Generalized Snaith Splittings

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Abstract

A Segal Γ-space $\mathbb{A}$ gives a homotopy functor $\mathbb{A}(X)$ and a connective homology theory $h_*(X;\mathbb{A}) = \pi_*(\mathbb{A}(X))$. The infinite symmetric product $\mathrm{SP}^\infty(X)$ and the configuration space $C(\mathbb{R}^\infty;X) \simeq Q(X)$ are well-known examples of Segal Γ-spaces; the former giving singular homology $\tilde{H}_*(X;\mathbb{Z})$ and the latter stable homotopy theory as their homotopy groups. Here we are concerned with another important example, the Segal Γ-space K leading to connective KO-theory: $\pi_*K(X) = \widetilde{\mathrm{ko}}(X)$.

Like the first two examples, such functors $\mathbb{A}$ come very often with a filtration $\mathbb{A}_n(X)$ which splits after applying another suitable homotopy functor, perhaps even a Segal Γ-space $\mathbb{B}$; in the first two examples one can take $\mathbb{B} = \mathbb{A}$ and obtain the well-known Dold-Puppe splitting of $\mathrm{SP}^\infty(X)$ resp. the Snaith splitting of $Q(X)$. Our main result is a splitting of $K(X)$ using the functor $\mathbb{B}(X_+) \simeq \Omega^{\infty-1}(\mathrm{MO} \wedge X_+)$ representing unoriented cobordism, namely

$$\mathbb{B}(K(X)_+) \simeq \mathbb{B}(\bigvee_{n=0}^{\infty} K_n(X)/K_{n-1}(X)).$$

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Chapter 1

Introduction

In the 1970's Segal [Se2] introduced the concept of Segal Γ-spaces and proved they give rise to a homotopy category equivalent to the usual homotopy category of connective (i.e. (-1)-connected) spectra. To describe his construction, let Γ be the category of finite pointed sets $\boldsymbol{n} = \{0, 1, \cdots, n\}$ with 0 the base point and morphisms the based functions. A Segal Γ-space is a covariant functor $\mathbb{A} : \Gamma \to \mathrm{Top}_*$ such that $p_n : \mathbb{A}(\boldsymbol{n}) \to \mathbb{A}(\boldsymbol{1})^n$ induced by $\pi_i : \boldsymbol{n} \to \boldsymbol{1}, \pi_i(j) = \delta_{ij}$ and $p_0 : \mathbb{A}(\boldsymbol{0}) \to \mathbb{A}(\boldsymbol{1})^0 = *$ are homotopy equivalences. The space $\mathbb{A}(\boldsymbol{1})$ is called the *underlying space* of $\mathbb{A}$. A Segal Γ-space $\mathbb{A}$ can be extended, by left Kan-extension, along the inclusion $\Gamma \to \mathrm{Top}_*$ to a functor $\mathbb{A} : \mathrm{Top}_* \to \mathrm{Top}_*$, i.e.

$$\mathbb{A}(X) = \int^{\boldsymbol{n} \in \Gamma} \mathbb{A}(\boldsymbol{n}) \times X^n = (\coprod_{\boldsymbol{n} \in \Gamma} \mathbb{A}(\boldsymbol{n}) \times X^n) / \sim .$$

The relation is generated by $(a, \alpha^*(x)) \sim (\alpha_*(a), x)$ for $a \in \mathbb{A}(\boldsymbol{n}), x \in X^m$ and $\alpha : \boldsymbol{n} \to \boldsymbol{m}$ in Γ. We say a Segal Γ-space $\mathbb{A}$ is *very special*, if $\pi_0(\mathbb{A})$ is a group. A very special Segal Γ-space determines an Ω-spectrum by applying it to the sphere spectrum, namely $\mathbb{A}(\underline{S}) = \{\mathbb{A}(S^0), \mathbb{A}(S^1), \mathbb{A}(S^2), \cdots\}$. Let $h_*(-; \mathbb{A})$ be the associated homology theory, i.e. $\tilde{h}_*(X; \mathbb{A}) := \pi_*(\mathbb{A}(\underline{S}) \wedge X) \cong \pi_*(\mathbb{A}(X))$ for $X \in \mathrm{Top}_*$. Segal [Se2] showed that this homology theory is connective and every connective homology theory arises from a Segal Γ-space.

Such functors come very often with a filtration

$$\cdots \subseteq \mathbb{A}_n(X) \subseteq \mathbb{A}_{n+1}(X) \cdots .$$

And in many examples we observe the following splitting phenomenon: there is a homotopy functor $\mathbb{B}$ with a weak equivalence

$$\mathbb{B}\mathbb{A}(X) \simeq \mathbb{B}(\bigvee_{n \geq 1} \mathbb{A}_n(X) / \mathbb{A}_{n-1}(X)). \qquad (1.0.1)$$

We call such functor $\mathbb{B}$ a *splitting functor* of $\mathbb{A}$. It is natural to ask if an arbitrary Segal Γ-space $\mathbb{A}$ admits such a splitting functor $\mathbb{B}$. Our motivation comes from several well-known homotopy functors.

The historically first example is the *infinite symmetric product* $\mathrm{SP}^\infty(X) = \bigcup_n \mathrm{SP}^n(X)$ with $\mathrm{SP}^n(X) = X^n / \Sigma_n$. As a Segal Γ-space $\mathbb{A}$ it arises as $\mathbb{A}(\boldsymbol{n}) = \mathbb{N}^n$. The juxtaposition $X^n \times X^m \to X^{n+m}$ gives a commutative multiplication $\mathrm{SP}^n(X) \times \mathrm{SP}^m(X) \to \mathrm{SP}^{n+m}(X)$ making $\mathrm{SP}^\infty(X)$ into an abelian monoid.

It was first proved by Dold and Thom [DoTh] that for X a CW-complex of finite type, $\mathrm{SP}^\infty(X)$ is a product of Eilenberg-MacLane spaces.

$$\mathrm{SP}^\infty(X) \simeq \prod_i K(\tilde{H}_i(X;\mathbb{Z});i).$$

This makes the functor SP^∞ a representative functor for singular homology. More precisely, they proved that $\pi_* \mathrm{SP}^\infty(X) \cong \tilde{H}_*(X;\mathbb{Z})$ for any connected space X. This is the first example in the history that the homology of a space has been written as the homotopy of a functor applied to that space. Later, Dold and Puppe [DoPu] proved that there is a splitting

$$\mathrm{SP}^\infty(\mathrm{SP}^\infty X) \simeq \mathrm{SP}^\infty(\bigvee_{k\geq 1} \mathrm{SP}^k(X)/\mathrm{SP}^{k-1}(X)). \tag{1.0.2}$$

Since in this example the splitting functor can be taken to be $\mathbb{B} = \mathbb{A}$ itself, we call this a *self-splitting*.

An element in $\mathrm{SP}^\infty(X)$ is a formal sum $\Sigma k_i x_i$ of points in X with multiplicatives $k_i \in \mathbb{N}$; written in this way the identifications in $\mathrm{SP}^\infty(X)$ are $0x = 0 = *$ (base point), $k* = *$ and $kx + k'x = (k+k')x$.

We can thus identify $\mathrm{SP}^\infty(X)$ as

$$\mathrm{SP}^\infty(X) \cong \coprod_n \mathbb{N}^n \times_{\Sigma_n} X^n/(0x = *, k* = *, kx + k'x = (k+k')x).$$

This definition has been generalized in [McC]. For any abelian monoid G with unit e, we define a *generalized symmetric product*,

$$\mathrm{SPG}(X) := \coprod_n G^n \times_{\Sigma_n} X^n/(ex = *, g* = *, gx + g'x = (g+g')x).$$

We studied two examples, namely $\mathbb{A} = \mathrm{SP}\,\mathbb{Z}_n$ associated to $G = \mathbb{Z}_n = \mathbb{Z}/n$, and $\mathbb{A} = \mathrm{SP}\,\mathbb{Z}$ associated to $G = \mathbb{Z}$. They are representative functors for mod-n homology and again for integral homology. That is, $\pi_* \mathrm{SP}\,\mathbb{Z}_n(X) \cong \tilde{H}_*(X;\mathbb{Z}_n)$ and $\pi_* \mathrm{SP}\,\mathbb{Z}(X) \cong \tilde{H}_*(X;\mathbb{Z})$. To describe their splittings, assume $n = p_1^{\epsilon_1}\cdots p_r^{\epsilon_r}$ is the prime decomposition of $n \in \mathbb{N}$. Set

$$N = \begin{cases} p_1 & \text{if } r = 1, \epsilon_1 \geq 1 \\ 1 & \text{else.} \end{cases}$$

We prove in Chapter 3 that there are weak homotopy equivalences

Theorem 3.4.2.

$$\mathrm{SP}\,\mathbb{Z}_N(\mathrm{SP}\,\mathbb{Z}_n(X)) \simeq \mathrm{SP}\,\mathbb{Z}_N(\bigvee_{k\geq 1} D_k \,\mathrm{SP}\,\mathbb{Z}_n(X)) \tag{1.0.3}$$

and

Theorem 3.5.1.

$$\mathrm{SP}\,\mathbb{Z}(\mathrm{SP}\,\mathbb{Z}(X)) \simeq \mathrm{SP}\,\mathbb{Z}(\bigvee_{k\geq 1} D_k \,\mathrm{SP}\,\mathbb{Z}(X)), \tag{1.0.4}$$

where $D_k\mathbb{A}(X)$ stands always for the filtration quotients $\mathbb{A}_k(X)/\mathbb{A}_{k-1}(X)$. In the last two examples the filtration is given by the sum of the coefficients. Note that in case $\mathbb{A} = \mathrm{SP}\,\mathbb{Z}_n$, it is not a self-splitting as in the case $\mathbb{A} = \mathrm{SP}^\infty$ or $\mathbb{A} = \mathrm{SP}\,\mathbb{Z}$.

The next example is the functor $C(X) = C(\mathbb{R}^\infty; X)$, the configuration space of $\mathbb{R}^\infty$ with labels in X defined as

$$C(\mathbb{R}^\infty; X) := (\coprod_{n\geq 1} \tilde{C}^n(\mathbb{R}^\infty) \times_{\Sigma_n} X^n)/\sim .$$

Here $\tilde{C}^n(\mathbb{R}^\infty)$ is the space of ordered configurations of n distinct points in $\mathbb{R}^\infty$ and the equivalence relation $\sim$ is generated by $(z_1, \cdots, z_n; x_1, \ldots, x_n) \sim (z_1, \ldots, \widehat{z_i}, \cdots, z_n; x_1, \ldots, \widehat{x_i}, \ldots, x_n)$ if $x_i = x_0$. There is an obvious filtration by the length n of a configuration. It is well-known that $C(X) \simeq \Omega^\infty\Sigma^\infty X = Q(X)$, i.e. $\pi_* C(X) = \pi_*^{\text{stab}}(X)$ is the stable homotopy theory of X.

The Snaith splitting [Sn] asserts

$$\Sigma^\infty C(\mathbb{R}^\infty; X) \simeq \Sigma^\infty \bigvee_{k\geq 1} D_k(\mathbb{R}^\infty; X), \tag{1.0.5}$$

or equivalently

$$QQX \simeq Q(\bigvee_{k\geq 1} D_k(\mathbb{R}^\infty; X)). \tag{1.0.6}$$

Before we describe our main result we digress to describe related splitting results. Denote by $C(M, M_0; X)$ the configuration space on a manifold pair (M, M_0) with labels in X. Here $M_0 \subset M$ is a codimension-zero submanifold of M; M_0 can be empty. Let $\tilde{C}^n(M)$ denote the space of ordered configurations $(z_1, \cdots, z_n)$ in M. For a space X with a base point x_0 we denote by

$$C(M, M_0; X) := (\coprod_{n\geq 1} \tilde{C}^n(M) \times_{\Sigma_n} X^n)/\sim$$

the space of *configurations in M modulo M_0 with labels in X*; here Σ_n is the symmetric group of rank n with the obvious permutation action on $\tilde{C}^n(M)$ and on X^n; and the equivalence relation $\sim$ is generated by the cancelations $(z_1, \cdots, z_n; x_1, \ldots, x_n) \sim (z_1, \ldots, \widehat{z_i}, \cdots, z_n; x_1, \ldots, \widehat{x_i}, \ldots, x_n)$ if $z_i \in M_0$ or if $x_i = x_0$.

Ever since the work of James, Segal, Cohen, Bödigheimer etc., it has become clear that configuration spaces can be used to model mapping spaces. A simple but useful construction given in terms of configurations with labels and due to Milgram, May and Segal gave very concrete models for iterated loop spaces. This model was later extended by Cohen, Bödigheimer and McDuff to various other mapping spaces by choosing configurations to lie not in disks but other parallelizable manifolds.

a) The May-Milgram model [Mi] says that the configurations on $\mathbb{R}^m$ with labels in a based connected space X is homotopy equivalent to the m-fold loop space of an m-fold suspension on X, that is

$C(\mathbb{R}^m; X) \xrightarrow{\simeq} \Omega^m\Sigma^m X$.

b) In the limit case this gives $C(\mathbb{R}^\infty; X) \simeq \Omega^\infty\Sigma^\infty X$, used above.

c) Cohen [Co] and Bödigheimer [Bö1] also studied the case $M = S^1, M_0 = \emptyset$ and proved $C(S^1; X) \simeq \Lambda\Sigma X$, the free loop space of a suspension of a connected space X.

d) Historically the first model is the James model $J(X)$ in [Ja], the free non-commutative topological monoid generated by X modulo its base point x_0. And $J(X) \simeq C(\mathbb{R}; X) \simeq \Omega\Sigma X$.

e) All these are special cases of [Bö1], where the based mapping spaces $\text{Map}(K, K_0; \Sigma^m X)$, here $K_0 \subset K \subset \mathbb{R}^m$ are finite polyhedra in $\mathbb{R}^m$; and X or the pair (K, K_0) must be connected. If $W \subset \mathbb{R}^m$ is open with $W \supset M \supset M_0$ such that $(M, M_0) \simeq (K, K_0)$, then there is a homotopy equivalence

$$C(M, M_0; X) \longrightarrow \text{Map}(W\backslash M_0, W\backslash M; \Sigma^m X). \tag{1.0.7}$$

One of the most important applications of configuration space models was the stable splitting of mapping spaces into bouquets of simpler spaces. These simpler spaces are the filtration quotients

$$D_k(M, M_0; X) = C_k(M, M_0; X)/C_{k-1}(M, M_0; X)$$

of the filtration

$$C_k(M, M_0; X) := (\coprod_{n=1}^{k} \tilde{C}^n(M) \times_{\Sigma_n} X^n)/\sim .$$

Note that the spaces $D_k(M; \mathbb{S}^q)$ are Thom spaces of the vector bundles $\tilde{C}^k(M) \times_{\Sigma_k} \mathbb{R}^{qk} \longrightarrow C^k(M)$. Bödigheimer [Bö1], [BöMa] proved that

$$\Sigma^\infty \operatorname{Map}(K, K_0; X) \simeq \Sigma^\infty \bigvee_{k \geq 1} D_k(M, M_0; X). \tag{1.0.8}$$

The first result of this kind was James unstable splitting [Ja] of $\Omega\Sigma X$. He proved in [Ja] that there is an unstable splitting of the James model, known as the James splitting,

$$\Sigma J(X) \simeq \Sigma\Omega\Sigma X \simeq \Sigma \bigvee_{k=1}^{\infty} X^{\wedge k}. \tag{1.0.9}$$

We note here that the free loop space $\Lambda\Sigma X$ can be split with just two suspensions, see [BöWa].

We return after this digression to our next example, which stands in the center of this work. The space $K(X)$, introduced by Segal [Se3], is obtained from the Segal Γ-space K of finite-dimensional Grassmannians in $\mathbb{R}^\infty$. More precisely,

$$K(\mathbf{1}) = \textstyle\coprod_k \mathrm{Gr}_k(\mathbb{R}^\infty),$$

$$K(\boldsymbol{n}) = \{(V_1, \cdots, V_n) \in K(\mathbf{1})^{\times n} | V_i \perp V_j, \text{ if } i \neq j\}.$$

So the extension $K(X)$ has the form

$$K(X) = (\coprod_{n \geq 0} K(\boldsymbol{n}) \times_{\Sigma_n} X^n)/\sim,$$

where

$$(V_1, \cdots, V_i, \cdots, V_n; x_1, \cdots, x_i, \cdots, x_n) \sim (V_1, \cdots, \hat{V_i}, \cdots, V_n; x_1, \cdots, \hat{x_i}, \cdots, x_n), \text{ if } x_i = x_0;$$

and

$$(\cdots, V_i, \cdots, V_j, \cdots; \cdots, x_i, \cdots, x_j, \cdots) \sim (\cdots, V_i \oplus V_j, \cdots, \hat{V_j}, \cdots; \cdots, x_i, \cdots, \hat{x_j}, \cdots), \text{ if } x_i = x_j.$$

Its filtration is given by the sum of dimensions of the vector spaces

$$K_n(X) = \left\{ \ \Sigma_i V_i x_i \in K(X) \ \Big| \ \Sigma \dim V_i \leq n \ \right\}.$$

Segal proved that $K(X)$ is a representing space for the connective real K-homology theory, namely $\pi_* K(X) \cong \widetilde{\mathrm{ko}}(X)$.

Our goal is to construct a splitting functor $\mathbb{B}$ for $K(X)$, that is, to find a homotopy functor $\mathbb{B}$ and a weak equivalence

$$\mathbb{B}(K(X)_+) \simeq \mathbb{B}(\bigvee_n K_n(X)/K_{n-1}(X)). \tag{1.0.10}$$

We prove in Chapter 5 that there exists such a $\mathbb{B}$ which represents the infinite loop space of the Thom spectrum MO for the universal real vector bundles, i.e. it represents unoriented cobordism.

The main idea to search for the functor $\mathbb{B}$ is implicit in the work of Randal-Williams [RW]. He defined in case $Y = S^0$ a topology on the set of equivalence classes:

$$B_d(M;Y) := (\coprod_F \mathrm{Emb}(F, M) \times_{\mathrm{Diff}(F)} \mathrm{Map}(F, Y))/\sim$$

where F varies over smooth d-dimensional manifolds without boundary (not necessarily compact or connected). The equivalence relation cancels a component of a manifold F if the labeling function is trivial on that component. He proved that this space is weakly equivalent to the space of sections of a certain fiber bundle. The space $B_d(M;Y)$ is a kind of configuration space of d-dimensional manifolds $\epsilon : F \hookrightarrow M$ in M with label functions $\varphi : F \to Y$.

Define $B^c_d(M \times \mathbb{R}^{d+1}; Y_+)$ to be the subspace of $B_d(M \times \mathbb{R}^{d+1}; Y_+)$ where $\epsilon(F) \subset M \times \mathbb{R}^{d+1}$ projects into a compact subspace of M. We use his idea and define a topology on $B_d(M;Y)$ for all Y. Then we apply Gromov's h-principle and prove that there is a weak homotopy equivalence

$$B^c_d(M \times \mathbb{R}^{d+1}; Y_+) \to \mathrm{Sect}^c(E_d(M \times \mathbb{R}^{d+1}; Y_+), M). \tag{1.0.11}$$

Here $\mathrm{Sect}^c(E_d(M \times \mathbb{R}^{d+1}; Y_+))$ is the space of compactly supported sections of the bundle

$$E_d(M;Y_+) := V_n(TM) \times_{O(n)} (\mathrm{Th}(U^\perp_{d,n}) \wedge Y_+) \xrightarrow{\pi} M. \tag{1.0.12}$$

$V_n(TM)$ is the frame bundle of M, $U^\perp_{d,n} := \{(V, v) \in Gr_d(\mathbb{R}^n) \times \mathbb{R}^n | V \perp v\}$, and $\mathrm{Th}(U^\perp_{d,n})$ is the corresponding Thom space. As n varies, all the Thom spaces $\mathrm{Th}(U^\perp_{d,n})$ form a spectrum, denoted by MTO_d.

We have for $M = \mathbb{R}^{n-1}$ a weak equivalence

$$\gamma : B^c_d(\mathbb{R}^{n-1} \times \mathbb{R}^{d+1}; Y_+) \simeq \Omega^{n-1}(\mathrm{Th}(U^\perp_{d,n}) \wedge Y_+). \tag{1.0.13}$$

For the limit case $n \to \infty$, we obtain a weak equivalence

$$B^c_d(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+) \simeq \Omega^{\infty-1}(\mathrm{MTO}_d \wedge Y_+) \tag{1.0.14}$$

which is the infinite loop space of the Thom spectrum $\mathrm{MTO}_d \wedge Y_+$.

By crossing a manifold with $\mathbb{R}^1$, we define a map

$$\begin{array}{ccc} B^c_d(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+) & \to & B^c_{d+1}(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+2}; Y_+) \\ \mathbb{R}^{\infty+d} \overset{\epsilon}{\hookleftarrow} F^d \xrightarrow{\varphi} Y_+ & \mapsto & \mathbb{R}^{\infty+d+1} \hookleftarrow F^d \times \mathbb{R}^1 \xrightarrow{\varphi'} Y_+. \end{array} \tag{1.0.15}$$

where $\varphi' : F \times \mathbb{R}^1 \to Y_+, (f, t) \mapsto \varphi(f)$.

Define

$$\mathbb{B}(Y_+) := \operatorname*{colim}_d B^c_d(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+).$$

This is the splitting functor we are looking for. Its homotopy type is

$$\mathbb{B}(Y_+) \simeq \mathrm{Sect}(Y_+) \simeq \Omega^{\infty-1} MO \wedge Y_+.$$

The main result of this thesis is the following splitting.

Theorem 5.3.6.

$$\mathbb{B}(K(X)_+) \simeq \mathbb{B}(\bigvee_{n=0}^{\infty} K_n(X)/K_{n-1}(X)). \tag{1.0.16}$$

We should remark that a splitting functor is in general not unique, as one can see from the example $\mathbb{A} = C(\mathbb{R}^\infty; -)$ which is split by $\mathbb{B}_1 = \mathbb{A}$ itself and by $\mathbb{B}_2 = \mathrm{SP}^\infty$. Whether there is and how to find for a given $\mathbb{A}$ the "best" (i.e. universal) splitting functor $\mathbb{B}$ is a difficult question.

Looking back at our proof we notice, that the functor $\mathbb{B}$ can also be used to split other Segal Γ-spaces $\mathbb{A}$, if some mild hypothesis is satisfied: the space $\mathbb{A}(\mathbf{1})$ is assumed to be disjoint unions of finite-dimensional manifolds and certain subspaces of $\mathbb{A}(\boldsymbol{n})$ are finite-dimensional manifolds. Under these conditions there is a weak equivalence

Theorem 6.4.1.

$$\mathbb{B}(\mathbb{A}(X)_+) \simeq \mathbb{B}(\bigvee_{n=0}^{\infty} \mathbb{A}_n(X)/\mathbb{A}_{n-1}(X)). \tag{1.0.17}$$

The plan of this paper is as follows.

In Chapter 2 the Segal Γ-spaces are defined and discussed.

Chapter 3 and 4 concentrate individually on the separate cases of Segal Γ-spaces: infinite symmetric product $\mathrm{SP}^\infty(X)$, configuration space $C(\mathbb{R}^\infty; X)$. Their homotopy types are well understood and we introduce the well-known Dold-Thom splitting and Snaith splitting. The work we present in these two chapters is a mixture of previously known results, new results and also previously known results in a new framework.

In Chapter 5 we study the example $K(X)$. This is the main part of the thesis. We find the splitting functor $\mathbb{B}$ for K and prove the main result Theorem 5.3.6.

Chapter 6 then deals with the splitting of an arbitrary Segal Γ-space $\mathbb{A}$. The proof is parallel to the proof in Chapter 5.

In Appendix A we explain the h-principle of Gromov; this is crucial for the proof of Theorem 5.3.6.

In Appendix B we outline the homotopy calculus of functors according to Goodwillie.

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Chapter 2

Segal Γ-Spaces

This chapter is concerned with G. Segal's notion of Γ-space [Se2], in which he showed that the homotopy category is equivalent to the usual homotopy category of connective spectra and therefore gives rise to a connective homology theory. Furthermore, every connective homology theory can be represented by a Segal Γ-space.

Throughout this chapter Top_* means the category of based compactly generated Hausdorff spaces and based maps, and let $\mathcal{A}b$ be the category of abelian topological monoids.

2.1 Segal Γ-Spaces - a Covariant Version

Let Γ denote the category of finite pointed sets represented by $\boldsymbol{n} = \{0, 1, \cdots, n\}$ with 0 as base point and the morphisms are pointed maps. It is isomorphic to the opposite of that category which was also called Γ in [Se2].

For each i, let π_i be the morphism $\pi_i : \boldsymbol{n} \to \mathbf{1}, \pi_i(j) = \delta_{ij}$ in Γ. And let $p_n : \mathbb{A}(\boldsymbol{n}) \to \mathbb{A}(\mathbf{1})^n$ be the map whose i-th component is induced by π_i.

Definition 2.1.1. A *Segal* Γ*-space* is a covariant functor $\mathbb{A} : \Gamma \to \mathrm{Top}_*$ such that

(1) $\mathbb{A}(\mathbf{0}) \simeq *$,

(2) $p_n : \mathbb{A}(\boldsymbol{n}) \to \mathbb{A}(\mathbf{1})^n$ is a homotopy equivalence.

A morphism of Segal Γ-spaces is a natural transformation of functors $F : \mathbb{A} \to \mathbb{A}'$, such that the diagram is homotopy commutative,

$$\begin{array}{ccc} \mathbb{A}(\boldsymbol{n}) & \xrightarrow{F(\boldsymbol{n})} & \mathbb{A}'(\boldsymbol{n}) \\ \downarrow & & \downarrow \\ \mathbb{A}(\mathbf{1})^n & \xrightarrow{\Pi F(\mathbf{1})} & \mathbb{A}'(\mathbf{1})^n \end{array}$$

We denote the category of Segal Γ-spaces by $\Gamma\,\mathrm{Top}_*$. Analogously, a contravariant functor $\mathbb{A}^\times : \Gamma^{\mathrm{op}} \to$

Top_* which satisfies (1) and (2) will be called a Γ^{op}-*space*, and they are objects in a topological category $\Gamma^{\mathrm{op}}\,\mathrm{Top}_*$.

For a Segal Γ-space $\mathbb{A}$, $\pi_0\mathbb{A}(\mathbf{1})$ is an abelian monoid with multiplication

$$\pi_0\mathbb{A}(\mathbf{1}) \times \pi_0\mathbb{A}(\mathbf{1}) \xrightarrow{(\pi_{1*}\times\pi_{2*})^{-1}} \pi_0\mathbb{A}(\mathbf{2}) \xrightarrow{\mu_*} \pi_0\mathbb{A}(\mathbf{1}),$$

where $\mu : \mathbf{2} \to \mathbf{1}$ is the fold map defined by $\mu(1) = \mu(2) = 1$. It also implies that $\mathbb{A}(\mathbf{1})$ is an H-space.

Example 2.1.2. Fix an abelian topological monoid A, written additively with neutral element 0. It determines a Segal Γ-space $\mathbb{A}$ be setting $\mathbb{A}(\boldsymbol{n}) := A^n$, and by setting for $\alpha : \boldsymbol{n} \to \boldsymbol{m}$ in Γ

$$\mathbb{A}(\alpha) = \alpha_* : A^n \to A^m, (a_1, \cdots, a_n) \mapsto (\Sigma_{j\in\alpha^{-1}(1)}a_j, \cdots, \Sigma_{j\in\alpha^{-1}(i)}a_j, \cdots, \Sigma_{j\in\alpha^{-1}(n)}a_j)$$

and if $\alpha^{-1}(k) = \emptyset$, we set $a_\emptyset = 0$. Note that it also works for the discrete monoid, for example the natural numbers $\mathbb{N}$, $\mathbb{N}(\boldsymbol{n}) := \mathbb{N}^n$.

Example 2.1.3. One interesting example is the configuration space. Define

$$C(\mathbf{1}) := \coprod_{n\geq 0} \tilde{C}^n(\mathbb{R}^\infty)$$

to be the disjoint union of ordered configuration spaces on $\mathbb{R}^\infty$. Define

$$C(\boldsymbol{k}) := \left\{ (\xi_1, \cdots, \xi_k) \in C(\mathbf{1})^k \;\middle|\; \xi_i \cap \xi_j = \emptyset \text{ in } \mathbb{R}^\infty \text{ for } i \neq j \right\}.$$

We shall prove in section 4.4 that this is a Segal Γ-space. Another example is the Grassmanniann of finite-dimensional real vector spaces. More precisely,

$$K(\mathbf{1}) = \textstyle\coprod_k Gr_k(\mathbb{R}^\infty),$$
$$\cdots$$
$$K(\boldsymbol{n}) = \{(V_1, \cdots, V_n) \in K(\mathbf{1})^{\times n} | V_1, \cdots, V_n \text{ pairwise orthogonal}\}.$$

We shall prove in section 5.1 that this also defines a Segal Γ-space.

Recall the *simplicial category* Δ whose objects are finite ordered sets $[m] = \{0, 1, \cdots, m\}$ and whose morphisms are non-decreasing maps. Note that the category Γ is larger than the simplicial category, because it has more morphisms. A simplicial space is a contravariant functor $\Delta \to \mathrm{Top}_*$. There is a contravariant functor $\Delta \to \Gamma$ taking $[n]$ to the corresponding unordered set $\boldsymbol{n}$ with base point 0 and an order preserving morphism $f : [m] \to [n]$ to $\theta : \boldsymbol{n} \to \boldsymbol{m}$ by

$$\theta(i) = \begin{cases} j, & f(j-1) < i \leq f(j); \\ 0, & \text{otherwise.} \end{cases}$$

Therefore a Segal Γ-space can be regarded as a simplicial space and the realization of a Segal Γ-space means the realization of the simplicial space it defines.

There is a natural extension of a Segal Γ-space $\mathbb{A} : \Gamma \to \mathrm{Top}_*$ to a functor which we also denote by $\mathbb{A} : \mathrm{Top}_* \to \mathrm{Top}_*$. Recall the coend construction: if $\mathbb{A}$ is a Segal Γ-space, X is a based topological space with x_0 as base point, i.e. a Γ^{op}-space, consider the contravariant functor $p_X : \Gamma \to \mathrm{Top}_*$, $\boldsymbol{n} \mapsto X^n$, a map $\alpha : \boldsymbol{n} \to \boldsymbol{m}$ in Γ induces $\alpha^*(x_1, \cdots, x_m) = (x_{\alpha(1)}, \cdots, x_{\alpha(n)})$, where all $x_{\alpha(i)}$ with $\alpha(i) = 0$ are the base points x_0. We let $\mathbb{A}(X)$ denote the quotient space

$$\mathbb{A}(X) := \coprod_{\boldsymbol{n} \in \Gamma} \mathbb{A}(\boldsymbol{n}) \times X^n / (a, \alpha^* x) \sim (\alpha_* a, x) = \int^{\boldsymbol{n} \in \Gamma} \mathbb{A}(\boldsymbol{n}) \times X^n.$$

where $a \in \mathbb{A}(\boldsymbol{n}), x \in X^m$. Note that the equivalence relation $\sim$ includes the action of the symmetric group Σ_n on $\mathbb{A}(\boldsymbol{n}) \times X^n$. In case $p_n : \mathbb{A}(\boldsymbol{n}) \to \mathbb{A}(\boldsymbol{1})^n$ is an inclusion one can view an element of $\mathbb{A}(X)$ as a formal sum $[a; x] = \Sigma_{i=1}^n a_i x_i$ with $p_n(a) = (a_1, \cdots, a_n)$. We do not distinguish the extention notationally from the original Segal Γ-space. The extended functor preserves homotopy equivalences in Top_*. Obviously, if S is a discrete finite space, this new definition of $\mathbb{A}(S)$ agrees with the old one.

Example 2.1.4. In the case $\mathbb{A} = \mathbb{N}$, we have $\mathbb{A}(\boldsymbol{n}) = \mathbb{N}^n$, and thus $\mathbb{A}(X) = \mathrm{SP}^\infty(X)$, the infinite symmetric product. In the case $\mathbb{A} = C$, it is $C(X) = C(\mathbb{R}^\infty; X)$, the labeled configuration space of $\mathbb{R}^\infty$. In the case $\mathbb{A} = K$, it is the space $K(X)$. This was mentioned in the introduction, which will be given more details in Chapter 5.

2.2 Segal Γ-Spaces Arising From Categories

Segal [Se2] demonstrated that the Segal Γ-spaces can be obtained naturally from categories with composition laws. In this section we are going to reformulate it in a covariant version. As in [Se1], "category" means that the set of objects and the set of morphisms have topologies for which the structural maps are continuous.

First we recall the *nerve* $N\mathscr{A}$ of a small category $\mathscr{A}$ [Se1]. It is a simplicial set with n-simplex $N\mathscr{A}_n = \hom_{cat}([n], \mathscr{A})$, set of functors from $[n]$ to $\mathscr{A}$. That is, an n-simplex is a string

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n$$

of composable arrows of length n in $\mathscr{A}$. The geometric realization of this simplicial set, denoted by $B\mathscr{A} := |N\mathscr{A}| = |\mathscr{A}|$ is called the *classifying space* of $\mathscr{A}$.

Here are some elementary properties:

(i) $|\mathscr{A} \times \mathscr{A}'| \simeq |\mathscr{A}| \times |\mathscr{A}'|$.

(ii) Equivalence of categories $\mathscr{A} \to \mathscr{A}'$ induces a homotopy equivalence $|\mathscr{A}| \to |\mathscr{A}'|$.

Example 2.2.1. Fix an a topological group G, one can associate a topological category $\mathscr{G}$ with one object $*$ and one morphism $g : * \to *$, for each $g \in G$, then $\mathrm{ob}(\mathscr{G}) = *, \mathrm{mor}(\mathscr{G}) = G, N\mathscr{G}_k = G^k$, thus

$$B\mathscr{G} = |N\mathscr{G}| = (\coprod_k \Delta^k \times G^k)/ \sim = BG,$$

which is the Milnor construction.

Definition 2.2.2. ([Se2], Definition 2.1.) A Γ-*category* is a covariant functor $\mathscr{A} : \Gamma \to$ categories, which satisfies

(i) $\mathscr{A}(\mathrm{pt})$ is equivalent to the category with one object and one morphism;

(ii) for any n, the functor $p_n : \mathscr{A}(\boldsymbol{n}) \to \mathscr{A}(\boldsymbol{1})^n$ induced by the morphisms $\pi_i : \boldsymbol{n} \to \boldsymbol{1}$ defined in Definition 2.1.1. is an equivalence of the categories.

Proposition 2.2.3. ([Se2], Corollary 2.2.) If $\mathscr{A}$ is a Γ-category, then $|\mathscr{A}|$ is a Segal Γ-space. $\square$

Here $|\mathscr{A}|$ means the functor $S \mapsto |\mathscr{A}(S)|$.

Let $\mathscr{C}$ be a category in which sums exist. For each object $S \in \Gamma$, associate a category $\mathscr{P}(S)$ whose objects are pointed subsets of S and inclusions as morphisms, an object of the category $\mathscr{C}(S)$ is a functor $\mathscr{P}(S) \to \mathscr{C}$ which takes wedge product of sets to sums in $\mathscr{C}$. Morphisms are isomorphisms of functors. Now consider the object of $\mathscr{C}(\mathbf{2})$, it is a diagram $A_1 \to A_{12} \leftarrow A_2$ with the universal property that regarding A_{12} as a coproduct of A_1 and A_2, i.e. $A_{12} := A_1 + A_2$. The morphisms are defined that the morphisms $S \to T \in \Gamma$ correspond to functors $\mathscr{P}(S) \to \mathscr{P}(T)$ which preserves wedge products. Then $S \mapsto \mathscr{C}(S)$ is a covariant functor from Γ to categories. And since the forgetful functor

$$\begin{array}{ccc} \mathscr{C}(\mathbf{2}) & \longrightarrow & \mathscr{C} \times \mathscr{C} \\ (A_1 \to A_{12} \leftarrow A_2) & \mapsto & (A_1, A_2) \end{array}$$

is an equivalence of categories, we obtain a Γ-category as defined.

Example 2.2.4. ([Se2], page 299) Consider the category $\mathscr{S}$ of pointed finite sets under wedge product $\vee$ as the sum operation, choose a model for $\mathscr{S}$ in which there is one object $\boldsymbol{n}$ for each natural number. For example, $\mathscr{S}(\mathbf{1})$ is a category with objects $F : \mathscr{P}(\mathbf{1}) \to \mathscr{S}, F(\{0\}) = \mathbf{0}, F(\{0,1\}) = \boldsymbol{n}$ and morphisms are pointed automorphism of $\boldsymbol{n}$, namely Σ_n. Segal showes that

$$|\mathscr{S}(\mathbf{1})| = \coprod_{n \geq 0} B\Sigma_n.$$

We denote this Segal Γ-space $|\mathscr{S}|$ by $B\Sigma$.

One can generalize $B\Sigma$ as follows: let F be a contravariant functor from the category of finite pointed sets with inclusions as morphisms to the category Top_*. Let $\mathscr{S}_F$ be the topological category whose objects are pairs (S, x) with S a finite set and $x \in F(S)$ and whose morphism $(S, x) \to (T, y)$ are injections $\theta : S \to T$ such that $\theta^*(y) = x$. One can then construct $\mathscr{S}_F(\boldsymbol{n})$ in a similar way as $\mathscr{S}(\boldsymbol{n})$. If for each S, T the map $F(S \vee T) \to F(S) \times F(T)$ is a homotopy equivalence, then $\boldsymbol{n} \mapsto |\mathscr{S}(\boldsymbol{n})|$ is a Segal Γ-space.

Example 2.2.5. ([Se2], Chapter 2) Define $F : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Top}_*$, $\boldsymbol{n} \mapsto X^n$ for a fixed space X. Then the category $\mathscr{S}_F(\mathbf{1})$ has objects to be functors $\mathscr{P}(\mathbf{1}) \to \mathscr{S}_F, \mathbf{1} \mapsto (S, x)$ with $x = (x_1, \cdots, x_n) \in X^n$ and morphisms are automorphisms $\sigma_* : (\boldsymbol{n}, (x_1, \cdots, x_n)) \to (\boldsymbol{n}, (x_{\sigma_{(1)}}, \cdots, x_{\sigma_{(n)}}))$ for $\sigma \in \Sigma_n$. We call the resulting Segal Γ-space $B\Sigma_X$, and especially

$$B\Sigma_X(\mathbf{1}) = (\coprod_{n \geq 0} E\Sigma_n \times X^n)/\Sigma_n,$$

which is the labeled configuration space of $\mathbb{R}^\infty$, i.e. $C(\mathbb{R}^\infty; X)$.

