case was carried out recently in [**Kos**].

Finally, the last chapter deals with the group $B_0^{\mathbb{Z}}$. As mentioned above, we prove that the von Neumann algebras generated by reducible right and left regular representations are type III_1 factors. These results will appear in [**DK**]

2. Von Neumann algebras

After their discovery by von Neumann in the thirties, von Neumann algebras have become a major mathematical area. In what follows we shall review some basic facts about von Neumann algebras and the classification of factors. A detailed exposition can be found in among others [**KR83, KR86, Tak02, Tak03a, Tak03b**] and [**Con94**]. Furthermore, the reader should be familiar with the basic theory of Hilbert spaces and operators (see e.g. [**KR83**]).

Let $\mathcal{H}$ be a Hilbert space (in this work we shall only consider the separable case). We denote by $B(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$.

Definition 1.1. A von Neumann algebra is a *-subalgebra $\mathcal{M}$ of $B(\mathcal{H})$ such that

$$\mathcal{M}'' = \mathcal{M},$$

where $\mathcal{M}' := \{a \in B(\mathcal{H}); am = ma, \ \forall m \in \mathcal{M}\}$ is the commutant of $\mathcal{M}$.

The von Neumann algebra $\mathcal{C}_{\mathcal{M}} = \mathcal{M} \cap \mathcal{M}'$ is called the *center* of $\mathcal{M}$. A von Neumann algebra with a trivial center, i.e. $\mathcal{C}_{\mathcal{M}} = \mathbb{C}.\mathbf{1}$, is called a *factor*.

With a von Neumann algebra is associated its dual, $\mathcal{M}^*$, which is the set of all continuous linear functionals on $\mathcal{M}$. The predual of $\mathcal{M}$ is a Banach subspace $\mathcal{M}_*$ of $\mathcal{M}^*$ of all σ-weakly continuous linear functionals (Appendix A). The positive functionals in $\mathcal{M}_*$ are called *normal*. The following holds:

$$(\mathcal{M}_*)^* = \mathcal{M}$$

([**KR86**], Theorem 7.4.2). Now we define a state.

Definition 1.2. A state on a von Neumann algebra $\mathcal{M}$ is a linear functional $\phi \in \mathcal{M}^*$ such that $\phi(aa^*) \geq 0, \forall a \in \mathcal{M}$ (positivity) and $\phi(\mathbf{1}) = 1$. A state ϕ is called *normal* if $\phi \in \mathcal{M}_*$.

A slightly more general notion is that of a *weight*.

Definition 1.3. A *weight* on a von Neumann algebra $\mathcal{M}$ is a mapping $\rho : \mathcal{M}_+ \mapsto [0, \infty]$, where $\mathcal{M}_+$ is the set of positive elements of $\mathcal{M}$ (i.e. elements of the form aa^*, $a \in \mathcal{M}$), such that:

$$\begin{aligned} \rho(a+b) &= \rho(a) + \rho(b), \quad a, b \in \mathcal{M}_+ \\ \rho(\lambda a) &= \lambda \rho(a), \quad \lambda \in \mathbb{R}_+. \end{aligned}$$

A weight (or a state) is called *tracial* (or simply a trace) if

$$\rho(a^*a) = \rho(aa^*), \quad \forall a \in \mathcal{M}.$$

We adopt the notation:

$$\begin{aligned} \mathcal{N}_\rho &:= \{a \in \mathcal{M}; \rho(a^*a) < \infty\}, \\ N_\rho &:= \{a \in \mathcal{M}; \rho(a^*a) = 0\}, \\ F_\rho &:= \{a \in \mathcal{M}_+; \rho(a) < \infty\}. \end{aligned}$$

When $N_\rho = \{0\}$, we say that ρ is *faithful*. We say that ρ is *semi-finite* if the linear span of F_ρ is weakly dense in $\mathcal{M}$. Finally, we say that ρ is normal if there exists a family $\{\rho_i; i \in I\}$ of positive normal functionals on $\mathcal{M}$ such that $\rho(a) = \sum_{i\in I} \rho_i(a)$, for each $a \in \mathcal{M}_+$

We also need the following definition.

Definition 1.4. A closed (generally unbounded) densely defined operator A with domain $D(A)$ is said to be affiliated to a von Neumann algebra $\mathcal{M}$ (write $A\eta\mathcal{M}$) if for all unitaries u in $\mathcal{M}'$

$$u\mathrm{D}(A) \subset \mathrm{D}(A),$$
$$uAu^*\xi = A\xi, \forall \xi \in \mathrm{D}(A).$$

Equivalently, $A\eta\mathcal{M}$ if all spectral projections of $|A|$ lie in M. If A is self-adjoint, then $A\eta\mathcal{M}$ iff the one-parameter group generated by A lies in $\mathcal{M}$ (see [**Dix69a**]). The following theorem states that the operators affiliated to an abelian von Neumann algebra form a *-algebra.

Theorem 1.5. [**KR83**] *If $\mathcal{M}$ is an abelian von Neumann algebra and A, B are operators affiliated to $\mathcal{M}$, then:*

(1) *Each finite set of operators affiliated to $\mathcal{M}$ have a common dense core.*
(2) *$A + B$ is densely defined and closable and its closure $A\hat{+}B := \overline{A+B}\eta\mathcal{M}$,*
(3) *AB is densely defined and closable and its closure $A\hat{\cdot}B := \overline{AB}\eta\mathcal{M}$,*
(4) *$A\hat{\cdot}B = B\hat{\cdot}A$ and $A^*.A = A.A^*$ $(= A\hat{\cdot}A^*)$,*
(5) *if $A \subseteq B$, then $A = B$; if A is symmetric, then A is self-adjoint.*

One of the big problems in von Neumann algebra theory is their classification. The *decomposition theory* of von Neumann algebras (see e.g. [**KR86**] Chapter 14) states that a von Neumann algebra can be decomposed as a direct integral of factors. In general, an abelian von Neumann algebra can be seen as the $L^\infty(X, d\mu)$ space for some measure space (X, μ). The decomposition in factors is then the direct integral over (X, μ). When a von Neumann algebra $\mathcal{M}$ is non-commutative, one considers a direct integral of factors $\mathcal{M}_x$, labeled by x, where x varies within the center of $\mathcal{M}$, which is an abelian von Neumann algebra. Hence, factors can be regarded as building blocks of von Neumann algebras. Unfortunately, a detailed discussion of this theory would go beyond the scope of this dissertation. Thus, a classification of factors is sufficient to classify von Neumann algebras. The first classification was carried out by Murray and von Neumann in [**MvN36, MvN37, vN40, MvN43**]. We briefly discuss this theory (the details can be found in e.g. [**KR86**]).

Let $\mathcal{M}$ be a factor. Then there is a unique (up to a constant) tracial weight τ on $\mathcal{M}$. Denote by D the restriction of τ to the projections in $\mathcal{M}$ (i.e. $e \in \mathcal{M}$ such that $e^2 = e$ and $e^* = e$). That is,

$$D := \tau_{|Proj\mathcal{M}} : Proj\mathcal{M} \mapsto [0, \infty]. \tag{1}$$

The set of projections can be equipped with an equivalence relation $\sim$:

$$p_1 \sim p_2 \Leftrightarrow \exists u \in \mathcal{M} \text{ such that } p_1 = u^*u, p_2 = uu^*, \tag{2}$$

where p_1, p_2 are projections in $\mathcal{M}$ and u is a partial isometry. The function D fulfills the following properties:

$$\begin{aligned} p_1 \sim p_2 \quad &\Leftrightarrow \quad D(p_1) = D(p_2), \\ p_1 p_2 = 0 \quad &\Rightarrow \quad D(p_1 + p_2) = D(p_1) + D(p_2), \\ p \text{ is finite} \quad &\Leftrightarrow \quad D(p) < \infty. \end{aligned}$$

For a projection p to be finite means that $p \sim q$ and $q \leq p$ imply $p = q$. The function D is called *the dimension function* and is an invariant of the factor $\mathcal{M}$. In the table below we list the possible types of factors according the the range of D (after normalization), together with some examples.

$\Im m D$	Type	Example
$\{1, ..., n\}$	I_n	$\mathrm{M}_n(\mathbb{C})$
$\{1, ..., \infty\}$	I_∞	$B(\mathcal{H})$
$[0, 1]$	II_1	$W^*(G)$, G is countable i.c.c. group
$[0, \infty]$	II_∞	$W^*(G)\overline{\otimes}B(\mathcal{H})$
$\{0, \infty\}$	III	R_∞

The type I_n factors are the only finite dimensional von Neumann algebras. They are all isomorphic to matrix algebras. The infinite type I factors are isomorphic to the algebra of all bounded operators on some Hilbert space. The factors for which D takes only finite values are called *finite*, whereas all the other are called *infinite*. The above example of a type II_1 factor is the von Neumann algebra generated by the left regular representation of an infinite, discrete i.c.c. group. The latter are groups, where all the non-trivial conjugacy classes $C(g) := \{hgh^{-1}; h \in G\}$, $g \neq e$ are infinite. One can obtain an infinite type II factor by just tensoring $W^*(G)$ with $B(\mathcal{H})$ ($\overline{\otimes}$ means the von Neumann algebra tensor product, see [**KR86**]). The factor R_∞ is the unique hyperfinite type III_1 factor (see later) discovered by Araki and Woods ([**AW69**], also see example below). The factors, generated by regular representations of $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$ in this dissertation, are all isomorphic to R_∞.

Definition 1.6. A von Neumann algebra which admits a semi-finite faithful normal trace is called *semi-finite*. Otherwise it is called *properly-infinite*

Hence, the only properly-infinite factors are the type III factors. The main tool for the classification of type III factors by Alain Connes ([**Con73**]) is the modular theory of Tomita and Takesaki, which we introduce below.

3. Tomita-Takesaki Modular Theory

The main theorem of Tomita-Takesaki theory ([**Tak70**]) proves that, given a von Neumann algebra together with a faithful normal semi-finite weight, there is a canonical one-parameter group of automorphisms of the algebra (which can be regarded as a time evolution). Furthermore, the theory provides a canonical conjugation, which maps the algebra into its commutant, by the adjoint action. Here we shall review the Tomita-Takesaki theory for states, since this is sufficient for the purposes of this dissertation. For the more general version for weights and left-Hilbert algebras, we refer to [**Tak70, Tak03a**].

Recall that to a closed operator T one can associate a polar decomposition,

$$T = J|T|,$$

where J is an anti-unitary operator and $|T| = \sqrt{T^*T}$ is a positive self-adjoint operator. This decomposition is unique.

Let $\mathcal{M}$ be a von Neumann algebra and ϕ a faithful normal state on $\mathcal{M}$. The GNS construction ([**KR83**], Theorem 4.5.2) provides us with a representation π_ϕ of $\mathcal{M}$ on a Hilbert space $\mathcal{H}_\phi$, and a cyclic and separating vector $\eta_\phi \in \mathcal{H}_\phi$. The cyclic property means that the set $\pi_\phi(\mathcal{M})\eta_\phi$ is dense in $\mathcal{H}_\phi$. We say that η_ϕ is separating for $\pi_\phi(\mathcal{M})$ if it is cyclic for $\mathcal{M}'$ (or equivalently, if $\pi_\phi(a)\eta_\phi = 0$ implies $a = 0$ for all $a \in \mathcal{M}$). From now on we assume that $\mathcal{M}$ is already in its GNS representation and omit the subscript ϕ in the notation. We define the following operator:

$$S : \mathcal{M}\eta \to \mathcal{H}, a\eta \mapsto a^*\eta. \tag{3}$$

This operator is closable ([**KR86**], Lemma 9.2.1) and we denote its closure by the same symbol. We consider the polar decomposition of S:

$$S = J\Delta^{1/2}. \tag{4}$$

Now we can state the theorem of Tomita and Takesaki (for states).

Theorem 1.7. *Let $\mathcal{M}$ be a von Neumann algebra together with a faithful normal state ϕ. Let S, J and Δ be the operators defined above. Then*

$$\begin{aligned} J\mathcal{M}J &= \mathcal{M}' \\ \Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M}, \quad \forall t \in \mathbb{R}. \end{aligned}$$

Now we define the canonical automorphism group associated to $(\mathcal{M}, \phi)$.

DEFINITION 1.8. The one-parameter automorphism group defined by

$$\sigma_t(a) := \Delta^{it} a \Delta^{-it}, \quad \forall t \in \mathbb{R}, \tag{5}$$

is called the *modular automorphism group* of associated to $(\mathcal{M}, \phi)$.

One of the first consequences of Theorem 1.7 is the *modular condition.* It is a special case of the *KMS*[2]*-condition* from quantum statistical mechanics ([**HHW67**]). Let $(\mathcal{M}, \alpha, \mathbb{R})$ be a W^**-dynamical system*, i.e. a von Neumann algebra together with a strongly continuous one-parameter automorphism group α_t of $\mathcal{M}$. Define the following strip in the complex plain.

$$\mathcal{D}_\beta := \{z \in \mathbb{C}; 0 < \Im z < \beta\},$$

where $\beta \geq 0$ a positive real number.

DEFINITION 1.9. A normal state ϕ is called a KMS$_\beta$-state w.r.t. the one-parameter group α_t if for any pair of operators $a, b \in \mathcal{M}$ there exists a complex function $F_{a,b}$, which is analytic in $\mathcal{D}$, and bounded and continuous in $\overline{\mathcal{D}}$, such that

$$F_{a,b}(t) = \phi(a\alpha_t(b)), \tag{6}$$

$$F_{a,b}(t + i\beta) = \phi(\alpha_t(b)a), \tag{7}$$

for all $t \in \mathbb{R}$.

An equivalent condition is the following. A normal state ϕ on $\mathcal{M}$ is a KMS$_\beta$ state w.r.t. a time evolution α_t, if the following twisted commutation rule holds for all elements a, b in a norm dense α-invariant subalgebra of $\mathcal{M}_\alpha$ (the algebra of analytic elements, see [**BR79**] section 2.5.3).

$$\phi(a\alpha_{i\beta}(b)) = \phi(ba) \tag{8}$$

The theory of KMS states on dynamical systems is a topic on its own and we refer to [**BR02**] for a detailed discussion.

The KMS-condition comes into play in Tomita-Takesaki theory as follows. Let $(\mathcal{M}, \phi)$ be a von Neumann algebra and a faithful normal state and let J, Δ be the corresponding modular data. Then ϕ fulfills the KMS-condition at $\beta = 1$ for the time evolution $\alpha_t := \sigma_{-t}$. Later we shall use this condition in the proof that the fixed point algebra of a von Neumann algebra being trivial implies the type III$_1$ factor property of the von Neumann algebra.

4. Connes' Classification of Type III factors

Although the version of Tomita-Takesaki theorem we presented in the previous section is in the context of states, there exists a more general theory, where instead of a state one considers a weight on the algebra (in general one works with the so-called left Hilbert algebras, see [**Tak03a**], Chapter VI). Instead of a faithful normal state one considers a semi-finite faithful normal

[2] after Kubo, Martin and Schwinger

weight ψ. Then there is a one-parameter group of automorphisms of the algebra implemented by the operator Δ_ψ^{it}, associated to the weight. Now, to what extend do those groups depend on the weight? The answer to this question is given by the non-commutative version of the Radon-Nikodym theorem ([**Con73**]).

THEOREM 1.10. *Let $\mathcal{M}$ be a von Neumann algebra let ϕ be a faithful semi-finite normal weight on $\mathcal{M}$, and let U be the unitary group of $\mathcal{M}$ equipped with the σ-weak operator topology. For every faithful semi-finite normal weight ψ on $\mathcal{M}$ there exists a unique continuous mapping u of $\mathbb{R}$ into U such that:*

(1)
$$\begin{aligned} u_{t+s} &= u_t\sigma_t^\phi(u_s), \quad \forall t,s\in\mathbb{R},\\ \sigma_t^\psi(x) &= u_t\sigma_t^\phi(x)u_t^*, \quad \forall t\in\mathbb{R}, x\in\mathcal{M},\\ \psi(x) &= \phi(u_{-i/2}^* x u_{-i/2}), \quad x\in\mathcal{M}. \end{aligned}$$
This is expressed by writing $u_t = (D\psi : D\phi)_t$.

(2) *Conversely, let $t\mapsto u_t$ be a continuous mapping of $\mathbb{R}$ into U such that*
$$u_{t+s} = u_t\sigma_t^\phi(u_s), \quad \forall t,s\in\mathbb{R}$$
Then there exists a unique faithful normal semi-finite weight ψ on $\mathcal{M}$ such that $(D\psi : D\phi) = u$.

Hence the class of modular groups does not vary with the weight, modulo inner automorphisms. Another question is when the modular group is inner. The following theorem of J. Dixmier and M. Takesaki gives the answer:

THEOREM 1.11 ([**Tak03a**], Theorem VIII.3.14). *For a von Neumann algebra $\mathcal{M}$, the following are equivalent:*

(1) *$\mathcal{M}$ is semi-finite.*
(2) *There exists a semi-finite faithful normal weight for which its modular automorphism group is inner.*
(3) *The modular automorphism group of every semi-finite normal faithful weight is inner.*

In general, Connes defined the following set ([**Con73**]), which is equal to $\mathbb{R}$ if and only if $\mathcal{M}$ is semi-finite:

$$T(\mathcal{M}) := \{T_0; \sigma_{T_0}^\phi \text{ is an inner automorphism for some weight } \phi\}$$

Let $\mathcal{M}$ be a factor. The classification of Connes relies on the following invariant of $\mathcal{M}$ ([**Con73**]):

$$S(\mathcal{M}) := \bigcap_\phi \{Sp\Delta_\phi; \phi \text{ is a semi-finite faithful normal weight on } \mathcal{M}\} \tag{9}$$

The above set is called *the modular spectrum* of $\mathcal{M}$. Now we can define the different type III factors.

DEFINITION 1.12 ([**Con73**]). Let $\mathcal{M}$ be a factor of type III. Then $\mathcal{M}$ can be of the following type according the invariant $S(\mathcal{M})$:

$S(\mathcal{M})$	Type
$\mathbb{R}$	III_1
$\{\lambda^n; n \in \mathbb{Z}\} \cup \{0\}, 0 < \lambda < 1$	III_λ
$\{0, 1\}$	III_0

It seems a lot of work to compute the spectra of all the modular operators above. However, there is another set which is usually more easy to compute, called *Connes' spectrum*. Let ϕ be a semi-faithful normal weight of $\mathcal{M}$ and we define the *centralizer* of ϕ as the fixed point algebra of $\mathcal{M}$ w.r.t. the modular group:

$$\mathcal{M}_\phi := \{a \in \mathcal{M}; \sigma_t^\phi(a) = a, \forall t \in \mathbb{R}\}$$

Then, Connes' spectrum of σ^ϕ is defined as follows ([**Con73**]):

$$\Gamma(\sigma^\phi) := \bigcap \{Sp(\Delta_{\phi_e}); e \in Proj(\mathcal{M}_\phi), e \neq 0\}, \tag{10}$$

where ϕ_e is the reduced weight on the reduced von Neumann algebra $\mathcal{M}_e = e\mathcal{M}e$, and $\phi_e(a) = \phi(a)$ for all $a \in e\mathcal{M}e$. Then by [**Con73**], Théorème 3.2.1.,

$$S(\mathcal{M}) \cap \mathbb{R}_+^* = \Gamma(\sigma^\phi) =: \Gamma(\mathcal{M}), \tag{11}$$

for some semi-finite faithful normal weight ϕ.

5. Crossed Products and Duality

Given a W^*-dynamical system $(\mathcal{M}, \alpha, G)$ on the Hilbert space $\mathcal{H}$, where G is a locally compact group and α a continuous homomorphism of G into $Aut(\mathcal{M})$, one can associate to it a new dynamical system on the Hilbert space $\tilde{\mathcal{H}} := L^2(G, \mathcal{H})$, called *the crossed product* of $\mathcal{M}$ with G w.r.t. α. Let us give the definition. Consider the following two representations, π_α and λ_G of $\mathcal{M}$ and G on $\tilde{\mathcal{H}}$:

$$(\pi_\alpha(a)\xi)(s) := \alpha_{s^{-1}}(a)\xi(s), \quad a \in \mathcal{M}, s \in G, \tag{12}$$

$$(\lambda_G(t)\xi)(s) := \xi(t^{-1}s), \quad t, s \in G, \tag{13}$$

where we assume that the Haar measure on G is left invariant. The representation $\{\pi_\alpha, \lambda_G\}$ is covariant, that is it fulfills the following identity:

$$\pi_\alpha(\alpha_s(a)) = \lambda_G(s)\pi_\alpha(a)\lambda_G(s)^*. \tag{14}$$

Then we define

DEFINITION 1.13. Let $(\mathcal{M}, \alpha, G)$ be a W^*-dynamical system on $\mathcal{H}$ and π_α and λ_G defined as above. Than the von Neumann algebra

$$\hat{\mathcal{M}} = \mathcal{M} \rtimes_\alpha G := (\pi_\alpha(\mathcal{M}) \cup \lambda_G(G))'' \tag{15}$$

is called *the crossed product* of $\mathcal{M}$ and G, w.r.t. α.

We can also define the crossed product for a covariant representation $\{\rho, V, \mathcal{K}\}$ of $\mathcal{M}$:

$$\begin{aligned} (\rho_\alpha(a)\xi)(s) &:= \rho(\alpha_{s^{-1}}(a))\xi(s), \quad \xi \in L^2(G, \mathcal{K}), \\ (\lambda_{\mathcal{K}}(t)\xi)(s) &:= \xi(t^{-1}s), \quad t, s \in G, \\ \mathcal{N}_{\rho,\mathcal{K}} &:= (\rho_\alpha(\mathcal{M}) \cup \lambda_{\mathcal{K}}(G))'' \end{aligned} \tag{16}$$

The crossed product is, however, independent (up to a cocycle permutation) of the representation $\mathcal{M}$ ([**Tak03a**], Theorem X.1.7).

The notion of crossed product for algebras is an analogue (or rather generalization) of the notion of semi-direct product in groups.

Example: Let G and H be two countable discrete groups and $\alpha : G \to Aut(H)$ is a homomorphism. Recall that the semi-direct product, which we shall denote by $H \rtimes_\alpha G$, is a group K whose underlying set is $H \times G$, where group-multiplication is given by

$$(h_1, g_1)(h_2, g_2) = (h_1\alpha_{g_1}(h_2), g_1g_2);$$

It is not difficult to prove (see e.g. [**Sun87**]), that the group von Neumann algebra of $H \rtimes_\alpha G$, i.e. the von Neumann algebra $W^*(H \rtimes_\alpha G)$, generated by the left regular representation of $H \rtimes_\alpha G$, is isomorphic to the crossed product of $W^*(H)$ with G w.r.t. an action $\tilde{\alpha}$ induced by α:

$$\begin{aligned} u_g\xi(h) &:= \xi(\alpha_{g^{-1}}(h)), \quad g \in G, h \in H, \\ \tilde{\alpha}_g(a) &:= u_g a u_g^{-1}, \text{ and hence} \\ W^*(H \rtimes_\alpha G) &\cong W^*(H) \rtimes_{\tilde{\alpha}} G. \end{aligned}$$

Now we state a theorem, which will be important later. It gives a convenient description of the commutant of the crossed product $\mathcal{N}$ of $\mathcal{M}$ with G. For a proof we refer to [**Tak03a**], Theorem X.1.21.

