The Analytic Torsion of Manifolds with Boundary and Conical Singularities

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Analytic Torsion of manifolds with boundary and conical singularities

Abstract

The analytic torsion was introduced by D.B. Ray and I.M. Singer as an analytic counterpart to the combinatorial Reidemeister torsion. In this thesis we are concerned with analytic torsion of manifolds with boundary and conical singularities. Our work is comprised basically of three projects.

In the first project we discuss a specific class of regular singular Sturm Liouville operators with matrix coefficients. Their zeta determinants were studied by K. Kirsten, P. Loya and J. Park on the basis of the Contour integral method, with general boundary conditions at the singularity and Dirichlet boundary conditions at the regular boundary.

Our main result in the first project is the explicit verification that the Contour integral method indeed applies in the regular singular setup, and the generalization of the zeta determinant computations by Kirsten, Loya and Park to generalized Neumann boundary conditions at the regular boundary. Moreover we apply our results to Laplacians on a bounded generalized cone with relative boundary conditions.

In the second project we derive a new formula for analytic torsion of a bounded generalized cone, generalizing the computational methods of M. Spreafico and using the symmetry in the de Rham complex, as established by M. Lesch. We evaluate our result in lower dimensions and further provide a separate computation of analytic torsion of a bounded generalized cone over S^1, since the standard cone over the sphere is simply a flat disc.

Finally, in the third project we discuss the refined analytic torsion, introduced by M. Braverman and T. Kappeler as a canonical refinement of analytic torsion on closed manifolds. Unfortunately there seems to be no canonical way to extend their construction to compact manifolds with boundary.

We propose a different refinement of analytic torsion, similar to Braverman and Kappeler, which does apply to compact manifolds with and without boundary. We establish a gluing formula for our construction, which in fact can also be viewed as a gluing law for the original definition of refined analytic torsion by Braverman and Kappeler.

Contents

1 Introduction

1.1 The Reidemeister-Franz Torsion

Torsion invariants for manifolds which are not simply connected were introduced by K. Reidemeister in [Re1, Re2] and generalized to higher dimensions by W. Franz in [Fr]. Using the introduced torsion invariants the authors obtained a full PL-classification of lens spaces. The Reidemeister-Franz torsion, short − the Reidemeister torsion, was the first invariant of manifolds which was not a homotopy invariant.

The Reidemeister-Franz definition of torsion invariants was extended later to smooth manifolds by J. H. Whitehead in [Wh] and G. de Rham in [Rh]. With their construction G. de Rham further proved that a spherical Clifford-Klein manifold is determined up to isometry by its fundamental group and its Reidemeister torsion.

The Reidemeister-Franz torsion is a combinatorial invariant and can be constructed using a cell-decomposition or a triangulation of the underlying manifold. The combinatorial invariance under subdivisions was established by J. Milnor in [Mi], see also [RS]. It is therefore a topological invariant of M, however not a homotopy invariant.

There is a series of results relating combinatorial and analytic objects, among them the Atiyah-Singer Index Theorem. In view of these results it is natural to ask for the analytic counterpart of the combinatorial Reidemeister torsion. Such an analytic torsion was introduced by D. B. Ray and I. M. Singer in [RS] in form of a weighted product of zeta-regularized determinants of Laplace operators on differential forms.

The zeta-regularized determinant of a Laplace Operator is a spectral invariant which very quickly became an object of interest on its own in differential and conformal geometry, studied in particular as a function of metrics for appropriate geometric operators. Further it plays a role in mathematical physics where it gives a regularization procedure of functional path integrals (partition function), see [H].

In their work D.B. Ray and I. M. Singer provided some motivation why the analytic torsion should equal the combinatorial invariant. The celebrated Cheeger-Müller Theorem, established independently by J. Cheeger in [Ch] and W. Müller in [Mu1], proved equality between the analytic Ray-Singer torsion and the combinatorial Reidemeister torsion for any smooth closed manifold with an orthogonal representation of its fundamental group.

The proofs of J. Cheeger and W. Müller use different approaches. The first author in principle studied the behaviour of the Ray-Singer torsion under surgery. The second author used combinatorial parametrices and approximation theory of Dodziuk [Do] to reduce the problem to trivial representations, treating this problem then by surgeries.

Note a different approach of Burghelea-Friedlander-Kappeler in [BFK] and Bismut-Zhang in [BZ1], who obtained a new proof of the result by J. Cheeger and W. Müller using Witten deformation of the de Rham complex via a Morse function.

The study of the analytic torsion of Ray and Singer has taken the following natural steps. The setup of a closed Riemannian manifold with its marking point − the Cheeger Müller Theorem, was followed by the discussion of compact manifolds with smooth boundary. In the context of smooth compact manifolds with boundary a Cheeger-Müller type result was established in the work of W. Lück [Lü] and S. Vishik [V].

While the first author reduced the discussion to the known Cheeger-Müller Theorem on closed manifolds via the closed double construction, the second author gave an independent proof of the Cheeger-Müller Theorem on smooth compact manifolds with and without boundary by establishing gluing property of the analytic torsion.

Both proofs work under the assumption of product metric structures near the boundary. However by the anomaly formula in [DF] the assumption of product metric structures can be relaxed.

1.2 Functional Determinants on a Generalized Cone

The next natural step in the study of analytic torsion is the treatment of Riemannian manifolds with singularities. We are interested in the simplest case, the conical singularity. The analysis and the geometry of spaces with conical singularities were developped in the classical works of J. Cheeger in [Ch1] and [Ch2]. This setup is modelled by a bounded generalized cone $M=(0,R]\times N, R>0$ over a closed Riemannian manifold (N,g^N) with the Riemannian metric

$$g^M=dx^2\oplus x^2g^N.$$

In Section 3 we study natural boundary conditions for Laplacians on differential forms, relevant in the context of analytic torsion. These are the relative or the absolute boundary conditions, arising from the maximal and minimal closed extensions of the exterior derivative, see [BL1, Section 3]. In the case of a model cone they are given at the cone base explicitly by a combination of Dirichlet and generalized Neumann boundary conditions.

The study of the relative boundary conditions at the cone singularity is interesting on its own. In [BL2], among other issues, the relative extension of the Laplacian on differential forms is shown to coincide with the Friedrich's extension at the cone singularity outside of the "middle degrees". We discuss the relative boundary conditions for Laplace operators on differential forms in any degree and obtain explicit results, relevant for further computations.

The main ingredient of the Ray-Singer analytic torsion is the zeta-regularized determinant of a Laplace operator. For the computation of zeta-regularized or so-called "functional" determinants of de Rham Laplacians on a bounded generalized cone it is necessary to note that the Laplacian admits a direct sum decomposition

$$\triangle = L \oplus \widetilde{\triangle},$$

which is compatible with the relative boundary conditions, such that $\widetilde{\triangle}$ is the maximal direct sum component, subject to compatibility condition, which is essentially self-adjoint at the cone singularity.

The direct sum component $\widetilde{\triangle}$ is discussed by K. Kirsten and J.S. Dowker in [DK] and [DK1] with general boundary conditions of Dirichlet and Neumann type at the cone base. The other component L is a differential operator with matrix coefficients and is addressed by K. Kirsten, P. Loya and J. Park in [KLP1] with general boundary conditions at the cone singularity but only with Dirichlet boundary conditions at the cone base.

The argumentation of Kirsten, Loya and Park in the preprints [KLP1] and [KLP2] is based on the Contour integral method, which gives a specific integral representation of the zeta-function. A priori the Contour integral method need not to apply in the regular-singular setup and is only formally a consequence of the Argument Principle.

One of the essential results of the Section 4 is the proof that the Contour integral method indeed applies in the regular-singular setup. Our proof is the basis for the integral representation of the zeta-function. Otherwise the Contour integral method would only give information on the pole structure of the zeta-function, but no results on the zeta-determinants.

In this thesis the proof is provided in the setup of generalized Neumann boundary conditions at the cone base, however for Dirichlet boundary conditions the arguments are similar. The author intends to publish the proof for the applicability of the Contour integral method in the regular-singular setup with Dirichlet boundary conditions as an appendix to [KLP2].

In view of the explicit form of the relative boundary conditions for the Laplace operator on differential forms, we extend in Section 4 the computations of [KLP1] to the setup of generalized Neumann boundary conditions at the cone base. Then, using the results of Section 3, we provide finally an explicit result for the functional determinant of L with relative boundary conditions.

1.3 Analytic Torsion of a Generalized Cone

The analytic Ray-Singer Torsion is defined as a weighted alternating product of functional determinants of Laplacians on differential forms, with relative or absolute boundary conditions. It is shown in [Dar] to exist on a bounded generalized cone. Unfortunately the methods of Section 4 for the calculation of functional determinants apply only in a finite-dimensional setup, so we could not continue to compute the analytic torsion on the basis of this approach.

In the actual computation of the analytic torsion of a bounded generalized cone in Section 5 we use the approach of M. Spreafico [S] together with an observation of symmetry in the de Rham complex by M. Lesch in [L3]. Moreover we apply some computational ideas of K. Kirsten, J.S. Dowker in [DK]. The computation is performed for simplicity under an additional assumption of a scaled metric g^M, such that the form-valued Laplacians are essentially self-adjoint at the cone singularity.

This apparent gap can be considered as closed by the preceeding discussion of the finite-dimensional parts of the Laplacians which are not essentially self-adjoint at the cone singularity and naturally appear in the general case.

Our explicit calculation of the analytic torsion of a bounded generalized cone can be viewed as an attempt towards a Cheeger-Müller Theorem for compact manifolds with conical singularities. Further details on this issue are provided in Subsection 5.8.

1.4 Refined Analytic Torsion and a Gluing Formula

Finally in Section 6 we turn our attention to a recent project of M. Braverman and T. Kappeler [BK1, BK2] − the refinement of the analytic torsion. In fact the Ray-Singer analytic torsion can be viewed as a norm on a determinant line. The refinement is a canonical construction of an element in the determinant line with the Ray-Singer norm one. The complex phase of the element is given by a rho-type invariant of the odd-signature operator.

The construction of Braverman and Kappeler essentially relies on Poincare duality on closed manifolds and hence unfortunately does not directly apply to manifolds with boundary. In this thesis we propose a refinement of the analytic torsion, in the spirit of Braverman and Kappeler, which does apply to compact manifolds with boundary.

An interesting feature of the analytic Ray-Singer Torsion is its nice behaviour under cut and paste operations, as established by S. Vishik in [V] for trivial representations, see also [Lü]. This "gluing property" is particularly surprising in view of the non-locality of higher spectral invariants. Such a feature of the torsion invariant is in many aspects an advantage, especially for computational reasons.

In view of the gluing property of analytic torsion, we derive in Section 7 using the Cheeger-Müller Theorem a gluing formula for our construction, which was natural to expect, since a refinement of the analytic torsion should resemble the central properties of the original construction. In particular we deduce a nice gluing formula for the scalar analytic torsion. In fact our result can also be viewed as a gluing formula for the original refined analytic torsion in the sense of Braverman and Kappeler.

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This work is dedicated to my mother Margarita Vertman in appreciation of her love, support and encouragement.

2 Regular-Singular Model Operators

In this section we discuss closed extensions of regular-singular model operators, (cf. [BS]) and self-adjoint extensions of the associated regular-singular model Laplacians. The explicit identification of the relevant domains is used later in the computation of functional determinants on a bounded generalized cone. The arguments and results of this section are well-known, however we give a balanced overview and adapt the presentation to later applications. The presented calculations go back to J. Brüning, R.T. Seeley in [BS] and J. Cheeger in [Ch1] and [Ch2]. For further reference see mainly [W], [BS] and [KLP1], but also [C] and [M].

2.1 Closed Extensions of Model Operators

Let $D: C_0^\infty(0,R) \to C_0^\infty(0,R), R>0$ be a differential operator acting on smooth $\mathbb{C}$-valued functions with compact support in $(0,R)$. The standard Hermitian scalar product on $\mathbb{C}$ and the standard measure dx on $\mathbb{R}$ define the natural L^2-structure on $C_0^\infty(0,R)$:

$$\forall f,g \in C_0^\infty(0,R): \ \langle f,g\rangle_{L^2} := \int_0^R \langle f(x), g(x)\rangle dx.$$

Denote the completion of $C_0^\infty(0,R)$ under the L^2-scalar product by $L^2(0,R)$. This defines a Hilbert space with the natural L^2-Hilbert structure.

We define the maximal extension $D_{\max}$ of D by

$$\mathcal{D}(D_{\max}) := \{f \in L^2(0,R) | Df \in L^2(0,R)\}, \quad D_{\max} f := Df,$$

where $Df \in L^2(0,R)$ is understood in the distributional sense. The minimal extension $D_{\min}$ of D is defined as the graph-closure of D in $L^2(0,R)$, more precisely:

$$\mathcal{D}(D_{\min}) := \{f \in L^2(0,R) | \exists (f_n) \subset C_0^\infty(0,R): \\ f_n \xrightarrow{L^2} f, \quad Df_n \xrightarrow{L^2} Df\} \subseteq \mathcal{D}(D_{\max}), \quad D_{\min} f := Df.$$

Analogously we can form the minimal and the maximal extensions of the formal adjoint differential operator D^t. Since $C_0^\infty(0,R)$ is dense in $L^2(0,R)$, the maximal and the minimal extensions provide densely defined operators in $L^2(0,R)$. In particular we can form their adjoints. The next result provides a relation between the maximal and the minimal extensions of D, D^t:

Theorem 2.1. [W, Section 3] *The maximal extensions* $D_{\max}, D^t_{\max}$ *and the minimal extensions* $D_{\min}, D^t_{\min}$ *are closed densely defined operators in the Hilbert space* $L^2(0,R)$ *and are related as follows*

$$D_{\max} = (D^t_{\min})^*, \quad D^t_{\max} = (D_{\min})^*. \tag{2.1}$$

The actual discussion in [W, Section 3] is in fact peformed in the setup of symmetric operators. But the arguments there transfer analogously to not necessarily symmetric differential operators.

Moreover we introduce the following notation. Let

$$C(L^2(0,R))$$

denote the set of all closed extensions $\widetilde{D}$ in $L^2(0,R)$ of differential operators D acting on $C_0^\infty(0,R)$, such that $D_{\min} \subseteq \widetilde{D} \subseteq D_{\max}$.

Below, we restrict our attention to the setup of symmetric differential operators with real coefficients. We are interested in the characterization of the space of possible closed extensions of D in $C(L^2(0,R))$, described by the von Neumann space

$$\mathcal{D}(D_{\max})/\mathcal{D}(D_{\min}).$$

For this purpose the following general concepts, introduced in the classical reference [W], become relevant:

Definition 2.2. *A symmetric differential operator* $D : C_0^\infty(0,R) \to C_0^\infty(0,R)$ *with real coefficients is said to be*

- *in the limit point case (l.p.c.) at* $x = 0$*, if for any* $\lambda \in \mathbb{C}$ *there is at least one solution* u *of* $(D - \lambda)u = 0$ *with* $u \notin L^2_{loc}[0,R)$*.*
- *in the limit circle case (l.c.c.) at* $x = 0$*, if for any* $\lambda \in \mathbb{C}$ *all solutions* u *of* $(D - \lambda)u = 0$ *are such that* $u \in L^2_{loc}[0,R)$*.*

Here, $L^2_{loc}[0,R)$ denotes elements that are L^2-integrable over any closed intervall $I \subset [0,R)$, but not necessarily L^2-integrable over $[0,R]$. Furthermore the result [W, Theorem 5.3] implies that if the limit point or the limit circle case holds for one $\lambda \in \mathbb{C}$, then it automatically holds for any complex number. Hence it suffices to check l.p.c or l.c.c. at any fixed $\lambda \in \mathbb{C}$. Similar definition holds at the other boundary $x = R$.

The central motivation for introducing the notions of limit point and limit circle cases is that it provides a characterization of the von Neumann space $\mathcal{D}(D_{\max})/\mathcal{D}(D_{\min})$ and in particular criteria for uniqueness of closed extensions of D in $C(L^2(0,R))$.

Theorem 2.3. [W, Theorem 5.4] *Let $D : C_0^\infty(0,R) \to C_0^\infty(0,R)$ be a symmetric differential operator with real coefficients. Let $\phi_0, \phi_R \in C^\infty[0,R]$ be smooth cut-off functions being identically one near $x=0, x=R$ and identically zero near $x=R, x=0$ respectively. Then*

(i) *If D is in the l.c.c. at $x=0, (x=R)$ and $\{u_j\}$ is the fundamental system of solutions to $Du=0$, then $\{\phi_0 \cdot u_j\}, (\{\phi_R \cdot u_j\})$ forms modulo $\mathcal{D}(D_{\min})$ a linearly independent set.*

(ii) *We have the following four possible cases*

- *If D is in the l.c.c. at $x=0$ and $x=R$, then*
$$\mathcal{D}(D_{\max})/\mathcal{D}(D_{\min}) = \mathrm{Lin}(\{\phi_0 \cdot u_j\}, \{\phi_R \cdot u_j\}).$$
- *If D is in the l.c.c. at $x=0$ and l.p.c. at $x=R$, then*
$$\mathcal{D}(D_{\max})/\mathcal{D}(D_{\min}) = \mathrm{Lin}(\{\phi_0 \cdot u_j\}).$$
- *If D is in the l.p.c. at $x=0$ and l.c.c. at $x=R$, then*
$$\mathcal{D}(D_{\max})/\mathcal{D}(D_{\min}) = \mathrm{Lin}(\{\phi_R \cdot u_j\}).$$
- *If D is in the l.p.c. at $x=0$ and $x=R$, then*
$$\mathcal{D}(D_{\max}) = \mathcal{D}(D_{\min}).$$

The first statement in Theorem 2.3 is precisely the claim of [W, Theorem 5.4 (a)]. The second statement in Theorem 2.3 is contained in the proof of [W, Theorem 5.4 (b)].

To simplify the language of the forthcoming discussion, we introduce at this point a notion, which will be used throughout the presentation.

Definition 2.4. *We say that two closed extensions D_1, D_2 of a differential operator $D : C_0^\infty(0,R) \to C_0^\infty(0,R)$* "coincide" at $x=0$, *if for any cut-off function $\phi \in C^\infty[0,R]$ vanishing identically at $x=R$ and being identically one at $x=0$, the following relation holds*

$$\phi\mathcal{D}(D_1) = \phi\mathcal{D}(D_2).$$

In particular we say that a formally self-adjoint differential operator is "essentially self-adjoint" at $x=0$ *if all its self-adjoint extensions in $C(L^2(0,R))$ coincide at $x=0$.*

Similar definition holds at $x = R$. These definitions hold similarly for closed operators in $L^2((0,R),H)$, where H is any Hilbert space. With the introduced notation we obtain as a corollary of Theorem 2.3.

Corollary 2.5. *Let D be a symmetric differential operator over $C_0^\infty(0,R)$ with real coefficients, in the limit point case at $x = 0$. Then all closed extensions of D in $C(L^2(0,R))$ coincide at $x = 0$ and in particular D is essentially self-adjoint at $x = 0$.*

2.2 First order Regular-Singular Model Operators

We consider the following regular-singular model operator

$$d_p := \frac{d}{dx} + \frac{p}{x} : C_0^\infty(0,R) \to C_0^\infty(0,R), \quad p \in \mathbb{R}.$$

Recall, its maximal closed extension $d_{p,\max}$ is defined by

$$\mathcal{D}(d_{p,\max}) = \{f \in L^2(0,R) | d_p f \in L^2(0,R)\}.$$

We find that any element of the maximal domain is square-integrable with its weak derivative in $L^2_{loc}(0,R]$, due to regularity of the coefficients of d_p at $x = R$. So we have (compare [W, Theorem 3.2])

$$\mathcal{D}(d_{p,\max}) \subset H^1_{loc}(0,R].$$

Consequently elements of the maximal domain $\mathcal{D}(d_{p,\max})$ are continuous at any $x \in (0,R]$. Further we derive by solving the inhomogeneous differential equation $d_p f = g \in L^2(0,R)$ via the variation of constants method (the solution to the homogeneous equation $d_p u = 0$ is simply $u(x) = c \cdot x^{-p}$), that elements of the maximal domain $f \in \mathcal{D}(d_{p,\max})$ are of the following form

$$f(x) = c \cdot x^{-p} - x^{-p} \cdot \int_x^R y^p (d_p f)(y) dy. \tag{2.2}$$

We now analyze the expression above in order to determine the asymptotic behaviour at $x = 0$ of elements in the maximal domain of d_p for different values of $p \in \mathbb{R}$.

Proposition 2.6. *Let $O(\sqrt{x})$ and $O(\sqrt{x|\log(x)|})$ refer to the asymptotic behaviour as $x \to 0$. Then the maximal domain of d_p is characterized explicitly as follows:*

(i) *For* $p < -1/2$ *we have*

$$\mathcal{D}(d_{p,\max}) = \{f \in H^1_{loc}(0,R] | f(x) = O(\sqrt{x}), d_p f \in L^2(0,R)\}.$$

(ii) *For* $p = -1/2$ *we have*

$$\mathcal{D}(d_{p,\max}) = \{f \in H^1_{loc}(0,R] | f(x) = O(\sqrt{x|\log x|}), d_p f \in L^2(0,R)\}.$$

(iii) *For* $p \in (-1/2; 1/2)$ *we have*

$$\mathcal{D}(d_{p,\max}) = \{f \in H^1_{loc}(0,R] | f(x) = c_f x^{-p} + O(\sqrt{x}), d_p f \in L^2(0,R)\},$$

where the constants c_f *depend only on* f.

(iv) *For* $p \geq 1/2$ *we have*

$$\mathcal{D}(d_{p,\max}) = \{f \in H^1_{loc}(0,R] | f(x) = O(\sqrt{x}), d_p f \in L^2(0,R)\}.$$

Proof. Due to similarity of arguments we prove the first statement only, in order to avoid repetition. Let $p < -1/2$ and consider any $f \in \mathcal{D}(d_{p,\max})$. By (2.2) this element can be expressed by

$$f(x) = c \cdot x^{-p} - x^{-p} \cdot \int_x^R y^p g(y) dy,$$

where $g = d_p f$. By the Cauchy-Schwarz inequality we obtain for the second term in the expression

$$\left| x^{-p} \int_x^R y^p g(y) dy \right| \leq x^{-p} \sqrt{\int_x^R y^{2p} dy} \cdot \sqrt{\int_x^R g^2} \leq$$
$$\leq c \cdot x^{-p} \sqrt{x^{2p+1} - R^{2p+1}} \|g\|_{L^2} = c \cdot \sqrt{x} \sqrt{1 - R^{2p+1} x^{-2p-1}} \|g\|_{L^2},$$

where $c = 1/\sqrt{-2p-1}$. Since $(-2p-1) > 0$ we obtain for the asymptotics as $x \to 0$

$$x^{-p} \int_x^R y^p g(y) dy = O(\sqrt{x}).$$

Observe further that for $p < -1/2$ we also have $x^{-p} = O(\sqrt{x})$. This shows the inclusion $\subseteq$ in the statement. To see the converse inclusion observe

$$\{f \in H^1_{loc}(0,R] | f(x) = O(\sqrt{x}), \text{ as } x \to 0\} \subset L^2(0,R).$$

This proves the statement. □

In order to analyze the minimal closed extension $d_{p,\min}$ of d_p, we need to derive an identity relating d_p to its formal adjoint d_p^t, the so-called Lagrange identity. With the notation of Proposition 2.6 we obtain the following result.

Lemma 2.7. *(Lagrange-Identity) For any $f \in \mathcal{D}(d_{p,\max})$ and $g \in \mathcal{D}(d^t_{p,\max})$*

$$\langle d_p f, g\rangle - \left\langle f, d_p^t g\right\rangle = f(R)\overline{g(R)} - c_f\overline{c_g}, \text{ for } |p| < 1/2,$$
$$\langle d_p f, g\rangle - \left\langle f, d_p^t g\right\rangle = f(R)\overline{g(R)}, \text{ for } |p| \geq 1/2.$$

Proof.
$$\langle d_p f, g\rangle - \left\langle f, d_p^t g\right\rangle = f(R)\overline{g(R)} - f(x)\cdot\overline{g(x)}|_{x\to 0}.$$
Applying Proposition 2.6 to $f \in \mathcal{D}(d_{p,\max})$ and $g \in \mathcal{D}(d^t_{p,\max}) = \mathcal{D}(d_{-p,\max})$ we obtain:

$$f(x)\cdot\overline{g(x)}|_{x\to 0} = c_f\overline{c_g}, \text{ for } |p| < 1/2,$$
$$f(x)\cdot\overline{g(x)}|_{x\to 0} = 0, \text{ for } |p| \geq 1/2,$$

This proves the statement of the lemma. □

Proposition 2.8.

$$\mathcal{D}(d_{p,\min}) = \{f \in \mathcal{D}(d_{p,\max}) | c_f = 0,\ f(R) = 0\}, \text{ for } |p| < 1/2,$$
$$\mathcal{D}(d_{p,\min}) = \{f \in \mathcal{D}(d_{p,\max}) | f(R) = 0\}, \text{ for } |p| \geq 1/2,$$

where the coefficient c_f refers to the notation in Proposition 2.6 (iii).

Proof. Fix some $f \in \mathcal{D}(d_{p,\min})$. Then for any $g \in \mathcal{D}(d^t_{p,\max})$ we obtain using $d_{p,\min} = (d^t_{p,\max})^*$ (see Theorem 2.1) the following relation:

$$\langle d_{p,\min} f, g\rangle - \left\langle f, d^t_{p,\max} g\right\rangle = 0.$$

Together with the Lagrange identity, established in Lemma 2.7 we find

$$f(R)\overline{g(R)} - c_f\overline{c_g} = 0, \text{ for } |p| < 1/2, \tag{2.3}$$
$$f(R)\overline{g(R)} = 0, \text{ for } |p| \geq 1/2. \tag{2.4}$$

Let now $|p| < 1/2$. Then for any $c, b \in \mathbb{C}$ there exists $g \in \mathcal{D}(d^t_{p,\max})$ such that $c_g = c$ and $g(R) = b$. By arbitrariness of $c, b \in \mathbb{C}$ we conclude from (2.3)

$$c_f = 0, \quad f(R) = 0.$$

For $|p| \geq 1/2$ similar arguments hold, so we get $f(R) = 0$. This proves the inclusion $\subseteq$ in the statements. For the converse inclusion consider some

$f \in \mathcal{D}(d_{p,\max})$ with $c_f = 0$ (for $|p| < 1/2$) and $f(R) = 0$. Now for any $g \in \mathcal{D}(d^t_{p,\max})$ we infer from Lemma 2.7

$$\langle d_{p,\max} f, g \rangle - \left\langle f, d^t_{p,\max} g \right\rangle = 0.$$

Thus f is automatically an element of $\mathcal{D}((d^t_{p,\max})^*) = \mathcal{D}(d_{p,\min})$. This proves the converse inclusion. □

Now by a direct comparison of the results in Propositions 2.6 and 2.8 we obtain the following corollary.

Corollary 2.9.

(i) *For $|p| \geq 1/2$ the closed extensions $d_{p,\min}$ and $d_{p,\max}$ coincide at $x = 0$.*

(ii) *For $|p| < 1/2$ the asymptotics of elements in $\mathcal{D}(d_{p,\max})$ differs from the asymptotics of elements in $\mathcal{D}(d_{p,\min})$ by presence of $u(x) := c \cdot x^{-p}$, solving $d_p u = 0$.*

Remark 2.10. *The calculations and results of this subsection are the one-dimensional analogue of the discussion in [BS]. In particular, the result of Corollary 2.9 corresponds to [BS, Lemma 3.2].*

2.3 Self-adjoint extensions of Model Laplacians

Let the model Laplacian be the following differential operator

$$\triangle := -\frac{d^2}{dx^2} + \frac{\lambda}{x^2} : C_0^\infty(0,R) \to C_0^\infty(0,R),$$

where we assume $\lambda \geq -1/4$. Put

$$p := \sqrt{\lambda + \frac{1}{4}} - \frac{1}{2} \geq -\frac{1}{2}.$$

In this notation we find

$$\triangle = d^t_p d_p =: \triangle_p.$$

Recall that the maximal domain $\mathcal{D}(\triangle_{p,\max})$ is defined as follows

$$\mathcal{D}(\triangle_{p,\max}) = \{f \in L^2(0,R) | \triangle_p f \in L^2(0,R)\}.$$

Hence any element of the maximal domain is square-integrable with its second and thus also its first weak-derivative in $L^2_{loc}(0,R]$. So we have (compare [W, Theorem 3.2])

$$\mathcal{D}(\triangle_{p,\max}) \subset H^2_{loc}(0,R]. \tag{2.5}$$

We determine the maximal domain $\mathcal{D}(\triangle_{p,\max})$ explicitly, see also the classical calculations provided in [KLP1].

Proposition 2.11. *Let $O(x^{3/2})$ and $O(x^{1/2})$ refer to the asymptotic behaviour as $x \to 0$. Then the maximal domain $\mathcal{D}(\triangle_{p,\max})$ of the Laplace operator $\triangle_p$ is characterized explicitly as follows:*

(i) *For $p = -1/2$ we have* (2.6)
$$\mathcal{D}(\triangle_{p,\max}) = \{f \in H^2_{loc}(0,R] | f(x) = c_1(f) \cdot \sqrt{x} + c_2(f) \cdot \sqrt{x}\log(x) + \\ + \widetilde{f}(x),\ \widetilde{f}(x) = O(x^{3/2}),\ \widetilde{f}'(x) = O(x^{1/2}),\ \triangle_p \widetilde{f}(x) \in L^2(0,R)\}.$$

(ii) *For $|p| < 1/2$ we have* (2.7)
$$\mathcal{D}(\triangle_{p,\max}) = \{f \in H^2_{loc}(0,R] |, f(x) = c_1(f) \cdot x^{p+1} + c_2(f) \cdot x^{-p} + \\ + \widetilde{f}(x),\ \widetilde{f}(x) = O(x^{3/2}),\ \widetilde{f}'(x) = O(x^{1/2}),\ \triangle_p \widetilde{f}(x) \in L^2(0,R)\}.$$

(iii) *For $p \geq 1/2$ we have* (2.8)
$$\mathcal{D}(\triangle_{p,\max}) = \{f \in H^2_{loc}(0,R] | f(x) = O(x^{3/2}), \\ f'(x) = O(x^{1/2}),\ \triangle_p f(x) \in L^2(0,R)\}.$$

The constants $c_1(f), c_2(f)$ depend only on the function f.

Proof. Consider any $f \in \mathcal{D}(\triangle_{p,\max}), p \geq -1/2$ and note that $\triangle_p = d_p^t d_p = -d_{-p}d_p$. Hence we have the inhomogeneous differential equation $d_{-p}(d_p f) = -g$ with $g \equiv \triangle_p f \in L^2(0,R)$.

Analogous situation has been considered in Proposition 2.6. Repeating the arguments there we obtain

$$(d_p f)(x) = c \cdot x^p + A(x), \tag{2.9}$$

where $A(x) = O(\sqrt{x}), x \to 0$ for $p \neq 1/2$ and $A(x) = O(\sqrt{x|\log(x)|}), x \to 0$ for $p = 1/2$. Note by (2.5) that the functions $d_p f(x)$ and $A(x)$ are continuous at any $x \in (0,R]$. Applying the variation of constants method to the differential equation in (2.9) we obtain

$$\begin{aligned} f(x) &= \text{const} \cdot x^{-p} - x^{-p} \int_x^R y^p (d_p f)(y) dy = \\ &= \text{const} \cdot x^{-p} - \text{const} \cdot x^{-p} \int_x^R y^{2p} dy - x^{-p} \int_x^R y^p A(y) dy = \\ &= \text{const} \cdot x^{-p} - \text{const} \cdot x^{-p} \int_x^R y^{2p} dy + x^{-p} \int_0^x y^p A(y) dy, \end{aligned} \tag{2.10}$$

where "const" denotes any constant depending only on f and the last equality follows from the fact that $id^p \cdot A \in L^1(0,R)$ for $p \geq -1/2$, due to the asymptotics of $A(y)$ as $y \to 0$. Put

$$\widetilde{f}(x) = x^{-p} \int_0^x y^p A(y) dy.$$

Using the asymptotic behaviour of $A(y)$ as $y \to 0$, we derive $\widetilde{f}(x) = O(x^{3/2})$ and $\widetilde{f}'(x) = O(x^{1/2})$ as $x \to 0$. Evaluating now explicitly the second integral in (2.10) for different values of $p \geq -1/2$ and noting for $p \geq 1/2$ the facts that $x^{p+1} = O(x^{3/2})$ and $id^{-p} \notin L^2(0,R)$, we prove the inclusion $\subseteq$ in the statement on the domain relations.

For the converse inclusion observe that any $f \in H^2_{loc}(0,R]$ with the asymptotic behaviour as $x \to 0$:

$$\begin{aligned} f(x) = c_1(f) \cdot \sqrt{x} + c_2(f) \cdot \sqrt{x} \log(x) + O(x^{3/2}), &\text{ for } p = -1/2, \\ f(x) = c_1(f) \cdot x^{p+1} + c_2(f) \cdot x^{-p} + O(x^{3/2}), &\text{ for } |p| < 1/2, \\ f(x) = O(x^{3/2}), &\text{ for } p \geq 1/2, \end{aligned}$$

is square integrable, $f \in L^2(0,R)$. It remains to observe why $\triangle_p f \in L^2(0,R)$ for any f in the domains on the right hand side of the statement. This becomes clear, once we note that the additional terms in the asymptotics of f other than $\widetilde{f}(x)$ are solutions to $\triangle_p u = 0$. □

In order to analyze the minimal closed extension $\triangle_{p,\min}$ we need to derive the Lagrange identity for $\triangle_p$, see also [KLP2, (3.2)].

Lemma 2.12. *[Lagrange-identity] For any $f, g \in \mathcal{D}(\triangle_{p,\max})$ the following identities hold.*

(i) *If $p = -1/2$, then we have in the notation of Proposition 2.11*

$$\begin{aligned} &\langle f, \triangle_p g \rangle_{L^2} - \langle \triangle_p f, g \rangle_{L^2} = \\ &= [c_1(f)\overline{c_2(g)} - c_2(f)\overline{c_1(g)}] + [f'(R)\overline{g(R)} - f(R)\overline{g'(R)}]. \end{aligned}$$

(ii) *If $|p| < 1/2$, then we have in the notation of Proposition 2.11*

$$\begin{aligned} &\langle f, \triangle_p g \rangle_{L^2} - \langle \triangle_p f, g \rangle_{L^2} = \\ &= -(2p+1)[c_1(f)\overline{c_2(g)} - c_2(f)\overline{c_1(g)}] + [f'(R)\overline{g(R)} - f(R)\overline{g'(R)}]. \end{aligned}$$

(iii) *If $p \geq 1/2$, then we have*

$$\langle f, \triangle_p g \rangle_{L^2} - \langle \triangle_p f, g \rangle_{L^2} = [f'(R)\overline{g(R)} - f(R)\overline{g'(R)}].$$

Proof. Let $f, g \in \mathcal{D}(\triangle_{p,\max})$ be any two elements of the maximal domain of $\triangle_p$. We compute:

$$\begin{aligned}
&\langle f, \triangle_p g\rangle_{L^2} - \langle \triangle_p f, g\rangle_{L^2} = \\
&= \lim_{\epsilon\to 0}\int_\epsilon^R [f(x)\overline{\triangle_p g(x)} - \triangle_p f(x)\overline{g(x)}]dx = \\
&= \lim_{\epsilon\to 0}\int_\epsilon^R \frac{d}{dx}[-f(x)\overline{g'(x)} + f'(x)\overline{g(x)}]dx = \\
&= \lim_{\epsilon\to 0}[f(\epsilon)\overline{g'(\epsilon)} - f'(\epsilon)\overline{g(\epsilon)}] + [f'(R)\overline{g(R)} - f(R)\overline{g'(R)}].
\end{aligned}$$

Now the statement follows by inserting the asymptotics at $x = 0$ of $f, g \in \mathcal{D}(\triangle_{p,\max})$ into the first summand of the expression above. □

Proposition 2.13. *The minimal domain of the model Laplacian $\triangle_p$ is given explicitly in the notation of Proposition 2.11 as follows*

$$\begin{aligned}
&\mathcal{D}(\triangle_{p,\min}) = \\
&= \{f \in \mathcal{D}(\triangle_{p,\max}) | c_1(f) = c_2(f) = 0, f(R) = f'(R) = 0\},\ p \in [-1/2, 1/2), \\
&\mathcal{D}(\triangle_{p,\min}) = \{f \in \mathcal{D}(\triangle_{p,\max}) | f(R) = f'(R) = 0\},\ p \geq 1/2.
\end{aligned}$$

Proof. Fix some $f \in \mathcal{D}(\triangle_{p,\min})$. Then for any $g \in \mathcal{D}(\triangle_{p,\max})$ we obtain with $\triangle_{p,\min} = \triangle^*_{p,\max}$ (see Theorem 2.1) the following relation

$$\langle f, \triangle_p g\rangle_{L^2} - \langle \triangle_p f, g\rangle_{L^2} = 0.$$

Together with the Lagrange-identity, established in Lemma 2.12, and the fact that for $p \in [-1/2, 1/2)$ and any arbitrary $c_1, c_2, b_1, b_2 \in \mathbb{C}$ there exists $g \in \mathcal{D}(\triangle_{p,\max})$ such that

$$c_1(g) = c_1,\ c_2(g) = c_2,\ g(R) = b_1,\ g'(R) = b_2,$$

we conclude for $f \in \mathcal{D}(\triangle_{p,\min}), p \in [-1/2, 1/2)$

$$c_1(f) = c_2(f) = 0, f(R) = f'(R) = 0. \tag{2.11}$$

Analogous arguments for $f \in \mathcal{D}(\triangle_{p,\min}), p \geq 1/2$ show $f(R) = f'(R) = 0$. This proves the inclusion $\subseteq$ in the statement. For the converse inclusion consider any $f \in \mathcal{D}(\triangle_{p,\max})$, satisfying (2.11), where the condition $c_1(f) = c_2(f) = 0$ is imposed only for $p \in [-1/2, 1/2)$. Now we obtain from the Lagrange-identity in Lemma 2.12

$$\forall g \in \mathcal{D}(\triangle_{p,\max}) :\ \langle f, \triangle_p g\rangle_{L^2} - \langle \triangle_p f, g\rangle_{L^2} = 0.$$

Hence f is automatically an element of $\mathcal{D}(\triangle^*_{p.\max}) = \mathcal{D}(\triangle_{p,\min})$. This proves the converse inclusion. □

Corollary 2.14. (i) *For* $\lambda \geq 3/4$, *equivalently for* $p = \sqrt{\lambda + 1/4} - 1/2 \geq 1/2$, *the model Laplacian* $\triangle_p$ *is in the limit point case at* $x = 0$ *and the closed extensions* $\triangle_{p,\max}$ *and* $\triangle_{p,\min}$ *coincide at* $x = 0$. *In particular* $\triangle_p$ *is essentially self-adjoint at* $x = 0$.

(ii) *For* $\lambda \in [-1/4, 3/4)$, *equivalently for* $p = \sqrt{\lambda + 1/4} - 1/2 \in [-1/2, 1/2)$, *the model Laplacian* $\triangle_p$ *is in the limit circle case at* $x = 0$ *and the asymptotics at zero of the elements in* $\mathcal{D}(\triangle_{p,\max})$ *differ from the asymptotics at zero of elements in* $\mathcal{D}(\triangle_{p,\min})$ *by presence of fundamental solutions to* $\triangle_p u = 0$.

Proof. On the one hand statements on the coincidence or the difference of maximal and minimal domains at $x = 0$ follow from a direct comparison of the results of Propositions 2.11 and 2.13. On the other hand, given the statements on the limit point and the limit circle cases, the comparison of the maximal and the minimal domains follows from Theorem 2.3.

It remains then to verify the limit point and the limit circle statements. They follow by definition from the study of the fundamental solutions $u_1, u_2 : (0, R) \to \mathbb{R}$ of $\triangle_p u = 0$:

$$\text{For } p = -1/2 \quad u_1(x) = \sqrt{x},\ u_2(x) = \sqrt{x}\log(x), \tag{2.12}$$

$$\text{For } p > -1/2 \quad u_1(x) = x^{p+1},\ u_2(x) = x^{-p}. \tag{2.13}$$

□

Next, since the model Laplacian $\triangle_p$ is shown to be essentially self-adjoint at $x = 0$ for $p \geq 1/2$, we are interested in the self-adjoint extensions of $\triangle_p$ for $p \in [-1/2, 1/2)$, since only there the boundary conditions at $x = 0$ are not redundant. In this subsection we determine for these values of p the two geometrically meaningsful extensions of the model Laplacian − the D-extension and N-extension:

$$\triangle_p^D := (d_{p,\min})^*(d_{p,\min}) = d^t_{p,\max} d_{p,\min},$$

$$\triangle_p^N := (d_{p,\max})^*(d_{p,\max}) = d^t_{p,\min} d_{p,\max}.$$

Corollary 2.15. *For* $|p| < 1/2$ *we have in the notation of Proposition 2.11*

$$\mathcal{D}(\triangle_p^D) = \{f \in \mathcal{D}(\triangle_{p,\max}) | c_2(f) = 0,\ f(R) = 0\}, \tag{2.14}$$

$$\mathcal{D}(\triangle_p^N) = \{f \in \mathcal{D}(\triangle_{p,\max}) | c_1(f) = 0,\ d_p f(R) = 0\}. \tag{2.15}$$

Proof. Let us consider the D−extension first. By definition $\mathcal{D}(\triangle_p^D) \subset \mathcal{D}(d_{p,\min})$ and thus by Proposition 2.8 we have for any $f \in \mathcal{D}(\triangle_p^D)$

$$f(x) = O(\sqrt{x}) \quad \text{and} \quad f(R) = 0.$$

Since $x^{-p} \neq O(\sqrt{x}), x \to 0$ for $|p| < 1/2$ we find in the notation of Proposition 2.11 that the constant $c_2(f)$ must be zero for $f \in \mathcal{D}(\triangle_p^D)$. This proves the inclusion $\subseteq$ in the first statement.

For the converse inclusion consider $f \in \mathcal{D}(\triangle_{p,\max})$ with $c_2(f) = 0,\ f(R) = 0$. By Proposition 2.8 we find $f \in \mathcal{D}(d_{p,\min})$. Now with $f \in \mathcal{D}(\triangle_{p,\max})$ we obtain $d_{p,\min} f \in \mathcal{D}(d^t_{p,\max})$ and hence

$$f \in \mathcal{D}(\triangle_p^D).$$

This proves the converse inclusion of the first statement.

For the second statement consider any $f \in \mathcal{D}(\triangle_p^N) = \mathcal{D}(d^t_{p,\min} d_{p,\max})$. There exists some $g \in \mathcal{D}(d^t_{p,\min})$ such that $d_p f = g$ with the general solution of this differential equation obtained by the variation of constants method.

$$f(x) = c \cdot x^{-p} - x^{-p} \int_0^x y^p g(y) dy.$$

Since $g \in \mathcal{D}(d^t_{p,\min})$ and thus in particular $g(x) = O(\sqrt{x})$, we find via the Cauchy-inequality that the second summand in the solution above behaves as $O(x^{3/2})$ for $x \to 0$. Hence $c_1(f) = 0$ in the notation of Proposition 2.11. Further $d_p f \in \mathcal{D}(d^t_{p,\min})$ and thus

$$d_p f(R) = 0.$$

This proves the inclusion $\subseteq$ in the second statement.

For the converse inclusion consider an element $f \in \mathcal{D}(\triangle_{p,\max})$ with $f(x) = c_2(f) \cdot x^{-p} + \widetilde{f}(x)$, where $\widetilde{f}(x) = O(x^{3/2}), \widetilde{f}'(x) = O(\sqrt{x})$, as $x \to 0$, and $d_p f(R) = 0$. The inclusion $f \in \mathcal{D}(d_{p,\max})$ is then clear by Proposition 2.6. Now by Proposition 2.8 we have $d_p f = d_p \widetilde{f} \in \mathcal{D}(d_{-p,\min})$ due to asymptotics of $\widetilde{f}(x)$ as $x \to 0$ and $d_p f(R) = 0$. □

Corollary 2.16. *For $p = -1/2$ we have in the notation of Proposition 2.11*

$$\mathcal{D}(\triangle_p^D) = \{f \in \mathcal{D}(\triangle_{p,\max}) | c_2(f) = 0,\ f(R) = 0\}.$$

$$\mathcal{D}(\triangle_p^N) = \{f \in \mathcal{D}(\triangle_{p,\max}) | c_2(f) = 0,\ d_p f(R) = 0\}.$$

Proof. The first statement is proved by similar arguments as in Corollary 2.14. The Corollary 2.9 asserts the equality of the D-extension and the N-extension at $x = 0$ in the sense of Definition 2.4. This determines the asymptotic behaviour of $f \in \mathcal{D}(\triangle_p^N)$ as $x \to 0$. For the boundary conditions of $\triangle_p^N$ at $x = R$ simply observe that for any $f \in \mathcal{D}(\triangle_p^N)$ one has in particular $d_p f \in \mathcal{D}(d_{-p,\min})$ and hence $d_p f(R) = 0$. □

Remark 2.17. *The naming "D-extension" and "N-extension" coincides with the convention chosen in [LMP, Section 2.3]. However the motivation for this naming is given here by the type of the boundary conditions at the regular end $x = R$. In fact $\mathcal{D}(\triangle_p^D)$ has Dirichlet boundary conditions at $x = R$ and $\mathcal{D}(\triangle_p^D)$ − generalized Neumann boundary conditions at $x = R$.*

So far we considered the self-adjoint extensions of $\triangle_p = d_p^t d_p$ with $p := \sqrt{\lambda + 1/4} - 1/2 \in [-1/2, 1/2)$. However for $r = -p - 1$ we have

$$d_r^t d_r = d_p^t d_p = -\frac{d^2}{dx^2} + \frac{\lambda}{x^2}, \text{ since } r(r+1) = p(p+1) = \lambda.$$

Hence for completeness it remains identify the D- and the N-extensions for $d_r^t d_r, r = p - 1 \in (-3/2, -1/2]$ as well. Note however that for $p = -1/2$ we get $r = p = -1/2$ and the D-, N-extensions are as established before. It remains to consider $r \in (-3/2, -1/2)$.

Corollary 2.18. *Let $p \in (-1/2; -1/2)$. Put $r = -p - 1 \in (-3/2, -1/2)$. Then we have in the notation of Proposition 2.11*

$$\mathcal{D}(\triangle_r^D) = \{f \in \mathcal{D}(\triangle_{p,\max}) | c_2(f) = 0, \ f(R) = 0\},$$

$$\mathcal{D}(\triangle_r^N) = \{f \in \mathcal{D}(\triangle_{p,\max}) | c_2(f) = 0, \ d_r f(R) = 0\}.$$

Proof. The first statement is proved by similar arguments as in Corollary 2.15. Further, Corollary 2.9 implies equality of the D-extension and the N-extension at $x = 0$ in the sense of Definition 2.4. This determines the asymptotic behaviour of $f \in \mathcal{D}(\triangle_r^N)$ at $x = 0$. For the boundary conditions of $\triangle_r^N$ at $x = R$ simply observe that for any $f \in \mathcal{D}(\triangle_r^N)$ one has in particular $d_r f \in \mathcal{D}(d_{-r,\min})$ and hence $d_r f(R) = 0$. □

3 Boundary Conditions for the de Rham Laplacian on a Bounded Generalized Cone

In this section we discuss the geometry of a bounded generalized cone and following [L3] decompose the associated de Rham Laplacian in a compatible way with respect to its relative self adjoint extension. This decomposition allows us to study the relative self-adjoint extension of the Laplace operator explicitly and provides a basis for the computation of the associated zeta-regularized determinants.

The question about the self-adjoint extensions of the Laplacians on differential forms of a fixed degree, on manifolds with conical singularities is addressed in [BL2, Theorems 3.7 and 3.8]. There, among many other issues, the relative extension is shown to coincide with the Friedrich's extension at the cone singularity, outside of the middle degrees.

Using the decomposition of the complex we obtain further explicit results without the degree limitations.

3.1 Regular-Singular Operators

Consider a bounded generalized cone $M=(0,R]\times N$ over a closed oriented Riemannian manifold (N,g^N) of dimension $\dim N = n$, with the Riemannian metric on M given by a warped product

$$g^M = dx^2 \oplus x^2 g^N.$$

The volume forms on M and N, associated to the Riemannian metrics g^M and g^N, are related as follows:

$$\mathrm{vol}(g^M) = x^n dx \wedge \mathrm{vol}(g^N).$$

Consider as in [BS, (5.2)] the following separation of variables map, which is linear and bijective:

$$\begin{aligned} \Psi_k : C_0^\infty((0,R),\Omega^{k-1}(N)\oplus\Omega^k(N)) &\to \Omega_0^k(M) \\ (\phi_{k-1},\phi_k) &\mapsto x^{k-1-n/2}\phi_{k-1}\wedge dx + x^{k-n/2}\phi_k, \end{aligned} \tag{3.1}$$

where ϕ_k,ϕ_{k-1} are identified with their pullback to M under the natural projection $\pi:(0,R]\times N\to N$ onto the second factor, and x is the canonical coordinate on $(0,R]$. Here $\Omega_0^k(M)$ denotes differential forms of degree $k=0,..,n+1$ with compact support in the interior of M. With respect to the L^2-scalar products, induced by the volume forms $\mathrm{vol}(g^M)$ and $\mathrm{vol}(g^N)$, we note the following relation for any $(\phi_k, \phi_{k-1}), (\psi_k, \psi_{k-1}) \in C_0^\infty((0,R), \Omega^{k-1}(N) \oplus \Omega^k(N))$:

$$\begin{aligned}
\langle \Psi_k(\phi_k, \phi_{k-1}), \Psi_k(\psi_k, \psi_{k-1})\rangle_{L^2(M)} = \\
&= \int_M x^{2(k-1)-n} g^M(\phi_{k-1}, \psi_{k-1}) x^n dx \wedge \mathrm{vol}(g^N) + \\
&+ \int_M x^{2k-n} g^M(\phi_k, \psi_k) x^n dx \wedge \mathrm{vol}(g^N) = \\
= \int_M g^N(\phi_{k-1}(x), \psi_{k-1}(x)) dx \wedge \mathrm{vol}(g^N) + \\
+ \int_M g^N(\phi_k(x), \psi_k(x)) dx \wedge \mathrm{vol}(g^N) = \\
&= \int_0^R \langle \phi_{k-1}(x), \psi_{k-1}(x)\rangle_{L^2(N)} dx = \\
&+ \int_0^R \langle \phi_k(x), \psi_k(x)\rangle_{L^2(N)} dx,
\end{aligned}$$

where we extended the Riemannian metrics to inner products on differential forms. The relation implies that the separation of variables map Ψ_k extends to an isometry on the L^2-completions, proving the proposition below.

Proposition 3.1. *The separation of variables map* (3.1) *extends to an isometric identification of L^2-Hilbert spaces*

$$\Psi_k : L^2([0,R], L^2(\wedge^{k-1}T^*N \oplus \wedge^k T^*N, \mathrm{vol}(g^N)), dx) \to L^2(\wedge^k T^*M, \mathrm{vol}(g^M)).$$

Under this identification we obtain for the exterior derivative, as in [BS, (5.5)]

$$\Psi_{k+1}^{-1} d_k \Psi_k = \begin{pmatrix} 0 & (-1)^k \partial_x \\ 0 & 0 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} d_{k-1,N} & c_k \\ 0 & d_{k,N} \end{pmatrix}, \tag{3.2}$$

where $c_k = (-1)^k (k - n/2)$ and $d_{k,N}$ denotes the exterior derivative on differential forms over N of degree k. Taking adjoints we find

$$\Psi_k^{-1} d_k^t \Psi_{k+1} = \begin{pmatrix} 0 & 0 \\ (-1)^{k+1}\partial_x & 0 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} d_{k-1,N}^t & 0 \\ c_k & d_{k,N}^t \end{pmatrix}. \tag{3.3}$$

Consider now the Gauss-Bonnet operator D_{GB}^+ mapping forms of even degree to forms of odd degree. The Gauss-Bonnet operator acting on forms of odd degree is simply the formal adjoint $D^-_{GB} = (D^+_{GB})^t$. With respect to $\Psi_+ := \oplus \Psi_{2k}$ and $\Psi_- := \oplus \Psi_{2k+1}$ the relevant operators take the following form:

$$\Psi_-^{-1} D^+_{GB} \Psi_+ = \frac{d}{dx} + \frac{1}{x} S_0, \quad \Psi_+^{-1} D^-_{GB} \Psi_- = -\frac{d}{dx} + \frac{1}{x} S_0, \tag{3.4}$$

$$\Psi_+^{-1} \triangle^+ \Psi_+ = \Psi_+^{-1} (D^+_{GB})^t \Psi_- \Psi_-^{-1} D^+_{GB} \Psi_+ = -\frac{d^2}{dx^2} + \frac{1}{x^2} S_0(S_0+1), \tag{3.5}$$

$$\Psi_-^{-1} \triangle^- \Psi_- = \Psi_-^{-1} (D^-_{GB})^t \Psi_+ \Psi_+^{-1} D^-_{GB} \Psi_- = -\frac{d^2}{dx^2} + \frac{1}{x^2} S_0(S_0-1).$$

where S_0 is a first order elliptic differential operator on $\Omega^*(N)$. It is given explicitly by the following matrix form (cf. [BL2, (2.12)]):

$$S_0 = \begin{pmatrix} c_0 & d^t_{0,N} & 0 & \cdots & 0 \\ d_{0,N} & c_1 & d^t_{1,N} & \cdots & 0 \\ \vdots & d_{1,N} & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & c_{n-1} & d^t_{n-1,N} \\ 0 & \cdots & 0 & d_{n-1,N} & c_n \end{pmatrix}, \ c_k = (-1)^k \left(k - \frac{n}{2} \right).$$

The transformed Gauss-Bonnet operators in (3.4) are regular singular in the sense of [BS] and [Br, Section 3]. Moreover, the Laplace Operator on k-forms over M transforms to

$$\Psi_k \triangle_k \Psi_k^{-1} = -\frac{d^2}{dx^2} + \frac{1}{x^2} A_k. \tag{3.6}$$

The operator A_k denotes the restriction of $S_0(S_0+(-1)^k)$ to $\Omega^{k-1}(N) \oplus \Omega^k(N)$ and is given explicitly by the following matrix form:

$$A_k = \begin{pmatrix} \triangle_{k-1,N} + c^2_{k-1} + (-1)^k c_{k-1} & (c_k + c_{k-1} + (-1)^k) d^t_{k,N} \\ (c_k + c_{k-1} + (-1)^k) d_{k-1,N} & \triangle_{k,N} + c^2_k + (-1)^{k+1} c_k \end{pmatrix}, \tag{3.7}$$

where $\triangle_{k,N}$ denotes the Laplacian on differential forms of degree k over N.

Note, that under the isometric identification Ψ_* the previous non-product situation of the bounded generalized cone M is now incorporated in the x-dependence of the tangential parts of the geometric Gauss-Bonnet and Laplace operators.

Next we take boundary conditions into account and consider their behaviour under the isometric identification Ψ_*. More precisely consider the exterior derivatives and their formal adjoints on differential forms with compact support in the interior of M:

$$\begin{aligned} d_k &: \Omega_0^k(M) \to \Omega_0^{k+1}(M), \\ d_k^t &: \Omega_0^{k+1}(M) \to \Omega_0^k(M). \end{aligned}$$

Define the minimal closed extensions $d_{k,\min}$ and $d^t_{k,\min}$ as the graph closures in $L^2(\bigwedge^* T^*M, \mathrm{vol}(g^M))$ of the differential operators d_k and d_k^t respectively.

The operators $d_{k,\min}$ and $d^t_{k,\min}$ are closed and densely defined. In particular we can form the adjoint operators and set for the maximal extensions:

$$d_{k,\max} := (d^t_{k,\min})^*, \quad d^t_{k,\max} := (d_{k,\min})^*.$$

These definitions correspond to Theorem 2.1. The following result is an easy consequence of the definitions of the minimal and maximal extensions and of Proposition 3.1.

Proposition 3.2.

$$\begin{aligned} \Psi_k^{-1}(\mathcal{D}(d_{k,\min})) &= \mathcal{D}([\Psi_{k+1}^{-1} d_k \Psi_k]_{\min}), \\ \Psi_k^{-1}(\mathcal{D}(d_{k,\max})) &= \mathcal{D}([\Psi_{k+1}^{-1} d_k \Psi_k]_{\max}). \end{aligned}$$

Similar statements hold for the minimal and maximal extensions of the formal adjoint operators d_k^t. The minimal and the maximal extensions of the exterior derivative give rise to self-adjoint extensions of the associated Laplace operator

$$\triangle_k = d_k^t d_k + d_{k-1} d_{k-1}^t.$$

It is important to note that there are self-adjoint extensions of $\triangle_k$ which do not come from closed extensions of d_k and d_{k-1}, compare the notion of "ideal boundary conditions" in [BL1]. However the most relevant self-adjoint extensions of the Laplacian indeed seem to come from closed extensions of the exterior derivatives.

We are interested in the relative and the absolute self-adjoint extensions of $\triangle_k$, defined as follows:

$$\begin{aligned} \triangle_k^{rel} &:= d^*_{k,\min} d_{k,\min} + d_{k-1,\min} d^*_{k-1,\min} = \\ &= d^t_{k,\max} d_{k,\min} + d_{k-1,\min} d^t_{k-1,\max}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \triangle_k^{abs} &:= d^*_{k,\max} d_{k,\max} + d_{k-1,\max} d^*_{k-1,\max} = \\ &= d^t_{k,\min} d_{k,\max} + d_{k-1,\max} d^t_{k-1,\min}. \end{aligned} \tag{3.9}$$

As a direct consequence of the previous proposition and Proposition 3.1 we obtain for the relative self-adjoint extension (absolute self-adjoint extension is discussed in a similar way):

Corollary 3.3. *Consider the following two complexes*

$$(\Omega^*_0(M), d_k), \quad (C^\infty_0((0,R), C^\infty(\wedge^{k-1}T^*N \oplus \wedge^k T^*N)), \widetilde{d}_k := \Psi^{-1}_{k+1} d_k \Psi_k).$$

Then the relative self-adjoint extensions of the associated Laplacians

$$\begin{aligned}\triangle^{rel}_k &= d^*_{k,\min} d_{k,\min} + d_{k-1,\min} d^*_{k-1,\min},\\ \widetilde{\triangle}^{rel}_k &= \widetilde{d}^*_{k,\min} \widetilde{d}_{k,\min} + \widetilde{d}_{k-1,\min} \widetilde{d}^*_{k-1,\min}\end{aligned}$$

are spectrally equivalent, with $\Psi^{-1}_k(\mathcal{D}(\triangle^{rel}_k)) = \mathcal{D}(\widetilde{\triangle}^{rel}_k)$ *and*

$$\widetilde{\triangle}^{rel}_k = \Psi^{-1}_k \triangle^{rel}_k \Psi_k.$$

As a consequence of Corollary 3.3 we can deal with the minimal extension of the unitarily transformed exterior differential $\Psi^{-1}_{k+1} d_k \Psi_k$ and the relative extension of the unitarily transformed Laplacian $\Psi^{-1}_k \triangle_k \Psi_k$ without loss of generality. By a small abuse of notation we denote the operators again by $d_{k,\min}$ and $\triangle^{rel}_k$, in order to keep the notation simple. This setup shall be fixed up to Section 6.

3.2 Decomposition of the de Rham Laplacian

Our goal is the explicit determination of the domain of $\triangle^{rel}_k, k = 0,..,m = \dim M$. We restrict ourselves to the relative extension, since the absolute extension is treated analogously.

By the convenient structure (3.6) of the Laplacian $\triangle_k$ one is tempted to write

$$\triangle_k = \bigoplus_{\lambda \in \mathrm{Sp}(A_k)} -\frac{d^2}{dx^2} + \frac{\lambda}{x^2},$$

and study the boundary conditions induced on each one-dimensional component. However this decomposition might be incompatible with the boundary conditions, so the discussion of the corresponding self-adjoint realization might not reduce to simple one-dimensional problems. This is in fact the case for the relative boundary conditions, which (by definition) determine the domain of the relative extension $\triangle^{rel}_k$.

Nevertheless we infer from the decomposition above and (2.5) the regularity properties for elements $\phi \in \mathcal{D}(\triangle_{k,\max})$, needed in the formulation of Proposition 3.5 below.

At the cone face $\{x = R\} \times N$ the relative boundary conditions are derived from the following trace theorem of L. Paquet:

Theorem 3.4. [P, Theorem 1.9] *Let K be a compact oriented Riemannian manifold with boundary ∂K and let $\iota : \partial K \hookrightarrow K$ be the natural inclusion. Then the pullback $\iota^* : \Omega^k(K) \to \Omega^k(\partial K)$ with $\Omega^k(\partial K) = \{0\}$ for $k = \dim K$, extends continuously to the following linear surjective map:*

$$\iota^* : \mathcal{D}(d_{k,\max}) \to \mathcal{D}(d^{-1/2}_{k,\partial K}),$$

where $d^{-1/2}_{k,\partial K}$ is the closure of the exterior derivative on ∂K in the Sobolev space $H^{-1/2}(\wedge^* T^* \partial K)$ and $d_{k,\max}$ the maximal extension of the exterior derivative on K. The domains $\mathcal{D}(d_{k,\max})$ and $\mathcal{D}(d^{-1/2}_{k,\partial K})$ are Hilbert spaces with respect to the graph-norms of the corresponding operators.

Proposition 3.5. *Let $\gamma \in C^\infty[0, R]$ be a smooth cut-off function, vanishing identically at $x = 0$ and being identically one at $x = R$. Then*

$$\begin{aligned}\gamma\mathcal{D}(\triangle^{rel}_k) = \{\Psi_k(\phi_{k-1}, \phi_k) \in \gamma\mathcal{D}(\triangle_{k,\max}) | \phi_k(R) = 0,\\ \phi'_{k-1}(R) - \frac{(k-1-n/2)}{R}\phi_{k-1}(R) = 0\}.\end{aligned}$$

Proof. Let $r \in (0, R)$ be fixed and consider the associated natural inclusions

$$\begin{aligned}\chi &: [0, R] \times N =: M_r \hookrightarrow M,\\ \iota &: \{R\} \times N \equiv N \hookrightarrow M,\\ \iota_r &: \{R\} \times N \equiv N \hookrightarrow M_r.\end{aligned}$$

We obviously have $\iota = \chi \circ \iota_r$. The inclusions above induce pullbacks of differential forms. The pullback map $\chi^* : \Omega^k(M) \to \Omega^k(M_r)$ is simply a restriction and extends to a continuous linear map

$$\chi^* : \mathcal{D}(d_{k,\max}) \to \mathcal{D}(d^r_{k,\max}),$$

where d^r_k is the $k-$th exterior derivative on $M_r \subset M$ and the domains are endowed with the graph norms of the corresponding operators. Applying Theorem 3.4 to the compact manifold M_r, we deduce that $\iota^* = \iota^*_r \circ \chi^*$ extends to a continuous linear map

$$\iota^* : \mathcal{D}(d_{k,\max}) \to \mathcal{D}(d^{-1/2}_{k,N}). \tag{3.10}$$

Now, continuity of ι^* together with the definition of the minimal domain $\mathcal{D}(d_{k,\min})$ implies

$$\gamma\mathcal{D}(d_{k,\min}) \subseteq \{\phi \in \gamma\mathcal{D}(d_{k,\max}) | \iota^*\phi = 0\}.$$

Equality in the relation above follows from the Lagrange identity for d_k. We obtain for the relative boundary conditions at the cone base:

$$\gamma\mathcal{D}(\triangle_k^{rel}) = \{\phi \in \gamma\mathcal{D}(\triangle_{k,\max}) | \iota^*\phi = 0, \iota^*(d_{k-1}^t\phi) = 0\}.$$

Now the statement of the proposition follows from the explicit action of d_{k-1}^t under the isometric identification Ψ_* and the fact that for $\Psi_k(\phi_{k-1}, \phi_k) \in \mathcal{D}(\triangle_{k,\max})$ we have $\iota^*(\Psi_k(\phi_{k-1}, \phi_k)) = R^{k-n/2}\phi_k(R)$. □

In order to identify the relative boundary conditions at the cone singularity, we decompose $\triangle_k$ into a direct sum of operators such that the decomposition is compatible with the relative self-adjoint extension.

Compatibility of a decomposition means explicitly the following in the context of our presentation.

Definition 3.6. *Let D be a closed operator in a Hilbert space H. Let H_1 be a closed subspace of H and $H_2 := H_1^{\perp}$. We say the decomposition $H = H_1 \oplus H_2$ is compatible with D if $D(H_j \cap \mathcal{D}(D)) \subset H_j, j = 1, 2$ and for any $\phi_1 \oplus \phi_2 \in \mathcal{D}(D)$ we get $\phi_1, \phi_2 \in \mathcal{D}(D)$.*

This definition corresponds to [W2, Exercise 5.39] where the subspaces $H_j, j = 1, 2$ are called the "reducing subspaces of D". We have the following result:

Proposition 3.7. [W2, Theorem 7.28] *Let D be a self-adjoint operator in a Hilbert space H. Let H_1 be a closed subspace of H and $H_2 := H_1^{\perp}$. Let the decomposition $H = H_1 \oplus H_2$ be compatible with D. Then each operator $D_i := D|H_i, i = 1, 2$ with domain*

$$\mathcal{D}(D_i) := \mathcal{D}(D) \cap H_i, \ i = 1, 2$$

is a self-adjoint operator in H_i. In other words, the induced decomposition $D = D_1 \oplus D_2$ is an orthogonal decomposition of D into sum of two self-adjoint operators.

Definition 3.8. *In the setup of Proposition 3.7 we say $D_i, i = 1, 2$ is a self-adjoint operator* "induced" *by D.*

In order to simplify notation, put:

$$\begin{aligned} H^k &:= L^2([0,R], L^2(\wedge^{k-1}T^*N \oplus \wedge^k T^*N, \mathrm{vol}(g^N)), dx), \\ H^* &:= \bigoplus_{k\geq 0} H^k, \end{aligned}$$

where H^k are mutually orthogonal in H^*. The following result gives a practical condition for compatibility of a decomposition of H^k with the self-adjoint realization $\triangle_k^{rel}$.

Proposition 3.9. *Let $H^k = H_1 \oplus H_2, H_2 := H_1^\perp$ be an orthogonal decomposition into closed subspaces, such that $\triangle_k^{rel}(H_j \cap \mathcal{D}(\triangle_k^{rel})) \subset H_j, j = 1,2$. Assume that for $D \in \{d_k, d_k^t, d_k^t d_k, d_{k-1}d_{k-1}^t\}$ the images $D_{\max}(H_j \cap \mathcal{D}(D_{\max})), j = 1,2$ are mutually orthogonal in H^*. Then the decomposition $H^k = H_1 \oplus H_2$ is compatible with the relative extension $\triangle_k^{rel}$.*

Proof. Consider $\phi = \phi_1 \oplus \phi_2 \in \mathcal{D}(\triangle_k^{rel})$. In particular $\phi \in \mathcal{D}(d_{k,\min})$, i.e. there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_0^\infty((0,R), \Omega^{k-1}N \oplus \Omega^k N)$ such that as $n \to \infty$

$$\phi_n \xrightarrow{L^2} \phi \text{ and } d_k\phi_n \xrightarrow{L^2} d_k\phi.$$

Under the decomposition $H^k = H_1 \oplus H_2$ we write

$$\begin{aligned} \phi_n = \phi_n^1 \oplus \phi_n^2 &\xrightarrow{L^2} \phi_1 \oplus \phi_2 \text{ and} \\ d_k\phi_n = d_k\phi_n^1 \oplus d_k\phi_n^2 &\xrightarrow{L^2} d_k\phi_1 \oplus d_k\phi_2. \end{aligned}$$

Since the decomposition $H^k = H_1 \oplus H_2$ is orthogonal and by assumption orthogonality is maintained under the action of $d_{k,\max}$, we find

$$\phi_n^i \xrightarrow{L^2} \phi_i \text{ and } d_k\phi_n^i \xrightarrow{L^2} d_k\phi_i, i = 1,2.$$

This implies $\phi_1, \phi_2 \in \mathcal{D}(d_{k,\min})$.

Further $\phi \in \mathcal{D}(\triangle_k^{rel})$ lies in particular in $\mathcal{D}(d_{k-1,\max}^t)$, i.e.

$$\phi = \phi_1 \oplus \phi_2 \in L^2, \ d_{k-1}^t\phi = d_{k-1}^t\phi_1 \oplus d_{k-1}^t\phi_2 \in L^2.$$

By orthogonality of the decompositions, each component must be square-integrable individually. Hence $\phi_1, \phi_2 \in \mathcal{D}(d_{k-1,\max}^t)$. Iterative application of these arguments proves the statement. $\square$

Now we can present, following [L3], a decomposition of H^*, compatible with $\triangle_*^{rel}$. To describe the decomposition in convenient terms, we denote by $\triangle_{k,ccl,N}$ the Laplace operator on coclosed $k-$forms on N and introduce some notation

$$\begin{aligned} &V_k := \{\lambda \in \mathrm{Spec}\triangle_{k,ccl,N}\}\backslash\{0\}, \\ &E^k_\lambda := \{\omega \in \Omega^k(N) | \triangle_{k,N}\omega = \lambda\omega, d^t_N\omega = 0\}, \\ &\widetilde{E}^k_\lambda := E^k_\lambda \oplus d_N E^k_\lambda, \quad \mathcal{H}^k(N) := E^k_0. \end{aligned}$$

Here $k = 0, .., \dim N = n$ and the eigenvalues of $\triangle_{k,ccl,N}$ in V_k are counted with their multiplicities, so that each single E^k_λ is a one-dimensional subspace. The eigenvectors for a $\lambda \in V_k, k = 0, .., n$ with multiplicity bigger than 1 are chosen to be mutually orthogonal with respect to the L^2-inner product on N. Further let for each $\mathcal{H}^k(N)$ choose an orthonormal basis of harmonic forms $\{u^k_i\}$ with $i = 1, .., \dim \mathcal{H}^k(N)$.

Then by the Hodge decomposition on N we obtain for any fixed degree $k = 0, .., n+1$ (put $\Omega^{n+1}(N) = \Omega^{-1}(N) = \{0\}$)

$$\begin{aligned} \Omega^{k-1}(N) \oplus \Omega^k(N) = &\left[\bigoplus_{i=1}^{\dim \mathcal{H}^{k-1}(N)} \langle u^{k-1}_i \rangle\right] \oplus \left[\bigoplus_{i=1}^{\dim \mathcal{H}^{k}(N)} \langle u^{k}_i \rangle\right] \\ &\oplus \left[\bigoplus_{\lambda \in V_{k-1}} \widetilde{E}^{k-1}_\lambda\right] \oplus \left[\bigoplus_{\lambda \in V_{k-2}} d_N E^{k-2}_\lambda\right] \oplus \left[\bigoplus_{\lambda \in V_k} E^k_\lambda\right]. \end{aligned} \tag{3.11}$$

Theorem 3.10. *The decomposition* (3.11) *induces an orthogonal decomposition of H^k, compatible with the relative extension $\triangle^{rel}_k$.*

Proof. The decomposition of H^k induced by (3.11) is orthogonal, since the decomposition (3.11) is orthogonal with respect to the L^2-inner product on N. Applying now $d_k, d^t_k d_k$ and $d_{k-1}, d_{k-1}d^t_{k-1}$ to each of the orthogonal components we find that the images remain mutually orthogonal, so we obtain with Proposition 3.9 the desired statement. □

3.3 The Relative Boundary Conditions

By Proposition 3.7 the orthogonal decomposition of H^k in Theorem 3.10 corresponds to a decomposition of $\triangle^{rel}_k$ into an orthogonal sum of self-adjoint operators. This decomposition is discussed by M. Lesch in [L3]. Using the decomposition we can now determine explicitly the boundary conditions for each of the self-adjoint components, up to the self-adjoint extension induced by $\triangle_k^{rel}$ over

$$L^2((0,R), \widetilde{E}_\lambda^{k-1}), \lambda \in V_{k-1}.$$

We do not determine the boundary conditions for these particular self-adjoint components. However even for these components we can reduce the zeta-determinant calculations, which we perform in the next section, to other well-understood problems.