2.3 Segal Γ-Spaces and Spectra

In this section we show the equivalence between the category of Segal Γ-spaces and the category of connective spectra. To associate a spectrum to a Segal Γ-space, there are two ways: (i) the classifying space construction; (ii) applying the extended Segal Γ-space to the spheres.

For a Segal Γ-space $\mathbb{A}$, we know from last section that $\pi_0 \mathbb{A}(\mathbf{1})$ is an abelian monoid.

Definition 2.3.1. A Segal Γ-space $\mathbb{A}$ is *very special* (or sometimes called *group complete*), if $\pi_0\mathbb{A}(\mathbf{1})$ is an abelian group.

Now we digress to the general group completion theory associated to a topological monoid M. Assume M is strictly associative and has a unit. Consider its classifying space BM. It is a based space, and the adjoint of the inclusion $\Sigma M \hookrightarrow BM$ is a map $i : M \longrightarrow \Omega BM$ which is a weak homotopy equivalence if the monoid of connected components $\pi_0(M)$ is a group. Quillen's group completion theorem ([May], [McD], [Ka1]) indicates the relationship between M and ΩBM generally.

A map $f : M_1 \to M_2$ between two topological monoids is a *group completion* if $\pi_0(f) : \pi_0 M_1 \to \pi_0 M_2$ is an algebraic group completion (i.e. $\pi_0(f)$ is universal with respect to morphisms of monoids from $\pi_0(M_2)$ to groups), and if $f_* : H_*(M_1) \to H_*(M_2)$ is a localization of the ring $H_*(M_1)$ at its multiplicative submonoid $\pi_0(M_1)$ for every commutative coefficient ring R.

Theorem 2.3.2. (*Quillen's group completion Theorem*, [May]), the natural inclusion $i : M \to \Omega BM$ is a group completion whenever M is homotopy commutative. □

Consider the example 1.2.4 in the last section. The Barratt-Priddy-Quillen theorem tells us that $B\Sigma$ group completes to QS^0.

A *spectrum* consists of a collection of pointed spaces $\underline{X} = \{X_n\}_{n\geq 0}$ together with maps $\sigma_n : \Sigma X_n \to X_{n+1}$. If all σ_n are weak equivalences, it is called a *suspension spectrum*. If the adjoint maps $\sigma_n^\sharp : X_n \to \Omega X_{n+1}$ are weak equivalences, it is called an Ω*-spectrum*. A map of spectra $\underline{X} \to \underline{Y}$ consists of maps $X_n \to Y_n$ strictly commuting with the suspension maps. We denote the category of spectra by Sp. The homotopy groups of a spectrum $\underline{X}$ are defined as

$$\pi_n \underline{X} = \operatorname{colim}_i \pi_{n+i} X_i.$$

A map of spectra is a *stable equivalence* if it induces isomorphisms on all homotopy groups.

Return to our case when $M = \mathbb{A}(\mathbf{1})$ with the discrete topology. Segal ([Se1]) showed that a Segal Γ-space $\mathbb{A}$ gives rise to a spectrum. For $X, Y \in \mathrm{Top}_*$, denote $\mathrm{Map}_*(X, Y)$ space of based maps from X into Y, the continuous map

$$\mathrm{Map}_*(X, Y) \to \mathrm{Map}_*(\mathbb{A}(X), \mathbb{A}(Y)), f \mapsto \mathbb{A}(f)$$

preserves base points. Hence there are natural maps, called *assembly maps*

$$X \wedge \mathbb{A}(Y) \longrightarrow \mathbb{A}(X \wedge Y)$$

which are the adjunctions of the composition

$$X \xrightarrow{\ l\ } \mathrm{Map}_*(Y, X \wedge Y) \xrightarrow{\ \mathbb{A}\ } \mathrm{Map}_*(\mathbb{A}(Y), \mathbb{A}(X \wedge Y)),$$

where l is given by $l(x)(y) = x \wedge y$.

Given an object $\underline{X} = \{X^n\}_{n\geq 0}$ in Sp, we define $\mathbb{A}(\underline{X}) \in \mathrm{Sp}$ by $\mathbb{A}(\underline{X})^n = \mathbb{A}(X^n)$ with the structure map

$$S^1 \wedge \mathbb{A}(X^n) \to \mathbb{A}(S^1 \wedge X^n) \to \mathbb{A}(X^{n+1}).$$

Analogously, for $\underline{X} \in \mathrm{Sp}$, $L \in \mathrm{Top}_*$, one has a natural map $\mathbb{A}(\underline{X}) \wedge L \to \mathbb{A}(\underline{X} \wedge L)$ in Sp. A Segal Γ-space determines a spectrum by applying $\mathbb{A}$ to the sphere spectrum $\underline{S}$, namely $\mathbb{A}(\underline{S}) = \{\mathbb{A}(S^0), \mathbb{A}(S^1), \mathbb{A}(S^2), \cdots\}$. Let $h_*(-;\mathbb{A})$ be the associated homology theory, i.e. $\tilde{h}_*(X;\mathbb{A}) = \pi_*(\mathbb{A}(\underline{S}) \wedge X)$ for $X \in \mathrm{Top}_*$.

An alternative construction of $\tilde{h}_*(X;\mathbb{A})$ is given as follows:

We first need to recall the concept of quasifibration, introduced by Dold and Thom [DoTh], which is made exactly in order to obtain the homotopy exact sequence which we have for the Serre fibrations.

Definition 2.3.3. (Dold-Thom) A map $p: E \to B$ is called a *quasi-fibration*, if for every $b \in B$ and for every $e \in p^{-1}(b)$ we have that

$$p_* : \pi_n(E, p^{-1}(b), e) \to \pi_n(B, b)$$

is an isomorphism for all $n \geq 0$.

It is equivalent to say that the fiber $p^{-1}(b)$ is weakly equivalent to the homotopy fiber of p over b. Thus, quasi-fibrations behave for homotopy theory very much like other types of fibrations since we have the following:

If $p: E \to B$ is a quasi-fibration, $b \in B$ and $e \in p^{-1}(b) = F$, then there is a long exact homotopy sequence

$$\cdots \longrightarrow \pi_n(F,e) \xrightarrow{i_*} \pi_n(E,e) \xrightarrow{p_*} \pi_n(B,b) \xrightarrow{\partial} \pi_{n-1}(F,e) \longrightarrow \cdots$$

Proposition 2.3.4. If Y is a path connected closed subspace of X with a well-based base point $x_0 \in Y \subset X$, then the cofibration $Y \hookrightarrow X \to X/Y$ induces a quasi-fibration $\mathbb{A}(X) \to \mathbb{A}(X/Y)$ with all its fibres homeomorphic to $\mathbb{A}(Y)$.

We will give the proof in Section 6.1.

Theorem 2.3.5. ([Se2], [Wo]) If $\mathbb{A}$ is a Segal Γ-space, then $\tilde{h}_n(X;\mathbb{A}) := \pi_n(\mathbb{A}(X))$ is a homology theory for connected space X. □

Example 2.3.6. (1) When $\mathbb{A} = \mathbb{N}$, then $\mathbb{A}(X) = \mathrm{SP}^\infty(X)$, by the Dold-Thom Theorem, the induced homology theory is $\tilde{h}_n(X;\mathbb{A}) \cong \tilde{H}_n(X;\mathbb{Z})$, the singular homology theory.

(2) If $\mathbb{A} = G$ a discrete abelian group, then $\mathbb{A}(X) = \mathrm{SPG}(X)$, and $\tilde{h}_n(X;\mathbb{A}) \cong \tilde{H}_n(X;G)$, singular homology with coefficients in G.

(3) When $\mathbb{A} = C$ the configuration space,

$$\mathbb{A}(X) \cong C(\mathbb{R}^\infty;X) \cong \varinjlim_{m} C(\mathbb{R}^m;X) \simeq \varinjlim_{m} \Omega^m\Sigma^m X =: \Omega^\infty\Sigma^\infty X = Q(X),$$

then $\tilde{h}_n(X;\mathbb{A}) = \pi_n^{\mathrm{stab}}(X)$, the stable homotopy theory.

Lemma 2.3.7. ([BoFr],[Wo]) If $\mathbb{A}$ is a Segal Γ-space and $X \in \mathrm{Top}_*$, then the map $\mathbb{A}(\underline{S}) \wedge X \to \mathbb{A}(\underline{S} \wedge X)$ is a weak equivalence, and thus $\tilde{h}_*(X;\mathbb{A}) \cong \mathrm{colim}_n\, \pi_{*+n}\mathbb{A}(S^n \wedge X)$. □

The Segal Γ-space $\Phi(\underline{S}, -)$ associated to a spectrum $\underline{X}$ is defined

$$\boldsymbol{n} \mapsto \mathrm{Hom}_{\mathrm{Sp}}(\underline{S}^{\times n}, \underline{X}) =: \Phi(\underline{S}, \underline{X})(\boldsymbol{n}).$$

This indeed defines a Segal Γ-space, because

$$\Phi(\underline{S},\underline{X})(\boldsymbol{n}) = \mathrm{Hom}(\underline{S} \times \cdots \times \underline{S}, \underline{X}) \simeq \mathrm{Hom}(\underline{S} \vee \cdots \vee \underline{S}, \underline{X}) \cong \mathrm{Hom}(\underline{S}, \underline{X})^n \cong \Phi(\underline{S},\underline{X})(\mathbf{1})^n.$$

The functor $\Phi(\underline{S}, -) : \mathrm{Sp} \to \Gamma\,\mathrm{Top}_*, \underline{X} \mapsto \Phi(\underline{S}, \underline{X})$ is actually right adjoint to the functor $\Gamma\,\mathrm{Top}_* \to \mathrm{Sp}, \mathbb{A} \mapsto \mathbb{A}(\underline{S})$, so it implies that

Proposition 2.3.8. ([Se2]) The homotopy category of very special Segal Γ-spaces is equivalent to the homotopy category of connective spectra. $\square$

Chapter 3

Infinite Symmetric Products and Singular Homology

3.1 Definition

The infinite symmetric product was first studied by Dold and Thom in the 1950s [DoTh]. It is the first example that a homology theory can be described as the homotopy group of a functor. It is also used to construct the classifying spaces for monoids, and to generalize the definition of Eilenberg-MacLane spaces of certain type. Furthermore, the infinite symmetric product $\mathrm{SP}^\infty(X)$ of a topological space X is a homotopically simpler space which reflects the topological properties of X, since $\mathrm{SP}^\infty(X)$ has the property of being an abelian topological monoid.

We assume in this section that all spaces are pointed, connected and all maps are base-point preserving.

Definition 3.1.1. The *n-th symmetric product* $\mathrm{SP}^n(X)$ of a based space X is the quotient X^n/Σ_n of n-th cartesian product of X by the permutation action of the symmetric group Σ_n on the coordinates.

We denote the equivalence class of $(x_1, \cdots, x_n)$ by $[x_1, \cdots, x_n]$. Sometimes we use the formal sum notation Σx_i. Note that these x_i are not necessarily distinct. Then there is a natural inclusion by adding the base point $*$,

$$\mathrm{SP}^n(X) \hookrightarrow \mathrm{SP}^{n+1}(X), [x_1, \cdots, x_n] \mapsto [x_1, \cdots, x_n, *].$$

The union $\mathrm{SP}^\infty(X) = \bigcup \mathrm{SP}^n(X)$ with the weak topology is called the *infinite symmetric product* of X.

Remark 3.1.2. • The elements of $\mathrm{SP}^\infty(X)$ can be viewed as unordered tuples $[x_1, \cdots, x_n]$ in M for some n(repetition is allowed). There is a unique smaller $n \geq 0$ with $x_i \neq *$. We denote the base point $[*] = 0$, represented by $*$ or 0.

• If X is a CW-complex, one can give the CW-structure to X^n such that each $\sigma \in \Sigma_n$ is either the identity on a cell or a homeomorphism of the cell onto some other cell. Hence the quotient space $\mathrm{SP}^n(X)$ has also a CW-structure. $\mathrm{SP}^{n-1}(X)$ is a sub-complex, and the colimit $\mathrm{SP}^\infty(X)$ is also a CW-complex.

• The juxtaposition of points $X^n \times X^m \to X^{n+m}$ induces a commutative diagram

$$\begin{array}{ccc} X^n \times X^m & \longrightarrow & X^{n+m} \\ \downarrow{\scriptstyle \Sigma_n \times \Sigma_m} & & \downarrow{\scriptstyle \Sigma_{n+m}} \\ \mathrm{SP}^n(X) \times \mathrm{SP}^m(X) & \longrightarrow & \mathrm{SP}^{n+m}(X) \end{array}$$

It follows that $\mathrm{SP}^\infty(X)$ is an abelian monoid with neutral element 0.

• A pointed map $f : X \to Y$ induces a map $\mathrm{SP}^\infty(f) : \mathrm{SP}^\infty(X) \to \mathrm{SP}^\infty(Y)$. This construction has the following functorial properties:

1) $\mathrm{SP}^\infty(\mathrm{id}_X) = \mathrm{id}_{\mathrm{SP}^\infty(X)}$.

2) Given $f : X \to Y$ and $g : Y \to Z$,then $\mathrm{SP}^\infty(g \circ f) = \mathrm{SP}^\infty(g) \circ \mathrm{SP}^\infty(f)$.

3) If $f \simeq g$, then $\mathrm{SP}^\infty(f) \simeq \mathrm{SP}^\infty(g)$. In particular if X is contractible, so is $\mathrm{SP}^\infty(X)$.

Example 3.1.3. (1) $\mathrm{SP}^2(S^1) \cong$ Möbius band. By definition, $\mathrm{SP}^2(S^1) = (S^1 \times S^1)/(x, y) \sim (y, x)$, by cutting and pasting along the rectangle, we get the Möbius band.

(2) $\mathrm{SP}^n(S^2) \cong \mathbb{CP}^n$.

View $S^2 = \mathbb{C} \bigcup \{\infty\}$ as the Riemann sphere $\mathbb{CP}^1$. The element of $\mathbb{CP}^n$ can be thought of as a homogeneous polynomial of degree n by its coefficients. So assume given $[z_1, \cdots, z_n] \in \mathrm{SP}(S^2) = \mathrm{SP}(\mathbb{C} \bigcup \{\infty\})$, there exists a nonzero polynomial $\prod_{1 \le i \le n}(z - z_i)$, unique up to a nonzero complex factor, of degree $\le n$ whose roots are precisely $z_1, \cdots, z_n$. If the degree is $k < n$, then there are only k complex roots and the remaining entries will be assigned the point at infinity, we can view ∞ as a root of the polynomial. Consider the coefficients of the polynomial as homogeneous coordinates on the complex projective space $\mathbb{CP}^n$, we get the homeomorphism by constructing the inverse map directly by

$$[a_0 : a_1 : \cdots : a_n] \to [z_1, \cdots, z_k, \infty, \cdots, \infty]$$

where z_i are the roots of $a_n z^n + \cdots a_1 z_1 + a_0$. The map is well-defined and bijective, hence a homeomorphism. Thus we can regard $\mathrm{SP}^n(S^2)$ as the space of nonzero polynomials $\sum_{i=0}^n a_i z^i$ of degree $\le n$.

(3) $\mathrm{SP}^n(S^1) \simeq S^1$ for all $n \ge 1$.

View $\mathbb{S}^1 \simeq S^2 - \{0, \infty\} = \mathbb{C}^*$ as the Riemann sphere punctured in its poles. Let $\mathcal{H}$ be the space of nonzero polynomials with nonzero roots. By looking at the coefficients $\mathcal{H}$ is homeomorphic to $\mathbb{C}^{n-1} \times \mathbb{C}^* \simeq \mathbb{S}^1$. Since $\mathrm{SP}^n(S^1) \simeq \mathrm{SP}^n(\mathbb{C}^*)$, and the map

$$\mathrm{SP}^n(\mathbb{C}^*) \to \mathcal{H}, \sum z_i \mapsto \prod_{1 \le i \le n} (z - z_i) \tag{3.1.1}$$

is a homeomorphism, so it follows that $\mathrm{SP}^n(S^1) \simeq S^1$.

3.2 Dold-Thom Theorem and Dold-Puppe Splitting

Most of the proofs in this section are omitted since they are either well-known or trivial. Good references for much of the materials are [AGP], [Ka1], [DoTh], [DoPu].

The following theorem is a key point in showing that the functor SP^∞ induces a homology theory, since it is a homotopy functor converting cofibrations into quasifibrations.

Theorem 3.2.1. (Dold-Thom) Suppose that X is a Hausdorff space with a closed path-connected subspace A and $A \to X$ is a closed cofibration. Then the quotient map $p : X \to X/A$ induces a quasifibration $\hat{p} : \mathrm{SP}^\infty(X) \to \mathrm{SP}^\infty(X/A)$ with the fiber homotopy equivalent to $\mathrm{SP}^\infty(A)$. $\square$

Corollary 3.2.2. For a pointed map $f : X \to Y$, the cofibration sequence

$$X \xrightarrow{f} Y \to C_f \xrightarrow{\rho} \Sigma X$$

induces a quasifibration $\hat{\rho} : \mathrm{SP}^\infty(C_f) \to \mathrm{SP}^\infty(\Sigma X)$ with fibre $\hat{\rho}^{-1}(\bar{x}) \simeq \mathrm{SP}^\infty(Y)$. $\square$

Particularly, from the cofibration sequence

$$X \xrightarrow{\mathrm{id}} X \hookrightarrow CX \to \Sigma X$$

we obtain the quasifibration $\mathrm{SP}^\infty(CX) \to \mathrm{SP}^\infty(\Sigma X)$ with fibre $\mathrm{SP}^\infty(X)$.

Corollary 3.2.3. If X is Hausdorff and path-connected, for every $n \geq 0$, we have an isomorphism $\pi_{n+1}(\mathrm{SP}^\infty(\Sigma X)) \cong \pi_n(\mathrm{SP}^\infty(X))$. $\square$

Suppose that X is a connected space, the canonical inclusion $i : X \hookrightarrow \mathrm{SP}^\infty(X)$ induces the *Hurewicz homomorphism* $\pi_*(X) \to H_*(X)$, it also induces that

Theorem 3.2.4. (Dold-Thom Theorem, [AGP], $A.3.$) $\pi_*(\mathrm{SP}^\infty(X)) \cong \tilde{H}_*(X;\mathbb{Z})$. $\square$

The following theorem shows that $\mathrm{SP}^\infty(X)$ is an generalized Eilenberg-Mac Lane space.

Theorem 3.2.5. (Dold-Thom [DoTh]) For a connected CW-complex X, there is a homotopy equivalence

$$\mathrm{SP}^\infty(X) \simeq \prod_i K(\tilde{H}_i(X;\mathbb{Z}), i). \tag{3.2.1}$$

$\square$

We are interested in the splitting property of symmetric product. Denote the *filtration quotient* by

$$D^n(X) := \mathrm{SP}^n(X)/\,\mathrm{SP}^{n-1}(X) = X^{\wedge n}/\Sigma_n,$$

where $X^{\wedge n} = X \wedge \cdots \wedge X$ is the n-fold smash product. The following Dold-Puppe splitting shows that SP is a self-splitting functor.

Theorem 3.2.6. (Dold-Puppe splitting, [DoPu], section 10) For a connected space X, there ia a homotopy equivalence

$$\mathrm{SP}^\infty(\mathrm{SP}^\infty(X)) \simeq \mathrm{SP}^\infty(\bigvee_{n=1}^{\infty} D^n(X)). \tag{3.2.2}$$

Proof. For each n and k, we first need to construct the map $h_k^n : \mathrm{SP}^n(X) \to SP^\infty(D^k(X))$. Given $\xi = [x_1, \cdots, x_n]$, let $I = j_1, \cdots, j_k \subset \underline{n} = \{1, \cdots, n\}$ with cardinality $\sharp I = k$. Let $- : \mathrm{SP}^k(X) \to D^k(X)$ be the quotient map. So we can define

$$h_k^n(\xi) = \sum_{\substack{I \subset \underline{n} \\ \sharp I = k}} \overline{x_{j_1} \cdots x_{j_k}}. \tag{3.2.3}$$

And these maps make the following triangle commutative

$$\begin{array}{ccc} \mathrm{SP}^{n+1}(X) & & \\ {\scriptstyle i_n}\Big\uparrow & \searrow^{h_k^{n+1}} & \\ \mathrm{SP}^n(X) & \xrightarrow[h_k^n]{} & \mathrm{SP}^\infty(D^k(X)) \end{array}$$

where $i_n : \mathrm{SP}^n(X) \hookrightarrow \mathrm{SP}^{n+1}(X)$ is the standard base point adjunction. Thus they induce a map $h^n : \mathrm{SP}^n(X) \to \mathrm{SP}^\infty(\bigvee_{k=1}^n D^k(X)) = \prod_{k=1}^n \mathrm{SP}(D^k(X))$. By taking the colimit of all h^n, we obtain the "power" map

$$h : \mathrm{SP}^\infty(X) \to \mathrm{SP}^\infty(\bigvee_{k=1}^{\infty} D^k(X)).$$

The natural inclusion $X = \mathrm{SP}^1(X) \hookrightarrow \mathrm{SP}^\infty(X)$ induces the inclusion $\iota : \mathrm{SP}^\infty(X) \hookrightarrow \mathrm{SP}^\infty(\mathrm{SP}^\infty(X))$.

The extension $\bar{h}$ of h is defined by $\bar{h} : \mathrm{SP}^\infty(\mathrm{SP}^\infty X) \to \mathrm{SP}^\infty(\bigvee_{k=1}^\infty D^k(X)), \xi_1 \cdots \xi_n \mapsto h(\xi_1) \cdots h(\xi_n)$.

It is easy to check that this map makes the triangle commutative:

$$\begin{array}{ccc} \mathrm{SP}^\infty(\mathrm{SP}^\infty(X)) & & \\ {\scriptstyle \iota}\Big\uparrow & \searrow^{\bar{h}} & \\ \mathrm{SP}^\infty(X) & \xrightarrow[h]{} & \mathrm{SP}^\infty(\bigvee_{k \geq 1} D^k(X)). \end{array}$$

Moreover, we claim that the following diagram is also commutative:

$$\begin{array}{ccc} \mathrm{SP}^\infty(D^n(X)) & = & \mathrm{SP}^\infty(D^n(X)) \\ \Big\uparrow & & \Big\uparrow \\ \mathrm{SP}^\infty(\mathrm{SP}^n(X)) & \xrightarrow{\bar{h}^n} & \mathrm{SP}^\infty(\bigvee_{k=1}^n D^k(X)) \\ \Big\uparrow & & \Big\uparrow \\ \mathrm{SP}^\infty(\mathrm{SP}^{n-1}(X)) & \xrightarrow{\bar{h}^{n-1}} & \mathrm{SP}^\infty(\bigvee_{k=1}^{n-1} D^k(X)) \end{array}$$

This is because in the case $n = 1$, $\mathrm{SP}^1(X) = D^1(X)$, so $\bar{h}^1 = \mathrm{id} : \mathrm{SP}^\infty(\mathrm{SP}^1(X)) \to \mathrm{SP}^\infty(D^1(X))$. For any $\xi \in \mathrm{SP}^n(X)$ and also $\xi \in \mathrm{SP}^{n-1}(X)$, $\bar{h}^n(\xi) = \bar{h}^{n-1}(\xi)$ lies in the $(n-1)$-th filtration. So if we project them to the n-th filtration quotient $D^n(X)$, they are the base point. Therefore we only need to consider the nontrivial case. In this case, given $\xi_1 \cdots \xi_m \in \mathrm{SP}^\infty(\mathrm{SP}^n(X))$, then it means each $\xi_i \in \mathrm{SP}^n(X)$. So by the construction of $\bar{h}^n$, we have to take all possible n subtuple of points out of an n-tuple. Then there is only one choice, namely each ξ_i itself. Thus $\bar{h}^n(\xi_1 \cdots \xi_m) = h(\xi_1) \cdots h(\xi_m) = \xi_1 \cdots \xi_m$, which proves the commutativity of the top diagram.

Since $\mathrm{SP}^{n-1}(X) \to \mathrm{SP}^n(X) \to D^n(X)$ and $\bigvee_{k=1}^{n-1} D^k(X) \to \bigvee_{k=1}^{n} D^k(X) \to D^n(X)$ are cofibrations, so two columns are quasi-fibrations. By induction and Five-lemma it follows that $\bar{h}$ is a weak equivalence. $\square$

3.3 Generalized Symmetric Product

The symmetric product construction is the first example which gives rise to a homology theory. Now we are going to see other similar constructions which also have this property.

Definition 3.3.1. For any abelian monoid G with identity 0, define the *generalized symmetric product* on a connected space X with base point x_0 to be the abelian topological monoid with $*$ as base point,

$$\mathrm{SPG}(X) := \coprod_{n \geq 1} G^n \times_{\Sigma_n} X^n/(\sim).$$

We use the formal sum $\sum g_i x_i$ to denote an element of $\mathrm{SPG}(X)$ with $g_i \in G, x_i \in X$. So $\sim$ is generated by $0x = *$, $gx_0 = *$, $gx + g'x = (g + g')x$.

It is easy to see that $\mathrm{SPG}(X) \cong \mathrm{SP}(G \wedge X)/(\sim)$, here the identification $\sim$ is

$$(g_i \wedge x) + (g_j \wedge x) = (g_i + g_j) \wedge x, 0 \wedge x = *, g \wedge x_0 = *.$$

And it is a homotopy functor in the first variable and a functor of monoids and monoids of maps in the second variable. It was McCord who first studied this construction [McC].

Example 3.3.2. When $G = \mathbb{N}$, then $\mathrm{SP}\,\mathbb{N}(X) = \mathrm{SP}^\infty(X)$. When $G = \mathbb{Z}_n$, we get the *truncated symmetric product*. It also can be represented by $\mathrm{SP}\,\mathbb{Z}_n(X) = \mathrm{SP}^\infty(X)/(nx \sim *)$. By [BCM], it is a functor that represents $\mod -n$ homology theory. In the next section we mainly focus on this example.

$\mathrm{SPG}(-)$ is a homotopy functor of pointed spaces, and converts cofibrations into fibrations, we skip the proof here, it is similar as the proof for SP^∞. Indeed for a cofibration sequence $X \hookrightarrow Y \xrightarrow{f} Y/X$, the extension $\mathrm{SPG}(Y) \xrightarrow{\hat{f}} \mathrm{SPG}(Y/X)$ is a homomorphism of groups with fibre $\hat{f}^{-1}(*) = \mathrm{SPG}(X)$. Note that for the cofibration sequence $X \hookrightarrow CX \to \Sigma X$, one can get an equivalence

$$\Omega(\mathrm{SPG}(\Sigma X)) \simeq \mathrm{SPG}(X)$$

given by $h : \mathrm{SPG}(X) \to \Omega(\mathrm{SPG}(\Sigma X)), h(\Sigma g_i x_i)(t) = \Sigma g_i[t \wedge x_i]$. This indicates $\mathrm{SPG}(\Sigma X)$ is a delooping of $\mathrm{SPG}(X)$, or equivalently a classifying space of $\mathrm{SPG}(X)$. In particular, one has $\mathrm{SPG}(S^0) = G$, thus $\mathrm{SPG}(S^1) \cong B\,\mathrm{SPG}(S^0) = BG$. It follows that $\mathrm{SPG}(S^n) \simeq K(G, n)$ for discrete abelian group G. Because for an abelian G, BG is also abelian, so $\mathrm{SPG}(S^n) \simeq B^n G$ is an n-fold classifying space for G, hence an Eilenberg-MacLane space of type $K(G, n)$.

The above statement implies that $\pi_*(\mathrm{SPG}(-))$ represents a reduced homology theory $\tilde{h}_*$. When G is discrete, this homology becomes ordinary in the sense that $\tilde{h}_n(S^0) = 0, n > 0$ and hence $\tilde{h}_*$ is in fact the singular homology with coefficients in $\tilde{h}_0(S^0) = G$.

3.4 Splitting of $\mathrm{SP}\,\mathbb{Z}_n$

Assume given an element $\Sigma g_i x_i \in \mathrm{SP}\,\mathbb{Z}_n(X)$, we write $\Sigma \tilde{g}_i x_i$ to denote its reduced representative such that $\tilde{g}_i \in \{0, 1, \cdots, n-1\}$.

Define the m-th filtration of $\mathrm{SP}\,\mathbb{Z}_n(X)$ by

$$\mathrm{SP}\,\mathbb{Z}_n(X)_m := \left\{ \ \textstyle\sum_{i=1}^r g_i x_i \in \mathrm{SP}\,\mathbb{Z}_n(X) \ \middle| \ \sum_{i=1}^r \tilde{g}_i \le m \in \mathbb{N} \ \right\}.$$

and we regard the unique element of $\mathrm{SP}\,\mathbb{Z}_n(X)_0$ as the base point $*$. In fact this is an infinite filtration, since the sum $\sum_{i=1}^r \tilde{g}_i$ can be greater than n.

One sees immediately that $\mathrm{SP}\,\mathbb{Z}_n(X)_{m-1} \subseteq \mathrm{SP}\,\mathbb{Z}_n(X)_m$, and we write

$$D_k\,\mathrm{SP}\,\mathbb{Z}_n(X) := (\mathrm{SP}\,\mathbb{Z}_n(X)_m)/(\mathrm{SP}\,\mathbb{Z}_n(X)_{m-1})$$

as the filtration quotient.

Definition 3.4.1. For $n = p_1^{\epsilon_1} \cdots p_r^{\epsilon_r}$, the prime decomposition of $n \in \mathbb{N}$. We define

$$N = \begin{cases} p_1 & \text{if } r = 1, \epsilon_1 \ge 1 \\ 1 & \text{else.} \end{cases}$$

as a function $N = N(n)$ of n. Note that $\gcd(\binom{n}{1}, \cdots, \binom{n}{n-1}) = N$.

Here is a table of the first few numbers of N and the binomial coefficients $\binom{n}{k}$.

n																						N
0											0											0
1										1		1										1
2									1		2		1									2
3								1		3		3		1								3
4							1		4		6		4		1							2
5						1		5		10		10		5		1						5
6					1		6		15		20		15		6		1					1
7				1		7		21		35		35		21		7		1				7
8			1		8		28		56		70		56		28		8		1			2
9		1		9		36		84		126		126		84		36		9		1		3
10	1		10		45		120		210		252		210		120		45		10		1	1

Theorem 3.4.2. There is a weak homotopy equivalence

$$\mathrm{SP}\,\mathbb{Z}_N(\mathrm{SP}\,\mathbb{Z}_n(X)) \simeq \mathrm{SP}\,\mathbb{Z}_N(\bigvee_{k \ge 1} D_k\,\mathrm{SP}\,\mathbb{Z}_n(X)).$$

In other words: The functor $\mathbb{B} = \mathrm{SP}\,\mathbb{Z}_N$ splits the functor $\mathbb{A} = \mathrm{SP}\,\mathbb{Z}_n$. In particular, $\mathrm{SP}\,\mathbb{Z}_p$ is a self-splitting if p is a prime.

Note that $\mathrm{SP}\,\mathbb{Z}_1(X)$ is a point, thus this splitting is only of value if $n = p^\epsilon$ is a prime power (and thus $N = p$).

Proof. First for each $m, k \ge 1$, we construct a map

$$\begin{array}{rcl} f_{m,k} : \mathrm{SP}\,\mathbb{Z}_n(X)_m & \to & \mathrm{SP}\,\mathbb{Z}_N(D_k\,\mathrm{SP}\,\mathbb{Z}_n(X)) \\ \sum_{i=1}^r g_i x_i & \mapsto & 1 \cdot \sum_{\binom{\Sigma \tilde{g}_i}{k}} (\sum_{(*)} a_i x_i). \end{array}$$

Where the sum $\Sigma_{(*)} a_i x_i$ mean all the formal sums $\Sigma_i a_i x_i$ such that each a_i is not greater than the reduced representative $\tilde{g}_i$ of g_i, i.e. $0 < a_i \leq \tilde{g}_i$ and $\Sigma a_i = k$. Here $\binom{\Sigma \tilde{g}_i}{k}$ gives the cardinality of all the possibilities of these a_i's.

All the $f_{m,k}$ together induces a map

$$f : \operatorname{SP} \mathbb{Z}_n(X) \to \operatorname{SP} \mathbb{Z}_N(\bigvee_{k=1}^{\infty} D_k \operatorname{SP} \mathbb{Z}_n(X))$$

Now we are going to extend this map to a map $\bar{f}$ as follows:

$$\begin{array}{ccccc} \xi & & \operatorname{SP} \mathbb{Z}_n(X) & \xrightarrow{f} & \operatorname{SP} \mathbb{Z}_N(\bigvee_{k=1}^{\infty} D_k \operatorname{SP} \mathbb{Z}_n(X)) \\ \downarrow & & \downarrow & \nearrow_{\bar{f}} & \\ 1 \cdot \xi & & \operatorname{SP} \mathbb{Z}_N(\operatorname{SP} \mathbb{Z}_n(X)) & & \end{array}$$

$$\bar{f} : \sum_{i=1}^{r} g_i \cdot \xi_i \longmapsto \sum_{i=1}^{r} g_i \cdot f(\xi_i), \text{ where } \xi_i \in \operatorname{SP} \mathbb{Z}_n(X).$$

This extension makes the above triangle commutative. To show that $\bar{f}$ is a homotopy equivalence, consider the following commutative diagram

$$\begin{array}{ccc} \operatorname{SP} \mathbb{Z}_N(D_m \operatorname{SP} \mathbb{Z}_n(X)) & \xrightarrow[=]{\tilde{f}_m} & \operatorname{SP} \mathbb{Z}_N(D_m \operatorname{SP} \mathbb{Z}_n(X)) \\ \uparrow q_* & & \uparrow \\ \operatorname{SP} \mathbb{Z}_N(\operatorname{SP} \mathbb{Z}_n(X)_m) & \xrightarrow{\bar{f}_m} & \operatorname{SP} \mathbb{Z}_N(\bigvee_{k=1}^{m} D_k \operatorname{SP} \mathbb{Z}_n(X)) \\ \uparrow & & \uparrow \\ \operatorname{SP} \mathbb{Z}_N(\operatorname{SP} \mathbb{Z}_n(X)_{m-1}) & \xrightarrow{\bar{f}_{m-1}} & \operatorname{SP} \mathbb{Z}_N(\bigvee_{k=1}^{m-1} D_k \operatorname{SP} \mathbb{Z}_n(X)) \end{array}$$

where q_* and $\tilde{f}_m$ are the induced maps of the quotients.

We prove it by induction. For $m = 1$, we have $\operatorname{SP} \mathbb{Z}_n(X)_1 = D_1 \operatorname{SP} \mathbb{Z}_n(X)$. So it implies that $\bar{f}_1 = \mathrm{id} : \sum g_i(1 \cdot x_i) \mapsto \sum g_i(1 \cdot x_i)$.

For the top arrow, we claim that $\tilde{f}_m = \mathrm{id}$. Because for any $\xi \in \operatorname{SP} \mathbb{Z}_N(D_m \operatorname{SP} \mathbb{Z}_n(X))$ and in particular if $\xi \in \operatorname{SP} \mathbb{Z}_N(\operatorname{SP} \mathbb{Z}_n(X)_{m-1})$, then $\bar{f}_m(\xi) = \bar{f}_{m-1}(\xi)$ lies in the $(m-1)$-th filtration. So if we project them to the m-th filtration quotient $D_m \operatorname{SP} \mathbb{Z}_n(X)$, they are the base point. Therefore we only need to consider the nontrivial case, in which ξ can be represented by $\xi = \sum_{i=1}^{r} g_i(\sum_{j=1}^{r'} g'_{ij} x_{ij})$ such that $\sum_{j=1}^{r'} \tilde{g}'_{ij} = m$. By the construction of $\bar{f}_m$, one need to find the coefficients a_{ij} such that the sum adds up exactly to m. Since $\binom{\Sigma \tilde{g}'_{ij}}{m} = \binom{m}{m} = 1$, so there is only one unique choice of the equivalence class, namely $a_{ij} = g'_{ij}$, because otherwise all the other choices would project to the trivial element in $D_m \operatorname{SP} \mathbb{Z}_n(X)$. Hence $\tilde{f}_m = \mathrm{id}$.

Moreover notice that the two sequences

$$\operatorname{SP} \mathbb{Z}_n(X)_{m-1} \to \operatorname{SP} \mathbb{Z}_n(X)_m \to D_m \operatorname{SP} \mathbb{Z}_n(X),$$

$$\bigvee_{k=1}^{m-1} D_k \operatorname{SP} \mathbb{Z}_n(X) \to \bigvee_{k=1}^{m} D_k \operatorname{SP} \mathbb{Z}_n(X) \to D_m \operatorname{SP} \mathbb{Z}_n(X)$$

are cofibrations. Therefore, the vertical sequences in the diagram are both quasifibrations. Passing to the homotopy groups, we know that $\tilde{f}_{m_*}$ is an isomorphism, so by induction and the Five-Lemma we get $\bar{f}_{m_*}$ is an isomorphism, for all m. This proves the Theorem 3.4.2. □

Corollary 3.4.3.

$$\tilde{H}_*(K(\mathbb{Z}_n,\ell);\mathbb{Z}_N) \cong \bigoplus_{k\geq 1} \tilde{H}_*(D_k K(\mathbb{Z}_n,\ell);\mathbb{Z}_N).$$

□

Example 3.4.4. In the case $X = S^1, n = 2$ thus $N = 2$, the Corollary becomes

$$\tilde{H}_*(K(\mathbb{Z}_2,1);\mathbb{Z}_2) \cong \bigoplus_{k\geq 1} \tilde{H}_* D_k K(\mathbb{Z}_2,1);\mathbb{Z}_2).$$

In fact there is a direct way to see it:

$LHS \cong \tilde{H}_*(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2 < u >, |u| = 1.$

On the RHS, we claim that $D_k \operatorname{SP}\mathbb{Z}_2(S^1) \cong S^k$, because by identifying $S^1 \cong \mathbb{I}/(0\sim 1)$,

$$D_k \operatorname{SP}\mathbb{Z}_2(S^1) = (\operatorname{SP}\mathbb{Z}_2(S^1)_k)/(\operatorname{SP}\mathbb{Z}_2(S^1)_{k-1}) \cong \Delta^k/\partial\Delta^{k-1} \cong S^k.$$

It implies that

$$RHS = \bigoplus_{k\geq 1} \tilde{H}_*(D_k K(\mathbb{Z}_2,1);\mathbb{Z}_2) \cong \bigoplus_{k\geq 1} \tilde{H}_*(S^k;\mathbb{Z}_2) \cong \bigoplus_{k\geq 1} \mathbb{Z}_2 < u^k >, |u^k| = k$$

The functor $\operatorname{SP}\mathbb{Z}_2$ has been studied by [MiLö] and [BCM]. In [BCM] they showed that

Theorem 3.4.5. ([BCM], Theorem 2.9.) Let Y be a locally finite based CW-complex, then there is an isomorphism

$$H_*(\operatorname{SP}\mathbb{Z}_2(Y)_n;\mathbb{Z}_2)) \cong \bigoplus_{r\leq n} H_*(\operatorname{SP}\mathbb{Z}_2(Y)_r, \operatorname{SP}\mathbb{Z}_2(Y)_{r-1};\mathbb{Z}_2).$$

□

So if we take the colimit in both sides, we get the special case of Corollary 3.4.2. for $n = 2$, namely

$$H_*(\operatorname{SP}\mathbb{Z}_2(Y);\mathbb{Z}_2)) \cong \bigoplus_{r\geq 1} H_*(\operatorname{SP}\mathbb{Z}_2(Y)_r, \operatorname{SP}\mathbb{Z}_2(Y)_{r-1};\mathbb{Z}_2).$$

This filtration quotient on the right hand side can be described as follows. For a compact manifold Y with base point $*$,

$$D_k \operatorname{SP}\mathbb{Z}_2(Y) = \operatorname{SP}\mathbb{Z}_2(Y)_k/\operatorname{SP}\mathbb{Z}_2(Y)_{k-1} \cong (C^k(Y,*)/C^{k-1}(Y,*))^\infty,$$

the one-point compactification of the filtration quotient of the relative configuration space.

