On Rho Invariants of Fiber Bundles

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Zusammenfassung

Der Gegenstand dieser Arbeit ist eine detaillierte Untersuchung von Rho-Invarianten auf Totalräumen von Faserbündeln. Die Grundidee ist, mit Hilfe der Theorie adiabatischer Grenzwerte von Eta-Invarianten die Berechnung von Rho-Invarianten weitestgehend nach Faser und Basis des Bündels zu trennen. Mit einer adiabatische Metrik auf einem Faserbündel wird die Metrik der Basismannigfaltigkeit so skaliert, dass sich die Geometrie des Bündels immer mehr einer Produktsituation annähert. Für die Eta-Invariante ist dieser Prozess in der Literatur weitreichend untersucht worden. Aus diesem Grunde beschäftigt sich ein gewisser Teil der Arbeit damit, die sehr technischen Aspekte der lokalen Indextheorie für Familien von Dirac Operatoren im Falle des ungeraden Signaturoperators zu formulieren und bekannte Resultate in einen Kontext zu setzen, der die Behandlung von Rho-Invarianten ermöglicht.

Die resultierende Formel drückt die Rho-Invariante als Summe dreier Terme aus, die jeweils sehr unterschiedlicher Natur sind. Zunächst wird ein höher dimensionales Analogon der Rho-Invariante der Faser über der Basis integriert. Dieser Term ist von lokaler Natur auf der Basis, enthält jedoch globale spektrale Information der Faser. Der nächste Term ist im Wesentlichen eine Rho-Invariante der Basis, wobei der zugrundeliegende flache Zusammenhang auf dem Bündel der Kohomologiegruppen der Faser definiert ist. Als letztes tritt ein rein topologischer Term auf, der mit Hilfe der Spektralsequenz des Faserbündels berechnet werden kann. Insgesamt stellt diese Formel also eine Zerlegung der Rho-Invariante des Totalraums dar, die der Struktur des Faserbündels gerecht wird.

Das Hauptaugenmerk dieser Arbeit liegt jedoch darauf, diese abstrakte Formel für Beispielklassen von gefaserten 3-Mannigfaltigkeiten explizit anzuwenden. Genauer beschäftigen wir uns mit S^1-Hauptfaserbündeln über kompakten Riemannschen Flächen und mit Abbildungstori, deren Fasern ebenfalls eine kompakte Riemannsche Fläche ist. Für die erste Beispielklasse lassen sich U(1)-Rho-Invariaten bereits ohne Anwendung dieser allgemeinen Formel ad hoc berechnen. Insbesondere ergibt sich so die Gelegenheit, diese verschiedenen Herangehensweisen miteinander zu vergleichen und den systematischen Vorteil der allgemeinen Formel zu testen.

Für 3-dimensionale Abbildungstori liefert die dargestellte Theorie ebenfalls die Möglichkeit, Rho-Invarianten explizit zu berechnen. Wir beschränken uns zunächst auf den Fall, dass die Monodromieabbildung von endlicher Ordnung ist. Hier lässt sich eine allgemeine Formel herleiten. Um eine weitere interessante Klasse von Abbildungstori zu untersuchen, werden U(1)-Rho-Invarianten in dem Fall betrachtet, dass die Faser ein 2-dimensionaler Torus ist. Insbesondere hyperbolische Monodromieabbildungen erfordern hier eine besondere Aufmerksamkeit. In diesem Fall treten in natürlicher Weise Logarithmen verallgemeinerter Dedekindscher Eta-Funktionen auf, aus deren Transformationsverhalten sich eine sehr zufriedenstellende Formel für U(1)-Rho-Invarianten herleiten lässt.

Summary

The content of this thesis is a detailed investigation of Rho invariants of the total spaces of fiber bundles. The main idea is to use adiabatic limits of Eta invariants to obtain a formula for Rho invariants that separates the contribution coming from the fiber and the one coming from the base. An adiabatic metric on a fiber bundle rescales the metric of the base manifold in such a way that the geometry of the fiber bundle approaches a product situation. Concerning the Eta invariant, this process has received a far-reaching treatment in the literature. For this reason, one concern of this thesis is to formulate the technical aspects of local index theory for families of Dirac operator in terms of the odd signature operator, and place known results in a context which permits the treatment of Rho invariants.

The resulting formula expresses the Rho invariant as a sum of three terms, each of which is of a very different nature. First of all, a higher dimensional analog of the Rho invariants of the fiber has to be integrated over the base. This term is of a local nature on the base, but contains global spectral information about the fiber. The next term is essentially a Rho invariant of the base, where the underlying flat connection is defined on the bundle of cohomology groups of the fiber. Lastly, there is a purely topological term, which can be computed from the spectral sequence of the fiber bundle. Together, this formula casts the Rho invariant of the total space into a form which incorporates the structure of the fiber bundle in a satisfactory way.

The main concern of this thesis is, however, to use this theoretical formula to compute Rho invariants for explicit classes of fibered 3-manifolds. More precisely, we consider principal S^1-bundles over closed, oriented surfaces as well as mapping tori with fiber a closed, oriented surface. For the first class of examples, one can compute U(1)-Rho invariants without using this general formula. In particular, this yields the opportunity to compare the different approaches and test the systematical advantage of the general formula.

For 3-dimensional mapping tori, the presented theory can also be used for explicit computations. We first consider the case that the monodromy map is of finite order. In this case, a general formula for Rho invariants can be derived. To investigate a further interesting class of mapping tori, we consider U(1)-Rho invariants in the case that the fiber is a 2-dimensional torus. Here, hyperbolic monodromy maps deserve particular attention. When discussing them, the logarithm of a generalized Dedekind Eta function naturally appears. A satisfactory formula for U(1)-Rho invariants of hyperbolic mapping tori can then be deduced from a transformation formula for these Eta functions.

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Introduction

The Rho Invariant for Closed Manifolds. In their famous series of articles [7, 8, 9], Atiyah, Patodi and Singer established an index theorem for manifolds with boundary. Part of their motivation was to find a generalization of Hirzebruch's Signature Theorem to manifolds with boundary and give a differential geometric explanation for the signature defect.

We recall this briefly. Let W be a closed, oriented 4-manifold, and let $\mathrm{Sign}(W)$ be its signature. Then the Hirzebruch's Signature Theorem states that

$$\mathrm{Sign}(W) = \frac{1}{3}\int_W p_1(TW). \tag{1}$$

Here, $p_1(TW)$ is the first Pontrjagin form, and since W is closed, it is a characteristic class independent of the connection used to compute it. Let us now assume that W has a boundary M. If $\alpha : \pi_1(W) \to \mathrm{U}(k)$ is a unitary representation of the fundamental group, one defines a twisted signature $\mathrm{Sign}_\alpha(W)$ using cohomology groups with local coefficients. Then an application of the signature formula (1) and its twisted version to the closed double $W \cup_M -W$ shows that the difference

$$\mathrm{Sign}_\alpha(W) - k \cdot \mathrm{Sign}(W)$$

depends only on the topology of the boundary $\partial W = M$ as well as the restriction of α to $\pi_1(M)$. This signature defect is in general non-trivial. Moreover, the signature itself fails to be multiplicative under finite coverings of manifolds with boundary. Both observations show that the signature of a manifold with boundary is in general not expressible as in (1).

The Atiyah-Patodi-Singer Index Theorem for manifolds with boundary identifies the correction term in great generality. For a formally self-adjoint elliptic differential operator D of first order, acting on sections of a vector bundle over a closed manifold M, one defines the Eta function

$$\eta(D, s) := \sum_{0 \neq \lambda \in \mathrm{spec}(D)} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}, \quad \mathrm{Re}(s) \text{ large}. \tag{2}$$

The function $\eta(D, s)$ admits a meromorphic continuation to the whole s-plane, and it is a remarkable fact that $s = 0$ is not a pole. The *Eta invariant* $\eta(D)$ is defined as this finite value. Then a special case of the Atiyah-Patodi-Singer Index Theorem for a compact, oriented 4-manifold W with boundary M is

$$\mathrm{Sign}_\alpha(W) = \frac{k}{3}\int_W p_1(TW, \nabla^g) - \eta(B_A^{\mathrm{ev}}). \tag{3}$$

Here, $p_1(TW, \nabla^g)$ is the first Pontrjagin form, computed with respect to a metric g in product form near the boundary, A is a flat $\mathrm{U}(k)$-connection over M whose holonomy coincides with $\alpha|_{\pi_1(M)}$, and—most importantly—B_A^{ev} is the *odd signature operator* on M. It is defined on differential forms of even degree with values in the flat bundle E as

$$B_A^{\mathrm{ev}}\omega = (-1)^p(*d_A - d_A*)\omega, \quad \omega \in \Omega^{2p}(M, E).$$

A simple consequence of (3) is that

$$\mathrm{Sign}_\alpha(W) - k \cdot \mathrm{Sign}(W) = \eta(B_A^{\mathrm{ev}}) - k \cdot \eta(B^{\mathrm{ev}}). \tag{4}$$

The right hand side of (4) is called the *Rho invariant* $\rho_A(M)$. It is defined for every odd dimensional manifold M and flat unitary connection A over M, without any reference to a bounding manifold. Moreover, $\rho_A(M)$ turns out to be independent of the choice of the metric used in defining the involved odd signature operators. Therefore, it is an intrinsically defined smooth invariant of M, which extends the signature defect.

Since the Eta invariants appearing in the definition of $\rho_A(M)$ are non-local spectral invariants, it is difficult to compute Rho invariants directly without using property (4). However, one cannot always find a bounding manifold in such a way that the flat connection extends. Therefore, intrinsic methods to compute Rho invariants are in demand. The concern of this thesis is to investigate an intrinsic approach to this problem in the case that the manifold M is the total space of an oriented fiber bundle of closed manifolds.

Adiabatic Limits of Eta Invariants. In a remarkable paper of Witten [97], Eta invariants appeared in the interpretation of anomalies in physics. Associated to a family of Dirac operators is a determinant line bundle over the parameter space, first described by in [83] by Quillen. It comes equipped with a natural connection, defined in terms of the Ray-Singer analytic torsion [87]. The topology of this line bundle encodes the obstruction to defining in a consistent way a regularized determinant associated to the family of Dirac operators. In the physicists terminology, there is no "local anomaly" if Quillen's connection on the determinant bundle is flat. However, the bundle might still not be trivial, and this "global anomaly" is encoded in the holonomy of Quillen's connection. Witten suggests an interpretation of this holonomy using the Eta invariant. For example, a family of spin Dirac operators is naturally associated to a fiber bundle over the parameter space B whose fiber F is a closed spin manifold. Here, we also restrict to the case that F is even dimensional. Pulling back this structure using a closed loop $c : S^1 \to B$ leads to a fiber bundle $F \hookrightarrow M \to S^1$. Now Witten considers an *adiabatic metric*, that is, a family of submersion metrics of the form

$$g_\varepsilon = \frac{g_{S^1}}{\varepsilon^2} \oplus g_v, \tag{5}$$

where g_v is a metric on the vertical tangent bundle of M. Then, if D_ε denotes the Dirac operator associated to (5) on the total space M, Witten suggests that the holonomy of Quillen's connection around the loop $c : S^1 \to B$ is given by

$$\lim_{\varepsilon \to 0} \exp\big(2\pi i \eta(D_\varepsilon)\big).$$

The mathematical treatment of this holonomy theorem is due to Bismut-Freed [18] and Cheeger [26].

Motivated by this geometric interpretation, Bismut and Cheeger [16] gave a formula for the *adiabatic limit* of the Eta invariant, and generalized it to fiber bundles $F \hookrightarrow M \to B$ with higher dimensional base spaces. Using ideas of Bismut's local index theory for families [14], they construct a differential form $\widehat{\eta}$ on B, whose value at each point $x \in B$ depends only on global information of the fiber over x. Under the assumption that the fiberwise Dirac operator is invertible, they prove that

$$\lim_{\varepsilon \to 0} \eta(D_\varepsilon) = \int_B \widehat{A}(TB, \nabla^{g_B})\widehat{\eta}, \tag{6}$$

where $A(TB, \nabla^{g_B})$ is the Hirzebruch $\widehat{A}$-form of B. Very roughly, a consequence of (6) is that the adiabatic limit of the Eta invariant is a simpler object than the Eta invariant itself, since it is local on the base. The main matter of this thesis is to analyze in which way this effect can be used to simplify explicit computations of Rho invariants of fiber bundles.

Dai's Adiabatic Limit Formula. The Rho invariants we are considering are associated to the odd signature operator. Here, the kernel of the vertical operator forms a vector bundle $\mathscr{H}_v^\bullet(M) \to B$ whose fiber over each point is isomorphic to the cohomology of F. Therefore, the invertibility hypothesis leading to (6) is too restrictive. Fortunately, the result of Bismut and Cheeger has been generalized by Dai [30] to a setting which applies in particular to the case we are interested in. The bundle $\mathscr{H}_v^\bullet(M) \to B$ of vertical cohomology groups is endowed with a natural flat connection $\nabla^{\mathscr{H}_v}$. Using this, one associates a twisted odd signature operator $D_B \otimes \nabla^{\mathscr{H}_v}$ over the base. Then Dai proves the following, very remarkable adiabatic limit formula.

Theorem 1 (Dai). *Let $F \hookrightarrow M \to B$ be an oriented fiber bundle of closed manifolds with odd dimensional total space, endowed with a submersion metric. Let $B_\varepsilon^{\mathrm{ev}}$ be the family of odd signature operators on M associated to an adiabatic metric. Then*

$$\lim_{\varepsilon \to 0} \eta(B_\varepsilon^{\mathrm{ev}}) = 2^{[\frac{b+1}{2}]} \int_B \widehat{L}(TB, \nabla^{g_B}) \wedge \widehat{\eta} + \tfrac{1}{2}\eta\big(D_B \otimes \nabla^{\mathscr{H}_v}\big) + \sigma.$$

Here, b is the dimension of B, and the differential forms $\widehat{L}(TB, \nabla^B)$ and $\widehat{\eta}$ are the Hirzebruch $\widehat{L}$-form and the Eta form of Bismut-Cheeger, respectively. Moreover, σ is a topological invariant computed from the Leray-Serre spectral sequence.

We will give more details on the terms appearing here in the main body of this thesis. However, we already want to stress that $\eta\big(D_B \otimes \nabla^{\mathscr{H}_v}\big)$ and σ are of a very different nature than the integral of the Eta form. Where the latter is local on the base and contains spectral information about the fiber, the twisted Eta term is a spectral invariant of the base which contains cohomological information of the fiber. Moreover, since σ arises from the Leray-Serre spectral sequence it is purely cohomological. In this respect, Theorem 1 is a very satisfactory decomposition of the adiabatic limit into contributions coming from the base and the fiber, respectively.

Concern of this Thesis. Since the treatment in [30] is more general than what we have stated in Theorem 1, Dai's adiabatic limit formula continues to hold for the odd signature operator twisted by a flat connection A over M. We will see that there are natural analogs

$\widehat{\eta}_A$, $D_B \otimes \nabla^{\mathscr{H}_{A,v}}$ and σ_A of the quantities appearing in Theorem 1. As the Rho invariant $\rho_A(M)$ is independent of the metric, it is immediate, that with respect to every adiabatic metric,

$$\rho_A(M) = \lim_{\varepsilon \to 0} \eta(B^{\mathrm{ev}}_{A,\varepsilon}) - k \cdot \lim_{\varepsilon \to 0} \eta(B^{\mathrm{ev}}_{\varepsilon}).$$

Then, Theorem 1 yields

Theorem 2. *Let A be a flat $\mathrm{U}(k)$-connection over M. Then with respect to every submersion metric*

$$\rho_A(M) = 2^{[\frac{b+1}{2}]} \int_B \widehat{L}(TB, \nabla^B) \wedge \big(\widehat{\eta}_A - k \cdot \widehat{\eta}\big) + \tfrac{1}{2}\eta\big(D_B \otimes \nabla^{\mathscr{H}_{A,v}}\big) - \tfrac{k}{2}\eta\big(D_B \otimes \nabla^{\mathscr{H}_v}\big) + \sigma_A - k \cdot \sigma.$$

Now, the main matter of this thesis is to investigate how this rather straightforward consequence of Theorem 1 can be used for explicit computations of Rho invariants. Due to the technical nature of local families index theory, our first concern is to assemble the building blocks we need, and specialize many known results to the case of the odd signature operator. The motivation here is certainly not to exhibit new results, but to present the theory in such a way that it becomes accessible for a treatment of Rho invariants of fiber bundles. Therefore, our perspective will usually be a geometric one rather than discussing analytical difficulties. For a discussion of these aspects, we usually refer to the wide variety of literature. Nevertheless, we include proofs of some folklore results, for instance a fibered version of the Hodge decomposition theorem, and a result about how to achieve that the mean curvature of a fiber bundle vanishes.

Apart from the theoretical discussion, our true focus is on explicit examples. We will examine two important classes of fibered 3-manifolds in detail.

Circle Bundles over Surfaces. The simplest class of fiber bundles for which a discussion of Rho invariants is meaningful, is given by principal S^1-bundles over Riemann surfaces. Nevertheless, this family of manifolds already deserves some attention as it is a model for two important classes of manifolds, namely 3-dimensional Seifert fibrations and higher dimensional principal bundles.

In this spirit, Nicolaescu [78, 79] has analyzed the Seiberg-Witten equations of Seifert manifolds, and parts of our discussion are influenced by his work. Given a closed, oriented surface Σ, and an oriented principal bundle $S^1 \hookrightarrow M \to \Sigma$, we will see that we can represent every flat $\mathrm{U}(1)$-connection A over M by pulling back a line bundle of degree k over Σ. In terms of this data, the Rho invariant associated to A is given as follows, see Theorem 2.3.18.

Theorem 3. *Assume that $l \neq 0$, and that $q_0 \in [0, 1)$ is such that $k/l \equiv q_0 \mod \mathbb{Z}$. Then*

$$\rho_A(M) = 2l(q_0^2 - q_0) + \operatorname{sgn}(l).$$

If $l = 0$, so that the fiber bundle is trivial, all Rho invariants vanish.

We shall include two proofs of this result. The first one in Chapter 2 uses only basic considerations about the geometry of fiber bundles, and the second one in Chapter 3 shows how Theorem 2 can be used for this class of examples.

3-dimensional Mapping Tori. The second family of manifolds we will consider are fiber bundles $\Sigma \hookrightarrow M \to S^1$, where Σ is again a closed, oriented surface. A manifold of this type is determined by an element f of the mapping class group of Σ. Due to the rich algebraic structure encoded in the latter, we have not attempted to treat the class of 3-dimensional mapping tori in full generality.

What we shall do instead, is to assume first that the monodromy f of the mapping torus M is of finite order. Under this assumption, the formula of Theorem 2 for the Rho invariant of a flat $\mathrm{U}(k)$-connection A over M reduces to

$$\rho_A(M) = \tfrac{1}{2}\eta\big(D_B \otimes \nabla^{\mathscr{H}_{A,v}}\big) - \tfrac{k}{2}\eta\big(D_B \otimes \nabla^{\mathscr{H}_v}\big).$$

In Theorem 4.2.4 we shall give an expression of the right hand side in terms of Hodge-de-Rham cohomology of Σ, thus obtaining a cohomological formula for the Rho invariant. Since the precise statement would need a longer preamble, we refer Chapter 4 for details.