Let $\psi \in E_\lambda^k, \lambda \in V_k, k = 1, .., n$ be a fixed non-zero generator of E_λ^k. Put

$$\begin{aligned}
\xi_1 &:= (0, \psi) \in \Omega^{k-1}(N) \oplus \Omega^k(N), \\
\xi_2 &:= (\psi, 0) \in \Omega^k(N) \oplus \Omega^{k+1}(N), \\
\xi_3 &:= (0, \frac{1}{\sqrt{\lambda}} d_N \psi) \in \Omega^k(N) \oplus \Omega^{k+1}(N), \\
\xi_4 &:= (\frac{1}{\sqrt{\lambda}} d_N \psi, 0) \in \Omega^{k+1}(N) \oplus \Omega^{k+2}(N).
\end{aligned}$$

Then $C_0^\infty((0,R), \langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle)$ is invariant under d, d^t and we obtain a subcomplex of the de Rham complex:

$$0 \to C_0^\infty((0,R), \langle \xi_1 \rangle) \xrightarrow{d_0} C_0^\infty((0,R), \langle \xi_2, \xi_3 \rangle) \xrightarrow{d_1} C_0^\infty((0,R), \langle \xi_4 \rangle) \to 0, \tag{3.12}$$

where d_0, d_1 take the following form with respect to the chosen basis:

$$d_0^\psi = \begin{pmatrix} (-1)^k \partial_x + \frac{c_k}{x} \\ x^{-1}\sqrt{\lambda} \end{pmatrix}, \quad d_1^\psi = \left(x^{-1}\sqrt{\lambda}, \ (-1)^{k+1}\partial_x + \frac{c_{k+1}}{x} \right).$$

By Proposition 3.7 and Theorem 3.10 we obtain for the induced (in the sense of Definition 3.8) self-adjoint extensions:

$$\mathcal{D}(\triangle_k^{rel}) \cap L^2((0,R), E_\lambda^k) = \mathcal{D}(d_{0,\max}^t d_{0,\min}) =: \mathcal{D}(\triangle_{0,\lambda}^k), \tag{3.13}$$

$$\mathcal{D}(\triangle_{k+2}^{rel}) \cap L^2((0,R), d_N E_\lambda^k) = \mathcal{D}(d_{1,\min} d_{1,\max}^t) =: \mathcal{D}(\triangle_{2,\lambda}^k), \tag{3.14}$$

$$\mathcal{D}(\triangle_{k+1}^{rel}) \cap L^2((0,R), \widetilde{E}_\lambda^k) = \mathcal{D}(d_{0,\min} d_{0,\max}^t + d_{1,\max}^t d_{1,\min}) =: \mathcal{D}(\triangle_\lambda^k). \tag{3.15}$$

Note further that $d_0^t d_0$ and $d_1 d_1^t$ both act as the following regular-singular model Laplacian

$$\triangle := -\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\lambda + \left[k + \frac{1}{2} - \frac{n}{2}\right]^2 - \frac{1}{4}\right), \tag{3.16}$$

under the identification of any $\phi = f \cdot \xi_i \in C_0^\infty((0,R), \langle \xi_i \rangle), i = 1,4$ with its scalar part $f \in C_0^\infty(0,R)$. We continue under this identification from here on, as asserted in the next remark.

Remark 3.11. *Let $V = \langle v \rangle$ denote any one-dimensional Hilbert space. Consider a particular type of a differential operator*

$$\mathcal{P} : C_0^\infty((0,R), V) \to C_0^\infty((0,R), V),$$
$$f \cdot v \mapsto (Pf) \cdot v,$$

where $f \in C_0^\infty(0,R)$ and P is a scalar differential operator on $C_0^\infty(0,R)$. We call f and P the "scalar parts" of $f \cdot v$ and $\mathcal{P}$, respectively.

We can reduce without loss of generality the spectral analysis of self-adjoint extensions of $\mathcal{P}$ to the spectral analysis of self-adjoint extensions of P by identifying the $V-$valued functions $f \cdot v$ with their scalar parts. We fix this identification henceforth.

In view of Corollary 2.14 we have to distinguish two cases. The first case is

$$\lambda + \left[k + \frac{1}{2} - \frac{n}{2}\right]^2 - \frac{1}{4} < \frac{3}{4}, \tag{3.17}$$

so that $\triangle$ is in the limit circle case at $x = 0$. Hence (note $\lambda \in V_k$ and so $\lambda \gneqq 0$)

$$p := \sqrt{\lambda + \left[k + \frac{1}{2} - \frac{n}{2}\right]^2} - \frac{1}{2} \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Then we get $\triangle = \triangle_p$ in the notation of Subsection 2.3. Since $p \in (-1/2, 1/2)$ we obtain from Proposition 2.11 for the asymptotics of elements $f \in \mathcal{D}(\triangle_{\max})$:

$$f(x) = c_1(f) \cdot x^{p+1} + c_2(f) \cdot x^{-p} + O(x^{3/2}). \tag{3.18}$$

In the second case

$$\lambda + \left[k + \frac{1}{2} - \frac{n}{2}\right]^2 - \frac{1}{4} \geq \frac{3}{4} \tag{3.19}$$

the Laplacian $\triangle$ is by Corollary 2.14 in the limit point case at $x = 0$ and hence in this case boundary conditions at zero are redundant. We can now compute the domains $\mathcal{D}(\triangle_{0,\lambda}^k)$ and $\mathcal{D}(\triangle_{2,\lambda}^k)$ explicitly.

Lemma 3.12. *Identify any* $\phi \in \mathcal{D}(\triangle^k_{0,\lambda})$ *with its scalar part* $f \in \mathcal{D}(\triangle_{p,\max})$. *Under this identification we obtain for* $\triangle_p$ *in the limit circle case* (3.17) *at* $x=0$, *in the notation of* (3.18)

$$\mathcal{D}(\triangle^k_{0,\lambda}) = \{f \in \mathcal{D}(\triangle_{p,\max}) | c_2(f) = 0, f(R) = 0\}.$$

In the limit point case (3.19) *at* $x=0$, *we obtain*

$$\mathcal{D}(\triangle^k_{0,\lambda}) = \{f \in \mathcal{D}(\triangle_{p,\max}) | f(R) = 0\}.$$

Proof. Consider $\phi \in \mathcal{D}(\triangle^k_{0,\lambda})$ with its scalar part $f \in \mathcal{D}(\triangle_{p,\max})$. We begin with the limit circle case at $x=0$. Since $\phi \in \mathcal{D}(d_{0,\min}) \subset \mathcal{D}(d_{0,\max})$ we deduce from the explicit form of d_0 that $f \in \mathcal{D}(1/x)_{\max}$, where $1/x$ is the obvious multiplication operator. Since with $p \in (-1/2, 1/2)$

$$\mathrm{id}^{-p} \notin \mathcal{D}(1/x)_{\max}$$

we deduce $c_2(f) = 0$. On the other hand we infer from Proposition 3.5

$$f(R) = 0.$$

This proves the inclusion $\subset$ in the first statement. Since both sides of the inclusion define self-adjoint extensions and these are maximally symmetric, the inclusion must be an equality.

For the limit point case at $x=0$ the argumentation is similar up to the fact that the boundary conditions at $x=0$ are redundant by Corollary 2.14. $\square$

Lemma 3.13. *Identify any* $\phi \in \mathcal{D}(\triangle^k_{2,\lambda})$ *with its scalar part* $f \in \mathcal{D}(\triangle_{p,\max})$. *Under this identification we obtain for* $\triangle_p$ *in the limit circle case* (3.17) *at* $x=0$, *in the notation of* (3.18)

$$\mathcal{D}(\triangle^k_{2,\lambda}) = \{f \in \mathcal{D}(\triangle_{p,\max}) | c_2(f) = 0, f'(R) - \frac{(k+1-n/2)}{R} f(R) = 0\}.$$

In the limit point case (3.19) *at* $x=0$, *we obtain*

$$\mathcal{D}(\triangle^k_{2,\lambda}) = \{f \in \mathcal{D}(\triangle_{p,\max}) | f'(R) - \frac{(k+1-n/2)}{R} f(R) = 0\}.$$

Proof. Consider $\phi \in \mathcal{D}(\triangle^k_{2,\lambda})$ with its scalar part $f \in \mathcal{D}(\triangle_{p,\max})$. We begin with the limit circle case at $x=0$. Since $\phi \in \mathcal{D}(d^t_{1,\max})$ we deduce from the explicit form of d^t_1 that $f \in \mathcal{D}(1/x)_{\max}$, where $1/x$ is the obvious multiplication operator. Since with $p \in (-1/2, 1/2)$

$$\mathrm{id}^{-p} \notin \mathcal{D}(1/x)_{\max}$$

we deduce $c_2(f) = 0$ as in the previous lemma. On the other hand we infer from Proposition 3.5 in the degree $k+2$

$$f'(R) - \frac{(k+1-n/2)}{R} f(R) = 0.$$

This proves the inclusion $\subset$ in the first statement. Since both sides of the inclusion define self-adjoint extensions and these are maximally symmetric, the inclusion must be an equality.

For the limit point case at $x = 0$ the argumentation is similar up to the fact that the boundary conditions at $x = 0$ are redundant by Corollary 2.14. $\square$

In contrary to $\mathcal{D}(\triangle^k_{0,\lambda})$ and $\mathcal{D}(\triangle^k_{2,\lambda})$, it is not straightforward to determine $\mathcal{D}(\triangle^k_{\lambda})$ explicitly. However for the purpose of later calculations of zeta determinants it is sufficient to observe that $\triangle^k_{0,\lambda}, \triangle^k_{2,\lambda}, \triangle^k_{\lambda}$ are Laplacians of the complex (3.12) with relative boundary conditions and hence satisfy the following spectral relation:

$$\mathrm{Spec}(\triangle^k_{\lambda})\backslash\{0\} = \mathrm{Spec}(\triangle^k_{0,\lambda})\backslash\{0\} \sqcup \mathrm{Spec}(\triangle^k_{2,\lambda})\backslash\{0\},$$

where the eigenvalues are counted with their multiplicities.

Next consider $\mathcal{H}^k(N)$ with the fixed orthonormal basis $\{u^k_i\}, i = 1,..,\dim \mathcal{H}^k(N)$. Observe that for any i the subspace $C^\infty_0((0,R), \langle 0 \oplus u^k_i, u^k_i \oplus 0\rangle)$ is invariant under d, d^t and we obtain a subcomplex of the de Rham complex

$$0 \to C^\infty_0((0,R), \langle 0 \oplus u^k_i, \rangle) \xrightarrow{d} C^\infty_0((0,R), \langle u^k_i \oplus 0\rangle) \to 0,$$
$$d = (-1)^k \partial_x + \frac{c_k}{x},$$

where the action of d is of scalar type under the identification fixed in Remark 3.11. We continue under this identification. By Proposition 3.7 and Theorem 3.10 we obtain for the induced self-adjoint extensions

$$\begin{aligned}\mathcal{D}(\triangle^{rel}_k) \cap L^2((0,R),&\langle 0 \oplus u^k_i\rangle) = \mathcal{D}(d^t_{\max} d_{\min}) = \\ &= \mathcal{D}\left((-1)^{k+1}\partial_x + \frac{c_k}{x}\right)_{\max}\left((-1)^k\partial_x + \frac{c_k}{x}\right)_{\min},\end{aligned} \tag{3.20}$$

$$\begin{aligned}\mathcal{D}(\triangle^{rel}_{k+1}) \cap L^2((0,R),&\langle u^k_i \oplus 0\rangle) = \mathcal{D}(d_{\min} d^t_{\max}) = \\ &= \mathcal{D}\left((-1)^{k}\partial_x + \frac{c_k}{x}\right)_{\min}\left((-1)^{k+1}\partial_x + \frac{c_k}{x}\right)_{\max}.\end{aligned} \tag{3.21}$$

Depending on the explicit value of $c_k = (-1)^k(k - n/2)$ these domains are self-adjoint extensions of regular-singular model Laplacians in limit point or limit circle case at $x = 0$. For model Laplacians in the limit circle case at $x = 0$ the domains are determined in Subsection 2.3. In the limit point case at $x = 0$ the boundary conditions at $x = 0$ are redundant by Corollary 2.14 and the boundary conditions at $x = R$ are determined in Proposition 3.5.

4 Functional Determinants for Regular-Singular Sturm-Liouville Operators

Different sources have analyzed zeta-determinants of Laplace operators over a bounded generalized cone. Under the condition that all self-adjoint extensions in the L^2-Hilbert space of the Laplace operator coincide at the cone singularity, calculations are provided in the joint work of J.S. Dowker and K. Kirsten in [DK1].

The self-adjoint extensions of the de Rham Laplacians on a bounded generalized cone however need not necessarily coincide at the cone singularity. This situation was considered by K. Kirsten, P. Loya and J. Park in [KLP1]. In their discussion, however, only pure Dirichlet boundary conditions at the cone base have been considered.

More precisely Kirsten, Loya and Park consider in [KLP1] a finite direct sum of regular-singular model Laplacians in the limit circle case at the singular end and compute their zeta-determinants for general boundary conditions at the singular and Dirichlet boundary conditions at the regular end.

The argumentation in the preprints [KLP1] and [KLP2] is based on the Contour integral method, which a priori need not to apply in the regular-singular setup and is only formally a consequence of the Argument Principle.

We verify explicitly that the Contour integral method indeed applies in the regular-singular setup, however for Neumann boundary conditions at the cone base. The Dirichlet boundary conditions can be discussed in a similar manner. The author intends to publish the proof for the applicability of the Contour integral method in the regular-singular setup with Dirichlet boundary conditions as an appendix to [KLP2].

Our proof is the basis for the integral representation of the zeta-function. Otherwise the Contour integral method only gives information on the pole structure of the zeta-function, but no results on the zeta-determinants.

The determinants of a more general class of operators, the regular-singular Sturm-Liouville operators, have been discussed in scalar setup by M. Lesch in [L].

We extend the treatment of [KLP1] to generalized Neumann boundary conditions at the cone base. This allows us to compute the zeta-determinants of the Laplace operators with the geometrically relevant relative boundary conditions, by a combination of our results with the results by [L] and [KLP1].

In view of [DK1] and [KLP1] this provides a complete picture on the Laplace operator on a bounded generalized cone.

4.1 Self-adjoint Realizations

We consider the following model setup. Let the operator L be the following regular-singular Sturm-Liouville operator

$$L = -\frac{d^2}{dx^2} + \frac{1}{x^2}A : C_0^\infty((0,R),\mathbb{C}^q) \to C_0^\infty((0,R),\mathbb{C}^q),$$

where for any fixed $q \in \mathbb{N}$, $C_0^\infty((0,R),\mathbb{C}^q)$ denotes the space of smooth $\mathbb{C}^q$-valued functions with compact support in $(0,R)$. Let the tangential operator A be a symmetric $q \times q$ matrix and choose on $\mathbb{C}^q$ an orthonormal basis of A-eigenvectors. Then we can write:

$$L = \bigoplus_{\lambda \in \mathrm{Spec}(A)} -\frac{d^2}{dx^2} + \frac{\lambda}{x^2}.$$

Following [KLP1, KLP2] we need a classification of boundary conditions at $x = 0$ for self-adjoint realizations of L. In view of Corollary 2.14 we restrict to the case

$$\mathrm{Spec}(A) \subset [-1/4, 3/4),$$

so that L is a finite direct sum of model Laplace operators in the limit circle case at $x = 0$ and $x = R$. In this case boundary conditions must be posed at both boundary components.

Fix a counting on Spec(A) as follows

$$-\frac{1}{4} = \lambda_1 = \cdots = \lambda_{q_0} < \lambda_{q_0+1} \leq \cdots \leq \lambda_{q=q_0+q_1} < \frac{3}{4}.$$

Denote by E_l the λ_l-eigenspace of A. We count the eigenvalues of A with their multiplicities, so E_l is understood to be one-dimensional with $E_l = \langle e_l \rangle$. Over $C_0^\infty((0,R),E_l), l = 1,...,q$ the differential operator L reduces to a model Laplace operator (recall the convention fixed in Remark 3.11)

$$-\frac{d^2}{dx^2} + \frac{\lambda_l}{x^2} : C_0^\infty(0,R) \to C_0^\infty(0,R).$$

Consider the maximal closed extension $L_{\max}$ of the differential operator L. Any $\phi_l \in \mathcal{D}(L_{\max}) \cap L^2((0,R),E_l), l = 1,...,q$ is given by $f_l \cdot e_l$ where e_l is the generator of the one-dimensional eigenspace E_l and by definition (see also (2.5))

$$f_l \in \mathcal{D}\left(-\frac{d^2}{dx^2}+\frac{\lambda_l}{x^2}\right)_{\max} \subset H^2_{loc}(0,R].$$

We identify any ϕ_l with its scalar part f_l and observe by Proposition 2.11 that $f_l(x)$ has the following asymptotics at $x=0$:

$$c_l\sqrt{x}+c_{q+l}\sqrt{x}\log x+O(x^{3/2}), \text{ as } l=1,...,q_0. \tag{4.1}$$

$$c_l x^{\nu_l+\frac{1}{2}}+c_{q+l}x^{-\nu_l+\frac{1}{2}}+O(x^{3/2}), \text{ as } l=q_0+1,...,q. \tag{4.2}$$

$$\text{with } \nu_l := \sqrt{\lambda_l+\frac{1}{4}}.$$

A general element $\phi \in \mathcal{D}(L_{\max})$ decomposes into a direct sum of such $\phi_l, l = 1,...,q$, each of them of the asymptotics above. This defines a vector for any $\phi \in \mathcal{D}(L_{\max})$

$$\vec{\phi} := (c_1, ...c_{2q})^T \in \mathbb{C}^{2q}.$$

Consider now any $\phi, \psi \in \mathcal{D}(L_{\max})$ and the associated vectors $\vec{\phi}, \vec{\psi}$. Each of the components $\phi_l, \psi_l, l = 1,..,q$ lies in the maximal domain of the corresponding model Laplace operator and thus is continuous over $(0,R]$ and differentiable over $(0,R)$ with the derivatives ϕ_l', ψ_l' extending continuously to $x=R$. We impose boundary conditions at $x=R$ as follows

$$\phi'(R)+\alpha\phi(R)=0, \quad \psi'(R)+\alpha\psi(R)=0,$$

where these equations are to be read componentwise and $\alpha \in \mathbb{R}$.

In [KLP2, Section 3] the classical results on self-adjoint extensions are reviewed and based on the Lagrange identity for L (similar to Lemma 2.12) the boundary conditions at $x=0$ for the self-adjoint extensions of L are classified in terms of Lagrangian subspaces.

As a consequence of [KLP2, Corollary 3.5, (4.2)] the self-adjoint realizations of L with fixed generalized Neumann boundary conditions at $x=R$ are characterized as follows:

$$\mathcal{D}(\mathcal{L}) = \{\phi \in \mathcal{D}(L_{\max}) | \phi'(R)+\alpha\phi(R)=0, \ (\mathcal{A}\ \mathcal{B})\vec{\phi}=0\}, \tag{4.3}$$

where the matrices $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{q\times q}$ are fixed according to the conditions of [KLP2, Corollary 3.5], i.e. $(\mathcal{A}\ \mathcal{B}) \in \mathbb{C}^{q\times 2q}$ is of full rank q and $\mathcal{A}'\mathcal{B}^*$ is self-adjoint, where $\mathcal{A}'$ is the matrix $\mathcal{A}$ with the first q_0 columns multiplied by (-1).

Remark 4.1. *Note that in general $\mathcal{L}$ does not decompose into a finite sum of one-dimensional boundary value problems, since the matrices $(\mathcal{A}, \mathcal{B})$ need not to be diagonal. Therefore the computations below do not reduce to a one-dimensional discussion.*

4.2 Functional Determinants

In this subsection we continue with the analysis of the self-adjoint realization $\mathcal{D}(\mathcal{L})$, fixed in (4.3). Our aim is to construct explicitly the analytic continuation of the associated zeta-function to $s = 0$ and to compute the zeta-regularized determinant of $\mathcal{L}$.

We follow the ideas of [KLP1, KLP2], where however only Dirichlet boundary conditions at the cone base $x = R$ have been considered. We extend their approach to generalized Neumann boundary conditions at the cone base, in order to apply the calculations to the relative self-adjoint extension of the Laplace Operator on a bounded generalized cone.

Furthermore we put the arguments on a thorough footing by proving applicability of the Contour integral method in the regular-singular setup.

We introduce the following $q \times q$ matrices in terms of Bessel functions of first and second kind:

$$J^{\pm}(\mu) := \begin{pmatrix} \left(\kappa J_{\pm 0}(\mu R) + \mu\sqrt{R} J'_{\pm 0}(\mu R)\right) \cdot Id_{q_0} & 0 \\ 0 & \operatorname{diag}\left[2^{\pm\nu_l}\Gamma(1 \pm \nu_l)\mu^{\mp\nu_l}\left(\kappa J_{\pm\nu_l}(\mu R) + \mu\sqrt{R} J'_{\pm\nu_l}(\mu R)\right)\right] \end{pmatrix},$$

where the diagonal block matrix in the right low corner has entries for $l = q_0 + 1, ..., q$. Further we have introduced new constants

$$\kappa := \frac{1}{2\sqrt{R}} + \alpha\sqrt{R}, \quad \nu_l := \sqrt{\lambda_l + \frac{1}{4}}, \ l = q_0 + 1, ..., q,$$

to simplify notation. Moreover the function $J_{-0}(\mu R)$ is defined as follows:

$$J_{-0}(\mu x) := \frac{\pi}{2} Y_0(\mu x) - (\log \mu - \log 2 + \gamma) J_0(\mu x)$$

with γ being the Euler constant and where we fix for the upcoming discussion the branch of logarithm in $\mathbb{C}\backslash\mathbb{R}^+$ with $0 \leq Im \log(z) < 2\pi$. With this notation we can now formulate the implicit eigenvalue equation for $\mathcal{L}$.

Proposition 4.2. *μ^2 is an eigenvalue of $\mathcal{L}$ if and only if the following equation is satisfied*

$$F(\mu) := \det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ J^+(\mu) & J^-(\mu) \end{pmatrix} \overset{!}{=} 0.$$

Proof. Any μ^2-eigenvector ϕ of $\mathcal{L}$ is given by a direct sum of scalar functions $\phi_l, l = 1, .., q$, which are in the λ_l-eigenspace of A for any fixed $x \in (0, R]$. Each ϕ_l arises as a solution to the Bessel equation

$$-\phi_l'' + \frac{\lambda_l}{x^2}\phi_l = \mu^2\phi_l.$$

Putting $\nu_l := \sqrt{\lambda_l + 1/4}$ we can rewrite the equation as follows:

$$-\phi_l'' + \frac{1}{x^2}\left(\nu_l - \frac{1}{4}\right)\phi_l = \mu^2\phi_l. \tag{4.4}$$

The general solution to this Bessel equation is given in terms of J_* and Y_*, Bessel functions of first and second kind, respectively.

For $l = 1, .., q_0$ we have $\lambda_l = -1/4$ and hence $\nu_l = 0$. In this case the Bessel equation (4.4) has two linearly independent solutions, $\sqrt{x}J_0(\mu x)$ and $\sqrt{x}Y_0(\mu x)$. Following [KLP2, Section 4.2] we write for the general solution

$$c_l\sqrt{x}J_0(\mu x) + c_{q+l}\sqrt{x}J_{-0}(\mu x), \text{ as } l = 1, ..., q_0, \tag{4.5}$$

where c_l, c_{q+l} are constants and $J_{-0}(\mu x) = \frac{\pi}{2}Y_0(\mu x) - (\log\mu - \log 2 + \gamma)J_0(\mu x)$ with γ being the Euler constant. Note from [AS, p.360] with $H_k = 1 + 1/2 + ... + 1/k$ and $z \in \mathbb{C}$:

$$\frac{\pi}{2}Y_0(z) = (\log z - \log 2 + \gamma)J_0(z) - \sum_{k=1}^{\infty}\frac{H_k(-z^2/4)^k}{(k!)^2},$$

with $H_k = 1 + 1/2 + .. + 1/k$. Thus by definition we obtain

$$J_{-0}(\mu x) = \log x \cdot J_0(\mu x) - \sum_{k=1}^{\infty}\frac{H_k(-(\mu x)^2/4)^k}{(k!)^2}.$$

For $l = q_0 + 1, .., q$ we have $\lambda_l \in (-1/4, 3/4)$ and hence $\nu_l \in (0, 1)$, in particular ν_l is non-integer. In this case the Bessel equation (4.4) has two linearly independent solutions $\sqrt{x}J_{\nu_l}(\mu x)$ and $\sqrt{x}J_{-\nu_l}(\mu x)$. Following [KLP2, Section 4.2] we write for the general solution

$$c_l 2^{\nu_l}\Gamma(1 + \nu_l)\mu^{-\nu_l}\sqrt{x}J_{\nu_l}(\mu x) + c_{q+l}2^{-\nu_l}\Gamma(1 - \nu_l)\mu^{\nu_l}\sqrt{x}J_{-\nu_l}(\mu x), \tag{4.6}$$
$$\text{as } l = q_0 + 1, ..., q.$$

Now we deduce from the standard series representation of Bessel functions [AS, p. 360] the following asymptotic behaviour as $x \to 0$:

$$\sqrt{x}J_0(\mu x) = \sqrt{x} + \sqrt{x}O(x^2), \qquad (4.7)$$

$$\sqrt{x}J_{-0}(\mu x) = \sqrt{x}\log x + \sqrt{x}\log x \cdot O(x^2) + \sqrt{x}O(x^2), \qquad (4.8)$$

$$2^{\pm\nu_l}\Gamma(1 \pm \nu_l)\mu^{\mp\nu_l}\sqrt{x}J_{\pm\nu_l}(\mu x) = x^{\pm\nu_l+1/2} + x^{\pm\nu_l+1/2}O(x^2), \qquad (4.9)$$

where $O(x^2)$ is given by power-series in $(x\mu)^2$ with no constant term. Hence the asymptotic behaviour at $x = 0$ of the general solutions (4.5) and (4.6) corresponds to the asymptotics (4.1) and (4.2), respectively. Organizing the constants $c_l, c_{q+l}, l = 1, .., q$ into a vector $\vec{\phi} = (c_1, ..., c_{2q})$, we obtain

$$(\mathcal{A}, \mathcal{B})\vec{\phi} = 0,$$

since by assumption, $\phi \in \mathcal{D}(\mathcal{L})$. We now evaluate the generalized Neumann boundary conditions at the regular boundary $x = R$.

$$\phi_l'(R) + \alpha\phi_l(R) = 0 \;\Rightarrow\; c_l \cdot \{(\frac{1}{2\sqrt{R}} + \alpha\sqrt{R})J_0(\mu R) + \mu\sqrt{R}J_0'(\mu R)\} +$$

$$+ \; c_{q+l} \cdot \{(\frac{1}{2\sqrt{R}} + \alpha\sqrt{R})J_{-0}(\mu R) + \mu\sqrt{R}J_{-0}'(\mu R)\} = 0, \text{ as } l = 1, ..., q_0.$$

$$c_l \cdot 2^{\nu_l}\Gamma(1+\nu_l)\mu^{-\nu_l}\{(\frac{1}{2\sqrt{R}} + \alpha\sqrt{R})J_{\nu_l}(\mu R) + \mu\sqrt{R}J_{\nu_l}'(\mu R)\} +$$

$$+ \; c_{q+l} \cdot 2^{-\nu_l}\Gamma(1-\nu_l)\mu^{\nu_l}\{(\frac{1}{2\sqrt{R}} + \alpha\sqrt{R})J_{-\nu_l}(\mu R) + \mu\sqrt{R}J_{-\nu_l}'(\mu R)\} = 0,$$

$$\text{as } l = q_0 + 1, ..., q,$$

We can rewrite this system of equations in a compact form as follows

$$(J^+(\mu); J^-(\mu))\vec{\phi} = 0,$$

where the matrices $J^{\pm}(\mu)$ are defined above.

We obtain equations which have to be satisfied by the μ^2-eigenvectors of the self-adjoint realization $\mathcal{L}$:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ J^+(\mu) & J^-(\mu) \end{pmatrix} \vec{\phi} = 0.$$

This equation has non-trivial solutions if and only if the determinant of the matrix in front of the vector is zero. Hence we finally arrive at the following implicit eigenvalue equation

$$F(\mu) := \det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ J^+(\mu) & J^-(\mu) \end{pmatrix} \stackrel{!}{=} 0.$$

□

Proposition 4.3. *With* $\nu_l = \sqrt{\lambda_l + 1/4}$ *and* $\kappa = \frac{1}{2\sqrt{R}} + \alpha\sqrt{R}$

$$F(0) = \det \begin{pmatrix} & \mathcal{A} & & \mathcal{B} \\ \kappa Id_{q_o} & 0 & (\kappa \log R + \frac{1}{\sqrt{R}}) Id_{q_o} & 0 \\ 0 & \mathrm{diag}(\kappa R^{\nu_l} + \nu_l R^{\nu_l - \frac{1}{2}}) & 0 & \mathrm{diag}(\kappa R^{-\nu_l} - \nu_l R^{-\nu_l - \frac{1}{2}}) \end{pmatrix}$$

Proof. The asymptotics (4.7), (4.8) and (4.9), where $O(x^2)$ is in fact power-series in $(x\mu)^2$ with no constant term, imply by straightforward computations:

$$\begin{aligned} \kappa J_0(\mu R) + \mu\sqrt{R} J_0'(\mu R) &\to \kappa, \text{ as } \mu \to 0, \\ \kappa J_{-0}(\mu R) + \mu\sqrt{R} J_{-0}'(\mu R) &\to \kappa \cdot \log R + \frac{1}{\sqrt{R}}, \text{ as } \mu \to 0, \\ 2^{\pm\nu_l}\Gamma(1 \pm \nu_l)\mu^{\mp\nu_l} J_{\pm\nu_l}(\mu R) &\to R^{\pm\nu_l}, \text{ as } \mu \to 0, \\ 2^{\pm\nu_l}\Gamma(1 \pm \nu_l)\mu^{\mp\nu_l}\mu\sqrt{R} J_{\pm\nu_l}'(\mu R) &\to \pm\nu_l R^{\pm\nu_l - \frac{1}{2}}, \text{ as } \mu \to 0, \end{aligned}$$

where $l = q_0 + 1, .., q$. These relations prove the statement. □

The next proposition is similar to [KLP2, Proposition 4.3] and we use the notation therein.

Proposition 4.4. *Let* $\Upsilon \subset \mathbb{C}$ *be a closed angle in the right half-plane. Then as* $|x| \to \infty, x \in \Upsilon$ *we can write*

$$\begin{aligned} F(ix) = \rho x^{|\nu| + \frac{q}{2}} e^{qxR} (2\pi)^{-\frac{q}{2}} (\widetilde{\gamma} - \log x)^{q_o} \times \\ p((\widetilde{\gamma} - \log x)^{-1}, x^{-1}) \left(1 + O\left(\frac{1}{x}\right)\right), \end{aligned}$$

where γ *is the Euler constant,* $|\nu| = \nu_{q_o+1} + ... + \nu_q$. *Moreover, as* $|x| \to \infty$ *with* $x \in \Upsilon$, $O(1/x)$ *is a power-series in* x^{-1} *with no constant term. Furthermore we have set:*

$$\widetilde{\gamma} := \log 2 - \gamma, \ \rho := \prod_{l=q_o+1}^{q} 2^{-\nu_l}\Gamma(1 - \nu_l),$$

$$p(x, y) := \det \begin{pmatrix} & \mathcal{A} & \mathcal{B} \\ x \cdot Id_{q_0} & 0 & \\ 0 & \mathrm{diag}\,[\tau_l y^{2\nu_l}] & Id_q \end{pmatrix}, \quad \text{with } \tau_l := \frac{\Gamma(1 + \nu_l)}{\Gamma(1 - \nu_l)} 2^{2\nu_l},$$

where the submatrix $\mathrm{diag}\,[\tau_l y^{2\nu_l}]$ *has entries for* $l = q_0 + 1, .., q$.

Proof. We present $F(ix)$ in terms of modified Bessel functions of first and second kind. We use following well-known relations

$$(iz)^{-\nu}J_\nu(iz)=z^{-\nu}I_\nu(z),\ J'_\nu(z)=J_{\nu-1}(z)-\frac{\nu}{z}J_\nu(z)$$

to analyze the building bricks of $F(ix)$ where we put with $l=q_0+1,..,q$

$$A_l^\pm:=2^{\pm\nu_l}\Gamma(1\pm\nu_l)(ix)^{\mp\nu_l}\left(\kappa J_{\pm\nu_l}(ixR)+ix\sqrt{R}J'_{\pm\nu_l}(ixR)\right)=2^{\pm\nu_l}\Gamma(1\pm\nu_l)\cdot\left((\kappa\mp\frac{\nu_l}{\sqrt{R}})x^{\mp\nu_l}I_{\pm\nu_l}(xR)+\sqrt{R}x^{\mp\nu_l+1}I_{\pm\nu_l-1}(xR)\right)$$

$$B:=\kappa J_0(ixR)+\sqrt{R}ixJ'_0(ixR)=\kappa I_0(xR)+\sqrt{R}xI'_0(xR),$$

and using the identity $J_{-0}(ixR)=-(\log x-\widetilde{\gamma})I_0(xR)-K_0(xR)$ from [KLP2, Section 4.3, p.20] where K_* denotes the modified Bessel function of second kind:

$$C:=\kappa J_{-0}(ixR)+\sqrt{R}ixJ'_{-0}(ixR)=\kappa J_{-0}(ixR)+\sqrt{R}\frac{d}{dR}J_{-0}(ixR)=$$
$$=\kappa(-(\log x-\widetilde{\gamma})I_0(xR)-K_0(xR))+\sqrt{R}\frac{d}{dR}(-(\log x-\widetilde{\gamma})I_0(xR)-$$
$$-K_0(xR))=\kappa(-(\log x-\widetilde{\gamma})I_0(xR)-K_0(xR))+$$
$$+\sqrt{R}(-(\log x-\widetilde{\gamma})xI'_0(xR)-xK'_0(xR)).$$

Now in order to compute the asymptotics of $F(ix)$ we use following property of the Bessel functions: as $x\to\infty$ with $x\in\Upsilon$ we have by [AS, p. 377]

$$I_\nu(x),I'_\nu(x)\sim\frac{e^x}{\sqrt{2\pi x}}\left(1+O(x^{-1})\right)$$
$$\Rightarrow\ \frac{I_\nu(xR)}{I_{-\nu}(xR)}\sim 1,\ \frac{I_{-\nu-1}(xR)}{I_{-\nu}(xR)}\sim 1,\ \frac{I_{\nu-1}(xR)}{I_{-\nu}(xR)}\sim 1,$$

where as $|x|\to\infty$ with $x\in\Upsilon$, $O(x^{-1})$ is a power-series in x^{-1} with no constant term. From here we obtain the asymptotics of the terms $A_l^\pm,B,C$ as $x\to\infty,x\in\Upsilon$ with the same meaning for $O(x^{-1})$:

$$A_l^+=2^{-\nu_l}\Gamma(1-\nu_l)x^{\nu_l}\frac{e^{xR}}{\sqrt{2\pi xR}}\left[2^{2\nu_l}\frac{\Gamma(1+\nu_l)}{\Gamma(1-\nu_l)}x^{-2\nu_l}\right]\times\left((\kappa-\frac{\nu_l}{\sqrt{R}})+x\sqrt{R}\right)\cdot(1+O(x^{-1}))=$$
$$=2^{-\nu_l}\Gamma(1-\nu_l)x^{\nu_l+1/2}\frac{e^{xR}}{\sqrt{2\pi}}\left[2^{2\nu_l}\frac{\Gamma(1+\nu_l)}{\Gamma(1-\nu_l)}x^{-2\nu_l}\right]\times(1+O(x^{-1})).$$

Similarly we compute

$$A_l^- = 2^{-\nu_l}\Gamma(1-\nu_l)x^{\nu_l+1/2}\frac{e^{xR}}{\sqrt{2\pi}}\cdot(1+O(x^{-1})),$$

$$B = \frac{e^{xR}}{\sqrt{2\pi xR}}(\kappa + x\sqrt{R})\cdot(1+O(x^{-1})) = \sqrt{x}\frac{e^{xR}}{\sqrt{2\pi}}\cdot(1+O(x^{-1})),$$

$$C = \frac{e^{xR}}{\sqrt{2\pi xR}}(\widetilde{\gamma} - \log x)(\kappa + x\sqrt{R})\cdot(1+O(x^{-1})) =$$
$$= \sqrt{x}\frac{e^{xR}}{\sqrt{2\pi}}(\widetilde{\gamma} - \log x)\cdot(1+O(x^{-1})),$$

where we have further used the fact that by [AS, p. 378] $K_0(xR)$ is exponentially decaying as $|x| \to \infty, x \in \Upsilon$. Now substitute these asymptotics into the definition of $F(ix)$ and obtain

$$F(ix) = \left[\prod_{l=q_0+1}^{q} 2^{-\nu_l}\Gamma(1-\nu_l)x^{\nu_l+\frac{1}{2}}\right]\left(\frac{e^{xR}}{\sqrt{2\pi}}\right)^q x^{q_0/2}(\widetilde{\gamma} - \log x)^{q_0}\times$$
$$\det\begin{pmatrix} \mathcal{A} & & \mathcal{B} \\ (\widetilde{\gamma}-\log x)^{-1}Id_{q_0} & 0 & \\ 0 & * & Id_q \end{pmatrix}(1+O(x^{-1})),$$
$$\text{where } * = \operatorname{diag}\left(2^{2\nu_l}\frac{\Gamma(1+\nu_l)}{\Gamma(1-\nu_l)}x^{-2\nu_l}\right).$$

□

Using the expansion in [KLP1, (4.9)] we evaluate the asymptotics of $p((\widetilde{\gamma}-\log x)^{-1}, x^{-1})$ and obtain in the notation introduced in the statement of Proposition 4.4

$$F(ix) = a_{j_o\alpha_o}\rho x^{|\nu|+\frac{q}{2}-2\alpha_o}\left(\frac{e^{xR}}{\sqrt{2\pi}}\right)^q(\widetilde{\gamma}-\log x)^{q_o-j_o}(1+G(x)), \qquad (4.10)$$

where $G(x) = O(\frac{1}{\log(x)})$ and $G'(x) = O(\frac{1}{x\log^2(x)})$ as $|x| \to \infty$ with x inside any fixed closed angle of the right half plane of $\mathbb{C}$. The coefficients $\alpha_0, j_0, a_{j_o\alpha_o}$ are defined in [KLP2, Section 2.1] and are characteristic values of the boundary conditions $(\mathcal{A}, \mathcal{B})$ at the cone singularity. We recall their definition here for convenience.

Definition 4.5. *The expression $p(x,y)$ defined in Proposition 4.4 can be written as a finite sum*

$$p(x,y) = \sum a_{j\alpha}x^j y^\alpha.$$

The characteristic values $\alpha_0, j_0, a_{\alpha_o j_o}$ are defined as follows:

(i) *The coefficient α_0 is the smallest of all exponents α with $a_{j\alpha} \neq 0$.*

(ii) *The coefficient j_0 is the smallest of all exponents j with $a_{j\alpha_o} \neq 0$.*

(iii) *The coefficient $a_{j_o\alpha_o}$ is the coefficient in the polynomial $p(x,y)$ of the summand $x^{j_o}y^{\alpha_o}$.*

Unfortunately the asymptotic expansion, obtained in Proposition 4.4, does not hold uniformly for arguments z of $F(z)$ in a fixed closed angle of the positive real axis. This gap is closed by the following proposition.

Proposition 4.6. *Fix any $\theta \in [0,\pi)$ and put $\Omega := \{z \in \mathbb{C} \| \arg(z)| \leq \theta\}$. Then for $|z| \to \infty, z \in \Omega$ we have the following uniform expansion:*

$$F(z) = \prod_{l=q_0+1}^{q} \left\{ 2^{-\nu_l}\Gamma(1-\nu_l) z^{\nu_l+1/2}\sqrt{\frac{2}{\pi}}\cos(zR + \frac{\nu_l\pi}{2} + \frac{\pi}{4}) \right\} \times \\ \times \left\{ \sqrt{\frac{2z}{\pi}}(\log z - \widetilde{\gamma})\cos(zR - \frac{3}{4}\pi) \right\}^{q_0} \cdot \det M(z).$$

Here the matrix $M(z)$ is given as follows:

$$M(z) = \left(\begin{array}{cccc} \multicolumn{2}{c}{\mathcal{A}} & \multicolumn{2}{c}{\mathcal{B}} \\ b(z)\cdot Id_{q_0} & 0 & c(z)\cdot Id_{q_0} & 0 \\ 0 & \mathrm{diag}\,[a_l^+(z)] & 0 & \mathrm{diag}\,[a_l^-(z)] \end{array} \right),$$

where for $l = q_0+1,..,q$ we have

$$a_l^+(z) = 2^{2\nu_l}\frac{\Gamma(1+\nu_l)}{\Gamma(1-\nu_l)} z^{-2\nu_l} \frac{\cos(zR - \frac{\nu_l\pi}{2} + \frac{\pi}{4})}{\cos(zR + \frac{\nu_l\pi}{2} + \frac{\pi}{4})} \cdot \left(1 + \frac{f_l^+(z)}{\cos(zR - \frac{\nu_l\pi}{2} + \frac{\pi}{4})}\right),$$
$$a_l^-(z) = 1 + \frac{f_l^-(z)}{\cos(zR + \frac{\nu_l\pi}{2} + \frac{\pi}{4})}, \quad b(z) = \frac{1}{\widetilde{\gamma} - \log z} \cdot \left(1 + \frac{f_b(z)}{\cos(zR - \frac{3}{4}\pi)}\right),$$
$$c(z) = 1 + \frac{f_c(z)}{\cos(zR - \frac{3}{4}\pi)},$$

and the functions $f_l^{\pm}(z), f_b(z), f_c(z)$ have the following asymptotic behaviour *as* $|z| \to \infty, z \in \Omega$

$$\begin{aligned} f_l^{\pm}(z) &= e^{|\mathrm{Im}(zR)|} O\left(\frac{1}{|z|}\right), \ \frac{d}{dz} f_l^{\pm}(z) = e^{|\mathrm{Im}(zR)|} O\left(\frac{1}{|z|}\right), \\ f_b(z) &= e^{|\mathrm{Im}(zR)|} O\left(\frac{1}{|z|}\right), \ \frac{d}{dz} f_b(z) = e^{|\mathrm{Im}(zR)|} O\left(\frac{1}{|z|}\right), \\ f_c(z) &= e^{|\mathrm{Im}(zR)|} O\left(\frac{1}{|\log z|}\right), \ \frac{d}{dz} f_c(z) = e^{|\mathrm{Im}(zR)|} O\left(\frac{1}{|\log z|}\right). \end{aligned}$$

Proof. The formulas [AS, 9.2.1, 9.2.2] provide the standard asymptotic behaviour of Bessel functions as $|z| \to \infty, z \in \Omega$

$$\begin{aligned} J_\nu(z) &= \sqrt{\frac{2}{\pi z}} \left(\cos(z - \frac{\nu\pi}{2} - \frac{\pi}{4}) + f(z)\right), \ f(z) = e^{|\mathrm{Im}(z)|} O\left(\frac{1}{|z|}\right), \\ Y_\nu(z) &= \sqrt{\frac{2}{\pi z}} \left(\sin(z - \frac{\nu\pi}{2} - \frac{\pi}{4}) + g(z)\right), \ g(z) = e^{|\mathrm{Im}(z)|} O\left(\frac{1}{|z|}\right). \end{aligned}$$

Here $\nu \in \mathbb{R}$ and the expansions are uniform in the closed angle Ω. Moreover we infer from the more explicit form of asymptotics in [GRA, 8.451]:

$$\frac{d}{dz} f(z) = e^{|\mathrm{Im}(z)|} O\left(\frac{1}{|z|}\right), \ \frac{d}{dz} g(z) = e^{|\mathrm{Im}(z)|} O\left(\frac{1}{|z|}\right).$$

We apply these asymptotics in order to analyze the asymptotic behaviour as $|z| \to \infty, z \in \Omega$ of the following building bricks of $F(z)$:

$$\begin{aligned} A_l^{\pm} &:= 2^{\pm\nu_l} \Gamma(1 \pm \nu_l) z^{\mp\nu_l} \left(\kappa J_{\pm\nu_l}(zR) + z\sqrt{R} J'_{\pm\nu_l}(zR)\right), \ l = q_0 + 1, .., q, \\ B &:= \kappa J_0(zR) + z\sqrt{R} J'_0(zR), \\ C &:= \kappa J_{-0}(zR) + z\sqrt{R} J'_{-0}(zR). \end{aligned}$$

Straightforward application of the asymptotics for $J_\nu(z)$ and $Y_\nu(z)$ as $|z| \to \infty, z \in \Omega$ and furthermore the use of the well-known formulas

$$\begin{aligned} J'_\nu(z) &= J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z), \\ J'_0(z) &= -J_1(z), \ Y'_0(z) = -Y_1(z), \end{aligned}$$

lead to the following intermediate results:

$$A_l^+ = 2^{\nu_l}\Gamma(1+\nu_l)z^{-\nu_l+1/2}\sqrt{\frac{2}{\pi}}\cos(zR-\frac{\nu_l\pi}{2}+\frac{\pi}{4})\cdot\left(1+\frac{f_l^+(z)}{\cos(zR-\frac{\nu_l\pi}{2}+\frac{\pi}{4})}\right),$$

$$A_l^- = 2^{-\nu_l}\Gamma(1-\nu_l)z^{\nu_l+1/2}\sqrt{\frac{2}{\pi}}\cos(zR+\frac{\nu_l\pi}{2}+\frac{\pi}{4})\cdot\left(1+\frac{f_l^-(z)}{\cos(zR+\frac{\nu_l\pi}{2}+\frac{\pi}{4})}\right),$$

$$B = -\sqrt{\frac{2z}{\pi}}\cos(zR-\frac{3}{4}\pi)\cdot\left(1+\frac{f_b(z)}{\cos(zR-\frac{3}{4}\pi)}\right),$$

$$C = \sqrt{\frac{2z}{\pi}}(\log z-\widetilde{\gamma})\cos(zR-\frac{3}{4}\pi)\cdot\left(1+\frac{f_c(z)}{\cos(zR-\frac{3}{4}\pi)}\right),$$

where the functions $f_l^\pm(z), f_b(z)$ and their derivatives are of the asymptotics $e^{|\mathrm{Im}(zR)|}O\left(1/|z|\right)$ as $|z|\to\infty, z\in\Omega$. The function $f_c(z)$ and its derivative are of the asymptotics $e^{|\mathrm{Im}(zR)|}O\left(1/|\log z|\right)$, as $|z|\to\infty, z\in\Omega$. Recall finally the definition of $F(z)$:

$$F(z) = \det\left(\begin{array}{cccc} \multicolumn{2}{c}{\mathcal{A}} & \multicolumn{2}{c}{\mathcal{B}} \\ B\cdot Id_{q_0} & 0 & C\cdot Id_{q_0} & 0 \\ 0 & \mathrm{diag}\,[A_l^+] & 0 & \mathrm{diag}\,[A_l^-] \end{array}\right).$$

Inserting the asymptotics for $A_l^\pm, B$ and C into the definition of $F(z)$ we obtain the statement of the proposition. □

The following result on the spectrum of $\mathcal{L}$ is a corollary of Proposition 4.4 and is necessary for the definition and discussion of certain contour integrals below.

Corollary 4.7. *The self-adjoint operator $\mathcal{L}$ is bounded from below. The zeros of its implicit eigenvalue function $F(\mu)$ are either real or purely imaginary, where the number of the purely imaginary zeros is finite.*

The positive eigenvalues of $\mathcal{L}$ are given by squares of the positive zeros of $F(\mu)$. The negative eigenvalues of $\mathcal{L}$ are given by squares of the purely imaginary zeros of $F(\mu)$ with positive imaginary part, i.e. counting the eigenvalues of $\mathcal{L}$ and zeros of $F(\mu)$ with their multiplicities we have

$$\mathrm{Spec}\mathcal{L}\backslash\{0\} = \{\mu^2\in\mathbb{R}|F(\mu)=0, \mu>0\wedge\mu=ix, x>0\} \qquad (4.11)$$

Proof. The relation between zeros of $F(\mu)$ and eigenvalues of $\mathcal{L}$ is established in Proposition 4.2. The self-adjoint operator $\mathcal{L}$ has real spectrum, hence the zeros of $F(\mu)$ are either real or purely imaginary, representing positive or negative eigenvalues of $\mathcal{L}$, respectively.

The standard infinite series representation of Bessel functions (see [AS, p.360]) implies that zeros of $F(\mu)$ are symmetric about the origin and any two symmetric zeros do not correspond to two linearly independent eigenfunctions of $\mathcal{L}$. Hence the non-zero eigenvalues of $\mathcal{L}$ are in one-to-one correspondence with zeros of $F(\mu)$ at the positive real and the positive imaginary axis.

The asymptotics (4.10) implies in particular that depending on the characteristic values $j_0, q_0, a_{j_o\alpha_o}$ of the boundary conditions $(\mathcal{A}, \mathcal{B})$, the implicit eigenvalue function $F(ix)$ goes either to plus or minus infinity as $x \in \mathbb{R}, x \to \infty$ and cannot become zero for $|x|$ sufficiently large. Since the zeros of the meromorphic function $F(\mu)$ are discrete, we deduce that $F(\mu)$ has only finitely many purely imaginary eigenvalues. Thus in turn, $\mathcal{L}$ has only finitely many negative eigenvalues, i.e. is bounded from below. $\square$

Next we fix an angle $\theta \in (0, \pi/2)$ and put for any $a \in \mathbb{R}^+$:

$$\begin{aligned}
&\delta(a) := \{z \in \mathbb{C} | \mathrm{Re}(z) = a, |\arg(z)| \leq \theta\}, \\
&\rho(a) := \{z \in \mathbb{C} || z| = a/\cos(\theta), |\arg(z)| \in [\theta, \pi/2]\}, \\
&\gamma(a) := \delta(a) \cup \rho(a),
\end{aligned}$$

where the contour $\gamma(a)$ is oriented counter-clockwise, as in the Figure 1 below:

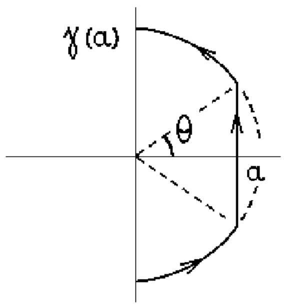

Figure 1: The contour $\gamma(a)$ for the fixed $\theta \in (0, \pi/2)$ and $a \in \mathbb{R}^+$.

Furthermore we fix the branch of logarithm in $\mathbb{C} \backslash \mathbb{R}^+$ with $0 \leq Im \log(z) < 2\pi$. In this setup, the following result is a central application of the asymptotic expansions in Proposition 4.4 and Proposition 4.6.

Proposition 4.8. *There exists a sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers with $a_n \to \infty$ as $n \to \infty$, such that $F(a_n) \neq 0$ for all $n \in \mathbb{N}$ and for $Re(s) > 1/2$ the following integrals*

$$\int_{\gamma(a_n)} z^{-2s} \frac{d}{dz} \log F(z) dz, \ n \in \mathbb{N}$$

are well-defined and the sequence of integrals converges to zero as $n \to \infty$.

Proof. Consider first the logarithmic form of the asymptotics (4.10)

$$\log F(ix) = \log\left(a_{j_o\alpha_o} \cdot \rho \cdot (2\pi)^{-q/2}\right) + (|\nu| + \frac{q}{2} - 2\alpha_0)\log x + qxR+ \\ +(q_0 - j_0)\log(\widetilde{\gamma} - \log x) + \log(1 + G(x)),$$

where $G(x) = O(\frac{1}{\log(x)})$ and $G'(x) = O(\frac{1}{x\log^2(x)})$ as $|x| \to \infty$ with $ix \in \{z \in \mathbb{C} || \arg(z)| \in [\theta, \pi/2], \ \mathrm{Im}(z) > 0\}$. Same asymptotics holds for $ix \in \{z \in \mathbb{C} || \arg(z)| \in [\theta, \pi/2], \ \mathrm{Im}(z) < 0\}$, since $F(ix) = F(-ix)$ by the standard infinite series representation of Bessel functions [AS, p.360]. By straightforward calculations we see for $\mathrm{Re}(s) > 1/2$:

$$\int_{\rho(a_n)} z^{-2s} \frac{d}{dz} \log F(z) dz \xrightarrow{n\to\infty} 0, \tag{4.12}$$

for any sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers with $a_n \to \infty$ as $n \to \infty$. Thus it remains to find a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ which goes to infinity and further ensures that

$$\int_{\delta(a_n)} z^{-2s} \frac{d}{dz} \log F(z) dz \xrightarrow{n\to\infty} 0, \tag{4.13}$$

where for each $n \in \mathbb{N}$ the integral is well-defined. In order to construct such a sequence, fix $a > 0$ subject to the following conditions

$$\cos(aR \pm \frac{\nu_l \pi}{2} + \frac{\pi}{4}) \neq 0, \ l = q_0 + 1, .., q; \tag{4.14}$$

$$\cos(aR - \frac{3}{4}\pi) \neq 0. \tag{4.15}$$

Such a choice is always possible, due to discreteness of zeros of the holomorphic function $\cos(z)$. Given such an $a > 0$, we define

$$\Delta(a) := \bigcup_{k\in\mathbb{N}} \delta(a + \frac{2\pi}{R} k).$$

Using $\cos(z) = (e^{iz} + e^{-iz})/2$ we find for any $\xi \in \mathbb{R}$ with $\cos(aR + \xi) \neq 0$ as $|z| \to \infty, z \in \Delta(a)$

$$\cos(zR + \xi) = e^{|\mathrm{Im}(zR)|} O(1), \tag{4.16}$$

where $|O(1)|$ is bounded away from zero with the bounds depending only on the sign of $\mathrm{Im}(zR)$, $a > 0$ and $\xi \in \mathbb{R}$. Putting $\vec{\alpha} = (\alpha_{q_0+1}, .., \alpha_q) \in \{0,1\}^{q_1}, q_1 = q - q_0$, we obtain for the asymptotic behaviour of $\det M(z)$, introduced in Proposition 4.6, as $|z| \to \infty, z \in \Delta(a)$:

$$\det M(z) = \sum_{j=0}^{q_0} \sum_{\vec{\alpha} \in \{0,1\}^{q_1}} \sum_{\beta=0}^{q} \mathrm{const}(j, \vec{\alpha}, \beta) \left[\frac{1}{\widetilde{\gamma} - \log z} \left(1 + O\left(\frac{1}{|z|} \right) \right) \right]^j \times$$

$$\times \prod_{l=q_0+1}^{q} \left[z^{-2\nu_l} \frac{\cos(zR - \frac{\nu_l \pi}{2} + \frac{\pi}{4})}{\cos(zR + \frac{\nu_l \pi}{2} + \frac{\pi}{4})} \cdot \left(1 + O\left(\frac{1}{|z|} \right) \right) \right]^{\alpha_l} \cdot \left[1 + O\left(\frac{1}{|\log z|} \right) \right]^{\beta},$$

where $\mathrm{const}(j, \vec{\alpha}, \beta)$ depends moreover on $\mathcal{A}$ and $\mathcal{B}$. In fact one has by construction

$$\sum_{\vec{\alpha} \in I_\alpha} \sum_{\beta=0}^{q} \mathrm{const}(j, \vec{\alpha}, \beta) = a_{j\alpha},$$

where $I_\alpha = \{ \vec{\alpha} \in \{0,1\}^{q_1} | \sum_{l=q_o+1}^{q} \nu_l \alpha_l = \alpha \}$ and $a_{j\alpha}$ are the coefficients in the Definition 4.5. Multiplying out the expression for $\det M(z)$ we compute:

$$\det M(z) = \sum_{j=0}^{q_0} \sum_{\vec{\alpha} \in \{0,1\}^{q_1}} \sum_{\beta=0}^{q} \mathrm{const}(j, \vec{\alpha}, \beta) \left[\frac{1}{\widetilde{\gamma} - \log z} \right]^j \times$$

$$\times \prod_{l=q_0+1}^{q} \left[z^{-2\nu_l} \frac{\cos(zR - \frac{\nu_l \pi}{2} + \frac{\pi}{4})}{\cos(zR + \frac{\nu_l \pi}{2} + \frac{\pi}{4})} \right]^{\alpha_l} \cdot [1 + f_{j,\vec{\alpha},\beta}(z)], \ f_{j,\vec{\alpha},\beta}(z) = O\left(\frac{1}{|\log z|} \right).$$

The asymptotic behaviour of $f_{j,\vec{\alpha},\beta}(z)$ under differentiation follows from Proposition 4.6

$$\frac{d}{dz} f_{j,\vec{\alpha},\beta}(z) = O\left(\frac{1}{|\log z|} \right). \tag{4.17}$$

Before we continue let us make an auxiliary observation, in the spirit of (4.16). Under the condition (4.14) on the choice of $a > 0$, we have for $z \in \Delta(a)$ and $l = q_0 + 1, .., q$:

$$\frac{\cos(zR - \frac{\nu_l \pi}{2} + \frac{\pi}{4})}{\cos(zR + \frac{\nu_l \pi}{2} + \frac{\pi}{4})} = C \cdot \left(1 + \frac{e^{-2|\mathrm{Im}(zR)|} C'}{1 + e^{-2|\mathrm{Im}(zR)|} C''} \right), \tag{4.18}$$

where the constants C, C', C'' are given explicitly as follows:

$$C = \exp\left(i \,\mathrm{sign}[\mathrm{Im}(z)](\nu_l \pi)\right),$$
$$C' = \exp\left(i \,\mathrm{sign}[\mathrm{Im}(z)](2aR - \nu_l\pi + \frac{\pi}{2})\right) - \exp\left(i \,\mathrm{sign}[\mathrm{Im}(z)](2aR + \nu_l\pi + \frac{\pi}{2})\right),$$
$$C'' = \exp\left(i \,\mathrm{sign}[\mathrm{Im}(z)](2aR + \nu_l\pi + \frac{\pi}{2})\right).$$

Note that the constants are non-zero, depend only on $\mathrm{sign}[\mathrm{Im}(z)]$, the choice of a and ν_l. Hence for $|\mathrm{Im}(zR)| \to \infty$ the quotient (4.18) tends to $C \neq 0$. Therefore, due to conditions (4.14) and (4.15), there exist constants $\mathfrak{C}_1 > 0$ and $\mathfrak{C}_2 > 0$, depending only on a, such that for $z \in \Delta(a)$ and for all $l = q_0 + 1, .., q$ we have:

$$\mathfrak{C}_1 \leq \left| \frac{\cos(zR - \frac{\nu_l\pi}{2} + \frac{\pi}{4})}{\cos(zR + \frac{\nu_l\pi}{2} + \frac{\pi}{4})} \right| \leq \mathfrak{C}_2. \tag{4.19}$$

In particular the cosinus terms in $\det M(z)$ are not relevant for its asymptotic behaviour as $|z| \to \infty, z \in \Delta(a)$. Now let us consider the summands in $\det M(z)$ of slowest decrease as $|z| \to \infty, z \in \Delta(a)$:

$$\left[\frac{1}{\widetilde{\gamma} - \log z}\right]^{j_0} z^{-2\alpha_0} \cdot \left\{ \sum_{\beta=0}^{q} \sum_{\vec{\alpha}\in I_{\alpha_o}} \mathrm{const}(j_0, \vec{\alpha}, \beta) \prod_{l=q_0+1}^{q} \left[\frac{\cos(zR - \frac{\nu_l\pi}{2} + \frac{\pi}{4})}{\cos(zR + \frac{\nu_l\pi}{2} + \frac{\pi}{4})}\right]^{\alpha_l} \right\}$$
$$=: \left[\frac{1}{\widetilde{\gamma} - \log z}\right]^{j_0} z^{-2\alpha_0} g(z),$$

where the coefficients j_0, α_0 correspond to those in Definition 4.5. By similar calculus as behind (4.18) we can write

$$g(z) = \widetilde{C}\left(1 + \frac{e^{-2|\mathrm{Im}(zR)|}C'(a,z)}{1 + e^{-2|\mathrm{Im}(zR)|}C''(a,z)}\right),$$

where $C'(a,z), C''(a,z)$ further depend on $\mathcal{A}, \mathcal{B}$ and $\nu_l, l = q_0 + 1, .., q$. Moreover they are bounded from above independently of $a > 0$ and $z \in \Delta(a)$. The factor $\widetilde{C}$ is given explicitly as follows:

$$\widetilde{C} = \sum_{\vec{\alpha}\in I_{\alpha_o}} \sum_{\beta=0}^{q} \mathrm{const}(j_0, \vec{\alpha}, \beta) \cdot \exp(i \,\mathrm{sign}[\mathrm{Im}(z)]\pi\alpha_0)$$
$$= a_{j_o\alpha_o} \cdot \exp(i \,\mathrm{sign}[\mathrm{Im}(z)]\pi\alpha_0) \neq 0,$$

since $a_{j_o\alpha_o} \neq 0$ by the definition of characteristic values in Definition 4.5. Since $g(z)$ is a meromorphic function with discrete zeros and poles, we can choose $a>0$ sufficiently large, still subject to conditions (4.14) and (4.15), such that $g(z)$ has no zeros and poles on $\delta(a)$ and

$$\left| \frac{e^{-2|\mathrm{Im}(zR)|}C'(a,z)}{1+e^{-2|\mathrm{Im}(zR)|}C''(a,z)} \right| << 1,$$

for $z \in \delta(a)$ with the highest possible absolute value of its imaginary part, i.e. with $|\mathrm{Im}(z)| = a \cdot \tan\theta$. This guarantees that there exist constants $\mathfrak{C}_1' > 0$ and $\mathfrak{C}_2' > 0$, depending only on $a>0$, such that for $z \in \Delta(a)$

$$\mathfrak{C}_1' \leq |g(z)| \leq \mathfrak{C}_2'. \tag{4.20}$$

By similar arguments we find that $|\frac{d}{dz}g(z)|$ is bounded from above for $z \in \Delta(a)$. Using (4.20) we finally obtain for $\det M(z)$ as $|z| \to \infty, z \in \Delta(a)$

$$\det M(z) = \left[\frac{1}{\widetilde{\gamma} - \log z} \right]^{j_0} z^{-\alpha_0} g(z)(1+f(z)), f(z) = O\left(\frac{1}{|\log(z)|} \right),$$

as $|z| \to \infty, z \in \Delta(a)$. Using (4.17), (4.19) and boundedness of $g(z), g'(z)$ we obtain

$$\frac{d}{dz}f(z) = O\left(\frac{1}{|\log z|} \right).$$

In total we have derived the following asymptotic behaviour of $F(z)$ as $|z| \to \infty, z \in \Delta(a)$:

$$F(z) = \prod_{l=q_0+1}^{q} \left\{ 2^{-\nu_l}\Gamma(1-\nu_l) z^{\nu_l+1/2} \sqrt{\frac{2}{\pi}} \cos(zR + \frac{\nu_l \pi}{2} + \frac{\pi}{4}) \right\} \times$$
$$\times \left\{ \sqrt{\frac{2z}{\pi}} (\log z - \widetilde{\gamma}) \cos(zR - \frac{3}{4}\pi) \right\}^{q_0} \left[\frac{1}{\widetilde{\gamma} - \log z} \right]^{j_0} z^{-\alpha_0} g(z)(1+f(z)),$$

where there exist positive constants $\mathfrak{C}_1', \mathfrak{C}_2', \mathfrak{C}''$, depending only on $a>0$, such that

$$\mathfrak{C}_1' \leq |g(z)| \leq \mathfrak{C}_2', \ |g'(z)| \leq \mathfrak{C}'',$$
$$f(z) = O\left(\frac{1}{|\log z|} \right), \ \frac{d}{dz}f(z) = O\left(\frac{1}{|\log z|} \right).$$

Note that for $N \in \mathbb{N}$ sufficiently large, the asymptotics above, together with the conditions (4.14), (4.15) and (4.20), imply that $F(a+2\pi k/R) \neq 0$ for all $k \in \mathbb{N}, k \geq N$ (note also that by construction $\sum_{l=q_o+1}^{q} \nu_l + q_1/2 - \alpha_0 > 0$).

Putting $a_n := a+2\pi(N+n)/R, n \in \mathbb{N}$ we obtain a sequence $(a_n)_{n\in\mathbb{N}}$ of positive numbers, going to infinity as $n \to \infty$ and we infer from the asymptotics of $F(z)$ above, that for $\mathrm{Re}(s) > 1/2$

$$\int_{\delta(a_n)} z^{-2s} \frac{d}{dz} \log F(z) dz \xrightarrow{n\to\infty} 0, \tag{4.21}$$

where by construction for each $n \in \mathbb{N}$ we have $F(a_n) \neq 0$, and hence the integrals are well-defined. Together with (4.12) this finally proves the statement of the proposition. □

Consider now the following contour

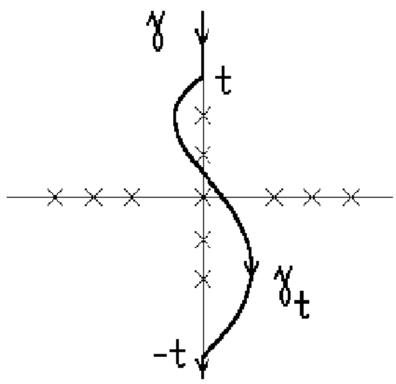

Figure 2: The contour γ. The $\times$'s represent the zeros of $F(\mu)$. The number of purely imaginary zeros is finite by Corollary 4.7. The $t \in i\mathbb{R}$ is chosen such that $|t|^2$ is larger than the largest absolute value of negative eigenvalues of $\mathcal{L}$ (if present). The contour $\gamma_t \subset \gamma$ goes from t to $-t$.

The asymptotics obtained in Proposition 4.4 implies that the contour integral

$$\frac{1}{2\pi i} \int_\gamma \mu^{-2s} \frac{d}{d\mu} \log F(\mu) d\mu$$

with the fixed branch of logarithm in $\mathbb{C}\backslash\mathbb{R}^+$ such that $0 \leq Im \log z < 2\pi$ and the contour γ defined in Figure 2 above, converges for $Re(s) > 1/2$.

The definition of γ corresponds to [KLP1, Figure 1]. We can view the contour

γ to be closed up at infinity on the right hand side of $\mathbb{C}$. Then by construction γ encircles the relevant zeros of $F(\mu)$ in (4.11). As a consequence of Proposition 4.8 we can apply the Argument Principle and finally arrive at

$$\zeta_{\mathcal{L}}(s) = \frac{1}{2\pi i}\int_\gamma \mu^{-2s}\frac{d}{d\mu}\log F(\mu)d\mu, \quad Re(s) > 1/2.$$

This integral representation of the zeta-function is referred by K. Kirsten, P. Loya and J. Park as the "Contour integral method". Thus, on the basis of Proposition 4.8, we have verified applicability of the Contour integral method in the regular-singular setup, which is the basis for further arguments in [KLP1] and [KLP2].

Breaking the integral into three parts $\gamma = \{ix|x \geq t\} \cup \gamma_t \cup \{ix|x \leq -t\}$ we obtain as in [KLP1, (4.10)]

$$\zeta_{\mathcal{L}}(s) = \frac{\sin(\pi s)}{\pi}\int_{|t|}^{\infty} x^{-2s}\frac{d}{dx}\log F(ix)dx + \frac{1}{2\pi i}\int_{\gamma_t}\mu^{-2s}\frac{F'(\mu)}{F(\mu)}d\mu. \quad (4.22)$$

Analytic continuation of the first integral to $s=0$, see [KLP1, (4.12)] allows computation of the functional determinant of $\mathcal{L}$ after subtracting possible logarithmic singularities. We have the following result.

Proposition 4.9. *Under the assumption that* $\ker \mathcal{L} = \{0\}$ *we obtain in the notation of Propositions 4.3 and 4.4*

$$\exp\left[-\lim_{s\to 0+}\frac{d}{ds}\left\{\frac{1}{2\pi i}\int_\gamma \mu^{-2s}\frac{d}{d\mu}\log F(\mu)d\mu - (j_0-q_0)s\cdot\log(s)\right\}\right] =$$
$$= \frac{(2\pi)^{q/2}}{a_{j_o\alpha_o}}(-2e^{\gamma})^{q_0-j_0}\prod_{l=q_0+1}^{q}\frac{2^{\nu_l}}{\Gamma(1-\nu_l)}\times$$
$$\det\begin{pmatrix} & \mathcal{A} & & \mathcal{B} & \\ \kappa Id_{q_o} & 0 & & (\kappa\log R + \frac{1}{\sqrt{R}})Id_{q_o} & 0 \\ 0 & \mathrm{diag}(\kappa R^{\nu_l} + \nu_l R^{\nu_l - \frac{1}{2}}) & & 0 & \mathrm{diag}(\kappa R^{-\nu_l} - \nu_l R^{-\nu_l-\frac{1}{2}})\end{pmatrix}.$$

Proof. Put $C := a_{j_o\alpha_o}\cdot\rho\cdot(2\pi)^{-q/2}$ and rewrite the asymptotic expansion (4.10) for $|x|\to\infty$ with x inside any fixed closed angle of the right half plane of $\mathbb{C}$ in its logarithmic form:

$$\log F(ix) = \log C + (|\nu| + \frac{q}{2} - 2\alpha_0)\log x + qxR +$$
$$+(q_0-j_0)\log(\widetilde{\gamma} - \log x) + \log(1+G(x)). \quad (4.23)$$

In fact the asymptotics differs from the result in [KLP1, Proposition 4.3] only by a presence of an additional summand:

$$\log(x\sqrt{R})^q.$$

Hence the same computations as those leading to [KLP1, p.16] give:

$$\begin{aligned}&\zeta(s,\mathcal{L})-(j_0-q_0)s\log s=\\&=\frac{\sin\pi s}{\pi}\left(|\nu|+\frac{q}{2}-2\alpha_0\right)\frac{|t|^{-2s}}{2s}+\frac{\sin\pi s}{\pi}qR\frac{|t|^{-2s+1}}{2s-1}+\\&+\frac{\sin\pi s}{\pi}(j_0-q_0)g(s)+\frac{\sin\pi s}{\pi}\int_{|t|}^{\infty}x^{-2s}\frac{d}{dx}\log(1+G(x))dx+\\&+\frac{1}{2\pi i}\int_{\gamma_t}\mu^{-2s}\frac{F'(\mu)}{F(\mu)}d\mu,\end{aligned}$$

where with [KLP1, (4.11)] the function $g(s)$ is entire and $g(0)=\gamma+\log(2(\log|t|-\widetilde{\gamma}))$. Explicit differentiation at $s\to 0+$ leads to the following result (compare [KLP1, p.16]):

$$\begin{aligned}&\lim_{s\to 0+}\frac{d}{ds}\left\{\frac{1}{2\pi i}\int_{\gamma}\mu^{-2s}\frac{d}{d\mu}\log F(\mu)d\mu-(j_0-q_0)s\cdot\log(s)\right\}\\&\quad=-\left(|\nu|+\frac{q}{2}-2\alpha_0\right)\log|t|-qR|t|+(j_0-q_0)\left(\gamma+\log(2(\log|t|-\widetilde{\gamma}))\right)\\&\quad-\ \log(1+G(|t|))-\frac{1}{\pi i}\int_{\gamma_t}\log\mu\frac{F'(\mu)}{F(\mu)}d\mu=:Q.\end{aligned}$$

Using (4.23) we can evaluate $\log(1+G(|t|))$ and by inserting it into the expression above we obtain

$$\begin{aligned}Q=-\log\left(\frac{F(i|t|)}{C(-1)^{q_o-j_o}}\right)+(j_0-q_0)(\gamma+\log 2)-\\-\frac{1}{\pi i}\int_{\gamma_t}\log\mu\frac{F'(\mu)}{F(\mu)}.\end{aligned}\tag{4.24}$$

The formula above is a priori derived for $t=i|t|$ being on the upper-half of the imaginary axis. At this point we continue with the trick of [KLP1, Figure 2] to take $|t|\to 0$, which works well under the assumption $\ker\mathcal{L}=\{0\}$.