And since $\operatorname{SP}\mathbb{Z}_2$ represents the mod -2 homology theory, there is a similar Dold-Thom theorem for $\operatorname{SP}\mathbb{Z}_2$.

Theorem 3.4.6. ([BCM], Theorem 2.6.) If Y is a based, locally finite CW-complex, then

$$\operatorname{SP}\mathbb{Z}_2(Y) \simeq \prod_i K(\tilde{H}_i(Y;\mathbb{Z}_2), i).$$

□

3.5 Splitting of $\mathrm{SP}\,\mathbb{Z}$

Now we consider the example $\mathrm{SP}\,\mathbb{Z}(X)$, which is the group completion of $\mathrm{SP}\,\mathbb{N}(X)$. First note that an element $\xi = \Sigma g_i \cdot x_i \in \mathrm{SP}\,\mathbb{Z}(X)$ can be written as $\xi = \Sigma g'_i \cdot x_i + \Sigma g''_j \cdot x_j$, where $g'_i \geq 0, g''_j < 0$, and $x_i \neq x_j$. Define a double-filtration $\mathrm{SP}\,\mathbb{Z}(X)_{p,q}$ by

$$\mathrm{SP}\,\mathbb{Z}(X)_{p,q} = \left\{ \; \Sigma g'_i \cdot x_i + \Sigma g''_j \cdot x_j \in \mathrm{SP}\,\mathbb{Z}(X) \;\middle|\; \Sigma g'_i \leq p, \Sigma |g''_j| \leq q, x_i \neq x_j \; \right\}.$$

Let the m-th filtration of $\mathrm{SP}\,\mathbb{Z}(X)$ be

$$\mathrm{SP}\,\mathbb{Z}(X)_m := \bigcup_{p+q \leq m} \mathrm{SP}\,\mathbb{Z}(X)_{p,q}$$

and the double filtration quotient be

$$D_{p,q}\,\mathrm{SP}\,\mathbb{Z}(X) := \mathrm{SP}\,\mathbb{Z}(X)_{p,q} / (\mathrm{SP}\,\mathbb{Z}(X)_{p-1,q} \cup \mathrm{SP}\,\mathbb{Z}(X)_{p,q-1}).$$

So we have the m-th filtration quotient

$$D_m\,\mathrm{SP}\,\mathbb{Z}(X) := \mathrm{SP}\,\mathbb{Z}(X)_m / \,\mathrm{SP}\,\mathbb{Z}(X)_{m-1} \cong \bigvee_{p+q=m} D_{p,q}\,\mathrm{SP}\,\mathbb{Z}(X).$$

Then in a similar way, we have a self-splitting of $\mathrm{SP}\,\mathbb{Z}(X)$.

Theorem 3.5.1. There is a weak homotopy equivalence

$$\mathrm{SP}\,\mathbb{Z}(\mathrm{SP}\,\mathbb{Z}(X)) \sim \mathrm{SP}\,\mathbb{Z}(\bigvee_{k \geq 1} D_k\,\mathrm{SP}\,\mathbb{Z}(X)).$$

Proof. Similar as in Theorem 3.4.2., we first construct a map restricting to the m-th filtration,

$$h_{m,k} : \mathrm{SP}\,\mathbb{Z}(X)_m \longrightarrow \mathrm{SP}\,\mathbb{Z}(D_k\,\mathrm{SP}\,\mathbb{Z}(X)) \cong \prod_{p+q=k} \mathrm{SP}\,\mathbb{Z}(D_{p,q}\,\mathrm{SP}\,\mathbb{Z}(X))$$

$$\sum g'_i \cdot x_i + \sum g''_j \cdot x_j \longmapsto 1 \cdot \sum_{\ell_{ij}} (\sum_{(*)} a_i x_i + \sum_{(**)} a_j x_j)$$

Here $\ell_{ij} = \binom{\Sigma g'_i}{p}\binom{\Sigma |g''_j|}{q}$. And $\underset{(*)}{\Sigma}$ means the formal sum of all $a_i x_i$ such that $0 < a_i \leq g'_i$ and $\Sigma a_i = p$. Also $\underset{(**)}{\Sigma}$ means the formal sum of all $a_j x_j$ such that $g''_j \leq a_j < 0$ and $\Sigma |a_j| = q$.

All the $h_{m,k}$ together induces a map $h : \mathrm{SP}\,\mathbb{Z}(X) \to \mathrm{SP}\,\mathbb{Z}(\bigvee_{k=1}^{\infty} D_k\,\mathrm{SP}\,\mathbb{Z}(X))$.

Then one can extend this map to $\bar{h}$:

$$\begin{array}{ccccc} \xi & & \mathrm{SP}\,\mathbb{Z}(X) & \xrightarrow{\;h\;} & \mathrm{SP}\,\mathbb{Z}(\bigvee_{k=1}^{\infty} D_k\,\mathrm{SP}\,\mathbb{Z}(X)) \\ \downarrow & & \downarrow & \overset{\bar{h}}{\nearrow} & \\ 1 \cdot \xi & & \mathrm{SP}\,\mathbb{Z}(\mathrm{SP}\,\mathbb{Z}(X)) & & \end{array}$$

$$\bar{h} : \Sigma_i z_i \cdot \xi_i \longmapsto \Sigma_i z_i \cdot h(\xi_i) = \Sigma_i z_i \cdot (\Sigma_j t_{ij} \cdot \zeta_j) = \Sigma_i \Sigma_j (z_i t_{ij}) \cdot \zeta_j,$$

where $h(\xi_i) = \Sigma_j t_{ij} \cdot \zeta_j$.

And this extension gives us the following commutative diagram

$$\begin{array}{ccc}
\operatorname{SP}\mathbb{Z}(D_m \operatorname{SP}\mathbb{Z}(X)) & = & \operatorname{SP}\mathbb{Z}(D_m \operatorname{SP}\mathbb{Z}(X)) \\
\uparrow & & \uparrow \\
\operatorname{SP}\mathbb{Z}(\operatorname{SP}\mathbb{Z}(X)_m) & \xrightarrow{\bar{h}_m} & \operatorname{SP}\mathbb{Z}(\bigvee_{k=1}^{m} D_k \operatorname{SP}\mathbb{Z}(X)) \\
\uparrow & & \uparrow \\
\operatorname{SP}\mathbb{Z}(\operatorname{SP}\mathbb{Z}(X)_{m-1}) & \xrightarrow{\bar{h}_{m-1}} & \operatorname{SP}\mathbb{Z}(\bigvee_{k=1}^{m-1} D_k \operatorname{SP}\mathbb{Z}(X))
\end{array}$$

Again we prove by induction and the five-lemma, passing to the homotopy groups we get the required weak homotopy equivalences. □

Chapter 4

Configuration Spaces and Stable Homotopy Theory

4.1 Properties of Configurations and Examples

Definition 4.1.1. The *ordered configuration space* of a space M is $\tilde{C}^n(M) := \{(x_1, \cdots, x_n) \in M^n | x_i \neq x_j$ for $i \neq j\}$. It is an open subspace of the cartesian product. The *unordered configuration space* $C^n(M) := \tilde{C}^n(M)/\Sigma_n$ is the quotient space of $\tilde{C}^n(M)$ by the action of the symmetric group Σ_n.

Example 4.1.2. Here are some well known examples which have some nice homotopy types (see for example [Ka1], 3.1.1.).

(1) $\tilde{C}^n(\mathbb{R}^1) \cong \tilde{C}^n(]0,1[) \cong \coprod_{n!} \overset{\circ}{\Delta}{}^n$ has $n!$ contractible component, one for each permutation of Σ_n, i.e. each $\sigma \in \Sigma_n$ defines a component containing the configuration $(\sigma(1), \cdots, \sigma(n))$, and obviously $C^n(\mathbb{R}^1) \simeq *$.

(2) The map $\tilde{C}^2(\mathbb{R}^n) \to \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}) : (x_1, x_2) \mapsto (x_1 + x_2, x_1 - x_2)$ is a homeomorphism, hence $\tilde{C}^2(\mathbb{R}^n) \simeq S^{n-1}$ and $C^2(\mathbb{R}^n) \simeq \mathbb{R}P^{n-1}$.

(3) $\tilde{C}^3(S^n)$ can be identified with the unit tangent bundle τS^n of S^n. There is a deformation retract of $\tilde{C}^3(S^n)$ into the subspace consisting of configurations $(x_1, -x_1, x_2)$ with $x_1 \neq -x_1 \neq x_2$. Then one can fix x_1, and project x_3 stereographically from x_1 onto the tangent plane of $-x_1$.

(4) If G is a (Lie)-group, acting transitively on itself, then the map

$$\tilde{C}^n(G) \to G \times \tilde{C}^{n-1}(G \setminus \{1\}) : (g_1, \cdots, g_n) \mapsto (g_1, (g_1^{-1}g_2, \cdots, g_1^{-1}g_n))$$

is a homeomorphism, for example $\tilde{C}^n(\mathbb{R}^m) \cong \mathbb{R}^m \times \tilde{C}^{n-1}(\mathbb{R}^m \setminus \{0\})$, and $\tilde{C}^n(S^1) \cong S^1 \times \tilde{C}^{n-1}(S^1 \setminus \{1\})$, thus $\tilde{C}^n(S^1)$ has $(n-1)!$ contractible components all of the form $S^1 \times \tilde{C}^{n-1}(]0,1[)$, one also obtains $\tilde{C}^3(\mathbb{R}^2) \simeq S^1 \times (S^1 \vee S^1)$.

(5) $C^2(\mathbb{R}P^n) = \{(\ell_1, \ell_2) |$ distinct lines in $\mathbb{R}^{n+1}\}$, there is a fibration of $C^2(\mathbb{R}P^n)$ into the Grassmanian $\mathrm{Gr}_2(\mathbb{R}^{n+1})$, namely the fibration

$$C^2(\mathbb{R}P^n) \to \mathrm{Gr}_2(\mathbb{R}^{n+1}) : (\ell_1, \ell_2) \mapsto \langle \ell_1, \ell_2 \rangle = \text{2-plane spanned by } \ell_1, \ell_2$$

with fibre homeomorphic to $C^2(\mathbb{R}P^1) \simeq S^1$.

4.2 Braid Groups

Theorem 4.2.1. (Fadell-Neuwirth Theorem) Let M be a connected manifold of dimension at least 2, and let $Q_r = \{x_1, \cdots, x_r\}$ be any set of r distinct points of M. Then for $r \leq n$, there are fibrations

$$\tilde{C}^{n-r}(M \backslash Q_r) \to \tilde{C}^n(M) \xrightarrow{\pi} \tilde{C}^r(M),$$

where π is any coordinate projection.

Take M to be a connected surface, one can iterate this construction as follows:

$$\begin{array}{ccccccccc} \tilde{C}^n(M) & \longleftarrow & \tilde{C}^{n-1}(M\backslash Q_1) & \longleftarrow & \tilde{C}^{n-2}(M\backslash Q_2) & \longleftarrow \cdots \longleftarrow & \tilde{C}^2(M\backslash Q_{n-2}) & \longleftarrow & M\backslash Q_{n-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ M & & M\backslash Q_1 & & M\backslash Q_2 & & M\backslash Q_{n-2} & & \end{array}$$

Then use the fact that $\pi_n(M\backslash Q_r) = 0$, for $n \geq 2, r \geq 1$ (note if $M \neq S^2, \mathbb{R}P^2$, then $\pi_n(M\backslash Q_r) = 0$, for $n \geq 2, r \geq 0$) and that $M\backslash Q_r$ has as fundamental group a free group F on r generators. It follows that the fibration $M\backslash Q_{n-1} \to \tilde{C}^n(M) \to \tilde{C}^{n-1}(M)$ yields a short exact sequence

$$0 \to F \to \mathrm{PBr}_n(M) := \pi_1(\tilde{C}^n(M)) \to \mathrm{PBr}_{n-1}(M) \to 0$$

and we have the following proposition

Proposition 4.2.2. If M is a surface without boundary and $\neq S^2, \mathbb{R}P^2$, then $\tilde{C}^n(M)$ is an Eilenberg-MacLane space of type $K(\pi_1\tilde{C}^n(M), 1)$. $\square$

Note when $M = \mathbb{R}^2$, $Br(n) := \pi_1 C^n(\mathbb{R}^2)$ is the classical *Artin's braid group* and $PBr(n) := \pi_1\tilde{C}^n(\mathbb{R}^2)$ the pure braid group.

There is a natural fibration from ordered into unordered configurations,

$$\Sigma_n \to \tilde{C}^n(M) \to C^n(M)$$

Now we focus on the cases $M = \mathbb{R}^n$, which derives the Barratt-Quillen-Priddy Theorem ([Ka1], Lemma 3.23.).

Lemma 4.2.3. For $n \geq 2, k \geq 1$, we have

$$\pi_1(C^k(\mathbb{R}^n)) = \begin{cases} \Sigma_k, & n > 2 \\ \mathrm{Br}(k), & n = 2 \end{cases}$$

where $\mathrm{Br}(k)$ is Artin's braid group on k-strings. In particular,

$$C^k(\mathbb{R}^\infty) = B\Sigma_k \text{ and } C^k(\mathbb{R}^2) = B\,\mathrm{Br}(k)$$

Proof. Note that when $n > 2, \pi_1 \tilde{C}^k(\mathbb{R}^n) = 0$ and more generally

$$\pi_r \tilde{C}^k(\mathbb{R}^n) = 0, r \leq n-2. \tag{4.2.1}$$

This is because $\tilde{C}^k(\mathbb{R}^n)$ is the complement of the fat diagonal of codimension n in $\mathbb{R}^{kn}$. Then one gets that for $n \geq 3, \tilde{C}^k(\mathbb{R}^n)$ is a universal cover of $C^k(\mathbb{R}^n)$ with fundamental group Σ_k. We have the short exact sequence

$$0 \to \Sigma_k \to \mathrm{PBr}(k) \to \mathrm{Br}(k) \to 0$$

where Σ_k acts by permuting the end points of strings. When $n = \infty$, it follows from (4.2.1) that $\pi_r \tilde{C}^k(\mathbb{R}^\infty) = 0$, for all r, hence contractible. Since the permutation action by Σ_k is free we therefore obtain that $C^k(\mathbb{R}^\infty)$ is a model for $B\Sigma_k$ and $\tilde{C}^k(\mathbb{R}^\infty) = E\Sigma_k$. □

4.3 Snaith Splitting and Stable Homotopy Theory

In this section we show that one can stably split the labeled configuration space $C(M, M_0; X)$ ([Bö1], [BCT]).

Definition 4.3.1. Let M be a manifold with M_0 a closed submanifold, X a space with base point x_0, the *labeled configuration space* of manifold M is

$$C(M, M_0; X) = \coprod_{n=0}^{\infty} \tilde{C}^n(M) \times_{\Sigma_n} X^n/(\sim)$$

where $\sim$ is generated by

$(z_1, \cdots, z_n; x_1, \cdots, x_n) = (z_1, \cdots, \hat{z}_i, \cdots, z_n; x_1, \cdots, \hat{x}_i, \cdots, x_n)$ if $z_i \in M_0$, or $x_i = x_0$.

There is a natural filtration of $C(M, M_0; X)$ by the closed subspaces:

$$C_n(M, M_0; X) := \coprod_{k=0}^{n} \tilde{C}^k(M) \times_{\Sigma_k} X^k/(\sim)$$

Denote the filtration quotient by

$$D_n(M, M_0; X) := C_n(M, M_0; X)/C_{n-1}(M, M_0; X)$$

for example $D_0 = *, D_1 = (M/M_0)_+ \wedge X, D_n(\mathbb{R}^\infty; X) = \tilde{C}^n(\mathbb{R}^\infty)_+ \wedge_{\Sigma_n} X^{\wedge n}$.

In the case $M_0 = \emptyset, X = S^n$ we have the following geometric description of $D_k(M; X)$ ([BCT], 1.6., [Ka1]). Consider the bundle projection

$$\tau_n : \tilde{C}^k(M) \times_{\Sigma_k} (\mathbb{R}^{nk}) \to C^k(M)$$

with fibre $\mathbb{R}^{nk}$. By construction τ_n is the n-fold Whitney sum $\tau_1^{\oplus n}$. It turns out that $D_k(M; S^n)$ is the Thom space of the vector bundle τ_n.

By the Thom isomorphism,

Corollary 4.3.2. If τ_n is an orientable vector bundle, $\tilde{H}_i(D_k(M; S^n); \mathbb{Z}) \cong \tilde{H}_{i-kn}(C^k(M); \mathbb{Z})$; if τ_n is not orientable, the isomorphism still holds by replacing the coefficients to be $\mathbb{Z}/2$. □

Remark 4.3.3. This indicates one way to compute the homology of classical configuration space with $\mathbb{Z}/2$-coefficients, that is

$$\tilde{H}_i(C^k(M);\mathbb{Z}/2) \cong \tilde{H}_{i+kn}(D_k(M;S^n);\mathbb{Z}/2).$$

Theorem 4.3.4. ([Bö1], [McD]) Let M be a smooth compact manifold and let M_0 and N be the smooth compact submanifolds of M with $\operatorname{codim} N = 0$. If $N/M_0 \cap N$ or X is path-connected, then

$$C(N, N \cap M_0; X) \to C(M, M_0; X) \to C(M, N \cup M_0; X)$$

is a quasifibration. □

Let M be an m-dimensional manifold and let W be an m-dimensional manifold without boundary which contains M, e.g. $W = M$ if M is closed, or $W = M \cup \partial M \times [0,1)$ if M has boundary. Let ξ be the principal $O(m)$-bundle of the tangent bundle of W. Let $\Gamma_{\xi[S^m X]}(B, B_0)$ be the space of cross sections of $\xi[S^m X]$ which are defined on B and take values at $\infty \wedge X$ on B_0 for each subspace pair (B, B_0) in W, where $\xi[S^m X]$ is the associated bundle and $O(m)$ acts trivially on X and canonically on $S^m \cong \mathbb{R}^m \cup \{\infty\}$, i.e.

$$E := \xi \times_{O(m)} S^m \wedge X \to W.$$

Example 4.3.5. Assume W is parallizable, i.e. $TW \cong W \times \mathbb{R}^m$, then $\xi \cong W \times O(m) \Rightarrow E \cong W \times \Sigma^m X$, which implies that

$$\Gamma_{\xi[S^m X]}(W - M_0, W - M) \cong \operatorname{Map}(W - M_0, W - M; \Sigma^m X, \infty).$$

Proposition 4.3.6. ([BCT], 2.5.) Let M be a smooth compact manifold and let M_0 be a smooth compact submanifold of M. If M/M_0 or X is path-connected, then there is a (weak) homotopy equivalence

$$C(M, M_0; X) \to \Gamma_{\xi[S^m X]}(W - M_0, W - M).$$

□

Remark 4.3.7. (1) By Proposition 4.3.4, there is a homotopy equivalence

$$C((M, M_0;) \times \mathbb{R}^n; X) \simeq \Omega^n C(M, M_0; S^n X)$$

if M/M_0 or X is path-connected.

(2) The interesting cases of Proposition 4.3.6 are:

i) the m-fold loop space of an m-fold suspension (take $W = \mathbb{R}^m$, $M = \mathbb{D}^m$, $M_0 = \emptyset$, X path-connected) ([May]):

$$C(\mathbb{R}^m; X) \xrightarrow{\sim} \Omega^m \Sigma^m X$$

ii) free loop space of a suspension(take $W = M = S^1$, $M_0 = \emptyset$, X path-connected) ([Bö1]):

$$C(S^1; X) \xrightarrow{\sim} \Lambda \Sigma X$$

The first example is called *May-Milgram Model* ([Ka1], 3.6.1.), we can realize the map

$$\alpha_m : C(\mathbb{R}^m; X) \to \Omega^m \Sigma^m X$$

as follows: a point $[v_1, \cdots, v_n; x_1, \cdots, x_n] \in C(\mathbb{R}^m; X)$ determines a map

$$\begin{aligned} \varphi: \quad \mathbb{S}^m \cong \mathbb{R}^m \cup \{\infty\} \quad &\to \Sigma^m X = D^m/\partial D^m \wedge X \\ v \quad &\mapsto \begin{cases} T_{v_i}^{-1}(v) \wedge x_i, & \text{if } v \in B_1(v_i) \\ *, & \text{otherwise} \end{cases} \end{aligned}$$

where $B_1(v_i) \subset \mathbb{R}^m$ denotes the ball of radius 1 centered at v_i, and $T_{v_i} : D^m \to B_1(v_i)$ is the translation by v_i. Moreover $* \in D^m/\partial D^m \wedge X$ is the basepoint. One can check that this map is well-defined and a weak homotopy quivalence for path-connected X. This example is special and builds up other configuration spaces in a natural way.

The first splitting result in the history was James unstable splitting [Ja] of $\Omega\Sigma X$ using the James model $J(X)$, the free non-commutative topological monoid generated by X modulo its base point x_0. Its element $(x_1, x_2, \cdots, x_n)$ are finite sequences of points in X with possible repetitions, $1 := * =$ base point of X and order matters; the topology on $J(X)$ is induced by the topology on X, and one can regard it equivalently as a quotient space of cartesian product of X, namely

$$J(X) = \coprod_{n\geq 1} X^n / \sim,$$

$$(x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_n) \sim (x_1, \cdots, x_{i-1}, *, x_{i+1}, \cdots, x_n).$$

It was known that the James model $J(X) \to \Omega\Sigma X$ is actually a weak homotopy equivalence, and Puppe [Pu] showed that under certain conditions on X, they are genuinely homotopy equivalent. There is an intermediate space $C(\mathbb{R}; X)$ between them making the diagram homotopy commutative,

$$\begin{array}{ccc} x_1 x_2 \cdots x_n & \xmapsto{\ \ell\ } & \omega_{x_1} * \cdots * \omega_{x_n} \\ \uparrow & & \\ \xi = \sum_{t_1 < \cdots < t_n} t_i x_i & & \end{array} \qquad \begin{array}{ccc} J(X) & \xrightarrow[\simeq]{\ \ell\ } & \Omega\Sigma X \\ \simeq \uparrow \pi & \nearrow_{\gamma} & \\ C(\mathbb{R}; X) & & \end{array}$$

Here $\omega_{x_i} : I \to \Sigma X,\ t \mapsto [t, x_i]$. The homotopy inverse of π is $x_1 x_2 \cdots x_n \mapsto \sum_{i=1}^n i x_i$. And $\gamma(\xi) = \omega'_{x_1} * \cdots * \omega'_{x_n}$, where ω'_{x_i} is defined by

$$\begin{aligned} \omega_{x_i}: \quad \mathbb{S}^1 \cong \mathbb{R} \cup \{\infty\} \quad &\to \quad \Sigma X \\ t \quad &\mapsto \quad \begin{cases} \frac{t - t_i}{\epsilon} \wedge x_i, & \text{if } t \in B_\epsilon(t_i), \\ *, & \text{otherwise.} \end{cases} \end{aligned}$$

Here $\epsilon := \min_i\{\frac{t_i - t_{i-1}}{2}\}$, and $B_\epsilon(t_i)$ is the ball of radius ϵ centered at $t_i, 1 \leq i \leq n$.

I. James [Ja] proved that there is an unstable splitting of this space, known as James splitting (see also [Mil]),

$$\Sigma J(X) \simeq \Sigma\Omega\Sigma X \simeq \Sigma \bigvee_{k=1}^{\infty} X^{\wedge k}. \tag{4.3.1}$$

Theorem 4.3.8. (Snaith-splitting, [Sn], [Bö1]) There is a stable homotopy equivalence for any pair (M, M_0) and X,

$$C(M, M_0; X) \xrightarrow{\sim_s} \bigvee_{k\geq 1} D_k(M, M_0; X)$$

This theorem can be reformulated in the following equivalent statements:

Proposition 4.3.9. The following are equivalent:

(i) the suspension spectra are stably equivalent, i.e.

$$\Sigma^\infty C(M, M_0; X) \simeq \Sigma^\infty \bigvee_{k\geq 1} D_k(M, M_0; X).$$

(ii) the infinite loop spaces are homotopy equivalent, i.e.

$$\Omega^\infty\Sigma^\infty C(M, M_0; X) \simeq \Omega^\infty\Sigma^\infty \bigvee_{k\geq 1} D_k(M, M_0; X).$$

(iii)

$$C(\mathbb{R}^\infty; C(M, M_0; X)) \simeq C(\mathbb{R}^\infty; \bigvee_{k\geq 1} D_k(M, M_0; X)).$$

In the case $M = \mathbb{R}^\infty, M_0 = \emptyset$, recall that $C(\mathbb{R}^\infty; X) \simeq \Omega^\infty\Sigma^\infty X$, hence we obtain

Corollary 4.3.10. The configuration functor is self-splitting, namely

$$C(\mathbb{R}^\infty; C(\mathbb{R}^\infty; X)) \simeq C(\mathbb{R}^\infty; \bigvee_{k\geq 1} D_k(\mathbb{R}^\infty; X)).$$

Or equivalently,

$$QQ(X) \simeq Q(\bigvee_{k\geq 1} D_k(\mathbb{R}^\infty; X)).$$

Therefore simultaneously we have,

$$\pi_q^{\mathrm{stab}} QX \cong \bigoplus_{k\geq 1} \pi_q^{\mathrm{stab}}(D_k(\mathbb{R}^\infty; X)).$$

Before proving the Snaith splitting Theorem 4.3.8., we introduce a notion of flag configuration space.

Definition 4.3.11. The *flag configuration space* is

$$C^{n,k}(M) = \{(\xi, \xi') \in C^n(M) \times C^k(M)) | \xi' \subseteq \xi\}.$$

Proof. We demonstrate here an elegant proof, which is known to specialists of configuration spaces ([Ka1], Theorem 3.74., [Böl]).

Let $\xi = \Sigma z_i x_i \in C_n(M, M_0; X)$ be a labeled configuration, denote by $\bar{\xi}$ its image in $D_k(M, M_0; X)$. We divide the proof into several steps.

First step: We define a "power" map h as follows:

$$h : C(M, M_0; X) \to C(\mathbb{R}^\infty; \bigvee_k D_k(M, M_0; X))$$

Set $I = \{1, \cdots, n\}$, and for each subset $J \subseteq I$ with $\sharp(J) = k$, we let $\xi_J = \sum_{i\in J} z_i x_i \in C_k(M, M_0; X)$ be the corresponding subterms, and $\bar{\xi}_J$ its image under the composition

$$C_k(M, M_0; X) \to D_k(M, M_0; X) \to \bigvee_k D_k(M, M_0; X).$$

Note that the flag configuration space is an $\binom{n}{k}$-fold covering space of $C^n(M)$, namely $C^{n,k}(M) \to C^n(M) : (\zeta, \zeta') \to \zeta$.

We write $\zeta := (z_1, \cdots, z_n)$. Choosing J means to pick a $\zeta_J := \{z_i | i \in J \subset I\} \in C^k(M)$, then $(\zeta, \zeta_J) \in C^{n,k}(M)$. By summing over all subsets $J \subset I$, we construct a map

$$C(M, M_0; X) \to C(\coprod_k C^k(M); \bigvee_k D_k(M, M_0; X)).$$

Second step: Since for each $k \geq 0$, $C^k(M)$ is a finite dimensional manifold, $\coprod_k C^k(M)$ can be embedded into $\mathbb{R}^\infty$. Pick any embedding $e : \coprod_k C^k(M) \hookrightarrow \mathbb{R}^\infty$. We obtain a map by compositions

$$h_n : \quad C_n(M, M_0; X) \longrightarrow C(\mathbb{R}^\infty; \bigvee_k D_k(M, M_0; X))$$

$$\xi = \sum_{i \in I} z_i x_i \longmapsto \sum_k \sum_{\substack{J \subset I \\ \sharp(J)=k}} (e(\zeta_J); \bar{\xi}_J).$$

Note that the following diagram is commutative

$$\begin{array}{ccc} C_n(M, M_0; X) & \xrightarrow{h_n} & C(\mathbb{R}^\infty; \bigvee_{k=1}^n D_k(M, M_0; X)) \\ \uparrow \iota & & \uparrow \\ C_{n-1}(M, M_0; X) & \xrightarrow{h_{n-1}} & C(\mathbb{R}^\infty; \bigvee_{k=1}^{n-1} D_k(M, M_0; X)) \end{array}$$

where $\iota(\xi)$ has one particle $z_n \in M_0$ or $x_n = x_0$. Now for $J \subset I = \{1, \cdots, n\}$ with $\sharp(J) = k$ there are only two possibilities:

(i) $n \notin J \Longrightarrow$ then $e_k(\zeta_J)$ and $\bar{\xi}_J$ agree on level $n-1$ and level n(here e_k is the restriction of e on $C^k(M)$).

(ii) $n \in J \Longrightarrow$ the $\bar{\xi}_J$ is the base point in $D_k(M, M_0; X) \subset \bigvee_k D_k(M, M_0; X)$.

Thus we get the map

$$h : C(M, M_0; X) \to C(\mathbb{R}^\infty; \bigvee_k D_k(M, M_0; X)).$$

This map is well-defined (disjointness, equivariance and base point conditions are preserved).

Third step: extend this map to $\bar{h}$,

$$\begin{array}{ccccc} a \in & & C(M, M_0; X)) & \xrightarrow{h} & C(\mathbb{R}^\infty; \bigvee_k D_k(M, M_0; X)) \\ \downarrow & & \downarrow & \nearrow_{\bar{h}} & \\ (0; a) \in & & C(\mathbb{R}^\infty; C(M, M_0; X)) & & \end{array}$$

Given $(z_1, \cdots, z_n; a_1, \cdots, a_n) \in C(\mathbb{R}^\infty; C(M, M_0; X))$, we write

$$h(a_1) = (z_{11}, \cdots, z_{1\ell_1}; b_{11}, \cdots, b_{1\ell_1}),$$
$$\vdots$$
$$h(a_n) = (z_{n1}, \cdots, z_{n\ell_n}; b_{n1}, \cdots, b_{n\ell_n}),$$

then we define

$$\bar{h}(\sum z_i a_i) = \begin{array}{l} ((z_{11}, z_1), \cdots, (z_{1\ell_1}, z_1), \cdots, (z_{n1}, z_n), \cdots, (z_{n\ell_n}, z_n); \\ b_{11}, \cdots, b_{1\ell_1}, \cdots, b_{n1}, \cdots, b_{n\ell_n}) \end{array}$$

One remark is that here one uses $(z_{ik}, z_i) \in \mathbb{R}^\infty \times \{z_i\} \subseteq \mathbb{R}^\infty \times \mathbb{R}^\infty \cong \mathbb{R}^\infty$.

Fourth step: We will show that $\bar{h}$ is a weak homotopy equivalence, Prove it by induction, to this end, we consider the commutative square for each n,

$$\begin{array}{ccc} C(\mathbb{R}^\infty; D_n(M, M_0; X)) & \xrightarrow{\tilde{h}_n} & C(\mathbb{R}^\infty; D_n(M, M_0; X)) \\ \uparrow & & \uparrow \\ C(\mathbb{R}^\infty; C_n(M, M_0; X)) & \xrightarrow{\bar{h}_n} & C(\mathbb{R}^\infty; \bigvee_{k=1}^n D_k(M, M_0; X)) \\ \uparrow & & \uparrow \\ C(\mathbb{R}^\infty; C_{n-1}(M, M_0; X)) & \xrightarrow{\bar{h}_{n-1}} & C(\mathbb{R}^\infty; \bigvee_{k=1}^{n-1} D_k(M, M_0; X)) \end{array}$$

where $\bar{h}_n$ is just the restriction of $\bar{h}$ on $C_n(M, M_0; X)$.

Since for $n = 1$, $C_1(M, M_0; X) = D_1(M, M_0; X) = (M/M_0)_+ \wedge X$, $e_1 : M = C^1(M) \hookrightarrow \mathbb{R}^\infty$, we have that

$$\begin{array}{rcl} C_1(M, M_0; X) & \to & C(\mathbb{R}^\infty; D_1(M, M_0; X)) \\ \xi = (m; x) & \mapsto & (e_1(m); \xi). \end{array}$$

$$\begin{array}{ccccccccccc} \cdots \to & \pi_{p-1}C(\mathbb{R}^\infty; D_n) & \to & \pi_p C(\mathbb{R}^\infty; C_{n-1}) & \to & \pi_p C(\mathbb{R}^\infty; C_n) & \to & \pi_p C(\mathbb{R}^\infty; D_n) & \to \cdots \\ & \downarrow \tilde{h}_{n*} & & \downarrow \bar{h}_{n-1*} & & \downarrow \bar{h}_{n*} & & \downarrow \tilde{h}_{n*} & \\ \cdots \to & \pi_{p-1}C(\mathbb{R}^\infty; D_n) & \to & \pi_p C(\mathbb{R}^\infty; \bigvee^{n-1} D_k) & \to & \pi_p C(\mathbb{R}^\infty; \bigvee^n D_k) & \to & \pi_p C(\mathbb{R}^\infty; D_n) & \to \cdots \end{array}$$

It follows that $\bar{h}_1 \simeq \mathrm{id}$. Also on the top arrow, a similar computation shows that $\tilde{h}_n \simeq \mathrm{id}$, for each n.

Then a theorem of Bödigheimer [Bö1] says that C has the property converting the cofibrations to quasi-fibrations, and since

$$\begin{array}{l} C(\mathbb{R}^\infty; C_{n-1}) \to C(\mathbb{R}^\infty; C_n) \to C(\mathbb{R}^\infty; D_n) \\ C(\mathbb{R}^\infty; \bigvee\limits_{1 \le k \le n-1} D_k) \to C(\mathbb{R}^\infty; \bigvee\limits_{1 \le k \le n} D_k) \to C(\mathbb{R}^\infty; D_n) \end{array}$$

are cofibrations, we obtain the long exact sequences

For brevity, we omit $(M, M_0; X)$ in the notation of this diagram.

By the induction and 5-lemma, it follows that $\bar{h}$ is a weak homotopy equivalence, which finishes the proof.

□

We remark in the end that the splitting functor for the configuration space $C(\mathbb{R}^\infty; X)$ is not unique. The infinite symmetric product SP^∞ is also a splitting functor for C. Its proof is left to the reader. Moreover, we have a commutative diagrams of these two splittings

$$\begin{array}{ccc} C(\mathbb{R}^\infty; C(\mathbb{R}^\infty; X)) & \xrightarrow{\simeq} & C(\mathbb{R}^\infty; \bigvee_k D_k(\mathbb{R}^\infty; X)) \\ \downarrow{\scriptstyle \mathrm{hur}} & & \downarrow{\scriptstyle \mathrm{hur}} \\ \mathrm{SP}^\infty(C(\mathbb{R}^\infty; X)) & \xrightarrow{\simeq} & \mathrm{SP}^\infty(\bigvee_k D_k(\mathbb{R}^\infty; X)) \end{array}$$

given by the Hurewicz map:

$$\mathrm{hur} : (z_1, \cdots, z_n; \xi_1, \cdots, \xi_n) \mapsto \xi_1 \cdots \xi_n.$$

We call it so because the induced map in homotopy groups is the Hurewicz homomorphism

$$\mathrm{hur}_* : \pi_*^{\mathrm{stab}} C(\mathbb{R}^\infty; X) \to H_*(C(\mathbb{R}^\infty; X); \mathbb{Z}).$$

4.4 Γ-spaces arising from $\tilde{C}(\mathbb{R}^\infty)$

Denote

$$C(\mathbf{1}) := \coprod_{n \geq 0} \tilde{C}^n(\mathbb{R}^\infty) = \tilde{C}(\mathbb{R}^\infty).$$

It has a partial monoid structure: call $\xi, \xi' \in C(\mathbf{1})$ *composable*, if they are disjoint, then declare the disjoint union $\xi \sqcup \xi'$ to be their "composition".

Define

$$C(\boldsymbol{k}) := \left\{ (\xi_1, \cdots, \xi_k) \in C(\mathbf{1})^k \;\middle|\; \xi_i \cap \xi_j = \emptyset \text{ in } \mathbb{R}^\infty \text{ for } i \neq j \right\}.$$

We have a composition

$$\begin{array}{rrcl} \sqcup : & C(\boldsymbol{k}) & \to & C(\mathbf{1}) \\ & (\xi_1, \cdots, \xi_k) & \mapsto & \xi = \xi_1 \sqcup \cdots \sqcup \xi_k. \end{array}$$

Note that this $\sqcup$ is associative, and the unique point $\emptyset \in C(\mathbf{0}) = \tilde{C}(\mathbb{R}^\infty)^0$ is the neutral element.

