Bounded Rationality, Heterogeneity and Learning

Inaugural-Dissertation
zur Erlangung des Grades eines Doktors
der Wirtschafts- und Gesellschaftswissenschaften
durch die
Rechts- und Staatswissenschaftliche Fakultät
der Rheinischen Friedrich-Wilhelms-Universität
Bonn

vorgelegt von
Christina Matzke
aus Mülheim an der Ruhr

Bonn 2009

Dekan: Prof. Dr. Christian Hillgruber
Erstreferent: Prof. Dr. Frank Riedel
Zweitreferent: Prof. Paul Heidhues, Ph. D.

Tag der mündlichen Prüfung: 14. Juli 2009

Diese Dissertation ist auf dem Hochschulschriftenserver der ULB Bonn (http://hss.ulb.uni-bonn.de/diss_online) elektronisch publiziert.

to my parents

Acknowledgments

Frank Riedel is a great supervisor. Although he unfortunately left Bonn at the beginning of my dissertation he was always reachable and took the time to talk about new ideas or problems whenever necessary.

Since Frank moved to Bielefeld, Paul Heidhues can be described as being my supervisor in situ. His assistance and multiple discussions throughout my dissertation were especially helpful and valuable for me. I am deeply grateful to both of them.

It was great working and discussing with my co-authors Damien Challet and Benedikt Wirth. I much benefitted from their knowledge and creativity which also made the collaboration immensely fruitful. Damien additionally taught me how to handle requests of journalists. I am most grateful for this experience. Thanks to Benedikt for proof-reading the dissertation.

For numerous discussions and ideas I especially thank Tymon Tatur, Lars Koch-Metzger, Philipp Wichardt, Johannes Münster and Christian Seel.

The financial support from the DFG (German Research Foundation, GRK 629 "Quantitative Ökonomie") is highly appreciated.

Special thanks go to Urs Schweizer, Jürgen von Hagen and Silke Kinzig for running the Bonn Graduate School of Economics as well as the Graduiertenkolleg in such an excellent and commendable way.

The time here at the Bonn Graduate School of Economics would not be half as exciting without my friends and colleagues here at university. Thanks a lot to all of you!

Last but not least, I would like to thank my family for supporting me in everything I do. Especially, I owe thanks to my parents, my brother and Ben who supported and advised me throughout my time at university. Moreover, I thank my grandparents as well as Uta and Bruno for their support and advice.

Contents

List of Figures

Motivation and Overview[1]

Agents who interact in economic markets generally are different in various aspects, for example concerning taste, reservation price, or their action space. Additionally, they often have imperfect information of the situation they face. In consumer markets, even if the consumers had perfect information, the large amount of it would already lead to a vast game theoretic problem, if approached rationally. Hence, it is worthwhile considering and analyzing heterogeneous agents with limited information as well as boundedly rational consumers in order to find out what consequences these assumptions have in economic modeling.

The question arises how agents approach the problem of finding the best possible solution for them, given the behavior of the other agents, when a standard game theoretic ansatz seems too complicated, unrealistic, or requires disproportionately high effort. Possible, often pursued strategies include the reliance on an inductive learning rule or a rule of thumb for the agents' behavior in order to come to a reasonable economic decision.

This dissertation deals with both a model in which agents with limited information conform to a learning rule, as well as boundedly rational consumers who follow

[1]The first chapter of this dissertation is based upon the paper "Taking a Shower in Youth Hostels: Risks and Delights of Heterogeneity" which was written in joint work with Damien Challet (Matzke & Challet 2008). It found public recognition in articles of online *Nature* news (18. Jan 2008), *PhysicsWorld* (Vol. 21, No. 2, Feb 2008), *FAZ* (12. Feb 2008), *New Scientist* (issue 2643, 16 Feb 2008), *Bayerischer Rundfunk* (Interview 09.06.2008), *forsch - Bonner Universitäts-Nachrichten* (Nr. 2/3, Juni 2008). The second and third chapters are based on the papers "The Evolution of Sales with Habitual and Imitative Consumers" and "Product Pricing when Demand Follows a Rule of Thumb", respectively, which were written together with Benedikt Wirth (Matzke & Wirth 2008*a*, Matzke & Wirth 2008*b*).

simple rules of thumb. The first case is treated within a generic, illustrative model situation, the so-called Shower Temperature Problem, in which the agents possess either the same (homogeneous) or an individual (heterogeneous) action space. The latter case is treated for consumer markets and additionally requires the modeling of an appropriate, strategic pricing of firms.

In summary, the main result of the Shower Temperature Problem is that action heterogeneity represents a robust solution for the agents with only few systematic deviations, but at the cost of a higher risk for the individual agent than in the homogeneous case.

Regarding the market of boundedly rational consumers, we will obtain interesting results how psychological and experimental results can be cast into a mathematical model with boundedly rational, habit-forming and imitative consumers; we will analyze this consumer model and investigate its consequences. From the firms' point of view, we will examine conditions under which firms operate profitably in the long-term. Furthermore, we will show that the considered market is in a sense well-behaved, since for a rising number of firms, the prices decrease, the prices of the weakest products converge against marginal costs, and the welfare rises. We will also prove for a monopoly that Nash equilibria are already found in the strategy space of all time-constant price paths. Finally, advertising is shown to be an effective method to sustain demand.

There exist many publications dealing with heterogeneity and lack of information. Those in the area of minority games are most related to the approach in the Shower Temperature Problem. The initial impulse for the theory of minority games came from Arthur (1994). It is argued that inductive reasoning is more realistic than deductive reasoning in complex and complicated decision making situations. The agents' decision process is characterized by bounded rationale and lack of information, and additionally it is based solely on the history of the game outcomes. Hence, the agents learn from their experiences. Challet & Zhang (1997*a*) proposed a mathematical model incorporating the agents' features described in Arthur (1994) and the theory of minority games was born. We will borrow some ideas from minority games concerning the agents' learning rule and the influence of the game history on future actions, that is, the inductive reasoning. In the classical minority game, heterogeneity of the agents is of considerable importance since it is worthwhile for the agents to behave differently. The Shower Temperature Problem opens a slightly different perspective on action heterogeneity.

Boundedly rational consumer behavior is often observed in consumer research, in particular imitation of group behavior or habitual purchase (cf. Assael 1984, p. 371ff, 53). Indications for such a behavior can also be found in psychological experiments such as Venkatesan (1966), Corriveau et al. (2009) or Pingle & Day (1996). Additionally, imitative behavior constitutes a concept frequently used in evolutionary game theory and social learning, as for instance in Schlag (1998), Ellison & Fudenberg (1993), and Banerjee (1992), and habit occurs in the habit formation literature (Heaton 1993) as well as implicitly in some industrial organization models (for example Smallwood & Conlisk (1979), which is closely related to the demand dynamic employed in this dissertation). Habit may also be interpreted as a special case of learning, since agents learn from past experience (Sobel 2000, p. 257) and positive experience with a good may cause habitual purchase behavior.

From the mathematical point of view, the consumer market model is stated in form of a population game as defined in Sandholm (2005) and Sandholm (2006), who makes use of the fact that for a large population size, the stochastic process generated by an evolutionary process can be approximated by solutions to ordinary differential equations (Benaïm & Weibull 2003).

The firms, analyzed in this dissertation, aim at maximizing their profits given the demand side and the pricing strategies of the competitors. In order to accomplish this, we employ a differential game as introduced by Isaacs (1954), that is, a time-continuous game where the state variables (here the consumer subpopulation sizes) follow first-order ordinary differential equations. More specifically, the chosen product prices (control variables) determine the rate of change in consumer subpopulation sizes (state variables), where consumers are grouped into subpopulations according to the product they own.

The model will be able to generate typical patterns observed in consumer markets such as product life cycles (de Kluyver 1977, Brockhoff 1967, Polli & Cook 1969).

The outline of the dissertation is as follows. Chapter 1 focuses on learning and the advantages of action heterogeneity: Tuning one's shower in some hotels may turn into a challenging coordination game with imperfect information. The temperature sensitivity increases with the number of agents, making the problem possibly unlearnable. Because there is in practice a finite number of possible tap positions, identical agents are unlikely to reach even approximately their favorite water temperature. Heterogeneity allows some agents to reach much better temperatures, at the cost of higher risk.

The Shower Temperature Problem is an easily accessible example for a wide range of other situations in economics which can be modeled by similar means. Another application is, for instance, the problem of how many employees each firm should optimally employ, depending on how many projects this firm has. One can also examine the question of how many products each firm should optimally produce, depending on the fluctuating demand.

In chapter 2 we analyze demand in markets where consumers follow simple behavioral decision rules based on imitation and habit as suggested in consumer research, social learning, and related fields. Demand can be viewed as the outcome of a population game whose revision protocol is determined by the consumers' behavioral rules. The evolution of sales is then analyzed in order to explore the demand side and first implications for a strategic supply side, as well as conditions under which a product survives on the market. For the two goods case, structural stability of the system can be proven.

In chapter 3 we analyze the strategic behavior of firms when demand is determined by the rule of thumb behavior from the previous chapter. We investigate monopoly and competition between firms, described via an open-loop differential game which in this setting is equivalent to but analytically more convenient than a closed-loop system. We derive a Nash equilibrium and examine the influence of advertising. We show for the monopoly case that a reduction of the space of all price paths in time to the space of time-constant prices is sensible since the latter in general contains Nash equilibria. We prove that the equilibrium price of the weakest active firm tends to marginal cost as the number of (non-identical) firms grows. Our model is consistent with observed market behavior such as product life cycles.

Chapter 1

Taking a Shower in Youth Hostels: Risks and Delights of Heterogeneity

1.1 Introduction

Taking a shower can turn into a painful tuning and retuning game when many people take a shower at the same time if the flux of hot water is insufficient. In this fascinating game, it is in the interest of everybody not only to reach an agreeable equilibrium temperature but also to avoid large fluctuations. These two goals are difficult to achieve because one inevitably not only has incomplete information about the behavior and personal preferences of the other bathers, but also about the nonlinear intricacies of the plumbing system.

The central issue of this paper is to find the conditions under which the agents are satisfied, which depends on the learning procedure and on its parameters.

The need to depart from rational representative agents was forcefully voiced among others by Kirman (2006) and Brian Arthur, for instance in his El Farol bar problem (Arthur 1994), subsequently simplified as minority game (Challet & Zhang 1997*b*, Challet et al. 2005), from which we shall borrow some ideas concerning the learning mechanism. In these models, the agents try to behave maximally differently from each other, hence the need for heterogeneous agents.

The Shower Temperature Problem is different in that the perfect equilibrium is obtained when all the agents behave exactly in the same optimal, unique way. A priori, it is a perfect example of a case where the representative agent approach fully applies. As we shall see, however, because in practice there is only a finite number of tap tuning settings, it may pay off to be heterogeneous with respect to the strategy sets. The reason lies in the fact that in a discrete world the optimal setting most probably is not part of the choice set.

Therefore, the problem we propose in this paper is another example of a situation where heterogeneity is tempting because potentially beneficial. The intrinsic and strong non-linearity of the temperature response function prevents the use of the mathematical machinery for heterogeneous systems that successfully solved the minority game (Challet et al. 2005, Coolen 2005), the El Farol bar problem (Challet et al. 2004) and the Clubbing problem (De Sanctis & Galla 2006).

1.2 The Shower Temperature Problem

One of the problems of poor plumbing systems is the interaction between the water temperatures of all the people taking a shower simultaneously. If one person changes her shower setting, she influences the temperature of all the other bathers. However, this only happens when a higher hot water flux is required than the plumbing system can provide. In that case, cascading shower tuning and retuning may follow. A key issue is how people can learn from past temperature fluctuations how to tune their own shower so as to obtain an average agreeable temperature $\hat{T}$, and also to avoid large temperature fluctuations.

Some rudimentary shower systems allow only for one degree of freedom, the desired fraction of hot water in one's shower water, denoted by $\phi \in [0, 1]$. Assuming that H and $C = H$ denote the maximal fluxes of hot and cold water available to a shower, the obtained temperature is equal to

$$T = \frac{\phi H T_H + C T_C (1 - \phi)}{\phi H + C(1 - \phi)}, \tag{1.1}$$

where T_H and T_C denote the constant temperatures of hot and cold water.

In the following, we shall consider the special case were $T_C = 0$ and $T_H = 1$, which amounts to expressing T in T_H units, i. e. to rescale T by T_H, which leads to $T = \phi$.

The situation may become more complex, however, if many people take a shower at the same time. Indeed, it sometimes happens that altogether the N bathers ask for a larger hot water flux than the plumbing system can provide, a feature more likely found in old-style youth hostels than in more upmarket hotels. Assume that the total available hot water flux for all bathers together is H while the cold water flux available at each single shower is $C = H$. We denote by $\Phi = \sum_{i=1}^{N} \phi_i$ the total fraction of desired hot water. If $\Phi > 1$, each agent will only receive the hot water amount $(\phi_i/\Phi)H$ instead of $\phi_i H$, and therefore the total flux of hot water she obtains is smaller than expected. The amount of cold water is still $(1 - \phi_i)C$, according to the agent's choice, since cold water is assumed to be unrestricted. Finally, agent i obtains

$$T_i = \frac{\phi_i}{\phi_i + \Psi(1 - \phi_i)}, \tag{1.2}$$

where $\Psi = \max(1, \Phi)$. Clearly, $T_i(\phi_i = 0) = 0$ and $T_i(\phi_i = 1) = 1$. When $\Phi \leq 1$, this equation reduces to the no-interaction case $T_i = \phi_i$. Therefore, provided that $\Phi > 1$, the agents interact through the temperature they each obtain, that is, via Φ. Assuming no inter-agent communication, the global quantity Φ is the only means of interaction. Therefore, this model is of mean-field nature. Henceforth, we consider the more involved case of interaction, i. e. $\Phi > 1$.

1.3 Tuning one's shower

1.3.1 Equilibrium and sensitivity: the homogeneous case

Before setting up the full adaptive agent model, we shall discuss the homogeneous case where $\phi_i = \phi$.

Assuming that all the agents have the same favorite temperature ($\hat{T}_i = \hat{T} \leq 1$),

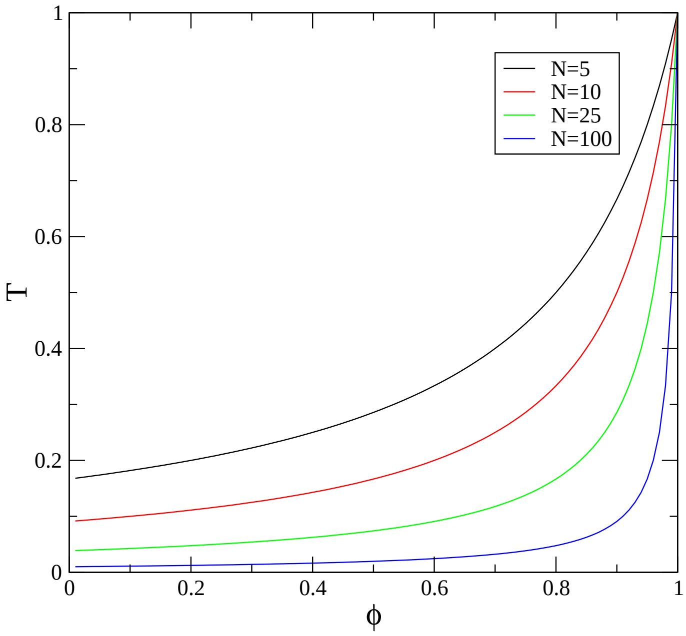

Figure 1.1: Individual temperature as a function of ϕ in the homogeneous case for increasing N (from top to bottom).

they do not interact if $N \leq 1/\phi$, in which case $\phi = \hat{T}$ and hence $N \leq 1/\hat{T}$. In the reverse case, if $N > 1/\hat{T}$ (or equivalently $N > 1/\phi$) the equilibrium in which all agents obtain their favorite temperature $\hat{T}$ can be deduced from equation (1.2) with $\Psi = N\phi$, i. e.

$$\hat{T} = \frac{\phi}{\phi + N\phi(1-\phi)}$$

which is equivalent to

$$\phi = \phi_{\mathrm{eq}} = 1 - \frac{1}{N}\left(\frac{1}{\hat{T}} - 1\right). \tag{1.3}$$

Hence, there is always a ϕ that satisfies everybody (for instance, setting $\hat{T} = 1/2$ leads to $\phi_{\mathrm{eq}} = 1 - 1/N$). In equilibrium each agent actually gets $\phi_{eq}H/(N \cdot \phi_{eq}) = C/N$ hot water instead of $\phi_{eq}H$ and thus a total water flux of $C/N + (1-\phi_{eq})C = C/(N\hat{T})$. Hence, indeed the desired temperature $\hat{T}$ is reached for every agent, but the total water flux per agent is quite small for large N.

The sensitivity of T to ϕ, defined as $\chi = \frac{\mathrm{d}T}{\mathrm{d}\phi} = \frac{N}{[1+N(1-\phi)]^2}$ is an increasing function of ϕ and maximal at $\phi = 1$ (a similar result also holds for $T_i = \frac{\phi_i}{\phi_i + \Phi(1-\phi_i)}$). The problem is that $\chi(\phi_{\mathrm{eq}}) = N\hat{T}^2 \propto N$; therefore, as N increases, tuning ϕ around ϕ_{eq} becomes more and more difficult, suggesting already that the agents might experience difficulties to learn how to tune their shower. Figure 1.1 illustrates this phenomenon: As N increases, the region in which most of the variation of T occurs shrinks substantially.

This problem is made worse by the fact that, in practice, there is only a finite number S of possible values for ϕ that can be effectively used by the agents, mostly because of internal tap static friction—the larger the friction, the smaller the number of different achievable values for ϕ. Assuming that the resolution in ϕ is $\delta\phi$, or equivalently that $S = 1/(\delta\phi)$ values of ϕ are usable, it becomes impossible to tune one's shower if $|T(\phi_{\rm eq} \pm \delta\phi) - \hat{T}| \simeq \chi(\phi_{\rm eq})\delta\phi$ is larger than some acceptable value. As $\chi \propto N$ around $\phi_{\rm eq}$, $S \propto N$ is needed; as a consequence, the ideal temperature is not learnable beyond a number of agents, which is for a large part pre-determined by the plumbing system.

1.3.2 Learning

The question is how to reach $\phi_{\rm eq}$. In this model, it is hoped that the agents have a common interest to avoid large fluctuations of T_i around their favorite temperature $\hat{T}_i$: The Shower Temperature Problem is a repeated coordination game (cf. Crawford & Haller (1990) and Bhaskar (2000)) with many agents and limited information.

The dynamics of the agents are fully determined by their possible tap settings, hereafter called strategies, and by the trust they have in them. Before the game begins, each agent i is equipped with S possible strategies $\phi_{i,s} \in [0, 1]$, $s = 1, ..., S$, which are kept constant during the game (how to choose the strategy ϕ is discussed in the next section). The typical resolution in ϕ is $1/S$; for the same reason, the typical maximal ϕ_i over all the agents is of order $1 - 1/S$. This paper follows the road of inductive behavior advocated by Brian Arthur: To each possible choice $\phi_{i,s}$ agent i attributes a score $U_{i,s}(t)$ (where t denotes the time step of the game), which describes its cumulated payoff at time t. The agents choose their $\phi_{i,s}$ probabilistically according to a logit model $P(\phi_i(t) = \phi_{i,s}) = \exp(\Gamma U_{i,s}(t))/Z$, where Z is a normalization factor and Γ is the rate of reaction to a relative change of $U_{i,s}$.

If one were to follow blindly El Farol bar problem and minority game literature, one would write

$$U_{i,s}(t+1) = U_{i,s}(t) + \phi_{i,s}\left[\hat{T}_i - T_i(t)\right].$$

When $S > 2$, such payoffs are not suitable any more; a payoff allowing for a gradual increase of $\phi_{i,s}$ is necessary. Absolute value-based payoffs are fit for this purpose, mathematically,

$$U_{i,s}(t+1) = U_{i,s}(t) - \left|\hat{T}_i - T_i(t)\right|.$$

This payoff however does not depend on $\phi_{i,s}$. As a consequence, all the strategies would have the same payoff. Therefore, one has to give more information to the agents. An agent that has perfect information about the plumbing system, the temperatures and fluxes of hot and cold water—for instance the plumber that built the whole installation—may know precisely which temperature she would have obtained, had she played $\phi_{i,s'}$ instead of her chosen action $\phi_{i,s_i(t)}$. Such people are probably not very frequent amongst the general population, however. This is why we shall consider an in-between case, where the agents' estimation of $T_{i,s}(t)$ is a linear interpolation between the temperature of the strategy currently in use, i. e. $T_i(t) = T_{i,s_i(t)}$ and its correct virtual value. The payoff is therefore

$$U_{i,s}(t+1) = U_{i,s}(t)(1-\lambda) - \lambda\left|\hat{T}_i - (1-\eta)T_i(t) - \eta T_{i,s}(t)\right|, \tag{1.4}$$

where $\eta \in [0,1]$ encodes the ability of the agents to infer the influence of $\phi_{i,s}$ on the real temperature and $0 \leq \lambda < 1$ introduces an exponential decay of cumulated payoffs, with typical score memory length $\propto 1/\lambda$. The parameter η is to some extent related to the difference between naive and sophisticated agents as defined by (Easley & Rustichini 1999). The first kind of agents believe that they are faced with an external process, i. e. that they do not contribute to Φ, whereas sophisticated agents are able to compute $\Phi_{-i} = \Phi - \phi_i$. In this model, perfect sophisticated agents have $\eta = 1$.

1.4 Results

It is natural to measure two collective quantities, the average temperature T obtained by the agents and its average distance from ideal temperature averaged over all the agents, denoted by $\Delta T = T - \hat{T}$; this characterizes the average temperature obtained by the agents, or how far the agents are collectively from their goal. The individual dissatisfaction is the distance from the ideal temperature for a given agent. One therefore measures it with $|\delta T| = \frac{1}{N}\sum_{i=1}^{N}|T_i - \hat{T}_i|$; it is a measure of the dispersion (and therefore positively correlated with the agents' risk).

Concerning the numerical simulations, all the quantities reported here are measured in the stationary state over $10,000$ time steps for $\hat{T} = 0.5$, $\eta = 1$, $\lambda = 0.001$ and if not stated differently $N = 20$, after an equilibration time of $30/(\lambda\Gamma)$. The stationary state does not depend much on λ. On the other hand, the performance of the population is of course improved as η increases and saturates for $\eta > 0.5$. The role of Γ is discussed below.

1.4.1 Homogeneous population

Since the equilibrium is reached when all the agents tune their shower in exactly the same way, trying first homogeneous agents (or equivalently a representative agent) makes sense *a priori*. We shall therefore set $\phi_{i,s} = \phi_s = \frac{s}{S+1}$, $s = 1, ..., S$ so that the agents avoid using only hot or cold water.

Agents with homogeneous strategies have a peculiar way of converging to their ideal temperature as S increases. Figure 1.2 displays the oscillations of the reached temperature with decreasing amplitude as a function of S. The asymmetric upward and downward slopes are due to the asymmetry of T around ϕ_{eq}, as seen in Figure 1.1. Theoretically, this can easily be explained by assuming that all the agents select the same s that gives T as close as possible to $\hat{T}$. Solving $\phi_{i,s} = \frac{s}{S+1} = \phi_{\text{eq}}$ for s, we obtain $\hat{s} = [1 - 1/N(1/\hat{T} - 1)](S+1)$. The agents therefore choose $[\hat{s}]$ or $[\hat{s}]+1$, where $[x]$ is the integer part of x (one may need to enforce $[\hat{s}] < S$ when $S < N$). Either $T([\hat{s}])$ or $T([\hat{s}]+1)$ is closest to $\hat{T}$, therefore the actually optimal temperature T_{th} (whichever $T([\hat{s}])$ or $T([\hat{s}]+1)$) oscillates around $\hat{T}$, as seen in Figure 1.2. The period of the oscillations is N, and their amplitude decreases as $1/S$. As expected, a very large value of Γ replicates closely the dented nature of the value of T_{th}, in which case all the agents take the same choice even close to the peak of T_{th}. Generally,

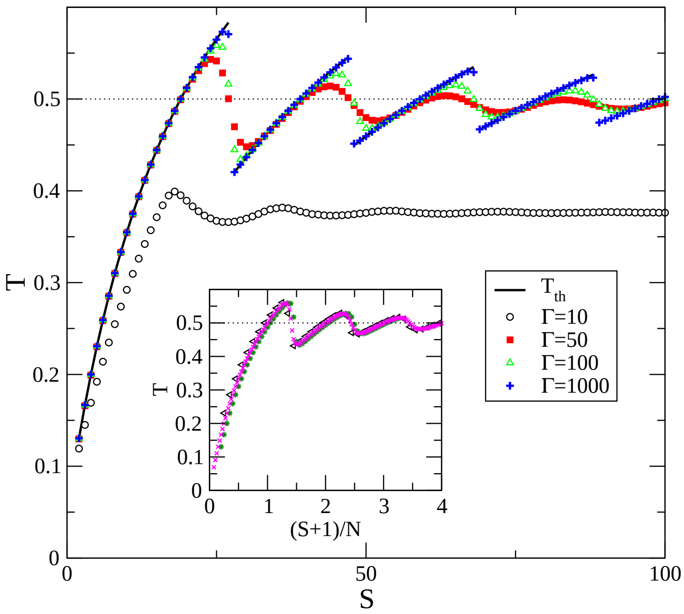

Figure 1.2: Average temperature T reached by homogeneous agents as a function of S for various Γ. Inset: T vs. $(S+1)/N$, showing the scaling property of T, with $N = 10, 20, 40$ (asterisks, triangles, crosses).

smaller Γ (at least to a certain degree) leads to better average temperatures as it allows to play mixed strategies and thus combine two temperatures so as to achieve a collective average result closer to $\hat{T}$. From that point of view, $\Gamma = 50$ is a better choice than $\Gamma = 1000$. Hence, there exists an optimal global value of Γ, leading to a mixed-strategy equilibrium. This is because taking stochastic decisions is a way to overcome the rigid structure imposed on the strategy space, whose inadequacy is reinforced by the strong non-linearity of $T(\phi)$. A too small Γ is detrimental as it allows for using ϕ further away from ϕ_{eq}. Because of the shape of $T(\phi)$, those with smaller ϕ are more likely to be selected. To find the optimal Γ, we may average $|\Delta T|$ over S in numerical simulations, for instance with $I = \sum_{S=N}^{5N} |\Delta T|/(4N)$.[1] The inset of Figure 1.3 reports that the minimum of I is at $\Gamma \simeq 42$ for the chosen parameters, which shows the existence of an optimal learning rate.

The individual dissatisfaction $|\delta T|$ unsurprisingly mirrors $|\Delta T|$ since all the play-

[1]Simulations show that the average temperature is in fact a function of $(S+1)/N$ (cf. Figure 1.2) (instead of a function of S and N), i. e. Figure 1.3 would look the same if S was fixed and N varied. Hence we may take the average over S instead of over N.

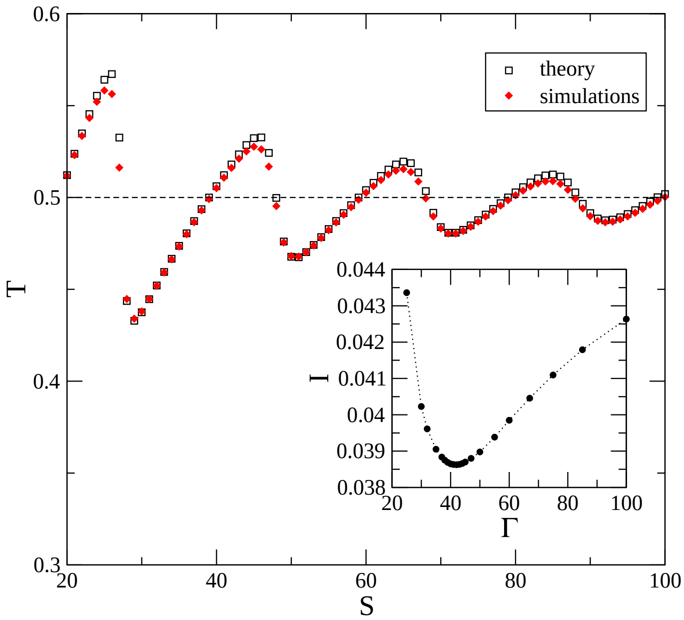

Figure 1.3: Average temperature T reached by homogeneous agents as a function of S for $\Gamma = 100$. Squares: theory, circles: numerical simulations. Inset: average deviation I from $\hat{T}$ versus Γ (same parameters); the dotted lines are for eye guidance only.

ers are identical. Both quantities are the same for large Γ as everybody plays the same fixed strategy. The amplitude of $|\delta T|$ also decreases as $1/S$ (see Figure 1.5, squares). However, the larger Γ, the smaller $|\delta T|$, as each agent manages to get closer to the optimal choice.