THEOREM 1.14. *Consider a W^*-dynamical system $(\mathcal{M}, G, \alpha)$ over a locally compact group G, represented by a covariant representation $\{\rho, V, \mathcal{K}\}$. Define*

$$(W\xi)(s) = V(s)^*\xi(s), \quad \xi \in L^2(G, \mathcal{K}).$$

Then

$$\begin{aligned} \mathcal{N}_{\rho,\mathcal{K}} &= (W\rho(\mathcal{M})W^* \cup \mathcal{R}_l(G))'', \\ \mathcal{N}'_{\rho,\mathcal{K}} &= \left(\rho(\mathcal{M})' \cup W\mathcal{R}_r(G)W^*\right)'', \end{aligned}$$

where $\mathcal{N}_{\rho,\mathcal{K}}$ is defined by (16) and $\mathcal{R}_l$ (resp. $\mathcal{R}_r$) is the right (resp. left) von Neumann algebra of G.

The next topic in the theory of crossed product we shall discuss is the *duality theory*. It was discovered by Connes ([**Con73**]) and Takesaki ([**Tak73**]). From now on we assume that G is abelian. Let $\hat{G}$ be the dual group of G. On the Hilbert space $L^2(G)$ we define

$$\begin{aligned} (\lambda_G(s)\xi)(r) &:= \xi(r - s), \quad \xi \in L^2(G), \quad r, s \in G, \\ (\mu_G(p)\xi)(r) &:= \overline{\langle r, p\rangle}\xi(r), \quad p \in \hat{G}. \end{aligned}$$

It then follows that with $U(s) = \lambda_G(s)$ and $V(p) = \mu_G(p)$, $s \in G, p \in \hat{G}$, U and V satisfy the following relation:

$$U(s)V(p)U(s)^*V(p)^* = \langle s, p\rangle. \tag{17}$$

Definition 1.15. In general, a pair of unitary representations U of G and V of $\hat{G}$ on the same Hilbert space $\mathcal{H}$ is said to be *covariant* if the commutation relation (17) is satisfied. The commutation relation (17) is called the *Weyl-Heisenberg* commutation relation.

The following result will play a crucial role in the poof of our main theorems.

Proposition 1.16. *[*[**Tak03a**]*, Proposition 2.2] The covariant representation $\{\lambda_G, \mu_G\}$ generates the factor $B(L^2(G))$ of all bounded operators. If $\{U, V, \mathcal{H}\}$ is a covariant representation, there exists a Hilbert space $\mathcal{H}_0$ such that*

$$\{\lambda_G \otimes \mathbf{1}, \mu_G \otimes \mathbf{1}, L^2(G) \otimes \mathcal{H}_0\} \cong \{U, V, \mathcal{H}\}.$$

The dimension of $\mathcal{H}_0$ is called the multiplicity *of $\mathcal{H}$.*

Proof. Due to the importance of this theorem we include the proof of the first part here, for the second part we refer to [**Tak03a**], Proposition 2.2. For each $f \in L^1(\hat{G})$, we define

$$V(f) := \int_{\hat{G}} f(p)V(p)dp.$$

Then V is a *-representation of $L^1(\hat{G})$, so that it can be extended to the enveloping C^*-algebra $C_0(G)$ (the algebra of continuous functions vanishing at infinity[3]). We shall denote the extended representation of $C_0(G)$ by V again. In the case when $V = \mu_G$, we have that $\mu_G(f)$ is the multiplication by f on $L^2(G)$ ($f \in C_0(G)$). Hence the von Neumann algebra $\mathcal{A}$ generated by $\{\mu_G(f); f \in C_0(G)\}$ is the multiplication algebra $L^\infty(G)$ on $L^2(G)$. So it is maximal abelian (i.e. $L^\infty(G)' = L^\infty(G)$). Now, we have

$$\lambda_G(s)\mu_G(f)\lambda_G(s)^* = \mu_G(\lambda_s f), \quad s \in G, \quad f \in L^\infty(G),$$

where $(\lambda_s f)(r) = f(r - s)$. Hence the operators of $\mathcal{A}$ commuting with $\lambda_G(G)$ are only scalars (the Haar measure dr is ergodic). Therefore,

$$\{\lambda_G(G), \mu_G(\hat{G})\}' = \mathbb{C},$$

so that $\{\lambda_G, \mu_G\}$ is irreducible. □

Now we again consider the W^*-dynamical system $(\mathcal{M}, \alpha, G)$. Let λ_G be the representation (12) on $L^2(G, \mathcal{H})$. Analogously to the previous case we define

$$(\mu_G(p)\xi)(s) = \overline{\langle s, p\rangle}\xi(s), \quad p \in \hat{G}. \tag{18}$$

[3]A function f on G is said to vanish at infinity if given any $\epsilon > 0$, there is a compact subset of G such that $|f(x)| < \epsilon$ for x outside this subset

We define the following action of $\hat{G}$ on $\mathcal{M} \rtimes_\alpha G$:

$$\hat{\alpha}_p(x) = \mu_G(p) x \mu_G(p)^*, \quad x \in \mathcal{M} \rtimes_\alpha G, \quad p \in \hat{G}. \tag{19}$$

DEFINITION 1.17. The representation μ_G of $\hat{G}$ on $L^2(G, \mathcal{H})$ defined above is called *the dual representation* to λ_G. The action $\hat{\alpha}$ of $\hat{G}$ on the crossed product $\hat{\mathcal{M}} = \mathcal{M} \rtimes_\alpha G$ is called *the dual action* and the resulting dynamical system $(\tilde{\mathcal{M}}, \hat{\alpha}, \hat{G})$, we call *the dual system.*

The above definition is justified by the following duality theorem.

THEOREM 1.18 ([**Con73, Tak73**]). *Let $(\mathcal{M}, \alpha, G)$ and $(\hat{\mathcal{M}}, \hat{\alpha}, \hat{G})$ be as above. Then we have*

$$(\mathcal{M} \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong \mathcal{M} \overline{\otimes} B(L^2(G)). \tag{20}$$

Although we discussed the general case of the above theory, from now on we will consider only the case $G = \mathbb{R} = \hat{G}$. For convenience we shall denote a dynamical system by only the von Neumann algebra and the action of $\mathbb{R}$.

Now we state the main theorem of the structure of type III von Neumann algebras. Recall that a von Neumann algebra is called properly infinite, if there is no semi-finite faithful normal trace on it.

THEOREM 1.19 ([**Tak73, CT77**]).

(1) *Let $(\mathcal{N}, \theta)$ be a W^*-dynamical system such that*
   - *$\mathcal{N}$ admits a faithful semi-finite normal trace τ;*
   - *θ transforms in such a way that*

$$\tau \circ \theta_s = e^{-s} \tau, \quad s \in \mathbb{R}.$$

Then the crossed product $\mathcal{M} = \mathcal{N} \rtimes_\theta \mathbb{R}$ is properly infinite and the center $\mathcal{C}_\mathcal{M}$ is precisely the fixed point algebra $\mathcal{C}_\mathcal{N}^\theta$ of the center of $\mathcal{N}$ under the canonical embedding of $\mathcal{N}$ into $\mathcal{M}$ (the representation π_θ). Furthermore, $\mathcal{M}$ is of type III (i.e. all the factors in the decomposition of $\mathcal{M}$ are of type III) if and only if the central dynamical system $(\mathcal{C}_\mathcal{N}, \theta)$ does not contain an invariant subalgebra $\mathcal{A}$, such that the subsystem $(\mathcal{A}, \theta)$ is isomorphic to $L^\infty(\mathbb{R})$ together with the translation action of $\mathbb{R}$. In the case that $\mathcal{M}$ is of type III, $\mathcal{N}$ is necessarily of type II_∞ (i.e. $\tau(\mathbf{1}) = \infty$).

(2) *If $\mathcal{M}$ is a von Neumann algebra of type III, then there exists a unique, up to conjugacy, covariant system $(\mathcal{N}, \theta)$ satisfying the conditions of (1).*

An immediate consequence of the above Theorem is that $\mathcal{M}$ is a factor if and only if $(\mathcal{N}, \theta)$ is centrally ergodic. Also, we see that the system $(\mathcal{N}, \theta)$ is an invariant for the algebraic type of $\mathcal{M}$.

DEFINITION 1.20 ([**Con73, Tak73**]). The dynamical system $(\mathcal{N}, \theta)$ associated to $\mathcal{M}$ is called *the non-commutative flow of weights* of $\mathcal{M}$, whereas the central system $(\mathcal{C}_\mathcal{N}, \theta)$ is called *the flow of weights* of $\mathcal{M}$.

The flow of weights turns out to be an invariant, which gives us the same types as the modular spectrum.

THEOREM 1.21 ([**CT77**]). *Let $\mathcal{M}$ be a factor of type III. The the following holds:*

(1) *$\mathcal{M}$ is of type III_λ, $0 < \lambda < 1$, if and only if the flow of weights has a period T, with $\lambda = e^{-T}$,*
(2) *$\mathcal{M}$ is of type III_0 if the flow of weights has no period,*
(3) *$\mathcal{M}$ is of type III_1 if the flow of weights is trivial, i.e. $\mathcal{N}$ is a factor.*
(4) *$p \in T(\mathcal{M})$ if and only if there exists $u \in U(\mathcal{C}_\mathcal{N})$ with $\theta_s(u) = e^{ips}u$.*

Although the invariants $S(\mathcal{M})$ and $(\mathcal{C}_\mathcal{M}, \theta)$ provide a finer classification of factors, they are not complete (i.e. classify the factors up to isomorphism). However, there is an important class of factors which has been classified completely ([**Con76, Haa87**]). These are the *hyperfinite* or *injective* factors (other names are amenable and AFD[4]). The definitions of each of the terms are different, but they were proven to be equivalent ([**Con76**]).

6. Classification of Hyperfinite Factors

DEFINITION 1.22. A von Neumann algebra with a separable predual is called hyperfinite if it is generated by an increasing family of finite-dimensional subalgebras.

Now we shall, very briefly, review the full classification of the hyperfinite factors, which was carried out by Connes, up to one case, in [**Con76**]. The remaining, type III_1 case was solved by Haagerup ([**Haa87**]).

The type I case is of course trivial. The type II situation is the following.

THEOREM 1.23 ([**Con76**]).
(1) *Any amenable (hyperfinite) factor of type II_1 is isomorphic to the Murray von Neumann hyperfinite factor R, where $R \cong W^*(G)$, G countable discrete i.c.c. group.*
(2) *Let F be a type I_∞ factor. There exists up to isomorphism, only one amenable factor of type II_∞, namely $R_{0,1} = R\overline{\otimes}F$.*

An important corollary is that von Neumann algebras of connected separable locally compact groups can have at most type II_∞ factors in their decomposition.

COROLLARY 1.24 ([**Con76**]). *Let G be a connected separable locally compact group and let λ be the left regular representation of G in $L^2(G)$. Then $W^*(G) := (\lambda(G))''$ is a direct integral of factors which are either of type I or isomorphic to $R_{0,1}$.*

As already mentioned in the introduction, von Neumann algebras generated by regular representations of infinite-dimensional groups which are inductive limits of connected locally compact groups, can be of type III.

[4] Approximately finite dimensional

In order to explain the type III case we have to mention infinite tensor products of von Neumann algebras. Let $(\mathcal{M}_\nu, \phi_\nu)_\nu$ be a sequence of pairs (matrix algebras, faithful state). Let $\mathcal{A}$ be the inductive limit of the C^*-algebras (for the precise definition we refer to [**Tak03b**], Chapter XIV)

$$\mathcal{A}_\nu = \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes ... \otimes \mathcal{M}_\nu,$$

where the embedding $\mathcal{A}_\nu \subset \mathcal{A}_{\nu+1}$ is by means of the mapping $x \mapsto x \otimes 1$. On $\mathcal{A}$, which is a C^*-algebra with unit one defines a state $\phi := \bigotimes_{\nu=1}^{\infty} \phi_\nu$ by the equality

$$\phi(x_1 \otimes x_2 \otimes ... \otimes x_\nu \otimes 1 \otimes ...) := \phi_1(x_1)\phi_2(x_2)...\phi_\nu(x_\nu).$$

Then the infinite tensor product $(\mathcal{M}, \phi)$ of $(\mathcal{M}_\nu, \phi_\nu)$ is defined as

$$(\mathcal{M}, \phi) := \pi_\phi(\mathcal{A}).$$

Example 1: The following construction was discovered by R. T. Powers in 1967 ([**Pow67**]). Let $(\mathcal{M}_\nu, \phi_\nu)$ be a sequence of factors and states, where $\mathcal{M}_\nu$ are all equal to $M_2(\mathbb{C})$ and the states are given by

$$\phi_\nu((a_{ij})) := \alpha a_{11} + (1-\alpha)a_{22}.$$

Define $R_\lambda := \bigotimes_{\nu=1}^{\infty}(\mathcal{M}_\nu, \phi_\nu)$, where $\lambda = \frac{\alpha}{\alpha - 1}$. Then R_λ are mutually non-isomorphic factors ([**Pow67**]). Moreover, R_λ is of type III$_\lambda$, $0 < \lambda < 1$ ([**Con73**]).

THEOREM 1.25 ([**Con76**]). *For each $0 < \lambda < 1$, there exists up to isomorphism only one hyperfinite factor of type III$_\lambda$, the Powers factor R_λ.*

After Powers discovered the above factors, Araki and Woods made a classification of factors which are infinite tensor products of matrix algebras in the above sense, called ITPFI factors ([**AW69**]). They defined two invariants

$$r_\infty(\mathcal{M}) := \{\lambda \in]0,1[; \mathcal{M}\overline{\otimes}R_\lambda \cong \mathcal{M}\}, \tag{21}$$

$$\rho(\mathcal{M}) := \{\lambda \in]0,1[; \mathcal{M}\overline{\otimes}R_\lambda \cong R_\lambda\}. \tag{22}$$

Later, Connes proved ([**Con73**]) that $r_\infty(\mathcal{M}) = S(\mathcal{M})$ and $T(\mathcal{M}) = \frac{2\pi}{Log\rho(\mathcal{M})}$.

The above example can be obtained in a different way, using the so-called group measure space construction, already introduced in [**MvN36, vN40**]. We give another example, which shows a procedure to obtain hyperfinite factors from ergodic theory.

Example 2: Let $X_0 = \{1, 2, ...N\}$ be a finite set and let μ_0 be a probability measure defined on the subsets of X_0. such that $\mu_0(\{j\}) = p_j > 0$ for $1 \le j \le N$ and $\sum p_j = 1$. Let $X = X_0^{\mathbb{N}} = \{\omega : \mathbb{N} \to X_0\}$. Equip X with the product σ-algebra $\mathcal{F}$ and the product measure

$$\mu := \bigotimes_{n=1}^{\infty} \mu_n,$$

with $\mu_n = \mu_0$ for all n. By a cylinder set in X (see also next section), we shall mean a set of the form $\{\omega \in X; (\omega(1), \omega(2), ..., \omega(n)) \in E_n, \}$, where E_n is any subset of

$$X_0^n := X_0 \times ... \times X_0,$$

where this is an n-fold product. Thus $\mathcal{F}$ is the σ-algebra generated by cylinder sets. By a elementary cylinder set we shall mean a set of the form $\{\omega \in X : \omega(n) = j_0\}$ for some n in $\mathbb{N}$ and $j_0 \in X_0$.

For each permutation σ of $\{1, ..., N\}$ and k in $\mathbb{N}$, let $T_{\sigma,k} : X \to X$ be defined by

$$(T_{\sigma,k}\omega)(m) \quad := \quad \begin{array}{ll} \omega(m), & \text{if } m \neq k, \\ \sigma(\omega(k)), & \text{if } m = k. \end{array}$$

Since $p_j > 0$ for all j, it is clear that each $T_{\sigma,k}$ is a automorphism of $(X, \mathcal{F}, \mu)$. Let G be the group generated by $\{T_{\sigma,k}; \sigma \in C_N, k \in \mathbb{N}\}$, where C_N is the cyclic subgroup of S_N generated by a full cycle, say $(12...N)$.

Now consider the algebra $\mathcal{M} := L^\infty(X, \mathcal{F}, \mu)$ associated to the above dynamical system. We define the action on $\mathcal{M}$, induced by the action of G on X, as

$$\alpha_g(f) := f \circ g^{-1}.$$

Definition 1.26. We define the von Neumann algebra associated with $(X, \mathcal{F}, \mu, G)$ by

$$\mathcal{R}(X, \mathcal{F}, \mu, G) := L^\infty(X, \mathcal{F}, \mu) \rtimes_\alpha G.$$

We say that the action of G on $(X, \mathcal{F}, \mu)$ is *free* if for any $g \in G$ and for any set $E \in \mathcal{F}$, $\mu(E) > 0$, there exists a set $F \in \mathcal{F}$ such that $F \subseteq E$, $\mu(F) > 0$ and $F \cap gF = \emptyset$. We say that the action α is *ergodic* if the only invariant sets are trivial (measure 0 or full measure). On the algebra level this is equivalent to the fixed point algebra w.r.t. the induced action α being trivial. For the following Theorem we refer to [**Tak03b**], Theorem 1.5 and Corollary 1.6.

Theorem 1.27.
(1) *The action of G on $(X, \mathcal{F}, \mu)$ is free $\Leftrightarrow \mathcal{M} = L^\infty(X, \mathcal{F}, \mu)$ is maximal abelian in $\mathcal{R}(X, \mathcal{F}, \mu, G)$.*
(2) *$\mathcal{R}(X, \mathcal{F}, \mu, G)$ is a factor $\Leftrightarrow \alpha$ is ergodic.*

For the above example one can prove that the action of G on $(X, \mathcal{F}, \mu)$ is free and ergodic (see e.g. [**Sun87**], Ex. 4.3.6, 4.3.7). W. Krieger proved ([**Kri70**]) that a system such as the one above induces an ITPFI factor in the sense of Araki and Woods. Moreover, he showed that every ITPFI factor can be obtained in this way. He introduced an invariant $r(\mathcal{M})$, called *the asymptotic ratio set*, which is equivalent to the $r_\infty(\mathcal{M})$ of Araki-Woods:

$$\begin{aligned} r(G) \quad := \quad & \{\lambda \in [0, +\infty); \forall \epsilon > 0, \forall A \subset X, \mu(A) > 0, \\ & \exists B \subset A, \mu(B) > 0, \text{ and } g \in G \text{ such that} \\ & gB \subset A \text{ and } \left| \frac{d\mu(gx)}{\mu(x)} - \lambda \right| \leq \epsilon, \forall x \in B\} \end{aligned}$$

If we set $X_0 = \{1, 2\}$ and $p_1 = 1/(1+\lambda), p_2 = \lambda/(1+\lambda)$ in example 2, one can prove that $r(\mathcal{R}(X, \mathcal{F}, \mu, G)) = \{\lambda^n; n \in \mathbb{Z}\} \cup \{0\}$ and hence we obtain the type III_λ factor R_λ defined above. For $X_0 = \{1, 2, 3\}$ and

$$p_1 = \frac{1}{1+\lambda_1+\lambda_2}, \quad p_2 = \frac{\lambda_1}{1+\lambda_1+\lambda_2}, \quad p_3 = \frac{\lambda_2}{1+\lambda_1+\lambda_2},$$

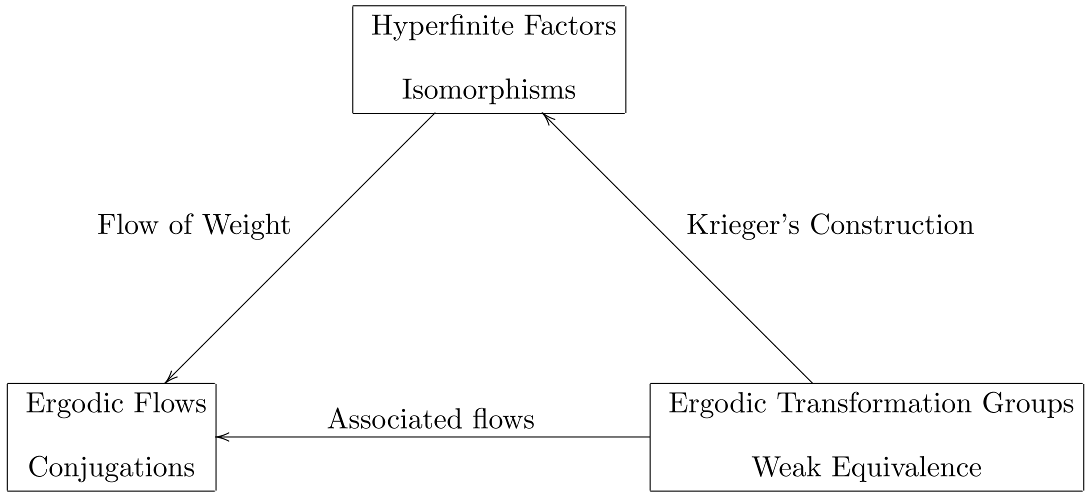
we obtain a hyperfinite type III_1 factor (for a general proof see e.g. [**Sun87**] Lemma 4.3.8). Let us call this factor R_∞.

THEOREM 1.28 ([**Haa87**]). *There is, up to isomorphism, only one hyperfinite type III_1 factor, namely the factor R_∞.*

In fact, W. Krieger studied the so-called weak equivalence of dynamical systems $(X, \mathcal{R}, \mu, T)$, where T is an ergodic transformation and μ is quasi-invariant under T and obtained certain factors (of which the above is an example), which we now call Krieger's factors. He proved that two transformations are weakly equivalent if and only if the corresponding Krieger factors are isomorphic. A detailed discussion, however, would go beyond the scope of this dissertation. We refer to [**Kri69, Kri70, Tak03b**] for more details. Krieger also proved that two Krieger's factors are isomorphic if and only if their flows if weights are isomorphic ([**Kri76**]). Moreover, Connes proved that

THEOREM 1.29 ([**Con76**]). *Any hyperfinite type III_0 factor is a Krieger factor.*

This also concludes the classification of hyperfinite factors. From the above discussion it follows that there is an equivalence of categories, which is implied by in the following diagram:



7. Gaussian measures on infinite-dimensional vector spaces

In this last section of this chapter, we shall review some basic facts, needed in the following chapters, on Gaussian measures. We start by recalling the definition of a Gaussian measure on $\mathbb{R}^1$.

DEFINITION 1.30. (1) A Borel probability measure γ on $\mathbb{R}^1$ is called *Gaussian*, if it is either the Dirac measure δ_a at a point or has density

$$p(., a, \sigma^2) : t \mapsto \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(t-a)^2}{2\sigma^2}\right)$$

with respect to the Lebesgue measure. In the latter case the measure γ is called *non-degenerate*. The measure is called *centered* if $a = 0$ in the above definition.

(2) A Borel measure γ on $\mathbb{R}^n$ is called Gaussian if for every functional f on $\mathbb{R}^n$, the induced measure $\gamma \circ f^{-1}$ is Gaussian.

Recall that the Fourier transform $\tilde{\mu}$ of a finite Borel measure μ on $\mathbb{R}^n$ is defined by the formula

$$\tilde{\mu} : \mathbb{R}^n \to \mathbb{C}^1, \quad \tilde{\mu}(y) = \int_{\mathbb{R}^n} \exp(i(y, x)) d\mu(x),$$

and that measures on $\mathbb{R}^n$ are uniquely determined by their Fourier transforms.