The integral over the finite contour γ_t in (4.24) vanishes as $t\to 0$. By triviality of $\ker\mathcal{L}$ we have $F(0)\neq 0$ and obtain

$$Q=-\log\left(\frac{F(0)}{C(-1)^{q_o-j_o}}\right)+(j_0-q_0)(\gamma+\log 2).\tag{4.25}$$

By Proposition 4.3 we arrive at the final result

$$Q = (j_0 - q_0)(\gamma + \log 2) + \log \left[\alpha_{j_o a_o} \rho (2\pi)^{-q/2} (-1)^{q_o - j_o}\right] -$$
$$- \log \det \begin{pmatrix} & \mathcal{A} & & \mathcal{B} & \\ \kappa Id_{q_o} & 0 & (\kappa \log R + \frac{1}{\sqrt{R}})Id_{q_o} & 0 \\ 0 & \mathrm{diag}(\kappa R^{\nu_l} + \nu_l R^{\nu_l - \frac{1}{2}}) & 0 & \mathrm{diag}(\kappa R^{-\nu_l} - \nu_l R^{-\nu_l - \frac{1}{2}}) \end{pmatrix}.$$

Exponentiating the expression proves the statement of the proposition. □

Remark 4.10. *In case* $\ker \mathcal{L} \neq \{0\}$ *we can't apply Proposition 4.9. However the intermediate relation* (4.24) *still holds. Further steps are possible if the asymptotics of* $F(\mu)$ *at zero is determined.*

4.3 Special Cases of Self-adjoint Extensions

We compute the zeta-regularized determinants of some particular self-adjoint extensions of the model Laplacian $\triangle_{\nu-1/2}, \nu \geq 0$ in the notation of Subsection 2.3, where we put $R = 1$ for simplicity:

$$\triangle_{\nu-1/2} = -\frac{d^2}{dx^2} + \frac{1}{x^2}\left[\nu^2 - \frac{1}{4}\right] : C_0^\infty(0,1) \to C_0^\infty(0,1), \ \nu \geq 0.$$

According to Proposition 2.11 we get for the asymptotics of any $f \in \mathcal{D}(\triangle_{\nu-1/2,\max})$ as $x \to 0$:

$$f(x) = c_1(f) \cdot \sqrt{x} + c_2(f) \cdot \sqrt{x} \log(x) + O(x^{3/2}), \ \nu = 0, \tag{4.26}$$
$$f(x) = c_1(f) \cdot x^{\nu + \frac{1}{2}} + c_2(f) \cdot x^{-\nu + \frac{1}{2}} + O(x^{3/2}), \ \nu \in (0,1), \tag{4.27}$$
$$f(x) = O(x^{3/2}), \ \nu \geq 1. \tag{4.28}$$

The results of the previous subsection apply directly to the model situation for $\nu \in [0,1)$. In order to obtain results for $\nu \geq 1$ as well, we need to apply [L, Theorem 1.2]. We compute now a sequence of results on zeta-determinants for particular self-adjoint extensions which will become relevant afterwards.

Corollary 4.11. *Let D be the self-adjoint extension of $\triangle_{\nu-1/2}, \nu \geq 0$ with*

$$\mathcal{D}(D) := \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | f(x) = O(\sqrt{x}), \ x \to 0; f'(1) + \alpha f(1) = 0\}.$$

Then for $\alpha \neq -\nu - 1/2$ the operator D is injective and

$$\det{}_\zeta(D) = \sqrt{2\pi} \frac{\alpha + \nu + 1/2}{\Gamma(1+\nu) 2^\nu}.$$

Proof. We first consider $\nu \in [0,1)$. In this case the extension D amounts to

$$\mathcal{D}(D) := \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_2(f) = 0, f'(1) + \alpha f(1) = 0\}.$$

Consider the polynomial $p(x,y)$ defined in Proposition 4.4. Its explicit form for the given extension is

$$p(x,y) = -x, \text{ for } \nu = 0,$$
$$p(x,y) = -\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} 2^{2\nu} y^{2\nu}, \text{ for } \nu \in (0,1).$$

Recall the definition of characteristic values $\alpha_0, j_0, a_{j_o \alpha_o}$ in Definition 4.5. For the given extension D we obtain

$$j_0 = q_0 = 1, \; q = 1, a_{j_o\alpha_o} = -1, \text{ for } \nu = 0,$$
$$j_0 = q_0 = 0, \; q = 1, a_{j_o\alpha_o} = -\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} 2^{2\nu}, \text{ for } \nu \in (0,1).$$

Evaluating with Proposition 4.3 the corresponding implicit eigenvalue function $F(\mu)$ at $\mu = 0$ we obtain for any $\nu \in [0,1)$

$$F(0) = -\frac{1}{2} - \alpha - \nu.$$

Thus the condition $\alpha \neq -\nu - 1/2$ guarantees $F(0) \neq 0$ and thus $\ker D = \{0\}$. Applying Proposition 4.9 we obtain the desired formula.

In order to obtain a result for all $\nu \geq 0$, we need to apply [L, Theorem 1.2]. Consider mappings $\phi, \psi : (0,1) \to \mathbb{R}$ such that

$$\triangle_{\nu-1/2}\phi = 0, \phi(x) = O(\sqrt{x}), x \to 0 \text{ and } \phi(x) = x^{\nu+1/2}\phi_0(x), \phi_0(0) = 1,$$
$$\triangle_{\nu-1/2}\psi = 0, \psi'(1) + \alpha\psi(1) = 0 \text{ and } \psi(1) = 1.$$

These solutions exist and are uniquely determined. In the sense of [L, (1.38a), (1.38b)] the solutions ϕ, ψ are "normalized" at $x = 0, x = 1$, respectively. In the present setup the normalized solution ϕ is given explicitly as follows

$$\phi(x) = x^{\nu+1/2}.$$

In particular we obtain for the Wronski determinant

$$W(\psi,\psi) = \phi'(1)\psi(1) - \phi(1)\psi'(1) = \alpha + \nu + 1/2.$$

By assumption $\alpha \neq -\nu - 1/2$ and hence $W(\psi,\phi) \neq 0$. Note from the fundamental system of solutions to $\triangle_{\nu-1/2} f = 0$ in (2.12) and (2.13) that

$\ker D = \{0\}$. Thus we can apply [L, Theorem 1.2] where in the notation therein $\nu_0 = \nu$ and $\nu_1 = -1/2$:

$$\det_\zeta D = \frac{\pi W(\psi,\phi)}{2^{\nu_0+\nu_1}\Gamma(1+\nu_0)\Gamma(1+\nu_1)} = \sqrt{2\pi}\frac{\alpha+\nu+1/2}{2^\nu\Gamma(1+\nu)}.$$

This proves the statement. □

Corollary 4.12. *Let D be the self-adjoint extension of $\triangle_{\nu-1/2}, \nu \geq 0$ with*

$$\mathcal{D}(D) := \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | f(x) = O(\sqrt{x}), x \to 0; f(1) = 0\}.$$

The zeta-regularized determinant of this self-adjoint realization is given by

$$\det_\zeta(D) = \frac{\sqrt{2\pi}}{\Gamma(1+\nu)2^\nu}.$$

Proof. We first consider $\nu \in [0,1)$. In this case the extension D amounts to

$$\mathcal{D}(D) := \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_2(f) = 0, f(1) = 0\}.$$

As in the proof of Corollary 4.11 we infer for the characteristic values of the extension D

$$\begin{aligned} &j_0 = q_0 = 1,\ q = 1, a_{j_o\alpha_o} = -1,\ \text{for } \nu = 0, \\ &j_0 = q_0 = 0,\ q = 1, a_{j_o\alpha_o} = -\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}2^{2\nu},\ \text{for } \nu \in (0,1). \end{aligned}$$

Consider now the following self-adjoint extension of $\triangle_{\nu-1/2}$:

$$\mathcal{D}(\triangle^{\mathcal{N}}_{\nu-1/2}) := \begin{cases} \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_2(f) = 0, f(1) = 0\},\ \nu = 0, \\ \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_1(f) = 0, f(1) = 0\},\ \nu \in (0,1). \end{cases}$$

This extension is referred to as "Neumann-extension" (with Dirichlet boundary conditions at $x = 1$) in [KLP1] and by [KLP1, Corollary 4.7] we have:

$$\det_\zeta \triangle^{\mathcal{N}}_{\nu-1/2} = \sqrt{2\pi}\frac{2^\nu}{\Gamma(1-\nu)},\ \nu \in [0,1).$$

Note that for $\nu = 0$ we have $\triangle^{\mathcal{N}}_{\nu-1/2} \equiv D$. Hence it remains to compute the zeta-determinant of D for $\nu \in (0,1)$. Using [KLP1, Theorem 2.3], where arbitrary extensions at the cone-singularity are expressed in terms of the "Neumann extension", we obtain

$$\det_\zeta(D) = \det_\zeta \triangle^{\mathcal{N}}_p \cdot \frac{(-2e^\gamma)^{q_o-j_o}}{a_{j_o\alpha_o}} \cdot \det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \frac{\sqrt{2\pi}}{\Gamma(1+\nu)2^\nu}.$$

This proves the statement for $\nu \in [0,1)$. In order to obtain the desired result for all $\nu \geq 0$, we apply [L, Theorem 1.2] by similar means as in the previous corollary. □

The remaining three results differ from the previous two corollaries by the fact that the self-adjoint realizations considered there are not injective. In this case we cannot apply [L, Theorem 1.2] and Proposition 4.9. We have to apply the intermediate result (4.24).

Corollary 4.13. *Consider a self-adjoint extension D of $\triangle_{\nu-1/2}$ with $\nu = 1/2$*

$$\mathcal{D}(D) := \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_2(f) = 0, f'(1) - f(1) = 0\}.$$

The zeta-regularized determinant of this self-adjoint realization is given by

$$\det_\zeta(D) = \frac{2}{3}.$$

Proof. As in the proof of Corollary 4.11 we infer for the characteristic values of the extension D

$$j_0 = q_0 = 0,\ q = 1, a_{j_o\alpha_o} = -\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} 2^{2\nu} = -1.$$

Evaluating with Proposition 4.3 the corresponding implicit eigenvalue function $F(\mu)$ at $\mu = 0$ we obtain as in Corollary 4.11

$$F(0) = -\frac{1}{2} + 1 - \nu = 0.$$

This implies $\ker D \neq \{0\}$ and so unfortunately we cannot apply Proposition 4.9 directly. Note however from the definition of $F(\mu)$ in Proposition 4.2

$$F(\mu) = -\sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{\mu}} \left[-\frac{1}{2} J_{1/2}(\mu) + \mu J'_{1/2}(\mu) \right].$$

With $J_{1/2}(\mu) = \sqrt{\frac{2}{\pi\mu}} \sin(\mu)$ we compute further

$$F(\mu) = \frac{\sin(\mu)}{\mu} - \cos(\mu).$$

From the Taylor expansion of $\sin(\mu), \cos(\mu)$ around zero we get

$$F(\mu) = \frac{1}{3}\mu^2 + O(\mu^4), \text{ as } |\mu| \to 0.$$

Thus we equivalently can consider a different implicit eigenvalue function

$$\tilde{F}(\mu) = \frac{F(\mu)}{\mu^2}, \tilde{F}(0) = 1/3.$$

By Remark 4.10 and the relation (4.24) we obtain with $j_0 - q_0 = 0, R = 1$

$$\zeta'(0,D) = -\log\left(\frac{\tilde{F}(i|t|)}{\tilde{C}}\right) - \frac{1}{\pi i}\int_{\gamma_t} \log\mu \frac{\tilde{F}'(\mu)}{\tilde{F}(\mu)}. \tag{4.29}$$

Note that the second summand is now an entire integral over a finite curve. Taking $|t| \to 0$ this integral vanishes. Recall that the constant C was introduced in (4.23) to describe the constant term in the asymptotics of $\log F(ix)$. By construction we have a relation between $\tilde{C}$ associated to $\tilde{F}(\mu)$ and C associated to the original $F(\mu)$

$$\tilde{C} = -C = -a_{j_o\alpha_o}\rho(2\pi)^{-q/2},$$

where ρ is defined in the statement of Proposition 4.4, leading in the present situation to $\rho = \Gamma(1-\nu)2^{-\nu} = \sqrt{\pi/2}$. Hence $\tilde{C}$ computes explicitly to $\tilde{C} = 1/2$. Inserting this now into (4.29) and taking $|t| \to 0$ we obtain

$$\zeta'(0,D) = -\log\left(\frac{\tilde{F}(0)}{\tilde{C}}\right) = -\log\left(\frac{2}{3}\right).$$

This proves the statement with $\det_\zeta(D) = \exp(-\zeta'(0,D))$. □

Corollary 4.14. *Consider a self-adjoint extension D of $\triangle_{\nu-1/2}$ with $\nu = 1/2$*

$$\mathcal{D}(D) := \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_1(f) = 0, f'(1) = 0\}.$$

The zeta-regularized determinant of this self-adjoint realization is given by

$$\det_\zeta(D) = 2.$$

Proof. Consider the polynomial $p(x,y)$ defined in Proposition 4.4. Its explicit form for the given extension is

$$p(x,y) = 1.$$

Thus we get for the characteristic values $\alpha_0, j_0, a_{j_o\alpha_o}$, defined in Definition 4.5

$$j_0 = q_0 = 0,\ q = 1, a_{j_o\alpha_o} = 1.$$

Evaluating with Proposition 4.3 the corresponding implicit eigenvalue function $F(\mu)$ at $\mu = 0$ we obtain

$$F(0) = \frac{1}{2} - \nu = 0.$$

This implies $\ker D \neq \{0\}$ and so unfortunately we cannot apply Proposition 4.9 directly. Note however from the definition of $F(\mu)$ in Proposition 4.2

$$F(\mu) = \sqrt{\frac{\pi}{2}}\sqrt{\mu}\left[\frac{1}{2}J_{-1/2}(\mu) + \mu J'_{-1/2}(\mu)\right].$$

With $J_{-1/2}(\mu) = \sqrt{\frac{2}{\pi\mu}}\cos(\mu)$ we compute further

$$F(\mu) = -\mu\sin(\mu).$$

From the Taylor expansion of $\sin(\mu)$ around zero we get

$$F(\mu) = -\mu^2 + O(\mu^4), \text{ as } |\mu| \to 0.$$

Thus we equivalently can consider a different implicit eigenvalue function

$$\tilde{F}(\mu) = \frac{F(\mu)}{\mu^2}, \tilde{F}(0) = -1.$$

By Remark 4.10 and the relation (4.24) we obtain with $j_0 - q_0 = 0, R = 1$

$$\zeta'(0, D) = -\log\left(\frac{\tilde{F}(i|t|)}{\tilde{C}}\right) - \frac{1}{\pi i}\int_{\gamma_t} \log\mu \frac{\tilde{F}'(\mu)}{\tilde{F}(\mu)}. \tag{4.30}$$

Note that the second summand is now an entire integral over a finite curve. Taking $|t| \to 0$ this integral vanishes. As in the previous corollary we find by construction a relation between $\tilde{C}$ associated to $\tilde{F}(\mu)$ and C associated to the original $F(\mu)$

$$\tilde{C} = -C = -a_{j_o\alpha_o}\rho(2\pi)^{-q/2},$$

where $\rho = \Gamma(1-\nu)2^{-\nu} = \sqrt{\pi/2}$ is defined in the statement of Proposition 4.4. The constant $\tilde{C}$ computes explicitly to $\tilde{C} = -1/2$. Inserting this now into (4.30) and taking $|t| \to 0$ we obtain

$$\zeta'(0, D) = -\log\left(\frac{\tilde{F}(0)}{\tilde{C}}\right) = -\log 2.$$

This proves the statement with $\det_\zeta(D) = \exp(-\zeta'(0, D))$. □

Corollary 4.15. *Consider a self-adjoint extension D of $\triangle_{\nu-1/2}$ with $\nu = 0$*

$$\mathcal{D}(D) := \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_2(f) = 0, f'(1) - \frac{1}{2}f(1) = 0\}.$$

The zeta-regularized determinant of this self-adjoint realization is given by

$$\det_\zeta(D) = \sqrt{\frac{\pi}{2}}.$$

Proof. Consider the polynomial $p(x,y)$ defined in Proposition 4.4. Its explicit form for the given extension is

$$p(x,y) = -x.$$

According to Definition 4.5 we obtain from above the characteristic values of the extension D

$$j_0 = q_0 = 1, \;\; q = 1, a_{j_o \alpha_o} = -1.$$

Evaluating with Proposition 4.3 the corresponding implicit eigenvalue function $F(\mu)$ at $\mu = 0$ we obtain

$$F(0) = \det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0.$$

This implies $\ker D \neq \{0\}$ and so unfortunately we cannot apply Proposition 4.9 directly. Note however from the definition of $F(\mu)$ in Proposition 4.2

$$F(\mu) = -\mu J_0'(\mu) = \mu J_1(\mu),$$

where we used the identity $J_0' = J_{-1} = -J_1$. Hence as $|\mu| \to 0$ we obtain the following asymptotics

$$F(\mu) = \mu J_1(\mu) \sim \frac{\mu^2}{2\Gamma(2)} = \frac{\mu^2}{2}.$$

Thus we equivalently can consider a different implicit eigenvalue function

$$\tilde{F}(\mu) = \frac{F(\mu)}{\mu^2}, \tilde{F}(0) = 1/2.$$

By Remark 4.10 and the relation (4.24) we obtain with $j_0 - q_0 = 0, R = 1$

$$\zeta'(0,D) = -\log \left(\frac{\tilde{F}(i|t|)}{\tilde{C}} \right) - \frac{1}{\pi i} \int_{\gamma_t} \log \mu \frac{\tilde{F}'(\mu)}{\tilde{F}(\mu)}. \tag{4.31}$$

Note that the second summand is now an entire integral over a finite curve. Taking $|t| \to 0$ this integral vanishes. As before we find by construction a relation between $\tilde{C}$ associated to $\tilde{F}(\mu)$ and C associated to the original $F(\mu)$

$$\tilde{C} = -C = -a_{j_o \alpha_o} \rho (2\pi)^{-q/2},$$

where ρ is defined in the statement of Proposition 4.4 and equals 1 in the present case. The constant $\tilde{C}$ computes explicitly to $\tilde{C} = 1/\sqrt{2\pi}$. Inserting this now into (4.31) and taking $|t| \to 0$ we obtain

$$\zeta'(0,D) = -\log \left(\frac{\tilde{F}(0)}{\tilde{C}} \right) = -\log \left(\sqrt{\frac{\pi}{2}} \right).$$

This proves the statement with $\det_\zeta(D) = \exp(-\zeta'(0,D))$. $\square$

4.4 Functional determinant of the Laplacian with relative Boundary Conditions

We continue in the setup and notation of Section 3 and consider the de Rham Laplacian $\triangle_k$ on differential forms of degree k. Its self-adjoint extension $\triangle_k^{rel}$ is defined in (3.8) and can be viewed as a self-adjoint operator in

$$H^k = L^2([0,1], L^2(\wedge^{k-1}T^*N \oplus \wedge^k T^*N, \mathrm{vol}(g^N)), dx).$$

We want to identify in each fixed degree k a decomposition

$$\triangle_k = L_k \oplus \widetilde{\triangle}_k, \quad H^k = H_1^k \oplus H_2^k,$$

compatible with the relative extension $\triangle_k^{rel}$, such that $\widetilde{\triangle}_k$ is the maximal direct sum component, subject to compatibility condition, which is essentially self-adjoint at the cone-singularity in the sense that all its self-adjoint extensions in H_2^k coincide at the cone-tip, in analogy to Definition 2.4.

The component $\widetilde{\triangle}_k$ is discussed in [DK1]. In this subsection our aim is to understand the structure of L_k and its self-adjoint extension $\mathcal{L}_k^{rel}$ induced in the sense of Definition 3.8 by the relative extension $\triangle_k^{rel}$. In particular we want to compute the zeta-regularized determinant of $\mathcal{L}_k^{rel}$ in degrees where it is present.

Consider the decomposition of

$$L^2((0,1), L^2(\wedge^{k-1}T^*N \oplus \wedge^k T^*N)),$$

induced by (3.11). By Theorem 3.10 it is compatible with $\triangle_k^{rel}$. Thus by Lemma 3.7 the relative extension $\triangle_k^{rel}$ induces self-adjoint extensions of $\triangle_k$ restricted to each of the orthogonal components of the decomposition. We consider each of the components distinctly.

Proposition 4.16. *The relative extension $\triangle_k^{rel}$ induces a self-adjoint extension of $\triangle_k$ restricted to $C_0^\infty((0,1), \mathcal{H}^k(N))$. This component contributes to L_k only for*

$$k \in \left(\frac{m}{2} - 2, \frac{m}{2}\right).$$

In these degrees the contribution of the component to the zeta-determinant of $\mathcal{L}_k^{rel}$ is given with $\nu := |k+1-m/2|$ by

$$\left[\frac{\sqrt{2\pi}}{\Gamma(1+\nu)} 2^{-\nu}\right]^{\dim \mathcal{H}^k(N)}.$$

Proof. Recall from (3.20) in the convention of Remark 3.11

$$\mathcal{D}(\triangle_k^{rel}) \cap L^2((0,1), \langle 0 \oplus u_i^k \rangle) =$$
$$= \mathcal{D}\left(\left[\partial_x + \frac{(-1)^k c_k}{x} \right]_{\max}^t \left[\partial_x + \frac{(-1)^k c_k}{x} \right]_{\min} \right) = \mathcal{D}(\triangle_{(-1)^k c_k}^D),$$

where $\{u_i^k\}$ is an orthonormal basis of $\mathcal{H}^k(N)$, $c_k = (-1)^k(k - n/2)$ and $\triangle_{(-1)^k c_k}^D$ denotes the self-adjoint D-extension of $\triangle_{(-1)^k c_k}$, as introduced in Subsection 2.3. Note

$$\triangle_{(-1)^k c_k} \equiv \triangle_{k-n/2} = -\frac{d^2}{dx^2} + \frac{1}{x^2}\left[(k+1-m/2)^2 - \frac{1}{4} \right] = \triangle_{\nu-1/2},$$

where we put $\nu := |k+1-m/2|$. We know from Corollary 2.14 that $\triangle_{\nu-1/2}$ is in the limit circle case at $x = 0$ and hence not essentially self-adjoint at $x = 0$ iff

$$\nu^2 - \frac{1}{4} = \left[k + 1 - \frac{m}{2} \right]^2 - \frac{1}{4} < \frac{3}{4}, \text{ i.e. } k \in \left(\frac{m}{2} - 2, \frac{m}{2} \right).$$

Thus we get a contribution to L_k in these degrees only, which is the first part of the statement.

Fix such a degree $k \in (m/2 - 2, m/2)$. Then the contribution to $\mathcal{L}_k^{rel}$ is given by

$$\left[\det_\zeta \triangle_{k-n/2}^D \right]^{\dim \mathcal{H}^k(N)}.$$

Note that for $k \in (m/2 - 2, m/2)$ we have $(-1)^k c_k = k - n/2 \in (-3/2, 1/2)$. Depending on the explicit value of $(-1)^k c_k$ we apply Corollaries 2.15, 2.16 and 2.18. We deduce in any case:

$$\mathcal{D}(\triangle_{k-n/2}^D) = \{ f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_2(f) = 0, f(1) = 0 \},$$

where $c_2(f)$ refers to the asymptotics in Proposition 2.11 or equivalently in (4.27).

We deduce the explicit value of $\det_\zeta \triangle_{k-n/2}^D$ from Corollary 4.12. □

Proposition 4.17. *The relative extension $\triangle_k^{rel}$ induces a self-adjoint extension of $\triangle_k$ restricted to $C_0^\infty((0,1), \mathcal{H}^{k-1}(N))$. This component contributes to L_k only for*

$$k \in \left(\frac{m}{2}, \frac{m}{2} + 2 \right).$$

In these degrees the contribution of the component to the zeta-determinant of $\mathcal{L}_k^{rel}$ *is given by*

$$\begin{aligned} \left[\sqrt{\pi/2}\right]^{\dim \mathcal{H}^{k-1}(N)}, &\quad \text{if } \dim M = m \text{ is even}, k = m/2+1, \\ [2]^{\dim \mathcal{H}^{k-1}(N)}, &\quad \text{if } \dim M = m \text{ is odd}, \ k = n/2+1, \\ [2/3]^{\dim \mathcal{H}^{k-1}(N)}, &\quad \text{if } \dim M = m \text{ is odd}, \ k = n/2+2. \end{aligned}$$

Proof. Recall from (3.21) in the convention of Remark 3.11

$$\mathcal{D}(\triangle_k^{rel}) \cap L^2((0,1), \langle u_i^{k-1} \oplus 0\rangle) = \\ = \mathcal{D}\left(\left[\partial_x + \frac{(-1)^k c_{k-1}}{x}\right]_{\min}^t \left[\partial_x + \frac{(-1)^k c_{k-1}}{x}\right]_{\max} \right) = \mathcal{D}(\triangle^N_{(-1)^k c_{k-1}}),$$

where $\{u_i^{k-1}\}$ is an orthonormal basis of $\mathcal{H}^{k-1}(N)$, $c_{k-1} = (-1)^{k-1}(k-1-n/2)$ and $\triangle^N_{(-1)^k c_{k-1}}$ denotes the self-adjoint N-extension of $\triangle_{(-1)^k c_{k-1}}$, as introduced in Subsection 2.3. Note

$$\triangle_{(-1)^k c_{k-1}} \equiv \triangle_{n/2+1-k} = -\frac{d^2}{dx^2} + \frac{1}{x^2}\left[(k-1-m/2)^2 - \frac{1}{4}\right] = \triangle_{\nu-1/2},$$

where we put $\nu := |k-1-m/2|$. We know from Corollary 2.14 that $\triangle_{\nu-1/2}$ is in the limit circle case at $x=0$ and hence not essentially self-adjoint at $x=0$ iff

$$\nu^2 - \frac{1}{4} = \left[k-1-\frac{m}{2}\right]^2 - \frac{1}{4} < \frac{3}{4}, \text{ i.e. } k \in \left(\frac{m}{2}, \frac{m}{2}+2\right).$$

Thus we get a contribution to L_k in these degrees only, which is the first part of the statement.

Fix such a degree $k \in (m/2, m/2+1)$. Then the contribution to $\mathcal{L}_k^{rel}$ is given by

$$\left[\det\nolimits_\zeta \triangle^N_{n/2+1-k}\right]^{\dim \mathcal{H}^{k-1}(N)}.$$

Unfortunately the explicit form of the domain $\mathcal{D}(\triangle^N_{n/2+1-k})$ cannot be presented in such a homogeneous way as in the previous proposition. So we need to discuss different cases separately.

If $\dim M = m$ is even, then the only degree $k \in (m/2, m/2+2)$ is $k = m/2+1$. Then $n/2+1-k = -1/2, \nu = 0$ and we obtain with Corollary 2.16

$$\mathcal{D}(\triangle^N_{n/2+1-k}) = \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_2(f) = 0, f'(1) - \frac{1}{2}f(1) = 0\},$$

where $c_2(f)$ refers to the asymptotics (4.26). The contribution to the zeta-determinant of $\mathcal{L}_k^{rel}$ follows now from Corollary 4.15.

If $\dim M = m$ is odd, then the only degrees $k \in (m/2, m/2+2)$ are $k = n/2+1, n/2+2$.

For $k = n/2+1$ we have $n/2+1-k = 0, \nu = 1/2$ and we obtain from Corollary 2.15

$$\mathcal{D}(\triangle^N_{n/2+1-k}) = \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_1(f) = 0, f'(1) = 0\},$$

where $c_1(f)$ refers to the asymptotics (4.27). The contribution to the zeta-determinant of $\mathcal{L}_k^{rel}$ in this case follows from Corollary 4.14.

For the second case $k = n/2+2$ we have $n/2+1-k = -1, \nu = 1/2$. So we obtain from Corollary 2.18

$$\mathcal{D}(\triangle^N_{n/2+1-k}) = \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | c_2(f) = 0, f'(1) - f(1) = 0\},$$

where $c_2(f)$ refers to the asymptotics (4.27). The contribution to the zeta-determinant of $\mathcal{L}_k^{rel}$ in this case follows from Corollary 4.13.

Now all the possible cases are discussed and the proof is complete. □

Proposition 4.18. *The relative extension $\triangle_k^{rel}$ induces a self-adjoint extension of $\triangle_k$ restricted to $C_0^\infty((0,1), \{0\} \oplus E_\lambda^k)$ for $\lambda \in V_k = \mathrm{Spec}\triangle_{k,ccl,N} \backslash \{0\}$. This component contributes to L_k only for*

$$k \in \left(\frac{m}{2} - 2, \frac{m}{2}\right), \ \lambda < 1 - \left[k + \frac{1}{2} - \frac{n}{2}\right]^2.$$

In this case the contribution of the component to the zeta-determinant of $\mathcal{L}_k^{rel}$ is given by

$$\frac{\sqrt{2\pi}}{\Gamma(1+\nu)2^\nu}, \text{ where } \nu := \sqrt{\lambda + \left[k + \frac{1}{2} - \frac{n}{2}\right]^2}.$$

Proof. We infer from (3.16) that $\triangle_k$ acts on $C_0^\infty((0,1), \{0\} \oplus E_\lambda^k)$ with $\lambda \in V_k$ as a rank-one model Laplacian

$$-\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\lambda + \left[k + \frac{1}{2} - \frac{n}{2}\right]^2 - \frac{1}{4}\right),$$

under the identification of elements in $C_0^\infty((0,1),\{0\}\oplus E^k_\lambda)$ with their scalar parts in $C_0^\infty(0,1)$ in the convention of Remark 3.11. This operator is in the limit circle case at $x=0$ and hence not essentially self-adjoint at $x=0$ iff

$$\lambda + \left[k+\frac{1}{2}-\frac{n}{2}\right]^2 - \frac{1}{4} < \frac{3}{4}, \text{ i.e. } k \in \left(\frac{m}{2}-2, \frac{m}{2}\right), \ \lambda < 1 - \left[k+\frac{1}{2}-\frac{n}{2}\right]^2.$$

This proves the first part of the statement. Fix such k and λ. Observe now by Lemma 3.12

$$\mathcal{D}(\triangle_k^{rel}) \cap L^2((0,1),\{0\}\oplus E^k_\lambda) = \{f \in \mathcal{D}(\triangle_{p,\max}) | c_2(f)=0, f(1)=0\}.$$

The result now follows from Corollary 4.12. □

Proposition 4.19. *The relative extension $\triangle_k^{rel}$ induces a self-adjoint extension of $\triangle_k$ restricted to $C_0^\infty((0,1),d_N E^{k-2}_\lambda)\oplus\{0\}$ for $\lambda \in V_{k-2} =$ Spec$\triangle_{k-2,ccl,N}\backslash\{0\}$. This component contributes to L_k only for*

$$k \in \left(\frac{m}{2}, \frac{m}{2}+2\right), \ \lambda < 1 - \left[k-\frac{3}{2}-\frac{n}{2}\right]^2.$$

In this case the contribution of the component to the zeta-determinant of $\mathcal{L}_k^{rel}$ is given by

$$\sqrt{2\pi}\frac{\nu+m/2+1-k}{\Gamma(1+\nu)\cdot 2^\nu}, \text{ where } \nu := \sqrt{\lambda + \left[k-\frac{3}{2}-\frac{n}{2}\right]^2}.$$

Proof. We infer from (3.16) that $\triangle_k$ acts on $C_0^\infty((0,1),d_N E^{k-2}_\lambda)$ with $\lambda \in V_{k-2}$ as a rank-one model Laplacian

$$-\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\lambda + \left[k-\frac{3}{2}-\frac{n}{2}\right]^2 - \frac{1}{4}\right),$$

under the identification of elements with their scalar parts as before. This operator is in the limit circle case at $x=0$ and hence not essentially self-adjoint at $x=0$ iff

$$\lambda + \left[k-\frac{3}{2}-\frac{n}{2}\right]^2 - \frac{1}{4} < \frac{3}{4}, \text{ i.e. } k \in \left(\frac{m}{2}, \frac{m}{2}+2\right), \ \lambda < 1 - \left[k-\frac{3}{2}-\frac{n}{2}\right]^2.$$

This proves the first part of the statement. Fix such k and λ. Observe now by Lemma 3.13

$$\mathcal{D}(\triangle_k^{rel}) \cap L^2((0,1),\{0\}\oplus E^k_\lambda) = \{f \in \mathcal{D}(\triangle_{p,\max}) | \\ c_2(f)=0, f'(1)-(k-1-n/2)f(1)=0\}.$$

The result now follows from Corollary 4.11. □

Proposition 4.20. *The relative extension $\triangle_k^{rel}$ induces a self-adjoint extension of $\triangle_k$ restricted to $C_0^\infty((0,1), \widetilde{E}_\lambda^{k-1})$ with $\lambda \in V_{k-1} = \mathrm{Spec}\triangle_{k-1,ccl,N}\backslash\{0\}$. This component contributes to L_k only for*

$$k \in \left(\frac{m}{2}-2, \frac{m}{2}+2\right), \ \lambda < 4 - \left[k - \frac{m}{2}\right]^2.$$

The contribution of the component to the zeta-determinant of $\mathcal{L}_k^{rel}$ is given by

$$2\pi \frac{(\nu - k + m/2)}{\Gamma(1+\nu)^2} 2^{-2\nu}, \text{where } \nu := \sqrt{\lambda + \left[k - \frac{m}{2}\right]^2}.$$

Proof. The space $\widetilde{E}_\lambda^{k-1}, \lambda \in V_{k-1}$ is an orthogonal sum of S_0-eigenspaces to eigenvalues (see [BL2, Section 2])

$$p_\pm^k(\lambda) := \frac{(-1)^k}{2} \pm \sqrt{\left(k - \frac{m}{2}\right)^2 + \lambda}.$$

Put

$$a_\pm^k(\lambda) := p_\pm^k(\lambda) \cdot (p_\pm^k(\lambda) + (-1)^k).$$

The restriction of $\triangle_k$ to $C_0^\infty((0,1), \widetilde{E}_\lambda^{k-1})$ decomposes into

$$\left(-\frac{d^2}{dx^2} + \frac{a_+^k(\lambda)}{x^2}\right) \oplus \left(-\frac{d^2}{dx^2} + \frac{a_-^k(\lambda)}{x^2}\right)$$

in correspondence to the decomposition of $\widetilde{E}_\lambda^{k-1}$ into the S_0-eigenspaces. This decomposition is not compatible with the relative boundary conditions, which is clear from the relative boundary conditions at the cone base. Nevertheless we infer from the decomposition, that the restriction of $\triangle_k$ to $C_0^\infty((0,1), \widetilde{E}_\lambda^{k-1})$ contributes to L_k only if

$$\left|k - \frac{m}{2}\right| < 2, \quad \lambda \leq 4 - \left(k - \frac{m}{2}\right)^2,$$

since for the complementary case both $a_+^k(\lambda)$ and $a_-^k(\lambda)$ are $\geq 3/4$. This proves the first part of the statement. In order to compute the contribution of the component to the determinant of $\mathcal{L}_k^{rel}$, we study as in (3.12) the associated de Rham complex:

$$0 \to C_0^\infty((0,1), \{0\} \oplus E_\lambda^{k-1}) \xrightarrow{d_0} C_0^\infty((0,1), \widetilde{E}_\lambda^{k-1}) \xrightarrow{d_1} C_0^\infty((0,1), d_N E_\lambda^{k-1} \oplus \{0\}) \to 0.$$

Note that $d_0^t d_0$ and $d_1 d_1^t$ both act as rank-one model Laplacians

$$-\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\lambda + \left[k - \frac{1}{2} - \frac{n}{2}\right]^2 - \frac{1}{4}\right),$$

under the identification of elements with their scalar parts, as before. The relative boundary conditions turn the complex into a Hilbert complex with the corresponding self-adjoint extensions of $d_0^t d_0$ and $d_1 d_1^t$ denoted by $\triangle_0$, $\triangle_1$ respectively. The contribution to the determinant of $\mathcal{L}_k^{rel}$ is then given by

$$\det_\zeta \triangle_0 \cdot \det_\zeta \triangle_1.$$

We obtain with $\nu := \sqrt{\lambda + (k - m/2)^2}$

$$\begin{aligned}\mathcal{D}(\triangle_0) = \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | f(x) = O(\sqrt{x}), x \to 0; f(1) = 0\},\\ \mathcal{D}(\triangle_1) = \{f \in \mathcal{D}(\triangle_{\nu-1/2,\max}) | f(x) = O(\sqrt{x}), x \to 0,\\ f'(1) - (k - n/2)f(1) = 0\}.\end{aligned}$$

For $\nu \in (0,1)$ these domains were determined in Lemma 3.12 and Lemma 3.13, with the asymptotics $f(x) = O(\sqrt{x}), x \to 0$ being expressed by $c_2(f) = 0$. The coefficient $c_2(f)$ refers to the relation (4.27).

For $\nu \geq 1$ the operator $\triangle_{\nu-1/2}$ is in the limit point case, see Corollary 2.14. So the condition on the asymptotic behaviour at $x = 0$ is redundant and the domains are determined only by the relative boundary conditions at $x = 1$, which are computed in Proposition 3.5.

Applying Corollaries 4.11 and 4.12 we obtain

$$\begin{aligned}\det_\zeta \triangle_0 = \frac{\sqrt{2\pi}}{\Gamma(1+\nu)2^\nu},\\ \det_\zeta \triangle_1 = \sqrt{2\pi}\frac{\nu - k + m/2}{\Gamma(1+\nu)2^\nu}.\end{aligned}$$

Multiplication of both expressions gives the result. □

Before we write down an explicit expression for $\det_\zeta \mathcal{L}_k^{rel}$, let us introduce some simplifying notation. Put:

$$\begin{aligned}A_k := \{\nu = \sqrt{\lambda + [k+1-m/2]^2} | \lambda \in \mathrm{Spec}\triangle_{k,ccl,N},\\ 0 \leq \lambda < 1 - [k+1-m/2]^2\}, \quad (4.32)\\ \widetilde{A}_k := \{\nu = \sqrt{\lambda + [k+1-m/2]^2} | \lambda \in \mathrm{Spec}\triangle_{k,ccl,N}\backslash\{0\},\\ 0 < \lambda < 1 - [k+1-m/2]^2\}, \quad (4.33)\\ B_k := \{\nu = \sqrt{\lambda + [k-m/2]^2} | \lambda \in \mathrm{Spec}\triangle_{k-1,ccl,N}\backslash\{0\},\\ 0 < \lambda < 4 - [k-m/2]^2\}. \quad (4.34)\end{aligned}$$

Moreover we write

$$P_k := \begin{cases} (\sqrt{\pi/2})^{\dim \mathcal{H}^{k-1}(N)}, & \text{for } k = m/2+1 \text{ if } \dim M = m \text{ even}, \\ 2^{\dim \mathcal{H}^{k-1}(N)}, & \text{for } k = n/2+1 \text{ if } \dim M = n+1 \text{ odd}, \\ (2/3)^{\dim \mathcal{H}^{k-1}(N)}, & \text{for } k = n/2+2 \text{ if } \dim M = n+1 \text{ odd}. \end{cases} \tag{4.35}$$

The preceeding computations imply that L_k is a finite direct sum of model Laplace operators, a regular-singular Sturm-Liouville operator with matrix coefficients, and in fact does not occur for $|k - m/2| \geq 2$. This corresponds to the general fact, see [BL2, Theorem 3.7, Theorem 3.8] that the Laplace operator on $k-$forms over a compact manifold with an isolated singularity is "essentially self-adjoint" at the cone tip outside of the middle degrees, i.e. for $|k - m/2| \geq 2$.

Therefore the complete determinant of $\mathcal{L}_k^{rel}$ is given simply by a product of finitely many contributions, determined in Propositions 4.16 − 4.20, depending on the choice of a degree. This proves the central result of this subsection.

Theorem 4.21. *The self-adjoint operator $\mathcal{L}_k^{rel}$ is non-trivial only for degrees*

$$k \in \left(\frac{m}{2} - 2, \frac{m}{2} + 2\right).$$

In these degrees the zeta-determinant of $\mathcal{L}_k^{rel}$ is given as follows, where we use the notation established in (4.32) − (4.35)*:*

(i) *For $k \in (m/2 - 2, m/2)$ we have*

$$\det\nolimits_\zeta \mathcal{L}_k^{rel} = \prod_{\nu \in A_k} \frac{\sqrt{2\pi}}{2^\nu \Gamma(1+\nu)} \prod_{\nu \in B_k} 2\pi \frac{\nu - k + m/2}{2^{2\nu}\Gamma(1+\nu)^2}.$$

(ii) *For $k \in (m/2, m/2 + 2)$ we have*

$$\det\nolimits_\zeta \mathcal{L}_k^{rel} = \prod_{\nu \in \widetilde{A}_{k-2}} \frac{\sqrt{2\pi}(\nu + m/2 + 1 - k)}{2^\nu \Gamma(1+\nu)} \prod_{\nu \in B_k} 2\pi \frac{\nu - k + m/2}{2^{2\nu}\Gamma(1+\nu)^2} \cdot P_k.$$

(iii) *For $\dim M = m$ even and $k = m/2$ we have*

$$\det\nolimits_\zeta \mathcal{L}_k^{rel} = \prod_{\nu \in B_k} 2\pi \frac{\nu}{2^{2\nu}\Gamma(1+\nu)^2}.$$

5 The Scalar Analytic Torsion of a Bounded Generalized Cone

The analytic torsion of closed Riemannian manifolds is well-understood and in particular the Cheeger-Müller Theorem is established. The analytic torsion of singular manifolds however, in particular of a bounded generalized cone, still lacks deep understanding.

In fact J.S. Dowker and K. Kirsten provided in [DK] some explicit results for a bounded generalized cone $M = (0,1] \times N$, giving formulas which related the zeta-determinants of form-valued Laplacians, essentially self-adjoint at the cone singularity and with Dirichlet or generalized Neumann conditions at the cone base, to the spectral information on the base manifold N. So, in the manner of [Ch2], they reduced analysis on the cone to that on its base.

Theoretically these results can be composed directly into a formula for the analytic torsion. However this approach would disregard the subtle symmetry of the de Rham complex of a bounded generalized cone, which was derived by M. Lesch in [L3]. Furthermore the formulas obtained this way turn out to be rather ineffective.

We present here an approach that does make use of the symmetry of the de Rham complex and leads to expressions that are easier to evaluate. The calculations use essentially the method of [S], combined with elements of [BKD].

The calculations are performed for any dimension ≥ 2 with an overall general result for the analytic torsion of a bounded generalized cone. Further calculations are possible only by specifying the base manifold N. In Subsection 5.6 we provide explicit results in three and in two dimensions.

However for a bounded generalized cone of dimension two, over a one-dimensional sphere, one needs to introduce an additional parameter in the Riemannian cone metric in order to deal with bounded generalized cone and not simply with a flat disc D^1. There is no need to evaluate the symmetry of the de Rham complex in this case. The calculations of [S] can be generalized to this setup in a straightforward way, which is done in Subsection 5.7.

5.1 Decomposition of the de Rham complex

Let $M^m := (0,1] \times N^n, g^M = dx^2 \oplus x^2 g^N$ be a bounded generalized cone of length one over a closed oriented Riemannian manifold (N, g^N) of dimension n. The Laplace operator $\triangle_k$ on k-forms over M transforms under the isometric identification in Proposition 3.1 to a regular-singular operator of the form (recall (3.6))

$$-\frac{d^2}{dx^2}+\frac{1}{x^2}A_k, \tag{5.1}$$

where A_k is a symmetric differential operator of order two on $\Omega^{k-1}(N)\oplus\Omega^k(N)$, see (3.7). The non-product situation on the bounded generalized cone M is pushed into the x-dependence of the tangential part of the Laplacian.

Consider the relative self-adjoint extension $\triangle_k^{rel}$, introduced in (3.8). We will only need the following well-known result, which is a direct application of [L1, Proposition 1.4.7]

Theorem 5.1. *The self-adjoint operator $\triangle_k^{rel}$ is discrete with the zeta-function*

$$\zeta(s,\triangle_k^{rel})=\sum_{\lambda\in Sp(\triangle_k^{rel})\backslash\{0\}}\lambda^{-s},\ \ Re(s)>m/2,$$

being holomorphic for $Re(s)>m/2$.

The meromorphic continuation of zeta-functions for general self-adjoint extensions of regular-singular operators is discussed in a series of sources, notably [L1, Theorem 2.4.1], [Ch2, Theorem 4.1] and [LMP, Theorem 5.7].

For a compact oriented Riemannian manifold X^m the scalar analytic torsion ([RS]) is defined by

$$\log T(X):=\frac{1}{2}\sum_{k=0}^{m}(-1)^k\cdot k\cdot\zeta_k'(0),$$

where $\zeta_k(s)$ denotes the zeta-function of the Laplacian on k-forms of X, with relative or the absolute boundary conditions posed at ∂X. On compact Riemannian manifolds the zeta-functions $\zeta_k(s)$ extend meromorphically to $\mathbb{C}$ with $s=0$ being regular, so the definition makes sense.

On the bounded generalized cone M the zeta-functions $\zeta(s,\triangle_k^{rel})$ possibly have a simple pole at $s=0$. However we have the following result of A.Dar:

Theorem 5.2. [Dar] *The meromorphic function*

$$T(M,s):=\frac{1}{2}\sum_{k=0}^{m}(-1)^k\cdot k\cdot\zeta(s,\triangle_k^{rel}) \tag{5.2}$$

is regular at $s=0$. Thus the analytic torsion $T^{rel}(M):=\exp(T'(M,s=0))$ of a bounded generalized cone exists.

Thus even though the zeta-functions $\zeta(s, \triangle_k^{rel})$ need not be regular at $s=0$, their residua at $s=0$ cancel in the alternating weighted sum $T(M,s)$.

Remark 5.3. *One can easily check via Poincare duality that the scalar analytic torsion $T^{rel}(M)$ defined with respect to the relative boundary conditions exists if and only if so does the scalar analytic torsion $T^{abs}(M)$, defined with respect to the absolute boundary conditions, and*

$$\log T^{rel}(M) = (-1)^{\dim M+1} \log T^{abs}(M).$$

Hence it suffices to consider the relative boundary conditions and we put

$$T(M) := T^{rel}(M).$$

The statement extends to general compact manifolds with isolated conical singularities. A compact manifold with a conical singularity is a Riemannian manifold $(M_1 \cup_N U, g)$ partitioned by a compact hypersurface N, such that M_1 is a compact manifold with boundary N and U is isometric to $(0,\epsilon] \times N$ with the metric over U being of the following form

$$g|_U = dx^2 \oplus x^2 g|_N.$$

In this section we compute for the bounded generalized cone M the analytic continuation of $\log T(M,s)$ to $s=0$ by means of a decomposition of the de Rham complex. We continue under the isometric identification Φ introduced in Subsection 3.1, which preserves all the spectral properties of the operators, as asserted by Corollary 3.3.

Following [L3], we decompose the de Rham complex of M into a direct sum of subcomplexes of two types. The first type of the subcomplexes is given as follows:

$$0 \to C_0^\infty((0,1), \langle \xi_1 \rangle) \xrightarrow{d_0} C_0^\infty((0,1), \langle \xi_2, \xi_3 \rangle) \xrightarrow{d_1} C_0^\infty((0,1), \langle \xi_4 \rangle) \to 0, \quad (5.3)$$

where $\psi \in \Omega^k(N), k \leq n-1$ is a coclosed normalized η-eigenform of $\triangle_{k,N}$ with $\eta > 0$ and

$$\begin{aligned}
\xi_1 &:= (0, \psi) \in \Omega^{k-1}(N) \oplus \Omega^k(N), \\
\xi_2 &:= (\psi, 0) \in \Omega^k(N) \oplus \Omega^{k+1}(N), \\
\xi_3 &:= (0, \frac{1}{\sqrt{\eta}} d_N \psi) \in \Omega^k(N) \oplus \Omega^{k+1}(N), \\
\xi_4 &:= (\frac{1}{\sqrt{\eta}} d_N \psi, 0) \in \Omega^{k+1}(N) \oplus \Omega^{k+2}(N).
\end{aligned}$$

Observe that under the action of d and d^t the subspace $C_0^\infty((0,1),\langle\xi_1,\xi_2,\xi_3,\xi_4\rangle)$ is invariant and in fact we indeed obtain a complex.

The second type of the subcomplexes comes from the harmonics on the base manifold N and is given as follows. Consider $\mathcal{H}^k(N)$ and fix an orthonormal basis $\{u_i\}, i=1,..,\dim\mathcal{H}^k(N)$ of $\mathcal{H}^k(N)$. Observe that for any i the subspace $C_0^\infty((0,1),\langle 0\oplus u_i, u_i\oplus 0\rangle)$ is invariant under d, d^t and we obtain a subcomplex of the de Rham complex

$$0\to C_0^\infty((0,1),\langle 0\,\oplus\, u_i,\rangle)\xrightarrow{d} C_0^\infty((0,1),\langle u_i\oplus 0\rangle)\to 0. \tag{5.4}$$

By the decomposition (3.11) the de Rham complex $(\Omega_0^*(M),d)$ decomposes completely into subcomplexes of the two types above. This decomposition gives in each degree k a compatible decomposition for $\triangle_k^{rel}$, as observed in Theorem 3.10. Hence the Laplacians $\triangle_k^{rel}$ are composed of the relative extensions of the Laplacians of the subcomplexes. In particular each subcomplex contributes to the function in (5.2) as follows.

The relative boundary conditions turn the complex (5.3) of the first type into a Hilbert complex (see [BL1]) of the following general form:

$$0\to H_k\xrightarrow{D} H_{k+1}\xrightarrow{D} H_{k+2}\to 0.$$

By the specific form of the subcomplex we have the following relation between the zeta-functions corresponding to the Laplacians of the subcomplex

$$\zeta_{k+1}(s)=\zeta_k(s)+\zeta_{k+2}(s). \tag{5.5}$$

From the spectral relation (5.5) we deduce that the contribution of the subcomplex H to the function $T(M,s)$ amounts to

$$\begin{aligned}\frac{(-1)^k}{2}&\left[k\zeta_k(s)-(k+1)\zeta_{k+1}(s)+(k+2)\zeta_{k+2}(s)\right]\\ &=\frac{(-1)^k}{2}(\zeta_{k+2}(s)-\zeta_k(s)).\end{aligned} \tag{5.6}$$

Since there are in fact infinitely many subcomplexes of the first type, we first have to add up the contributions for $Re(s)$ large and then continue the sum analytically to $s=0$. Then the derivative at zero gives the contribution to $T(M)$.

For the contribution of the subcomplexes (5.4) of the second kind to the analytic torsion, note that the relative boundary conditions turn the complex of second type into a Hilbert complex of the following general form:

$$0\to H_k\xrightarrow{D} H_{k+1}\to 0.$$

There are only finitely many such subcomplexes, since $\dim \mathcal{H}^*(N) < \infty$. Hence we obtain directly for the contribution to $\log T(M)$ from each of such subcomplexes

$$\frac{(-1)^{k+1}}{2}\zeta'(D^*D, s=0). \tag{5.7}$$

5.2 Symmetry in the Decomposition

In this section we present a symmetry of the de Rham complex on a model cone, as elaborated by M. Lesch in [L3]. Consider the subcomplexes (5.3) of the first type

$$0 \to C_0^\infty((0,1), \langle\xi_1\rangle) \xrightarrow{d_0} C_0^\infty((0,1), \langle\xi_2, \xi_3\rangle) \xrightarrow{d_1} C_0^\infty((0,1), \langle\xi_4\rangle) \to 0.$$

where $\psi \in \Omega^k(N), k \leq n-1$ is a coclosed normalized η-eigenform of $\triangle_{k,N}$ with $\eta > 0$ and

$$\begin{aligned}
\xi_1 &:= (0, \psi) \in \Omega^{k-1}(N) \oplus \Omega^k(N), \\
\xi_2 &:= (\psi, 0) \in \Omega^k(N) \oplus \Omega^{k+1}(N), \\
\xi_3 &:= (0, \frac{1}{\sqrt{\eta}} d_N \psi) \in \Omega^k(N) \oplus \Omega^{k+1}(N), \\
\xi_4 &:= (\frac{1}{\sqrt{\eta}} d_N \psi, 0) \in \Omega^{k+1}(N) \oplus \Omega^{k+2}(N),
\end{aligned}$$

By computing explicitly the action of the exterior derivative (3.2) on the basis elements ξ_i we obtain with $c_j := (-1)^j(j - n/2), j = 0, .., n$

$$d_0^\psi = \begin{pmatrix} (-1)^k \partial_x + \frac{c_k}{x} \\ x^{-1}\sqrt{\eta} \end{pmatrix}, \quad d_1^\psi = \left(x^{-1}\sqrt{\eta},\ (-1)^{k+1}\partial_x + \frac{c_{k+1}}{x}\right).$$

Next we compute the associated Laplacians (identified with their scalar action as in Remark 3.11)

$$\triangle_0^\psi := (d_0^\psi)^t d_0^\psi = -\partial_x^2 + \frac{1}{x^2}\left[\eta + \left(k + \frac{1}{2} - \frac{n}{2}\right)^2 - \frac{1}{4}\right] = d_1^\psi (d_1^\psi)^t =: \triangle_2^\psi. \tag{5.8}$$

From the detailed discussion in Subsection 3.2 we infer that separating out the subcomplex above provides a compatible decomposition of $\triangle_k^{rel}, \triangle_{k+1}^{rel}, \triangle_{k+2}^{rel}$. Hence the relative boundary conditions induce self-adjoint extensions of the Laplacians $\triangle_0^\psi, \triangle_2^\psi$

$$\triangle_{0,rel}^\psi = (d_0^\psi)^t_{\max} d_{0,\min}^\psi, \quad \triangle_{2,rel}^\psi = d_{1,\min}^\psi (d_1^\psi)^t_{\max}$$

Now we discuss the relative boundary conditions for $\triangle_0^\psi, \triangle_2^\psi$. Assume that the lowest non-zero eigenvalue η of $\triangle_{k,N}$ is $\eta > 1$. This can always be achieved by an appropriate scaling of the metric on N

$$g^{N,c} := c^{-2} g^N, \quad c > 0 \text{ large enough.} \tag{5.9}$$

More precisely, the Laplacian $\triangle_N^c$ defined on $\Omega^*(N)$ with respect to $g^{N,c}$ is related to the original Laplacian $\triangle_N$ as follows

$$\triangle_N^c = c^2 \triangle_N.$$

Hence indeed for $c > 0$ sufficiently large we achieve that the Laplacian $\triangle_N^c$ has no "small" non-zero eigenvalues.

This guarantees in view of Corollary 2.14 that $\triangle_0^\psi$ and $\triangle_2^\psi$ are in the limit point case at $x = 0$ and hence all their self-adjoint extensions in $L^2(0,1)$ coincide at $x = 0$. Hence we only need to consider the relative boundary conditions at $x = 1$. With Proposition 3.5 (see also Lemma 3.12 and Lemma 3.13) we obtain

$$\mathcal{D}(\triangle_{0,rel}^\psi) = \{f \in \mathcal{D}(\triangle_{0,\max}^\psi) | f(1) = 0\},$$
$$\mathcal{D}(\triangle_{2,rel}^\psi) = \{f \in \mathcal{D}(\triangle_{2,\max}^\psi) | (-1)^k f'(1) + c_{k+1} f(1) = 0\}.$$

The values $f(1), f'(1)$ are well-defined since by (2.5) we know $\mathcal{D}(\triangle_{0,2,\max}^\psi) \subset H^2_{\text{loc}}(0,1]$.

Remark 5.4. *The assumption on the lower bound of the non-zero eigenvalues of $\triangle_{k,N}$ can be dropped. Then the discussion of a finite direct sum of model Laplacians in the limit circle case enters the calculations. The associated zeta-determinants were determined in Theorem 4.21.*

Next we consider the twin-subcomplex, associated to the subcomplex discussed above. Let $\phi := *_N \psi \in \Omega^{n-k}(N)$. Put

$$\widetilde{\xi_1} := (0, \frac{1}{\sqrt{\eta}} d_N^t \phi) \in \Omega^{n-k-2}(N) \oplus \Omega^{n-k-1}(N),$$
$$\widetilde{\xi_2} := (\frac{1}{\sqrt{\eta}} d_N^t \phi, 0) \in \Omega^{n-k-1}(N) \oplus \Omega^{n-k}(N),$$
$$\widetilde{\xi_3} := (0, \phi) \in \Omega^{n-k-1}(N) \oplus \Omega^{n-k}(N),$$
$$\widetilde{\xi_4} := (\phi, 0) \in \Omega^{n-k}(N) \oplus \Omega^{n-k+1}(N).$$

Again the subspace $C_0^\infty((0,1),\langle\widetilde{\xi_1},\widetilde{\xi_2},\widetilde{\xi_3},\widetilde{\xi_4}\rangle)$ is invariant under the action of d and d^t and in fact we obtain a complex

$$0 \to C_0^\infty((0,1),\langle\widetilde{\xi_1}\rangle) \xrightarrow{d_0^\phi} C_0^\infty((0,1),\langle\widetilde{\xi_2},\widetilde{\xi_3}\rangle) \xrightarrow{d_1^\phi} C_0^\infty((0,1),\langle\widetilde{\xi_4}\rangle) \to 0.$$

By computing explicitly the action of the exterior derivative (3.2) on the basis elements $\widetilde{\xi_i}$ we obtain

$$d_0^\phi = \begin{pmatrix} (-1)^{n-k-1}\partial_x + \frac{c_{n-k-1}}{x} \\ x^{-1}\sqrt{\eta} \end{pmatrix}, \quad d_1^\phi = \left(x^{-1}\sqrt{\eta},\ (-1)^{n-k}\partial_x + \frac{c_{n-k}}{x}\right).$$

As for the first subcomplex we compute the relevant Laplacians:

$$\triangle_0^\phi = \triangle_2^\phi = -\partial_x^2 + \frac{1}{x^2}\left[\eta + \left(k+\frac{1}{2}-\frac{n}{2}\right)^2 - \frac{1}{4}\right] = \triangle_0^\psi = \triangle_2^\psi, \tag{5.10}$$

where the operators are identified with their scalar actions. As before, separating out the subcomplex above, we decompose $\triangle_{n-k\pm 1}^{rel}, \triangle_{n-k}^{rel}$ compatibly. Hence the relative boundary conditions induce self-adjoint extensions

$$\triangle_{0,rel}^\phi = (d_0^\phi)^t_{\max} d_{0,\min}^\phi, \quad \triangle_{2,rel}^\phi = d_{1,\min}^\phi (d_1^\phi)^t_{\max}$$

of the Laplacians $\triangle_0^\phi, \triangle_2^\phi$ respectively. Under the scaling assumption of (5.9) the relative boundary conditions for this pair of operators are computed to

$$\begin{aligned} \mathcal{D}(\triangle_{0,rel}^\phi) &= \{f \in \mathcal{D}(\triangle_{0,\max}^\phi) | f(1) = 0\}, \\ \mathcal{D}(\triangle_{2,rel}^\phi) &= \{f \in \mathcal{D}(\triangle_{2,\max}^\phi) | (-1)^{n-k+1} f'(1) + c_{n-k} f(1) = 0\}, \end{aligned}$$

with Proposition 3.5. As before the values $f(1), f'(1)$ are well-defined since $\mathcal{D}(\triangle_{0,2,\max}^\phi) \subset H^2_{\text{loc}}(0,1]$.

So in total we obtain four self-adjoint operators, which differ only by their boundary conditions. Unfortunately the differences in the domains do not allow to cancel the contribution of the two twin-subcomplexes to the analytic torsion. However the symmetry still allows us to perform explicit computations.

Recall that ψ was chosen to be a normalized coclosed η-eigenform on N of degree k and $\phi = *_N\psi$. Denote the dependence of the generating forms ψ and ϕ on the eigenvalue η by $\psi(\eta)$ and $\phi(\eta)$. Introduce further the notation

$$\begin{aligned} D(k) :=& \{\lambda \in \text{Spec}\triangle_{0,rel}^{\psi(\eta)} | \eta \in \text{Spec}\triangle_{k,ccl,N}\backslash\{0\}\} \\ =& \{\lambda \in \text{Spec}\triangle_{0,rel}^{\phi(\eta)} | \eta \in \text{Spec}\triangle_{k,ccl,N}\backslash\{0\}\}, \\ N_1(k) :=& \{\lambda \in \text{Spec}\triangle_{2,rel}^{\psi(\eta)} | \eta \in \text{Spec}\triangle_{k,ccl,N}\backslash\{0\}\}, \\ N_2(k) :=& \{\lambda \in \text{Spec}\triangle_{2,rel}^{\phi(\eta)} | \eta \in \text{Spec}\triangle_{k,ccl,N}\backslash\{0\}\}, \end{aligned}$$

where all eigenvalues are counted according to their multiplicities. Using this notation we now introduce the following zeta-functions for $Re(s) \gg 0$

$$\zeta^k_D(s) := \sum_{\lambda \in D(k)} \lambda^{-s}, \ \zeta^k_{N_1}(s) := \sum_{\lambda \in N_1(k)} \lambda^{-s}, \ \zeta^k_{N_2}(s) := \sum_{\lambda \in N_2(k)} \lambda^{-s}, Re(s) \gg 0.$$

The D-subscript is aimed to point out that the zeta-functions in the sum are associated to Laplacians with Dirichlet boundary conditions at $x = 1$. Similarly the N-subscripts point out the generalized Neumann boundary conditions at $x = 1$, which are however different for $\triangle^{\phi}_2$ and $\triangle^{\psi}_2$.

The zeta-functions $\zeta^k_D(s), \zeta^k_{N_1}(s)$ and $\zeta^k_{N_2}(s)$ are by Theorem 5.1 holomorphic for $Re(s)$ sufficiently large, since they sum over eigenvalues of $\triangle^{rel}_*$ but with lower multiplicities. In view of (5.6), which describes the contribution to analytic torsion from the subcomplexes, we set for $Re(s)$ large

Definition 5.5. $\zeta_k(s) := \zeta^k_{N_1}(s) - \zeta^k_D(s)) + (-1)^{n-1}(\zeta^k_{N_2}(s) - \zeta^k_D(s))$.

Remark 5.6. *Note that $\zeta^k_D(s)$ in the definition of $\zeta_k(s)$ cancel for $m = \dim M$ odd, simplifying the expression for $\zeta_k(s)$ considerably. Further simplifications (notably Proposition 5.18) take place throughout the discussion, so that an effective result can be obtained in the end.*

Below we provide the analytic continuation of $\zeta_k(s)$ to $s = 0$ for any fixed degree $k < \dim N - 1$ and compute $(-1)^k \zeta'_k(0)$. The contribution coming from the subcomplexes of the second type (5.4), induced by the harmonic forms on the base N, is not included in $\zeta_k(s)$ and will be determined explicitly in a separate discussion.

Remark 5.7. *The total contribution of subcomplexes* (5.3) *of first type to the logarithmic scalar analytic torsion* $\log T(M)$ *of the odd-dimensional bounded generalized cone M is given by*

$$\frac{1}{2} \sum_{k=0}^{n/2-1} (-1)^k \cdot \zeta'_k(0).$$

For an even-dimensional cone M the zeta-function $\zeta_k(s)$ counts in the degree $k = (n-1)/2$ each subcomplex of type (5.3) *twice. Thus the total contribution of subcomplexes of first type to* $\log T(M)$ *is given by*

$$\frac{1}{2} \left(\sum_{k=0}^{(n-3)/2} (-1)^k \cdot \zeta'_k(0) + \frac{(-1)^{\frac{(n-1)}{2}}}{2} \cdot \zeta'_{\frac{n-1}{2}}(0) \right),$$

where the first sum is set to zero for $\dim M = n + 1 = 2$.

5.3 Some auxiliary analysis

Fix a real number $\nu > 1$ and consider the following differential operator

$$l_\nu := -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2} : C_0^\infty(0,1) \to C_0^\infty(0,1).$$

By the choice $\nu > 1$ we infer from Corollary 2.14 (i) that the maximal and the minimal extensions of l_ν coincide at $x = 0$ and hence we only need to fix boundary conditions at $x = 1$ to define a self-adjoint extension of l_ν. Put for $\alpha \in \mathbb{R}^*$:

$$\mathcal{D}(L_\nu(\alpha)) := \{f \in \mathcal{D}(l_{\nu,\max}) | (\alpha - 1/2)^{-1} f'(1) + f(1) = 0\},$$

where $\alpha = \infty$ defines the Dirichlet boundary conditions at $x = 1$ and $\alpha = 1/2$ – the pure Neumann boundary conditions at $x = 1$.

Proposition 5.8. *The self-adjoint operator $L_\nu(\alpha), \alpha \in \mathbb{R}^*$ is discrete and bounded from below. For $\alpha^2 < \nu^2$ and $\alpha = \infty$ the operator $L_\nu(\alpha)$ is positive.*

Proof. The discreteness of $L_\nu(\alpha)$ is asserted in [BS2], see also [L, Theorem 1.1] where this result is restated. For semi-boundedness of $L_\nu(\alpha)$ note that the potential $(\nu^2 - 1/4)/x^2$ is positive. Hence it suffices to discuss semi-boundedness of $-d^2/dx^2$ under different boundary conditions. By [W2, Theorem 8.24] all the self-adjoint extensions of

$$-\frac{d^2}{dx^2} : C_0^\infty(0,1) \to C_0^\infty(0,1)$$

are bounded from below, since $-d^2/dx^2$ on $C_0^\infty(0,1)$ is semi-bounded. Indeed for any $f \in C_0^\infty(0,1)$

$$\langle -f'', f\rangle_{L^2(0,1)} = - f'(x)\overline{f(x)}\Big|_0^1 + \int_0^1 |f'(x)|^2 dx \geq 0.$$

Hence $L_\nu(\alpha)$ is indeed semi-bounded. It remains to identify the lower bound in the case $\alpha^2 < \nu^2$ and $\alpha = \infty$. For this consider any $f \in \mathcal{D}(l_{\nu,\max}), f \not\equiv 0$. Recall the relation (2.5), which implies that f is continuously differentiable at $(0,1)$ and f, f' extend continuously to $x = 1$. Moreover we infer from the proof of Proposition 2.11 (iii) the asymptotic behaviour $f(x) = O(x^{3/2}), f'(x) = O(x^{1/2})$, as $x \to 0$. We compute via integration by parts for any $\epsilon \in \mathbb{R}$ with $\epsilon^2 < \nu^2$:

$$\langle l_\nu f, f\rangle_{L^2(0,1)} = \int_0^1 \left[\left(-\frac{d}{dx}+\frac{\epsilon-1/2}{x}\right)\left(\frac{d}{dx}+\frac{\epsilon-1/2}{x}\right)f(x) + \right.$$
$$\left. + \frac{\nu^2-\epsilon^2}{x^2}f(x)\right]\cdot\overline{f(x)}dx = -\left(f'(x)+\frac{\epsilon-1/2}{x}f(x)\right)\overline{f(x)}\Big|_{x\to 0+}^{1} +$$
$$+\left\|f'+\frac{\epsilon-1/2}{x}f\right\|^2_{L^2(0,1)} + \left\|\frac{\sqrt{\nu^2-\epsilon^2}}{x}f\right\|^2_{L^2(0,1)}.$$

Now the asymptotics of $f(x)$ and $f'(x)$ as $x \to 0$ implies together with $f \not\equiv 0$:

$$\langle l_\nu f, f\rangle_{L^2(0,1)} > -\overline{f(1)}\cdot(f'(1)+(\epsilon-1/2)f(1)). \tag{5.11}$$

Evaluation of the conditions at $x = 1$ for $L_\nu(\alpha)$ with $\alpha = \infty$ or $\alpha^2 < \nu^2$ proves the statement. □

Corollary 5.9. *Let $J_\nu(z)$ denote the Bessel function of first kind and put for any fixed $\alpha \in \mathbb{R}^*$*

$$\widetilde{J}^\alpha_\nu(z) := \alpha J_\nu(z) + zJ'_\nu(z),$$

where for $\alpha = \infty$ we put $\widetilde{J}^\alpha_\nu(z) := J_\nu(z)$. Then for $\nu > 1$ and $\alpha = \infty$ or $\alpha^2 < \nu^2$, the zeros of $\widetilde{J}^\alpha_\nu(z)$ are real, discrete and symmetric about the origin. The eigenvalues of the positive operator $L_\nu(\alpha)$ are simple and given by squares of positive zeros of $\widetilde{J}^\alpha_\nu(z)$, i.e.

$$\mathrm{Spec}L_\nu(\alpha) = \{\mu^2 | \widetilde{J}^\alpha_\nu(\mu) = 0, \mu > 0\}.$$

Proof. The general solution to $l_\nu f = \mu^2 f, \mu \neq 0$ is of the following form

$$f(x) = c_1\sqrt{x}J_\nu(\mu x) + c_2\sqrt{x}Y_\nu(\mu x),$$

where c_1, c_2 are constants and J_ν, Y_ν denote Bessel functions of first and second kind, respectively. For $\nu > 1$ the asymptotic behaviour of $f \in \mathcal{D}(l_{\nu,\max})$ is given by $f(x) = O(x^{3/2}), x \to 0$. Hence a solution to $l_\nu f = \mu^2 f, \mu \neq 0$ with $f \in \mathcal{D}(l_{\nu,\max})$ must be of the form

$$f(x) = c_1\sqrt{x}J_\nu(\mu x).$$

Taking in account the boundary conditions for $L_\nu(\alpha)$ with at $\alpha = \infty$ or $\alpha^2 < \nu^2$, we deduce correspondence between zeros of $\widetilde{J}^\alpha_\nu(z)$ and eigenvalues of $L_\nu(\alpha)$. Hence by Proposition 5.8 we deduce the statements about the zeros of $\widetilde{J}^\alpha_\nu(z)$, up to the statement on the symmetry of zeros, which follows simply from the standard infinite series representation of Bessel functions.

Furthermore, $J_\nu(-\mu x) = (-1)^\nu J_\nu(\mu x), \mu \neq 0$ and hence each eigenvalue μ^2 of $L_\nu(\alpha)$ is simple with the unique (up to a multiplicative constant) eigenfunction $f(x) = \sqrt{x} J_\nu(\mu x), \mu > 0$ This completes the proof. □

Similar statements can be deduced for more general values of $\alpha \in \mathbb{R}^*$, but are not relevant in the present discussion. Finally note as a direct application of Proposition 5.8 that the Laplacians $\triangle^\psi_{0,2,rel}$ and $\triangle^\phi_{0,2,rel}$, introduced in the previous subsection, are positive.

Corollary 5.10. *For all degrees $k = 0, .., \dim N$ we have*

$$D(k) \subset \mathbb{R}^+, \ N_i(k) \subset \mathbb{R}^+, \ i = 1, 2.$$

Next consider the zeta-function $\zeta(s, L_\nu(\alpha)), \alpha \in \mathbb{R}^*$ associated to the self-adjoint realization $L_\nu(\alpha)$ of l_ν. It is well-known, see [L, Theorem 1.1] that the zeta-function extends meromorphically to $\mathbb{C}$ with the analytic representation given by the Mellin transform of the heat trace:

$$\zeta(s, L_\nu(\alpha)) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}_{L^2}(e^{-tL_\nu(\alpha)} P) dt,$$

where P is the projection on the orthogonal complement of the null space of $L_\nu(\alpha)$. The heat operator $\exp(-tL_\nu(\alpha))$ is defined by the spectral theorem and is a bounded smoothing operator with finite trace $\mathrm{Tr}_{L^2}(e^{-tL_\nu(\alpha)} P)$ of standard polylogarithmic asymptotics as $t \to 0+$, see [Ch, Theorem 2.1]. We can write for $t > 0$

$$\mathrm{Tr}(e^{-tL_\nu(\alpha)}) = \frac{1}{2\pi i} \int_\Lambda e^{-\lambda t} \mathrm{Tr}(\lambda - L_\nu(\alpha))^{-1} d\lambda,$$

where the contour Λ shall encircle all non-zero eigenvalues of the semi-bounded $L_\nu(\alpha), \alpha \in \mathbb{R}^*$ and be counter-clockwise oriented, in analogy to Figure 3 below.

Now, following [S], we obtain an integral representation for $\zeta(s, L_\nu(\alpha))$ in a computationally convenient form. Introduce a numbering (λ_n) of the eigenvalues of $L_\nu(\alpha)$ and observe

$$\mathrm{Tr}(\lambda - L_\nu(\alpha))^{-1} = \sum_n \frac{1}{\lambda - \lambda_n} = \sum_n \frac{d}{d\lambda} \log\left(1 - \frac{\lambda}{\lambda_n}\right),$$

where we fix henceforth the branch of logarithm in $\mathbb{C}\backslash\mathbb{R}^+$ with $0 \leq Im \log z < 2\pi$. We continue with this branch of logarithm throughout the section. Integrating now by parts first in λ, then in t we obtain

$$\zeta(s, L_\nu(\alpha)) = \frac{s^2}{\Gamma(s+1)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_\Lambda \frac{e^{-\lambda t}}{-\lambda} \left[-\sum_n \log \left(1 - \frac{\lambda}{\lambda_n} \right) \right] d\lambda dt. \tag{5.12}$$

5.4 Contribution from the Subcomplexes I

We continue in the setting and in the notation of Section 5.2 and fix any degree $k \leq \dim N - 1$. We define the following contour:

$$\Lambda_c := \{\lambda \in \mathbb{C} | |\arg(\lambda - c)| = \pi/4\} \tag{5.13}$$

oriented counter-clockwise, with $c > 0$ a fixed positive number, smaller than the lowest non-zero eigenvalue of $\triangle_*^{rel}$. The contour is visualized in the Figure 3:

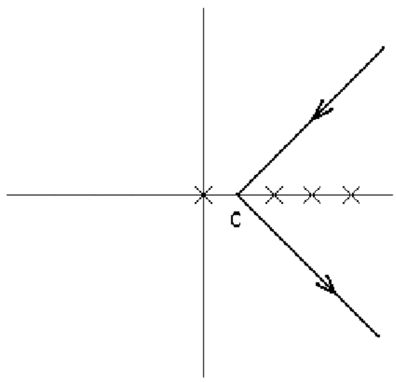

Figure 3: The contour Λ_c. The $\times$'s represent the eigenvalues of $\triangle_*^{rel}$.