It is easy to obtain analytical insights by solving the stationary state equations for $U_{i,s}$. For the sake of simplicity, assuming that $\eta = 1$ and that only the two strategies around $\phi_{\rm eq}$, i. e. $[\hat{s}]$ and $[\hat{s}]+1$, denoted by $-$ and $+$ respectively, are used, one obtains the set of equations (independent from λ and i)

$$U_{i,\pm} = U_\pm = -|T_\pm - \hat{T}| \tag{1.5}$$

where

$$T_{i,\pm} = T_\pm = \frac{1}{1 + \frac{N_+\phi_+ + N_-\phi_-}{\phi_\pm}(1-\phi_\pm)} \tag{1.6}$$

with $N_\pm = N \cdot P(s = \pm)$, where $P(s=+) = \frac{\exp(\Gamma U_{i,+})}{\exp(\Gamma U_{i,+}) + \exp(\Gamma U_{i,-})}$ and $P(s=-) = 1 - P(s=+)$ is a logit model for the two-strategy case $S = 2$. Figure 1.3 shows the good agreement between numerical simulations and this simple theory, especially in

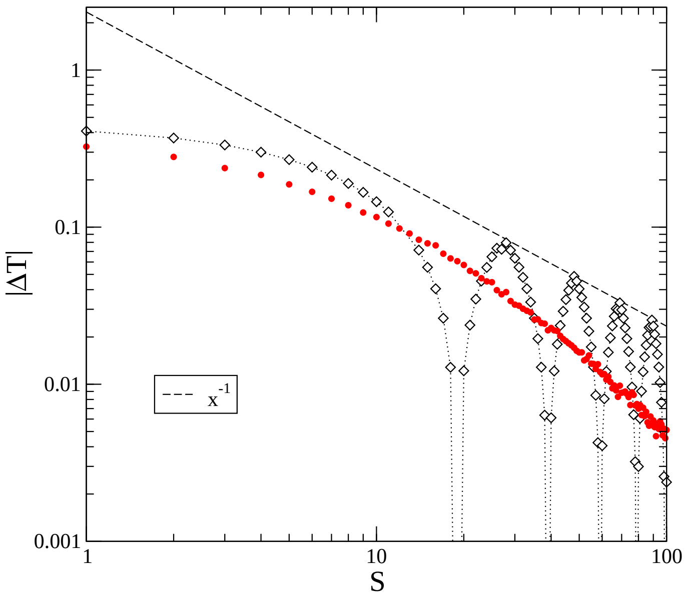

Figure 1.4: Absolute temperature deviation $|\Delta T|$ reached by homogeneous (squares) and heterogeneous (circles) agents as a function of S for $\Gamma = 100$. Average over 500 samples for heterogeneous agents.

the convex part of the oscillations, as long as Γ is large enough (about 50) to prevent the use of more than 2 strategies.

Being faced with oscillations is problematic since the agents do not know N a priori and because N may vary with time. In addition, since all the agents select the same ϕ for large Γ, not a single agent is ever likely to reach a temperature close to $\hat{T}$. However, the individual satisfaction is maximal in the limit $\Gamma \to \infty$ (see above) since $|\delta T|$ gets minimal there.

1.4.2 Heterogeneous population

There are many ways for agents to be heterogeneous. One could imagine to vary S, Γ, η, λ or $\hat{T}$ amongst the agents. Here we focus on strategy heterogeneity, i. e. the agents face showers with different tap settings: The strategy space of agent i is no longer $\frac{1}{S+1}, \dots, \frac{S}{S+1}$, but now each agent has an individual strategy space where each strategy $\phi_{i,s}$, $s = 1, \dots, S$, is assigned a random number from the uniform distribution on $[0, 1]$ before the simulation.

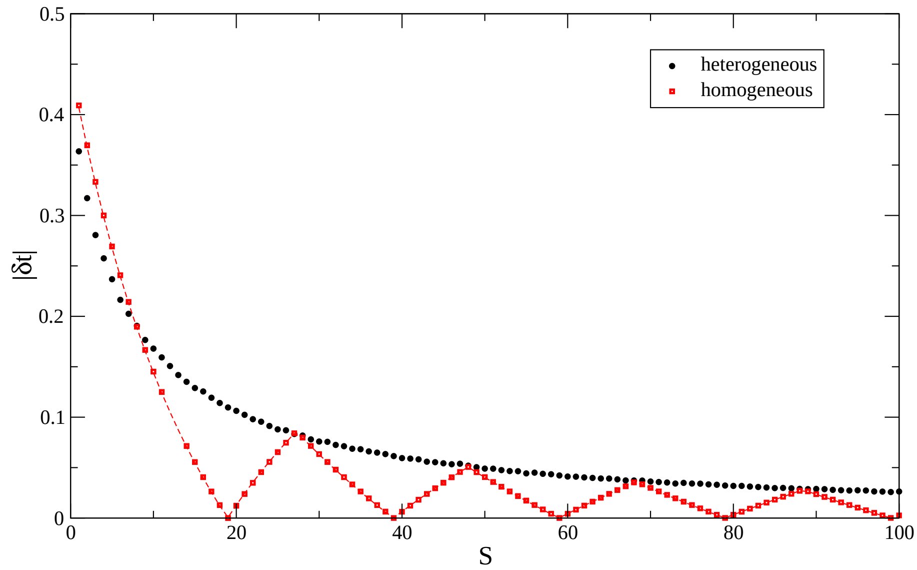

Figure 1.5: Individual dissatisfaction $|\delta T|$ reached by homogeneous (squares) and heterogeneous agents (circles) as a function of S for $\Gamma = 1000$. Average over 500 samples for heterogeneous agents. Dashed line: theoretical predictions.

Thus, each heterogeneous agent has a different strategy set she can choose from. This heterogeneity may, for instance, emerge from the internal tap friction and how each agent deals with it (e. g. big pushes versus tapping on the tap).

Intuitively, the effect of heterogeneity is to break the structural rigidity of the strategy set of a representative agent. Figure 1.4 reports that $|\Delta T|$ does not oscillate, and converges to zero faster than S^{-1}. Homogeneous agents might achieve a better average temperature depending on N and S, but on the whole perform collectively worse. In addition, heterogeneous agents expect to have a smaller than ideal temperature, but on average *predictably smaller*, with no strong dependence on S. Thus, the expectation over the temperature of the agents is much improved by heterogeneity.

However, looking at the average absolute individual deviation from $\hat{T}$ reveals that the uncertainty brought by heterogeneity is considerably worse *on average*. Plotting $|\delta T|$ for both types of agents shows that $|\delta T|$ is always smaller for homogeneous agents (Figure 1.5). This means that being heterogeneous is more risky. Which agent (or equivalently, shower) performs better depends not only on N, but also on the tuning settings of all the agents.

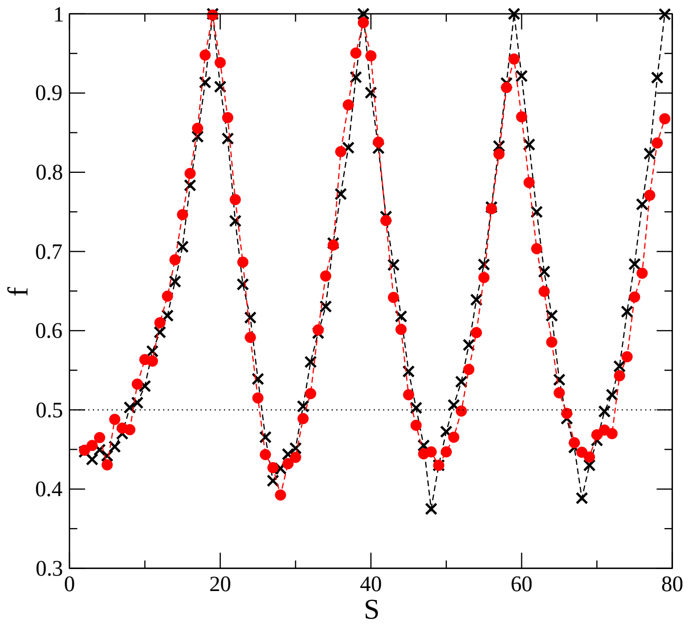

Figure 1.6: Fraction of the simulations for which a single heterogeneous agent is worse off than the other $N-1$ homogeneous agents; $\Gamma = 1000$ (crosses) and $\Gamma = 30$ (circles). Average over 2000 samples.

1.5 Discussion and conclusions

Heterogeneity may be tempting as it suppresses the systematic abrupt oscillations experienced by homogeneous populations and is collectively better on average. However, it seems that heterogeneous showers are potentially more risky. However, it is human nature to take a risk in the hope to be better off, even if the probability is small, think of the lottery, for instance. Hence, the information how good the best agent (or the few best ones) feels is more relevant than how poor the worst agent feels. At least some agents will try, and some of them will succeed, hence sticking to being heterogeneous. Thus, the fact that there are some chances that by being different one can improve the temperature means that heterogeneity will appear because of temptation.

In other words, the agents must consider the trade-off between the temptation of an expected better temperature and a potentially larger deviation.

The situation discussed above is only global. Does it pay off to be heterogeneous for a single agent? An answer comes from a system consisting of $N-1$ homogeneous agents as defined above and a single random one with random $\phi_{i,s}$. The fraction f of the simulations in which the homogeneous showers have a smaller $|T_i - \hat{T}_i|$ is reported in Figure 1.6; this quantity indicates that the majority of heterogeneous agents are not worse off for about a quarter of the values of S. This finding is not in contradiction with the fact that the average personal dissatisfaction of heterogeneous agents is always larger than that of homogeneous agents: $|\delta T|$ is much influenced by large deviations contributed by a minority of agents because of large temperature sensitivity to small deviations in ϕ.

The results of the Shower Temperature Problem also allow some first implications for other economic situations which can be modeled by similar means. For instance, consider the problem of how many employees each firm should optimally employ, depending on how many orders this firm has. Just as the hot water supply was the restricting factor in the Shower Temperature Problem, this role is now played by the overall order situation, i. e. the cumulative number of all orders available on the market. The profit of each firm now depends on the received orders (corresponding to the fraction of hot water received) and the number of employees who draw a salary. In turn, the maximum number of orders that can be processed and thus accepted depends on the number of employees. Transferring the result of the Shower Temperature Problem suggests that it might be tempting for a firm to be different in the sense of employment strategies. Such a heterogeneity can for instance result from different production technologies or different corporate structures.

One can also examine the question of how many (perishable or other) products each firm should optimally produce, depending on the fluctuating demand.

As a final note, minimizing $|\Delta T|$ is equivalent to solving a number partitioning problem (Garey & Johnson 1979) in which one splits a set of N numbers $a_i > 0$ into two subsets so that the sums of the numbers in the subsets are as close as possible, which amounts to minimizing $C = |\sum_i s_i a_i|$ where $s_i = \pm 1$; it is an NP-complete problem; in other words, the only way to find the absolute minimum of C is to sample all the 2^N configurations. Let us consider an even simpler version of the Shower Temperature Problem that makes more explicit its NP-complete nature. Each agent i is given a_i and plays $\phi_{\text{eq}} + s_i a_i$, $s_i = \pm 1$. Neglecting the self-impact on the resulting temperature and the non-linearity of the temperature response, the analogy between the Shower Temperature Problem and the number partitioning problem is straightforward. Methods borrowed from statistical mechanics show that the average optimal C scales as 2^{-N}, which requires to enumerate the 2^N possible configurations (Mertens 1998). This is much better than what the agents achieve; the reason for this discrepancy is that the agents do not reach a stationary state in $O(\exp N)$ time steps, hence, they cannot sample all the possible configurations. Another reason is that the globally optimal solution may require some agents to use a strategy that would yield a worse temperature than their optimal choice.

In conclusion, the Shower Temperature Problem shows the subtle trade-offs between a homogeneous population with equally spaced actions and a fully random one. In a system where the agents' action space is not likely to include the optimal equilibrium choice, heterogeneity is a way to solve this kind of problem more robustly and with less systematic deviation, at the expense of a higher risk for individual agents.

Chapter 2

The Evolution of Sales in a Market with Habit-Forming and Imitative Consumers

2.1 Introduction

Optimal economic choices are often very hard to make for consumers. Especially if a product is new on the market, its quality can only be guessed. Consider, for example, an innovative and new liquid crystal display (LCD) TV shortly after its product introduction. A potential consumer does not know the true product quality (cf. Smallwood & Conlisk 1979, p. 2), though she can have some (not necessarily true) idea about it. In such a situation, as supported by studies in social psychology and consumer research (e.g. Assael 1984, Venkatesan 1966), consumers often base their product choices on imitation of others who already own the product.

Such imitative behavior will form one of the two building blocks of the model proposed here, the second is habitual behavior, which we will motivate shortly. We shall try to construct a formal model incorporating just these two basic behavioral rules, which have been suggested by psychological and experimental studies. The model will be phrased within the framework of population games and will allow to analyze the demand side and to deduce market implications such as feasibility, i. e. "survival" conditions for successfully introducing a new product in a single or multi-goods market.

Typically, in a consumer–seller relationship the degrees of freedom (e. g. price, quality, output) are determined by the supply side. On the consumer side, any influencing parameter (such as personal preference or reservation price) is fixed, i. e. exogenous in a market model. Hence, the consumer can be assumed to be—as is common practice for instance in Cournot's standard model—purely reacting, while the firm is acting, determining the degrees of freedom. It therefore makes sense to model the demand and supply side separately: Only after the demand side has been described (e. g. the demand function in the Cournot setting), a strategic behavior of determining the degrees of freedom can be devised for the firms. This is exactly the approach we follow, and in this chapter we deal with the demand side. Due to their reactive role, consumers naturally do not compete actively with each other or with firms. Hence, we have to refrain from conventional game theoretic modeling such as the Bayesian ansatz, and we employ a population game instead, which offers an adequate approach in this context.

This chapter aims at providing an insight into the market dynamics generated by consumer behavior. The modeling approach is related to Smallwood & Conlisk (1979) as well as von Thadden (1992) in that the consumers are unable or not willing to act strategically and thus act adaptively. We extend these authors' approaches by examining a time-continuous model, which complicates the mathematical analysis a little but in fact may be seen as just the next logical step after time-discrete modeling and also resolves the problem of choosing an appropriate time period (Dockner et al. 2001). Even more so, the time-continuous setting gives rise to novel interesting issues such as conditions under which demand is ensured for firms over time. A comparable analysis is impossible in time-discrete models. Furthermore, as opposed to von Thadden (1992) we allow for a multi-goods case, and in contrast to Smallwood & Conlisk (1979) we introduce the possibility of not buying any good. Most importantly, we elaborate explicitly the factors habit and imitation, which only implicitly occur in Smallwood & Conlisk (1979). Hence, despite some analogies between our model and Smallwood & Conlisk (1979) or von Thadden (1992) our approach is substantially different.

We believe that our modeling approach for the demand side captures the true mechanisms and is therefore worthwhile elaborating on. It contains a natural mathematical description of psychology-based consumer behavior and more generally provides a route to incorporate psychological insights into economic modeling. Boundedly rational consumer behavior and continuity in time are inherent in the model structure, and both allow insight into novel issues. For instance, only time-continuity will enable time-continuous price paths (which will be dealt with in the next chapter). Furthermore, the analysis opens a different perspective on standard results in industrial organization and offers new explanations. It will be shown that the market under bounded rationality performs well in the sense that e. g. competition gets harder the more competing products enter the market. The model also explains e. g. the long-lasting survival of a bad quality product among some high quality products.

The next paragraphs will include a brief review of the most related modeling work as well as of the wider range of literature that either employs a concept of imitation or habit or yields experimental and psychological support for both.

2.1.1 Further motivation and related literature

Imitation can take place either in an implicit or explicit manner. For instance, when a person encounters someone owning the LCD TV, the familiarity with this product increases, implicitly raising the probability to buy it. On the other hand, explicit or deliberate imitation may occur if a consumer understands the product's popularity as a hint to its past performance (cf. Ellison & Fudenberg 1993).

Imitation represents a well-known concept in areas such as evolutionary game theory or social learning. For example, Schlag (1998) shows that a proportional imitation rule is better than any other (well-performing or so-called improving) behavioral rule in a multi-armed bandit environment. Ellison & Fudenberg (1993) employ imitation through the concept of popularity weighting. For a high degree of popularity weighting the agents choose the most popular choice, independent of its payoff.

An extreme result of such imitative behavior is herding. In this context, Bikhchandani et al. (1992) explain how social conventions, norms, fashion or new behavior can arise through informational cascades. Banerjee (1992) shows that imitative behavior yields socially inefficient equilibria: Herding can emerge since the agents rather imitate others than following the information contained in their own signal. Another approach to modeling conventions, where the own choice depends also on past actions of others, is due to Young (1993). In his model n agents, drawn randomly from a population, play an n-person game. They act as in fictitious play, except that their information about the predecessors is incomplete and that the agents are not free from errors.

Word-of-mouth learning is another field closely related to imitation and the herding literature and is said to be an important mechanism for consumers to choose a product brand. Banerjee & Fudenberg (2004), for example, examine rational word-of-mouth learning. They search for aspects that affect the characteristics of word-of-mouth learning in the long run by investigating several sampling rules for imitation. Ellison & Fudenberg (1995) represents a nice example where boundedly rational word-of-mouth learning from a few other agents may lead to socially efficient results. Similar forerunners of word-of-mouth learning are for instance Kirman (1993), where a vivid example of herding (ants eating only from one of two exactly identical food sources) is explained within a recruiting model, and the seminal work of Smallwood & Conlisk (1979). In order to identify market (share) equilibria, they investigate a model in which consumers respond to breakdowns of their products. As in our model, they consider consumers who are uncertain about the product quality. Their consumers buy a new product each period—either of the same brand as before or of another one, depending on whether a breakdown happened. After breakdown, the consumer makes a decision taking the population shares into account, i. e. how many agents bought each brand in the last period. The analysis is then divided into that of weakly dissatisfied and strongly dissatisfied consumers, who will not buy the product again. Furthermore, Smallwood & Conlisk (1979) explore the equilibria for different values of a parameter representing the consumers' degree of confidence in the significance of brand popularity. Only the best quality brands are used in equilibrium (if the model only incorporates weak, but not strong dissatisfaction) or additionally some non-best (if the model only incorporates strong, but not weak dissatisfaction) if there are moderate consumers' beliefs about brand popularity. In a second step, Smallwood and Conlisk consider the supply side of the market and assume that the firms set their products' breakdown probabilities each period in order to maximize their profit. Due to the high mathematical complexity, they only obtain approximate firm strategies. For the demand side, we use an approach quite similar to Smallwood's and Conlisk's paper, extending and refining some ideas, for example, including a weighting of imitation according to (anticipated) product quality as well as introducing a habit mechanism.

Experiments in consumer research have revealed further important mechanisms of decision making besides imitation. In particular, consumers obviously also take their own experience and satisfaction into account, resulting in a habitual behavior (Assael 1984). Let us return to the TV example: A consumer who owned an LCD TV for a while and is satisfied with it rather sticks to LCD TVs instead of following the majority of other agents buying a plasma TV. Consequently, we introduce habit as a second mechanism.

An indication of habitual purchases was found in several studies, summarized in Assael (1984, p. 53). Implicitly, the amount of habit induced by a product is also modeled in Smallwood & Conlisk (1979) via its breakdown probability or so-called "dissatisfaction probability". The consumers buy the same brand—hence follow their habit—until they encounter a problem. Expressed in terms of our model, if the consumers' habit is disturbed, the consumers imitate the buying decision of other consumers. A habit-related aspect has also already been advanced in the early papers of bounded rationality (for instance Simon 1955), where satisficing as opposed to optimizing is motivated. Satisficing describes the consumer practice to already accept the first satisfying choice instead of extensively searching for the optimal choice. Similarly, in our model we will assume that consumers who own a satisfying good will continue to buy it.

Generally, our approach can be classified as belonging to the bounded rationality field as it is surveyed in at least three excellent reviews, Ellison (2006), Sobel (2000), and Conlisk (1996), with focuses on industrial organization, learning, and general motivations, respectively. There seems to exist quite some work taking a basically similar perspective as our approach. Early precursors of bounded rationality in industrial organization focus on firms' rather than consumers' bounded rationality, for instance Rothschild (1947) and Cyert & March (1956). Heiner (1983) hypothesizes that predictable behavior in economics results mostly from rules of thumb and not from optimizing agents, since maximizing agents would lead to irregularities on the market. More recently, von Thadden (1992) introduced a model in which the firms act strategically, whereas the consumers do not.

In order to describe markets, we will define a game with demand and supply side, of which the demand is the content of this chapter. A continuous time consumer model based on behavioral rules which are reasonable according to psychologic and experimental studies (Assael 1984, Venkatesan 1966) is used. The model is stated in form of a population game as defined in Sandholm (2005) and Sandholm (2006), who makes use of the fact that for a large population size, the stochastic process generated by the evolutionary process can be approximated by solutions to ordinary differential equations (Benaïm & Weibull 2003). The supply side, which will be the topic of the next chapter, describes the strategic behavior of firms, which base their decisions on the results of the bounded rational consumer model. Altogether, it will be seen that two simple rule of thumb ingredients, imitation and habit, are sufficient to generate typical patterns observed in consumer markets, such as product life cycles (as described in de Kluyver 1977, Brockhoff 1967, Polli & Cook 1969). Concerning the methodology, we employ a new approach to describe interacting consumers in continuous time as well as a new method to describe supply and demand side of a market.

Although a complete market description has to take into account supply and demand side, the model presented in this chapter stands on its own. It is, for example, applicable in cases where the supply side is (temporarily) almost inactive, as, more specifically, in marginal cost pricing (Bertrand, perfect competition), supply sides with strong inertia, sticky pricing, regularity constraints, or markets with expired patents and low entry costs which can be approximated by perfect competition (e. g. generics in Africa). Furthermore, some implications on product survival on the market can already be obtained, if prices and production costs are implicitly encoded in the model parameters.

The outline of the chapter is as follows. We begin section 2.2 with the introduction of the used methodology. The general methodology is then applied to a market where demand follows a simple rule of thumb. Subsequently, we deduce the important sales equation, motivate an appropriate revision protocol (similar to a transition probability), and find the corresponding mean dynamic which determines the demand evolution dependent on two parameters per product, a convincement factor and a habit coefficient. In section 2.3 the demand evolution is analyzed for varying numbers of different products (or brands), where the focus lies on product feasibility. Finally, in sections 2.4 and 2.5 we discuss further model implications (in particular for the supply side) and conclude.

2.2 Model and microfoundation of sales

This section proposes a mathematical model for the evolution of sales based on a microfoundation. A microfoundation describes the economic dynamics on the lowest level, i. e. on the level of the decisions by single agents. Here, the agents behave boundedly rationally and myopically change their activities. On that basis, interaction of a large number of agents can be described by a so-called modified population game. The mean over all agents then provides the governing differential equations to describe the overall, global system behavior.

After briefly introducing the methodology of a microfoundation following Sandholm (2005) and (2006), it is applied to a goods market, yielding a first simple result about the relation between the evolution of product distribution and sales. Finally, we suggest an exemplary model of consumer behavior and derive the resulting sales evolution.

2.2.1 Methodology

A typical normal form game consists of a population of agents, their actions, and the payoff function, which rewards or penalizes the agents' actions depending on the other players' behavior. Since the agents are usually assumed to be rational, their behavior directly follows from maximizing payoff. However, in the case of boundedly rational agents the payoff function is insufficient to determine their actions; a revision protocol, for example, similar to Sandholm (2006), seems more adequate to describe the agents' behavior.

Definition 2.2.1 (Revision protocol)**.** *A* revision protocol ρ *is a map*

$$\rho : X \times \mathbb{R} \to \mathbb{R}_+^{(n+1)\times(n+1)}, \tag{2.1}$$

where $X = \{x = (x_0, \ldots, x_n) \in [0,1]^{n+1} \,|\sum_{i=0}^{n} x_i = 1\}$ *is called the* state space*. The scalar* $\rho_{ij}(x,t)$ *is called the* conditional switch rate *from activity* i *to activity* j *at time* t *and state* x*. The sum*

$$R_i = \sum_{j=0}^{n} \rho_{ij}(x,t) \tag{2.2}$$

is called the alarm clock rate *of subpopulation* i *and the scalar* $p_{ij} = \frac{\rho_{ij}}{R_i}$ *the* switching probability*.*

Before motivating this definition, we use the revision protocol to specify the model framework.

Definition 2.2.2 (Modified population game)**.** *A* modified population game *is defined by the triple* $\mathcal{G} = (N, \mathcal{A}, \rho)$*, where* N *is the agent* population size*,* $\mathcal{A} = \{0, \ldots, n\}$ *is the* activity set*, and* $\rho \in \mathbb{R}_+^{(n+1)\times(n+1)}$ *is a* revision protocol*.*

The motivation is as follows: In a modified population game a situation with N agents is considered, each of whom plays precisely one activity $i \in \mathcal{A}$ at a time. We use the term "activity" as opposed to the game theoretic term "action" in order to emphasize the continuity of the activity in time, whereas actions take place at discrete times (e. g. in repeated games). The subpopulation of those agents playing activity i amounts to the time-varying fraction x_i of the total population so that the set of all possible states is given by $X = \{x \in [0,1]^{n+1} | \sum_{i=0}^{n} x_i = 1\}$. Each agent playing activity i is equipped with an independent Poisson alarm clock of rate R_i, i. e. an alarm clock which rings after an exponentially distributed time with expected value R_i^{-1}. Each time the alarm goes off the agent revises her activity and switches to activity $j \in \mathcal{A}$ with probability p_{ij}.

The objective of studying the macroscopic behavior of a system requires averaging over all agents. In order to limit complexity, we make use of the continuum hypothesis so that differential equations can be applied. Hence, we assume N very large with the effect that the state space X can be approximated as continuous and that the mean agent behavior can be approximated by expected behavior according to the law of large numbers. Under these conditions, Sandholm (2006) derives the macroscopic system behavior:

Definition 2.2.3 (Mean dynamic). *Let $\mathcal{G}$ be a modified population game, and let ρ be its revision protocol. The* mean dynamic *corresponding to $\mathcal{G}$ is*

$$\dot{x}_i = \sum_{j=0}^{n} x_j \rho_{ji}(x,t) - x_i \sum_{j=0}^{n} \rho_{ij}(x,t), \quad i = 0, \dots, n, \tag{2.3}$$

where $\dot{x}_i$ denotes the time derivative of x_i.

2.2.2 Application to a consumer market

Let us consider a market consisting of N consumers and n different products produced by some firms. Our focus is on the consumer side of the market, i. e. given the firms' behavior in the sense of product properties[1] (e. g. price and quality) the evolution of the market is governed by the consumers' purchase decisions. Though in principle the application of modified population games allows for time-dependent product properties (i. e. time-varying firm strategies) and consumer characteristics, we concentrate on a market with static parameters.

The consumers, who take the role of the agents in the modified population game, own at most one good each, i. e. the different products are substitutes to them (but not perfect substitutes, since they might differ in quality, appearance et cetera). Their activity set comprises the activities "not owning any product", denoted 0,

[1] Exogenously given prices can for instance be applied in a Bertrand competition for oligopolies.

"owning good 1", denoted 1, and so forth. The alarm clock of each consumer announces the time when she can decide to buy a new product and hence to switch her activity. Reasons can be a breakdown or a defect when owning a product or a sudden interest in a product when currently being without any good. This suddenly arising interest can come for instance from the desire to follow a fashion trend. Due to the memorylessness of such incidents, the use of a Poisson alarm clock seems most adequate to model their occurrence. Since the old products are broken, they are disposed and therefore reselling is assumed to be impossible. Finally, let the firms have—as is standard practice in industrial organization—perfect knowledge about the consumer behavior depending on the product properties, and hence they produce exactly as many goods as are demanded at any time.

2.2.3 The sales equation

We define $S_i(t)$ as the number of units of product $i \in \{1, \ldots, n\}$, sold per time at time t. S_i shall be called the sales of product i.

The sales of product i correspond to the rate of consumers switching to or rather buying this good,

$$S_i(t) = N\left[x_i(t)\rho_{ii}(t) + \sum_{\substack{j=0\\j\neq i}}^{n} x_j(t)\rho_{ji}(t)\right], \quad i = 0, \ldots, n. \tag{2.4}$$

Using the mean dynamic (2.3), equation (2.4) can be transformed into the sales equation, which describes the relationship between sales and consumer subpopulations,

$$\frac{S_i}{N} = \dot{x}_i + x_i R_i, \quad i = 0, \ldots, n. \tag{2.5}$$

To enhance intuition, the rate of change $\dot{x}_i$ can be interpreted as the difference between consumers adopting and abandoning good i, while the second summand represents all those owners of good i who currently reorientate themselves and either abandon or stick with[2] product i. As the leaving consumers cancel, the net effect consists of those consumers buying product i.

[2]Note that these consumers replace the broken unit they own and buy a new one.

Let us provide a brief analysis of the sales equation: As indicated above, the equation decomposes into a pair of additive terms, leading to two cases, each corresponding to one term outweighing the other. In the case of small x_i, e. g. during product launch, sales precisely reflect the increasing distribution among the population, e. g. due to growing product awareness. On the other hand, for negligible $\dot{x}_i$ (after reaching a comparatively stable situation) sales of good i solely originate from replacements of both defect units of i and different products. The rate of replacements is equivalent to the rate of consumer reorientation or revision, $x_i R_i$.

2.2.4 A consumer revision protocol and the resulting mean dynamic

The specific revision protocol that we employ borrows well-established ideas from social learning (Smallwood & Conlisk 1979, Ellison & Fudenberg 1995), psychological, experimental, and consumer research literature (Assael 1984). On a goods market, there are mainly two factors influencing the buying behavior of a consumer, the goods' properties and their perceived distribution among other consumers. It is, for example, beyond question that any purchase decision depends on the anticipated product quality, including functionality, reliability, value for money, and many further properties. Also, the life span of a product plays a role as it affects the frequency of purchases. On the other hand, the product distribution among other consumers may have an effect by simply determining the level of product awareness or inducing a fashion or even networking. The revision protocol ought to reflect these influences on the consumer behavior.

A revision protocol $\rho(x, t)$ is uniquely defined by the product-dependent alarm clock rates R_i and the switching probabilities $p_{ij}(x, t)$, for which we shall suggest a specification in the following.

The alarm clock rate represents the average frequency at which consumers think about buying a specific product and hence replacing their old one—unless they do not yet possess any good. Since this frequency depends on the durability of the currently owned product, alarm clock rates in general differ from product to product. For simplicity, we shall assume the rates R_i to be given for each product i and to be time-invariant. Typically, the frequency R_0 of revisions without any good is larger than the rate R_i of possible replacements of good i, since the goods usually survive longer than the consumers without any good are satisfied.