A map $\alpha : \boldsymbol{m} \to \boldsymbol{n}$ in Γ induces a map

$$\begin{array}{rccc} \alpha_* : & C(\boldsymbol{m}) & \to & C(\boldsymbol{n}) \\ & (\xi_1, \cdots, \xi_m) & \mapsto & (\sqcup_{j \in \alpha^{-1}(1)} \xi_j, \cdots, \sqcup_{j \in \alpha^{-1}(n)} \xi_j). \end{array}$$

We claim that the natural inclusion $p_k : C(\boldsymbol{k}) \hookrightarrow C(\mathbf{1})^k$ is a homotopy equivalence by using the following trick: endow $\mathbb{R}^\infty$ with the weak topology, then we have a homeomorphism

$$\begin{array}{rccc} h : & \mathbb{R}^\infty \times \mathbb{R} & \to & \mathbb{R}^\infty \\ & ((x_1, \cdots, x_n, \cdots), y) & \mapsto & (y, x_1, x_2, \cdots). \end{array}$$

Also for $i = \pm 1$ there is a homeomorphism $\mathbb{R}^\infty \to \mathbb{R}^\infty \times \{i\}, (x_1, \cdots, x_n, \cdots) \mapsto (i, x_1, \cdots, x_n, \cdots)$. They induce a homeomorphism $\tilde{C}(\mathbb{R}^\infty) \cong \tilde{C}(\mathbb{R}^\infty \times \{i\})$. Given an arbitrary configuration $\xi \in \tilde{C}(\mathbb{R}^\infty)$, we denote its image under this homeomorphism by ξ_i.

It suffices to look at the p_2:

$$\begin{array}{ccccc}
(\xi, \xi') \in & \tilde{C}(\mathbb{R}^\infty) \times \tilde{C}(\mathbb{R}^\infty) = C(\mathbf{1})^2 & & \\
\downarrow & \downarrow \cong & & \\
(\xi_{-1}, \xi'_{+1}) \in & \tilde{C}(\mathbb{R}^\infty \times \{-1\}) \times \tilde{C}(\mathbb{R}^\infty \times \{+1\}) & \subset & \tilde{C}(\mathbb{R}^\infty \times \mathbb{R})^2 \\
\downarrow & \downarrow & & \downarrow h_* \times h_* \\
(h(\xi_{-1}), h(\xi'_{+1})) \in & C(\mathbf{2}) & \subset & \tilde{C}(\mathbb{R}^\infty)^2
\end{array}$$

One can show that this compostion is homotopy inverse to the inclusion, which implies that the functor C is a Γ-space. To compare its extension

$$C(Y) := (\coprod_k C(\boldsymbol{k}) \times Y^k)/\sim$$

with the original configuration space

$$C(\mathbb{R}^\infty; Y) = (\coprod_k \tilde{C}^k(\mathbb{R}^\infty) \times_{\Sigma_k} Y^k)/\sim,$$

one can see directly that $C(Y)$ is just a reformulation of $C(\mathbb{R}^\infty; Y)$. Namely in the second case, its elements are distinct points $x_1, \cdots, x_n$ in $\mathbb{R}^\infty$ with labels $y_1, \cdots, y_n$ in Y. And in $C(Y)$ its elements are points $y_1, \cdots, y_m$ in Y with configurations $\xi_1, \cdots, \xi_m$ in $C(\boldsymbol{m})$ as coefficients of each y_i. The natural filtration of $C(\mathbb{R}^\infty, Y)$ gives rise to a filtration of $C(Y)$. Namely, denote by $|\xi_i|$ the sum of cardinalities of all the configurations represented by ξ_i, and define

$$C_n(\boldsymbol{k}) := \left\{ (\xi_1, \cdots, \xi_k) \in C(\boldsymbol{k}) \;\middle|\; \textstyle\sum_i |\xi_i| \leq n \right\}.$$

The filtration

$$C_n(Y) := \coprod_k C_n(\boldsymbol{k}) \times Y^k/(\sim)$$

corresponds to

$$C_n(\mathbb{R}^\infty; Y) = \coprod_{k \leq n} \tilde{C}^k(\mathbb{R}^\infty) \times_{\Sigma_k} Y^k/(\sim).$$

Remark 4.4.1. For an arbitrary manifold M, the configuration functor $C(M \times \mathbb{R}^\infty; -)$ is a Γ-space as well, the proof is similar.

Chapter 5

Grassmannians and Connective K-Theory

In this chapter following an idea of G. Segal [Se3] we introduce a Γ-space K, built out of Grassmannians, and use it to model connective KO-theory. We then exhibit a functor $\mathbb{B}$ which splits K in the sense of the introduction (1.0.9). This functor $\mathbb{B}$ relates to the work of [GMTW] and [RW] and represents the infinite loop space of the Thom spectrum MO. Whereas in previous chapters we have split spaces with discrete labels, the main novelty here is that the splitting "space" is not a finite discrete set but a topological space. We will have a space of choices of the splitting "space".

5.1 Connective K-Homology Theory

Let $\mathrm{Gr}_k(\mathbb{R}^\infty) = \mathrm{colim}_n \mathrm{Gr}_k(\mathbb{R}^n)$ denote the Grassmannian of k-planes in $\mathbb{R}^\infty$ and $\mathrm{Gr}(\mathbb{R}^\infty) = \coprod_{k\geq 0} \mathrm{Gr}_k(\mathbb{R}^\infty)$ the disjoint union of all these Grassmannians. This is a partial abelian monoid under direct sum.

Define a functor $K : \Gamma \to \mathrm{Top}_*$ as follows:

$K(\mathbf{0}) = *$,

$K(\mathbf{1}) = \mathrm{Gr}(\mathbb{R}^\infty)$, and in general

$K(\boldsymbol{n}) = \left\{ (V_1, \cdots, V_n) \in K(\mathbf{1})^n \;\middle|\; V_i \perp V_j, \text{ if } i \neq j \right\}.$

Lemma 5.1.1. *K is a Segal Γ-space.*

Proof. We need to show that the inclusion $K(\boldsymbol{n}) \hookrightarrow K(\mathbf{1})^n$ is a homotopy equivalence. We prove this for $n = 2$. Let an arbitrary $(V_1, V_2) \in K(\mathbf{1}) \times K(\mathbf{1})$ be given. For each $i = 1, 2$, let p_i be the composite

$$p_i : V_i \hookrightarrow \mathbb{R}^\infty \xrightarrow{\iota_i} \mathbb{R}^\infty \times \mathbb{R}^\infty \xrightarrow{\tau} \mathbb{R}^\infty.$$

where $\iota_i : \mathbb{R}^\infty \to \mathbb{R}^\infty \times \mathbb{R}^\infty$ is the inclusion into the i-th factor. And τ is defined by

$$\tau : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty, ((x_1, x_2, \cdots), (y_1, y_2, \cdots)) \mapsto (x_1, y_1, x_2, y_2, \cdots). \tag{5.1.1}$$

Obviously $p_1(V_1) \perp p_2(V_2)$ because their inner product is 0. So we obtain a map

$$\mu : K(\mathbf{1}) \times K(\mathbf{1}) \to K(\mathbf{2}), (V_1, V_2) \mapsto (p_1(V_1), p_2(V_2)).$$

μ is a homotopy inverse to the inclusion. To see this, we are going to construct a homotopy rotating $p_1(V_1)$ continuously into V_1. In order to get the general formula, we first show what we do at time $t \in [0, \frac{1}{2}]$. Denote an arbitrary element of $\mathbb{R}^\infty$ by $(x_1, x_2, x_3, \cdots)$. We define a rotation map $\rho_{2,3} : \mathbb{R}^\infty \times [0, \frac{1}{2}] \to \mathbb{R}^\infty$ which exchanges x_2, x_3 and keeps the other vectors fixed.

$$\rho_{2,3}((x_1, x_2, x_3, \cdots), t) = (x_1, x_2 \cos(\frac{\pi}{2} \cdot 2t) + x_3 \sin(\frac{\pi}{2} \cdot 2t), x_2 \cos(\frac{\pi}{2} \cdot 2(\frac{1}{2} - t)) + x_3 \sin(\frac{\pi}{2} \cdot 2(\frac{1}{2} - t)), x_4, \dots)$$

At time $t \in [\frac{1}{2}, \frac{3}{4}]$, we can apply $\rho_{3,5}$ (resp. $\rho_{2,4}$) to exchange x_3 and x_5 (resp. x_2 and x_4). So a general formula for the i-th step $\rho_{i+1,2i+1} : \mathbb{R}^\infty \times [1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}] \to \mathbb{R}^\infty$ will be

$$\rho_{i+1,2i+1}((\cdots, x_{i+1}, \cdots, x_{2i+1}, \cdots), t) = (\cdots, x_{i+1} \cos(\frac{\pi}{2} \cdot 2^i(t - 1 + \frac{1}{2^{i-1}})) + x_{2i+1} \sin(\frac{\pi}{2} \cdot 2^i(t - 1 + \frac{1}{2^{i-1}})), \cdots, x_{i+1} \cos(\frac{\pi}{2} \cdot 2^i(1 - \frac{1}{2^i} - t)) + x_{2i+1} \sin(\frac{\pi}{2} \cdot 2^i(1 - \frac{1}{2^i} - t)), \cdots). \tag{5.1.2}$$

The composition of all the $\rho_{i+1,2i+1}$'s defines a continuous homotopy

$$\rho_{\text{odd}} : \mathbb{R}^\infty \times I \to \mathbb{R}^\infty, ((x_1, x_2, x_3, \cdots), t) \mapsto \rho_{i+1,2i+1}((x_1, x_2, x_3, \cdots), t) \text{ for } t \in [1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}] \tag{5.1.3}$$

which moves all the odd-indexed vectors forwards. One needs to do infinite many rotations in this process, but since all the vector spaces are of finite dimensions, so for i big enough, $x_{i+1} = x_{2i+1} = 0$. Therefore $\rho_{i+1,2i+1} = \text{id}$, as $i \gg 0$. In particular, after finitely many rotations, $p_1(V_1)$ is mapped into V_1. One can define a similar function $\rho_{i,2i}$ to exchanges x_i and x_{2i} at each step and fix other vectors, therefore $p_2(V_2)$ is sent to V_2. So this proves that K is a Γ-space. $\square$

Let X be well pointed in the sense that the inclusion from the base point $x_0 \hookrightarrow X$ is a cofibration. The extended Γ-space K has the following form:

$$K(X) = (\coprod_{n \geq 0} K(\boldsymbol{n}) \times_{\Sigma_n} X^n)/\sim$$

where

$$(V_1, \cdots, V_i, \cdots, V_n; x_1, \cdots, x_i, \cdots, x_n) \sim (V_1, \cdots, \hat{V_i}, \cdots, V_n; x_1, \cdots, \hat{x_i}, \cdots, x_n), \text{ if } x_i = x_0;$$

and

$$(\cdots, V_i, \cdots, V_j, \cdots; \cdots, x_i, \cdots, x_j, \cdots) \sim (\cdots, V_i \oplus V_j, \cdots, \hat{V_j}, \cdots; \cdots, x_i, \cdots, \hat{x_j}, \cdots), \text{ if } x_i = x_j.$$

As usual, we write an equivalence class $\xi = [V_1, \cdots, V_r; x_1, \cdots, x_r] \in K(X)$ as $\xi = \Sigma_i V_i x_i$.

Segal ([Se3]) described this model and proved that the functor K converts cofibrations into quasifibrations. In the case $X = S^0$,

$$K(S^0) = \coprod_{m \geq 0} BO(m) = \coprod_m \text{Gr}_m(\mathbb{R}^\infty) = \text{Gr}(\mathbb{R}^\infty).$$

Define the m-th filtration of $K(X)$ to be

$$K_m(X) = \left\{ \ \Sigma_i V_i x_i \in K(X) \ \Big| \ \Sigma \dim V_i \leq m \ \right\}.$$

We now assume given $\xi = [V_1, \cdots, V_r; x_1, \cdots, x_r] \in K_{m-1}(X)$ with $\sum_{i=1}^r \dim V_i = \ell \leq m-1$. There is a natural inclusion

$$\iota_{m-1} : K_{m-1}(X) \to K_m(X), \xi \mapsto \xi + (V_{r+1}, x_0)$$

with some V_{r+1} such that V_{r+1} is the orthogonal complement of $V_1 \oplus \cdots \oplus V_r$ in $\mathbb{R}^{\bar{m}}$, where $\bar{m}$ is the smallest k with $V_1 \oplus \cdots \oplus V_r \subset \mathbb{R}^k$.

Define $K_m(\boldsymbol{n}) := \{(V_1, \cdots, V_n) \in K(\boldsymbol{n}) | \sum_i \dim V_i \leq m\}$.

Lemma 5.1.2. Let X be well pointed, then $\iota_{m-1} : K_{m-1}(X) \to K_m(X)$ is a cofibration.

Proof. Let σ_q the natural inclusion

$$\sigma_q : X^{n-1} \to X^n : (x_1, \cdots, x_{n-1}) \mapsto (x_1, \cdots, x_{q-1}, x_0, x_q, \cdots, x_{n-1}).$$

So we have a push out diagrams with $\sigma X^{n-1} = \cup_{q=0}^{n-1} \sigma_q X^{n-1}$:

$$\begin{array}{ccc} K_m(\boldsymbol{n}) \times_{\Sigma_n} \sigma X^{n-1} & \longrightarrow & K_{m-1}(X) = \coprod_n K_{m-1}(\boldsymbol{n}) \times_{\Sigma_n} X^n / \sim \\ \downarrow & & \downarrow \\ K_m(\boldsymbol{n}) \times_{\Sigma_n} X^n & \longrightarrow & K_m(X) = \coprod_n K_m(\boldsymbol{n}) \times_{\Sigma_n} X^n / \sim \end{array}$$

Since X is well pointed and the left inclusion in this diagram is a cofibration, so is the right map, because push out preserves cofibrations. ☐

Remark 5.1.3. There is a natural map

$$K(X) \to \mathrm{SP}^\infty(X) : \Sigma V_i x_i \mapsto \Sigma(\dim V_i) x_i. \tag{5.1.4}$$

It induces the Hurewicz map $\pi_* K(X) \to \pi_* \mathrm{SP}^\infty(X)$.

A theorem of Segal asserts that

Theorem 5.1.4. ([Se3]) If X is connected, then $\pi_*(K(X)) \cong \widetilde{\mathrm{ko}}_*(X)$. ☐

Here ko_* is the connective K-homology theory associated to periodic real K-theory KO_*. The following is a table of the Ω-spectrum $\underline{\mathrm{KO}}$, where $\mathrm{BO}, \mathrm{BSp}$ are the classifying spaces of the orthogonal group O and the sympletic group Sp. $\underline{KO}$ fulfills Bott periodicity: $\mathrm{KO}_n = \mathrm{KO}_{n+8}$.

q mod 8	0	1	2	3	4	5	6	7
KO_q	$\mathrm{BO} \times \mathbb{Z}$	$\Omega^3 \mathrm{BSp}$	$\Omega^2 \mathrm{BSp}$	$\Omega \mathrm{BSp}$	$\mathrm{BSp} \times \mathbb{Z}$	$\Omega^3 \mathrm{BO}$	$\Omega^2 \mathrm{BO}$	$\Omega \mathrm{BO}$

By definition,

$$\widetilde{\mathrm{KO}}^0(X) = [X, \mathrm{BO} \times \mathbb{Z}] \cong \widetilde{\mathrm{KO}}(X),$$
$$\mathrm{KO}^{-q}(X) = \mathrm{KO}(\Sigma^q X) = [\Sigma^q X, \mathrm{KO}_0] = [X, \Omega^q \mathrm{KO}_0] = [X, \mathrm{KO}_{-q}].$$

The spectrum $\underline{\mathrm{ko}}$ is connective means there is a natural transformation $\mathrm{ko}_* \to \mathrm{KO}_*$ such that $\mathrm{ko}_q(\mathrm{pt}) \to \mathrm{KO}_q(\mathrm{pt})$ is an isomorphism for $q \geq 0$, and $\mathrm{ko}_q(X) = 0$ for all X when $q < 0$. More precisely, the q-th space in the spectrum $\underline{\mathrm{ko}}$ is the following

$$\mathrm{ko}_q = \begin{cases} \mathrm{KO}_q & q \leq 0; \\ \mathrm{KO}_q < q > & q > 0. \end{cases}$$

$\mathrm{KO}_q < q >$ is the $(q-1)$-fold connective cover of KO_q, that is

$$\pi_i(\mathrm{KO}_q < q >) = \begin{cases} 0 & i < q; \\ \pi_i(\mathrm{KO}_q) & i \geq q. \end{cases}$$

However KO_* is fully determined by ko_* because $\mathrm{KO}_*(X)$ is the direct limit of the sequence $\mathrm{ko}_q(X) \to \mathrm{ko}_{q+8}(X) \to \mathrm{ko}_{q+16}(X) \to \cdots$, here the maps are Bott periodicity. In the case $X = \mathrm{pt}$, we have

$$\begin{CD}
@>>> \mathrm{ko}_q(\mathrm{pt}) @>>> \mathrm{ko}_{q+8}(\mathrm{pt}) @>>> \mathrm{ko}_{q+16}(\mathrm{pt}) @>>> \cdots \\
@. @| {\scriptstyle q<0} @| {\scriptstyle \cong,\ q+8\geq 0} @V{\cong}VV \\
0 @>>> 0 @>>> \mathrm{KO}_{q+8}(\mathrm{pt}) @>{\cong}>> \mathrm{KO}_{q+16}(\mathrm{pt}) @>>> \cdots
\end{CD}$$

Let X be a connected space, we recall how to get the connective cover of X via the Postnikov towers. Denote $K_s = K(\pi_s(X), s)$, we have a tower of the form

$$\begin{CD}
X @<<< X_2 @<<< X_3 @<<< X_4 @<<< \cdots \\
@V{f_1}VV @V{f_2}VV @V{f_3}VV @V{f_4}VV \\
K_1 @. K_2 @. K_3 @. K_4
\end{CD}$$

f_s is a map inducing an isomorphism in the bottom homotopy group. X_{s+1} is the fiber of f_s, and X_{s+1} is the s-th connective cover of X.

Take BO for example. We have two coverings $\mathbb{Z}/2 \to \mathrm{SO} \to O$ and $\mathbb{Z}/2 \to \mathrm{Spin} \to \mathrm{SO}$. Note that $\pi_1(\mathrm{SO}(k)) = \mathbb{Z}/2, k \geq 3$ and $\mathrm{Spin}(k)$ is simply connected for $k \geq 3$. So Spin is the universal cover of SO, which implies that $B\,\mathrm{Spin}$ is the 2-connective cover of $B\,\mathrm{SO}$. We can identify $B\,\mathrm{Spin}$ with the total space of the fibration over $B\,\mathrm{SO}$ induced from the path-loop fibration over $K(\mathbb{Z}/2, 2)$ via the map $f : B\,\mathrm{SO} \to K(\mathbb{Z}/2, 2)$ realizing $\omega_2 \in H^2(B\,\mathrm{SO}; \mathbb{Z}/2)$, the second Stiefel-Whitney class.

$$\begin{CD}
K(\mathbb{Z}/2, 1) @>{\simeq}>> \Omega K(\mathbb{Z}/2, 2) \\
@VVV @VVV \\
B\,\mathrm{Spin} @>>> PK(\mathbb{Z}/2, 2) \\
@V{\pi}VV @VVV \\
B\,\mathrm{SO} @>{f}>> K(\mathbb{Z}/2, 2)
\end{CD}$$

We obtain an interesting tower of fibrations

$$\begin{CD}
\mathrm{BO} @<<< B\,\mathrm{SO} = \mathrm{BO} < 2 > @<<< B\,\mathrm{Spin} = \mathrm{BO} < 4 > @<<< \mathrm{BO} < 8 > \\
@VVV @V{f}VV @V{f'}VV \\
K(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty @. K(\mathbb{Z}/2, 2) @. K(\mathbb{Z}, 4)
\end{CD}$$

Here f' realizes $p_1 \in H^4(B\,\mathrm{Spin};\mathbb{Z})$, the first Pontrjagin class.

This tower characterizes different group structures $(O, \mathrm{SO}, \mathrm{Spin})$ on manifolds, hence it gives a corresponding Thom spectra $\underline{\mathrm{MO}<8>} \to \underline{\mathrm{MSpin}} \to \underline{\mathrm{MSO}} \to \underline{\mathrm{MO}}$.

The significance of Theorem 5.1.4. and the description of $\underline{\mathrm{ko}}$ in terms of labeled configurations is that it yields an obvious Hurewicz homomorphism between connective K-theory and singular homology theory by the map (5.1.1). Passing to homotopy groups of this map yields a map

$$\widetilde{\mathrm{ko}}_*(X) \to \tilde{H}_*(X;\mathbb{Z}).$$

5.2 The Sheaf of Parameterized Embeddings

5.2.1 Topology on the Sheaf $B_d(-;Y)$

The flavor of the topology defined here is due to Galatius [Ga] and Randal-Williams [RW] presented in the cases of graphs and manifolds with tangential structures. We applied their ideas and generalized it to the case of manifolds with labels.

Let M be an n-dimensional manifold (possibly with boundary), and Y a connected space with base point $*$. We define a class $\mathcal{F}_d$ of manifolds F which are smooth d-dimensional manifolds without boundary (not necessarily compact or connected). Note that the dimension n is arbitrarily large and $n \gg d$. Let $\mathrm{Emb}(F, M)$ be the space of all smooth embeddings $\epsilon : F \hookrightarrow M$ of a manifold $F \in \mathcal{F}_d$ into M with closed image in M. And $\epsilon(F)$ needs to be disjoint from ∂M if $\partial M \neq \emptyset$.

We then define a sheaf of sets of equivalence classes on M by setting for any open $U \subseteq M$,

$$B_d(U;Y) := \Big(\coprod_{F\in\mathcal{F}_d} \mathrm{Emb}(F,U) \times_{\mathrm{Diff}(F)} \mathrm{Map}(F,Y)\Big)/\sim \tag{5.2.1}$$

Denote at moment elements of $B_d(U;Y)$ by (ϵ, F, φ) with $\epsilon : F \hookrightarrow U$ an embedding and $\varphi : F \to Y$ continuous. We say (ϵ, F, φ) is equivalent to $(\epsilon', F', \varphi')$, if $F = F' \sqcup F''$, $\varphi(F'') = *$ and there is a commutative diagram:

$$\begin{array}{ccccc} U & \xleftarrow{\ \epsilon\ } & F & \xrightarrow{\ \varphi\ } & Y \\ \| & & \uparrow & & \| \\ U & \xleftarrow{\ \epsilon'\ } & F' & \xrightarrow{\ \varphi'\ } & Y \end{array}$$

Denote by $\xi = [\epsilon, F, \varphi]$ the equivalence class of (ϵ, F, φ).

Let $V \subseteq U$ be open subsets of M. There exists a restriction map $\mathrm{rest} : B_d(U;Y) \to B_d(V;Y)$ that we shall explain later in Lemma 5.2.8.. We wish to assign a topology to the sheaf making these restriction maps continuous. Throughout this section we will use dashed arrows for partially defined maps. That is, the notation $f : X \dashrightarrow Y$ means that f is a function $f : U \to Y$ for some subset $U \subseteq X$. The following notion of morphisms between elements of $B_d(U;Y)$ is important for defining the topology on $B_d(U;Y)$.

Definition 5.2.1. For $\xi = [\epsilon, F, \varphi], \xi' = [\epsilon', F', \varphi'] \in B_d(U;Y)$, a morphism is a triple

$$\lambda = (L, L', D) : \xi \dashrightarrow \xi'$$

where

(i) $L \subseteq F$ is an open subset that is the interior of its compact closure $\bar{L}$ in F, where F is the minimal representative, i.e. no component has the constant trivial label. And L' similarly for F'.

(ii) $D : L \hookrightarrow L'$ is a smooth embedding.

Definition 5.2.2. Given a $\xi = [\epsilon, F, \varphi]$, we say a quintuple $\alpha = (K, L, W, N, Q)$ is ξ-*allowable* if

(i) K is a compact subset of U.

(ii) $L \subset F$ is as above, $\epsilon(F) \cap K \subseteq \epsilon(L)$.

(iii) $W \subseteq \mathrm{Emb}(L, U)$ is a neighborhood of $\epsilon|_L$ in the strong C^∞-topology.

(iv) $N \subseteq \mathrm{Map}(L, Y)$ is a neighborhood of $\varphi|_L$ in the compact-open topology.

(v) $Q \subset Y$ is a neighborhood of the base point $*$.

Definition 5.2.3. For a ξ-allowable quintuple α, we say a morphism $\lambda = (L, L', D)$ is α-*small* if

(i) $\epsilon'(F') \cap K \subseteq \epsilon'(L')$.

(ii) $L \xhookrightarrow{D} L' \xhookrightarrow{\epsilon'|_{L'}} U$ is in W.

(iii) $L \overset{D}{\hookrightarrow} L' \xrightarrow{\varphi'|_{L'}} Y$ is in N.

(iv) $\varphi'(L' \backslash D(L)) \subseteq Q$.

Definition 5.2.4. Given a ξ-allowable α, define a subset $\mathcal{N}_\alpha(\xi) \subseteq B_d(U;Y)$ by

$$\mathcal{N}_\alpha(\xi) := \left\{ \ \xi' = [\epsilon', F', \varphi'] \in B_d(U;Y) \ \middle| \ \text{there exists an } \alpha-\text{small morphism } \lambda : \xi \dashrightarrow \xi' \ \right\}.$$

We use the collection

$$\mathcal{N}(\xi) := \left\{ \ \mathcal{N}_\alpha(\xi) \ \middle| \ \alpha \text{ is } \xi-\text{allowable} \ \right\}$$

to generate the topology of $B_d(U;Y)$.

Lemma 5.2.5. For a fixed $\xi = [\epsilon, F, \varphi]$, the collection $\mathcal{N}(\xi)$ forms a neighborhood basis at ξ.

Proof. We shall show the 3 axioms a neighborhood basis needs to satisfy.

1) Note that $\xi \in \mathcal{N}_\alpha(\xi)$ because (L, L, id) is α-small for all α.

2) For any $\alpha = (K, L, W, N, Q)$ and $\beta = (\tilde{K}, \tilde{L}, \tilde{W}, \tilde{N}, \tilde{Q})$, let $N' = \{(\varphi' : L' \to Y) \in N \cap \tilde{N} | \ \varphi'(L' \backslash D(L)) \subseteq Q \cap \tilde{Q}\}$ and $\tilde{N}' = \{(\varphi' : L' \to Y) \in N \cap \tilde{N} | \ \varphi'(L' \backslash D(L)) \subseteq Q \cap \tilde{Q}\}$. Write $\gamma = (K \cup \tilde{K}, L \cup \tilde{L}, W \cap \tilde{W}, N' \cap \tilde{N}', Q \cap \tilde{Q})$. We claim that $\mathcal{N}_\gamma(\xi) \subseteq \mathcal{N}_\alpha(\xi) \cap \mathcal{N}_\beta(\xi)$. Assume given any $\xi' = [\epsilon', F', \varphi'] \in \mathcal{N}_\gamma(\xi)$, in other words $(L \cap \tilde{L}, L', D) : \xi \dashrightarrow \xi'$ is γ-small. That is, $\epsilon'(F') \cap (K \cup \tilde{K}) \subseteq \epsilon'(L')$, $L \cup \tilde{L} \overset{D}{\hookrightarrow} L' \xrightarrow{\epsilon'} U$ is in $W \cap \tilde{W}$, $L \cup \tilde{L} \overset{D}{\hookrightarrow} L' \xrightarrow{\varphi'} Y$ is in $N' \cap \tilde{N}'$, $\varphi'(L' \backslash D(L \cup \tilde{L})) \subseteq Q \cap \tilde{Q}$. So it implies that $\epsilon'(F') \cap K \subseteq \epsilon'(L')$, $L \hookrightarrow L \cup \tilde{L} \overset{D}{\hookrightarrow} L' \xrightarrow{\epsilon'} U$ is in W. $L \hookrightarrow L \cup \tilde{L} \overset{D}{\hookrightarrow} L' \xrightarrow{\varphi'} Y$ is in N', $\varphi'(L' \backslash D(L)) \subseteq Q$, thus $\xi' \in \mathcal{N}_\alpha(\xi)$. Similarly we can show $\xi' \in \mathcal{N}_\beta(\xi)$. So $\mathcal{N}_\gamma(\xi) \subseteq \mathcal{N}_\alpha(\xi) \cap \mathcal{N}_\beta(\xi)$.

3) Let $\xi' \in \mathcal{N}_\alpha(\xi)$. Then this consists of the following data: $D : L \hookrightarrow L'$ is an embedding, $\epsilon(F) \cap K \subseteq \epsilon(L), \epsilon'(F') \cap K \subseteq \epsilon'(L'), \varphi'(L' \backslash D(L)) \subseteq Q$. So D induces a continuous map

$$\begin{array}{rcl} D^* : \mathrm{Emb}(L', U) \times \mathrm{Map}(L', Y) & \to & \mathrm{Emb}(L, U) \times \mathrm{Map}(L, Y) \\ (L' \overset{\epsilon'|_{L'}}{\hookrightarrow} U, L' \overset{\varphi'|_{L'}}{\longrightarrow} Y) & \mapsto & (L \overset{D}{\hookrightarrow} L' \overset{\epsilon'|_{L'}}{\hookrightarrow} U, L \overset{D}{\hookrightarrow} L' \overset{\varphi'|_{L'}}{\longrightarrow} Y). \end{array}$$

The map D^* sends $(\epsilon'|_{L'}, \varphi'|_{L'})$ to an element which lies in $W \times N$. Thus $W' \times N' := {D^*}^{-1}(W \times N)$ is an open neighborhood of $(\epsilon'|_{L'}, \varphi'|_{L'})$.

Denote by $\tilde{W}' := \{(\epsilon' : L' \hookrightarrow U) \in W' | \epsilon'(L' \backslash D(L)) \subseteq W\} \subseteq W'$ and $\tilde{N}' := \{(\tilde{\varphi} : L' \to Y) \in N' | \tilde{\varphi}(L' \backslash D(L)) \subseteq Q\} \subseteq N'$.

Let $\alpha' = (K, L', \tilde{W}', \tilde{N}', Q)$ and $\xi'' = [\epsilon'', F'', \varphi''] \in \mathcal{N}_{\alpha'}(\xi')$, so we get $D' : L' \hookrightarrow L''$ is an embedding, $\epsilon''(F'') \cap K \subseteq \epsilon''(L'')$. Consider the composite

$$(L, L'', D' \circ D) : \xi \dashrightarrow \xi''.$$

The pair

$$L \overset{D}{\hookrightarrow} \underbrace{L' \overset{D'}{\hookrightarrow} L'' \overset{\epsilon''|_{L''}}{\hookrightarrow} U}_{\in W'},\ L \overset{D}{\hookrightarrow} \underbrace{L' \overset{D'}{\hookrightarrow} L'' \overset{\varphi''|_{L''}}{\hookrightarrow} Y}_{\in \tilde{N}'}$$

is in the image of $W \times \tilde{N}'$ under D^*, so in $W \times N$ and $\varphi''(L'' \backslash D' \circ D(L)) \subseteq Q$.

Thus the composite morphism $(L, L'', D' \circ D)$ is α-small, so $\xi'' \in \mathcal{N}_\alpha(\xi)$. In particular, if we take $\xi' = \xi$, it follows that

$$\mathcal{N}_{\alpha'}(\xi) \subseteq \mathcal{N}_\alpha(\xi).$$

is a subneighborhood of ξ. Moreover, for any $\xi'' \in \mathcal{N}_{\alpha'}(\xi)$, we claim $\mathcal{N}_{\alpha'}(\xi) \in \mathcal{N}(\xi'')$. Take an arbitrary $\tilde{\xi} = [\tilde{\epsilon}, \tilde{F}, \tilde{\varphi}] \in \mathcal{N}_{\alpha'}(\xi)$, we need to show that $\lambda : \xi'' \dashrightarrow \tilde{\xi}$ is α'-small. Since $\tilde{\xi} \in \mathcal{N}_{\alpha'}(\xi)$, so we have $\tilde{\epsilon}(\tilde{F}) \cap K \subseteq \tilde{\epsilon}(\tilde{L})$, $L \hookrightarrow \tilde{L} \to U$ is in $\tilde{W}'$. Since $\xi'' \in \mathcal{N}_{\alpha'}(\xi)$, so $L \hookrightarrow L'' \to U$ is also in $\tilde{W}'$. It implies that $L'' \hookrightarrow \tilde{L} \to U$ is in $\tilde{W}'$. Similarly we have $L'' \hookrightarrow \tilde{L} \to Y$ is in $\tilde{N}'$ and $\tilde{\varphi}(\tilde{L} \backslash D(L'')) \subseteq Q$. Thus $\mathcal{N}_{\alpha'}(\xi) \in \mathcal{N}(\xi'')$. Hence the result follows. □

Example 5.2.6. This topology is easily understood in the case $d = 0$. In chapter 4 we explained the relative configuration space $C(M, M_0; Y)$ with labels in Y. Let $V \subseteq U \subseteq M$ be open subsets, we claim that $B_0(U; Y) \cong C(M, M \backslash U; Y)$ and the diagram is commutative (The map rest will be defined below in Lemma 5.2.8.):

$$\begin{array}{ccc} B_0(U;Y) & \xrightarrow{\text{rest}} & B_0(V;Y) \\ \downarrow f & & \downarrow f \\ C(M, M\backslash U; Y) & \longrightarrow & C(M, M\backslash V; Y). \end{array}$$

Proof of the claim. Given an arbitrary $\xi = [z_1, z_2, \cdots, z_n; y_1, y_2, \cdots, y_n] \in C(M, M \backslash U; Y)$. The map $f : B_0(U; Y) \to C(M, M \backslash U; Y) : \xi \to \xi$ as a map of sets is an isomorphism. We assume it is a minimal representative, which means there is no $z_i \in M \backslash U$ and no $y_i = *$. Recall that a neighborhood basis at ξ was indexed by a natural number k and $\mu = (W_1, \cdots, W_n, W_0; N_1, \cdots, N_n, N_0)$ such that $W_i \cap W_j = \emptyset$, where $W_i \subseteq M$ is a neighborhood of z_i, $W_0 \subset M$ is a neighborhood of $M \backslash U$, $N_i \subseteq Y$ is a neighborhood of y_i and $N_0 \subseteq Y$ is a neighborhood of the base point $*$.

Write $z' = (z'_1, \cdots, z'_n, \cdots, z'_{n+k}), y' = (y'_1, \cdots, y'_n, \cdots, y'_{n+k})$. Set $\mathcal{W}_{k,\mu}(\xi) :=$

$$\left\{ (z';y') \in \tilde{C}^{n+k}(M) \times Y^{n+k} \;\middle|\; \begin{array}{l} \text{either } z' \in W_1 \times \cdots W_n \times M^k, y' \in N_1 \times N_n \times N_0^k; \\ \text{or } z' \in W_1 \times \cdots W_n \times W_0, y' \in N_1 \times N_n \times Y^k \end{array} \right\}.$$

Then the family $\{\mathcal{W}_{k,\mu}(\xi)\}$ is a neighborhood basis at ξ. Let $K \subset \overline{M\backslash W_0}$ be a compact subset of U containing $z_1, z_2, \cdots, z_n$. Take $L = \boldsymbol{n}, \epsilon \in \mathrm{Emb}(L, M)$ is the map assigning each i to z_i. Actually we can identify ϵ with $(z_1, \cdots, z_n)$ and set $W = W_1 \times \cdots \times W_n, N = N_1 \times \cdots \times N_n, Q = N_0$. Denote $\alpha = (K, L, W, N, Q)$, then $f(\mathcal{N}_\alpha(\xi)) \subset \mathcal{W}_{k,\mu}(\xi)$. It follows that f is continuous. Vice versa, given a neighborhood $\mathcal{N}_\alpha(\xi)$ of $\xi \in B_0(U;Y)$ with $\alpha = (K, L, W, N, Q)$. Write $L = \boldsymbol{n}$, then $W \subset \mathrm{Emb}(\boldsymbol{n}, U)$ is a neighborhood of $\epsilon|_{\boldsymbol{n}}$. That is, W is a neighborhood of $(z_1, \cdots, z_n) \in C^n(U)$ where $z_i = \epsilon(i)$. So there exists a neighborhood $W_1 \times \cdots \times W_n \subset W$ of $(z_1, \cdots, z_n)$ such that W_i is a neighborhood of z_i. By choosing each W_i small enough, we can assume that $W_i \cap W_j = \emptyset$. Similarly, $N \subseteq \mathrm{Map}(\boldsymbol{n}, Y)$ is a neighborhood of $\varphi|_{\boldsymbol{n}}$. That is, N is a neighborhood of $(y_1, \cdots, y_n) \in Y^n$ where $y_i = \varphi(i)$. So there exists a neighborhood $N_1 \times \cdots \times N_n \subset N$ of $(y_1, \cdots, y_n)$. Let $W_0 = M\backslash K, N_0 = Q$ and write $\mu = (W_1, \cdots, W_n, W_0; N_1, \cdots, N_n, N_0)$, then $\mathcal{W}_{n,\mu}(\xi)$ is a neighborhood of ξ in $C(M, M\backslash U; Y)$. Moreover $f^{-1}(\mathcal{W}_{n,\mu}(\xi)) \subset \mathcal{N}_\alpha(\xi)$,which implies that f^{-1} is continuous. Thus the homeomorphism follows. The commutativity of the diagram is left to the reader. □

Example 5.2.7. The base point in $B_d(U;Y)$ is represented by

$$\xi_\emptyset = [U \overset{\epsilon_\emptyset}{\hookleftarrow} \emptyset \xrightarrow{\varphi_\emptyset} Y] = [U \hookleftarrow F' \to *],$$

where $*$ is the base point in Y and F' is arbitrarily chosen. We discuss this example which will illustrate the role of the compact set K in its neighborhood basis.

Any morphism $(L, L', D) : [\epsilon, F, \varphi] \dashrightarrow [\epsilon_\emptyset, \emptyset, \varphi_\emptyset]$ must have $L = L' = \emptyset$ because $L' \subseteq \emptyset$ and $D : L \to L'$ is a smooth embedding. Then $W = \emptyset, N = \emptyset$. A neighborhood of $\xi_\emptyset$ is indexed by a compact set K. And the morphism (L, L', D) is K-small if and only if $\epsilon(F) \cap K = \emptyset$. That is,

$\mathcal{N}_K(\xi_\emptyset) := \{[\epsilon, F, \varphi] | [\epsilon, F, \varphi]$ is the minimal representative and $\epsilon(F) \cap K = \emptyset\}$.

In particular

- If X is a topological space and a map $f : X \to B_d(U;Y)$ is continuous at a point $x \in X$ with $f(x) = \xi_\emptyset$ if and only if for all compact subsets $K \subseteq U$ there exists a neighborhood $H \subseteq X$ of x such that $\epsilon(F) \cap K = \emptyset$ for all $y \in H$, where $f(y) = [\epsilon, F, \varphi]$ is the minimal representative.
- If $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of elements of $B_d(U;Y)$ with $\xi_n = [\epsilon_n, F_n, \varphi_n]$, then $\xi_n \to \xi_\emptyset$ if and only if for all compact subsets $K \subseteq U$, there exists $N_0 \in \mathbb{N}$ such that $\epsilon_n(F_n) \cap K = \emptyset$ for all $n > N_0$.