In analogy to the constructions of [S] we obtain for the zeta-functions $\zeta_D^k(s), \zeta_{N_1}^k(s), \zeta_{N_2}^k(s)$ the following results.

Proposition 5.11. *Let* $M = (0,1] \times N, g^M = dx^2 \oplus x^2 g^N$ *be a bounded generalized cone. Let the metric on the base manifold* N *be scaled as in* (5.9) *such that the non-zero eigenvalues of the form-valued Laplacians on* N *are* *bigger than* 1*. Denote by* $\triangle_{k,ccl,N}$ *the Laplace Operator on coclosed k-forms on* N*. Let*

$$F_k := \{\xi \in \mathbb{R}^+ \mid \xi^2 = \eta + (k+1/2-n/2)^2 \,, \eta \in \mathrm{Spec}\triangle_{k,ccl,N}\backslash\{0\}\}.$$

Then we obtain with $(j_{\nu,i})_{i\in\mathbb{N}}$ *being the positive zeros of the Bessel function* $J_\nu(z)$

$$\zeta^k_D(s) = \frac{s^2}{\Gamma(s+1)}\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\frac{e^{-\lambda t}}{-\lambda}T^k_D(s,\lambda)d\lambda dt, \qquad (5.14)$$

$$T^k_D(s,\lambda) = \sum_{\nu\in F_k} t^{D,k}_\nu(\lambda)\,\nu^{-2s}, \quad t^{D,k}_\nu(\lambda) = -\sum_{i=1}^\infty \log\left(1-\frac{\nu^2\lambda}{j^2_{\nu,i}}\right). \qquad (5.15)$$

Proof. Consider for $\eta \in \mathrm{Spec}\triangle_{k,ccl,N}\backslash\{0\}$ the operators $\triangle_0^{\psi(\eta)}$ and $\triangle_0^{\phi(\eta)}$, defined in (5.8) and (5.10). Under the identification with their scalar parts, as in Remark 3.11 we have

$$\triangle_0^{\phi(\eta)} = \triangle_0^{\psi(\eta)} = -\partial_x^2 + \frac{1}{x^2}\left[\nu^2 - \frac{1}{4}\right],$$

where $\nu := \sqrt{\eta + (k+1/2-n/2)^2}$. By scaling of the metric on N we have $\nu > 1$ and hence the self-adjoint extensions $\triangle_{0,rel}^{\phi(\eta)}$ and $\triangle_{0,rel}^{\psi(\eta)}$ are determined only by their Dirichlet boundary conditions at $x=1$. By Corollary 5.9 we obtain:

$$\zeta^k_D(s) = \sum_{\nu\in F_k}\sum_{i=1}^\infty j_{\nu,i}^{-2s} = \sum_{\nu\in F_k}\nu^{-2s}\sum_{i=1}^\infty\left(\frac{j_{\nu,i}}{\nu}\right)^{-2s}, \quad Re(s) \gg 0,$$

where $j_{\nu,i}$ are the positive zeros of $J_\nu(z)$. This series is well-defined for $Re(s)$ large by Theorem 5.1, since $\triangle_{0,rel}^{\phi(\eta)}(\equiv \triangle_{0,rel}^{\psi(\eta)})$ as direct sum components of $\triangle_*^{rel}$ have the same spectrum as $\triangle_*^{rel}$, but with lower multiplicities in general.

Due to the uniform convergence of integrals and series we obtain with similar computations as for (5.12) an integral representation for this sum:

$$\zeta^k_D(s) = \frac{s^2}{\Gamma(s+1)}\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_c}\frac{e^{-\lambda t}}{-\lambda}T^k_D(s,\lambda)d\lambda dt, \qquad (5.16)$$

$$T^k_D(s,\lambda) = \sum_{\nu\in F_k} t^{D,k}_\nu(\lambda)\,\nu^{-2s}, \quad t^{D,k}_\nu(\lambda) = -\sum_{i=1}^\infty \log\left(1-\frac{\nu^2\lambda}{j^2_{\nu,i}}\right). \qquad (5.17)$$

Note that the contour Λ_c, defined in (5.13) encircles all eigenvalues of $\triangle_{0,rel}^{\phi(\eta)} \equiv \triangle_{0,rel}^{\psi(\eta)}$ by construction, since the operators are positive by Corollary 5.10. □

Proposition 5.12. *Let $M = (0,1] \times N, g^M = dx^2 \oplus x^2 g^N$ be a bounded generalized cone. Let the metric on the base manifold N be scaled as in* (5.9) *such that the non-zero eigenvalues of the form-valued Laplacians on N are bigger than* 1*. Denote by $\triangle_{k,ccl,N}$ the Laplace Operator on coclosed k-forms on N. Let*

$$F_k := \{\xi \in \mathbb{R}^+ \mid \xi^2 = \eta + (k+1/2-n/2)^2, \eta \in \mathrm{Spec}\triangle_{k,ccl,N}\backslash\{0\}\}.$$

Then we obtain for $l = 1,2$

$$\zeta^k_{N_l}(s) = \frac{s^2}{\Gamma(s+1)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\wedge_c} \frac{e^{-\lambda t}}{-\lambda} T^k_{N_l}(s,\lambda) d\lambda dt, \tag{5.18}$$

$$T^k_{N_l}(s,\lambda) = \sum_{\nu \in F_k} t^{N_l,k}_\nu(\lambda)\, \nu^{-2s}, \quad t^{N_l,k}_\nu(\lambda) = -\sum_{i=1}^\infty \log\left(1 - \frac{\nu^2 \lambda}{\widetilde{j}^2_{\nu,l,i}}\right), \tag{5.19}$$

where $(\widetilde{j}_{\nu,l,i})_{i\in\mathbb{N}}$ are the positive zeros of $\widetilde{J}^{N_l,k}_\nu(z)$ for $l=1,2$. The functions $\widetilde{J}^{N_l}_\nu(z)$ are defined as follows

$$\widetilde{J}^{N_1,k}_\nu(z) := \left(\frac{1}{2} + (-1)^k c_{k+1}\right) J_\nu(z) + zJ'_\nu(z),$$
$$\widetilde{J}^{N_2,k}_\nu(z) := \left(\frac{1}{2} + (-1)^k c_k\right) J_\nu(z) + zJ'_\nu(z).$$

Proof. Consider for $\eta \in \mathrm{Spec}\triangle_{k,ccl,N}\backslash\{0\}$ the operators $\triangle^{\psi(\eta)}_2$ and $\triangle^{\phi(\eta)}_2$, defined in (5.8) and (5.10), which contribute to the zeta-functions $\zeta^k_{N_1}(s)$ and $\zeta^k_{N_2}(s)$ correspondingly. Under the identification with their scalar parts, as in Remark 3.11 we have

$$\triangle^{\phi(\eta)}_2 = \triangle^{\psi(\eta)}_2 = -\partial^2_x + \frac{1}{x^2}\left[\nu^2 - \frac{1}{4}\right],$$

where $\nu := \sqrt{\eta + (k+1/2-n/2)^2}$. By scaling of the metric on N we have $\nu > 1$ and hence the self-adjoint extensions $\triangle^{\phi(\eta)}_{2,rel}$ and $\triangle^{\psi(\eta)}_{2,rel}$ are determined only by their generalized Neumann boundary conditions at $x=1$. Recall

$$\mathcal{D}(\triangle^\psi_{2,rel}) = \{f \in \mathcal{D}(\triangle^\psi_{2,\max}) | f'(1) + (-1)^k c_{k+1} f(1) = 0\},$$
$$\mathcal{D}(\triangle^\phi_{2,rel}) = \{f \in \mathcal{D}(\triangle^\phi_{2,\max}) | f'(1) + (-1)^{n-k+1} c_{n-k} f(1) = 0\}.$$

Observe $(-1)^{n-k+1}c_{n-k} = (-1)^k c_k$ and put

$$\widetilde{J}_\nu^{N_1,k}(\mu) := \left(\frac{1}{2} + (-1)^k c_{k+1}\right) J_\nu(\mu) + \mu J'_\nu(\mu),$$
$$\widetilde{J}_\nu^{N_2,k}(\mu) := \left(\frac{1}{2} + (-1)^k c_k\right) J_\nu(\mu) + \mu J'_\nu(\mu).$$

Note for any degree k and any $\nu \in F_k$

$$\left|\frac{1}{2} + (-1)^k c_{k+1}\right| = \left|\frac{1}{2} + (-1)^k c_k\right| = \left|\frac{n}{2} - \frac{1}{2} - k\right| < \nu.$$

Hence by Corollary 5.9 we obtain for $l = 1, 2$:

$$\zeta^k_{N_l}(s) = \sum_{\nu \in F_k} \sum_{i=1}^{\infty} \widetilde{j}_{\nu,l,i}^{-2s} = \sum_{\nu \in F_k} \nu^{-2s} \sum_{i=1}^{\infty} \left(\frac{\widetilde{j}_{\nu,l,i}}{\nu}\right)^{-2s}, \quad Re(s) \gg 0,$$

where $\widetilde{j}_{\nu,l,i}$ are the positive zeros of $\widetilde{J}_\nu^{N_l,k}(z)$ for $l = 1, 2$. This series is well-defined for $Re(s)$ large by Theorem 5.1, since $\triangle^{\phi(\eta)}_{2,rel}, \triangle^{\psi(\eta)}_{2,rel}$ as direct sum components of $\triangle^{rel}_*$ have the same spectrum as $\triangle^{rel}_*$, but with lower multiplicities in general.

Due to the uniform convergence of integrals and series we obtain with similar computations as for (5.12) an integral representation for this sum:

$$\zeta^k_{N_l}(s) = \frac{s^2}{\Gamma(s+1)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\wedge_c} \frac{e^{-\lambda t}}{-\lambda} T^k_{N_l}(s,\lambda) d\lambda dt, \tag{5.20}$$

$$T^k_{N_l}(s,\lambda) = \sum_{\nu \in F_k} t_\nu^{N_l,k}(\lambda)\, \nu^{-2s}, \quad t_\nu^{N_l,k}(\lambda) = -\sum_{i=1}^{\infty} \log\left(1 - \frac{\nu^2 \lambda}{\widetilde{j}^2_{\nu,l,i}}\right). \tag{5.21}$$

Note that the contour Λ_c encircles all the possible eigenvalues of $\triangle^{\phi(\eta)}_{2,rel}, \triangle^{\psi(\eta)}_{2,rel}$ by construction, since the operators are positive by Corollary 5.10. □

Corollary 5.13. *Let $M = (0,1] \times N, g^M = dx^2 \oplus x^2 g^N$ be a bounded generalized cone. Let the metric on N be scaled as in (5.9) such that the non-zero eigenvalues of the form-valued Laplacians on N are bigger than 1. Then we obtain with Definition 5.5 in the notation of Propositions 5.11 and 5.12*

$$\zeta_k(s) = \frac{s^2}{\Gamma(s+1)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\wedge_c} \frac{e^{-\lambda t}}{-\lambda} T^k(s,\lambda) d\lambda dt, \tag{5.22}$$

$$T^k(s,\lambda) := \sum_{\nu \in F_k} t^k_\nu(\lambda)\, \nu^{-2s},$$

$$t^k_\nu(\lambda) := \left[t^{N_1,k}_\nu(\lambda) - t^{D,k}_\nu(\lambda)) + (-1)^{n-1}(t^{N_2,k}_\nu(\lambda) - t^{D,k}_\nu(\lambda)) \right].$$

If $\dim M$ *is odd we obtain with* $z := \sqrt{-\lambda}$ *and* $\alpha_k := n/2 - 1/2 - k$

$$\begin{aligned} t^k_\nu(\lambda) = \Big[&- \log(\alpha_k I_\nu(\nu z) + \nu z I'_\nu(\nu z)) + \log\left(1 + \frac{\alpha_k}{\nu}\right) + \\ &+ \log(-\alpha_k I_\nu(\nu z) + \nu z I'_\nu(\nu z)) - \log\left(1 - \frac{\alpha_k}{\nu}\right)\Big]. \end{aligned}$$

For $\dim M$ *even we compute with* $z := \sqrt{-\lambda}$

$$\begin{aligned} t^k_\nu(\lambda) = \Big[&- \log(\alpha_k I_\nu(\nu z) + \nu z I'_\nu(\nu z)) + \log\left(1 + \frac{\alpha_k}{\nu}\right) - \\ &- \log(-\alpha_k I_\nu(\nu z) + \nu z I'_\nu(\nu z)) + \log\left(1 - \frac{\alpha_k}{\nu}\right) + \\ &+ 2\log(I_\nu(\nu z)) + 2\log \nu \Big]. \end{aligned}$$

Proof. Recall for convenience the definition of $\zeta_k(s)$ in Definition 5.5

$$\zeta_k(s) := \zeta^k_{N_1}(s) - \zeta^k_D(s)) + (-1)^{n-1}(\zeta^k_{N_2}(s) - \zeta^k_D(s)).$$

The integral representation and the definition of $t^k_\nu(\lambda)$ are then a direct consequence of Propositions 5.11 and 5.12. It remains to present $t^k_\nu(\lambda)$ in terms of special functions.

In order to simplify notation we put (recall $c_j := (-1)^j(j - n/2)$)

$$\alpha_k := \frac{1}{2} + (-1)^k c_{k+1} = \frac{n}{2} - \frac{1}{2} - k = -\left(\frac{1}{2} + (-1)^k c_k\right).$$

Now we present $t^{D,k}_\nu(\lambda)$ and $t^{N_l,k}_\nu(\lambda), l = 1,2$ in terms of special functions. This can be done by referring to tables of Bessel functions in [GRA] or [AS]. However in the context of the paper it is more appropriate to derive the presentation from results on zeta-regularized determinants. Here we follow the approach of [L, Section 4.2] in a slightly different setting.

The original setting of [L, (4.22)] provides an infinite product representation for $I_\nu(z)$. We apply its approach in order to derive the corresponding result for $\widetilde{I}^N_\nu(z) := \alpha I_\nu(z) + z I'_\nu(z)$, with $\alpha \in \{\pm\alpha_k\}$ and $\nu \in F_k$.

Consider now the following regular-singular Sturm-Liouville operator and its self-adjoint extension with $\alpha \in \{\pm\alpha_k\}$ and $\nu \in F_k$

$$l_\nu := -\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\nu^2 - \frac{1}{4}\right) : C^\infty_0(0,1) \to C^\infty_0(0,1),$$

$$\mathcal{D}(L_\nu(\alpha)) := \{f \in \mathcal{D}(l_{\nu,\max}) | f'(1) + (\alpha - 1/2)f(1) = 0\}.$$

Note we have $\alpha^2 < \nu^2$ by construction and in particular $\alpha \neq -\nu$. Thus we find by Corollary 4.11 or also by Proposition 5.8 that $\ker L_\nu(\alpha) = \{0\}$ and

$$\det\nolimits_\zeta(L_\nu(\alpha)) = \sqrt{2\pi}\frac{\alpha+\nu}{2^\nu\Gamma(\nu+1)}. \tag{5.23}$$

Denote by $\phi(x,z), \psi(x,z)$ the solutions of $(l_\nu + z^2)f = 0$, normalized in the sense of [L, (1.38a), (1.38b)] at $x=0$ and $x=1$, respectively. The general solution to $(l_\nu + z^2)f = 0$ is of the following form

$$f(x) = c_1\sqrt{x}I_\nu(zx) + c_2\sqrt{x}K_\nu(zx).$$

Applying the normalizing conditions of [L, (1.38a), (1.38b)] we obtain straightforwardly

$$\begin{aligned}
&\psi(1,z) = 1, \quad \psi'(1,z) = 1/2 - \alpha,\\
&\phi(1,z) = 2^\nu\Gamma(\nu+1)z^{-\nu}I_\nu(z) \text{ with } \phi(1,0) = 1,\\
&\phi'(1,z) = 2^\nu\Gamma(\nu+1)z^{-\nu}(I_\nu(z)\cdot 1/2 + zI_\nu'(z)) \text{ with } \phi'(1,0) = \nu + 1/2.
\end{aligned}$$

Finally by [L, Proposition 4.6] we obtain with $\{\lambda_n\}_{n\in\mathbb{N}}$ being a counting of the eigenvalues of $L_\nu(\alpha)$:

$$\det\nolimits_\zeta(L_\nu(\alpha) + z^2) = \det\nolimits_\zeta(L_\nu(\alpha)) \cdot \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{\lambda_n}\right). \tag{5.24}$$

Since $\ker L_\nu(\alpha) = \{0\}$, for all $n \in \mathbb{N}$ we have $\lambda_n \neq 0$. Denote the positive zeros of $\widetilde{J}_\nu^N(z) := \alpha J_\nu(z) + zJ_\nu'(z)$ by $(\widetilde{j}_{\nu,i})_{i\in\mathbb{N}}$. Note in the notation of Proposition 5.12 that for $\alpha = \alpha_k$, $\widetilde{J}_\nu^N(z) = \widetilde{J}_\nu^{N_1,k}(z)$ and for $\alpha = -\alpha_k$, $\widetilde{J}_\nu^N(z) = \widetilde{J}_\nu^{N_2,k}(z)$. Observe by Corollary 5.9:

$$\mathrm{Spec}(L_\nu(\alpha)) = \{\widetilde{j}_{\nu,i}^2 | i \in \mathbb{N}\}.$$

Using the product formula (5.24) and [L, Theorem 1.2] applied to $L_\nu(\alpha) + z^2$, we compute in view of (5.23)

$$\begin{aligned}
\prod_{i=1}^{\infty}\left(1 + \frac{z^2}{\widetilde{j}_{\nu,i}^2}\right) &= \frac{W(\phi(\cdot,z);\psi(\cdot,z))}{\alpha+\nu} = \frac{2^\nu\Gamma(\nu)}{z^\nu(1+\alpha/\nu)}(\alpha I_\nu(z) + zI_\nu'(z))\\
\Rightarrow \quad \widetilde{I}_\nu^N(z) &\equiv \alpha I_\nu(z) + zI_\nu'(z) = \frac{z^\nu}{2^\nu\Gamma(\nu)}\left(1 + \frac{\alpha}{\nu}\right)\prod_{i=1}^{\infty}\left(1 + \frac{z^2}{\widetilde{j}_{\nu,i}^2}\right).
\end{aligned} \tag{5.25}$$

The original computations of [L, (4.22)] provide an analogous result for $I_\nu(z)$

$$I_\nu(z)=\frac{z^\nu}{2^\nu\Gamma(\nu+1)}\prod_{i=1}^{\infty}\left(1+\frac{z^2}{j_{\nu,i}^2}\right),$$

where $j_{\nu,i}$ are the positive zeros of $J_\nu(z)$. Finally in view of the series representations for $t_\nu^{D,k}(\lambda)$ and $t_\nu^{N_l,k}(\lambda), l=1,2$ derived in Propositions 5.11 and 5.12 we obtain with $z=\sqrt{-\lambda}$

$$t_\nu^{D,k}(\lambda)=-\log I_\nu(\nu z)+\log\left(\frac{(\nu z)^\nu}{2^\nu\Gamma(\nu+1)}\right), \quad (5.26)$$

$$t_\nu^{N_l,k}(\lambda)=-\log(\alpha_l I_\nu(\nu z)+\nu z I'_\nu(\nu z))+\log\left(\frac{(\nu z)^\nu}{2^\nu\Gamma(\nu)}\left(1+\frac{\alpha_l}{\nu}\right)\right), \quad (5.27)$$

where $\alpha_l=\alpha_k$ if $l=1$ and $\alpha_l=-\alpha_k$ if $l=2$. Putting together these two results we obtain with Definition 5.5 the statement of the corollary. □

Now we turn to the discussion of $T^k(s,\lambda)$. For this we introduce the following zeta-function for $Re(s)$ large:

$$\zeta_{k,N}(s):=\sum_{\nu\in F_k}\nu^{-s}=\sum_{\nu\in F_k}(\nu^2)^{-s/2},$$

where $\nu\in F_k$ are counted with their multiplicities and the second equality is clear, since $\nu\in F_k$ are positive. Recall that $\nu\in F_k$ solves

$$\nu^2=\eta+(k+1/2-n/2)^2,\ \eta\in\mathrm{Spec}\triangle_{k,ccl,N}\backslash\{0\}$$

and hence $\zeta_{k,N}(2s)$ is simply the zeta-function of $\triangle_{k,ccl,N}+(k+1/2-n/2)^2$. By standard theory $\zeta(2s)$ extends (note that $\zeta(2s)$ can be presented by an alternating sum of zeta functions of $\triangle_{j,N}+(k+1/2-n/2)^2, j=0,..,k$) to a meromorphic function with possible simple poles at the usual locations $\{(n-p)/2|p\in\mathbb{N}\}$ and $s=0$ being a regular point. Thus the $1/\nu^r$ dependence in $t_\nu^k(\lambda)$ causes a non-analytic behaviour of $T^k(s,\lambda)$ at $s=0$ for $r=1,..,n$, since

$$\sum_{\nu\in F_k}\nu^{-2s}\frac{1}{\nu^r}=\zeta_{k,N}(2s+r)$$

possesses possibly a pole at $s=0$. Therefore the first $n=\dim N$ leading terms in the asymptotic expansion of $t_\nu^k(\lambda)$ for large orders ν are to be removed. We put

$$t_\nu^k(\lambda)=:p_\nu^k(\lambda)+\sum_{r=1}^{n}\frac{1}{\nu^r}f_r^k(\lambda), \quad P^k(s,\lambda):=\sum_{\nu>1}p_\nu^k(\lambda)\,\nu^{-2s}. \quad (5.28)$$

In order to get explicit expressions for $f_r^k(\lambda)$ we need following expansions of Bessel-functions for large order ν, see [O, Section 9]:

$$I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \left[1 + \sum_{r=1}^{\infty} \frac{u_r(t)}{\nu^r}\right],$$

$$I'_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{z(1+z^2)^{-1/4}} \left[1 + \sum_{r=1}^{\infty} \frac{v_r(t)}{\nu^r}\right],$$

where we put $z := \sqrt{-\lambda}, t := (1+z^2)^{-1/2}$ and $\eta := 1/t + \log(z/(1+1/t))$. Recall that $\lambda \in \Lambda_c$, defined in (5.13). The induced $z = \sqrt{-\lambda}$ is contained in $\{z \in \mathbb{C} | |\arg(z)| < \pi/2\} \cup \{ix | x \in (-1,1)\}$. This is precisely the region of validity for these asymptotic expansions, determined in [O, (7.18)].

Same expansions are quoted in [BKD, Section 3]. In particular we have as in [BKD, (3.15)] the following expansion in terms of orders

$$\log\left[1 + \sum_{r=1}^{\infty} \frac{u_r(t)}{\nu^r}\right] \sim \sum_{r=1}^{\infty} \frac{D_r(t)}{\nu^r}, \tag{5.29}$$

$$\log\left[\left(1 + \sum_{k=1}^{\infty} \frac{v_r(t)}{\nu^r}\right) \pm \frac{\alpha_k}{\nu} t \left(1 + \sum_{r=1}^{\infty} \frac{u_r(t)}{\nu^r}\right)\right] \sim \sum_{r=1}^{\infty} \frac{M_r(t, \pm\alpha_k)}{\nu^r}, \tag{5.30}$$

where $D_r(t)$ and $M_r(t, \pm\alpha_k)$ are polynomial in t. Using these series representations we prove the following result.

Lemma 5.14. *For* $\dim M$ *being odd we have with* $z := \sqrt{-\lambda}$, $t := (1+z^2)^{-1/2} = 1/\sqrt{1-\lambda}$ *and* $\alpha_k = n/2 - 1/2 - k$

$$f_r^k(\lambda) = M_r(t, -\alpha_k) - M_r(t, +\alpha_k) + (-1)^{r+1} \frac{\alpha_k^r - (-\alpha_k)^r}{r}.$$

For $\dim M$ *being even we have in the same notation*

$$f_r^k(\lambda) = -M_r(t, -\alpha_k) - M_r(t, +\alpha_k) + 2D_r(t) + (-1)^{r+1} \frac{\alpha_k^r + (-\alpha_k)^r}{r}.$$

Proof. We get by the series representation (5.29) and (5.30) the following expansions for large orders ν:

$$\log(\pm\alpha_k I_\nu(\nu z) + \nu z I'_\nu(\nu z)) \sim \log\left(\frac{\nu}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{z(1+z^2)^{-1/4}}\right) + \sum_{r=1}^{\infty} \frac{M_r(t, \pm\alpha_k)}{\nu^r},$$

$$\log(I_\nu(\nu z)) \sim \log\left(\frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}}\right) + \sum_{r=1}^{\infty} \frac{D_r(t)}{\nu^r}.$$

Furthermore, with $\nu > |\alpha_k|$ for $\nu \in F_k$ we obtain

$$\log(1 \pm \frac{\alpha_k}{\nu}) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{(\pm\alpha_k)^r}{r\nu^r}.$$

Hence in total we obtain an expansion for $t_\nu^k(\lambda)$ in terms of orders ν:

$$t_\nu^k(\lambda) \sim \sum_{r=1}^{\infty} \frac{1}{\nu^r} \left(M_r(t, -\alpha_k) - M_r(t, +\alpha_k) + (-1)^{r+1} \frac{\alpha_k^r - (-\alpha_k)^r}{r} \right), \quad \text{for } \dim M \text{ odd},$$

$$t_\nu^k(\lambda) \sim \sum_{r=1}^{\infty} \frac{1}{\nu^r} \Big(2D_r(t) - M_r(t, -\alpha_k) - M_r(t, +\alpha_k) + (-1)^{r+1} \frac{\alpha_k^r + (-\alpha_k)^r}{r} \Big) + \log\left(\frac{\lambda}{\lambda - 1}\right), \quad \text{for } \dim M \text{ even}.$$

From here the explicit result for $f_r^k(\lambda)$ follows by its definition. □

From the integral representation (5.22) we find that the singular behaviour enters the zeta-function in form of

$$\sum_{r=1}^{n} \frac{s^2}{\Gamma(s+1)} \zeta_{k,N}(2s + r) \int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\wedge_c} \frac{e^{-\lambda t}}{-\lambda} f_r^k(\lambda) d\lambda dt.$$

We compute explicitly this contribution coming from $f_r^k(\lambda)$ in terms of the polynomial structure of M_r and D_r. It can be derived from (5.29) and (5.30), see also [BKD, (3.7), (3.16)], that the polynomial structure of M_r and D_r is given by

$$D_r(t) = \sum_{b=0}^{r} x_{r,b} t^{r+2b}, \quad M_r(t, \pm\alpha_k) = \sum_{b=0}^{r} z_{r,b}(\pm\alpha_k) t^{r+2b}.$$

Lemma 5.15. *For* $\dim M$ *odd we obtain*

$$\int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\wedge_c} \frac{e^{-\lambda t}}{-\lambda} f_r^k(\lambda) d\lambda dt =$$
$$= \sum_{b=0}^{r} (z_{r,b}(-\alpha_k) - z_{r,b}(\alpha_k)) \frac{\Gamma(s + b + r/2)}{s\Gamma(b + r/2)}.$$

For $\dim M$ *even we obtain*

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\wedge_c}\frac{e^{-\lambda t}}{-\lambda}f_r^k(\lambda)d\lambda dt =$$

$$= \sum_{b=0}^{r}(2x_{r,b} - z_{r,b}(-\alpha_k) - z_{r,b}(\alpha_k))\frac{\Gamma(s+b+r/2)}{s\Gamma(b+r/2)}.$$

Proof. Observe from [GRA, 8.353.3] by substituting the new variable $x = \lambda - 1$, with $a > 0$:

$$\frac{1}{2\pi i}\int_{\wedge_c}\frac{e^{-\lambda t}}{-\lambda}\frac{1}{(1-\lambda)^a}d\lambda = \frac{1}{2\pi i}e^{-t}\int_{\wedge_{c-1}} -\frac{e^{-xt}}{x+1}\frac{1}{(-x)^a}dx =$$
$$= \frac{1}{\pi}\sin(\pi a)\Gamma(1-a)\Gamma(a,t).$$

Using now the relation between the incomplete Gamma function and the probability integral

$$\int_0^\infty t^{s-1}\Gamma(a,t)dt = \frac{\Gamma(s+a)}{s}$$

we obtain

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\wedge_c}\frac{e^{-\lambda t}}{-\lambda}\frac{1}{(1-\lambda)^a}d\lambda dt =$$
$$\frac{1}{\pi}\sin(\pi a)\,\Gamma(1-a)\frac{\Gamma(s+a)}{s} = \frac{\Gamma(s+a)}{s\Gamma(a)}.$$

Further note for $t > 0$

$$\frac{1}{2\pi i}\int_{\wedge_c}\frac{e^{-\lambda t}}{-\lambda}d\lambda = 0,$$

since the contour Λ_c does not encircle the pole $\lambda = 0$ of the integrand. Hence the $\lambda-$independent part of $f_r^k(\lambda)$ vanishes after integration. The statement is now a direct consequence of Lemma 5.14. □

Next we derive asymptotics of $p_\nu^k(\lambda) := t_\nu^k(\lambda) - \sum_{r=1}^n \frac{1}{\nu^r} f_r^k(\lambda)$ for large arguments λ and fixed order ν

Proposition 5.16. *For large arguments* λ *and fixed order* ν *we have the following asymptotics*

$$p_\nu^k(\lambda) = a_\nu^k \log(-\lambda) + b_\nu^k + O\left((-\lambda)^{-1/2}\right),$$

where for $\dim M$ *odd*

$$a_\nu^k = 0, \quad b_\nu^k = \left((\log\left(1+\frac{\alpha_k}{\nu}\right) - \log\left(1-\frac{\alpha_k}{\nu}\right) - \sum_{r=1}^{n} (-1)^{r+1} \frac{\alpha_k^r - (-\alpha_k)^r}{r\nu^r} \right),$$

and for $\dim M$ *even*

$$a_\nu^k = -1, \quad b_\nu^k = \left(\log\left(1+\frac{\alpha_k}{\nu}\right) + \log\left(1-\frac{\alpha_k}{\nu}\right) - \sum_{r=1}^{n} (-1)^{r+1} \frac{\alpha_k^r + (-\alpha_k)^r}{r\nu^r} \right).$$

Proof. For large argument λ we obtain

$$t = \frac{1}{\sqrt{1+z^2}} = \frac{1}{\sqrt{1-\lambda}} = O\left((-\lambda)^{-1/2}\right).$$

Therefore the polynomials $M_r(t, \pm\alpha_k)$ and $D_r(t)$, having no constant terms, are of asymptotics $O\left((-\lambda)^{-1/2}\right)$ for large λ. Hence directly from Lemma 5.14 we obtain in odd dimensions for large λ

$$\frac{f_r^k(\lambda)}{\nu^r} \sim (-1)^{r+1} \frac{(\alpha_k)^r - (-\alpha_k)^r}{r\nu^r} + O\left((-\lambda)^{-1/2}\right). \tag{5.31}$$

In even dimensions we get

$$\frac{f_r^k(\lambda)}{\nu^r} \sim (-1)^{r+1} \frac{(\alpha_k)^r + (-\alpha_k)^r}{r\nu^r} + O\left((-\lambda)^{-1/2}\right). \tag{5.32}$$

It remains to identify explicitly the asymptotics of $t_\nu^k(\lambda)$. Note by [AS, p. 377] the following expansions for large arguments and fixed order:

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left(1 + O\left(\frac{1}{z}\right)\right), \quad I'_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left(1 + O\left(\frac{1}{z}\right)\right).$$

These expansions hold for $|\arg(z)| < \pi/2$ and in particular for $z = \sqrt{-\lambda}$ with $\lambda \in \Lambda_c$ large, where Λ_c is defined in (5.13). Further observe for such $z = \sqrt{-\lambda}, \lambda \in \Lambda_c$ large:

$$\log\left(1 + O\left(\frac{1}{z}\right)\right) = O\left((-\lambda)^{-1/2}\right),$$

$$\Rightarrow \log(\pm\alpha_k + \nu z) = \log z + \log\nu + \log\left(1 \pm \frac{\alpha_k}{\nu z}\right) =$$

$$= \log z + \log\nu + O\left((-\lambda)^{-1/2}\right).$$

Together with the expansions of the Bessel-functions we obtain for $t_\nu^k(\lambda)$ defined in Corollary 5.13

$$t_\nu^k(\lambda) = \log\left(1+\frac{\alpha_k}{\nu}\right) - \log\left(1-\frac{\alpha_k}{\nu}\right) + O\left((-\lambda)^{-1/2}\right), \quad \text{for } \dim M \text{ odd},$$

$$t_\nu^k(\lambda) = -\log(-\lambda) + \log\left(1+\frac{\alpha_k}{\nu}\right) + \log\left(1-\frac{\alpha_k}{\nu}\right) + O\left((-\lambda)^{-1/2}\right), \quad \text{for } \dim M \text{ even}.$$

Recall the definition of $p_\nu^k(\lambda)$ in (5.28). Combining this with (5.31) and (5.32) we obtain the desired result. □

Definition 5.17. *With the coefficients a_ν^k and b_ν^k defined in Proposition 5.16, we set for $Re(s) \gg 0$*

$$A^k(s) := \sum_{\nu \in F_k} a_\nu^k \, \nu^{-2s}, \quad B^k(s) := \sum_{\nu \in F_k} b_\nu^k \, \nu^{-2s}.$$

Now the last step towards the evaluation of the zeta-function of Corollary 5.13 is the discussion of

$$P^k(s,\lambda) := \sum_{\nu \in F_k} p_\nu^k(\lambda) \, \nu^{-2s}, \; Re(s) \gg 0.$$

At this point the advantage of taking in account the symmetry of the de Rham complex is particularly visible:

Proposition 5.18.

$$P^k(s,0) = 0.$$

Proof. As $\lambda \to 0$ we find that $t = (1-\lambda)^{-1/2}$ tends to 1. Since as in [BGKE, (4.24)]

$$M_r(1, \pm\alpha_k) = D_r(1) + (-1)^{r+1}\frac{(\pm\alpha_k)^r}{r} \tag{5.33}$$

we find with Lemma 5.14 that in both the even- and odd-dimensional case $f_r^k(\lambda) \to 0$ as $\lambda \to 0$. Thus we simply need to study the behaviour of $t_\nu^k(\lambda)$ defined in Corollary 5.13 for small arguments. The results follow from the asymptotic behaviour of Bessel functions of second order for small arguments which holds without further restrictions on z

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu, \; |z| \to 0.$$

Using the relation $I'_\nu(z) = \frac{1}{2}(I_{\nu+1}(z) + I_{\nu-1}(z)$ we compute as $|z| \to 0$

$$\pm\alpha_k I_\nu(\nu z) + \nu z I'_\nu(\nu z) \sim \frac{\nu}{\Gamma(\nu+1)}\left(\frac{\nu z}{2}\right)^\nu \left[1 \pm \frac{\alpha_k}{\nu} + \frac{\nu z^2}{4(\nu+1)}\right],$$
$$\nu I_\nu(\nu z) \sim \frac{\nu}{\Gamma(\nu+1)}\left(\frac{\nu z}{2}\right)^\nu.$$

The result now follows from the explicit form of $t^k_\nu(\lambda)$. □

Remark 5.19. *The statement of Proposition 5.18 shows an obvious advantage of taking in account the symmetry of the de Rham complex.*

Now we have all the ingredients together, since by analogous arguments as in [S, Section 4.1] the total zeta-function of Corollary 5.13 is given as follows:

$$\zeta_k(s) = \frac{s}{\Gamma(s+1)}[\gamma A^k(s) - B^k(s) - \frac{1}{s}A^k(s) + P^k(s,0)] +$$
$$+ \sum_{r=1}^n \frac{s^2}{\Gamma(s+1)}\zeta_{k,N}(2s+r)\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\wedge_c}\frac{e^{-\lambda t}}{-\lambda}f^k_r(\lambda)d\lambda dt + \frac{s^2}{\Gamma(s+1)}h(s),$$

where the last term vanishes with its derivative at $s = 0$. Simply by inserting the results of Lemma 5.15, Proposition 5.16, Proposition 5.18 together with Definition 5.17 into the above expression we obtain the following proposition:

Proposition 5.20. *Continue in the setting of Corollary 5.13. Up to a term of the form $s^2h(s)/\Gamma(s+1)$, which vanishes with its derivative at $s = 0$, the zeta-function $\zeta_k(s)$ from Definition 5.5 is given in odd dimensions by*

$$\frac{s}{\Gamma(s+1)}\left[\sum_{\nu\in F_k}\nu^{-2s}\log\left(1 - \frac{\alpha_k}{\nu}\right) - \sum_{\nu\in F_k}\nu^{-2s}\log\left(1 + \frac{\alpha_k}{\nu}\right) + \right.$$
$$\left. + \sum_{r=1}^n \zeta_{k,N}(2s+r)\,(-1)^{r+1}\frac{\alpha_k^r - (-\alpha_k)^r}{r}\right] +$$
$$+ \sum_{r=1}^n \zeta_{k,N}(2s+r)\frac{s}{\Gamma(s+1)}\left[\sum_{b=0}^r (z_{r,b}(-\alpha_k) - z_{r,b}(\alpha_k))\frac{\Gamma(s+b+r/2)}{\Gamma(b+r/2)}\right].$$

In even dimensions we obtain

$$\frac{s}{\Gamma(s+1)}\left[-\sum_{\nu\in F_k}\nu^{-2s}\log\left(1-\frac{\alpha_k}{\nu}\right)-\sum_{\nu\in F_k}\nu^{-2s}\log\left(1+\frac{\alpha_k}{\nu}\right)+\right.$$
$$\left.+\sum_{\nu\in F_k}\nu^{-2s}\left(\frac{1}{s}-\gamma\right)+\sum_{r=1}^{n}\zeta_{k,N}(2s+r)(-1)^{r+1}\frac{\alpha_k^r+(-\alpha_k)^r}{r}\right]+$$
$$+\sum_{r=1}^{n}\zeta_{k,N}(2s+r)\frac{s}{\Gamma(s+1)}\left[\sum_{b=0}^{r}(2x_{r,b}-z_{r,b}(-\alpha_k)-\right.$$
$$\left.-z_{r,b}(\alpha_k))\frac{\Gamma(s+b+r/2)}{\Gamma(b+r/2)}\right].$$

Corollary 5.21. *With $\zeta_{k,N}(s,a):=\sum_{\nu\in F_k}(\nu+a)^{-s}$ we deduce for odd dimensions*

$$\zeta'_k(0)=\zeta'_{k,N}(0,\alpha_k)-\zeta'_{k,N}(0,-\alpha_k)+$$
$$+\sum_{i=1}^{n}(-1)^{i+1}\frac{\alpha_k^i-(-\alpha_k)^i}{i}\mathrm{Res}\zeta_{k,N}(i)\left\{\frac{\gamma}{2}+\frac{\Gamma'(i)}{\Gamma(i)}\right\}+$$
$$+\sum_{i=1}^{n}\frac{1}{2}\mathrm{Res}\zeta_{k,N}(i)\sum_{b=0}^{i}\left(z_{i,b}(-\alpha_k)-z_{i,b}(\alpha_k)\right)\frac{\Gamma'(b+i/2)}{\Gamma(b+i/2)}.$$

and for even dimensions

$$\zeta'_k(0)=\zeta'_{k,N}(0,\alpha_k)+\zeta'_{k,N}(0,-\alpha_k)+$$
$$+\sum_{i=1}^{n}(-1)^{i+1}\frac{\alpha_k^i+(-\alpha_k)^i}{i}\mathrm{Res}\zeta_{k,N}(i)\left\{\frac{\gamma}{2}+\frac{\Gamma'(i)}{\Gamma(i)}\right\}+$$
$$+\sum_{i=1}^{n}\frac{1}{2}\mathrm{Res}\zeta_{k,N}(i)\sum_{b=0}^{i}\left(2x_{i,b}-z_{i,b}(-\alpha_k)-z_{i,b}(\alpha_k)\right)\frac{\Gamma'(b+i/2)}{\Gamma(b+i/2)}.$$

Proof. First we consider a major building brick of the expressions in Proposition 5.20. Here we follow the approach of [BKD, Section 11]. Put for $\alpha\in\{\pm\alpha_k\}$

$$K(s):=\sum_{\nu\in F_k}\nu^{-2s}\left[-\log\left(1+\frac{\alpha}{\nu}\right)+\sum_{r=1}^{n}(-1)^{r+1}\frac{1}{r}\left(\frac{\alpha}{\nu}\right)^r\right].$$

Since the zeta-function $\zeta_{k,N}(s)=\sum_{\nu\in F_k}\nu^{-s}$ converges absolutely for $Re(s)\geq n+1, n=\dim N$, the sum above converges for $s=0$. In order to evaluate

$K(0)$, introduce a regularization parameter z as follows:

$$\begin{aligned} K_0(z) &:= \sum_{\nu \in F_k} \int_0^\infty t^{z-1} e^{-\nu t} \left(e^{-\alpha t} + \sum_{i=0}^{n} (-1)^{i+1} \frac{\alpha^i t^i}{i!} \right) dt \\ &= \Gamma(z) \cdot \zeta_{k,N}(z,\alpha) + \sum_{i=0}^{n} (-1)^{i+1} \frac{\alpha^i}{i!} \Gamma(z+i) \zeta_{k,N}(z+i), \end{aligned}$$

where we have introduced

$$\zeta_{k,N}(z,\alpha) := \frac{1}{\Gamma(z)} \sum_{\nu \in F_k} \int_0^\infty t^{z-1} e^{-(\nu+\alpha)t} \, dt.$$

For $Re(s)$ large enough $\zeta_{k,N}(z,\alpha) = \sum_{\nu \in F_k} (\nu + \alpha)^{-z}$, is holomorphic and extends meromorphically to $\mathbb{C}$, since it is the zeta-function of $\triangle_{k,ccl,N} + \alpha$. Note that for $\alpha \in \{\pm \alpha_k\}$ and $\nu \in F_k$ we have $\alpha \neq -\nu$, so no zero mode appears in the zeta function $\zeta_{k,N}(z,\alpha)$. In particular $K_0(z)$ is meromorphic in $z \in \mathbb{C}$ and by construction

$$K_0(0) = K(0).$$

With the same arguments as in [BKD, Section 11] we arrive at

$$\begin{aligned} K(0) =& \zeta'_{k,N}(0,\alpha) - \zeta'_{k,N}(0) + \\ &+ \sum_{i=1}^{n} (-1)^{i+1} \frac{\alpha^i}{i} \left[\text{Res} \zeta_{k,N}(i) \left\{ \gamma + \frac{\Gamma'(i)}{\Gamma(i)} \right\} + \text{PP} \zeta_{k,N}(i) \right], \end{aligned}$$

where $\text{PP}\zeta_{k,N}(r)$ denotes the constant term in the asymptotics of $\zeta_{k,N}(s)$ near the pole singularity $s = r$. This result corresponds to the result obtained in [BKD, p.388], where the factors $1/2$ in front of $\zeta'_{k,N}(0)$ and 2 in front of $\text{Res}\, \zeta_{k,N}(i)$, as present in [BKD], do not appear here because of a different notation: here we have set $\zeta_{k,N}(s) = \sum \nu^{-s}$ instead of $\sum \nu^{-2s}$.

In fact $K(0)$ enters the calculations twice: with $\alpha = \alpha_k$ and $\alpha = -\alpha_k$. In the odd-dimensional case both expressions are subtracted from each other, in the even-dimensional case they are added up. Furthermore we compute straightforwardly

$$\begin{aligned} &\left. \frac{d}{ds} \right|_{s=0} \zeta_{k,N}(2s+r) \frac{s}{\Gamma(s+1)} \frac{\Gamma(s+b+r/2)}{\Gamma(b+r/2)} = \\ &= \frac{1}{2} \text{Res} \zeta_{k,N}(r) \left[\frac{\Gamma'(b+r/2)}{\Gamma(b+r/2)} + \gamma \right] + \text{PP} \zeta_{k,N}(r). \end{aligned}$$

We infer from (5.33)

$$\sum_{b=0}^{r}(z_{r,b}(-\alpha_k) - z_{r,b}(\alpha_k)) = (-1)^r \frac{\alpha_k^r - (-\alpha_k)^r}{r},$$

$$\sum_{b=0}^{i}(2x_{i,b} - z_{i,b}(-\alpha_k) - z_{i,b}(\alpha_k)) = (-1)^r \frac{\alpha_k^r + (-\alpha_k)^r}{r}.$$

This leads after several cancellations to the desired result in odd dimensions. In even dimensions the result follows by a straightforward evaluation of the derivative at zero for the remaining component:

$$\left.\frac{d}{ds}\right|_{s=0} \frac{s}{\Gamma(s+1)}\zeta_{k,N}(2s)\left(\frac{1}{s} - \gamma\right) = 2\zeta'_{k,N}(0).$$

□

5.5 Contribution from the Subcomplexes II

It remains to identify the contribution to the analytic torsion coming from the subcomplexes (5.4) of second type, induced by the harmonics on the base manifold N. The necessary calculations are provided in [L3] and are repeated here for completeness. Recall the explicit form of these subcomplexes

$$0 \to C_0^\infty((0,1), \langle 0 \oplus u_i\rangle) \xrightarrow{d} C_0^\infty((0,1), \langle u_i \oplus 0\rangle) \to 0, \tag{5.34}$$

where $\{u_i\}$ is an orthonormal basis of $\dim \mathcal{H}^k(N)$. With respect to the generators $0 \oplus u_i$ and $u_i \oplus 0$ we obtain for the action of the exterior derivative

$$d = (-1)^k \partial_x + \frac{c_k}{x}, \quad c_k = (-1)^k(k - n/2).$$

By compatibility of the induced decomposition we have (cf. (3.20))

$$\mathcal{D}(\triangle_k^{rel}) \cap L^2((0,R), \langle 0 \oplus u_i\rangle) = \mathcal{D}(d^t_{\max} d_{\min}) = \\ = \mathcal{D}\left((-1)^{k+1}\partial_x + \frac{c_k}{x}\right)_{\max}\left((-1)^k\partial_x + \frac{c_k}{x}\right)_{\min}.$$

Consider, in the notation of Subsection 5.3, for any $\nu \in \mathbb{R}$ and $\alpha \in \mathbb{R} \cup \{\infty\}$ the operator $l_\nu = -\partial_x^2 + x^{-2}(\nu^2 - 1/4)$ with the following self-adjoint extension:

$$\mathcal{D}(L_\nu(\alpha)) = \{f \in \mathcal{D}(l_{\nu,\max}) | (\alpha - 1/2)^{-1} f'(1) + f(1) = 0, \\ f(x) = O(\sqrt{x}), x \to 0\}.$$

Here $L_\nu(\alpha = 1/2)$ denotes the self-adjoint extension of l_ν with pure Neumann boundary conditions at $x = 1$. Furthermore $L_\nu(\infty)$ is the extension with Dirichlet boundary conditions at $x = 1$. As a consequence of Proposition 2.8 we have

$$\Big((-1)^{k+1}\partial_x + \frac{c_k}{x}\Big)_{\max}\Big((-1)^k\partial_x + \frac{c_k}{x}\Big)_{\min} = L_{|k-(n-1)/2|}(\infty).$$

It is well-known, see also [L, Theorem 1.1] and [L, (1.37)], that the zeta-function of $L_\nu(\alpha)$ extends meromorphically to $\mathbb{C}$ and is regular at the origin. We abbreviate

$$T(L_\nu(\alpha)) := \log\det L_\nu(\alpha) = -\zeta'(s=0, L_\nu(\alpha)).$$

Put $b_k := \dim\mathcal{H}^k(N)$. Then the contribution to the analytic torsion coming from harmonics on the base manifold is given due to the formula (5.7) as follows:

$$\frac{1}{2}\sum_{k=0}^{\dim M}(-1)^k b_k\, T(L_{|k-(n-1)/2|}(\infty)) \tag{5.35}$$

Proposition 5.22.
For $\nu \geq 0$ we have $\mathrm{Spec}(L_\nu(\infty)) \cup \{0\} = \mathrm{Spec}(L_{\nu+1}(\nu+1)) \cup \{0\}$.

Proof. Put $d_p := \partial_x + x^{-1}p$. We get

$$l_{p+1/2} = d_p^t d_p, \quad l_{p-1/2} = d_p d_p^t.$$

By a combination of Propositions 2.6, 2.8, which determine the maximal and the minimal domains of d_p, we obtain for $\nu \geq 0$

$$\begin{aligned}\mathcal{D}(d_{\nu+1/2,\max}d^t_{\nu+1/2,\min}) &= \{f \in \mathcal{D}(l_{\nu,\max}) | f(x) = O(\sqrt{x}), x \to 0, f(1) = 0\},\\ \mathcal{D}(d^t_{\nu+1/2,\min}d_{\nu+1/2,\max}) &= \{f \in \mathcal{D}(l_{\nu+1,\max}) | f(x) = O(\sqrt{x}), x \to 0,\\ &\qquad f'(1) + (\nu+1/2)f(1) = 0\}.\end{aligned}$$

Hence we find

$$\begin{aligned}L_\nu(\infty) &= d_{\nu+1/2,\max}d^t_{\nu+1/2,\min} = d_{\nu+1/2,\max}(d_{\nu+1/2,\max})^*,\\ L_{\nu+1}(\nu+1) &= d^t_{\nu+1/2,\min}d_{\nu+1/2,\max} = (d_{\nu+1/2,\max})^* d_{\nu+1/2,\max}.\end{aligned}$$

Comparing both operators we deduce the statement on the spectrum, since all non-zero eigenvalues of the operators are simple by similar arguments as in Corollary 5.9. □

Proposition 5.23. *Let $\alpha+\nu\neq 0$. Then*

$$T(L_\nu(\infty)) = T(L_\nu(\alpha)) - \log(\alpha+\nu).$$

Proof. The assumption $\alpha+\nu\neq 0$ implies with Corollary 4.11

$$\det{}_\zeta(L_\nu(\alpha)) = \sqrt{2\pi}\frac{\alpha+\nu}{2^\nu\Gamma(1+\nu)}.$$

Moreover we have by Corollary 4.12

$$\det{}_\zeta(L_\nu(\infty)) = \frac{\sqrt{2\pi}}{\Gamma(1+\nu)2^\nu}.$$

Consequently we obtain for $\alpha+\nu\neq 0$

$$\frac{\det_\zeta(L_\nu(\infty))}{\det_\zeta(L_\nu(\alpha))} = \frac{1}{\alpha+\nu}.$$

Taking logarithms we get the result. □

Proposition 5.24.

$$T(L_{k+1/2}(\infty)) = \log 2 - \sum_{l=0}^{k}\log(2l+1).$$

Proof. Apply Proposition 5.23 to $L_{\nu+1}(\nu+1), \nu\geq 0$. We obtain

$$T(L_{\nu+1}(\infty)) = T(L_{\nu+1}(\nu+1)) - \log(2\nu+2) = T(L_\nu(\infty)) - \log(2\nu+2),$$

where for the second equality we used Proposition 5.22. We iterate the equality with $\nu = k-1/2$ and obtain

$$T(L_{k+1/2}(\infty)) = T(L_{1/2}(\infty)) - \sum_{l=0}^{k}\log(2l+1).$$

The operator $L_{1/2}(\infty)$ is simply $-\partial_x^2$ on [0,1] with Dirichlet boundary conditions. Its spectrum is given by $(n^2\pi^2)_{n\in\mathbb{N}}$. Thus we obtain with $\zeta_R(0) = -1/2$ and $\zeta_R'(0) = -1/2\log 2\pi$

$$\zeta_{L_{1/2}(\infty)}(s) = \sum_{n=1}^{\infty}\pi^{-2s}n^{-2s}$$
$$\Rightarrow\ \zeta'_{L_{1/2}(\infty)}(0) = -2(\log\pi)\zeta_R(0) + 2\zeta_R'(0) = -\log 2.$$

□

Now we finally compute the contribution from harmonics on the base:

Theorem 5.25. *Let M be a bounded generalized cone of length one over a closed oriented Riemannian manifold N of dimension n. Let $\chi(N)$ denote the Euler characteristic of N and $b_k := \dim \mathcal{H}^k(N)$ be the Betti numbers.*

Then the contribution to the analytic torsion coming from harmonics on the base manifold is given as follows. For $\dim M$ odd the contribution amounts to

$$\frac{\log 2}{2}\chi(N) - \sum_{k=0}^{n/2-1}(-1)^k b_k \sum_{l=0}^{n/2-k-1}\log(2l+1) - \\ -\frac{1}{2}\sum_{k=0}^{n/2-1}(-1)^k b_k \log(n-2k+1).$$

For $\dim M$ even the contribution amounts to

$$\frac{1}{2}\sum_{k=0}^{(n-1)/2}(-1)^k b_k \log(n-2k+1).$$

Proof. We infer from (5.35) for the contribution of the harmonics on the base manifold

$$\frac{1}{2}\sum_{k=0}^{\dim M}(-1)^k b_k\, T(L_{|k-(n-1)/2|}(\infty)).$$

We obtain by Poincare duality on the base manifold N

$$\text{For } \dim M = n+1 \text{ odd: } \frac{1}{2}\sum_{k=0}^{\dim M}(-1)^k b_k\, T(L_{|k-(n-1)/2|}(\infty)) = \\ = \frac{1}{2}\sum_{k=0}^{n/2-1}(-1)^k b_k\, (T(L_{n/2-k-1/2}(\infty)) + T(L_{n/2-k+1/2}(\infty))),$$

$$\text{For } \dim M = n+1 \text{ even: } \frac{1}{2}\sum_{k=0}^{\dim M}(-1)^k b_k\, T(L_{|k-(n-1)/2|}(\infty)) = \\ = \frac{1}{2}\sum_{k=0}^{(n-1)/2}(-1)^k b_k\, (T(L_{n/2-k-1/2}(\infty)) - T(L_{n/2-k+1/2}(\infty))).$$

Inserting the result of Proposition 5.24 into the expressions above, we obtain the statement. □

5.6 Total Result and Formulas in lower Dimensions

Patching together the results of the both preceeding sections we can now provide a complete formula for the analytic torsion of a bounded generalized cone. In fact we simply have to add up the results of Theorem 5.25 and Corollary 5.21. In even dimensions one has to be careful in the middle degree, as explained in Remark 5.7.

Theorem 5.26. *Let $M = (0,1] \times N, g^M = dx^2 \oplus x^2 g^N$ be an odd-dimensional bounded generalized cone over a closed oriented Riemannian manifold (N, g^N). Let the metric on the base manifold N be scaled such that the non-zero eigenvalues of the form-valued Laplacians on N are bigger than one. Introduce the notation $n = \dim N$, $\alpha_k = (n-1)/2 - k$ and $b_k = \dim \mathcal{H}^k(N)$. Put*

$$F_k := \{\xi \in \mathbb{R}^+ \mid \xi^2 = \eta + (k + 1/2 - n/2)^2, \eta \in \mathrm{Spec}\triangle_{k,ccl,N} \backslash \{0\}\},$$
$$\zeta_{k,N}(s) = \sum_{\nu \in F_k} \nu^{-s}, \quad \zeta_{k,N}(s,\alpha) := \sum_{\nu \in F_k} (\nu + \alpha)^{-s}, \quad Re(s) \gg 0.$$

Then the logarithm of the scalar analytic torsion of M is given by

$$\log T(M) = \frac{\log 2}{2}\chi(N) - \sum_{k=0}^{n/2-1} (-1)^k b_k \sum_{l=0}^{n/2-k-1} \log(2l+1) -$$
$$-\frac{1}{2}\sum_{k=0}^{n/2-1}(-1)^k b_k \log(n-2k+1) + \sum_{k=0}^{n/2-1} \frac{(-1)^k}{2}(\zeta'_{k,N}(0,\alpha_k) - \zeta'_{k,N}(0,-\alpha_k)) +$$
$$+\sum_{k=0}^{n/2-1} \frac{(-1)^k}{2} \sum_{i=1}^{n} (-1)^{i+1} \frac{\alpha_k^i - (-\alpha_k)^i}{i} \mathrm{Res}\zeta_{k,N}(i) \left\{\frac{\gamma}{2} + \frac{\Gamma'(i)}{\Gamma(i)}\right\} +$$
$$+\sum_{k=0}^{n/2-1} \frac{(-1)^k}{2} \sum_{i=1}^{n} \frac{1}{2}\mathrm{Res}\zeta_{k,N}(i) \sum_{b=0}^{i} (z_{i,b}(-\alpha_k) - z_{i,b}(\alpha_k)) \frac{\Gamma'(b+i/2)}{\Gamma(b+i/2)}.$$

Theorem 5.27. *Let $M = (0,1] \times N, g^M = dx^2 \oplus x^2 g^N$ be an even-dimensional bounded generalized cone over a closed oriented Riemannian manifold (N, g^N). Let the metric on the base manifold N be scaled such that the non-zero eigenvalues of the form-valued Laplacians on N are bigger than one. Introduce the notation $n = \dim N$, $\alpha_k = (n-1)/2 - k$ and*

$b_k = \dim \mathcal{H}^k(N)$. *Put*

$$F_k := \{\xi \in \mathbb{R}^+ \mid \xi^2 = \eta + (k + 1/2 - n/2)^2, \eta \in \mathrm{Spec}\triangle_{k,ccl,N}\backslash\{0\}\},$$

$$\zeta_{k,N}(s) := \sum_{\nu \in F_k} \nu^{-s}, \quad \zeta_{k,N}(s,\alpha) := \sum_{\nu \in F_k} (\nu + \alpha)^{-s}, \quad Re(s) \gg 0.$$

$$\delta_k := \begin{cases} 1/2 & \text{if } k = (n-1)/2, \\ 1 & \text{otherwise.} \end{cases}$$

Then the logarithm of the scalar analytic torsion of M is given by

$$\begin{aligned}\log T(M) = &\sum_{k=0}^{(n-1)/2} \frac{(-1)^k}{2} \left[b_k \log(n-2k+1) + \delta_k \zeta'_{k,N}(0,\alpha_k) + \delta_k \zeta'_{k,N}(0,-\alpha_k)\right] \\ &+ \sum_{k=0}^{(n-1)/2} \frac{(-1)^k}{2} \delta_k \sum_{i=1}^{n} (-1)^{i+1} \frac{\alpha_k^i + (-\alpha_k)^i}{i} \mathrm{Res}\zeta_{k,N}(i) \left\{\frac{\gamma}{2} + \frac{\Gamma'(i)}{\Gamma(i)}\right\} + \\ &+ \sum_{k=0}^{(n-1)/2} \frac{(-1)^k}{2} \delta_k \sum_{i=1}^{n} \frac{1}{2} \mathrm{Res}\zeta_{k,N}(i) \sum_{b=0}^{i} (2x_{i,b} - z_{i,b}(-\alpha_k) - \\ &- z_{i,b}(\alpha_k)) \frac{\Gamma'(b+i/2)}{\Gamma(b+i/2)}.\end{aligned}$$

The formula could not be made further explicit due to presence of coefficients $x_{r,b}$ and $z_{r,b}(\pm\alpha_k)$, arising from the polynomials

$$D_r(t) = \sum_{b=0}^{r} x_{r,b} t^{r+2b}, \quad M_r(t, \pm\alpha) = \sum_{b=0}^{r} z_{r,b}(\pm\alpha) t^{r+2b},$$

which were introduced in the expansions (5.29) and (5.30). These polynomials can be computed explicitly for any given order $r \in \mathbb{N}$. To point out the applicability of the general results we pursue explicit computations in dimension two and three. We continue in the notation of the theorems above.

Corollary 5.28. *Let M be a two-dimensional bounded generalized cone of length one over a closed oriented manifold N with a metric scaled as in Theorem 5.27. Then the analytic torsion of M is given by*

$$\log T(M) = \frac{1}{2} \dim H^0(N) \log 2 + \frac{1}{2} \zeta'_{0,N}(0) - \frac{1}{4} \mathrm{Res}\zeta_{0,N}(s=1).$$

In the special case of $N = S^1$ we obtain

$$\log T(M) = \frac{1}{2}\left(-\log \pi - 1\right).$$

Proof. In the two-dimensional case the general formula of Theorem 5.27 reduces to the following expression:

$$\log T(M) = \frac{1}{2}\dim H^0(N)\log 2 + \frac{1}{4}\zeta'_{0,N}(0,\alpha_0) + \frac{1}{4}\zeta'_{0,N}(0,-\alpha_0) +$$
$$+\frac{1}{8}\mathrm{Res}\,\zeta_{0,N}(1)\left[\sum_{b=0}^{1}(2x_{1,b} - z_{1,b}(-\alpha_k) - z_{1,b}(\alpha_k))\frac{\Gamma'(b+1/2)}{\Gamma(b+1/2)}\right].$$

Now we evaluate the combinatorial factor of $\mathrm{Res}\,\zeta_{0,N}(1)$ by considering the following formulas, encountered in [BGKE, Section 2-3]

$$D_1(t) = \sum_{b=0}^{1} x_{1,b}t^{1+2b} = \frac{1}{8}t - \frac{5}{24}t^3,$$
$$M_1(t,\alpha) = \sum_{b=0}^{1} z_{1,b}(\pm\alpha)t^{1+2b} = \left(-\frac{3}{8}+\alpha\right)t + \frac{7}{24}t^3. \qquad (5.36)$$

Further one needs the following values (calculated from the known properties of Gamma functions)

$$\frac{\Gamma'(1/2)}{\Gamma(1/2)} = -(\gamma + 2\log 2), \quad \frac{\Gamma'(3/2)}{\Gamma(3/2)} = 2 - (\gamma + 2\log 2).$$

Finally one observes $\alpha_0 = 0$ in this setting. This easily leads to the first formula in the statement of corollary. The second formula follows from the first by

$$\zeta_{0,N}(s) = 2\zeta_R(s),$$

where the factor 2 comes from the fact that the eigenvalues n^2 of the Laplacian $\triangle_{k=0,S^1}$ are of multiplicity two for $n \neq 0$. The Riemann zeta function has the following special values

$$\zeta'_R(0) = -\frac{1}{2}\log 2\pi, \quad \mathrm{Res}\,\zeta_R(1) = 1,$$

which gives the second formula. □

Corollary 5.29. *Let M be a three-dimensional bounded generalized cone of length one over a closed oriented manifold N with a metric scaled as in Theorem 5.26. Then the analytic torsion of M is given by*

$$\log T(M) = \frac{\log 2}{2}\chi(N) - \frac{\log 3}{2}\dim H^0(N) + \frac{1}{2}\zeta'_{0,N}(0,1/2) -$$
$$-\frac{1}{2}\zeta'_{0,N}(0,-1/2) + \frac{\log 2}{2}\mathrm{Res}\,\zeta_{0,N}(1) + \frac{1}{16}\mathrm{Res}\,\zeta_{0,N}(2).$$

Proof. In the three-dimensional case the general formula of Theorem 5.26 reduces to the following expression:

$$\log T(M) = \frac{\log 2}{2}\chi(N) - \frac{\log 3}{2}\dim H^0(N) +$$
$$+\frac{1}{2}\left(\zeta'_{0,N}(0,\alpha_0) - \zeta'_{0,N}(0,-\alpha_0)\right) + \alpha_0 \mathrm{Res}\,\zeta_{0,N}(1)\left[\frac{\gamma}{2} + \frac{\Gamma'(1)}{\Gamma(1)}\right] +$$
$$+\frac{1}{4}\sum_{i=1}^{2}\mathrm{Res}\,\zeta_{0,N}(i)\sum_{b=0}^{i}\left(z_{i,b}(-\alpha_k) - z_{i,b}(\alpha_k)\right)\frac{\Gamma'(b+i/2)}{\Gamma(b+i/2)}.$$

Now we simply evaluate the last combinatorial sum by considering formulas from [BGKE, (3.6), (3.7)]

$$M_1(t,\alpha) = \sum_{b=0}^{1} z_{1,b}(\pm\alpha)t^{1+2b} = \left(-\frac{3}{8} + \alpha\right)t + \frac{7}{24}t^3,$$
$$M_2(t,\alpha) = \sum_{b=0}^{2} z_{2,b}(\pm\alpha)t^{2+2b} = \left(-\frac{3}{16} + \frac{\alpha}{2} - \frac{\alpha^2}{2}\right)t^2 + \left(\frac{5}{8} - \frac{\alpha}{2}\right)t^4 - \frac{7}{16}t^6.$$

We further need the values

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma, \quad \frac{\Gamma'(1/2)}{\Gamma(1/2)} = -(\gamma + 2\log 2),$$
$$\frac{\Gamma'(2)}{\Gamma(2)} = 1-\gamma, \quad \frac{\Gamma'(3/2)}{\Gamma(3/2)} = 2 - (\gamma + 2\log 2).$$

This leads together with $\alpha_0 = 1/2$ in the three-dimensional case to the following formula

$$\log T(M) = \frac{\log 2}{2}\chi(N) - \frac{\log 3}{2}\dim H^0(N) +$$
$$+\frac{1}{2}\left(\zeta'_{0,N}(0,1/2) - \zeta'_{0,N}(0,-1/2)\right) - \frac{\gamma}{4}\mathrm{Res}\,\zeta_{0,N}(1) +$$
$$+\frac{1}{4}\left(\mathrm{Res}\,\zeta_{0,N}(1)[\gamma + 2\log 2] + \frac{1}{4}\mathrm{Res}\,\zeta_{0,N}(2)\right). \tag{5.37}$$

Obvious cancellations in the formula above prove the result. □

5.7 Analytic torsion of a cone over S^1

The preceeding computations reduce in the two-dimensional case simply to the computation of the analytic torsion of a disc. In order to deal with a generalized bounded cone in two dimensions, which is not simply a flat disc, we need to introduce an additional parameter in the Riemannian metric. So in two dimensions the setup is as follows.

Let $M := (0, R] \times S^1$ with

$$g^M = dx^2 \oplus \nu^{-2} x^2 g^{S^1}$$

be a bounded generalized cone over S^1 of angle $\mathrm{arcsec}(\nu)$ and length 1, with a fixed orientation and with a fixed parameter $\nu \geq 1$.

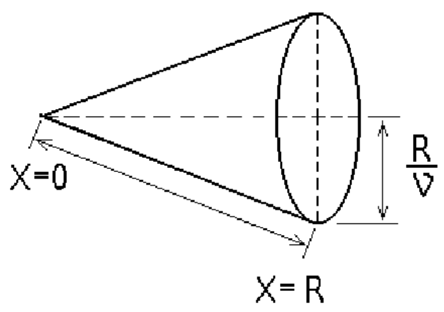

Figure 4: A bounded cone of angle $\mathrm{arcsec}(\nu), \nu \geq 1$ and length R.

The main result of our discussion in this part of the presentation is then the following theorem:

Theorem 5.30. *The analytic torsion $T(M)$ of a bounded generalized cone M of length R and angle* $\mathrm{arcsec}\,\nu > 0$ *over S^1 is given by*

$$2 \log T(M) = -\log(\pi R^2) + \log \nu - \frac{1}{\nu}.$$

This result corresponds precisely to the result obtained in Corollary 5.28 for the special case $\nu = 1$ (for $R = 1$). In fact this result can also be derived from [BGKE, Section 5]. This setup was considered by Spreafico in [S]. However [S] deals only with Dirichlet boundary conditions at the cone base. So we extend his approach to the Neumann boundary conditions in order to obtain an overall result for the analytic torsion of this specific cone manifold. We proceed as follows.

Denote forms with compact support in the interior of M by $\Omega_0^*(M)$. The associated de Rham complex is given by

$$0 \to \Omega_0^0(M) \xrightarrow{d_0} \Omega_0^1(M) \xrightarrow{d_1} \Omega_0^2(M) \to 0.$$

Consider the following maps

$$\begin{aligned}
\Psi_0 : C_0^\infty((0,R),\Omega^0(S^1)) &\to \Omega_0^0(M), \\
\phi \quad &\mapsto \quad x^{-1/2}\phi. \\
\Psi_2 : C_0^\infty((0,R),\Omega^1(S^1)) &\to \Omega_0^2(M), \\
\phi \quad &\mapsto \quad x^{1/2}\phi \wedge dx,
\end{aligned}$$

where ϕ is identified with its pullback to M under the natural projection $\pi : (0,R] \times N \to N$ onto the second factor, and x is the canonical coordinate on $(0,R]$. We find

$$\triangle^0 := \Psi_0^{-1} d_0^t d_0 \Psi_0 = -\frac{d^2}{dx^2} + \frac{1}{x^2}\left(-\nu^2\partial_\theta^2 - \frac{1}{4}\right) \quad \text{on } C_0^\infty((0,R),\Omega^0(S^1)),$$
$$\triangle^2 := \Psi_2^{-1} d_1 d_1^t \Psi_2 = -\frac{d^2}{dx^2} + \frac{1}{x^2}\left(-\nu^2\partial_\theta^2 - \frac{1}{4}\right) \quad \text{on } C_0^\infty((0,R),\Omega^1(S^1)).$$

where θ is the local variable on the one-dimensional sphere. In fact both maps Ψ_0 and Ψ_2 extend to isometries on the L^2-completion of the spaces, by similar arguments as behind Proposition 3.1. Now consider the minimal extensions $D_k := d_{k,\min}$ of the boundary operators d_k in the de Rham complex $(\Omega_0^*(M), d)$. This defines by [BL1, Lemma 3.1] a Hilbert complex

$$(\mathcal{D}, D), \text{ with } \mathcal{D}^k := \mathcal{D}(D_k).$$

Put

$$\begin{aligned}
\triangle_{\text{rel}}^0 &:= \Psi_0^{-1} D_0^* D_0 \Psi_0, \\
\triangle_{\text{rel}}^2 &:= \Psi_2^{-1} D_1 D_1^* \Psi_2.
\end{aligned}$$

The Laplacians $\triangle_{\text{rel}}^0, \triangle_{\text{rel}}^2$ are spectrally equivalent to $D_0^*D_0, D_1D_1^*$, respectively. The boundary conditions for $\triangle_{\text{rel}}^0$ and $\triangle_{\text{rel}}^2$ at the cone base $\{1\} \times S^1$ are determined in Proposition 3.5.

In order to identify the boundary conditions for $\triangle^0_{\mathrm{rel}}$ and $\triangle^2_{\mathrm{rel}}$ at the cone singularity, observe that by [BL2, Theorem 3.7] the ideal boundary conditions for the de Rham complex are uniquely determined at the cone singularity. Further [BL2, Lemma 3.1] shows that the corresponding extension coincides with the Friedrich's extension at the cone singularity. We infer from [BS3, Theorem 6.1] that the elements in the domain of the Friedrich's extension are of the asymptotics $O(\sqrt{x})$ as $x \to 0$. Hence we find

$$\begin{aligned}
&\mathcal{D}(\triangle^0_{\mathrm{rel}}) = \\
=&\{\phi \in H^2_{loc}((0,R] \times S^1) | \phi(R) = 0,\ \phi(x) = O(\sqrt{x}) \text{ as } x \to 0\}, \\
&\mathcal{D}(\triangle^2_{\mathrm{rel}}) = \\
=&\{\phi \in H^2_{loc}((0,R] \times S^1) | \phi'(R) - \frac{1}{2R}\phi(R) = 0,\ \phi(x) = O(\sqrt{x}) \text{ as } x \to 0\}.
\end{aligned}$$

The first operator with Dirichlet boundary conditions at the cone base is already elaborated in [S]. We adapt their approach to deal with the second operator with generalized Neumann boundary conditions at the cone base. The scalar analytic torsion of the bounded generalized cone is then given in terms of both results

$$2 \log T(M) = \zeta'_{\triangle^2_{\mathrm{rel}}}(0) - \zeta'_{\triangle^0_{\mathrm{rel}}}(0).$$

Note that the Laplacian $(-\partial^2_\theta)$ on S^1 has a discrete spectrum $n^2, n \in \mathbb{Z}$, where the eigenvalues n^2 are of multiplicity two, up to the eigenvalue $n^2 = 0$ of multiplicity one.