PROPOSITION 1.31 ([**Bog91**], Proposition 1.2.2.). *A measure γ on $\mathbb{R}^n$ is Gaussian if and only if its Fourier transform has the form*

$$\tilde{\gamma}(y) = \exp\left(i(y,a) - \frac{1}{2}(Ky,y)\right),$$

where a is a vector in $\mathbb{R}^n$ and K is a non-negative, matrix. It has a density if K is non-degenerate, in which case it is given by

$$x \mapsto \frac{1}{\sqrt{(2\pi)^n \det K}} \exp\left(-\frac{1}{2}(K^{-1}(x-a), x-a)\right)$$

For a Gaussian measure γ, a basis of $L^2(\mathbb{R}^n, d\gamma)$ is given by the so-called Hermite polynomials (see [**Bog91**] Section 1.3.):

$$H_\alpha(x_1, x_2, ..., x_n) := H_{k_1} H_{k_2} ... H_{k_n},$$

where α is a multi index and H_k is a Hermite polynomial on $\mathbb{R}^1$, defined by

$$H_k(x) := \frac{(-1)^k}{\sqrt{k!}} \exp\left(\frac{x^2}{2}\right) \frac{d^k}{dx^k} \exp\left(\frac{-x^2}{2}\right). \tag{23}$$

Now we turn to Gaussian measures on infinite-dimensional spaces. The first thing to do is to define the σ-algebra on an infinite dimensional space. Already in the last section we mentioned the definition of cylindrical sets. We repeat it in a more general context:

DEFINITION 1.32. Let X be a locally convex space with dual X^*. The sets of the following form are called *cylindrical sets*:

$$C = \{x \in X; (l_1(x), l_2(x), ... l_n(x)) \in C_0\}, \quad l_i \in X^*,$$

where $C_0 \in \mathcal{B}(\mathbb{R}^n)$ is called the base of C.

We denote by $\mathcal{F}(X)$ the σ-algebra generated by cylindrical subsets of X.

LEMMA 1.33. *The sets of the form*

$$\{x \in \mathbb{R}^\infty; (x_1, ..., x_n \in B\}, \quad B \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N},$$

generate $\mathcal{B}(\mathbb{R}^\infty) = \mathcal{F}(\mathbb{R}^\infty)$.

Now we define the Gaussian measure.

DEFINITION 1.34. Let X be a locally convex space. A probability measure on γ on the σ-algebra $\mathcal{F}(X)$, is called Gaussian if, for any $f \in X^*$, the induced measure $\gamma \circ f^{-1}$ on $\mathbb{R}^1$ is Gaussian. The measure γ is called centered if all the measures $\gamma \circ f^{-1}$ are centered.

THEOREM 1.35 ([**Bog91**], Theorem 2.2.4). *A measure γ on a locally convex space X is Gaussian if and only if its Fourier transform has the form*

$$\tilde{\gamma}(f) = \exp\left(iL(f) - \frac{1}{2}B(f,f)\right),$$

where L is a linear function on X^* and B is a symmetric bilinear form on X^* such that the quadratic form $f \mapsto B(f,f)$ is non-negative.

A Gaussian measure is centered if $L = 0$. If X is a Hilbert space then we can identify X^* with X, by Riesz' theorem. Then $L(x) = (a,x)$, $B(x,x) = (Kx,x)$, for some vector $a \in X$ and a bounded self-adjoint operator K on X ([**Bog91**], Theorem 2.3.1).

Let $(X_n, \mathcal{F}_n, \mu_n)$ be a sequence of measure spaces. Than the σ-algebra on $\Pi_{n=1}^{\infty} X_n$ is generated by cylindrical sets of the form

$$C = B_1 \times B_2 \times ... \times B_n \times X_{n+1} \times ..., \quad B_i \in \mathcal{F}_i.$$

The measure defined by

$$\mu(C) := \mu_1(B_1)...\mu_n(B_n)$$

is called the *product measure* and denoted by

$$\mu = \bigotimes_{n=1}^{\infty} \mu_n.$$

This measure is well defined, since it is countably additive and extends to the product σ-algebra. A product of Gaussian measures is also a Gaussian measure.

Next we turn to the question of singularity or equivalence of two Gaussian measures. Recall that two measures are equivalent (denoted by $\sim$) if their null sets coincide. A measure μ is called quasi-invariant w.r.t. a transformation T if the transformed measure μ^T defined by

$$\mu^T(A) = \mu(T^{-1}(A))$$

is equivalent to μ. Of course we assume that T maps measurable sets into measurable sets (i.e. a measurable transformation). Two measures are said to be mutually singular (denoted by $\perp$), if they are supported on different subsets. The following theorem of Hajec and Feldman is important for our work.

THEOREM 1.36 ([**Bog91**], Theorem 2.7.2). *Any two Gaussian measures on the same locally convex space are either equivalent or mutually singular.*

Let μ and ν be two probability measures on a measure space $(X, \mathcal{F})$ and let λ be a measure such that $\mu << \lambda$ and $\nu << \lambda$ (i.e. the μ and ν-null sets are also λ-null sets). Then we define the Hellinger integral:

$$H(\mu,\nu) := \int \sqrt{\frac{d\mu}{d\lambda}}\sqrt{\frac{d\nu}{d\lambda}}d\lambda.$$

It does not depend on the choice of λ ([**Bog91**], Proposition 2.12.6) and the following holds:

$$0 \leq H(\mu, \nu) \leq 1.$$

Moreover, $\mu \sim \nu$ implies $H(\mu, \nu) > 0$ (and the converse is true for Gaussian measures), $H(\mu, \nu) = 0$ iff $\mu \perp \nu$ and $H(\mu, \nu) = 1$ iff $\mu = \nu$. In fact, the above Theorem holds in general for product measures.

THEOREM 1.37 ([**Kak48**]). *For two product-measures $\mu = \otimes_{n=1}^{\infty} \mu_n$ and $\nu = \otimes_{n=1}^{\infty} \nu_n$, where $\mu_n \sim \nu_n$ for all n, the following alternative holds: either $\mu \sim \nu$ or $\mu \perp \nu$. In addition, $\mu \sim \nu$ precisely when the following product converges:*

$$\Pi_{n=1}^{\infty} \int \sqrt{\rho_n} d\nu_n,$$

where ρ_n is the density of μ_n w.r.t. ν_n.

The above integral is nothing else than the Hellinger integral $H(\mu, \nu)$.

Now we shortly discuss measures on groups. In the theory of locally compact groups we know that such a group has a Haar measure, which is unique up to constant. Moreover, the following theorem of Weil holds.

THEOREM 1.38 ([**Wei65**]). *A group admits a left (or right) invariant measure if and only if it is locally compact.*

Moreover, a similar result, by Xia Dao-Xing, holds for quasi-invariance of measures.

THEOREM 1.39 ([**DX72**], Corollary 3.1.14). *Let G be a topological group of the second category. Then, the local compactness of G is a necessary and sufficient condition for the existence of a regular measure space $(G, \mathcal{F}, \mu)$ which is left (and right) quasi-invariant under G.*

According to the above theorem it is impossible to find a G-quasi-invariant measure on an infinite-dimensional group G. However, in certain cases, one can find a topological group $\tilde{G}$ in which G is dense and construct a measure μ on $\tilde{G}$ which is G-quasi-invariant. For non-abelian G a general framework was proposed in the PhD dissertation of Kosyak in 1985 ([**Kos85**]). In the case when G is a Hilbert space and μ the standard Gaussian measure, the problem of defining the appropriate $\tilde{G}$ was solved by Gross in 1965 ([**Gro65**]). In the next Chapters we shall consider the special cases, where $G = B_0^{\mathbb{N}}$ (resp. $B_0^{\mathbb{Z}}$), $\tilde{G} = B^{\mathbb{N}}$ (resp. $B^{\mathbb{Z}}$) and μ is a Gaussian measure.

CHAPTER 2

Type III_1 factors generated by regular representations of the group $B_0^{\mathbb{N}}$

1. Regular representations

Let us consider the group $\tilde{G} = B^{\mathbb{N}}$ of all upper-triangular real matrices of infinite order with units on the diagonal

$$\tilde{G} = B^{\mathbb{N}} = \{I + x \mid x = \sum_{1 \le k < n} x_{kn} E_{kn}\},$$

and its subgroup

$$G = B_0^{\mathbb{N}} = \{I + x \in B^{\mathbb{N}} \mid \ x \text{ is finite}\},$$

where E_{kn} is an infinite-dimensional matrix with 1 at the place $k, n \in \mathbb{N}$ and zeros elsewhere, $x = (x_{kn})_{k<n}$ is *finite* means that $x_{kn} = 0$ for all (k, n) except for a finite number of indices $k, n \in \mathbb{N}$.

$$\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & \cdots \\ 0 & 1 & x_{23} & x_{24} & \cdots \\ 0 & 0 & 1 & x_{34} & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ & & & & \ddots \end{pmatrix}$$

Obviously, $B_0^{\mathbb{N}} = \varinjlim_n B(n, \mathbb{R})$ is the inductive limit of the group $B(n, \mathbb{R})$ of real upper-triangular matrices with units on the principal diagonal

$$B(n, \mathbb{R}) = \{I + \sum_{1 \le k < r \le n} x_{kr} E_{kr} \mid x_{kr} \in \mathbb{R}\}$$

with respect to the embedding $B(n, \mathbb{R}) \ni x \mapsto x + E_{n+1n+1} \in B(n+1, \mathbb{R})$.

We define the Gaussian measure μ_b on the group $B^{\mathbb{N}}$ in the following way

$$d\mu_b(x) = \bigotimes_{1 \le k < n} (b_{kn}/\pi)^{1/2} \exp(-b_{kn} x_{kn}^2) dx_{kn} = \bigotimes_{k<n} d\mu_{b_{kn}}(x_{kn}), \tag{24}$$

where $b = (b_{kn})_{k<n}$ is some set of positive numbers.

Let us denote by R and L the right and the left action of the group $B^{\mathbb{N}}$ on itself: $R_s(t) = ts^{-1}$, $L_s(t) = st$, $s, t \in B^{\mathbb{N}}$ and by $\Phi : B^{\mathbb{N}} \mapsto B^{\mathbb{N}}$, $\Phi(I + x) := (I + x)^{-1}$ the inverse mapping. It is known [**Kos92**] that

LEMMA 2.1. *$\mu_b^{R_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{N}}$ for any set $b = (b_{kn})_{k<n}$.*

LEMMA 2.2. *$\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{N}}$ if and only if $S_{kn}^L(b) < \infty$, $\forall k < n$, where*

$$S_{kn}^L(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{b_{nm}}.$$

LEMMA 2.3. *$\mu_b^{L_t} \perp \mu_b$ $\forall t \in B_0^{\mathbb{N}} \backslash \{e\} \Leftrightarrow S_{kn}^L(b) = \infty$ $\forall k < n$.*

LEMMA 2.4. [**Kos00**] *If $E(b) = \sum_{k<n} S_{kn}^L(b)(b_{kn})^{-1} < \infty$, then $\mu_b^{\Phi} \sim \mu_b$.*

LEMMA 2.5. [**Kos00**] *The measure μ_b on $B^{\mathbb{N}}$ is $B_0^{\mathbb{N}}$ ergodic with respect to the right action.*

Let $\alpha : G \to \mathrm{Aut}(X)$ be a measurable action of a group G on the measurable space X. We recall that a measure μ on the space X is G-ergodic if $f(\alpha_t(x)) = f(x)$ $\forall t \in G$ implies $f(x) = const$ μ a.e. for all functions $f \in L^1(X, \mu)$.

REMARK 2.6. [**KZ00**] If $\mu_b^{\Phi} \sim \mu_b$ then $\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{N}}$.

PROOF. This follows from the fact that the inversion Φ interchanges the right and the left action: $R_t \circ \Phi = \Phi \circ L_t$ $\forall t \in B^{\mathbb{N}}$. Indeed, if we denote $\mu^f(\cdot) = \mu(f^{-1}(\cdot))$ we have $(\mu^f)^g = \mu^{f\circ g}$. Hence

$$\mu_b \sim \mu_b^{R_t} \sim (\mu_b^{R_t})^{\Phi} = \mu_b^{R_t\circ\Phi} = \mu_b^{\Phi\circ L_t} = (\mu_b^{\Phi})^{L_t} \sim \mu_b^{L_t}.$$

□

If $\mu_b^{R_t} \sim \mu_b$ and $\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{N}}$, one can define in a natural way (see [**Kos92**]), an analogue of the right $T^{R,b}$ and left $T^{L,b}$ regular representation of the group $B_0^{\mathbb{N}}$ in the Hilbert space $\mathcal{H}_b = L^2(B^{\mathbb{N}}, d\mu_b)$

$$T^{R,b},\ T^{L,b} : B_0^{\mathbb{N}} \to U(\mathcal{H}_b = L_2(B^{\mathbb{N}}, d\mu_b)),$$

$$(T_t^{R,b} f)(x) = (d\mu_b(xt)/d\mu_b(x))^{1/2} f(xt),$$

$$(T_s^{L,b} f)(x) = (d\mu_b(s^{-1}x)/d\mu_b(x))^{1/2} f(s^{-1}x).$$

2. Von Neumann algebras generated by the regular representations

Let $\mathfrak{A}^{R,b} = (T_t^{R,b} \mid t \in B_0^{\mathbb{N}})''$ (resp. $\mathfrak{A}^{L,b} = (T_s^{L,b} \mid s \in B_0^{\mathbb{N}})''$) be the von Neumann algebras generated by the right $T^{R,b}$ (resp. the left $T^{L,b}$) regular representation of the group $B_0^{\mathbb{N}}$.

THEOREM 2.7. [**Kos00**] *If $E(b) < \infty$ then $\mu_b^{\Phi} \sim \mu_b$. In this case the left regular representation is well defined and the commutation theorem holds:*

$$(\mathfrak{A}^{R,b})' = \mathfrak{A}^{L,b}. \tag{25}$$

Moreover, the operator J_{μ_b} given by

$$(J_{\mu_b} f)(x) = (d\mu_b(x^{-1})/d\mu_b(x))^{1/2}\overline{f(x^{-1})} \tag{26}$$

is an intertwining operator:

$$T_t^{L,b} = J_{\mu_b} T_t^{R,b} J_{\mu_b},\ t \in B_0^{\mathbb{N}} \quad \textit{and}\ J_{\mu_b} \mathfrak{A}^{R,b} J_{\mu_b} = \mathfrak{A}^{L,b}.$$

If $\mu_b^{L_t} \perp \mu_b\ \forall t \in B_0^{\mathbb{N}} \backslash \{e\}$ one can't define the left regular representation of the group $B_0^{\mathbb{N}}$. Moreover the following theorem holds ([**Kos92**])

Theorem 2.8. *The right regular representation $T^{R,b} : B_0^{\mathbb{N}} \to U(\mathcal{H}_b)$ is irreducible if and only if $\mu_b^{L_s} \perp \mu_b\ \forall s \in B_0^{\mathbb{N}} \backslash \{0\}$.*

Corollary 2.9. *The von Neumann algebra $\mathfrak{A}^{R,b}$ is a type I_∞ factor if $\mu_b^{L_s} \perp \mu_b\ \forall s \in B_0^{\mathbb{N}} \backslash \{0\}$.*

Let us assume now that $\mu_b^{L_t} \sim \mu_b\ \forall t \in B_0^{\mathbb{N}} \backslash \{e\}$. In this case the right regular representation and the left regular representation of the group $B_0^{\mathbb{N}}$ are well defined.

In this Chapter we shall prove that if $E(b) < \infty$, the von Neumann algebras $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are always factors. This is implied by the triviality of the centralizer of our algebras w.r.t. the vector $1 \in L^2(B^{\mathbb{N}}, d\mu)$, which is the main step in proving the type III_1 property.

Until now there existed sufficient conditions on the measure μ_b, for the factor property ([**KZ00**]). We give a short review.

Since $T_t^{L,b} \in (\mathfrak{A}^{R,b})'\ \forall t \in B_0^{\mathbb{N}}$, we have $\mathfrak{A}^{L,b} \subset (\mathfrak{A}^{R,b})'$, hence

$$\mathfrak{A}^{R,b} \cap (\mathfrak{A}^{R,b})' \subset (\mathfrak{A}^{L,b})' \cap (\mathfrak{A}^{R,b})' = (\mathfrak{A}^{R,b} \cup \mathfrak{A}^{L,b})'. \tag{27}$$

The last relation shows that $\mathfrak{A}^{R,b}$ is factor if the representation

$$B_0^{\mathbb{N}} \times B_0^{\mathbb{N}} \ni (t,\, s) \mapsto T_t^{R,b} T_s^{L,b} \in U(H_b)$$

is irreducible.

Let us denote by $\mathfrak{A}^{R,L,b}$ the the von Neumann algebras generated by the right $T^{R,b}$ and the left $T^{L,b}$ regular representations of the group $B_0^{\mathbb{N}}$:

$$\mathfrak{A}^{R,L,b} = (T_t^{R,b}, T_s^{L,b} \mid t, s \in B_0^{\mathbb{N}})'' = (\mathfrak{A}^{R,b} \cup \mathfrak{A}^{L,b})''.$$

Let us denote

$$S_{kn}^{R,L}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{S_{nm}^{L}(b)},\ \ k < n. \tag{28}$$

Theorem 2.10. [**KZ00**] *The representation*

$$B_0^{\mathbb{N}} \times B_0^{\mathbb{N}} \ni (t,\, s) \mapsto T_t^{R,b} T_s^{L,b} \in U(H_b)$$

is irreducible if $S_{kn}^{R,L}(b) = \infty,\ \forall k < n$.

Corollary 2.11. *The von Neumann algebra $\mathfrak{A}^{R,b}$ is a factor if $S_{kn}^{R,L}(b) = \infty\ \forall k < n$.*

3. Modular operator

In this section we review the construction of the modular operator for locally compact groups and extend it to the case of inductive limits of such groups. First we establish that the constant function 1 is cyclic and separating for $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ (assuming $E(b) < \infty$). We prove this by showing that set $\mathfrak{A}^{R,b}1$ contains all the polynomials, in the variables $x_{kn}, k < n \in \mathbb{N}$, which is dense in $\mathcal{H}_b$. Using the intertwining operator J_b, we show that the same holds for the commutant of $\mathfrak{A}^{R,b}$, which implies the separating property.

LEMMA 2.12. *Assume that $E(b) < \infty$. Then the function $1 \in L^2(B^{\mathbb{N}}, \mu_b)$ is cyclic and separating for $\mathfrak{A}^{R,b}$.*

PROOF. (1) First we prove the cyclic property. Consider the one-parameter groups in $B_0^{\mathbb{N}}$,

$$G_{kn}(t) := \{1 + tE_{kn}, t \in \mathbb{R}\}. \tag{29}$$

The corresponding one parameter groups $T_{kn}^{R,b}(t) := \{T_u^{R,b}; u \in G_{kn}(t)\}$ have generators (see [**Kos92**], here for convenience, we omit the superscript b)

$$A_{kn}^R = \sum_{r=1}^{k-1} x_{rk} D_{rn} + D_{kn}, \tag{30}$$

where $D_{pq} = \partial_{pq} - b_{pq}x_{pq}$ and $\partial_{pq} = \frac{\partial}{\partial x_{pq}}$.

Suppose that $f \in L^2(B^{\mathbb{N}}, d\mu_b)$ and

$$(f, T_t^{R,b}1) = \int f(x) T_t^{R,b}1(x) d\mu_b(x) = 0, \quad \forall t \in B_0^{\mathbb{N}}. \tag{31}$$

We want to prove that $f = 0$, which implies that the linear span of the set $\{T_t^{R,b}1; t \in B_0^{\mathbb{N}}\}$ is dense in $L^2(B^{\mathbb{N}}, d\mu_b)$, since we chose f arbitrarily. We shall prove that (31) implies

$$(f, P) = 0, \tag{32}$$

where $P(x)$ are polynomials of finite order in the variables x_{kn}. Since the set $\mathcal{P}$ of polynomials P is dense in $L^2(B^{\mathbb{N}}, d\mu_b)$ (for example by the fact that the Hermite Polynomials (23) span $L^2(B^{\mathbb{N}}, d\mu_b)$), this proves that $f = 0$.

Now we shall prove the above property. First of all from (31) follows that

$$(f, \prod T_{k_i n_i}^{R,b}(t_{i,\alpha_j})1) = (f, \prod e^{(\sum_{j=1}^{m_i} t_{k_i n_i \alpha_j})A_{k_i n_i}^R} 1) = 0,$$

for some finite product of $T_{kn}^{R,b}(t)$, where the index α_j varies according to the multiplicity of i. Since $T_{kn}^{R,b}(t)$ are strongly continuous one-parameter groups with generators A_{kn}^R and $\mathcal{P}(\supset \{1\})$ is a common dense domain of these generators ([**Kos01**]), we conclude (after taking derivatives in all the parameters and setting them to 0) that

$$(f, \prod_{i=1}^{p} (A^R_{k_i n_i})^{m_i} 1) = 0,$$

for all finite products of different $A^R_{k_i n_i}$.

Now we show that the set $\langle A^R_{kn}; k < n \in \mathbb{N}\rangle 1$, where $\langle A^R_{kn}; k < n \in \mathbb{N}\rangle$ is the algebraic linear span generated by the generators A^R_{kn}, contains all the polynomials of finite order in the independent variables x_{kn}. We make use of multiple nested inductions as follows. The main induction is the following:

(a) We prove that the span of

$$\{x^{\alpha_1}_{1n_1} ... x^{\alpha_k}_{1n_k}; 1 \leq k, \alpha_i \in \mathbb{N}, i = 1..k\},$$

where all the indices n_i are mutually different, is contained in $\langle A^R_{kn}; k < n \in \mathbb{N}\rangle 1$.

(b) The second step is to prove that the span of

$$\{x^{\beta_1}_{2m_1} ... x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1} ... x^{\alpha_k}_{1n_k}; 1 \leq k, l, \alpha_i, \beta_j \in \mathbb{N}, i = 1..k, j = 1..l\},$$

is contained in $\langle A^R_{kn}; k < n \in \mathbb{N}\rangle 1$

(c) Now comes the induction step. Assume that the span of

$$\{x^{\gamma_1}_{p-1 s_1} ... x^{\gamma_r}_{p-1 s_r} ... x^{\beta_1}_{2m_1} ... x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1} ... x^{\alpha_k}_{1n_k};$$

$$1 \leq k, l, r, \alpha_\mu, \beta_\nu, \gamma_\eta \in \mathbb{N}, \mu = 1..k, \nu = 1..l, \eta = 1..r\}$$

is contained in $\langle A^R_{kn}; k < n \in \mathbb{N}\rangle 1$ for some $p > 2$. Then we prove that the span of

$$\{x^{\delta_1}_{ps_1} ... x^{\delta_u}_{ps_u} ... x^{\beta_1}_{2m_1} ... x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1} ... x^{\alpha_k}_{1n_k};$$

$$1 \leq k, l, u, \alpha_\mu, \beta_\nu, \delta_\eta \in \mathbb{N}, \mu = 1..k, \nu = 1..l, \eta = 1..u\}$$

is also contained in $\langle A^R_{kn}; k < n \in \mathbb{N}\rangle 1$. This implies that any polynomial of finite order is in the latter set.

(1a) We prove the first step. Again we use induction, this time on the number of factors in the monomials.