Of those people, who do not own any product, the fraction of consumers deciding to buy product i can be described by the switching probability p_{0i}. When consumers start thinking about buying a specific product they scan the market and become susceptible to various types of information about possible alternatives. In the accompanied decision process, passive and active decision mechanisms can be distinguished: When consumers passively encounter a product, its level of familiarity rises, thus increasing the possibility for this product to be bought. On the other hand, consumers may actively imitate others in buying the same good since the popularity of a product might give information about the product's past performance (cf. Ellison & Fudenberg 1993), so that according to Smallwood & Conlisk (1979) the consumers' choices are sensitive to market shares or popularities of the products. Additionally, studies in social psychology support the individual's conformity to group norms, i. e. that consumers imitate group behavior (Assael 1984, p. 371ff).

Purely active, purely passive, and intermediate decision mechanisms are categorized in the following. They imply a proportionality between the switching probability p_{0i} and the fraction x_i of consumers currently owning product i. The mechanisms are ordered from rather passive to rather active.

- In daily life, consumers encounter product i at a frequency proportional to its distribution x_i among the population. Hence, familiarity with the good and proneness to buy it rise accordingly.
- A consumer's idea of an ideal product is partly shaped by the surroundings. Product characteristics often observed are commonly desired. The intensity of influence by good i and thus proneness to buy it may be assumed proportional to its frequency x_i.
- Information about products can be obtained from various media (including the Internet), e. g. experience is exchanged on product evaluation websites.

The media in total roughly reflect the real world including the market; the more widespread a product is, the more frequently the media report about it, thereby increasing product awareness proportionally to x_i.

- The expected quality of a product is often inferred from the number of sales, assuming that superior products always find a ready market. The prevalently perceived indicator is thus the observed distribution x_i.
- Consumers also actively ask around their friends and let their purchase decision be influenced by the found product distribution, which on average matches the x_i. For certain goods, networking may play a role as well, intrinsically implying the positive effect of high x_i on the sales of product i.
- Finally, a fashion might be induced by a high distribution of a specific product, leading to even higher sales of that product.

Motivated by the above, we assume a linear relation between switching probability p_{0i} and x_i as a first approximation, that is $p_{0i} \sim x_i$. The proportionality factor, denoted φ_i, still remains to be determined. Generally, it differs from product to product (Assael 1984, p. 432, 414) and can even be time dependent. We need $\varphi_i \in [0, 1]$, since

$$1 = \sum_{i=0}^{n} p_{0i} = p_{00} + \sum_{i=1}^{n} \varphi_i x_i,$$

must also hold for any state $x_i = 1, \ x_{j \neq i} = 0$. φ_i constitutes the accumulated influence of product frequency on the consumers' purchase decision via all different mechanisms. It can be interpreted as the intensity by which a consumer is convinced during an encounter with the product and is therefore termed convincement factor in the following. It is similar to an anticipated product quality. Of course, φ_i does depend on the good properties as there are the price, the (expected) quality, the strength of networking, and fashion effects for that product etc. To summarize,

$$p_{0i} = \varphi_i x_i, \quad i \neq 0. \tag{2.6}$$

Let us now turn to those people owning product i. Someone who is content with that good, tends to buy a new unit of that good when the alarm clock rings, even though a better product might exist. Surveying the market is time-consuming, and furthermore, consumers usually act conservatively and avoid changes, so that the same product i is bought. Assael (1984, p. 53) summarizes several studies on the topic and comes to the conclusion that a form of habit evolves, leading to repeat purchases of a product without further information search or evaluating brand alternatives.

We may deduce that the fraction p_{ii} of consumers sticking to product i only depends on the habit induced by this good or consumer sluggishness and is independent of any other factors. Also, we assume this habit level to represent a characteristic of the product, as justified by the fact that a consumer usually gets the more discontent with a product the more often it breaks down or the less satisfactory its functionality seems. In our model, a fixed, product-specific percentage of consumers will develop a buying habit, so that finally

$$p_{ii} = s_i \in [0, 1], \quad i \neq 0. \tag{2.7}$$

Obviously, the fraction of switching consumers $(1 - p_{ii})$ divides up into the fractions p_{ij} of people switching to product $j \neq i$. They behave just like those consumers not yet owning any good, except that they do not purchase product i again. Therefore,

$$\begin{aligned} p_{ij} &= (1 - p_{ii})p_{0j} \\ &= (1 - s_i)\varphi_j x_j, \quad i \neq 0 \wedge j \neq 0, i. \end{aligned} \tag{2.8}$$

The switching probabilities p_{i0} and p_{00} are now uniquely determined by the constraints $\sum_{j=0}^{n} p_{ij} = 1$, $i = 0, \ldots, n$,

$$p_{00} = 1 - \sum_{j=1}^{n} p_{0j} = 1 - \sum_{j=1}^{n} \varphi_j x_j, \tag{2.9}$$

$$p_{i0} = 1 - \sum_{j=1}^{n} p_{ij} = (1 - p_{ii})\left(p_{00} + p_{0i}\right) = (1 - s_i)\Bigg(1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \varphi_j x_j\Bigg). \tag{2.10}$$

For $i = 1, \ldots, n$, the mean dynamic (2.3) eventually takes the form

$$\begin{aligned} \dot{x}_i &= \rho_{0i} + \sum_{j=1}^{n} x_j(\rho_{ji} - \rho_{0i}) - x_i R_i \\ &= x_i \left(\varphi_i R_0 - (1 - s_i) R_i - \varphi_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \left[R_0 - (1 - s_j) R_j \right] x_j - \varphi_i R_0 x_i \right). \quad (2.11) \end{aligned}$$

This specific mean dynamic has Lotka-Volterra form[3], $\dot{x}_i = x_i \left(a_i - \sum_{j=1}^{n} b_{ij} x_j \right)$, of competitive species with coefficients $a_i = \varphi_i R_0 - (1 - s_i) R_i$, $b_{ii} = \varphi_i R_0$, and for $j \neq i$, $b_{ij} = \varphi_i \left[R_0 - (1 - s_j) R_j \right]$.

All constants R_i, φ_i, and s_i may in principle be time-dependent so that product modifications or fashion trends can be modeled. However, for our theoretical analysis, we assume the coefficients to be constant. Non-constant coefficients will be dealt with in the next chapter. In particular, we will show how to interpret the convincement factor φ_i via a demand function that is influenced by the product's price.

φ_i and s_i should be seen as the mean parameters over the entire population of heterogeneous, boundedly rational agents. The probability φ_i, for example, may be seen as that fraction of the heterogeneous population that is convinced by product i. Even more, φ_i may also be seen as the probability of an individual to be convinced, i. e. due to the individual's bounded rationality her decision is not deterministic. Hence, the parameters incorporate both an individual variation and a global variation over the population.

Modified population games with the consumer revision protocol motivated above will frequently be used in the later model analysis. For simplicity we shall therefore define the following:

Definition 2.2.4 (Habitual imitative consumers)**.** *Agents who follow the above revision protocol* (2.6) *to* (2.10) *are called* habitual imitative consumers*. A modified population game with such agents is called the demand side of a* market with habitual imitative consumers.

[3]The Lotka-Volterra equations represent a first-order, nonlinear system of differential equations that is frequently used to model competing species in biology.

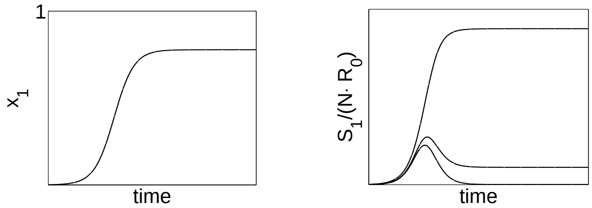

Figure 2.1: Sketch of a typical behavior of x_1 (left) and $\frac{S_1}{N \cdot R_0}$ (right) in time.

2.3 Model analysis

2.3.1 Single good market

For the single good case, the mean dynamic has the following form

$$\dot{x}_1 \quad = \quad x_1 \varphi_1 R_0 \left(\Psi - x_1 \right), \tag{2.12}$$

where $\Psi = 1 - \frac{R_1}{R_0} \frac{1-s_1}{\varphi_1}$ encodes the "quality" of the good (Ψ is large for high durability, convincement, and habituation factor). The ordinary differential equation (2.12) can be solved analytically. Using partial fraction expansion and separation of variables, we find

$$\begin{aligned}
\frac{dx_1}{dt} = x_1 \varphi_1 R_0 (\Psi - x_1) &\Rightarrow \frac{dx_1}{x_1(\Psi - x_1)} = \varphi_1 R_0 dt \\
&\Rightarrow \int \frac{1}{x_1(\Psi - x_1)} dx_1 = \int \varphi_1 R_0 dt \\
&\Rightarrow \ln \left(\frac{x_1}{\Psi - x_1} \right) = \Psi \varphi_1 R_0 t + C_1 \\
&\Rightarrow x_1 = \frac{\Psi}{1 + C_2 \exp(-\Psi \varphi_1 R_0 t)},
\end{aligned}$$

where via an initial condition at $t = 0$ the constants C_1, C_2 can be resolved according to

$$x_1(\tau) = \frac{\Psi}{1 + \left(\frac{\Psi}{x_1(0)} - 1 \right) \exp\left[-\tau \Psi \varphi_1 \right]} \tag{2.13}$$

with dimensionless time $\tau = t R_0$. The evolution of the population x_1 either has a sigmoidal shape with initial exponential growth and saturation value Ψ (figure 2.1) or—in case of an undesirable product—immediately decays to zero.

Definition 2.3.1 (Feasibility). *We call a good* feasible *if a positive number of consumers owns the product in the steady state.*

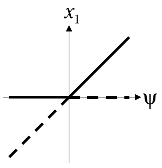

Figure 2.2: Transcritical bifurcation with stable (solid line) and unstable (dashed line) steady state values of x_1.

In this case, the number of sales is positive at least within a certain time period. Mathematically, feasibility of a good is determined by the parameter Ψ, which can readily be shown by investigating the stability of the (unique) steady states $x_1 = 0$ and $x_1 = \Psi$ respectively. Figure 2.2 shows a transcritical bifurcation to occur at $\Psi = 0$, which renders the steady state $x_1 = 0$ instable and $x_1 = \Psi$ stable. Hence we obtain:

Proposition 2.3.1. *The single product on a market with habitual imitative consumers is feasible if and only if* $\Psi > 0$. $\square$

The proposition states that for a product to survive its quality has to be sufficiently high.[4]

The sales equation for the single good market with a feasible good takes the explicit form

$$\frac{S_1(\tau)}{N \cdot R_0} = \frac{\Psi}{1 + \left(\frac{\Psi}{x_1(0)} - 1\right) \exp\left[-\tau\Psi\varphi_1\right]} \left[\frac{R_1}{R_0} + \varphi_1\Psi - \frac{\varphi_1\Psi}{1+\left(\frac{\Psi}{x_1(0)}-1\right)\exp[-\tau\Psi\varphi_1]}\right] \quad (2.14)$$

with saturation value

$$\lim_{\tau\to\infty} \frac{S_1(\tau)}{N \cdot R_0} = \Psi\frac{R_1}{R_0}. \quad (2.15)$$

We may assume that the initial condition $x_1(0) > 0$ is near zero. Depending on $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1}$, the sales can take qualitatively different shapes (figure 2.1 right) corresponding to different patterns of monopoly product life cycles:

For $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1} < 1$ we find an initial exponential growth, a maximum, and then a decay with saturation. Intuitively, the stronger the effect of habituation s_1 or imitation φ_1 (i. e. the more convincing the product is), the stronger is the increase

[4]Note that $\Psi \in (-\infty, 1]$.

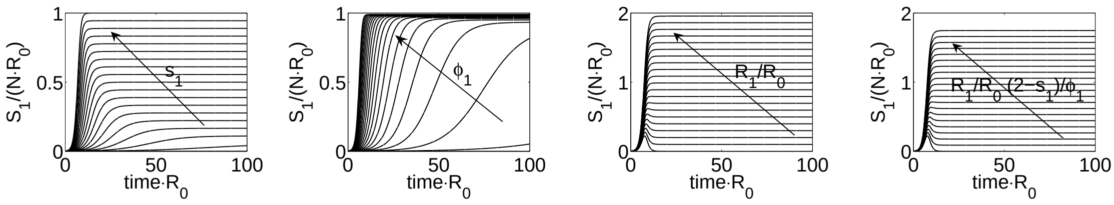

Figure 2.3: Comparative statics of $\frac{S_1}{N \cdot R_0}$ with respect to the parameters s_1, φ_1, $\frac{R_1}{R_0}$, and $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1}$. In each graph, the fixed parameters have the values $\frac{R_1}{R_0} = 1$, $x_1(0) = 0.001$, $\varphi_1 = 0.9$, $s_1 = 0.99$. The arrows indicate the variation of the curve for an increasing parameter value, where the parameter values of s_1 and φ_1 are varied from zero to one in steps of 0.05 and the values of $\frac{R_1}{R_0}$ and $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1}$ are varied from zero to two in steps of 0.1.

of the subpopulation. After a maximum is reached, however, the effect of product durability gains in importance, since most purchases are replacements of broken products. Hence, the sales decrease and approach a saturation value. In the special case $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1} = 0$ of an infinitely durable good ($R_1 = 0$), the saturation value is zero since once the good is bought, the consumer owns it forever and does not purchase another unit of this product.

For $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1} \geq 1$ the curve is sigmoidal. In this case habit and imitation cause an initial increase, then the good has to be replaced quite often and a saturation value is approached.

S_1 is uniquely determined by three parameters, s_1, φ_1, and $\frac{R_1}{R_0}$, which together with $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1}$ lend themselves for a comparative statics analysis (figure 2.3). Generally, the time scale inherent in the sales evolution decreases for rising φ_1, s_1, $\frac{R_1}{R_0}$, $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1}$ (i. e. the system becomes faster), and the saturation value increases, which can also be directly inferred from equation (2.15) and the mean dynamic (2.12). The effect is especially sensitive to changes in φ_1. These results are intuitive, since rising habit, imitation, and replacement rates are indeed expected to speed up the system and to cause more frequent purchases.

A single product market can be a model of different firm constellations: Either a monopoly (at a regulated price) produces the good or an oligopoly, where all firms produce exactly the same good and the brands do not influence the consumers' decision (the consumers do not even differentiate between them by any means).

2.3.2 Two goods market

In the two goods market, the mean dynamic has the form

$$\dot{x}_i \;=\; x_i\varphi_i R_0 \left[\Psi_i - x_i - \Phi_j x_j\right], \quad i = 1, 2,\ j \neq i, \tag{2.16}$$

where the $\Phi_j = 1-\frac{R_j}{R_0}(1-s_j)$ are the slopes of the nullclines (or zero-growth isoclines) in the corresponding phase plane (i. e. the lines along which $\dot{x}_i = 0$, see figure 2.4) and the $\Psi_i = 1 - \frac{R_i}{R_0}\frac{1-s_i}{\varphi_i}$ are the intersections of nullclines with the axes and thus represent the respective saturation values of the single good case. Obviously, the Ψ_i have to be positive for feasible goods, which we will assume in the following.

Phase planes are a representation of the dynamic system in which the nullclines divide the state space into regions with $\dot{x}_i > 0$ and $\dot{x}_i < 0$. Steady states are given by the intersection points $x = (x_1, x_2)$ of nullclines $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. Obviously, in the two goods case there are four possible distinct steady states,

$$\mathbf{x}^A \;=\; \left(\frac{\Psi_1 - \Phi_2\Psi_2}{1 - \Phi_1\Phi_2}, \frac{\Psi_2 - \Phi_1\Psi_1}{1 - \Phi_1\Phi_2}\right), \quad \mathbf{x}^B = (0, \Psi_2)\,, \quad \mathbf{x}^C = (\Psi_1, 0)\,, \quad \mathbf{x}^D = (0, 0).$$

$\mathbf{x}^D$, the trivial steady state, is instable. For a stability analysis of the other steady states, two cases have to be distinguished, $\mathbf{x}^A \in \mathbb{R}^2_{\geq 0} := \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0\}$ and $\mathbf{x}^A \notin \mathbb{R}^2_{\geq 0}$ respectively (cf. figure 2.4). For the description of the time evolution of populations x_1 and x_2 and the according product life cycles we may again assume $x_1(0), x_2(0) > 0$ to be near zero.

1. $\mathbf{x}^A \in \mathbb{R}^2_{\geq 0}$. This is possible if either the denominators and numerators of x_1^A and x_2^A are all positive (1a) or all zero (1b), in which case the nullclines lie on top of each other.

    (a) $1 > \Phi_1\Phi_2$, $\Psi_1 > \Phi_2\Psi_2$, $\Psi_2 > \Phi_1\Psi_1$ (figure 2.4 left). The only stable steady state is $\mathbf{x}^A$. Hence the products coexist lastingly on the market. Depending on the initial condition, x_1 or x_2 may have a maximum before approaching the saturation value; the form of the trajectory strongly depends on its starting point.

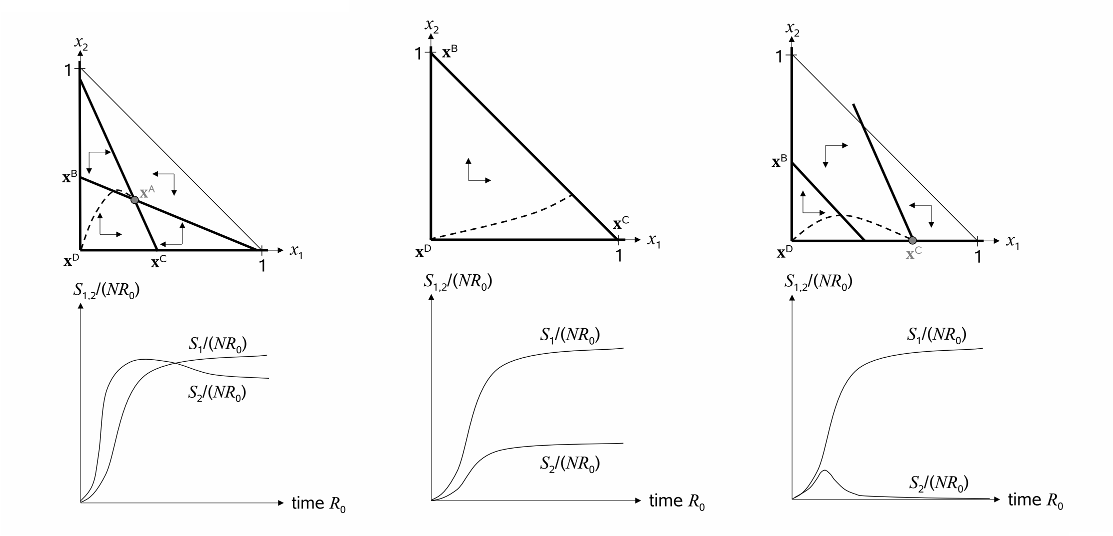

Figure 2.4: Phase planes for all three possible cases 1a, 1b, 2 (from left to right), with nullclines (solid lines) and one trajectory (dotted line) starting near zero each. The (asymptotically) stable steady states are indicated by filled circles. Resulting product life cycles are displayed beneath each diagram.

For the product life cycles—similarly to the single good case—a maximum may or may not exist depending on the parameter values. The number of products sold initially and overall strongly depends on the initial condition.

(b) $1 = \Phi_1\Phi_2, \quad \Psi_1 = \Phi_2\Psi_2, \quad \Psi_2 = \Phi_1\Psi_1$ (figure 2.4 middle). This case occurs for either $R_i = 0$ or $s_i = 1$ for $i = 1, 2$, i. e. for infinitely durable goods or perfect habituation. $\mathbf{x}^A$ degenerates to a continuous line of (not asymptotically) stable steady states. $x_1(t)$ and $x_2(t)$ are monotonous, while the sales evolution may exhibit a maximum depending on the R_i. For infinitely durable goods ($R_i = 0$, $i = 1, 2$) the sales eventually approach zero.

2. $\mathbf{x}^A \notin \mathbb{R}^2_{\geq 0}$ (figure 2.4 right). Without loss of generality let $\Psi_1 > \Phi_2\Psi_2$ (otherwise simply renumber the species; $\Psi_1 < \Phi_2\Psi_2$ and $\Psi_2 < \Phi_1\Psi_1$ is not possible for $\Phi_i \leq 1$). Since either x_1^A or x_2^A has to be negative to prevent an intersection of nullclines in $\mathbb{R}^2_{\geq 0}$, this implies $1 = \Phi_1\Phi_2$ or $\Psi_2 < \Phi_1\Psi_1$. $\mathbf{x}^C$ is stable and $\mathbf{x}^B$ is instable (reverse for $\Psi_1 < \Phi_2\Psi_2$ and $\Psi_2 > \Phi_1\Psi_1$). Hence, one product (here product 2) dies out and the other product survives on the market. For

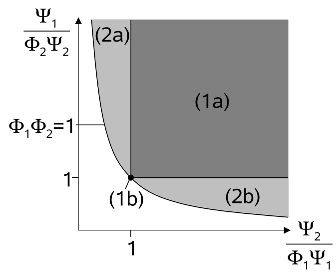

Figure 2.5: Bifurcation diagram. (1a) represents the coexistence region of both products, good 2 dies out in (2a) and good 1 in (2b). The region below $\Phi_1\Phi_2 = 1$ cannot be reached.

> the vanishing product, both the subpopulation and sales reach a maximum and then approach zero. The corresponding curves of the surviving product qualitatively behave as in the single good case. The total sales of the inferior product strongly depend on the initial condition.

The bifurcation diagram in figure 2.5 illustrates the parameter regions for the different cases, in particular the region of coexistence (1a,1b) and the region of exclusion of a product (2). The following proposition can immediately be inferred:

Proposition 2.3.2. *Product 1 is feasible on a two goods market with habitual imitative consumers, if and only if either $\Psi_1 > 0$ and $\Psi_2 \leq 0$ or $\Psi_1 > \Phi_2\Psi_2$. The analogous result holds for product 2.* □

Since Ψ_i encodes the "quality" of the good (Ψ_i is large for high durability, convincement, and habituation factor), the previous proposition basically performs a comparison of product qualities. However, due to the consumers' heterogeneity and bounded rationality, the "quality" of one good need only be larger than the "quality" of the other downscaled by a factor Φ_j, which is monotonously increasing with the habit induction by the competing product. That is, also the weaker product can survive since $\Phi_i \leq 1$.

A further, quite intuitive result follows directly from the phase planes in figure 2.4:

Proposition 2.3.3. *If a new product enters a single good market with habitual imitative consumers, the steady state market share of the incumbent decreases, while* *the joint market share of entrant and incumbent increases. Formally,*

$$x_i^* \leq \Psi_i = x_i^\star \leq x_1^* + x_2^*, \quad i = 1, 2, \tag{2.17}$$

where x_i^* is the (or a) stable steady state value of x_i in the two goods market and $x_i^\star$ the steady state value in the single good market. □

As in the single good case, a two product market can model different firm constellations: There might be a monopoly producing both goods, or the products are offered by a duopoly or even oligopoly, but with only two distinguishable goods. Equation (2.17) suggests that it might be beneficial for a monopoly to offer a variety of goods instead of just one (due to their bounded rationality this even holds for completely homogeneous consumers, in which case a single, optimally fitted product would seem more profitable at first glance).

2.3.3 Structural stability

The question arises how robust the dynamic system (2.11) is when small perturbations occur, for instance when consumers change their behavior slightly. The idea of structural stability is the appropriate tool to tackle this question. A structurally stable system maintains its qualitative properties despite small perturbations (Guckenheimer & Holmes 1990). In fact, structural stability of a dynamical system means that the structure and topology of the phase portrait (e. g. the number and type of steady states) stay the same if its parameters are varied. This is a general concept: Mathematical models may not be too sensitive to variations of their input data since input data is never completely reliable so that the results would be questionable.

Let us here focus on the case (2.16) of a two goods market. Structural stability of a two-dimensional system was examined in the pioneer work Andronov & Pontrjagin (1937), which includes a necessary and sufficient condition for structural stability:

1. finite number of equilibrium points and closed orbits which are all hyperbolic

2. no trajectories connecting saddle points

To show structural stability of the dynamic system given in (2.16), we have to check both conditions. We can directly conclude from figure 2.4 middle, that the rather academic case 1b is not structurally stable: Changing the parameters a bit, we directly arrive at the structurally different cases 1a or 2 (figure 2.4 left and right). As shown in section 2.3.2, for all other cases, a finite number (maximum four, depending on the habit and imitation parameter values) of steady states exists, and figure 2.4 obviously shows that there are no trajectories between saddle points and no closed orbits (limit cycles). Additionally, all steady states are hyperbolic, since all eigenvalues are real and nonzero, except for the two cases $\Psi_i = \Phi_{3-i}\Psi_{3-i}$, $i = 1, 2$, which represent the bifurcation point between the cases 1a and 2. Hence, away from these cases, the dynamic system (2.16) is structurally stable.

2.3.4 n goods market

Extension to multi-goods markets, where the mean dynamic takes the form

$$\dot{x}_i \;=\; x_i\varphi_i R_0\Bigg[\Psi_i - x_i - \sum_{\substack{j=1\\ j\neq i}}^{n} \Phi_j x_j\Bigg], \quad i = 1, \dots, n, \tag{2.18}$$

is obvious and will therefore be kept brief here. We will only derive the feasibility condition for the nth good (and hence for any good after renumbering) and take a look at the special case of a symmetric market. In the following, we will abbreviate vectors of scalars according to $(\sigma_i)_{i=1,\dots,m} = \vec{\sigma}$. Also, we will need the following lemma, whose proof is given in the appendix:

Lemma 2.3.4. *Let $0 < \Phi_i < 1$ and let A_n for $n \in \mathbb{N}$ be the matrix defined as*

$$(A_n)_{ij} = \begin{cases} 1, & i = j, \\ \Phi_j, & i \neq j, \end{cases} \qquad i, j = 1 \dots n.$$

Then $\det(A_n) > 0$.

Now we can characterize the feasibility of a good:

Proposition 2.3.5. *Consider an n-product market with habitual imitative consumers on which the products i, $i = 1, \dots, n-1$, coexist with $0 < \Phi_i < 1$. Then* *product n is feasible if and only if*

$$\Psi_n > A_{n-1}^{-1}\vec{\Psi} \cdot \vec{\Phi} = \vec{\tilde{x}} \cdot \vec{\Phi} = \sum_{i=1}^{n-1} \Phi_i \tilde{x}_i \text{ for } (A_{n-1})_{ij} = \begin{cases} 1, & i = j, \\ \Phi_j, & i \neq j, \end{cases} \quad i, j = 1, \ldots, n-1$$

where $\vec{\tilde{x}}$ is the vector of market shares on the $(n-1)$-goods market (i. e. without product n).

In this proposition, the identity $A_{n-1}^{-1}(\Psi_1, \ldots, \Psi_{n-1})^{\mathrm{T}} = (\tilde{x}_1, \ldots, \tilde{x}_{n-1})^{\mathrm{T}}$ results from the steady state equations of the mean dynamic (2.18) for the $(n-1)$-goods market if all market shares are non-zero. Hence, intuitively, the above proposition implies that the (hypothetic) monopoly market share Ψ_n has to be larger than the weighted sum of market shares of products 1 to $n-1$, where the weights Φ_i increase with habit induction. The proof is deferred to the appendix.

For a sensible market model, we postulate that it be consistent with economic intuition. The next few lines are devoted to such a proof of consistency: We aim to show the intuitive fact that competition becomes harder when another competitor enters the market. For this purpose, let $\bar{\Psi}_{n-1}$ be the feasibility boundary from proposition 2.3.5 for an nth product to enter a market with $n-1$ existing products and with habitual imitative consumers. Furthermore, let $\bar{\Psi}_{n-2,j}$ be the same feasibility boundary, however, only for entering a market with just $n-2$ existing products, namely all products from the $(n-1)$-goods market except for product j. Then the following lemma can be proven:

Lemma 2.3.6. *Under the conditions of the previous proposition 2.3.5 and with $\bar{\Psi}_{n-1}$ and $\bar{\Psi}_{n-2,j}$ as just defined,*

$$\bar{\Psi}_{n-1} - \bar{\Psi}_{n-2,j} = \prod_{\substack{i=1 \\ i \neq j}}^{n-1} (1 - \Phi_i)(\Psi_j - \bar{\Psi}_{n-2,j}) \frac{\Phi_j}{\det(A_{n-1})}.$$

□

We shall not give the tedious proof here, however, the reader may easily verify the relation by trying out different n. This lemma now yields the desired result:

Proposition 2.3.7. *Under the conditions of lemma 2.3.6, competition gets harder the more competitors enter the market, i. e.*

$$\bar{\Psi}_{n-1} > \bar{\Psi}_{n-2,j} \text{ for all } j = 1, \ldots n-1.$$

Hence, it is harder to enter the $(n-1)$-goods market than it is to enter the same market, but with any one of the products removed.