Consider now a μ-eigenform ϕ of $\triangle^2_{\mathrm{rel}}$. Since eigenforms of $(-\partial^2_\theta)$ on S^1 are smooth, the projection of ϕ for any fixed $x \in (0,R]$ onto some n^2-eigenspace of $(-\partial^2_\theta)$ maps again to $H^2_{loc}((0,R] \times S^1)$, still satisfies the boundary conditions for $\mathcal{D}(\triangle^2_{2,\mathrm{rel}})$ and hence gives again an eigenform of $\mathcal{D}(\triangle^2_{2,\mathrm{rel}})$.

Hence for the purpose of spectrum computation we can assume without loss of generality the $\mu-$eigenform ϕ to lie in a n^2-eigenspace of $(-\partial^2_\theta)$ for any fixed $x \in (0,R]$. This element ϕ, identified with its scalar part as in Remark 3.11, is a solution to

$$-\frac{d^2}{dx^2}\phi(x) + \frac{1}{x^2}\left(\nu^2 n^2 - \frac{1}{4}\right)\phi(x) = \mu^2 \phi(x),$$

subject to the relative boundary conditions. The general solution to the equation above is

$$\phi(x) = c_1 \sqrt{x} J_{\nu n}(\mu x) + c_2 \sqrt{x} Y_{\nu n}(\mu x),$$

where $J_{\nu n}(z)$ and $Y_{\nu n}(z)$ denote the Bessel functions of first and second kind. The boundary conditions at $x=0$ are given by $\phi(x)=O(\sqrt{x})$ as $x\to 0$ and consequently $c_2=0$. The boundary conditions at the cone base give

$$\phi'(R)-\frac{1}{2R}\phi(R)=c_1\mu\sqrt{R}J'_{\nu n}(\mu R)=0.$$

Since we are not interested in zero-eigenvalues, the relevant eigenvalues are by Corollary 5.9 given as follows:

$$\lambda_{n,k}=\left(\frac{\widetilde{j}_{\nu n,k}}{R}\right)^2$$

with $\widetilde{j}_{\nu n,k}$ being the positive zeros of $J'_{\nu n}(z)$. We obtain in view of the multiplicities of the n^2-eigenvalues of $(-\partial^2_\theta)$ on S^1 for the zeta-function

$$\begin{aligned}\zeta_{\triangle^2_{\text{rel}}}(s)=\sum_{k=1}^{\infty}\lambda_{0,k}^{-s}+2\sum_{n,k=1}^{\infty}\lambda_{n,k}^{-s}=\\ =\sum_{k=1}^{\infty}\left(\frac{\widetilde{j}_{0,k}}{R}\right)^{-2s}+2R^{2s}\sum_{n,k=1}^{\infty}\widetilde{j}_{\nu n,k}^{-2s}.\end{aligned}$$

The derivative at zero for the first summand follows by a direct application of [S, Section 3]:

Lemma 5.31.

$$K:=\frac{d}{ds}\Big|_0\sum_{k=1}^{\infty}\left(\widetilde{j}_{0,k}/R\right)^{-2s}=-\frac{1}{2}\log 2\pi-\frac{3}{2}\log R+\log 2.$$

Proof. The values $\widetilde{j}_{0,k}$ are zeros of $J'_0(z)$. Since $J'_0(z)=-J_1(z)$ they are also zeros of $J_1(z)$. Using [S, Lemma 1 (b)] and its application on [S, p.361] we obtain in the notation therein

$$\begin{aligned}\frac{d}{ds}\Big|_0\sum_{k=1}^{\infty}\left(\widetilde{j}_{0,k}/R\right)^{-2s}&=-B(1)+T(0,1)\\ &=-\frac{1}{2}\log 2\pi-\frac{3}{2}\log R+\log 2.\end{aligned}$$

□

Now we turn to the discussion of the second summand. We put $z(s) = \sum_{n,k=1}^{\infty} \widetilde{j}_{\nu n,k}^{-2s}$ for $Re(s) \gg 0$. This series is well-defined for $Re(s)$ sufficiently large by the general result in Theorem 5.1. Due to uniform convergence of integrals and series we obtain with computations similar to (5.12) the following integral representation

$$z(s) = \frac{s^2}{\Gamma(s+1)} \int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\wedge_c} \frac{e^{-\lambda t}}{-\lambda} T(s,\lambda) d\lambda dt, \tag{5.38}$$

$$T(s,\lambda) = \sum_{n=1}^{\infty} (\nu n)^{-2s} t_n(\lambda), \quad t_n(\lambda) = -\sum_{k=1}^{\infty} \log\left(1 - \frac{(\nu n)^2 \lambda}{\widetilde{j}_{\nu n,k}^2}\right), \tag{5.39}$$

where $\Lambda_c := \{\lambda \in \mathbb{C} | |arg(\lambda - c)| = \pi/4\}$ with $c > 0$ being any fixed positive number, smaller than the lowest non-zero eigenvalue of $\triangle_{\text{rel}}^2$.

We proceed with explicit calculations by presenting $t_n(\lambda)$ in terms of special functions. Using the infinite product expansion (5.25) we obtain the following result for the derivative of the modified Bessel function of first kind:

$$I'_{\nu n}(\nu n z) = \frac{(\nu n z)^{\nu n - 1}}{2^{\nu n}\Gamma(\nu n)} \prod_{k=1}^{\infty} \left(1 + \frac{(\nu n z)^2}{\widetilde{j}_{\nu n,k}^2}\right),$$

where $\widetilde{j}_{\nu n,k}$ denotes the positive zeros of $J'_{\nu n}(z)$. Putting $z = \sqrt{-\lambda}$ we get

$$\begin{aligned} t_n(\lambda) = -\sum_{k=1}^{\infty} \log\left(1 - \frac{(\nu n)^2 \lambda}{\widetilde{j}_{\nu n,k}^2}\right) &= -\log\left[\prod_{k=1}^{\infty}\left(1 + \frac{(\nu n z)^2}{\widetilde{j}_{\nu n,k}^2}\right)\right] \\ &= -\log I'_{\nu n}(\nu n z) + \log(\nu n z)^{\nu n - 1} - \log 2^{\nu n}\Gamma(\nu n). \end{aligned} \tag{5.40}$$

The associated function $T(s,\lambda)$ from (5.39) is however not analytic at $s = 0$. The $1/\nu n$-dependence in $t_n(\lambda)$ causes non-analytic behaviour. We put

$$t_n(\lambda) =: p_n(\lambda) + \frac{1}{\nu n} f(\lambda), \quad P(s,\lambda) = \sum_{n=1}^{\infty} (\nu n)^{-2s} p_n(\lambda). \tag{5.41}$$

To get explicit expressions for $P(s,\lambda)$ and $f(\lambda)$ we use asymptotic expansion of the Bessel-functions for large order from [O], in analogy to Lemma 5.14. We obtain in the notation of (5.30) with $z = \sqrt{-\lambda}$ and $t = 1/\sqrt{1-\lambda}$:

$$f(\lambda) = -M_1(t,0) = \frac{3}{8}t - \frac{7}{24}t^3,$$

where we inferred the explicit form of $M_1(t,0)$ from (5.36). We obtain for $p_n(\lambda)$

$$p_n(\lambda) = -\log I'_{\nu n}(\nu n z) + \log(\nu n z)^{\nu n-1} - \log 2^{\nu n}\Gamma(\nu n) - \\ -\frac{1}{\nu n}\left(\frac{3}{8}t - \frac{7}{24}t^3\right). \qquad (5.42)$$

As in Lemma 5.15 we compute the contribution coming from $f(\lambda)$.

Lemma 5.32.

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\wedge_c}\frac{e^{-\lambda t}}{-\lambda}f(\lambda)d\lambda dt = \frac{1}{12\sqrt{\pi}}\Gamma\left(s+\frac{1}{2}\right)\left(\frac{1}{s}-7\right).$$

Proof. Observe from [GRA, 8.353.3] by substituting the new variable $x = \lambda - 1$

$$\frac{1}{2\pi i}\int_{\wedge_c}\frac{e^{-\lambda t}}{-\lambda}\frac{1}{(1-\lambda)^a}d\lambda = \frac{1}{2\pi i}e^{-t}\int_{\wedge_{c-1}} -\frac{e^{-xt}}{x+1}\frac{1}{(-x)^a}dx = \\ = \frac{1}{\pi}\sin(\pi a)\Gamma(1-a)\Gamma(a,t).$$

Using now the relation between the incomplete Gamma function and the probability integral

$$\int_0^\infty t^{s-1}\Gamma(a,t)dt = \frac{\Gamma(s+a)}{s}$$

we finally obtain

$$\begin{aligned}
&\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\wedge_c}\frac{e^{-\lambda t}}{-\lambda}f(\lambda)d\lambda dt \\
=&\frac{3}{8\pi}\sin\left(\frac{\pi}{2}\right)\Gamma\left(1-\frac{1}{2}\right)\frac{\Gamma(s+1/2)}{s} - \\
-&\frac{7}{24\pi}\sin\left(\frac{3\pi}{2}\right)\Gamma\left(1-\frac{3}{2}\right)\frac{\Gamma(s+3/2)}{s} \\
=&\frac{3}{8\sqrt{\pi}}\frac{\Gamma(s+1/2)}{s} - \frac{7}{12\sqrt{\pi}}\frac{\Gamma(s+3/2)}{s} = \\
=&\frac{1}{\sqrt{\pi}}\Gamma\left(s+\frac{1}{2}\right)\left\{\frac{3}{8s} + \frac{7}{12s}\left(s+\frac{1}{2}\right)\right\} = \\
=&\frac{1}{12\sqrt{\pi}}\Gamma\left(s+\frac{1}{2}\right)\left(\frac{1}{s}-7\right).
\end{aligned}$$

□

By classical asymptotics of Bessel functions for large arguments and fixed order

$$I'_{\nu n}(\nu n z) = \frac{e^{\nu n z}}{\sqrt{2\pi \nu n z}} \left(1 + O\left(\frac{1}{z}\right)\right),$$

where the region of validity is preserved (see the discussion in the higher-dimensional case in Proposition 5.16), we obtain for $p_n(\lambda)$ from (5.42)

$$\begin{aligned} p_n(\lambda) = -\nu n \sqrt{\lambda} + \left(\frac{1}{4} + (\nu n - 1)\frac{1}{2}\right) \log(-\lambda) + \frac{1}{2} \log 2\pi \nu n \\ +(\nu n - 1) \log \nu n - \log(2^{\nu n} \Gamma(\nu n)) + O((-\lambda)^{-1/2}). \end{aligned}$$

Following [S, Section 4.2] we reorder the summands in the above expression to get

$$p_n(\lambda) = -\nu n \sqrt{\lambda} + a_n \log(-\lambda) + b_n + O((-\lambda)^{-1/2}),$$

where the interesting terms are clear from above. We set

$$\begin{aligned} A(s) &:= \sum_{n=1}^{\infty} (\nu n)^{-2s} a_n = \frac{1}{2} \nu^{-2s+1} \zeta_R(2s-1) - \frac{1}{4} \nu^{-2s} \zeta_R(2s), \\ B(s) &:= \sum_{n=1}^{\infty} (\nu n)^{-2s} b_n = \frac{1}{2} \nu^{-2s} \log\left(\frac{2\pi}{\nu}\right) \zeta_R(2s) + \\ &\quad + \nu^{-2s+1} \log\left(\frac{\nu}{2}\right) \zeta_R(2s-1) - \nu^{-2s+1} \zeta'_R(2s-1) + \\ &\quad + \frac{1}{2} \nu^{-2s} \zeta'_R(2s) - \sum_{n=1}^{\infty} (\nu n)^{-2s} \log \Gamma(\nu n). \end{aligned}$$

Following the approach of M. Spreafico it remains to evaluate $P(s, 0)$ defined in (5.41) in order to obtain a closed expression for the function $z(s)$.

Lemma 5.33.

$$P(s, 0) = -\frac{1}{12} \nu^{-2s-1} \zeta_R(2s+1).$$

Proof. Recall the asymptotic behaviour of Bessel functions of second order for small arguments

$$I_{\nu n}(x) \sim \frac{1}{\Gamma(\nu n + 1)} \left(\frac{x}{2}\right)^{\nu n} \Rightarrow I'_{\nu n}(x) \sim \frac{\nu n}{2\Gamma(\nu n + 1)} \left(\frac{x}{2}\right)^{\nu n - 1}.$$

Further observe that as $\lambda \to 0$ we obtain with $z = \sqrt{-\lambda}$ and $t = 1/\sqrt{1 + z^2}$

$$M_1(t, 0) = -\frac{3}{8} t + \frac{7}{24} t^3 \xrightarrow{\lambda \to 0} -\frac{3}{8} + \frac{7}{24} = -\frac{1}{12}.$$

Using these two facts we obtain from (5.42) for $p_n(0)$

$$p_n(0) = -\log \nu n + \log \Gamma(\nu n + 1) - \log \Gamma(\nu n) - \frac{1}{12\nu n} = -\frac{1}{12\nu n}$$
$$\Rightarrow P(s,0) = \sum_{n=1}^{\infty} (\nu n)^{-2s} p_n(0) = -\frac{1}{12}\nu^{-2s-1}\zeta_R(2s+1).$$

□

Now we have all the ingredients together, since by [S, p. 366] and Lemma 5.32 the function $z(s)$ is given as follows:

$$z(s) = \frac{s}{\Gamma(s+1)}[\gamma A(s) - B(s) - \frac{1}{s}A(s) + P(s,0)] +$$
$$+ \frac{s^2}{\Gamma(s+1)}\nu^{-2s-1}\zeta_R(2s+1)\frac{1}{12\sqrt{\pi}}\Gamma\left(s+\frac{1}{2}\right)\left(\frac{1}{s}-7\right) + \frac{s^2}{\Gamma(s+1)}h(s),$$

where the last term vanishes with its derivative at $s = 0$. We are interested in the value of the function itself $z(0)$ and its derivative $z'(0)$. In order to compute the value of $z(0)$ recall the fact that close to 1 the Riemann zeta function behaves as follows

$$\zeta_R(2s+1) = \frac{1}{2s} + \gamma + o(s), \quad s \to 0.$$

This implies

$$\frac{s^2}{\Gamma(s+1)}\nu^{-2s-1}\zeta_R(2s+1)\frac{1}{12\sqrt{\pi}}\Gamma\left(s+\frac{1}{2}\right)\left(\frac{1}{s}-7\right) \to \frac{1}{24\nu}, s \to 0.$$

Furthermore note that the function

$$\eta(s,\nu) := \sum_{n=1}^{\infty} (\nu n)^{-2s} \log \Gamma(\nu n + 1) - \frac{1}{12}\nu^{-2s-1}\zeta_R(2s+1),$$

introduced in [S, p.366] is regular at $s = 0$, cf. [S, Section 4.3]. Hence $\gamma A(s) - B(s) + P(s,0)$ is regular at $s = 0$ and we obtain straightforwardly:

$$z(0) = -A(0) + \frac{1}{24\nu} = -\frac{1}{2}\nu\zeta_R(-1) + \frac{1}{4}\zeta_R(0) + \frac{1}{24\nu}.$$

In view of the explict values $\zeta_R(-1) = -\frac{1}{12}$ and $\zeta_R(0) = -\frac{1}{2}$ we find

$$z(0) = \frac{\nu}{24} + \frac{1}{24\nu} - \frac{1}{8}. \tag{5.43}$$

Lemma 5.34.

$$z'(0) = \eta(0,\nu) + \frac{1}{2}\log\nu - \frac{1}{4}\log 2\pi - \frac{1}{12}\nu\log 2 + \frac{1}{12\nu}(\gamma - \log 2\nu - \frac{7}{2}),$$

where $\eta(s,\nu) = \sum_{n=1}^{\infty}(\nu n)^{-2s}\log\Gamma(\nu n+1) - \frac{1}{12}\nu^{-2s-1}\zeta_R(2s+1)$.

Proof. We compute $z'(0)$ from the above expression for $z(s)$, using

$$\Gamma'(1/2) = -\sqrt{\pi}(\gamma + 2\log 2).$$

Straightforward computations lead to:

$$z'(0) = P(0,0) - A'(0) - B(0) + \frac{1}{12\nu}(\gamma - \log 2\nu - \frac{7}{2}). \tag{5.44}$$

The statement follows with $\eta(s,\nu)$ being defined precisely as in [S, Section 4.2]. $\square$

Now we are able to provide a result for the derivative of the zeta function $\zeta'_{\triangle^2_{\text{rel}}}(0)$. Recall

$$\zeta_{\triangle^2_{\text{rel}}}(s) = \sum_{k=1}^{\infty}\left(\frac{\widetilde{j}_{0,k}}{R}\right)^{-2s} + 2R^{2s}\sum_{\nu n,k=1}^{\infty}\widetilde{j}_{\nu n,k}^{-2s}.$$

With K defined in Lemma 5.31 and $z(s) = \sum_{n,k=1}^{\infty}\widetilde{j}_{\nu n,k}^{-2s}$ we get

$$\zeta'_{\triangle^2_{\text{rel}}}(0) = K + 4z(0)\log R + 2z'(0).$$

It remains to compare each summand to the corresponding results for $\zeta'_{\triangle^0_{\text{rel}}}(0)$ obtained in [S]. Using Lemma 5.31, (5.43) and (5.34) we finally arrive after several cancellations at Theorem 5.30

$$2\log T(M) = \zeta'_{\triangle^2_{\text{rel}}}(0) - \zeta'_{\triangle^0_{\text{rel}}}(0) = -\log(\pi R^2) + \log\nu - \frac{1}{\nu}.$$

5.8 Open Problems

The presented computation of analytic torsion on a bounded generalized cone solves problem posed in [L, Problem 5.3]. We have provided the general answer to the question in Theorems 5.26 and 5.27 and obtained as an example explicit results in two and in three dimensions in Corollaries 5.28 and 5.29.

The question of [L, Problem 5.3] is motivated by the vision of a Cheeger-Müller Theorem for compact manifolds with conical singularities. The idea is to reduce via the gluing formula of Vishik [V] the comparison of Ray-Singer and $\bar{p}$-Reidemeister torsion (intersection torsion, cf. [Dar]) on compact manifolds with conical singularities to a comparison on a bounded generalized cone.

After the computation of the analytic torsion of a bounded generalized cone one faces the problem of comparing it to the intersection torsion in the "right" perversity $\bar{p}$. However the complex form of the result for the analytic torsion at least complicates the comparison with the topological counterpart.

6 Refined Analytic Torsion

The refined analytic torsion, defined by M. Braverman and T. Kappeler in [BK1] and [BK2] on closed manifolds, can be viewed as a refinement of the Ray-Singer torsion, since it is a canonical choice of an element with Ray-Singer norm one, in case of unitary representations.

The complex phase of the refinement is given by the rho-invariant of the odd-signature operator. Hence one can expect the refined analytic torsion to give more geometric information than the Ray-Singer torsion.

Indeed, let us consider the setup of lens spaces with explicit formulas for the associated Ray-Singer torsion and eta-invariants, see [RH, Section 5] and the references therein. Then it is easy to find explicit examples of lens spaces which are not distinguished by the Ray-Singer torsion, however have different rho-invariants of the associated odd-signature operators.

An important property of the Ray-Singer torsion norm is its gluing property, as established by W. Lück in [Lü] and S. Vishik in [V]. It is natural to expect a refinement of the Ray-Singer torsion to admit an analogous gluing property.

Unfortunately there seems to be no canonical way to extend the construction of Braverman and Kappeler to compact manifolds with boundary. In particular a gluing formula seems to be out of reach.

We propose a different refinement of analytic torsion, similar to Braverman and Kappeler, which does apply to compact manifolds with and without boundary. We establish a gluing formula for our construction, which in fact can also be viewed as a gluing law for the original definition of refined analytic torsion by Braverman and Kappeler.

The presented construction is analogous to the definition in [BK1] and [BK2], but applies to any smooth compact Riemannian manifold, with or without boundary. For closed manifolds the construction differs from the original definition in [BK2]. Nevertheless we still refer to our concept as "refined analytic torsion" within the present discussion.

6.1 Motivation for the generalized construction

The essential ingredient in the definition of the refined analytic torsion in [BK2] is the twisted de Rham complex with a chirality operator and the elliptic odd-signature operator associated to the complex, viewed as a map between the even forms. Hence in the case of a manifold with boundary we are left with the task of finding elliptic boundary conditions for the odd-signature operator which preserve the complex structure and provide a Fredholm complex, in the sense of [BL1].

The notions of a Hilbert and a Fredholm complex were studied systematically in [BL1] and will be provided for convenience in the forthcoming section. The boundary conditions, that give rise to a Hilbert complex are referred to as "ideal boundary conditions". It is important to note that the most common self-adjoint extensions of the odd-signature operator between the even forms do not come from ideal boundary conditions.

The existence and explicit determination of elliptic boundary conditions for the odd-signature operator between the even forms, arising from ideal boundary conditions, is an open question. However, it is clear that the absolute and relative boundary conditions do not satisfy these requirements.

On the other hand the gluing formula in [V] and [Lü] for the Ray-Singer torsion makes essential use of the relative and absolute boundary conditions. Since the establishment of a corresponding gluing formula for the refined analytic torsion is a motivation for our discussion, these boundary conditions seem to be natural choices.

We are left with a dilemma, since neither the relative nor the absolute boundary conditions are invariant under the Hodge operator. We resolve this dilemma by combining the relative and absolute boundary conditions. This allows us to apply the concepts of [BK2] in a new setting and to establish the desired gluing formula.

6.2 Definition of Refined analytic torsion

Let (M^m, g^M) be a smooth compact connected odd-dimensional oriented Riemannian manifold with boundary ∂M, which may be empty. Let (E, ∇, h^E) be a flat complex vector bundle with any fixed Hermitian metric h^E, which need not to be flat with respect to ∇.

The flat covariant derivative ∇ is a first order differential operator

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E),$$

satisfying the Leibniz rule

$$\nabla_X(fs) = (Xf)s + f\nabla_X s, \quad s \in \Gamma(E), X \in \Gamma(TM), f \in C^\infty(M).$$

The covariant derivative ∇ extends by the Leibniz rule to the twisted exterior differential $\nabla : \Omega_0^k(M,E) \to \Omega_0^{k+1}(M,E)$ on $E-$valued differential forms with compact support in the interior of the manifold $\Omega_0^k(M,E)$. The exterior differential satisfies the (generalized) Leibniz rule

$$\nabla_X(w \wedge \eta) = (\nabla_X w) \wedge \eta + (-1)^p w \wedge \nabla_X \eta,$$

for any $w \in \Omega_0^p(M), \eta \in \Omega_0^q(M,E), X \in \Gamma(TM)$.

Due to flatness of (E,∇) the twisted exterior differential gives rise to the twisted de Rham complex $(\Omega_0^*(M,E),\nabla)$. The metrics g^M, h^E induce an L^2-inner product on $\Omega_0^*(M,E)$. We denote the L^2-completion of $\Omega_0^*(M,E)$ by $L^2_*(M,E)$.

Next we introduce the notion of the dual covariant derivative ∇'. It is defined by requiring:

$$dh^E(u,v)[X] = h^E(\nabla_X u, v) + h^E(u, \nabla'_X v), \tag{6.1}$$

to hold for all $u,v \in C^\infty(M,E)$ and $X \in \Gamma(TM)$. In the special case that the Hermitian metric h^E is flat with respect to ∇, the dual ∇' and the original covariant derivative ∇ coincide. More precisely the Hermitian metric h^E can be viewed as a section of $E^* \otimes E^*$. The covariant derivative ∇ on E gives rise to a covariant derivative on the tensor bundle $E^* \otimes E^*$, also denoted by ∇ by a minor abuse of notation.

For u, v, X as above one has:

$$\nabla h^E(u,v)[X] = dh^E(u,v)[X] - h^E(\nabla_X u, v) - h^E(u, \nabla_X v).$$

In view of (6.1) we find

$$\nabla h^E = 0 \ \Leftrightarrow \nabla = \nabla'.$$

As before, the dual ∇' gives rise to a twisted de Rham complex. Consider the differential operators ∇, ∇' and their formal adjoint differential operators ∇^t, ∇'^t. The associated minimal closed extensions $\nabla_{\min}, \nabla'_{\min}$ and $\nabla^t_{\min}, \nabla'^t_{\min}$ are defined as the graph-closures in $L^2_*(M,E)$ of the respective differential operators. The maximal closed extensions are defined by

$$\nabla_{\max} := (\nabla^t_{\min})^*, \quad \nabla'_{\max} := (\nabla'^t_{\min})^*.$$

The definition of the maximal and the minimal closed extensions of course corresponds to the discussion in Subsection 2.1. These extensions define Hilbert complexes in the following sense, as introduced in [BL1].

Definition 6.1. [BL1] *Let the Hilbert spaces $H_i, i = 0, .., m, H_{m+1} = \{0\}$ be mutually orthogonal. For each $i = 0, .., m$ let $D_i \in C(H_i, H_{i+1})$ be a closed operator with domain $\mathcal{D}(D_i)$ dense in H_i and range in H_{i+1}. Put $\mathcal{D}_i := \mathcal{D}(D_i)$ and $R_i := D_i(\mathcal{D}_i)$ and assume*

$$R_i \subseteq \mathcal{D}_{i+1}, \quad D_{i+1} \circ D_i = 0.$$

This defines a complex $(\mathcal{D}, D)$

$$0 \to \mathcal{D}_0 \xrightarrow{D_0} \mathcal{D}_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{m-1}} \mathcal{D}_m \to 0.$$

Such a complex is called a Hilbert complex. If the homology of the complex is finite, i.e. if R_i is closed and $\ker D_i / \mathrm{im} D_{i-1}$ is finite-dimensional for all $i = 0, ..., m$, the complex is referred to as a Fredholm complex.

Indeed, by [BL1, Lemma 3.1] the extensions define Hilbert complexes as follows

$$(\mathcal{D}_{\min}, \nabla_{\min}), \text{ where } \mathcal{D}_{\min} := \mathcal{D}(\nabla_{\min}),$$
$$(\mathcal{D}_{\max}, \nabla_{\max}), \text{ where } \mathcal{D}_{\max} := \mathcal{D}(\nabla_{\max})$$

$$(\mathcal{D}'_{\min}, \nabla'_{\min}), \text{ where } \mathcal{D}'_{\min} := \mathcal{D}(\nabla'_{\min}),$$
$$(\mathcal{D}'_{\max}, \nabla'_{\max}), \text{ where } \mathcal{D}'_{\max} := \mathcal{D}(\nabla'_{\max}).$$

Note the following well-known central result on these complexes.

Theorem 6.2. *The Hilbert complexes $(\mathcal{D}_{\min}, \nabla_{\min})$ and $(\mathcal{D}_{\max}, \nabla_{\max})$ are Fredholm with the associated Laplacians $\triangle_{\mathrm{rel}}$ and $\triangle_{\mathrm{abs}}$ being strongly elliptic in the sense of [Gi]. The de Rham isomorphism identifies the homology of the complexes with the relative and absolute cohomology with coefficients:*

$$H^*(\mathcal{D}_{\min}, \nabla_{\min}) \cong H^*(M, \partial M, E),$$
$$H^*(\mathcal{D}_{\max}, \nabla_{\max}) \cong H^*(M, E).$$

Furthermore the cohomology of the Fredholm complexes $(\mathcal{D}_{\min}, \nabla_{\min})$ and $(\mathcal{D}_{\max}, \nabla_{\max})$ can be computed from the following smooth subcomplexes,

$$(\Omega^*_{\min}(M, E), \nabla), \quad \Omega^*_{\min}(M, E) := \{\omega \in \Omega^*(M, E) | \iota^*(\omega) = 0\},$$
$$(\Omega^*_{\max}(M, E), \nabla), \quad \Omega^*_{\max}(M, E) := \Omega^*(M, E),$$

respectively, where we denote by $\iota : \partial M \hookrightarrow M$ the natural inclusion of the boundary.

In the untwisted setup this theorem is essentially the statement of [BL1, Theorem 4.1]. The theorem remains true in the general setup. An analogue of the trace theorem [P, Theorem 1.9], in case of flat vector bundles, allows an explicit computation of the boundary conditions for $\triangle_{\mathrm{rel}}$ and $\triangle_{\mathrm{abs}}$. Then [Gi, Lemma 1.11.1] implies strong ellipticity of the Laplacians. Note that this result in the reference [Gi] is proved explicitly, even though other aspects of [Gi, Section 1.11] are rather expository.

By strong ellipticity the Laplacians $\triangle_{\mathrm{rel}}$ and $\triangle_{\mathrm{abs}}$ are Fredholm and by [BL1, Theorem 2.4] the complexes $(\mathcal{D}_{\min}, \nabla_{\min})$ and $(\mathcal{D}_{\max}, \nabla_{\max})$ are Fredholm as well. By [BL1, Theorem 3.5] their cohomology indeed can be computed from the smooth subcomplexes $(\Omega^*_{\min}(M,E), \nabla)$ and $(\Omega^*_{\max}(M,E), \nabla)$, respectively.

Finally, the relation to the relative and absolute cohomolgy (the twisted de Rham theorem) is proved in [RS, Section 4] for flat Hermitian metrics, but an analogous proof works in the general case. Corresponding results hold also for the complexes associated to the dual connection ∇'.

Furthermore, the Riemannian metric g^M and the fixed orientation on M give rise to the Hodge-star operator for any $k = 0,..,m = \dim M$:

$$* : \Omega^k(M,E) \to \Omega^{m-k}(M,E).$$

Define

$$\Gamma := i^r(-1)^{\frac{k(k+1)}{2}} * : \Omega^k(M,E) \to \Omega^{m-k}(M,E), \quad r := (\dim M + 1)/2.$$

This operator extends to a well-defined self-adjoint involution on $L^2_*(M,E)$, which we also denote by Γ. The following properties of Γ are essential for the later construction.

Lemma 6.3. *The self-adjoint involution Γ relates the minimal and maximal closed extensions of ∇ and ∇' as follows*

$$\Gamma\nabla_{\min}\Gamma = (\nabla'_{\max})^*, \quad \Gamma\nabla_{\max}\Gamma = (\nabla'_{\min})^*.$$

Proof. One first checks explicitly, cf. [BGV, Proposition 3.58]

$$\Gamma\nabla\Gamma = (\nabla')^t, \quad \Gamma\nabla'\Gamma = \nabla^t.$$

Recall that the maximal domain of ∇, ∇' can also be characterized as a subspace of $L^2_*(M,E)$ with its image under ∇, ∇' being again in $L^2_*(M,E)$. Since Γ gives an involution on $L^2_*(M,E)$, we obtain:

$$\begin{aligned} \Gamma\nabla_{\max}\Gamma &= (\nabla')^t_{\max}, \quad \Gamma\nabla'_{\max}\Gamma = \nabla^t_{\max}, \\ \text{i.e.} \quad \Gamma\nabla_{\max}\Gamma &= (\nabla'_{\min})^*, \quad \Gamma\nabla'_{\max}\Gamma = \nabla^*_{\min}. \end{aligned}$$

Taking adjoints on both sides of the last relation, we obtain the full statement of the lemma, since Γ is self-adjoint. $\square$

Now we can introduce the following central concepts.

Definition 6.4. $(\widetilde{\mathcal{D}}, \widetilde{\nabla}) := (\mathcal{D}_{\min}, \nabla_{\min}) \oplus (\mathcal{D}_{\max}, \nabla_{\max})$. *The chirality operator* $\widetilde{\Gamma}$ *on* $(\widetilde{\mathcal{D}}, \widetilde{\nabla})$ *by definition acts anti-diagonally with respect to the direct sum of the components*

$$\widetilde{\Gamma} := \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}. \tag{6.2}$$

The Fredholm complex $(\widetilde{\mathcal{D}}, \widetilde{\nabla})$ with the chirality operator $\widetilde{\Gamma}$ is in case of a flat Hermitian metric a complex with Poincare duality, in the sense of [BL1, Lemma 2.16], i.e.

$$\nabla h^E = 0 \;\Rightarrow\; \widetilde{\Gamma}\widetilde{\nabla} = \widetilde{\nabla}^* \widetilde{\Gamma},$$

which follows directly from Lemma 6.3. We now apply the concepts of Braverman and Kappeler to our new setup.

Definition 6.5. *The odd-signature operator of the Hilbert complex* $(\widetilde{\mathcal{D}}, \widetilde{\nabla})$ *is defined as follows*

$$\mathcal{B} := \widetilde{\Gamma}\widetilde{\nabla} + \widetilde{\nabla}\widetilde{\Gamma}.$$

Before we can state some basic properties of the odd signature operator, let us recall the notions of the Gauss-Bonnet operator and its relative and absolute self-adjoint extensions. The Gauss-Bonnet operator

$$D^{GB} := \nabla + \nabla^t,$$

admits two natural self-adjoint extensions

$$D^{GB}_{\text{rel}} = \nabla_{\min} + \nabla^*_{\min}, \; D^{GB}_{\text{abs}} = \nabla_{\max} + \nabla^*_{\max}, \tag{6.3}$$

respectively called the relative and the absolute self-adjoint extensions. Their squares are correspondingly the relative and the absolute Laplace operators:

$$\triangle_{\text{rel}} = (D^{GB}_{\text{rel}})^* D^{GB}_{\text{rel}}, \quad \triangle_{\text{abs}} = (D^{GB}_{\text{abs}})^* D^{GB}_{\text{abs}}.$$

Similar definitions, of course, hold for the Gauss-Bonnet Operator associated to the dual covariant derivative ∇'. Now we can state the following basic result.

Lemma 6.6. *The leading symbols of $\mathcal{B}$ and $\widetilde{\Gamma}\left(D^{GB}_{\text{rel}} \oplus D'^{GB}_{\text{abs}}\right)$ coincide and moreover*

$$\mathcal{D}(\mathcal{B}) = \mathcal{D}\left(D^{GB}_{\text{rel}} \oplus D'^{GB}_{\text{abs}}\right).$$

Proof. First recall the relations

$$\Gamma\nabla\Gamma = (\nabla')^t, \quad \Gamma\nabla^t\Gamma = \nabla'.$$

All connections differ by an endomorphism-valued differential form of degree one, which can be viewed as a differential operator of order zero. This implies the statement on the leading symbol of $\mathcal{B}$ and $\widetilde{\Gamma}\left(D^{GB}_{\text{rel}} \oplus D'^{GB}_{\text{abs}}\right)$

A differential operator of zero order naturally extends to a bounded operator on the L^2-Hilbert space, and hence does not pose additional restrictions on the domain, in particular we obtain (compare Lemma 6.3)

$$\mathcal{D}(\nabla^*_{\min}) = \mathcal{D}(\Gamma\nabla_{\max}\Gamma), \quad \mathcal{D}(\nabla^*_{\max}) = \mathcal{D}(\Gamma\nabla_{\min}\Gamma).$$

Using these domain relations we find:

$$\mathcal{D}(\mathcal{B}) = \mathcal{D}\left(\widetilde{\Gamma}(D^{GB}_{\text{rel}} \oplus D'^{GB}_{\text{abs}})\right) = \mathcal{D}\left(D^{GB}_{\text{rel}} \oplus D'^{GB}_{\text{abs}}\right).$$

□

Note by the arguments of the lemma above that $\mathcal{B}$ is a bounded perturbation of a closed operator $\widetilde{\Gamma}\left(D^{GB}_{\text{rel}} \oplus D'^{GB}_{\text{abs}}\right)$ and hence is closed, as well. Before we continue analyzing the spectral properties of the odd-signature operator $\mathcal{B}$, let us introduce some concepts and notation.

Definition 6.7. *Let D be a closed operator in a separable Hilbert space. An angle $\theta \in [0, 2\pi)$ is called an "Agmon angle" for D, if for $R_\theta \subset \mathbb{C}$ being the cut in $\mathbb{C}$ corresponding to θ*

$$R_\theta := \{z \in \mathbb{C} | z = |z| \cdot e^{i\theta}\}$$

we have the following spectral relation

$$R_\theta \cap \text{Spec}(D) \backslash \{0\} = \emptyset.$$

Theorem 6.8. [S. Agmon, R. Seeley] *Let (K, g^K) be a smooth compact oriented Riemannian manifold with boundary ∂K. Let (F, h^F) be a Hermitian vector bundle over K. The metric structures (g^K, h^F) define an L^2-inner product. Let*

$$D : C^\infty(K, F) \to C^\infty(K, F)$$

be a differential operator of order ω such that $\omega \cdot \text{rank} F$ is even. Consider a boundary value problem (D, B) strongly elliptic with respect to $\mathbb{C}\backslash\mathbb{R}^*$ in the sense of [Gi]. Then

(i) D_B *is a Fredholm operator with compact resolvent and discrete spectrum of eigenvalues of finite (algebraic) multiplicity, accumulating only at infinity.*

(ii) *The operator* D_B *admits an Agmon angle* $\theta \in (-\pi, 0)$ *and the associated zeta-function*

$$\zeta(s, D_B) := \sum_{\lambda \in \mathrm{Spec}(D_B)\backslash\{0\}} m(\lambda) \cdot \lambda_\theta^{-s}, \quad \mathrm{Re}(s) > \frac{\dim K}{\omega},$$

where $\lambda_\theta^{-s} := \exp(-s \cdot \log_\theta \lambda)$ *and* $m(\lambda)$ *denotes the multiplicity of the eigenvalue* λ*, is holomorphic for* $\mathrm{Re}(s) > \dim K/\omega$ *and admits a meromorphic extension to the whole complex plane* $\mathbb{C}$ *with* $s = 0$ *being a regular point.*

For the proof of the theorem note that the notion of strong ellipticity in the sense of [Gi] in fact combines ellipticity with Agmon's conditions, as in the treatment of elliptic boundary conditions by R.T. Seeley in [Se1, Se2]. The statement of the theorem above follows then from [Ag] and [Se1, Se2].

Remark 6.9. *The definition of a zeta-function, as in Theorem 6.8 (ii), also applies to any operator* D *with finite spectrum* $\{\lambda_1, .., \lambda_n\}$ *and finite respective multiplicities* $\{m_1, .., m_n\}$*. For a given Agmon angle* $\theta \in [0, 2\pi)$ *the associated zeta-function*

$$\zeta_\theta(s, D) := \sum_{i=1, \lambda_i \neq 0}^{n} m_i \cdot (\lambda_i)_\theta^{-s}$$

is holomorphic for all $s \in \mathbb{C}$*, since the sum is finite and the eigenvalue zero is excluded.*

Now we return to our specific setup. The following result is important in view of the relation between $\mathcal{B}$ and the Gauss-Bonnet operators with relative and absolute boundary conditions, as established in Lemma 6.6.

Proposition 6.10. *The operators*

$$D = \widetilde{\Gamma}(D_{\mathrm{rel}}^{GB} \oplus D_{\mathrm{abs}}^{\prime GB}), \quad D^2 = \triangle_{\mathrm{rel}} \oplus \triangle_{\mathrm{abs}}'$$

are strongly elliptic with respect to $\mathbb{C}\backslash\mathbb{R}^*$ *and* $\mathbb{C}\backslash\mathbb{R}^+$*, respectively, in the sense of P. Gilkey [Gi].*

The fact that $D^2 = \triangle_{\text{rel}} \oplus \triangle'_{\text{rel}}$ is strongly elliptic with respect to $\mathbb{C}\backslash\mathbb{R}^+$ is already encountered in Theorem 6.2. The strong ellipticity of D now follows from [Gi, Lemma 1.11.2]. Note that this result in the reference [Gi] is proved explicitly, even though other aspects of [Gi, Section 1.11] are rather expository.

Since Lemma 6.6 asserts the equality between the leading symbols of the differential operators $\mathcal{B}, D$ and moreover the equality of the associated boundary conditions, the odd signature operator $\mathcal{B}$ and its square $\mathcal{B}^2$ are strongly elliptic as well. This proves together with Theorem 6.8 the next proposition.

Proposition 6.11. *The operators $\mathcal{B}$ and $\mathcal{B}^2$ are strongly elliptic with respect to $\mathbb{C}\backslash\mathbb{R}^*$ and $\mathbb{C}\backslash\mathbb{R}^+$, respectively, in the sense of P. Gilkey [Gi]. The operators $\mathcal{B}, \mathcal{B}^2$ are discrete with their spectrum accumulating only at infinity.*

Let now $\lambda \geq 0$ be any non-negative real number. Denote by $\Pi_{\mathcal{B}^2,[0,\lambda]}$ the spectral projection of $\mathcal{B}^2$ onto eigenspaces with eigenvalues of absolute value in the interval $[0, \lambda]$:

$$\Pi_{\mathcal{B}^2,[0,\lambda]} := \frac{i}{2\pi} \int_{C(\lambda)} (\mathcal{B}^2 - x)^{-1} dx,$$

with $C(\lambda)$ being any closed counterclockwise circle surrounding eigenvalues of absolute value in $[0, \lambda]$ with no other eigenvalue inside. One finds using the analytic Fredholm theorem that the range of the projection lies in the domain of $\mathcal{B}^2$ and that the projection commutes with $\mathcal{B}^2$.

Since $\mathcal{B}^2$ is discrete, the spectral projection $\Pi_{\mathcal{B}^2,[0,\lambda]}$ is of finite rank, i.e. with a finite-dimensional image. In particular $\Pi_{\mathcal{B}^2,[0,\lambda]}$ is a bounded operator in $L^2_*(M, E \oplus E)$. Hence with [K, Section 4, p.155] the decomposition

$$L^2_*(M, E \oplus E) = \text{Image}\Pi_{\mathcal{B}^2,[0,\lambda]} \oplus \text{Image}(\mathbf{1} - \Pi_{\mathcal{B}^2,[0,\lambda]}), \tag{6.4}$$

is a direct sum decomposition into closed subspaces of the Hilbert space $L^2_*(M, E \oplus E)$.

Note that if $\mathcal{B}^2$ is self-adjoint, the decomposition is orthogonal with respect to the fixed L^2-Hilbert structure, i.e. the projection $\Pi_{\mathcal{B}^2,[0,\lambda]}$ is an orthogonal projection, which is the case only if the Hermitian metric h^E is flat with respect to ∇.

The decomposition induces by restriction a decomposition of $\widetilde{\mathcal{D}}$, which was introduced in Definition 6.4:

$$\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_{[0,\lambda]} \oplus \widetilde{\mathcal{D}}_{(\lambda,\infty)}.$$

Since $\widetilde{\nabla}$ commutes with $\mathcal{B}, \mathcal{B}^2$ and hence also with $\Pi_{\mathcal{B}^2,[0,\lambda]}$, we find that the decomposition above is in fact a decomposition into subcomplexes:

$$(\widetilde{\mathcal{D}}, \widetilde{\nabla}) = (\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]}) \oplus (\widetilde{\mathcal{D}}_{(\lambda,\infty)}, \widetilde{\nabla}_{(\lambda,\infty)})$$
$$\text{where } \widetilde{\nabla}_{\mathcal{I}} := \widetilde{\nabla}|_{\widetilde{\mathcal{D}}_{\mathcal{I}}} \text{ for } \mathcal{I} = [0,\lambda] \text{ or } (\lambda,\infty). \tag{6.5}$$

Further $\widetilde{\Gamma}$ also commutes with $\mathcal{B}, \mathcal{B}^2$ and hence also with $\Pi_{\mathcal{B}^2,[0,\lambda]}$. Thus as above we obtain

$$\widetilde{\Gamma} = \widetilde{\Gamma}_{[0,\lambda]} \oplus \widetilde{\Gamma}_{(\lambda,\infty)}.$$

Consequently the odd-signature operator of the complex $(\widetilde{\mathcal{D}}, \widetilde{\nabla})$ decomposes correspondingly

$$\mathcal{B} = \mathcal{B}^{[0,\lambda]} \oplus \mathcal{B}^{(\lambda,\infty)}$$
$$\text{where} \quad \mathcal{B}^{\mathcal{I}} := \widetilde{\Gamma}_{\mathcal{I}}\widetilde{\nabla}_{\mathcal{I}} + \widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}} \text{ for } \mathcal{I} = [0,\lambda] \text{ or } (\lambda,\infty). \tag{6.6}$$

The closedness of the subspace $\text{Image}(1-\Pi_{\mathcal{B}^2,[0,\lambda]})$ implies that the domain of $\mathcal{B}^{(\lambda,\infty)}$

$$\mathcal{D}(\mathcal{B}^{(\lambda,\infty)}) := \mathcal{D}(\mathcal{B}) \cap \text{Image}(1-\Pi_{\mathcal{B}^2,[0,\lambda]})$$

is closed under the graph-norm, hence the operator $\mathcal{B}^{(\lambda,\infty)}$ is a closed operator in the Hilbert space $\text{Image}(1-\Pi_{\mathcal{B}^2,[0,\lambda]})$.

We need to analyze the direct sum component $\mathcal{B}^{(\lambda,\infty)}$. For this we proceed with the following general functional analytic observations.

Proposition 6.12. *Let D be a closed operator in a separable Hilbert space $(H, \langle\cdot,\cdot\rangle)$. The domain $\mathcal{D}(D)$ is a Hilbert space with the graph-norm*

$$\langle x, y\rangle_D = \langle x, y\rangle + \langle Dx, Dy\rangle$$

for any $x, y \in \mathcal{D}(D)$. Let $ResD \neq \emptyset$. Then the following statements are equivalent
1) The inclusion $\iota : \mathcal{D}(D) \hookrightarrow H$ is a compact operator
2) D has a compact resolvent, i.e. for some (and thus for all) $z \in Res(D)$ the resolvent operator $(D-z)^{-1}$ is a compact operator on H.

Proof. Assume first that the inclusion $\iota : \mathcal{D}(D) \hookrightarrow H$ is a compact operator. Since $\text{Spec}D \neq \mathbb{C}$ the resolvent set $\text{Res}(D)$ is not empty. For any $z \in \text{Res}(D)$ the resolvent operator

$$(D-z)^{-1} : H \to \mathcal{D}(D)$$

exists and is bounded, by definition of the resolvent set. With the inclusion ι being a compact operator we find directly that $(D-z)^{-1}$ is compact as an operator from H to H. Finally, if $(D-z)^{-1}$ is compact for some $z \in \mathrm{Res}(D)$, then by the second resolvent identity it is compact for all $z \in \mathrm{Res}(D)$, see also [K, p.187].

Conversely assume that for some (and therefore for all) $z \in \mathrm{Res}(D)$ the resolvent operator $(D-z)^{-1}$ is compact as an operator from H into H. Observe

$$\iota = (D-z)^{-1} \circ (D-z) : \mathcal{D}(D) \hookrightarrow H.$$

By compactness of the resolvent operator, ι is compact as an operator between the Hilbert spaces $\mathcal{D}(D)$ and H. □

Proposition 6.13. *Let D be a closed operator in a separable Hilbert space H with Res$(D) \neq \emptyset$ and compact resolvent. Then D is a Fredholm operator with*

$$\operatorname{index} D = 0.$$

Proof. By closedness of D the domain $\mathcal{D}(D)$ turns into a Hilbert space equipped with the graph norm. By Proposition 6.12 the natural inclusion

$$\iota : \mathcal{D}(D) \hookrightarrow H$$

is a compact operator. Therefore, viewing $\mathcal{D}(D)$ as a subspace of H, i.e. endowed with the inner-product of H, the inclusion

$$\iota : \mathcal{D}(D) \subset H \hookrightarrow H$$

is relatively D-compact in the sense of [K, Section 4.3, p.194]. More precisely this means, that if for a sequence $\{u_n\} \subset \mathcal{D}(D)$ both $\{u_n\}$ and $\{Du_n\}$ are bounded sequences in H, then $\{\iota(u_n)\} \subset H$ has a convergent subsequence.

Now for any $\lambda \in \mathbb{C} \backslash \mathrm{Spec}(D)$ the operator

$$(D - \lambda\iota) : \mathcal{D}(D) \subset H \to H$$

is invertible and hence trivially a Fredholm operator with trivial kernel and closed range H. In particular

$$\operatorname{index}(D - \lambda\iota) = 0.$$

Now, from stability of the Fredholm index under relatively compact perturbations (see [K, Theorem 5.26] and the references therein) we infer with the inclusion ι being relatively compact, that D is a Fredholm operator of zero index:

$$\operatorname{index} D = \operatorname{index}(D - \lambda\iota) = 0.$$

□

Corollary 6.14. *The operator* $\mathcal{B}^{(\lambda,\infty)} : \mathcal{D}(\mathcal{B}^{(\lambda,\infty)}) \to \mathrm{Image}(1-\Pi_{\mathcal{B}^2,[0,\lambda]})$ *of the complex* $(\widetilde{\mathcal{D}}_{(\lambda,\infty)}, \widetilde{\nabla}_{(\lambda,\infty)})$ *with* $\lambda \geq 0$ *is bijective.*

Proof. Consider any $\lambda \in \mathbb{C}\backslash \mathrm{Spec}\mathcal{B}$. By the strong ellipticity of $\mathcal{B}$, the operator

$$(\mathcal{B}-\lambda) : \mathcal{D}(\mathcal{B}) \to L^2_*(M, E \oplus E)$$

is bijective with compact inverse. Hence we immediately find that the restriction

$$(\mathcal{B}^{(\lambda,\infty)} - \lambda) \equiv (\mathcal{B}-\lambda) \restriction \mathrm{Im}(1-\Pi_{\mathcal{B}^2,[0,\lambda]}) : \mathcal{D}(\mathcal{B}^{(\lambda,\infty)}) \to \mathrm{Im}(1-\Pi_{\mathcal{B}^2,[0,\lambda]})$$

is bijective with compact inverse, as well. Now we deduce from Proposition 6.13 that $\mathcal{B}^{(\lambda,\infty)}$ is Fredholm with

$$\mathrm{index}\, \mathcal{B}^{(\lambda,\infty)} = 0.$$

The operator $\mathcal{B}^{(\lambda,\infty)}$ is injective, by definition. Combining injectivity with the vanishing of the index, we derive surjectivity of $\mathcal{B}^{(\lambda,\infty)}$. This proves the statement. □

Note, that in case of a flat Hermitian metric the assertion of the previous corollary is simply the general fact that a self-adjoint Fredholm operator is invertible if and only if its kernel is trivial.

Corollary 6.15. *The subcomplex* $(\widetilde{\mathcal{D}}_{(\lambda,\infty)}, \widetilde{\nabla}_{(\lambda,\infty)})$ *is acyclic and*

$$H^*((\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]})) \cong H^*(\widetilde{\mathcal{D}}, \widetilde{\nabla}).$$

Proof. Corollary 6.14 allows us to apply the purely algebraic result [BK2, Lemma 5.8]. Consequently $(\widetilde{\mathcal{D}}_{(\lambda,\infty)}, \widetilde{\nabla}_{(\lambda,\infty)})$ is an acyclic complex. Together with the decomposition (6.5) this proves the assertion. □

Observe that since the spectrum of $\mathcal{B}^2$ is discrete accumulating only at infinity, $(\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]})$ is a complex of finite-dimensional complex vector spaces with $\widetilde{\Gamma}_{[0,\lambda]} : \widetilde{\mathcal{D}}^k_{[0,\lambda]} \to \widetilde{\mathcal{D}}^{m-k}_{[0,\lambda]}$ being the chirality operator on the complex in the sense of [BK2, Section 1.1].

We also use the notion of determinant lines of finite dimensional complexes in [BK2, Section 1.1], which are given for any finite complex of finite-dimensional vector spaces (C^*, ∂_*) as follows:

$$\mathrm{Det}H^*(C^*,\partial_*) = \bigotimes_k \det H^k(C^*,\partial_*)^{(-1)^k},$$

where $\det H^k(C^*, \partial_*)$ is the top exterior power of $H^k(C^*, \partial_*)$ and $\det H^k(C^*, \partial_*)^{-1} \equiv \det H^k(C^*, \partial_*)^*$. We follow [BK2, Section 1.1] and form the "refined torsion" (note the difference to "refined analytic torsion") of the complex $(\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]})$

$$\begin{aligned}\rho_{[0,\lambda]} := c_0 \otimes (c_1)^{-1} \otimes \cdots \otimes (c_r)^{(-1)^r} \otimes (\widetilde{\Gamma}_{[0,\lambda]} c_r)^{(-1)^{r+1}} \otimes \cdots \\ \cdots \otimes (\widetilde{\Gamma}_{[0,\lambda]} c_1) \otimes (\widetilde{\Gamma}_{[0,\lambda]} c_0)^{(-1)} \in \mathrm{Det}(H^*(\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]})),\end{aligned} \tag{6.7}$$

where $c_k \in \det H^k(\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]})$ are arbitrary elements of the determinant lines, $\widetilde{\Gamma}_{[0,\lambda]}$ denotes the chirality operator $\widetilde{\Gamma}_{[0,\lambda]} : \widetilde{\mathcal{D}}^{\bullet}_{[0,\lambda]} \to \widetilde{\mathcal{D}}^{m-\bullet}_{[0,\lambda]}$ extended to determinant lines and for any $v \in \det H^k(\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]})$ the dual $v^{-1} \in \det H^k(\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]})^{-1} \equiv \det H^k(\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]})^*$ is the unique element such that $v^{-1}(v) = 1$.

By Corollary 6.15 we can view $\rho_{[0,\lambda]}$ canonically as an element of $\mathrm{Det}(H^*(\widetilde{\mathcal{D}}, \widetilde{\nabla}))$, which we do henceforth.

The second part of the construction is the graded determinant. The operator $\mathcal{B}^{(\lambda,\infty)}, \lambda \geq 0$ is bijective by Corollary 6.14 and hence by injectivity (put $\mathcal{I} = (\lambda, \infty)$ to simplify the notation)

$$\ker(\widetilde{\nabla}_{\mathcal{I}} \widetilde{\Gamma}_{\mathcal{I}}) \cap \ker(\widetilde{\Gamma}_{\mathcal{I}} \widetilde{\nabla}_{\mathcal{I}}) = \{0\}. \tag{6.8}$$

Further the complex $(\widetilde{\mathcal{D}}_{\mathcal{I}}, \widetilde{\nabla}_{\mathcal{I}})$ is acyclic by Corollary 6.15 and due to $\widetilde{\Gamma}_{\mathcal{I}}$ being an involution on $\mathrm{Im}(1 - \Pi_{\mathcal{B}^2,[0,\lambda]})$ we have

$$\ker(\widetilde{\nabla}_{\mathcal{I}} \widetilde{\Gamma}_{\mathcal{I}}) = \widetilde{\Gamma}_{\mathcal{I}} \ker(\widetilde{\nabla}_{\mathcal{I}}) = \widetilde{\Gamma}_{\mathcal{I}} \mathrm{Im}(\widetilde{\nabla}_{\mathcal{I}}) = \mathrm{Im}(\widetilde{\Gamma}_{\mathcal{I}} \widetilde{\nabla}_{\mathcal{I}}), \tag{6.9}$$

$$\ker(\widetilde{\Gamma}_{\mathcal{I}} \widetilde{\nabla}_{\mathcal{I}}) = \ker(\widetilde{\nabla}_{\mathcal{I}}) = \mathrm{Im}(\widetilde{\nabla}_{\mathcal{I}}) = \mathrm{Im}(\widetilde{\nabla}_{\mathcal{I}} \widetilde{\Gamma}_{\mathcal{I}}). \tag{6.10}$$

We have $\mathrm{Im}(\widetilde{\Gamma}_{\mathcal{I}} \widetilde{\nabla}_{\mathcal{I}}) + \mathrm{Im}(\widetilde{\nabla}_{\mathcal{I}} \widetilde{\Gamma}_{\mathcal{I}}) = \mathrm{Im}(\mathcal{B}^{\mathcal{I}})$ and by surjectivity of $\mathcal{B}^{\mathcal{I}}$ we obtain from the last three relations above

$$\mathrm{Im}(1 - \Pi_{\mathcal{B}^2,[0,\lambda]}) = \ker(\widetilde{\nabla}_{\mathcal{I}} \widetilde{\Gamma}_{\mathcal{I}}) \oplus \ker(\widetilde{\Gamma}_{\mathcal{I}} \widetilde{\nabla}_{\mathcal{I}}). \tag{6.11}$$

Note that $\mathcal{B}$ leaves $\ker(\widetilde{\nabla}\widetilde{\Gamma})$ and $\ker(\widetilde{\Gamma}\widetilde{\nabla})$ invariant. Put

$$\begin{aligned}\mathcal{B}^{+,(\lambda,\infty)}_{\mathrm{even}} &:= \mathcal{B}^{(\lambda,\infty)} \restriction \widetilde{\mathcal{D}}^{\mathrm{even}} \cap \ker(\widetilde{\nabla}\widetilde{\Gamma}), \\ \mathcal{B}^{-,(\lambda,\infty)}_{\mathrm{even}} &:= \mathcal{B}^{(\lambda,\infty)} \restriction \widetilde{\mathcal{D}}^{\mathrm{even}} \cap \ker(\widetilde{\Gamma}\widetilde{\nabla}).\end{aligned}$$

We obtain a direct sum decomposition

$$\mathcal{B}^{(\lambda,\infty)}_{\mathrm{even}} = \mathcal{B}^{+,(\lambda,\infty)}_{\mathrm{even}} \oplus \mathcal{B}^{-,(\lambda,\infty)}_{\mathrm{even}}.$$

As a consequence of Theorem 6.8 (ii) and Proposition 6.11 there exists an Agmon angle $\theta \in (-\pi, 0)$ for $\mathcal{B}$, which is clearly an Agmon angle for the restrictions above, as well.

By Theorem 6.8 and Proposition 6.11 the zeta function $\zeta_\theta(s, \mathcal{B})$ is holomorphic for $\mathrm{Re}(s)$ sufficiently large. The zeta-functions $\zeta_\theta(s, \mathcal{B}^{\pm,(\lambda,\infty)}_{\text{even}})$ of $\mathcal{B}^{\pm,(\lambda,\infty)}_{\text{even}}$, defined with respect to the given Agmon angle θ, are holomorphic for $\mathrm{Re}(s)$ large as well, since the restricted operators have the same spectrum as $\mathcal{B}$ but in general with lower or at most the same multiplicities.

We define the *graded zeta-function*

$$\zeta_{gr,\theta}(s, \mathcal{B}^{(\lambda,\infty)}_{\text{even}}) := \zeta_\theta(s, \mathcal{B}^{+,(\lambda,\infty)}_{\text{even}}) - \zeta_\theta(s, -\mathcal{B}^{-,(\lambda,\infty)}_{\text{even}}), \; Re(s) \gg 0.$$

In the next subsection we prove in Theorem 6.21 that the graded zeta-function extends meromorphically to $\mathbb{C}$ and is regular at $s = 0$. For the time being we shall assume regularity at zero and define the graded determinant.

Definition 6.16. *[Graded determinant] Let $\theta \in (-\pi, 0)$ be an Agmon angle for $\mathcal{B}^{(\lambda,\infty)}$. Then the "graded determinant" associated to $\mathcal{B}^{(\lambda,\infty)}$ and its Agmon angle θ is defined as follows:*

$$\det_{gr,\theta}(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}) := \exp(-\left.\frac{d}{ds}\right|_{s=0} \zeta_{gr,\theta}(s, \mathcal{B}^{(\lambda,\infty)}_{\text{even}})).$$

Proposition 6.17. *The element*

$$\rho(\nabla, g^M, h^E) := \det_{gr,\theta}(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}) \cdot \rho_{[0,\lambda]} \in \mathrm{Det}(H^*(\widetilde{\mathcal{D}}, \widetilde{\nabla}))$$

is independent of the choice of $\lambda \geq 0$ and choice of Agmon angle $\theta \in (-\pi, 0)$ for the odd-signature operator $\mathcal{B}^{(\lambda,\infty)}$.

Proof. Let $0 \leq \lambda < \mu < \infty$. We obtain $\widetilde{\mathcal{D}}_{[0,\mu]} = \widetilde{\mathcal{D}}_{[0,\lambda]} \oplus \widetilde{\mathcal{D}}_{(\lambda,\mu]}$ and also $\widetilde{\mathcal{D}}_{(\lambda,\infty)} = \widetilde{\mathcal{D}}_{(\lambda,\mu]} \oplus \widetilde{\mathcal{D}}_{(\mu,\infty)}$. Since the odd-signature operator respects this spectral direct sum decomposition (see (6.6)), we obtain

$$\det_{gr}(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}) = \det_{gr}(\mathcal{B}^{(\mu,\infty)}_{\text{even}}) \cdot \det_{gr}(\mathcal{B}^{(\lambda,\mu]}_{\text{even}}).$$

Further the purely algebraic discussion behind [BK2, Proposition 5.10] implies

$$\rho_{[0,\mu]} = \det_{gr}(\mathcal{B}^{(\lambda,\mu]}_{\text{even}}) \cdot \rho_{[0,\lambda]}.$$

This proves the following equality

$$\det_{gr}(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}) \cdot \rho_{[0,\lambda]} = \det_{gr}(\mathcal{B}^{(\mu,\infty)}_{\text{even}}) \cdot \rho_{[0,\mu]}.$$

To see independence of $\theta \in (-\pi, 0)$ note that the strongly elliptic operator (cf. Lemma 6.6)

$$D := \widetilde{\Gamma}(D^{GB}_{rel} \oplus D'^{GB}_{abs})$$

is self-adjoint and $\mathcal{B}$ differs from D by a bounded perturbation. By a Neumann-series argument and the asymptotics of the resolvent for D (see [Se1, Lemma 15]) we get:

$$\forall \theta \in (-\pi, 0): \quad \text{Spec}(\mathcal{B}) \cap R_\theta \quad \text{is finite.} \tag{6.12}$$

By discreteness of $\mathcal{B}$ we deduce that if $\theta, \theta' \in (-\pi, 0)$ are both Agmon angles for $\mathcal{B}^{(\lambda,\infty)}$, there are only finitely many eigenvalues of $\mathcal{B}^{(\lambda,\infty)}$ in the solid angle between θ and θ'. Hence

$$\left.\frac{d}{ds}\right|_{s=0} \zeta_{gr,\theta}(s, \mathcal{B}^{(\lambda,\infty)}_{\text{even}})) \equiv \left.\frac{d}{ds}\right|_{s=0} \zeta_{gr,\theta'}(s, \mathcal{B}^{(\lambda,\infty)}_{\text{even}})) \mod 2\pi i,$$
$$\text{and therefore} \quad \det_{gr,\theta}(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}) = \det_{gr,\theta'}(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}).$$

This proves independence of the choice of $\theta \in (-\pi, 0)$ and completes the proof. □

The element $\rho(\nabla, g^M, h^E)$ is well-defined but a priori not independent of the choice of metrics g^M, h^E and so does not provide a differential invariant. In the next subsection we determine the metric anomaly of $\rho(\nabla, g^M, h^E)$ in order to construct a differential invariant, which will be called the refined analytic torsion.

6.3 Metric Anomaly and Refined Analytic Torsion

We introduce the notion of the eta-function leading to the notion of the eta-invariant of an elliptic operator. The eta-invariant was first introduced by Atiyah-Patodi-Singer in [APS] as the boundary correction term in their index formula.

Theorem 6.18. *[P.B. Gilkey, L. Smith] Let (K, g^K) be a smooth compact oriented Riemannian manifold with boundary ∂K. Let (F, h^F) be a Hermitian vector bundle and let the metric structures (g^K, h^F) define an L^2-scalar product. Let*

$$D : C^\infty(K, F) \to C^\infty(K, F)$$

be a differential operator of order ω such that $\omega \cdot \mathrm{rank} F$ is even. Let a boundary value problem (D, B) be strongly elliptic with respect to $\mathbb{C} \backslash \mathbb{R}^*$ and an Agmon angle $\theta \in (-\pi, 0)$. Then we have

(i) *D_B is a discrete Fredholm operator in the Hilbert space $L^2(K, F)$ and its eta-function*

$$\eta_\theta(s, D_B) := \sum_{\mathrm{Re}(\lambda)>0} m(\lambda) \cdot \lambda_\theta^{-s} - \sum_{\mathrm{Re}(\lambda)<0} m(\lambda) \cdot (-\lambda)_\theta^{-s},$$

where $m(\lambda)$ denotes the finite (algebraic) multiplicity of the eigenvalue λ , is holomorphic for Re(s) large and extends meromorphically to $\mathbb{C}$ with at most simple poles.

(ii) *If D is of order one with the leading symbol $\sigma_D(x, \xi), x \in K, \xi \in T_x^* K$ satisfying*

$$\sigma_D(x, \xi)^2 = |\xi|^2 \cdot I,$$

where I is $\mathrm{rank} F \times \mathrm{rank} F$ identity matrix, and the boundary condition B is of order zero, then the meromorphic extension of $\eta_\theta(s, D_B)$ is regular at $s = 0$.

The proof of the theorem follows from the results in [GS1] and [GS2] on the eta-function of strongly elliptic boundary value problems. The fact that $\eta_\theta(s, D_B)$ is holomorphic for $\mathrm{Re}(s)$ sufficiently large is asserted in [GS1, Lemma 2.3 (c)]. The meromorphic continuation with at most isolated simple poles is asserted in [GS1, Theorem 2.7].

The fact that $s = 0$ is a regular point of the eta-function is highly non-trivial and cannot be proved by local arguments. Using homotopy invariance of the residue at zero for the eta-function, P. Gilkey and L. Smith [GS2] reduced the discussion to a certain class of operators with constant coefficients in the collar neighborhood of the boundary and applied the closed double manifold argument. The reduction works for differential operators of order one with 0-th order boundary conditions under the assumption on the leading symbol of the operator as in the second statement of the theorem. The regularity statement of Theorem 6.18 follows directly from [GS2, Theorem 2.3.5] and [GS2, Lemma 2.3.4].

Remark 6.19. *The definition of an eta-function, as in Theorem 6.18 (i), also applies to any operator D with finite spectrum $\{\lambda_1, .., \lambda_n\}$ and finite* *respective multiplicities* $\{m_1, .., m_n\}$. *For a given Agmon angle* $\theta \in [0, 2\pi)$ *the associated eta-function*

$$\eta_\theta(s, D) := \sum_{\mathrm{Re}(\lambda)>0} m(\lambda) \cdot \lambda_\theta^{-s} - \sum_{\mathrm{Re}(\lambda)<0} m(\lambda) \cdot (-\lambda)_\theta^{-s},$$

is holomorphic for all $s \in \mathbb{C}$, *since the sum is finite and the zero-eigenvalue is excluded.*

Proposition 6.20. *The eta-function* $\eta_\theta(s, \mathcal{B}_{\mathrm{even}})$ *associated to the even part* $\mathcal{B}_{\mathrm{even}}$ *of the odd-signature operator and its Agmon angle* $\theta \in (-\pi, 0)$, *is holomorphic for Re*(s) *large and extends meromorphically to* $\mathbb{C}$ *with* $s = 0$ *being a regular point.*

The statement of the proposition on the meromorphic extension of the eta-function is a direct consequence of Theorem 6.18 (i) and Proposition 6.11. The regularity statement follows from Theorem 6.18 (ii) and an explicit computation of the leading symbol of the odd-signature operator, compare also [GS2, Example 2.2.4].

Using Proposition 6.20 we can define the eta-invariant in the manner of [BK2] for $\mathcal{B}_{\mathrm{even}}$:

$$\eta(\mathcal{B}_{\mathrm{even}}) := \frac{1}{2}\left(\eta_\theta(s = 0, \mathcal{B}_{\mathrm{even}}) + m_+ - m_- + m_0\right), \tag{6.13}$$

where $m_\pm$ is the number of $\mathcal{B}_{\mathrm{even}}-$eigenvalues on the positive, respectively the negative part of the imaginary axis and m_0 is the dimension of the generalized zero-eigenspace of $\mathcal{B}_{\mathrm{even}}$.

Implicit in the notation is also the fact, that $\eta(\mathcal{B}_{\mathrm{even}})$ does not depend on the Agmon angle $\theta \in (-\pi, 0)$. This is due to the fact that, given a different Agmon angle $\theta' \in (-\pi, 0)$, there are by (6.12) and discreteness of $\mathcal{B}$ only finitely many eigenvalues of $\mathcal{B}_{\mathrm{even}}$ in the acute angle between θ and θ'.

Similarly we define the eta-invariants of $\mathcal{B}_{\mathrm{even}}^{(\lambda,\infty)}$ and $\mathcal{B}_{\mathrm{even}}^{[0,\lambda]}$ and in particular we get

$$\eta(\mathcal{B}_{\mathrm{even}}) = \eta(\mathcal{B}_{\mathrm{even}}^{(\lambda,\infty)}) + \eta(\mathcal{B}_{\mathrm{even}}^{[0,\lambda]}).$$

Before we prove the next central result, let us make the following observation.

Consider the imaginary axis $i\mathbb{R} \subset \mathbb{C}$. By (6.12) there are only finitely many eigenvalues of $\mathcal{B}$ on $i\mathbb{R}$. Further by the discreteness of $\mathcal{B}$ small rotation of the imaginary axis does not hit any further eigenvalue of $\mathcal{B}$ and in particular of

$\mathcal{B}_{\text{even}}^{(\lambda,\infty)}, \lambda \geq 0$. More precisely this means that there exists an $\epsilon > 0$ sufficiently small such that the angle

$$\theta := -\frac{\pi}{2} + \epsilon$$

is an Agmon angle for $\mathcal{B}_{\text{even}}^{(\lambda,\infty)}$ and the solid angles

$$L_{(-\pi/2,\theta]} := \{z \in \mathbb{C} | z = |z| \cdot e^{i\phi}, \phi \in (-\pi/2, \theta]\},$$
$$L_{(\pi/2,\theta+\pi]} := \{z \in \mathbb{C} | z = |z| \cdot e^{i\phi}, \phi \in (\pi/2, \theta + \pi]\}$$

do not contain eigenvalues of $\mathcal{B}_{\text{even}}^{(\lambda,\infty)}$. With this observation we can state the following central result:

Theorem 6.21. *Let $\theta \in (-\pi/2, 0)$ be an Agmon angle for $\mathcal{B}_{\text{even}}^{(\lambda,\infty)}$ such that there are no eigenvalues of $\mathcal{B}_{\text{even}}^{(\lambda,\infty)}$ in the solid angles $L_{(-\pi/2,\theta]}$ and $L_{(-\pi/2,\theta+\pi]}$. Then 2θ is an Agmon angle for $(\mathcal{B}_{\text{even}}^{(\lambda,\infty)})^2$. Then the graded zeta-function $\zeta_{gr,\theta}(s, \mathcal{B}_{\text{even}}^{(\lambda,\infty)}), Re(s) \gg 0$ extends meromorphically to $\mathbb{C}$ and is regular at $s = 0$ with the following derivative at zero:*

$$\left.\frac{d}{ds}\right|_{s=0} \zeta_{gr,\theta}(s, \mathcal{B}_{\text{even}}^{(\lambda,\infty)})) = \frac{1}{2}\sum_{k=0}^{m}(-1)^{k+1} \cdot k \cdot \left.\frac{d}{ds}\right|_{s=0} \zeta_{2\theta}(s, \mathcal{B}^2 \restriction \widetilde{\mathcal{D}}^k_{(\lambda,\infty)}) + \\ + \frac{i\pi}{2}\sum_{k=0}^{m}(-1)^k \cdot k \cdot \zeta_{2\theta}(0, \mathcal{B}^2 \restriction \widetilde{\mathcal{D}}^k_{(\lambda,\infty)}) + i\pi\eta(\mathcal{B}_{\text{even}}^{(\lambda,\infty)}).$$

Proof. For $Re(s) \gg 0$ the general identities [BK1 (4.10), (4.11)] imply the following relation between holomorphic functions:

$$\zeta_{gr,\theta}(s, \mathcal{B}_{\text{even}}^{(\lambda,\infty)})) = \frac{1 + e^{-i\pi s}}{2}\left[\zeta_{2\theta}\left(\frac{s}{2}, \left(\mathcal{B}_{\text{even}}^{+,(\lambda,\infty)}\right)^2\right) - \zeta_{2\theta}\left(\frac{s}{2}, \left(\mathcal{B}_{\text{even}}^{-,(\lambda,\infty)}\right)^2\right)\right] + \\ + \frac{1}{2}(1 - e^{-i\pi s})\left[\eta(s, \mathcal{B}_{\text{even}}^{(\lambda,\infty)}) + f(s)\right],$$

where $f(s)$ is a holomorphic function (combination of zeta-functions associated to finite-dimensional operators) with

$$f(0) = m_+(\mathcal{B}_{\text{even}}^{(\lambda,\infty)}) - m_-(\mathcal{B}_{\text{even}}^{(\lambda,\infty)}),$$

where $m_\pm(\cdot)$ denotes the number of eigenvalues of the operator in brackets, lying on the positive, respectively the negative part of the imaginary axis.