After this we will consider $\mathrm{U}(1)$-Rho invariants of a mapping torus T^2_M, where the fiber is the 2-dimensional torus T^2, and $M \in \mathrm{SL}_2(\mathbb{Z})$—the mapping class group of T^2. One naturally has to distinguish between three cases, depending of whether M is elliptic, parabolic or hyperbolic. The first two cases are rather special, and we shall not discuss them here. The explicit formulæ for the corresponding Rho invariants can be found in Theorem 4.4.4 and Theorem 4.4.8, respectively.

The case of a hyperbolic monodromy matrix requires more background material. Here we will use ideas of Atiyah [3], who gives a far-reaching treatment of the untwisted Eta invariant for mapping tori with fiber T^2. In particular, he uses the relation to Hirzebruch's signature defect to show that for a hyperbolic element $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the Eta invariant of the odd signature operator with respect to a natural metric on T^2_M, is given by

$$\eta(B^{\mathrm{ev}}) = \frac{a+d}{3c} - 4\,\mathrm{sgn}(c)s(a,c) - \mathrm{sgn}\,\big(c(a+d)\big). \tag{7}$$

Here, $s(a,c)$ is the *Dedekind sum*

$$s(a,c) = \sum_{k=1}^{|c|-1} P_1\big(\tfrac{ak}{c}\big)P_1\big(\tfrac{k}{c}\big), \tag{8}$$

where for $x \in \mathbb{R}$,

$$P_1(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ x - [x] - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}. \end{cases}$$

Atiyah also relates the Eta invariant to the classical *Dedekind Eta function*, which is defined for a point σ in the upper half plane as

$$\boldsymbol{\eta}(\sigma) := q_\sigma^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q_\sigma^n), \quad q_\sigma := e^{2\pi i \sigma}.$$

One can define a natural logarithm of $\boldsymbol{\eta}$, and the transformation property of $\log \boldsymbol{\eta}$ under the action of elements of $\mathrm{SL}_2(\mathbb{Z})$ has a long history, going back to Dedekind [34]. For an element $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$, this transformation formula states that

$$\log \boldsymbol{\eta}(M\sigma) - \log \boldsymbol{\eta}(\sigma) = \frac{1}{2}\log\Big(\frac{c\sigma + d}{\mathrm{sgn}(c)i}\Big) + \pi i\Big(\frac{a+d}{12c} - \mathrm{sgn}(c)s(a,c)\Big). \tag{9}$$

Here, the logarithm on the right hand side is the standard branch on $\mathbb{C} \setminus \mathbb{R}^-$, and $s(a, c)$ is the Dedekind sum (8). Atiyah's explanation of the relation between (9) and the formula (7) makes essential use of the idea of taking the adiabatic limit.

Motivated by this, we will study in detail the expression $\int_{S^1} \widehat{\eta}_A$ appearing in Theorem 2 for the case of a flat U(1)-connection over a hyperbolic mapping torus over S^1. Using ideas related to Kronecker's second limit formula, we will cast it into a form, where the logarithm of a generalized Dedekind Eta function naturally appears. We will then employ a transformation formula due to Dieter [35], to obtain the analog of (9) for the logarithm of this generalized Dedekind Eta function. From this we shall deduce the main result of Chapter 4, see Theorem 4.4.20.

Theorem 4. *Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic, and let $(\nu_1, \nu_2) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ with $\nu_1 \in [0, 1)$ satisfy*

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = (\mathrm{Id} - M^t) \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \in \mathbb{Z}^2.$$

This defines a flat connection A over T^2_M, and

$$\begin{aligned} \rho_A(T^2_M) = \tfrac{2(a+d)-4}{c}(\nu_1^2 - \nu_1) - 4\sum_{k=1}^{|c|-r} P_1\big(\tfrac{dk}{c}\big) + \operatorname{sgn}\big(c(a+d)\big) - \operatorname{sgn}(c)\delta(\nu_1)\big(1 - \delta(\tfrac{m_1}{c})\big) \\ - 2P_1\big(\tfrac{dm_1}{c}\big) - 2\delta(\nu_1)\Big(P_1\big(\tfrac{m_1}{c}\big) - P_1\big(\tfrac{dm_1}{c}\big)\Big), \end{aligned}$$

where $r \in \{0, \dots |c| - 1\}$ is such that $m_1 \equiv r \mod c$, and δ is the characteristic function of $\mathbb{R} \setminus \mathbb{Z}$.

Although this formula might appear to be somewhat involved, it is satisfactory in two ways. First of all, the involved terms are easy to compute for explicit choices of $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and (ν_1, ν_2). Secondly, we shall see that it contains previous computations of Chern-Simons invariants by Freed and Vafa [42] as a special case. In this respect, the author hopes that a possible generalization to SU(2)-connections will reprove results of Kirk and Klassen [57] and might shed a new light on Jeffrey's conjecture [54] concerning the spectral flow associated to twisted odd signature operators on a mapping torus of the form considered here.

Outline of this Thesis. We end the introduction with a very brief outline of this thesis. We will keep this rather short since the beginning of each chapter contains a more detailed outline of its contents.

- Chapter 1 is a survey of results from index theory that we will need. In particular, we shall introduce the signature of a manifold, discuss its relation to index theory, and recall the Atiyah-Singer Index Theorem in its cohomological version for geometric Dirac operators. Then we introduce the Eta and Rho invariant, and discuss how they appear in the index theorem for manifolds with boundary. We also place some emphasis on variation formulæ and sketch how they are related to local index theory.

- In Chapter 2 we start with the discussion of fiber bundles. We will introduce the geometric setup, paying close attention to the structure of the odd signature operator. Then, we shall encounter the basic idea of adiabatic limits and use this to give an elementary proof of Theorem 3 above.

- Chapter 3 contains the main theoretical part of this thesis. Here, the main objective is to introduce all quantities appearing in Theorem 1. After discussing the bundle of vertical cohomology groups in some detail, we will give a heuristic derivation of Theorem 1. For this, we also include a short survey of local families index theory. All this discussion will lead to Theorem 2, which we will then use to reprove Theorem 3 in a more abstract way.

- The content of Chapter 4 is the discussion of 3-dimensional mapping tori along the lines we have already outlined above.

- For the reader's convenience, and to keep our discussion more self-contained, we have also included a couple of appendices, which contain material that we freely use, but that would lead to far afield if discussed in the main body of this thesis.

  - Appendix A contains a discussion of Chern-Weil theory in a way which is particularly well-suited for the applications we need. Moreover, we have included some aspects concerning Chern-Simons invariants, as they appear throughout our discussion.
  - Since the Rho invariants we are interested in depend only on the gauge equivalence class of the involved flat connection, we include some remarks concerning the moduli space of flat connections in Appendix B. We start giving some details on the relation to representations of the fundamental group. Moreover, the moduli space of flat connections over a mapping torus is discussed, since we need this in Chapter 4. We end this appendix with a brief survey of the moduli space of holomorphic line bundles over a Riemann surface, which is an ingredient for discussing flat U(1)-connections over principal S^1-bundles over surfaces.
  - Appendix C contains some computations. On the one hand, we need explicit values of basic Eta and Zeta functions to which we reduce most computations in the main body of the thesis. On the other hand, we shall discuss the Dedekind sum in (8) and its generalization, establishing a relation among them which we need to prove Theorem 4.
  - Finally, Appendix D includes some more analytical details concerning the heat operator. Specifically, we will give some remarks concerning families of heat operators, and derive the variation formula for the Eta invariant.

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Chapter 1

The Signature Operator and the Rho Invariant

In this chapter we will survey some results from index theory, which we need in our discussion later on. The objective is not to give a systematic treatment but merely to introduce the objects we are interested in and to fix notation. For this reason we shall include proofs only if they enrich the discussion and do not lead too far afield.

We start with a brief discussion of the signature of a closed manifold, placing emphasis on the generalized version where the intersection form is associated to a flat unitary vector bundle. Introducing the signature operator relates the signature to the index of an elliptic operator, and this leads us to a discussion of the main facts concerning the heat equation on closed manifolds. Although our focus is on the signature operator and its odd dimensional analog, we present the Atiyah-Singer Index Theorem in its version for geometric Dirac operators. This is because many ideas in the later chapters are influenced by local index theory which is more transparent when formulated in terms of Clifford modules and Dirac operators.

Then we will introduce the object which constitutes the main topic of this thesis—namely the Eta invariant of an elliptic operator on a closed manifold. Variation formulæ for Eta invariants will play a prominent role in the discussion in the next chapters. Therefore, we discuss this topic in some detail. Most notably, the behaviour under the variation of a flat twisting connection leads naturally to the first appearance of a Rho invariant as the difference of certain Eta invariants.

After this we describe—briefly leaving the realm of closed manifolds—how the Eta invariant arises as a correction term in the index theorem for a manifold with boundary. From then on the focus will be on the case that the elliptic operator in question is the odd signature operator. From the signature theorem for manifolds with boundary we derive some general and well-known properties of the Eta invariant. In particular, the relation of Rho invariants to Chern-Simons invariants will be exhibited.

We close the general discussion of this chapter with a short outline of how local index theory methods for odd dimensional manifolds can be used to obtain important properties of Rho invariants without referring to the Atiyah-Patodi-Singer Index Theorem. Our interest in this is not only of a purely academic nature, as similar ideas are the ones underlying local families index theory, which we will need in the context of fiber bundles in Chapter 3.

1.1 The Signature of a Manifold

1.1.1 Intersection Forms and Local Coefficients

Let M be a closed, oriented and connected manifold of dimension m. On the real cohomology groups there exists the *intersection pairing*

$$H^p(M,\mathbb{R}) \times H^{m-p}(M,\mathbb{R}) \to \mathbb{R}, \quad (a,b) \mapsto \big\langle a \cup b\,,\, [M]\big\rangle,$$

where $\langle .,. \rangle$ is the Kronecker pairing, $\cup$ is the cup product, and $[M]$ denotes the fundamental class of M determined by the orientation. Expressing $H^p(M,\mathbb{R})$ in terms of de Rham cohomology groups, the intersection pairing is induced by

$$\Omega^p(M) \times \Omega^{m-p}(M) \mapsto \mathbb{R}, \quad (\alpha,\beta) \mapsto \int_M \alpha \wedge \beta.$$

As a consequence of the Poincaré duality theorem, the above pairing is non-degenerate. In particular, if $\dim M = m$ is even, there is a non-degenerate bilinear form

$$Q : H^{m/2}(M,\mathbb{R}) \times H^{m/2}(M,\mathbb{R}) \to \mathbb{R}, \tag{1.1}$$

which is called the *intersection form* of M. If $(m \equiv 0 \mod 4)$ the intersection form is symmetric, and if $(m \equiv 2 \mod 4)$ it is skew. Recall that the signature of a symmetric form is the number of positive minus the number of negative eigenvalues.

Convention. We also use the convention that the signature of a skew form is the number of positive imaginary eigenvalues minus the number of negative imaginary ones.

Definition 1.1.1. Let M be a closed, oriented and connected manifold of even dimension m. Then the *signature* of M is defined as

$$\mathrm{Sign}(M) := \mathrm{Sign}(Q).$$

Remark. In topology, the intersection form is usually considered as a form over $\mathbb{Z}$. If one uses cohomology groups with integer coefficients, one has to divide out the torsion subgroup of $H^{m/2}(M,\mathbb{Z})$ to get a non-degenerate form. Moreover, note that if we want to work with complex coefficients, we have to extend Q anti-linearly in, say, the first variable to get a (skew) Hermitian form. Then the signature is also well-defined and agrees with the signature of the underlying real form. Note, however, that if a skew form comes from a form over $\mathbb{R}$ it has zero signature since its eigenvalues come in conjugate pairs.

Cohomology with Local Coefficients. We will also need a twisted version of the intersection form. For this we briefly recall the construction of cohomology groups with local coefficients. We refer to [33, Ch. 5] for more details and proofs.

Let M be a connected manifold, not necessarily closed, and let $\widetilde{M}$ be the universal cover of M. Let $\pi = \pi_1(M)$ be the fundamental group of M, and let $\mathbb{C}[\pi]$ denote the group algebra of π. The fundamental group acts from the right on $\widetilde{M}$, so that the cellular chain groups $C_p(\widetilde{M})$ are $\mathbb{C}[\pi]$ right modules in a natural way. This is because a cell decomposition of M induces via lifting of cells a cell decomposition of $\widetilde{M}$ which is compatible with the action of π on $\widetilde{M}$.

Now let $\alpha : \pi \to \mathrm{U}(k)$ be a unitary representation, and let

$$C^p(M, E_\alpha) := \mathrm{Hom}_{\mathbb{C}[\pi]}\big(C_p(\widetilde{M}), \mathbb{C}^k\big).$$

Here, the action of $\mathbb{C}[\pi]$ on $\mathbb{C}^k$ is by matrix multiplication $x \mapsto \alpha(x)^{-1}$. The differential on cochains on $\widetilde{M}$ turns $C^\bullet(M, E_\alpha)$ into a complex. As for untwisted cellular cohomology, the homology of this complex does not depend on the particular cell decomposition and the chosen lifts.

Definition 1.1.2. Let M be a compact, connected manifold, and $\alpha : \pi_1(M) \to \mathrm{U}(k)$ a representation. Then the homology of $C^\bullet(M, E_\alpha)$ is denoted by $H^\bullet(M, E_\alpha)$ and is called the cohomology of M with *local coefficients* given by α.

If we now assume that M is also closed and oriented, then there exists a non-degenerate pairing

$$H^p(M, E_\alpha) \times H^{m-p}(M, E_\alpha) \to \mathbb{C} \tag{1.2}$$

induced by the cup product on the cohomology of $\widetilde{M}$ and the scalar product on $\mathbb{C}^k$. If M is of even dimension m this yields a bilinear form

$$Q_\alpha : H^{m/2}(M, E_\alpha) \times H^{m/2}(M, E_\alpha) \to \mathbb{C},$$

which we call the *twisted intersection form.*

Definition 1.1.3. Let M be a closed, oriented and connected manifold of even dimension m, and let $\alpha : \pi_1(M) \to \mathrm{U}(k)$ be a unitary representation of the fundamental group of M. Then the *twisted signature* of M is defined as

$$\mathrm{Sign}_\alpha(M) := \mathrm{Sign}(Q_\alpha),$$

where we use again the convention that the signature of a skew form is the number of positive imaginary eigenvalues minus the number of negative imaginary ones.

Remark. As we will see soon, the twisted signatures we have just defined give no new topological information for a closed manifold M. However, their version for manifolds with boundary are a non-trivial generalization of topological importance. Nevertheless, we have included the definition here to keep the discussion parallel.

Local Coefficients and Flat Bundles. Twisted cohomology groups can also be defined in terms of differential forms. Let $E \to M$ be a Hermitian vector bundle of rank k, endowed with a unitary connection A. This gives rise to a twisted version of the exterior differential

$$d_A : \Omega^p(M, E) \to \Omega^{p+1}(M, E), \quad d_A(\omega \otimes e) = d\omega \otimes e + \omega \wedge Ae.$$

The square of d_A is given by exterior multiplication with the curvature $F_A \in \Omega^2\big(M, \mathrm{End}(E)\big)$. Therefore, we get a complex $\big(\Omega^\bullet(M, E), d_A\big)$ precisely if A is flat.

Definition 1.1.4. Let $E \to M$ be a Hermitian vector bundle of rank k, endowed with a unitary flat connection A. Then we denote the homology of $\big(\Omega^\bullet(M, E), d_A\big)$ by $H^\bullet(M, E_A)$ and call it the cohomology of M *with values in the flat bundle* (E, A).

We briefly sketch the relation between cohomology with local coefficients and cohomology with values in a flat bundle. As a general reference, we refer to [86, Sec. 5.5]. In addition, we have included a detailed discussion concerning the equivalence of flat connections and representations of the fundamental group in Appendix B.1. The language there is in terms of principal bundles, but the translation to Hermitian vector bundles is done without effort.

Let M be a connected manifold, not necessarily closed, and let E be a flat unitary bundle over M with connection A. As explained in Appendix B.1 we can lift every closed loop c in M horizontally to E with respect to A. The starting point and the end point of the lifted loop lie in the same fiber of E and since A is unitary they differ by the action of an element in $\mathrm{U}(k)$. This construction gives rise to the holonomy representation of the based loop group of M, see (B.3) on p. 192. Since A is flat, the holonomy representation depends only on homotopy classes. Thus, we obtain a representation

$$\mathrm{hol}_A : \pi_1(M) \to \mathrm{U}(k),$$

which is precisely the object we need to define cohomology with local coefficients.

Conversely, let us start with a representation $\alpha : \pi_1(M) \to \mathrm{U}(k)$ of the fundamental group. Via the action of $\pi_1(M)$ as the group of deck transformations we may interpret the universal cover $\widetilde{M}$ as a $\pi_1(M)$-principal bundle over M. The representation α defines an associated vector bundle

$$E_\alpha = \widetilde{M} \times_\alpha \mathbb{C}^k \to M.$$

Since α is unitary, one can define a natural Hermitian metric on E_α. Moreover, the trivial connection on $\widetilde{M} \times \mathbb{C}^k$ descends to a unitary, flat connection A_α on E_α.

When taking suitable equivalence classes, the above constructions are inverses of each other, see Appendix B.1. Then there is the following twisted version of the de Rham Theorem, see for example the discussion in [86, p. 154].

Proposition 1.1.5. *Let M be a connected manifold, and let E be a Hermitian vector bundle with flat connection A. If $\alpha : \pi_1(M) \to \mathrm{U}(k)$ is the holonomy representation of A, then there is a natural isomorphism*

$$H^\bullet(M, E_A) \xrightarrow{\cong} H^\bullet(M, E_\alpha).$$

Moreover, if M is closed, the twisted intersection pairing of (1.2) *corresponds under this isomorphism to the bilinear form on $H^\bullet(M, E_A)$ induced by*

$$\Omega^p(M, E) \times \Omega^{m-p}(M, E) \mapsto \mathbb{C}, \quad (\omega, \eta) \mapsto \int_M \langle \omega \wedge \eta \rangle,$$

The notation $\langle \omega \wedge \eta \rangle$ is shorthand for taking the exterior product in the differential form part and pairing with the Hermitian metric in the bundle part.

Having the above canonical isomorphism in mind, we will henceforth not carefully distinguish between $H^\bullet(M, E_A)$ and $H^\bullet(M, E_\alpha)$. Similarly, when concerned with the intersection form, we will also write Q_A and $\mathrm{Sign}_A(M)$ if the focus is on a flat connection rather than a representation of the fundamental group.

1.1.2 Twisted Signature Operators

Historically, one of the starting points of index theory is the observation that the signature can be described as the index of an elliptic operator. Before we can define the signature operator, we need to fix some notation and conventions. Since they are of a purely linear algebraic nature, we formulate them for an oriented vector space V of dimension m over $\mathbb{R}$ which plays the role of the cotangent space T_x^*M.