Lemma 5.2.8. Let $V \subseteq U \subseteq M$ be open subsets. Then the restriction map

$$\mathrm{rest} : \quad B_d(U;Y) \to B_d(V;Y) \tag{5.2.2}$$

$$[\epsilon, F, \varphi] \mapsto [\epsilon|, \epsilon^{-1}(\epsilon(F) \cap V), \varphi|] \tag{5.2.3}$$

is continuous (For reasons of brevity, we write $\epsilon|$ for the map ϵ restricted on $\epsilon^{-1}(\epsilon(F)$, that is $\epsilon|_{\epsilon^{-1}(\epsilon(F)\cap V)} : \epsilon^{-1}(\epsilon(F) \cap V) \to V$. And similarly for $\varphi|$).

Proof. Given an arbitrary $\xi = [\epsilon, F, \varphi] \in B_d(U;Y)$, denote its image $[\epsilon|, \epsilon^{-1}(\epsilon(F) \cap V), \varphi|]$ by $\mathrm{rest}(\xi)$. Let $\mathcal{N}_\alpha(\mathrm{rest}(\xi))$ be an open neighborhood of $\mathrm{rest}(\xi)$ with $\alpha = (K, L, W, N, Q)$. It consists of the following data: $K \subseteq V$ is compact, $L \subset \epsilon^{-1}(\epsilon(F)\cap V)$ is an open subset, $(\epsilon(F)\cap V)\cap K \subseteq \epsilon(L)$, $W \subseteq \mathrm{Emb}(L,V)$ is a neighborhood of $\epsilon|_L$, $N \subseteq \mathrm{Map}(L,Y)$ is a neighborhood of $\varphi|_L$, and Q is a neighborhood of $*$. Choose a metric on U, let $\delta := \mathrm{dist}(\epsilon(L), U\backslash V) > 0$. There is a neighborhood $\overline{W}$ of $\epsilon|_L$ in $\mathrm{Emb}(L,U)$ such that

$$e \in \overline{W} \Rightarrow \| e(x) - \epsilon(x) \| < \frac{1}{2}\delta, \text{ for all } x \in L.$$

then it follows that $e(L) \subseteq V$.

So we have a continuous map $\theta : \overline{W} \to \mathrm{Emb}(L,V), e \mapsto e$. Denote by $W' := \theta^{-1}(W)$, it is an open neighborhood of $\epsilon|_L$ in $\mathrm{Emb}(L,U)$.

Set $\alpha' = (K, L, W', N, Q)$, and take an arbitrary $\xi' = [\epsilon', F', \varphi'] \in \mathcal{N}_{\alpha'}(\xi)$. There exists the following data

(i) $D : L \hookrightarrow L' \subseteq F'$ is an embedding,
(ii) $\epsilon'(F') \cap K \subseteq \epsilon'(L')$,
(iii) $L \overset{D}{\hookrightarrow} L' \overset{\epsilon'|}{\hookrightarrow} U$ is in W', in particular $\epsilon' \circ D(L) \subseteq V$

constituting an α'-small morphism $\lambda = (L, L', D) : \xi \dashrightarrow \xi'$. Thus we obtain $\epsilon'(L') \subseteq \epsilon'(F) \cap V$, which implies that $L' \subseteq \epsilon'^{-1}(\epsilon'(F) \cap V)$. And $L \overset{D}{\hookrightarrow} L' \overset{\epsilon'}{\hookrightarrow} V$ is in W, $\varphi'(L'\backslash D(L)) \subseteq Q$.

This gives an α-small morphism in $B_d(V;Y)$

$$(L, L', D) : \mathrm{rest}(\xi') \dashrightarrow \mathrm{rest}(\xi),$$

which implies rest sends $\mathcal{N}_{\alpha'}(\xi)$ into $\mathcal{N}_\alpha(\mathrm{rest}(\xi))$. □

The sheaf property of $B_d(-;Y)$ means that the continuity of a map $f : X \to B_d(U;Y)$ is *local* in $X \times U$ in the following sense. Let U_α be a cover of U, and let $T = \coprod_\alpha U_\alpha$ and $T' = \coprod_{\alpha\neq\beta} U_\alpha \cap U_\beta$. There are two maps $T' \to T$ given by inclusion into the first and second terms of each intersection

$$\begin{array}{ccc} T' & \xrightarrow{i_1} & T \\ \downarrow{\scriptstyle i_2} & & \downarrow \\ T & \longrightarrow & U \end{array}$$

and U is the pushout of this diagram. Applying the sheaf $B_d(-;Y)$ we get two restriction maps $B_d(T;Y) \to B_d(T';Y)$ and a pullback diagram

$$\begin{array}{ccc} B_d(U;Y) & \longrightarrow & B_d(T;Y) \\ \downarrow & & \downarrow{\scriptstyle i_1^*} \\ B_d(T;Y) & \xrightarrow{i_2^*} & B_d(T';Y). \end{array}$$

In this diagram all the maps involved are restrictions and so are continuous.

Thus if we have a map $f : X \to B_d(U;Y)$ such that for any point $u \in U$ there is a neighborhood $U_u \subseteq U$ such that $X \xrightarrow{f} B_d(U;Y) \to B_d(U_u;Y)$ is continuous, then taking $\{U_u\}$ as the cover we have a continuous map from X to the diagram $B_d(T;Y) \to B_d(T';Y)$ and so a continuous map to the pullback. This map must be the original f, thus f is continuous. In other words, f is continuous if for each $x \in X$ and $u \in U$ there is a neighborhood $V_x \times U_u \subseteq X \times U$ such that the composition

$$V_x \to X \xrightarrow{f} B_d(U;Y) \xrightarrow{\mathrm{rest}} B_d(U_u;Y)$$

is continuous. In particular, $U \mapsto \mathrm{Map}(X, B_d(U;Y))$ is a sheaf for every space X.

The sheaf $B_d(-;Y)$ is an example of an equivariant, quasi-continuous sheaf explained in the Appendix A. This means that $B_d(-;Y)$ is continuously functorial with respect to embeddings (not just inclusions) of open subsets of M.

Theorem 5.2.9. Let $V \subseteq U$ be open subsets of M and $\mathrm{Emb}(V,U)$ the space of embeddings of V into U with weak C^∞-topology, then the action map

$$f_V : \mathrm{Emb}(V,U) \times B_d(U;Y) \rightarrow B_d(V;Y) \tag{5.2.4}$$

$$(e : V \hookrightarrow U, [\epsilon, F, \varphi]) \mapsto [e^{-1} \circ \epsilon|, \epsilon^{-1}(\epsilon(F) \cap e(V)), \varphi|] \tag{5.2.5}$$

is continuous.

Proof. We only need to show the continuity of f_V being local in $V \times \mathrm{Emb}(V,U) \times B_d(U;Y)$. The proof is almost the same as in Theorem 2.4.5. of [RW].

Choose a point $(x \in V, e : V \hookrightarrow U, [\epsilon, F, \varphi])$ and a neighborhood V'_x of x that is a coordinate patch, and that we shall identify $V'_x \cong \mathbb{R}^n$. It has a proper subneighborhood $V_x \subset V'_x$ such that $\bar{V}_x \subset V'_x$. Thus $e(V_x) \subset e(\bar{V}_x) \subset e(V'_x)$ and $e(\bar{V}_x)$ is compact because $\bar{V}_x$ is closed and bounded.

Define $U' := e(V'_x) \cong \mathbb{R}^n, K' := \bar{V}_x = \bar{D}^n$ which is the closed disk in $\mathbb{R}^n$. There exists an open set $M(K', U')$ in the weak topology consisting of smooth maps sending the compact K' into the open U'. $M(K', U')$ is an open neighborhood of e. For any $\phi \in M(K', U'), \phi(V_x) \subseteq \phi(K') \subset U'$. So $M(K', U') \subset \mathrm{Emb}(V_x, U')$. The following diagram commutes.

$$\begin{array}{ccc} M(K',U') \times B_d(U;Y) & \xrightarrow{f_{K'}} & B_d(V;Y) \\ \Big\downarrow g & & \Big\downarrow \text{rest} \\ \mathrm{Emb}(V_x,U') \times B_d(U';Y) & \xrightarrow{f_{V_x}} & B_d(V_x;Y) \end{array}$$

The horizontal maps $f_{K'}, f_{V_x}$ are actions maps as in (5.2.4). And g is induced by the inclusion of $M(K, U')$ and the restriction. Then f_K is continuous if and only if $\mathrm{rest} \circ f_K$ is continuous for any $x \in V$, thus if and only if $f_{V_x} \circ g$ is continuous. Since restrictions are continuous by Lemma 5.2.6., the vertical maps are continuous. Thus f_K will be continuous if f_{V_x} is continuous for any $x \in V$. Therefore we reduce to the case where U and V are diffeomorphic to $\mathbb{R}^n$.

Let $(e, \xi) \in \mathrm{Emb}(V,U) \times B_d(U;Y)$ with $\xi = [\epsilon, F, \varphi]$. Its image is $f_V(e,\xi) = [e^{-1} \circ \epsilon|, \epsilon^{-1}(\epsilon(F) \cap e(V)), \varphi|] \in B_d(V;Y)$. Let $\mathcal{N}_\beta(f_V(e,\xi))$ be a neighborhood of $f_V(e,\xi)$ with $\beta = (K, L, W, N, Q)$. This means the following data is given:

- (i) $K \subseteq V$ is compact;
- (ii) $L \subseteq \epsilon^{-1}(\epsilon(F) \cap e(V)) \subseteq F$ is an open subset that is the interior of its compact closure $\bar{L}$ of $\epsilon^{-1}(\epsilon(F) \cap e(V))$;
- (iii) $e^{-1}(\epsilon(F) \cap e(V)) \cap K \subseteq e^{-1}\epsilon(L)$;
- (iv) $W \subseteq \mathrm{Emb}(L, V)$;
- (v) $N \subseteq \mathrm{Map}(L, Y)$;
- (vi) Q is a neighborhood of $*$.

The standard metric on $\mathbb{R}^n$ gives a metric on U. Use e to determine a metric on V such that e is an isometry. Choose a compact ball $C \subseteq V$ such that $K \cup e^{-1} \circ \epsilon(\bar{L}) \subseteq C$ and $\operatorname{dist}(K \cup e^{-1} \circ \epsilon(\bar{L}), V\backslash C) =: \alpha_1 > 0$.

It follows that

$$\operatorname{dist}(e(K \cup e^{-1}(\epsilon(\bar{L}))), U\backslash e(C)) = \alpha_1 > 0.$$

So $e^{-1}(\epsilon(F) \cap e(V)) \cap K \subseteq e^{-1}(\epsilon(L)) \subseteq V$ is a compact subset of an open set.

Define

$$K^{+\delta} := \{v \in V| \ \exists k \in K, \text{ such that } \|v-k\| < \delta\} \supset K, \text{ it is an open neighborhood of } K,$$
$$L^{-\delta} := \{l \subseteq \epsilon^{-1}(\epsilon(F) \cap e(V)) | \text{for any } z \in \epsilon^{-1}(\epsilon(F) \cap e(V))\backslash L, \|l-z\| \geq \delta\} \subset L, \text{ a closed subset of } L.$$

Since $L^{-\delta}$ is a compact subset of an open set L, there exists $\alpha_2 > 0$ such that if $\delta < \alpha_2$, then

$$e^{-1}(\epsilon(F) \cap e(V)) \cap K^{+\delta} \subseteq e^{-1}(\epsilon(L^{-\delta})).$$

Let $\alpha = \min(\alpha_1, \alpha_2)$, then

$$e(K^{+\frac{1}{3}\alpha}) \cap \epsilon(F) = e(K^{+\frac{1}{3}\alpha} \cap e^{-1}(\epsilon(F) \cap e(V))) \subseteq \epsilon(L^{-\frac{1}{3}\alpha}).$$

Take an open neighborhood $N' \subseteq \operatorname{Emb}(V, U)$ of e such that for any $\phi \in N'$,

(i) $\phi(K) \subseteq e(K)^{+\frac{1}{3}\alpha} \subset e(C)$,

(ii) $e(C)^{-\frac{1}{3}\alpha} \subseteq \phi(C) \subseteq e(C)^{+\frac{1}{3}\alpha}$.

Let W_1 be a strong neighborhood of ϵ in $\operatorname{Emb}(L, U)$ such that

$$e' \in W_1 \Rightarrow \|e'(l) - \epsilon(l)\| < \frac{1}{3}\alpha.$$

Then $e'(L) \subseteq e(C)^{-\frac{2}{3}\alpha} \subseteq e(C)^{-\frac{1}{3}\alpha}$.

For $\phi \in N'$ and $e' \in W_1$, note that $e'(L) \subseteq e(C)^{-\frac{1}{3}\alpha} \subseteq \phi(V)$. Since $e(C)^{-\frac{1}{3}\alpha} = e(C^{-\frac{1}{3}\alpha}) \subset U$ a compact submanifold of U, we can apply Lemma 2.4.4. in [RW] to obtain the inverse of ϕ over $e(C)^{-\frac{1}{3}\alpha}$. So the map $I : N' \to \operatorname{Emb}(e(C)^{-\frac{1}{3}\alpha}, V), \phi \mapsto \phi|_{e(C)^{-\frac{1}{3}\alpha}}$ defined in Lemma 2.4.4. of [RW] is continuous.

Now consider the composite

$$\operatorname{comp} : W_1 \times N' \xrightarrow{\operatorname{id} \times I} W_1 \times \operatorname{Emb}(e(C)^{-\frac{1}{3}\alpha}, V) \xrightarrow{\circ} \operatorname{Emb}(L, V), \tag{5.2.6}$$

which is a composition of continuous maps so continuous.

Denote $W' \times \tilde{W} := \operatorname{comp}^{-1}(W) \subset W_1 \times N'$. Let $\beta' = (e(C), L, W', N, Q)$, we claim that $\tilde{W} \times \mathcal{N}_{\beta'}(\xi)$ is sent under the action map f_V into $\mathcal{N}_\beta(f_V(e, \xi))$.

Let $\phi \in \tilde{W}$ and $\xi' = [\epsilon', F', \varphi'] \in \mathcal{N}_{\beta'}(\xi)$, then we have an embedding $L \overset{d}{\hookrightarrow} L' \subseteq F'$ such that

$$e^{-1} \circ \epsilon'(F') \cap C \subseteq e^{-1} \circ \epsilon'(L'),$$
$$L \overset{D}{\hookrightarrow} L' \overset{\epsilon'|}{\hookrightarrow} U \text{ in } W' \subseteq W_1 \subseteq \operatorname{Emb}(L, U).$$

In particular $\epsilon'(L') \subseteq \phi(V)$.

Now consider

$$L \overset{D}{\hookrightarrow} L' \overset{\epsilon'|}{\to} \epsilon'(L') \xrightarrow{\phi^{-1}} \phi^{-1}(\epsilon'(L')) \hookrightarrow V,$$

which is obtained by applying comp in (5.2.2) to $(\epsilon' \circ D, \phi) \in W' \times \tilde{W}$. And the map comp sends $(\epsilon' \circ D, \phi)$ into W.

$$K \cap \phi^{-1}(\epsilon'(F')) = \phi^{-1}(\phi(K) \cap \epsilon'(F')) \subseteq \phi^{-1}(e(C) \cap \epsilon'(F')) \subseteq \phi^{-1}(\epsilon'(F')).$$

Denote $f_V(\phi, \xi') = [\phi^{-1} \circ \epsilon'|, \epsilon'^{-1}(\epsilon'(F') \cap \phi(V)), \varphi'|]$. It follows that (L, L', D) is a β-small morphism in $B_d(V;Y)$

$$f_V(e, \xi) \dashrightarrow f_V(\phi, \xi'),$$

which finishes the proof. □

Lemma 5.2.10. If $h : Y \times I \to Y$ is a homotopy, then the induced map

$$\begin{aligned} H : B_d(U;Y) \times I &\to B_d(U;Y) \\ ([\epsilon, F, \varphi], t) &\mapsto [\epsilon, F, h \circ (\varphi \times \{t\})] \end{aligned}$$

is continuous. Therefore $B_d(U;Y)$ is homotopy invariant of Y.

Proof. Let $\mathcal{N}_\alpha[\epsilon, F, h \circ (\varphi \times \{t\})]$ be an open neighborhood of $[\epsilon, F, h \circ (\varphi \times \{t\})]$ with $\alpha = (K, L, W, N, Q)$. Thus $K \subseteq U$ is compact, $L \subseteq F$ is an open subset, $\epsilon(F) \cap K \subseteq \epsilon(L)$, $W \subseteq \text{Emb}(L, U)$ is a neighborhood of $\epsilon|_L$, $N \subseteq \text{Map}(L, Y)$ is a neighborhood of $h \circ (\varphi \times \{t\})$. Let K' be a compact subset of L, and let O' be an open subset of Y, such that $h \circ (\varphi \times \{t\})(K') \subset O'$. There is an open set $M(K', O')$ consisting of continuous maps in N sending the compact set K' into the open set O'. So $h \circ (\varphi \times \{t\}) \in M(K', O') \subset N$. Since h is continuous, there exists an open subset $\tilde{O}' \times \tilde{I} \subset Y \times I$, such that $h(\tilde{O}' \times \tilde{I}) \subset O'$. Then for any $(\varphi', t') \in M(K', \tilde{O}') \times \tilde{I}$, we have

$$K' \subset L \xrightarrow{\varphi' \times \{t\}} Y \times \{t'\} \xrightarrow{\text{id} \times \{t'\}} Y \times I \xrightarrow{h} Y \supset O'.$$

Thus $h \circ (\varphi' \times \{t'\}) \in N$.

Let $\beta = (K, L, W, M(K', \tilde{O}'), Q)$, then $\mathcal{N}_\beta[\epsilon, F, \varphi] \times \tilde{I}$ is sent under H into $\mathcal{N}_\alpha[\epsilon, F, h \circ (\varphi \times \{t\})]$. Therefore H is continuous. □

The topology we defined on $B_d(U;Y)$ is notationally complex. If we consider the space $Y_+ = Y \sqcup +$, the disjoint union of a space Y and a discrete base point, the topology on $B_d(U;Y_+)$ in Definition 5.2.1. can be simplified by $Q = \{+\}$ and the map $d : L \to L'$ is a diffeomorphism. One can ignore the effect of Q. Since there are no restrictions of the connectivity of Y, all the results in this section also work for $B_d(M;Y_+)$. We shall describe the homotopy type of $B_d(M;Y_+)$ in section 5.2.3.

Lemma 5.2.11. If Y is connected, then $B_d(\mathbb{R}^n;Y_+)$ is path connected.

Proof. Let $\xi = [\epsilon, F, \varphi] \in B_d(U;Y_+)$ be a given minimal representative. We are going to construct an explicit path from ξ to the base point $\xi_\emptyset = [U \overset{\epsilon_\emptyset}{\hookleftarrow} \emptyset \xrightarrow{\varphi_\emptyset} Y_+]$.

Choose a point $p \in \mathbb{R}^n \backslash \epsilon(F)$ and associate the label $+$ to p. Let $\theta_t : \mathbb{R}^n \to \mathbb{R}^n, t \in [0,1]$ be the map given by $\theta_t(x) = (1-t)x + tp$. Then θ_t is a diffeomorphism for $t < 1$ and $\theta_1(x) = p$ for all x. We can identify $[\epsilon, F, \varphi]$ with $[\text{inc}, \epsilon(F), \varphi \circ \epsilon^{-1}]$. That is, regard $\epsilon(F)$ as a submanifold of $\mathbb{R}^n$ and the embeddings will always be inclusions. Let $F_t = \theta_t^{-1}(\epsilon(F))$ and φ_t be the composite $\varphi_t : F_t \xrightarrow{\theta_t} \epsilon(F) \xrightarrow{\varphi \circ \epsilon^{-1}} Y_+$. This defines a map $t \mapsto [\text{inc}, F_t, \varphi_t] \in B_d(\mathbb{R}^n;Y_+)$. By Theorem 5.2.8. this map is continuous on $[0,1)$. Continuity at 1 can be seen as follows. We need to use the neighborhood of $\xi_\emptyset$ as explained in Example 5.2.6.. For a given compact $K \subseteq \mathbb{R}^n$, choose $\delta > 0$ such that $K \subseteq B_{\delta^{-1}}(p)$, then $F_t \cap K = \emptyset$ for all $t > 1 - \delta$. □

5.2.2 Section Space $\mathrm{Sect}_d(M; Y_+)$

Let A be an $m-$dimensional real vector space. Define $V_k(A)$ $(1 \le k \le m)$ to be the Stiefel manifold of k-frames in A, i.e.

$$V_k(A) := \{(v_1, \cdots, v_k) \in A^k |\ v_1, \cdots, v_k \text{ are linearly independent } \} \cong \mathrm{LinEmb}(\mathbb{R}^k, A).$$

$\mathrm{LinEmb}(\mathbb{R}^k, A)$ denotes the space of linear embeddings of $\mathbb{R}^k$ into A with the compact-open topology. The homeomorphism is given by $f \mapsto (f(e_1), \cdots, f(e_k))$, where e_i is the i-th standard basis vector of $\mathbb{R}^m$.

In the case $k = m$, $V_m(A) = \mathrm{LinIsom}(\mathbb{R}^m, A) \cong \mathrm{LinIsom}(\mathbb{R}^m, \mathbb{R}^m) \cong \mathrm{GL}(\mathbb{R}^m)$.

What we did for a single vector space A, can be generalized to an m-dimensional vector bundle $A \to E \xrightarrow{\pi} B$. Define $V_k(\pi)(1 \le k \le m)$ to be the space of k-frames in a fibre, i.e.

$$V_k(\pi) := \left\{ \ (b; v_1, \cdots, v_k) \in B \times E^k \ \middle|\ (v_1, \cdots, v_k) \text{ is a k-frame in the fibre } \pi^{-1}(b) \ \right\}.$$

Remark 5.2.12. We obtain a new vector bundle called the frame bundle $V_k(\pi) \to B$. The total space $V_k(\pi)$ can be described as the space of bundle maps of the trivial bundle $\mathbb{R}^k$ into π, namely $V_k(\pi) \cong \mathrm{Emb}(\mathbb{R}^k, \pi)$.

Example 5.2.13. *(i)* $k = 1, V_1(\pi) = E \backslash E_0$, where E_0 is the zero section;

(ii) $V_m(\pi) = \mathrm{Prin}(\pi)$, the principal bundle of π with structure group $O(m)$.

Assume W is an n-dimensional manifold without boundary and containing M. For example, $W = M$ if M is closed, or $W = M \cup (\partial M \times [0, 1))$ if M has boundary.

Let $\mathrm{Gr}_d(\mathbb{R}^{n+d})$ be the Grassmannian of d-dimensional vector spaces in $\mathbb{R}^{n+d}$. We are interested in the orthogonal complement of the tautological bundle $U_{d,n} \to \mathrm{Gr}_d(\mathbb{R}^{n+d})$, namely

$$U_{d,n}^{\perp} = \{(V, v) \in \mathrm{Gr}_d(\mathbb{R}^{n+d}) \times \mathbb{R}^{n+d} |\ V \perp v\}.$$

The direct sum $U_{d,n} \oplus U_{d,n}^{\perp}$ is the trivial bundle $\mathrm{Gr}_d(\mathbb{R}^{n+d}) \times \mathbb{R}^{n+d}$. Since $U_{d,n+1}^{\perp}$ restricts over $\mathrm{Gr}_d(\mathbb{R}^{n+d})$ to the direct sum of $U_{d,n}^{\perp}$ and a trivial line bundle $\mathbb{R}^1$, there is an induced map

$$\mathrm{Th}(U_{d,n}^{\perp}) \wedge S^1 \to \mathrm{Th}(U_{d,n+1}^{\perp}). \tag{5.2.7}$$

For a fixed d, the Thom spaces $\mathrm{Th}(U_{d,n}^{\perp})$ define a spectrum MTO_d, where the n-th space is $\mathrm{Th}(U_{d,n}^{\perp})$. The associated infinite loop space is

$$\Omega^{\infty} \mathrm{MTO}_d = \operatorname*{colim}_{n \to \infty} \Omega^n \mathrm{Th}(U_{d,n}^{\perp}),$$

where the maps in the colimit

$$\Omega^n \mathrm{Th}(U_{d,n}^{\perp}) \to \Omega^{n+1} \mathrm{Th}(U_{d,n+1}^{\perp})$$

are the n-fold loops of the adjoint of (5.2.7). Note that our notation is different from those in [GMTW], where they denote the Thom spectrum to be $\mathrm{MTO}(d)$ and the $(n + d)$-th space in $\mathrm{MTO}(d)$ is $\mathrm{Th}(U_{d,n}^{\perp})$.

Let $V_n(TW)$ be the frame bundle of W. Replace the fibre $O(n)$ by $\mathrm{Th}(U_{d,n}^{\perp})$. We obtain a new bundle over W, namely

$$E_d(W) := V_n(TW) \times_{O(n)} \mathrm{Th}(U_{d,n}^{\perp}) \xrightarrow{\pi} W. \tag{5.2.8}$$

The action of $O(n)$ on the new fibre $\mathrm{Th}(U_{d,n}^{\perp})$ has a fixed point, namely the infinite section $s_\infty : M \to E_d(M)$ sending $w \in W$ to ∞ in the fibre $\mathrm{Th}(U_{d,n}^{\perp})$ over w.

Define $\mathrm{Sect}_d(W, W\backslash M)$ to be the space of sections $s : W \to E_d(W)$ such that s agrees with s_∞ on $W\backslash M$. This section space has the compact-open topology.

Example 5.2.14. Assume W is parallalizable, i.e. $TW \cong W \times \mathbb{R}^n$. Then $V_n(TW) \cong W \times O(n)$ and $E_d(W) \cong W \times \mathrm{Th}(U_{d,n}^{\perp})$. Therefore, $\mathrm{Sect}_d(W, W\backslash M) \cong \mathrm{Map}_*(W, W\backslash M; \mathrm{Th}(U_{d,n}^{\perp}), \infty)$.

Next we want to amplify this construction by a label space Y. Let Y be a connected space. We replace the fibre of $V_n(TW) \to W$ by $\mathrm{Th}(U_{d,n}^{\perp}) \wedge Y_+$. The action of $O(n)$ on Y_+ is trivial. Denote this new bundle by

$$E_d(W; Y_+) := V_n(TW) \times_{O(n)} (\mathrm{Th}(U_{d,n}^{\perp}) \wedge Y_+) \xrightarrow{\pi} W. \tag{5.2.9}$$

Define $\mathrm{Sect}_d(W, W\backslash M; Y_+) := \mathrm{Sect}_d(E_d(W; Y_+), M)$, space of sections of the fiber bundle $E_d(W; Y_+) \to W$.

In the next sections we are mainly focusing on the case that M is open and has no boundary. In this case $W = M$. So we shall abbreviate $\mathrm{Sect}_d(W, W\backslash M; Y_+)$ by $\mathrm{Sect}_d(M; Y_+)$.

5.2.3 A Scanning Construction for the Thom Spectrum MTO_d

There exists a map

$$\gamma : B_d(M; Y_+) \to \mathrm{Sect}_d(M; Y_+).$$

This map is the so called "scanning map" introduced by G. Segal. It appears in some other forms as well as in the h-principle construction of Gromov. We indicate the construction of this map:

(i) Assume M has a Riemannian metric. Given $\xi = [\epsilon, F, \varphi] \in B_d(M; Y)$ with $M \xleftarrow{\epsilon} F \xrightarrow{\varphi} Y$, take an "observer" $z \in M$. Let $D(TM_z)$ be the unit disc in the tangent space TM_z to M at z. Suppose that $\rho > 0$ is small enough so that for each $z \in M$ the *exponential map* $\exp_z : D(TM_z) \to B_\rho(z) = \{z' : d(z, z') \leq \rho\} \subset M$ is a diffeomorphism. We "scan" with respect to this chosen family of neighborhoods $B_\rho(z)$ in the ground manifold M as follows.

(ii) Write $N = \{z \in M | B_\rho(z) \cap \epsilon(F) \neq \emptyset\}$, it is a tubular neighborhood of $\epsilon(F)$. Identify N with the normal bundle $\nu(\epsilon(F))$ of $\epsilon(F)$. If the point z lies in N, it is mapped to the corresponding point in $\nu(\epsilon(F))$, denote this point again by z. Let $\nu(z)$ be the image of z under the bundle projection $\nu : \nu(\epsilon(F)) \to \epsilon(F)$. If z is outside of N, it is mapped to $\infty \in \mathrm{Th}(U_{d,n}^{\perp})$.

We first choose a frame $\overrightarrow{v} = (v_1, \cdots, v_n) \subset T(M)_z$. Then define the section $\gamma(\xi)$ by the formula:

$$\gamma(\xi)(z) := [(v_1/\rho(z), \cdots, v_n/\rho(z); (T_{\nu(z)}(\epsilon(F)), \nu^{-1}(\nu(z))) \wedge \varphi(\epsilon^{-1}(\nu(z))].$$

$\gamma(\xi)$ is continuous, because the exponential map exp is continuous on $D(T(M)_z)$. And γ is continuous as well, because its adjoint

$$\gamma^\sharp : B_d(M; Y_+) \times M \to E_d(M; Y_+)$$

is the Thom-Pontrjagin construction, which is continuous.

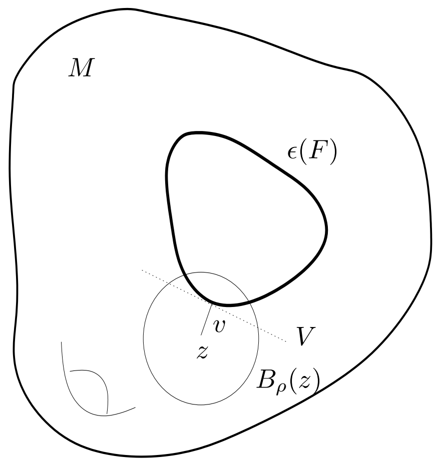


Figure 5.1: scanning

Proposition 5.2.15. Let M be a n-dimensional open manifold (i.e. no component of M is compact), $n \geq d$, then

$$\gamma : B_d(M; Y_+) \to \mathrm{Sect}_d(M; Y_+)$$

is a weak homotopy equivalence.

The rest of this section is dedicated to the proof of Proposition 5.2.15.. First we do not claim any originality for this proof. The main work has been done in [RW] by Randal-Williams in the case $Y = \mathrm{pt}$ and in [Ga] by Galatius in the case of graph spaces. We applied their ideas and generalized it to an arbitrary space Y_+. We use the h-principle as stated in Appendix A. Namely we need to show $B_d^c(M \times \mathbb{R}^{d+1}; Y_+)$ is both $\mathrm{Diff}(M \times \mathbb{R}^{d+1})$-invariant and microflexible. That $B_d^c(M \times \mathbb{R}^{d+1}; Y_+)$ is $\mathrm{Diff}(M \times \mathbb{R}^{d+1})$ -invariant is implied by Theorem 5.2.9.. So it remains only to show the sheaf is microflexible. The proof is based on the following lemmas and propositions.

Lemma 5.2.16. Let $U \subseteq V$ be an open subset of V, let X be a smooth k-dimensional manifold and Y_+ a topological space with discrete base point $+$. For a smooth $(d + k)$-dimensional submanifold $\Gamma \subset X \times U$, closed as a subspace, consider the smooth projection $\pi_1 : \Gamma \to X$ and a continuous map $\varphi : \Gamma \to Y_+$. Set $\Gamma_0 = \Gamma \backslash \varphi^{-1}(+)$ and assume that $\pi_1|_{\Gamma_0}$ is a submersion. Then there is a continuous map $f : X \to B_d(U; Y_+)$ such that Γ_0 is the adjoint graph Γ_f of f, i.e. $\Gamma_f = \{(x, u) \in X \times U | u \in \epsilon(F)$ if $f(x) = [\epsilon, F, \varphi]$ the minimal representative$\}$.

Proof. Note that $[\epsilon, F, \varphi]$ and $[\mathrm{inc}, \epsilon(F), \varphi \circ \epsilon^{-1}]$ represent the same equivalence class, where inc means the inclusions and $\epsilon(F)$ is a submanifold of U. We shall not distinguish them. Now define

$$\begin{aligned} f : X &\to B_d(U; Y_+) \\ x &\mapsto [U \overset{\mathrm{inc}}{\hookleftarrow} A_x \xrightarrow{\varphi|_{A_x}} Y_+] \end{aligned}$$

where $A_x := \pi_1^{-1}(x) \cap \Gamma_0 \subset \{x\} \times U$. We identify $\{x\} \times U$ with U and regard A_x as a submanifold of U.

We need to show that f is continuous. Let an arbitrary $(x, u) \in \Gamma_0 \subseteq X \times U$ be given, by locality, we only need to show f is continuous at (x, u) . Since Γ_0 is a manifold and $\pi_1 : \Gamma_0 \to X$ is a submersion, there are neighborhoods $V_x \subseteq X$ of x and $U_u \subseteq U$ of u such that

$$\pi_1| : \Gamma_0 \cap (V_x \times U_u) \to V_x \tag{5.2.10}$$

has fibers diffeomorphic to $\mathbb{R}^d$, and it is surjective. We may assume V_x is contractible. Like in Lemma 4.2.5. of [RW], we know that $\pi_1|$ in (5.2.10) is a trivial vector bundle. Choose a bundle diffeomorphism $D : \Gamma_0 \cap (V_x \times U_u) \to V_x \times \mathbb{R}^d$.

Consider

$$\xi = [U_u \overset{\mathrm{inc}|}{\hookleftarrow} A_x \cap \Gamma_0 \cap U_u \overset{\varphi|}{\to} Y_+] \in B_d(U_u; Y_+),$$

and let $\mathcal{N}_\alpha(\xi)$ be a neighborhood of ξ with $\alpha = (K, L, W, N)$. The trivilization D gives a diffeomorphism $t_y : A_x \cap \Gamma_0 \cap U_u \to A_y \cap \Gamma_0 \cap U_u$, for all $y \in V_x$.

Since $L \subseteq A_x \cap \Gamma_0 \cap U_u$ is an open subset, we may restrict t_y to L and obtain $t_y : L \to t_y(L) \to U_u$. Therefore we have a continuous map $t : V_x \to \mathrm{Emb}(L, U_u)$.

Since $W \subseteq \mathrm{Emb}(L, U_u)$ is a neighborhood of $\mathrm{inc}|_L$, let $W_1 := t^{-1}(W) \subseteq V_x$. There is a subneighborhood $W_2 \subseteq V_x$ of x such that $A_y \cap \mathrm{inc}^{-1}(K) \subseteq t_y(L)$. For each $y \in V_x$, we also obtain a map $L \to t_y(L) \overset{\varphi|_{t_y(L)}}{\longrightarrow} Y_+$. Thus we obtain a continuous map $s : V_x \to \mathrm{Map}(L, Y_+)$.

Note that $N \subseteq \mathrm{Map}(L, Y_+)$ is an open neighborhood of $\varphi|_L$, denote $N_1 := s^{-1}(N) \subseteq V_x$ and $\bar{W} := W_1 \cap W_2 \cap N_1$. So if $y \in \bar{W}$, then $t_y : L \to t_y(L) \subseteq A_y \cap \Gamma_0 \cap U_u$ and $\mathrm{inc}(A_x) \cap K \subseteq \mathrm{inc}(L), \mathrm{inc}(A_y) \cap K \subseteq \mathrm{inc}(t_y(L))$, $L \overset{t_y}{\longrightarrow} t_y(L) \hookrightarrow U_u$ is in W, $L \overset{t_y}{\longrightarrow} t_y(L) \to Y_+$ is in N. So it implies that

$$[U_u \overset{\mathrm{inc}|}{\hookleftarrow} A_y \cap (\Gamma_0 \cap U_u) \overset{\varphi|}{\longrightarrow} Y_+] \in \mathcal{N}_\alpha(\xi).$$

which means $f(\bar{W}) \subseteq \mathcal{N}_\alpha(\xi)$. Thus we have continuity at (x, u). □

We should remind the reader, if space Y has a non-discrete base point, this lemma will not be true. It might happen that $\pi_1| : \Gamma \cap V_x \times U_u \to V_x$ is not a vector bundle. The advantage of a discrete base point $+$ is that if any point of a component in Γ has the trivial label $+$, then the entire component must have the trivial label $+$. So we can remove this entire component. The resulting submanifold of Γ_0 can be described as an adjoint-graph of some function f.

We collect some definitions from [RW].

Definition 5.2.17. For a smooth manifold X, we say a continuous map $f : X \to B_d(U; Y_+)$ is *smooth near* (x, u) if there are neighborhoods $x \in V_x \subseteq X$ and $u \in U_u \subseteq U$ such that the composite

$$V_x \hookrightarrow X \overset{f}{\to} B_d(U; Y_+) \overset{\mathrm{rest}}{\longrightarrow} B_d(U_u; Y_+)$$

has a Γ that satisfies the conditions of Lemma 5.2.15: Γ is a smooth $(k+d)$-dimensional manifold, $\pi_1 : \Gamma \to X$ is smooth, $\Gamma_0 \to X$ is a submersion and $\varphi : \Gamma \to Y_+$ is continuous.

For a closed submanifold $A \subseteq X \times U$, we say f is *smooth near* A if it is smooth near each point of A.

Our fundamental problem is to find conditions when one can deform this adjoint-graph and still obtain an adjoint-graph of a new function.

Let $\tau : X \times U \to [0,1]$ and $F_\tau : [0,1] \times X \times U \times Y_+ \to [0,1] \times X \times U \times Y_+, (t, x, u, y) \mapsto (t\tau(x, u), x, u, y)$. If $f : [0,1] \times X \to B_d(U; Y_+)$ is a homotopy, then $\Gamma_f \subseteq [0,1] \times X \times U \times Y_+$. Let $\Gamma := F_\tau^{-1}(\Gamma_f) \subset$

$[0,1] \times X \times U \times Y_+$. There exists a map $\varphi_f : \Gamma \to Y_+$ induced by f and F_τ. Because f and F_τ are continuous, so is φ_f.

The following lemma gives a criterion for $\Gamma_0 = \Gamma \backslash \varphi_f^{-1}(+)$ to be the adjoint-graph Γ_{f_τ} for some continuous map $f_\tau : [0,1] \times X \to B_d(U; Y_+)$.

Lemma 5.2.18. Γ_0 is the adjoint graph Γ_{f_τ} for f_τ, if one of the two conditions holds

i) τ is independent of u, or

ii) f is smooth and $F_\tau|_{\{t\} \times X \times U \times Y_+}$ is transversal to Γ_f for all t.