Put $\mathcal{I} = (\lambda, \infty)$ to simplify notation. Recall (6.10) and show that

$$\widetilde{\nabla}_{\mathcal{I}} : \ker(\widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}) \rightarrow \ker(\widetilde{\Gamma}_{\mathcal{I}}\widetilde{\nabla}_{\mathcal{I}}) = \text{Im}(\widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}) \tag{6.14}$$

is bijective. Indeed, injectivity is clear by (6.8). For surjectivity let $x = \widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}v \in \mathrm{Im}(\widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}})$ with (recall (6.11))

$$v = v' \oplus v'' \in \mathrm{Im}(\widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}) \oplus \mathrm{Im}(\widetilde{\Gamma}_{\mathcal{I}}\widetilde{\nabla}_{\mathcal{I}}) = \mathrm{Im}(1 - \Pi_{\mathcal{B}^2,[0,\lambda]}).$$

In particular $v'' \in \mathrm{Im}(\widetilde{\Gamma}_{\mathcal{I}}\widetilde{\nabla}_{\mathcal{I}}) = \ker \widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}$ and $v' = \widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}\omega$ for some ω. Hence we obtain

$$x = \widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}v = \widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}v' = \widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}\widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}\omega,$$
$$\text{and} \quad \widetilde{\Gamma}_{\mathcal{I}}\widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}\omega \in \ker \widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}.$$

In other words we have found a preimage of any $x \in \mathrm{Im}(\widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}})$ under $\widetilde{\nabla}_{\mathcal{I}}$. This proves bijectivity of the map in (6.14) and consequently, since $\widetilde{\nabla}_{\mathcal{I}}$ commutes with $\mathcal{B}^{\mathcal{I}}$ and $(\mathcal{B}^{\mathcal{I}})^2$, we obtain in any degree $k = 0, .., m$

$$\zeta_{2\theta}(s, (\mathcal{B}^{+,\mathcal{I}})^2 \restriction \widetilde{\mathcal{D}}^k) = \zeta_{2\theta}(s, (\mathcal{B}^{-,\mathcal{I}})^2 \restriction \widetilde{\mathcal{D}}^{k+1}). \tag{6.15}$$

Using this relation we compute straightforwardly for $Re(s)$ sufficiently large:

$$\zeta_{2\theta}(s, (\mathcal{B}^{+,\mathcal{I}}_{\mathrm{even}})^2) - \zeta_{2\theta}(s, (\mathcal{B}^{-,\mathcal{I}}_{\mathrm{even}})^2) = \sum_{k=0}^{m}(-1)^{k+1} \cdot k \cdot \zeta_{2\theta}(s, (\mathcal{B}^{\mathcal{I}})^2 \restriction \widetilde{\mathcal{D}}^k). \tag{6.16}$$

We arive at the following preliminary result for $Re(s) \gg 0$

$$\begin{aligned}\zeta_{gr,\theta}(s, \mathcal{B}^{\mathcal{I}}_{\mathrm{even}})) = \frac{1}{2}(1 + e^{-i\pi s}) \sum_{k=0}^{m}(-1)^{k+1} \cdot k \cdot \zeta_{2\theta}(s, (\mathcal{B}^{\mathcal{I}})^2 \restriction \widetilde{\mathcal{D}}^k) + \\ + \frac{1}{2}(1 - e^{-i\pi s}) \left[\eta(s, \mathcal{B}^{\mathcal{I}}_{\mathrm{even}}) + f(s)\right].\end{aligned} \tag{6.17}$$

We find with Theorem 6.8 and Proposition 6.20 that the right hand side of the equality above is a meromorphic function on the entire complex plane and is regular at $s = 0$. Hence the left hand side of the equality, the graded zeta-function, is meromorphic on $\mathbb{C}$ and regular at $s = 0$, as claimed and as anticipated in Definition 6.16. Computing the derivative at zero, we obtain the statement of the theorem. □

As a consequence of the theorem above, we obtain for the element $\rho(\nabla, g^M, h^E)$ defined in Proposition 6.17 the following relation

$$\rho(\nabla, g^M, h^E) = e^{\xi_\lambda(\nabla, g^M)} e^{-i\pi\xi'_\lambda(\nabla, g^M)} e^{-i\pi\eta(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}(g^M))} \cdot \rho_{[0,\lambda]}, \tag{6.18}$$

$$\xi_\lambda(\nabla, g^M) = \frac{1}{2}\sum_{k=0}^{m}(-1)^k \cdot k \cdot \frac{d}{ds}\bigg|_{s=0} \zeta_{2\theta}(s, (\mathcal{B}^2 \restriction \widetilde{\mathcal{D}}^k_{(\lambda,\infty)})) \tag{6.19}$$

$$\xi'_\lambda(\nabla, g^M) = \frac{1}{2}\sum_{k=0}^{m}(-1)^k \cdot k \cdot \zeta_{2\theta}(s=0, (\mathcal{B}^2 \restriction \widetilde{\mathcal{D}}^k_{(\lambda,\infty)})). \tag{6.20}$$

Now we can identify explicitly the metric dependence of $\rho(\nabla, g^M, h^E)$ using the formula (6.18).

First note that the construction is in fact independent of the choice of a Hermitian metric h^E. Indeed, a variation of h^E does not change the odd-signature operator $\mathcal{B}$ as a differential operator. However it enters a priori the definition of $\mathcal{D}(\mathcal{B})$, since h^E defines the L^2-Hilbert space.

Recall that different Hermitian metrics give rise to equivalent L^2-norms over compact manifolds. Hence a posteriori the domain $\mathcal{D}(\mathcal{B})$ is indeed independent of the particular choice of h^E.

Independence of the choice of a Hermitian metric h^E is essential, since for non-unitary flat vector bundles there is no canonical choice of h^E and Hermitian metric is fixed arbitrarily.

Consider a smooth family $g^M(t), t \in \mathbb{R}$ of Riemannian metrics on M. Denote by $\widetilde{\Gamma}_t$ the corresponding chirality operator in the sence of Definition 6.2 and denote the associated refined torsion (recall (6.7)) of the complex $(\widetilde{\mathcal{D}}_{t,[0,\lambda]}, \widetilde{\nabla}_{t,[0,\lambda]})$ by $\rho_{t,[0,\lambda]}$.

Let $\mathcal{B}(t) = \mathcal{B}(\nabla, g^M(t))$ be the odd-signature operator corresponding to the Riemannian metric $g^M(t)$. Fix $t_0 \in \mathbb{R}$ and choose $\lambda \geq 0$ such that there are no eigenvalues of $\mathcal{B}(t_0)^2$ of absolute value λ. Then there exists $\delta > 0$ small enough such that the same holds for the spectrum of $\mathcal{B}(t)^2$ for $|t - t_0| < \delta$. Under this setup we obtain:

Proposition 6.22. *Let the family $g^M(t)$ vary only in a compact subset of the interior of M. Then $\exp(\xi_\lambda(\nabla, g^M(t))) \cdot \rho_{t,[0,\lambda]}$ is independent of $t \in (t_0 - \delta, t_0 + \delta)$.*

Proof. The arguments of [BK2, Lemma 9.2] are of local nature and transfer ad verbatim to the present situation for metric variations in the interior of the manifold. Hence the assertion follows for Riemannian metric remaining fixed in an open neighborhood of the boundary. □

Proposition 6.23. *Denote the trivial connection on the trivial line bundle $M \times \mathbb{C}$ by ∇_{trivial}. Consider the even part of the associated odd-signature operator (recall Definition 6.5)*

$$\mathcal{B}_{\text{trivial}} = \mathcal{B}_{\text{even}}(\nabla_{\text{trivial}}).$$

Indicate the metric dependence by $\mathcal{B}_{\text{trivial}}(t) := \mathcal{B}_{\text{trivial}}(g^M)$. Then

$$\eta(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}(t)) - \text{rank}(E)\eta(\mathcal{B}_{\text{trivial}}(t)) \mod \mathbb{Z}$$

is independent of $t \in (t_0 - \delta, t_0 + \delta)$.

Proof. Indicate the dependence of $\widetilde{\mathcal{D}}^*_{[0,\lambda]}$ on $g^M(t)$ by

$$\widetilde{\mathcal{D}}^k_{[0,\lambda]}(t) := \text{Image}\, \Pi_{\mathcal{B}(t)^2,[0,\lambda]} \cap \widetilde{\mathcal{D}}^k.$$

Note first the by the choice of $\delta > 0$

$$\dim \widetilde{\mathcal{D}}^k_{[0,\lambda]}(t) = \text{const}, \quad t \in (t_0 - \delta, t_0 + \delta).$$

Since $\mathcal{B}^{[0,\lambda]}_{\text{even}}(t)$ is finite-dimensional, we infer from the definition of the eta-invariant (cf. [BK2, (9.11)])

$$\eta(\mathcal{B}^{[0,\lambda]}_{\text{even}}(t)) \equiv \frac{1}{2} \dim \widetilde{\mathcal{D}}^k_{[0,\lambda]}(t) \equiv \text{const} \mod \mathbb{Z}, \quad t \in (t_0 - \delta, t_0 + \delta). \quad (6.21)$$

By construction

$$\eta(\mathcal{B}_{\text{even}}(t)) = \eta(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}(t)) + \eta(\mathcal{B}^{[0,\lambda]}_{\text{even}}(t)).$$

Hence, in view of (6.21), it suffices (modulo $\mathbb{Z}$) to study the metric dependence of the eta-invariant of $\eta(\mathcal{B}_{\text{even}}(t))$.

View $\mathcal{B}_{\text{even}}(t)$ as a pair of a differential operator $P_E(t)$ with its boundary conditions $Q_E(t)$. Similarly view $\mathcal{B}_{\text{trivial}}(t)$ as a pair $(P_{\mathbb{C}}(t), Q_{\mathbb{C}}(t))$. Note that by construction the pair $(P_E(t), Q_E(t))$ is locally isomorphic to $(P_{\mathbb{C}}(t), Q_{\mathbb{C}}(t)) \times \mathbf{1}^k$, since the flat connection ∇ is locally trivial in appropriate local trivializations.

Since the variation of the eta-invariants is computed from the local information of the symbols (cf. [GS1, Theorem 2.8, Lemma 2.9]), we find that the difference

$$\begin{aligned} &\eta(\mathcal{B}_{\text{even}}(t)) - \text{rank}(E)\eta(\mathcal{B}_{\text{trivial}}(t)) = \\ &= \eta(P_E(t), Q_E(t)) - \text{rank}(E)\eta(P_{\mathbb{C}}(t), Q_{\mathbb{C}}(t)) \end{aligned}$$

is independent of $t \in \mathbb{R}$ modulo $\mathbb{Z}$. The modulo $\mathbb{Z}$ reduction is needed to annihilate discontinuity jumps arising from eigenvalues crossing the imaginary axis. This proves the statement of the proposition. □

Proposition 6.24. *Let $\mathcal{B}(\nabla_{\text{trivial}})$ denote the odd-signature operator (Definition 6.5) associated to the trivial line bundle $M \times \mathbb{C}$ with the trivial connection ∇_{trivial}. Consider in correspondence to* (6.20) *the expression*

$$\xi'(\nabla_{\text{trivial}}, g^M(t)) = \frac{1}{2}\sum_{k=0}^{m}(-1)^k \cdot k \cdot \zeta_{2\theta}(s=0, (\mathcal{B}(\nabla_{\text{trivial}}, g^M(t))^2 \upharpoonright \widetilde{\mathcal{D}}^k).$$

Then

$$\xi'_\lambda(\nabla, g^M(t)) - \operatorname{rank}(E) \cdot \xi'(\nabla_{\text{trivial}}, g^M(t)) \mod \mathbb{Z}$$

is independent of $t \in \mathbb{R}$.

Proof. We show first that modulo $\mathbb{Z}$ it suffices to study the metric dependence of

$$\xi'(\nabla, g^M(t)) := \frac{1}{2}\sum_{k=0}^{m}(-1)^k \cdot k \cdot \zeta_{2\theta}(s=0, (\mathcal{B}(\nabla, g^M(t))^2 \upharpoonright \widetilde{\mathcal{D}}^k).$$

Indeed, by construction we have

$$\xi'(\nabla, g^M(t)) = \xi'_\lambda(\nabla, g^M(t)) + \frac{1}{2}\sum_{k=0}^{m}(-1)^k \cdot k \cdot \dim \widetilde{\mathcal{D}}^k_{[0,\lambda]}(t).$$

Anticipating the auxiliary result of Lemma 6.25 (iii) below, we obtain

$$\xi'(\nabla, g^M(t)) \equiv \xi'_\lambda(\nabla, g^M(t)) \mod \mathbb{Z}.$$

Recall that $\mathcal{B}(\nabla_{\text{trivial}}, g^M) \times \mathbf{1}^{\text{rk}E}$ and $\mathcal{B}(\nabla, g^M)$ are locally isomorphic, as already encountered in the proof of Proposition 6.23. Now the statement of the proposition follows from the fact that the value of a zeta function at zero is given, modulo $\mathbb{Z}$ in order to avoid $\dim\ker\mathcal{B}(t) \in \mathbb{Z}$, by integrands of local invariants of the operator and its boundary conditions. □

Lemma 6.25. *Let $\mathcal{I} \subset \mathbb{R}$ denote any bounded intervall. Then*

(i) $\frac{1}{2}\sum_{k=0}^{m}(-1)^{k+1} \cdot k \cdot \dim \widetilde{\mathcal{D}}^k_{\mathcal{I}} \equiv \frac{\dim M}{2} \dim \widetilde{\mathcal{D}}^{\text{even}}_{\mathcal{I}} \mod 2\mathbb{Z}$.

(ii) *If $0 \notin \mathcal{I}$, then* $\dim \widetilde{\mathcal{D}}^{\text{even}}_{\mathcal{I}} \equiv 0 \mod 2\mathbb{Z}$,

(iii) *If $0 \notin \mathcal{I}$, then* $\frac{1}{2}\sum_{k=0}^{m}(-1)^{k+1} \cdot k \cdot \dim \widetilde{\mathcal{D}}^k_{\mathcal{I}} \equiv 0 \mod \mathbb{Z}$.

Proof. Note first the following relation

$$\mathcal{B}_k^2 = \widetilde{\Gamma} \circ \mathcal{B}_{m-k}^2 \circ \widetilde{\Gamma}.$$

Hence with $r = (m+1)/2$ we obtain:

$$\frac{1}{2}\sum_{k=0}^{m}(-1)^{k+1} \cdot k \cdot \dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{k} = \frac{1}{2}\sum_{k=0}^{r-1}(m-4k) \cdot \dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{2k} = \tag{6.22}$$

$$= \frac{m}{2} \dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{\text{even}} - 2\sum_{k=0}^{r-1} k \cdot \dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{2k}. \tag{6.23}$$

This proves the first statement. For the second statement assume $0 \notin \mathcal{I}$ till the end of the proof. Consider the operators

$$\mathcal{B}_k^{+,\mathcal{I}} = \widetilde{\Gamma}_{\mathcal{I}}\widetilde{\nabla}_{\mathcal{I}} : \widetilde{\mathcal{D}}_{\mathcal{I}}^{k} \cap \ker(\widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}) \to \widetilde{\mathcal{D}}_{\mathcal{I}}^{m-k-1} \cap \ker(\widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}}), \tag{6.24}$$

$$\mathcal{B}_k^{-,\mathcal{I}} = \widetilde{\nabla}_{\mathcal{I}}\widetilde{\Gamma}_{\mathcal{I}} : \widetilde{\mathcal{D}}_{\mathcal{I}}^{k} \cap \ker(\widetilde{\Gamma}_{\mathcal{I}}\widetilde{\nabla}_{\mathcal{I}}) \to \widetilde{\mathcal{D}}_{\mathcal{I}}^{m-k+1} \cap \ker(\widetilde{\Gamma}_{\mathcal{I}}\widetilde{\nabla}_{\mathcal{I}}). \tag{6.25}$$

Since $0 \notin \mathcal{I}$, the maps $\mathcal{B}_k^{\pm,\mathcal{I}}$ are isomorphisms by bijectivity of the map in (6.14). Furthermore they commute with $(\mathcal{B}^{\pm,\mathcal{I}})^2$ in the following way

$$\mathcal{B}_k^{\pm,\mathcal{I}} \circ [(\mathcal{B}^{\pm,\mathcal{I}})^2 \restriction \widetilde{\mathcal{D}}^k] = [(\mathcal{B}^{\pm,\mathcal{I}})^2 \restriction \widetilde{\mathcal{D}}^{m-k\mp 1}] \circ \mathcal{B}_k^{\pm,\mathcal{I}}. \tag{6.26}$$

Hence we obtain with $\widetilde{\mathcal{D}}_{\mathcal{I}}^{\pm,k}$ denoting the span of generalized eigenforms of $(\mathcal{B}^{\pm,\mathcal{I}})^2 \restriction \widetilde{\mathcal{D}}^k$ the following relations

$$\dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{+,k} = \dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{+,m-k-1},$$
$$\dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{-,k} = \dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{-,m-k+1}.$$

Due to $\dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{\text{even}} = \dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{+,\text{even}} + \dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{-,\text{even}}$ this implies (recall M is odd-dimensional)

$$\dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{\text{even}} \equiv \dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{+,2p} \mod 2\mathbb{Z}, \text{if } \dim M = 4p+1, \tag{6.27}$$

$$\dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{\text{even}} \equiv \dim \widetilde{\mathcal{D}}_{\mathcal{I}}^{-,2p} \mod 2\mathbb{Z}, \text{if } \dim M = 4p-1. \tag{6.28}$$

Finally recall the explicit form of $(\mathcal{B}^{\pm})^2$:

$$(\mathcal{B}^+)^2 = \begin{pmatrix} \Gamma\nabla_{\max}\Gamma\nabla_{\min} & 0 \\ 0 & \Gamma\nabla_{\min}\Gamma\nabla_{\max} \end{pmatrix} =: \begin{pmatrix} D_1^+ & 0 \\ 0 & D_2^+ \end{pmatrix},$$

$$(\mathcal{B}^-)^2 = \begin{pmatrix} \nabla_{\min}\Gamma\nabla_{\max}\Gamma & 0 \\ 0 & \nabla_{\max}\Gamma\nabla_{\min}\Gamma \end{pmatrix} =: \begin{pmatrix} D_1^- & 0 \\ 0 & D_2^- \end{pmatrix}.$$

Moreover we put

$$(\mathcal{B}^{\pm,\mathcal{I}})^2 \restriction \widetilde{\mathcal{D}}^k = D^{\pm,\mathcal{I}}_{1,k} \oplus D^{\pm,\mathcal{I}}_{2,k}.$$

Note the following relations

$$\begin{aligned}(\Gamma\nabla_{\min}) \circ D^+_1 &= D^+_2 \circ (\Gamma\nabla_{\min}),\\ D^+_1 \circ (\Gamma\nabla_{\max}) &= (\Gamma\nabla_{\max}) \circ D^+_2;\\ (\nabla_{\max}\Gamma) \circ D^-_1 &= D^-_2 \circ (\nabla_{\max}\Gamma),\\ D^-_1 \circ (\nabla_{\min}\Gamma) &= (\nabla_{\min}\Gamma) \circ D^-_2.\end{aligned}$$

Due to $0 \notin \mathcal{I}$ these relations imply, similarly to (6.26), spectral equivalence of $D^{\pm,\mathcal{I}}_{1,k}$ and $D^{\pm,\mathcal{I}}_{2,k}$ in the middle degree $k = 2p$ for $\dim M = 4p \pm 1$, respectively. This finally yields the desired relations

$$\begin{aligned}\dim \widetilde{\mathcal{D}}^{\text{even}}_{\mathcal{I}} &\equiv \dim \widetilde{\mathcal{D}}^{+,2p}_{\mathcal{I}} \equiv 0 \bmod 2\mathbb{Z}, \text{if } \dim M = 4p+1,\\ \dim \widetilde{\mathcal{D}}^{\text{even}}_{\mathcal{I}} &\equiv \dim \widetilde{\mathcal{D}}^{-,2p}_{\mathcal{I}} \equiv 0 \bmod 2\mathbb{Z}, \text{if } \dim M = 4p-1.\end{aligned}$$

□

Propositions 6.22, 6.23 and 6.24 determine together the metric anomaly of $\rho(\nabla, g^M, h^E)$ up to a sign and we deduce the following central corollary.

Corollary 6.26. *Let M be an odd-dimensional oriented compact Riemannian manifold. Let (E, ∇, h^E) be a flat complex vector bundle over M. Denote by ∇_{trivial} the trivial connection on $M \times \mathbb{C}$ and let $\mathcal{B}_{\text{trivial}}$ denote the even part of the associated odd-signature operator. Then*

$$\rho_{\text{an}}(\nabla) := \rho(\nabla, g^M, h^E) \cdot \exp\left[i\pi \operatorname{rk}(E)(\eta(\mathcal{B}_{\text{trivial}}(g^M)) + \xi'(\nabla_{\text{trivial}}, g^M))\right]$$

is modulo sign independent of the choice of g^M in the interior of M.

In view of the corollary above we can now define the "refined analytic torsion". It will be a differential invariant in the sense, that even though defined by geometric data in form of the metric structures, it is shown to be independent of their form in the interior of the manifold.

Definition 6.27. *Let M be an odd-dimensional oriented Riemannian manifold. Let (E, ∇) be a flat complex vector bundle over M. Then the refined analytic torsion is defined as the equivalence class of $\rho_{\text{an}}(\nabla)$ modulo multiplication by $\exp[i\pi]$:*

$$\rho_{\text{an}}(M, E) := \rho_{\text{an}}(\nabla)/_{e^{i\pi}}.$$

Note that the sign indeterminacy is also present in the original construction by Braverman and Kappeler, see [BK2, Remark 9.9 and Remark 9.10]. In the presentation below, we refer to the representative $\rho_{\text{an}}(\nabla)$ of the class $\rho_{\text{an}}(M, E)$ as refined analytic torsion, as well.

6.4 Ray-Singer norm of Refined analytic torsion

Recall first the construction of the Ray-Singer torsion as a norm on the determinant line bundle for compact oriented Riemannian manifolds. Let (M, g^M) and (E, ∇, h^E) be as in Subsection 6.2.

Let $\triangle_{\text{rel}}$ be the Laplacian associated to the Fredholm complex $(\mathcal{D}_{\min}, \nabla_{\min})$ defined at the beginning of Section 6.2. As in (6.5) in case of the squared odd-signature operator $\mathcal{B}^2$, it induces a spectral decomposition into a direct sum of subcomplexes for any $\lambda \geq 0$.

$$(\mathcal{D}_{\min}, \nabla_{\min}) = (\mathcal{D}_{\min}^{[0,\lambda]}, \nabla_{\min}^{[0,\lambda]}) \oplus (\mathcal{D}_{\min}^{(\lambda,\infty)}, \nabla_{\min}^{(\lambda,\infty)}).$$

The scalar product on $\mathcal{D}_{\min}^{[0,\lambda]}$ induced by g^M and h^E, induces a norm on the determinant line $\mathrm{Det}(\mathcal{D}_{\min}^{[0,\lambda]}, \nabla_{\min}^{[0,\lambda]})$ (we use the notation of determinant lines of finite dimensional complexes in [BK2, Section 1.1]). There is a canonical isomorphism

$$\phi_\lambda : \mathrm{Det}(\mathcal{D}_{\min}^{[0,\lambda]}, \nabla_{\min}^{[0,\lambda]}) \to \mathrm{Det} H^*(\mathcal{D}_{\min}, \nabla_{\min}),$$

induced by the Hodge-decomposition in finite-dimensional complexes. Choose on $\mathrm{Det} H^*(\mathcal{D}_{\min}, \nabla_{\min})$ the norm $\| \cdot \|_\lambda^{\text{rel}}$ such that ϕ_λ becomes an isometry. Further denote by $T^{RS}_{(\lambda,\infty)}(\nabla_{\min})$ the scalar analytic torsion associated to the complex $(\mathcal{D}_{\min}^{(\lambda,\infty)}, \nabla_{\min}^{(\lambda,\infty)})$:

$$T^{RS}_{(\lambda,\infty)}(\nabla_{\min}) := \exp\left(\frac{1}{2} \sum_{k=1}^{m} (-1)^{k+1} \cdot k \cdot \zeta'(s=0, \triangle_{k,\text{rel}}^{(\lambda,\infty)}) \right),$$

where $\triangle_{\text{rel}}^{(\lambda,\infty)}$ is the Laplacian associated to the complex $(\mathcal{D}_{\min}^{(\lambda,\infty)}, \nabla_{\min}^{(\lambda,\infty)})$. Note the difference to the sign convention of [RS]. However we are consistent with [BK2].

The Ray-Singer norm on $\mathrm{Det} H^*(\mathcal{D}_{\min}, \nabla_{\min})$ is then defined by

$$\| \cdot \|^{RS}_{\mathrm{Det} H^*(\mathcal{D}_{\min}, \nabla_{\min})} := \| \cdot \|_\lambda^{\text{rel}} \cdot T^{RS}_{(\lambda,\infty)}(\nabla_{\min}). \tag{6.29}$$

With a completely analogous construction we obtain the Ray-Singer norm on the determinant line $\mathrm{Det} H^*(\mathcal{D}_{\max}, \nabla_{\max})$

$$\| \cdot \|^{RS}_{\mathrm{Det} H^*(\mathcal{D}_{\max}, \nabla_{\max})} := \| \cdot \|_\lambda^{\text{abs}} \cdot T^{RS}_{(\lambda,\infty)}(\nabla_{\max}). \tag{6.30}$$

Both constructions turn out to be independent of the choice of $\lambda \geq 0$, which follows from arguments analogous to those in the proof of Proposition 6.17. In fact we get for $0 \leq \lambda < \mu$:

$$\| \cdot \|_\mu^{\text{rel/abs}} = \| \cdot \|_\lambda^{\text{rel/abs}} \cdot T^{RS}_{(\lambda,\mu]}(\nabla_{\min/\max}),$$

which implies that the Ray-Singer norms are well-defined. Furthermore by the arguments in [Mu, Theorem 2.6] the norms do not depend on the metric structures in the interior of the manifold.

Remark 6.28. *Note that the Ray-Singer analytic torsion considered in [V] and [Lü] differs from our setup in the sign convention and by the absence of factor* 1/2*.*

We can apply the same construction to the Laplacian of the complex $(\widetilde{\mathcal{D}}, \widetilde{\nabla})$ introduced in Definition 6.4

$$(\widetilde{\mathcal{D}}, \widetilde{\nabla}) = (\mathcal{D}_{\min}, \nabla_{\min}) \oplus (\mathcal{D}_{\max}, \nabla_{\max}).$$

Similarly we obtain

$$\| \cdot \|^{RS}_{\mathrm{Det} H^*(\widetilde{\mathcal{D}}, \widetilde{\nabla})} := \| \cdot \|_\lambda \cdot T^{RS}_{(\lambda,\infty)}(\widetilde{\nabla}). \tag{6.31}$$

This "doubled" Ray-Singer norm is naturally related to the previous two norms in (6.29) and (6.30). There is a canonical "fusion isomorphism", cf. [BK2, (2.18)] for general complexes of finite dimensional vector spaces

$$\mu : \mathrm{Det} H^*(\mathcal{D}_{\min}, \nabla_{\min}) \oplus \mathrm{Det} H^*(\mathcal{D}_{\max}, \nabla_{\max}) \to \mathrm{Det} H^*(\widetilde{\mathcal{D}}, \widetilde{\nabla}),$$
$$\text{such that } \|\mu(h_1 \otimes h_2)\|_\lambda = \|h_1\|^{\mathrm{rel}}_\lambda \cdot \|h_2\|^{\mathrm{abs}}_\lambda, \tag{6.32}$$

where we recall $(\widetilde{\mathcal{D}}, \widetilde{\nabla}) = (\mathcal{D}_{\min}, \nabla_{\min}) \oplus (\mathcal{D}_{\max}, \nabla_{\max})$ by definition. Further we have by the definition of $(\widetilde{\mathcal{D}}, \widetilde{\nabla})$ following relation between the scalar analytic torsions:

$$T^{RS}_{(\lambda,\infty)}(\widetilde{\nabla}) = T^{RS}_{(\lambda,\infty)}(\nabla_{\min}) \cdot T^{RS}_{(\lambda,\infty)}(\nabla_{\max}). \tag{6.33}$$

Combining (6.32) and (6.33) we end up with a relation between norms

$$\|\mu(h_1 \otimes h_2)\|^{RS}_{\mathrm{Det} H^*(\widetilde{\mathcal{D}}, \widetilde{\nabla})} = \|h_1\|^{RS}_{\mathrm{Det} H^*(\mathcal{D}_{\min}, \nabla_{\min})} \cdot \|h_2\|^{RS}_{\mathrm{Det} H^*(\mathcal{D}_{\max}, \nabla_{\max})}. \tag{6.34}$$

The next theorem provides a motivation for viewing $\rho_{\mathrm{an}}(\nabla)$ as a refinement of the Ray-Singer torsion.

Theorem 6.29. *Let M be a smooth compact odd-dimensional oriented Riemannian manifold. Let (E, ∇, h^E) be a flat complex vector bundle over M with a flat Hermitian metric h^E. Then*

$$\|\rho_{\mathrm{an}}(\nabla)\|^{RS}_{\mathrm{Det} H^*(\widetilde{\mathcal{D}}, \widetilde{\nabla})} = 1.$$

Proof. Recall from the assertion of Theorem 6.21

$$\det_{gr}(\mathcal{B}_{\mathrm{even}}^{(\lambda,\infty)}) = e^{\xi_\lambda(\nabla,g^M)} \cdot e^{-i\pi\xi'_\lambda(\nabla,g^M)} \cdot e^{-i\pi\eta(\mathcal{B}_{\mathrm{even}})},$$

Flatness of h^E implies by construction that $\mathcal{B}^2 = \triangle_{\mathrm{rel}} \oplus \triangle_{\mathrm{abs}}$ and hence

$$\xi_\lambda(\nabla, g^M) = -\log T^{RS}_{(\lambda,\infty)}(\widetilde{\nabla}).$$

Further $\mathcal{B}_{\mathrm{even}}$ is self-adjoint and thus has a real spectrum. Hence $\eta(\mathcal{B}_{\mathrm{even}})$ and $\xi'_\lambda(\nabla, g^M)$ are real-valued, as well. Thus we derive

$$\left|\det_{gr}(\mathcal{B}_{\mathrm{even}}^{(\lambda,\infty)})\right| = \frac{1}{T^{RS}_{(\lambda,\infty)}(\widetilde{\nabla})}. \tag{6.35}$$

Furthermore we know from [BK2, Lemma 4.5], which is a general result for complexes of finite-dimensional vector spaces,

$$\|\rho_{[0,\lambda]}\|_\lambda = 1. \tag{6.36}$$

Now the assertion follows by combining the definition of the refined analytic torsion with (6.35), (6.36) and the fact that the additional terms annihilating the metric anomaly are all of norm one. In fact we have:

$$\|\rho_{\mathrm{an}}(\nabla)\|^{RS}_{\mathrm{Det}H^*(\widetilde{\mathcal{D}},\widetilde{\nabla})} = \left|\det_{gr}(\mathcal{B}_{\mathrm{even}}^{(\lambda,\infty)})\right| \cdot T^{RS}_{(\lambda,\infty)}(\widetilde{\nabla}) \cdot \|\rho_{[0,\lambda]}\|_\lambda = 1.$$

□

If the Hermitian metric is not flat, the situation becomes harder. In the setup of closed manifolds M. Braverman and T. Kappeler performed a deformation procedure in [BK2, Section 11] and proved in this way the relation between the Ray-Singer norm and the refined analytic torsion in [BK2, Theorem 11.3].

Unfortunately the deformation argument is not local and the arguments in [BK2] do not apply in the setup of manifolds with boundary. Nevertheless we can derive appropriate result by relating our discussion to the closed double manifold.

Assume the metric structures (g^M, h^E) to be product near the boundary ∂M. The issues related to the product structures are discussed in detail in [BLZ, Section 2]. More precisely, we identify using the inward geodesic flow a collar neighborhood $U \subset M$ of the boundary ∂M diffeomorphically with $[0,\epsilon) \times \partial M, \epsilon > 0$. Explicitly we have the diffeomorphism

$$\begin{aligned} \phi^{-1} : [0,\epsilon) \times \partial M &\to U, \\ (t,p) &\mapsto \gamma_p(t), \end{aligned}$$

where γ_p is the geodesic flow starting at $p \in \partial M$ and $\gamma_p(t)$ is the geodesics from p of length $t \in [0, \epsilon)$. The metric g^M is product near the boundary, if over U it is given under the diffeomorphism $\phi : U \to [0, \epsilon) \times \partial M$ by

$$\phi_* g^M|_U = dx^2 \oplus g^M|_{\partial M}. \tag{6.37}$$

The diffeomorphism $U \cong [0, \epsilon) \times \partial M$ shall be covered by a bundle isomorphism $\widetilde{\phi} : E|_U \to [0, \epsilon) \times E|_{\partial M}$. The fiber metric h^E is product near the boundary, if it is preserved by the bundle isomorphism, i.e.

$$\widetilde{\phi}_* h^E|_{\{x\} \times \partial M} = h^E|_{\partial M}. \tag{6.38}$$

The assumption of product structures guarantees that the closed double manifold

$$\mathbb{M} = M \cup_{\partial M} M$$

is a smooth closed Riemannian manifold and the Hermitian vector bundle (E, h^E) extends to a smooth Hermitian vector bundle $(\mathbb{E}, h^{\mathbb{E}})$ over the manifold $\mathbb{M}$.

Moreover we assume the flat connection ∇ on E to be in *temporal gauge*. The precise definition of a connection in temporal gauge and the proof of the fact that each flat connection is gauge-equivalent to a flat connection in temporal gauge, are provided in Subsection 7.2.

The assumption on ∇ to be a flat connection in temporal gauge is required in the present context to guarantee that ∇ extends to a smooth flat connection $\mathbb{D}$ on $\mathbb{E}$, with

$$\mathbb{D}|_M = \nabla.$$

Theorem 6.30. *Let (M^m, g^M) be an odd-dimensional oriented and compact smooth Riemannian manifold with boundary ∂M. Let (E, ∇, h^E) be a flat Hermitian vector bundle with the Hermitian metric h^E, not necessarily flat.*

Assume the metric structures (g^M, h^E) to be product and the flat connection ∇ to be in temporal gauge near the boundary ∂M. Then

$$\|\rho_{\mathrm{an}}(\nabla)\|^{RS}_{\det H^*(\widetilde{\mathcal{D}}, \widetilde{\nabla})} = \exp[\pi \mathrm{Im}\, \eta(\mathcal{B}_{\mathrm{even}}(g^M))].$$

Proof. By assumption we obtain a closed Riemannian double manifold $(\mathbb{M}, g^{\mathbb{M}})$ and a flat Hermitian vector bundle $(\mathbb{E}, \mathbb{D}, h^{\mathbb{E}})$ over $\mathbb{M}$ with a flat Hermitian metric $h^{\mathbb{E}}$. Denote by $(\mathcal{D}, \mathbb{D})$ the unique boundary conditions (see [BL1]) of the twisted de Rham complex $(\Omega^*(\mathbb{M}, \mathbb{E}), \mathbb{D})$. Denote the closure of $\Omega^*(\mathbb{M}, \mathbb{E})$ with respect to the L^2-scalar product defined by $g^{\mathbb{M}}$ and $h^{\mathbb{E}}$, by

$L^2_*(\mathbb{M},\mathbb{E})$.

The Riemannian metric $g^{\mathbb{M}}$ gives rise to the Hodge star operator $*$ and we set

$$\mathbb{G} := i^r(-1)^{\frac{k(k+1)}{2}} * : \Omega^k(\mathbb{M},\mathbb{E}) \to \Omega^{k-1}(\mathbb{M},\mathbb{E}), \quad r := (\dim M + 1)/2$$

which extends to a self-adjoint involution on $L^2_*(\mathbb{M},\mathbb{E})$. We define the odd signature operator $\mathbb{B}$ of the Hilbert complex $(\mathcal{D},\mathbb{D})$:

$$\mathbb{B} := \mathbb{G}\mathbb{D} + \mathbb{D}\mathbb{G}.$$

This is precisely the odd-signature operator associated to the closed manifold $\mathbb{M}$, as used in the construction of [BK1, BK2].

Note that we now have two triples: the triple $(\mathbb{D},\mathbb{G},\mathbb{B})$ associated to the closed manifold $\mathbb{M}$ and the triple $(\widetilde{\nabla},\widetilde{\Gamma},\mathcal{B})$ associated to $(M,\partial M)$, as defined in Subsection 6.2.

Consider now the diffeomorphic involution on the closed double

$$\alpha : \mathbb{M} \to \mathbb{M},$$

interchanging the two copies of M. It gives rise to an isomorphism of Hilbert complexes

$$\alpha^* : (\mathcal{D},\mathbb{D}) \to (\mathcal{D},\mathbb{D}),$$

which is an involution as well. We get a decomposition of $(\mathcal{D},\mathbb{D})$ into the (± 1)-eigenspaces of α^*, which form subcomplexes of the total complex:

$$(\mathcal{D},\mathbb{D}) = (\mathcal{D}^+,\mathbb{D}^+) \oplus (\mathcal{D}^-,\mathbb{D}^-), \tag{6.39}$$

where the upper-indices $\pm$ refer to the (± 1)-eigenspaces of α^*, respectively.

The central property of the decomposition, by similar arguments as in [BL1, Theorem 4.1], lies in the following observation

$$\mathcal{D}^+|_M = \mathcal{D}_{\max}, \quad \mathcal{D}^-|_M = \mathcal{D}_{\min}. \tag{6.40}$$

By the symmetry of the elements in $\mathcal{D}^\pm$ we obtain the following natural isomorphism of complexes:

$$\Phi : (\mathcal{D},\mathbb{D}) = (\mathcal{D}^+,\mathbb{D}^+) \oplus (\mathcal{D}^-,\mathbb{D}^-) \to (\mathcal{D}_{\max},\nabla_{\max}) \oplus (\mathcal{D}_{\min},\nabla_{\min}),$$
$$\omega = \omega^+ \oplus \omega^- \mapsto 2\omega^+|_M \oplus 2\omega^-|_M,$$

which extends to an isometry with respect to the natural L^2-structures. Using the relations

$$\Phi \circ \mathbb{D} \circ \Phi^{-1} = \widetilde{\nabla}, \quad \Phi \circ \mathbb{G} \circ \Phi^{-1} = \widetilde{\Gamma}, \tag{6.41}$$

we obtain with Δ and $\widetilde{\triangle}$, denoting respectively the Laplacians of the complexes $(\mathcal{D},\mathbb{D})$ and $(\widetilde{\mathcal{D}},\widetilde{\nabla}) \equiv (\mathcal{D}_{\min},\nabla_{\min}) \oplus (\mathcal{D}_{\max},\nabla_{\max})$:

$$\begin{aligned}\Phi\mathcal{D}(\mathbb{B}) = \mathcal{D}(\mathcal{B}), &\quad \Phi \circ \mathbb{B} \circ \Phi^{-1} = \mathcal{B},\\ \Phi\mathcal{D}(\Delta) = \mathcal{D}(\widetilde{\triangle}), &\quad \Phi \circ \Delta \circ \Phi^{-1} = \widetilde{\triangle}.\end{aligned}$$

Hence the odd-signature operators $\mathbb{B},\mathcal{B}$ as well as the Laplacians $\Delta,\widetilde{\triangle}$ are spectrally equivalent. Consider the spectral projections $\Pi_{\mathbb{B}^2,[0,\lambda]}$ and $\Pi_{\mathcal{B}^2,[0,\lambda]}, \lambda \geq 0$ of $\mathbb{B}$ and $\mathcal{B}$ respectively, associated to eigenvalues of absolute value in $[0,\lambda]$. By the spectral equivalence $\mathbb{B}$ and $\mathcal{B}$ we find

$$\Phi \circ \Pi_{\mathbb{B}^2,[0,\lambda]} = \Pi_{\mathcal{B}^2,[0,\lambda]} \circ \Phi.$$

Hence the isomorphism Φ reduces to an isomorphism of finite-dimensional complexes:

$$\begin{aligned}\Phi_\lambda : (\mathcal{D}_{[0,\lambda]}, \mathbb{D}_{[0,\lambda]}) &\xrightarrow{\sim} (\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]}),\\ \text{where} \quad \mathcal{D}_{[0,\lambda]} &:= \mathcal{D} \cap \text{Image}\Pi_{\mathbb{B}^2,[0,\lambda]},\\ \widetilde{\mathcal{D}}_{[0,\lambda]} &:= \widetilde{\mathcal{D}} \cap \text{Image}\Pi_{\mathcal{B}^2,[0,\lambda]}.\end{aligned}$$

Moreover Φ_λ induces an isometric identification of the corresponding determinant lines, which we denote again by Φ_λ, by a minor abuse of notation

$$\Phi_\lambda : \det(\mathcal{D}_{[0,\lambda]}, \mathbb{D}_{[0,\lambda]}) \xrightarrow{\sim} \det(\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]}),$$

where we use the notation for determinant lines of finite-dimensional complexes in [BK2, Section 1.1]. By Corollary 6.15 we have the canonical identifications of determinant lines

$$\det(\mathcal{D}_{[0,\lambda]}, \mathbb{D}_{[0,\lambda]}) \cong \det H^*(\mathcal{D},\mathbb{D}), \tag{6.42}$$

$$\det(\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]}) \cong \det H^*(\widetilde{\mathcal{D}},\widetilde{\nabla}), \tag{6.43}$$

The determinant lines on the left hand side of both identifications carry the natural L^2-Hilbert structure. Denote the norms on $\det H^*(\mathcal{D},\mathbb{D})$ and $\det H^*(\widetilde{\mathcal{D}},\widetilde{\nabla})$ which turn both identifications into isometries, by $\|\cdot\|_\lambda$ and $\|\cdot\|^{\sim}_\lambda$, respectively. Then we can view Φ_λ as

$$\Phi_\lambda : \det H^*(\mathcal{D},\mathbb{D}) \xrightarrow{\sim} \det H^*(\widetilde{\mathcal{D}},\widetilde{\nabla}),$$

isometric with respect to the Hilbert structures induced by $\|\cdot\|_\lambda$ and $\|\cdot\|_{\widetilde{\lambda}}$.

Finally, consider the refined torsion elements (not the refined analytic torsion) of the determinant lines, as defined in [BK2, Section 1.1], see also (6.7)

$$\rho^{\mathbb{G}}_{[0,\lambda]} \in \det(\mathcal{D}_{[0,\lambda]}, \mathbb{D}_{[0,\lambda]}) \cong \det H^*(\mathcal{D}, \mathbb{D}),$$
$$\rho^{\widetilde{\Gamma}}_{[0,\lambda]} \in \det(\widetilde{\mathcal{D}}_{[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]}) \cong \det H^*(\widetilde{\mathcal{D}}, \widetilde{\nabla}).$$

We infer from (6.41) the following relation:

$$\Phi_\lambda\left(\rho^{\mathbb{G}}_{[0,\lambda]}\right) = \rho^{\widetilde{\Gamma}}_{[0,\lambda]}, \quad \text{hence: } \|\rho^{\mathbb{G}}_{[0,\lambda]}\|_\lambda = \|\rho^{\widetilde{\Gamma}}_{[0,\lambda]}\|_{\widetilde{\lambda}}.$$

Together with spectral equivalence of Δ and $\widetilde{\triangle}$, as well as of $\mathbb{B}$ and $\mathcal{B}$, with similar statements for constructions on trivial line bundles $M \times \mathbb{C}$ and $\mathbb{M} \times \mathbb{C}$, we finally obtain

$$\|\rho_{\mathrm{an}}(\mathbb{D})\|^{RS}_{\det H^*(\mathcal{D},\mathbb{D})} = \|\rho_{\mathrm{an}}(\nabla)\|^{RS}_{\det H^*(\widetilde{\mathcal{D}},\widetilde{\nabla})}, \tag{6.44}$$

where $\rho_{\mathrm{an}}(\mathbb{D})$ denotes the refined analytic torsion as defined by M. Braverman and T. Kappeler in [BK2] and $\rho_{\mathrm{an}}(\nabla)$ denotes the refined analytic torsion in the sense of the present discussion.

The statement now follows from [BK2, Theorem 11.3]. □

In the setup of the previous theorem we can improve the sign indeterminacy of $\rho_{\mathrm{an}}(\nabla)$ as follows:

Proposition 6.31. *Let M be an odd-dimensional oriented compact Riemannian manifold. Let (E, ∇, h^E) be a flat complex vector bundle over M. Denote by $\nabla_{\mathrm{trivial}}$ the trivial connection on $M \times \mathbb{C}$ and let $\mathcal{B}_{\mathrm{trivial}}$ denote the even part of the associated odd-signature operator.*

Assume the metric structures (g^M, h^E) to be product and the flat connection ∇ to be in temporal gauge near the boundary ∂M. Then

$$\rho_{\mathrm{an}}(\nabla) = \rho(\nabla, g^M, h^E) \cdot \exp\left[i\pi\, \mathrm{rk}(E)(\eta(\mathcal{B}_{\mathrm{trivial}}(g^M)) + \xi'(\nabla_{\mathrm{trivial}}, g^M))\right]$$

is independent of the choice of g^M in the interior of M, up to multiplication by

$$\exp[i\pi \mathrm{rank}(E)].$$

In particular it is independent of g^M in the interior of M for E being a complex vector bundle of even rank.

Proof. Consider a smooth family $g^M(t), t \in \mathbb{R}$ of Riemannian metrics, variing only in the interior of M and being of fixed product structure near ∂M. By arguments in Theorem 6.30 we can relate $\mathcal{B}(g^M(t))$ to operators on the closed double $\mathbb{M}$ and deduce from [BK1, Theorem 5.7] that $\rho(\nabla, g^M(t), h^E)$ is continuous in t. However

$$\exp\left[i\pi \,\mathrm{rk}(E)\eta(\mathcal{B}_{\mathrm{trivial}}(g^M(t)))\right]$$

is continuous in $t \in \mathbb{R}$ only up to multiplication by $e^{i\pi \mathrm{rk} E}$. Hence the element $\rho_{\mathrm{an}}(\nabla)$, where we denote the a priori metric dependence by $\rho_{\mathrm{an}}(\nabla, g^M(t))$, is continuous in t only modulo multiplication by $e^{i\pi \mathrm{rk}(E)}$. For $g^M(t)$ varying only in the interior of M and any $t_0, t_1 \in \mathbb{R}$ we infer from the mod $\mathbb{Z}$ metric anomaly considerations in Propositions 6.23 and 6.24:

$$\rho_{\mathrm{an}}(\nabla, g^M(t_0)) = \pm \rho_{\mathrm{an}}(\nabla, g^M(t_1)).$$

For $\mathrm{rk}(E)$ odd this is already the desired statement, since $\exp(i\pi \mathrm{rk}(E)) = -1$. For $\mathrm{rk}(E)$ even, $\rho_{\mathrm{an}}(\nabla, g^M(t))$ is continuous in t and nowhere vanishing, so the sign in the last relation must be positive. This proves the statement. $\square$

In view of the corollary above we can re-define the refined analytic torsion in the setup of product metric structures and flat connection in temporal gauge as follows:

$$\rho_{\mathrm{an}}(M, E) := \rho_{\mathrm{an}}(\nabla)/_{e^{i\pi \mathrm{rank}(E)}}. \tag{6.45}$$

Remark 6.32. *The interdeterminacy of $\rho_{\mathrm{an}}(\nabla)$ modulo multiplication by the factor $e^{i\pi \mathrm{rk} E}$ in fact corresponds and is even finer than the general indeterminacy in the construction of M. Braverman and T. Kappeler on closed manifolds, see [BK2, Remark 9.9 and Remark 9.10].*

6.5 Open Problems

Ideal Boundary Conditions

As explained in the introduction, the approach of Braverman and Kappeler in [BK1, BK2] requires ideal boundary conditions for the twisted de Rham complex, which turn it into a Fredholm complex with Poincare duality and further provide elliptic boundary conditions for the associated odd-signature operator, viewed as a map between the even forms. In our construction we pursued a different strategy, however the question about existence of such boundary conditions remains.

This question was partly discussed in [BL1]. In view of [BL1, Lemma 4.3] it is not even clear whether ideal boundary conditions exist, satisfying Poincare duality and providing a Fredholm complex. For the approach of Braverman and Kappeler we need even more: the ideal boundary conditions need to provide elliptic boundary conditions for the odd-signature operator. We arrive at the natural open question, whether such boundary conditions exist.

Conical Singularities

Another possible direction for the discussion of refined analytic torsion is the setup of compact manifolds with conical singularities. At the conical singularity the question of appropriate boundary conditions is discussed in [Ch2], as well as in [BL2].

It turns out that on odd-dimensional manifolds with conical singularities the topological obstruction is given by $H^\nu(N)$, where N is the base of the cone and $\nu = \dim N/2$. If

$$H^\nu(N) = 0$$

then all ideal boundary conditions coincide and the construction of Braverman and Kappeler [BK1, BK2] goes through. Otherwise, see [Ch2, p.580] for the choice of ideal boundary conditions satisfying Poincare duality.

Combinatorial Counterpart

Let us recall that the definition of the refined analytic torsion in [BK1, BK2] was partly motivated by providing analytic counterpart of the refined combinatorial torsion, introduced by V. Turaev in [Tu1].

In his work V. Turaev introduced the notion of Euler structures and showed how it is applied to refine the concept of Reidemeister torsion by removing the ambiguities in choosing bases needed for construction. Moreover, Turaev observed in [Tu2] that on three-manifolds a choice of an Euler structure is equivalent to a choice of a Spin^c-structure.

Both, the Turaev-torsion and the Braverman-Kappeler refined torsion are holomorphic functions on the space of representations of the fundamental group on $GL(n, \mathbb{C})$, which is a finite-dimensional algebraic variety. Using methods of complex analysis, Braverman and Kappeler computed the quotient between their and Turaev's construction.

A natural question is whether this procedure has an appropriate equivalent for our proposed refined analytic torsion on manifolds with boundary. In our view this question can be answered affirmatively.

Indeed, by similar arguments as in [BK1, BK2] the proposed refined analytic torsion on manifolds with boundary can also be viewed as an analytic function on the finite-dimensional variety of representations of the fundamental group.

For the combinatorial counterpart note that M. Farber introduced in [Fa] the concept of Poincare-Reidemeister metric, where using Poincare-duality in the similar spirit as in our construction, he constructed an invariantly defined Reidemeister torsion norm for non-unimodular representations. Further M. Farber and V. Turaev elaborated jointly in [FaTu] the relation between their concepts and introduced the refinement of the Poincare-Reidemeister scalar product.

The construction in [Fa] extends naturally to manifolds with boundary by similar means as in our definition of refined analytic torsion. This provides a combinatorial torsion norm on compact manifolds, well-defined without unimodularity assumption. It can then be refined in the spirit of [FaTu]. This would naturally provide the combinatorial counterpart for the presented refined analytic torsion.

7 Gluing Formula for Refined Analytic Torsion

In this section we turn to the main motivation for the proposed construction of refined analytic torsion − a gluing formula. A gluing formula allows to compute the torsion invariant by cutting the manifold into elementary pieces and performing computations on each component. Certainly, the general fact of existence of such gluing formulas is remarkable from the analytic point of view, since the secondary spectral invariants are uppermost non-local.

We establish a gluing formula for the refined analytic torsion in three steps. First we establish a splitting formula for the eta-invariant of the even part of the odd-signature operator. This is essentially an application of the results in [KL].

Secondly we establish a splitting formula for the refined torsion $\rho_{[0,\lambda]}$ in the special case $\lambda = 0$. This is the most intricate part and is done by a careful analysis of long exact sequences in cohomology und the Poincare duality on manifolds with boundary. The discussion is subdivided into several subsections.

Finally we are in the position to establish the desired gluing formula for the refined analytic torsion, as a consequence of the Cheeger-Müller Theorem and a gluing formula for the combinatorial torsion by M. Lesch [L2]. As a byproduct we also obtain a splitting formula for the scalar analytic torsion in terms of combinatorial torsion of a long exact sequence on cohomology.

In our discussion we do not rely on the gluing formula of S. Vishik in [V], where only the case of trivial representations is treated. In particular we use a different isomorphism between the determinant lines, which is more convenient in the present setup.

We perform the proof under the assumption of a flat Hermitian metric, in other words in case of unitary representations. This is done partly because the Cheeger-Müller Theorem for manifolds with boundary and unimodular representations is not explicitly established for the time being. It seems, however, that the appropriate result can be established by an adaptation of arguments in [Lü] and [Mu].

Finally it should be emphasized that the result of this section can also be viewed as a gluing formula for the refined analytic torsion in the sense of Braverman and Kappeler.

7.1 Setup for the Gluing Formula

Let $M = M_1 \cup_N M_2$ be an odd-dimensional oriented closed Riemannian manifold where N is an embedded closed hypersurface of codimension one which separates M into two pieces M_1 and M_2 such that $M_j, j = 1, 2$ are compact bounded Riemannian manifolds with $\partial M_j = N$ and orientations induced from M. The setup is visualized in the Figure 5 below:

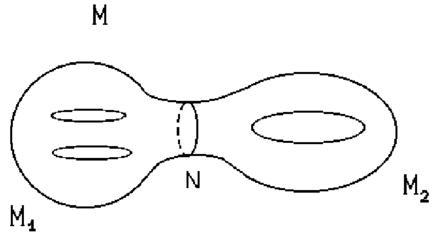

Figure 5: A compact closed split-manifold $M = M_1 \cup_N M_2$.

Let $\rho : \pi_1(M) \to U(n, \mathbb{C})$ be a unitary representation of the fundamental group of M. Denote by $\widetilde{M}$ the universal cover of M. It is a principal bundle over M with the structure group $\pi_1(M)$, cf. [KN, Proposition 5.9 (2)]. Consider the complex vector bundle E associated to the principal bundle $\widetilde{M}$ via the representation ρ.

$$E = \widetilde{M} \times_\rho \mathbb{C}^n.$$

The vector bundle is naturally endowed with a canonical flat connection ∇, induced by the exterior derivative on $\widetilde{M}$. The holonomy representation of ∇ is given by the representation ρ.

Note that all flat vector bundles arise this way. In fact flatness of a given connection on a vector bundle implies that the associated holonomy map gives rise to a well-defined representation of the fundamental group of the base manifold, and the representation is related to the vector bundle as above.

By unitariness of the representation ρ the standard Hermitian inner product on $\mathbb{C}^n$ gives rise to a smooth Hermitian metric h^E on E, compatible with the flat connection ∇. In other words the canonically induced Hermitian metric h^E is flat.

Assume the metric structures (g^M, h^E) to be product near the hypersurface N. The issues related to the product structures are discussed in detail in [BLZ, Section 2]. More precisely, we identify using the inward geodesic flow an open collar neighborhood $U \subset M$ of the hypersurface N diffeomorphically with $(-\epsilon,\epsilon)\times N, \epsilon>0$, where the hypersurface N is identified with $\{0\}\times N$. The metric g^M is product over the collar neighborhood of N, if over U it is given under the diffeomorphism $\phi : U \to (-\epsilon,\epsilon)\times \partial M$ by

$$\phi_* g^M|_U = dx^2 \oplus g^M|_N. \tag{7.1}$$

The diffeomorphism $U \cong (-\epsilon,\epsilon)\times N$ shall be covered by a bundle isomorphism $\widetilde{\phi} : E|_U \to (-\epsilon,\epsilon)\times E|_N$. The fiber metric h^E is product near the boundary, if it is preserved by the bundle isomorphism, i.e. if for all $x\in(-\epsilon,\epsilon)$

$$\widetilde{\phi}_* h^E|_{\{x\}\times N} = h^E|_N. \tag{7.2}$$

The restrictive assumption of product metric structures is necessary to apply the splitting formula of [KL] to our setup, which works only on Dirac type operators in product form over the collar with constant tangential part.

Furthermore we use the product metric structures in order to apply the Cheeger-Müller Theorem for manifolds with boundary, (cf. [Lü], [V]). However with the anomaly formulas in [BZ1] and [DF] the product structures are not essential here.

By Leibniz rule the connection ∇ gives rise to flat twisted exterior differential on smooth E-valued differential forms. The restrictions of (E,∇) to $M_j, j=1,2$ give rise to twisted de Rham complexes $(\Omega^*_0(M_j,E),\nabla_j)$. According to notation of Subsection 6.2 we denote their minimal and maximal extensions by

$$(\mathcal{D}_{j,\min/\max}, \nabla_{j,\min/\max}),$$

respectively. By Theorem 6.2 these complexes are Fredholm and their cohomologies can be computed from smooth subcomplexes as follows. Consider for $j=1,2$ the natural inclusions $\iota_j : N \hookrightarrow M_j$ and put

$$\Omega^*_{\min}(M_j,E) := \{\omega\in\Omega^*(M,E) | \iota_j^*\omega = 0\},$$
$$\Omega^*_{\max}(M_j,E) := \Omega^*(M,E).$$

The operators ∇_j yield exterior derivatives on $\Omega^*_{\min}(M_j,E)$ and $\Omega^*_{\max}(M_j,E)$. The complexes $(\Omega^*_{\min\,/\,\max}(M_j,E),\nabla_j)$ are by Theorem 6.2 smooth subcomplexes of the Fredholm complexes $(\mathcal{D}_{j,\min/\max},\nabla_{j,\min/\max})$ with

$$\begin{aligned} H^*_{\mathrm{rel}\,/\,\mathrm{abs}}(M_j,E) &:= H^*(\Omega^*_{\min/\max}(M_j,E),\nabla_j) \\ &\cong H^*(\mathcal{D}_{j,\min/\max},\nabla_{j,\min/\max}), \quad j=1,2. \end{aligned}$$

Finally corresponding to Definition 6.5 we define for $j = 1, 2$

$$(\widetilde{\mathcal{D}}_j, \widetilde{\nabla}_j) := (\mathcal{D}_{j,\min}, \nabla_{j,\min}) \oplus (\mathcal{D}_{j,\max}, \nabla_{j,\max}),$$
$$(\widetilde{\mathcal{D}}, \widetilde{\nabla}) := (\mathcal{D}, \nabla) \oplus (\mathcal{D}, \nabla),$$

where $(\mathcal{D}, \nabla)$ denotes the unique ideal boundary conditions of $(\Omega^*(M, E), \nabla)$. We denote the associated odd-signature operators of the complexes by $\mathcal{B}^j$, $j = 1, 2$ and $\mathcal{B}$ respectively. The upper index j will not pose any confusion with the square of the odd signature operator, since the squared odd-signature operator does not appear in the arguments below.

The presented notation remains fixed throughout the discussion below, unless stated otherwise. However for convenience, some setup and notation will be repeated for clarification.

7.2 Temporal Gauge Transformation

Consider the closed oriented Riemannian split-manifold (M, g^M) and the flat Hermitian vector bundle (E, ∇, h^E) with the structure group $G := U(n, \mathbb{C})$ as introduced in Subsection 7.1. Denote the principal G-bundle associated to E by P. G acts on P from the right.

Consider $U \cong (-\epsilon, \epsilon) \times N$ the collar neighborhood of the splitting hypersurface N. We view the restrictions $P|_U, P|_N$ as G-bundles, where the structure group can possibly be reduced to a subgroup of G.

By the setup of Subsection 7.1 the bundle structures are product over U. More precisely let $\pi : (-\epsilon, \epsilon) \times \partial X \to \partial X$ be the natural projection onto the second component. We have a bundle isomorphisms $E|_U \cong \pi^* E|_N$ and for the associated principal bundles

$$P|_U \cong \pi^* P|_N \xrightarrow{f} P|_N,$$

where f is the principal bundle homomorphism, covering π, with the associated homomorphism of the structure groups $G \to G$ being the identity automorphism.

Now let ω_N denote a flat connection one-form on $P|_N$. Then

$$\omega_U := f^* \omega_N$$

gives a connection one-form on $P|_U$ which is flat again.

In order to understand the structure of $\omega_U = f^* \omega_N$, let $\{\widetilde{U}_\alpha, \widetilde{\Phi}_\alpha\}$ be a set of local trivializations for $P|_N$. Then $P|_U \cong \pi^* P|_N$ is equipped with a set of naturally induced local trivializations $\{U_\alpha := (-\epsilon, \epsilon) \times \widetilde{U}_\alpha, \Phi_\alpha\}$. The local trivializations define local sections $\widetilde{s}_\alpha$ and s_α as follows. For any $p \in \widetilde{U}_\alpha$, normal variable $x \in (-\epsilon, \epsilon)$ and for $e \in G$ being the identity matrix we put

$$\widetilde{s}_\alpha(p) := \widetilde{\Phi}_\alpha^{-1}(p, e),$$
$$s_\alpha(x, p) := \Phi_\alpha^{-1}((x, p), e).$$

We use the local sections to obtain local representations for the connection one-forms ω_U and ω_N:

$$\widetilde{\omega}^\alpha := \widetilde{s}_\alpha^* \omega_N \in \Omega^1(\widetilde{U}_\alpha, \mathcal{G}),$$
$$\omega^\alpha := s_\alpha^* \omega_U \in \Omega^1(U_\alpha, \mathcal{G}),$$

where $\mathcal{G}$ denotes the Lie Algebra of G. By construction both local representations are related as follows. Let $(x, \underline{y}) = (x, y_1, .., y_n)$ denote the local coordinates on $U_\alpha = (-\epsilon, \epsilon) \times \widetilde{U}_\alpha$ with $x \in (-\epsilon, \epsilon)$ being the normal coordinate and $\underline{y}$ denoting the local coordinates on $\widetilde{U}_\alpha$. Then

$$\widetilde{\omega}^\alpha = \sum_{i=1}^n \omega_i^\alpha(\underline{y}) dy_i,$$
$$\omega^\alpha = \omega_0^\alpha(x, \underline{y}) dx + \sum_{i=1}^n \omega_i^\alpha(x, \underline{y}) dy_i,$$
$$\text{with} \quad \omega_0^\alpha \equiv 0, \text{ and } \omega_i^\alpha(x, \underline{y}) \equiv \omega_i^\alpha(\underline{y}). \tag{7.3}$$

We call a flat connection ω on P a connection in temporal gauge, if there exists a flat connection ω_N on P_N such that over the collar neighborhood U

$$\omega|_U = f^* \omega_N.$$

The local properties of a connection in temporal gauge and in particular its independence of the normal variable $x \in (-\epsilon, \epsilon)$ are discussed in (7.3). Our aim is to show that any flat connection one-form on a principal bundle P can be gauge transformed to a flat connection in temporal gauge. Recall that a gauge-transformation of P is a principal bundle automorphism $g \in \text{Aut}(P)$ covering identity on M with $g(p \cdot u) = g(p) \cdot u$ for any $u \in G$ and $p \in P$.

A gauge transformation can be viewed interchangedly as a transformation from one system of local trivializations into another. Hence the action of a gauge transformation on a connection one-form is determined by the transformation law for connections under change of coordinates.

We have the following result.

Proposition 7.1. *Any flat connection on the principal bundle P is gauge equivalent to a flat connection in temporal gauge.*

Proof. By a partition of unity argument it suffices to discuss the problem locally over a trivializing neighborhood $(U_\alpha := (-\epsilon, \epsilon) \times \widetilde{U}_\alpha, \Phi_\alpha)$.

Let ω be a flat connection on $P|_U$. Let g be any gauge transformation on $P|_U$. Denote the gauge transform of ω under g by ω_g.

Over the trivializing neighborhood U_α the connections ω, ω_g and the gauge tranformation g are given by local $\mathcal{G}$-valued one-forms $\omega^\alpha, \omega_g^\alpha$ and a G-automorphism g^α respectively. They are related in correspondence to the transformation law of connections as follows

$$\omega_g^\alpha = (g^\alpha)^{-1} \circ \omega^\alpha \circ g^\alpha + (g^\alpha)^{-1} dg^\alpha,$$

where the action $\circ$ is the concatenation of matrices ($G \subset GL(n, \mathbb{C})$), after evaluation at a local vector field and a base point in U_α. The local one form ω^α writes as

$$\omega^\alpha = \omega_0^\alpha(x, \underline{y}) dx + \sum_{i=1}^{n} w_i^\alpha(x, \underline{y}) dy_i.$$

In order to gauge-transform ω into temporal gauge, we need to annihilate ω_0^α and the x-dependence in ω_i^α. For this reason we consider the following initial value problem with parameter $\underline{y} \in \widetilde{U}_\alpha$

$$\begin{aligned} \partial_x g^\alpha(x, \underline{y}) &= -\omega_0^\alpha(x, \underline{y}) g^\alpha(x, \underline{y}), \\ g^\alpha(0, \underline{y}) &= \mathbf{1} \in GL(n, \mathbb{C}). \end{aligned} \tag{7.4}$$

In order to identify the solution to (7.4) consider for any fixed $\underline{y} \in \widetilde{U}_\alpha$ the following $x-$time dependent vector field $V_{x,\underline{y}}^\alpha, x \in (-\epsilon, \epsilon)$ on G:

$$\forall u \in G \quad V_{x,\underline{y}}^\alpha u := -(R_u)_* \omega_0^\alpha(x, \underline{y}) = -\omega_0^\alpha(x, \underline{y}) \cdot u,$$

where R_u is the right multiplication on G and the second equality follows from the fact that $G \in GL(n, \mathbb{C})$ is a matrix Lie group.

Let $\widetilde{g}^\alpha(x, \underline{y})$ be the unique integral curve of the time-dependent vector field $V_{x,\underline{y}}^\alpha$ with $\widetilde{g}^\alpha(0, \underline{y}) = \mathbf{1} \in G$. It satisfies

$$\partial_x \widetilde{g}^\alpha(x, \underline{y}) = V_{x,\underline{y}}^\alpha \widetilde{g}^\alpha(x, \underline{y}) = -\omega_0^\alpha(x, \underline{y}) \widetilde{g}^\alpha(x, \underline{y}).$$

Hence the integral curve $\widetilde{g}^\alpha(x, \underline{y})$ solves (7.4). By the fundamental theorem for ordinary linear differential equations (cf. [KN, Appendix 1]) we know that the initial value problem (7.4) has a unique solution, smooth in $x \in (-\epsilon, \epsilon)$ and $\underline{y} \in \widetilde{U}_\alpha$. Since $\widetilde{g}^\alpha(x, \underline{y})$ solves (7.4) we find that the solution is moreover $G-$valued.

With gauge transformation g being locally the solution to (7.4) we find for the gauge transformed connection ω_g

$$\begin{aligned} \omega_g^\alpha = (g^\alpha)^{-1} \circ \omega^\alpha \circ g^\alpha + (g^\alpha)^{-1} dg^\alpha = \\ = (g^\alpha)^{-1} \circ \omega_0^\alpha \circ g^\alpha dx + \sum_{i=1}^n (g^\alpha)^{-1} \circ \omega_i^\alpha \circ g^\alpha dy_i + \\ +(g^\alpha)^{-1} \partial_x g^\alpha dx + \sum_{i=1}^n (g^\alpha)^{-1} \partial_{y_i} g^\alpha dy_i = \\ = \sum_{i=1}^n (g^\alpha)^{-1} \circ \omega_i^\alpha \circ g^\alpha dy_i + \sum_{i=1}^n (g^\alpha)^{-1} \partial_{y_i} g^\alpha dy_i. \end{aligned}$$

where in the last equality we cancelled two summands due to g^α being the solution to (7.4). So far we didn't use the fact that ω is a flat connection. A gauge transformation preserves flatness, so ω_g is flat again. Put

$$\omega_g^\alpha = \omega_{g,0}^\alpha(x, \underline{y}) dx + \sum_{i=1}^n \omega_{g,i}^\alpha(x, \underline{y}) dy_i,$$

where by the previous calculation

$$\omega_{g,0}^\alpha \equiv 0, \quad \omega_{g,i}^\alpha \equiv (g^\alpha)^{-1} \circ \omega_i^\alpha \circ g^\alpha + (g^\alpha)^{-1} \partial_{y_i} g^\alpha.$$

Flatness of ω_g implies

$$\partial_x \omega_{g,i}^\alpha(x, \underline{y}) = \partial_{y_i} \omega_{g,0}^\alpha(x, \underline{y}) = 0.$$

Hence the gauge transformed connection is indeed in temporal gauge. This completes the proof. □

A gauge transformation, viewed so far as a principal bundle automorphism on the $G-$principal bundle P, can equivalently be viewed as a $G-$valued bundle automorphism on the vector bundle E associated to P. We adopt this point of view for the forthcoming discussion.

Take the given flat connection ∇ on the Hermitian vector bundle (E, h^E) with the structure group $G = U(n, \mathbb{C})$ and the canonical metric h^E induced by the standard inner product on $\mathbb{C}^n$. Proposition 7.1 asserts existence of a temporal gauge transformation $g \in Aut_G(E)$ such that the gauge transformed covariant derivative $g\nabla g^{-1}$ is in temporal gauge (a covariant derivative is said to be in temporal gauge if the associated connection one-form is in temporal gauge).

The temporal gauge transformation g gives rise to a map on sections in a natural way

$$\mathfrak{G} : \Omega^*(M, E \otimes E) \to \Omega^*(M, E \otimes E).$$

Due to the fact that g takes locally values in $U(n, \mathbb{C})$ and the Hermitian metric h^E is canonically induced by the standard inner product on $\mathbb{C}^n$, we obtain the following result:

Proposition 7.2. *$\mathfrak{G}$ extends to a unitary transformation*

$$\mathfrak{G} : L^2_*(M, E \otimes E, g^M, h^E) \to L^2_*(M, E \otimes E, g^M, h^E).$$

Corollary 7.3. *The odd-signature operators $\mathcal{B} = \mathcal{B}(\nabla)$ and $\mathcal{B}^j = \mathcal{B}^j(\nabla_j), j = 1, 2$ are spectrally equivalent to $\mathcal{B}(g\nabla g^{-1})$ and $\mathcal{B}^j(g\nabla g^{-1}|_{M_j}), j = 1, 2$ respectively.*

The statement of the corollary above follows from invariance of minimal and maximal extensions under unitary transformations and from the fact that unitary transformations preserve spectral properties of operators, compare also Proposition 3.2 and Corollary 3.3.

The statement of the corollary implies that in the setup of this section (for unitary vector bundles) the assumption of temporal gauge is done without loss of generality, which we do henceforth. In this particular geometric setup we obtain the following specific result for refined analytic torsion.