Algebraic Preliminaries. Let V be an Euclidean vector space with scalar product g, and let $\Lambda^\bullet V_{\mathbb{C}}$ denote the complexified exterior algebra. We endow it with the Hermitian metric g_h given by extending g antilinearly in the first variable. Then

$$g_h(\alpha,\beta)\,\mathrm{vol}(g) = \overline{\alpha}\wedge *\beta, \quad \alpha,\beta\in\Lambda^\bullet V_{\mathbb{C}},$$

where $\mathrm{vol}(g)$ is the volume element given by g and the orientation of V and $*$ is the complex linear extension of the Hodge $*$ operator. V acts on $\Lambda^\bullet V$ via exterior multiplication

$$\mathrm{e}(v)\alpha = v\wedge\alpha, \quad \alpha\in\Lambda^\bullet V.$$

Using the metric g, one defines an interior multiplication by requiring that

$$\mathrm{i}(v)w = g(v,w) \quad\text{and}\quad \mathrm{i}(v)(\alpha\wedge\beta) = \mathrm{i}(v)(\alpha)\wedge\beta + (-1)^{|\alpha|}\alpha\wedge\mathrm{i}(v)\beta,$$

for every $v,w\in V$ and $\alpha,\beta\in\Lambda^\bullet V$ with α homogeneous of degree $|\alpha|$. For $v\in V$ we extend $\mathrm{i}(v)$ and $\mathrm{e}(v)$ complex linearly to $\Lambda^\bullet V_{\mathbb{C}}$, and define the *Clifford multiplication*

$$c : V\to \mathrm{End}_{\mathbb{C}}\big(\Lambda^\bullet V_{\mathbb{C}}\big), \quad c(v) := \mathrm{e}(v) - \mathrm{i}(v). \tag{1.3}$$

Then, for all $v\in V$,

$$c(v)^2 = -g(v,v), \quad c(v)^* = -c(v).$$

This means that c extends to a complex representation of the Clifford algebra[1]

$$c : \mathrm{Cl}(V,g)\to\mathrm{End}_{\mathbb{C}}\big(\Lambda^\bullet V_{\mathbb{C}}\big).$$

Equivalently, this is a representation of the complexified Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(V)$. We define the *symbol map*

$$\boldsymbol{\sigma} : \mathrm{Cl}_{\mathbb{C}}(V)\to\Lambda^\bullet V_{\mathbb{C}}, \quad a\mapsto c(a)1. \tag{1.4}$$

The symbol map $\boldsymbol{\sigma}$ is an isomorphism of vector spaces, but certainly not of algebras. The inverse $\boldsymbol{\sigma}^{-1}$ is called the *quantization map*. Using this we define the *chirality operator*

$$\tau := i^{[\frac{m+1}{2}]}\, c\circ\boldsymbol{\sigma}^{-1}\big(\mathrm{vol}(g)\big) : \Lambda^\bullet V_{\mathbb{C}}\to\Lambda^\bullet V_{\mathbb{C}}. \tag{1.5}$$

Here, $[\frac{m+1}{2}]$ denotes the integral part of $\frac{m+1}{2}$. The following result is straightforward, see [13, Prop. 3.58].

[1] Recall (e.g. from [13, Sec. 1.3]) that the *Clifford algebra* $\mathrm{Cl}(V,g)$ of an $\mathbb{R}$ vector space V with a metric g is the $\mathbb{R}$ algebra generated by V and the relations

$$c(v)c(w) + c(w)c(v) = -2g(v,w), \quad v,w\in V.$$

Lemma 1.1.6. *The chirality operator* τ *satisfies*

$$\tau^2 = \mathrm{Id} \quad \textit{and} \quad \tau^* = \tau = \tau^{-1}.$$

On $\Lambda^p V_{\mathbb{C}}$ *it is explicitly given by*

$$\tau = (-1)^{\frac{p(p-1)}{2} + mp} \, i^k *_p, \tag{1.6}$$

where $k := [\frac{m+1}{2}]$ *and* $*_p$ *is the complex linear Hodge* $*$ *operator on* $\Lambda^p V_{\mathbb{C}}$*. Moreover,*

$$\tau \circ c(v) = (-1)^{m+1} c(v) \circ \tau \quad \textit{and} \quad \tau \circ \mathrm{i}(v) = (-1)^m \, \mathrm{e}(v) \circ \tau.$$

Convention. From now on we will always drop the subscripts $\mathbb{C}$. Thus, for a real vector space V, we will use $\Lambda^\bullet V$ to denote the complexified exterior algebra, and $\mathrm{Cl}(V)$ will denote the complexified Clifford algebra.

The Signature an the Index. Now let M be an oriented manifold of dimension m, endowed with a Riemannian metric g. We fix a Hermitian vector bundle $E \to M$ of rank k, endowed with a unitary connection A. Let $\Omega^\bullet(M, E)$ denote differential forms with values in E. The Riemannian metric g and the bundle metric on E define an L^2 scalar product on $\Omega^\bullet(M, E)$. With respect to this, the formal adjoint of the twisted exterior differential

$$d_A : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E)$$

is given in terms of the chirality operator $\tau = \tau_M$ from (1.5) by

$$d_A^t = (-1)^{m+1} \tau \circ d_A \circ \tau, \tag{1.7}$$

see [13, Prop. 3.58]. Here, τ_M acts only on the differential form part. Note that we are using that A is unitary. Now (1.7) implies that the twisted de Rham operator $d_A + d_A^t$ satisfies

$$\tau(d_A + d_A^t) = (-1)^{m+1} (d_A + d_A^t) \tau. \tag{1.8}$$

Let us assume from now on that m is even. Since τ is an involution, we may decompose $\Omega^\bullet(M, E)$ into the ± 1 eigenspaces of τ,

$$\Omega^\bullet(M, E) = \Omega^+(M, E) \oplus \Omega^-(M, E).$$

It follows from (1.8) that we can decompose

$$d_A + d_A^t = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix}, \quad \text{where } D_A^+ : \Omega^+(M, E) \to \Omega^-(M, E), \quad D_A^- = (D_A^+)^t.$$

Definition 1.1.7. Let M be an even dimensional, oriented Riemannian manifold, and let $E \to M$ be a Hermitian vector bundle with a unitary connection A. Then

$$D_A^+ : \Omega^+(M, E) \to \Omega^-(M, E)$$

is called the *twisted signature operator* of M twisted by A.

Since the signature operator is an elliptic operator of first order and M is a closed manifold, we have a well-defined index problem. The name *signature operator* is only justified if A is a flat connection, since then the index of D_A^+ is indeed the twisted signature $\mathrm{Sign}_A(M)$.

Proposition 1.1.8. *Let M be a closed, oriented and connected Riemannian manifold of even dimension m. Let E be a Hermitian vector bundle, endowed with a unitary flat connection A. Then*

$$\mathrm{Sign}_A(M) = \mathrm{ind}(D_A^+).$$

Proof. Although this result can be found in many textbooks, we include its proof as related arguments will appear again. First of all, since $(D_A^+)^t = D_A^-$,

$$\mathrm{ind}(D_A^+) = \dim\big(\ker D_A^+\big) - \dim\big(\ker D_A^-\big) \tag{1.9}$$

To identify $\mathrm{Ker}\, D_A^\pm$ in cohomological terms consider the twisted Laplacian

$$\Delta_A = (d_A + d_A^t)^2 : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E). \tag{1.10}$$

The twisted version of the Hodge isomorphism identifies $H^p(M, E_A)$ with the space of harmonic forms

$$\mathscr{H}^p(M, E_A) := \big(\ker \Delta_A\big) \cap \Omega^p(M,E).$$

It follows from (1.8) that the chirality operator τ commutes with Δ_A, hence it induces an involution on $\mathscr{H}^\bullet(M, E_A)$. On $\mathscr{H}^{m/2}(M, E_A)$ the chirality operator is $\tau = i^k *$, where $k := m^2/4$. Using this the intersection form can be expressed in terms of harmonic forms as

$$Q_A\big(\alpha, \beta\big) = \int_M \langle \alpha \wedge \beta \rangle = \big(\alpha\,,\, i^k \tau\beta\big)_{L^2}, \quad \alpha, \beta \in \mathscr{H}^{m/2}(M, E_A).$$

Therefore,

$$\mathrm{Sign}(Q_A) = \mathrm{Sign}(i^k \tau|_{\mathscr{H}^{m/2}}),$$

where we use the same convention as before regarding the signature of a skew endomorphism. We deduce that

$$\mathrm{Sign}(Q_A) = \dim \mathscr{H}^{m/2}(M, E_A)^+ - \dim \mathscr{H}^{m/2}(M, E_A)^-.$$

If $p \neq m/2$, there are isomorphisms

$$\Phi^\pm : \mathscr{H}^p \xrightarrow{\cong} \big(\mathscr{H}^p \oplus \mathscr{H}^{m-p}\big)^\pm, \quad \Phi^\pm(\alpha) := \tfrac{1}{2}(\alpha \pm \tau\alpha).$$

From this and from the fact that $\mathrm{Ker}(d_A + d_A^t) = \mathrm{Ker}\,\Delta_A$ we find

$$\dim\big(\ker D_A^\pm\big) = \dim\big(\mathscr{H}^{m/2}(M, E_A)^\pm\big) + \sum_{p<m/2} \dim\big(\mathscr{H}^p(M, E_A)\big)$$

Thus, all terms in (1.9) are cancelled except the one in the middle degree so that

$$\mathrm{ind}(D_A^+) = \dim \mathscr{H}^{m/2}(M, E_A)^+ - \dim \mathscr{H}^{m/2}(M, E_A)^- = \mathrm{Sign}(Q_A) \qquad \square$$

Remark. We have already pointed out that for closed manifolds the twisted signatures do not carry interesting topological information. If the flat twisting bundle is trivial, this can be deduced from the above result: Let A be a flat connection on the trivial vector bundle $E = M \times \mathbb{C}^k$, and let D_A^+ be the associated signature operator. Furthermore, let $D_{\oplus k}^+$ denote the signature operator associated to the trivial connection on E. Clearly, $D_A^+ - D_{\oplus k}^+$ is an operator of order 0. On a closed manifold adding a 0-order perturbation to an elliptic operator of first order does not change the index. This is because it is a compact perturbation of a Fredholm operator in the appropriate Hilbert space setting. Therefore,

$$\mathrm{Sign}_A(M) = \mathrm{ind}(D_A^+) = \mathrm{ind}(D_{\oplus k}^+) = k \cdot \mathrm{ind}(D^+) = k \cdot \mathrm{Sign}(M).$$

This means that the only new information encoded in the twisted signature is the rank of the twisting bundle. We will see in Corollary 1.2.10 below that this is also true for flat twisting bundles which are topologically non-trivial.

1.2 Dirac Operators and the Atiyah-Singer Index Theorem

The famous index theorem equates the index of an elliptic operator over a closed manifold M with the integral over certain characteristic classes over M. In this section we briefly recall the definitions occurring in the index theorem for Dirac type operators.

1.2.1 The Index and the Heat Equation

We first recall some facts about the spectral theory of formally self-adjoint elliptic operators on closed manifolds, see e.g. [49, Sec.'s. 1.3 & 1.6].

Definition 1.2.1. Let M be a Riemannian manifold, and let $E \to M$ be a Hermitian vector bundle. We denote by

$$\mathscr{P}_{s,e}^d = \mathscr{P}_{s,e}^d(M, E)$$

the space of formally self-adjoint elliptic differential operators of order d.

Theorem 1.2.2. *Let $D \in \mathscr{P}_{s,e}^d(M, E)$, and assume that M closed.*

(i) *The operator D extends to an unbounded self-adjoint operator in $L^2(M, E)$ with domain the Sobolev space $L_d^2(M, E)$. It defines Fredholm operators*

$$D : L_{s+d}^2(M, E) \to L_s^2(M, E), \quad s \in \mathbb{R},$$

with Fredholm index independent of s.

(ii) *There exists a constant C such that for all $\varphi \in C^\infty(M, E)$*

$$\|\varphi\|_{L_d^2} \le C\big(\|\varphi\|_{L^2} + \|D\varphi\|_{L^2}\big). \tag{1.11}$$

(iii) *The spectrum* $\mathrm{spec}(D)$ *is a discrete subset of $\mathbb{R}$ consisting of eigenvalues with finite multiplicities. There is an orthonormal basis of $L^2(M, E)$ consisting of smooth eigenvectors.*

(iv) *If we define*

$$N(\lambda) := \#\{\lambda_n \in \operatorname{spec}(D) \mid |\lambda_n| \leq \lambda\}, \quad \lambda \geq 0,$$

then for some constant $C > 0$

$$N(\lambda) \sim C\lambda^{m/d}, \quad \textit{as } \lambda \to \infty. \tag{1.12}$$

(v) *If* D *has a positive definite leading symbol, then the spectrum* $\operatorname{spec}(D)$ *is bounded from below.*

The Heat Kernel. We now specialize to a second order operator, $H \in \mathscr{P}^2_{s,e}(M, E)$, and assume that H has positive definite leading symbol. Clearly, the model we are having in mind is a generalized Laplacian, i.e., an operator $H \in \mathscr{P}^2_{s,e}(M, E)$ such that its principal symbol $\sigma(H)$ satisfies

$$\sigma(H)(x, \xi) = |\xi|^2_g \operatorname{id}_{E_x}, \quad \xi \in T^*_x M.$$

Recall that the heat equation with initial condition $\varphi \in L^2(M, E)$ in terms of H is the partial differential equation

$$\big(\tfrac{d}{dt} + H\big)\varphi(t) = 0, \quad t \geq 0, \quad \varphi(0) = \varphi. \tag{1.13}$$

Formally, if $\{\lambda_n\}_{n \geq -n_0}$ denotes the set of eigenvalues of H with eigenvectors φ_n, the solution to (1.13) is

$$\begin{aligned}
\varphi(t) = e^{-tH}\varphi &= \sum_{n \geq -n_0} e^{-t\lambda_n}\varphi_n\big(\varphi_n\,,\,\varphi\big)_{L^2} \\
&= \int_M \sum_{n \geq -n_0} e^{-t\lambda_n}\varphi_n(x)\langle\varphi_n(y), \varphi(y)\rangle \operatorname{vol}_M(y).
\end{aligned}$$

Thus, e^{-tH} is an integral operator with kernel

$$k_t(x, y) = e^{-tH}(x, y) = \sum_{n \geq -n_0} e^{-t\lambda_n}\varphi_n(x) \otimes \varphi_n(y)^* \in C^\infty\big(M \times M, E \boxtimes E^*\big). \tag{1.14}$$

Here, for vector bundles $E \to M$ and $F \to N$, we employ the standard notation

$$E \boxtimes F := \pi^*_M E \otimes \pi^*_N F \to M \times N, \tag{1.15}$$

where π_M and π_N are the natural projections. The formal expression (1.14) can be made precise using the following basic estimate.

Lemma 1.2.3. *Let* $0 < \lambda_0 \leq \lambda_1 \leq \dots$ *denote the positive eigenvalues of* H*, and let* $\lambda_{-1} \leq \dots \leq \lambda_{-n_0}$ *denote the finite number of eigenvalues of* H *which are less or equal than 0. Let* $\{\varphi_n\}_{n \geq -n_0}$ *be a basis of smooth eigenvectors and for* $N \in \mathbb{N}$ *consider*

$$k^N_t(x, y) := \sum_{n=-n_0}^{N} e^{-t\lambda_n}\varphi_n(x) \otimes \varphi^*_n(y) \in C^\infty\big(M \times M, E \boxtimes E^*\big).$$

Then for every $k \in \mathbb{N}$ *and* $t_0 > 0$ *there exists a constant* C *such that for every* $N \in \mathbb{N}$ *and* $t \geq t_0$ *we can estimate*

$$\Big\| k^N_t(x, y) - \sum_{n<0} e^{-t\lambda_n}\varphi_n(x) \otimes \varphi^*_n(y) \Big\|_{C^k} \leq Ce^{-t\lambda_0/2}.$$

Proof. Sobolev embedding [49, Lem. 1.3.5] and the elliptic estimate (1.11) imply that for $l > k + m/2$ there exist constants C_1, C_2 such that for all $n \geq 0$

$$\|\varphi_n\|_{C^k} \leq C_1 \|\varphi_n\|_{L^2_l} \leq C_2\big(\|\varphi_n\|_{L^2} + \|H^{l/2}\varphi_n\|_{L^2}\big) = C_2\big(1 + \lambda_n^{l/2}\big).$$

Thus, for some other constants C_1 and C_2,

$$\big\|e^{-t\lambda_n}\varphi_n(x) \otimes \varphi_n^*(y)\big\|_{C^k} \leq C_1 e^{-t\lambda_n}\big(1 + \lambda_n^l\big) \leq C_2 e^{-t\lambda_n/2}(1 + t^{-l}),$$

where we have used that for $x = \lambda_n t > 0$

$$x^l \leq C_l\, e^{x/2} \quad \text{and thus,} \quad x^l e^{-x} \leq C_l\, e^{-x/2}.$$

Now for each $n \geq 0$ we have $\lambda_n \geq \lambda_0$ and thus for $t \geq t_0$

$$e^{-t\lambda_n/2}(1 + t^{-l}) \leq e^{-t_0\lambda_n/2}(1 + t_0^{-l})e^{-(t-t_0)\lambda_0/2} = Ce^{-t_0\lambda_n/2}e^{-t\lambda_0/2},$$

where the constant C depends only on t_0 and l. Putting the pieces together we find that there exists a constant C such that for every $N \geq 0$ and $t \geq t_0$

$$\Big\|\sum_{n=0}^{N} e^{-t\lambda_n}\varphi_n(x) \otimes \varphi_n^*(y)\Big\|_{C^k} \leq Ce^{-t\lambda_0/2}\sum_{n=0}^{N} e^{-t_0\lambda_n/2}$$

Now the eigenvalue asymptotics (1.12) shows that $\sum_{n=0}^{N} e^{-t_0\lambda_n/2}$ is absolutely convergent for $N \to \infty$. Absorbing this to the constant, we get the desired result. $\square$

The above result shows that the kernels $k_t^N(x, y)$ converge for $N \to \infty$ to the expression (1.14), uniformly with respect to all C^k. Hence, we can define e^{-tH} for $t > 0$ by

$$(e^{-tH}\varphi)(x) = \int_M k_t(x, y)\varphi(y)\, \mathrm{vol}_M(y), \quad \varphi \in L^2(M, E).$$

In Appendix D.1 we give an expression for e^{-tH} using the spectral theorem. It is then easy to check that the collection e^{-tH} forms a strongly continuous semi-group in each Sobolev space $L^2_s(M, E)$, i.e.,

$$e^{-(s+t)H} = e^{-sH}e^{-tH}, \quad \text{and} \quad \lim_{t\to 0}\|e^{-tH}\varphi - \varphi\|_{L^2_s} = 0 \quad \text{for each } \varphi \in L^2_s(M, E),$$

see also Proposition D.1.2. Moreover, e^{-tH} is smooth in t and does indeed solve the heat equation (1.13). In addition, each e^{-tH} is trace class, and

$$\mathrm{Tr}\, e^{-tH} = \int_M \mathrm{tr}_E\, \big[k_t(x, x)\big]\, \mathrm{vol}_M(x),$$

where $\mathrm{tr}_E : C^\infty\big(M, \mathrm{End}(E)\big) \to C^\infty(M)$ denotes the fiberwise trace.

More generally, given an auxiliary differential operator $D : C^\infty(M, E) \to C^\infty(M, E)$ of order $d \geq 0$, the uniform bound of Lemma 1.2.3 ensures that we can apply D under the integral to get

$$De^{-tH}\varphi(x) = D\Big(\int_M k_t(x, y)\varphi(y)\, \mathrm{vol}_M(y)\Big) = \int_M D_x k_t(x, y)\varphi(y)\, \mathrm{vol}_M(y),$$

where D_x means applying D with respect to the x variable. Thus, the operator De^{-tH} has a smooth kernel

$$D_x k_t(x,y) \in C^\infty\big(M \times M, E \boxtimes E^*\big),$$

so that De^{-tH} is trace class. The estimate in Lemma 1.2.3 implies the following basic estimate on $\mathrm{Tr}(De^{-tH})$.