- $x^{\alpha_1}_{1n_1} \in \langle A^R_{kn}; k < n \in \mathbb{N}\rangle 1$. Indeed (again we use induction),

$$A^R_{1n_1} 1 = -b_{1n_1} x_{1n_1}.$$

Furthermore, assume that $Span\{x^{\alpha_1 - 1}_{1n_1}\} \subset \langle A^R_{kn}; k < n \in \mathbb{N}\rangle 1$, then also the desired property holds, since

$$A^R_{1n_1} x^{\alpha_1 - 1}_{1n_1} = D_{1n_1} x^{\alpha_1 - 1}_{1n_1} = (\alpha_1 - 1) x^{\alpha_1 - 2}_{1n_1} - b_{1n_1} x^{\alpha_1}_{1n_1}.$$

- The induction step is as follows. Assume that $x_{1n_{k-1}}^{\alpha_{k-1}} x_{1n_{k-2}}^{\alpha_{k-2}} ... x_{1n_1}^{\alpha_1} \in \langle A_{kn}^R; k < n \in \mathbb{N}\rangle 1$. The following equations show (using induction on α_k), that also
$$x_{1n_k}^{\alpha_k} x_{1n_{k-1}}^{\alpha_{k-1}} ... x_{1n_1}^{\alpha_1} \in \langle A_{kn}^R; k < n \in \mathbb{N}\rangle 1 :$$
$$A_{1n_k}^R x_{1n_{k-1}}^{\alpha_{k-1}} x_{1n_{k-2}}^{\alpha_{k-2}} ... x_{1n_1}^{\alpha_1} = -b_{1n_k} x_{1n_k} x_{1n_{k-1}}^{\alpha_{k-1}} x_{1n_{k-2}}^{\alpha_{k-2}} ... x_{1n_1}^{\alpha_1},$$
$$A_{1n_k}^R x_{1n_k}^{\alpha_k - 1} x_{1n_{k-1}}^{\alpha_{k-1}} x_{1n_{k-2}}^{\alpha_{k-2}} ... x_{1n_1}^{\alpha_1} = (\alpha_k - 1) x_{1n_k}^{\alpha_k - 2} ... x_{1n_1}^{\alpha_1}$$
$$-b_{1n_k} x_{1n_k}^{\alpha_k} x_{1n_{k-1}}^{\alpha_{k-1}} x_{1n_{k-2}}^{\alpha_{k-2}} ... x_{1n_1}^{\alpha_1}$$

(1b) Now we continue with the second step of the main induction.

- Consider the following equation:
$$A_{2m_1}^R 1 = (x_{12} D_{1m_1} + D_{2m_1})\, 1 = -b_{1m_1} x_{12} x_{1m_1} - b_{2m_1} x_{2m_1}.$$
Since the first term is in $\langle A_{kn}^R; k < n \in \mathbb{N}\rangle 1$, so is the second term. It is also easy to see that
$$x_{2m_1}^{\beta_1} x_{1n_k}^{\alpha_k} x_{1n_{k-1}}^{\alpha_{k-1}} x_{1n_{k-2}}^{\alpha_{k-2}} ... x_{1n_1}^{\alpha_1} \in \langle A_{kn}^R; k < n \in \mathbb{N}\rangle 1$$
and
$$x_{2m_{l-1}}^{\beta_{l-1}} ... x_{2m_1}^{\beta_1} x_{1n_k}^{\alpha_k} ... x_{1n_1}^{\alpha_1} \in \langle A_{kn}^R; k < n \in \mathbb{N}\rangle 1$$
for some l. This is proved in the same way as the previous case.
- Now suppose that
$$x_{2m_l}^{\beta_l - 1} ... x_{2m_1}^{\beta_1} x_{1n_k}^{\alpha_k} ... x_{1n_1}^{\alpha_1} \in \langle A_{kn}^R; k < n \in \mathbb{N}\rangle 1$$
Then
$$A_{2m_l}^R x_{2m_{l-1}}^{\beta_l - 1} ... x_{2m_1}^{\beta_1} x_{1n_k}^{\alpha_k} ... x_{1n_1}^{\alpha_1}$$
$$= x_{12}(\partial_{1m_l} - b_{1m_l} x_{1m_l}) x_{2m_{l-1}}^{\beta_l - 1} ... x_{2m_1}^{\beta_1} x_{1n_k}^{\alpha_k} ... x_{1n_1}^{\alpha_1}$$
$$+(\partial_{2m_l} - b_{2m_l} x_{2m_l}) x_{2m_{l-1}}^{\beta_l - 1} ... x_{2m_1}^{\beta_1} x_{1n_k}^{\alpha_k} ... x_{1n_1}^{\alpha_1}.$$
The first term in the right hand side contains only monomials of order lower than β_l in the variable x_{2m_l} and in the second term we see the monomials
$$x_{2m_l}^{\beta_l} ... x_{2m_1}^{\beta_1} x_{1n_k}^{\alpha_k} ... x_{1n_1}^{\alpha_1}.$$

(1c) Finally we turn to the main induction step. So assume that
$$Span\{x_{p-1s_1}^{\gamma_1} ... x_{p-1s_r}^{\gamma_r} ... x_{2m_1}^{\beta_1} ... x_{2m_l}^{\beta_l} x_{1n_1}^{\alpha_1} ... x_{1n_k}^{\alpha_k};$$
$$1 \leq k, l, r, \alpha_\mu, \beta_\nu, \gamma_\eta \in \mathbb{N}, \mu = 1..k, \nu = 1..l, \eta = 1..r\}$$
is contained in $\langle A_{kn}^R; k < n \in \mathbb{N}\rangle 1$ for some $p > 2$.

- Again we see that

$$A^R_{pu_1} x^{\gamma_1}_{p-1s_1}...x^{\gamma_r}_{p-1s_r}...x^{\beta_1}_{2m_1}...x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1}...x^{\alpha_k}_{1n_k}$$

$$= (\textstyle\sum_{i=1}^{p-1} x_{ip}(\partial_{iu_1} - b_{iu_1} x_{iu_1})) x^{\gamma_1}_{p-1s_1}...x^{\gamma_r}_{p-1s_r}...x^{\beta_1}_{2m_1}...x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1}...x^{\alpha_k}_{1n_k}$$

$$+(\partial_{pu_1} - b_{pu_1} x_{pu_1}) x^{\gamma_1}_{p-1s_1}...x^{\gamma_r}_{p-1s_r}...x^{\beta_1}_{2m_1}...x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1}...x^{\alpha_k}_{1n_k}.$$

  The summation in the first term gives rise only to monomials containing x_{in} for $i < p$, which are in $\langle A^R_{kn}; k < n \in \mathbb{N}\rangle 1$ by the induction hypothesis. Thus the latter set also contains the second term and hence

$$x_{pu_1} x^{\gamma_1}_{p-1s_1}...x^{\gamma_r}_{p-1s_r}...x^{\beta_1}_{2m_1}...x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1}...x^{\alpha_k}_{1n_k}.$$

  The same holds for

$$x^{\delta_1}_{pu_1} x^{\gamma_1}_{p-1s_1}...x^{\gamma_r}_{p-1s_r}...x^{\beta_1}_{2m_1}...x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1}...x^{\alpha_k}_{1n_k}.$$

- Finally suppose that also

$$x^{\delta_v - 1}_{pu_v}...x^{\delta_1}_{pu_1} x^{\gamma_1}_{p-1s_1}...x^{\gamma_r}_{p-1s_r}...x^{\beta_1}_{2m_1}...x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1}...x^{\alpha_k}_{1n_k}$$

  is in $\langle A^R_{kn}; k < n \in \mathbb{N}\rangle 1$. We calculate

$$A^R_{pu_v} x^{\delta_v - 1}_{pu_v}...x^{\delta_1}_{pu_1} x^{\gamma_1}_{p-1s_1}...x^{\gamma_r}_{p-1s_r}...x^{\beta_1}_{2m_1}...x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1}...x^{\alpha_k}_{1n_k}$$

$$= (\textstyle\sum_{i=1}^{p-1} x_{ip}(\partial_{iu_v} - b_{iu_v} x_{iu_v})) x^{\delta_v - 1}_{pu_v}...x^{\delta_1}_{pu_1} x^{\gamma_1}_{p-1s_1}...x^{\gamma_r}_{p-1s_r}...x^{\beta_1}_{2m_1}...x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1}...x^{\alpha_k}_{1n_k}$$

$$+(\partial_{pu_v} - b_{pu_v} x_{pu_v}) x^{\delta_v - 1}_{pu_v}...x^{\delta_1}_{pu_1} x^{\gamma_1}_{p-1s_1}...x^{\gamma_r}_{p-1s_r}...x^{\beta_1}_{2m_1}...x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1}...x^{\alpha_k}_{1n_k}.$$

  Again, the summation in the first term gives rise to a polynomial of order smaller than δ_v in the x_{pu_v} variable, which by the last induction hypothesis is contained in our span of generators acting on 1. So does the first monomial in the second term (after expanding the brackets). The last monomial gives us the final statement:

$$x^{\delta_v}_{pu_v}...x^{\delta_1}_{pu_1} x^{\gamma_1}_{p-1s_1}...x^{\gamma_r}_{p-1s_r}...x^{\beta_1}_{2m_1}...x^{\beta_l}_{2m_l} x^{\alpha_1}_{1n_1}...x^{\alpha_k}_{1n_k} \in \langle A^R_{kn}; k < n \in \mathbb{N}\rangle 1,$$

  for any of the parameters $p, \delta_i, \gamma_i, \beta_i, \alpha_i \in \mathbb{N}$ and $v, r, l, k \in \mathbb{N}$.

It follows that the set $\langle A^R_{kn}, k < n\rangle$ acting on 1 generates the set $\mathcal{P}$ of all possible polynomials in the independent variables x_{kn}. Thus equation (32) holds for the function f. $\mathcal{P}$ is dense in $L^2(B^{\mathbb{N}}, d\mu_b)$ and hence f must be equal to 0. Since $f \in L^2(B^{\mathbb{N}}, d\mu_b)$ was arbitrary and the equation (31) holds for all $t \in B_0^{\mathbb{N}}$, the span of $\{T^{R,b}_t 1; t \in B_0^{\mathbb{N}}\}$ must be dense in $L^2(B^{\mathbb{N}}, d\mu_b)$ and hence 1 is cyclic for $\mathfrak{A}^{R,b}$.

(2) Now we turn to the separating property. In this case we have to prove that 1 is cyclic for $(\mathfrak{A}^{R,b})' = \mathfrak{A}^{L,b}$. Thus, again consider $f \in L^2(B^{\mathbb{N}}, d\mu_b)$ and assume

$$(f, b1) = 0, \forall b \in \mathfrak{A}^{L,b}. \tag{33}$$

Recall that $E(b) < \infty$ implies the existence of the intertwining operator J, which is anti-unitary. Then the following calculation holds:

$$\begin{aligned}(f, T_t^{R,b}1) &= (JT_t^{R,b}1, Jf)\\ &= \int \sqrt{\frac{d\mu_b(x^{-1})}{d\mu_b(x)}}\sqrt{\frac{d\mu_b((xt)^{-1})}{d\mu_b(x^{-1})}}\sqrt{\frac{d\mu_b(x^{-1})}{d\mu_b(x)}}\overline{f(x^{-1})}d\mu_b(x)\\ &= \int \sqrt{\frac{d\mu_b(t^{-1}x^{-1})}{d\mu_b(x^{-1})}}\overline{f(x^{-1})}d\mu(x^{-1}).\end{aligned}$$

If we replace x^{-1} by x in the above integral we obtain $(f, T_t^{R,b}1) = (f, T_t^{L,b}1)$ for all $t \in B_0^{\mathbb{N}}$. From (1) we know that $(f, T_t^{R,b}1) = 0$ for all $t \in B_0^{\mathbb{N}}$ implies that $f = 0$. Hence $(f, T_t^{L,b}1) = 0$ for all $t \in B_0^{\mathbb{N}}$ also implies that $f = 0$ and hence 1 is cyclic for $\mathfrak{A}^{L,b}$, since we chose f arbitrarily.

□

We recall how to find the modular operator and the operator of canonical conjugation for the von Neumann algebra $\mathfrak{A}_G^{\rho}$, generated by the right regular representation ρ of a locally compact Lie group G. Let h be a right invariant Haar measure on G and

$$\rho, \lambda : G \mapsto U(L^2(G, h))$$

be the right and the left regular representations of the group G defined by

$$(\rho_t f)(x) = f(xt),\ (\lambda_t f)(x) = (dh(t^{-1}x)/dh(x))^{-1/2} f(t^{-1}x).$$

To define the right Hilbert algebra on G we can proceed as follows. Let $M(G)$ be algebra of all probability measures on G with convolution determined by

$$\int f d\mu * \nu = \int\int f(st)d\mu(s)d\nu(t).$$

We define the homomorphism

$$M(G) \ni \mu \mapsto \rho^{\mu} = \int_G \rho_t d\mu(t) \in B(L^2(G, h)).$$

We have $\rho^{\mu}\rho^{\nu} = \rho^{\mu*\nu}$, indeed

$$\rho^{\mu}\rho^{\nu} = \int_G \rho_t d\mu(t)\int_G \rho_s d\nu(s) = \int_G\int_G \rho_{ts} d\mu(t)\nu(s) = \int_G \rho_t d(\mu*\nu)(t) = \rho^{\mu*\nu}.$$

Let us consider a subalgebra $M_h(G) := (\nu \in M(G) \mid \nu \sim h)$ of the algebra $M(G)$ In the case when $\mu \in M_h(G)$ we can associate with the measure μ its Rodon-Nikodim derivative $d\nu(t)/dh(t) = f(t)$. When $f \in C_0^\infty(G)$ or $f \in L^1(G)$ we can write

$$\rho^f = \int_G f(t)\rho_t dh(t),$$

hence we can replace the algebra $M_h(G)$ by its subalgebra identified with algebra of functions $C_0^\infty(G)$ or $L^1(G,h)$ with convolutions. If we replace the Haar measure h with some measure $\mu \in M_h(G)$ we obtain the isomorphic image $T^{R,\mu}$ of the right regular representation ρ in the space $L^2(G,\mu)$: $T_t^{R,\mu} = U\rho_t U^{-1}$ where $U : L^2(G,h) \mapsto L^2(G,\mu)$ defined by $(Uf)(x) = \left(\frac{dh(x)}{d\mu(x)}\right)^{1/2} f(x)$. We have

$$(T_t^{R,\mu} f)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt),$$

and

$$T^f = \int_G f(t) T_t^{R,\mu} d\mu(t).$$

We have (see [**Con94**], p.462) (we shall write T_t instead of $T_t^{R,\mu}$)

$$S(T^f) := (T^f)^* = \int_G \overline{f(t)} T_{t^{-1}} d\mu(t) = \int_G \overline{f(t)} T_{t^{-1}} \frac{d\mu(t)}{d\mu(t^{-1})} d\mu(t^{-1})$$
$$\int_G \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1})} T_t d\mu(t).$$

Hence

$$(Sf)(t) = \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1})}.$$

To calculate S^* we use the fact that S is anti-linear so $(Sf,g) = (S^*g,f)$. We have

$$(Sf,g) = \int_G \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1}) g(t)} d\mu(t) = \int_G \overline{f(t^{-1}) g(t)} d\mu(t^{-1}) =$$
$$\int_G \overline{g(t^{-1}) f(t)} d\mu(t) = (S^*g,f),$$

hence $(S^*g)(t) = \overline{g(t^{-1})}$. Finally the modular operator Δ defined by $\Delta = S^*S$ has the following form $(\Delta f)(t) = \frac{d\mu(t)}{d\mu(t^{-1})} f(t)$. Indeed we have

$$f(t) \overset{S}{\mapsto} \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1})} \overset{S^*}{\mapsto} \frac{d\mu(t)}{d\mu(t^{-1})} f(t).$$

Finally, since $J = S\Delta^{-1/2}$ (see [**Con94**] p.462) we get

$$f(t) \overset{\Delta^{-1/2}}{\mapsto} \left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2} f(t) \overset{J}{\mapsto} \frac{d\mu(t^{-1})}{d\mu(t)} \left(\frac{d\mu(t)}{d\mu(t^{-1})}\right)^{1/2} \overline{f(t^{-1})}$$

$$= \left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2} \overline{f(t^{-1})}.$$

Hence

$$(Jf)(t) = \left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2} \overline{f(t^{-1})}, \text{ and } (\Delta f)(t) = \frac{d\mu(t)}{d\mu(t^{-1})} f(t).$$

To prove that $JT_t^{R,\mu}J = T_t^{L,\mu}$ we get

$$f(t) \overset{J}{\mapsto} \left(\frac{d\mu(x^{-1})}{d\mu(x)}\right)^{1/2} \overline{f(x^{-1})} \overset{T_t^{R,\mu}}{\mapsto} \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} \left(\frac{d\mu((xt)^{-1})}{d\mu(xt)}\right)^{1/2} \overline{f((xt)^{-1})} =$$

$$\left(\frac{d\mu(t^{-1}x^{-1})}{d\mu(x)}\right)^{1/2} \overline{f(t^{-1}x^{-1})} \overset{J}{\mapsto} \left(\frac{d\mu(x^{-1})}{d\mu(x)}\right)^{1/2} \left(\frac{d\mu(t^{-1}x)}{d\mu(x^{-1})}\right)^{1/2} f(t^{-1}x) =$$

$$\left(\frac{d\mu(t^{-1}x)}{d\mu(x)}\right)^{1/2} f(t^{-1}x) = (T_t^{L,\mu} f)(x).$$

Remark 2.13. The representation $T^{R,b}$ is the inductive limit of the representations T^{R,μ_b^m} of the group $B(m,\mathbb{R})$ where the measure μ_b^m is the projection of the measure μ_b onto subgroup $B(m,\mathbb{R})$. Obviously μ_b^m is equivalent with the Haar measure h_m on $B(m,\mathbb{R})$.

Hence, we conclude that the modular operator of $(\mathfrak{A}^{R,b}, 1)$ is defined by

$$\Delta(f)(x) = \frac{d\mu_b(x)}{d\mu_b(x^{-1})} f(x). \tag{34}$$

4. Examples

We still need to verify whether the von Neumann algebras $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ exist. Here we give an example of a measure μ_b for which the conditions $S_{kn}^L(b) < \infty$, for all $k < n \in \mathbb{N}$ and $E(b) < \infty$ are fulfilled.

In the example 1 below for the particular case $b_{kn} = (a_k)^n$ we give some sufficient conditions on the sequence a_n.

Example 1. Let us take $b_{kn} = (a_k)^n,\ k, n \in \mathbb{N}$.

We have

(35)

$$S_{kn}^L(b) = \sum_{m=n+1}^{\infty} \left(\frac{a_k}{a_n}\right)^m = \left(\frac{a_k}{a_n}\right)^{n+1} \sum_{m=0}^{\infty} \left(\frac{a_k}{a_n}\right)^m = \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} < \infty.$$

For $E(b)$ holds:

$$E(b) = \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \frac{S_{kn}^L(b)}{b_{kn}} = \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} \frac{1}{a_k^n} =$$

$$\sum_{k=1}^{\infty} a_k \sum_{n=k+1}^{\infty} \left(\frac{1}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} < \sum_{k=1}^{\infty} \frac{a_k}{1-\frac{a_k}{a_{k+1}}} \sum_{n=k+1}^{\infty} \left(\frac{1}{a_n}\right)^{n+1}$$

$$< \sum_{k=1}^{\infty} \frac{a_k}{1-\frac{a_k}{a_{k+1}}} \sum_{n=k+1}^{\infty} \left(\frac{1}{a_{k+1}}\right)^{n+1} = \sum_{k=1}^{\infty} \frac{a_k}{1-\frac{a_k}{a_{k+1}}} \left(\frac{1}{a_{k+1}}\right)^{k+2} \frac{1}{1-\frac{1}{a_{k+1}}} =$$
$$\sum_{k=1}^{\infty} \frac{\frac{a_k}{a_{k+1}}}{\left(1-\frac{a_k}{a_{k+1}}\right)^2} \left(\frac{1}{a_{k+1}}\right)^{k+1}.$$

Example 2. Let us take $b_{kn} = (a_k)^n,\ k,n \in \mathbb{N}$ where $a_k = s^k,\ k \in \mathbb{N}$ with $s > 1$.

Conditions (35) hold for $a_k = s^k$. We have

$$S_{kn}^L(b) = \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} = \left(\frac{1}{s^{n-k}}\right)^{n+1} \frac{1}{1-\frac{1}{s^{n-k}}} \sim \left(\frac{1}{s^{n-k}}\right)^{n+1} < \infty$$

Using the latter equivalence we conclude that $E(b) < \infty$. Indeed we have

$$E(b) = \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \frac{S_{kn}^L(b)}{b_{kn}} \sim \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \frac{1}{s^{(n-k)(n+1)}} \frac{1}{s^{kn}}$$
$$= \sum_{k=1}^{\infty} s^k \sum_{n=k+1}^{\infty} \frac{1}{s^{n(n-1)}} = \sum_{k=1}^{\infty} s^k \sum_{n=0}^{\infty} \frac{1}{s^{(n+k+1)(n+k+2)}}$$
$$< \sum_{k=1}^{\infty} s^k \frac{1}{s^{(k+1)(k+2)}} \sum_{n=0}^{\infty} \frac{1}{s^n} = \frac{1}{1-\frac{1}{s}} \sum_{k=1}^{\infty} \frac{1}{s^{(k+1)^2+1}} < \infty.$$

5. The type III$_1$ factors

Let $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ be the von Neumann algebras defined in Section 2 and assume that $E(b) < \infty$. In this section we prove that $\mathfrak{A}^{L,b}$ (and hence $\mathfrak{A}^{R,b}$) are type III$_1$ factors, with no further assumptions. The main step is to prove that the fixed point algebra of $\mathfrak{A}^{L,b}$ w.r.t. the modular group is trivial. From the last section follows that the state $\phi(.) = (1, .1)$ is a faithful normal state on $\mathfrak{A}^{L,b}$ (and $\mathfrak{A}^{R,b}$). Note that in this case, $\mathcal{M}_\phi = \mathcal{M}^{\sigma^\phi}$, where σ_ϕ is the modular group of ϕ and $\mathcal{M}_\phi$ is the centralizer of $\mathcal{M}$ w.r.t. ϕ ([**BR02**], Prop. 5.3.28). We want to prove that $\mathcal{M}_\phi = \mathbb{C}.\mathbf{1}$ implies that $\mathcal{M}$ is of type III$_1$.

Lemma 2.14 ([**Bau95**]). *Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ and η a cyclic and separating vector for $\mathcal{M}$, with σ its associated modular group. Assume that σ is inner, i.e. $\sigma_t(a) = U(t)aU(t)^*$, with a one-parameter group $U(t) \in \mathcal{M}$ for all $t \in \mathbb{R}$. Then $U(t) \in \mathcal{M}^\sigma$ for all $t \in \mathbb{R}$.*

Proof. The state $\phi(a) := (\eta, a\eta)$ is invariant w.r.t. σ (follows for example from the modular condition, see Chapter 1). Thus $(\eta, U(t)aU(-t)\eta) = (\eta, a\eta)$. Since $U(t) \in \mathcal{M}$ we can replace a by $aU(t)$ and obtain

$$(\eta, U(t)a\eta) = (\eta, aU(t)\eta), \quad a \in \mathcal{M}.$$

Since $\mathcal{M}_\phi = \mathcal{M}^\sigma$, we have proven the result. □

Assume now that $\mathcal{M}^\sigma = \mathbb{C}.\mathbf{1}$. First of all we note that $\mathcal{C}_\mathcal{M} \subseteq \mathcal{M}_\phi = \mathbb{C}\mathbf{1}$, which follows from the definition. Thus in this case $\mathcal{M}$ is a factor. It also follows that if $\mathcal{M}$ is semi-finite and hence σ is inner (Theorem 1.11 from Chapter 1), that $\sigma_t(a) = a$, since σ is implemented by a scaling operator. Thus $\mathbb{C}\mathbf{1} = \mathcal{M}^\sigma = \mathcal{M}$. Hence a non-trivial von Neumann algebra with an ergodic modular automorphism group must be type III factor. In fact, by the following theorem, it is a type III_1 factor!

THEOREM 2.15. [**Bau95**] *Let $\mathcal{M}$ be a von Neumann algebra ($\mathcal{M} \neq \mathbb{C}.\mathbf{1}$) and ϕ a faithful normal state. Assume that the centralizer of ϕ is trivial, i.e.*

$$\mathcal{M}_\phi := \{a \in \mathcal{M}; \phi(ab) = \phi(ba), \forall b \in \mathcal{M}\} = \mathbb{C}.\mathbf{1}.$$

Then $\mathcal{M}$ is a type III_1 factor.

PROOF. The strategy of the proof is to exclude the other cases. Recall the definition (11) of Connes spectrum $\Gamma(\mathcal{M})$.