Proof. $(1-\Phi_i)$ is positive due to $0 < \Phi_i < 1$. $(\Psi_j - \bar{\Psi}_{n-2,j})$ is positive since otherwise product j would not be feasible and would hence not exist. Thus the result follows from lemma 2.3.6 together with lemma 2.3.4. $\square$

We shall now briefly provide a further interpretation of the feasibility condition in proposition 2.3.5: Analogously to $\bar{\Psi}_{n-2,j}$, let the matrix $A_{n-2,j}$ be defined by omitting the jth row and the jth column of matrix A_{n-1}. Furthermore, let matrix $(A_n)_{i\to\vec{\Psi}}$ be equal to A_n with column i replaced by $\vec{\Psi}$, let the vector of ones be denoted by $\vec{1}$ and let $M_{\not{i},\not{j}}$ denote a matrix M with the ith row and jth column removed. Then

$$
\begin{aligned}
\det(A_{n-1})\bar{\Psi}_{n-1} &= \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} (-1)^{i+j}\Phi_i\Psi_i\Phi_j \det(((A_{n-1})_{j\to\vec{1}})_{\not{j},\not{i}}) + \sum_{i=1}^{n-1} \Phi_i\Psi_i \det((A_{n-1})_{\not{i},\not{i}}) \\
&= \sum_{j=1}^{n-1} \left[-\sum_{\substack{i=1\\ i\neq j}}^{n-1} (-1)^{i+j}\Phi_i\Psi_i\Phi_j \left((-1)^{i+j}\det((A_{n-2,j})_{i\to\vec{1}})\right) + \Phi_j\Psi_j\det(A_{n-2,j})\right] \\
&= \sum_{j=1}^{n-1} \left[-\bar{\Psi}_{n-2,j}\Phi_j\det(A_{n-2,j}) + \Phi_j\Psi_j\det(A_{n-2,j})\right] \\
&= \sum_{j=1}^{n-1} \Phi_j \det(A_{n-2,j})\left[\Psi_j - \bar{\Psi}_{n-2,j}\right].
\end{aligned}
$$

Hence, the lower feasibility bound $\bar{\Psi}_{n-1}$ can be interpreted as the weighted sum of the feasibilities $(\Psi_j - \bar{\Psi}_{n-2,j})$ of all existing products, as $(\Psi_j - \bar{\Psi}_{n-2,j})$ indeed expresses how much more feasible good j is in comparison with the minimum to exist on the market.

In the special case of a symmetric market with $n-1$ identical products, A_{n-1}^{-1} has a simple form, which allows to look at proposition 2.3.5 from a slightly different viewpoint. The following corollary compares the quality of the entrant with the quality of the incumbents and enables us to examine this relation for growing numbers of products.

Corollary 2.3.8. *Consider a symmetric $(n-1)$-product market with identical firms ($\Phi_i = \Phi$, $\Psi_i = \Psi$) and habitual imitative consumers on which the products coexist* *with* $0 < \Phi < 1$. *Then a new (not necessarily identical) nth product is feasible if and only if*

$$\Psi_n > \frac{(n-1)\Phi\Psi}{1+(n-2)\Phi}.$$

Proof. By calculating $A_{n-1}^{-1}A_{n-1}$ we can readily verify

$$(A_{n-1}^{-1})_{ij} = \begin{cases} \frac{(n-3)\Phi+1}{(1-\Phi)[1+(n-2)\Phi]}, & i = j, \\ \frac{-\Phi}{(1-\Phi)[1+(n-2)\Phi]}, & i \neq j. \end{cases}$$

Hence, by proposition 2.3.5,

$$\Psi_n > A_{n-1}^{-1}\vec{\Psi} \cdot \vec{\Phi} = \frac{(n-1)\Phi\Psi}{1+(n-2)\Phi}.$$

□

Obviously, the "quality" Ψ_n of the new nth product may always be smaller than the "quality" Ψ of the $n-1$ incumbents. However, the factor $\frac{(n-1)\Phi\Psi}{1+(n-2)\Phi}$ monotonously increases with the number of products n, and in the limit

$$\Psi_n > \frac{(n-1)\Phi\Psi}{1+(n-2)\Phi} \overset{n\to\infty}{\longrightarrow} \Psi$$

so that the entrant's "quality" has to approach the incumbents' "quality" Ψ.

2.4 Discussion

So far we have worked out how the sales dynamics of products or brands evolve in an environment with habitual imitative consumers and how this evolution is influenced by the product-dependent habit and imitation parameters. In this section we shall briefly mention further implications for firms deduced from the model.

For non-varying products and thus fixed habit and imitation parameter values, the final steady state cannot be influenced by the firms (unless the parameter values are functions $\varphi_i(x_1, \ldots, x_n)$, $s_i(x_1, \ldots, x_n)$ of the state so that the mean dynamic (2.11) no longer has Lotka-Volterra form). Nevertheless, a moderately large initial share $x_i(t = 0)$ generally is profitable for firms (i. e. it pays off to give away a number of product units for free initially) in order to sell more product units on the whole. The effect arises, because a larger initial product share leads to a faster approach to the steady state. Similarly, it pays off for a firm to launch a product early in comparison to the competitors in order to have a large market share when the other products enter the market. This causes sales advantages such as the initial hump in figure 2.4 (left).

Of course any real market which is to be modeled needs careful choice of habit, imitation, and durability parameters. Such parameters can be obtained from matching empirical curves with simulated sales development, for instance using the observed time scales and steady state market shares. As an example consider the evolution of consumption of filter cigarettes as found in Polli & Cook (1969, p. 389). Here we will regard filter cigarettes as a product class within which brand differences (at least initially) do not matter much so that a monopoly approximation may be appropriate (i. e. there are only two options: buying or not buying filter cigarettes). The fact that habit plays a strong role here is unquestionable (and rumor has it that imitation takes place as well). To fit the data, we roughly estimated a high (packet) consumption rate of $R_1 = 200$ per year and a rather low consideration frequency of $R_0 = 20$ per year as well as a low imitation of $\varphi_1 = 0.01$, which seem reasonable values. Habit on the other hand is close to one with $s_1 = 0.9992$. This results in $x_1 = 20\%$ of the population smoking filter cigarettes in the steady state, which seems a reasonable estimate considering that according to Polli & Cook (1969) in the 1960s filter cigarettes consumption amounted to roughly 40% of total cigarette consumption (and nearly half the US population were smokers). The resulting sales evolution is given in figure 2.6. It closely imitates the empirical curve for plain filter cigarettes in Polli & Cook (1969, p. 389, figure 3).

The next chapter will additionally model the supply side of the market and analyze firms' pricing or advertising strategies as well as the welfare in such a market. Moreover, we will show how product life cycles of the typically empirically supported form can be generated in a market with habitual imitative consumers.

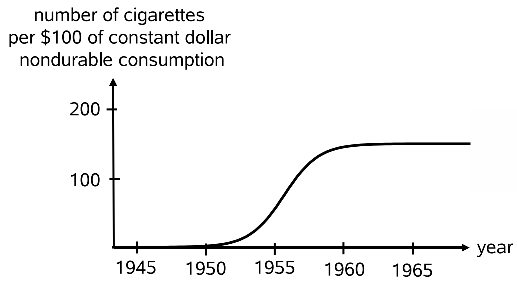

Figure 2.6: Simulation results, reproducing the sales evolution of plain filter cigarettes in the USA from Polli & Cook (1969).

2.5 Conclusion

We examined the evolution of a consumer market, where boundedly rational consumers follow rules of thumb, basing their decisions on imitation and habit. To achieve this goal we set up a modified population game with a corresponding revision protocol as framework. We then analyzed the resulting sales evolution of products and investigated product feasibility in a market with habitual imitative consumers. The behavioral parameters can be adapted to match observed sales evolutions of products or product classes, and the introduced methodology allows for broad applications and qualitative theoretical analysis. In particular, demand by habitual imitative consumers will serve as the basis for modeling a strategic supply side in the next chapter.

One of the main achievements of this chapter consists in having cast psychological and experimental results into a mathematical model with boundedly rational and habitual imitative consumers, as well as investigation of the consequences. For example, such a market model is shown to be consistent with standard economic intuition in that it becomes more difficult for products to survive on the market the more competitors enter the market.

Chapter 3

Product Pricing when Demand Follows a Rule of Thumb

3.1 Introduction

In many situations, strategic pricing represents a difficult task for firms, especially if the consumers are not known to strictly follow a given demand function. In reality, consumers behave boundedly rationally as observed in numerous psychological and experimental investigations (cf. Conlisk 1996) and appreciated in some areas of industrial organization (cf. Ellison 2006). In this chapter, we examine a model which describes how firms shall optimally, i. e. strategically, set their prices or advertising levels when confronted with habitual imitative consumers. Such consumers imitate popular product choices and form a habit to repeatedly purchase the same product. This rule of thumb behavior is psychologically supported (Assael 1984) and acknowledged in the economic literature (e. g. Stigler & Becker 1977, Schlag 1998).

The demand side of a market with habitual imitative consumers has been examined in the previous chapter. There we have shown—using imitation and habit as the only model ingredients—under which conditions a product is feasible (i. e. has sufficient demand to survive lastingly on the market), which types of sales curves can be generated, and how the imitative and habitual parameters influence feasibility and sales evolution. The corresponding model has been stated in form of a population game as defined in Sandholm (2005), using the fact that for a large population size the stochastic process generated by the evolutionary process can be approximated by solutions to ordinary differential equations (Benaïm & Weibull 2003).

For a complete market description, the demand side is here complemented with a supply side model. The demand side consists of the continuous-time consumer population game from the previous chapter. The game of the supply side describes the strategic behavior of firms, which anticipate the consumers' behavior and thus the demand dynamics. With such a description at hand, we can then transfer the concept of welfare into this framework, examine how advertising might influence the results, and show that the model is consistent with observed market patterns such as product life cycles.

As motivated earlier, in a consumer–seller relationship the strategic variables (e. g. price, quality, output) are determined by the firms. Any influencing parameter on the demand side (such as personal preference or reservation price) is fixed, i. e. exogenous in a market model. Hence, due to their reactive role, consumers naturally do not compete actively with each other or with firms, which prohibits a conventional game theoretic demand side model. In contrast, for the firms a strategic behavior of determining the degrees of freedom can be devised using game theoretic approaches. In particular, we apply a differential game in order to model the firms' behavior.

This chapter aims at providing an insight into the strategic response of firms when confronted with demand dynamics generated by the imitative and habitual consumer behavior. The consumer model is in the spirit of Smallwood & Conlisk (1979) as well as von Thadden (1992) in that the consumers are unable or not willing to act strategically and thus act adaptively. Our rational supply side approach differs in methodology. We employ a normal form game in which firms choose a time-continuous price path at the beginning of the game and gain an according profit. (This could easily be extended to repeated choices of price paths.) A more classic normal form game, which describes a single discrete time step, would not exploit the full richness of the model since the time-continuous demand side calls for at least a continuous-time price path. At the other extreme, a steady price adjustment at all points in time is rather unrealistic since a firm can hardly react continuously without time lag (though, nonetheless, the proposed model will be able to capture even this case). By letting the firms fix a price path at the beginning of the game (e. g. for a certain time period, which is represented by the duration of the game), we strike a balance between both extremes.

We employ a differential game, i. e. a time-continuous game where the state variables (here the consumer subpopulation sizes) follow first-order ordinary differential equations. More specifically, the chosen product prices (control variables) determine the rate of change in consumer subpopulation sizes (state variables), where consumers are grouped into subpopulations according to the product they own. Technically, there is no effect in the opposite direction, which renders the dynamics an open-loop system. However, in a deterministic setting we have equivalence of open and closed loops. This is convenient since closed loops are considered—in contrast to static open loops—as being genuinely strategic since they comprise a feedback in which the control variables are affected by the state variables.

In this context, the following result turns out to be very interesting: The firms' action space generally contains all possible price paths and thus is very complex. However, for a monopoly we will show that a Nash equilibrium often lies in the reduced space of time-constant prices. This justifies that most of the time we return to time-constant price paths and steady state analysis. Nevertheless, we additionally analyze some exemplary cases with general price paths. Moreover, we will see that markets with imitative and habitual consumers behave naturally in that e. g. an increasing number of firms enhances competition and reduces prices. However, perfect competition is generally only achieved in symmetric markets. Finally, advertising is shown to be an effective method to sustain demand, and a welfare definition is suggested.

3.1.1 Further motivation and related literature

Boundedly rational consumer behavior—as advocated by Ellison (2006), Conlisk (1996), and many others—is often observed in consumer research, in particular imitation of group behavior or habitual purchase (cf. Assael 1984, p. 371ff, 53). Just to mention some exemplary laboratory experiments, Venkatesan (1966) shows that consumers generally conform to group norms and Corriveau et al. (2009) find a sensibility to group consensus for young children. Pingle & Day (1996) summarize experiments which show that boundedly rational behavior such as imitation and habit (which they call "economic choices in reality") represents means in order to get well-performing economic choices in presence of decision costs. Our focus here lies on markets with boundedly rational consumers that follow habitual imitative decision rules as introduced in the previous chapter.

Closely related to the demand dynamic employed here is the model by Smallwood & Conlisk (1979). They consider consumers who buy the same product each period until a breakdown occurs. Then, they choose another product depending on its market share. It is examined how strongly the consumers should rely on product popularities. Despite having been published in 1979 already, there is still social learning literature building on this model, for example Ellison & Fudenberg (1995).

Imitative behavior in general constitutes a well-known and frequently used concept in evolutionary game theory and social learning, compare for instance Schlag (1998), Ellison & Fudenberg (1993), and Banerjee (1992), just to name a few notable papers. Habit, on the other hand, occurs in the habit formation literature (Heaton 1993) as well as implicitly in some industrial organization models (for instance in Smallwood & Conlisk 1979, where habit is implicitly formed as long as no breakdown occurs). Habit may also be interpreted as a special case of learning, since agents learn from past experience (Sobel 2000, p. 257) and positive experience with a good may cause habitual purchase behavior.

Firms are usually more rational than consumers can or aim to be. The reason lies in the large number of people and equipment that are employed in order to avoid costly wrong decisions. The approximation of rational firms seems reasonable, even though some early work in the field of bounded rationality assumes the opposite, i. e. boundedly rational firms (e.g. Rothschild 1947, Cyert & March 1956). However, in line with most of the recent literature, we restrict bounded rationality to the consumers and assume fully rational firms (Ellison 2006, p. 4), i. e. our firms aim at maximizing their profits given the demand side and the pricing strategies of the competitors, modeled via a differential game as introduced by Isaacs (1954). In combination with the two previously mentioned simple rule of thumb ingredients, imitation and habit by consumers, the model will be able to generate typical patterns observed in consumer markets such as product life cycles (de Kluyver 1977, Brockhoff 1967, Polli & Cook 1969).

The outline of this chapter is as follows. Section 3.2 introduces the competition game played by the supply side. The game is first applied to exemplary monopoly or oligopoly settings, after which more general results and conclusions are drawn for the monopoly and oligopoly case. In section 3.3, an adequate welfare definition is provided, and a possible generation of product life cycles is described. Additionally, a model extension by advertising is suggested. Finally, we conclude in section 3.4.

3.2 Strategic pricing in a monopoly & oligopoly

Let us examine markets in which firms anticipate the consumers' actions (indeed, companies do try to predict consumer behavior) and set their prices accordingly.

Naturally, the imitation factor φ_i and habit coefficient s_i (see previous chapter) depend on the good prices, i. e. $\varphi_i = \varphi_i(\xi_1(t), \dots, \xi_n(t))$, $s_i = s_i(\xi_1(t), \dots, \xi_n(t))$, $i = 1, \dots, n$, where the price of good j at time t is denoted $\xi_j(t)$. To keep things simple while staying sufficiently realistic, we shall assume φ_i and s_i to depend on ξ_i only. The consumers see the prices of all goods, and the probability to buy product i (encoded by φ_i and s_i) rises with falling price ξ_i. They behave like many small iron particles which are attracted by different magnets, representing the products. The strength of a magnet relative to its competitors determines the eventual amount of trapped particles, which illustrates the mechanism of competition. Competing firms will seek a compromise between large margins and sufficiently low prices to attract consumers more strongly than their competitors (via high φ_i and s_i).

Recall that the imitation function $\varphi_i(\xi_i)$ has the following interpretation: A consumer owning no good or switching products imitates the population of consumers owning good i with probability $\varphi_i(\xi_i)$. Equivalently, the fraction $\varphi_i(\xi_i)$ of the whole population would purchase good i at a price of ξ_i. Obviously, $\varphi_i(\xi_i)$ represents the normalized demand function of product i, or in probabilistic terms, $\varphi_i(\xi_i)$ is the demand distribution of product i. Hence, let us agree upon the following

Condition 3.2.1. *In a market with habitual imitative consumers, let $\varphi_i(\vec{\xi})$ and $s_i(\vec{\xi})$ denote the imitation and habit coefficient for product i, depending on the vector $\vec{\xi}=(\xi_1,\dots,\xi_n)$ of product prices. Then φ_i and s_i are monotonously decreasing in ξ_i.*

In the following we will abbreviate vectors of scalars according to $(\sigma_i)_{i=1,\dots,m} = \vec{\sigma}$. We are now able to describe a normal form competition game of the firms.

Definition 3.2.2 (Normal form competition game)**.** *The* normal form competition game *in a market with habitual imitative consumers is a normal form game $G = (n, \mathfrak{S}, \Pi)$ with*

- *the number of agents n being the number of firms, where each firm produces one product and the products are understood to be characterized by the functions $s_i(\xi_1, \dots, \xi_n)$ and $\varphi_i(\xi_1, \dots, \xi_n)$,*
- *the set $\mathfrak{S}$ of all possible strategy combinations with $\mathfrak{S} \subseteq [\mathbb{R}_+^{\mathbb{R}_+}]^n = \mathbb{R}_+^{\mathbb{R}_+} \times \cdots \times \mathbb{R}_+^{\mathbb{R}_+}$ (a subset of the space of n-tuples over maps $\xi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto \xi_i(t)$, where $\xi_i(t)$ denotes the price of good i at time t),*
- *the utility function*

$$\Pi : \mathfrak{S} \to \mathbb{R}_+^n, \quad \Pi_i(\xi_1, \dots, \xi_n) = F\left[(\xi_i(t) - c_i) S_i\left(\vec{s}(\vec{\xi}(t)), \vec{\varphi}(\vec{\xi}(t)), \vec{x}(t)\right)\right],$$

  being the firms' profit, where there are no fixed costs, c_i denotes the (time-independent) marginal cost of production for good i, and $\vec{S}(\vec{s}(\vec{\xi}(t)), \vec{\varphi}(\vec{\xi}(t)), \vec{x}(t)) \equiv \vec{S}(t, \vec{x}(0))$ is the sales vector (cf. (2.4)*) belonging to the population game* (2.18) *with ρ defined by* (2.6) *to* (2.10)*. The operator $F : \mathbb{R}^{\mathbb{R}_+} \to \mathbb{R}_+$, which assigns a nonnegative real number to each map $f : \mathbb{R}_+ \to \mathbb{R}$, may assume different forms.*

The imitation and habit function $\vec{\varphi}(\vec{\xi})$ and $\vec{s}(\vec{\xi})$ may in general be time dependent. The operator F can e. g. constitute the cumulated discounted profit over a certain time period $[0, T]$,

$$F_T\left[\pi(\cdot)\right] = \int_0^T \exp[-rt]\pi(t)\mathrm{d}t,$$

where $\pi(t)$ represents the firm's profit at time t. For an infinite time horizon this is extended to

$$F_\infty\left[\pi(\cdot)\right] = \int_0^\infty \exp[-rt]\pi(t)\mathrm{d}t.$$

For a zero discount rate r, the latter definition is not well-defined. In this case we resort to the long-term profit rate,

$$F_\partial\left[\pi(\cdot)\right] = \lim_{T\to\infty} \frac{1}{T} \int_0^T \pi(t)\mathrm{d}t = \lim_{t\to\infty} \pi(t),$$

where the last expression only holds for time-invariant prices in the steady state. In this case (which is for example of interest when the firms would like to validate their prices in an equilibrated market), we will also denote F_∂ as the steady state profit.

Note that definition 3.2.2 in conjunction with (2.18) defines a differential game as in Isaacs (1954).

Definition 3.2.2 allows for a time-dependent price. The firms set their price paths initially and then strictly follow these. The lack of opportunities for price path revisions implies no disadvantage for the firms, since the consumer model is deterministic and consumer behavior thus predictable. Therefore, even if the firms would be able to change their price paths during the game, they would not do so unless the market conditions were changed by an external event. (Such price path revisions could readily be modeled by extending the normal form competition game to a repeated game.) Put differently, assume the optimal pricing strategies to be functions of the state also, i. e. $\vec{\xi} = \vec{\xi}(\vec{x}, t)$, which would correspond to a closed-loop model in which the price path is continuously adapted according to the current state. Then, the corresponding optimal path in state space, $\vec{x}(t)$, could be computed by solving the mean dynamic (2.18). Choosing $\hat{\xi}(t) = \vec{\xi}(\vec{x}(t), t)$, we obtain the optimal open-loop control path, which is exactly equivalent to the corresponding closed-loop path. (Note that this would be different for models with non-Markovian strategies and positive time lags between a firm's action and the others' reaction, in which case trigger strategies, i. e. punishment after path deviation of a competitor, might exist, cf. Dockner et al. (2001).)

Obviously, firms do not at each time maximize their current profit, but choose their price paths in order to obtain an optimal overall profit in the long run.

3.2.1 Monopoly

As a first illustrative example, let us consider the simplest case possible, the monopoly with fixed prices (as in equilibrated or price-restricted markets). The monopoly strategy space thus reduces to the space $\mathfrak{S} = \mathbb{R}_+$ of time-constant price functions $\xi_1(t) = \xi_1$. For given $\varphi_1(\xi_1)$, $s_1(\xi_1)$ the Nash equilibrium with steady state profit now yields an optimum monopoly price (corresponding oligopoly prices are addressed in the next section).

Example 3.2.1 (Steady state monopoly price)**.** *Consider the normal form competition game*

$$G = (1, \mathbb{R}_+, F_\partial[\pi(\cdot)]) = \left(1, \mathbb{R}_+, \lim_{t\to\infty} (\xi_1 - c_1) S_1(s_1, \varphi_1(\xi_1), t)\right),$$

with $\mathfrak{S} = \mathbb{R}_+$ representing the set of all constant price functions in $\mathbb{R}_+$. Let us assume a constant habit function $s_1(\xi_1) = s_1$ and the generic piecewise affine imitation function $\varphi_1(\xi_1) = \max\left(0, 1 - \frac{\xi_1}{\Xi_1}\right)$ with maximum reservation price $\Xi_1 \geq c_1$. The price in the (unique) Nash-equilibrium is the maximizer of

$$\max_{\xi_1 \in \mathbb{R}_+} \left[\lim_{t\to\infty} (\xi_1 - c_1) S_1(s_1, \varphi_1(\xi_1), t)\right],$$

where S_1 follows from equation (2.5) with $x_1(t) = \frac{\Psi_1}{1+\left(\frac{\Psi_1}{x_1(0)}-1\right)\exp[-tR_0\Psi_1\varphi_1]}$ solving the ordinary differential equation (2.12). The optimal price is given by

$$\xi_1^* = \Xi_1 - \sqrt{\frac{R_1}{R_0}(1 - s_1)\Xi_1(\Xi_1 - c_1)} \quad \text{if } \xi_1^* > c_1.$$

For $\xi_1^* < c_1$ the firm is recommended not to produce. The product therefore is feasible, if and only if $c_1 \leq \Xi_1 \left[1 - \frac{R_1}{R_0}(1 - s_1)\right]$.[1]

[1]This is just a different representation of the feasibility condition $\Psi_1[\xi_1 = c_1] > 0$ from the previous chapter.

Of course, the constant s_1 in the above example is a very crude approximation (though reasonable if a certain range of prices is not exceeded), since then, obviously, the monopolist could gain infinite profit by suddenly charging infinite prices (which the consumers will pay due to their habit). Hence, when looking at time dependent prices, one should for example set $s_1(\xi_1) = \varphi_1(\xi_1)$.

A product is feasible if it persists in the steady state and the firm does not make any losses when selling the good. The feasibility condition from the above example naturally extends to a result for general $\varphi_1(\xi_1)$ and $s_1(\xi_1)$.

Proposition 3.2.1. *The single product on the market with habitual imitative consumers is feasible if and only if* $\Psi_1[\xi_1 = c_1] > 0$.

Proof. Due to condition 3.2.1, φ_1 and s_1 are monotonously decreasing with ξ_1. Hence, Ψ_1 is so as well. Also, in proposition 2.3.1 we have shown that a single product on a market is feasible if and only if $\Psi_1 > 0$. Hence, if and only if $\Psi_1[\xi_1 = c_1] > 0$, there exist prices ξ_1 for which the firm makes a positive profit. □

The previous example of an equilibrated monopoly market did not exploit the possibility of time-varying prices. Yet, it might very well be that non-constant prices result in a higher profit: A widely observed pricing strategy consists in charging an elevated price most of the time with (more or less regular) intermittent special offers. This strategy probably aims at making people buy the product during the low-price period and thereby inducing a habit for the high-price period. However, for periodic price changes we will show that in a market with habitual imitative consumers a Nash equilibrium is found to lie in the space of time-constant prices, which in many cases justifies to a priori confine ourselves to steady states and constant pricing. To show this, we will proceed in steps and first prove the criticality, later the optimality of a constant price.

Proposition 3.2.2. *Let us consider a monopoly market with habitual imitative consumers in which the firm has a periodic price path, i. e. in each time period* $[kT, kT + T], k \in \mathbb{N}$, *the same price path* $\xi_1(t) = \xi_1(t + T) = \xi_1(t + 2T) = \cdots$ *is pursued. The appropriate normal form competition game reads*

$$G = \left(1, \mathbb{R}_+^{[0,T]}, \frac{1}{T}\int_0^T (\xi_1(t) - c_1) S_1(s_1(\xi_1(t)), \varphi_1(\xi_1(t)), t)\mathrm{d}t\right),$$

assuming that a periodic state, i. e. a state with $x_1(t) = x_1(t+T) = \cdots$, has been reached so that the proposed profit operator indeed yields the average profit per time. (Instead, profit F_∂ could equivalently be used.) Then, if the good is feasible, a constant price is a critical value for the monopoly.

Proof. For simplicity, we abbreviate $R := \frac{R_1}{R_0}$ and skip the index 1 for all other variables. Also, we will introduce the non-dimensional time $\hat{t} = R_0 t$ and $\hat{T} = R_0 T$, where for ease of notation, the hats are dropped in the following.

Let $\xi(t)$ be periodic with period T and assume that after some equilibration time, all other system variables also behave periodically with same period.

The non-dimensionalized ordinary differential equation $\dot{x} = x\varphi(\Psi - x)$ is of Riccati type and thus readily solved for x,

$$x = \frac{\exp\left(\int_0^t \varphi\Psi \mathrm{d}\tau\right)}{\int_0^t \varphi \exp\left(\int_0^\tau \varphi\Psi \mathrm{d}\theta\right) \mathrm{d}\tau + \frac{1}{x(0)}}, \quad x(0) = \frac{\exp\left(\int_0^T \varphi\Psi \mathrm{d}\tau\right) - 1}{\int_0^T \varphi \exp\left(\int_0^\tau \varphi\Psi \mathrm{d}\theta\right) \mathrm{d}\tau},$$

where the expression for $x(0)$ follows from the periodicity condition $x(0) = x(T)$.

Using $S(t)/(NR_0) = \dot{x} + Rx = x[R + \varphi(\Psi - x)]$, the normalized long-term profit rate can be expressed as

$$\frac{\Pi}{NR_0} = \frac{1}{T}\int_0^T (\xi - c)x[R + \varphi(\Psi - x)]\mathrm{d}t.$$

Finally, in appendix B.2.1, a lengthy sequence of non-trivial transformations proves that the Gâteaux derivative of Π with respect to ξ is zero for all test directions ϑ, if for ξ we substitute the constant price ξ^* which is implicity defined by $\xi^* - c = -\frac{\Psi(\xi^*)}{\Psi'(\xi^*)}$, where $\Psi' \equiv \frac{\mathrm{d}\Psi}{\mathrm{d}\xi}$. In other words, the Euler-Lagrange equation for Π is fulfilled for the constant price ξ^*, and hence ξ^* is critical. $\square$

For the constant price to be a Nash equilibrium, the second variation of the long-term profit rate Π with respect to the price is required to be negative definite. To show this, we need the following lemma, whose proof is given in appendix B.2.2.

Lemma 3.2.3. *Let $H : \mathbb{R} \to \{0, 1\}$ be the Heaviside function. The following inequality holds for all $\alpha \in \mathbb{R}$ and Lebesgue-integrable functions $\vartheta : [0, T] \to \mathbb{R}$:*

$$\int_0^T \int_0^T \vartheta(t)\vartheta(\tau)\exp[\alpha(\tau - t + TH(t - \tau))]\mathrm{d}\tau\mathrm{d}t \leq \frac{\exp(\alpha T) - 1}{\alpha}\int_0^T \vartheta^2 \mathrm{d}t.$$

Under fairly mild conditions on the functions φ and Ψ at the critical point ξ^* we now obtain the optimality result. We will use the same abbreviations as in the previous proof.