Proposition 1.2.4. *For every $t_0 > 0$ there exists a constant C such that for all $t \geq t_0$*

$$\big|\,\mathrm{Tr}\left(De^{-tH}P_{(0,\infty)}\right)\big| \leq Ce^{-t\lambda_0/2},$$

where λ_0 is the smallest positive eigenvalue of H and $P_{(0,\infty)}$ is the spectral projection of H associated to the interval $(0,\infty)$.

Proof. The kernel of $e^{-tH}P_{(0,\infty)}$ is given by

$$\tilde{k}_t(x,y) := k_t(x,y) - \sum_{n<0} e^{-t\lambda_n}\varphi_n(x) \otimes \varphi_n^*(y).$$

Then we find that for $t \geq t_0$

$$\begin{aligned}\big|\,\mathrm{Tr}\left(De^{-tH}P_{(0,\infty)}\right)\big| &= \Big|\int_M \mathrm{tr}_E\left[D_x\tilde{k}_t(x,y)\right]_{y=x}\mathrm{vol}_M(x)\Big| \\ &\leq \mathrm{rk}(E)\,\mathrm{vol}(M)\,\big\|D_x\tilde{k}_t(x,y)\big\|_{C^0} \\ &\leq C_1\,\big\|\tilde{k}_t(x,y)\big\|_{C^d} \leq C_2\,e^{-t\lambda_0/2},\end{aligned}$$

where in the last line we have used that D is a differential operator of order d and then Lemma 1.2.3. $\square$

The McKean-Singer Formula. We now turn our attention to first order differential operators. To be able to restrict to the formally self-adjoint case, we use the following construction: Let $D^+ : C^\infty(M,E^+) \to C^\infty(M,E^-)$ be an elliptic differential operator of first order, acting between Hermitian vector bundles E^+ and E^-. We define $E := E^+ \oplus E^-$, and consider

$$D := \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} : C^\infty(M,E) \to C^\infty(M,E), \quad \text{where} \quad D^- := (D^+)^t. \tag{1.16}$$

Certainly, an operator of this form is formally self-adjoint and elliptic.

Definition 1.2.5. Let $E \to M$ be a Hermitian vector bundle endowed with a splitting $E = E^+ \oplus E^-$.

(i) Let $\sigma : E \to E$ be the involution on E given by $\sigma|_{E^\pm} = \pm\,\mathrm{id}$. Then σ is called the *grading operator* of E.

(ii) An operator $D \in \mathscr{P}^1_{s,e}(M,E)$ is called $\mathbb{Z}_2$*-graded* if

$$\{D,\sigma\} = D\sigma + \sigma D = 0.$$

Note that—unless stated otherwise—we are using commutators and anti-commutators in an *ungraded* sense. Clearly, D is $\mathbb{Z}_2$-graded if and only if it is of the form (1.16).

(iii) For $T \in C^\infty\big(M, \mathrm{End}(E)\big)$, we define *fiberwise supertrace* of T as

$$\mathrm{str}_E(T) := \mathrm{tr}_E(\sigma T) \in C^\infty(M).$$

(iv) If $D \in \mathscr{P}^1_{e,s}(M, E)$ is $\mathbb{Z}_2$-graded, and M is closed, then the *heat supertrace* associated to D is defined as

$$\mathrm{Str}(e^{-tD^2}) := \mathrm{Tr}\left(\sigma e^{-tD^2}\right).$$

Note that in (iv) the operator $H = D^2$ is positive and splits as $H = H_+ \oplus H_-$, where $H_\pm$ are formally self-adjoint, positive operator as well. Thus, we are in the situation considered before so that the respective heat traces exist, and

$$\mathrm{Str}(e^{-tD^2}) = \mathrm{Tr}(e^{-tH_+}) - \mathrm{Tr}(e^{-tH_-}) = \mathrm{Tr}(e^{-t(D^+)^t D^+}) - \mathrm{Tr}(e^{-tD^+(D^+)^t}).$$

Now, as an elliptic operator on a closed manifold D^+ has a well-defined Fredholm index

$$\mathrm{ind}(D^+) = \dim\big(\ker D^+\big) - \dim\big(\ker(D^+)^t\big) = \mathrm{Str}(P_0),$$

where $P_0 : L^2(M, E) \to \ker(D)$ is the orthogonal projection on the kernel of D. The famous McKean-Singer formula [69] relates this index and the heat supertrace.

Theorem 1.2.6 (McKean-Singer)**.** *Let M be a closed manifold, $E \to M$ a Hermitian vector bundle, and let $D \in \mathscr{P}^1_{e,s}(M, E)$ be $\mathbb{Z}_2$-graded. Then for all $t > 0$*

$$\mathrm{ind}(D^+) = \mathrm{Str}(e^{-tD^2}).$$

Proof. Since D^2 has no negative eigenvalues, we have $P_0 + P_{(0,\infty)} = \mathrm{Id}$. Then the estimate in Proposition 1.2.4 implies that there exist constants c and C such that for large t

$$\big|\,\mathrm{Str}(e^{-tD^2}) - \mathrm{Str}(P_0)\big| = \big|\,\mathrm{Str}(e^{-tD^2} P_{(0,\infty)})\big| \le Ce^{-ct}.$$

Thus,

$$\lim_{t\to\infty} \mathrm{Str}(e^{-tD^2}) = \mathrm{Str}(P_0) = \mathrm{ind}(D^+).$$

It remains to check that $\mathrm{Str}(e^{-tD^2})$ is independent of $t > 0$. For this, we note that the heat equation yields

$$\tfrac{d}{dt} e^{-tD^2} = -D^2 e^{-tD^2}.$$

The basic trace estimate then implies that $\mathrm{Str}(e^{-tD^2})$ is differentiable in t with

$$\tfrac{d}{dt}\,\mathrm{Str}(e^{-tD^2}) = -\,\mathrm{Str}(D^2 e^{-tD^2}).$$

However, since D anti-commutes with σ, we infer from the trace property that

$$\mathrm{Str}(D^2 e^{-tD^2}) = \mathrm{Tr}(\sigma D^2 e^{-tD^2}) = -\,\mathrm{Tr}(D\sigma D e^{-tD^2}) = -\,\mathrm{Tr}(\sigma e^{-tD^2} D^2) = -\,\mathrm{Str}(D^2 e^{-tD^2}),$$

so that indeed $\mathrm{Str}(D^2 e^{-tD^2}) = 0$. $\square$

Heat Kernel Asymptotics. So far, the treatment of the heat kernel has been of a functional analytic nature. The only input are the basic properties of elliptic differential operators on closed manifolds as in Theorem 1.2.1. However, one important missing piece is the analysis of the heat trace as $t \to 0$, which requires further work. We summarize the following from [49, Sec.'s 1.8 & 1.9].

Theorem 1.2.7. *Let M be a closed manifold, and let $H \in \mathscr{P}^2_{s,e}(M, E)$ have positive definite leading symbol. Let $D : C^\infty(M, E) \to C^\infty(M, E)$ be an auxiliary differential operator of order $d \geq 0$, and let $k_t(x, y)$ denote the kernel of De^{-tH}.*

(i) *There exists an asymptotic expansion*

$$k_t(x, x) \sim \sum_{n=0}^{\infty} t^{\frac{n-m-d}{2}} e_n(x), \quad \text{as } t \to 0,$$

with $e_n \in C^\infty\big(M, \mathrm{End}(E)\big)$ such that $e_n(x)$ is locally computable from the total symbols of H and D near x. If $n + d$ is odd, then $e_n = 0$.

(ii) *The trace of De^{-tH} admits an asymptotic expansion*

$$\mathrm{Tr}(De^{-tH}) = \int_M \mathrm{tr}_E \big[k_t(x, x)\big] \operatorname{vol}_M(x) \sim \sum_{n=0}^{\infty} t^{\frac{n-m-d}{2}} a_n(D, H), \quad \text{as } t \to 0.$$

The asymptotic expansion can be differentiated in t, and the a_n are given by

$$a_n(D, H) = \int_M \mathrm{tr}_E \big[e_n\big] \operatorname{vol}_M .$$

The Index Density. As a consequence of the McKean-Singer formula and the asymptotic expansion of the heat trace, we get the following result of [69] and [5].

Theorem 1.2.8. *Let $e_n(x)$ be the coefficient appearing in the asymptotic expansion of $e^{-tD^2}(x, x)$ as in Theorem 1.2.7. Then*

$$\int_M \mathrm{str}_E[e_n] \operatorname{vol}_M = \begin{cases} \mathrm{ind}(D^+), & \text{if } n = \dim M, \\ 0, & \text{if } n < \dim M. \end{cases}$$

Remark.

(i) For aparent reasons, the differential form $\mathrm{str}_E[e_m] \operatorname{vol}_M$ with $m = \dim M$ is called the *index density* of D^+.

(ii) As mentioned in Theorem 1.2.7, the coefficients e_n vanish if n is odd. This implies that the index of D^+ vanishes if M is odd dimensional.

We also want to note an important consequence of the local nature of the e_n. Assume that two operators $D \in \mathscr{P}^1_{e,s}(M, E)$ and $D' \in \mathscr{P}^1_{e,s}(M, E')$ are *locally equivalent.* This means that for every $x \in M$, there exists a neighbourhood U of x and a local isometry $\Phi : E|_U \to E'|_U$ such that

$$\Phi \circ D \circ \Phi^{-1} = D' \quad \text{over } U. \tag{1.17}$$

Since the sections e_n and e'_n in the respective asymptotic expansions of the heat kernels are computable from the total symbols over U, relation (1.17) implies that $\mathrm{str}_E[e_n]|_U = \mathrm{str}_{E'}[e'_n]|_U$. This has the following consequence.

Corollary 1.2.9. *Let M be a closed manifold, $E \to M$ a Hermitian vector bundle, and let $D \in \mathscr{P}^1_{e,s}(M,E)$ be a $\mathbb{Z}_2$-graded operator.*

(i) *If $\pi : \widehat{M} \to M$ is a k-fold regular cover, and*

$$\widehat{D} : C^\infty(\widehat{M}, \pi^* E) \to C^\infty(\widehat{M}, \pi^* E)$$

is the natural lift of D to $\widehat{M}$, then

$$\operatorname{ind}(\widehat{D}^+) = k \cdot \operatorname{ind}(D^+).$$

(ii) *If A is a flat connection on a Hermitian vector bundle F of rank k over M, and if D_A is the operator D twisted by F and A, then*

$$\operatorname{ind}\left(D_A^+\right) = k \cdot \operatorname{ind}(D^+).$$

Proof. For (i) one notes that the $\widehat{e}_n$ are the lifts to $\widehat{M}$ of the e_n. Thus,

$$\int_{\widehat{M}} \operatorname{str}_{\pi^* E}[\widehat{e}_n] \operatorname{vol}_{\widehat{M}} = \int_{\widehat{M}} \pi^*\big(\operatorname{str}_E[e_n] \operatorname{vol}_M\big) = k \cdot \int_M \operatorname{str}_E[e_n] \operatorname{vol}_M,$$

where we have used that $\operatorname{vol}(\widehat{M}) = k \cdot \operatorname{vol}(M)$. For (ii) note that choosing local trivializations for F which are parallel with respect to A, one finds that the flat bundle F is locally isomorphic to $M \times \mathbb{C}^k$ endowed with the trivial connection. This yields that D_A is locally equivalent to $D^{\oplus k}$. □

As we have seen in Proposition 1.1.8, the signature of a closed manifold equals the index of an elliptic differential operator of first order. Hence, Corollary 1.2.9 proves our earlier assertion that the twisted signatures do not contain new information other than the rank of the twisting bundle.

Corollary 1.2.10. *Let M be a closed, even dimensional manifold.*

(i) *If $\pi : \widehat{M} \to M$ is a k-fold regular cover, then*

$$\operatorname{Sign}(\widehat{M}) = k \cdot \operatorname{Sign}(M)$$

(ii) *If A is a flat connection on a Hermitian vector bundle E of rank k over M, then*

$$\operatorname{Sign}_A(M) = k \cdot \operatorname{Sign}(M).$$

1.2.2 Geometric Dirac Operators and the Local Index Theorem

Theorem 1.2.8 is the starting point for local index theory. Since the coefficients e_n are—in principal—locally computable, a strategy to prove the Atiyah-Singer Index Theorem is to identify the index density $\operatorname{str}_E[e_m] \operatorname{vol}_M$ with a Chern-Weil representative of an appropriate characteristic class. Note, however, that Chern-Weil classes are expressions in the curvature, whereas the e_n a priori contain higher order derivatives of the connection. Nevertheless, this strategy works for an important class of differential operators, which we describe briefly.

Geometric Dirac Operators. Let (M, g) be an oriented, Riemannian manifold, and let $E \to M$ be a Hermitian vector bundle. E is called a *Clifford module* if it is endowed with a bundle map $c : T^*M \to \mathrm{End}(E)$ such that for every $\xi \in T^*M$

$$c(\xi)^2 = -|\xi|_g^2 \,\mathrm{id}_E \quad \text{and} \quad c(\xi)^* = -c(\xi). \tag{1.18}$$

A Clifford module is called $\mathbb{Z}_2$-graded if E is $\mathbb{Z}_2$-graded and if $c(\xi)$ is an odd element of $\mathrm{End}(E)$ for all $\xi \in T^*M$. Provided that E is endowed with suitable connection, we can construct a natural first order differential operator:

Definition 1.2.11. Let E be a Clifford module over a Riemannian manifold (M, g).

(i) A connection ∇^E on E which is compatible with the metric is called a *Clifford connection* if for all $e \in C^\infty(M, E)$ and $\xi \in \Omega^1(M)$

$$\big[\nabla^E, c(\xi)\big]e = \nabla^E\big(c(\xi)e\big) - c(\xi)\nabla^E e = c(\nabla^g \xi)e,$$

where ∇^g is the Levi-Civita connection acting on forms.

(ii) If ∇^E is a Clifford connection, we define the associated *geometric Dirac operator* by

$$D := c \circ \nabla^E : C^\infty(M, E) \to C^\infty(M, E).$$

Here, we are viewing the Clifford structure as a bundle map $c : T^*M \otimes E \to E$.

(iii) A geometric Dirac operator is called $\mathbb{Z}_2$*-graded* if E is a $\mathbb{Z}_2$-graded Clifford module, and the Clifford connection respects the splitting $E = E^+ \oplus E^-$.

Remark 1.2.12. One often defines a *Dirac operator* to be a formally self-adjoint elliptic operator whose square is a generalized Laplacian. It is straightforward to check that geometric Dirac operators have this property. However, not every Dirac operator is a geometric one. In Section 3.2 we sketch how to associate Dirac operators to generalized connections, in particular Clifford *superconnections*, thereby obtaining a larger class of Dirac operators, see also [13, Prop. 3.42].

Canonical Structures on Clifford Modules. As pointed out, the index density is a purely local object. The reason for studying geometric Dirac operators is that Clifford modules have a canonical local structure which we now describe briefly. For proofs we refer to [13, Sec.'s 3.2 & 3.3].

We assume from now on that $m = \dim M$ is even. Some aspects of the odd dimensional case are contained in Section 1.2.3 and Section 1.5.2 in a special case. Let us further assume for the moment that M is *spin*. Without going into the details of the definition and the topological restrictions that this imposes on M, we note that it implies that there exists a unique irreducible Clifford module S of rank $2^{m/2}$ over M, called the *spinor module*. It follows from the representation theory of Clifford algebras that

$$\mathrm{End}(S) = \mathrm{Cl}(T^*M). \tag{1.19}$$

The Clifford module S is naturally $\mathbb{Z}_2$-graded, and (1.19) is an isomorphism of $\mathbb{Z}_2$-graded algebras. Here, the grading on $\mathrm{Cl}(T^*M)$ is the one induced via the symbol map from the even/odd grading on differential forms. Now, every $\mathbb{Z}_2$-graded Clifford module E can be decomposed as $E = S \otimes W$, where W carries a trivial Clifford structure. Moreover,

$$\mathrm{End}(W) = \mathrm{End}_{\mathrm{Cl}}(E) := \Big\{ T \in \mathrm{End}(E) \;\Big|\; [T, c(\alpha)]_s = 0 \quad \text{for all } \alpha \in T^*M \Big\}, \tag{1.20}$$

and

$$\mathrm{End}(E) = \mathrm{Cl}(T^*M) \otimes_s \mathrm{End}_{\mathrm{Cl}}(E). \tag{1.21}$$

Here, $[.,.]_s$ and $\otimes_s$ are the commutator and the tensor product in the $\mathbb{Z}_2$-graded sense. If σ denotes the grading operator of E, there exists a unique decomposition

$$\sigma = \tau \otimes \sigma_W \in \mathrm{Cl}(T^*M) \otimes_s \mathrm{End}_{\mathrm{Cl}}(E), \tag{1.22}$$

where $\tau := i^{m/2} c(\mathrm{vol}_M)$ is the chirality operator, and σ_W is a grading operator on W. Now, if $T \in \mathrm{End}(W)$, then one verifies that

$$\mathrm{str}_W(T) = \mathrm{tr}_W[\sigma_W T] = \frac{1}{\mathrm{rk}(S)} \mathrm{tr}_E \big[(\tau^2 \otimes \sigma_W) T \big] = \frac{1}{2^{m/2}} \mathrm{str}_E[\tau T]. \tag{1.23}$$

Moreover, the spinor module S comes equipped with a canonical Clifford connection ∇^S which is induced by the Levi-Civita connection ∇^g via (1.19). From this one gets a 1-1 correspondence between Clifford connections ∇^E on E and connections of the form $\nabla^S \otimes 1 + 1 \otimes \nabla^W$, where ∇^W is a Hermitian connection on W. The curvature F_{∇^W} satisfies

$$F_{\nabla^W} = F_{\nabla^E} - R^S \in \Omega^2\big(M, \mathrm{End}_{\mathrm{Cl}}(E)\big), \tag{1.24}$$

where for any orthonormal frame $\{e_i\}$ of TM

$$R^S := \tfrac{1}{8} g\big(R^g(e_i, e_j)e_k, e_l\big) e^i \wedge e^j \otimes c(e^k) c(e^l). \tag{1.25}$$

Here, $R^g \in \Omega^2\big(M, \mathrm{End}(TM)\big)$ is the curvature tensor of the Levi-Civita connection.

We now note that the right hand sides of (1.20), (1.23) and (1.24) can be defined globally on M without referring to the spinor module S. Thus, we can introduce corresponding objects also in the case that M is not spin. In particular, the definition of $\mathrm{End}_{\mathrm{Cl}}(E)$ in (1.20) is meaningful for every Clifford module E.

Definition 1.2.13. Let M be an m-dimensional manifold, where m is even, and let E be a $\mathbb{Z}_2$-graded Clifford module over M, endowed with a Clifford connection ∇^E.

(i) Let R^S be defined as in (1.25). Then we can decompose

$$F_{\nabla^E} = R^S + F^{E/S}, \quad \text{where} \quad F^{E/S} \in \Omega^2\big(M, \mathrm{End}_{\mathrm{Cl}}(E)\big).$$

The term $F^{E/S}$ is called the *twisting curvature* of E.