(1) We assert that $\Gamma = \{1\}$ is impossible. If $Sp\Delta\backslash\{0\} = \{1\}$ then either $Sp\Delta = \{1\}$ or $Sp\Delta = \{1, 0\}$. $Sp\Delta = \{1\}$ means that $\Delta = \mathbf{1}$, which implies $\mathcal{M} = \mathcal{M}_\phi = \mathbb{C}.\mathbf{1}$, and this is not the case. $Sp\Delta = \{1, 0\}$ means that 0 is an isolated point of $Sp\Delta$, therefore the eigenprojection of the point 0 does not vanish, hence there is a vector $e \neq 0$ with $\Delta e = 0$ which is impossible since Δ is invertible.

(2) We assert that $\Gamma = \lambda^{\mathbb{Z}}, 0 < \lambda < 1$ is impossible. On the contrary, let $Sp\Delta\backslash\{0\} = \lambda^{\mathbb{Z}}$. This implies, among other things, that $0 < \lambda < 1$ is an isolated point of $Sp\Delta$. Then $\ln\lambda$ is an isolated point of $Sp\ln\Delta$. Now we use some results from spectral analysis of automorphism groups (see Appendix B). Since $Sp\ln\Delta = Sp\sigma^\phi$, we have $\ln\lambda \in Sp\sigma^\phi$ and $\ln\lambda$ is isolated. Now, by Corollary B.2 from Appendix B, there is an $0 \neq a \in \mathcal{M}$, such that

$$\Delta^{it} a \Delta^{-it} = \lambda^{it} a. \tag{36}$$

Now we use the KMS condition (see Chapter 1) to obtain a contradiction. Let $b \in \mathcal{M}$ and define the functions

$$\begin{aligned} F_{a,b}(t) &= \phi(\sigma_t^\phi(a)b) = \lambda^{it}\phi(ab), \\ F_{a,b}(t+i) &= \phi(b\sigma_t^\phi(a)) = \lambda^{it}(ba). \end{aligned}$$

On the other hand, $F_{a,b}(t+i) = \lambda^{i(t+i)}\phi(ab)$. It follows that

$$\phi(ab) = \lambda\phi(ba). \tag{37}$$

Now we use (36), in the adjoint form:

$$\Delta^{it} a^* \Delta^{-it} = \lambda^{-it} a^*.$$

This implies, using the automorphism property,

$$\Delta^{it} a^* a \Delta^{-it} = a^* a,$$

i.e. $a^*a \in \mathbb{C}.\mathbf{1}$ and even $a^*a = \mu\mathbf{1}$, $\mu > 0$, because $a \neq 0$. The same argument holds for aa^*, which yields $aa^* = \kappa\mathbf{1}$, $\kappa > 0$. Now, we can normalize a such that $a^*a = \mathbf{1}$, then aa^* is a projection, hence $aa^* = \mathbf{1}$ follows and a is unitary. But if we put $b := a^*$ in equation (37), we obtain

$$\phi(aa^*) = \lambda\phi(a^*a),$$

or $\lambda = 1$ which is a contradiction.

□

From Section 3 we know that the state $\phi(.) := (1, .1)$ (here we consider it on $\mathfrak{A}^{L,b}$) is faithful, since 1 is cyclic and separating for $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ and that its modular group is defined by

$$\sigma_\phi(a) := \mathrm{Ad}\Delta^{it}(a), a \in \mathfrak{A}^{L,b},$$
$$\Delta(f)(x) = \frac{d\mu_b(x)}{d\mu_b(x^{-1})} f(x), f \in L^2(B^{\mathbb{N}}, d\mu_b).$$

Now we state the first main theorem.

THEOREM 2.16. *Consider the von Neumann algebra* $\mathfrak{A}^{L,b}$ *generated by the left regular representation* $T^{L,b}$ *of* $B_0^{\mathbb{N}}$. *Assume that* $E(b) < \infty$. *Let* $\phi(a) = (1, a1)$ *be the faithful normal state associated to the cyclic and separating vector* 1 *and* Δ *the corresponding modular operator. Then* $\mathfrak{A}^{L,b}_\phi = \mathbb{C}.\mathbf{1}$ *and hence* $\mathfrak{A}^{L,b}$ *is a type III$_1$ factor. The same holds for* $\mathfrak{A}^{R,b}$.

We need some intermediate results.

LEMMA 2.17. *Let* g *be a multiplication on* $L^2(B^{\mathbb{N}}, d\mu_b)$ *by a measurable function* g *on* $B^{\mathbb{N}}$, *then*

$$(T_t^{R,b} g T_{t^{-1}}^{R,b} f)(x) = g(xt)f(x), \quad \forall x \in B^{\mathbb{N}}, t \in B_0^{\mathbb{N}}, f \in L^2(B^{\mathbb{N}}, d\mu_b).$$

PROOF. Let $f \in L^2(B^{\mathbb{N}}, d\mu_b)$. The following calculation holds:

$$\begin{aligned}(T_t^{R,b} g T_{t^{-1}}^{R,b} f)(x) &= \sqrt{\frac{d\mu_b(xt)}{d\mu_b(x)}} (g T_{t^{-1}}^{R,b} f)(xt) \\ &= \sqrt{\frac{d\mu_b(xt)}{d\mu_b(x)}} g(xt) \sqrt{\frac{d\mu_b(x)}{d\mu_b(xt)}} f(x) \\ &= g(xt)f(x).\end{aligned}$$

□

PROPOSITION 2.18. *Let* $\mathcal{M}$ *be a von Neumann algebra on* $\mathcal{H}_b = L^2(B^{\mathbb{N}}, d\mu_b)$. *If* $e^{isx_{kn}} \in \mathcal{M}', k < n$, $T_t^{R,b} \in \mathcal{M}', \forall t \in B_0^{\mathbb{N}}, s \in \mathbb{R}$ *and the measure* μ_b *is ergodic, then* $\mathcal{M} = \mathbb{C}.\mathbf{1}$.

PROOF. From Chapter 1, Proposition 1.16 we know that if $\mathcal{H}_b$ were $L^2(\mathbb{R}, d\mu_b)$, then the result would hold. The space $L^2(B^{\mathbb{N}}, d\mu_b)$ is isomorphic to $L^2(\mathbb{R}^\infty, d\mu_b) = \otimes_{k<n\in\mathbb{N}} L^2(\mathbb{R}^1, d\mu_{b_{kn}})$. Since the variables x_{kn} are independent, the condition $e^{ix_{kn}s} \in \mathcal{M}'$, for all $k < n$ and $s \in \mathbb{R}$, means that $L^\infty(\mathbb{R}, d\mu_{b_{kn}}) \subset \mathcal{M}'$ for all $k < n$. This implies that the von Neumann algebra generated by $(L^\infty(\mathbb{R}, d\mu_{b_{kn}}))_{k<n}$ is contained in $\mathcal{M}'$. The latter is isomorphic to $L^\infty(B^{\mathbb{N}}, d\mu_b)$, which is maximally abelian. Hence, $\mathcal{M} \subset L^\infty(B^{\mathbb{N}}, d\mu_b)' = L^\infty(B^{\mathbb{N}}, d\mu_b)$. Moreover, since we assume that $T_t^{R,b} \in \mathcal{M}'$ for all $t \in B_0^{\mathbb{N}}$, all functions in $\mathcal{M}$ are $B_0^{\mathbb{N}}$-right invariant, by Lemma 2.17. By the ergodicity of the measure, they are constant μ_b.a.e. □

Let $\mathcal{M} := \mathfrak{A}_\phi^{L,b}$. Then

$$\mathcal{M}' = (\mathfrak{A}^{L,b} \cap \{\Delta^{is}; s \in \mathbb{R}\}')' = \{T_t^{R,b}, \Delta^{is}; t \in B_0^{\mathbb{N}}, s \in \mathbb{R}\}''.$$

LEMMA 2.19. *Let $Q_{kn} f(x) := x_{kn} f(x)$, where $f \in L^2(B^{\mathbb{N}}, d\mu_b)$. Then*

$$Q_{kn} \;\eta\; \mathcal{M}', \quad \forall k < n \in \mathbb{N},$$

which is equivalent to $e^{iQ_{kn}s} \in \mathcal{M}'$ for all $s \in \mathbb{R}$ (by Chapter 1, Section 2).

PROOF. We shall give two possible methods to prove the Lemma. However, only one of them will lead to a rigorous proof.

Some useful formulas[**Kos02**]

Let us denote by X^{-1} the inverse matrix to the upper triangular matrix $X = I + x = I + \sum_{k<n} x_{kn} E_{kn} \in B^{\mathbb{N}}$

$$X^{-1} = (I + x)^{-1} = I + \sum_{k<n} x_{kn}^{-1} E_{kn} \in B^{\mathbb{N}}.$$

We have by definition $X^{-1}X = XX^{-1} = I$ hence

(38)

$$\left(XX^{-1}\right)_{kn} = \sum_{r=k}^{n} X_{kr} X_{rn}^{-1} = \delta_{kn} = \sum_{r=k}^{n} X_{kr}^{-1} X_{rn} = \left(X^{-1}X\right)_{kn}, \quad k \le n,$$

hence

$$x_{kn}^{-1} + \sum_{r=k+1}^{n-1} x_{kr} x_{rn}^{-1} + x_{kn} = 0 = x_{kn} + \sum_{r=k+1}^{n-1} x_{kr} x_{rn}^{-1} + x_{kn}^{-1}, \quad k < n,$$

and

$$x_{kn}^{-1} = -x_{kn} - \sum_{r=k+1}^{n-1} x_{kr} x_{rn}^{-1} = -x_{kn} - \sum_{r=k+1}^{n-1} x_{kr}^{-1} x_{rn}. \tag{39}$$

From $x_{kk+1}^{-1} = -x_{kk+1}$, by induction follows

(40)

$$x_{kn}^{-1} = -x_{kn} + \sum_{r=1}^{n-k-1} (-1)^{r+1} \sum_{k \le i_1 < i_2 < \ldots < i_r \le n} x_{ki_1} x_{i_1 i_2} \ldots x_{i_r n}, \quad k < n-1.$$

REMARK 2.20. Using (40) we see that x^{-1}_{kn} depends only on x_{rs} with $k \le r < s \le n$.

We have

$$x^{-1}_{kn} + x_{kn} = -\sum_{r=k+1}^{n-1} x_{kr}x^{-1}_{rn}, \quad x^{-1}_{kn} - x_{kn} = 2x_{kn} - \sum_{r=k+1}^{n-1} x_{kr}x^{-1}_{rn}. \tag{41}$$

Let us denote

$$w_{kn} := w_{kn}(x) := (x_{kn} + x^{-1}_{kn})(x_{kn} - x^{-1}_{kn}).$$

Using (63) we get

$$\Delta(x) = \frac{d\mu_b(x)}{d\mu_b(x^{-1})} = \exp\left[\sum_{k,n\in\mathbb{N},\, k<n} b_{kn}\left((x^{-1}_{kn})^2 - x^2_{kn}\right)\right]. \tag{42}$$

$$-\ln\Delta(x) = \sum_{k,n\in\mathbb{N},\, k<n} b_{kn}\left[x^2_{kn} - (x^{-1}_{kn})^2\right] = \sum_{k,n\in\mathbb{N},\, k<n} b_{kn}(x_{kn}+x^{-1}_{kn})(x_{kn}-x^{-1}_{kn})$$

$$\sum_{k,n\in\mathbb{N},\, k<n} b_{kn}(x_{kn} + x^{-1}_{kn})[2x_{kn} - (x_{kn} + x^{-1}_{kn})] = \sum_{k,n\in\mathbb{N},\, k<n} b_{kn}w_{kn}(x).$$

Method 1: Consider the one-parameter groups in $B_0^{\mathbb{N}}$,

$$G_{kn}(t) := \{1 + tE_{kn}, t \in \mathbb{R}\}. \tag{43}$$

Recall that the corresponding one parameter groups $T^{R,b}_{kn}(t) := \{T^{R,b}_u; u \in G_{kn}(t)\}$ have generators

$$A^R_{kn} = \sum_{r=1}^{k-1} x_{rk}D_{rn} + D_{kn}, \tag{44}$$

where $D_{pq} = \frac{\partial}{\partial x_{pq}} - b_{pq}x_{pq}$. Hence we have two types of generators, affiliated to $\mathcal{M}'$ at our disposal, namely A^R_{kn} and $i\ln\Delta$. Using certain commutator relations involving these operators, we shall obtain the independent variables x_{kn}. This suggests that these variables are also affiliated to $\mathcal{M}'$, since this would certainly be true if A^R_{kn} and $\ln\Delta$ were bounded. However, since we are dealing with unbounded operators, a more rigorous argument is needed in order to prove the Lemma.

First, we study the commutators of A^R_{kn} on $\ln\Delta$. In order to do this we need to know the action of D_{pq} on x^{-1}_{kn}.

LEMMA 2.21. *We have*

(45)

$$[D_{pq}, x^{-1}_{kn}] = \begin{cases} -x^{-1}_{kp}x^{-1}_{qn} - \delta_{kp}x^{-1}_{qn} - \delta_{qn}x^{-1}_{kp} - \delta_{kp}\delta_{qn}, & \text{if } k \le p < q \le n, \\ 0, & \text{otherwise}. \end{cases}$$

PROOF. We prove (45) by induction in $n-k$, such that $k \le p < q \le n$. Since $x_{kn}^{-1} = -x_{kn}$, when $n-k=1$,

$$[D_{pq}, x_{kn}] = -\delta_{pk}\delta_{qn}$$

in this case. Next we consider the induction step.

Let us suppose that (45) holds for all (p,q) with $r \le p < q \le n,\ k < r$. We prove that then (45) holds also for $(p,q),\quad k \le p < q \le n$. Indeed we have

$$[D_{pq}, x_{kn}^{-1}] = -[D_{pq}, x_{kn} + \sum_{r=k+1}^{n-1} x_{kr}x_{rn}^{-1}] = -\delta_{kp}\delta_{pn} - \delta_{kp}x_{kn}^{-1} - \sum_{r=k+1}^{n-1} x_{kr}[D_{sq}, x_{rn}^{-1}]$$

By induction, the last term equals to

$$\sum_{r=k+1}^{n-1} x_{kr}(x_{rp}^{-1}x_{qn}^{-1} + \delta_{pr}x_{qn}^{-1} + \delta_{qn}x_{jp}^{-1} + \delta_{pr}\delta_{qn})$$

Hence, by (39) $[D_{pq}, x_{kn^{-1}}] =$

$-\delta_{kp}\delta_{pn} - \delta_{kp}x_{kn}^{-1} - (x_{kp}^{-1}x_{qn}^{-1} + x_{kp}x_{qn}^{-1}) + x_{kp}x_{qn}^{-1} - x_{kp}^{-1}\delta_{qn} - x_{kp}\delta_{qn} + x_{kp}\delta_{qn}$,

which reduces to the desired result. □

Using (45) we get

(46)
$$[D_{pq}, x_{kn} + x_{kn}^{-1}] = \begin{cases} -x_{kp}^{-1}x_{qn}^{-1} - \delta_{kp}x_{qn}^{-1} - \delta_{qn}x_{kp}^{-1}, & \text{if } k \le p < q \le n, \\ 0, & \text{otherwise}. \end{cases}$$

Using (46) we have $[D_{pq}, (x_{kn} + x_{kn}^{-1})(x_{kn} - x_{kn}^{-1})] =$

(47)
$$\begin{cases} 2\,x_{kp}^{-1}x_{qn}^{-1}x_{kn}^{-1} + 2\,\delta_{kp}x_{qn}^{-1}x_{kn}^{-1} + 2\,\delta_{qn}x_{kp}^{-1}x_{kn}^{-1}, & \text{if } k \le p < q \le n,\ (p,q) \ne (k,n), \\ 2(x_{kn} + x_{kn}^{-1}), & \text{if } (p,q) = (k,n), \\ 0, & \text{otherwise}. \end{cases}$$

Indeed, if $k \le p < q \le n$, $(p,q) \ne (k,n)$ we have

$$[D_{pq}, (x_{kn} + x_{kn}^{-1})(x_{kn} - x_{kn}^{-1})] = [D_{pq}, (x_{kn} + x_{kn}^{-1})(2x_{kn} - (x_{kn} + x_{kn}^{-1}))]$$

$$= [D_{pq}, (x_{kn} + x_{kn}^{-1})](2x_{kn} - (x_{kn} + x_{kn}^{-1})) - (x_{kn} + x_{kn}^{-1})[D_{pq}, (x_{kn} + x_{kn}^{-1})] =$$

$$-2x_{kn}^{-1}[D_{pq}, (x_{kn} + x_{kn}^{-1})] \overset{(46)}{=} 2x_{kp}^{-1}x_{qn}^{-1}x_{kn}^{-1} + 2\delta_{kp}x_{qn}^{-1}x_{kn}^{-1} + 2\delta_{qn}x_{kp}^{-1}x_{kn}^{-1}.$$

LEMMA 2.22. *We have*

$$[A_{pq}^R,\ w_{kn}] = \begin{cases} 0, & \text{if } k < n \le p, \\ 2x_{kp}x_{kq}, & \text{if } n = q,\ k \le p-1, \\ 0, & \text{if } 1 \le k \le m-1,\ m+1 < n, \\ 2x_{pn}^{-1}x_{qn}^{-1}, & \text{if } k = p,\ n \ge q+1, \\ 0, & \text{if } q \le k < n. \\ 2(x_{pq} + x_{pq}^{-1}), & \text{if } k = p, q = n. \end{cases} \tag{48}$$

hence

$$(49)\quad -[A^R_{pq},\,\ln\Delta(x)] = 2\sum_{r=1}^{p-1} b_{rq}x_{rp}x_{rq} + 2\sum_{n=q+1}^{\infty} b_{pn}x^{-1}_{pn}x^{-1}_{qn} + 2(x_{pq}+x^{-1}_{pq}).$$

PROOF. Since

$$A^R_{pq} = \sum_{r=1}^{p-1} x_{rp}D_{rq} + D_{pq}$$

and $w_{kn},\ k<n\le p$ does not depend on $x_{rq},\ 1\le r\le q$ we conclude that $[A^R_{pq},\,w_{kn}]=0$ for $k<n\le p$ and $q\le k<n$.

Let $n=q$, since $[D_{rq},\,w_{kq}]=0$ for $1<r<k$ we get

$$[A^R_{pq},\,w_{kq}] = \sum_{r=k}^{p-1} x_{rp}[D_{rq},\,w_{kq}] + [D_{pq},\,w_{kq}] =$$

$$2\left(x_{kp}(x_{kq}+x^{-1}_{kq}) + \sum_{r=k+1}^{p-1} x_{rp}x^{-1}_{kr}x^{-1}_{kq} + x^{-1}_{kp}x^{-1}_{kq}\right) =$$

$$2\left(x_{kp}x_{kq} + \left(x_{kp} + \sum_{r=k+1}^{p-1} x^{-1}_{kr}x_{rp} + x^{-1}_{kp}\right)x^{-1}_{kq}\right) \stackrel{(39)}{=} 2x_{kp}x_{kq}.$$

Similarly, for $1\le k\le p-1,\ q<n$ we get

$$[A^R_{pq},\,w_{kn}] = \sum_{r=k}^{p-1} x_{rp}[D_{rq},\,w_{kn}] + [D_{pq},\,w_{kn}] =$$

$$2\left(x_{kp}x^{-1}_{qn} + \sum_{r=k+1}^{p-1} x_{rp}x^{-1}_{kr}x^{-1}_{qn} + x^{-1}_{kp}x^{-1}_{qn}\right)$$

$$2\left(x_{kp} + \sum_{r=k+1}^{p-1} x_{rp}x^{-1}_{kr} + x^{-1}_{kp}\right)x^{-1}_{qn} \stackrel{(39)}{=} 0.$$

If $k=p$ and $n\ge q+1$ we have as before

$$[A^R_{pq},\,w_{pn}] = [D_{pq},\,w_{pn}] \stackrel{(47)}{=} 2x^{-1}_{pn}x^{-1}_{qn}.$$

Finally if $(p,q)=(k,n)$,

$$[A^R_{pq}, w_{pq}] = 2(x_{pq}+x^{-1}_{pq})$$

□

For $p=m, q=m+1$ the last term of equation (49) vanishes, and we obtain the following formula.

$$(50)\quad -[A^R_{mm+1},\,\ln\Delta] = 2\sum_{r=1}^{m-1} b_{rm+1}x_{rm}x_{rm+1} + 2\sum_{n=m+2}^{\infty} b_{mn}x^{-1}_{mn}x^{-1}_{m+1n}.$$

Next, we shall use induction to obtain the independent variables. First, we want to act with $A^R_{m-1,m+1}$ and A^R_{m-1m} on the above formula. From (45) we conclude that the terms involving components of x^{-1} vanish. The action of A^R_{pq} on x_{kn} is easily computed:

$$[A^R_{pq}, x_{kn}] = \sum_{r=1}^{p-1} x_{rp}[D_{rq}, x_{kn}] + [D_{pq}, x_{kn}] = (x_{kp} + \delta_{kp})\delta_{qn}. \tag{51}$$

We get the following formulas:

$$\begin{gathered} [A^R_{m-1m+1}, [A^R_{mm+1}, \ln\Delta]] = \\ -2\textstyle\sum_{r=1}^{m-2} b_{rm+1}x_{rm-1}x_{rm} - 2b_{m-1m+1}x_{m-1m}, \\ [A^R_{m-1m}, [A^R_{mm+1}, \ln\Delta]] = \\ -2\textstyle\sum_{r=1}^{m-2} b_{rm+1}x_{rm-1}x_{rm+1} - 2b_{m-1m+1}x_{m-1m+1} \end{gathered} \tag{52}$$

To get x_{1m} we compute

$$[A^R_{1m-1}, [A^R_{m-1m+1}, [A^R_{mm+1}, \ln\Delta]]] = -2b_{1m+1}x_{1m}.$$

Next we consider the induction step. Suppose that we have obtained all variables $x_{rn}, r < p < m-1$ for some p. The variables x_{m-1m} and x_{m-1m+1} are deduced from (52). To get $x_{pm+1}, p < m-1$ we compute

$$[A^R_{pm-1}, [A^R_{m-1m+1}, [A^R_{mm+1}, \ln\Delta]]] = -2\sum_{r=1}^{p-1} b_{rm+1}x_{rp}x_{rm-1} - 2b_{pm+1}x_{pm+1}.$$

This method looks nice at the first glance, and it would lead to a rigorous proof if we were to find a common invariant domain for the operators A^R_{kn} and $\ln\Delta$. Indeed, suppose that D is such a domain and A and B are two operators on D, affiliated to $\mathcal{M}'$. Than, by Definition 1.4, $uD \subseteq D$, $u \in \mathcal{M}$ and

$$\begin{aligned} u[A,B]u^*\xi &= (uAu^*uBu^* - uBu^*uAu^*)\xi \\ &= (uAu^*B - uBu^*A)\xi \\ &= [A,B]\xi, \end{aligned}$$

since $A\xi, B\xi \in D$. Hence the closure of $[A,B]$ is also affiliated to $\mathcal{M}'$.

A possible candidate for a common invariant domain for A^R_{pq} and $\ln\Delta$ is the set of functions

$$\langle A^R_{pq}, \ln\Delta; p < q \in \mathbb{N}\rangle 1,$$

where $\langle . \rangle$ denotes the linear algebraic span (we already know that the above set contains the set $\mathcal{P}$ of polynomials). However, proving that the above functions are indeed in $L^2(B^{\mathbb{N}}, d\mu_b)$ is a very difficult task and hence another approach seems to be more plausible. The second method, which we describe below, will give us the solution.