Proof. In case i), write $\tau(x,u) = \sigma(x)$. Define

$$f_\tau : [0,1] \times X \to [0,1] \times X \xrightarrow{f} B_d(U; Y_+).$$

where the first map is $(t,x) \mapsto (t\sigma(x), x)$. Note for each $[\epsilon, F, \varphi] \in B_d(U; Y_+)$, we identify F with $\epsilon(F)$ as a submanifold of U and the embedding to be inclusion since F and $\epsilon(F)$ are of the same diffeomorphism type. Let $(t,x,u) \in \Gamma_0$ and $\varphi_f(t,x,u) = y$, so $(t\sigma(x), x, u, y) \in \Gamma_f$, thus $(u,y) \in f(t\sigma(x), x)$. It implies that $(u,y) \in f_\tau(t,x)$, so $(t,x,u,y) \in \Gamma_{f_\tau}$. The reverse inclusion holds similarly.

In case ii), by transversality, $F_\tau^{-1}(\Gamma_f) = \Gamma_0$ is a smooth manifold. Γ_0 can be identified with the core manifolds of Γ_f, that is, all the manifolds without the trivial label $+$. This completely characterizes the map f. So we can identify Γ_0 with Γ_f via F_τ. And $\Gamma_0 \to [0,1] \times X$ is also a submersion. $\Gamma \to Y_+$ is given by the composition $\Gamma \to \Gamma_f \to Y_+$, thus $\Gamma_0 = \Gamma_{f_\tau}$. ☐

Proposition 5.2.19. Let $K \subseteq U$ be compact and P be a polyhedron. Let $f : [0,1] \times P \to B_d(U; Y_+)$ be continuous. Then there exists an $\epsilon > 0$ and a continuous map $g : [0,\epsilon] \times P \to B_d(U; Y_+)$ such that

i) $f|_{[0,\epsilon] \times P}$ agrees with g on a neighborhood of K. Namely, $f = g : [0,\epsilon] \times P \to B_d(U; Y_+) \to B_d(\mathcal{N}(K); Y_+)$.

ii) $g|_{\{0\} \times P} = f|_{\{0\} \times P}$;

iii) there exists a compact subset $C \subseteq U$ such that

$$[0,\epsilon] \times P \xrightarrow{g} B_d(U; Y_+) \xrightarrow{\text{rest}} B_d(U \backslash C; Y_+)$$

factors through the projection $\text{pr} : [0,\epsilon] \times P \to P$, i.e. the following diagram commutes

$$\begin{array}{ccccc} \{0\} \times P & & \searrow & & \\ \uparrow \text{pr} & & & & \\ [0,\epsilon] \times P & \xrightarrow{g} & B_d(U; Y_+) & \xrightarrow{\text{rest}} & B_d(U \backslash C; Y_+). \end{array}$$

Proof. Choose $\tau : P \times U \to [0,1]$ with compact support and constant to 1 on a neighborhood $P \times B$ of $P \times K$. Let $A \subseteq U \backslash K$ be a closed set such that τ is locally constant on $P \times (U \backslash A)$. Assume $B \subseteq U \backslash A$ is a closed neighborhood of K. By lemma 4.2.8. in [RW], we may assume that f is smooth near $P \times A$ and unchanged near $P \times B$.

Let $F_\tau : [0,1] \times P \times U \times Y_+ \to [0,1] \times P \times U \times Y_+, (t,p,u,y) \to (t\tau(p,u), p, u, y)$. The transversality condition is satisfied on $P \times A$ at $t = 0$, and so it is satisfied for $t \in [0,\epsilon]$ for some $\epsilon > 0$, as transversality is an open condition. On $P \times (U \backslash A)$, τ is locally constant, so the first condition of Lemma 5.2.19. is satisfied. Thus $(F_\tau^{-1}(\Gamma_f) \backslash \varphi_f^{-1}(+)) \cap [0,\epsilon] \times P \times U \times Y_+ = \Gamma_{f_\tau}$ for continuous f_τ.

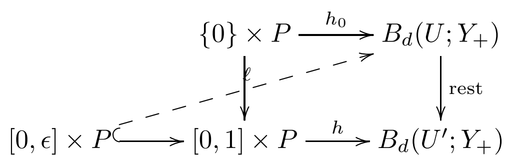
Choose $g = f_\tau : [0,\epsilon] \times P \to B_d(U; Y_+)$. Part i) is satisfied since $\tau(p,u) = 1$, for all $(p,u) \in P \times B$. It follows that $F_\tau|_{[0,1]\times P\times B\times Y_+} = \mathrm{id}$, so $f|_{[0,\epsilon]\times P}$ and f_τ agree on $B \supseteq K$.

Part ii) is also true, because $F_\tau(0,p,u,y) = (0,p,u,y)$, so $f|_{\{0\}\times P} = g|_{\{0\}\times P}$.

For part iii), take C to be such that $\mathrm{supp}(\tau) \subseteq P \times C$, then $\tau|_{P\times(U\backslash C)} = 0$. For any $u \notin C$, $F_\tau(t,p,u,y) = (0,p,u,y)$, so $\Gamma_{f_\tau} \cap ([0,\epsilon] \times P \times (U\backslash C) \times Y_+) = [0,\epsilon] \times \Gamma_{f_\tau(0,p,u,y)}$, so the factorization is satisfied. □

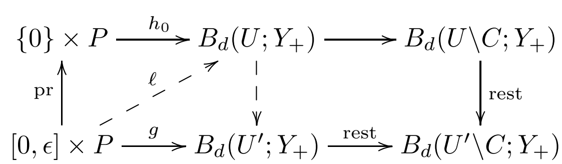
Microflexibility. Let $K' \subseteq K \subseteq V$ be compact subsets of V. A sheaf on a closed set is defined by $B_d(K; Y_+) := \mathrm{colim}_{K\subset U} B_d(U; Y_+)$, the colimit is taken over all open sets containing K, partially ordered by inclusion.

It suffices to show that for all open sets $U \supset K, U' \supset K'$ with $U' \subseteq U$ and for all the squares of the form



there is an $\epsilon > 0$, and a lifting $\ell : [0,\epsilon] \times P \to B_d(U; Y_+)$ extending h_0 over $h|_{[0,\epsilon]\times P}$.

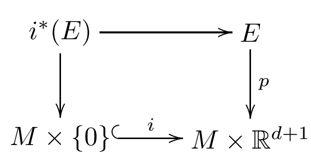
Since $K' \subseteq U'$ is compact and $h : [0,1] \times P \to B_d(U'; Y_+)$ is continuous, there is an $\epsilon > 0$ and a continuous map $g : [0,\epsilon] \times P \to B_d(U'; Y_+)$ satisfying the properties of Proposition 5.2.20.. Let $C \subseteq U'$ be the compact set given in this proposition. We have the commutative diagram



Regard $B_d(U; Y)$ as the pull back of $B_d(U'; Y_+) \to B_d(U'\backslash C; Y_+) \leftarrow B_d(U\backslash C; Y_+)$ in the diagram. By universal property, we obtain a continuous map $\ell : [0,\epsilon] \times P \to B_d(U; Y_+)$ and its restriction to U' is g. Since g agrees with $h|_{[0,\epsilon]\times P}$ near K', we can pass to a smaller $K' \subset \tilde{U}' \subset U'$ on which they agree. Then ℓ is a lifting extending h_0 and covering $[0,\epsilon] \times P \to B_d(U'; Y_+) \to B_d(\tilde{U}'; Y_+)$.

We are interested in manifolds of the form $M \times \mathbb{R}^{d+1}$. We will see that the homotopy type of $B_d(M \times \mathbb{R}^{d+1}; Y)$ is connected to the Thom spectrum MTO_d.

Let $p : E \to M \times \mathbb{R}^{d+1}$ be a fiber bundle with a "zero" element in each fiber, so we can define the support of sections. If a section $s : M \times \mathbb{R}^{d+1} \to E$ is supported in $K \times \mathbb{R}^{d+1}$ for some compact $K \subset M$, then its restriction $i^*(s) : M \times \{0\} \to i^*(E)$ is supported in K.



Thus the map $\mathrm{Sect}_d(M \times \mathbb{R}^{d+1}) \xrightarrow{i^*} \mathrm{Sect}_d(M)$ is a homotopy equivalence. A homotopy inverse is induced by the projection to the center.

Let M be an arbitrary $(n-1)$-dimensional manifold, $n \geq d$, let $B_d^c(M \times \mathbb{R}^{d+1}; Y_+) \subseteq B_d(M \times \mathbb{R}^{d+1}; Y_+)$ consist of those d-dimensional submanifolds such that the projection to $\mathbb{R}^{d+1}$ is proper, i.e.

$$B_d^c(M \times \mathbb{R}^{d+1}; Y_+) = \left\{ (M \times \mathbb{R}^{d+1} \overset{\epsilon}{\hookleftarrow} F \overset{\varphi}{\to} Y_+) \in B_d(M \times \mathbb{R}^{d+1}; Y_+) \;\middle|\; \begin{array}{l} \text{there exists a compact set } K \subset M \\ \text{such that } \epsilon(F) \subseteq K \times \mathbb{R}^{d+1} \end{array} \right\}.$$

Let $\mathrm{Sect}_d^c(M \times \mathbb{R}^{d+1}; Y_+) \subseteq \mathrm{Sect}_d(M \times \mathbb{R}^{d+1}; Y_+)$ consist of all the compactly-supported sections, i.e.

$$\mathrm{Sect}_d^c(M \times \mathbb{R}^{d+1}; Y_+) = \left\{ s : M \to E_d(M \times \mathbb{R}^{d+1}; Y_+) \;\middle|\; \begin{array}{l} s \text{ agrees with } s_\infty \text{ on } M \backslash K \text{ for a} \\ \text{compact subset } K \subset M \end{array} \right\}.$$

Corollary 5.2.20. The scanning map

$$B_d^c(M \times \mathbb{R}^{d+1}; Y_+) \to \mathrm{Sect}_d^c(M \times \mathbb{R}^{d+1}; Y_+)$$

is a weak homotopy equivalence. □

Here we write $B_d^c(M \times M'; Y_+)$ and $\mathrm{Sect}_d^c(M \times M'; Y_+)$ to mean that the compactness condition only refers to the first variable M. If we take $M = \mathbb{R}^{n-1}$, we have

Corollary 5.2.21. $\gamma : B_d^c(\mathbb{R}^{n-1} \times \mathbb{R}^{d+1}; Y_+) \to \mathrm{Sect}_d^c(\mathbb{R}^{n-1} \times \mathbb{R}^{d+1}; Y_+) \simeq \Omega^{n-1}(\mathrm{Th}(U_{d,n}^\perp) \wedge Y_+)$ is a weak homotopy equivalence. □

The natural inclusions $\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$ induce maps

$$B_d^c(\mathbb{R}^{n-1} \times \mathbb{R}^{d+1}; Y_+) \to B_d^c(\mathbb{R}^n \times \mathbb{R}^{d+1}; Y_+).$$

We define the colimit of this sequence to be

$$B_d^c(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+) := \lim_n B_d^c(\mathbb{R}^{n-1} \times \mathbb{R}^{d+1}; Y_+).$$

By corollary 5.2.21. we obtain

$$B_d^c(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+) \sim \mathrm{Sect}_d^c(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+) \simeq \Omega^{\infty-1}\, \mathrm{MTO}_d \wedge Y_+$$

which is the infinite loop space of the Thom spectrum $\mathrm{MTO}_d \wedge Y_+$.

5.3 Splitting of the Functor K

5.3.1 Homotopy Type of the Splitting Space $\mathbb{B}(Y_+)$

[GMTW] identifies the homotopy type of the cobodism category for a fixed d. We are interested in understanding some further properties as d varies. Namely, in our case, the dimension d is not fixed and can tend to infinity.

We now collect some facts from section 3.1 in [GMTW]. For two vector bundles E_1 and E_2 over the same base space X, let $p : S(E_2) \to X$ be the bundle projection of the unit sphere bundle. There is a cofiber sequence

$$\mathrm{Th}(p^* E_1) \to \mathrm{Th}(E_1) \to \mathrm{Th}(E_1 \oplus E_2). \tag{5.3.1}$$

Apply this to $X = \mathrm{Gr}_d(\mathbb{R}^{d+n}), E_1 = U_{d,n}^{\perp}, E_2 = U_{d,n}$. Then the cofiber sequence induces a direct system of spectra

$$\mathrm{MTO}_0 \to \mathrm{MTO}_1 \to \cdots \to \mathrm{MTO}_{d-1} \to \mathrm{MTO}_d \to \cdots \tag{5.3.2}$$

whose direct limit MTO is weakly equivalent to the Thom spectrum MO, because $U_{d,n}^{\perp} \cong U_{n,d}$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Gr}_d(\mathbb{R}^{n+d}) & \xrightarrow{\cong} & \mathrm{Gr}_n(\mathbb{R}^{n+d}) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_d(\mathbb{R}^{n+d+1}) & \xrightarrow{\cong} & \mathrm{Gr}_{n+1}(\mathbb{R}^{n+d+1}) \end{array}$$

The direct system can be thought of as a filtration of MTO, with filtration quotient the suspension spectrum $\underline{\Sigma}(\Sigma^d \mathrm{BO}(d)_+)$. In particular, the maps in the direct system induce an isomorphism, namely the homotopy groups $\pi_* \mathrm{MTO}_d$ can be computed by the homotopy groups of MO, the unoriented bordism ring Ω^O. That is, $\pi_* \mathrm{MTO}_d = \Omega^O_*$, for $* < d$.

Recall that the Pontrjagin-Thom construction gives a geometric description of the homotopy groups $\pi_n \mathrm{MTO}_d$, which agrees with $\pi_n(\Omega^\infty \mathrm{MTO}_d)$, for $n > 0$. And $\pi_n \mathrm{MTO}_d$ is isomorphic to the group of the bordism classes of pairs (M, ϕ), where M is a closed smooth $(n-d)$-dimensional manifold, ϕ is a map of stable vector bundles

$$\begin{array}{ccc} \nu_M & \xrightarrow{\phi} & U_{d,*}^{\perp} \\ \downarrow & & \downarrow \\ M & \longrightarrow & Gr_d(\mathbb{R}^{d+*}). \end{array}$$

Let $E_* = \{E_0 \subseteq E_1 \subseteq \cdots\}$ be a sequence of topological spaces and $E_* \to B_*$ be a *stable vector bundle* of dimension d, i.e. a sequence $E_n \to B_n$ of real vector bundles of dimension $n+d$ together with isomorphisms $\epsilon_n : E_n \oplus \mathbb{R} \cong E_{n+1}|_{B_n}$. The stable normal bundle ν_M of a closed m-dimensional manifold is an example.

On the other hand, the sequence (5.3.2) induces a sequence of infinite loop spaces

$$\Omega^{\infty-1} \mathrm{MTO}_0 \to \Omega^{\infty-1} \mathrm{MTO}_1 \to \cdots \to \Omega^{\infty-1} \mathrm{MTO}_d \to \Omega^{\infty-1} \mathrm{MTO}_{d+1} \to \cdots \tag{5.3.3}$$

We would like to understand this sequence in the geometric point of view.

An element of $\Omega^{\infty-1} \mathrm{MTO}_d$ is represented by some loop $\omega : S^{n-1} \times \mathbb{R}^{d+1} \to \mathrm{Th}(U_{d,n}^{\perp})$. Note that ω is homotopic to a map that is transverse to the zero section; denote this loop again by ω. And $F^d = \omega^{-1}(Gr_d(\mathbb{R}^{n+d})) \subset S^{n-1} \times \mathbb{R}^{d+1}$ is a submanifold of codimension n. So we have the following commutative diagram

$$\begin{array}{ccccccc} F^d \subset & & S^{n-1} \times \mathbb{R}^{d+1} & \xrightarrow{\omega} & \mathrm{Th}(U_{d,n}^{\perp}) & & \supset V \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F^d \times \mathbb{R} \subset & & S^{n-1} \times \mathbb{R}^{d+2} & \longrightarrow & \mathrm{Th}(U_{d+1,n}^{\perp}) & & \supset V \oplus \mathbb{R}. \end{array} \tag{5.3.4}$$

Therefore we obtain the following geometric stabilization

$$\begin{array}{ccc} B_d^c(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+) & \to & B_{d+1}^c(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+2}; Y_+) \\ \mathbb{R}^{\infty+d} \overset{\epsilon}{\hookleftarrow} F^d \xrightarrow{\varphi} Y_+ & \mapsto & \mathbb{R}^{\infty+d+1} \hookleftarrow F^d \times \mathbb{R}^1 \xrightarrow{\varphi'} Y_+. \end{array} \tag{5.3.5}$$

where $\varphi' : F \times \mathbb{R}^1 \to Y_+, (f,t) \mapsto \varphi(f)$.

This stabilization is compatible with its algebraic correspondence. Namely for $d \geq 0$, (5.3.4) and (5.3.5) combined together, we obtain a commutative diagram

$$\begin{array}{ccc} B^c_d(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+) & \longrightarrow & B^c_{d+1}(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+2}; Y_+) \\ \downarrow & & \downarrow \\ \Omega^{\infty-1}\operatorname{MTO}_d \wedge Y_+ & \longrightarrow & \Omega^{\infty-1}\operatorname{MTO}_{d+1} \wedge Y_+. \end{array} \tag{5.3.6}$$

Define

$$\mathbb{B}(Y_+) := \operatorname*{colim}_d B^c_d(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+),$$
$$\operatorname{Sect}(Y_+) := \operatorname*{colim}_d \operatorname{Sect}^c_d(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+).$$

Consequently it follows that there is an induced map $\mathbb{B}(Y_+) \to \operatorname{Sect}(Y_+) \simeq \Omega^{\infty-1}\operatorname{MO} \wedge Y_+$,

$$\begin{array}{ccccccc} B^c_0(\mathbb{R}^{\infty-1} \times \mathbb{R}; Y_+) & \longrightarrow \cdots \longrightarrow & B^c_d(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+) & \longrightarrow \cdots \xrightarrow{\text{colim}} & \mathbb{B}(Y_+) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \\ \operatorname{Sect}^c_0(\mathbb{R}^{\infty-1} \times \mathbb{R}; Y_+) & \longrightarrow \cdots \longrightarrow & \operatorname{Sect}^c_d(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; Y_+) & \longrightarrow \cdots \xrightarrow{\text{colim}} & \operatorname{Sect}(Y_+) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \Omega^{\infty-1}\operatorname{MTO}_0 \wedge Y_+ & \longrightarrow \cdots \longrightarrow & \Omega^{\infty-1}\operatorname{MTO}_d \wedge Y_+ & \longrightarrow \cdots \xrightarrow{\text{colim}} & \Omega^{\infty-1}\operatorname{MO} \wedge Y_+. \end{array}$$

The levelwise weak equivalences imply that the induced map on colimits is a weak equivalence.

Proposition 5.3.1. $\mathbb{B}(Y_+) \simeq \operatorname{Sect}(Y_+) \simeq \Omega^{\infty-1}\operatorname{MO} \wedge Y_+$.

5.3.2 Proof of the Main Theorem

Our goal in this section is to define a splitting of the Grassmannian functor K. This asserts that the functor $\mathbb{B}$ we constructed in the last section splits K. The proof may shed light on the understanding of the splitting problem for arbitrary Γ-spaces.

Definition 5.3.2. We call a topological space Z a $\mathbb{B}$-*module* if there exists a collection of maps

$$\rho_F : \operatorname{Emb}(F, \mathbb{R}^\infty) \times Z^F \to Z$$

such that

- 1) for any F', F lie in some $\mathcal{F}_d$ and any diffeomorphism $s : F \to F'$, the diagram

$$\begin{array}{ccc} \operatorname{Emb}(F, \mathbb{R}^\infty) \times Z^F & \xrightarrow{\rho_F} & Z \\ \uparrow s^* \times s^* & \nearrow \rho_{F'} & \\ \operatorname{Emb}(F', \mathbb{R}^\infty) \times Z^{F'} & & \end{array}$$

  commutes.

- 2) in the case $F = \{\mathrm{pt}\}$, the diagram

$$\begin{array}{ccc} \mathrm{Emb}(\mathrm{pt}, \mathbb{R}^\infty) \times Z^{\mathrm{pt}} & \xrightarrow{\rho_{\mathrm{pt}}} & Z \\ \downarrow \cong & \nearrow_{\mathrm{proj}} & \\ \mathbb{R}^\infty \times Z & & \end{array}$$

  commutes.

There exists a natural inclusion $\iota : Y_+ \to \mathbb{B}(Y_+)$ by taking F to be the one-point space $\{*\}$, namely

$$\begin{array}{rcl} \iota: & Y_+ \hookrightarrow & \mathbb{B}(Y_+) \\ & y \mapsto & \begin{pmatrix} \mathbb{R}^\infty & \leftarrow & \{*\} & \rightarrow & Y_+ \\ 0 & \leftarrow\!\shortmid & * & \mapsto & y \end{pmatrix}. \end{array}$$

One can see directly ι is well-defined and continuous.

Lemma 5.3.3. Let Y be an arbitrary space and Z a $\mathbb{B}$-module. Then any map $f : Y_+ \to Z$ can be extended to a map $\bar{f} : \mathbb{B}(Y_+) \to Z$ making the following diagram homotopy commutative.

$$\begin{array}{ccc} Y_+ & \xrightarrow{f} & Z \\ \downarrow \iota & \nearrow_{\bar{f}} & \\ \mathbb{B}(Y_+) & & \end{array} \tag{5.3.7}$$

Proof. Given $[\epsilon, F, \varphi]$ with $\mathbb{R}^\infty \xleftarrow{\epsilon} F \xrightarrow{\varphi} Y_+$, $\bar{f}$ is defined by $\bar{f}([\epsilon, F, \varphi]) = \rho_F(\epsilon, f \circ \varphi)$. This map is well-defined because of the first condition in Definition 5.3.2. and therefore also continuous. The second condition implies the homotopy commutativity. □

Lemma 5.3.4. $\mathbb{B}(W_+)$ is a $\mathbb{B}$-module.

Proof. Define $E_d(W_+) := (\coprod_{F \in \mathcal{F}_d} \mathrm{Emb}(F, \mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}) \times_{\mathrm{Diff}(F)} \mathrm{Map}(F, W_+) \times F)/\sim$.
Two elements

$$(M \xleftarrow{\epsilon} F \xrightarrow{\varphi} W_+, z) \sim (M \xleftarrow{\epsilon_1} F_1 \xrightarrow{\varphi_1} W_+, z'),$$

are equivalent, if $F = F_1 \sqcup F_2, \varphi(F_2) = +$ and $z = z' \in F_1$. One can analogously define a subspace $E^c_d(W_+) \subseteq E_d(W_+)$ by considering all the embeddings $\epsilon : F \to \mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}$ such that $\epsilon(F) \subseteq K \times \mathbb{R}^{d+1}$ for some compact subset $K \subset \mathbb{R}^{\infty-1}$.
Similar to $\mathbb{B}(W_+)$, we denote $\mathbb{E}(W_+) := \operatorname{colim}_d E^c_d(\mathbb{R}^{\infty-1} \times \mathbb{R}^{d+1}; W_+)$, define the map

$$\zeta : \mathbb{E}(W_+) \to \mathbb{B}(W_+), [\epsilon, F, \varphi, z] \mapsto [\epsilon, F, \varphi].$$

For the given continuous map $\theta : F' \to \mathbb{B}(W_+)$, let $\hat{F}$ be the pull-back of the diagram

$$\begin{array}{ccc} \hat{F} & \longrightarrow & \mathbb{E}(W_+) \\ \downarrow & & \downarrow \zeta \\ F' & \xrightarrow{\theta} & \mathbb{B}(W_+) \end{array}$$

By composing with the evaluation map $\mathrm{ev} : \mathbb{E}(W_+) \to W_+, [\epsilon, F, \varphi, z] \mapsto \varphi(z)$, we obtain a continuous map $\hat{\varphi} : \hat{F} \to W_+$. Set $\hat{F}_0 := \hat{F} \backslash \hat{\varphi}^{-1}(+)$. Note that $\hat{F}_0 \to F'$ is a locally trivial fibre bundle, since ζ in the right column is so. Thus each component of $\hat{F}_0$ is a manifold. The embedding is given by

$$\begin{array}{cccc} \hat{\epsilon} : & \hat{F}_0 & \hookrightarrow & \mathbb{R}^\infty \times \mathbb{R}^\infty \xrightarrow{\cong} \mathbb{R}^\infty \\ & (\ell, \epsilon_F, \varphi_F, z) & \mapsto & (\epsilon_{F'}(\ell), \epsilon_F(z)). \end{array}$$

Here the second homeomorphism is

$$\tau : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty, ((x_1, x_2, \cdots), (y_1, y_2, \cdots)) \mapsto (x_1, y_1, x_2, y_2, \cdots). \tag{5.3.8}$$

The restriction gives the morphism

$$\begin{array}{cccc} \hat{\varphi}| : & \hat{F}_0 & \to & W_+ \\ & (\ell, \epsilon_F, \varphi_F, z) & \mapsto & \varphi_F(z). \end{array}$$

We can now define $\rho_{F'}$ to be

$$\begin{array}{ccccc} \mathrm{Emb}(F', \mathbb{R}^\infty) & \times & \mathrm{Map}(F', \mathbb{B}(W_+)) & \xrightarrow{\rho_{F'}} & \mathbb{B}(W_+) \\ (\epsilon_{F'} : F' \to \mathbb{R}^\infty & , & \theta : F' \to \mathbb{B}(W_+)) & \mapsto & [\mathbb{R}^\infty \overset{\hat{\epsilon}}{\hookleftarrow} \hat{F}_0 \xrightarrow{\hat{\varphi}|} W_+]. \end{array}$$

It is well-defined, since if given an $\alpha \in \mathrm{Diff}(F)$, write $\hat{\ell} = (\ell, \epsilon_F, \varphi_F, z)$, then

$$\alpha(\hat{\ell}) = (\ell, \epsilon_F \circ \alpha^{-1}, \varphi_F \circ \alpha^{-1}, \alpha(z)),$$

which implies that

$$\begin{aligned} \hat{\epsilon}(\alpha(\hat{\ell})) &= (\epsilon_{F'}(\ell), \epsilon_F \circ \alpha^{-1}(\alpha(z))) = \hat{\epsilon}(\hat{\ell}), \\ \hat{\varphi}(\alpha)(\hat{\ell}) &= \varphi_F \circ \alpha^{-1}(\alpha(z)) = \hat{\varphi}(\hat{\ell}). \end{aligned}$$

The composite $\bar{f} \circ \iota : Y_+ \to \mathbb{B}(W_+)$ is homotopic to f. Because for any $y \in Y_+$, assume we can write $f(y) = [\epsilon, F, \varphi]$ for some $\epsilon : F \hookrightarrow \mathbb{R}^\infty$. By construction $\bar{f} \circ \iota(y) = [\epsilon', F \times \{0\}, \varphi']$ with $\epsilon' : F \times \{0\} \hookrightarrow \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$. Since all the F are of finite dimensions, we can use the homotopy ρ_{odd} defined in section 5.1 to rotate $\epsilon'(F \times \{0\})$ by an isotopy into $\epsilon(F)$, because ρ_{odd} is also smooth. So the triangle (5.3.7) is homotopy commutative. □

Corollary 5.3.5. $\mathbb{B}(\bigvee_{k \geq 0} D_k K(X))$ is a $\mathbb{B}$-module. □

We can now state the main theorem of this chapter.

Theorem 5.3.6. There is a weak homotopy equivalence

$$\mathbb{B}(K(X)_+) \simeq \mathbb{B}(\bigvee_{k \geq 0} D_k K(X)).$$

Proof. For each m, k, recall first the m-th filtration $K_m(X) := \{\Sigma_i V_i x_i \in K(X) | \Sigma \dim V_i \leq m\}$. Now we construct the map

$$\gamma_m : K_m(X)_+ \to \mathbb{B}(\bigvee_{k=0}^{m} D_k K(X)).$$

Given $\xi = \sum_{i=1}^r V_i x_i \in K_m(X)_+$, we set

$$F_\xi = \coprod_{(k_1,\cdots,k_r)} \mathrm{Gr}_{k_1}(V_1) \times \cdots \times \mathrm{Gr}_{k_r}(V_r), \tag{5.3.9}$$

with the disjoint union ranging over all ordered partitions of k of any length $r \geq 1$. Note that different components of F_ξ may have different dimensions. We want to thicken those components with dimensions lower than the maximal one such that all the components of F_ξ have the same dimension.

Let

$$d := \max_{(k_1,\cdots,k_n)} \dim(\mathrm{Gr}_{k_1}(V_1) \times \cdots \times \mathrm{Gr}_{k_r}(V_r)),$$
$$\tilde{F}_\xi := \coprod_{(k_1,\cdots,k_r)} \mathrm{Gr}_{k_1}(V_1) \times \cdots \times \mathrm{Gr}_{k_r}(V_r) \times \mathbb{R}^{d-\alpha(k_1,\cdots,k_r)},$$

where

$$\begin{aligned} \alpha(k_1,\cdots,k_r) : &= d - \dim(\mathrm{Gr}_{k_1}(V_1) \times \cdots \times \mathrm{Gr}_{k_r}(V_r)) \\ &= d - \prod_{i=1}^r k_i \times (\dim V_i - k_i). \end{aligned}$$

This new manifold $\tilde{F}_\xi$ has only finitely many components. We now fix an embedding

$$e : \coprod_k \coprod_{(k_1,\cdots,k_r)} \coprod_\ell \mathrm{Gr}_{k_1}(\mathbb{R}^\infty) \times \cdots \times \mathrm{Gr}_{k_r}(\mathbb{R}^\infty) \times \mathbb{R}^\ell \hookrightarrow \mathbb{R}^\infty.$$

Define

$$\gamma_m(\xi) = \begin{pmatrix} \mathbb{R}^\infty & \hookleftarrow & \coprod_k \tilde{F}_\xi & \to & \bigvee_{k=0}^m D_k K(X) \\ e(W_1,\cdots,W_r,t) & \leftarrow\!\shortmid & (W_1,\cdots,W_r,t) & \mapsto & \overline{\Sigma W_i x_i} \end{pmatrix}.$$

Here $W_i \in \mathrm{Gr}_{k_i}(V_i) \subset \mathrm{Gr}_{k_i}(\mathbb{R}^\infty)$, $t \in \mathbb{R}^{d-\alpha(k_1,\cdots,k_r)}$ and we denote the quotient map by

$$- : K_k(X) \to D_k K(X),\ \Sigma W_i x_i \mapsto \overline{\Sigma W_i x_i}.$$

Furthermore we have a commutative square

$$\begin{array}{ccc} K_{m-1}(X)_+ & \xrightarrow{\gamma_{m-1}} & \mathbb{B}(\bigvee_{k=0}^{m-1} D_k K(X)) \\ \downarrow{\scriptstyle \iota_{m-1}} & & \downarrow \\ K_m(X)_+ & \xrightarrow{\gamma_m} & \mathbb{B}(\bigvee_{k=0}^{m} D_k K(X)). \end{array}$$

Here $\iota_{m-1} : K_{m-1}(X)_+ \to K_m(X)_+$ has been defined in Section 5.1 .

Thus all γ_m together induce a map

$$\gamma : K(X)_+ \to \mathbb{B}(\bigvee_{k\geq 0} D_k K(X)).$$

Notice that we have already defined the inclusion $\iota : K(X)_+ \hookrightarrow \mathbb{B}(K(X)_+)$ in Lemma 5.3.3. And $\mathbb{B}(\bigvee_k D_k K(X))$ being a $\mathbb{B}$-module by Corollary 5.3.7. implies that there exists an extension $\bar{\gamma}$ of γ:

$$\begin{array}{ccc} K(X)_+ & \xrightarrow{\gamma} & \mathbb{B}(\bigvee_{k\geq 0} D_k K(X)) \\ \big\downarrow{\scriptstyle \iota} & \nearrow{\scriptstyle \bar{\gamma}} & \\ \mathbb{B}(K(X)_+) & & \end{array}$$

And this $\bar{\gamma}$ also preserves the filtration. We denote the following composite by

$$\bar{\gamma}_m : \mathbb{B}(K_m(X)_+) \hookrightarrow \mathbb{B}(K(X)_+) \xrightarrow{\bar{\gamma}} \mathbb{B}(\bigvee_{k=0}^{\infty} D_k K(X)).$$

By construction, this composite factors through $\mathbb{B}(K_m(X)_+) \to \mathbb{B}(\bigvee_{k=0}^{m} D_k K(X))$, we will denote this map by $\bar{\gamma}_m$ as well.

Next we claim that the following diagram is weakly homotopy commutative (i.e. after applying π_*, it is commutative):

$$\begin{array}{ccc} \mathrm{Sect}(D_m K(X)) & = & \mathrm{Sect}(D_m K(X)) \\ \uparrow (*) & & \uparrow \\ \mathbb{B}(K_m(X)_+) & \xrightarrow{\bar{\gamma}_m} & \mathbb{B}(\bigvee_{k=0}^{m} D_k K(X)) \\ \uparrow \iota_{m-1*} & & \uparrow \\ \mathbb{B}(K_{m-1}(X)_+) & \xrightarrow{\bar{\gamma}_{m-1}} & \mathbb{B}(\bigvee_{k=0}^{m-1} D_k K(X)) \end{array} \tag{5.3.10}$$

The map $(*)$ is the composite $\mathbb{B}(K_m(X)_+) \to \mathrm{Sect}(K_m(X)_+) \to \mathrm{Sect}(D_m K(X))$.

We should emphasize here we actually use the commutative diagram

$$\begin{array}{ccc} \mathrm{Sect}(D_m K(X)) & = & \mathrm{Sect}(D_m K(X)) \\ \uparrow & & \uparrow \\ \mathrm{Sect}(K_m(X)_+) & \longrightarrow & \mathrm{Sect}(\bigvee_{k=0}^{m} D_k K(X)) \\ \uparrow & & \uparrow \\ \mathrm{Sect}(K_{m-1}(X)_+) & \longrightarrow & \mathrm{Sect}(\bigvee_{k=0}^{m-1} D_k K(X)) \end{array} \tag{5.3.11}$$

Using Proposition 5.3.1. we replace $\mathbb{B}(Y_+)$ by $\mathrm{Sect}(Y_+)$ and study instead the diagram (5.3.11) because of the commutativity of the square

$$\begin{array}{ccc} \pi_* \mathrm{Sect}(K_m(X)_+) & \longrightarrow & \pi_* \mathrm{Sect}(\bigvee_{k=0}^{m} D_k K(X)) \\ \cong \uparrow & & \cong \uparrow \\ \pi_* \mathbb{B}(K_m(X)_+) & \xrightarrow{\bar{\gamma}_{m*}} & \pi_* \mathbb{B}(\bigvee_{k=0}^{m} D_k K(X)). \end{array} \tag{5.3.12}$$

We shall explain later there exists a splitting map on the top of this square similar to $\bar{\gamma}_{m_*}$.

The possible partition $(k_1, \cdots, k_r) \subset k$ fall into two cases:

(i) $k_r = 0$, then $\bar{\gamma}_{m-1}$ and $\bar{\gamma}_m$ in (5.3.10) agree on the level $m-1$ and m;

(ii) $k_r \neq 0$, then for any $(W_1, \cdots, W_r) \in \mathrm{Gr}_{k_1}(V_1) \times \cdots \times \mathrm{Gr}_{k_r}(V_r)$, by the construction of $\bar{\gamma}$, $\sum_{i=1}^{r} W_i x_i$ with $x_r = x_0$ is trivial in $D_k K(X)$, which implies that the lower square is commutative. For the upper square, given a manifold F which embedds in $\mathbb{R}^{\infty+d}$, $F \hookrightarrow \mathbb{R}^{\infty+d}$, one obtains a section by the scanning. So we only need to check the commutativity in terms of the sections.

We assume given a compactly supported section s in $\mathrm{Sect}(K_m(X)_+)$

$$s : \mathbb{R}^{\infty+d} \to \mathrm{Th}(U_{d,\infty}^{\perp}) \wedge K_m(X)_+.$$

Then the composition

$$\mathrm{pr}_1 \circ s : \mathbb{R}^{\infty+d} \to \mathrm{Th}(U_{d,\infty}^{\perp}) \wedge K_m(X)_+ \to \mathrm{Th}(U_{d,\infty}^{\perp})$$

is continuous. And it is homotopic to a map which is transverse to the zero section. Without loss of generality, we may assume s itself transverse to the zero section $\mathrm{BO}(d) \subseteq \mathrm{Th}(U_{d,\infty}^{\perp})$. Let $F := (\mathrm{pr}_1 \circ s)^{-1}(\mathrm{BO}(d))$, then the projection $F \to \mathrm{BO}(d) \wedge K_m(X)_+ \to K_m(X)$ is also continuous. This is because $\mathrm{BO}(d)$ is the zero section of $\mathrm{Th}(U_{d,\infty}^{\perp})$, it will not tend to the base point $\infty \in \mathrm{Th}(U_{d,\infty}^{\perp})$. This projection means that each point in F corresponds to a label ξ in $K_m(X)_+$.

Define the multiplication

$$\begin{array}{rccc} \circ_{n,n'} : & \mathrm{Th}(U_{d,n}^{\perp}) \wedge \mathrm{Th}(U_{d',n'}^{\perp}) & \to & \mathrm{Th}(U_{d+d',n+n'}) \\ & ((V_1, v_1), (V_2, v_2)) & \mapsto & (V_1 \oplus V_2, v_1 \oplus v_2) \end{array}$$

The family of all $\circ_{n,n'}$ induces a natural map

$$\circ : \mathrm{Th}(U_{d,\infty}^{\perp}) \wedge \mathrm{Th}(U_{d',\infty}^{\perp}) \to \mathrm{Th}(U_{d+d',\infty+\infty}^{\perp}) \xrightarrow{\cong} \mathrm{Th}(U_{d+d',\infty}^{\perp})$$

The second map is induced by $\tau : \mathbb{R}^{\infty} \times \mathbb{R}^{\infty} \to \mathbb{R}^{\infty}$ which was defined in (5.3.8).