Proposition 3.2.4. *Let the conditions of proposition 3.2.2 hold. Moreover, assume $R(\varphi\Psi)' \geq \Psi\Psi'\varphi^2$ at the constant critical price ξ^*. Then, for a feasible good, if $2(\Psi'(\xi^*))^2 > \Psi''(\xi^*)\Psi(\xi^*)$, a constant price is a (local) optimum for the monopoly.*

Proof. Only the negative definiteness of the second variation of Π with respect to ξ remains to be shown. Indeed, using

$$\begin{aligned}\int_0^T \left\langle \frac{\partial^2 x(t)}{\partial \xi^2}, \vartheta, \vartheta \right\rangle \mathrm{d}t = &-\frac{2\Psi'\varphi'\Psi}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta \exp(\varphi\Psi t)dt \int_0^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t \\ &+\frac{(\varphi\Psi)''-\varphi''\Psi}{\varphi}\int_0^T \vartheta^2 dt - 2\Psi'\varphi'\Psi\int_0^T \frac{\vartheta}{\exp(\varphi\Psi t)}\int_0^t \vartheta\exp(\varphi\Psi\tau)\mathrm{d}\tau\mathrm{d}t\end{aligned}$$

for any test direction ϑ, appendix B.2.3 derives

$$\begin{aligned}\left\langle \frac{\partial^2 \Pi/R_0}{\partial \xi^2}, \vartheta, \vartheta \right\rangle = \frac{N}{T}2\Psi\Bigg[&\frac{R(\varphi\Psi)'-\Psi\Psi'\varphi^2}{\exp(\varphi\Psi T)-1}\Bigg(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\int_0^\Theta \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t\mathrm{d}\Theta \\ +\exp(\varphi\Psi T)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\int_\Theta^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t\mathrm{d}\Theta\Bigg) &+ \left(\varphi\Psi' - R\frac{\varphi'}{\varphi} - R\frac{\Psi''}{2\Psi'}\right)\int_0^T \vartheta^2\mathrm{d}t\Bigg].\end{aligned}$$

By lemma 3.2.3 we obtain

$$\begin{aligned}\left\langle \frac{\partial^2 \Pi/R_0}{\partial \xi^2}, \vartheta, \vartheta \right\rangle &\leq \frac{N}{T}2\Psi\left[\frac{R(\varphi\Psi)'-\Psi\Psi'\varphi^2}{\varphi\Psi}\int_0^T \vartheta^2\mathrm{d}t + \left(\varphi\Psi' - R\frac{\varphi'}{\varphi} - R\frac{\Psi''}{2\Psi'}\right)\int_0^T \vartheta^2\mathrm{d}t\right] \\ &= \frac{N}{T}2R\left(\Psi' - \frac{\Psi''\Psi}{2\Psi'}\right)\int_0^T \vartheta^2\mathrm{d}t\end{aligned}$$

for $R(\varphi\Psi)' \geq \Psi\varphi^2\Psi'$. Hence, under the assumption $2(\Psi')^2 > \Psi''\Psi$ (and $\Psi' < 0$ and $\Psi \geq 0$ for a feasible product) we have the desired negative definiteness. □

The first condition holds for R small enough, i. e. at least for long-lasting products, the latter condition holds for instance for affine Ψ. Hence it indeed makes sense for some cases to reduce the complex price space $\mathbb{R}_+^{[0,T]}$ by focussing on time-constant prices.

For the issues dealt with so far, the general mathematical system was analytically treatable. For other questions, we have to resort to specific examples in order to obtain a qualitative insight into the characteristics of a market with habitual

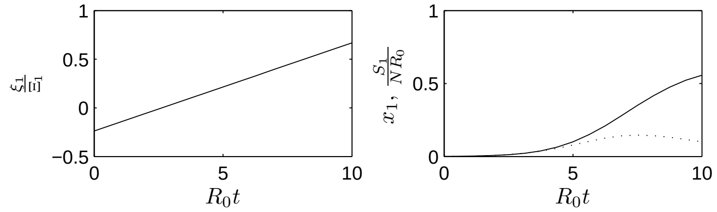

Figure 3.1: Optimal affine price evolution (left) as well as subpopulation (solid line) and sales (dotted line) evolution (right) for parameter values $R_1 = 0.1R_0$, $T = 10/R_0$, $r = 0$, $c_1 = 0.1/\Xi_1$, and $x_1(0) = 0.001$ (cf. example 3.2.2).

imitative consumers. Clearly, in such cases it is instructive to consider only very simple forms of s, φ, and especially ξ that just capture the necessary features for the discussed problem at hand. In particular, affine functions (the simplest case possible) are well-suited to study trends (e. g. price trends). The following example is meant to examine the optimal price trend for a good that is sold during a finite time period. It illustrates that proposition 3.2.4 does not hold for bounded time intervals.

Example 3.2.2 (Cumulated discounted profit in a monopoly setting)**.** *For simplicity, let us assume* $\varphi_1 = s_1 = 1 - \frac{\xi_1}{\Xi_1}$*, and let us only allow for affine price functions* $\xi_1(\cdot) \in \mathfrak{L}([0,T]) := \{f : [0,T] \to \mathbb{R} \,|\, \exists\, a, b : f(t) = a + bt\}$*. Consider the normal form competition game*

$$G = \left(1, \mathfrak{L}([0,T]), \int_0^T \exp[-rt](\xi_1(t) - c_1)S_1\left(s_1(\xi_1(t)), \varphi_1(\xi_1(t)), t\right) \mathrm{d}t\right).$$

For given parameters $R_1, T, r, c_1, x_1(0)$*, the optimal price path* $\xi_1(t)$ *can be found numerically (an analytical solution turns out to be too complex to provide any insight). As a result, for a whole range of realistic parameters we obtain that the product is initially sold below marginal cost, and then the price rises. One example calculation is depicted in figure 3.1.*

Of course, when r is chosen extremely large, this trend is reversed. However, this only happens for values of $r \sim R_0$ to $2R_0$. This would correspond to an interest rate of above 100 % within the time R_0^{-1}, i. e. if on average a consumer thinks of the good only once a year, the interest rate would have to be above 100 % per annum!

From this example, we may conclude that on a market with habitual imitative consumers a beneficial pricing strategy consists in starting at a low price and then increasing the price steadily. It might even be advantageous to initially give away products for free. The underlying idea is to initially strongly increase the market share in order to exploit habitual behavior.

3.2.2 Hamilton-Jacobi-Bellman equation as an alternative solution concept

The Hamilton-Jacobi-Bellman approach is an alternative approach to solve the optimization problem incorporated in a differential game. The idea behind this solution technique for optimal control problems is based on embedding and recursion (Dockner et al. 2001). That is, the problem of solving a certain optimal control problem P for a given initial state x^0 and initial time $t = 0$ is interpreted as a special case in a class of subproblems $\{P(x,t)|x \in X, t \in [0,T]\}$ with X the state space and $[0,T]$ the considered time interval, which are solved in a backward induction manner, finding the optimum step by step and backwards in time.

To solve a monopoly normal form competition game $(1, \mathfrak{S}, \Pi) = (1, \mathbb{R}^{\mathbb{R}_+}, F_T(\pi(\cdot)))$ subject to the mean dynamic

$$\dot{x}_1 = \varphi_1(\xi_1) R_0 x_1 \left(\Psi_1(\xi_1) - x_1\right) =: \mathcal{F}(x_1, \xi_1),$$

define $V(\hat{x}_1, t)$"$=$" $\max_{\xi_1} \int_t^T \mathcal{C}(x_1(\tau), \xi_1(\tau)) \mathrm{d}\tau$ to be the prospective optimal profit starting at time t from an initial state $\hat{x}_1$, where

$$\begin{aligned}
\mathcal{C}(x_1, \xi_1) &= \frac{\exp(-r\tau)}{T} (\xi_1 - c_1) S_1(s_1(\xi_1), \varphi_1(\xi_1), x_1) \\
&= \frac{\exp(-r\tau)}{T} (\xi_1 - c_1) N[\varphi_1 R_0 x_1 (\Psi_1 - x_1) + x_1 R_1]
\end{aligned}$$

represents the integrand of the payoff Π. Then the Hamilton-Jacobi-Bellman equation is given by

$$\frac{\partial V(x_1, t)}{\partial t} + \max_{\xi_1} \left\{ \frac{\partial V(x_1, t)}{\partial x_1} \mathcal{F}(x_1, \xi_1) + \mathcal{C}(x_1, \xi_1) \right\} = 0.$$

Compared to the variational approach from the previous section, we have replaced the combination of an ordinary differential equation and a maximization problem by a partial differential equation in $V(x_1, t)$. While in proposition 3.2.2, we could first solve the ordinary differential equation and subsequently the maximization for arbitrary $s_1(\xi_1)$ and $\varphi_1(\xi_1)$, the Hamilton-Jacobi-Bellman equation does not allow for an analytical solution, which is associated with its larger complexity due to the attempt to solve for all different initial conditions $x_1(t=0)$ at the same time.

However, one can pursue a numerical simulation for special cases. For instance, choose $r = 0$, $\varphi_1 = 1 - \frac{\xi_1}{\Xi_1}$, $s_1 = 1 - \frac{\xi_1}{\Xi_1}$, $R_0 = 2$, $R_1 = 1$, $c_1 = 0$, and thus $\Psi_1 = 1 - \frac{R_1}{R_0}\frac{1-s_1}{\varphi_1} = 1 - \frac{1}{2}\frac{\xi_1}{1-\xi_1}$, where $\Xi_1 = 1$ is the maximum reservation price and hence the Hamilton-Jacobi-Bellman differential equation takes the form

$$\frac{\partial V}{\partial t} = -\max_{\xi_1} f(\xi_1)$$

for $f(\xi_1) := \left(\frac{\partial V}{\partial x_1} + N\xi_1\right) 2(1-\xi_1)x_1\left(1 - \frac{\xi_1}{2-2\xi_1} - x_1\right) + N\xi_1 x_1$. By the first order condition we obtain the optimal price

$$\xi_1^* = \frac{1}{2}\left(1 - \frac{1}{N}\frac{\partial V}{\partial x_1}\right)$$

and hence

$$\max_{\xi_1} f(\xi_1) = f(\xi_1^*) = \frac{x_1}{N}\left(\frac{3}{4} - \frac{x}{2}\right)\left(N + \frac{\partial V}{\partial x_1}\right)^2 - x_1\frac{\partial V}{\partial x_1}.$$

The Hamilton-Jacobi-Bellman equation therefore turns into

$$\frac{\partial V}{\partial t} = -\frac{x_1}{N}\left(\frac{3}{4} - \frac{x}{2}\right)\left(N + \frac{\partial V}{\partial x_1}\right)^2 + x_1\frac{\partial V}{\partial x_1}.$$

The simulation result for the time period $[0,2]$ is displayed in figure 3.2. Obviously, the optimal price ξ_1 and the prospective profit V increase with x_1 since for a high market share habit and imitation can fully be exploited. Note that for low market shares x_1 the optimal price is initially zero in order to quickly achieve a state with higher x_1.

3.2.3 Oligopoly and polygopoly

In this section we turn to oligopoly and polygopoly markets. As for the monopoly, we will begin with an introductory example and then prove a feasibility result analogous to the result for a single firm. Afterwards, we examine the firms' behavior for an increasing number of competitors.

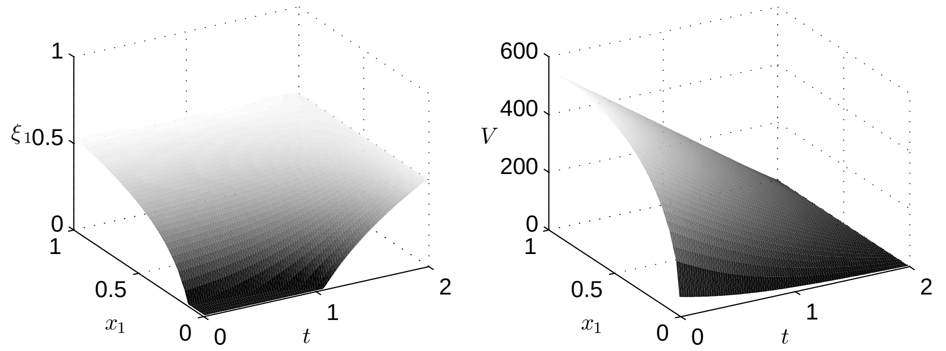

Figure 3.2: Solution of the Hamilton-Jacobi-Bellman equation (left: $\xi_1(x_1,t)$, right: $V(x_1,t)$) for the example in the text.

Example 3.2.3 (Steady state oligopoly prices)**.** *Consider the situation of example 3.2.1, this time with n firms. Moreover, we now assume* $\varphi_i(\xi_i) = \frac{1}{1+\frac{\xi_i}{\Xi_i}}$ *(this choice renders the system analytically solvable and is an approximation to an affine* $\varphi_i(\xi_i)$ *for low prices). The corresponding normal form competition game reads*

$$G = \left(n, \mathbb{R}^n_+, \lim_{t\to\infty}\Big((\xi_i - c_i)S_i(\vec{s},\vec{\varphi}(\vec{\xi}),t)\Big)_{i=1,\dots,n}\right).$$

If all n products are feasible, the steady state Nash equilibrium oligopoly prices ξ_i^* *can be computed analytically (cf. appendix B.2.4),*

$$\vec{\xi}^* = \begin{pmatrix} \frac{\Xi_1}{1-\Phi_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{\Xi_n}{1-\Phi_n} \end{pmatrix} \left[\Lambda + \begin{pmatrix} \Lambda_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Lambda_{nn} \end{pmatrix}\right]^{-1} \left[\Lambda \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_n \end{pmatrix} + \begin{pmatrix} (1-\Phi_1)\Lambda_{11}\frac{c_1}{\Xi_1} \\ \vdots \\ (1-\Phi_n)\Lambda_{nn}\frac{c_n}{\Xi_n} \end{pmatrix}\right], \qquad (3.1)$$

where Λ *is the inverse of matrix* A_n*, defined as*

$$(A_n)_{ij} := \begin{cases} 1, & j = i, \\ \Phi_j, & j \neq i, \end{cases} \quad i,j = 1,\dots,n.$$

This example is one of the very few cases, where the optimal prices can indeed be calculated analytically. It is not of great importance, but serves to illustrate few general features of oligopoly markets. First of all, we observe that the reservation prices Ξ_i and the marginal costs c_i have a positive effect on ξ_i^*. The reservation price Ξ_i even acts as a kind of proportionality factor on ξ_i^* via the first diagonal matrix in equation (3.1). Also, due to $1-\Phi_i = \frac{R_i}{R_0}(1-s_i)$, the matrix entries and hence the prices tend to infinity as the habit factor s_i approaches one, the value where consumers blindly purchase habitually. A further intuitive fact consists in the shrinking significance of the marginal costs with rising reservation price. Finally, let us note that the different good parameters Φ_i, c_i, Ξ_i affect all prices $\xi_j^*, j = 1, \ldots, n$, and not just the price of that good which they describe.

Let us now turn to the feasibility of goods in an oligopoly.

Proposition 3.2.5. *Consider an n-product market with habitual imitative consumers on which the products i, $i = 1, \ldots, n-1$, coexist with $0 < \Phi_i < 1$. Then product n is feasible if and only if*

$$\Psi_n[\xi_n = c_n] > \vec{\tilde{x}} \cdot \Phi(\vec{\tilde{\xi}}) = \sum_{i=1}^{n-1} \Phi_i(\tilde{\xi}_i)\tilde{x}_i$$

where $\vec{\tilde{x}}$ is the vector of market shares on the $(n-1)$-goods market (i. e. without product n) in the steady state and $\vec{\tilde{\xi}}$ the corresponding price vector.

Proof. Due to condition 3.2.1, Ψ_n is monotonously decreasing with ξ_n. Also, in the previous chapter we have shown an nth product to be feasible on a market with habitual imitative consumers, if and only if $\Psi_n > \vec{\tilde{x}} \cdot \vec{\Phi}$. Let us assume, $\hat{\xi}$ is such that $\Psi_n[\hat{\xi}] = \vec{\tilde{x}} \cdot \vec{\Phi}(\vec{\tilde{\xi}})$. Then, according to the result just cited, for $\xi_n = \hat{\xi}$ the good would just die out on the market so that the n-goods market behaves like the $(n-1)$-goods market and firms 1 to $n-1$ choose the prices $\vec{\tilde{\xi}}$. If $\hat{\xi}$ is smaller than c_n, then due to condition 3.2.1, $\Psi_n[\hat{\xi}]$ is larger than $\Psi_n[\xi_n]$ for $\xi_n \geq c_n$ so that for a profitable price good n still does not persist on the market. If on the other hand $\hat{\xi}$ is larger than c_n, then by decreasing ξ_n a little (to which the other firms react by choosing prices slightly different from $\vec{\tilde{\xi}}$) we obtain a situation in which good n has a non-zero market share and is sold above marginal costs. □

According to the previous chapter, $\Psi_n[\xi_n = c_n]$ represents the hypothetic monopoly market share when the price equals the marginal costs. Hence, intuitively, the above proposition implies that this hypothetic monopoly market share has to be larger than the weighted sum of market shares of products 1 to $n-1$, where the weights $\Phi_i \leq 1$ are the larger the stronger the corresponding goods induce habit.

Next, we shall study market implications from rising numbers of competitors. To start with, let us return to example 3.2.3 with identical firms.

Proposition 3.2.6. *In example 3.2.3, assume a symmetric oligopoly with n identical firms, where each firm optimally chooses the same price $\xi^{*,n}$. Then, for an increasing number of firms the price $\xi^{*,n}$ decreases. In the limit $n \to \infty$, it converges to the marginal cost c.*

Proof. Due to the symmetry of the market, we can skip the indices in equation (3.1). Also, we readily verify $\Lambda_{ij} = \frac{-\Phi}{(1-\Phi)(1+(n-1)\Phi)}$ for $i \neq j$ and $\Lambda_{ii} = \frac{1+(n-2)\Phi}{(1-\Phi)(1+(n-1)\Phi)}$ so that equation (3.1) yields

$$\xi^{*,n} = \frac{\Xi\Phi + (n-1)c - (n-2)c(1-\Phi)}{(n-1)-(n-3)(1-\Phi)} \overset{n\to\infty}{\longrightarrow} c.$$

Furthermore,

$$\frac{\xi^{*,n+1}}{\xi^{*,n}} = \frac{n + \left(\frac{\Xi}{c} + \frac{1-\Phi}{\Phi}\right)}{n-1+\underbrace{\left(\frac{\Xi}{c} + \frac{1-\Phi}{\Phi}\right)}_{=:C_1>0}} \cdot \frac{n+\frac{2-3\Phi}{\Phi}}{n+1+\underbrace{\frac{2-3\Phi}{\Phi}}_{=:C_2\geq -1}} = \frac{(n+C_1)(n+C_2)}{\underbrace{(n+C_1)}_{\geq 1}\underbrace{(n+C_2)}_{\geq 0}+C_1-C_2-1}.$$

Also, since for feasible goods, $\xi^{*,n} \geq c$ and $\Psi[\xi^{*,n}] > 0$, we have

$$0 < \Xi\left[\Psi[\xi^{*,n}]\left(1+\frac{\Xi}{c}\right) + \frac{\xi^{*,n}}{c} - 1\right] = \frac{\Xi}{c}[\Xi + \xi^{*,n} + (\Psi[\xi^{*,n}]-1)\Xi] + (\Psi[\xi^{*,n}]-1)\Xi$$
$$\Leftrightarrow 0 < \frac{\Xi}{c} + \frac{(\Psi[\xi^{*,n}]-1)\Xi}{\Xi + \xi^{*,n} + (\Psi[\xi^{*,n}]-1)\Xi} = C_1 - C_2 - 1,$$

where in the last step we have used $1-\Phi = (1-\Psi[\xi^{*,n}])\frac{\Xi}{\Xi+\xi^{*,n}}$. Together with the above equation, this yields $\frac{\xi^{*,n+1}}{\xi^{*,n}} < 1$. $\square$

Apparently, competition gets harder the more competitors coexist on the market. In the limit, we obtain perfect competition. This result actually holds for more general symmetric markets, which we will prove step by step. We will first show that steady state Nash equilibrium prices decrease for rising numbers n of firms. Later we will analyze the limit $n \to \infty$.

Lemma 3.2.7. *Consider a symmetric oligopoly with n identical firms and with habitual imitative consumers. Let $\Phi(\xi)$ and $\Psi(\xi)$ be differentiable. Then, the derivative of the steady state profit rate Π_i^n (of the ith firm in the n-goods market) with respect* *to the price ξ_i (of the ith product), evaluated at the steady state Nash equilibrium price $\xi^{*,n-1}$ of the $(n-1)$-goods market, is negative,*

$$\frac{\partial \Pi_i^n}{\partial \xi_i}\bigg|_{\xi_i=\xi_j=\xi^{*,n-1}} < 0.$$

Proof. If all n products coexist on the market, the steady state of mean dynamic (2.18) can be written as $A_n\vec{x} = \vec{\Psi}$ and the steady state profit of firm i as

$$\Pi_i^n = NR_i(\xi_i - c_i)x_i = NR_i(\xi_i - c_i)\Lambda_i\vec{\Psi},$$

with matrices A_n and Λ defined as in example 3.2.3 and Λ_i being the ith row of Λ. Since Ψ_j does not depend on ξ_i for $i \neq j$, we obtain

$$\frac{\partial \Pi_i^n/(NR_i)}{\partial \xi_i} = \sum_{j=1}^{n} \Lambda_{ij}\Psi_j + (\xi_i - c_i)\Lambda_{ii}\frac{\mathrm{d}\Psi_i}{\mathrm{d}\xi_i} + (\xi_i - c_i)\sum_{j=1}^{n}\Psi_j\frac{\partial \Lambda_{ij}}{\partial \xi_i}.$$

Due to the market symmetry, we may write $\Psi_i = \Psi$, $\xi_i = \xi$, and $c_i = c$. The derivative $\frac{\partial \Lambda_{ij}}{\partial \xi_i}$ at $\xi_i = \xi_j = \xi$ can be rewritten as $\frac{\partial \tilde{\Lambda}_{ij}}{\partial \xi_i}$, where $\tilde{\Lambda}$ is the inverse of matrix

$$(\tilde{A}_n)_{kl} = \begin{cases} 1, & l = k, \\ \Phi, & l \neq k, i, \\ \Phi_i[\xi_i], & l = i, l \neq k. \end{cases}$$

We readily verify $\tilde{\Lambda}_{ij} = \frac{-\Phi}{1+(n-2)\Phi-(n-1)\Phi\Phi_i[\xi_i]}$ for $i \neq j$ and $\tilde{\Lambda}_{ii} = \frac{1+(n-2)\Phi}{1+(n-2)\Phi-(n-1)\Phi\Phi_i[\xi_i]}$ so that

$$\frac{\partial}{\partial \xi_i}\left(\sum_{j=1}^{n}\Lambda_{ij}\right)\Bigg|_{\Phi_i=\Phi} = \frac{\partial}{\partial \Phi_i}\left(\frac{\Phi - 1}{(n-1)\Phi_i\Phi - (n-2)\Phi - 1}\right)\Bigg|_{\Phi_i=\Phi}\frac{\mathrm{d}\Phi_i}{\mathrm{d}\xi_i},$$

$$\text{and} \quad \frac{\partial \Pi_i^n/(NR_i)}{\partial \xi_i} = \frac{\Psi}{1+(n-1)\Phi}\left[1+\frac{\xi-c}{1-\Phi}\left(\frac{1+(n-2)\Phi}{\Psi}\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}+\frac{(n-1)\Phi}{1+(n-1)\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\right)\right].$$

We would like to show that this is negative at $\xi_i = \xi_j = \xi^{*,n-1}$, which is the Nash equilibrium price on the $(n-1)$-firms market and hence satisfies $\frac{\partial \Pi_i^{n-1}}{\partial \xi_i}\big|_{\xi_i=\xi_j=\xi^{*,n-1}} = 0$. Solving $\frac{\partial \Pi_i^{n-1}/(NR_i)}{\partial \xi_i}\big|_{\xi_i=\xi_j=\xi^{*,n-1}} = 0$ for $(\xi^{*,n-1} - c)$, we obtain

$$\xi^{*,n-1} - c = -\frac{\Psi(1-\Phi)[1+(n-2)\Phi]}{[1+(n-3)\Phi][1+(n-2)\Phi]\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\big|_{\xi^{*,n-1}} + (n-2)\Phi\Psi\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\big|_{\xi^{*,n-1}}}.$$

This can be inserted into $\frac{\partial\Pi_i^n/(NR_i)}{\partial\xi_i}\Big|_{\xi_i=\xi_j=\xi^{*,n-1}}$, which yields

$$\left.\frac{\partial\Pi_i^n/(NR_i)}{\partial\xi_i}\right|_{\xi_i=\xi_j=\xi^{*,n-1}}$$
$$=\frac{\Psi}{1+(n-1)\Phi}\left[1-\frac{[1+(n-2)\Phi]^2\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\big|_{\xi^{*,n-1}}+(n-1)\Phi\Psi\frac{1+(n-2)\Phi}{1+(n-1)\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\big|_{\xi^{*,n-1}}}{[1+(n-3)\Phi][1+(n-2)\Phi]\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\big|_{\xi^{*,n-1}}+(n-2)\Phi\Psi\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\big|_{\xi^{*,n-1}}}\right].$$

This is indeed negative, since the fraction in brackets is larger than one: Due to $\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}, \frac{\mathrm{d}\Psi}{\mathrm{d}\xi} < 0$ (follows from condition 3.2.1), all summands in the numerator and denominator are negative. Furthermore, the first summand of the numerator is smaller (i. e. more negative) than the first summand of the denominator. The same holds for the second summands, if we use that for feasible products $0 < \Psi \leq \Phi \leq 1$ and hence $\frac{1+(n-2)\Phi}{1+(n-1)\Phi} > \frac{n-1}{n}$. □

The previous lemma now implies the desired result.

Proposition 3.2.8. *Consider a symmetric oligopoly with n identical firms and with habitual imitative consumers. Let $\Phi(\xi)$ and $\Psi(\xi)$ be differentiable. If $\Phi(\xi)$ and $\Psi(\xi)$ are such that there exists exactly one steady state Nash equilibrium, then the equilibrium price $\xi^{*,n}$ decreases as the number of firms n rises.*

Proof. We show $\xi^{*,n} \leq \xi^{*,n-1}$. We may assume

$$\left.\frac{\partial\Pi_i^n}{\partial\xi_i}\right|_{\xi_i=\xi_j=c} \geq 0,$$

since otherwise all firms would choose $\xi_i = c$ as the unique Nash equilibrium price, and $\xi^{*,n-1}$ must have been greater than or equal to marginal cost c so that the proposition would already be proven. Also, due to lemma 3.2.7,

$$\left.\frac{\partial\Pi_i^n}{\partial\xi_i}\right|_{\xi_i=\xi_j=\xi^{*,n-1}} < 0.$$

Define $f : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto f(\xi) = \frac{\partial\Pi_i^n}{\partial\xi_i}\Big|_{\xi_i=\xi_j=\xi}$. Then f is continuous with $f(\xi^{*,n-1}) < 0$ and $f(c) \geq 0$ so that by Rolle's theorem there exists a price $\xi^{*,n} \in [c, \xi^{*,n-1})$ with $f(\xi^{*,n}) = 0$ at which f changes sign to negative. Hence, $\xi^{*,n} < \xi^{*,n-1}$ is a local maximizer of Π_i^n and thus the unique Nash equilibrium price. □

Hence, despite the consumers' bounded rationality, our model has intuitively competitive features. Also, it provides a foundation for perfect competitive equilibrium prices as the number of firms tends to infinity: We will next show that the prices of the weakest products on the market converge against their marginal costs as the number of competitors rises to infinity. This holds for a general polygopoly and directly implies that on the symmetric market all prices converge against marginal costs.

Lemma 3.2.9. *Consider a polygopoly with n firms and with habitual imitative consumers, where all n products coexist in the steady state Nash equilibrium. For given n, let i_n denote the index of the "weakest" good, i. e. the one with lowest market share $x_{i_n} = \min_{j=1,\dots,n}\{x_j\}$ in the steady state Nash equilibrium. Let $\Phi_{i_n}(\xi_{i_n})$ and $\Psi_{i_n}(\xi_{i_n})$ be differentiable. If there is $\nu > 0$ such that in the steady state Nash equilibrium $\frac{\partial x_{i_n}}{\partial \xi_{i_n}} < -\varepsilon < 0$ for all $n > \nu$, then as the number of firms n tends to infinity, the price of good i_n converges to marginal cost, i. e. $(\xi^{*,n}_{i_n} - c_{i_n}) \to 0$.*

Proof. The steady state profit rate of good i_n is given by $\Pi^n_{i_n} = NR_{i_n}(\xi_{i_n} - c_{i_n})x_{i_n}$ so that

$$\left.\frac{\partial \Pi^n_{i_n}}{\partial \xi_{i_n}}\right|_{\xi_j=\xi^{*,n}_j} = NR_{i_n}\left(x_{i_n}\big|_{\xi_j=\xi^{*,n}_j} + (\xi_{i_n} - c_{i_n})\left.\frac{\partial x_{i_n}}{\partial \xi_{i_n}}\right|_{\xi_j=\xi^{*,n}_j}\right),$$

where $\xi_j = \xi^{*,n}_j$ indicates evaluation at the steady state Nash equilibrium prices.

For a contradiction, assume there exists $\delta > 0$ such that for all $\mu > \nu$ there is $n(\mu) > \mu$ with $(\xi^{*,n}_{i_n} - c_{i_n}) > \delta$. Hence,

$$\left.\frac{\partial \Pi^{n(\mu)}_{i_{n(\mu)}}}{\partial \xi_{i_{n(\mu)}}}\right|_{\xi_j=\xi^{*,n(\mu)}_j} < NR_{i_{n(\mu)}}\left(x_{i_{n(\mu)}} - \varepsilon\delta\right) \leq NR_{i_{n(\mu)}}\left(\frac{1}{n(\mu)} - \varepsilon\delta\right)$$

which is strictly negative for μ large enough. However, this contradicts the Nash equilibrium condition $\left.\frac{\partial \Pi^{n(\mu)}_{i_{n(\mu)}}}{\partial \xi_{i_{n(\mu)}}}\right|_{\xi_j=\xi^{*,n(\mu)}_j} = 0$ so that we obtain $\limsup_{n\to\infty}(\xi^{*,n}_{i_n} - c_{i_n}) \leq 0$.

Since trivially, $(\xi^{*,n}_{i_n} - c_{i_n}) \geq 0$, we finally find $\lim_{n\to\infty}(\xi^{*,n}_{i_n} - c_{i_n}) = 0$. □

This lemma is almost what we aimed at, however, it depends on conditions on state variables ($\frac{\partial x_{i_n}}{\partial \xi_{i_n}} < -\varepsilon$) which might not be satisfied. A shrinking market share for rising prices is indeed economically plausible but not necessarily true. Hence, we would like to express all conditions in terms of the control variables Φ_i and Ψ_i, for which we need the following lemma, whose proof is given in appendix B.2.5.