Method 2: In this method we circumvent the problem of dealing with unbounded operators, by directly working with elements in $\mathcal{M}'$. By certain combinations of elements in $\mathcal{M}'$ we would like to obtain the one-parameter groups generated by the multiplication operators Q_{kn}, $k < n \in \mathbb{N}$. First we consider a special case, which leads to a better understanding of the situation. Consider the restriction of the group $B^{\mathbb{N}}$ to the first two lines. That is, let $X^{(2)}$ be the space

$$X^{(2)} := \{1 + x; x = \sum_{n=2}^{\infty} x_{1n}E_{1n} + \sum_{n=3}^{\infty} x_{2n}E_{2n}\}$$

$$X^{(2)} = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & \cdots \\ 0 & 1 & x_{23} & x_{24} & \cdots \end{pmatrix}$$

We restrict the action of $B_0^{\mathbb{N}}$ to $X^{(2)}$ and keep the previous notation for the restricted operators $T_t^{R,b}$. We get the following generators of the groups $\{T_t^{R,b}; t \in G_{kn}(t)\}$:

$$\begin{aligned} A_{1n}^{R,2} &= D_{1n} = \partial_{1n} - b_{1n}x_{1n}, \\ A_{2n}^{R,2} &= x_{12}D_{1n} + D_{2n}, \\ A_{kn}^{R,2} &= x_{1k}D_{1n} + x_{2k}D_{2n}. \end{aligned}$$

Define the restriction of the measure μ_b on $X^{(2)}$ by

$$\mu_b^2 := \bigotimes_{k=1,2;n>k} \mu_{b_{kn}}.$$

The formulas for the inverse matrix are the following:

$$\begin{aligned} x_{1n}^{-1} &= -x_{1n} + x_{12}x_{2n}, \\ x_{2n}^{-1} &= -x_{2n}. \end{aligned}$$

Furthermore, the modular operator has the following form. Note that the terms with w_{12} and w_{kn} for $k > 1$ vanish in this case.

(53)

$$\ln \Delta(x) = \sum_{n=3}^{\infty} b_{1n}(x_{1n}^{-1} - x_{1n})(x_{1n}^{-1} + x_{1n}) = \sum_{n=3}^{\infty} b_{1n}((x_{12})^2(x_{2n})^2 - 2x_{12}x_{1n}x_{2n}).$$

Let $\mathfrak{A}_2^{L,b}$ be the von Neumann algebra on $L^2(X^{(2)}, d\mu_b^2)$ defined by:

$$\mathfrak{A}_2^{L,b} := \{T_t^{L,b}; t \in B_0^{\mathbb{N}}\}''.$$

We remind that also in this case the centralizer of $\mathfrak{A}_2^{L,b}$ w.r.t. the vector 1 is given by

$$\mathcal{M}_2 = \{T_t^{R,b}, \Delta^{is}; t \in B_0^{\mathbb{N}}, s \in \mathbb{R}\}'.$$

Now we want to prove that the one-parameter groups generated by the multiplication operators $Q_{12}, Q_{13}, ...$ and $Q_{23}, Q_{24}, ...$, are contained in $\mathcal{M}_2'$.

First we try to obtain Q_{12}. Let $X' = XG_{23}(s)$, where $X = 1 + x$. We calculate the following expression, where we use Lemma 2.17:

$$\Delta^{it}(x)T_{23}^{R,b}(s)\Delta^{-it}(x)T_{23}^{R,b}(-s) = \Delta^{it}(x)\Delta^{-it}(x') = e^{it(\ln\Delta(x)-\ln\Delta(x'))} \tag{54}$$

Note that only x_{13} and x_{23} change under the above transformation. We formally compute $\ln\Delta(x')$:

$$\begin{aligned}
\ln\Delta(x') &= b_{13}\left((x'_{12})^2(x'_{23})^2 - 2x'_{12}x'_{13}x'_{23}\right) \\
&\quad + \sum_{n=4}^{\infty} b_{1n}\left((x_{12})^2(x_{2n})^2 - 2x_{12}x_{1n}x_{2n}\right) \\
&= b_{13}\{(x_{12})^2(x_{23})^2 + 2s(x_{12})^2x_{23} + s^2(x_{12})^2 \\
&\quad -2x_{12}x_{13}x_{23} - 2s((x_{12})^2x_{23} + x_{12}x_{23}) - 2s^2(x_{12})^2\} \\
&\quad + \sum_{n=4}^{\infty} b_{1n}\left((x_{12})^2(x_{2n})^2 - 2x_{12}x_{1n}x_{2n}\right) \\
&= -b_{13}(2sx_{12}x_{13} + s^2(x_{12})^2) + \ln\Delta(x).
\end{aligned}$$

Hence, expression (54) becomes

$$U_{23}(s,t)(x) := e^{it\{b_{13}(2sx_{12}x_{13}+s^2(x_{12})^2)\}}$$

REMARK 2.23. It is important to say that the exponentials of infinite sums such as the ones above are well defined, since functional calculus in multiple variables applies to the set of commuting operators $\{Q_{kn}\}$. Indeed, the above formulas with infinite sums (and all formulas in this dissertation, involving sums of functions in the variables x_{kn}) consist of terms, which act as multiplication operators on mutually independent Hilbert spaces in the decomposition

$$L^2(B^{\mathbb{N}}, d\mu_b) = \bigotimes_{k<n} L^2(\mathbb{R}^1, d\mu_{b_{kn}}).$$

Hence we do not need to verify any convergence properties.

It follows that the multiplication operator of the function $U_{23}(s,t)(x)$ is in $\mathcal{M}_2'$ for all $s,t \in \mathbb{R}$. Let now $X' = XG_{13}(s')$. To obtain x_{12} we compute:

$$\begin{aligned}
T_{13}^{R,b}(s')U_{23}(1,1)(x)T_{13}^{R,b}(-s') &= e^{ib_{13}\left(2x_{12}x'_{13}+(x_{12})^2\right)} \\
&= e^{ib_{13}\left(2x_{12}x_{13}+(x_{12})^2+2s'x_{12}\right)}
\end{aligned}$$

Upon multiplying the above expression by $U_{23}(1,-1)$ we obtain the one parameter group $e^{2b_{13}s'Q_{12}}$, $s' \in \mathbb{R}$. Similarly, we can obtain the one-parameter groups generated by the operators Q_{1m}, $m > 2$. In order to do this we compute

$$U_{mm+1}(s,t) = \Delta^{it}(x) T^{R,b}_{mm+1}(s) \Delta^{-it}(x) T^{R,b}_{mm+1}(-s)$$

instead of U_{23}. Again only the term of $\ln \Delta(x)$, (53) with $n = m+1$ changes under the above transformation and we extract

(55)

$$U_{mm+1}(s,t)(x) := e^{it\{b_{1m+1}(2sx_{1m}x_{1m+1}+s^2(x_{1m})^2)+b_{2m+1}(2sx_{2m}x_{2m+1}+s^2(x_{2m})^2)\}}$$

and hence we obtain x_{1m} analogously to x_{12}, by transforming the x_{1m+1} term and subtracting the initial part.

In order to obtain the operators Q_{2m}, we operate in a similar way. Again consider formula (55). Now substitute $X' = XG_{2m+1}(s)$ for X and subtract the initial part:

(56)

$$U_{mm+1}(1,-1) T^{R,b}_{2m+1}(s') U_{mm+1}(1,1)(x) T^{R,b}_{2m+1}(-s') = e^{2is'(b_{1m+1}x_{1m}x_{12}+b_{2m+1}x_{2m})}.$$

We already proved that the one-parameter groups generated by the self-adjoint operators Q_{1n} are in $\mathcal{M}_2'$. From Chapter 1 we know that this is equivalent to $Q_{1n}\,\eta\,\mathcal{M}_2'$. According to Theorem 1.5 in Chapter 1, this implies that also closure of products and sums of these operators are affiliated to the abelian von Neumann algebra generated by Q_{1n}, which is contained in $\mathcal{M}_2'$. Equation (56) then implies that also $Q_{2m}\ \eta\ \mathcal{M}_2'$. Hence we obtained $Q_{kn}\ \eta\ \mathcal{M}_2'$ for $k = 1, 2$, $n > k$.

Now we generalize the above discussion to the case $(B_0^{\mathbb{N}}, B^{\mathbb{N}})$.

LEMMA 2.24. *Fix an $m \in \mathbb{N}$ and let*

$$U_{mm+1}(s,t) := \Delta^{it} T^{R,b}_{mm+1}(s) \Delta^{-it} T^{R,b}_{mm+1}(-s) \in \mathcal{M}' \text{ for } s,t \in \mathbb{R}.$$

Then $(U_{mm+1}(s,t)f)(x) =$

(57)

$$\exp\Big(it \textstyle\sum_{k=1}^{m-1} b_{km+1}\left(2sx_{km}x_{km+1} + s^2(x_{km})^2\right) + it \sum_{n=m+2}^{\infty} b_{mn}\left(2sx^{-1}_{mn}x^{-1}_{m+1n} - s^2(x^{-1}_{m+1n})^2\right)\Big) f(x)$$

Furthermore, the following identities hold for all $s' \in \mathbb{R}$ and $p < m$:

(58)

$$\left(U_{mm+1}(1,-1) T^R_{1m+1}(s') U_{mm+1}(1,1) T^R_{1m+1}(-s') f\right)(x) = \exp(2is' b_{1m+1}x_{1m}) f(x),$$

(59)

$$\left(U_{mm+1}(1,-1) T^R_{pm+1}(s') U_{mm+1}(1,1) T^R_{pm+1}(-s') f\right)(x) = \exp\left(2is'(\textstyle\sum_{r=1}^{p-1} b_{rm+1}x_{rm}x_{rp} + b_{pm+1}x_{pm})\right) f(x).$$

PROOF. First of all, we would like to know the right action of $G_{pq}(t)$ (43) on $x \in B^{\mathbb{N}}$ and on x^{-1}.

$$x'_{kn} = (XG_{pq}(t))_{kn} = \sum_{i=k}^{n} X_{ki}(\delta_{in} + t\delta_{pi}\delta_{qn}) = x_{kn} + \delta_{qn}(tx_{kp} + t\delta_{kp}) \tag{60}$$

Note that only the qth column of x is affected by this transformation.

$$XG_{pq}(t) = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & \cdots & x_{1q} + tx_{1p} & \cdots \\ 0 & 1 & x_{23} & x_{24} & \cdots & x_{2q} + tx_{2p} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & x_{pq} + t & \cdots \\ & & & & & & \ddots \end{pmatrix}$$

From (60) we now can deduce the inverse of x'. In order to do this we note the following:

$$x'^{-1}_{kn} = (XG_{pq}(t))^{-1}_{kn} = (G_{pq}(-t)X^{-1})_{kn}.$$

Hence we compute: $x'^{-1}_{kn} =$

$$\sum_{j=k}^{n}(G_{pq}(-t))_{kj}X^{-1}_{jn} = \sum_{j=k}^{n}(\delta_{kj} - t\delta_{pk}\delta_{jq})(x^{-1}_{jn} + \delta_{jn}) = x^{-1}_{kn} - t\delta_{kp}(x^{-1}_{qn} + \delta_{qn}),$$

where of course x' depends on $t \in \mathbb{R}$.

To prove the formula (57), we set $X' = XG_{mm+1}(s)$. We note that the following identity holds:

$$(U_{mm+1}(s,t)f)(x) = \Delta^{it}(x)\Delta^{-it}(x')f(x).$$

Hence we have to formally compute $-\ln\Delta(x')$. From the above discussion follows that $\ln\Delta$ can be written as the sum of three different terms:

$$-\ln\Delta = \sum_{1 \le k < m, n > m+1} b_{kn}w_{kn}(x) + \sum_{k<m} b_{km+1}w_{km+1}(x') + \sum_{n>m+1} b_{mn}w_{mn}(x')$$

Note that the term with $k = m, n = m+1$ vanishes, because $w_{mm+1}(x) = 0$. First we consider the second term: $\sum_{k<m} b_{km+1}w_{km+1}(x') =$

(61)

$$\sum_{k<m} b_{km+1}(x'_{km+1} - x'^{-1}_{km+1})(x'_{km+1} + x'^{-1}_{km+1}) =$$

$$\sum_{k<m} b_{km+1}(x_{km+1} + sx_{km} - x^{-1}_{km+1})(x_{km+1} + sx_{km} + x^{-1}_{km+1}) =$$

$$\sum_{k<m} b_{km+1}w_{km+1}(x) + \sum_{k<m} b_{km+1}\left(2sx_{km}x_{km+1} + s^2(x_{km})^2\right).$$

The third term is as follows: $\sum_{n>m+1} b_{mn} w_{mn}(x') =$
(62)

$$\sum_{n>m+1} b_{mn}(x'_{mn} - x'^{-1}_{mn})(x'_{mn} + x'^{-1}_{mn}) =$$

$$\sum_{n>m+1} b_{mn}(x_{mn} - x^{-1}_{mn} + sx^{-1}_{m+1n})(x_{mn} + x^{-1}_{mn} - sx^{-1}_{m+1n}) =$$

$$\sum_{n>m+1} b_{mn} w_{mn}(x) + \sum_{n>m+1} b_{mn}\left(2sx^{-1}_{mn}x^{-1}_{m+1n} - s^2(x^{-1}_{m+1n})^2\right).$$

After adding up the three terms we see formula (57) appear.

In order to get formulas (58) and (59) we note that multiplication from the right by $G_{pm+1}(s')$ does not affect the formulas in (62). Hence only the first term in (57) is transformed and namely (we set $s = 1, t = 1, x'' = x'G_{1m+1}(s')$):

$$i\sum_{k<m} b_{km+1}\left(2x''_{km}x''_{km+1} + (x''_{km})^2\right) =$$

$$i\sum_{k<m} b_{km+1}\left(2x_{km}x_{km+1} + (x_{km})^2\right) + 2is'b_{1m+1}x_{1m},$$

and finally setting $x'' = xG_{pm+1}(s'), 1 < p < m$, we obtain

$$i\sum_{k<m} b_{km+1}\left(2x''_{km}x''_{km+1} + (x''_{km})^2\right) =$$

$$i\sum_{k<m} b_{km+1}\left(2x_{km}x_{km+1} + (x_{km})^2\right) + 2is'\left(\sum_{r=1}^{p-1} b_{rm+1}x_{rm}x_{rp} + b_{pm+1}x_{pm}\right).$$

By subtracting the first term in each of the above formulas and exponentiating the outcome, we can deduce (58) and (59), and conclude the proof. □

To finish the proof of Lemma 2.19 we use induction. From equation (58), we conclude that $Q_{1m}\eta\mathcal{M}'$ for any $1 < m$. We proceed with the induction step. Assume that $Q_{rm}\eta\mathcal{M}'$, for $1 \leq r < p$. By Theorem 1.5 we conclude that also $2i\sum_{r=1}^{p-1} b_{rm+1}Q_{rm}\hat{\cdot}Q_{rp}\eta\mathcal{M}'$ (the sum being interpreted as $\hat{+}$). To be more precise, we consider the abelian algebra generated by spectral projections of the operators Q_{rm}, which is contained in $\mathcal{M}'$. Hence $\mathcal{M}'$ contains one parameter groups

$$\exp\left(-2is'(\sum_{r=1}^{p-1} b_{rm+1}Q_{rm}\hat{\cdot}Q_{rp})\right),$$

where $p < m$ and hence, by multiplying with (59), we conclude that (after rescaling the parameter) also $\exp(isQ_{pm}) \in \mathcal{M}'$ for all $s \in \mathbb{R}$. It follows that $Q_{pm}\eta\mathcal{M}'$. This proves Lemma 2.19. □

Since, $T^{R,b}_t \in \mathcal{M}'$ for all $t \in B_0^{\mathbb{N}}$ and the measure μ_b is ergodic, it follows from Proposition 2.18 that $\mathcal{M}$ is trivial. Finally, Theorem 2.15 implies that $\mathfrak{A}^{L,b}$ and hence $\mathfrak{A}^{R,b}$ are type III_1 factors. The interesting fact is that this property does not depend on any further conditions of the measure μ_b other than $S^L_{kn}(b) < \infty$ for all $k < n \in \mathbb{N}$ and $E(b) < \infty$. This also shows that the previously known sufficient conditions (28) (see Theorem 2.10), for $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ to be factors, are not necessary.

6. Uniqueness of the constructed factor

THEOREM 2.25. *The von Neumann algebras $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are hyperfinite type III_1 factors and hence isomorphic to the factor R_∞ of Araki and Woods.*

PROOF. Let G be a solvable separable locally compact group or a connected locally compact group. Then any representation π of G in a Hilbert space generates a hyperfinite von Neumann algebra (see Chapter 1, Corollary 1.24).
The group $B_0^{\mathbb{N}}$ is the inductive limit of groups of finite dimensional upper-triangular matrices (with units on the diagonal), which are of course solvable, connected locally compact groups. Hence their group algebras are hyperfinite (and by a theorem of Dixmier ([**Dix59**]) even type I algebras). Thus the von Neumann algebra $\mathfrak{A}^{R,b}$ is the inductive limit of hyperfinite von Neumann algebras and hence itself hyperfinite. From the theorem of Haagerup (Chapter 1 Theorem 1.28) follows that $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are all isomorphic to the Araki-Woods factor R_∞ (and hence are also Krieger factors). □

CHAPTER 3

Type III_1 factors generated by regular representations of the group $B_0^{\mathbb{Z}}$

The second group that we investigate here is similar to the group $B_0^{\mathbb{N}}$, with the difference that the matrices can also have negative indices and their limits go not only into the $+\infty$ direction but also to $-\infty$. This causes additional problems in the proofs, which sometimes have to be solved with different methods. In this chapter, we shall also prove that the corresponding von Neumann algebras are type III_1 factors. However, the method from the previous Chapter, using the centralizer, does not work that well anymore and we have to come up with another approach. Indeed, since we are dealing with matrices which can also have negative indices, the sum in the formula (57), will go down to $k = -\infty$, instead of $k = 1$. This implies that the operators (58) and (59) have infinite sums in the exponentials and can not be used to extract the independent variables. To circumvent this problem, in this Chapter rely on another invariant, namely the *flow of weights* invariant from Definition 1.20 and prove that it is trivial for the von Neumann algebras associated to the new group. Theorems 1.19 and 1.21 in Chapter 1 then imply that $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are type III_1 factors.

1. Regular representations

Let us consider the group $\tilde{G} = B^{\mathbb{Z}}$ of all upper-triangular real matrices of infinite order with units on the diagonal

$$\tilde{G} = B^{\mathbb{Z}} = \{I + x \mid x = \sum_{k,n\in\mathbb{Z}\, k<n} x_{kn}E_{kn}\},$$

and its subgroup

$$G = B_0^{\mathbb{Z}} = \{I + x \in B^{\mathbb{Z}} \mid \ x \text{ is finite}\},$$

where E_{kn} is an infinite-dimensional matrix with 1 at the place $k, n \in \mathbb{Z}$ and zeros elsewhere, $x = (x_{kn})_{k<n}$ is *finite* means that $x_{kn} = 0$ for all (k, n)

except for a finite number of indices $k,n\in\mathbb{Z}$.

$$\begin{pmatrix} \ddots & & & & & \\ & 1 & x_{01} & x_{02} & x_{03} & \cdots \\ & 0 & 1 & x_{12} & x_{13} & \cdots \\ & 0 & 0 & 1 & x_{23} & \cdots \\ & 0 & 0 & 0 & 1 & \cdots \\ & & & & & \ddots \end{pmatrix}$$

Obviously, $B_0^{\mathbb{Z}}=\varinjlim_n B(2n-1,\mathbb{R})$ is the inductive limit of the group $B(2n-1,\mathbb{R})$ of real upper-triangular matrices with units on the principal diagonal realized in the following form

$$B(2n-1,\mathbb{R})=\{I+\sum_{-n+1\le k<r\le n-1}x_{kr}E_{kr}\mid x_{kr}\in\mathbb{R}\}\quad n\in\mathbb{N}$$

with respect to the embedding $B(2n-1,\mathbb{R})\ni x\mapsto x+E_{-n,-n}+E_{nn}\in B(2n+1,\mathbb{R})$.

We define the Gaussian measure μ_b on the group $B^{\mathbb{Z}}$ in the following way

(63)
$$d\mu_b(x)=\bigotimes_{k,n\in\mathbb{Z},\,k<n}(b_{kn}/\pi)^{1/2}\exp(-b_{kn}x_{kn}^2)dx_{kn}=\bigotimes_{k,n\in\mathbb{Z},\,k<n}d\mu_{b_{kn}}(x_{kn}),$$

where $b=(b_{kn})_{k<n}$ is some set of positive numbers $b_{kn}>0,\ k,n\in\mathbb{Z}$.

Let us denote by R and L the right and the left action of the group $B^{\mathbb{Z}}$ on itself: $R_t(s)=st^{-1},\ L_t(s)=ts,\ s,t\in B^{\mathbb{Z}}$ and by $\Phi:B^{\mathbb{Z}}\mapsto B^{\mathbb{Z}},\ \Phi(I+x):=(I+x)^{-1}$ the inverse mapping. It is known [**Kos01**] that

Lemma 3.1. *$\mu_b^{R_t}\sim\mu_b\ \forall t\in B_0^{\mathbb{Z}}$ if and only if $S_{kn}^R(b)<\infty,\ \forall k,n\in\mathbb{Z},\ k<n$ where*

$$S_{kn}^R(b)=\sum_{r=-\infty}^{k-1}\frac{b_{rn}}{b_{rk}}.$$

Lemma 3.2. *$\mu_b^{L_t}\sim\mu_b\ \forall t\in B_0^{\mathbb{Z}}$ if and only if $S_{kn}^L(b)<\infty,\ \forall k,n\in\mathbb{Z},\ k<n$, where*

$$S_{kn}^L(b)=\sum_{m=n+1}^{\infty}\frac{b_{km}}{b_{nm}}.$$

Lemma 3.3. *$\mu_b^{L_{I+tE_{kn}}}\perp\mu_b\ \forall t\in\mathbb{R}\backslash\{0\}\Leftrightarrow S_{kn}^L(b)=\infty,\ \ k,n\in\mathbb{Z},\ k<n$.*

Let us denote

(64)
$$E(b)=\sum_{k<n<r}\frac{b_{kr}}{b_{kn}b_{nr}},\ E_m(b)=\sum_{k<n<r\le m}\frac{b_{kr}}{b_{kn}b_{nr}},\ m\in\mathbb{Z}.$$

Lemma 3.4. [**KZ00**] *If $E(b)<\infty$, then $\mu_b^{\Phi}\sim\mu_b$.*

REMARK 3.5. [**KZ00**] If $\mu_b^\Phi \sim \mu_b$ then $\mu_b^{L_t} \sim \mu_b \Leftrightarrow \mu_b^{R_t} \sim \mu_b \ \forall t \in B_0^{\mathbb{Z}}$.