Since $\gamma(\xi) = s_{\xi} : \mathbb{R}^{\infty+d'} \to \mathrm{Th}(U_{d',\infty}^{\perp}) \wedge \bigvee_{k\geq 0} D_k K(X)$ is a section, we obtain a map $\bar{\gamma}_m$:

$$(x, z) \longmapsto (\mathrm{pr}_1 \circ s(x), s_{\xi}(z)) \tag{5.3.13}$$

$$\begin{array}{ccc} \mathbb{R}^{\infty+d} \times \mathbb{R}^{\infty+d'} & \longrightarrow & \mathrm{Th}(U_{d,\infty}^{\perp}) \wedge \mathrm{Th}(U_{d',\infty}^{\perp}) \wedge \bigvee_{k\geq 0} D_k K(X) \\ \cong \big\uparrow & & \big\downarrow \circ\wedge\mathrm{id} \\ \mathbb{R}^{\infty+d+d'} & \overset{\bar{\gamma}_m(s)}{\dashrightarrow} & \mathrm{Th}(U_{d+d',\infty}^{\perp}) \wedge \bigvee_{k\geq 0} D_k K(X) \end{array}$$

Then $\bar{\gamma}_m(s) = *$ for $s \in \mathrm{Sect}(K_{m-1}(X)_+)$. And $\bar{\gamma}_m$ makes the top square in (5.3.11) commutative. The reason is as follows: assume given a section $s \in \mathrm{Sect}(D_m K(x))$, we write $\xi = \Sigma V_i x_i \in D_m K(X)$, then $\Sigma \dim V_i = m$. In order to get a nontrivial element in $D_m K(X)$, the only choice of F is $F = Gr_{k_1}(V_1) \times \cdots \times Gr_{k_r}(V_r)$ with $k_i = \dim V_i$,which is a one-point space.

We should first understand the scanning map for a one-point space. This is essentially the embedding

$$\iota : K(X)_+ \to \mathrm{Sect}(K(X)_+).$$

We now identify $S^N \cong \mathbb{R}^N \cup \{\infty\}$ and denote by D^N the unit disc in $\mathbb{R}^N$. Then we define for each N a section determined by ξ, namely

$$\begin{array}{rccl} s_N^{\xi} : & \mathbb{R}^N & \to & \mathrm{Th}(U_{0,N}^{\perp}) \wedge K(X) \cong S^N \wedge K(X) \\ & z & \mapsto & \begin{cases} \frac{z}{1-||z||} \wedge \xi, & z \in D^N; \\ \infty \wedge \xi, & z \in \mathbb{R}^N \backslash D^N. \end{cases} \end{array}$$

In the following diagram, the two vertical maps are the natural inclusions.

$$\begin{array}{ccc} \mathbb{R}^N & \xrightarrow{s_N^{\xi}} & \mathrm{Th}(U_{0,N}^{\perp}) \wedge K(X) \cong S^N \wedge K(X) \\ \big\downarrow & & \big\downarrow \\ \mathbb{R}^{N+1} & \xrightarrow{s_{N+1}^{\xi}} & \mathrm{Th}(U_{0,N+1}^{\perp}) \wedge K(X) \cong S^{N+1} \wedge K(X) \end{array}$$

The diagram is commutative, and we define $\iota(\xi) = \operatorname{colim}_N s_N^\xi$. Now we return to the map $\mathrm{Sect}(D_m K(X)) \to \mathrm{Sect}(D_m K(X))$ in (5.3.11). We claim that this map is the identity. Given $s \in \mathrm{Sect}(D_m K(X))$, let s_N be the composite of the following maps

$$(x, z) \longmapsto (\mathrm{pr}_1 \circ s(x), \gamma(\xi)(z))$$

$$\begin{array}{ccc}
\mathbb{R}^N \times \mathbb{R}^\infty & \longrightarrow & \mathrm{Th}(U_{0,N}^\perp) \wedge \mathrm{Th}(U_{d',\infty-d'}^\perp) \wedge \bigvee_{k\geq 1} D_k K(X) \\
\uparrow \cong & & \downarrow \circ\wedge \mathrm{id} \\
\mathbb{R}^{N+\infty} & \xrightarrow{s_N} & \mathrm{Th}(U_{d',\infty+N-d'}^\perp) \wedge \bigvee_{k\geq 1} D_k K(X)
\end{array}$$

By construction of $\mathrm{Sect}(K_m(X)_+) \to \mathrm{Sect}(\bigvee_{k=0}^m D_k K(X))$, the stabilization of s_N is the image of s under $\tilde{\gamma}_m$. Namely on the top $\bar{\gamma}_m(s) = \operatorname{colim} s_N = s$, therefore it is the identity. So the diagram (5.3.11) is commutative, thus the top square in (5.3.10) is weakly homotopy commutative.

We prove by induction that $\bar{\gamma}$ is a weak equivalence. Note $K_0(X)_+ = K_0(X) \sqcup +$, $D_0 K(X) = K_0(X)/K_{-1}(X) = K_0(X) \sqcup +$. For $m = 1$, $K_1(X) = D_1 K(X)$, the map $\bar{\gamma}_1 : \mathbb{B}(K_1(X)_+) \to \mathbb{B}(D_0 K(X) \wedge D_1 K(X))$ is the identity. Namely, given

$$s : \mathbb{R}^\infty \to \mathrm{Th}(U_{d,\infty-d}^\perp) \wedge K_1(X)_+, x \mapsto \mathrm{pr}_1 \circ s(x) \wedge \xi.$$

Write $\xi = (V, x)$, then $\dim V \leq 1$, $F = \mathbb{P}(V) = \{*\}$ the projective space of a line V, which is a one-point space, so $\gamma(\xi)$ is the stable map of $\mathbb{R}^N \to \mathrm{Th}(U_{0,N}^\perp) \wedge D_1 K(X)$.

Thus $\bar{\gamma}_1(s)$ is the stabilization of $\mathbb{R}^{\infty+N} \to \mathrm{Th}(U_{d,\infty+N-d}^\perp) \wedge D_1 K(X)$, which is s.

Since $\mathbb{B}$ represents an infinite loop space of the Thom spectrum, which is a linear functor, so it converts cofibrations into quasi-fibration, the assertion now follows by induction on m and the 5-lemma. □

Corollary 5.3.7. There is a weak homotopy equivalence

$$\Omega^{\infty-1} MO \wedge K(X)_+ \simeq \Omega^{\infty-1} MO \wedge \bigvee_{k\geq 0} D_k K(X).$$

□

Remark 5.3.8.

- In the construction of the map $\bar{\gamma} : \mathbb{B}(K(X)_+) \to \mathbb{B}(\bigvee_{k\geq 0} D_k K(X))$, we can see that the splitting $\bar{\gamma}$ factors through

  $$\mathbb{B}(K(X)_+) \to B(\coprod_k \coprod_{(k_1,\cdots,k_r)} \coprod_\ell \mathrm{Gr}_{k_1}(\mathbb{R}^\infty) \times \cdots \times \mathrm{Gr}_{k_r}(\mathbb{R}^\infty) \times \mathbb{R}^\ell; \bigvee_{k\geq 0} D_k K(X)) \to B(\mathbb{R}^\infty; \bigvee_{k\geq 0} D_k K(X)).$$

  Since we do not know the homotopy type of $B(\coprod_k \coprod_{(k_1,\cdots,k_r)} \coprod_\ell \mathrm{Gr}_{k_1}(\mathbb{R}^\infty) \times \cdots \times \mathrm{Gr}_{k_r}(\mathbb{R}^\infty) \times \mathbb{R}^\ell; \bigvee_{k\geq 0} D_k K(X))$, we replace it instead by $B(\mathbb{R}^\infty; \bigvee_{k\geq 0} D_k K(X)) = \mathbb{B}(\bigvee_{k\geq 0} D_k K(X)))$, whose homotopy type is already studied in the last sections.

- This factorization phenomenon is already apparent in the case of configurations, namely in the proof of the Snaith splitting we saw a factorization of the "power" map (in this case, with a non-discrete base point it is possible to construct the splitting map).

  $$C(\mathbb{R}^\infty; C(\mathbb{R}^\infty; X)) \longrightarrow C(\sqcup_n C^n(\mathbb{R}^\infty); \bigvee_{k\geq 1} D_k(\mathbb{R}^\infty; X)) \longrightarrow C(\mathbb{R}^\infty; \bigvee_{k\geq 1} D_k(\mathbb{R}^\infty; X).$$

Also in this case we replace $C(\sqcup_n C^n(\mathbb{R}^\infty); -)$ by $C(\mathbb{R}^\infty; -)$ because the homotopy type of $C(\mathbb{R}^\infty; -)$ is well understood.

- The proof indicates that if $\mathbb{A}(\mathbf{1})$ is well-behaved, for example it decomposes into manifolds of finite-dimensions, the main theorem might be possibly generalized to an arbitrary Γ-space $\mathbb{A}$, namely there is a weak equivalence

$$\mathbb{B}(\mathbb{A}(X)_+) \to \mathbb{B}(\bigvee_{k \geq 0} D_k \mathbb{A}(X)).$$

We shall explain this in the next chapter.

Chapter 6

Splitting of Segal Γ-Spaces

We have seen several examples of splittings of Segal Γ-spaces. One might ask if there is a functor which splits an arbitrary Γ-space. From what we have done, one might get some sort of hint how to generalize these results to an arbitrary one. In this chapter, we will work with arbitrary Segal Γ-spaces and give an answer in general.

6.1 Weight Filtration of $\mathbb{A}(X)$

The configuration space $C(M, M_0; X)$ has a natural filtration given by the closed subspaces

$$C_n(M, M_0; X) := (\coprod_{k=0}^{n} \tilde{C}^k(M) \times_{\Sigma_k} X^k)/\sim .$$

And $K(X)$ has filtration given by

$$K_n(X) = \{\Sigma V_i x_i \in K(X) | \Sigma \dim V_i \leq n\}.$$

The filtration of these two examples can be generalized to an arbitrary Segal Γ-space $\mathbb{A}$. We construct a weight filtration of $\mathbb{A}(X)$.

We assume that there exists a natural transformation ϑ between two Segal Γ-spaces, $\vartheta : \mathbb{A} \to \mathbb{N}$, i.e. for each n we have maps $\vartheta_n : \mathbb{A}(\boldsymbol{n}) \to \mathbb{N}^n$ such that the following diagrams commutes,

$$\begin{array}{ccc} \mathbb{A}(\boldsymbol{n}) & \xrightarrow{\vartheta_n} & \mathbb{N}^n \\ \downarrow \simeq & & \downarrow = \\ \mathbb{A}(\mathbf{1})^n & \xrightarrow{\vartheta_1^n} & \mathbb{N}^n. \end{array}$$

We call this ϑ the *weight transformation*. By convention, $\mathbb{N}^0 = 0$, so $\mathbb{A}(\mathbf{0})$ has weight 0. For example in the case when $\mathbb{A} = SP$, $\vartheta_n = \mathrm{id}$, and when $\mathbb{A} = K$, $\vartheta_n(V_1, \cdots, V_n) = (\dim V_1, \cdots, \dim V_n)$.

Note that all the ϑ_n induce a map ϑ_*:

$$\begin{array}{c} \mathbb{A}(X) := (\coprod_n \mathbb{A}(\boldsymbol{n}) \times_{\Sigma_n} X^n)/\sim \\ \Big\downarrow \vartheta_* \\ SP(X) := (\coprod_n \mathbb{N}(\boldsymbol{n}) \times_{\Sigma_n} X^n)/\sim . \end{array}$$

We take the inverse images of all the ordered partitions of k under ϑ_*, and define a subset of $\mathbb{A}(\boldsymbol{n})$ consisting of all the elements of *weight* k,

$$\mathbb{A}_k(\boldsymbol{n}) := \coprod_{(k_1,\cdots,k_n)\in\mathbb{N}^n, 0\le\Sigma k_i\le k} \vartheta_n^{-1}(k_1,\cdots,k_n).$$

So $\mathbb{A}(X)$ admits a *k-th filtration*

$$\mathbb{A}_k(X) := (\coprod_{\boldsymbol{n}} \mathbb{A}_k(\boldsymbol{n}) \times_{\Sigma_n} X^n)/\sim,$$

and the filtration quotient are denoted by

$$D_k(\mathbb{A}(X)) := \mathbb{A}_k(X)/\mathbb{A}_{k-1}(X).$$

Take an arbitrary $(a; x_1, \cdots, x_n) \in \mathbb{A}_k(X)$, so we have $a \in \mathbb{A}_k(\boldsymbol{n})$. In the case $k < n$, the map ϑ_n : $\mathbb{A}_k(\boldsymbol{n}) \to \mathbb{N}^n$ has the form $\vartheta_n(a) = (k_1, \cdots, k_n)$. Since each $k_i \geq 0$ and $\Sigma k_i \leq k < n$, so some k_i must be 0. Assume then $\vartheta_n(a) = (k_{j_1}, \cdots, k_{j_t}, 0, \cdots, 0)$ with $\sum k_{j_i} = \ell \leq k$. The projection to the first k elements $\mathrm{pr} : \boldsymbol{n} \to \boldsymbol{k}, k \geq j \mapsto j, k \leq j \mapsto 0$ gives a commutative diagram

$$\begin{array}{ccc} \mathbb{A}(\boldsymbol{n}) & \longrightarrow & \mathbb{N}^n \\ \Big\downarrow \mathrm{pr}_* & & \Big\downarrow \mathrm{pr}_* \\ \mathbb{A}(\boldsymbol{k}) & \longrightarrow & \mathbb{N}^k. \end{array}$$

Let $a' = \mathrm{pr}_*(a)$, then there exists $(a'; x'_1, \cdots, x'_k) \in \mathbb{A}_k(X)$, such that

$$\mathbb{A}_k(\boldsymbol{n}) \times_{\Sigma_n} X^n \ni (a; x_1, \cdots, x_n) \sim (a'; x'_1, \cdots, x'_k) \in \mathbb{A}_k(\boldsymbol{k}) \times_{\Sigma_k} X^k.$$

This means in the case $k < n$ we can always find a representative $(a'; x'_1, \cdots, x'_k)$ in the same equivalence class as $(a; x_1, \cdots, x_n)$, such that its length is not greater than its weight.

Before introducing the functor B, it is necessary to first give the sketch of the proof of Proposition 2.3.4. in Chapter 2.

Proof of Proposition 2.3.4. (1) We denote by p the projection $\mathbb{A}(X) \to \mathbb{A}(X/Y)$, filter the base space $\mathbb{A}(X/Y)$ by $P_k = \mathbb{A}_k(X/Y)$ as above, and filter the total space $\mathbb{A}(X)$ by $Q_k = p^{-1}(P_k)$. Let R be an open neighborhood deformation retract of Y in X, i.e. $r : X \to X$ is a deformation leaving Y invariant and retracting R into Y. It is clear $X-Y$ can be identified with $X/Y-*$, so we have $\mathbb{A}_k(X-Y) \cong \mathbb{A}_k(X/Y-*)$.

(2) For each $k \geq 0$, let

$$\mathrm{pr}_1 : \boldsymbol{n}+\boldsymbol{k} \to \boldsymbol{n}, i \mapsto \begin{cases} i, & i \leq n; \\ 0, & i > n. \end{cases}$$

$$\mathrm{pr}_2 : \boldsymbol{n}+\boldsymbol{k} \to \boldsymbol{k}, j \mapsto \begin{cases} 0, & j \leq 0; \\ j-n, & j > n. \end{cases}$$

They induce a homotopy equivalence $\mathrm{pr}_{1*} \times \mathrm{pr}_{2*} : \mathbb{A}(\boldsymbol{n}+\boldsymbol{k}) \to \mathbb{A}(\boldsymbol{n}) \times \mathbb{A}(\boldsymbol{k})$.

The fibre of $p : \mathbb{A}(X) \to \mathbb{A}(X/Y)$ at $c = (a'; x_1, \cdots, x_n) \in \mathbb{A}(X/Y)$ consists of all the elements $(a; x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+k}) \in \mathbb{A}(\boldsymbol{n}+\boldsymbol{k}) \times_{\Sigma_{n+k}} X^{n+k}$ for some $k \geq 0$ such that $\mathrm{pr}_{1*}(a) = a'$ and $x_{n+1}, \cdots, x_{n+k} \in Y$. This fibre can be identified with the set of all $(\mathrm{pr}_{2*}(a); x_{n+1}, \cdots, x_{n+k}) \in \mathbb{A}(\boldsymbol{k}) \times_{\Sigma_k} X^k$, for all $k \geq 0$. That is, $p^{-1}(c)$ is homeomorphic to $\mathbb{A}(Y)$.

For each $\xi \in Q_k - Q_{k-1} = p^{-1}(P_k - P_{k-1})$, assume that ξ is represented by $[a; x_1, \cdots, x_i, y_{i+1}, \cdots, y_n]$ for some i. Let $\mathrm{pr}_1 : \boldsymbol{n} \to \boldsymbol{i}, \mathrm{pr}_2 : \boldsymbol{n} \to \boldsymbol{n}-\boldsymbol{i}$ be the projections defined similarly as above. Observe that the weight of a satisfies that $\vartheta_i \circ \mathrm{pr}_{1*}(a) = (k_1, \cdots, k_i)$ and $k_1 + \cdots + k_i = k$. For each k there is a homeomorphism

$\tau : Q_k - Q_{k-1} \to (P_k - P_{k-1}) \times \mathbb{A}(Y)$,

$$[a; x_1, \cdots, x_i, y_{i+1}, \cdots, y_n] \mapsto ([\mathrm{pr}_{1*}(a); x_1, \cdots, x_i], [\mathrm{pr}_{2*}(a); y_{i+1}, \cdots, y_n]).$$

The inverse of τ is obtained by taking the homotopy inverse of $\mathrm{pr}_{1*} \times \mathrm{pr}_{2*} : \mathbb{A}(\boldsymbol{n}) \to \mathbb{A}(\boldsymbol{i}) \times \mathbb{A}(\boldsymbol{n}-\boldsymbol{i})$.

(3) Take U to be the set consisting of those points in $\mathbb{A}(X/Y)$ that have at least one label in R, then define

$$U_k := \{[a; x_1, \cdots, x_n] \in \mathbb{A}_k(X/Y) | \text{ at least one } x_i \text{ lies in } R\} \subseteq P_k.$$

for each k. This is a neighborhood of P_{k-1} in P_k, i.e. $P_{k-1} \subset U_k \subset P_k$. Moreover, r induces a retraction $\bar{r}_k : p^{-1}(U_k) \to p^{-1}(P_{k-1}) = Q_{k-1}$ and a retraction $r_k : U_k \to P_{k-1}$ which satisfies $p \circ \bar{r}_k = r_k \circ p$.

(4) Let $b = [a; x_1, \cdots, x_n] \in P_k$ with $x_1, \cdots, x_n \in X - Y$. Then $b \in U_k$ if at least one of the x_i is in R. Consider the restriction of $\bar{r}_k$ to the fibre:

$$\bar{r}_k : p^{-1}(b) \to p^{-1}(r(b))$$

we write $b = [a; x_1, \cdots, x_m, z_{m+1}, \cdots, z_n] \in U_k$ such that $x_1, \cdots, x_m \in X - R$ and $z_{m+1}, \cdots, z_n \in R - Y$. Let $w \in p^{-1}(b)$ be represented by

$$w = [\alpha_*(a); x_1, \cdots, x_m, z_{m+1}, \cdots, z_n, y_1, \cdots, y_s]$$

for some $s \in \mathbb{N}$, $y_1, \cdots, y_s \in Y$ and $\alpha : \boldsymbol{n} \to \boldsymbol{n}+\boldsymbol{s}$ is the natural inclusion.

$\bar{r}_k(w) = [\alpha_*(a); r(x_1), \cdots, r(z_1), \cdots, r(y_1), \cdots, r(y_s)]$ sending $r(z_i)$ in R to Y. Since $\bar{r}_k$ is homotopic to the identity, when restricting to fibres, the induced map is also homotopic to identity. It follows that p is a quasifibration. □

6.2 Duality Theorem

Let $\mathbb{A}$ be a group complete Segal Γ-space (i.e.$\pi_0\mathbb{A}(\mathbf{1})$ is an abelian) and M be a compact parallelizable n-dimensional manifold. Let us assume for now that $\partial M \neq \emptyset$. Assume W is an n-dimensional manifold without boundary and containing M.

Define $\mathbb{A}(X, Y) := \mathbb{A}(X/Y)$. We have known that $h_*(X, X_0; \mathbb{A}) = \pi_*(\mathbb{A}(X, X_0))$ is a homology theory for all topological pairs. And recall that the cohomology represented by $\mathbb{A}$ is given by $h^n(X, X_0; \mathbb{A}) = \mathrm{colim}_k[\Sigma^k(X, X_0), \mathbb{A}(S^{n+k})]$. If $\mathbb{A}$ represents an Ω-spectrum, then $h^n(X, X_0; \mathbb{A}) = [(X, X_0), \mathbb{A}(S^n)]$.

Theorem 6.2.1. Assume $\mathbb{A}$ is a group complete Segal Γ-space, $M_0 \subseteq M \subseteq W$ are manifolds of dimension m, M is compact and W has no boundary, then there is a homotopy equivalence

$$s : \mathbb{A}(M, M_0) \to \mathrm{Sect}(W\backslash M_0, W\backslash M; \mathbb{A}(\mathbb{D}^m, \partial\mathbb{D}^m)).$$

Thus we have Poincaré duality:

$$s_* : h_p(M, M_0; \mathbb{A}) \cong h^{m-p}(W\backslash M_0, W\backslash M; \mathbb{A}).$$

This is a special case of Satz 4.5 in [Bö2]. The idea is as follows. Suppose that M has a Riemannian metric d. If $\epsilon > 0$, we write

$\mathbb{A}_\epsilon(M, M_0) := \{(a; x_1, \cdots, x_n) \in \mathbb{A}(M) |$ each x_i is at least 2ϵ away from the boundary $M_0\}$.

By scanning $M\backslash M_0$ using discs of radius ϵ, $s(\xi)$ maps x to the base point for x sufficiently near M_0. This gives rise to a map $\mathbb{A}_\epsilon(M, M_0) \to \mathrm{Sect}(W\backslash M_0, W\backslash M; \mathbb{A}(\mathbb{D}^m, \partial\mathbb{D}^m))$. As $\epsilon \to 0$, one obtains in the limit a map $s : \mathbb{A}(M, M_0) \to \mathrm{Sect}(W\backslash M_0, W\backslash M; \mathbb{A}(\mathbb{D}^m, \partial\mathbb{D}^m))$.

Since $\mathbb{A}$ converts cofibrations to quasi-fibrations, $h_*(X; \mathbb{A}) := \pi_*(\mathbb{A}(X))$ is a homology theory for any connected X. By Theorem 2 in [McD], we have

$$\mathbb{A}(M, M_0) \xrightarrow{\simeq} \mathrm{Sect}(W\backslash M_0, W\backslash M; \mathbb{A}(\mathbb{D}^m, \partial\mathbb{D}^m)).$$

In the case W is parallelizable, the diagram is commutative, and we have the isomorphism

$$\begin{array}{ccc}
\pi_p\mathbb{A}(M, M_0) & \longrightarrow & \pi_p \,\mathrm{Map}(W\backslash M_0, W\backslash M; \mathbb{A}(\mathbb{D}^m, \partial\mathbb{D}^m)) \\
 & & \downarrow \cong \\
 & & [S^p \wedge (W\backslash M_0, W\backslash M), \mathbb{A}(S^m)] \\
\downarrow \cong & & \downarrow \cong \\
 & & h^m(S^p \wedge (W\backslash M_0, W\backslash M); \mathbb{A}) \\
 & & \downarrow \cong \\
h_p(M, M_0; \mathbb{A}) & \xrightarrow{\cong} & h^{m-p}(W\backslash M_0, W\backslash M; \mathbb{A}).
\end{array}$$

Example 6.2.2.

$$\begin{array}{ccccc}
\pi_p K(M) & \xrightarrow{\cong} & \pi_p \,\mathrm{Map}(M, K(S^m)) & \xrightarrow{\cong} & [S^p \wedge M, K(S^m)] \\
\downarrow \cong & & & & \downarrow \cong \\
ko_p(M) & & \xrightarrow{\cong} & & ko^{m-p}(M)
\end{array}$$

The left hand side is the connective real K-homology theory, the right hand side is the real connective K-cohomology theory.

6.3 Splitting Spaces

Consider the addition map μ and the projection map p:

$$\mu : \mathbf{2} \to \mathbf{1} : 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1;$$
$$s : \mathbf{2} \to \mathbf{1} : 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 0.$$

They induce the following maps $\mu_{2n} := s_n^{-1} \circ \mathbb{A}(\mu)^n \circ s_{2n}$, $p_{2n} := s_n^{-1} \circ \mathbb{A}(p)^n \circ s_{2n}$ defined by the compositions in the diagrams

$$\begin{CD} \mathbb{A}(\boldsymbol{2n}) @>{\mu_{2n}}>> \mathbb{A}(\boldsymbol{n}) \\ @V{s_{2n}}V{\simeq}V @V{s_n}V{\simeq}V \\ \mathbb{A}(\mathbf{2}) \times \cdots \times \mathbb{A}(\mathbf{2}) @>{\mathbb{A}(\mu)^n}>> \mathbb{A}(\mathbf{1}) \times \cdots \times \mathbb{A}(\mathbf{1}) \end{CD} \qquad \begin{CD} \mathbb{A}(\boldsymbol{2n}) @>{p_{2n}}>> \mathbb{A}(\boldsymbol{n}) \\ @V{s_{2n}}V{\simeq}V @V{s_n}V{\simeq}V \\ \mathbb{A}(\mathbf{2}) \times \cdots \times \mathbb{A}(\mathbf{2}) @>{\mathbb{A}(s)^n}>> \mathbb{A}(\mathbf{1}) \times \cdots \times \mathbb{A}(\mathbf{1}) \end{CD}$$

For each $a \in \mathbb{A}(\boldsymbol{n})$, we define the *splitting space* of a to be $\mathrm{spl}(a) := \mu_{2n}^{-1}(a) \subseteq \mathbb{A}(\boldsymbol{2n})$ and the *projective splitting space* of a to be $\mathrm{pspl}(a) := p_{2n}\,\mathrm{spl}(a)$.

In case of $\mathbb{A}(\mathbf{2}) = \mathbb{A}(\mathbf{1}) \times \mathbb{A}(\mathbf{1})$, for example when $\mathbb{A}$ is an abelian group, we have the equation

$$\begin{array}{ccc} (a_1, \cdots, a_n) = a \in & & \mathbb{A}(\boldsymbol{n}) \\ \Big\uparrow & & \Big\uparrow{\scriptstyle \mu_{2n}} \\ ((b_1', b_1''), \cdots, (b_n', b_n'')) = b \in & & \mathbb{A}(\boldsymbol{2n}) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p_{2n}} \\ (b_1', \cdots, b_n') = b' \in & & \mathbb{A}(\boldsymbol{n}) \end{array}$$

where each a_i can be regarded as the "sum" of b_i' and b_i'', particularly in most examples b_i' and b_i'' are uniquely determined by each other.

Example 6.3.1. Typical examples of this very good case are $\mathbb{A} = \mathbb{N}$ or K. For $\mathbb{A} = K$, given $b = ((V_1', V_1''), \cdots, (V_n', V_n'')) \in \mathrm{spl}(a)$, then $a = \mu(b) = (V_1' \oplus V_1'', \cdots, V_n' \oplus V_n'')$ and $b' = (V_1', \cdots, V_n')$.

6.4 Splitting of Segal Γ-Spaces

Let $\mathbb{B}$ be the functor defined in Chapter 5. Our main theorem in this chapter now goes as follows.

Theorem 6.4.1. Assume $\mathbb{A}(\boldsymbol{n})$ is a manifold for each n and $\mathrm{pspl}(a) \subset \mathbb{A}(\boldsymbol{n})$ is a finite-dimensional manifold for each $a \in \mathbb{A}(\boldsymbol{n})$. Then for any connected space X, $\mathbb{B}$ splits the Segal Γ-space $\mathbb{A}$, i.e. there is a weak homotopy equivalence

$$\mathbb{B}(\mathbb{A}(X)_+) \to \bigvee_{k \geq 0} \mathbb{B}(D_k \mathbb{A}(X)).$$

Proof. As before, we are going to construct the inclusion map ι and the "power" map f such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} \mathbb{A}(X)_+ & \xrightarrow{\ f\ } & \mathbb{B}(\bigvee_{k=0}^{\infty} D_k \mathbb{A}(X)) \\ {\scriptstyle \iota}\Big\downarrow & \nearrow{\scriptstyle \bar{f}} & \\ \mathbb{B}\mathbb{A}(X)_+ & & \end{array}$$

First we have the inclusion

$$\begin{array}{rccl} \iota: & \mathbb{A}(X)_+ & \hookrightarrow & \mathbb{B}(\mathbb{A}(X)_+) \\ & (a;x_1,\cdots,x_n) & \mapsto & \left(\begin{array}{ccccc} \mathbb{R}^\infty & \leftarrow & F=\{*\} & \rightarrow & \mathbb{A}(X)_+ \\ 0 & \leftarrow\!\shortmid & * & \mapsto & (a;x_1,\cdots,x_n) \end{array}\right) \end{array}$$

We fix an embedding $e:\coprod_n \mathbb{A}(\boldsymbol{n}) \to \mathbb{R}^\infty$. For each s, we define f_s on the s-th filtration $\mathbb{A}_s(X)_+$,

$$\begin{array}{rccl} f_s: & \mathbb{A}_s(X)_+ & \longrightarrow & \mathbb{B}(\bigvee_{k=0}^s D_k\mathbb{A}(X)) \\ & (a;x_1,\cdots,x_n) & \mapsto & \left(\begin{array}{ccccc} \mathbb{R}^\infty & \overset{\epsilon}{\leftarrow} & \coprod_k F & \overset{\varphi}{\longrightarrow} & \bigvee_{k=0}^s D_k\mathbb{A}(X) \\ e(a') & \leftarrow\!\shortmid & a' & \mapsto & \overline{(a';x_1,\cdots,x_n)} \end{array}\right) \end{array}$$

Here F varies over all the $\mathrm{pspl}(a) \subseteq \mathbb{A}_k(\boldsymbol{n}) \subset \mathbb{A}(\boldsymbol{n})$ for all n. That is, F has weight k for $0 \le k \le s$. And $\overline{(a';x_1,\cdots,x_n)}$ is the image of $(a';x_1,\cdots,x_n)$ under the quotient $\mathbb{A}_k(X) \to D_k\mathbb{A}(X)$. So all the f_s induces a map

$$f:\mathbb{A}(X)_+ \to \mathbb{B}(\bigvee_{k=0}^{\infty} D_k\mathbb{A}(X)).$$

To construct $\bar f : \mathbb{B}\mathbb{A}(X)_+ \to B(\bigvee_{k=0}^\infty D_k\mathbb{A}(X))$, we shall apply the similar idea as above to define

$$\bar f_s : \mathbb{B}(\mathbb{A}_s(X)_+) \to \mathbb{B}(\bigvee_{k=0}^{s} D_k\mathbb{A}(X)).$$

Assume given $\beta = (\mathbb{R}^\infty \xleftarrow{\epsilon} F' \xrightarrow{\varphi} \mathbb{A}_s(X)) \in \mathbb{B}(\mathbb{A}_s(X))$. For the given continuous composition $f_s\circ\varphi : F' \to \mathbb{A}_s(X) \to \mathbb{B}(\bigvee_{k=0}^s D_k\mathbb{A}(X)$, we let $\hat F$ be the pull-back of the diagram

$$\begin{array}{ccc} \hat F & \longrightarrow & \mathbb{E}(\bigvee_{k=0}^s D_k\mathbb{A}(X)) \\ \downarrow & & \downarrow \\ F' & \longrightarrow & \mathbb{B}(\bigvee_{k=0}^s D_k\mathbb{A}(X)). \end{array}$$

Denote the composite map by $\hat\varphi : \hat F \to \mathbb{E}(\bigvee_{k=0}^s D_k\mathbb{A}(X)) \to \bigvee_{k=0}^s D_k\mathbb{A}(X)$, where the second map is the evaluation map $\mathrm{ev} : \mathbb{E}(\bigvee_{k=0}^s D_k\mathbb{A}(X)) \to \bigvee_{k=0}^s D_k\mathbb{A}(X), [\epsilon_F, F, \varphi_F, z] \mapsto \varphi_F(z)$. Set $\hat F_0 := \hat F\backslash\hat\varphi^{-1}(+)$. The morphisms are given by

$$\bar f_s(\beta) = \left(\begin{array}{ccccc} \mathbb{R}^\infty \cong \mathbb{R}^\infty\times\mathbb{R}^\infty & \overset{\hat\epsilon}{\leftarrow} & \hat F_0 & \overset{\hat\varphi}{\longrightarrow} & \bigvee_{k=0}^s D_k\mathbb{A}(X) \\ (\epsilon(\ell),\epsilon_F(z)) & \leftarrow\!\shortmid & (\ell,\epsilon_F,F,\varphi_F,z) & \mapsto & \varphi_F(z) \end{array}\right)$$

It is straightforward to check $\bar f_s$ is a well-defined map and hence induces the required map $\bar f$.

Consider the following homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{Sect}(D_m\mathbb{A}(X)) & = & \mathrm{Sect}(D_m\mathbb{A}(X)) \\ \uparrow & & \uparrow \\ \mathbb{B}(\mathbb{A}_m(X)_+) & \xrightarrow{\bar f_m} & \mathbb{B}(\bigvee_{k=0}^m D_k\mathbb{A}(X)) \\ \uparrow & & \uparrow \\ \mathbb{B}(\mathbb{A}_{m-1}(X)_+) & \xrightarrow{\bar f_{m-1}} & \mathbb{B}(\bigvee_{k=0}^{m-1} D_k\mathbb{A}(X)) \end{array}$$

We claim that each $\bar{f}_m$ is a homotopy equivalence. When $m = 1$, the bottom map $\bar{f}_1 \simeq \mathrm{id}$, since in this case $\mathbb{A}_1(X) = D_1\mathbb{A}(X)$, given

$$\beta = (\mathbb{R}^\infty \xleftarrow{\epsilon} F' \xrightarrow{\varphi} \mathbb{A}_1(X)_+) \in \mathbb{B}(\mathbb{A}_1(X)_+)$$

$\hat{F} \simeq F' \times \mathbb{R}^\infty \times D_1\mathbb{A}(X)_+$, we write $\varphi(\ell) = (a; x_1, \cdots, x_n) \in \mathbb{A}_1(X)$, by construction, $F = \mathrm{pspl}(a) \subseteq \mathbb{A}_1(\boldsymbol{n})$. For each $(a; x_1, \cdots, x_n) \in \mathbb{A}_1(X)_+$, we know from the last section that there is an equivalence class $(a'; x') \in \mathbb{A}_1(X)$ of length 1. Therefore $F \simeq \mathrm{pspl}(a') = a' \in \mathbb{A}_1(\mathbf{1})$ which is contractible, similar as the proof in the last chapter, it follows that $\bar{f}_1 \simeq \mathrm{id}$.

It is almost the same argument as in the proof of Theorem 5.3.6., so we skip the rest of the proof. □

Remark 6.4.2. (i) In the construction of the map $\bar{f} : \mathbb{B}\mathbb{A}(X)_+ \to \mathbb{B}(\bigvee_{k=0}^\infty D_k\mathbb{A}(X))$, it is clear for each $m \geq 1$, the composite $\mathbb{B}\mathbb{A}_m(X)_+ \hookrightarrow \mathbb{B}\mathbb{A}(X)_+ \xrightarrow{\bar{f}} \mathbb{B}(\bigvee_{k=0}^\infty D_k\mathbb{A}(X))$ factors up to homotopy through the inclusion

$$\mathbb{B}(\bigvee_{k=0}^{m} D_k\mathbb{A}(X)) \hookrightarrow \mathbb{B}(\bigvee_{k=0}^{\infty} D_k\mathbb{A}(X)).$$

This is equivalent to say that the composite is null-homotopic,

$$\mathbb{B}\mathbb{A}_m(X) \hookrightarrow B\mathbb{A}(X) \xrightarrow{\bar{f}} \mathbb{B}(\bigvee_{k=1}^{\infty} D_k\mathbb{A}(X)) \to \mathbb{B}(\bigvee_{k=m+1}^{\infty} D_k\mathbb{A}(X)).$$

(ii) If we denote $\bar{f}_k$ the composite map

$$\bar{f}^k : \mathbb{B}\mathbb{A}(X)_+ \to \mathbb{B}(\bigvee_{k=0}^{\infty} D_k\mathbb{A}(X)) \to \mathbb{B}D_k\mathbb{A}(X),$$

where the second map is the projection onto the k-th wedge summand, then the composite $\mathbb{B}\mathbb{A}_k(X)_+ \hookrightarrow \mathbb{B}\mathbb{A}(X)_+ \xrightarrow{\bar{f}^k} \mathbb{B}D_k\mathbb{A}(X)$ is homotopic to the natural projection

$$\mathbb{B}\mathbb{A}_k(X)_+ \to \mathbb{B}(\mathbb{A}_k(X)_+/\mathbb{A}_{k-1}(X)_+) = \mathbb{B}D_k\mathbb{A}(X).$$

6.5 Homotopy Calculus of Segal Γ-Spaces

We generalize Segal Γ-space and investigate Goodwillie's Taylor tower of the generalized Γ-spaces $\mathbb{A} : \Gamma \to \mathrm{Top}_*$. Note we denote its extension also by $\mathbb{A} : \mathrm{Top}_* \to \mathrm{Top}_*$. And we try to find examples of quadratic Γ-functors. Recall that a functor is of degree n if it sends a strongly cocartesian $(n+1)$-cube to a cartesian cube, more details see Appendix B.

Lemma 6.5.1. Segal Γ-spaces are linear.

Proof. Regard Γ as a subcategory of Top_* and its objects as discrete spaces. For the original Γ space, we consider first the second cross effect $cr_2\mathbb{A}$,

$$cr_2\mathbb{A}(\boldsymbol{n_1}, \boldsymbol{n_2}) = \mathrm{thofib}\begin{pmatrix} \mathbb{A}(\boldsymbol{n_1} \vee \boldsymbol{n_2}) & \to & \mathbb{A}(\boldsymbol{n_1}) \\ \downarrow & & \downarrow \\ \mathbb{A}(\boldsymbol{n_2}) & \to & \mathbb{A}(\mathbf{0}) \end{pmatrix}$$

Since $\mathbb{A}(\mathbf{0}) \simeq *$, we have $\mathbb{A}(\boldsymbol{n_1} \vee \boldsymbol{n_2}) \simeq \mathbb{A}(\boldsymbol{n_1}) \times \mathbb{A}(\boldsymbol{n_2})$. It implies that $cr_2\mathbb{A} \simeq *$, and analogously $cr_k\mathbb{A} \simeq *$, for $k \geq 3$. So we see that $\mathbb{A}$ is linear.