Proposition 7.4. *Let $T^{RS}(\widetilde{\nabla})$ and $T^{RS}(\widetilde{\nabla}_j), j = 1, 2$ denote the scalar analytic torsions associated to the complexes $(\widetilde{\mathcal{D}}, \widetilde{\nabla}), (\widetilde{\mathcal{D}}_j, \widetilde{\nabla}_j)$, respectively. Furthermore let $\rho_\Gamma(M, E)$ and $\rho_\Gamma(M_j, E)$ denote the associated refined torsion elements in the sense of (6.7) for $\lambda = 0$. Then we have*

$$\begin{aligned}\rho_{\text{an}}(\nabla) = \frac{1}{T^{RS}(\widetilde{\nabla})} \cdot \exp\left[-i\pi\eta(\mathcal{B}_{\text{even}}) + i\pi \text{rk}(E)\eta(\mathcal{B}_{\text{trivial}})\right] \times \\ \times \exp\left[-i\pi\frac{m-1}{2}\dim\ker\mathcal{B}_{\text{even}} + i\pi\text{rk}(E)\frac{m}{2}\dim\ker\mathcal{B}_{\text{trivial}}\right]\rho_\Gamma(M, E),\end{aligned}$$

$$\begin{aligned}\rho_{\text{an}}(\nabla_j) = \frac{1}{T^{RS}(\widetilde{\nabla}_j)} \cdot \exp\left[-i\pi\eta(\mathcal{B}^j_{\text{even}}) + i\pi \text{rk}(E)\eta(\mathcal{B}^j_{\text{trivial}})\right] \times \\ \times \exp\left[-i\pi\frac{m-1}{2}\dim\ker\mathcal{B}^j_{\text{even}} + i\pi\text{rk}(E)\frac{m}{2}\dim\ker\mathcal{B}^j_{\text{trivial}}\right]\rho_\Gamma(M_j, E).\end{aligned}$$

Proof. Recall from the definition of refined analytic torsion in Corollary 6.26

$$\begin{aligned}\rho_{\text{an}}(\nabla) = e^{\xi_\lambda(\nabla)} &\exp[-i\pi(\eta(\mathcal{B}_{\text{even}}^{(\lambda,\infty)}) + \xi'_\lambda(\nabla))]\times \\ &\exp[+i\pi\text{rk}(E)(\eta(\mathcal{B}_{\text{trivial}}) + \xi'(\nabla_{\text{trivial}}))]\cdot\rho_{[0,\lambda]}, \\ \rho_{\text{an}}(\nabla_j) = e^{\xi_\lambda(\nabla_j)} &\exp[-i\pi(\eta(\mathcal{B}_{\text{even}}^{j,(\lambda,\infty)}) + \xi'_\lambda(\nabla_j))]\times \\ &\exp[+i\pi\text{rk}(E)(\eta(\mathcal{B}^j_{\text{trivial}}) + \xi'(\nabla_{j,\text{trivial}}))]\cdot\rho^j_{[0,\lambda]}, j = 1,2.\end{aligned}$$

The assumption of product metric structures and the temporal gauge allow a reduction to closed double manifolds, as performed explicitly in Theorem 6.30. This yields by similar arguments, as in [BK2, Proposition 6.5]:

$$\begin{aligned}\xi'_\lambda(\nabla) &= \frac{1}{2}\sum_{k=0}^{m}(-1)^{k+1}\cdot k\cdot\dim\widetilde{\mathcal{D}}^k_{[0,\lambda]}, \\ \xi'_\lambda(\nabla_j) &= \frac{1}{2}\sum_{k=0}^{m}(-1)^{k+1}\cdot k\cdot\dim\widetilde{\mathcal{D}}^k_{j,[0,\lambda]}.\end{aligned}$$

Now, via Lemma 6.25 we obtain

$$\begin{aligned}\xi'_\lambda(\nabla) &\equiv \frac{m}{2}\dim\widetilde{\mathcal{D}}^{\text{even}}_{[0,\lambda]} \mod 2\mathbb{Z}, \\ \xi'_\lambda(\nabla_j) &\equiv \frac{m}{2}\dim\widetilde{\mathcal{D}}^{\text{even}}_{j,[0,\lambda]} \mod 2\mathbb{Z}.\end{aligned}$$

Similar arguments show

$$\begin{aligned}\xi'(\nabla_{\text{trivial}}) &\equiv \frac{m}{2}\dim\ker\mathcal{B}_{\text{trivial}} \mod 2\mathbb{Z}, \\ \xi'(\nabla_{j,\text{trivial}}) &\equiv \frac{m}{2}\dim\ker\mathcal{B}^j_{\text{trivial}} \mod 2\mathbb{Z}.\end{aligned}$$

Fix $\lambda = 0$ and observe for $j = 1,2$ from (6.21):

$$\begin{aligned}\eta(\mathcal{B}^{(0,\infty)}_{\text{even}}) + \xi'_0(\nabla) &\equiv \eta(\mathcal{B}_{\text{even}}) + \frac{m-1}{2}\dim\ker\mathcal{B}_{\text{even}} \mod 2\mathbb{Z}, \\ \eta(\mathcal{B}^{j,(0,\infty)}_{\text{even}}) + \xi'_0(\nabla_j) &\equiv \eta(\mathcal{B}^j_{\text{even}}) + \frac{m-1}{2}\dim\ker\mathcal{B}^j_{\text{even}} \mod 2\mathbb{Z}.\end{aligned}$$

Now the statement of the proposition follows from the fact that flatness of the Hermitian metric h^E implies equality between the squared odd-signature operator and the Laplacians of the corresponding complexes, and hence

$$e^{\xi_0(\nabla)} = \frac{1}{T^{RS}(\widetilde{\nabla})}, \quad e^{\xi_0(\nabla_j)} = \frac{1}{T^{RS}(\widetilde{\nabla}_j)}.$$

This proves the proposition. □

7.3 Splitting formula for the eta-invariant

This subsection is an application of [KL, Theorem 7.7]. For the setup of that result consider $\mathcal{U} \subset M$ the collar neighborhood of the hypersurface N together with a mapping

$$\Phi : C^\infty(\mathcal{U}, F|_\mathcal{U}) \to C^\infty((-\epsilon, \epsilon), C^\infty(N, F|_N)),$$

where F is any Hermitian vector bundle over M and Φ extends to an isometry on the L^2-completions of the spaces. Now let D be a self-adjoint operator of Dirac-type with the following product form over the collar neighborhood:

$$\Phi \circ D|_\mathcal{U} \circ \Phi^{-1} = \gamma \left[\frac{d}{dx} + A \right],$$

where γ is a bundle homomorphism on $C^\infty(N, F|_N)$ and the tangential operator A is a self-adjoint operator of Dirac-type over $C^\infty(N, F|_N)$. The essence of the product form lies in the x-independence of the coefficients γ and A. Moreover we assume

$$\gamma^2 = -I, \quad \gamma^* = -\gamma, \quad \gamma A = -A\gamma. \tag{7.5}$$

By restriction to M_1, M_2 we obtain Dirac operators D^1, D^2 with product structure (under the identification of Φ) as above over the collar neighborhoods $\mathcal{U} \cap M_j$ of the boundaries $\partial M_j = N, j = 1, 2$.

Let $P : L^2(N, F|_N) \to L^2(N, F|_N)$ satisfy the following conditions:

- P is pseudo-differential of order zero, (7.6)
- P is an orthogonal projection, i.e. $P^* = P, P^2 = P$, (7.7)
- $\gamma P \gamma^* = I - P$, (7.8)
- $(P_{>0}, P)$ form a Fredholm pair, i.e. $P_{>0}|_{\mathrm{im}P}$ is Fredholm. (7.9)

Here $P_{>0}$ denotes the positive spectral projection associated to the self-adjoint tangential operator A. The boundary value problems $(D^j, P), j = 1, 2$ are well-posed in the sense of R. T. Seeley, by [BL3, Theorem 7.2]. We know, see [KL, Theorem 3.1] and the references therein, that the eta-functions $\eta(D^j_P, s)$ extend meromorphically to $\mathbb{C}$. Assume for simplicity that the eta-functions are regular at $s = 0$ and set for $j = 1, 2$:

$$\eta(D^j_P) := \frac{1}{2} \left[\eta(D^j_P, s = 0) + \dim \ker D^j_P \right],$$
$$\eta(D) := \frac{1}{2} \left[\eta(D, s = 0) + \dim \ker D \right].$$

This definition coincides with (6.13) for $D_P^j = \mathcal{B}_{\text{even}}^j$ and $D = \mathcal{B}_{\text{even}}$, since in the setup of the present section the odd-signature operators are self-adjoint, hence have real spectrum.

The same holds for D_{I-P}^j as well, and the splitting formula in the version of [KL, Theorem 7.7] is given as follows:

$$\eta(D) = \eta(D_P^1) + \eta(D_{I-P}^2) - \tau_\mu(I - P_1, P, P_1), \tag{7.10}$$

where P_1 denotes the Calderon projector for D^1, which is the orthogonal projection of sections in $F|_N$ onto the Cauchy-data space of D^1 consisting of the traces at N of elements in the kernel of D^1. For further details see [BW].

The third summand τ_μ in (7.10) refers to the Maslov triple index defined in [KL, Definition 6.8]. The Maslov triple index is integer-valued and thus the result above leads in particular to a mod $\mathbb{Z}$ splitting formula for eta-invariants.

Leaving for the moment these general constructions aside, we continue in the setup of the Subsection 7.1. We adapt the constructions of [KL, Section 8.1] to the present situation. Let $\iota : N \hookrightarrow M$ denote the inclusion of the splitting hypersurface N into the closed split-manifold M. Define the restriction map:

$$\begin{aligned} R : \Omega^{\text{even}}(M, E \oplus E) &\to \Omega^*(N, (E \oplus E)|_N), \\ \beta &\mapsto \iota^*(\beta) + \iota^*(\widetilde{*}\beta), \end{aligned}$$

where $\widetilde{*}$ is the Hodge-star operator on the oriented Riemannian manifold M acting antidiagonally on $E \oplus E$ with the following matrix form:

$$\widetilde{*} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

It is further related as follows to the chirality operator $\widetilde{\Gamma}$ of the Hilbert complex $(\widetilde{\mathcal{D}}, \widetilde{\nabla})$ (see Definition 6.4):

$$\begin{aligned} \widetilde{\Gamma} := i^r (-1)^{\frac{k(k+1)}{2}} \widetilde{*} : \Omega^k(M, E \oplus E) \to \Omega^{m-k}(M, E \oplus E), \\ \text{where } r := (\dim M + 1)/2. \end{aligned}$$

The restriction map R induces with $\mathcal{U} \cong (-\epsilon, \epsilon) \times N$ the following identification:

$$\Phi : \Omega^{\text{even}}(\mathcal{U}, (E \oplus E)|_{\mathcal{U}}) \to C^\infty((-\epsilon, \epsilon), \Omega^*(N, (E \oplus E)|_N)),$$

which extends to an isometry on the L^2-completions of the spaces due to the product structure of the metrics. The isometric transformation preserves the spectral properties of the transformed operators. Hence we can equivalently deal with the even part $\mathcal{B}_{\mathrm{even}}$ of the odd-signature operator associated to the Hilbert complex $(\widetilde{\mathcal{D}}, \widetilde{\nabla})$ over M, under the isometric transformation Φ.

The assumption of temporal gauge for the connection ∇ implies with the same calculations as in [KL, Section 8.1]:

$$\Phi \circ \mathcal{B}_{\mathrm{even}} \circ \Phi^{-1} = \gamma \left[\frac{d}{dx} + A \right], \tag{7.11}$$

where the operators γ and A are of the following form (compare [KL, Section 8.1]):

$$\gamma(\beta) = \begin{cases} i^r(-1)^{p-1} \widetilde{*}_N \beta, & \text{if } \beta \in \Omega^{2p}(N, (E \oplus E)|_N), \\ i^r(-1)^{r-p-1} \widetilde{*}_N \beta, & \text{if } \beta \in \Omega^{2p+1}(N, (E \oplus E)|_N). \end{cases}$$

$$A(\beta) = \begin{cases} -(\widetilde{\nabla}_N \widetilde{*}_N + \widetilde{*}_N \widetilde{\nabla}_N)\beta, & \text{if } \beta \in \Omega^{2p}(N, (E \oplus E)|_N), \\ (\widetilde{\nabla}_N \widetilde{*}_N + \widetilde{*}_N \widetilde{\nabla}_N)\beta, & \text{if } \beta \in \Omega^{2p+1}(N, (E \oplus E)|_N). \end{cases}$$

Here $\widetilde{\nabla}_N = \nabla_N \oplus \nabla_N$ where ∇_N is the flat connection on $E|_N$ whose pullback to $E|_{\mathcal{U}}$ gives $\nabla|_{\mathcal{U}}$. Further $\widetilde{*}_N$ is the Hodge-star operator on N acting anti-diagonally on $(E \oplus E)|_N$. We write

$$\widetilde{\nabla}_N = \begin{pmatrix} \nabla_N & 0 \\ 0 & \nabla_N \end{pmatrix}, \ \widetilde{*}_N = \begin{pmatrix} 0 & *_N \\ *_N & 0 \end{pmatrix}.$$

Consider next the odd signature operators $\mathcal{B}^j_{\mathrm{even}}, j = 1, 2$ viewed as boundary value problems for $\mathcal{B}_{\mathrm{even}}|_{M_j}, j = 1, 2$ where the boundary conditions are to be identified. To visualize the structure involved, we distinguish notationally each direct sum component in $E \oplus E$:

$$E \oplus E \equiv E^+ \oplus E^-.$$

Decompose now $\Omega^*(N, (E \oplus E)|_N)$ as follows:

$$\begin{aligned} &\Omega^*(N, (E \oplus E)|_N) = \\ &\left[\Omega^{\mathrm{even}}(N, E^+|_N) \oplus \Omega^{\mathrm{odd}}(N, E^+|_N)\right] \oplus \left[\Omega^{\mathrm{even}}(N, E^-|_N) \oplus \Omega^{\mathrm{odd}}(N, E^-|_N)\right]. \end{aligned} \tag{7.12}$$

The restriction map R acts with respect to this decomposition as follows:

$$\begin{aligned} R(\beta^+ \oplus \beta^-) = [\iota^*(\beta^+) \oplus \iota^*(*\beta^-)] \oplus [\iota^*(\beta^-) \oplus \iota^*(*\beta^+)], \\ \text{where } \beta^+ \oplus \beta^- \in \Omega^{\mathrm{even}}(M, E^+ \oplus E^-). \end{aligned} \tag{7.13}$$

Furthermore with respect to this decomposition operators γ and A are given by the following matrix form:

$$\gamma = \begin{pmatrix} 0 & \overline{\gamma} \\ \overline{\gamma} & 0 \end{pmatrix}, \ A = \begin{pmatrix} 0 & \overline{A} \\ \overline{A} & 0 \end{pmatrix}, \tag{7.14}$$

$$\overline{\gamma}(\beta) = \begin{cases} i^r(-1)^{p-1} *_N \beta, & \text{if } \beta \in \Omega^{2p}(N, E|_N), \\ i^r(-1)^{r-p-1} *_N \beta, & \text{if } \beta \in \Omega^{2p+1}(N, E|_N). \end{cases}$$

$$\overline{A}(\beta) = \begin{cases} -(\nabla_N *_N + *_N \nabla_N)\beta, & \text{if } \beta \in \Omega^{2p}(N, E|_N), \\ (\nabla_N *_N + *_N \nabla_N)\beta, & \text{if } \beta \in \Omega^{2p+1}(N, E|_N). \end{cases}$$

Note that γ and A satisfy the conditions (7.5). Recall now from Lemma 6.6

$$\mathcal{D}(\mathcal{B}^j) = \mathcal{D}\left(D^{GB}_{j,\text{rel}} \oplus D^{GB}_{j,\text{abs}}\right),$$

where D^{GB}_j is the Gauss-Bonnet operator on M_j associated to the connection ∇_j. Hence the boundary conditions for $\mathcal{B}^j_{\text{even}}$ are given as follows (see [BL1, Theorem 4.1], where the arguments are performed in the untwisted setup, but transfer analogously to the twisted case, provided product metric structures and a flat connection in temporal gauge)

$$\begin{aligned} & \beta = \beta^+ \oplus \beta^- \in \mathcal{D}(\mathcal{B}^j_{\text{even}}) \cap \Omega^{\text{even}}(M_j, E^+ \oplus E^-), \\ \text{hence} \quad & \beta^+ \in \mathcal{D}(D^{GB}_{\text{rel}}), \quad \beta^- \in \mathcal{D}(D^{GB}_{\text{abs}}), \\ \text{hence} \quad & \iota_j^*(\beta^+) = 0, \quad \iota_j^*(*\beta^-) = 0. \end{aligned}$$

According to (7.13) we obtain under the isometric identification Φ over $U \cap M_j$ and with respect to the decomposition (7.12) the following matrix form for the boundary operators of $\mathcal{B}^j_{\text{even}}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.15}$$

Note that $(I - P)$ again provides boundary conditions for $\mathcal{B}^j_{\text{even}}$ with the components E^+, E^- interchanged. The boundary operator P obviously extends to a pseudo-differential operator of order zero. One checks explicitly by matrix calculations that P satisfies the conditions (7.7) and (7.8). The condition (7.9) remains to be verified.

Being elliptic and self-adjoint, A has discrete real spectrum with finite multiplicities. Discreteness of A together with self-adjointness of $\mathcal{B}^j_{\text{even}}$ implies with [BL3, Corollary 4.6 and Theorem 5.6] that P indeed satisfies (7.9).

Thus the conditions for the application of (7.10) are satisfied and we obtain

$$\eta(\mathcal{B}_{\mathrm{even}}) = \eta(\mathcal{B}^1_{\mathrm{even}}) + \eta(\mathcal{B}^2_{\mathrm{even}}) - \tau_\mu(I - P_1, P, P_1), \tag{7.16}$$

where the operators $\mathcal{B}^1_{\mathrm{even}}, \mathcal{B}^2_{\mathrm{even}}$ denote the even parts of the odd-signature operators associated in the sense of Definition 6.5 to the Hilbert complexes $(\widetilde{\mathcal{D}}_1, \widetilde{\nabla}_1), (\widetilde{\mathcal{D}}_2, \widetilde{\nabla}_2)$ respectively. Equivalently they are the self-adjoint realizations of the differential operators $\mathcal{B}_{\mathrm{even}}|_{M_1}, \mathcal{B}_{\mathrm{even}}|_{M_2}$ with the boundary conditions P and $(I-P)$ respectively. Further P_1 denotes the Calderon projector of $\mathcal{B}^1_{\mathrm{even}}$ and τ_μ the Maslov triple index.

Due to self-adjointness of the odd-signature operator, the notion of (reduced) eta-invariant in [KL] for $\mathcal{B}_{\mathrm{even}}$ and $\mathcal{B}^j_{\mathrm{even}}, j = 1, 2$ coincides with the setup of (6.13).

We obtain an analogous splitting formula in case of a trivial line bundle $M \times \mathbb{C}$ in the notation of Proposition 6.23:

$$\eta(\mathcal{B}_{\mathrm{trivial}}) = \eta(\mathcal{B}^1_{\mathrm{trivial}}) + \eta(\mathcal{B}^2_{\mathrm{trivial}}) - \tau_\mu(I - P_{1,\mathrm{trivial}}, P_{\mathrm{trivial}}, P_{1,\mathrm{trivial}}), \tag{7.17}$$

with the obvious notation. This formula will be necessary in order to obtain a splitting for the metric-anomaly term in refined analytic torsion.

In other words, the phase of refined analytic torsion is given in part by the rho invariant of $\mathcal{B}_{\mathrm{even}}$, which is defined (cf. [KL, Definition 8.17]) as the eta-invariant of the operator minus the metric anomaly correction term. The results (7.16) and (7.17) give together a splitting formula for the rho-invariant, which constitutes to the complex phase of refined analytic torsion.

7.4 Poincare Duality for manifolds with boundary

We continue in the setup and the notation, fixed in Subsection 7.1. Denote by $\triangle_{j,\mathrm{rel/abs}}$ the Laplacians of the Hilbert complexes $(\mathcal{D}_{j,\min/\max}, \nabla_{j,\min/\max})$ respectively. The coefficient j refers to the base manifold $M_j, j = 1, 2$. Consider the Hodge star operator $*$ on M and the associated chirality operator

$$\Gamma := i^r(-1)^{\frac{k(k+1)}{2}} * : \Omega^k(M, E) \to \Omega^{m-k}(M, E), \quad r := (\dim M + 1)/2.$$

By the properties of the chirality operator Γ (we do not indicate the restriction of Γ to $M_j, j = 1, 2$ by a subscript j, since it will always be clear from the action), established in Lemma 6.3, we infer:

$$\triangle_{j,\mathrm{abs/rel}} = \Gamma \circ \triangle_{j,\mathrm{rel/abs}} \circ \Gamma, \tag{7.18}$$

$$\text{and hence } \Gamma : \ker \triangle_{j,\mathrm{rel/abs}} \xrightarrow{\sim} \ker \triangle_{j,\mathrm{abs/rel}}. \tag{7.19}$$

Note the Hodge isomorphisms

$$\ker \triangle_{j,\text{rel/abs}} \xrightarrow{\sim} H^*_{\text{rel/abs}}(M_j, E), \quad \phi \longmapsto [\phi],$$

where $H^*_{\text{rel/abs}}(M_j, E)$ denote the de Rham cohomologies of the Fredholm complexes $(\mathcal{D}_{j,\min/\max}, \nabla_{j,\min/\max}), j = 1, 2$ respectively. Hence the chirality operator induces under the Hodge isomorphisms the so-called Poincare duality on manifolds with boundary:

$$\Gamma : H^k_{\text{rel/abs}}(M_j, E) \xrightarrow{\sim} H^{m-k}_{\text{abs/rel}}(M_j, E), \quad k = 0, .., m = \dim M.$$

Next we introduce two pairings. Let $\omega_1 = s_1 \otimes \chi_1$ and $\omega_2 = s_2 \otimes \chi_2$ be two differential forms in $\Omega^*(M_j, E)$ with $s_1, s_2 \in C^\infty(M_j, E)$ and $\chi_1, \chi_2 \in \Omega^*(M_j)$. Put:

$$h^E(\omega_1 \wedge \omega_2) := h^E(s_1, s_2) \cdot \chi_1 \wedge \chi_2.$$

This action extends by linearity to arbitrary differential forms in $\Omega^*(M_j, E)$. With this notation we define a pairing, which is the Hilbert structure on $H^*_{\text{rel/abs}}(M_j, E)$ induced by the L^2-structure on the harmonic forms:

$$\begin{aligned} \langle \cdot \, , \cdot \rangle_{L^2_j} : H^k_{\text{rel/abs}}(M_j, E) \times H^k_{\text{rel/abs}}(M_j, E) &\to \mathbb{C} \\ [\omega], [\eta] &\longmapsto \int_{M_j} h^E(\omega \wedge *\eta), \end{aligned} \tag{7.20}$$

where ω and η are the harmonic representatives of $[\omega]$ and $[\eta]$ respectively. The second pairing is the Poincare duality on Riemannian manifolds with boundary and is in fact independent of a choice of representatives:

$$\begin{aligned} \langle \cdot \, , \cdot \rangle_{P_j} : H^k_{\text{rel/abs}}(M_j, E) \times H^{m-k}_{\text{abs/rel}}(M_j, E) &\to \mathbb{C} \\ [\omega], [\eta] &\longmapsto \int_{M_j} h^E(\omega \wedge \eta). \end{aligned} \tag{7.21}$$

Both pairings are non-degenerate and induce canonical identifications between cohomology and its dual:

$$\#_{L^2_j} : H^k_{\text{rel/abs}}(M_j, E) \xrightarrow{\sim} \left(H^k_{\text{rel/abs}}(M_j, E)\right)^*, \quad [\omega] \longmapsto \langle \cdot \, , \, [\omega] \rangle_{L^2_j},$$
$$\#_{P_j} : H^k_{\text{rel/abs}}(M_j, E) \xrightarrow{\sim} \left(H^{m-k}_{\text{abs/rel}}(M_j, E)\right)^*, \quad [\omega] \longmapsto \langle \cdot \, , \, [\omega] \rangle_{P_j}.$$

Both maps are linear, with the Hermitian metric h^E set to be linear in the second component. The next proposition puts the constructions above into relation:

Proposition 7.5. *The action of* $\#_{L^2_j} \circ \Gamma$ *and* $\#_{P_j}$ *on* $H^k_{\text{rel/abs}}(M_j, E)$ *satisfies*

$$\#_{L^2_j} \circ \Gamma = i^r(-1)^{\frac{k(k+1)}{2}} \#_{P_j}, \quad r = (\dim M + 1)/2.$$

Proof. Let $[\omega] \in H^k_{\text{rel/abs}}(M_j, E)$ and $[\eta] \in H^k_{\text{abs/rel}}(M_j, E)$ with ω, η being the harmonic representatives of $[\omega], [\eta]$ respectively. Using $\Gamma^2 = \mathbf{1}$ we get

$$\begin{aligned} *\Gamma\omega &= i^{-r}(-1)^{\frac{(m-k)(m-k+1)}{2}} \Gamma \circ \Gamma\omega = \\ &= i^{-r}(-1)^{\frac{k(k+1)}{2} + \frac{m(m+1)}{2}} \omega = i^r(-1)^{\frac{k(k+1)}{2}} \omega. \end{aligned}$$

Due to linearity of the Hermitian metric in the second component we finally obtain:

$$\begin{aligned} (\#_{L^2_j} \circ \Gamma)([\omega])[\eta] &= \langle [\eta], \Gamma[\omega] \rangle_{L^2_j} = \int_{M_j} h^E(\eta \wedge *\Gamma\omega) = \\ &= i^r(-1)^{\frac{k(k+1)}{2}} \int_{M_j} h^E(\eta \wedge \omega) = i^r(-1)^{\frac{k(k+1)}{2}} \#_{P_j}([\omega])[\eta]. \end{aligned}$$

□

A similar discussion works also on the closed Riemannian split manifold M. In particular we obtain as before the pairings $\langle \cdot, \cdot \rangle_{L^2}$ and $\langle \cdot, \cdot \rangle_P$ with the associated identifications $\#_{L^2}$ and $\#_P$ respectively, over the manifold M. As in Proposition 7.5 we obtain for the action of $\#_{L^2} \circ \Gamma$ and $\#_P$ on $H^k(M, E)$ the following relation

$$\#_{L^2} \circ \Gamma = i^r(-1)^{\frac{k(k+1)}{2}} \#_P, \quad r = (\dim M + 1)/2. \tag{7.22}$$

Next we consider a complex, that takes the splitting $M = M_1 \cup_N M_2$ into account. Let $\iota_j : N \hookrightarrow M_j, j = 1, 2$ be the natural inclusions. Put

$$\Omega^*(M_1 \# M_2, E) := \{(\omega_1, \omega_2) \in \Omega^*(M_1, E) \oplus \Omega^*(M_2, E) | \iota_1^* \omega_1 = \iota_2^* \omega_2\}.$$

Denote the restrictions of the flat connection ∇ to $M_j, j = 1, 2$ by ∇_j, and extend the restrictions by Leibniz rule to operators on the complexes $\Omega^*(M_j, E), j = 1, 2$. We put further

$$\nabla_S(\omega_1, \omega_2) := (\nabla_1 \omega_1, \nabla_2 \omega_2).$$

This operation respects the transmission condition of $\Omega^*(M_1 \# M_2, E)$ and further its square is obviously zero. Therefore ∇_S turns the graded vector space $\Omega^*(M_1 \# M_2, E)$ into a complex, denoted by

$$(\Omega^*(M_1 \# M_2, E), \nabla_S). \tag{7.23}$$

The natural L^2-structure on $\Omega^*(M_1, E) \oplus \Omega^*(M_2, E)$, induced by the metrics g^M and h^E is defined on any $\omega = (\omega_1, \omega_2), \eta = (\eta_1, \eta_2)$ as follows

$$\langle \omega, \eta \rangle_{L^2} := \sum_{j=1}^{2} \langle \omega_j, \eta_j \rangle_{L^2_j}. \tag{7.24}$$

In order to analyze the associated Laplace operators, consider first the adjoint to ∇_S operator ∇_S^* in $\Omega^*(M_1, E) \oplus \Omega^*(M_2, E)$ with domain of definition $\mathcal{D}(\nabla_S^*)$ consisting of elements $\omega = (\omega_1, \omega_2) \in \Omega^*(M_1, E) \oplus \Omega^*(M_2, E)$ such that the respective linear functionals on any $\eta = (\eta_1, \eta_2) \in \Omega^*(M_1 \# M_2, E)$

$$L_\omega(\eta) = \langle \omega, \nabla_S \eta \rangle_{L^2}$$

are continuous in $\Omega^*(M_1 \# M_2, E)$ with respect to the natural L^2-norm of η. As a consequence of Stokes' formula we find for such elements $\omega \in \mathcal{D}(\nabla_S^*)$ that the following transmission condition has to hold

$$*\omega = (*\omega_1, *\omega_2) \in \Omega^*(M_1 \# M_2, E), \tag{7.25}$$

where $*$ also denotes the restrictions of the usual Hodge star operator on M to M_1 and M_2. The Laplace operator $\triangle_S = \nabla_S^* \nabla_S + \nabla_S \nabla_S^*$ of the complex (7.23) acts on the obvious domain of definition

$$\begin{aligned} \mathcal{D}(\triangle_S) = \{ & \omega \in \Omega^*(M_1 \# M_2, E) | \\ & \omega \in \mathcal{D}(\nabla_S^*), \nabla_S \omega \in \mathcal{D}(\nabla_S^*), \nabla_S^* \omega \in \Omega^*(M_1 \# M_2, E) \}. \end{aligned} \tag{7.26}$$

The $Dom(\triangle_S)$ is defined as the completion of $\mathcal{D}(\triangle_S)$ with respect to the graph topology norm. The Laplacian $\triangle_S$ with domain $Dom(\triangle_S)$ is a self-adjoint operator in the L^2-completion of $\Omega^*(M_1, E) \oplus \Omega^*(M_2, E)$.

For the spectrum of $\triangle_S$ we refer to the theorem below, established essentially by S. Vishik in [V, Proposition 1.1] in the untwisted setup.

Theorem 7.6. *The generalized eigenforms of the Laplacian $\triangle_S$ and the generalized eigenforms of the Laplacian $\triangle$ associated to the twisted de Rham complex $(\Omega^*(M, E), \nabla)$ coincide.*

Proof. The conditions for $(\omega_1, \omega_2) \in \mathcal{D}(\triangle_S)$ translate with (7.25) equivalently to

$$\begin{aligned} & \iota_1^* \omega_1 = \iota_2^* \omega_2, \quad \iota_1^*(*\omega_1) = \iota_2^*(*\omega_2), \\ & \iota_1^*(*\nabla_1 \omega_1) = \iota_2^*(*\nabla_2 \omega_2), \quad \iota_1^*(\nabla_1^t \omega_1) = \iota_2^*(\nabla_2^t \omega_2). \end{aligned} \tag{7.27}$$

Any eigenform ω of $\triangle$ is smooth and thus $(\omega|_{M_1}, \omega|_{M_2})$ satisfies the conditions (7.27). Thus any eigenform $\omega \equiv (\omega|_{M_1}, \omega|_{M_2})$ of $\triangle$ belongs to $\mathcal{D}(\triangle_S)$ and hence is an eigenform of $\triangle_S$. We need to show the converse statement.

Let $(\omega_1, \omega_2) \in \mathcal{D}(\triangle_S)$ be an eigenform of $\triangle_S$. Then for any $k \in \mathbb{N}$ the element $(\triangle_1^k \omega_1, \triangle_2^k \omega_2)$ satisfies the conditions (7.27). Fix local coordinates (x, y) in the collar neighborhood $(-\epsilon, \epsilon) \times N$ of $N \subset M$ with $x \in (-\epsilon, \epsilon)$ being the normal coordinate and $y \in N$ the local coordinates on N. Then the conditions (7.27) imply for $k = 1$

$$\frac{\partial \omega_1(x=0,y)}{\partial x} = \frac{\partial \omega_2(x=0,y)}{\partial x}.$$

Iterative application of the conditions (7.27) to $(\triangle_1^k \omega_1, \triangle_2^k \omega_2)$ for $k \in \mathbb{N}$ shows

$$\forall k \in \mathbb{N}: \ \frac{\partial^k \omega_1(x=0,y)}{\partial x^k} = \frac{\partial^k \omega_2(x=0,y)}{\partial x^k}. \tag{7.28}$$

The eigenform (ω_1, ω_2) consists of smooth eigenforms ω_j over $M_j, j = 1, 2$. The result (7.28) shows smoothness on $N \subset M$. Thus (ω_1, ω_2) can be viewed as a smooth differential form over M and so lies in $\mathcal{D}(\triangle)$ and hence is an eigenform of $\triangle$ as well. This proves the theorem. □

Corollary 7.7. *The Laplacian $\triangle_S$ on Dom$(\triangle_S)$ is a Fredholm operator and*

$$H^*(M_1 \# M_2, E) := H^*(\Omega^*(M_1 \# M_2, E), \nabla_S) \cong H^*_{\mathrm{dR}}(M, E).$$

The corollary is an obvious consequence of Theorem 7.6 and the Hodge-isomorphism. Therefore the pairings $\langle \cdot, \cdot \rangle_{L^2}$ and $\langle \cdot, \cdot \rangle$ with the associated identifications $\#_{L^2}$ and $\#_P$ respectively, over the manifold M give rise to pairings and maps on $H^*(M_1 \# M_2, E)$. We do not introduce a distinguished notation for these induced constructions

$$\begin{aligned} \langle \cdot, \cdot \rangle_{L^2} : H^k(M_1 \# M_2, E) \times H^k(M_1 \# M_2, E) &\to \mathbb{C} \\ [(\omega_1, \omega_2)], [(\eta_1, \eta_2)] &\longmapsto \sum_{j=1}^2 \int_{M_j} h^E(\omega_j \wedge * \eta_j), \end{aligned} \tag{7.29}$$

$$\begin{aligned} \langle \cdot, \cdot \rangle_P : H^k(M_1 \# M_2, E) \times H^{m-k}(M_1 \# M_2, E) &\to \mathbb{C} \\ [(\omega_1, \omega_2)], [(\eta_1, \eta_2)] &\longmapsto \sum_{j=1}^2 \int_{M_j} h^E(\omega_j \wedge \eta_j), \end{aligned} \tag{7.30}$$

where $(\omega_1, \omega_2), (\eta_1, \eta_2)$ are a priori harmonic representatives of the corresponding cohomology classes, due to the Hodge isomorphisms applied in the identification of Corollary 7.7. A posteriori we find by the next lemma that the pairing $\langle \cdot\,,\cdot\rangle_P$ like the pairings $\langle \cdot\,,\cdot\rangle_{P_j}, j=1,2$ is well-defined on cohomology classes, i.e. need not be evaluated on harmonic representatives only.

Lemma 7.8. *The pairing $\langle\cdot,\cdot\rangle_P$ is a well-defined pairing on cohomology.*

Proof. Let $[(\omega_1,\omega_2)] \in H^k(M_1\#M_2,E)$ be a cohomology class with a representative $(\omega_1,\omega_2)+\nabla_S(\gamma_1,\gamma_2)$ where $(\omega_1,\omega_2)\in\ker\nabla_S$ and $(\omega_1,\omega_2),(\gamma_1,\gamma_2)\in \Omega^*(M_1\#M_2,E)$, in particular

$$\iota_1^*\omega_1 = \iota_2^*\omega_2,\ \ \iota_1^*\gamma_1=\iota_2^*\gamma_2.$$

Similarly let $[(\eta_1,\eta_2)] \in H^{m-k}(M_1\#M_2,E)$. Choose a representative $(\eta_1,\eta_2)+\nabla_S(\xi_1,\xi_2)$ with $(\eta_1,\eta_2)\in\ker\nabla_S$ and $(\eta_1,\eta_2),(\xi_1,\xi_2)\in \Omega^*(M_1\#M_2,E)$. We compute

$$\begin{aligned}\sum_{j=1}^2\int_{M_j} h^E((\omega_j+\nabla_j\gamma_j)\wedge(\eta_j+\nabla_j\xi_j)) - \sum_{j=1}^2\int_{M_j} h^E(\omega_j\wedge\eta_j) = \\ =\sum_{j=1}^2\int_{M_j} h^E(\nabla_j\gamma_j\wedge\eta_j)+\sum_{j=1}^2\int_{M_j} h^E(\omega_j\wedge\nabla_j\xi_j)+ \\ +\sum_{j=1}^2\int_{M_j} h^E(\nabla_j\gamma_j\wedge\nabla_j\xi_j). \qquad (7.31)\end{aligned}$$

In order to verify that the pairing $\langle \cdot\,,\cdot\rangle_P$ is a well-defined pairing on cohomology we need to show that the last three summands in (7.31) are zero. Consider the first summand, the other two are dealt with analogously. Under the assumption of flatness of ∇ we get

$$\begin{aligned} dh^E(\gamma_j\wedge\eta_j) = h^E(\nabla_j\gamma_j\wedge\eta)+(-1)^{k-1}h^E(\gamma_j\wedge\nabla_j\eta) = h^E(\nabla_j\gamma_j\wedge\eta) \\ \Rightarrow \sum_{j=1}^2\int_{M_j} h^E(\nabla_j\gamma_j\wedge\eta_j) = \sum_{j=1}^2\int_{M_j} dh^E(\gamma_j\wedge\eta_j) = \\ = \sum_{j=1}^2\int_{\partial M_j}\iota_j^* h^E(\gamma_j\wedge\eta_j).\end{aligned}$$

Since $\iota_1^*\gamma_1=\iota_2^*\gamma_2$ and $\iota_1^*\eta_1=\iota_2^*\eta_2$ we find

$$\iota_1^*h^E(\gamma_1\wedge\eta_1)=\iota_2^*h^E(\gamma_2\wedge\eta_2).$$

However the orientations on $N=\partial M_1=\partial M_2$ induced from M_1 and M_2 are opposite. Hence the two integrals over M_1 and M_2 cancel. This completes the argumentation. □

7.5 Commutative diagramms in cohomological algebra

Consider the short exact sequences of complexes:

$$0 \to (\Omega^*_{\min}(M_1,E),\nabla_1) \xrightarrow{\alpha} (\Omega^*(M_1\#M_2,E),\nabla_S) \xrightarrow{\beta} (\Omega^*_{\max}(M_2,E),\nabla_2) \to 0,$$
$$0 \to (\Omega^*_{\min}(M_2,E),\nabla_2) \xrightarrow{\alpha'} (\Omega^*(M_1\#M_2,E),\nabla_S) \xrightarrow{\beta'} (\Omega^*_{\max}(M_1,E),\nabla_1) \to 0,$$

where $\alpha(\omega)=(\omega,0), \alpha'(\omega)=(0,\omega)$ and $\beta(\omega_1,\omega_2)=\omega_2, \beta'(\omega_1,\omega_2)=\omega_1$. The exactness at the first and the second complex of both sequences is clear by construction. The surjectivity of β and β' is clear, since $\Omega^*_{\max}(M_j,E), j=1,2$ consists of smooth differential forms over M_j which are in particular smooth at the boundary. These short exact sequences of complexes induce long exact sequences on cohomology:

$$\begin{aligned}\mathcal{H}:\ & ...H^k_{\text{rel}}(M_1,E) \xrightarrow{\alpha^*} H^k(M_1\#M_2,E) \xrightarrow{\beta^*} H^k_{\text{abs}}(M_2,E) \xrightarrow{\delta^*} H^{k+1}_{\text{rel}}(M_1,E)...\\ \mathcal{H}':\ & ..H^k_{\text{rel}}(M_2,E) \xrightarrow{\alpha'^*} H^k(M_1\#M_2,E) \xrightarrow{\beta'^*} H^k_{\text{abs}}(M_1,E) \xrightarrow{\delta'^*} H^{k+1}_{\text{rel}}(M_2,E)\end{aligned} \tag{7.32}$$

The first long exact sequence is related to the dual of the second long exact sequence by the diagramm below, where $\alpha'_*, \beta'_*, \delta'_*$ denote the dualizations of $\alpha'^*, \beta'^*, \delta'^*$ respectively.

$$\begin{array}{ccccccc} H^k_{\text{rel}}(M_1,E) & \xrightarrow{\alpha^*} & H^k(M_1\#M_2,E) & \xrightarrow{\beta^*} & H^k_{\text{abs}}(M_2,E) & \xrightarrow{\delta^*} & H^{k+1}_{\text{rel}}(M_1,E) \\ \#_{L^2_1}\circ\Gamma\downarrow & & \#_{L^2}\circ\Gamma\downarrow & & \#_{L^2_2}\circ\Gamma\downarrow & & \#_{L^2_1}\circ\Gamma\downarrow \\ H^{m-k}_{\text{abs}}(M_1,E)^* & \xrightarrow{\beta'_*} & H^{m-k}(M_1\#M_2,E)^* & \xrightarrow{\alpha'_*} & H^{m-k}_{\text{rel}}(M_2,E)^* & \xrightarrow{\delta'_*} & H^{m-k-1}_{\text{abs}}(M_1,E)^* \end{array} \tag{7.33}$$

Theorem 7.9. *The diagramm* (7.33) *is commutative.*

Proof. We need to verify commutativity of three types of squares in the diagramm. Consider the first type of squares:

$$\begin{array}{ccc} H^k_{\text{rel}}(M_1,E) & \xrightarrow{\alpha^*} & H^k(M_1\#M_2,E) \\ \#_{L^2_1}\circ\Gamma\Big\downarrow & & \Big\downarrow\#_{L^2}\circ\Gamma \\ H^{m-k}_{\text{abs}}(M_1,E)^* & \xrightarrow[\beta'_*]{} & H^{m-k}(M_1\#M_2,E)^*. \end{array}$$

Let $[\omega] \in H^k_{\text{rel}}(M_1, E)$. Recall

$$\begin{aligned}\#_{L^2_1} \circ \Gamma &= i^r (-1)^{\frac{k(k+1)}{2}} \#_{P_1} \text{ on } H^k_{\text{rel}}(M_1, E),\\ \#_{L^2} \circ \Gamma &= i^r (-1)^{\frac{k(k+1)}{2}} \#_{P} \text{ on } H^k(M_1 \# M_2, E).\end{aligned}$$

The maps $\#_{P_1}, \#_P$ are well-defined identifications on cohomology, due to Lemma 7.8. Let $[(\eta_1, \eta_2)] \in H^{m-k}(M_1 \# M_2, E)$ and compute:

$$\begin{aligned}&(\beta'_* \circ \#_{L^2_1} \circ \Gamma)[\omega]([\eta_1, \eta_2]) - (\#_{L^2} \circ \Gamma \circ \alpha^*)[\omega]([\eta_1, \eta_2]) =\\ =&i^r (-1)^{\frac{k(k+1)}{2}} \{\langle \beta'(\eta_1, \eta_2), \omega\rangle_{P_1} - \langle (\eta_1, \eta_2), \alpha\omega\rangle_P\} =\\ =&i^r (-1)^{\frac{k(k+1)}{2}} \left\{ \int_{M_1} h^E(\eta_1 \wedge \omega) - \int_{M_1} h^E(\eta_1 \wedge \omega) \right\} = 0.\end{aligned}$$

Consider now the second type of squares in the diagramm (7.33).

$$\begin{array}{ccc} H^k(M_1 \# M_2, E) & \xrightarrow{\beta^*} & H^k_{\text{abs}}(M_2, E) \\ \downarrow{\scriptstyle \#_{L^2} \circ \Gamma} & & \downarrow{\scriptstyle \#_{L^2_2} \circ \Gamma} \\ H^{m-k}(M_1 \# M_2, E)^* & \xrightarrow{\alpha'_*} & H^{m-k}_{\text{rel}}(M_2, E)^*. \end{array}$$

Let $[(\omega_1, \omega_2)] \in H^k(M_1 \# M_2, E)$ and $[\eta] \in H^{m-k}_{\text{rel}}(M_2, E)$. As before the maps in the diagramm are independent of particular choices of representatives, so we compute:

$$\begin{aligned}(\alpha'_* \circ \#_{L^2} \circ \Gamma)[(\omega_1, \omega_2)]([\eta]) - (\#_{L^2_2} \circ \Gamma \circ \beta^*)[(\omega_1, \omega_2)]([\eta]) =&\\ = i^r (-1)^{\frac{k(k+1)}{2}} \{\langle \alpha'\eta, (\omega_1, \omega_2)\rangle_P - \langle \eta, \beta(\omega_1, \omega_2)\rangle_{P_2}\} =&\\ = i^r (-1)^{\frac{k(k+1)}{2}} \left\{ \int_{M_2} h^E(\eta \wedge \omega_2) - \int_{M_2} h^E(\eta \wedge \omega_2) \right\} = 0.&\end{aligned}$$

Consider finally the third type of squares.

$$\begin{array}{ccc} H^k_{\text{abs}}(M_2, E) & \xrightarrow{\delta^*} & H^{k+1}_{\text{rel}}(M_1, E) \\ \downarrow{\scriptstyle \#_{L^2_2} \circ \Gamma} & & \downarrow{\scriptstyle \#_{L^2_1} \circ \Gamma} \\ H^{m-k}_{\text{rel}}(M_2, E)^* & \xrightarrow{\delta'_*} & H^{m-k-1}_{\text{abs}}(M_1, E)^*. \end{array} \tag{7.34}$$

To prove commutativity of this diagramm, we need a precise understanding of the connecting homomorphisms δ^*, δ'^*. Note for this the following diagramm of short exact sequences of complexes:

$$\begin{array}{ccccccccc}
0 \to & (\Omega^*_{\min}(M_1,E),\nabla_1) & \xrightarrow{\alpha} & (\Omega^*(M_1\#M_2,E),\nabla_S) & \xrightarrow{\beta} & (\Omega^*_{\max}(M_2,E),\nabla_2) & \to 0 \\
& \| & & \| & & \beta_\pi \uparrow & \\
0 \to & (\Omega^*_{\min}(M_1,E),\nabla_1) & \xrightarrow{\alpha} & (\Omega^*(M_1\#M_2,E),\nabla_S) & \xrightarrow{\pi} & \overline{(\Omega^*_{\max}(M_2,E),\nabla_2)} & \to 0.
\end{array} \tag{7.35}$$

The complex $\overline{(\Omega^*_{\max}(M_2,E),\nabla_2)}$ in the lower short exact sequence is the natural quotient of complexes

$$\overline{(\Omega^*_{\max}(M_2,E),\nabla_2)} := \frac{(\Omega^*(M_1\#M_2,E),\nabla_S)}{\alpha(\Omega^*_{\min}(M_1,E),\nabla_1)}.$$

The complex map π is the natural projection. The map β_π is an isomorphism of complexes:

$$\begin{aligned}\beta_\pi : \overline{(\Omega^*_{\max}(M_2,E),\nabla_2)} &\to (\Omega^*_{\max}(M_2,E),\nabla_2)\\ [(\omega_1,\omega_2)] &\mapsto \beta(\omega_1,\omega_2)=\omega_2.\end{aligned}$$

The diagramm (7.35) of short exact sequences of complexes obviously commutes. Hence the associated diagramm of long exact sequences on cohomology is also commutative and in particular we obtain the following commutative diagramm:

$$\begin{array}{ccc}
H^k_{\text{abs}}(M_2,E) & \xrightarrow{\delta^*} & H^{k+1}_{\text{rel}}(M_1,E) \\
\beta^*_\pi \uparrow & & \| \\
H^k(\overline{(\Omega^*_{\max}(M_2,E),\nabla_2)}) & \xrightarrow{d^*} & H^{k+1}_{\text{rel}}(M_1,E).
\end{array} \tag{7.36}$$

The vertical map β^*_π is the isomorphism induced by β_π and δ^*, d^* are the connecting homomorphisms of the long exact sequences associated to the lower and upper short exact sequence of complexes of (7.35), respectively.

The connecting homorphism d^* is easily defined. Let namely $[(\omega_1,\omega_2)] \in H^k(\overline{(\Omega^*_{\max}(M_2,E),\nabla_2)})$. Any of its representatives $(\omega_1,\omega_2)\in\Omega^k(M_1\#M_2,E)$ satisfies $\nabla_S(\omega_1,\omega_2)=(\nabla\omega_1,0)\in\alpha(\Omega^*_{\min}(M_1,E),\nabla_1)$ by definition. Then

$$d^*[(\omega_1,\omega_2)] = [\nabla_1\omega_1] \in H^{k+1}_{\text{rel}}(M_1,E).$$

Consider now the next diagramm of short exact sequences of complexes:

$$\begin{array}{ccccccccc} 0 \to & (\Omega^*_{\min}(M_2,E),\nabla_2) & \xrightarrow{\alpha'} & (\Omega^*(M_1\#M_2,E),\nabla_S) & \xrightarrow{\beta'} & (\Omega^*_{\max}(M_1,E),\nabla_1) & \to 0 \\ & \| & & \| & & \beta'_\pi \uparrow & \\ 0 \to & (\Omega^*_{\min}(M_2,E),\nabla_2) & \xrightarrow{\alpha'} & (\Omega^*(M_1\#M_2,E),\nabla_S) & \xrightarrow{\pi'} & \overline{(\Omega^*_{\max}(M_1,E),\nabla_1)} & \to 0. \end{array} \tag{7.37}$$

The complex $\overline{(\Omega^*_{\max}(M_1,E),\nabla_1)}$ in the lower short exact sequence is the natural quotient of complexes

$$\overline{(\Omega^*_{\max}(M_1,E),\nabla_1)} := \frac{(\Omega^*(M_1\#M_2,E),\nabla_S)}{\alpha(\Omega^*_{\min}(M_2,E),\nabla_2)}.$$

The complex map π' is the natural projection. The map β'_π is an isomorphism of complexes:

$$\begin{aligned} \beta'_\pi : \overline{(\Omega^*_{\max}(M_1,E),\nabla_1)} &\to (\Omega^*_{\max}(M_1,E),\nabla_1) \\ [(\omega_1,\omega_2)] &\mapsto \beta'(\omega_1,\omega_2) = \omega_1. \end{aligned}$$

The diagramm (7.37) of short exact sequences of complexes obviously commutes. Hence the associated diagramm of long exact sequences on cohomology is also commutative and in particular we obtain the following commutative diagramm:

$$\begin{array}{ccc} H^k_{abs}(M_1,E) & \xrightarrow{\delta'^*} & H^{k+1}_{rel}(M_2,E) \\ \beta'^*_\pi \uparrow & & \| \\ H^k(\overline{(\Omega^*_{\max}(M_1,E),\nabla_1)}) & \xrightarrow{d'^*} & H^{k+1}_{rel}(M_2,E). \end{array} \tag{7.38}$$

The vertical map β'^*_π is the isomorphism induced by β'_π and δ'^*, d'^* are the connecting homomorphisms of the long exact sequences associated to the lower and upper short exact sequence of complexes of (7.37), respectively.

The connecting homomorphism d'^* is easily defined. Let namely any $[(\eta_1,\eta_2)] \in H^{m-k-1}(\overline{(\Omega^*_{\max}(M_1,E),\nabla_1)})$. For any representative $(\eta_1,\eta_2) \in \Omega^{m-k-1}(M_1\#M_2,E)$ we have

$$\nabla_S(\eta_1,\eta_2) = (0,\nabla_2\eta_2) \in \alpha'(\Omega^*_{\min}(M_2,E),\nabla_2)$$

by definition. We obtain for the connecting homomorphism d'^*

$$d'^*[(\eta_1,\eta_2)] = [\nabla_2\eta_2] \in H^{m-k}_{rel}(M_2,E).$$

Now the pairings $\langle \cdot , \cdot \rangle_{P_1}, \langle \cdot , \cdot \rangle_{P_2}$, introduced in Section 7.4 induce via the isomorphisms on cohomology $\beta^*_\pi, \beta'^*_\pi$ the analogous pairings:

$$\overline{\langle \cdot , \cdot \rangle}_{P_2} := \langle \cdot , \beta^*_\pi(\cdot) \rangle_{P_2} : H^{m-k}_{\text{rel}}(M_2, E) \times H^k(\overline{(\Omega^*_{\max}(M_2, E), \nabla_2)}) \to \mathbb{C},$$
$$\overline{\langle \cdot , \cdot \rangle}_{P_1} := \langle \beta'^*_\pi(\cdot) , \cdot \rangle_{P_1} : H^{m-k-1}(\overline{(\Omega^*_{\max}(M_1, E), \nabla_1)}) \times H^{k+1}_{\text{rel}}(M_1, E) \to \mathbb{C}.$$

These pairings induce the following identifications

$$\begin{aligned} \overline{\#}_{P_2} : H^k(\overline{(\Omega^*_{\max}(M_2, E), \nabla_2)}) &\xrightarrow{\sim} \left(H^{m-k}_{\text{rel}}(M_2, E)\right)^*, \\ [\omega] &\longmapsto \overline{\langle \cdot , \, [\omega] \rangle}_{P_2} \equiv \langle \cdot , \beta^*_\pi([\omega]) \rangle_{P_2}, \\ \overline{\#}_{P_1} : H^{k+1}_{\text{rel}}(M_1, E) &\xrightarrow{\sim} \left(H^{m-k-1}(\overline{(\Omega^*_{\max}(M_1, E), \nabla_1)})\right)^*, \\ [\omega] &\longmapsto \overline{\langle \cdot , \, [\omega] \rangle}_{P_1} \equiv \langle \beta'^*_\pi(\cdot) , [\omega] \rangle_{P_1}. \end{aligned}$$

Due to commutativity of the previous two diagramms (7.36) and (7.38), the commutativity of (7.34) is equivalent to commutativity of the following diagramm:

$$\begin{array}{ccc} H^k(\overline{(\Omega^*_{\max}(M_2, E), \nabla_2)}) & \xrightarrow{d^*} & H^{k+1}_{\text{rel}}(M_1, E) \\ \Big\downarrow {\scriptstyle i^r(-1)^{\frac{k(k+1)}{2}} \overline{\#}_{P_2}} & & \Big\downarrow {\scriptstyle i^r(-1)^{\frac{(k+1)(k+2)}{2}} \overline{\#}_{P_1}} \\ H^{m-k}_{\text{rel}}(M_2, E)^* & \xrightarrow{d'_*} & H^{m-k-1}(\overline{(\Omega^*_{\max}(M_1, E), \nabla_1)})^*. \end{array} \tag{7.39}$$

Using the explicit form of the connecting homomorphisms d^* and d'^* we finally compute for any $[(\omega_1, \omega_2)] \in H^k(\overline{(\Omega^*_{\max}(M_2, E), \nabla_2)})$ and $[(\eta_1, \eta_2)] \in H^{m-k-1}(\overline{(\Omega^*_{\max}(M_1, E), \nabla_1)})$:

$$\begin{aligned} &\left(i^r(-1)^{\frac{(k+1)(k+2)}{2}} \overline{\#}_{P_1} \circ d^*\right)[(\omega_1, \omega_2)]([(\eta_1, \eta_2)]) - \\ &\quad \left(i^r(-1)^{\frac{k(k+1)}{2}} d'_* \circ \overline{\#}_{P_2}\right)[(\omega_1, \omega_2)]([(\eta_1, \eta_2)]) = \\ &= i^r(-1)^{\frac{(k+1)(k+2)}{2}} \langle \beta'^*_\pi[(\eta_1, \eta_2)], d^*[(\omega_1, \omega_2)] \rangle_{P_1} - \\ &\quad i^r(-1)^{\frac{k(k+1)}{2}} \langle d'^*[(\eta_1, \eta_2)], \beta^*_\pi[(\omega_1, \omega_2)] \rangle_{P_2} = \\ &= i^r(-1)^{\frac{(k+1)(k+2)}{2}} \int_{M_1} h^E(\eta_1 \wedge \nabla_1 \omega_1) - \\ &\quad - i^r(-1)^{\frac{k(k+1)}{2}} \int_{M_2} h^E(\nabla_2 \eta_2 \wedge \omega_2) =: A. \end{aligned}$$

Now we apply the following formula for $j = 1, 2$:

$$dh^E(\eta_j \wedge \omega_j) = h^E(\nabla_j \eta_j \wedge \omega_j) + (-1)^{m-k-1} h^E(\eta_j \wedge \nabla_j \omega_j).$$

Since $\nabla_1 \eta_1 = 0$ and $\nabla_2 \omega_2 = 0$ we find

$$\begin{aligned} A = i^r(-1)^{\frac{(k+1)(k+2)}{2}} \int_{M_1} (-1)^{m-k-1} dh^E(\eta_1 \wedge \omega_1) - \\ i^r(-1)^{\frac{k(k+1)}{2}} \int_{M_2} dh^E(\eta_2 \wedge \omega_2) = \\ i^r(-1)^{\frac{(k+1)(k+2)}{2}} (-1)^{m-k-1} \int_{\partial M_1} \iota_1^* h^E(\eta_1 \wedge \omega_1) - \\ i^r(-1)^{\frac{k(k+1)}{2}} \int_{\partial M_2} \iota_2^* h^E(\eta_2 \wedge \omega_2). \end{aligned}$$

Note $(-1)^{m-k-1} = (-1)^{-k}$ since m is odd. Further

$$\frac{(k+1)(k+2)}{2} - k = \frac{k(k+1)}{2}.$$

Hence we compute further

$$A = i^r(-1)^{\frac{k(k+1)}{2}+1} \left[\int_{\partial M_1} \iota_1^* h^E(\eta_1 \wedge \omega_1) + \int_{\partial M_2} \iota_2^* h^E(\eta_2 \wedge \omega_2) \right]. \qquad (7.40)$$

Since $\iota_1^* \omega_1 = \iota_2^* \omega_2$ and $\iota_1^* \eta_1 = \iota_2^* \eta_2$ by construction, we find

$$\iota_1^* h^E(\eta_1 \wedge \omega_1) = \iota_2^* h^E(\eta_2 \wedge \omega_2).$$

However the orientations on $N = \partial M_1 = \partial M_2$ induced from M_1 and M_2 are opposite, thus the two integrals in (7.40) cancel. This shows commutativity of (7.39) and completes the proof of the theorem. □

7.6 Canonical Isomorphisms associated to Long Exact Sequences

We first introduce some concepts and notations on finite-dimensional vector spaces. Let V be an finite-dimensional complex vector space. Given a basis $\{v\} := \{v_1, .., v_n\}, n = \dim V$, denote the induced element of the determinant line $\det V$ as follows

$$[v] := v_1 \wedge .. \wedge v_n \in \det V.$$

Given any two bases $\{v\} := \{v_1, .., v_n\}$ and $\{w\} := \{w_1, .., w_n\}$ of V, we have the corresponding coordinate change matrix

$$v_i = \sum_{j=1}^{n} l_{ij} w_j, \quad L := (l_{ij}).$$

We put

$$[v/w] := \det L \in \mathbb{C},$$

and obtain the following relation

$$[v] = [v/w][w]. \tag{7.41}$$

In general the determinant is a complex number (we don't take the mode), but later it will be convenient to have a relation between bases such that the determinant of the coordinate change matrix is real-valued and positive. We will use the result of the following lemma.

Lemma 7.10. *Let V be a complex finite-dimensional Hilbert space and $\{v\}$ any fixed basis, not necessarily orthogonal. Let $V = W \oplus W^{\perp}$ be an orthogonal decomposition into Hilbert subspaces. Then there exist orthonormal bases $\{w\} \equiv \{w_1, ..w_{\dim W}\}, \{u\} \equiv \{u_1, ..u_{\dim W^{\perp}}\}$ of $W, W^{\perp}$ respectively, such that the determinant of the coordinate change matrix between $\{w, u\}$ and $\{v\}$ is positive, i.e.*

$$[w, u/v] \in \mathbb{R}^{+}.$$

Proof. Consider any orthonormal bases $\{w\}$ and $\{u\}$ of W and $W^{\perp}$, respectively. This gives us two bases $\{v\}$ and $\{w, u\}$ of V. Denote the corresponding coordinate change matrix by L. We have

$$[w, u/v] = \det L = e^{i\phi}|\det L|,$$

for some $\phi \in [0, 2\pi)$. We replace $\{w\}$ and $\{u\}$ by new bases

$$\{w^v\} \equiv \{w^v_1, .., w^v_{\dim W}\}, \; w^v_i := w_i \cdot \exp\left(\frac{-i\phi}{\dim V}\right),$$

$$\{u^v\} \equiv \{u^v_1, .., u^v_{\dim W^{\perp}}\}, \; u^v_i := u_i \cdot \exp\left(\frac{-i\phi}{\dim V}\right).$$

Note that $\{w^v\}$ and $\{u^v\}$ are still orthonormal bases of complex Hilbert spaces W and $W^{\perp}$, respectively. By construction $[w^v, u^v/w, u] = \exp(-i\phi)$ and

$$[w^v, u^v/v] = [w^v, u^v/w, u][w, u/v] = e^{-i\phi} \cdot e^{i\phi}|\det L| = |\det L| \in \mathbb{R}^{+}.$$

Thus $\{w^v, u^v\}$ indeed provides the desired example of an orthonormal basis of V, respecting the given orthogonal decomposition, with positive determinant of the coordinate change $[w^v, u^v/v]$ relative to any given basis $\{v\}$. □

The decomposition $V = W \oplus W^{\perp}$ in the lemma above is of course not essential for the statement itself. However we presented the result precisely in the form how it will be applied later. We will also need the following purely algebraic result:

Proposition 7.11. *Let V and W be two finite-dimensional Hilbert spaces with some orthonormal bases $\{v\}$ and $\{w\}$ respectively. Let $f : V \to W$ be an isomorphism of vector spaces. Then $\{f(v)\}$ is also a basis of W, not necessarily orthonormal. As Hilbert spaces V and W are canonically identified with their duals V^* and W^*. Then $\{v^*\}, \{w^*\}$ are bases of V^*, W^* respectively and $\{f^*(w^*)\}$ is another basis of V^*. Under this setup the following relation holds*

$$[f(v)/w] = [f^*(w^*)/v^*].$$

Proof. Denote the scalar products on the Hilbert spaces V and W by $\langle \cdot , \cdot \rangle_V$ and $\langle \cdot , \cdot \rangle_W$, respectively. Let the scalar products be linear in the second component. They induce scalar products on $\det V$ and $\det W$, denoted by $\langle \cdot , \cdot \rangle_{\det V}$ and $\langle \cdot , \cdot \rangle_{\det W}$ respectively. Since the bases $\{v\}, \{w\}$ are orthonormal, we obtain for the elements $[v], [w]$ of the determinant lines $\det V, \det W$

$$\langle [v], [v] \rangle_{\det V} = \langle [w], [w] \rangle_{\det W} = 1.$$

The dual bases $\{v^*\}, \{w^*\}$ induce elements on the determinant lines $\det V^* \cong (\det V)^*$, and $\det W^* \cong (\det W)^*$ and under these identifications we have

$$[v^*] = [v]^* = \langle [v], \cdot \rangle_{\det V},$$
$$[w^*] = [w]^* = \langle [w], \cdot \rangle_{\det W}.$$

Now we compute

$$\begin{aligned}
[f^*(w^*)]([v]) = \langle [w], [f(v)] \rangle_{\det W} = [f(v)/w] \langle [w], [w] \rangle_{\det W} = \\
= [f(v)/w] \cdot 1 = [f(v)/w] \cdot [v^*]([v]), \\
\Rightarrow [f^*(w^*)] = [f(v)/w][v^*].
\end{aligned}$$

This implies the statement of the proposition. □

Next we consider the long exact sequences (7.32), introduced in Subsection 7.5.

$$\mathcal{H} : \ ...H^k_{\text{rel}}(M_1, E) \xrightarrow{\alpha^*_k} H^k(M_1 \# M_2, E) \xrightarrow{\beta^*_k} H^k_{\text{abs}}(M_2, E) \xrightarrow{\delta^*_k} H^{k+1}_{\text{rel}}(M_1, E) ...$$

$$\mathcal{H}' : ..H^k_{\text{rel}}(M_2, E) \xrightarrow{\alpha'^*_k} H^k(M_1 \# M_2, E) \xrightarrow{\beta'^*_k} H^k_{\text{abs}}(M_1, E) \xrightarrow{\delta'^*_k} H^{k+1}_{\text{rel}}(M_2, E) ...$$

The long exact sequences induce isomorphisms on determinant lines (cf. [Nic]) in a canonical way

$$\Phi : \det H^*_{\text{rel}}(M_1, E) \otimes \det H^*_{\text{abs}}(M_2, E) \otimes [\det H^*(M_1 \# M_2, E)]^{-1} \to \mathbb{C}, \tag{7.42}$$

$$\Phi' : \det H^*_{\text{rel}}(M_2, E) \otimes \det H^*_{\text{abs}}(M_1, E) \otimes [\det H^*(M_1 \# M_2, E)]^{-1} \to \mathbb{C}. \tag{7.43}$$

More precisely , the action of the isomorphisms Φ, Φ' is explicitly given as follows. Fix any bases $\{\widetilde{a}_k\}$, $\{\widetilde{b}_k\}$ and $\{\widetilde{c}_k\}$ of $\text{Im}\delta^*_{k-1}$, $\text{Im}\alpha^*_k$ and $\text{Im}\beta^*_k$ respectively. Here the lower index k indexes the entire basis and is not a counting of the elements in the set. Choose now any linearly independent elements $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ such that $\{\widetilde{a}_k\} = \delta^*_{k-1}(c_{k-1})$, $\{\widetilde{b}_k\} = \alpha^*_k(a_k)$ and $\{\widetilde{c}_k\} = \beta^*_k(b_k)$.

We make the same choices on the long exact sequence $\mathcal{H}'$. The notation is the same up to an additional apostroph. Since the sequences $\mathcal{H}, \mathcal{H}'$ are exact, the choices above provide us with bases of the cohomology spaces.

Under the Knudson-Mumford sign convention [KM] we define the action of the isomorphisms Φ and Φ' as follows:

$$\Phi \left\{ \left(\bigotimes_{k=0}^{m} [a_k, \widetilde{a}_k]^{(-1)^k} \right) \otimes \left(\bigotimes_{k=0}^{m} [c_k, \widetilde{c}_k]^{(-1)^k} \right) \otimes \left(\bigotimes_{k=0}^{m} [b_k, \widetilde{b}_k]^{(-1)^{k+1}} \right) \right\} \mapsto (-1)^{\nu}, \tag{7.44}$$

$$\Phi' \left\{ \left(\bigotimes_{k=0}^{m} [a'_k, \widetilde{a}'_k]^{(-1)^k} \right) \otimes \left(\bigotimes_{k=0}^{m} [c'_k, \widetilde{c}'_k]^{(-1)^k} \right) \otimes \left(\bigotimes_{k=0}^{m} [b'_k, \widetilde{b}'_k]^{(-1)^{k+1}} \right) \right\} \mapsto (-1)^{\nu'}. \tag{7.45}$$

The definition turns out to be independent of choices. The numbers ν, ν' count the pairwise reorderings in the definition of the isomorphisms. They are given explicitly by the following formula:

$$\begin{aligned}\nu = &\frac{1}{2}\sum_{k=0}^{m}\left(\dim \mathrm{Im}\alpha_k^* \cdot (\dim \mathrm{Im}\alpha_k^* + (-1)^k)\right) + \\ &\frac{1}{2}\sum_{k=0}^{m}\left(\dim \mathrm{Im}\beta_k^* \cdot (\dim \mathrm{Im}\beta_k^* + (-1)^k)\right) + \\ &\frac{1}{2}\sum_{k=0}^{m}\left(\dim \mathrm{Im}\delta_k^* \cdot (\dim \mathrm{Im}\delta_k^* + (-1)^k)\right) + \\ &\sum_{k=0}^{m}\left(\dim H^k_{\mathrm{rel}}(M_1,E)\cdot \sum_{i=0}^{k-1}\dim H^i(M_1\# M_2,E)\right) + \\ &\sum_{k=0}^{m}\left(\dim H^k_{\mathrm{rel}}(M_1,E)\cdot \sum_{i=0}^{k-1}\dim H^i_{\mathrm{abs}}(M_2,E)\right) + \\ &\sum_{k=0}^{m}\left(\dim H^i_{\mathrm{abs}}(M_2,E)\cdot \sum_{i=0}^{k-1}\dim H^i_{\mathrm{abs}}(M_2,E)\right). \end{aligned} \tag{7.46}$$

The first three lines in the formula are standard terms for "cancellations" of images and cokernels of the homomorphisms in an acyclic sequence of vector spaces. The last three lines are due to reordering of the cohomology groups into determinant lines. The number ν' is given by an analogous formula as ν. As a consequence of Theorem 7.9 which relates both sequences $\mathcal{H}$ and $\mathcal{H}'$ we have

$$\nu = \nu'.$$

Let the cohomology spaces in the long exact sequences $\mathcal{H}$ and $\mathcal{H}'$ be endowed with Hilbert structures naturally induced by the L^2-scalar products on harmonic elements. We have an orthogonal decomposition of each cohomology space in the long exact sequences:

$$\begin{aligned} H^k_{\mathrm{rel}}(M_1,E) &= \mathrm{Im}\delta^*_{k-1} \oplus (\mathrm{Im}\delta^*_{k-1})^{\perp}, \\ H^k(M_1\# M_2,E) &= \mathrm{Im}\alpha^*_k \oplus (\mathrm{Im}\alpha^*_k)^{\perp}, \\ H^k_{\mathrm{abs}}(M_2,E) &= \mathrm{Im}\beta^*_k \oplus (\mathrm{Im}\beta^*_k)^{\perp}, \\ H^k_{\mathrm{rel}}(M_2,E) &= \mathrm{Im}\delta'^*_{k-1} \oplus (\mathrm{Im}\delta'^*_{k-1})^{\perp}, \\ H^k(M_1\# M_2,E) &= \mathrm{Im}\alpha'^*_k \oplus (\mathrm{Im}\alpha'^*_k)^{\perp}, \\ H^k_{\mathrm{abs}}(M_1,E) &= \mathrm{Im}\beta'^*_k \oplus (\mathrm{Im}\beta'^*_k)^{\perp}. \end{aligned} \tag{7.47}$$

We can assume the bases $\{a_k, \widetilde{a}_k\}, \{b_k, \widetilde{b}_k\}, \{c_k, \widetilde{c}_k\}$ on $\mathcal{H}$ as well as the corresponding bases on $\mathcal{H}'$ to respect the orthogonal decomposition above, i.e. with respect to the orthogonal decompositions in (7.47) we have

$$\begin{aligned}
H^k_{\text{rel}}(M_1, E) &= \langle\{\widetilde{a}_k\}\rangle \oplus \langle\{a_k\}\rangle, \\
H^k(M_1 \# M_2, E) &= \langle\{\widetilde{b}_k\}\rangle \oplus \langle\{b_k\}\rangle, \\
H^k_{\text{abs}}(M_2, E) &= \langle\{\widetilde{c}_k\}\rangle \oplus \langle\{c_k\}\rangle, \\
H^k_{\text{rel}}(M_2, E) &= \langle\{\widetilde{a}'_k\}\rangle \oplus \langle\{a'_k\}\rangle, \\
H^k(M_1 \# M_2, E) &= \langle\{\widetilde{b}'_k\}\rangle \oplus \langle\{b'_k\}\rangle, \\
H^k_{\text{abs}}(M_1, E) &= \langle\{\widetilde{c}'_k\}\rangle \oplus \langle\{c'_k\}\rangle.
\end{aligned}$$

By Lemma 7.10 we can choose for any $k = 0, .., \dim M$ orthonormal bases of $H^k_{\text{rel}}(M_1, E), H^k(M_1 \# M_2, E), H^k_{\text{abs}}(M_2, E)$ with respect to orthogonal decomposition (7.47)

$$\begin{aligned}
H^k_{\text{rel}}(M_1, E) &= \langle\{\widetilde{v}_k\}\rangle \oplus \langle\{v_k\}\rangle, \\
H^k(M_1 \# M_2, E) &= \langle\{\widetilde{w}_k\}\rangle \oplus \langle\{w_k\}\rangle, \\
H^k_{\text{abs}}(M_2, E) &= \langle\{\widetilde{u}_k\}\rangle \oplus \langle\{u_k\}\rangle,
\end{aligned} \tag{7.48}$$

such that

$$[v_k, \widetilde{v}_k / a_k, \widetilde{a}_k], \; [u_k, \widetilde{u}_k / c_k, \widetilde{c}_k], \; [w_k, \widetilde{w}_k / b_k, \widetilde{b}_k] \; \in \mathbb{R}^+. \tag{7.49}$$

These bases induce bases of the cohomology spaces of the sequence $\mathcal{H}'$ by the action of the Poincare duality map Γ. Since the map is an isometry, the induced bases are still orthonormal. Furthermore commutativity of the diagramm (7.33), established in Theorem 7.9 implies that the induced bases still respect the orthogonal decomposition (7.47) of the cohomology spaces.

$$\begin{aligned}
H^{m-k}_{\text{rel}}(M_2, E) &= \langle\{\Gamma u_k\}\rangle \oplus \langle\{\Gamma \widetilde{u}_k\}\rangle, \\
H^{m-k}(M_1 \# M_2, E) &= \langle\{\Gamma w_k\}\rangle \oplus \langle\{\Gamma \widetilde{w}_k\}\rangle, \\
H^{m-k}_{\text{abs}}(M_1, E) &= \langle\{\Gamma v_k\}\rangle \oplus \langle\{\Gamma \widetilde{v}_k\}\rangle.
\end{aligned}$$

We obtain for the action of the canonical isomorphisms on the elements induced by these orthonormal bases the following central result, which relates the action of the isomorphisms to the combinatorial torsion of the long exact sequences.

Theorem 7.12.

$$\Phi\left\{\left(\bigotimes_{k=0}^{m}[v_k,\widetilde{v}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[u_k,\widetilde{u}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[w_k,\widetilde{w}_k]^{(-1)^{k+1}}\right)\right\}=$$
$$\Phi'\left\{\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{v}_k,\Gamma v_k]^{(-1)^{m-k}}\right)\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{u}_k,\Gamma u_k]^{(-1)^{m-k}}\right)\otimes\right.$$
$$\left.\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{w}_k,\Gamma w_k]^{(-1)^{m-k+1}}\right)\right\}=(-1)^{\nu}\cdot\tau(\mathcal{H})=(-1)^{\nu}\tau(\mathcal{H}').$$

Remark 7.13. *The statement of the theorem corresponds to the fact that the combinatorial torsions $\tau(\mathcal{H}),\tau(\mathcal{H}')$ are defined as modes of the complex numbers obtained by the action of the isomorphisms Φ,Φ' on the volume elements, induced by the Hilbert structures.*

However the value of the theorem for our purposes is firstly the equality $\tau(\mathcal{H})=\tau(\mathcal{H}')$ and most importantly the fact that it provides explicit volume elements on the determinant lines, which are mapped to the real-valued positive combinatorial torsions without additional undetermined complex factors of the form $e^{i\phi}$.