PROOF. This follows from the fact that the inversion Φ interchanges the right and the left action: $R_t \circ \Phi = \Phi \circ L_t \ \forall t \in B^{\mathbb{Z}}$. Indeed, if we denote $\mu^f(\cdot) = \mu(f^{-1}(\cdot))$ we have $(\mu^f)^g = \mu^{f\circ g}$. Hence

$$\mu_b \sim \mu_b^{R_t} \sim (\mu_b^{R_t})^\Phi = \mu_b^{R_t\circ\Phi} = \mu_b^{\Phi\circ L_t} = (\mu_b^\Phi)^{L_t} \sim \mu_b^{L_t}, \ \forall t \in B_0^{\mathbb{Z}}.$$

□

REMARK 3.6. We have

$$E(b) = \sum_{k<n} \frac{S_{kn}^L(b)}{b_{kn}} = \sum_{k<n} \frac{S_{kn}^R(b)}{b_{kn}}, \ E_m(b) = \sum_{k<n\leq m} \frac{S_{kn}^R(b)}{b_{kn}}. \tag{65}$$

Indeed

$$\sum_{k<n} \frac{S_{kn}^L(b)}{b_{kn}} = \sum_{k<n} \sum_{r=n+1}^{\infty} \frac{b_{kr}}{b_{kn}b_{nr}} = \sum_{k<n<r} \frac{b_{kr}}{b_{kn}b_{nr}} = E(b)$$

$$= \sum_{n<r} \frac{1}{b_{nr}} \sum_{k=-\infty}^{n-1} \frac{b_{kr}}{b_{kn}} = \sum_{n<r} \frac{S_{nr}^R(b)}{b_{nr}}.$$

If $\mu_b^{R_t} \sim \mu_b$ and $\mu_b^{L_t} \sim \mu_b \ \forall t \in B_0^{\mathbb{Z}}$, one can define in a natural way (see [**Kos92**]), an analogue of the right $T^{R,b}$ and the left $T^{L,b}$ regular representations of the group $B_0^{\mathbb{Z}}$ in the Hilbert space $\mathcal{H}_b = L^2(B^{\mathbb{Z}}, \mu_b)$

$$T^{R,b}, \ T^{L,b} : B_0^{\mathbb{Z}} \to U(H_b = L_2(B^{\mathbb{Z}}, \mu_b)),$$

$$(T_t^{R,b} f)(x) = (d\mu_b(xt)/d\mu_b(x))^{1/2} f(xt),$$

$$(T_s^{L,b} f)(x) = (d\mu_b(s^{-1}x)/d\mu_b(x))^{1/2} f(s^{-1}x).$$

2. Von Neuman algebras generated by regular representations

Let $\mathfrak{A}^{R,b} = (T_t^{R,b} \mid t \in B_0^{\mathbb{Z}})''$ (resp. $\mathfrak{A}^{L,b} = (T_s^{L,b} \mid s \in B_0^{\mathbb{Z}})''$) be the von Neumann algebras generated by the right $T^{R,b}$ (resp. the left $T^{L,b}$) regular representation of the group $B_0^{\mathbb{Z}}$.

THEOREM 3.7. [**Kos01**] *If $E(b) < \infty$ then $\mu_b^\Phi \sim \mu_b$. In this case the right and the left regular representations are well defined and the commutation theorem holds:*

$$(\mathfrak{A}^{R,b})' = \mathfrak{A}^{L,b}. \tag{66}$$

Moreover, the operator J_{μ_b} given by

$$(J_{\mu_b} f)(x) = (d\mu_b(x^{-1})/d\mu_b(x))^{1/2} \overline{f(x^{-1})} \tag{67}$$

is an intertwining operator:

$$T_t^{L,b} = J_{\mu_b} T_t^{R,b} J_{\mu_b}, \ t \in B_0^{\mathbb{Z}} \quad \text{and} \ J_{\mu_b} \mathfrak{A}^{R,b} J_{\mu_b} = \mathfrak{A}^{L,b}.$$

If $\mu_b^{R_t} \sim \mu_b \; \forall t \in B_0^{\mathbb{Z}}$ but $\mu_b^{L_t} \perp \mu_b \; \forall t \in B_0^{\mathbb{Z}} \backslash \{e\}$ one can't define the left regular representation of the group $B_0^{\mathbb{Z}}$. Moreover the following theorem holds

THEOREM 3.8. [**Kos01**] *The right regular representation* $T^{R,b} : B_0^{\mathbb{Z}} \to U(H_b)$ *is irreducible if*

1) $\mu_b^{L_s} \perp \mu_b \; \forall s \in B_0^{\mathbb{Z}} \backslash \{0\}$,

2) the measure μ_b *is* $B_0^{\mathbb{Z}}$ *right-ergodic.*

3) $\sigma_{kn}(b) = \infty, \; \forall k < n, \; k, n \in \mathbb{Z}$ *where*

$$\sigma_{kn}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}^2}{[S_{km}^R(b) + b_{km}][S_{nm}^R(b) + b_{nm}]}.$$

REMARK 3.9. We do not know whether the Ismagilov conjecture holds in this case, namely, whether conditions 1) and 2) of the theorem are the criteria of the irreducibility of the representation $T^{R,b}$ of the group $B_0^{\mathbb{Z}}$ as holds for example for the group $B_0^{\mathbb{N}}$ (see [**Kos92**]).

REMARK 3.10. We do not know the criterion of the $B_0^{\mathbb{Z}}$-ergodicity of the measure μ_b on the space $B^{\mathbb{Z}}$. Sufficient conditions are $E_m(b) < \infty, \; \forall m \in \mathbb{Z}$ ([**Kos01**]).

COROLLARY 3.11. *The von Neumann algebra* $\mathfrak{A}^{R,b}$ *is a type* I_∞ *factor if the conditions of Theorem 3.8 are valid.*

Let us assume now that $\mu_b^{L_t} \sim \mu_b \sim \mu_b^{R_t} \; \forall t \in B_0^{\mathbb{Z}}$. In this case the right regular representation and the left regular representation of the group $B_0^{\mathbb{Z}}$ are well defined.

In the case when the representation $T^{R,b}$ is reducible, we shall prove in this Chapter that all the corresponding algebras are also factors (in the case that $E(b) < \infty$). Until now there existed only sufficient conditions on the measure for the factor property to hold ([**KZ00**]). Let us review them briefly.

Since $T_t^{L,b} \in (\mathfrak{A}^{R,b})' \; \forall t \in B_0^{\mathbb{Z}}$, we have $\mathfrak{A}^{L,b} \subset (\mathfrak{A}^{R,b})'$, hence

$$\mathfrak{A}^{R,b} \cap (\mathfrak{A}^{R,b})' \subset (\mathfrak{A}^{L,b})' \cap (\mathfrak{A}^{R,b})' = (\mathfrak{A}^{R,b} \cup \mathfrak{A}^{L,b})'. \tag{68}$$

The last relation shows that $\mathfrak{A}^{R,b}$ is factor if the representation

$$B_0^{\mathbb{Z}} \times B_0^{\mathbb{Z}} \ni (t, s) \to T_t^{R,b} T_s^{L,b} \in U(\mathcal{H}_b)$$

is irreducible. Let us denote

$$S_{kn}^{R,L}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}^2}{[S_{km}^R(b) + b_{km}][S_{nm}^L(b) + S_{nm}^R(b)]}, \quad k < n.$$

THEOREM 3.12. [**KZ00**] *The representation*

$$B_0^{\mathbb{Z}} \times B_0^{\mathbb{Z}} \ni (t, s) \to T_t^{R,b} T_s^{L,b} \in U(\mathcal{H}_b)$$

is irreducible if $S_{kn}^{R,L}(b) = \infty, \; \forall k < n$ *and the measure* μ_b *is* $B_0^{\mathbb{Z}}$ *right-ergodic.*

Corollary 3.13. *The von Neumann algebra* $\mathfrak{A}^{R,b}$ *is a factor if* $S^{R,L}_{kn}(b) = \infty \, \forall k < n$ *and the measure* μ_b *is* $B_0^{\mathbb{Z}}$ *right-ergodic.*

3. Examples

In this section we give an example of a measure μ_b, $b = (b_{kn})_{k<n}$ for which the the representations $T^{R,b}$ and $T^{L,b}$ are reducible and the von Neumann algebras $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are well defined.

We show that the set $b = (b_{kn})_{k<n}$ for which

$$S^R_{kn}(b) < \infty,\ S^L_{kn}(b) < \infty,\ E(b) < \infty,\ k < n, \tag{69}$$

where

$$S^R_{kn}(b) = \sum_{r=-\infty}^{k-1} \frac{b_{rn}}{b_{rk}},\ S^L_{kn}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{b_{nm}},\ E(b) = \sum_{k<n} \frac{S^L_{kn}(b)}{b_{kn}},$$

is not empty.

In the example 1 below for the particular case $b_{kn} = (a_k)^n$ we give some sufficient conditions on the sequence a_n implying conditions (69).

Example 1. Let us take $b_{kn} = (a_k)^n,\ k, n \in \mathbb{Z}$.

We have

$$S^R_{kn}(b) = \sum_{r=-\infty}^{k-1} \frac{b_{rn}}{b_{rk}} = \sum_{r=-\infty}^{k-1} a_r^{n-k} < \infty \ \text{ if } \sum_{r=-\infty}^{0} a_r < \infty, \tag{70}$$

(71)

$$S^L_{kn}(b) = \sum_{m=n+1}^{\infty} \left(\frac{a_k}{a_n}\right)^m = \left(\frac{a_k}{a_n}\right)^{n+1} \sum_{m=0}^{\infty} \left(\frac{a_k}{a_n}\right)^m = \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} < \infty,$$

iff $a_k < a_{k+1},\ k \in \mathbb{Z}$. Finally we get

$$E(b) = \sum_{k=-\infty}^{\infty} \sum_{n=k+1}^{\infty} \frac{S^L_{kn}(b)}{b_{kn}} = \sum_{k=-\infty}^{\infty} \sum_{n=k+1}^{\infty} \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} \frac{1}{a_k^n} =$$

$$\sum_{k=-\infty}^{\infty} a_k \sum_{n=k+1}^{\infty} \left(\frac{1}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} < \sum_{k=-\infty}^{\infty} \frac{a_k}{1-\frac{a_k}{a_{k+1}}} \sum_{n=k+1}^{\infty} \left(\frac{1}{a_n}\right)^{n+1}$$

$$< \sum_{k=-\infty}^{\infty} \frac{a_k}{1-\frac{a_k}{a_{k+1}}} \sum_{n=k+1}^{\infty} \left(\frac{1}{a_{k+1}}\right)^{n+1} = \sum_{k=-\infty}^{\infty} \frac{a_k}{1-\frac{a_k}{a_{k+1}}} \left(\frac{1}{a_{k+1}}\right)^{k+2} \frac{1}{1-\frac{1}{a_{k+1}}} =$$

$$\sum_{k=-\infty}^{\infty} \frac{\frac{a_k}{a_{k+1}}}{\left(1-\frac{a_k}{a_{k+1}}\right)^2} \left(\frac{1}{a_{k+1}}\right)^{k+1}.$$

Example 2. Let us take $b_{kn} = (a_k)^n,\ k, n \in \mathbb{Z}$ where $a_k = s^k,\ k \in \mathbb{Z}$ with $s > 1$.

Conditions (69) hold for $a_k = s^k$. By (70) and (71) we have

$$S_{kn}^L(b) = \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} = \left(\frac{1}{s^{n-k}}\right)^{n+1} \frac{1}{1-\frac{1}{s^{n-k}}} \sim \left(\frac{1}{s^{n-k}}\right)^{n+1},$$

$$S_{kn}^R(b) = \sum_{r=-\infty}^{k-1} a_r^{n-k} = \sum_{r=-\infty}^{k-1} s^{r(n-k)} = \sum_{r=1-k}^{\infty} \frac{1}{s^{r(n-k)}} =$$

$$\left(\frac{1}{s^{n-k}}\right)^{1-k} \frac{1}{1-\frac{1}{s^{n-k}}} \sim s^{(n-k)(k-1)},$$

since

$$1 < \frac{1}{1-\frac{1}{s^{n-k}}} < \frac{1}{1-\frac{1}{s}}.$$

Using the latter equivalence we conclude that $E(b) < \infty$. Indeed we have

$$E(b) = \sum_{k=-\infty}^{\infty} \sum_{n=k+1}^{\infty} \frac{S_{kn}^L(b)}{b_{kn}} \sim \sum_{k=-\infty}^{\infty} \sum_{n=k+1}^{\infty} \frac{1}{s^{(n-k)(n+1)}} \frac{1}{s^{kn}}$$

$$= \sum_{k=-\infty}^{\infty} s^k \sum_{n=k+1}^{\infty} \frac{1}{s^{n(n-1)}} = \sum_{k=-\infty}^{\infty} s^k \sum_{n=0}^{\infty} \frac{1}{s^{(n+k+1)(n+k+2)}}$$

$$< \sum_{k=-\infty}^{\infty} s^k \frac{1}{s^{(k+1)(k+2)}} \sum_{n=0}^{\infty} \frac{1}{s^n} = \frac{1}{1-\frac{1}{s}} \sum_{k=-\infty}^{\infty} \frac{1}{s^{(k+1)^2+1}} < \infty.$$

4. The type III$_1$ factors

Let $\mathfrak{A}^{R,b}$ (resp. $\mathfrak{A}^{L,b}$) be the von Neumann algebra generated by the right regular representation $T^{R,b}$ (resp. the left regular representation $T^{L,b}$) of $B_0^{\mathbb{Z}}$. Moreover, assume that the conditions (69) hold. In this section we want again to prove that $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are type III$_1$ factors.

LEMMA 3.14. *Assume $E(b) < \infty$. The constant function $1 \in L^2(B^{\mathbb{Z}}, d\mu_b)$ is cyclic and separating for $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$.*

PROOF. First we note that if 1 is cyclic for one of the algebras it is necessarily cyclic for its commutant, since $E(b) < \infty$ (by the same argument as in Lemma 2.12 in Chapter 2). Now we prove that 1 is cyclic for $\mathfrak{A}^{R,b}$. We use a method similar to the one in the proof of the ergodicity of the measure μ_b in [**Kos01**], Lemma 4.

For any $m \in \mathbb{Z}$ we define the subgroups B^m and $B_{(m)}$ of the group $B^{\mathbb{Z}}$ as follows:

$$B^m := \{1 + x \in B^{\mathbb{Z}}; x = \sum_{k<n\leq m} x_{kn} E_{kn}\},$$

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ & 1 & x_{m-3m-2} & x_{m-3m-1} & x_{m-3m} & 0 & 0 & \cdots \\ & 0 & 1 & x_{m-2m-1} & x_{m-2m} & 0 & 0 & \cdots \\ & 0 & 0 & 1 & x_{m-1m} & 0 & 0 & \cdots \\ & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ & & & & & & & \ddots \end{pmatrix}$$

$$B_{(m)} := \{1 + x \in B^{\mathbb{Z}}; x = \sum_{k<n, n>m} x_{kn} E_{kn}\}.$$

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ & 1 & 0 & 0 & x_{m-2m+1} & x_{m-2m+2} & x_{m-2m+3} & \cdots \\ & 0 & 1 & 0 & x_{m-1m+1} & x_{m-1m+2} & x_{m-1m+3} & \cdots \\ & 0 & 0 & 1 & x_{mm+1} & x_{mm+2} & x_{mm+3} & \cdots \\ & 0 & 0 & 0 & 1 & x_{m+1m+2} & x_{m+1m+3} & \cdots \\ & 0 & 0 & 0 & 0 & 1 & x_{m+2m+3} & \cdots \\ & & & & & & & \ddots \end{pmatrix}$$

Obviously, $B^{\mathbb{Z}}$ is a semi-direct product of the two groups above ($B_{(m)}$ is a normal subgroup of $B^{\mathbb{Z}}$) for any m:

$$B^{\mathbb{Z}} = B_{(m)} \rtimes B^m.$$

Let $\mu_{b,(m)}, \mu_b^m$ be the projections of the measure μ_b on the above groups:

$$\mu_b^m := \bigotimes\nolimits_{k<n\leq m} \mu_{b_{kn}}, \quad \mu_{b,(m)} := \bigotimes\nolimits_{k<n,n>m} \mu_{b_{kn}}.$$

Furthermore let $B_{0,(m)}, B_0^m$ be the intersection of the above groups with $B_0^{\mathbb{Z}}$.

Now, fix an $m \in \mathbb{Z}$ and consider a function $f \in L^2(B^{\mathbb{Z}}, d\mu_b)$. Further, suppose that

$$(f, a1) = 0, \quad \forall a \in \mathfrak{A}^{R,b}. \tag{72}$$

First we note that the points of $B_{(m)}$ are invariant under the right action R_t for all $t \in B_0^m$. Indeed, we have for $t \in B_0^m$

$$(xt)_{kn} = \sum_{j=k+1}^{n-1} x_{kj} t_{jn} = x_{kn}, \quad n > m,$$

since $t_{kn} = \delta_{kn}$ for $n > m$.

We have for $t \in B_0^m$

$$\begin{aligned} 0 &= (f, T_t^{R,b}1) \\ &= \int\int f(yx) T_t^{R,b}1(yx) d\mu_{b,(m)}(y) d\mu^m(x) \\ &= \int_{B^m} f_m(x) T_t^{R,b}1(x) d\mu_m(x), \end{aligned}$$

where

$$f_m(x) := \int_{B_{(m)}} f(yx) d\mu_{b,(m)}(y).$$

We have

$$(f_m, T_t^{R,b}1)_m = 0, \quad \forall t \in B_0^m, \tag{73}$$

where $(.,.)_m$ denotes the restriction of the inner product $(.,.)$ to $L^2(B^m, d\mu_b^m)$.

Next, we define a bijection $\Psi : B^m \mapsto B_m$, where B_m is the group

$$B_m := \{1 + x; x = \sum_{m \le k < n} x_{kn} E_{kn}\} \cong B^{\mathbb{N}},$$

$$x'_{kn} = (\Psi(x))_{kn} := x_{2m-n2m-k}.$$

Note that Ψ are reflections around the axis $k = -n + m$ and if $m = 0$, $x'_{kn} = x_{-n-k}$. Now we continue with the equation (73):

$$\begin{aligned} 0 &= (f_m, T_t^{R,b}1)_m \\ &= \int_{B^m} f_m(x) \sqrt{\frac{d\mu_b^m(xt)}{d\mu_b^m(x)}} d\mu_b^m(x) \\ &= \int_{B_m} f_m^{\Psi}(x') \sqrt{\frac{d\mu_b^{m,\Psi}(t'x')}{d\mu_b^{m,\Psi}(x')}} d\mu_b^{m,\Psi}(x'), \end{aligned}$$

where $f_m^{\Psi} := f_m \circ \Psi$ and $\mu_b^{m,\Psi}(I) = \mu_b^m(\Psi(I))$ for each Borel set I. Since this holds for all $t \in B_0^m$ and hence all $t' \in B_{0,m}$ and $B_m \cong B^{\mathbb{N}}$,

$$0 = \int_{B^{\mathbb{N}}} f_m^{\Psi}(x) T_t^{L,b}1 d\mu_b(x) = (f^{\Psi}, T_t^{L,b}1)_{\mathbb{N}},$$

where more precisely, f_m^{Ψ} is interpreted as its image under the isomorphism form B_m to $B^{\mathbb{N}}$ and $(.,.)_{\mathbb{N}}$ is the inner product on $L^2(B^{\mathbb{N}}, d\mu_b)$. It also follows (after taking the linear span and weak limits) that

$$(f_m^{\Psi}, a1)_{\mathbb{N}} = 0, \quad \forall a \in \mathfrak{A}^{L,b,\mathbb{N}},$$

where $\mathfrak{A}^{L,b,\mathbb{N}}$ are the algebras from the previous Chapter. But 1 is cyclic for $\mathfrak{A}^{L,b,\mathbb{N}}$, by Lemma 2.12 in Chapter 2 and hence $f_m^{\Psi}(x') = 0$ for all $x' \in B_m$.

Since Ψ is a bijection, also $f_m = 0$ follows. This holds for all $m \in \mathbb{Z}$, because we chose m arbitrarily.

From the definition of f_m follows that $f_m \to f$, when $m \to \infty$ in $L^2(B^{\mathbb{Z}}, d\mu_b)$ (see [**Kos01**], Corollary 1). Thus $f_m = 0$ for all $m \in \mathbb{Z}$ implies $f = 0$. Since f was arbitrary, from equation (72) follows that the set $\mathfrak{A}^{R,b}1$ is dense in $L^2(B^{\mathbb{Z}}, d\mu_b)$ and hence 1 is cyclic for $\mathfrak{A}^{R,b}$.

□

Thus we can define the modular operator and conjugation using the method in Chapter 2, Section 3:

$$\Delta(f)(x) = \frac{d\mu_b(x)}{d\mu_b(x^{-1})} f(x),$$

$$(Jf)(x) = \sqrt{\frac{d\mu_b(x^{-1})}{d\mu_b(x)}}\,\overline{f(x^{-1})},$$

for $f \in L^2(B^{\mathbb{Z}}, d\mu_b)$. Now we state the main theorem of this section.

THEOREM 3.15. *Consider the von Neumann algebra $\mathfrak{A}^{R,b}$ generated by the right regular representation $T^{R,b}$ of $B_0^{\mathbb{Z}}$. Assume that $E(b) < \infty$. Let $\phi(a) = (1, a1)$ be the faithful normal state on $\mathfrak{A}^{R,b}$, associated to the cyclic and separating vector* 1*, and Δ the corresponding modular operator. Then $\mathfrak{A}^{R,b}$ is a type III_1 factor. The same holds for $\mathfrak{A}^{L,b}$.*

First we note that we can not, strait forwardly, implement the method used for the $B_0^{\mathbb{N}}$ case. Of course we can consider the centralizer $\mathfrak{A}_\phi^{L,b}$ which will be equal to

$$\{\Delta^{is}, T_t^{R,b}; s \in \mathbb{R}, t \in B_0^{\mathbb{Z}}\}'.$$

However, since we are dealing with matrices which can also have negative indices, the sum in the formula (57), will go down to $k = -\infty$, instead of $k = 1$. This will imply that the operators (58) and (59) will have infinite sums in the exponential and can not be used to extract the independent variables. Hence we have to find another method in order to prove the above theorem. Luckily, there is another approach to the classification of type III factors, namely using the flow of weights invariant introduced in Chapter 1, definition 1.20.

We define $\mathcal{N}$ to be the crossed product of $\mathfrak{A}^{R,b}$ with $\mathbb{R}$, w.r.t. the modular evolution σ_t. Note that it acts on the Hilbert space $L^2(B^{\mathbb{Z}}, d\mu_b) \otimes L^2(\mathbb{R}, dm) = L^2(B^{\mathbb{Z}} \times \mathbb{R}, d\mu_b \otimes dm)$, where m is the Lebesgue measure on $\mathbb{R}$. Then the non-commutative flow of weights is given by the pair $(\mathcal{N}, \hat{\sigma})$ where $\hat{\sigma}$ is the dual action of the modular group σ (Chapter 1, Definition 1.17).

In what follows, we shall prove that the flow of weights $(\mathcal{C}_{\mathcal{N}}, \theta)$ is trivial. This implies that $\mathcal{N}$ is a factor. First of all, since $\mathcal{C}_{\mathfrak{A}^{R,b}} = \mathcal{C}_{\mathcal{N}}^{\theta}$, where $\theta = \hat{\sigma}$ (Theorem 1.19 in Chapter 1), the triviality of $\mathcal{C}_{\mathcal{N}}$ implies that $\mathfrak{A}^{R,b}$ (and hence also $\mathfrak{A}^{L,b}$) is a factor. Moreover, by Theorem 1.21 in Chapter 1, $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are of type III$_1$.

REMARK 3.16. To be more precise, the factor property of $\mathcal{N}$ implies that the factor $\mathfrak{A}^{R,b}\overline{\otimes}B(L^2(\mathbb{R}))$ is of type III$_1$. Indeed, from the definition of $\mathcal{N}$ above and the duality theorem of Connes and Takesaki stated in Chapter 1 (theorem 1.18), follows that $\mathcal{N} \rtimes_\theta \mathbb{R} \cong \mathfrak{A}^{R,b}\overline{\otimes}B(L^2(\mathbb{R}))$. However, the same must hold for $\mathfrak{A}^{R,b}$ as well, since a type III factor can not be obtained from a non-type III by tensoring it with a type I factor.