(ii) If $T \in C^\infty\big(M, \mathrm{End}_{\mathrm{Cl}}(E)\big)$, then its *relative supertrace* is defined as

$$\mathrm{str}_{E/S}(T) := \frac{1}{2^{m/2}} \mathrm{str}_E(\tau T).$$

(iii) The *relative Chern character form* of E is given by

$$\mathrm{ch}_{E/S}(E,\nabla^E) := \mathrm{str}_{E/S}\left[\exp\left(\tfrac{i}{2\pi}F^{E/S}\right)\right] \in \Omega^{ev}(M). \tag{1.26}$$

Remark. It follows from (1.23) and (1.24) that if M is spin so that $E = S \otimes W$, then

$$\mathrm{ch}_{E/S}(E,\nabla^E) = \mathrm{str}_W\left[\exp\left(\tfrac{i}{2\pi}F^W\right)\right].$$

In particular, if W is ungraded this coincides with the Chern character form of W as in Definition A.1.3. In general, if $W = W^+ \oplus W^-$, the relative Chern character is the difference of the Chern characters of W^+ and W^-, see also (3.16).

The Local Index Theorem. We can now state a version of the local index theorem as in [13, Thm. 4.2].

Theorem 1.2.14 (Patodi, Gilkey)**.** *Let M be a closed, oriented Riemannian manifold of even dimension m, and let $E \to M$ be a $\mathbb{Z}_2$-graded Clifford module with Clifford connection ∇^E and Dirac operator D. Let $e_n(x)$ be the coefficient appearing in the asymptotic expansion of $e^{-tD^2}(x,x)$ as in Theorem 1.2.7. Then*

$$\mathrm{str}_E[e_n]\,\mathrm{vol}_M(x) = \begin{cases} \left(\widehat{A}(TM,\nabla^g) \wedge \mathrm{ch}_{E/S}(E,\nabla^E)\right)_{[m]}, & \text{if } n = \dim M, \\ 0, & \text{if } n < \dim M, \end{cases} \tag{1.27}$$

where the Hirzebruch $\widehat{A}$-form is as in Definition A.1.4, and $(\ldots)_{[m]}$ means taking the m-form part of a differential form.

Remark 1.2.15. There is a stronger version of the local index theorem due to E. Getzler, see [44] and [13, Thm. 4.1], which we also want to recall. Let $\boldsymbol{\sigma} : \mathrm{Cl}(T^*M) \to \Lambda^\bullet T^*M$ be the symbol map (1.4), and use this to endow $\mathrm{Cl}(T^*M)$ with a $\mathbb{Z}$-grading. With respect to this let $\mathrm{Cl}_n(T^*M)$ be the subbundle of $\mathrm{Cl}(T^*M)$ of elements of degree $\leq n$. Then it can be shown that

$$e_n \in C^\infty\big(M, \mathrm{Cl}_n(T^*M) \otimes_s \mathrm{End}_{\mathrm{Cl}}(E)\big).$$

The stronger version of the local index theorem is the formula

$$(4\pi)^{m/2} \sum_{n \leq m} \boldsymbol{\sigma}(e_n) = \det{}^{1/2}\left(\frac{R^g/2}{\sinh(R^g/2)}\right) \wedge \exp(-F^{E/S}),$$

where R^g is the Riemann curvature tensor. For the definition of the right hand side, see Appendix A, in particular (A.1) and Definition A.1.4. Now the supertrace of elements in $\mathrm{Cl}(T^*M)$ vanishes away from degree m. Hence, Theorem 1.2.14 follows by computing the supertrace of τ and taking into account the powers of $\frac{i}{2\pi}$ appearing in our definition of $\widehat{A}$ and $\mathrm{ch}_{E/S}$. In Section 1.5.2, we will sketch a proof of a variation formula for the Eta invariant based on this more general local index theorem.

A direct consequence of the local index theorem 1.2.14 and Theorem 1.2.8 is the famous Atiyah-Singer Index Theorem for geometric Dirac operators in its cohomological version.

Theorem 1.2.16 (Atiyah-Singer). *Let M be a closed, oriented Riemannian manifold of even dimension m, and let $E \to M$ be a $\mathbb{Z}_2$-graded Clifford module with Clifford connection ∇^E and Dirac operator D. Then*

$$\operatorname{ind}(D^+) = \int_M \widehat{A}(TM, \nabla^g) \wedge \operatorname{ch}_{E/S}(E, \nabla^E).$$

In terms of characteristic classes,

$$\operatorname{ind}(D^+) = \big\langle \widehat{A}(TM) \cup \operatorname{ch}_{E/S}(E), [M] \big\rangle.$$

1.2.3 Hirzebruch's Signature Theorem

A special case of the Atiyah-Singer Index Theorem is one of its predecessors, the Hirzebruch Signature Theorem. It arises if the geometric Dirac operator in question is a twisted signature operator. Therefore, we now collect some details about the structure of the exterior algebra as a Clifford module.

Clifford Structures on the Exterior Algebra. Let M be an m-dimensional closed, oriented Riemannian manifold. For the moment we do not assume that m is even. We consider the Clifford structure

$$c : T^*M \to \operatorname{End}\big(\Lambda^\bullet T^*M\big), \quad c(\xi) = \mathrm{e}(\xi) - \mathrm{i}(\xi),$$

see (1.3). The Levi-Civita connection ∇^g acting on forms is a Clifford connection, and we get a geometric Dirac operator

$$d + d^t = c \circ \nabla^g : \Omega^\bullet(M) \to \Omega^\bullet(M).$$

There are two natural gradings on $\Lambda^\bullet T^*M$, one given by the even/odd grading and one given by the chirality operator τ. We know from Lemma 1.1.6 that if m is even, Clifford multiplication is odd with respect to both gradings. However, if m is odd, Clifford multiplication commutes with τ so that in this case we do not get a $\mathbb{Z}_2$-graded Clifford module. Moreover, if M is spin and even dimensional, then

$$\Lambda^* T^*M \cong \mathrm{Cl}(T^*M) = \operatorname{End}(S) = S \otimes S^*,$$

which means that the twisting bundle is isomorphic to the dual bundle S^* of S. This motivates the following

Definition 1.2.17. Let M be a manifold of dimension m, not necessarily even. We define a *transposed Clifford multiplication*

$$\widehat{c} : T^*M \to \operatorname{End}\big(\Lambda^\bullet T^*M\big), \quad \widehat{c}(\xi) := \mathrm{e}(\xi) + \mathrm{i}(\xi).$$

The transposed Clifford multiplication has the following properties.

Lemma 1.2.18. *Let $\{e^i\}$ be a local orthonormal frame for TM.*

(i) *With the obvious abbreviations, we have*

$$\widehat{c}^i \widehat{c}^j + \widehat{c}^j \widehat{c}^i = 2\delta^{ij}, \quad \textit{and} \quad c^i \widehat{c}^j + \widehat{c}^j c^i = 0.$$

(ii) *If we define* $\widehat{\tau} := i^{[\frac{m+1}{2}]}\widehat{c}^1 \dots \widehat{c}^m$, *then*

$$\widehat{\tau}^2 = (-1)^m, \quad \textit{and} \quad \widehat{\tau} = \tau \circ (-1)^\nu,$$

where $\nu : \Lambda^\bullet T^*M \to \mathbb{N}$ *is the number operator given by* $\nu(\omega) = k$ *if* $\omega \in \Lambda^k T^*M$.

(iii) *Let* $\widehat{\mathrm{Cl}}(T^*M)$ *denote the subbundle of* $\mathrm{End}\big(\Lambda^\bullet T^*M\big)$ *generated by transposed Clifford multiplication. Then*

$$\widehat{\mathrm{Cl}}(T^*M) = \mathrm{End}_{\mathrm{Cl}}\big(\Lambda^\bullet T^*M\big),$$

where $\mathrm{End}_{\mathrm{Cl}}$ *is defined as in* (1.20) *with respect to the even/odd grading.*

We shall not include the proof which is straightforward but a bit tedious. Part (i) and (iii) of Lemma 1.2.18 can be found in [13, p. 144]. Part (ii) can be easily proved by induction on m. However, we want to point out that part (iii) implies that we have an isomorphism of $\mathbb{Z}_2$-graded algebras,

$$\mathrm{End}\big(\Lambda^\bullet T^*M\big) = \mathrm{Cl}(T^*M) \otimes_s \widehat{\mathrm{Cl}}(T^*M), \tag{1.28}$$

which is (1.21) translated to the case at hand. Moreover, part (ii) of Lemma 1.2.18 shows that the decomposition (1.22) in the case at hand is

$$(-1)^\nu = \tau \otimes \widehat{\tau}.$$

Remark. In the case that m is even, one might expect that the decomposition (1.21) for $\mathrm{End}\big(\Lambda^\bullet T^*M\big)$ with respect to the τ-grading is given by (1.28) together with the grading operator $\tau \otimes 1$. However, with respect to this, the endomorphism $\widehat{c}(\alpha)$ for $\alpha \in T^*M$ is even, and this is incompatible with part (i) of Lemma 1.2.18. To stay in the $\mathbb{Z}_2$-graded formalism, one would have to consider yet another kind of Clifford multiplication, namely

$$\widetilde{c} := \widehat{c} \circ (-1)^\nu : T^*M \to \mathrm{End}\big(\Lambda^\bullet T^*M\big).$$

This generates a subalgebra $\widetilde{\mathrm{Cl}}(T^*M)$ of $\mathrm{End}\big(\Lambda^\bullet T^*M\big)$ of purely even degree with respect to τ so that

$$\mathrm{End}\big(\Lambda^\bullet T^*M\big) = \mathrm{Cl}(T^*M) \otimes_s \widetilde{\mathrm{Cl}}(T^*M).$$

Fortunately, in the discussion to follow, we are interested only in elements of $\widetilde{\mathrm{Cl}}(T^*M)$, respectively $\widehat{\mathrm{Cl}}(T^*M)$, which are of even with respect to the even/odd grading. For elements of this form, $\widetilde{c}$ and $\widehat{c}$ coincide up to sign. More precisely, if $\{e_i\}$ is a local frame for TM, then for all $k \leq m/2$

$$\widetilde{c}^{i_1} \dots \widetilde{c}^{i_{2k}} = (-1)^k \widehat{c}^{i_1} \dots \widehat{c}^{i_{2k}}.$$

Hence, even if it is incorrect from a formal point of view, we use the transposed Clifford multiplication $\widehat{c}$ also in the case that $\Lambda^\bullet T^*M$ is graded by τ.

Traces of the Exterior Algebra. Let $\pi_0 : \mathrm{Cl}(T^*M) \to \mathbb{C}$ be the projection onto the subalgebra $\mathbb{C} \subset \mathrm{Cl}(T^*M)$. One easily verifies that $\big[\mathrm{Cl}(T^*M), \mathrm{Cl}(T^*M)\big] \cap \mathbb{C} = \{0\}$, so that we can define a trace on $\mathrm{Cl}(T^*M)$ by

$$\mathrm{tr}_{\mathrm{Cl}} := 2^{m/2}\pi_0 : \mathrm{Cl}(T^*M) \to \mathbb{C},$$

see [44, Thm. 1.8]. In the same way, we get a trace $\widehat{\mathrm{tr}_{\mathrm{Cl}}}$ on $\widehat{\mathrm{Cl}}(T^*M)$. Note that in the case that m is even, the natural supertrace of [13, Prop. 3.21] is given by

$$\mathrm{str}_{\mathrm{Cl}} := \mathrm{tr}_{\mathrm{Cl}} \circ \tau : \mathrm{Cl}(T^*M) \to \mathbb{C}.$$

Proposition 1.2.19. *With respect to the decomposition* (1.28) *we have*

$$\mathrm{tr}_{\Lambda^\bullet} = \mathrm{tr}_{\mathrm{Cl}} \otimes \widehat{\mathrm{tr}_{\mathrm{Cl}}},$$

where $\mathrm{tr}_{\Lambda^\bullet}$ *is the natural trace on* $\mathrm{End}(\Lambda^\bullet T^*M)$.

Remark. We include a proof, since the treatment in [13] considers only the case that m is even. There are some non-trivial sign difficulties involved, since (1.28) involves the graded tensor product, whereas $\mathrm{tr}_{\mathrm{Cl}}$ and $\widehat{\mathrm{tr}_{\mathrm{Cl}}}$ are traces rather than supertraces. Yet, for elements of pure degree,

$$\begin{aligned}\mathrm{tr}_{\mathrm{Cl}} \otimes \widehat{\mathrm{tr}_{\mathrm{Cl}}}\big((a\otimes\hat a)(b\otimes\hat b)\big) &= (-1)^{|\hat a||b|}\,\mathrm{tr}_{\mathrm{Cl}}(ab)\widehat{\mathrm{tr}_{\mathrm{Cl}}}(\hat a\hat b)\\ &= (-1)^{|\hat a||b|}\,\mathrm{tr}_{\mathrm{Cl}}(ba)\widehat{\mathrm{tr}_{\mathrm{Cl}}}(\hat b\hat a)\\ &= (-1)^{|\hat a||b|+|\hat b||a|}\,\mathrm{tr}_{\mathrm{Cl}} \otimes \widehat{\mathrm{tr}_{\mathrm{Cl}}}\big((b\otimes\hat b)(a\otimes\hat a)\big).\end{aligned} \tag{1.29}$$

Moreover, for $\mathrm{tr}_{\mathrm{Cl}}(ab)$ and $\widehat{\mathrm{tr}_{\mathrm{Cl}}}(\hat a\hat b)$ to be non-zero it is necessary that $|a| = |b|$ and $|\hat a| = |\hat b|$. In this case the sign in (1.29) is always $+1$. Hence, $\mathrm{tr}_{\mathrm{Cl}} \otimes \widehat{\mathrm{tr}_{\mathrm{Cl}}}$ is indeed a trace.

Proof of Proposition 1.2.19. Since the assertion is local, it suffices to consider an m-dimensional Euclidean vector space V. Let us first consider the case $V = \mathbb{R}$, and let e be a unit vector. Then

$$\Lambda^\bullet\mathbb{R} = \mathbb{C} \oplus \mathbb{C}e \cong \mathbb{C}^2.$$

Consider the following elements of $\mathrm{End}(\mathbb{C}^2)$

$$c := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \widehat{c} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad n := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathrm{Cl}(\mathbb{R}) \subset \mathrm{End}(\mathbb{C}^2)$ is the algebra generated by Id and c, and $\widehat{\mathrm{Cl}}(\mathbb{R})$ is generated by Id and $\widehat{c}$. Moreover, $c\widehat{c} = -\widehat{c}c = -n$, which implies that all monomials in c and $\widehat{c}$ have vanishing trace except $c^0\widehat{c}^0 = \mathrm{Id}$. In this case,

$$\mathrm{tr}(c^0\widehat{c}^0) = \mathrm{tr}(\mathrm{Id}) = 2 = \mathrm{tr}_{\mathrm{Cl}}(c^0)\widehat{\mathrm{tr}_{\mathrm{Cl}}}(\widehat{c}^0),$$

which yields the claimed formula in the case that $V = \mathbb{R}$.

We now assume that the claim holds for and m-dimensional vector space V, and want to prove it for $V \oplus \mathbb{R}$. Let $\{e_i\}$ be an orthonormal basis for V, and let e be a unit vector in $\mathbb{R}$. For an ordered multi-index $A = (i_1 < \ldots < i_k)$ we consider the following sets of generators

$$e_A := e_{i_1} \wedge \ldots \wedge e_{i_k} \in \Lambda^\bullet V, \quad c_A := c(e_{i_1}) \ldots c(e_{i_k}) \in \mathrm{Cl}(V),$$

and for $\alpha \in \{0,1\}$,

$$e_{A,\alpha} := e_A \wedge e^\alpha \in \Lambda^\bullet(V \oplus \mathbb{R}), \quad c_{A,\alpha} := c_A c^\alpha \in \mathrm{Cl}(V \oplus \mathbb{R}),$$

where $c := c(e)$. In the same way we define $\widehat{c}_A \in \widehat{\mathrm{Cl}}(V)$ and $\widehat{c}_{A,\alpha} \in \widehat{\mathrm{Cl}}(V \oplus \mathbb{R})$. Then a short computation shows that

$$\begin{aligned}\mathrm{tr}_{\Lambda^\bullet(V\oplus\mathbb{R})}(c_{A,\alpha}\widehat{c}_{B,\beta}) &= \sum_{|C|\le m}\sum_{\gamma\in\{0,1\}} \big\langle c_{A,\alpha}\widehat{c}_{B,\beta}e_{C,\gamma}\,,\, e_{C,\gamma}\big\rangle\\ &= \sum_{|C|\le m}\sum_{\gamma\in\{0,1\}} (-1)^{|B||\alpha|}\big\langle (c_A\widehat{c}_B)(c^\alpha\widehat{c}^\beta)e_{C,\gamma}\,,\, e_{C,\gamma}\big\rangle\\ &= \sum_{|C|\le m} (-1)^{|B||\alpha|+|C|(|\alpha|+|\beta|)}\big\langle (c_A\widehat{c}_B)e_C\,,\, e_C\big\rangle\,\mathrm{tr}_{\Lambda^\bullet\mathbb{R}}(c^\alpha\widehat{c}^\beta).\end{aligned}$$

From the case $m = 1$ we know that $\operatorname{tr}_{\Lambda^\bullet\mathbb{R}}(c^\alpha\widehat{c}^\beta) = 0$ if $(\alpha,\beta) \neq (0,0)$. In this case the factor in the last line above is $+1$ and so

$$\operatorname{tr}_{\Lambda^\bullet(V\oplus\mathbb{R})}(c_{A,\alpha}\widehat{c}_{B,\beta}) = \operatorname{tr}_{\Lambda^\bullet V}(c_A\widehat{c}_B)\operatorname{tr}_{\Lambda^\bullet\mathbb{R}}(c^\alpha\widehat{c}^\beta). \tag{1.30}$$

Now by induction we have

$$\operatorname{tr}_{\Lambda^\bullet V} = \operatorname{tr}_{\mathrm{Cl}(V)} \otimes \widehat{\operatorname{tr}_{\mathrm{Cl}(V)}}, \quad \operatorname{tr}_{\Lambda^\bullet\mathbb{R}} = \operatorname{tr}_{\mathrm{Cl}(\mathbb{R})} \otimes \widehat{\operatorname{tr}_{\mathrm{Cl}(\mathbb{R})}}. \tag{1.31}$$

Moreover, as in (1.29) one checks that with respect to

$$\mathrm{Cl}(V\oplus\mathbb{R}) \otimes_s \widehat{\mathrm{Cl}}(V\oplus\mathbb{R}) \cong \big(\mathrm{Cl}(V)\otimes_s \mathrm{Cl}(V)\big) \otimes_s \big(\widehat{\mathrm{Cl}}(\mathbb{R}) \otimes_s \widehat{\mathrm{Cl}}(\mathbb{R})\big)$$

one has

$$\operatorname{tr}_{\mathrm{Cl}(V\oplus\mathbb{R})} \otimes \widehat{\operatorname{tr}_{\mathrm{Cl}(V\oplus\mathbb{R})}} = \big(\operatorname{tr}_{\mathrm{Cl}(V)} \otimes \widehat{\operatorname{tr}_{\mathrm{Cl}(V)}}\big) \otimes \big(\operatorname{tr}_{\mathrm{Cl}(\mathbb{R})} \otimes \widehat{\operatorname{tr}_{\mathrm{Cl}(\mathbb{R})}}\big),$$

which together with (1.30) and (1.31) proves the assertion for $V \oplus \mathbb{R}$. □

Local Index Density and the Signature Theorem. As in the above proof let $V := T_x^*M$ for some $x \in M$, and let R be an element in the Lie algebra $\mathfrak{so}(V) \subset \mathrm{End}(V)$. Let $V_\mathbb{C}$ be the complexification of V. Then $iR \in \mathrm{End}(V_\mathbb{C})$ is a self-adjoint endomorphism, and we can define $\cosh(iR) \in \mathrm{End}(V_\mathbb{C})$ via the spectral theorem. Since the eigenvalues of iR are real, and cosh is a positive function on $\mathbb{R}$, we can define

$$\det{}^{1/2}\big(\cosh(iR)\big) := \sqrt{\det\big(\cosh(iR)\big)}.$$

It follows from the spectral theorem that

$$\det{}^{1/2}\big(\cosh(iR)\big) = \exp\big(\tfrac{1}{2}\operatorname{tr}\big[\log\cosh(iR)\big]\big),$$

which agrees with the definition in (A.1). Note, however, that the context here is slightly different since we are considering elements in $\mathfrak{so}(V)$ whereas in (A.1) we are considering elements of the algebra $(\Lambda^{\mathrm{ev}}\mathbb{C}^m) \otimes \mathrm{End}(V_\mathbb{C})$.