Lemma 3.2.10. *Let $0 < \Phi_i < 1$, $i = 1, \ldots, n$, and let Λ be the inverse of matrix A_n defined as*

$$(A_n)_{ij} = \begin{cases} 1, & i = j, \\ \Phi_j, & i \neq j, \end{cases} \qquad i, j = 1 \ldots n.$$

Then,

$$\Lambda_{ii} \geq 1 \quad \text{and} \quad \Lambda_{ij} \leq 0 \quad \text{for all } i, j = 1, \ldots, n \text{ with } i \neq j.$$

Now we can prove an estimate for the change of market shares.

Lemma 3.2.11. *Consider a polygopoly with n firms and with habitual imitative consumers, where all n products coexist in the steady state Nash equilibrium. Let $\Phi_i(\xi_i)$ and $\Psi_i(\xi_i)$ be differentiable. Then*

$$\left.\frac{\partial x_i}{\partial \xi_i}\right|_{\xi_j = \xi_j^{*,n}} \leq \left.\frac{\partial \Psi_i}{\partial \xi_i}\right|_{\xi_j = \xi_j^{*,n}}$$

for all $1 \leq i \leq n$, where x_i is understood as the steady state market share.

Proof. Without loss of generality let $i = 1$. Also, in the following, let all parameters, equations, and derivatives be evaluated at the steady state Nash equilibrium (i. e. at $\xi_j = \xi_j^{*,n}$). Then, mean dynamic (2.18) yields $A_n \vec{x} = \vec{\Psi}$, which can be differentiated with respect to ξ_1 to give

$$\frac{\partial A_n}{\partial \xi_1}\vec{x} + A_n \frac{\partial \vec{x}}{\partial \xi_1} = \frac{\partial \vec{\Psi}}{\partial \xi_1} \quad \Leftrightarrow \quad \frac{\partial \vec{x}}{\partial \xi_1} = A_n^{-1}\left(\frac{\partial \vec{\Psi}}{\partial \xi_1} - \frac{\partial A_n}{\partial \xi_1}\vec{x}\right) = A_n^{-1}\begin{pmatrix} \frac{\partial \Psi_1}{\partial \xi_1} \\ -x_1 \frac{\partial \Phi_1}{\partial \xi_1} \\ \vdots \\ -x_1 \frac{\partial \Phi_1}{\partial \xi_1} \end{pmatrix}.$$

Denoting the inverse of A_n by Λ, the first row of the above equation becomes

$$\frac{\partial x_1}{\partial \xi_1} = \Lambda_{11}\frac{\partial \Psi_1}{\partial \xi_1} - x_1 \frac{\partial \Phi_1}{\partial \xi_1}\sum_{j=2}^{n} \Lambda_{1j},$$

which together with the previous lemma and $\frac{\partial \Psi_1}{\partial \xi_1}, \frac{\partial \Phi_1}{\partial \xi_1} \leq 0$ (due to condition 3.2.1) yields the desired result. ☐

The previous lemma can be interpreted as follows: The change of steady state market shares x_i is larger than the change of the corresponding "product qualities" Ψ_i, i. e. shares are quite sensitive to quality changes. Lemmata 3.2.9 and 3.2.11 can now be combined to yield the following.

Proposition 3.2.12. *Consider a polygopoly with n firms and habitual imitative consumers, where all n products coexist in the steady state Nash equilibrium. For given n, let i_n denote the index of the "weakest" good, i. e. the one with lowest market share $x_{i_n} = \min_{j=1,\dots,n}\{x_j\}$. Let $\Phi_{i_n}(\xi_{i_n})$ and $\Psi_{i_n}(\xi_{i_n})$ be differentiable. If there is $\nu > 0$ such that $\frac{\partial \Psi_{i_n}}{\partial \xi_{i_n}}\big|_{\xi_{i_n}=\xi^{*,n}_{i_n}} < -\varepsilon < 0$ for all $n > \nu$, then as the number of firms tends to infinity, the price of good i_n converges to marginal cost, i. e. $(\xi^{*,n}_{i_n} - c_{i_n}) \to 0$.* □

The proof of this result can inductively be repeated for the second "weakest" good, the third "weakest" one, and so on up to the mth "weakest" good, where m is any positive integer.

Corollary 3.2.13. *Consider a polygopoly with n firms and let the assumptions from proposition 3.2.12 hold. As the number of firms tends to infinity, the m "weakest" products' prices converge to their marginal costs, where m is any positive integer.* □

As a consequence, since on a symmetric market any good is the "weakest" one, the prices of all goods converge to marginal costs. In general, however, there may be products so superior to the rest of the market that their prices stay away from marginal costs while all other prices converge against marginal costs. The proof of lemma 3.2.9 can be slightly adapted to show that this may only be true for a finite number of products.

3.3 Extensions: welfare, product life cycle generation, and advertising

To point into possible directions of further research we will briefly discuss three extensions to our model as there are welfare, the generation of product life cycles, and advertising.

3.3.1 A welfare definition

In the following, we shall propose a suitable welfare definition for our setting to allow for theories of social implications. The producer surplus can be calculated as usual

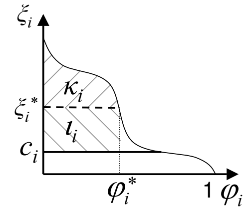

Figure 3.3: An arbitrary imitation function, plotted as normalized inverse demand function. The consumer surplus equals $CS_i = \frac{\kappa_i}{\varphi_i^*} S_i$, and the producer surplus is given by $PS_i = \frac{\iota_i}{\varphi_i^*} S_i$, where ξ_i^* denotes the current price of product i and $\varphi_i^* = \varphi_i(\xi_i^*)$ is the resulting imitation coefficient.

from the Marshallian definition, whereas the consumer surplus has to be obtained differently as a consequence of the non-standard consumer behavior.

Definition 3.3.1 (Welfare)**.** *Given the imitation function $\varphi_i(\xi_i)$ for good i, the contribution of that good to the producer and consumer surplus at time t are defined as*

$$\begin{aligned} PS_i(t) &= (\xi_i(t) - c_i) S_i(t), \qquad (3.2)\\ CS_i(t) &= \frac{S_i(t)}{\varphi_i(\xi_i(t))} \left(\int_0^{\varphi_i(\xi_i(t))} \left[\varphi_i^{-1}(\varphi) - \xi_i(t) \right] \mathrm{d}\varphi \right). \qquad (3.3) \end{aligned}$$

The social welfare takes the form

$$W(t) = \sum_{i=1}^{n} PS_i(t) + CS_i(t). \qquad (3.4)$$

For a single good with an arbitrary imitation function, consumer and producer surplus at a specific time are illustrated in figure 3.3. Before motivating this choice, note that the imitation function $\varphi_i(\xi_i)$ can be interpreted as the demand distribution for good i. In other words, for a consumer the probability P to have a reservation price ξ_i^{rp} for good i larger than or equal to some price ξ_i^* is given by the demand distribution, i. e.

$$\varphi_i(\xi_i^*) = \mathrm{P}[\xi_i^{\mathrm{rp}} \geq \xi_i^*].$$

Phrased differently, the imitation function $\varphi_i(\xi_i)$ constitutes the probability distribution of the reservation price ξ_i^{rp} of a set of heterogeneous consumers, and the reservation price ξ_i^{rp} is distributed according to the density $-\frac{\mathrm{d}\varphi_i(\xi_i)}{\mathrm{d}\xi_i}$. (If instead we assume fickle homogeneous consumers, where each individual's reservation price changes from time to time, $\varphi_i(\xi_i)$ can be interpreted as the reservation price probability of a single consumer.) Against this background, the following alternative characterization of the consumer surplus may serve as motivation.

Proposition 3.3.1. *Let $\varphi_i(\xi_i)\xi_i \to 0$ for $\xi_i \to \infty$, and let ξ_i^* be the current price of product i and $\varphi_i^* = \varphi_i(\xi_i^*)$ the resulting imitation coefficient. Let the consumers' reservation price ξ_i^{rp} for good i have probability density $-\frac{\mathrm{d}\varphi_i(\xi_i)}{\mathrm{d}\xi_i}\big|_{\xi_i^{\mathrm{rp}}} =: -\varphi_i'(\xi_i^{\mathrm{rp}})$.*

(i) Pick one consumer arbitrarily and give her the option to buy either product i or none. The expected value of her utility, $U_i = \max(\xi_i^{rp} - \xi_i^*, 0)$, is then given by

$$\mathrm{E}[U_i] = \int_{\xi_i^*}^{\infty} \varphi_i(\xi_i)\mathrm{d}\xi_i.$$

(For simplicity, we here also allow infinite values of the expected utility.)

(ii) Pick one consumer, who has reservation price $\xi_i^{\mathrm{rp}} \geq \xi_i^*$, i. e. who would buy product i. The expected value of her utility, u_i, is given as

$$\mathrm{E}[u_i] = \frac{\mathrm{E}[U_i]}{\varphi_i^*}.$$

(iii) Assume that those consumers who actually buy product i are uniformly distributed among all potential buyers (i. e. those with $\xi_i^{\mathrm{rp}} \geq \xi_i^*$), then the expected consumer surplus (3.3) *is given by*

$$CS_i = \mathrm{E}[u_i]S_i = \frac{\mathrm{E}[U_i]}{\varphi_i^*}S_i.$$

Proof. (i) The consumer's utility U_i is a random variable depending on the reservation price ξ_i^{rp}. Hence,

$$\begin{aligned}\mathrm{E}[U_i] &= \int_0^1 U_i(\xi_i^{\mathrm{rp}})\mathrm{dP}(\xi_i^{\mathrm{rp}}) = \int_0^{\infty} U_i(\xi_i^{\mathrm{rp}})(-\varphi_i'(\xi_i^{\mathrm{rp}}))\mathrm{d}\xi_i^{\mathrm{rp}} \\ &= -\int_{\xi_i^*}^{\infty} (\xi_i^{\mathrm{rp}} - \xi_i^*)\varphi_i'(\xi_i^{\mathrm{rp}})\mathrm{d}\xi_i^{\mathrm{rp}} = -\left[(\xi_i^{\mathrm{rp}} - \xi_i^*)\varphi_i(\xi_i^{\mathrm{rp}})\right]_{\xi_i^{\mathrm{rp}}=\xi_i^*}^{\infty} + \int_{\xi_i^*}^{\infty} \varphi_i(\xi_i^{\mathrm{rp}})\mathrm{d}\xi_i^{\mathrm{rp}}.\end{aligned}$$

The left summand of the last expression equals zero, since $\xi_i^{\mathrm{rp}}\varphi_i(\xi_i^{\mathrm{rp}}) \overset{\xi_i^{\mathrm{rp}}\to\infty}{\longrightarrow} 0$. ($U_i$ is a so-called integrable random variable, if and only if $\varphi_i \in L^1([0,\infty))$, in which case the integral is bounded.)

(ii) The consumers with $\xi_i^{\mathrm{rp}} \geq \xi_i^*$ make up $\varphi_i(\xi_i^*)$ of the total population. Among them, ξ_i^{rp} is distributed according to

$$\mathrm{P}[\xi_i^{\mathrm{rp}} \geq \xi] = \begin{cases} 1, & \xi < \xi_i^*, \\ \frac{\varphi_i(\xi_i^{\mathrm{rp}})}{\varphi_i(\xi_i^*)}, & \text{else,} \end{cases}$$

so that analogously to the above, $\mathrm{E}[u_i] = \frac{\mathrm{E}[U_i]}{\varphi_i(\xi_i^*)}$.

(iii) By a change of variables $\varphi = \varphi_i(\xi_i^{\mathrm{rp}})$ and integration by parts we obtain

$$\int_0^{\varphi_i(\xi_i^*)} [\varphi_i^{-1}(\varphi) - \xi_i^*]\mathrm{d}\varphi = -\int_{\xi_i^*}^{\infty} (\xi_i^{\mathrm{rp}} - \xi_i^*)\varphi_i'(\xi_i^{\mathrm{rp}})\mathrm{d}\xi_i^{\mathrm{rp}} = \int_{\xi_i^*}^{\infty} \varphi_i(\xi_i^{\mathrm{rp}})\mathrm{d}\xi_i^{\mathrm{rp}} = \mathrm{E}[U_i],$$

which together with the definition of CS_i yields the desired result.

□

Our definition thus resembles the standard approach: CS_i approximates the average utility of all buyers (given as the difference between average reservation price among all buyers and the actual price).

As a brief application, we show that for a symmetric oligopoly, the welfare rises with the number of firms.

Proposition 3.3.2. *Consider a symmetric oligopoly with n identical firms and with habitual imitative consumers. Let $\varphi(\xi)$ and $s(\xi)$ be differentiable and such that there exists exactly one steady state Nash equilibrium with equilibrium price $\xi^{*,n}$. If the total steady state sales $nS^n := \sum_{i=1}^n S_i(\xi^{*,n})$ increase more strongly in n than $\varphi^{*,n} := \varphi(\xi^{*,n})$, i. e. $\frac{\mathrm{d}(nS^n)}{\mathrm{d}n}/(nS^n) > \frac{\mathrm{d}\varphi^{*,n}}{\mathrm{d}n}/\varphi^{*,n}$, then the welfare increases for a rising number of firms.*

Proof. As before, since all firms are identical we skip the indices. As illustrated in figure 3.3, the welfare is given by $\frac{nS^n}{\varphi^{*,n}}(\kappa + \iota)$. Define $\Omega^n = \frac{nS^n}{\varphi^{*,n}}$, then

$$\frac{\mathrm{d}\Omega^n}{\mathrm{d}n} = \frac{\frac{\mathrm{d}(nS^n)}{\mathrm{d}n}\varphi^{*,n} - nS^n\frac{\mathrm{d}\varphi^{*,n}}{\mathrm{d}n}}{(\varphi^{*,n})^2},$$

which is larger than 0 by assumption. Also, $\xi^{*,n}$ decreases for a rising number of firms by proposition 3.2.8, and hence, $\kappa + \iota$ increases due to the monotonicity of $\varphi(\xi)$ (condition 3.2.1). Altogether, the welfare rises for rising n. □

Apparently, despite the consumers' bounded rationality, we obtain the standard result of an increasing welfare. This implies a certain amount of market efficiency, comparable to a market with rational consumers.

3.3.2 The generation of product life cycles

In this paragraph, we briefly illustrate how a realistic product life cycle may emerge from our model. As a simple example, consider a consecutive introduction of many products, all competing with each other. Think for instance of the mobile phone market, where new (innovative) mobile phones frequently enter the market. The maximum reservation price for a product may be assumed highest at its introduction on the market when it still represents the state of the art, and then it decreases in time, as innovation goes on. Hence, also habit and imitation function are highest at the time of product introduction.

The most simple setting is to assume fixed prices ξ_i, simple imitation and habit functions $\varphi_i(\xi_i) = s_i(\xi_i) = 1 - \frac{\xi_i}{\Xi_i}$, new product introductions equally distributed over time, and a simple evolution of the maximum reservation price Ξ_i in time, e. g. $\Xi_i = \frac{\Xi}{1+\alpha R_0(t-t_i)}$, where t_i is the time of introduction of product i. Figure 3.4 shows the resulting product life cycles of the successively introduced products, obtained from an exemplary simulation.

Notice the classical pattern with a gentle increase of the sales right after product launch, a broad maturity period and a quite steep decline until the product vanishes (cf. for example de Kluyver (1977), Polli & Cook (1969) and others).

3.3.3 Marketing strategies: advertisement

In section 2.2.4, in order to employ specific switching probabilities, we used the mechanism of imitation (2.6), that is, the probability of buying good i is proportional to the amount x_i of people who already own it. x_i may here be interpreted as the probability that the consumer gets to know the product from other consumers. The multiplicative imitation factor φ_i represents how strongly the consumer is con-

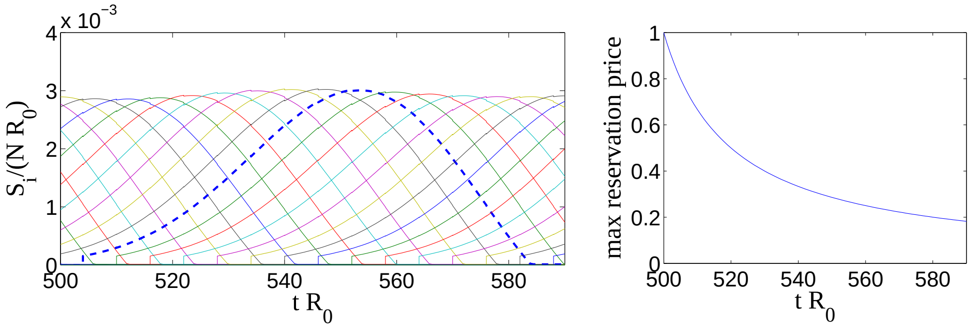

Figure 3.4: Product life cycles (left) when many products enter the market successively and the maximum reservation price Ξ_i of each product decreases in time (right). In this simulation we chose $\varphi_i(\xi_i) = s_i(\xi_i) = 1 - \frac{\xi_i}{\Xi_i}$ with maximum reservation price $\Xi_i = \frac{\Xi}{1+\alpha R_0(t-t_i)}$, $\alpha = 5 \cdot 10^{-2}$. Furthermore, we have a time interval $T = \frac{6}{R_0}$ between the introduction of products, an alarm clock rate $R_i = 0.025 R_0$ for all goods, an initial subpopulation $x_i(0) = 10^{-3}$, and constant product prices $\xi_i = 0.1\Xi$.

vinced to buy the good when she knows it. However, consumers can also get to know the good via advertisements, which constitute an effective tool for firms to influence the consumers' buying behavior. The probability to see the product's commercial is given by $a_i \in [0, 1]$, where a_i depends positively on the advertising budget. The overall probability to become aware of product i (via commercials or other consumers) hence is $a_i + x_i - a_i x_i$ so that (2.6) and (2.8) change to

$$p_{0i} = \varphi_i(x_i + a_i - x_i a_i), \quad i \neq 0, \tag{3.5}$$

$$p_{ij} = (1 - s_i)\varphi_j(x_j + a_j - x_j a_j), \quad i \neq 0 \wedge j \neq 0, i. \tag{3.6}$$

For the same motivation as in section 2.2.4, equation (2.7) remains unchanged,

$$p_{ii} = s_i \in [0, 1], \quad i \neq 0. \tag{3.7}$$

We shall in the following always assume $\varphi_i, s_i > 0$. An interesting question would be whether a non-feasible product can be made feasible by advertising. The following proposition provides an answer for a single good market (where we disregard advertising costs and only examine whether a demand for that good exists).

Proposition 3.3.3. *The single product on a market with habitual imitative consumers is always feasible if it is advertised, i. e. $a_1 > 0$.*

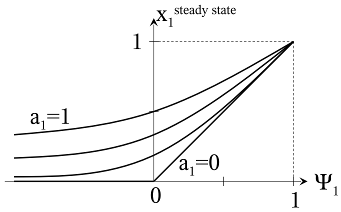

Figure 3.5: Stable steady state value of the market share x_1 for different advertising levels a_1.

Proof. The mean dynamic for the single good market takes the form

$$\begin{aligned} \dot{x}_1 &= \varphi_1 R_0(x_1 + a_1 - x_1 a_1) + x_1(R_1 s_1 - \varphi_1 R_0(x_1 + a_1 - x_1 a_1)) - R_1 x_1 \\ &= x_1 \varphi_1 R_0[\Psi_1 - 2a_1 - x_1(1 - a_1)] + \varphi_1 R_0 a_1. \end{aligned}$$

In the stationary state $\dot{x}_1 = 0$ we thus obtain

$$x_1 = \begin{cases} \frac{1}{2-\Psi_1}, & a_1 = 1, \\ \frac{\Psi_1 - 2a_1 + \sqrt{(\Psi_1 - 2a_1)^2 + 4a_1(1-a_1)}}{2-2a_1}, & a_1 \neq 1, \end{cases}$$

which is positive for $a_1 > 0$, irrespective of the value of Ψ_1. □

Apparently, commercials help the good to survive on the market. This statement is illustrated in figure 3.5 where the steady state market share is shown for different advertising levels. For a positive level, the market share is always positive and hence the product feasible.

Note that in proposition 3.3.3 we only consider the demand side of the market, i. e. we examine whether the product is demanded by consumers in the steady state. We ignore that the firm might not be able to operate in the black because of immense advertising costs.

An analogous result can be shown for an oligopoly.

Proposition 3.3.4. *On an n-product market with habitual imitative consumers, product i is always feasible if it is advertised, i. e. $a_i > 0$ (unless there is a good j with $\Phi_j = 1$).*

Proof. The mean dynamic reads

$$\dot{x}_i = \varphi_i R_0 x_i \left[\Psi_i - 2a_i - (1 - a_i)\left(x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \Phi_j x_j \right) \right] + \varphi_i R_0 a_i \left(1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \Phi_j x_j \right).$$

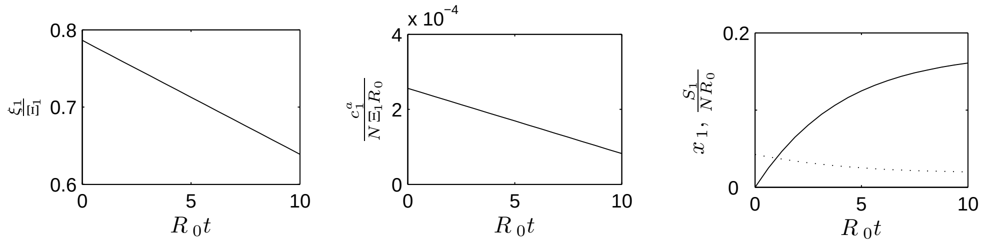

Figure 3.6: Optimal affine price evolution (left) and optimal affine advertising expenses (middle), as well as subpopulation (solid line) and sales (dotted line) evolution (right) for parameter values $R_1 = 0.1R_0$, $T = 10/R_0$, $r = 0$, $c_1 = 0.1/\Xi_1$, $K = 2 \cdot 10^{-6}$, $x_1(0) = 0$ (cf. example 3.3.1).

For $x_i = 0$ we obtain $\dot{x}_i > 0$. Hence, the system trajectory can never approach $x_i = 0$ so that the market share of product i always stays strictly away from zero. $\square$

An advertising campaign is usually associated with costs that depend on its reach. Let us therefore introduce advertising costs c_i^a for good i. Obviously, the derivative $\frac{\partial a_i}{\partial c_i^a}$ has to be non-negative. With this altered model at hand, various simulations can be performed for specific functions $\varphi_i(\xi_i)$, $s_i(\xi_i)$ and $a_i(c_i^a)$. One could for instance examine the product feasibility including advertising costs, whether advertising is profitable at all, how large a_i should optimally be, or whether there is a threshold value for x_i above which advertising is no longer beneficial. For illustration, we pick up example 3.2.2 and add advertising. We will compute the optimal affine pricing and advertising strategy. Before, however, we need to extend the definition of the normal form competition game.

With advertising, firms have a second strategic variable besides their product's price which represents the advertising expenses. Hence, the set $\mathfrak{S}$ of all possible strategy combinations now is a subset of the space of n-tuples over maps $(\xi_i, c_i^a) : \mathbb{R}_+ \to \mathbb{R}_+^2$, $t \mapsto (\xi_i(t), c_i^a(t))$, $\mathfrak{S} \subseteq \left[(\mathbb{R}_+^2)^{\mathbb{R}_+}\right]^n = (\mathbb{R}_+^2)^{\mathbb{R}_+} \times \ldots \times (\mathbb{R}_+^2)^{\mathbb{R}_+}$. The components of profit $\Pi : \mathfrak{S} \to \mathbb{R}_+^n$ now become $\Pi_i(\xi_1, \ldots, \xi_n, c_1^a, \ldots, c_n^a) = F\left[(\xi_i(t) - c_i) S_i\left(\vec{s}(\vec{\xi}(t)), \vec{\varphi}(\vec{\xi}(t)), \vec{a}(\vec{c^a}(t)), t\right) - \vec{c^a}(t)\right]$, where $\vec{S}(\vec{s}(\vec{\xi}(t)), \vec{\varphi}(\vec{\xi}(t)), \vec{a}(\vec{c^a}(t)), t)$ denotes the sales vector produced by the population game including advertising according to equations (3.5) and (3.6).

Example 3.3.1 (Cumulated discounted profit in a monopoly setting with advertising). *For simplicity, let us assume* $\varphi_1 = s_1 = 1 - \frac{\xi_1}{\Xi_1}$ *and* $a_1 = \frac{1}{1+K/c_1^a}$, *and let us only allow for affine price and advertising cost functions* $\xi_1(\cdot), c_1^a(\cdot) \in \mathfrak{L}([0,T]) := \{f : [0,T] \to \mathbb{R} \,|\, \exists\, a, b : f(t) = a + bt\}$. *Consider the normal form competition game*

$$G = \left(1, \mathfrak{L}^2([0,T]), \int_0^T \exp[-rt]\,[(\xi_1(t) - c_1) S_1\,(s_1(\xi_1(t)), \varphi_1(\xi_1(t)), a_1(c_1^a(t)), t) - c_1^a(t)]\mathrm{d}t\right).$$

For given parameters $R_1, T, r, c_1, x_1(0)$, *the optimal price paths* $\xi_1(t)$ *and* $c_1^a(t)$ *can be found numerically. As a result, for a whole range of realistic parameters we obtain the reverse of example 3.2.2: The price decreases with time. One example calculation is depicted in figure 3.6.*

Apparently, a firm is recommended to start an advertising campaign in parallel to the product launch and steadily decrease the product price as well as the advertising expenses during the lifespan of the product. Due to the initial advertising, the market share is rapidly increased with brute force. Via the subsequent price decrease, habit purchases can be kept on a high level, and reluctant customers are attracted. Thereby, the market is optimally exploited by initially letting customers with a high reservation price pay high prices and only later reducing the price to make people with low reservation prices buy the product (similarly to the concept of price discrimination). Advertising becomes less crucial when the market share has already reached a certain level (the product sells itself) and is therefore reduced.

3.4 Conclusion

We examined the optimal strategic pricing for firms when the demand evolution is generated by the behavior of boundedly rational consumers who follow a rule of thumb and base their decisions on imitation and habit. The demand dynamic is described within the framework of a population game with associated switching probabilities, and it serves as a basis for strategic pricing of a monopoly or oligopoly in a differential game. The optimal price paths correspond to Nash equilibria of a normal form competition game.

The modeling approach is supported by psychological and experimental studies, and the introduced methodology allows for broad applications and qualitative theoretical analysis.

We investigated product feasibility (i. e. the conditions under which firms operate profitably in the long-term) and expressed it with the help of the hypothetical popularity of the product if it was sold for a price equal to the marginal cost. Furthermore, we showed that markets with habitual imitative consumers are in a sense well-behaved: For a rising number of firms, the prices decrease, the prices of the weakest products (but not necessarily of all products) converge against marginal costs, and the welfare rises (at least for a symmetric market). Such results (despite the boundedly rational consumer behavior) prove once more the existence of some kind of efficiency in not totally rational markets.

We also proved for the monopoly that under certain conditions, Nash equilibria are found in the strategy space of all time-constant price paths so that a reduction of the (quite complex and untractable) strategy space of all possible price paths is at least sometimes sensible.

Finally, the assumed boundedly rational consumer behavior was shown to lead to observed market patterns such as product life cycles, and extensions to the model were proposed and examined such as an adequate definition of welfare, which allows for analysis of social implications, and the introduction of advertising, which allows to explore optimal advertising strategies.

Bibliography

Andronov, A. & Pontrjagin, L. (1937), 'Systèmes grossiers. dokl. akad. nauk.', *SSSR* **14**, 247–251.

Arthur, B. W. (1994), 'Inductive reasoning and bounded rationality: the El Farol problem', *Am. Econ. Rev.* **84**, 406–411.

Assael, H. (1984), *Consumer Behavior and Marketing Action*, 2^{nd} edn, Kent Publishing Company, Boston, Massachusetts.

Banerjee, A. & Fudenberg, D. (2004), 'Word-of-mouth learning', *Games and Economic Behavior* **46**, 1–22.

Banerjee, A. V. (1992), 'A simple model of herd behavior', *The Quarterly Journal of Economics* **107**, 797–817.

Benaïm, M. & Weibull, J. W. (2003), 'Deterministic approximation of stochastic evolution in games', *Econometrica* **71**, 873–903.

Bhaskar, V. (2000), 'Egalitarianism and efficiency in repeated symmetric games', *Games and Economic Behavior* **32**, 247–262.

Bikhchandani, S., Hirshleifer, D. & Welch, I. (1992), 'A theory of fads, fashion, custom, and cultural change as informational cascades', *The Journal of Political Economy* **100**, 992–1026.

Brockhoff, K. (1967), 'A test for the product life cycle', *Econometrica* **35**, 472–484.

Challet, D., Marsili, M. & Zhang, Y.-C. (2005), *Minority Games*, Oxford University Press, Oxford.

Challet, D., Ottino, G. & Marsili, M. (2004), 'Shedding light on El Farol', *Physica A* **332**, 469. preprint cond-mat/0306445.

Challet, D. & Zhang, Y.-C. (1997*a*), 'Emergence of cooperation and organization in an evolutionary game', *Physica A* **246**, 407 – 418.

Challet, D. & Zhang, Y.-C. (1997*b*), 'Emergence of cooperation and organization in an evolutionary game', *Physica A* **246**, 407. adap-org/9708006.

Conlisk, J. (1996), 'Why bounded rationality?', *Journal of Economic Literature* **XXXIV**, 669–700.

Coolen, A. A. C. (2005), *The Mathematical Theory of Minority Games*, Oxford University Press, Oxford.

Corriveau, K. H., Fusaro, M. & Harris, P. L. (2009), 'Going with the flow: Preschoolers prefer nondissenters as informants', *Psychological Science* **20**, 372–377.

Crawford, V. P. & Haller, H. (1990), 'Learning how to cooperate: Optimal play in repeated coordination games', *Econometrica* **58**, 571–595.

Cyert, R. M. & March, J. G. (1956), 'Organizational factors in the theory of oligopoly', *The Quarterly Journal of Economics* **70**, 44–64.

de Kluyver, C. A. (1977), 'Innovation and industrial product life cycles', *California Management Review* **20**.