Let W be the operator defined by

$$Wf(x,t) = \Delta^{-it}(x)f(x,t). \tag{74}$$

Then $\pi_\sigma(\mathfrak{A}^{R,b}) = W\mathfrak{A}^{R,b}W^*$ and $\pi_\sigma(\rho) = W\rho W^*$. Therefore

$$\mathcal{N} := \mathfrak{A}^{R,b} \rtimes_\sigma \mathbb{R} = (W\mathfrak{A}^{R,b}W^* \cup \lambda(\mathbb{R}))''.$$

By Theorem 1.14 in Chapter 1 (here we set $\{\rho, V, \mathcal{K}\} = \{1, \Delta^{it}, \mathcal{H}\}$), and since $\mathcal{C}_\mathcal{N} = \mathcal{N} \cap \mathcal{N}' = (\mathcal{N}' \cup \mathcal{N})'$, we have:

$$\mathcal{C}'_\mathcal{N} = (\pi_\sigma(\mathfrak{A}^{R,b}) \cup \mathfrak{A}^{L,b} \cup \lambda(\mathbb{R}) \cup \pi_\sigma(\rho(\mathbb{R})))'' \tag{75}$$

Again, we would like to prove the triviality of $\mathcal{C}_\mathcal{N}$ by showing that the independent variables x_{kn} and now also t, in the space $L^2(B^{\mathbb{Z}} \times \mathbb{R}, d\mu_b \otimes dm)$, are affiliated to $\mathcal{C}'_\mathcal{N}$. This, by Proposition 2.18, would imply the triviality of $\mathcal{C}_\mathcal{N}$.

LEMMA 3.17. *Let Q_{kn} and Q_t be the multiplication operators*

$$\begin{aligned} (Q_{kn}f)(x,t) &:= x_{kn}f(x,t), \\ (Q_t f)(x,t) &:= tf(x,t), \quad f \in L^2(B^{\mathbb{Z}} \times \mathbb{R}, d\mu_b \otimes m_t). \end{aligned}$$

Then

$$e^{iQ_{kn}s}, e^{iQ_t s} \in \mathcal{C}'_\mathcal{N},$$

for all $s \in \mathbb{R}$, $k < n \in \mathbb{Z}$.

PROOF. From (75) we see that $\mathcal{C}'_\mathcal{N}$ contains the following set of elements:

$$\left(WT_u^{R,b}W^*, T_u^{L,b}, \lambda(s), W\rho(s)W^*; u \in B_0^{\mathbb{Z}}, s \in \mathbb{R}\right). \tag{76}$$

Now again there are two possible methods to prove the Lemma. As already discussed in the proof of Lemma 2.19 from Chapter 2, using generators of one-parameter groups and their commutators needs extra attention when it comes to dealing with domains of unbounded operators. Hence we shall use the second approach for the proof of the Lemma. First, we shall prove some intermediate results. Denote $T_{pq}^{L,b}(s) := T_{1+sE_{pq}}^{L,b}$, $T_{pq}^{R,b}(s) := T_{1+sE_{pq}}^{R,b}$ and $V_{pq}(s) := WT_{1+sE_{pq}}^{R,b}W^*$.

LEMMA 3.18. *Let $U_{rm}(s) \in \mathcal{C}'_{\mathcal{N}}$ be the operators defined by*

$$\left(U_{rm}(s)f\right)(x,t) := \left(T^{L,b}_{rm+1}(-1)V^{R,b}_{mm+1}(s)T^{L,b}_{rm+1}(1)V^{R,b}_{mm+1}(-s)f\right)(x,t),$$

where $f \in L^2(B^{\mathbb{Z}} \times \mathbb{R}, d\mu_b \otimes dm)$ and $s \in \mathbb{R}$. Then

$$\left(U_{rm}(s)f\right)(x,t) = \exp\left(istx_{rm}\right) f(x,t), \forall t,s \in \mathbb{R}, x \in B^{\mathbb{Z}}.$$

Thus, by applying $U_{rm}(-s)Ad_{\lambda(1)}$ and $U_{rm}(-s)Ad_{T^{L,b}_{rm}(1)}$ to $U_{rm}(s)$ one obtains one-parameter groups generated by the independent variables t and $x_{rm}, r < m \in \mathbb{Z}$, where $Ad_T(X) = TXT^{-1}$.

PROOF. We begin by showing that $U_{rm}(s)$ only involves transforms of Δ^{it}. Fix $s \in \mathbb{R}$ and define:

$$\begin{aligned} x' &:= x(1+sE_{mm+1}), \\ {}'x &:= (1+E_{rm+1})x, \\ {}'x' &:= (1+E_{rm+1})x(1+sE_{mm+1}). \end{aligned}$$

We have $(U_{rm}(s)f)(x,t) =$

$$\left(T^{L,b}_{rm+1}(-1)WT^{R,b}_{mm+1}(s)W^*T^{L,b}_{rm+1}(1)WT^{R,b}_{mm+1}W^*f\right)(x,t)$$

$$= \sqrt{\tfrac{d\mu_b('x)}{d\mu_b(x)}}\Delta^{-it}('x)\sqrt{\tfrac{d\mu_b('x')}{d\mu_b('x)}}\Delta^{it}('x')\sqrt{\tfrac{d\mu_b(x')}{d\mu_b('x')}}\Delta^{-it}(x')\sqrt{\tfrac{d\mu_b(x)}{d\mu_b(x')}}\Delta^{it}(x)f(x,t)$$

$$= \Delta^{-it}('x)\Delta^{it}('x')\Delta^{-it}(x')\Delta^{it}(x)f(x,t).$$

Recall that

$$-\ln\Delta(x) = \sum_{k,n\in\mathbb{Z},k<n} b_{kn}w_{kn}(x).$$

The formal computation of $\Delta^{-it}(x')$ is the same as in Lemma 2.24 in Chapter 2, except that the sums here are also infinite in the negative direction. Hence

$$\Delta^{it}(x)\Delta^{-it}(x') =$$

$$\exp\left(it\sum_{k=-\infty}^{m-1} b_{km+1}(2sx_{km}x_{km+1} + s^2(x_{km})^2\right) \tag{77}$$

$$+it\sum_{n=m+2}^{\infty} b_{mn}\left(2sx^{-1}_{mn}x^{-1}_{m+1n} - s^2(x^{-1}_{m+1n})^2\right)$$

The next step is to compute $\Delta^{-it}('x)\Delta^{it}('x')$. Hence we have to compute the left action of the one-parameter groups $G_{pq}(s), s \in \mathbb{R}$ on $X \in B^{\mathbb{Z}}$, $X = 1+x$.

$$\left(G_{pq}(s)X\right)_{kn} = \sum_{r=k+1}^{n-1}\left(\delta_{kn} + s\delta_{pk}\delta_{qr}\right)\left(x_{rn} + \delta_{rn}\right) = x_{kn} + s\delta_{kp}\left(x_{qn} + \delta_{qn}\right). \tag{78}$$

In order to caculate $('x)^{-1}$ we note that $\left(G_{pq}(s)X\right)^{-1} = X^{-1}G_{pq}(-s)$ ($G_{pq}(s)$ are one-parameter groups). Thus

$$('x)^{-1}_{kn} = x^{-1}_{kn} - s\delta_{qn}(x^{-1}_{kp} + \delta_{kp}) \tag{79}$$

In order to get $\Delta^{-it}('x)\Delta^{it}('x')$, we have to substitute $'x$ for x and $-t$ for t in formula (77). According to equation (78) only the row with number r of x is affected by the left $G_{rm+1}(1)$-action. Moreover, $(G_{pq}(s)X)_{kn} = X_{kn}$, for $q > n$. Similarly, only the column with number $m+1$ of the inverse of $G_{rm+1}(1)X$ is affected. Hence we obtain

$$\Delta^{-it}('x)\Delta^{it}('x') =$$

$$\exp\{-it\sum_{k=-\infty}^{m-1} b_{km+1}(2s('x)_{km}('x)_{km+1} + s^2(('x)_{km})^2)$$

$$-it\sum_{n=m+2}^{\infty} b_{mn}(2s('x)^{-1}_{mn}('x)^{-1}_{m+1n} - s^2(('x)^{-1}_{m+1n})^2\} =$$

$$\exp\{-it\sum_{k=-\infty}^{m-1} b_{km+1}(2sx_{km}x_{km+1} + s^2(x_{km})^2)$$

$$-it\sum_{n=m+2}^{\infty} b_{mn}(2sx^{-1}_{mn}x^{-1}_{m+1n} - s^2(x^{-1}_{m+1n})^2)$$

$$-2isb_{rm+1}tx_{rm}\}.$$

Therefore

$$\left(U_{rm}(s)f\right)(x,t) = e^{-2ib_{rm+1}stx_{rm}}f(x,t).$$

Since the left regular representation of $\mathbb{R}$, $\lambda(s)$, is in $\mathcal{C}'_{\mathcal{N}}$ for all $s \in \mathbb{R}$ we can translate the operator $U_{rm}(s)$ by 1 and obtain the variables x_{rm}. Similarly, we can use $T^{L,b}_{rm}(1)$ to get the variable t:

$$\left(U_{rm}(s)\lambda(1)U_{rm}(-s)\lambda(-1)f\right)(x,t) = e^{isb_{rm+1}x_{rm}}f(x,t), \tag{80}$$

$$\left(U_{rm}(s)T^{L,b}_{rm}(1)U_{rm}(-s)T^{L,b}_{rm}(-1)f\right)(x,t) = e^{isb_{rm+1}t}f(x,t). \tag{81}$$

□

From the equations above we see that (after rescaling the parameter) the one-parameters groups generated by multiplications with the independent variables $x_{rm}, r, m \in \mathbb{Z}, r < m$ and t are contained in $\mathcal{C}'_{\mathcal{N}}$. Hence $Q_{rm}, Q_t \;\eta\; \mathcal{C}'_{\mathcal{N}}$, which proves Lemma 3.17. □

Again, we note that $T^{L,b}_t \in \mathcal{C}'_{\mathcal{N}}$ for all $t \in B_0^{\mathbb{Z}}$ and that the measure μ_b is ergodic (this is implied by the condition $E(b) < \infty$, see Remark 3.10). From Lemma 3.17 above and Proposition 2.18 in Chapter 2 follows that $\mathcal{C}_{\mathcal{N}} = \mathbb{C}.\mathbf{1}$. Hence $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are type III$_1$ factors.

5. Uniqueness of the constructed factor

THEOREM 3.19. *The von Neumann algebras $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are hyperfinite type III$_1$ factors and hence isomorphic to the factor R_∞ of Araki and Woods.*

PROOF. See Theorem 2.25 from Chapter 2 (and replace $B_0^{\mathbb{N}}$ by $B_0^{\mathbb{Z}}$ there). □

APPENDIX A

Locally convex Topologies on $B(\mathcal{H})$

In this appendix we give a quick review of different locally convex topologies on $B(\mathcal{H})$. For more details and proofs we refer to e.g. [**BR79**]. Recall that a locally convex topology on a vector space X is defined in terms of a family of semi-norms, that are real valued functions p on X satisfying

$$\begin{aligned} p(a+b) &\leq p(a)+p(b), \quad a,b \in X, \\ p(\lambda a) &= \lambda p(a), \quad \lambda \geq 0 \end{aligned}$$

Let $\{p_i\}$ be a family of semi-norms. Then the neighborhoods of zero that define the locally convex topology are give by

$$N(p_1,...,p_n;0) := \{x \in X; p_i(x) < 1, i = 1,...,n\}.$$

The existence of continuous functionals in such a topology is guaranteed by the well-known Hahn-Banach theorem (e.g. [**Rud73**], Theorem 3.2).

THEOREM A.1 (Hahn-Banach). *Let X be a real vector space and p a semi-norm. Further, let Y be a subspace of X and f a real-valued functional on Y satisfying*

$$f(a) \leq p(a), \quad a \in Y$$

It follows that f has a real linear extension F to X such that

$$F(a) \leq p(a), \quad a \in X.$$

Now we define a number of locally convex topologies on $B(\mathcal{H})$. They are the *strong, σ-strong, weak, σ-weak* and the *uniform* topologies. The uniform topology is of course the norm topology defined by the norm $\|a\| = \sup_{\|\xi\|=1} \|a\xi\|$. We shall discuss these topologies from a practical point of view needed to study von Neumann algebras. There is a beautiful theory of locally convex topologies on vector spaces, a treatise of which can be found in [**Gro73**].

Strong and σ-strong topologies. If ξ is in $\mathcal{H}$ then $p_\xi : a \mapsto \|a\xi\|$ is a semi-norm on $B(\mathcal{H})$. The family of semi-norms $\{p_\xi\}_{\xi\in\mathcal{H}}$ defines a locally convex topology called the *strong* topology on $B(\mathcal{H})$.

Now consider the function

$$p_{\xi_n} : a \mapsto \left(\sum_n \|a\xi_n\|^2 \right)^{1/2}, \tag{82}$$

where $\{\xi_n\} \in l^2(\mathcal{H})$. The topology defined by $\{p_{\xi_n}\}$ is called the *σ-strong* topology. The latter topology is finer than the strong one, but they coincide on the unit sphere of $B(\mathcal{H})$.

Weak and σ-weak topologies. To define the next two topologies we consider on one hand pairs of vectors (η, ξ) and on the other hand sequences $(\{\eta_n\}, \{\xi_n\})$, where $\eta, \xi \in \mathcal{H}$ and $\{\eta_n\}, \{\xi_n\} \in l^2(\mathcal{H})$. Then we define the *weak* and *σ-weak* topologies by the semi-norms:

$$\begin{aligned} \text{Weak:} \quad & p_{\eta,\xi}(a) && = (\eta, a\xi) \\ \sigma\text{-weak:} \quad & p_{\eta_n,\xi_n}(a) && = \sum_n |(\eta_n, a\xi_n)| \end{aligned}$$

Again, the σ-weak topology is finer than the weak, but they coincide on the unit sphere, which is then compact in these topologies. Moreover, the mappings $a \mapsto ab$, $a \mapsto ba$ and $a \mapsto a^*$ are continuous in the weak topology, but the multiplication is not jointly continuous (for $\mathcal{H}$ infinite-dimensional).

If we consider a locally convex space X, there is a dual space X^* associated to it. We can then introduce a topology on X^* coming from the X topology. In such a way we define the *weak** topology on X^* by the neighborhoods of a point $\omega \in X^*$

$$N(\omega, a_1, ..., a_n, \epsilon) := \{\omega' \in X^*; |\omega'(a_i) - \omega(a_i)| < \epsilon, i = 1, 2, ..., n\},$$

for $a_1, a_2, ..., a_n \in X$ and $\epsilon > 0$.

One can prove the following duality result:

THEOREM A.2 ([**BR79**]). *Let Tr be the usual trace on $B(\mathcal{H})$, and let $B^1(\mathcal{H})$ be the Banach space of trace-class operators on $\mathcal{H}$ equipped with the norm $\|.\|_{Tr} : t \mapsto Tr(|t|)$. Then it follows that $B(\mathcal{H})$ is the dual $B^1(\mathcal{H})^*$ of $B^1(\mathcal{H})$ by the duality*

$$a \times t \in B(\mathcal{H}) \times B^1(\mathcal{H}) \mapsto Tr(at).$$

The weak topology on $B(\mathcal{H})$ arising from this duality is just the σ-weak topology.*

DEFINITION A.3. The space $B^1(\mathcal{H})$ is called the *predual* of $B(\mathcal{H})$ and is denoted by $B(\mathcal{H})_*$. By the above theorem it is the space of all σ-weakly continuous functionals. Moreover, $(B(\mathcal{H})_*)^* = B(\mathcal{H})$.

Although, the definitions of the above topologies are different, they give rise to the same closures of *-subalgebras of $B(\mathcal{H})$ (except the uniform topology). A *-subalgebra of $B(\mathcal{H})$, closed in the uniform topology, is called a (concrete) C^*-algebra. A weakly closed *-subalgebra of $B(\mathcal{H})$ is called a *von Neumann algebra*. Moreover the following theorem, also refered to as the von Neumann bicommutant theorem, holds:

THEOREM A.4. *Let $\mathcal{M}$ be a *-subalgerba of $B(\mathcal{H})$. Then the following are equivalent*

(1) $\mathcal{M}'' = \mathcal{M}$,

(2) $\mathcal{M}$ *is weakly closed,*
(3) $\mathcal{M}$ *is strongly closed,*
(4) $\mathcal{M}$ *is σ-weakly closed,*
(5) $\mathcal{M}$ *is σ-strongly closed,*

where $\mathcal{M}' := \{a \in B(\mathcal{H}); am = ma, \forall m \in \mathcal{M}\}$ *is the commutant of* $\mathcal{M}$*.*

The theorem above allows us to give an algebraic definition of a von Neumann algebra, namely a *-subalgebra $\mathcal{M}$ of $B(\mathcal{H})$ which is closed under taking the double commutant:

$$\mathcal{M}'' = \mathcal{M}.$$

APPENDIX B

Spectral theory of Automorphism Groups

In this Appendix we review some Spectral theory of automorphism groups, which is needed for this dissertation. A detailed discussion can be found in e.g. [**Tak03a**] Chapter XI. The following discussion is mainly based on [**Bau95**], Section 1.8 .

First we define the spectrum of a unitary representation of a locally compact group, also called the *Arveson spectrum.* Let G be a locally compact group, $\mathcal{M}$ a von Neumann algebra on $\mathcal{H}$ and α an automorphic action of G on $\mathcal{M}$. Then we define the Arveson spectrum as follows:

$$Sp(\alpha_G) := \{\xi \in \hat{G};\ \text{if } \alpha(f) = 0 \text{ then } \hat{f}(\chi) = 0, \forall f \in L^1(G)\},$$

where

$$\hat{f}(\chi) := \int_G f(g)\chi(g)dg,$$
$$\alpha(f) := \int f(g)\alpha_g dg$$

Hence, $\alpha(f)(a) = \int f(g)\alpha_g(a)dg, a \in \mathcal{M}$ and $\alpha(f) = 0$ means that $\alpha(f)(a) = 0$ for all $a \in \mathcal{M}$.

Assume that α is implemented by unitary operators $U(g)$. Let $\eta \in \mathcal{H}$ be cyclic and separating w.r.t. $\mathcal{M}$ and assume $U(g)\eta = \eta$ for all $g \in G$. In this case the definition of the spectrum above becomes that of the spectrum of $U(G)$, which is just the support of the spectral measure $E(.)$ of $U(G)$ where

$$U(g) := \int_{\hat{G}} \chi(g)E(d\chi).$$

Moreover $Sp(\alpha_G)$ has also the following properties:

(1) $j \in Sp(\alpha_G)$, where j is the unit character of G, $j(g) = 1$ for all $g \in G$,
(2) $Sp(\alpha_G) = (Sp(\alpha_G))^{-1}$.

Assume now that we are dealing with the group $G = \mathbb{R}$ and α is the modular automorphism group of $\mathcal{M}$ w.r.t. a cylic and separating vector η. We introduce two new concepts, which are important for the study of Arveson spectrum of $\alpha_{\mathbb{R}}$. They are respectively *the spectrum of an element of* $\mathcal{M}$ w.r.t. α and that of spectral subspaces in $\mathcal{M}$ w.r.t. α and depending on a Borel set in $\mathbb{R}$:

$$\begin{aligned} Sp_\alpha(m) &:= \{\lambda \in \mathbb{R}; \alpha(f) = 0 \Rightarrow \hat{f}(\lambda) = 0, f \in L^1(\mathbb{R}, dt)\}, \quad m \in \mathcal{M} \\ M(\alpha, I) &:= \{m \in \mathcal{M}; Sp_\alpha(m) \subseteq I\}, \quad I \subset \mathbb{R}. \end{aligned}$$

$Sp_\alpha(m)$ has the following properties:

(1) For each $m \in \mathcal{M}$, $Sp_\alpha(m) = supp(m\eta, E(.)m\eta)$,
(2) $Sp(\alpha) = \overline{\{\cup_{m\in\mathcal{M}} Sp_\alpha(m)\}}$
(3) One has $Sp_\alpha(m) = Sp_\alpha(\alpha_t(m))$ for all $t \in \mathbb{R}$,

where $E(.)$ is the projection valued spectral measure of Δ^{it}.

We recall the concept of spectral subspaces $E(I)\mathcal{H}$ of a unitary group $U(\mathbb{R})$ depending on a Borel set $I \subset \mathbb{R}$. The spectral subspaces in a von Neumann algebra defined above are closely related to the latter:

$$M(\alpha, I) = \{m \in \mathcal{M}; m\eta \in E(I)\mathcal{H}\}. \tag{83}$$

Indeed, we have

$$\begin{aligned} m \in M(\alpha, I) \quad &\Leftrightarrow \quad supp(m\eta, E(.)m\eta) \subseteq I \\ &\Leftrightarrow \quad \int_{\mathbb{R}\setminus I} (m\eta, dE(\lambda)m\eta) = 0 \\ &\Leftrightarrow \quad \|(1 - E(I))m\eta\|^2 = 0. \end{aligned}$$

Thus, $E(I)m\eta = m\eta$ and this means $m\eta \in E(I)\mathcal{H}$.

The spectral subspaces of α are related to the spectrum of α as follows:

Lemma B.1. *The following holds:* $\lambda \in Sp(\alpha)$ *if and only if* $\{0\} \subsetneq M(\alpha, V(\lambda))$ *for each open neighborhood* $V(\lambda)$ *of* λ.

For a set $I \subset \mathbb{R}$ that does not intersect $Sp(\alpha)$, $M(\alpha, I) = \{0\}$ holds. Indeed, in the this case $E(I) = 0$ and hence $(m\eta, E(I)m\eta) = 0$ for all $m \in \mathcal{M}$ and hence there is no $m \neq 0$ such that $supp(m\eta, E(.)m\eta) \subseteq I$.

Moreover, for a single point λ in $Sp(\alpha)$, which is isolated $\{0\} \subsetneq M(\alpha, \{\lambda\})$ still holds. This follows from the fact that $\{\lambda\}$ is the intersection of all the open sets containing it, equation (83) and the lemma above. Finally we establish the following corollary.

Corollary B.2. *If* λ *is an isolated point of* $Sp(\alpha)$*, there is* $0 \neq m \in \mathcal{M}$ *such that*

$$\alpha_t(m) := \Delta^{it} m \Delta^{-it} = e^{it\lambda} m, \quad t \in \mathbb{R}$$

Proof. By the above considerations we have that $\{0\} \subsetneq M(\alpha, \{\lambda\})$, i.e. there is an $0 \neq m \in \mathcal{M}$ such that $m\eta \in E(\{\lambda\})\mathcal{H}$. But $E(\{\lambda\})\mathcal{H}$ is the subspace of all eigenvectors of $\ln \Delta$, w.r.t. λ. Hence

$$\Delta^{it} m \Delta^{-it} \eta = \Delta^{it} m \eta = e^{it\lambda} m\eta.$$

Now, $\Delta^{it} m \Delta^{-it} \in \mathcal{M}$ for all $t \in \mathbb{R}$ and since η is separating for $\mathcal{M}$ (which is equivalent to $a\eta = o \Rightarrow a = 0$), the proof is concluded. □

To conclude this Appendix we mention a sufficient condition for the additivity of $Sp(\alpha)$ (which is an important fact for the classification of type III factors).

THEOREM B.3. *Let* $(\mathcal{M}, \eta)$ *be as above and* α *an automorphic action of* $\mathbb{R}$ *on* $\mathcal{M}$. *Assume that* α *acts ergodically on* $\mathcal{M}$, *i.e.* $\mathcal{M}^\alpha = \mathbb{C}.\mathbf{1}$. *Then* $Sp(\alpha)$ *is additive, i.e.* $\lambda_1, \lambda_2 \in Sp(\alpha)$ *implies* $\lambda_1 + \lambda_2 \in Sp(\alpha)$.

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