We then have the following version of [13, Lem. 4.5]

Lemma 1.2.20. *Let V be an m-dimensional oriented Euclidean vector space, and let $R \in \mathfrak{so}(V)$. Define*

$$\widehat{R}^S := -\tfrac{1}{4}\langle Re_i, e_j\rangle \widehat{c}^i\widehat{c}^j \in \widehat{\mathrm{Cl}}(V^*),$$

where $\{e_i\}$ is any orthonormal basis for V, and $\widehat{c}^i = \widehat{c}(e^i)$. Then

$$\widehat{\operatorname{tr}_{\mathrm{Cl}}}\big[\exp(i\widehat{R}^S)\big] = 2^{m/2}\det{}^{1/2}\big(\cosh(iR/2)\big).$$

Proof. Let $k \in \mathbb{N}$ be such that $m = 2k$ or $m = 2k+1$. Since $R \in \mathfrak{so}(V)$ we can find an orthonormal basis $\{e_i\}$ such that

$$R(e_{2j-1}) = \theta_j e_{2j}, \quad R(e_{2j}) = -\theta_j e_{2j-1}, \quad j = 1,\dots,k, \quad \text{and} \quad R(e_{2k+1}) = 0, \tag{1.32}$$

where the last condition has to be considered as empty if m is even. Then

$$\widehat{R}^S = -\tfrac{1}{2}\sum_{j\le k}\theta_j\widehat{c}^{2j-1}\widehat{c}^{2j}.$$

Since $c^{2i-1}\widehat{c}^{2i}$ and $c^{2j-1}\widehat{c}^{2j}$ commute for $i \neq j$, one finds

$$\exp(i\widehat{R}^S) = \prod_{j\leq k} \exp\left((-i\theta_j/2)\widehat{c}^{2j-1}\widehat{c}^{2j}\right) = \prod_{j\leq k}\Big(\cosh(\theta_j/2) - i\sinh(\theta_j/2)\widehat{c}^{2j-1}\widehat{c}^{2j}\Big),$$

where the last equality follows from the relation $(\widehat{c}^{2j-1}\widehat{c}^{2j})^2 = -1$. By definition of the trace on $\widehat{\mathrm{Cl}}(V^*)$, we find

$$\widehat{\mathrm{tr}_{\mathrm{Cl}}}\big[\exp(i\widehat{R}^S)\big] = 2^{m/2}\prod_{j\leq k}\cosh(\theta_j/2).$$

On the other hand, for all $z \in \mathbb{C}$

$$\cosh\left[\begin{pmatrix} 0 & -z \\ z & 0\end{pmatrix}\right] = \cos(z)\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \quad \cosh(0) = 1,$$

from which it follows that

$$\sqrt{\det\big(\cosh(iR/2)\big)} = \prod_{j\leq k}\sqrt{\cosh(\theta_j/2)^2} = \prod_{j\leq k}\cosh(\theta_j/2). \qquad \square$$

The next result can be found in [13, p. 145]. Although the treatment there is only for m even, one verifies without effort that it holds for odd m as well.

Lemma 1.2.21. *Let (M, g) be an oriented Riemannian manifold with Riemann curvature tensor $R^g \in \Omega^2\big(M, \mathrm{End}(TM)\big)$. In a local orthonormal frame $\{e_i\}$ for TM write*

$$R^g = \tfrac{1}{2}R^l_{kij}e^i \wedge e^j \otimes (e_l \otimes e^k), \quad \text{and} \quad R_{lkij} = g_{ln}R^n_{kij} = g\big(R(e_i, e_j)e_k, e_l\big).$$

Define $R^S \in \Omega^2\big(M, \mathrm{Cl}(T^*M)\big)$ as in (1.25), and

$$\widehat{R}^S := -\tfrac{1}{8}R_{lkij}e^i \wedge e^j \otimes \widehat{c}^k\widehat{c}^l \in \Omega^2\big(M, \widehat{\mathrm{Cl}}(T^*M)\big).$$

Then, the curvature $R^{\Lambda^\bullet T^*M}$ of the induced connection on $\Lambda^\bullet T^*M$ decomposes as

$$R^{\Lambda^\bullet T^*M} = R^S + \widehat{R}^S.$$

Lemma 1.2.20 extends to the case $R \in \Lambda^2 V \otimes \mathfrak{so}(V)$, where (A.1) is used to define the right hand side, see [13, pp. 144–146]. Then Lemma 1.2.20 and Lemma 1.2.21 imply the following

Proposition 1.2.22. *Let (M, g) be an oriented Riemannian manifold of dimension m, and let R^g be the Riemann curvature tensor. Then*

$$\widehat{A}(TM, \nabla^g) \wedge \widehat{\mathrm{tr}_{\mathrm{Cl}}}\big[\exp(\tfrac{i}{2\pi}\widehat{R}^S)\big] = 2^{m/2}\widehat{L}(TM, \nabla^g).$$

with the Hirzebruch $\widehat{L}$-form, see Definition A.1.4.

From Proposition 1.2.22 and Theorem 1.2.16, we obtain the index theorem for twisted signature operators.

Theorem 1.2.23 (Atiah-Singer, Hirzebruch)**.** *Let M be a closed, oriented Riemannian manifold of even dimension m. Let $E \to M$ be a Hermitian vector bundle with connection A.*

(i) *The index of the twisted signature operator* D_A^+ *is given by*

$$\operatorname{ind}(D_A^+) = 2^{m/2} \int_M \widehat{L}(TM, \nabla^g) \wedge \operatorname{ch}(E, A).$$

(ii) *If* A *is a flat connection, and* E *has rank* k, *then*

$$\operatorname{Sign}_A(M) = k \int_M L(TM, \nabla^g) = k \cdot \big\langle L(TM), [M] \big\rangle, \tag{1.33}$$

where $L(TM, \nabla^g)$ *is the Hirzebruch L-form as in* (A.4).

1.3 Manifolds with Boundary and the Eta Invariant

1.3.1 The Eta Function

Let (M, g) be a closed, oriented Riemannian manifold of dimension m. Let $E \to M$ be a Hermitian vector bundle, and let

$$D : C^\infty(M, E) \to C^\infty(M, E)$$

be a formally self-adjoint elliptic differential operator of first order, i.e., $D \in \mathscr{P}^1_{s,e}(M, E)$. As M is closed, the growth of the eigenvalues of D are controlled by (1.12). This allows us to make the following definition.

Definition 1.3.1. The *Eta function* of D is defined for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > m$ as

$$\eta(D, s) := \sum_{0 \neq \lambda \in \operatorname{spec}(D)} \frac{\operatorname{sgn}(\lambda)}{|\lambda|^s}.$$

Via a Mellin transform, the Eta function is related to the heat operator e^{-tD^2} in the following way

$$\eta(D, s) = \frac{1}{\Gamma\big(\frac{s+1}{2}\big)} \int_0^\infty \operatorname{Tr}\big(De^{-tD^2}\big) t^{\frac{s-1}{2}} dt, \quad \operatorname{Re}(s) > m, \tag{1.34}$$

where $\Gamma(s)$ is the Gamma function[2],

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt, \quad \operatorname{Re}(s) > 0.$$

Note that for (1.34) to exist we are using that De^{-tD^2} is trace class, and that there exist constants c and C such that for large t

$$|\operatorname{Tr}(De^{-tD^2})| \leq Ce^{-ct}.$$

This follows from Proposition 1.2.4, because using the notation introduced there, we have $\operatorname{Tr}(De^{-tD^2}) = \operatorname{Tr}\big(De^{-tD^2} P_{(0,\infty)}\big)$. Thus, (1.34) yields a holomorphic function in the half plane $\operatorname{Re}(s) > m$.

[2] See Appendix C.1 for some facts about the Gamma function which we will use freely.

As we have noted in Theorem 1.2.7, there is an asymptotic expansion

$$\operatorname{Tr}\left(De^{-tD^2}\right) \sim \sum_{n=0}^{\infty} t^{\frac{n-m-1}{2}} a_n(D), \quad \text{as } t \to 0, \tag{1.35}$$

where $a_n(D)$ is an integral over a quantity locally computable from the total symbol of D. Substituting the asymptotic expansion into (1.34) and dividing the integration into $\int_0^1 + \int_1^\infty$ one easily verifies that for each $N \geq 0$

$$\eta(D,s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \Big(\sum_{n=0}^{N} \frac{2a_n(D)}{n-m+s} + h_N(s) \Big), \tag{1.36}$$

where $h_N(s)$ is holomorphic in the half plane $\operatorname{Re}(s) > m - (N+1)$. Since $\Gamma\left(\frac{s+1}{2}\right)^{-1}$ is an entire function, one can use this to deduce

Proposition 1.3.2. *The Eta function $\eta(D,s)$ extends uniquely to a meromorphic function on the whole plane with possible simple poles for $s \in \{m - n \,|\, n \in \mathbb{N}\}$.*

Regularity at $s = 0$. We note that $\Gamma\left(\frac{s+1}{2}\right)^{-1}$ has no zeros to cancel the possible poles. This is an important difference between the Eta function and the Zeta function of, say, a generalized Laplacian, see e.g. [13, Prop. 9.35]. Therefore, the following result is very remarkable. For references we refer to Remark 1.3.5 below.

Theorem 1.3.3 (Atiyah-Patodi-Singer, Gilkey)**.** *Let D be a formally self-adjoint elliptic differential operator of first order on a closed, Riemannian manifold M. Then the Eta function $\eta(D,s)$ has no pole at $s = 0$. If D is a geometric Dirac operator, $\eta(D,s)$ is holomorphic for* $\operatorname{Re}(s) > -1/2$.

Definition 1.3.4. Using the result of Theorem 1.3.3, we can define the *Eta invariant* of D as

$$\eta(D) := \eta(D,0).$$

Moreover, we define the *ξ-invariant* and the *reduced ξ-invariant* by

$$\xi(D) := \frac{\eta(D) + \dim(\ker D)}{2}, \quad \text{and} \quad [\xi(D)] := \xi(D) \mod \mathbb{Z}.$$

Remark 1.3.5. To further stress the non-triviality of Theorem 1.3.3, we want to give some historical remarks.

(i) Atiyah, Patodi and Singer first deduced the regularity of the Eta function at 0 from their proof of the index theorem for elliptic differential operators of first order on manifolds with boundary, see [7, Thm. 3.10]. The improved regularity for geometric Dirac operators is [7, Thm. 4.2].

(ii) Later, the same authors generalized the result to pseudo-differential operators of arbitrary order on closed, odd dimensional manifolds using K-theoretic arguments and the regularity results for geometric Dirac operators, see [9, Thm. 4.5].

(iii) In [48], Gilkey was able to generalize Theorem 1.3.3 to the case of even dimensional manifolds, again for the much larger class of formally self-adjoint elliptic pseudo-differential operators, see also [49, Sec. 3.8].

(iv) Gilkey [47] also initiated the study of the regularity of the *local Eta function* of an elliptic operator D,

$$\eta(D, s, x) := \frac{1}{\Gamma\big(\frac{s+1}{2}\big)} \int_0^\infty \operatorname{tr}\big(k_t(x,x)\big) t^{\frac{s-1}{2}} \, dt, \tag{1.37}$$

where $k_t(x, y)$ is the kernel of De^{-tD^2}. Studying various examples, Gilkey found that $\eta(D, s, x)$ is in general not regular at $s = 0$.

(v) Later Bismut and Freed were able to refine the result of Atiyah-Patodi-Singer for geometric Dirac operators. They showed using local index theory techniques, that the local Eta function $\eta(D, s, x)$ of a geometric Dirac operator is holomorphic for $\mathrm{Re}(s) > -2$, see [18, Thm. 2.6]. Their result implies that for a geometric Dirac operator one can define the Eta invariant directly by

$$\eta(D) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}\big(De^{-tD^2}\big) dt. \tag{1.38}$$

Lemma 1.3.6. *Let M and N be closed, oriented Riemannian manifolds, with Hermitian vector bundles E over M and F over N. Let $D \in \mathscr{P}^1_{s,e}(M, E)$ and $B \in \mathscr{P}^1_{s,e}(N, F)$.*

(i) *Assume that there exists and isometry $\varphi : M \to N$, and a unitary bundle map $\Phi : E \to F$ covering φ such that*

$$\Phi \circ D = B \circ \Phi.$$

Then $\eta(D) = \eta(B)$. In particular, if $M = N$, $E = F$ and $\{D, \Phi\} = 0$, then $\eta(D) = 0$.

(ii) *If $M = N$, then*

$$\eta(D \oplus B) = \eta(D) + \eta(B).$$

(iii) *Assume that D is $\mathbb{Z}_2$-graded with grading operator σ on E. Consider the operator*

$$D \otimes 1 + \sigma \otimes B \quad \text{on} \quad C^\infty\big(M \times N, E \boxtimes F\big),$$

with the fiber product $E \boxtimes F$ as in (1.15). Then

$$\eta\big(D \otimes 1 + \sigma \otimes B\big) = \operatorname{ind}(D^+) \cdot \eta(B).$$

Sketch of proof. Part (i) and (ii) of the above result are immediate for $\eta(D, s)$ for $\mathrm{Re}(s)$ large, since the whole spectrum has the respective properties. By meromorphic continuation, they continue to hold for $s = 0$. We sketch a proof of (iii).

First note that $D \otimes 1$ and $\sigma \otimes B$ anti-commute as operators on $C^\infty\big(M \times N, E \boxtimes F\big)$. Therefore,

$$\big(D \otimes 1 + \sigma \otimes B\big)e^{-t(D\otimes 1+\sigma\otimes B)^2} = (D \otimes 1)e^{-t(D^2\otimes 1+1\otimes B^2)} + (\sigma \otimes B)e^{-t(D^2\otimes 1+1\otimes B^2)}.$$

To compute traces one may choose an orthonormal basis of $L^2\big(M \times N, E \boxtimes F\big)$ of the form $\{\varphi_i \otimes \psi_j\}$ with $\varphi_i \in C^\infty(M,E)$ and $\psi_j \in C^\infty(N,F)$. Then one easily finds that

$$\begin{aligned}\operatorname{Tr}\Big((D \otimes 1)e^{-t(D^2\otimes 1 + 1 \otimes B^2)}\Big) &+ \operatorname{Tr}\Big((\sigma \otimes B)e^{-t(D^2\otimes 1 + 1 \otimes B^2)}\Big) \\ &= \operatorname{Tr}_{L^2(M,E)}\big(De^{-tD^2}\big)\operatorname{Tr}_{L^2(N,F)}\big(e^{-tB^2}\big) \\ &\qquad + \operatorname{Str}_{L^2(M,E)}\big(e^{-tD^2}\big)\operatorname{Tr}_{L^2(N,F)}\big(Be^{-tB^2}\big).\end{aligned}$$

Since D is $\mathbb{Z}_2$-graded, $\sigma De^{-tD^2} = -De^{-tD^2}\sigma$ and thus,

$$\operatorname{Tr}\big(De^{-tD^2}\big) = \operatorname{Tr}\big(\sigma^2 De^{-tD^2}\big) = -\operatorname{Tr}\big(\sigma De^{-tD^2}\sigma\big) = -\operatorname{Tr}\big(De^{-tD^2}\big),$$

where we use the trace property in the last equality. Therefore, $\operatorname{Tr}\big(De^{-tD^2}\big) = 0$. Moreover, Theorem 1.2.6 asserts that for all $t > 0$,

$$\operatorname{Str}\big(e^{-tD^2}\big) = \operatorname{ind}(D^+).$$

We conclude that for $\operatorname{Re}(s)$ large,

$$\begin{aligned}\eta\big(D \otimes 1 + \sigma \otimes B, s\big) &= \frac{1}{\Gamma\big(\frac{s+1}{2}\big)}\int_0^\infty \operatorname{Str}\big(e^{-tD^2}\big)\operatorname{Tr}\big(Be^{-tB^2}\big)t^{\frac{s-1}{2}}\,dt \\ &= \operatorname{ind}(D^+)\cdot\eta(B,s).\end{aligned}$$

By meromorphic continuation, part (iii) follows. $\square$

The Rho Function. We now want to use the local nature of the coefficients $a_n(D)$ appearing in (1.36) to introduce the Rho function—respectively, the Rho invariant.

Definition 1.3.7. Let $D \in \mathscr{P}^1_{s,e}(M,E)$, and let A be a flat connection on a Hermitian vector bundle of rank k. Denote by D_A the operator D twisted by A, and use the notation $D^{\oplus k}$ for the operator D twisted by the trivial flat bundle $\mathbb{C}^k$. We then define the *Rho function* of D_A as

$$\rho(D_A, s) = \eta(D_A, s) - \eta(D^{\oplus k}, s), \quad s \in \mathbb{C}.$$

Moreover, we define the *Rho invariant* of D_A as

$$\rho(D_A) := \rho(D_A, 0).$$

From Theorem 1.3.3 we know that the meromorphic functions $\eta(D_A,s)$ and $\eta(D^{\oplus k},s)$ have no pole in 0. Thus, the Rho invariant is well-defined. However, unlike in the case for the individual Eta invariants, this already follows from the local nature of heat trace asymptotics.

Proposition 1.3.8. *Let D_A and $D^{\oplus k}$, where A is a flat $\mathrm{U}(k)$-connection. Then the Rho function is holomorphic on the whole plane, and for all $s \in \mathbb{C}$, we have*

$$\rho(D_A, s) = \frac{1}{\Gamma\big(\frac{s+1}{2}\big)}\int_0^\infty \big[\operatorname{Tr}\big(D_A e^{-tD_A^2}\big) - \operatorname{Tr}\big(D^{\oplus k}e^{-t(D^{\oplus k})^2}\big)\big]t^{\frac{s-1}{2}}\,dt, \tag{1.39}$$

in particular,

$$\rho(D_A) = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\big[\operatorname{Tr}\big(D_A e^{-tD_A^2}\big) - \operatorname{Tr}\big(D^{\oplus k}e^{-t(D^{\oplus k})^2}\big)\big]dt.$$

Proof. For $\mathrm{Re}(s) > m$, we already know from (1.34) that $\rho(D_A, s)$ is holomorphic and that (1.39) is the correct formula. Moreover, as for the Eta function, we can split up the integral and use that $\int_1^\infty$ extends to a holomorphic function on $\mathbb{C}$. Concerning $\int_0^1$, we now make the following observation: As we have already seen in the proof of Corollary 1.2.9, the operators D_A and $D^{\oplus k}$ are locally equivalent. Then Theorem 1.2.7 implies that all coefficients of the asymptotic expansions of $\mathrm{Tr}\left(D_A e^{-tD_A^2}\right)$ and $\mathrm{Tr}\left(D^{\oplus k} e^{-t(D^{\oplus k})^2}\right)$ as $t \to 0$ agree, since they are local in the total symbols of the involved operators. Thus, for all N there exists a constant C such that as $t \to 0$

$$\left| \mathrm{Tr}\left(D_A e^{-tD_A^2}\right) - \mathrm{Tr}\left(D^{\oplus k} e^{-t(D^{\oplus k})^2}\right)\right| \leq Ct^N.$$

This shows that the integral

$$\int_0^1 \left[\mathrm{Tr}\left(D_A e^{-tD_A^2}\right) - \mathrm{Tr}\left(D^{\oplus k} e^{-t(D^{\oplus k})^2}\right)\right] t^{\frac{s-1}{2}}\, dt$$

exists and defines a holomorphic function for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -(N+1)$. Continuing in this way, one finds that $\rho(D_A, s)$ is holomorphic for $s \in \mathbb{C}$. $\square$

1.3.2 Variation of the Eta Invariant

While the Eta invariant is a spectral invariant and thus encodes global information about the manifold and the operator, its deformation theory turns out to be expressible in terms of heat trace asymptotics, thus being a local quantity. We have included further details in Appendix D, and summarize only briefly what we need here.