De Sanctis, L. & Galla, T. (2006), 'Adapting to heterogeneous comfort levels', *J. Stat. Mech.* .

Dockner, E., Jørgensen, S., Van Long, N. & Sorger, G. (2001), *Differential Games in Economics and Management Science*, Cambridge University Press.

Easley, D. & Rustichini, A. (1999), 'Choice without beliefs', *Econometrica* **67**, 1157 – 1184.

Ellison, G. (2006), *Bounded Rationality in Industrial Organization*, Blundell, Newey and Persson (eds.), Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress, Cambridge University Press.

Ellison, G. & Fudenberg, D. (1993), 'Rules of thumb for social learning', *The Journal of Political Economy* **101**.

Ellison, G. & Fudenberg, D. (1995), 'Word-of-mouth communication and social learning', *The Quarterly Journal of Economics* **110**, 93–125.

Garey, M. & Johnson, D. (1979), *Computers and Intractability: A Guide to NP-Completeness Freeman*, Freeman, New-York.

Guckenheimer, J. & Holmes, P. (1990), *Nonlinear Oscillations, Dynamical Systems, and Bifurctions of Vector Fields*, Springer.

Heaton, J. (1993), 'The interaction between time-nonseparable preferences and time aggregation', *Econometrica* **61**, 353–385.

Heiner, R. A. (1983), 'The origin of predictable behavior', *The American Economic Review* **73**, 560–595.

Isaacs, R. (1954), 'Differential games, i, ii, iii, iv', *Rand Corporation* **Reports RM-1391, 1399, 1411, 1486**.

Kirman, A. (1993), 'Ants, rationality, and recruitment', *The Quarterly Journal of Economics* **108**, 137–156.

Kirman, A. (2006), 'Heterogeneity in economics', *Journal of Economic Interaction and Coordination* **1**, 89–117.

Matzke, C. & Challet, D. (2008), 'Taking a shower in youth hostels: risks and delights of heterogeneity', *submitted, Discussion Paper 1/2008 (Bonn Econ Discussion Papers archive)* .

Matzke, C. & Wirth, B. (2008*a*), 'The evolution of sales with habitual and imitative consumers', *submitted, Discussion Paper 10/2008 (Bonn Econ Discussion Papers archive)* .

Matzke, C. & Wirth, B. (2008*b*), 'Product pricing when demand follows a rule of thumb', *submitted, Discussion Paper 03/2009 (Bonn Econ Discussion Papers archive)* .

Mertens, S. (1998), ‘Phase transition in the number partitioning problem’, *Phys. Rev. Lett.* **81**, 4281.

Pingle, M. & Day, R. H. (1996), ‘Modes of economizing behavior: Experimental evidence’, *Journal of Economic Behavior and Organization* **29**, 191–209.

Polli, R. & Cook, V. (1969), ‘Validity of the product life cycle’, *The Journal of Business* **42**.

Rothschild, K. W. (1947), ‘Price theory and oligopoly’, *The Economic Journal* **57**, 299–320.

Sandholm, W. H. (2005), ‘Excess payoff dynamics and other well-behaved evolutionary dynamics’, *Journal of Economic Theory* .

Sandholm, W. H. (2006), *Population Games and Evolutionary Dynamics*, Draft version 5/9/06. To be published by MIT Press.

Schlag, K. H. (1998), ‘Why imitate, and if so, how? a boundedly rational approach to multi-armed bandits’, *Journal of Economic Theory* **78**, 130–156.

Simon, H. A. (1955), ‘A behavioral model of rational choice’, *The Quarterly Journal of Economics* **69**, 99–118.

Smallwood, D. E. & Conlisk, J. (1979), ‘Product quality in markets where consumers are imperfectly informed’, *The Quarterly Journal of Economics* **93**.

Sobel, J. (2000), ‘Notes, comments, and letters to the editor: Economists’ model of learning’, *Journal of Economic Theory* **94**, 241–261.

Stigler, G. J. & Becker, G. S. (1977), ‘De gustibus non est disputandum’, *The American Economic Review* **67**, 76–90.

Venkatesan, M. (1966), ‘Experimental study of consumer behavior, conformity and independence’, *Journal of Marketing Research* **3**.

von Thadden, E.-L. (1992), ‘Optimal pricing against a simple learning rule’, *Games and Economic Behavior* **4**, 627–649.

Young, H. P. (1993), 'The evolution of conventions', *Econometrica* **61**, 57–84.

Appendices

A.1 Appendix to chapter 2

A.1.1 Proof of lemma 2.3.4

Proof. Consider the matrix function $M(t) := I + t(A_n - I)$ such that $M(0) = I$ and $M(1) = A_n$, where I denotes the identity matrix. Since $\det(M)$ is continuous in M, it is also continuous in t. If $\det(M(1))$ were non-positive, then according to Rolle's theorem there would exist some $t \in (0, 1]$ with $\det(M(t)) = 0$. However, $\det(M(t))$ cannot be zero due to the following reasoning: For a contradiction, assume the columns of A_n to be linearly dependent, i. e.

$$\vec{0} = \sum_{i=1}^{n} \alpha_i \left(\Phi_i, \ldots, 1, \Phi_i, \ldots\right)^{\mathrm{T}}$$

for $\alpha_i \neq 0$. This can be rewritten as

$$\vec{0} = \begin{pmatrix} \alpha_1(1 - \Phi_1) \\ \vdots \\ \alpha_n(1 - \Phi_n) \end{pmatrix} + \sum_{i=1}^{n} \alpha_i \begin{pmatrix} \Phi_i \\ \vdots \\ \Phi_i \end{pmatrix},$$

which together with $\Phi_i < 1$ implies $\alpha_i = \frac{k}{1-\Phi_i}$, for some number k. Plugging this back into the equation we obtain

$$0 = k\left(1 + \sum_{i=1}^{n} \frac{\Phi_i}{1 - \Phi_i}\right),$$

which is impossible for $\frac{\Phi_i}{1-\Phi_i} > 0$. □

A.1.2 Proof of proposition 2.3.5

Proof. For the computation, assume that also product n exists in the steady state.

1. *Linear system of equations:* Since by assumption $x_i > 0$ in the steady state for $i = 1, \ldots, n$ we obtain a linear system of steady state equations from the mean dynamic (2.18):

$$A_n \vec{x} = \vec{\Psi}$$

2. *Cramer's rule:* Due to the previous step, A_n is invertible. According to Cramer's rule the solution to $A_n \vec{x} = \vec{\Psi}$ is given by

$$x_i = \frac{\det\left((A_n)_{i \to \vec{\Psi}}\right)}{\det\left(A_n\right)}$$

where matrix $(A_n)_{i \to \vec{\Psi}}$ equals A_n with column i replaced by $\vec{\Psi}$.

3. *Cramer's rule backwards:* Let us denote the vector of ones by $\vec{1}$ and let $M_{\not{i},\not{j}}$ be matrix M with the ith row and jth column removed. Using Laplace expansion for the last column,

$$\begin{aligned}
&\det\left((A_n)_{n \to \vec{\Psi}}\right) \\
&= \sum_{i=1}^{n} (-1)^{i+n} \Psi_i \det((A_n)_{\not{i},\not{n}}) \\
&= \sum_{i=1}^{n-1} (-1)^{i+n} \Phi_i \Psi_i \det(((A_n)_{\not{i},\not{n}})_{i \to \vec{1}}) + \Psi_n \det(A_{n-1}) \\
&= \sum_{i=1}^{n-1} (-1) \Phi_i \Psi_i \det((A_{n-1})_{i \to \vec{1}}) + \Psi_n \det(A_{n-1}) \\
&= -\sum_{i,j=1}^{n-1} (-1)^{i+j} \Phi_i \Psi_i \det((A_{n-1})_{\not{j},\not{i}}) + \Psi_n \det(A_{n-1}) \\
&= -\sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} (-1)^{i+j} \Phi_i \Psi_i \Phi_j \det(((A_{n-1})_{j \to \vec{1}})_{\not{j},\not{i}}) - \sum_{i=1}^{n-1} \Phi_i \Psi_i \det((A_{n-1})_{\not{i},\not{i}}) + \Psi_n \det(A_{n-1}) \\
&= -\sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} (-1)^{i+j} \Phi_i \Psi_i \Phi_j \det(((A_{n-1})_{i \to \vec{1}})_{\not{i},\not{j}}) - \sum_{i=1}^{n-1} \Phi_i \Psi_i \det((A_{n-1})_{\not{i},\not{i}}) + \Psi_n \det(A_{n-1}) \\
&= -\sum_{i,j=1}^{n-1} (-1)^{i+j} \Phi_j \Psi_i \det((A_{n-1})_{\not{i},\not{j}}) + \Psi_n \det(A_{n-1})
\end{aligned}$$

$$= -\sum_{j=1}^{n-1} \Phi_j \det((A_{n-1})_{j\to\vec{\Psi}}) + \Psi_n \det(A_{n-1})$$
$$= \det(A_{n-1})\left(\Psi_n - \sum_{i=1}^{n-1} \Phi_i \left(A_{n-1}^{-1}\vec{\Psi}\right)_i\right),$$

where the last step follows from Cramer's rule and Laplace expansion has been applied various times.

Overall, we obtain

$$\begin{aligned}
\text{product } n \text{ feasible} \quad &\Leftrightarrow \quad x_n > 0 \\
&\overset{\text{lemma2.3.4}}{\Leftrightarrow} \quad x_n \det(A_n) > 0 \\
&\overset{\text{step2}}{\Leftrightarrow} \quad \det((A_n)_{n\to\vec{\Psi}}) > 0 \\
&\overset{\text{step3}}{\Leftrightarrow} \quad \det(A_{n-1})\left(\Psi_n - \sum_{i=1}^{n-1} \Phi_i \left(A_{n-1}^{-1}\vec{\Psi}\right)_i\right) > 0 \\
&\overset{\text{lemma2.3.4}}{\Leftrightarrow} \quad \Psi_n > \sum_{i=1}^{n-1} \Phi_i \left(A_{n-1}^{-1}\vec{\Psi}\right)_i
\end{aligned}$$

$\square$

B.2 Appendix to chapter 3

B.2.1 Criticality of a constant monopoly price (proposition 3.2.2)

Using the relation (obtained via integration by parts and Fubini's theorem)

$$\int_0^t f(\tau)\int_0^\tau g(\theta)\mathrm{d}\theta\mathrm{d}\tau = \left[\int_0^t f(\tau)\mathrm{d}\tau \int_0^t g(\tau)\mathrm{d}\tau\right]_0^t - \int_0^t\int_0^\tau f(\theta)\mathrm{d}\theta g(\tau)\mathrm{d}\tau = \int_0^t g(\tau)\int_\tau^t f(\theta)\mathrm{d}\theta\mathrm{d}\tau,$$

we can write the variation of $x(t)$ with respect to ξ in some test direction ϑ as

$$\begin{aligned}
\left\langle \frac{\partial x(t)}{\partial\xi},\vartheta\right\rangle &= \frac{\exp(\varphi\Psi t)\left[\int_0^t(\varphi\Psi)'\vartheta\mathrm{d}\tau\left(\frac{\exp(\varphi\Psi t)-1}{\Psi}+\frac{1}{\Psi}\right)-\int_0^t\exp(\varphi\Psi\tau)\left(\varphi'\vartheta+\varphi\int_0^\tau(\varphi\Psi)'\vartheta(\theta)\mathrm{d}\theta\right)\mathrm{d}\tau-\frac{\int_0^T\exp(\varphi\Psi\tau)(\varphi'\vartheta+\varphi\int_0^\tau(\varphi\Psi)'\vartheta\mathrm{d}\theta)\mathrm{d}\tau(\exp(\varphi\Psi T)-1)-\int_0^T(\varphi\Psi)'\vartheta\mathrm{d}\tau\exp(\varphi\Psi T)\int_0^T\varphi\exp(\varphi\Psi\tau)\mathrm{d}\tau}{(\exp(\varphi\Psi T)-1)^2}\right]}{\left(\frac{\exp(\varphi\Psi t)-1}{\Psi}+\frac{1}{\Psi}\right)^2}\\
&= \frac{\exp(\varphi\Psi t)\left[(\varphi\Psi)'\int_0^t\vartheta\mathrm{d}\tau\frac{\exp(\varphi\Psi t)}{\Psi}-\int_0^t\exp(\varphi\Psi\tau)\left(\varphi'\vartheta+\varphi(\varphi\Psi)'\int_0^\tau\vartheta\mathrm{d}\theta\right)\mathrm{d}\tau-\frac{\int_0^T\exp(\varphi\Psi\tau)(\varphi'\vartheta+\varphi(\varphi\Psi)'\int_0^\tau\vartheta\mathrm{d}\theta)\mathrm{d}\tau(\exp(\varphi\Psi T)-1)-(\varphi\Psi)'\int_0^T\vartheta\mathrm{d}\tau\exp(\varphi\Psi T)\int_0^T\varphi\exp(\varphi\Psi\tau)\mathrm{d}\tau}{(\exp(\varphi\Psi T)-1)^2}\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2}\\
&= \frac{\exp(\varphi\Psi t)\left[(\varphi\Psi)'\frac{\exp(\varphi\Psi t)}{\Psi}\int_0^t\vartheta\mathrm{d}\tau-\varphi'\int_0^t\exp(\varphi\Psi\tau)\vartheta\mathrm{d}\tau-\varphi(\varphi\Psi)'\int_0^t\exp(\varphi\Psi\tau)\int_0^\tau\vartheta\mathrm{d}\theta\mathrm{d}\tau-\frac{\int_0^T\exp(\varphi\Psi\tau)(\varphi'\vartheta+\varphi(\varphi\Psi)'\int_0^\tau\vartheta\mathrm{d}\theta)\mathrm{d}\tau(\exp(\varphi\Psi T)-1)-(\varphi\Psi)'\int_0^T\vartheta\mathrm{d}\tau\exp(\varphi\Psi T)\int_0^T\varphi\exp(\varphi\Psi\tau)\mathrm{d}\tau}{(\exp(\varphi\Psi T)-1)^2}\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2}\\
&= \frac{\exp(\varphi\Psi t)\left[(\varphi\Psi)'\frac{\exp(\varphi\Psi t)}{\Psi}\int_0^t\vartheta\mathrm{d}\tau-\varphi'\int_0^t\exp(\varphi\Psi\tau)\vartheta\mathrm{d}\tau-\varphi(\varphi\Psi)'\int_0^t\exp(\varphi\Psi\tau)\int_0^\tau\vartheta\mathrm{d}\theta\mathrm{d}\tau-\frac{\int_0^T\exp(\varphi\Psi\tau)(\varphi'\vartheta+\varphi(\varphi\Psi)'\int_0^\tau\vartheta\mathrm{d}\theta)\mathrm{d}\tau(\exp(\varphi\Psi T)-1)-(\varphi\Psi)'\int_0^T\vartheta\mathrm{d}\tau\exp(\varphi\Psi T)\int_0^T\varphi\exp(\varphi\Psi\tau)\mathrm{d}\tau}{(\exp(\varphi\Psi T)-1)^2}\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2}\\
&= \frac{\exp(\varphi\Psi t)\left[(\varphi\Psi)'\frac{\exp(\varphi\Psi t)}{\Psi}\int_0^t\vartheta\mathrm{d}\tau-\varphi'\int_0^t\exp(\varphi\Psi\tau)\vartheta\mathrm{d}\tau-\varphi(\varphi\Psi)'\int_0^t\vartheta\int_\theta^t\exp(\varphi\Psi\tau)\mathrm{d}\tau\mathrm{d}\theta-\frac{\int_0^T\exp(\varphi\Psi\tau)(\varphi'\vartheta+\varphi(\varphi\Psi)'\int_0^\tau\vartheta\mathrm{d}\theta)\mathrm{d}\tau(\exp(\varphi\Psi T)-1)-(\varphi\Psi)'\int_0^T\vartheta\mathrm{d}\tau\exp(\varphi\Psi T)\int_0^T\varphi\exp(\varphi\Psi\tau)\mathrm{d}\tau}{(\exp(\varphi\Psi T)-1)^2}\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2}\\
&= \frac{\exp(\varphi\Psi t)\left[(\varphi\Psi)'\frac{\exp(\varphi\Psi t)}{\Psi}\int_0^t\vartheta\mathrm{d}\tau-\varphi'\int_0^t\exp(\varphi\Psi\theta)\vartheta\mathrm{d}\theta-\varphi(\varphi\Psi)'\int_0^t\vartheta\frac{\exp(\varphi\Psi t)-\exp(\varphi\Psi\theta)}{\varphi\Psi}\mathrm{d}\theta-\frac{\int_0^T\exp(\varphi\Psi\tau)(\varphi'\vartheta+\varphi(\varphi\Psi)'\int_0^\tau\vartheta\mathrm{d}\theta)\mathrm{d}\tau(\exp(\varphi\Psi T)-1)-(\varphi\Psi)'\int_0^T\vartheta\mathrm{d}\tau\exp(\varphi\Psi T)\int_0^T\varphi\exp(\varphi\Psi\tau)\mathrm{d}\tau}{(\exp(\varphi\Psi T)-1)^2}\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2}\\
&= \frac{\exp(\varphi\Psi t)\left[-\varphi'\int_0^t\exp(\varphi\Psi\theta)\vartheta\mathrm{d}\theta+\frac{(\varphi\Psi)'}{\Psi}\int_0^t\vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta-\frac{\int_0^T\exp(\varphi\Psi\tau)(\varphi'\vartheta+\varphi(\varphi\Psi)'\int_0^\tau\vartheta\mathrm{d}\theta)\mathrm{d}\tau(\exp(\varphi\Psi T)-1)-(\varphi\Psi)'\int_0^T\vartheta\mathrm{d}\tau\exp(\varphi\Psi T)\int_0^T\varphi\exp(\varphi\Psi\tau)\mathrm{d}\tau}{(\exp(\varphi\Psi T)-1)^2}\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2}\\
&= \frac{\exp(\varphi\Psi t)\left[\frac{\varphi\Psi'}{\Psi}\int_0^t\vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta-\frac{\varphi'\int_0^T\exp(\varphi\Psi\tau)\vartheta\mathrm{d}\tau+\varphi(\varphi\Psi)'\int_0^T\exp(\varphi\Psi\tau)\int_0^\tau\vartheta\mathrm{d}\theta\mathrm{d}\tau}{\exp(\varphi\Psi T)-1}+\frac{(\varphi\Psi)'\exp(\varphi\Psi T)\int_0^T\varphi\exp(\varphi\Psi\tau)\mathrm{d}\tau\int_0^T\vartheta\mathrm{d}\tau}{(\exp(\varphi\Psi T)-1)^2}\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2}\\
&= \frac{\exp(\varphi\Psi t)\left[\frac{\varphi\Psi'}{\Psi}\int_0^t\vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta-\frac{\varphi'}{\exp(\varphi\Psi T)-1}\int_0^T\vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta-\frac{\varphi(\varphi\Psi)'}{\exp(\varphi\Psi T)-1}\int_0^T\exp(\varphi\Psi\tau)\int_0^\tau\vartheta\mathrm{d}\theta\mathrm{d}\tau+\frac{(\varphi\Psi)'\exp(\varphi\Psi T)\int_0^T\varphi\exp(\varphi\Psi\tau)\mathrm{d}\tau}{(\exp(\varphi\Psi T)-1)^2}\int_0^T\vartheta\mathrm{d}\tau\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2}\\
&= \frac{\exp(\varphi\Psi t)\left[\frac{\varphi\Psi'}{\Psi}\int_0^t\vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta-\frac{\varphi'\Psi}{\Psi(\exp(\varphi\Psi T)-1)}\int_0^T\vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta+\frac{(\varphi\Psi)'}{\Psi(\exp(\varphi\Psi T)-1)}\int_0^T\vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2} = \frac{\Psi\left[\varphi\Psi'\int_0^t\vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta+\frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T\vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta\right]}{\exp(\varphi\Psi t)}.
\end{aligned}$$

Using the price ξ implicitly defined by $\xi - c = -\Psi(\xi)/\Psi'(\xi)$ (which follows from maximizing the profit for a constant price and a steady state, i. e. no periodic oscillations), the variation of the profit rate then is given as

$$
\begin{aligned}
&\left\langle \frac{\partial \Pi/R_0}{\partial \xi}, \vartheta \right\rangle = \frac{1}{R_0}\frac{d}{d\varepsilon}\Pi(\xi(\cdot)+\varepsilon\vartheta(\cdot))\Big|_{\varepsilon=0} \\
&= \frac{N}{T}\int_0^T x(R+\varphi(\Psi-x))\vartheta + (\xi-c)\left(x(\varphi\Psi)'\vartheta - x^2\varphi'\vartheta + \left\langle \frac{\partial x(t)}{\partial \xi}, \vartheta \right\rangle (R+\varphi\Psi-2\varphi x)\right)\mathrm{d}t \\
&= \frac{N}{T}\int_0^T \Psi(R+\varphi(\Psi-\Psi))\vartheta - \frac{\Psi}{\Psi'}\left(\Psi(\varphi\Psi)'\vartheta - \Psi^2\varphi'\vartheta + \left[\frac{\exp(\varphi\Psi t)\left[\frac{\varphi\Psi'}{\Psi}\int_0^t \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta + \frac{\varphi\Psi'}{\Psi(\exp(\varphi\Psi T)-1)}\int_0^T \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2}\right](R+\varphi\Psi-2\varphi\Psi)\right)\mathrm{d}t \\
&= \frac{N}{T}(\Psi R-\Psi^2\varphi)\int_0^T \vartheta\mathrm{d}t - \frac{N}{T}\int_0^T \frac{\Psi}{\Psi'}(R-\varphi\Psi)\left[\frac{\exp(\varphi\Psi t)\left[\frac{\varphi\Psi'}{\Psi}\int_0^t \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta + \frac{\varphi\Psi'}{\Psi(\exp(\varphi\Psi T)-1)}\int_0^T \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta\right]}{\left(\frac{\exp(\varphi\Psi t)}{\Psi}\right)^2}\right]\mathrm{d}t \\
&= \frac{N}{T}(\Psi R-\Psi^2\varphi)\int_0^T \vartheta\mathrm{d}t - \frac{N}{T}\frac{\Psi^3}{\Psi'}(R-\varphi\Psi)\int_0^T \frac{\exp(\varphi\Psi t)}{(\exp(\varphi\Psi t))^2}\frac{\varphi\Psi'}{\Psi}\int_0^t \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta\mathrm{d}t - \frac{N}{T}\frac{\Psi^3}{\Psi'}(R-\varphi\Psi)\int_0^T \frac{\exp(\varphi\Psi t)}{(\exp(\varphi\Psi t))^2}\frac{\varphi\Psi'}{\Psi\left(\exp(\varphi\Psi T)-1\right)}\int_0^T \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta\mathrm{d}t \\
&= \frac{N}{T}(\Psi R-\Psi^2\varphi)\int_0^T \vartheta\mathrm{d}t - \frac{N}{T}\Psi^2\varphi(R-\varphi\Psi)\int_0^T \frac{1}{\exp(\varphi\Psi t)}\int_0^t \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta\mathrm{d}t - \frac{N}{T}\Psi^2\varphi(R-\varphi\Psi)\int_0^T \frac{1}{\exp(\varphi\Psi t)}\frac{1}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta\mathrm{d}t \\
&= \frac{N}{T}(\Psi R-\Psi^2\varphi)\int_0^T \vartheta\mathrm{d}t - \frac{N}{T}\Psi^2\varphi(R-\varphi\Psi)\int_0^T \vartheta\exp(\varphi\Psi\theta)\int_\theta^T \frac{1}{\exp(\varphi\Psi t)}\mathrm{d}t\mathrm{d}\theta - \frac{N}{T}\Psi^2\varphi(R-\varphi\Psi)\int_0^T \frac{1}{\exp(\varphi\Psi t)}\frac{1}{\exp(\varphi\Psi T)-1}\mathrm{d}t\int_0^T \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta \\
&= \frac{N}{T}(\Psi R-\Psi^2\varphi)\int_0^T \vartheta\mathrm{d}t - \frac{N}{T}\Psi^2\varphi(R-\varphi\Psi)\int_0^T \vartheta\exp(\varphi\Psi\theta)\left[-\frac{1}{\varphi\Psi\exp(\varphi\Psi T)} + \frac{1}{\varphi\Psi\exp(\varphi\Psi\theta)}\right]\mathrm{d}\theta - \frac{\frac{N}{T}\Psi^2\varphi(R-\varphi\Psi)}{\exp(\varphi\Psi T)-1}\left[-\frac{1}{\varphi\Psi\exp(\varphi\Psi T)} + \frac{1}{\varphi\Psi}\right]\int_0^T \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta \\
&= \frac{N}{T}(\Psi R-\Psi^2\varphi)\int_0^T \vartheta\mathrm{d}t - \frac{N}{T}\Psi(R-\varphi\Psi)\int_0^T \vartheta\exp(\varphi\Psi\theta)\left[-\frac{1}{\exp(\varphi\Psi T)} + \frac{1}{\exp(\varphi\Psi\theta)}\right]\mathrm{d}\theta - \frac{\frac{N}{T}\Psi(R-\varphi\Psi)}{\exp(\varphi\Psi T)-1}\left[1-\frac{1}{\exp(\varphi\Psi T)}\right]\int_0^T \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta \\
&= \frac{N}{T}(\Psi R-\Psi^2\varphi)\left[\int_0^T \vartheta\mathrm{d}t - \int_0^T \vartheta\exp(\varphi\Psi\theta)\left[-\frac{1}{\exp(\varphi\Psi T)} + \frac{1}{\exp(\varphi\Psi\theta)}\right]\mathrm{d}\theta - \frac{1}{\exp(\varphi\Psi T)-1}\left[1-\frac{1}{\exp(\varphi\Psi T)}\right]\int_0^T \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta\right] \\
&= \frac{N}{T}(\Psi R-\Psi^2\varphi)\left[\int_0^T \vartheta\mathrm{d}t - \int_0^T \vartheta\exp(\varphi\Psi\theta)\left[-\frac{1}{\exp(\varphi\Psi T)} + \frac{1}{\exp(\varphi\Psi\theta)}\right]\mathrm{d}\theta - \frac{1}{\exp(\varphi\Psi T)}\int_0^T \vartheta\exp(\varphi\Psi\theta)\mathrm{d}\theta\right] \\
&= \frac{N}{T}(\Psi R-\Psi^2\varphi)\left[\int_0^T \vartheta\mathrm{d}t - \int_0^T \vartheta\mathrm{d}\theta\right] \\
&= 0.
\end{aligned}
$$

B.2.2 Proof of lemma 3.2.3

Proof. For Lebesgue-integrable functions ϑ, θ, let us define the bilinear forms

$$\begin{aligned}\langle \vartheta, \theta\rangle_L &= \int_0^T \int_0^T \vartheta(t)\theta(\tau)\exp[\alpha(\tau - t + TH(t-\tau))]\mathrm{d}\tau\mathrm{d}t \\ \langle \vartheta, \theta\rangle_R &= \frac{\exp(\alpha T) - 1}{\alpha}\int_0^T \vartheta(t)\theta(t)\mathrm{d}t.\end{aligned}$$

For $\vartheta \in L^1([0,T])\backslash L^2([0,T])$, $\langle \vartheta, \vartheta\rangle_R$ is unbounded so that $\langle \vartheta, \vartheta\rangle_L \leq \langle \vartheta, \vartheta\rangle_R$ trivially. Hence, let us assume $\vartheta \in L^2([0,T])$ from now on.

$\langle \cdot, \cdot\rangle_L$ and $\langle \cdot, \cdot\rangle_R$ are (sequentially) continuous in both arguments in $L^2([0,T])$: let $\theta_k \xrightarrow{k\to\infty} \theta$ in $L^2([0,T])$. By Hölder's inequality and the continuous Sobolev embedding $L^2([0,T]) \hookrightarrow L^1([0,T])$,

$$\begin{aligned}|\langle \vartheta, \theta\rangle_R - \langle \vartheta, \theta_k\rangle_R| &= \left|\int_0^T \vartheta(\theta - \theta_k)\mathrm{d}t\right| \leq ||\vartheta||_{L^2}||\theta - \theta_k||_{L^2} \xrightarrow{k\to\infty} 0 \\ |\langle \vartheta, \theta\rangle_L - \langle \vartheta, \theta_k\rangle_L| &= \left|\int_0^T\int_0^T \vartheta(\theta - \theta_k)\mathrm{e}^{\alpha(\tau - t + TH(t-\tau))}\mathrm{d}\tau\mathrm{d}t\right| \leq \mathrm{e}^{|\alpha T|}\int_0^T |\vartheta|\mathrm{d}t \int_0^T |\theta - \theta_k|\mathrm{d}t \\ &= \mathrm{e}^{|\alpha T|}||\vartheta||_{L^1}||\theta - \theta_k||_{L^1} \leq C||\vartheta||_{L^2}||\theta - \theta_k||_{L^2} \xrightarrow{k\to\infty} 0\end{aligned}$$

for some constant C. Continuity in the first argument follows analogously. By

$$\left|\langle \theta, \theta\rangle_{L/R} - \langle \theta_k, \theta_k\rangle_{L/R}\right| \leq \left|\langle \theta, \theta\rangle_{L/R} - \langle \theta, \theta_k\rangle_{L/R}\right| + \left|\langle \theta, \theta_k\rangle_{L/R} - \langle \theta_k, \theta_k\rangle_{L/R}\right| \xrightarrow{k\to\infty} 0$$

we immediately have continuity of $\vartheta \mapsto \langle \vartheta, \vartheta\rangle_{L/R}$.