Proof of Theorem 7.12. Consider first the action of the canonical isomorphism Φ. By the action (7.44) we obtain

$$\Phi\left\{\left(\bigotimes_{k=0}^{m}[v_k,\widetilde{v}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[u_k,\widetilde{u}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[w_k,\widetilde{w}_k]^{(-1)^{k+1}}\right)\right\}=$$
$$(-1)^{\nu}\prod_{k=0}^{m}[v_k,\widetilde{v}_k/a_k,\widetilde{a}_k]^{(-1)^k}\cdot[u_k,\widetilde{u}_k/c_k,\widetilde{c}_k]^{(-1)^k}\cdot[w_k,\widetilde{w}_k/b_k,\widetilde{b}_k]^{(-1)^{k+1}}=$$
$$=(-1)^{\nu}\tau(\mathcal{H}), \tag{7.50}$$

where the second equality follows from the definition of combinatorial torsion and the particular choice of bases such that (7.49) holds. On the other hand we can rewrite the action of Φ as follows:

$$\Phi\left\{\left(\bigotimes_{k=0}^{m}[v_k,\widetilde{v}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[u_k,\widetilde{u}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[w_k,\widetilde{w}_k]^{(-1)^{k+1}}\right)\right\}=$$
$$(-1)^{\nu}\prod_{k=0}^{m}[v_k,\widetilde{v}_k/a_k,\widetilde{a}_k]^{(-1)^k}\cdot[u_k,\widetilde{u}_k/c_k,\widetilde{c}_k]^{(-1)^k}\cdot[w_k,\widetilde{w}_k/b_k,\widetilde{b}_k]^{(-1)^{k+1}}=$$
$$(-1)^{\nu}\prod_{k=0}^{m}[v_k/a_k]^{(-1)^k}[\widetilde{v}_k/\widetilde{a}_k]^{(-1)^k}\cdot[u_k/c_k]^{(-1)^k}\cdot[\widetilde{u}_k/\widetilde{c}_k]^{(-1)^k}$$
$$\cdot[w_k/b_k]^{(-1)^{k+1}}\cdot[\widetilde{w}_k/\widetilde{b}_k]^{(-1)^{k+1}}. \tag{7.51}$$

Observe now the following useful relations:

$$[\alpha_k^*(v_k)]=[\alpha_k^*(v_k)/\alpha_k^*(a_k)][\alpha_k^*(a_k)]=[v_k/a_k][\widetilde{b}_k]=\frac{[v_k/a_k]}{[\widetilde{w}_k/\widetilde{b}_k]}\cdot[\widetilde{w}_k]$$
$$\text{and hence}\quad[\alpha_k^*(v_k)/\widetilde{w}_k]=\frac{[v_k/a_k]}{[\widetilde{w}_k/\widetilde{b}_k]},$$
$$[\beta_k^*(w_k)/\widetilde{u}_k]=\frac{[w_k/b_k]}{[\widetilde{u}_k/\widetilde{c}_k]},$$
$$[\delta_k^*(u_k)/\widetilde{v}_{k+1}]=\frac{[u_k/c_k]}{[\widetilde{v}_{k+1}/\widetilde{a}_{k+1}]},$$

where the last two identities are derived in the similar manner as the first one. With these relations we can rewrite the action (7.51) of Φ as follows:

$$\Phi\left\{\left(\bigotimes_{k=0}^{m}[v_k,\widetilde{v}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[u_k,\widetilde{u}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[w_k,\widetilde{w}_k]^{(-1)^{k+1}}\right)\right\}=$$
$$(-1)^{\nu}\prod_{k=0}^{m}[\alpha_k^*(v_k)/\widetilde{w}_k]^{(-1)^k}\cdot[\beta_k^*(w_k)/\widetilde{u}_k]^{(-1)^{k+1}}\cdot[\delta_k^*(u_k)/\widetilde{v}_{k+1}]^{(-1)^k}. \tag{7.52}$$

Analogous argumentation for the canonical isomorphism Φ' shows

$$\Phi'\left\{\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{v}_k,\Gamma v_k]^{(-1)^{m-k}}\right)\otimes\right. \tag{7.53}$$

$$\left.\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{u}_k,\Gamma u_k]^{(-1)^{m-k}}\right)\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{w}_k,\Gamma w_k]^{(-1)^{k}}\right)\right\}$$

$$=(-1)^{\nu}\prod_{k=0}^{m}[\alpha'^{*}_{m-k}(\Gamma\widetilde{u}_k)/\Gamma w_k]^{(-1)^{m-k}}\cdot[\beta'^{*}_{m-k}(\Gamma\widetilde{w}_k)/\Gamma v_k]^{(-1)^{k}}$$

$$\cdot[\delta'^{*}_{m-k}(\Gamma\widetilde{v}_k)/\Gamma u_{k-1}]^{(-1)^{m-k}}. \tag{7.54}$$

Now using the fact that the diagramm (7.33) is commutative with vertical maps being linear, we obtain

$$[\delta^*_k(u_k)/\widetilde{v}_{k+1}]=[(\#_{L^2_1}\circ\Gamma)\delta^*_k(u_k)/(\#_{L^2_1}\circ\Gamma)\widetilde{v}_{k+1}]=[\delta'^{m-k}_{*}(\Gamma u_k)^*/(\Gamma\widetilde{v}_{k+1})^*],$$

$$[\alpha^*_k(v_k)/\widetilde{w}_k]=[\beta'^{m-k}_{*}(\Gamma v_k)^*/(\Gamma\widetilde{w}_k)^*],$$

$$[\beta^*_k(w_k)/\widetilde{u}_k]=[\alpha'^{m-k}_{*}(\Gamma w_k)^*/(\Gamma\widetilde{u}_k)^*],$$

where the last two identities are derived in a similar manner as the first one. Now with the following purely algebraic result of Proposition 7.11 we obtain

$$[\delta^*_k(u_k)/\widetilde{v}_{k+1}]=[\delta'^{m-k}_{*}(\Gamma u_k)^*/(\Gamma\widetilde{v}_{k+1})^*]=[\delta'^{*}_{m-k}(\Gamma\widetilde{v}_{k+1})/(\Gamma u_k)],$$

$$[\alpha^*_k(v_k)/\widetilde{w}_k]=[\beta'^{m-k}_{*}(\Gamma v_k)^*/(\Gamma\widetilde{w}_k)^*]=[\beta'^{*}_{m-k}(\Gamma\widetilde{w}_k)/(\Gamma v_k)],$$

$$[\beta^*_k(w_k)/\widetilde{u}_k]=[\alpha'^{m-k}_{*}(\Gamma w_k)^*/(\Gamma\widetilde{u}_k)^*]=[\alpha'^{*}_{m-k}(\Gamma\widetilde{u}_k)/(\Gamma w_k)].$$

These identities allow us to compare the actions (7.52) and (7.54) and derive the equality:

$$\Phi\left\{\left(\bigotimes_{k=0}^{m}[v_k,\widetilde{v}_k]^{(-1)^{k}}\right)\otimes\left(\bigotimes_{k=0}^{m}[u_k,\widetilde{u}_k]^{(-1)^{k}}\right)\otimes\left(\bigotimes_{k=0}^{m}[w_k,\widetilde{w}_k]^{(-1)^{k+1}}\right)\right\}=$$

$$=\Phi'\left\{\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{v}_k,\Gamma v_k]^{(-1)^{m-k}}\right)\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{u}_k,\Gamma u_k]^{(-1)^{m-k}}\right)\otimes\right.$$

$$\left.\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{w}_k,\Gamma w_k]^{(-1)^{m-k+1}}\right)\right\}=(-1)^{\nu}\cdot\tau(\mathcal{H}). \tag{7.55}$$

On the other hand, since Γ is an isometry, we have in (7.55) the Φ'-action on a volume element, induced by the Hilbert structures on $\mathcal{H}'$. The combinatorial torsion $\tau(\mathcal{H}')$ is defined as the mode of the complex-valued Φ'-image of the volume element. Hence

$$\Phi' \left\{ \left(\bigotimes_{k=0}^{m} [\Gamma\widetilde{v}_k, \Gamma v_k]^{(-1)^{m-k}} \right) \otimes \right. \tag{7.56}$$

$$\left. \otimes \left(\bigotimes_{k=0}^{m} [\Gamma\widetilde{u}_k, \Gamma u_k]^{(-1)^{m-k}} \right) \otimes \left(\bigotimes_{k=0}^{m} [\Gamma\widetilde{w}_k, \Gamma w_k]^{(-1)^{k}} \right) \right\}$$

$$= (-1)^{\nu} e^{i\psi} \cdot \tau(\mathcal{H}'). \tag{7.57}$$

The phase $e^{i\psi}$ can be viewed as the total rotation angle needed to rotate the orthonormal bases $\{\Gamma\widetilde{v}_k, \Gamma v_k\}, \{\Gamma\widetilde{u}_k, \Gamma u_k\}, \{\Gamma\widetilde{w}_k, \Gamma w_k\}$ to orthonormal bases with positive determinants of coordinate change matrices with respect to bases fixed in (7.45) (cf. Lemma 7.10).

Since the combinatorial torsions are positive real numbers, comparison of (7.57) with (7.55) leads to

$$\tau(\mathcal{H}) = \tau(\mathcal{H}').$$

This completes the statement of the theorem. □

The canonical isomorphisms Φ, Φ' induce isomorphisms

$$\Psi : \det H^*_{\text{rel}}(M_1, E) \otimes \det H^*_{\text{abs}}(M_2, E) \to \det H^*(M_1 \# M_2, E), \tag{7.58}$$

$$\Psi' : \det H^*_{\text{rel}}(M_2, E) \otimes \det H^*_{\text{abs}}(M_1, E) \to \det H^*(M_1 \# M_2, E) \tag{7.59}$$

by the following formula. Consider any $x \in \det H^*_{\text{rel}}(M_1, E)$, $y \in \det H^*_{\text{abs}}(M_2, E)$ and $z \in \det H^*(M_1 \# M_2, E)$. Then we set

$$\Psi(x \otimes y) := \Phi(x \otimes y \otimes z^{-1})z.$$

The definition of Ψ' is analogous. Then with the result and notation of Theorem 7.12 we obtain:

Corollary 7.14.

$$\Psi \left\{ \left(\bigotimes_{k=0}^{m} [v_k, \widetilde{v}_k]^{(-1)^{k}} \right) \otimes \left(\bigotimes_{k=0}^{m} [u_k, \widetilde{u}_k]^{(-1)^{k}} \right) \right\} =$$

$$= (-1)^{\nu} \tau(\mathcal{H}) \left(\bigotimes_{k=0}^{m} [w_k, \widetilde{w}_k]^{(-1)^{k}} \right),$$

$$\Psi' \left\{ \left(\bigotimes_{k=0}^{m} [\Gamma\widetilde{v}_k, \Gamma v_k]^{(-1)^{m-k}} \right) \otimes \left(\bigotimes_{k=0}^{m} [\Gamma\widetilde{u}_k, \Gamma u_k]^{(-1)^{m-k}} \right) \right\} =$$

$$= (-1)^{\nu} \tau(\mathcal{H}) \left(\bigotimes_{k=0}^{m} [\Gamma\widetilde{w}_k, \Gamma w_k]^{(-1)^{m-k}} \right).$$

7.7 Splitting formula for Refined Torsion of complexes

We continue in the setup of Subsection 7.1. Consider the refined analytic torsions of the manifolds $M_j, j = 1, 2$ and the split manifold $M = M_1 \cup_N M_2$. According to Proposition 7.4 we can write for the refined analytic torsions

$$\rho_{\mathrm{an}}(\nabla) = \frac{1}{T^{RS}(\widetilde{\nabla})} \cdot \exp\left[-i\pi\eta(\mathcal{B}_{\mathrm{even}}) + i\pi \mathrm{rk}(E)\eta(\mathcal{B}_{\mathrm{trivial}})\right]$$
$$\times \exp\left[-i\pi\frac{m-1}{2}\dim\ker\mathcal{B}_{\mathrm{even}} + i\pi\mathrm{rk}(E)\frac{m}{2}\dim\ker\mathcal{B}_{\mathrm{trivial}}\right]\rho_\Gamma(M, E),$$
$$\rho_{\mathrm{an}}(\nabla_j) = \frac{1}{T^{RS}(\widetilde{\nabla}_j)} \cdot \exp\left[-i\pi\eta(\mathcal{B}^j_{\mathrm{even}}) + i\pi \mathrm{rk}(E)\eta(\mathcal{B}^j_{\mathrm{trivial}})\right]$$
$$\times \exp\left[-i\pi\frac{m-1}{2}\dim\ker\mathcal{B}^j_{\mathrm{even}} + i\pi\mathrm{rk}(E)\frac{m}{2}\dim\ker\mathcal{B}^j_{\mathrm{trivial}}\right]\rho_\Gamma(M_j, E),$$

where $j = 1, 2$ and $T^{RS}(\widetilde{\nabla}), T^{RS}(\widetilde{\nabla}_j)$ denote the scalar analytic torsions associated to the complexes $(\widetilde{\mathcal{D}}, \widetilde{\nabla}), (\widetilde{\mathcal{D}}_j, \widetilde{\nabla}_j)$ respectively. Furthermore $\rho_\Gamma(M, E), \rho_\Gamma(M_j, E)$ denote the respective refined torsion elements in the sense of (6.7) for $\lambda = 0$. The refined torsion elements are elements of the determinant lines:

$$\rho_\Gamma(M_1, E) \in \det(H^*_{\mathrm{rel}}(M_1, E) \oplus H^*_{\mathrm{abs}}(M_1, E)),$$
$$\rho_\Gamma(M_2, E) \in \det(H^*_{\mathrm{rel}}(M_2, E) \oplus H^*_{\mathrm{abs}}(M_2, E)),$$
$$\rho_\Gamma(M, E) \in \det(H^*(M, E) \oplus H^*(M, E)).$$

These elements are in the sense of [BK2, Section 4] the refined torsions $\rho_{[0,\lambda]}, \lambda = 0$ (see also (6.7)) of the corresponding complexes:

$$H^*_{\mathrm{rel}}(M_1, E) \oplus H^*_{\mathrm{abs}}(M_1, E),$$
$$H^*_{\mathrm{rel}}(M_2, E) \oplus H^*_{\mathrm{abs}}(M_2, E),$$
$$H^*(M, E) \oplus H^*(M, E).$$

Note that up to the identification of Corollary 7.7 the refined torsion $\rho_\Gamma(M, E)$ corresponds to the refined torsion of the complex $H^*(M_1\#M_2, E) \oplus H^*(M_1\#M_2, E)$

$$\rho_\Gamma(M_1\#M_2, E) \in \det(H^*(M_1\#M_2, E) \oplus H^*(M_1\#M_2, E)). \tag{7.60}$$

With the preceeding three sections we can now relate the refined torsions $\rho_\Gamma(M_1, E)$, $\rho_\Gamma(M_2, E)$ and $\rho_\Gamma(M_1\#M_2, E)$ together. For this we first rewrite the refined torsions in convenient terms. We restrict the neccessary arguments to $\rho_\Gamma(M_1, E)$, since the discussion of the other elements is completely analogous.

Let for $k = 0, .., \dim M$ the sets $\{e_k\}$ and $\{\theta_k\}$ be the bases for $H^k_{\text{rel}}(M_1, E)$ and $H^k_{\text{abs}}(M_1, E)$ respectively. Then the refined torsion element $\rho_\Gamma(M_1, E)$ is given by:

$$\begin{aligned}\rho_\Gamma(M_1, E) = (-1)^{R_1}([e_0] \wedge [\theta_0]) \otimes ([e_1] \wedge [\theta_1])^{(-1)} \otimes ...\\ ... \otimes ([e_{r-1}] \wedge [\theta_{r-1}])^{(-1)^{r-1}} \otimes ([\Gamma\theta_{r-1}] \wedge [\Gamma e_{r-1}])^{(-1)^r} \otimes ...\\ ... \otimes ([\Gamma\theta_1] \wedge [\Gamma e_1]) \otimes ([\Gamma\theta_0] \wedge [\Gamma e_0])^{(-1)},\end{aligned} \tag{7.61}$$

where $r = (\dim M + 1)/2$. The sign R_1 is given according to [BK2, (4.2)] by

$$\begin{aligned}R_1 = \frac{1}{2}\sum_{k=0}^{r-1}(\dim H^k_{\text{rel}}(M_1, E) + \dim H^k_{\text{abs}}(M_1, E))\cdot\\ \cdot\left(\dim H^k_{\text{rel}}(M_1, E) + \dim H^k_{\text{abs}}(M_1, E) + (-1)^{r-k}\right).\end{aligned}$$

The formula for $\rho_\Gamma(M_1, E)$ is independent of the particular choice of bases $\{e_k\}$ and $\{\theta_k\}$. Hence, since $\{\Gamma e_k\}$ is also a basis of $H^{m-k}_{\text{abs}}(M_1, E)$ for any k, we can write equivalently, replacing in the formula (7.61) the basis $\{\theta_k\}$ by $\{\Gamma e_{m-k}\}$:

$$\begin{aligned}\rho_\Gamma(M_1, E) = (-1)^{R_1}([e_0] \wedge [\Gamma e_m]) \otimes ([e_1] \wedge [\Gamma e_{m-1}])^{(-1)} \otimes ...\\ ... \otimes (e_{m-1}] \wedge [\Gamma e_1]) \otimes ([e_m] \wedge [\Gamma e_0])^{(-1)}.\end{aligned}$$

With the "fusion isomorphism" for graded vector spaces (cf. [BK2, (2.18)])

$$\begin{aligned}\mu_{(M_1,E)} : \det H^*_{\text{rel}}(M_1, E) \otimes \det H^*_{\text{abs}}(M_1, E)\\ \xrightarrow{\sim} \det(H^*_{\text{rel}}(M_1, E) \oplus \det H^*_{\text{abs}}(M_1, E))\end{aligned}$$

we obtain

$$\mu^{(-1)}_{(M_1,E)}(\rho_\Gamma(M_1, E)) = \left(\bigotimes_{k=0}^{m}[e_k]^{(-1)^k}\right) \otimes \left(\bigotimes_{k=0}^{m}[\Gamma e_k]^{(-1)^{m-k}}\right) \cdot (-1)^{\mathcal{M}(M_1,E)+R_1},$$

where with [BK2, (2.19)]

$$\mathcal{M}(M_1, E) = \sum_{0 \le k < i \le m} \dim H^i_{\text{rel}}(M_1, E) \cdot \dim H^k_{\text{abs}}(M_1, E).$$

Analogous result holds for the refined torsions $\rho_\Gamma(M_2, E)$ and $\rho_\Gamma(M_1\#M_2, E)$, where the analogous quantities R, R_2 and $\mathcal{M}(M, E)$ and $\mathcal{M}(M_2, E)$ are introduced respectively. Using now the fact that the refined torsion elements are independent of choices, we find with bases, fixed in (7.48):

$$\mu^{(-1)}_{(M_1,E)}\left(\rho_\Gamma(M_1,E)\right) = \\ = (-1)^{\mathcal{M}(M_1,E)+R_1}\left(\bigotimes_{k=0}^{m}[v_k,\widetilde{v}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{v}_k,\Gamma v_k]^{(-1)^{m-k}}\right),$$

$$\mu^{(-1)}_{(M_2,E)}\left(\rho_\Gamma(M_2,E)\right) = \\ = (-1)^{\mathcal{M}(M_2,E)+R_2}\left(\bigotimes_{k=0}^{m}[u_k,\widetilde{u}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{u}_k,\Gamma u_k]^{(-1)^{m-k}}\right),$$

$$\mu^{(-1)}_{(M_1\# M_2,E)}\left(\rho_\Gamma(M_1\# M_2,E)\right) = \\ = (-1)^{\mathcal{M}(M_1\# M_2,E)+R}\left(\bigotimes_{k=0}^{m}[w_k,\widetilde{w}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{w}_k,\Gamma w_k]^{(-1)^{m-k}}\right).$$

Now combine the canonical isomorphisms Ψ,Ψ', introduced in (7.58) and (7.59), together with the fusion isomorphisms into one single canonical isomorphism:

$$\begin{aligned}\Omega :=&\mu_{(M_1\# M_2,E)}\circ(\Psi\otimes\Psi')\circ(\mu^{-1}_{(M_1,E)}\otimes\mu^{-1}_{(M_2,E)}) : \\ &\det(H^*_{\mathrm{rel}}(M_1,E)\oplus H^*_{\mathrm{abs}}(M_1,E))\otimes \\ &\det(H^*_{\mathrm{rel}}(M_2,E)\oplus H^*_{\mathrm{abs}}(M_2,E))\to \\ &\qquad\to\det(H^*(M_1\# M_2,E)\oplus H^*(M_1\# M_2,E)),\end{aligned} \tag{7.62}$$

where we employed implicitly flip-isomorphisms in order to reorder the determinant lines appropriately. Due to the Knudson-Momford sign convention this leads to an additional sign. We obtain by Corollary 7.14 for the action of this canonical isomorphism

$$\begin{aligned}&\Omega(\rho_\Gamma(M_1,E)\otimes\rho_\Gamma(M_2,E)) = (-1)^{\mathcal{M}(M_1,E)+\mathcal{M}(M_2,E)+R_1+R_2+1}\times \\ &\mu_{(M_1\# M_2,E)}\left(\Psi\left[\left(\bigotimes_{k=0}^{m}[v_k,\widetilde{v}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[u_k,\widetilde{u}_k]^{(-1)^k}\right)\right]\otimes\right. \\ &\left.\Psi'\left[\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{v}_k,\Gamma v_k]^{(-1)^{k+1}}\right)\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{u}_k,\Gamma u_k]^{(-1)^{k+1}}\right)\right]\right) = \\ &= (-1)^{\mathcal{M}(M_1,E)+\mathcal{M}(M_2,E)+R_1+R_2+1}\tau(\mathcal{H})^2\times \\ &\mu_{(M_1\# M_2,E)}\left(\bigotimes_{k=0}^{m}[w_k,\widetilde{w}_k]^{(-1)^k}\right)\otimes\left(\bigotimes_{k=0}^{m}[\Gamma\widetilde{w}_k,\Gamma w_k]^{(-1)^{k+1}}\right) = \\ &= (-1)^{\mathrm{sign}}\tau(\mathcal{H})^2\rho_\Gamma(M_1\# M_2,E),\end{aligned}$$

where we have set

$$\text{sign} := \mathcal{M}(M_1, E) + \mathcal{M}(M_2, E) - \mathcal{M}(M_1 \# M_2, E) + R_1 + R_2 - R + 1. \tag{7.63}$$

Summarizing, we have derived a relation between the refined torsion elements of the splitting problem under the canonical isomorphism Ω:

Proposition 7.15.

$$\Omega(\rho_\Gamma(M_1, E) \otimes \rho_\Gamma(M_2, E)) = (-1)^{\text{sign}} \tau(\mathcal{H})^2 \rho_\Gamma(M_1 \# M_2, E).$$

This is an important result in the derivation of the actual gluing formula for refined analytic torsion and the final outcome of the preceeding three sections on cohomological algebra.

7.8 Combinatorial complexes

Before we finally prove a gluing formula for refined analytic torsion, consider a general situation with Z being a smooth compact manifold and $Y \subset Z$ a smooth compact submanifold with the natural inclusion $\iota : Y \hookrightarrow Z$. The inclusion induces a group homomorphism

$$\iota^* : \pi_1(Y) \to \pi_1(Z).$$

Fix any representation $\rho : \pi_1(Z) \to GL(n, \mathbb{C})$. It naturally gives rise to further two representations

$$\begin{aligned} &\rho_Y := \rho \circ \iota^* : \pi_1(Y) \to GL(n, \mathbb{C}), \\ &\bar{\rho}_Y : \frac{\pi_1(Y)}{\ker \iota^*} \cong im\, \iota^* \to GL(n, \mathbb{C}), \quad [\gamma] \mapsto \rho_Y(\gamma), \end{aligned}$$

where the second map is well-defined since by construction $\rho_Y \restriction \ker \iota^* \equiv id$.

Denote by $\widetilde{Z}$ and $\widetilde{Y}$ the universal covering spaces of Z and Y respectively, which are (cf. [KN, Proposition 5.9 (2)]) principal bundles over Z, Y with respective structure groups $\pi_1(Z), \pi_1(Y)$. Denote by p_Z the bundle projection of the principal bundle $\widetilde{Z}$ over Z. By locality the covering space $p_Z^{-1}(Y)$ over Y is a principal bundle over Y with the structure group $im\, \iota^*$. Hence the universal cover $\widetilde{Y}$ is a principal bundle over $p_Z^{-1}(Y)$ with the structure group $\ker \iota^*$. Summarizing we have:

$$\frac{p_Z^{-1}(Y)}{im\, \iota^*} \cong Y, \quad \frac{\widetilde{Y}}{\ker \iota^*} \cong p_Z^{-1}(Y). \tag{7.64}$$

Next we consider any triangulation $\mathcal{Z}$ of Z, such that it leaves Y invariant, i.e. $\mathcal{Y} := \mathcal{Z} \cap Y$ provides a triangulation of the submanifold Y. Fix an embedding of Z into $\widetilde{Z}$ as the fundamental domain. Then we obtain a triangulation $\widetilde{\mathcal{Z}}$ of $\widetilde{Z}$ by applying deck transformations of $\pi_1(Z)$ to $\mathcal{Z}$. Put

$$p_Z^{-1}(\mathcal{Y}) := \widetilde{\mathcal{Z}} \cap p_Z^{-1}(Y)$$

which gives a triangulation of $p_Z^{-1}(Y)$ invariant under deck transformations of $im\, \iota^*$. Embed $p_Z^{-1}(Y)$ into its universal cover $\widetilde{Y}$ as the fundamental domain. By applying deck transformations of $\ker \iota^*$ to $p_Z^{-1}(\mathcal{Y})$ we get a triangulation $\widetilde{\mathcal{Y}}$ of $\widetilde{Y}$. Note by construction, in analogy to (7.64)

$$\frac{p_Z^{-1}(\mathcal{Y})}{im\, \iota^*} \cong \mathcal{Y}, \quad \frac{\widetilde{\mathcal{Y}}}{\ker \iota^*} \cong p_Z^{-1}(\mathcal{Y}). \tag{7.65}$$

We form now the combinatorial chain complexes $C_*(.)$ of the triangulations and arrive at the central result of this subsection.

Theorem 7.16. *Consider the following combinatorial cochain complexes*

$$\begin{aligned}\mathrm{Hom}_{\rho_Y}(C_*(\widetilde{\mathcal{Y}}), \mathbb{C}^n) &:= \{f \in \mathrm{Hom}(C_*(\widetilde{\mathcal{Y}}), \mathbb{C}^n) | \\ &\quad \forall x \in C_*(\widetilde{\mathcal{Y}}), \gamma \in \pi_1(Y) : f(x \cdot \gamma) = \rho_Y(\gamma)^{-1} f(x)\}, \\ \mathrm{Hom}_{\bar{\rho}_Y}(C_*(p_Z^{-1}(\mathcal{Y})), \mathbb{C}^n) &:= \{f \in \mathrm{Hom}(C_*(p_Z^{-1}(\mathcal{Y})), \mathbb{C}^n) | \\ &\quad \forall x \in C_*(p_Z^{-1}(\mathcal{Y})), \gamma \in im\, \iota^* : f(x \cdot \gamma) = \bar{\rho}_Y(\gamma)^{-1} f(x)\}.\end{aligned}$$

These complexes are isomorphic:

$$\mathrm{Hom}_{\rho_Y}(C_*(\widetilde{\mathcal{Y}}), \mathbb{C}^n) \cong \mathrm{Hom}_{\bar{\rho}_Y}(C_*(p_Z^{-1}(\mathcal{Y})), \mathbb{C}^n).$$

Proof. The relations in (7.65) imply in particular

$$C_*(p_Z^{-1}(\mathcal{Y})) \cong C_*(\widetilde{\mathcal{Y}}) / \ker \iota^*.$$

Hence to any $x \in C_*(\widetilde{\mathcal{Y}})$ we can associate its equivalence class $[x] \in C_*(p_Z^{-1}(\mathcal{Y}))$ and define

$$\begin{aligned}\phi : \mathrm{Hom}_{\rho_Y}(C_*(\widetilde{\mathcal{Y}}), \mathbb{C}^n) &\to \mathrm{Hom}(C_*(p_Z^{-1}(\mathcal{Y})), \mathbb{C}^n), \\ f &\mapsto \phi f, \quad \phi f[x] := f(x).\end{aligned}$$

This construction is well-defined, since for any other representative $x' \in [x]$ there exists $\gamma \in \ker \iota^*$ with $x' = x \cdot \gamma$ and since $f \in \mathrm{Hom}_{\rho_Y}(C_*(\widetilde{\mathcal{Y}}), \mathbb{C}^n)$ we get

$$f(x') = \rho_Y(\gamma)^{-1} f(x) = [\rho \circ \iota^*(\gamma)]^{-1} f(x) = f(x).$$

Note further with f and x as above and $[\gamma] \in \pi_1(Y)/\ker \iota^* \cong im\, \iota^*$:

$$(\phi f)([x]\cdot[\gamma]) = (\phi f)[x\cdot\gamma] = f(x\cdot\gamma) = \\ = \rho_Y(\gamma)^{-1} f(x) = \bar{\rho}_Y([\gamma])^{-1}(\phi f)[x].$$

Hence in fact we have a well-defined map:

$$\phi : \mathrm{Hom}_{\rho_Y}(C_*(\widetilde{\mathcal{Y}}), \mathbb{C}^n) \to \mathrm{Hom}_{\bar{\rho}_Y}(C_*(p_Z^{-1}(\mathcal{Y})), \mathbb{C}^n).$$

Now we denote the boundary operators on $C_*(\widetilde{\mathcal{Y}})$ and $C_*(p_Z^{-1}(\mathcal{Y}))$ by $\widetilde{\delta}$ and δ respectively. They give rise to coboundary operators $\widetilde{d}$ and d on the cochain complexes. Observe

$$d(\phi f)[x] = (\phi f)(\delta[x]) = (\phi f)[\widetilde{\delta} x] = \\ = f(\widetilde{\delta} x) = \widetilde{d} f(x) = (\phi \widetilde{d} f)[x].$$

This shows

$$d\phi = \phi \widetilde{d}.$$

Thus ϕ is a well-defined homomorphism of complexes. It is surjective by construction. Injectivity of ϕ is also obvious. Thus ϕ is an isomorphism of complexes, as desired. □

7.9 Gluing formula for Refined Analytic Torsion

We now finally are in the position to derive a gluing formula for refined analytic torsion. As a byproduct we obtain a splitting formula for the scalar analytic torsion in terms of combinatorial torsions of long exact sequences.

We derive the gluing formula by relating the Ray-Singer analytic torsion norm to the Reidemeister combinatorial torsion norm and applying the gluing formula on the combinatorial side, established by M. Lesch in [L2]. This makes it necessary to use the Cheeger-Müller Theorem on manifolds with and without boundary.

Continue in the setup of Subsection 7.1. Consider a smooth triangulation X of the closed smooth split manifold

$$M = M_1 \cup_N M_2$$

that leaves the compact submanifolds M_1, M_2, N invariant, i.e. with $X_j := X \cap M_j, j = 1, 2$ and $W := X_j \cap N$ we have subcomplexes of X providing smooth triangulations of $M_j, j = 1, 2$ and N respectively, and

$$X = X_1 \cup_W X_2.$$

Denote by $\widetilde{M_j}$ the universal covering spaces of $M_j, j = 1, 2$. Fix embeddings of M_j into $\widetilde{M_j}$ as fundamental domains. Then the triangulation X_j of M_j induces under the action of $\pi_1(M_j)$, viewed as the group of deck-transformations of $\widetilde{M_j}$, smooth triangulation $\widetilde{X}_j$ of the universal cover $\widetilde{M_j}$ for each $j = 1, 2$.

The complex chain group $C_*(\widetilde{X}_j)$ is generated by simplices of $\widetilde{X}_j$ and is a module over the group algebra $\mathbb{C}[\pi_1(X_j)]$. The simplices of X_j form a preferred base for $C_*(\widetilde{X}_j)$ as a $\mathbb{C}[\pi_1(X_j)]$-module.

Furthermore the given unitary representation $\rho : \pi_1(M) \to U(n, \mathbb{C})$ gives rise to the associated unitary representations $\rho_j := \rho \circ \iota_j^*$ of the fundamental groups $\pi_1(M_j)$, where $\iota_j^* : \pi_1(M_j) \to \pi_1(M)$ are the natural group homomorphisms induced by the inclusions $\iota_j : M_j \hookrightarrow M, j = 1, 2$. We can now define for each j

$$\begin{aligned} &C^*(X_j, \rho_j) := \mathrm{Hom}_{\rho_j}(C_*(\widetilde{X}_j), \mathbb{C}^n) = \\ &\{f \in \mathrm{Hom}(C_*(\widetilde{X}_j), \mathbb{C}^n) | \forall x \in C_*(\widetilde{X}_j), \gamma \in \pi_1(M_j) : \ f(x \cdot \gamma) = \rho_j(\gamma)^{-1} f(x)\} \\ &\qquad \cong C^*(\widetilde{X}_j) \otimes_{\mathbb{C}[\pi_1(M_j)]} \mathbb{C}^n, \end{aligned}$$

where the $\mathbb{C}[\pi_1(M_j)]$-module structure of $C^*(\widetilde{X}_j)$ comes from the module structure of the dual space $C_*(\widetilde{X}_j)$ and the $\mathbb{C}[\pi_1(M_j)]$ module structure on $\mathbb{C}^n$ is obtained via the representation ρ_j.

The boundary operator on $C_*(\widetilde{X}_j)$ induces a coboundary operator on $C^*(X_j, \rho_j)$. Further the preferred base on $C_*(\widetilde{X}_j)$ together with a fixed volume on $\mathbb{C}^n$ yields a Hilbert structure on $C^*(X_j, \rho_j)$. So $C^*(X_j, \rho_j)$ becomes a finite Hilbert complex.

Next we consider again the universal coverings $p_j : \widetilde{M_j} \to M_j$. Then the preimage $p_j^{-1}(N) \subset \widetilde{M_j}$ is a covering space of N with the group of deck transformations

$$\mathrm{im}(\pi_1(N) \xrightarrow{\iota^*} \pi_1(M_j)) \subset \pi_1(M_j).$$

Here ι^* is the natural homomorphism of groups induced by the inclusion $N \hookrightarrow M_j$. We do not distinguish the inclusions of N into $M_j, j = 1, 2$ at this point, since it will always be clear from the context.

The triangulation $W \subset X_j$ induces with a fixed embedding of M_j into $\widetilde{M_j}$ a triangulation $p_j^{-1}(W)$ of $p_j^{-1}(N)$ by the deck tranformations of $\iota^*\pi_1(N) \subset \pi_1(M_j)$. The chain complex $C_*(p_j^{-1}(W))$ is generated by simplices of $p_j^{-1}(W)$ and is a module over the group subalgebra $\mathbb{C}[\iota^*\pi_1(N)] \subset \mathbb{C}[\pi_1(M_j)]$. It is a subcomplex of $C_*(\widetilde{X}_j)$.

The following observation follows from Theorem 7.16 and is central for the later constructions:

$$\begin{aligned} &f \in \mathrm{Hom}_{\rho_j}(C_*(\widetilde{X}_j), \mathbb{C}^n) \\ \Rightarrow &f|_{C_*(p_j^{-1}(W))} \in \mathrm{Hom}_{\bar{\rho}_N}(C_*(p_j^{-1}(W)), \mathbb{C}^n) \cong C^*(W, \rho_N), \end{aligned} \tag{7.66}$$

where $\rho_N = \rho_j \circ \iota^* : \pi_1(N) \to U(n, \mathbb{C})$ is the natural representation of $\pi_1(N)$. It is induced by ρ and is trivial over $\ker\ \iota^*$ by definition. The homomorphism $\bar{\rho}_N$ is obtained from ρ_N by dividing out the trivial part:

$$\bar{\rho}_N : \frac{\pi_1(N)}{\ker\ \iota^*} \cong im\, \iota^* \to U(n, \mathbb{C}).$$

The isomorphism in (7.66) in particular implies that we can compare the restrictions to $C_*(p_j^{-1}(W))$ for elements of both complexes $C^*(X_j, \rho_j), j = 1, 2$. We can now define:

$$\begin{aligned} &C^*(X_j, W, \rho_j) := \{f \in \mathrm{Hom}_\rho(C_*(\widetilde{X}_j), \mathbb{C}^n) | f|_{C_*(p_j^{-1}(W))} = 0\}, \\ &C^*(X_1 \# X_2, \rho) := \{(f, g) \in C^*(X_1, \rho_1) \oplus C^*(X_2, \rho_2) | f|_{C_*(p_j^{-1}(W))} = g|_{C_*(p_j^{-1}(W))}\}. \end{aligned}$$

These complexes inherit structure of finite Hilbert complexes from $C^*(X_j, \rho_j)$ for $j = 1, 2$. Fix the naturally induced Hilbert structure on the cohomology, which gives rise to norms on the determinant lines of cohomology, and define the combinatorial Reidemeister norms

$$\begin{aligned} &\| \cdot \|^R_{\det H^*(C^*(X_1 \# X_2, \rho))} := \tau(C^*(X_1 \# X_2, \rho))^{-1} \| \cdot \|_{\det H^*(C^*(X_1 \# X_2, \rho))}, \\ &\| \cdot \|^R_{\det H^*(C^*(X_j, W, \rho))} := \tau(C^*(X_j, W, \rho))^{-1} \| \cdot \|_{\det H^*(C^*(X_j, W, \rho))}, \\ &\| \cdot \|^R_{\det H^*(C^*(X_j, \rho))} := \tau(C^*(X_j, \rho))^{-1} \| \cdot \|_{\det H^*(C^*(X_j, \rho))}, \end{aligned}$$

where we have put for any finite Hilbert complex (C^*, ∂_*) with the naturally induced Hilbert structure on cohomology $H^*(C^*, \partial_*)$ and the associated Laplacians denoted by $\triangle_*$:

$$\log \tau(C^*, \partial_*) = \frac{1}{2} \sum_j (-1)^j \cdot j \cdot \zeta'(0, \triangle_j).$$

This definition corresponds to the sign convention for the Ray-Singer norms in Subsection 6.4. The Reidemeister norms do not depend on choices made for the construction and are in particular invariant under subdivisions, see [Mi, Theorem 7.1] and [RS, Section 4]. Since any two smooth triangulations admit a common subdivision, see [Mun], the Reidemeister norms do not depend on the choice of a smooth triangulation X.

Consider now the following short exact sequences of finite Hilbert complexes:

$$0 \to C^*(X_1, W, \rho) \xrightarrow{\alpha_c} C^*(X_1 \# X_2, \rho) \xrightarrow{\beta_c} C^*(X_2, \rho) \to 0, \tag{7.67}$$

$$0 \to C^*(X_2, W, \rho) \xrightarrow{\alpha'_c} C^*(X_1 \# X_2, \rho) \xrightarrow{\beta'_c} C^*(X_1, \rho) \to 0, \tag{7.68}$$

where α_c, α'_c are the natural inclusions and β_c, β'_c the natural restrictions. Both sequences are exact by definition of the corresponding homomorphisms of complexes. The associated long exact sequences in cohomology, with the Hilbert structures being naturally induced by the Hilbert structures of the combinatorial complexes as defined above, shall be denoted by $\mathcal{H}_c$ and $\mathcal{H}'_c$ respectively.

Consider further the following complexes

$$0 \to (\Omega^*_{\min}(M_1, E), \nabla_1) \xrightarrow{\alpha} (\Omega^*(M_1 \# M_2, E), \nabla_S) \xrightarrow{\beta} (\Omega^*_{\max}(M_2, E), \nabla_2) \to 0,$$

$$0 \to (\Omega^*_{\min}(M_2, E), \nabla_2) \xrightarrow{\alpha'} (\Omega^*(M_1 \# M_2, E), \nabla_S) \xrightarrow{\beta'} (\Omega^*_{\max}(M_1, E), \nabla_1) \to 0,$$

which were already introduced in Subsection 7.5. Their associated long exact sequences (cf. (7.32)) are denoted by $\mathcal{H}$ and $\mathcal{H}'$ respectively. The short exact sequences commute under the de Rham maps with the short exact sequences (7.67) and (7.68), respectively.

Thus the corresponding diagramms of the long exact sequences $\mathcal{H}, \mathcal{H}_c$ and $\mathcal{H}', \mathcal{H}'_c$ commute. The de Rham maps induce isomorphisms on cohomology, as established in [RS, Section 4] with arguments for orthogonal representations which work for unitary representations as well:

$$H^*_{\text{abs}}(M_j, E) \cong H^*(C^*(X_j, \rho_j)), \ H^*_{\text{rel}}(M_j, E) \cong H^*(C^*(X_j, W, \rho_j)). \tag{7.69}$$

From these identifications we obtain with the five-lemma in algebra applied to the commutative diagramms of long exact sequences $\mathcal{H}$ and $\mathcal{H}_c$ or $\mathcal{H}'$ and $\mathcal{H}'_c$:

$$H^*(M_1 \# M_2, E) \cong H^*(C^*(X_1 \# X_2, \rho)), \tag{7.70}$$

induced by the de Rham integration maps as well. Thus under the de Rham isomorphisms the long exact sequences $\mathcal{H}_c, \mathcal{H}'_c$ correspond to $\mathcal{H}, \mathcal{H}'$ respectively, and differ only in the fixed Hilbert structures.

Furthermore the long exact sequences $\mathcal{H}_c, \mathcal{H}'_c$ give rise to isomorphisms on determinant lines in a canonical way (recall the definition of Ψ, Ψ' in (7.58) and (7.59))

$$\Psi_c : \det H^*(C^*(X_1, W, \rho)) \otimes \det H^*(C^*(X_2, \rho)) \to \det H^*(C^*(X_1 \# X_2, \rho)),$$
$$\Psi'_c : \det H^*(C^*(X_2, W, \rho)) \otimes \det H^*(C^*(X_1, \rho)) \to \det H^*(C^*(X_1 \# X_2, \rho)).$$

These maps correspond to the canonical identifications Ψ, Ψ' introduced in Subsection 7.6 up to the de Rham isomorphisms. We can now prove an appropriate gluing result for the combinatorial Reidemeister norms.

Theorem 7.17.
Let x, y be elements of $\det H^*(C^*(X_1, W, \rho)), \det H^*(C^*(X_2, \rho))$ and x', y' elements of $\det H^*(C^*(X_2, W, \rho)), \det H^*(C^*(X_1, \rho))$, respectively. Then we obtain for the combinatorial Reidemeister norms the following relation:

$$\|\Psi_c(x \otimes y)\|^R_{\det H^*(C^*(X_1 \# X_2, \rho))} = \tag{7.71}$$
$$= 2^{\chi(N)/2} \|x\|^R_{\det H^*(C^*(X_1, W, \rho))} \|y\|^R_{\det H^*(C^*(X_2, \rho))},$$
$$\|\Psi'_c(x' \otimes y')\|^R_{\det H^*(C^*(X_1 \# X_2, \rho))} = \tag{7.72}$$
$$= 2^{\chi(N)/2} \|x'\|^R_{\det H^*(C^*(X_2, W, \rho))} \|y'\|^R_{\det H^*(C^*(X_1, \rho))}.$$

Proof. First apply the gluing formula in [L2], derived by introducing transmission boundary conditions depending on a parameter, in the spirit of [V]:

$$\tau(C^*(X_1 \# X_2, \rho)) = \tau(C^*(X_1, W, \rho)) \cdot \tau(C^*(X_2, \rho)) \cdot \tau(\mathcal{H}_c) \cdot 2^{-\chi(N)/2}, \tag{7.73}$$
$$\tau(C^*(X_1 \# X_2, \rho)) = \tau(C^*(X_2, W, \rho)) \cdot \tau(C^*(X_1, \rho)) \cdot \tau(\mathcal{H}'_c) \cdot 2^{-\chi(N)/2}. \tag{7.74}$$

By the definition of the combinatorial torsions $\tau(\mathcal{H}_c)$ and $\tau(\mathcal{H}'_c)$ we obtain the following relation to the action of Φ_c, Φ'_c, by an appropriate version of Corollary 7.14:

$$\|\Psi_c(x \otimes y)\|_{\det H^*(C^*(X_1 \# X_2, \rho))} = \tag{7.75}$$
$$= \tau(\mathcal{H}_c) \cdot \|x\|_{\det H^*(C^*(X_1, W, \rho))} \|y\|_{\det H^*(C^*(X_2, \rho))},$$
$$\|\Psi'_c(x' \otimes y')\|_{\det H^*(C^*(X_1 \# X_2, \rho))} = \tag{7.76}$$
$$= \tau(\mathcal{H}'_c) \cdot \|x'\|_{\det H^*(C^*(X_2, W, \rho))} \|y'\|_{\det H^*(C^*(X_1, \rho))}.$$

Now a combination of the relations above, together with the gluing formulas (7.73) and (7.74) gives the desired statement. □

We can now prove the following gluing result for the analytic Ray-Singer torsion norms.

Theorem 7.18. *Let Ω be the canonical isomorphism of determinant lines, defined in* (7.62).

$$\Omega : \det H^*(\widetilde{\mathcal{D}}_1, \widetilde{\nabla}_1) \otimes \det H^*(\widetilde{\mathcal{D}}_2, \widetilde{\nabla}_2) \to \\ \det(H^*(M_1\#M_2, E) \oplus H^*(M_1\#M_2, E)).$$

For any γ_1, γ_2 in $\det H^*(\widetilde{\mathcal{D}}_1, \widetilde{\nabla}_1), \det H^*(\widetilde{\mathcal{D}}_2, \widetilde{\nabla}_2)$ respectively, we have in terms of the analytic Ray-Singer torsion norms on the determinant lines:

$$\|\Omega(\gamma_1 \otimes \gamma_2)\|^{RS}_{\det(H^*(M_1\#M_2,E)\oplus H^*(M_1\#M_2,E))} = \\ =2^{\chi(N)}\|\gamma_1\|^{RS}_{\det H^*(\widetilde{\mathcal{D}}_1,\widetilde{\nabla}_1)}\|\gamma_2\|^{RS}_{\det H^*(\widetilde{\mathcal{D}}_2,\widetilde{\nabla}_2)}.$$

Proof. Under the de Rham isomorphisms we can relate the combinatorial Reidemeister norms to the analytic Ray-Singer torsion norms. We get by an appropriate version of [Lü]

$$\begin{aligned} \|\cdot\|^{R}_{\det H^*(C^*(X_j,\rho))} &= 2^{\chi(N)/4}\|\cdot\|^{RS}_{\det H^*_{\text{abs}}(M_j,E)}, \\ \|\cdot\|^{R}_{\det H^*(C^*(X_j,W,\rho))} &= 2^{\chi(N)/4}\|\cdot\|^{RS}_{\det H^*_{\text{rel}}(M_j,E)}, \end{aligned} \tag{7.77}$$

where $\chi(N)$ is the Euler characteristic of the closed manifold N with the representation ρ_N of its fundamental group, hence defined in terms of the twisted cohomology groups $H^*(N, E|_N)$. Furthermore we need the following relation:

$$\|\cdot\|^{R}_{\det H^*(C^*(X_1\#X_2,\rho))} = 2^{\chi(N)/2}\|\cdot\|^{RS}_{\det H^*(M_1\#M_2,E)}. \tag{7.78}$$

This result is proved for trivial representations in [V, Theorem 1.5]. This is done by discussing a family of elliptic transmission value problems and doesn't rely on the Cheeger-Müller theorem. However in the setup of the present discussion we provide below in Proposition 7.19 a simple proof for general unitary representations, using the Cheeger-Müller Theorem on closed manifolds

It is important to note that the Ray-Singer analytic and combinatorial torsion considered in [V] and [Lü] are squares of the torsion norms in our convention and further differ in the sign convention (we adopted the sign convention of [BK2, Section 11.2]). Therefore we get factors $2^{\chi(N)/4}, 2^{\chi(N)/2}$ in (7.77) and (7.78) respectively, instead of $2^{-\chi(N)/2}, 2^{-\chi(N)}$ as asserted in [Lü, Theorem 4.5] and [V, Theorem 1.5].

By definition the canonical maps Ψ_c and Ψ'_c correspond under the de Rham isomorphism to the canonical maps Ψ and Ψ' respectively. In view of Theorem 7.17, the identities (7.77) and the relation (7.78) we obtain the following gluing formulas:

$$\begin{aligned} &\|\Psi(x\otimes y)\|^{RS}_{\det H^*(M_1\#M_2,E))} = \\ =&2^{\chi(N)/2}\|x\|^{RS}_{\det H^*_{\mathrm{rel}}(M_1,E))}\|y\|^{RS}_{\det H^*_{\mathrm{abs}}(M_2,E))}, \end{aligned} \tag{7.79}$$

$$\begin{aligned} &\|\Psi'(x'\otimes y')\|^{RS}_{\det H^*(M_1\#M_2,E))} = \\ =&2^{\chi(N)/2}\|x'\|^{RS}_{\det H^*_{\mathrm{rel}}(M_2,E))}\|y'\|^{RS}_{\det H^*_{\mathrm{abs}}(M_1,E))}. \end{aligned} \tag{7.80}$$

The fusion isomorphisms $\mu_{(M_1,E)}, \mu_{(M_2,E)}$ and $\mu_{(M_1\#M_2,E)}$, used in the construction of the canonical isomorphism Ω, are by construction isometries with respect to the analytic Ray-Singer norms and hence in total we obtain for any γ_1,γ_2 in $\det H^*(\widetilde{\mathcal{D}}_1,\widetilde{\nabla}_1), \det H^*(\widetilde{\mathcal{D}}_2,\widetilde{\nabla}_2)$ respectively,

$$\begin{aligned} &\|\Omega(\gamma_1\otimes\gamma_2)\|^{RS}_{\det(H^*(M_1\#M_2,E)\oplus H^*(M_1\#M_2,E)} = \\ &\quad = 2^{\chi(N)}\|\gamma_1\|^{RS}_{\det H^*(\widetilde{\mathcal{D}}_1,\widetilde{\nabla}_1)}\|\gamma_2\|^{RS}_{\det H^*(\widetilde{\mathcal{D}}_2,\widetilde{\nabla}_2)}, \end{aligned}$$

where we recall the following facts by construction:

$$\begin{aligned} H^*(\widetilde{\mathcal{D}}_1,\widetilde{\nabla}_1) &= H^*_{\mathrm{rel}}(M_1,E)\oplus H^*_{\mathrm{abs}}(M_1,E), \\ H^*(\widetilde{\mathcal{D}}_2,\widetilde{\nabla}_2) &= H^*_{\mathrm{rel}}(M_2,E)\oplus H^*_{\mathrm{abs}}(M_2,E). \end{aligned}$$

This proves the statement of the theorem. □

Now we prove the result (7.78) on comparison of the torsion norms, anticipated in the argumentation above. The proof uses ideas behind [V, Theorem 1.5].

Proposition 7.19.

$$\|\cdot\|^{R}_{\det H^*(C^*(X_1\#X_2,\rho))} = 2^{\chi(N)/2}\|\cdot\|^{RS}_{\det H^*(M_1\#M_2,E)}.$$

Proof. Consider the following short exact sequence of finite Hilbert complexes (recall that the Hilbert structures on the complexes were induced by the triangulation X and the fixed volume on $\mathbb{C}^n$)

$$0\to\bigoplus_{j=1}^{2}C^*(X_j,W,\rho_j)\xrightarrow{\alpha}C^*(X_1\#X_2,\rho)\xrightarrow{\beta}C^*(W,\rho_N)\to 0,$$

with $\alpha(\omega_1 \oplus \omega_2) = (\omega_1, \omega_2)$ and $\beta(\omega_1, \omega_2) = \frac{1}{\sqrt{2}}(\omega_1|_{C_*(\widetilde{W})} + \omega_2|_{C_*(\widetilde{W})})$. Note further

$$\theta : C^*(W, \rho_N) \to C^*(X_1 \# X_2, \rho),\ \theta(\omega) = \frac{1}{\sqrt{2}}(\omega, \omega)$$

is an isometry between $C^*(W, \rho_N)$ and $\mathrm{Im}\theta$, where $\mathrm{Im}\theta$ is moreover the orthogonal complement in $C^*(X_1 \# X_2, \rho)$ to the image of α. Furthermore $\beta \circ \theta = id$. Hence β is an isometry between the orthogonal complement of its kernel and $C^*(W, \rho_N)$. Here a volume on $\mathbb{C}^n$ is fixed for all combinatorial complexes.

The map of complexes α is also an isometry onto its image and hence the induced identification

$$\phi^R_{\#} : \det H^*(C^*(X_1 \# X_2, \rho)) \to \bigotimes_{j=1}^{2} \det H^*(C^*(X_j, W, \rho_j)) \otimes \det H^*(C^*(W, \rho_N))$$

is an isometry of combinatorial Reidemeister norms. Similarly we consider the next short exact sequence of finite complexes:

$$0 \to \bigoplus_{j=1}^{2} C^*(X_j, W, \rho_j) \xrightarrow{\alpha} C^*(X, \rho) \xrightarrow{r} C^*(W, \rho_N) \to 0,$$

where the third arow is the restriction as in (7.66). By similar arguments as before the induced identification

$$\phi^R : \det H^*(C^*(X, \rho)) \to \bigotimes_{j=1}^{2} \det H^*(C^*(X_j, W, \rho_j)) \otimes \det H^*(C^*(W, \rho_N))$$

is an isometry of combinatorial Reidemeister norms. Now we consider the following commutative diagramms of short exact sequences.

$$\begin{array}{ccccccccc}
0 & \to & \bigoplus_{j=1}^{2} \Omega^*_{\min}(M_j, E) & \hookrightarrow & \Omega^*(M_1 \# M_2, E) & \xrightarrow{\sqrt{2}\iota^*_N} & \Omega^*(N, E) & \to & 0 \\
 & & \downarrow R & & \downarrow R & & \downarrow R & & \\
0 & \to & \bigoplus_{j=1}^{2} C^*(X_j, W, \rho_j) & \xrightarrow{\alpha} & C^*(X, \rho) & \xrightarrow{\sqrt{2}r} & C^*(W, \rho_N) & \to & 0,
\end{array}$$

where R denotes the natural de Rham integration quasi-isomorphisms. The second commutative diagramm is as follows:

$$\begin{array}{ccccccccc}
0 & \to & \bigoplus_{j=1}^{2} \Omega^*_{\min}(M_j, E) & \hookrightarrow & \Omega^*(M_1 \# M_2, E) & \xrightarrow{\sqrt{2}\iota^*_N} & \Omega^*(N, E) & \to & 0 \\
 & & \downarrow R & & \downarrow R_{\#} & & \downarrow R & & \\
0 & \to & \bigoplus_{j=1}^{2} C^*(X_j, W, \rho_j) & \xrightarrow{\alpha} & C^*(X_1 \# X_2, \rho) & \xrightarrow{\beta} & C^*(W, \rho_N) & \to & 0,
\end{array}$$

with the vertical maps as before given by the natural de Rham integration quasi-isomorphisms. Note that $R_{\#}$ is a quasi-isomorphism as well, which is clear from the five-lemma applied to the commutative diagramm of the associated long exact sequences.

The lower sequences in both of the diagramms were discussed above. The upper short exact sequence in both diagramms induces the identification:

$$\phi^{RS}_{\#} : \det H^*(M_1\# M_2, E) \to \bigotimes_{j=1}^{2} \det H^*_{\text{rel}}(M_j, E) \otimes \det H^*(N, E).$$

By commutativity of the two diagramms we obtain:

$$2^{\chi(N)/2}\phi^R \circ R = R \circ \phi^{RS}_{\#}, \tag{7.81}$$

$$\phi^R_{\#} \circ R_{\#} = R \circ \phi^{RS}_{\#}. \tag{7.82}$$

Now let $x \in \det H^*(M_1\# M_2, E)$ be an arbitrary element, identified via Corollary 7.7 with $x \in \det H^*(M, E)$. We compute:

$$\begin{aligned} \|x\|^{RS}_{\det H^*(M_1\# M_2,E)} = \|x\|^{RS}_{\det H^*(M,E)} = \|R(x)\|^{R}_{\det H^*(X,\rho)} = \\ = \|\phi^R \circ R(x)\| = 2^{-\chi(N)/2}\|R \circ \phi^{RS}_{\#}(x)\| = \\ = 2^{-\chi(N)/2}\|\phi^R_{\#} \circ R_{\#}(x)\| = 2^{-\chi(N)/2}\|R_{\#}(x)\|^{R}_{\det H^*(C^*(X_1\# X_2,\rho))}, \end{aligned}$$

where we have put

$$\|\cdot\| := \|\cdot\|^{R}_{\det H^*(C^*(X_1,W,\rho_1))} \cdot \|\cdot\|^{R}_{\det H^*(C^*(X_2,W,\rho_2))} \cdot \|\cdot\|^{R}_{\det H^*(C^*(W,\rho_N))}.$$

The steps in the sequence of equalities need to be clarified. The first equation is due to Theorem 7.6 on the spectral equivalence of $\triangle_S$ and $\triangle$. The second equation is simply the Cheeger-Müller Theorem for closed Riemannian manifolds. The third equation is a consequence of the fact that ϕ^R is an isometry with respect to the combinatorial Reidemeister norms. Now the fourth and the fifth equation are consequences of (7.81) and (7.82) respectively. Using in the last equation again the isometry $\phi^R_{\#}$ we obtain the result. The sequence of equalities proves in total:

$$\|x\|^{RS}_{\det H^*(M_1\# M_2,E)} = 2^{-\chi(N)/2}\|R_{\#}(x)\|^{R}_{\det H^*(C^*(X_1\# X_2,\rho))}.$$

□

Next we recall that by Theorem 7.6 the complexes $\Omega^*(M,E)\oplus\Omega^*(M,E)$ and

$\Omega^*(M_1\#M_2,E)\oplus\Omega^*(M_1\#M_2,E)$ have spectrally equivalent Laplacians with identifyable eigenforms. This implies

$$\begin{aligned}T^{RS}(\Omega^*(M,E)\oplus\Omega^*(M,E)) &= T^{RS}(\Omega^*(M_1\#M_2,E)\oplus\Omega^*(M_1\#M_2,E)),\\ H^*(\Omega^*(M,E)\oplus\Omega^*(M,E)) &\cong H^*(\Omega^*(M_1\#M_2,E)\oplus\Omega^*(M_1\#M_2,E)).\end{aligned} \tag{7.83}$$

The identification (7.83) is in fact an isometry with respect to the natural Hilbert structures, since in both cases the Hilbert structure is induced by the L^2−scalar product on harmonic forms and the harmonic forms of both complexes coincide, see Theorem 7.6. This implies

$$\begin{aligned}&\|\cdot\|^{RS}_{\det(H^*(M_1\#M_2,E)\oplus H^*(M_1\#M_2,E))} =\\ &\|\cdot\|^{RS}_{\det(H^*(M,E)\oplus H^*(M,E))} \equiv \|\cdot\|^{RS}_{\det H^*(\widetilde{\mathcal{D}},\widetilde{\nabla})}.\end{aligned}$$

Moreover under the identification (7.83) we can view the canonical isomorphism Ω as

$$\Omega : \det H^*(\widetilde{\mathcal{D}}_1,\widetilde{\nabla}_1)\otimes\det H^*(\widetilde{\mathcal{D}}_2,\widetilde{\nabla}_2)\to\det H^*(\widetilde{\mathcal{D}},\widetilde{\nabla}).$$

Then we obtain as a corollary of Theorem 7.18:

Corollary 7.20. *Denote by $\rho_{\mathrm{an}}(\nabla)$ and $\rho_{\mathrm{an}}(\nabla_j), j=1,2$ the refined analytic torsions on M and $M_j, j=1,2$ respectively. There exists some $\phi\in[0,2\pi)$ such that*

$$\Omega(\rho_{\mathrm{an}}(\nabla_1)\otimes\rho_{\mathrm{an}}(\nabla_2)) = e^{i\phi}2^{\chi(N)}\rho_{\mathrm{an}}(\nabla).$$

Proof. Applying Theorem 7.18 to $\rho_{\mathrm{an}}(\nabla_1)\otimes\rho_{\mathrm{an}}(\nabla_2)$, we obtain with the identification (7.83) and Theorem 6.29:

$$\begin{aligned}&\|\Omega(\rho_{\mathrm{an}}(\nabla_1)\otimes\rho_{\mathrm{an}}(\nabla_2))\|^{RS}_{\det(H^*(\widetilde{\mathcal{D}},\widetilde{\nabla})} = 2^{\chi(N)}\times\\ &\times\|\rho_{\mathrm{an}}(\nabla_1)\|^{RS}_{\det H^*(\widetilde{\mathcal{D}}_1,\widetilde{\nabla}_1)}\|\rho_{\mathrm{an}}(\nabla_2)\|^{RS}_{\det H^*(\widetilde{\mathcal{D}}_2,\widetilde{\nabla}_2)} = 2^{\chi(N)}.\end{aligned}$$

On the other hand we have again by Theorem 6.29

$$\|\rho_{\mathrm{an}}(\nabla)\|^{RS}_{\det(H^*(\widetilde{\mathcal{D}},\widetilde{\nabla})} = 1.$$

This proves the corollary. □

In order to establish a gluing formula it remains to identify this phase ϕ explicitly. Under the identification of Corollary 7.7 the refined torsion

$\rho_\Gamma(M_1\# M_2, E)$ corresponds to the refined torsion element $\rho_\Gamma(M,E)$, as already encountered in (7.60). Hence with Proposition 7.15 we can write

$$\Omega(\rho_\Gamma(M_1,E)\otimes\rho_\Gamma(M_2,E)) = (-1)^{\mathrm{sign}}\tau(\mathcal{H})^2\rho_\Gamma(M,E).$$

Consequently we obtain using the splitting formulas (7.16) and (7.17) for the eta-invariants and using Proposition 7.4

$$\begin{aligned}\Omega(\rho_{\mathrm{an}}(\nabla_1)\otimes\rho_{\mathrm{an}}(\nabla_2)) = \frac{T^{RS}(\widetilde{\mathcal{D}},\widetilde{\nabla})}{T^{RS}(\widetilde{\mathcal{D}}_1,\widetilde{\nabla}_1)T^{RS}(\widetilde{\mathcal{D}}_2,\widetilde{\nabla}_2)}\times \\ \exp\left(-i\pi\cdot \mathrm{Err}\eta(\mathcal{B}_{\mathrm{even}}) + i\pi\cdot\mathrm{rank}(E)\mathrm{Err}\eta(\mathcal{B}_{\mathrm{trivial}})\right)\times \\ (-1)^{\mathrm{sign}}\tau(\mathcal{H})^2\cdot\rho_{\mathrm{an}}(\nabla),\end{aligned}\tag{7.84}$$

where we have put

$$\begin{aligned}\mathrm{Err}\eta(\mathcal{B}_{\mathrm{even}}) :=&\tau_\mu(I-P_1,P,P_1)+ \\ &+\frac{m-1}{2}\left(\dim\ker\mathcal{B}^1_{\mathrm{even}}+\dim\ker\mathcal{B}^2_{\mathrm{even}}-\dim\ker\mathcal{B}_{\mathrm{even}}\right),\end{aligned}\tag{7.85}$$

$$\begin{aligned}\mathrm{Err}\eta(\mathcal{B}_{\mathrm{trivial}}) :=&\tau_\mu(I-P_{1,\mathrm{trivial}},P_{\mathrm{trivial}},P_{1,\mathrm{trivial}})+ \\ &+\frac{m}{2}\left(\dim\ker\mathcal{B}^1_{\mathrm{trivial}}+\dim\ker\mathcal{B}^2_{\mathrm{trivial}}-\dim\ker\mathcal{B}_{\mathrm{trivial}}\right).\end{aligned}\tag{7.86}$$

Comparing this relation with the statement of Corollary 7.20 we obtain for the phase ϕ in Corollary 7.20:

$$\phi = \pi\left[-\mathrm{Err}\eta(\mathcal{B}_{\mathrm{even}}) + \mathrm{rank}(E)\mathrm{Err}\eta(\mathcal{B}_{\mathrm{trivial}}) + \mathrm{sign}\right].$$

Due to the definition of the refined analytic torsion in (6.45), it makes sense to reduce the phase ϕ modulo $\pi\mathrm{rank}(E)\mathbb{Z}$.

Lemma 7.21.

$$\phi\equiv\pi\left[\mathrm{sign}-\mathrm{Err}\eta(\mathcal{B}_{\mathrm{even}})\right] \mod \pi\mathrm{rank}(E)\mathbb{Z}.$$

Proof. We need to verify

$$\mathrm{Err}\eta(\mathcal{B}_{\mathrm{trivial}})\equiv 0 \mod \mathbb{Z}.$$

Denote the Laplace operators of the complexes $(\Omega^*_{\min/\max}(M_j),\nabla_{j,\mathrm{trivial}})$ by $\triangle^j_{\mathrm{rel/abs}}$, respectively. Let $\triangle$ denote the Laplacian of the complex $(\Omega^*(M),\nabla_{\mathrm{trivial}})$. We have by construction

$$\begin{aligned}\mathcal{B}(\nabla_{\mathrm{trivial}})^2 &= \triangle\oplus\triangle, \\ \mathcal{B}(\nabla_{j,\mathrm{trivial}})^2 &= \triangle^j_{\mathrm{rel}}\oplus\triangle^j_{\mathrm{abs}}.\end{aligned}$$

Since the Maslov-triple index τ_μ is integer-valued, we obtain via Lemma 6.25 the following mod $\mathbb{Z}$ calculation:

$$\begin{aligned}\mathrm{Err}\eta(\mathcal{B}_{\mathrm{trivial}}) = &\frac{1}{2}\sum_{k=0}^{m}(-1)^{k+1}\cdot k\cdot(\dim\ker\triangle^1_{k,\mathrm{rel}}+\dim\ker\triangle^1_{k,\mathrm{abs}})+\\ &+\frac{1}{2}\sum_{k=0}^{m}(-1)^{k+1}\cdot k\cdot(\dim\ker\triangle^2_{k,\mathrm{rel}}+\dim\ker\triangle^2_{k,\mathrm{abs}})-\\ &-\frac{1}{2}\sum_{k=0}^{m}(-1)^{k+1}\cdot k\cdot(\dim\ker\triangle_{k}+\dim\ker\triangle^1_{k}).\end{aligned}$$

The Poincare duality implies:

$$\begin{aligned}\dim\ker\triangle^j_{k,\mathrm{rel}} &= \dim\ker\triangle^j_{m-k,\mathrm{abs}},\\ \dim\ker\triangle_k &= \dim\ker\triangle_{m-k}.\end{aligned}$$

Hence we compute further modulo $\mathbb{Z}$

$$\mathrm{Err}\eta(\mathcal{B}_{\mathrm{trivial}}) = \frac{m}{2}\sum_{k=0}^{m}(-1)^k\dim\ker\triangle^1_{k,\mathrm{rel}}+$$

$$+\frac{m}{2}\sum_{k=0}^{m}(-1)^k\dim\ker\triangle^2_{k,\mathrm{abs}}-\frac{m}{2}\sum_{k=0}^{m}(-1)^k\dim\ker\triangle_{k}.$$

Finally, exactness of the long exact sequence $\mathcal{H}$ in (7.32) (in the setup of a trivial line bundle) implies $\mathrm{Err}\eta(\mathcal{B}_{\mathrm{trivial}})\equiv 0 \mod \mathbb{Z}$. □

We finally arrive at the following central result: a gluing formula for refined analytic torsion.

Theorem 7.22. [Gluing formula for Refined Analytic Torsion]
Let $M=M_1\cup_N M_2$ be an odd-dimensional oriented closed Riemannian split-manifold where $M_j, j=1,2$ are compact bounded Riemannian manifolds with $\partial M_j=N$ and orientation induced from M. Denote by (E,∇,h^E) a complex flat vector bundle induced by an unitary representation $\rho:\pi_1(M)\to U(n,\mathbb{C})$. Assume product structure for the metrics and the vector bundle. Set:

$$\begin{aligned}(\widetilde{\mathcal{D}}_j,\widetilde{\nabla}_j)&:=(\mathcal{D}_{j,\min},\nabla_{j,\min})\oplus(\mathcal{D}_{j,\max},\nabla_{j,\max}),\ \ j=1,2,\\ (\widetilde{\mathcal{D}},\widetilde{\nabla})&:=(\Omega^*(M,E),\nabla)\oplus(\Omega^*(M,E),\nabla).\end{aligned}$$

The canonical isomorphism

$$\Omega:\det H^*(\widetilde{\mathcal{D}}_1,\widetilde{\nabla}_1)\otimes\det H^*(\widetilde{\mathcal{D}}_2,\widetilde{\nabla}_2)\to\det H^*(\widetilde{\mathcal{D}},\widetilde{\nabla})$$

is induced by the long exact sequences on cohomologies

$$\mathcal{H}: \ ...H^k_{\text{rel}}(M_1,E) \to H^k(M,E) \to H^k_{\text{abs}}(M_2,E) \to H^{k+1}_{\text{rel}}(M_1,E)...$$
$$\mathcal{H}': ...H^k_{\text{rel}}(M_2,E) \to H^k(M,E) \to H^k_{\text{abs}}(M_1,E) \to H^{k+1}_{\text{rel}}(M_2,E)...$$

and fusion isomorphisms. The isomorphism Ω *is linear, hence well-defined on equivalence classes modulo multiplication by* $\exp[i\pi \text{rk} E]$. *Then the gluing formula for refined analytic torsion in* (6.45) *is given as follows:*

$$\begin{aligned}&\Omega(\rho_{\text{an}}(M_1,E)\otimes\rho_{\text{an}}(M_2,E)) = K(M,M_1,M_2,\rho)\cdot\rho_{\text{an}}(M,E),\\&K(M,M_1,M_2,\rho) := 2^{\chi(N)}\exp(i\phi),\\&\phi := \pi(\text{sign} - \text{Err}\eta(\mathcal{B}_{\text{even}})).\end{aligned}$$

The term $\text{Err}\eta(\mathcal{B}_{\text{even}})$ *is an error term in the gluing formula for eta-invariants*

$$\begin{aligned}\text{Err}\eta(\mathcal{B}_{\text{even}}) :=&\tau_\mu(I-P_1,P,P_1)+\\&+\frac{m-1}{2}\left(\dim\ker\mathcal{B}^1_{\text{even}} + \dim\ker\mathcal{B}^2_{\text{even}} - \dim\ker\mathcal{B}_{\text{even}}\right),\end{aligned}$$

where $\mathcal{B}$ *and* $\mathcal{B}^j, j=1,2$ *are the odd-signature operators associated to the Fredholm complexes* $(\widetilde{\mathcal{D}},\widetilde{\nabla})$ *and* $(\widetilde{\mathcal{D}}_j,\widetilde{\nabla}_j), j=1,2$ *respectively. Further* P,P_1 *denote the boundary conditions and the Calderon projector associated to* $\mathcal{B}^1_{\text{even}}$*, respectively.* τ_μ *is the Maslov triple index.*

The sign $\in\{\pm1\}$ *is a combinatorial sign, explicitly defined in* (7.63).

Corollary 7.23. [Gluing formula for scalar analytic torsion]

$$\frac{T^{RS}(M,E)}{T^{RS}_{\text{rel}}(M_1,E)\cdot T^{RS}_{\text{abs}}(M_2,E)} = \tau(\mathcal{H})^{-1}\cdot 2^{\chi(N)/2}.$$

Proof. Comparison of the statement of Corollary 7.20 with the relation (7.84) we obtain along the result of Theorem 7.22 the following formula as a byproduct:

$$\frac{T^{RS}(\widetilde{\mathcal{D}},\widetilde{\nabla})}{T^{RS}(\widetilde{\mathcal{D}}_1,\widetilde{\nabla}_1)T^{RS}(\widetilde{\mathcal{D}}_2,\widetilde{\nabla}_2)} = \tau(\mathcal{H})^{-2}\cdot 2^{\chi(N)}.$$

By construction and the Poincare duality on odd-dimensional manifolds with or without boundary (7.18) we know

$$\begin{aligned}T^{RS}(\widetilde{\mathcal{D}}_1,\widetilde{\nabla}_1) &= T^{RS}_{\text{rel}}(M_1,E)\cdot T^{RS}_{\text{abs}}(M_1,E) = T^{RS}_{\text{rel}}(M_1,E)^2,\\T^{RS}(\widetilde{\mathcal{D}}_2,\widetilde{\nabla}_2) &= T^{RS}_{\text{abs}}(M_2,E)\cdot T^{RS}_{\text{rel}}(M_2,E) = T^{RS}_{\text{abs}}(M_2,E)^2,\\T^{RS}(\widetilde{\mathcal{D}},\widetilde{\nabla}) &= T^{RS}(M,E)^2.\end{aligned}$$

Taking now square-roots gives the result. □

Note that this result refines the result of [Lee, Theorem 1.7 (2)] on the adiabatic decomposition of the scalar analytic torsion.

Note finally that in view of Theorem 6.30 the gluing formula in Theorem 7.22 can be viewed as a gluing formula for refined analytic torsion in the version of Braverman-Kappeler.

8 References

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