Families of Operators. Let M be a closed manifold of dimension m, and let $E \to M$ be a Hermitian vector bundle. Let $U \in \mathbb{R}^p$ be open, and let $(D_u)_{u \in U}$ be a p-parameter family of differential operators on $C^\infty(M, E)$ of order d. Choosing local frames for E we can write locally

$$D_u = \sum_{|\alpha| \leq d} a_\alpha(x, u) \partial_x^\alpha, \tag{1.40}$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multi-index, $x = (x_1, \dots, x_m)$ is a local coordinate chart on M, and the $a_\alpha(x, u)$ are matrix valued functions.

Definition 1.3.9. The p-parameter family $(D_u)_{u \in U}$ is called *smooth* if for every local frame and coordinate chart, the functions $a_\alpha(x, u)$ as in (1.40) are smooth, jointly in x and u.

We also need a functional analytic consequence of the geometric notion of smoothness we have just defined. Let $s, s' \in \mathbb{R}$ and denote by $\mathscr{B}(L^2_s, L^2_{s'})$ the space of bounded linear operators $L^2_s \to L^2_{s'}$ endowed with the operator norm $\|.\|_{s,s'}$. The proof of the following result is straightforward.

Lemma 1.3.10. *Let $(D_u)_{u \in U}$ be a p-parameter family of differential operator of order $d \geq 0$ which is smooth in the sense of Definition 1.3.9. Then for all $s \in \mathbb{R}$, we have a smooth map*

$$U \to \mathscr{B}(L^2_{s+d}, L^2_s), \quad u \mapsto D_u.$$

For more details on the following results we refer to Appendix D, in particular Proposition D.2.4 and Proposition D.2.5.

Proposition 1.3.11. *Let $(D_u)_{u\in\mathbb{R}}$ be a smooth one-parameter family of formally self-adjoint elliptic operators of first order on $C^\infty(M,E)$, and let $a_m(\frac{dD_u}{du}, D_u^2)$ denote the constant term in the asymptotic expansion of*

$$\sqrt{t}\ \mathrm{Tr}\left(\tfrac{dD_u}{du} e^{-tD_u^2}\right), \quad \text{as } t \to 0. \tag{1.41}$$

Then the following holds.

(i) *Assume that $\dim(\ker D_u)$ is constant. Then the meromorphic extension of $\eta(D_u,s)$ is continuously differentiable in u, and*

$$\tfrac{d}{du}\eta(D_u) = -\tfrac{2}{\sqrt{\pi}}\, a_m(\tfrac{dD_u}{du}, D_u^2). \tag{1.42}$$

(ii) *Without the assumption on $\ker(D_u)$, the reduced ξ-invariant $[\xi(D_u)] \in \mathbb{R}/\mathbb{Z}$ is continuously differentiable in u, and*

$$\tfrac{d}{du}[\xi(D_u)] = -\tfrac{1}{\sqrt{\pi}}\, a_m(\tfrac{dD_u}{du}, D_u^2). \tag{1.43}$$

An immediate consequence of the local nature of $a_m(\frac{dD_u}{du}, D_u^2)$ is the following result, see [8, Thm. 3.3].

Corollary 1.3.12. *Let $(D_u)_{u\in\mathbb{R}}$ be a smooth one-parameter family of operators in $\mathscr{P}^1_{s,e}(M,E)$, and let A be a flat connection on a Hermitian vector bundle of rank k. Denote by $(D_{A,u})_{u\in\mathbb{R}}$ the one-parameter family of operators obtained by twisting with A.*

(i) *If the kernels of D_u and $D_{A,u}$ are of constant dimensions, then the Rho invariant $\rho(D_{A,u})$ is independent of u.*

(ii) *In the general case, only the reduced Rho invariant,*

$$\big[\rho(D_{A,u})\big] := \big[\xi(D_{A,u})\big] - k\cdot\big[\xi(D_u)\big],$$

is independent of u.

Proof. As in the proof of Proposition 1.3.8, the operators $D_{A,u}$ and $D_u^{\oplus k}$ are locally equivalent smooth families in the sense of (1.17). Since the twisting connection is independent of u, the families $\big(\frac{dD_u^{\oplus k}}{du}\big)_{u\in\mathbb{R}}$ and $\big(\frac{dD_{A,u}}{du}\big)_{u\in\mathbb{R}}$ are locally equivalent as well. Now, the a_n in the asymptotic expansion of (1.41) can be computed locally from the total symbols of the involved operators, and so

$$a_m\big(\tfrac{dD_u^{\oplus k}}{du}, (D_u^{\oplus k})^2\big) = a_m\big(\tfrac{dD_{A,u}}{du}, (D_{A,u})^2\big).$$

Since $\eta(D_u^{\oplus k}) = k\cdot\eta(D_u)$, the result follows from (1.42) respectively (1.43). $\square$

Remark. One can also use Proposition 1.3.8 and proceed as in Appendix D to differentiate under the integral to proof part (i) in Corollary 1.3.12. This has the advantage that it is a bit less involved than the proof of the variation formula of the individual Eta invariant, since one does not have to deal with meromorphic continuations. However, the main steps remain the same.

Spectral Flow. The smoothness of $[\xi(D_u)]$ shows that the discontinuities of $\xi(D_u)$ as u varies are only integer jumps. Heuristically, this is due to the fact that the Eta invariant is a regularized signature so that whenever an eigenvalue of D_u crosses 0, it changes by an integer multiple of 2. This can be made more precise using the notion of *spectral flow*, which we now introduce briefly.

For a smooth one-parameter family of formally self-adjoint elliptic operators $(D_u)_{u\in[a,b]}$, it can be shown that the associated family of compact resolvents varies smoothly with u in the operator norm on $L^2(M,E)$, see the proof of Theorem D.1.7 for some related ideas. This implies that the eigenvalues of D_u can be arranged in such a way that they vary continuously with u. In particular, we can find a partition $a = u_0 < u_1 < \ldots < u_n = b$ such that for each $i \in \{1,\ldots,n\}$ there is $c_i > 0$ with

$$c_i \notin \bigcup_{u\in[u_{i-1},u_i]} \operatorname{spec}(D_u). \tag{1.44}$$

For $u \in [u_{i-1}, u_i]$ denote by $P_{[0,c_i]}(u)$ the finite-rank spectral projection associated to eigenvalues in the interval $[0, c_i]$. We then define the spectral flow in the spirit of [82].

Definition 1.3.13. Let $(D_u)_{u\in[a,b]}$ be a smooth one-parameter family in $\mathscr{P}^d_{s,e}(M,E)$, and let $a = u_0 < u_1 < \ldots < u_n = b$ be a partition such that there exist c_i as in (1.44). Then the *spectral flow* between D_a and D_b is defined as

$$\mathrm{SF}(D_u)_{u\in[a,b]} := \sum_{i=1}^{n} \operatorname{rk}\left(P_{[0,c_i]}(u_i)\right) - \operatorname{rk}\left(P_{[0,c_i]}(u_{i-1})\right).$$

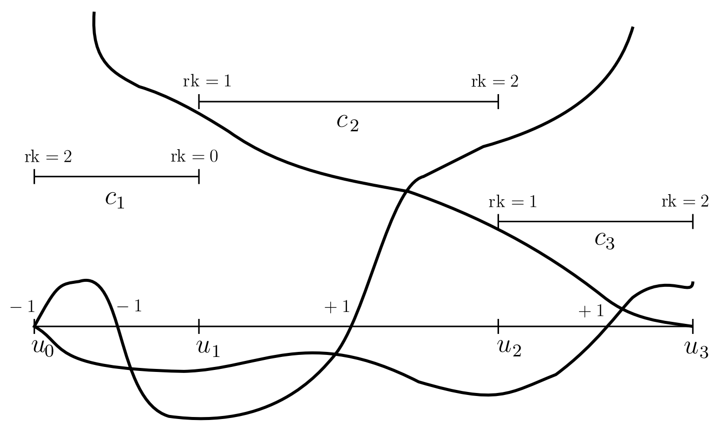

Figure 1.1: Spectral Flow

Without going into further details, we note that the spectral flow is well-defined and independent of the choices made. Moreover, as indicated in Figure 1.1, there is a built-in convention of how to count zero eigenvalues at the end points. For more details and generalizations we refer to [20, 65, 82]. In the situation at hand, there is also the approach as in [89] using Kato's selection theorem.

The Variation Formula. With the notion of spectral flow at hand, we can now state a result on the variation of the Eta invariant, due to Atiyah, Patodi and Singer in [9]. We sketch a proof in Corollary D.2.6, see also [61, Lem. 3.4].

Proposition 1.3.14. *Let $(D_u)_{u\in[a,b]}$ be a smooth one-parameter family of operators in $\mathscr{P}^1_{e,s}(M,E)$, where M is closed. Then*

$$\xi(D_b) - \xi(D_a) = \mathrm{SF}(D_u)_{u\in[a,b]} + \int_a^b \tfrac{d}{du}[\xi(D_u)]du.$$

1.3.3 The Atiyah-Patodi-Singer Index Theorem

To relate the Eta invariant to the index theorem, we have to leave the realm of closed manifolds briefly.

Product Structures at the Boundary. Let N be compact manifold with boundary $\partial N = M$. We equip N with a metric g_N of *product type* near the boundary, i.e., we assume that a collar of the boundary is isometric to $(-1,0]\times M$ endowed with the metric $g = dt^2 + g_M$, where g_M is a metric on M. If N is oriented, we get an induced orientation on $(-1,0]\times M$ which we use to orient M. This is the *outward normal first* convention, see Figure 1.2. Atiyah, Patodi and Singer in [7] use a different convention, which explains some sign differences.

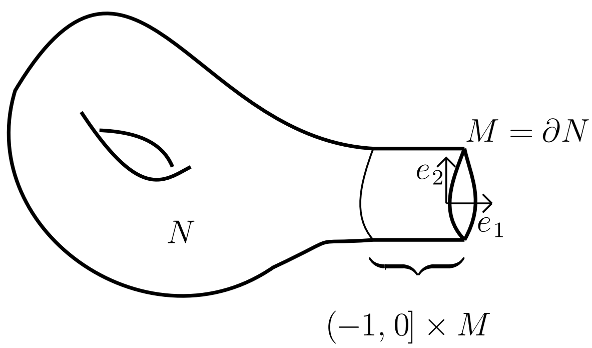

Figure 1.2: Collar and orientation convention

Definition 1.3.15. A $\mathbb{Z}_2$-graded, formally self-adjoint elliptic differential operator of first order $D : C^\infty(N,E)\to C^\infty(N,E)$ is called *in product form* if the following holds.

(i) On a collar of M

$$E^+|_{(-1,0]\times M} = \pi^* E_M,$$

where $\pi : (-1,0]\times M \to M$ is the projection, and E_M is a Hermitian vector bundle over M.

(ii) If γ is the bundle isomorphism $E^+|_M \to E^-|_M$, given by applying the symbol of D^+ to the outward normal unit vector, then

$$D^+ = \gamma\big(\tfrac{d}{dt} - D_M\big), \quad \text{over} \quad (-1,0]\times M,$$

where D_M is a formally self-adjoint elliptic differential operator of first order on E_M, called the *tangential operator*.

An operator of the above form can be extended canonically to an operator on the *closed double*

$$X = N \cup_M -N.$$

Here, $-N$ denotes N with the reversed orientation. One uses γ to glue $E|_{\partial N}$ to $E|_{\partial(-N)}$ to get a bundle $E_X \to X$, and the operator D extends naturally over X. For more details and a detailed proof of the following we refer to [21, Thm. 9.1].

Proposition 1.3.16. *If D is in product form, then the natural extension*

$$D_X : C^\infty(X, E_X) \to C^\infty(X, E_X)$$

is invertible and extends D in the sense that $D_X|_{C^\infty(N,E)} = D$.

Boundary Conditions and the Index Theorem. To formulate an index theorem for D one needs to introduce suitable boundary conditions in order to render D Fredholm. As observed by Atiyah and Bott [4], most geometric Dirac operators do not admit local boundary conditions that set up a well-posed boundary value problem. Atiyah, Patodi and Singer [7] solved this by introducing the following global boundary projection.

Definition 1.3.17. Let $D \in \mathscr{P}^1_{s,e}(N, E)$ be $\mathbb{Z}_2$-graded and in product form, with tangential operator D_M. The *Atiyah-Patodi-Singer projection*

$$P_{\geq}(D_M) : C^\infty(M, E_M) \to C^\infty(M, E_M)$$

is defined as the spectral projection onto the subspace spanned by the eigenvectors of D_M corresponding to eigenvalues ≥ 0.

Then the index theorem for manifolds with boundary in [7, Thm. 3.10] reads

Theorem 1.3.18 (Atiyah-Patodi-Singer)**.** *Let N be compact, oriented Riemannian manifold of dimension n with boundary $\partial N = M$, and let*

$$D : C^\infty(N, E) \to C^\infty(N, E)$$

be a $\mathbb{Z}_2$-graded, formally self-adjoint elliptic differential operator of first order. Assume that the metric and D are in product form in a collar of M. Let

$$C^\infty\big(N, E^+; P_{\geq}\big) := \big\{\varphi \in C^\infty(N, E^+) \;\big|\; P_{\geq}(D_M)(\varphi|_M) = 0\big\},$$

where $P_{\geq}(D_M)$ is the projection of Definition 1.3.17. Then

$$D^+ : C^\infty\big(N, E^+; P_{\geq}\big) \to C^\infty\big(N, E^-\big)$$

has a natural Fredholm extension with

$$\operatorname{ind}\big(D^+; P_{\geq}\big) = \int_N \operatorname{str}_E[\widetilde{e}_n] \operatorname{vol}_N - \xi(D_M). \tag{1.45}$$

Here, $\xi(D_M)$ is the ξ-invariant as in Definition 1.3.4, and $\widetilde{e}_n(x)$ is the coefficient of the constant term in the asymptotic expansion as $t \to 0$ of the heat kernel $e^{-tD_X^2}(x, x)$ associated to the extension D_X to the closed double X.

Remark. We are following [7] and use the operator D_X on the closed double to define the index density. Clearly, this is an ad hoc method which allows to use the asymptotic expansion of the heat trace in its version for closed manifolds as in Theorem 1.14. However, the trace expansion can be formulated in a much more abstract functional analytic setting to incorporate the case of manifolds with boundary, see [24] for an expository account.

If $\dim N$ is even and D is a $\mathbb{Z}_2$-graded geometric Dirac operator, the index density can be made explicit using the local index theorem for closed manifolds. One obtains the index theorem for geometric Dirac operators on manifolds with boundary, see [7, Thm. 4.2].

Theorem 1.3.19 (Atiyah-Patodi-Singer)**.** *Let N be compact, oriented Riemannian manifold of even dimension n with boundary $\partial N = M$, and let $D : C^\infty(N,E) \to C^\infty(N,E)$ be a $\mathbb{Z}_2$-graded, geometric Dirac operator which is in product form near M. Then the index of*

$$D^+ : C^\infty\big(N, E^+; P_\geq\big) \to C^\infty\big(N, E^-\big)$$

is given by

$$\operatorname{ind}\big(D^+; P_\geq\big) = \int_N \widehat{A}(TN, \nabla^g) \wedge \operatorname{ch}_{E/S}(E, \nabla^E) - \xi(D_M), \tag{1.46}$$

where $\widehat{A}(TN,\nabla^g)$ is the $\widehat{A}$-form of Definition A.1.4 with respect to the Levi-Civita connection ∇^g and $\operatorname{ch}_{E/S}$ is the realtive Chern character (1.26).

1.4 The Atiyah-Patodi-Singer Rho Invariant

In Proposition 1.1.8 we have seen that the signature of closed manifolds equals the index of the signature operator. One motivation that led Atiyah, Patodi and Singer to the discovery of Theorem 1.3.18 was the search for a generalization of this to the case for manifolds with boundary. We thus briefly recall the definition of the signature of a manifold with boundary.

1.4.1 The Signature of Manifolds with Boundary

Let N be a compact, oriented and connected manifold of dimension n with boundary ∂N. Then ∂N is closed and naturally oriented, but we allow that it consists of several connected components. Let $\alpha : \pi_1(N) \to \mathrm{U}(k)$ be a representation of the fundamental group. Via the map induced by the inclusion $\partial N \hookrightarrow N$, the representation α restricts to $\pi_1(\partial N)$. There is a relative version $H^\bullet(N, \partial N, E_\alpha)$ of cohomology with local coefficients, whose construction we will not describe in detail. We only note that the machinery of algebraic topology—like cup and cap products, Poincaré duality, and exact sequence of pairs—extends to this context[3]. In particular, there is a relative intersection pairing

$$H^p(N, E_\alpha) \times H^{n-p}(N, \partial N, E_\alpha) \to \mathbb{C}, \quad (a,b) \mapsto \big\langle a \cup b\,,\, [N, \partial N]\big\rangle,$$

where the fundamental class $[N, \partial N]$ is the generator of $H_n(N, \partial N)$ determined by the orientation. Via the de Rham isomorphism, the relative cohomology groups are isomorphic to de Rham cohomology with compact support in the interior of N,

$$H^p(N, \partial N, E_\alpha) \cong H^p_c(N, E_\alpha).$$

[3]However, one has to be careful with the Mayer-Vietoris sequence and excision, where one has to use van Kampen's Theorem to relate the representations of the involved fundamental groups.

Here, we also use E_α to denote the flat vector bundle determined by α. With respect to this identification, the intersection pairing is induced by

$$\Omega^p(N, E_\alpha) \times \Omega_c^{n-p}(N, E_\alpha) \mapsto \mathbb{C}, \quad (\omega, \eta) \mapsto \int_N \langle \omega \wedge \eta \rangle.$$

Now assume that n is even. There is a natural map $H^{n/2}(N, \partial N, E_\alpha) \to H^{n/2}(N, E_\alpha)$ which we can combine with the intersection pairing to define a *twisted intersection form*

$$Q_\alpha : H^{n/2}(N, \partial N, E_\alpha) \times H^{n/2}(N, \partial N, E_\alpha) \to \mathbb{C}.$$

As for closed manifolds, the intersection form is skew for $(n \equiv 2 \mod 4)$ and symmetric for $(n \equiv 0 \mod 4)$, but it is in general degenerate.