We would like to show $\langle \vartheta, \vartheta\rangle_L \leq \langle \vartheta, \vartheta\rangle_R$ for regular step functions $\vartheta \in \mathcal{T} = \left\{\sum_{i=0}^{N-1} a_i \chi_{[\frac{i}{N}T, \frac{i+1}{N}T]} \middle| N \in \mathbb{N}, a_i \in \mathbb{R}\right\}$. The lemma then follows by the above-shown continuity and the density of regular step functions in $L^2([0,T])$: It is well-known that step functions are dense in $L^2([0,T])$, so density of $\mathcal{T}$ in the space of step functions remains to be shown. Let $f = \sum_{j=0}^{n-1} b_j \chi_{[t_j, t_{j+1}]}$, $0 = t_0 \leq \cdots \leq t_n = T$. Set $f_k = \sum_{j=0}^{k-1} f(\frac{j}{k}T)\chi_{[\frac{j}{k}T, \frac{j+1}{k}T]} \in \mathcal{T}$. f_k equals f on all intervals $[\frac{j}{k}T, \frac{j+1}{k}T]$ which lie completely in an interval $[t_i, t_{i+1}]$. This is not the case for a maximum number of n intervals $[\frac{j}{k}T, \frac{j+1}{k}T]$. Hence, $||f_k - f||_{L^2} \leq n\frac{T}{k}(\max_{i,j}|b_i - b_j|)^2 \xrightarrow{k\to\infty} 0$.

Now, let $\vartheta = \sum_{i=0}^{N-1} a_i \chi_{[\frac{i}{N}T, \frac{i+1}{N}T]}$ and assume $\alpha \neq 0$. Then,

$$\begin{aligned}
\langle \vartheta, \vartheta \rangle_R &= \frac{e^{\alpha T} - 1}{\alpha} \sum_{j=0}^{N-1} a_j^2 \int_{\frac{j}{N}T}^{\frac{j+1}{N}T} 1 \mathrm{d}t = \frac{e^{\alpha T} - 1}{\alpha} \sum_{j=0}^{N-1} a_j^2 \frac{T}{N}, \\
\langle \vartheta, \vartheta \rangle_L &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} a_i a_j \int_{\frac{i}{N}T}^{\frac{i+1}{N}T} \int_{\frac{j}{N}T}^{\frac{j+1}{N}T} \mathrm{e}^{\alpha(\tau - t + TH(t-\tau))} \mathrm{d}\tau \mathrm{d}t \\
&= \frac{\left(1 - e^{\alpha \frac{T}{N}}\right)\left(1 - e^{-\alpha \frac{T}{N}}\right)}{\alpha^2} aAa^{\mathrm{T}} + \frac{e^{\alpha T} - 1}{\alpha} \sum_{j=0}^{N-1} a_j^2 \frac{T}{N},
\end{aligned}$$

where $a = (a_0, \ldots, a_{N-1})^{\mathrm{T}}$ and the matrix A is given by

$$A_{ij} = \begin{cases} -e^{\alpha \frac{i-j}{N} T}, & i > j, \\ -e^{\alpha \frac{N+i-j}{N} T}, & i < j, \\ \frac{e^{\alpha \frac{T}{N}} - e^{\alpha T}}{1 - e^{\alpha \frac{T}{N}}}, & i = j. \end{cases}$$

Hence, $\langle \vartheta, \vartheta \rangle_L \leq \langle \vartheta, \vartheta \rangle_R$ is equivalent to

$$\left(1 - e^{\alpha \frac{T}{N}}\right)\left(1 - e^{-\alpha \frac{T}{N}}\right) aAa^T \leq 0 \quad \Leftrightarrow \quad aAa^{\mathrm{T}} \geq 0 \quad \Leftrightarrow \quad aBa^T := \frac{1}{2} a(A + A^{\mathrm{T}}) a^{\mathrm{T}} \geq 0.$$

However, this is true since B is weakly diagonally dominant with non-negative diagonal entries,

$$\sum_{j \neq i} |B_{ij}| = -\frac{1}{2} \left(\sum_{j \neq i} A_{ij} + \sum_{j \neq i} A_{ji} \right) = \frac{1}{2} 2 \sum_{j=1}^{N-1} e^{j \alpha \frac{T}{N}} = \frac{1 - e^{\alpha T}}{1 - e^{\alpha \frac{T}{N}}} - 1 = \frac{e^{\alpha \frac{T}{N}} - e^{\alpha T}}{1 - e^{\alpha \frac{T}{N}}} = B_{ii},$$

and thus by Gershgorin's theorem, B is positive semi-definite. For $\alpha = 0$, the inequality $\langle \vartheta, \vartheta \rangle_L \leq \langle \vartheta, \vartheta \rangle_R$ follows from continuity in $\alpha = 0$ (using de l'Hôpital's rule). □

B.2.3 Negative definite second variation of profit for constant price (proposition 3.2.4)

$$
\begin{aligned}
&\left\langle \frac{\partial^2 \Pi/R_0}{\partial\xi^2}, \vartheta, \vartheta \right\rangle = \left\langle \frac{\partial \left\langle \frac{\partial \Pi/R_0}{\partial\xi}, \vartheta\right\rangle}{\partial\xi}, \vartheta\right\rangle = \frac{N}{T}\int_0^T \left\langle \frac{\partial x(t)}{\partial \xi}, \vartheta\right\rangle [R + \varphi(\Psi - 2x)]\vartheta + x[(\varphi\Psi)' - x\varphi']\vartheta\vartheta + \left[x(\varphi\Psi)'\vartheta - x^2\varphi'\vartheta + \left\langle \frac{\partial x(t)}{\partial\xi},\vartheta\right\rangle (R+\varphi\Psi - 2\varphi x)\right]\vartheta \\
&+(\xi - c)\left[\left\langle \frac{\partial x(t)}{\partial\xi},\vartheta\right\rangle(\varphi\Psi)'\vartheta + x(\varphi\Psi)''\vartheta\vartheta - 2x\left\langle \frac{\partial x(t)}{\partial\xi},\vartheta\right\rangle\varphi'\vartheta - x^2\varphi''\vartheta\vartheta + \left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle(R+\varphi\Psi-2\varphi x) + \left\langle \frac{\partial x(t)}{\partial\xi},\vartheta\right\rangle\left((\varphi\Psi)'\vartheta - 2\varphi' x\vartheta - 2\varphi\left\langle\frac{\partial x(t)}{\partial\xi},\vartheta\right\rangle\right)\right]\mathrm{d}t \\
&= \frac{N}{T}\int_0^T \frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}[R+\varphi(\Psi-2x)]\vartheta + x[(\varphi\Psi)' - x\varphi']\vartheta\vartheta + \left[x(\varphi\Psi)'\vartheta - x^2\varphi'\vartheta + \frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}(R+\varphi\Psi-2\varphi x)\right]\vartheta \\
&-\frac{\Psi}{\Psi'}\left[\frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}(\varphi\Psi)'\vartheta + x(\varphi\Psi)''\vartheta\vartheta - 2x\frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}\varphi'\vartheta - x^2\varphi''\vartheta\vartheta + \left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle(R+\varphi\Psi-2\varphi x)\right. \\
&\left.+\frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}\left((\varphi\Psi)'\vartheta - 2\varphi'x\vartheta - 2\varphi\frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}\right)\right]\mathrm{d}t \\
&= \frac{N}{T}\int_0^T \frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}(R-\varphi\Psi)\vartheta + \Psi\varphi\Psi'\vartheta\vartheta + \left[\Psi\varphi\Psi'\vartheta + \frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}(R-\varphi\Psi)\right]\vartheta \\
&-\frac{\Psi}{\Psi'}\left[\frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}(\varphi\Psi)'\vartheta + \Psi(\varphi\Psi)''\vartheta\vartheta - 2\Psi\frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}\varphi'\vartheta - \Psi^2\varphi''\vartheta\vartheta + \left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle(R-\varphi\Psi)\right. \\
&\left.+\frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}\left(\varphi\Psi'\vartheta - \varphi'\Psi\vartheta - 2\varphi\frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}\right)\right]\mathrm{d}t \\
&= \frac{N}{T}\int_0^T \frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}(R-\varphi\Psi)\vartheta + 2\Psi\varphi\Psi'\vartheta\vartheta \\
&+\frac{\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}\left[R\vartheta - 2\Psi\varphi\vartheta + \frac{\Psi^2}{\Psi'}\varphi'\vartheta + 2\varphi\frac{\Psi^2}{\Psi'}\frac{\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}\right] \\
&-\frac{\Psi^2}{\Psi'}\frac{\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}(\varphi\Psi'\vartheta - \Psi\varphi'\vartheta) - 2\Psi^2\varphi'\vartheta\vartheta - \frac{\Psi^2}{\Psi'}\varphi\Psi''\vartheta\vartheta - \frac{\Psi}{\Psi'}\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle(R-\varphi\Psi)\mathrm{d}t \\
&= \frac{N}{T}\int_0^T \frac{2\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}\left(R\vartheta - 2\varphi\Psi\vartheta + \frac{\Psi^2}{\Psi'}\varphi'\vartheta + \frac{\varphi^2\Psi^2}{\exp(\varphi\Psi t)}\left[\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{1}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]\right) \\
&-2\Psi^2\varphi'\vartheta^2 - \frac{\Psi^2}{\Psi'}\varphi\Psi''\vartheta^2 + 2\Psi\varphi\Psi'\vartheta^2 - \frac{\Psi}{\Psi'}(R-\varphi\Psi)\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t \\
&= \frac{N}{T}\int_0^T \frac{2\Psi\left[\varphi\Psi'\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]}{\exp(\varphi\Psi t)}\left(R\vartheta - 2\varphi\Psi\vartheta + \frac{\Psi^2}{\Psi'}\varphi'\vartheta + \frac{\varphi^2\Psi^2}{\exp(\varphi\Psi t)}\left[\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{1}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right]\right)\mathrm{d}t \\
&+\frac{N}{T}\left(-2\Psi^2\varphi' - \frac{\Psi^2}{\Psi'}\varphi\Psi'' + 2\Psi\varphi\Psi'\right)\int_0^T \vartheta^2\mathrm{d}t - \frac{N}{T}\frac{\Psi}{\Psi'}(R-\varphi\Psi)\int_0^T\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{N}{T} 2\Psi\varphi\Psi' \int_0^T \frac{1}{\exp(\varphi\Psi t)} \int_0^t \vartheta \exp(\varphi\Psi\Theta)\mathrm{d}\Theta \left[\left(R - 2\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\vartheta + \frac{\varphi^2\Psi^2}{\exp(\varphi\Psi t)}\left(\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{1}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)\right]\mathrm{d}t \\
&\quad + \frac{N}{T}\frac{2\Psi\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\left[\int_0^T \frac{1}{\exp(\varphi\Psi t)}\left(R - 2\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\vartheta\mathrm{d}t + \int_0^T \frac{1}{\exp(\varphi\Psi t)}\frac{\varphi^2\Psi^2}{\exp(\varphi\Psi t)}\left(\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{1}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)\mathrm{d}t\right] \\
&\quad + \frac{N}{T}\left(-2\Psi^2\varphi' - \frac{\Psi^2}{\Psi'}\varphi\Psi'' + 2\Psi\varphi\Psi'\right)\int_0^T \vartheta^2\mathrm{d}t - \frac{N}{T}\frac{\Psi}{\Psi'}(R-\varphi\Psi)\int_0^T\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t \\
&= \frac{N}{T}2\Psi\varphi\Psi'\left[\left(R - 2\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \frac{\vartheta}{\exp(\varphi\Psi t)}\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\mathrm{d}t + \varphi^2\Psi^2\int_0^T \frac{1}{\exp(\varphi\Psi t)^2}\left(\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)^2\mathrm{d}t + \frac{\varphi^2\Psi^2}{\exp(\varphi\Psi T)-1}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)\int_0^T \frac{1}{\exp(\varphi\Psi t)^2}\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\mathrm{d}t\right] \\
&\quad + \frac{N}{T}\frac{2\Psi\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\left[\left(R - 2\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t + \varphi^2\Psi^2\int_0^T \frac{1}{\exp(\varphi\Psi t)^2}\int_0^t \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\mathrm{d}t + \frac{\varphi^2\Psi^2}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\int_0^T \frac{1}{\exp(\varphi\Psi t)^2}\mathrm{d}t\right] \\
&\quad + \frac{N}{T}\left(-2\Psi^2\varphi' - \frac{\Psi^2}{\Psi'}\varphi\Psi'' + 2\Psi\varphi\Psi'\right)\int_0^T \vartheta^2\mathrm{d}t - \frac{N}{T}\frac{\Psi}{\Psi'}(R-\varphi\Psi)\int_0^T\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t \\
&= \frac{N}{T}2\Psi\varphi\Psi'\left[\left(R - 2\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\int_\Theta^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t\mathrm{d}\Theta + \frac{\varphi\Psi}{2}\int_0^T \vartheta\exp(\varphi\Psi\varpi)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\exp(-2\varphi\Psi\max\{\Theta,\varpi\})\mathrm{d}\Theta\mathrm{d}\varpi - \frac{\varphi\Psi}{2\exp(\varphi\Psi T)^2}\left(\int_0^T \vartheta\exp(\varphi\Psi\varpi)\mathrm{d}\varpi\right)^2\right. \\
&\quad \left. + \frac{\varphi\Psi}{2(\exp(\varphi\Psi T)-1)}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)\int_0^T \frac{\vartheta}{\exp(\varphi\Psi\Theta)}\mathrm{d}\Theta - \frac{\varphi\Psi\exp(-2\varphi\Psi T)}{2(\exp(\varphi\Psi T)-1)}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)^2\right] + \frac{N}{T}\frac{2\Psi\varphi\Psi'}{\exp(\varphi\Psi T)-1}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\left[\left(R - \frac{3}{2}\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t\right. \\
&\quad \left. - \frac{\varphi\Psi}{2\exp(\varphi\Psi T)^2}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta + \frac{\varphi\Psi}{2(\exp(\varphi\Psi T)-1)}\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\,(1-\exp(-2\varphi\Psi T))\right] + \frac{N}{T}\left(-2\Psi^2\varphi' - \frac{\Psi^2}{\Psi'}\varphi\Psi'' + 2\Psi\varphi\Psi'\right)\int_0^T \vartheta^2\mathrm{d}t - \frac{N}{T}\frac{\Psi}{\Psi'}(R-\varphi\Psi)\int_0^T\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t \\
&= \frac{N}{T}2\Psi\varphi\Psi'\left(R - 2\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\int_\Theta^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t\mathrm{d}\Theta + \frac{N}{T}(\Psi'\varphi^2\Psi^2)\int_0^T \vartheta\exp(\varphi\Psi\varpi)\int_0^T \frac{\vartheta\exp(\varphi\Psi\Theta)}{\exp(\varphi\Psi\max\{\Theta,\varpi\})^2}\mathrm{d}\Theta\mathrm{d}\varpi - \frac{N}{T}\frac{\Psi'\varphi^2\Psi^2}{\exp(\varphi\Psi T)^2}\left(\int_0^T \vartheta\exp(\varphi\Psi\varpi)\mathrm{d}\varpi\right)^2 \\
&\quad + \frac{N}{T}\frac{\Psi'\varphi^2\Psi^2}{(\exp(\varphi\Psi T)-1)}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)\int_0^T \frac{\vartheta}{\exp(\varphi\Psi\Theta)}\mathrm{d}\Theta - \frac{N}{T}\frac{\Psi'\varphi^2\Psi^2}{\exp(\varphi\Psi T)^2(\exp(\varphi\Psi T)-1)}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)^2 + \frac{N}{T}\frac{2\Psi\varphi\Psi'}{\exp(\varphi\Psi T)-1}\left(R - \frac{3}{2}\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\int_0^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t \\
&\quad + \frac{N}{T}\frac{\Psi^2\varphi^2\Psi'}{\exp(\varphi\Psi T)^2 - \exp(\varphi\Psi T)}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)^2 + \frac{N}{T}\left(-2\Psi^2\varphi' - \frac{\Psi^2}{\Psi'}\varphi\Psi'' + 2\Psi\varphi\Psi'\right)\int_0^T \vartheta^2\mathrm{d}t - \frac{N}{T}\frac{\Psi}{\Psi'}(R-\varphi\Psi)\int_0^T\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t \\
&= \frac{N}{T}2\Psi\varphi\Psi'\left(R - 2\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\int_\Theta^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t\mathrm{d}\Theta + \frac{N}{T}(\Psi'\varphi^2\Psi^2)\int_0^T \vartheta\exp(\varphi\Psi\varpi)\int_0^T \frac{\vartheta\exp(\varphi\Psi\Theta)}{\exp(\varphi\Psi\max\{\Theta,\varpi\})^2}\mathrm{d}\Theta\mathrm{d}\varpi \\
&\quad + \frac{N}{T}2\Psi\varphi\frac{\Psi'R - \Psi\varphi\Psi' + \Psi^2\varphi'}{\exp(\varphi\Psi T)-1}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)\left(\int_0^T \frac{\vartheta}{\exp(\varphi\Psi\Theta)}\mathrm{d}\Theta\right) + \frac{N}{T}\left(-2\Psi^2\varphi' - \frac{\Psi^2}{\Psi'}\varphi\Psi'' + 2\Psi\varphi\Psi'\right)\int_0^T \vartheta^2\mathrm{d}t - \frac{N}{T}\frac{\Psi}{\Psi'}(R-\varphi\Psi)\int_0^T\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t \\
&= \frac{N}{T}2\Psi\varphi\Psi'\left(R - 2\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\int_\Theta^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t\mathrm{d}\Theta + \frac{N}{T}(\Psi'\varphi^2\Psi^2)\int_0^T \vartheta\exp(\varphi\Psi\varpi)\left(\int_\varpi^T \frac{\vartheta}{\exp(\varphi\Psi\Theta)}\mathrm{d}\Theta + \int_0^\varpi \frac{\vartheta\exp(\varphi\Psi\Theta)}{\exp(2\varphi\Psi\varpi)}\mathrm{d}\Theta\right)\mathrm{d}\varpi \\
&\quad + \frac{N}{T}2\Psi\varphi\frac{\Psi'R - \Psi\varphi\Psi' + \Psi^2\varphi'}{\exp(\varphi\Psi T)-1}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)\left(\int_0^T \frac{\vartheta}{\exp(\varphi\Psi\Theta)}\mathrm{d}\Theta\right) + \frac{N}{T}\left(-2\Psi^2\varphi' - \frac{\Psi^2}{\Psi'}\varphi\Psi'' + 2\Psi\varphi\Psi'\right)\int_0^T \vartheta^2\mathrm{d}t - \frac{N}{T}\frac{\Psi}{\Psi'}(R-\varphi\Psi)\int_0^T\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t \\
&= \frac{N}{T}2\Psi\varphi\Psi'\left(R - 2\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\int_\Theta^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t\mathrm{d}\Theta + \frac{N}{T}(\Psi'\varphi^2\Psi^2)\int_0^T \vartheta\exp(\varphi\Psi\varpi)\int_\varpi^T \frac{\vartheta}{\exp(\varphi\Psi\Theta)}\mathrm{d}\Theta\mathrm{d}\varpi + \frac{N}{T}(\Psi'\varphi^2\Psi^2)\int_0^T \frac{\vartheta}{\exp(\varphi\Psi\varpi)}\int_0^\varpi \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\mathrm{d}\varpi \\
&\quad + \frac{N}{T}2\Psi\varphi\frac{\Psi'R - \Psi\varphi\Psi' + \Psi^2\varphi'}{\exp(\varphi\Psi T)-1}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)\left(\int_0^T \frac{\vartheta}{\exp(\varphi\Psi\Theta)}\mathrm{d}\Theta\right) + \frac{N}{T}\left(-2\Psi^2\varphi' - \frac{\Psi^2}{\Psi'}\varphi\Psi'' + 2\Psi\varphi\Psi'\right)\int_0^T \vartheta^2\mathrm{d}t - \frac{N}{T}\frac{\Psi}{\Psi'}(R-\varphi\Psi)\int_0^T\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t \\
&= \frac{N}{T}2\Psi\varphi\Psi'\left(R - 2\varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\int_\Theta^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t\mathrm{d}\Theta + \frac{N}{T}(\Psi'\varphi^2\Psi^2)\int_0^T \vartheta\exp(\varphi\Psi\varpi)\int_\varpi^T \frac{\vartheta}{\exp(\varphi\Psi\Theta)}\mathrm{d}\Theta\mathrm{d}\varpi + \frac{N}{T}(\Psi'\varphi^2\Psi^2)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\int_\Theta^T \frac{\vartheta}{\exp(\varphi\Psi\varpi)}\mathrm{d}\varpi\mathrm{d}\Theta \\
&\quad + \frac{N}{T}2\Psi\varphi\frac{\Psi'R - \Psi\varphi\Psi' + \Psi^2\varphi'}{\exp(\varphi\Psi T)-1}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)\left(\int_0^T \frac{\vartheta}{\exp(\varphi\Psi\Theta)}\mathrm{d}\Theta\right) + \frac{N}{T}\left(-2\Psi^2\varphi' - \frac{\Psi^2}{\Psi'}\varphi\Psi'' + 2\Psi\varphi\Psi'\right)\int_0^T \vartheta^2\mathrm{d}t - \frac{N}{T}\frac{\Psi}{\Psi'}(R-\varphi\Psi)\int_0^T\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t \\
&= \frac{N}{T}2\Psi\varphi\Psi'\left(R - \varphi\Psi + \frac{\Psi^2}{\Psi'}\varphi'\right)\int_0^T \vartheta\exp(\varphi\Psi\Theta)\int_\Theta^T \frac{\vartheta}{\exp(\varphi\Psi t)}\mathrm{d}t\mathrm{d}\Theta + \frac{N}{T}2\Psi\varphi\Psi'\left(R - \Psi\varphi + \frac{\Psi^2}{\Psi'}\varphi'\right)\frac{1}{\exp(\varphi\Psi T)-1}\left(\int_0^T \vartheta\exp(\varphi\Psi\Theta)\mathrm{d}\Theta\right)\left(\int_0^T \frac{\vartheta}{\exp(\varphi\Psi\Theta)}\mathrm{d}\Theta\right) \\
&\quad + \frac{N}{T}2\Psi\varphi\Psi'\left(-\frac{\Psi\varphi'}{\varphi\Psi'} - \frac{\Psi\Psi''}{2(\Psi')^2} + 1\right)\int_0^T \vartheta^2\mathrm{d}t - \frac{N}{T}\frac{\Psi}{\Psi'}(R-\varphi\Psi)\int_0^T\left\langle\frac{\partial^2 x(t)}{\partial\xi^2},\vartheta,\vartheta\right\rangle\mathrm{d}t
\end{aligned}
$$

B.2.4 Oligopoly price for example 3.2.3

Proof. Let A_n and $\Lambda = (A_n^{-1})$ be as in example 3.2.3. Furthermore, let us write vectors $(a_i)_{i=1,\dots,n}$ as $\vec{a}$ and diagonal $n \times n$ matrices with diagonal entries $(a_i)_{i=1,\dots,n}$ as $\bar{a}$. If all n goods coexist on the market, then for the steady state the mean dynamic (2.18) can be rewritten as

$$\vec{\Psi} = A_n \vec{x}.$$

According to equation (2.5), the sales $\vec{S}$ and the profit $\vec{\Pi}$ then adopt the form

$$\vec{S} = N\bar{R}\vec{x} = N\bar{R}A_n^{-1}\vec{\Psi} \quad \text{and} \quad \vec{\Pi} = (\bar{\xi} - \bar{c})\vec{S} = (\bar{\xi} - \bar{c})N\bar{R}A_n^{-1}\vec{\Psi}.$$

Letting $(A_n^{-1})_i$ denote the ith row of matrix A_n^{-1}, the Nash equilibrium is found by solving the system of first order optimality conditions

$$0 = \frac{\mathrm{d}\Pi_i}{\mathrm{d}\xi_i} = NR_i \left(A_n^{-1}\right)_i \left[\vec{\Psi} + (\xi_i - c_i)\frac{\mathrm{d}\vec{\Psi}}{\mathrm{d}\xi_i}\right], \quad i = 1, \dots, n,$$

where $\frac{\mathrm{d}\vec{\Psi}}{\mathrm{d}\xi_i} = \left(0, \dots, 0, \frac{1-\Psi_i}{\varphi_i}, 0, \dots, 0\right)^T \frac{\mathrm{d}\varphi_i}{\mathrm{d}\xi_i}$. For the system to be analytically solvable, we chose $\varphi_i = \left(1 + \frac{\xi_i}{\bar{\xi}_i}\right)^{-1}$ and hence $\frac{d\varphi_i}{d\xi_i} = -\frac{\varphi_i^2}{\bar{\xi}_i}$. Then, for $i = 1, \dots, n$ the optimality condition can be equivalently transformed into

$$\Lambda_i \begin{pmatrix} 1 - \frac{1-\Phi_1}{\bar{\xi}_1}(\bar{\xi}_1 + \xi_1) \\ \vdots \\ 1 - \frac{1-\Phi_i}{\bar{\xi}_i}(\bar{\xi}_i - c_i + 2\xi_i) \\ 1 - \frac{1-\Phi_{i+1}}{\bar{\xi}_{i+1}}(\bar{\xi}_{i+1} + \xi_{i+1}) \\ \vdots \end{pmatrix} = 0,$$

or expressed as a sum,

$$\sum_{j=1}^{n} \Lambda_{ij} \left(\Phi_j - \frac{1 - \Phi_j}{\bar{\xi}_j}\xi_j\right) - \Lambda_{ii}\left(\frac{1 - \Phi_i}{\bar{\xi}_i}\xi_i - \frac{(1 - \Phi_i)c_i}{\bar{\xi}_i}\right) = 0, \quad i = 1, \dots, n.$$

In matrix notation this reads

$$\left[\Lambda + \begin{pmatrix} \Lambda_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Lambda_{nn} \end{pmatrix}\right] \begin{pmatrix} \frac{1-\Phi_1}{\bar{\xi}_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1-\Phi_n}{\bar{\xi}_n} \end{pmatrix} \vec{\xi} = \Lambda\vec{\Phi} + \begin{pmatrix} (1 - \Phi_1)\Lambda_{11}\frac{c_1}{\bar{\xi}_1} \\ \vdots \\ (1 - \Phi_n)\Lambda_{nn}\frac{c_n}{\bar{\xi}_n} \end{pmatrix},$$

which directly implies the solution for the optimal prices $\vec{\xi}^*$. $\square$

B.2.5 Proof of lemma 3.2.10

Proof. In chapter 2 we have shown

$$\det \begin{pmatrix} 1 & \alpha_2 & \cdots & \alpha_m \\ \alpha_1 & 1 & & \vdots \\ \vdots & \vdots & \ddots & \alpha_m \\ \alpha_1 & \alpha_2 & \dots & 1 \end{pmatrix} > 0 \tag{8}$$

for any $m > 0$ and $0 < \alpha_i < 1$, $i = 1, \dots, m$, which we will need later.

According to Cramer's rule

$$\Lambda_{ij} = \frac{\det(A_n)_{i \to e_j}}{\det(A_n)},$$

where $(A_n)_{i \to e_j}$ denotes matrix A_n with the ith column replaced by the jth unit vector e_j. For $i \neq j$, using Laplace expansion along the ith column and subsequent column and row interchanges of the (i,j)-minor matrix of A_n, we obtain

$$\det(A_n)_{i \to e_j} = -\det B_0,$$

where (without loss of generality assuming $i < j$)

$$B_0 = \begin{pmatrix}
\Phi_j & \Phi_1 & \cdots & \Phi_{i-1} & \Phi_{i+1} & \cdots & \Phi_{j-1} & \Phi_{j+1} & \cdots & \Phi_n \\
\Phi_j & 1 & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & \Phi_1 & \ddots & \Phi_{i-1} & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & 1 & \Phi_{i+1} & & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \Phi_{i-1} & 1 & & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & \Phi_{i+1} & \ddots & \Phi_{j-1} & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & \vdots & & 1 & \Phi_{j+1} & & \vdots \\
\vdots & \vdots & & \vdots & \vdots & & \Phi_{j-1} & 1 & & \vdots \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \Phi_n \\
\Phi_j & \Phi_1 & \cdots & \Phi_{i-1} & \Phi_{i+1} & \cdots & \Phi_{j-1} & \Phi_{j+1} & \cdots & 1
\end{pmatrix}.$$

In order to show $\det B_0 \leq 0$, let us define B_t for $0 \leq t \leq 1$ by replacing the $(1,1)$-entry in B_0 by $\Phi_j + t(1 - \Phi_j)$. Now, $t \mapsto \det B_t$ is continuous with $\det B_1 > 0$ due to (8). Hence, if we had $\det B_0 < 0$, then by Rolle's theorem there would be some $t \in (0,1)$ with $\det B_t = 0$. However,

$$\det B_t = (\Phi_j + t(1-\Phi_j)) \det \begin{pmatrix} 1 & \Phi_1 & \cdots & \Phi_n \\ \frac{\Phi_j}{\Phi_j + t(1-\Phi_j)} & 1 & & \vdots \\ \vdots & \vdots & \ddots & \Phi_n \\ \frac{\Phi_j}{\Phi_j + t(1-\Phi_j)} & \Phi_1 & \cdots & 1 \end{pmatrix},$$

which due to (8) is larger than zero, since $0 < \frac{\Phi_j}{\Phi_j + t(1-\Phi_j)} < 1$. This contradicts $\det B_t = 0$ so that our initial assumption $\det B_0 < 0$ is wrong. Also, we have $\det A_n > 0$ due to (8) so that finally, $\Lambda_{ij} = \frac{-\det B_0}{\det A_n} \leq 0$. Moreover, from $1 = \sum_{j=1}^{n} \Lambda_{ij}(A_n)_{ji} = \Lambda_{ii} + \sum_{j \neq i} \Lambda_{ij}(A_n)_{ji} \leq \Lambda_{ii}$ we obtain $\Lambda_{ii} \geq 1$. $\square$