To show its extension $\mathbb{A} : \mathrm{Top}_* \to \mathrm{Top}_*$ is linear corresponds to show that $\mathbb{A}(X)$ is of the form $\Omega^\infty(\underline{E} \wedge X)$ for some spectrum $\underline{E}$. From Proposition 1.3.4. in Chapter 1 we know that $\mathbb{A}(X)$ has the weak homotopy type of $\Omega\mathbb{A}(\Sigma X)$, which implies that $\mathbb{A}(X) \simeq \Omega^\infty \mathbb{A}(\Sigma^\infty X) \simeq \Omega^\infty(\mathbb{A}(\underline{S}) \wedge X)$. □

Note that this results only works for connected X or group complete $\mathbb{A}$. Now we want to generalize Segal's notion of Γ-spaces by removing from now on the condition $\mathbb{A}(\boldsymbol{n}) \simeq \mathbb{A}(\mathbf{1})^n$. Assume only that $\mathbb{A}$ is reduced, namely $\mathbb{A}(\mathbf{0}) \simeq *$. And assume the generalized Γ-space $\mathbb{A}$ takes value in the category of spaces which have the homotopy type of finite CW complexes and the π_0 are groups. Then the following proposition tells us the relationship between the degree of Γ-spaces and that of their extensions generally, i.e. the above lemma works as well for higher degrees of $\mathbb{A}$.

Proposition 6.5.2. If $\mathbb{A}$ is a Γ-space of degree k, its extension is also of degree k.

Proof. Assume $\deg(\mathbb{A}) \leq k$, it follows that $cr_k\mathbb{A}(\boldsymbol{n_1}, \cdots, \boldsymbol{n_k}) : \Gamma^k \to \mathrm{Top}_*$ is symmetric multi-linear.

In ([Wo], Theorem 1.5.), Woolfson proved that if X has the homotopy type of finite CW-complex, $\mathbb{A}(X)$ is homeomorphic to $\mathbb{A}'(X)$, where $\mathbb{A}'(X)$ is the realization of the simplicial space whose space of k-simplices is

$$\coprod_{S_i \in \Gamma} \mathbb{A}(S_0) \times \mathrm{Mor}(S_0, S_1) \times \cdots \times \mathrm{Mor}(S_{k-1}, S_k) \times X^{S_k}.$$

By ([Wo], Lemma 1.9.), if X is the simplicial space $[k] \mapsto X_k$ and $\mathbb{A}(X)$ the simplicial space $[k] \mapsto \mathbb{A}(X_k)$, then $\mathbb{A}(|X|) = |\mathbb{A}(X)|$.

Since $\mathbb{A}$ lands in the category of spaces with π_0 being groups, $\mathbb{A}$ is a π_*-Kan functor in the sense of Oats ([Oa], Definition 4.4., Lemma 4.8.), hence $cr_k \operatorname{Ext}\mathbb{A} \simeq \operatorname{Ext}^k(cr_k\mathbb{A})$.

It follows that we can calculate the cross effects dimensionwise. By the co-end construction, the extension of $cr_k\mathbb{A}(\boldsymbol{n_1}, \cdots, \boldsymbol{n_k})$ on spaces has the form $cr_k\mathbb{A} : \mathrm{Top}_*^k \longrightarrow \mathrm{Top}_*$:

$$\begin{aligned}(X_1, \cdots, X_k) \mapsto \int^{(\boldsymbol{n_1}, \cdots, \boldsymbol{n_k}) \in \Gamma^k} & cr_k\mathbb{A}(\boldsymbol{n_1}, \cdots, \boldsymbol{n_k}) \times (X_1^{n_1} \times \cdots X_k^{n_k}) \\ = \int^{\boldsymbol{n_k} \in \Gamma} X_k^{n_k} \times \cdots \int^{\boldsymbol{n_1} \in \Gamma} & cr_k\mathbb{A}(\boldsymbol{n_1}, \cdots, \boldsymbol{n_k}) \times X_1^{n_1}.\end{aligned}$$

We first fix $(X_2, \cdots, X_k)$, and let X_1 vary, then the n-fold integral can be written as the n-fold iteration of a 1-fold integral. And since $cr_k\mathbb{A}(\boldsymbol{n_1}, \cdots, \boldsymbol{n_k})$ is multi-linear, in particular it is linear with respect to $\boldsymbol{n_1}$, by the same trick as lemma 6.5.1., $cr_k\mathbb{A}(X_1, \cdots, X_k)$ is linear with respect to X_1, and analogously for X_i's. Therefore $cr_k\mathbb{A}(X_1, \cdots, X_k)$ is also symmetric multi-linear, which implies that the extension $\mathbb{A}$ is of degree k. □

Example 6.5.3. Our motivating example derives from the generalization of configuration spaces. Define

$$T(\mathbf{0}) := \{+1, -1\}, \text{ a single-point space of a pair,}$$

$$T(\mathbf{1}) := \coprod_n \tilde{C}^n(C^2(\mathbb{R}^\infty)) = \coprod_n \tilde{C}^n(B\Sigma_2), \text{ordered configuration space of pairs in } \mathbb{R}^\infty,$$

$$\vdots$$

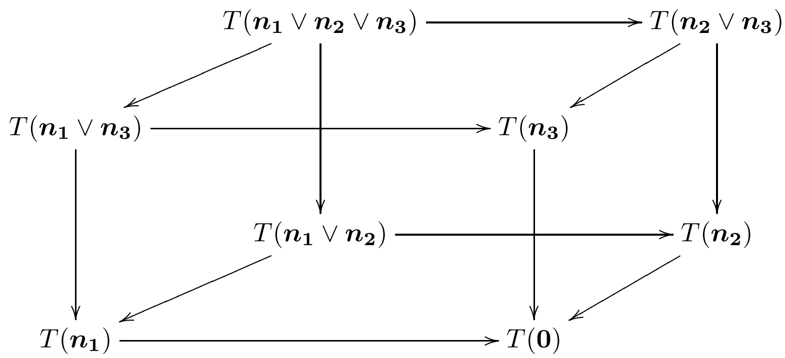
$$T(\boldsymbol{k}) := \left\{ \ (\xi_1, \cdots, \xi_k) \in T(\mathbf{1})^k \ \middle| \ \xi_i \cap \xi_j = \emptyset \text{ for } \ i \neq j \ \right\}.$$

One can check this is a Segal Γ-space and we claim that T is a polynomial functor of degree 2:

$$cr_2T(\boldsymbol{n_1}, \boldsymbol{n_2}) = \text{thofib} \begin{pmatrix} T(\boldsymbol{n_1} \vee \boldsymbol{n_2}) & \to & T(\boldsymbol{n_1}) \\ \downarrow & & \downarrow \\ T(\boldsymbol{n_2}) & \to & T(\boldsymbol{0}) \end{pmatrix}$$

is nontrivial, because the total homotopy fiber is equivalent to the configurations of all the pairs such that the pair does not lie in the same $T(\boldsymbol{n_i}), i = 1, 2$.

$cr_3T(\boldsymbol{n_1}, \boldsymbol{n_2}, \boldsymbol{n_3}) = \text{thofib}$



is trivial, since the configuration of pairs in $T(\boldsymbol{n_1} \vee \boldsymbol{n_2} \vee \boldsymbol{n_3})$ lies either in $T(\boldsymbol{n_2} \vee \boldsymbol{n_3})$, $T(\boldsymbol{n_1} \vee \boldsymbol{n_3}), T(\boldsymbol{n_1} \vee \boldsymbol{n_2})$ or their common parts, so we see that the 3-cube is a homotopy pullback , which follows that $\deg(T) \leq 2$, and so is its extension.

The quadratic functor $D_2T(X)$ will have the form

$$D_2T(X) \simeq \coprod_n C^n(X^{\wedge 2} \wedge_{\Sigma_2} E\Sigma_{2+}) \hookrightarrow \Omega^\infty \Sigma^\infty (X \wedge X \wedge E\Sigma_{2+})_{\Sigma_2} = \Omega^\infty \Sigma^\infty (X \wedge X)_{h\Sigma_2}.$$

Example 6.5.4. Another example is a generalization of the infinite symmetric product. Let X be a based connected space, define

$$Qu(X) := \left\{ \{(x_1, y_1), \cdots, (x_n, y_n)\} \in \bigcup_n \mathrm{SP}^n(\mathrm{SP}^2(X)) \right\} / \sim .$$

$$\{(x_1, y_1), \cdots, (x_n, y_n)\} \sim \{(x_1, y_1), \cdots, \widehat{(x_i, y_i)}, \cdots, (x_n, y_n)\}, \text{ if } x_i = * \text{ or } y_i = *.$$

It is straigtforward to check that $Qu(*) = *$ and $Qu(f) : Qu(X) \to Qu(Y)$ is a homotopy equivalence if $f : X \to Y$ is so. And in particular there is an isomorphism $Qu(S^0) \cong \mathbb{N}$ which is given by the cardinality of nonbasepoint elements.

Denote $S^0 \vee S^0 = \{*, u_1, u_2\}$, we then have the resulting map

$$Qu(S^0 \vee S^0) \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}, \ n_1(u_1, u_1) + n_2(u_1, u_2) + n_3(u_2, u_2) \mapsto (n_1, n_2, n_3).$$

It implies that $Qu(X)$ is not linear, but a quadratic functor.

Appendix A

Gromov's h-principle

We will give an introduction and recall some results of Gromov [Gr] which will be necessary for the understanding of the proof of Proposition 5.2.5.

In [Gr] Section 2.2 Gromov considered a general case, namely the set-theoretic sheaf. For a topological sheaf, it is understandable that the restriction map induced by inclusions is continuous. For a set-theoretic sheaf, there are no open sets, but still he can define the notion of quasi-continuous sheaves.

Recall that a *sheaf* over an n-dimensional manifold M, by definition, assigns a set $\Psi(U)$ to each open subset $U \subset M$ and a map $\Psi(\iota) : \Psi(U) \to \Psi(U')$ to each inclusion $\iota : U' \subset U$ such that the following axioms are satisfied.

(1) *Functoriality.* If $\iota' : U'' \subset U'$ and $\iota : U' \subset U$ are two inclusions, then $\Psi(\iota \circ \iota') = \Psi(\iota') \circ \Psi(\iota)$.

(2) *Locality* (Uniqueness). If two elements ψ_1 and ψ_2 of Ψ over U are *locally equal*, then they are equal. Here the local equality means that there exists a neighborhood $U' \subset U$ of every point $u \in U$ such that $\psi_1|_{U'} = \psi_2|_{U'}$.

(2') *Locality* (Existence). If $\{U_i\}$ is an open covering of U, and if we have elements $\psi_i \in \Psi(U_i)$ satisfying $\psi_i|_{U_i \cap U_j} = \psi_j|_{U_i \cap U_j}$ for each i, j, then there exists an element $\psi \in \Psi(U)$ such hat $\psi|_U = \psi_i$ for each i.

The axiom (2) and (2') show that every sheaf Ψ is uniquely defined by $\Psi(U_i)$ for any base of open subsets $U_i \subset M$. Next we can extend Ψ to non-open subsets $C \subset M$. We define $\Psi(C)$ to be the direct limit of $\Psi(U)$ over all neighborhoods U of C. In particular, one can also define the *stalk* $\Psi(v)$ for all $v \in M$. Then one can restrict Ψ to a sheaf over C, denoted by $\Psi|_C$ and define $(\Psi|_C)(D)$ to be the direct limit of $\Psi(O)$ for all open subsets $D \subset C$ and for $O \subset M$. Thus the sheaf $\Psi|_C$ has the same stalks over the point $c \in C$ as Ψ.

Definition A.0.5. (Spanier-Whitehead [SpWh], page 336; [Gr], page 36) Let A be any set. By a *quasi-topology* on A we mean a rule which, for every topological space P, selects a class of functions $P \to A$, to be called *quasi-continuous* in [SpWh], subject to the following formal properties.

(i) If $\mu : P \to A$ is quasi-continuous and $\varphi : Q \to P$ is a continuous map, then the composite $\mu \circ \varphi : Q \to A$ is quasi-continuous. And every constant function $P \to A$ is quasi-continuous.

(ii) If a map $\mu : P \to A$ is locally quasi-continuous, then it is quasi-continuous. Locally quasi-continuous means that for every point in P there exists a neighborhood $U \subset P$ such that $\mu|_U : U \to A$ is quasi-continuous.

(iii) Let P be covered by two closed (or two open) subsets P_1 and P_2. If a map μ is quasi-continuous on P_1 and P_2, then it is quasi-continuous on P. Therefore, if $P = \bigcup_{i=1}^k P_i$ is a covering of P by finitely many closed (or open) subsets, then a map $\mu : P \to A$ is quasi-continuous if and only if $\mu|_{P_i} : P_i \to A$ is quasi-continuous for all $i = 1, \cdots, k$.

In general, if $\mathbb{A}$ has a topology, by assigning the functions $P \to A$ to be the ordinary continuous maps, we obtain a quasi-topology on A. It is a much weaker structure. A map between quasi-topological spaces $\alpha : A \to B$ is called *quasi-continuous* if $\alpha \circ \mu : P \to B$ is quasi-continuous for all quasi-continuous map $P \to A$ from a topological spaces P.

Definition A.0.6. A set-valued sheaf Ψ on M is called *quasi-continuous* if every set $\Psi(U)$ for $U \subset M$ is endowed with a quasi-topology such that the restriction maps $\Psi(\iota)$ induced by inclusions $\iota : U \subseteq V$ are quasi-continuous maps $\Psi(V) \to \Psi(U)$.

Let Open(M) be the category of open submanifolds of M with inclusions as morphisms and QTOP be the category of quasi-topological spaces with quasi-continuous maps as morphisms. Then we can regard a quasi-continuous sheaf Ψ as a contravariant functor $\Psi : \text{Open}(M) \to \text{QTOP}$. A *homomorphism* between quasi-continuous sheaves over M, say $\alpha : \Phi \to \Psi$, is a collection of quasi-continuous maps $\alpha_U : \Phi(U) \to \Psi(U)$ for all open $U \subset M$ which commute with the restriction maps, that is $\alpha_{U'} \circ \Phi(\iota) = \Psi(\iota) \circ \alpha_U$ for all $\iota : U' \subset U$. Finally, one defines a *subsheaf* $\Psi' \subset \Psi$ by given a subset $\Psi'(U) \subset \Psi(U)$ for all $U \subset V$ such that Ψ' satisfies (2) and (2').

The standard definitions of homotopy theory (e.g. the weak homotopy equivalence) obviously generalize to quasi-topological spaces.

We write $\mathcal{D}\!\mathit{iff}(M)$ for the pseudogroup of diffeomorphisms of M, which is the set Λ of triples (U, f, U') with U an open set of V and $f : U \to U'$ a diffeomorphism. This pseudogroup Λ satisfies the following properties:

(i) For every open set U in V, (U, id, U) is in Λ.

(ii) If (U, f, U') is in Λ, then so is (U', f^{-1}, U).

(iii) If (U, f, U_1) and (U', f', U_1') are in Λ, and the intersection $U_1 \cap U'$ is not empty, then the restricted composition $(f^{-1}(U_1 \cap U'), f' \circ f)$ with $f' \circ f : f^{-1}(U_1 \cap U') \to f'(U_1 \cap U')$ is in Λ.

By a $\mathcal{D}\!\mathit{iff}(M)$-action on the sheaf Ψ we mean there is a family of morphisms $\{\Psi_f\}$, where $\Psi_f : \Psi(U') \to \Psi(U)$. If all Ψ_f are homeomorphisms, we say Ψ is $\mathcal{D}\!\mathit{iff}(M)$-invariant, or equivariant. Note that if we take U to be M itself, then the ordinary diffeomorphism group $\mathrm{Diff}(M)$ is a proper subset of $\mathcal{D}\!\mathit{iff}(M)$.

For any topological space P and a topological sheaf Ψ we define a new quasi-continuous sheaf Ψ^P on $M \times P$, which we call the *parametric sheaf*: its elements are the continuous families of elements of Ψ parametrized by P. To give the definition we only need to specify $\Psi^P(U \times R)$ for open sets $U \subseteq M$, $R \subseteq P$. Set $\Psi^P(U \times R) := (\Psi(U))^R$, the set of quasi-continuous maps with the following quasi-topology: a map $Q \to (\Psi(U))^R$ is quasi-continuous if and only if the map $Q \times R \to \Psi(U)$ is continuous.

Next we apply this to $P = M$ and restrict the parametric sheaf Ψ^M over $M \times M$ to the diagonal $\Delta \subseteq M \times M$. The resulting sheaf over $\Delta \cong M$ is denoted by $\Psi^\sharp$. So $\Psi^\sharp$ associates for each $z \in U$ a germ ψ_z in the stalk $\Psi(z)$. Every element of Ψ corresponds to a unique constant family of elements with the parameter space M. We obtain an injective homomorphism $D : \Psi \to \Psi^\sharp$ which makes Ψ a subsheaf $D(\Psi) = \Psi$ in $\Psi^\sharp$.

For example, $\Psi(U) = \coprod_F \operatorname{Emb}(F,U)/\operatorname{Diff}(F)$, where F varies over d-dimensional manifolds without boundary. This sheaf has a topology defined in [RW]. Note that an element in $\Psi^\sharp(U)$ assigns to each $z \in U$ a germ ψ_z of d-submanifolds in U. That is, ψ_z is represented by an equivalence class $\alpha \in \Psi(U)$, where U is a neighborhood of z. The equivalence relation is as follows. Let U' be another neighborhood of z, then an element $\beta \in \Psi(U')$ is equivalent to α if there exists a neighborhood U'' of z which lies in $U \cap U'$ such that the restriction of α and β on U'' are the same. For each z, we can choose a specific U, which is under the exponential map the image of open disc $D_zU \subset T_zU$ in the tangent space to z. This neighborhood $U \cong \mathbb{R}^n$ depends on the choice of the metric. Then $\Psi^\sharp(U)$ can be identified with the sheaf of sections of certain bundle on U. The fibre over each $z \in U$ is the space of germs of submanifolds of $\mathbb{R}^n$ at 0, namely $\Psi(0 \in \mathbb{R}^n)$. By definition, $\Psi(0 \in \mathbb{R}^n) = \operatorname*{colim}_{0 \in U \subseteq \mathbb{R}^n} \Psi(U)$. Since any ϵ-neighborhood of $0 \in \mathbb{R}^n$ can be stretched to all of $\mathbb{R}^n$, the restriction map $\Psi(\mathbb{R}^n) \to \Psi(0 \in \mathbb{R}^n)$ is a homotopy equivalence.

If $E \to X$ is an n-dimensional vector bundle and if Ψ is a topological sheaf, then there is an associated fiber bundle $\Psi^{\text{fib}}(E) \to X$ whose fiber over x is $\Psi(E_x)$ where E_x is the fiber of x in the vector bundle $E \to X$. So we can apply this construction to the tangent bundle $TM \to M$. Then $\Psi^{\text{fib}}(TM)$ can be constructed by letting $V_n(TM)$ be the frame bundle of M, a principal GL_n-bundle. Note that there is a group homomorphism $\mathrm{GL}_n \to \operatorname{Diff}(\mathbb{R}^n)$ into the ordinary group of diffeomorphism and GL_n acts continuously on $\Psi(\mathbb{R}^n)$ if Ψ is $\operatorname{Diff}(\mathbb{R}^n)$-invariant. So we can form

$$\Psi^{\text{fib}}(TM) := V_n(TM) \times_{\mathrm{GL}_n} \Psi(\mathbb{R}^n). \tag{A.0.1}$$

Since $\Psi(\mathbb{R}^n) \simeq \Psi(0 \in \mathbb{R}^n)$, $\Psi^\sharp(M)$ can be identified with the space of sections of the bundle $\Psi^{\text{fib}}(TM)$ obtained by applying Ψ to the tangent bundle. That is

$$\Psi^\sharp(M) \simeq \operatorname{Sect}(\Psi^{\text{fib}}(TM) \to M).$$

The composition

$$\Psi(M) \xrightarrow{D} \Psi^\sharp(M) \simeq \operatorname{Sect}(\Psi^{\text{fib}}(TM) \to M) \tag{A.0.2}$$

is homotopic to a "scanning map", more details are given in Chapter 5.

Definition A.0.7. A sheaf Ψ satisfies the *h-principle* if $D : \Psi \to \Psi^\sharp$ is a weak homotopy equivalence.

Definition A.0.8. A topological sheaf Ψ on M is called *microflexible* if for each pair of compact sets $C' \subset C$ in M, the restriction map $\Psi(C) \to \Psi(C')$ is a microfibration.

A *microfibration* is a map $p : E \to B$ such that for each commutative right square in the diagram with A a compact polyhedron, there is an $\epsilon > 0$ and a lifting $L : A \times [0,\epsilon] \to E$, such that $p \circ L = H|_{A\times[0,\epsilon]}, L|_{A\times\{0\}} = \ell$ and the diagram commutes.

$$\begin{array}{ccccc} A\times\{0\} & = & A\times\{0\} & \xrightarrow{\ell} & E \\ \downarrow & & \downarrow & \overset{L}{\nearrow} & \downarrow p \\ A\times[0,\epsilon] & \hookrightarrow & A\times[0,1] & \xrightarrow{H} & B. \end{array}$$

Theorem A.0.9. ([Gr], section 2.2.2) D is a weak homotopy equivalence if M is open (i.e. no component is compact), Ψ is $\mathcal{D}\!\mathit{iff}(M)$-invariant and microflexible. □

This theorem is crucial for the proof of our main result Theorem 5.3.6.

Appendix B

Homotopy Calculus of Functors: an Overview

The homotopy calculus of functors is a method of describing spaces up to weak homotopy equivalence by using categories and functors (developed by T. Goodwillie). Namely, one obtains information about a space by viewing the space as a special value of suitable functor, analyzes the functor using "calculus". We give a brief summary of the homotopy calculus of functors which we used in Chapter 6. For a more detailed discussion see [Go1],[Go2] and [Go3].

We assume given a homotopy functor $F : \mathcal{C} \to \mathcal{D}$ between two nice topological model categories in which one can do homotopy. One wishes to understand the homotopy type of $F(X)$, perhaps for some particular $X \in \mathcal{C}$. The idea is to use the *functoriality* as X varies, to construct a canonical *polynomial* "resolution" of $F(X)$ as a functor of X. In the ordinary differential calculus, the central idea is to approximate the functions by linear functions, and here analogously, the central idea is to approximate functors by "linear" functors.

A homotopy functor F is called *linear* if the following holds:

1) F takes homotopy pushout squares to homotopy pullback squares (call F *excisive*);

2) $F(*) \to *$ is a weak homotopy equivalence (call F *reduced*).

Each linear functor from Top_* to Top_* has the form $L(X) = \Omega^\infty(\underline{C} \wedge X)$ for a spectrum $\underline{C}$ (it is called the *coefficient of the linear functor*).

There is a standard process, which is called the *linearization*, for turning a reduced functor F into a linear functor L. Roughly speaking, there is a natural map $F(X) \to \Omega F(\Sigma X)$, and one iterates this process to form the linearization, the homotopy colimit of $\Omega^n F(\Sigma^n X)$ as n runs to infinity. If F is linear then L is equivalent to F. We shall explain this linearization process in the next part.

One the other hand, one can view the excision condition as a property of functors defined on 2-dimensional cubical diagrams. Generalizing this, we call a functor *polynomial of degree at most* n if it satisfies similar condition on $(n+1)$-dimensional cubical diagrams. And it turns out that for any F there is a universal n-excisive functor under F. We denote this functor $P_n F$ and call it the *n-th Taylor polynomial* of F.

We recall first the notion of cubical diagrams ([Go2], [Ku]). Let S be a finite pointed set. The poset of pointed subsets of S is $\mathcal{P}(S) = \{T \subset S \text{ and } T \text{ contains the base point}\}$. It is a partially ordered set via pointed inclusion, hence is a small category. Also, write $\mathcal{P}_0(S)$ for the full subcategory of all subsets T of S such that the complementary of the base point in T is nonempty, and $\mathcal{P}_1(S)$ for the full subcategory of all proper subsets of S. Often S is given by the concrete set $\underline{n} = \{0, 1, \cdots, n\}$ with 0 as base point.

Definition B.0.10. (1) An n-cube in $\mathcal{C}$ is a functor $\mathcal{X} : \mathcal{P}(\underline{n}) \to \mathcal{C}$.

(2) An n-cube $\mathcal{X}$ is *(homotopy) cartesian* if the natural composition

$$\mathcal{X}(\underline{0}) \to \lim_{T \in \mathcal{P}_0(\underline{n})} \mathcal{X}(T) \to \operatorname*{holim}_{T \in \mathcal{P}_0(\underline{n})} \mathcal{X}(T)$$

is a weak equivalence. Dually we say that an n-cube in $\mathcal{D}$ is *(homotopy) cocartesian* if the natural composition

$$\operatorname*{hocolim}_{T \in \mathcal{P}_1(\underline{n})} \mathcal{X}(T) \to \operatorname*{colim}_{T \in \mathcal{P}_1(\underline{n})} \mathcal{X}(T) \to \mathcal{X}(\underline{n})$$

is a weak equivalence.

(3)A *strongly (homotopy) cocartesian* n-cube $\mathcal{X}$ is one in which every 2-dimensional face is cocartesian. (This definition implies that the n-cube itself as well as every face of dim ≥ 2 is also cocartesian.)

Definition B.0.11. $F : \mathcal{C} \to \mathcal{D}$ is called *n-excisive* (or *polynomial of degree at most n)* , denoted by $\deg(F) \leq n$) if, whenever $\mathcal{X}$ is a strongly cocartesian $(n+1)$-cube in $\mathcal{C}$, $F(\mathcal{X})$ is a Cartesian cube in $\mathcal{D}$.

Example B.0.12. $F : \mathcal{C} \to \mathcal{D}$ is 1-excisive (linear) if and only if F takes (homotopy) pushout squares to (homotopy) pullback squares.

Excisive approximations are constructed by making for every $X \in \mathcal{C}$ a "nice" strongly cocartesian n-cube $\mathcal{X}(X) : \mathcal{P}(\boldsymbol{n}) \to \mathcal{C}$ and then investigating the resulting n-cubes $F(\mathcal{X}(X))$. If all the cubes $F(\mathcal{X}(X))$ are Cartesian, then F is $(n-1)$-excisive; otherwise $F(\mathcal{X}(X))$ gives rise to a new functor $T_{n-1}(F)$ which is slightly closer to being Cartesian on $\mathcal{X}(X)$.

We review the construction of Kuhn [Ku] and define the join of an object of $\mathcal{C}$ and a finite set.

Definition B.0.13. For $X \in \text{ob}(\mathcal{C})$ and T a finite set, define $X * T$, the *join* of X and T to be the homotopy cofiber of the folding map $X * T = \text{hocof}(\coprod_T X \to X)$.

Note that for $T \subset \underline{n}$, the assignment $T \mapsto X * T$ defines a strongly cocartesian n-cube $\mathcal{X}$. In the case $n = 2$, we have the pushout square

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

Definition B.0.14. Define $T_{n-1}F : \mathcal{C} \to \mathcal{D}$ to be $T_{n-1}F(X) := \operatorname*{holim}_{T \in \mathcal{P}_0(\underline{n})} F(X * T)$.

For example,

$$T_1F(X)) = \text{holim} \begin{pmatrix} F(CX) \\ \downarrow \\ F(\Sigma X) \\ \uparrow \\ F(CX) \end{pmatrix} \simeq \text{holim} \begin{pmatrix} * \\ \downarrow \\ F(\Sigma X) \\ \uparrow \\ * \end{pmatrix} \simeq \Omega F(\Sigma X).$$

There is a natural transformation $t_{n-1}(F) : F \to T_{n-1}F$. If F is $(n-1)$-excisive, this is an equivalence. If not, the $(n-1)$-excisive approximation to F is given by the homotopy colimit of the diagram

$$P_{n-1}F := \text{hocolim}\left\{F \to T_{n-1}F \to T_{n-1}^2 F \to \cdots\right\}.$$

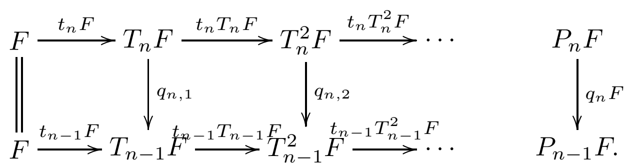
From the last example we know that

$$P_1F(X) \simeq \text{hocolim}_n\, \Omega^n F(\Sigma^n X).$$

In particular, if F is the identity functor $\text{id} : \text{Top}_* \to \text{Top}_*$, it follows that

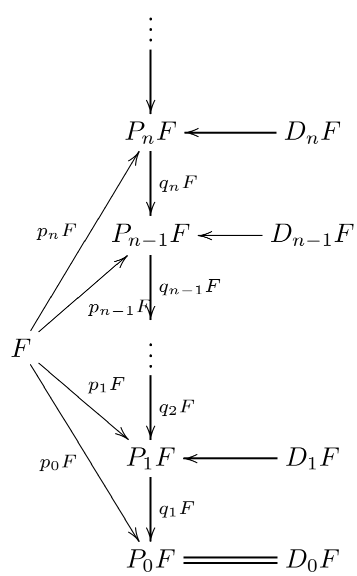
$$P_1(\text{id})(X) \simeq \Omega^\infty \Sigma^\infty X = QX.$$

Since $F = T_{n-1}^0 F$, the functor $P_{n-1}F$ comes equipped with a natural transformation $F \xrightarrow{p_{n-1}F} P_{n-1}F$. Furthermore, there are transformations $T_nF \to T_{n-1}F$ induced by the inclusion of categories $\mathcal{P}_0(\underline{n}) \to \mathcal{P}_0(\underline{n+1})$, which extends to give a commutative diagram ([Go3])



Therefore it defines a natural transformation between the homotopy colimits $q_nF : P_nF \to P_{n-1}F$, since all the $q_{n,i}F$ are the natural maps from the homotopy limit of a diagram to the homotopy limit of a restriction of the diagram.

Theorem B.0.15. $[Go3, \text{Theorem}1.13.]$ A homotopy functor $F : \mathcal{C} \to \mathcal{D}$ determines a tower of functors $P_nF : \mathcal{C} \to \mathcal{D}$ with natural transformations p_nF and q_nF:



such that P_nF are polynomial of degree n, for each $X \in \mathcal{C}$, the maps $q_nF(X) : P_nF(X) \to P_{n-1}F(x)$ are fibrations, the functors $D_nF := \text{hofib}(q_nF : P_nF \to P_{n-1}F)$ are n-homogeneous [2]. □

[2]A functor is *n-homogeneous* if it is both polynomial of degree at most n and n-reduced, i.e. $\deg(D_nF) \leq n$ and $P_{n-1}D_nF \simeq *$.

If F is degree n, then the Goodwillie tower of F is truncated, i.e. D_nF is the largest nontrivial layer of F and P_kF is equivalent to F for all $k \geq n$.

Definition B.0.16. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, we define $cr_nF : \mathcal{C}^n \to \mathcal{D}$, the n^{th}-*cross effect* of F, to be the functor of n variables given by

$$\begin{aligned}(cr_nF)(X_1, \cdots, X_n) :&= \text{hofib}\{F(\bigvee_{i \in \underline{n}} X_i) \to \operatorname*{holim}_{T \in \mathcal{P}_0(\underline{n})} F(\bigvee_{i \in \underline{n}-T} X_i)\} \\ &= \text{thofib}\, F(\mathcal{X}(X_1, \cdots, X_n)).\end{aligned}$$

where the n-cube $\mathcal{X}(X_1, \cdots, X_n) : \mathcal{P}(\underline{n}) \to \mathcal{C}$ is given by

$$\mathcal{X}(X_1, \cdots, X_n) : T \to \bigvee_{i \in \underline{n}-T} X_i$$

with maps induced by the maps $X_i \to *$.

The n^{th}-cross effect measures the extent to which F fails to be degree $n-1$. Furthermore, F is degree n if and only if cr_nF is linear in each variable. Analogously[Go3, Lemma 3.3], if F is degree n, then $cr_kF \simeq *$ for every $k \geq n+1$.

Lemma B.0.17. ([Go2, Proposition 3.4]) if $F : \mathcal{C}^n \to \mathcal{D}$ is $(x_1, \cdots, x_n)$-excisive, then the diagonalization functor Δ_n composed with F, namely Δ_nF is $(\sum x_i)$-excisive. □

Bibliography

[AGP] **M. Aguilar, S. Gitler and C. Prieto**, *Algebraic Topology from a Homotopical Viewpoint*, Universitext, Springer-Verlag, 2002.

[BoFr] **A. K. Bousfield and E. M. Friedlander**, *Homotopy theory of* Γ*-spaces, spectra, and bisimplicial sets*, Springer LNM **658** Geometric applications of homotopy theory. (Proc. Conf., Evanston, Ill., 1977), pp. 80-130.

[Bö1] **C. -F. Bödigheimer**, *Stable splitting of mapping space*, in Algebraic Topology. (Proc. Seattle 1985), Springer LNM **1286** (1987), pp. 174-187.

[Bö2] **C. -F. Bödigheimer**, *Gefärbte Konfigurationen*, Ph.D. thesis, Universität Heidelberg, 1984.

[BöMa] **C. -F. Bödigheimer and I. Madsen**, *Homotopy quotients of mapping spaces and their stable splitting*, Quart. J. Math. Oxford (2), **39** (1988), pp. 401-409.

[BCM] **C. -F. Bödigheimer, F. R. Cohen and R. J. Milgram**, *Truncated symmetric products and configuration spaces*, Math. Zeit. **214**, 179-216 (1993).

[BCT] **C. -F. Bödigheimer, F. R. Cohen and L. Taylor**, *On the homology of configuration spaces*, Topology **28** (1989), 111-123.

[BöWa] **C. -F. Bödigheimer and J. Wang**, *An unstable splitting of the free loop space of a suspension*, in preparation.

[BökOtt] **M. Bökstedt and I. Ottosen**, *A splitting result for the free loop space of spheres and projectve spaces.*, preprint arXiv:math.AT/0411594 (2004).

[Co] **R. L. Cohen**, *A model for the free loop space of a suspension*, in Algebraic Topology, Proc. Seattle 1985, Springer LNM **1286** (1987), pp. 193-207.

[Do] **A. Dold**, *Homology of symmetric products and other functors of complex*, Ann. of Math. (2) **68** (1958), pp. 54-80

[DoPu] **A. Dold and D. Puppe**, *Homologie nicht-additiver Funktoren. Anwendungen*, Annales de l' institut Fourier, **11** (1961), pp. 201-312.

[DoTh] **A. Dold and R. Thom**, *Quasifaserungen und unendliche symmetrische Produkte*, Annals of Math. **67** (1958), pp. 239-281.

[Ga] **S. Galatius**, *Stable homology of automorphism groups of free groups*, arXiv preprint math/0610216, 2006.

[GMTW] **S.Galatius, I. Madsen, U. Tillmann and M. Weiss**, *The homotopy type of the cobordism category*, arXiv: math/0605249v2, 2007.

[Go1] **T. G. Goodwillie**, *Calculus I: The first derivative of pseudoisotopy theory*, K-Theory, **4** (1990), pp. 1-27.

[Go2] **T. G. Goodwillie**, *Calculus II: Analytic functors*, K-Theory, **5** (1992), pp. 295-332.

[Go3] **T. G. Goodwillie**, *Calculus III: Taylor series*, Geometry and Topology, **7** (2003), pp. 645-711.

[Gr] **M. Gromov**, *Partial Differential Relations.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 9, Springer-Verlag (1986).

[Ja] **I. M. James**, *Reduced product spaces*, Annals of Math (2) **62** (1953), pp. 170-197.

[Ka1] **S. Kallel**, *Homotopy Theory of Mapping Spaces, Methods and Applications.* Habilitation, not available any more.

[Ka2] **S. Kallel**, *An interpolation between homology and stable homotopy*, Preprint available at arXiv: math. AT/9810068 v1.

[Ku] **N. J. Kuhn**, *Goodwillie towers and chromatic homotopy: an overview*, Preprint available at arXiv: math.AT/0410342 v1.

[Mar] **E. Markert**, *Connective 1-dimensional Euclidean field theories*, PhD Thesis, University of Notre Dame, 2005. Available at http://www.math.uni-bonn.de/people/markert/papers/thesis.pdf

[May] **J. P. May**, *The Geometry of Iterated Loop Spaces*, Springer LNM, **271** (1972).

[McC] **M. C. McCord**, *Classifying Spaces and Infinite Symmetric Products* , Transactions of the American Mathematical Society, Vol. **146**, (Dec., 1969), pp. 273-298.

[McD] **D. McDuff**, *Configuration spaces*, Springer LNM , Athens, Georgia, conference in K-theory and Operator Algebras, Athens, Georgia **575** (1975).

[Mi] **R. J. Milgram**, *Iterated loop spaces*, Annals of Math. **84** (1966), pp. 386-403.

[MiLö] **R. J. Milgram and P. Löffler**, *The structure of deleted symmetric products*, Braids (Santa Cruz, CA, 1986), Contemp. Math., **78**, Amer. Math. Soc., Providence, RI, 1988, pp. 415–424.

[Mil] **J. Milnor**, *On the construction FK*, In: J. F. Adams: Algebraic topology, a student's guide. London Math. Soc. Lecture Note Series **4** (1972), pp. 119-136.

[Pu] **D. Puppe**, *Some well known weak homotopy equivalences are genuine homotopy equivalences*, In: Symposia Mathematica, Vol. V (INDAM, Rome, 1969/70), pp. 363-374. London: Academic Press, 1971.

[Oa] **A. M-Oats**, *Algebraic Goodwillie calculus and a cotriple model for the remainder*, Trans. AMS. Vol **358**, Nr 5, pp. 1869-1895.

[RW] **O. Randal-Williams**, *Embedded Cobordism*, Transfer Thesis, University of Oxford, 2007.

[Se1] **G. Segal**, *Classifying spaces and spectral sequences*, Publ. Math. inst. des Hautes Etudes Scient. (Paris) **34** (1968), pp. 105-112.

[Se2] **G. Segal**, *Categories and cohomology theories*, Topology **13** (1974), pp. 293-312.

[Se3] **G. Segal**, *K-homology theory and algebraic K-theory*, K-theory and operator algebras (Athens, Georgia 1975), Springer LNM **575**, pp 113-127.

[Sh] **K. Shimakawa**, *Configuration spaces with partially summable labels and homology theories*, http://www.math.okayama-u.ac.jp/∼kazu/archive/config2.pdf

[Sn] **V. P. Snaith**, *A stable decomposition of* $\Omega^n S^n X$, J. Lond. Math. Soc.(2), **7** (1974), pp. 577-583.

[SpWh] **E. H. Spanier and J. H. C. Whitehead**, *The theory of carriers and S-theory*, Algebraic Geometry and Topology. (A symp. in Honour of S. Lefschets) (1957), Princeton University Press, Princeton N. J.

[Wo] **R. Woolfson**, *Hyper -Γ-spaces and hyperspectra*, Quart. J. Math. Oxford (2), **30** (1979), pp. 229-255.

[Wu] **J. Wu**, *On the homology of configuration spaces* $C((M, M_0) \times \mathbb{R}^n; X)$, Math. Z., **22** (1998), 235-248.