Definition 1.4.1. Let N be a compact, connected manifold with boundary of even dimension n, and let $\alpha : \pi_1(N) \to \mathrm{U}(k)$ be a representation of the fundamental group . Then the *twisted signature* of M is defined as

$$\mathrm{Sign}_\alpha(N) := \mathrm{Sign}(Q_\alpha),$$

where we use the earlier convention regarding the signature of a skew endomorphism.

Remark. To turn Q_α into a non-degenerate form, one needs to restrict to

$$\widehat{H}^{n/2}(N, E_\alpha) := \mathrm{im}\big(H^{n/2}(N, \partial N, E_\alpha) \to H^{n/2}(N, E_\alpha)\big).$$

Then Poincaré duality ensures that we get a non-degenerate form

$$\widehat{Q}_\alpha : \widehat{H}^{n/2}(N, E_\alpha) \times \widehat{H}^{n/2}(N, E_\alpha) \to \mathbb{C}.$$

Since this process just eliminates the radical of Q_α, it is immediate that

$$\mathrm{Sign}\, Q_\alpha = \mathrm{Sign}\, \widehat{Q}_\alpha.$$

1.4.2 Twisted Odd Signature Operators

To state the relation of the signature on a manifold with boundary to the index of the signature operator, we first need to understand the structure of the signature operator near the boundary. Therefore, we now consider the model case of a cylinder, and derive the formula for the odd signature operator.

Let M be a closed, oriented manifold of odd dimension m and consider the cylinder $N := \mathbb{R} \times M$. We use the natural splitting $T^*N = \mathbb{R} \oplus T^*M$ to orient N and make the identification

$$\Phi : C^\infty\big(\mathbb{R}, \Omega^\bullet(M)\big) \oplus \mathbb{C}^\infty\big(\mathbb{R}, \Omega^\bullet(M)\big) \xrightarrow{\cong} \Omega^\bullet(N), \quad \Phi(\omega_0, \omega_1) := dt \wedge \omega_0 + \omega_1,$$

where t denotes the $\mathbb{R}$ coordinate. Clearly,

$$d_N \Phi\big(\omega_0, \omega_1\big) = dt \wedge \big(\partial_t \omega_1 - d_M \omega_0\big) + d_M \omega_1 = \Phi\big(\partial_t \omega_1 - d_M \omega_0, d_M \omega_1\big). \tag{1.47}$$

We now endow N with a metric of product form $g = dt^2 + g_M$, and denote by τ_N and τ_M the chirality operators on $\Omega^\bullet(N)$ and $\Omega^\bullet(M)$, respectively. Then one checks that

$$\tau_N \Phi\big(\omega_0, \omega_1\big) = \Phi\big(\tau_M \omega_1, \tau_M \omega_0\big).$$

From this we obtain isomorphisms

$$\Phi_\pm : C^\infty\big(\mathbb{R}, \Omega^\bullet(M)\big) \xrightarrow{\cong} \Omega^\pm(N), \quad \omega \mapsto \Phi(\omega, \pm\tau_M\omega).$$

Let $D_N^+ : \Omega^+(N) \to \Omega^-(N)$ be the signature operator. Then a short computation using (1.47) and the formula (1.7) for the adjoint differential yields

$$\Phi_-^{-1} \circ D_N^+ \circ \Phi_+ = \tau_M\big(\partial_t - \tau_M d_M - d_M \tau_M\big).$$

The same continues to hold if $E \to M$ is a Hermitian vector bundle, and when we twist with a unitary connection A on π^*E in *temporal gauge*, i.e., a connection of the form $A := \pi^*a$. Here, $\pi : N \to M$ is the natural projection and a is a unitary connection on E. We summarize what we have observed so far.

Proposition 1.4.2. *Let D_A^+ be the signature operator on the cylinder $N = \mathbb{R} \times M$ twisted by a unitary connection $A = \pi^*a$ in temporal gauge. Then D_A^+ is isometric to*

$$\tau_M\big(\partial_t - B_a\big) : C^\infty\big(\mathbb{R}, \Omega^\bullet(M,E)\big) \to C^\infty\big(\mathbb{R}, \Omega^\bullet(M,E)\big),$$

where

$$B_a := \tau_M\big(d_a + d_a^t\big) = \tau_M d_a + d_a \tau_M. \tag{1.48}$$

Definition 1.4.3. Let a be a unitary connection over an oriented Riemannian manifold M of odd dimension. The operator

$$B_a^{\mathrm{ev}} := B_a|_{\Omega^{\mathrm{ev}}(M,E)} : \Omega^{\mathrm{ev}}(M,E) \to \Omega^{\mathrm{ev}}(M,E)$$

is called the *odd signature operator* on M twisted by a.

Remark 1.4.4.

(i) Note that the operator B_a does indeed preserve the even/odd grading,

$$B_a = B_a^{\mathrm{ev}} \oplus B_a^{\mathrm{odd}} : \Omega^{\mathrm{ev}}(M,E) \oplus \Omega^{\mathrm{odd}}(M,E) \to \Omega^{\mathrm{ev}}(M,E) \oplus \Omega^{\mathrm{odd}}(M,E).$$

Moreover, B_a^{ev} and B_a^{odd} are conjugate via τ_M. The kernels of B_a and B_a^{ev} are of a topological nature, since

$$\ker(B_a) = \ker(d_a + d_a^t) = \mathscr{H}^\bullet(M, E_a), \quad \ker(B_a^{\mathrm{ev}}) = \mathscr{H}^{\mathrm{ev}}(M, E_a).$$

(ii) It is straightforward to check that the odd signature operator is a geometric Dirac operator in the sense of Definition 1.2.11. Here, one defines the Clifford structure by

$$c^{\mathrm{ev}} : T^*M \otimes \Lambda^{\mathrm{ev}} T^*M \to \Lambda^{\mathrm{ev}} T^*M, \quad c^{\mathrm{ev}}(\xi)\omega := \tau_M\big(\xi \wedge \omega - \mathrm{i}(\xi)\omega\big). \tag{1.49}$$

Paying close attention to the various identifications made, one also verifies that the map

$$\tau_M : \Lambda^\bullet T^*M \to \Lambda^\bullet T^*M$$

corresponds to Clifford multiplication

$$c_N(dt) : \Lambda^+ T^*N|_M \to \Lambda^- T^*N|_M.$$

Hence, Proposition 1.4.2 shows that D_A^+ is of product form in the sense of Definition 1.3.15.

(iii) To give an explicit formula, let $m = 2k - 1$. Then one can check, using the formula (1.6) for τ_M, that for all $\omega \in \Omega^p$

$$B_a\omega = -i^{k+p(p+1)}\big((-1)^p * d_a - d_a *\big)\omega.$$

In particular for $p = 2q$,

$$B_a^{\mathrm{ev}}\omega = i^k(-1)^{q+1}(*d_a - d_a*)\omega.$$

1.4.3 The Signature Theorem for Manifolds with Boundary

Having identified the tangential operator, the APS projection sets up a well-defined index problem for the signature operator on manifolds with boundary. Moreover, the Atiyah-Patodi-Singer Index Theorem for geometric Dirac operators 1.3.19 and the index theorem for the twisted signature operator in Theorem 1.2.23 calculates its index. If the twisting bundle is flat, this index is related to the signature for manifolds with boundary as introduced in Section 1.4.1. However, this relation is more difficult to establish than in Proposition 1.1.8 for the case that the manifold is closed. We only state the result and refer to [7, Sec. 4] for the proof. A concise discussion also be found in [21, Sec. 23].

Theorem 1.4.5 (Atiyah-Patodi-Singer). *Let N be a compact, oriented Riemannian manifold of even dimension n with boundary $\partial N = M$, and let $E \to N$ be a Hermitian vector bundle of rank k with a unitary connection A. Assume that the metric is in product form, and that A is in temporal gauge $A = \pi^*a$ on a collar of M. Let D_A^+ be the twisted signature operator on N, and let*

$$P_\geq(a) : \Omega^\bullet(M, E|_M) \to \Omega^\bullet(M, E|_M)$$

be the APS projection in Definition 1.3.17 of the tangential operator B_a. Then

$$\operatorname{ind}\big(D_A^+; P_\geq(a)\big) = 2^{n/2}\int_N \widehat{L}(TN, \nabla^g) \wedge \operatorname{ch}(E, A) - \xi(B_a).$$

Moreover, if A is flat, then

$$\operatorname{ind}\big(D_A^+; P_\geq(a)\big) = \operatorname{Sign}_A(N) - \tfrac{1}{2}\dim(\ker B_a),$$

and therefore,

$$\operatorname{Sign}_A(N) = k \cdot \int_N L(TN, \nabla^g) - \eta(B_a^{\mathrm{ev}}).$$

Remark. In the last equation, the occurrence of B_a^{ev} stems from the relation

$$\xi(B_a) = \eta(B_a^{\text{ev}}) + \tfrac{1}{2}\dim(\ker B_a),$$

which follows from Remark 1.4.4 (i).

Rho Invariants. Motivated by Theorem 1.4.5, Atiyah, Patodi and Singer [8] introduced the Rho invariant, which we have already briefly considered in Proposition 1.3.8 and Corollary 1.3.12. We now treat the Rho invariant associated to the odd signature operator in more detail.

Definition 1.4.6. Let M be a closed, oriented Riemannian manifold of odd dimension m, and let A be a flat unitary connection on a Hermitian bundle E of rank k. Then the *Rho invariant of A* is defined as

$$\rho_A(M) := \rho(B_A^{\text{ev}}) = \eta(B_A^{\text{ev}}) - k \cdot \eta(B^{\text{ev}}).$$

We have the following immediate consequences of what we have discussed so far.

Proposition 1.4.7.

(i) *If A' is a flat connection, unitarily equivalent to A, then $\rho_{A'}(M) = \rho_A(M)$. In particular, the Rho invariant depends only on the holonomy representation*

$$\text{hol}_A : \pi_1(M) \to \mathrm{U}(k).$$

For this reason, we also use the notation $\rho_\alpha(M)$ if the focus is on representations of the fundamental group.

(ii) *The Rho invariant is independent of the metric used to define $\eta(B_A^{\text{ev}})$ and $\eta(B^{\text{ev}})$. Therefore, it is a smooth invariant of M and A.*

(iii) *If N is a compact, oriented manifold with boundary $\partial N = M$, and the representation $\alpha : \pi_1(M) \to \mathrm{U}(k)$ extends to a unitary representation $\beta : \pi_1(N) \to \mathrm{U}(k)$, then*

$$\rho_\alpha(M) = \mathrm{Sign}_\beta(N) - k \cdot \mathrm{Sign}(N).$$

Proof. Part (i) follows from the fact that the Eta invariant does not change, if we transform with a unitary bundle isomorphism. Part (ii) is a consequence of Corollary 1.3.12 (i), since the dimensions of the kernels of B_α^{ev} and B^{ev} are independent of the metric. Part (iii) follows immediately from the signature formula of Theorem 1.4.5. □

Remark 1.4.8.

(i) Part (i) of Proposition 1.4.7 shows that the Rho invariant can be interpreted as a map, defined on the moduli space of flat connections

$$\rho : \mathcal{M}\big(M, \mathrm{U}(k)\big) \to \mathbb{R}, \quad [A] \mapsto \rho_A(M).$$

In the explicit examples in Section 2.3 and Chapter 4 we will use this and consider particularly well-suited representatives for gauge equivalence classes of flat connections.

(ii) Proposition 1.4.7 (iii) gives a negative answer to the question of whether Corollary 1.2.10 continues to hold for the signature of manifolds with boundary. As mentioned in the introduction, an explanation of this *signature defect* was one of the motivations leading to the discovery of Theorem 1.4.5, and the Rho invariant is indeed an intrinsic characterization of this.

(iii) Rho invariants do in general give non-trivial invariants. As an easy example we consider $M = S^1$. We view S^1 as a subset of $\mathbb{C}$, endowed with the metric of length 2π. A flat $\mathrm{U}(1)$-connection on the trivial line bundle is determined by its holonomy $e^{2\pi i a} \in \mathrm{U}(1)$ with $a \in \mathbb{R}$. The corresponding odd signature operator is easily seen to be

$$B_a^{\mathrm{ev}} = -i(\mathscr{L}_e - ia) : C^\infty(S^1) \to C^\infty(S^1),$$

where with $\varphi \in C^\infty(S^1)$ and $z \in S^1$,

$$\mathscr{L}_e \varphi(z) = \tfrac{d}{dt}\big|_{t=0} \varphi(z e^{it}).$$

Therefore, $B_a^{\mathrm{ev}} \varphi = \lambda \varphi$ if and only if

$$\lambda + a \in \mathbb{Z} \quad \text{and} \quad \varphi(z) = z^{\lambda + a} \varphi(1).$$

This implies

$$\eta(B_a^{\mathrm{ev}}, s) = \sum_{\substack{n \in \mathbb{Z} \\ n \neq a}} \frac{\operatorname{sgn}(n - a)}{|n - a|^s}, \quad \operatorname{Re}(s) > 1.$$

In Proposition C.1.2 (i), we have included a computation of the value of the meromorphic continuation of this expression at 0. The result is, see also Definition C.1.1,

$$\eta(B_a^{\mathrm{ev}}) = 2P_1(a) = \begin{cases} 0, & \text{for } a \equiv 0 \mod \mathbb{Z}, \\ 2a_0 - 1, & \text{for } a_0 \in (0,1) \text{ and } a \equiv a_0 \mod \mathbb{Z}. \end{cases}$$

(iv) We want to point out that the Rho invariant is a true extension of the signature defect. More explicitly, there are Rho invariants which cannot be calculated using the formula of Proposition 1.4.7 (iii). As an example, consider a compact oriented surface Σ with one boundary component $\partial\Sigma = S^1$. Then the fundamental class of S^1 is a commutator in $\pi_1(\Sigma)$, see for example the discussion in Section 2.3.1. This implies that a non-trivial $\mathrm{U}(1)$-representations of $\pi_1(S^1)$ cannot extend to a representation of $\pi_1(\Sigma)$.

1.5 Rho Invariants and Local Index Theory

1.5.1 Relation to Chern-Simons Invariants

As seen in Proposition 1.4.7, the Rho invariant can be computed in a purely topological way if the representation α extends over a bounding manifold. However, we have already pointed out that this situation is often too restrictive. Therefore, intrinsic methods to compute Rho invariants are of great interest. One observation for applying topological tools is the relation of Rho invariants to Chern-Simons invariants. We refer to Appendix A for definitions and basic properties of Chern-Simons invariants, and make the relation more precise now. The following result goes back to [8, Sec. 4], see also [60, Sec. 7].

Proposition 1.5.1. *Assume that M is a closed, oriented Riemannian manifold of odd dimension m. Let A_t be a smooth path of connections on a fixed Hermitian vector bundle $E \to M$. Then the reduced ξ-invariant satisfies*

$$\int_0^1 \tfrac{d}{dt}[\xi(B_{A_t})]dt = 2^{\frac{m+1}{2}} \int_M \widehat{L}(TM, \nabla^g) \wedge \mathrm{cs}(A_0, A_1),$$

where $\mathrm{cs}(A_0, A_1)$ *is the transgression form of the Chern character, see Definition A.1.6.*

We postpone the proof and mention some consequences.

Corollary 1.5.2. *Let A_t be a smooth path of connections on E.*

(i) *The following variation formula holds*

$$\xi(B_{A_1}) - \xi(B_{A_0}) = \mathrm{SF}(B_{A_t})_{t\in[0,1]} + 2^{\frac{m+1}{2}} \int_M \widehat{L}(TM, \nabla^g) \wedge \mathrm{cs}(A_0, A_1).$$

(ii) *If A_0 and A_1 are flat, and M is 3-dimensional, then*

$$\rho_{A_1}(M) = \rho_{A_0}(M) + 4\,\mathrm{CS}(A_0, A_1) \mod \mathbb{Z}.$$

Here, $\mathrm{CS}(A_0, A_1)$ *is the Chern-Simons invariant associated to the Chern character, see Definition A.2.3 and* (A.7).

(iii) *Assume that A_t is a path of flat connections and that either A_0 and A_1 reduce to* $\mathrm{SU}(k)$*-connections or* $(\dim M \equiv 3 \mod 4)$. *Then*

$$\eta(B^{\mathrm{ev}}_{A_1}) - \eta(B^{\mathrm{ev}}_{A_0}) = 2\,\mathrm{SF}(B^{\mathrm{ev}}_{A_t})_{t\in[0,1]} - \dim\ker B^{\mathrm{ev}}_{A_1} + \dim\ker B^{\mathrm{ev}}_{A_0}.$$

In particular,

$$\rho_{A_1}(M) = \rho_{A_0}(M) \mod \mathbb{Z}.$$

Remark 1.5.3.

(i) Corollary 1.5.2 (ii) shows that on a 3-manifold the reduction mod $\mathbb{Z}$ of the Rho invariant is up to a constant the Chern-Simons invariant of the corresponding flat connection. Therefore, the reduced Rho invariant is the integral over local invariants of the connections. Now, the unreduced version is essentially this "local" contribution plus a spectral flow term which encodes "global" topological information.

(ii) Under the hypothesis of part (iii) the reduction mod $\mathbb{Z}$ of $\rho_{A_t}(M)$ is constant. In other words, the Chern-Simons invariant is constant on connected components of the moduli space of flat connections. Therefore, unreduced Rho invariants associated to one-parameter families of flat connections have only integer jumps, which occur precisely at the points where the rank of the twisted cohomology groups changes. Independently, Farber-Levine [39] and Kirk-Klassen [58, 59] have developed powerful methods to compute this spectral flow term in purely cohomological terms.

We will now give a proof of Proposition 1.5.1 based on the signature theorem for manifolds with boundary in Theorem 1.4.5. We shall also sketch a different proof using Getzler's approach to local index theory in Section 1.5.2 below.

Proof of Proposition 1.5.1. Consider the cylinder $N := [0,1] \times M$, endowed with the product metric $g_N = du^2 + \pi^* g$. Here, we are using u to denote the coordinate on $[0,1]$, and $\pi : N \to M$ is the natural projection. We endow $\pi^* E$ with the connection

$$\widetilde{A}_t := du \wedge \tfrac{d}{du} + \pi^* A_{t\varphi(u)},$$

where $\varphi : [0,1] \to [0,1]$ is a smooth function such that for some $\varepsilon > 0$,

$$\varphi(u) = \begin{cases} 0, & \text{if } u < \varepsilon, \\ 1, & \text{if } u > 1-\varepsilon, \end{cases} \tag{1.50}$$

see Figure 1.3. For fixed t, the connection $\widetilde{A}_t$ is in temporal gauge on a collar of ∂N. Therefore, we can apply Theorem 1.4.5 for each t to the signature operator $D^+_{\widetilde{A}_t}$ and conclude that

$$\xi(B_{A_t}) - \xi(B_{A_0}) = 2^{\frac{m+1}{2}} \int_N \widehat{L}(TN, \nabla^{g_N}) \wedge \mathrm{ch}(\pi^* E, \widetilde{A}_t) \mod \mathbb{Z}.$$

Since g_N is the product metric on N it is straightforward to check that

$$\widehat{L}(TN, \nabla^{g_N}) = \pi^* \widehat{L}(TM, \nabla^g).$$

Then, if $\int_{N/M}$ denotes integration along the fiber, see Proposition 2.1.12 below, we observe that

$$\begin{aligned} \int_N \widehat{L}(TN, \nabla^{g_N}) \wedge \mathrm{ch}(\pi^* E, \widetilde{A}_t) &= \int_M \widehat{L}(TM, \nabla^g) \wedge \int_{N/M} \mathrm{ch}(\pi^* E, \widetilde{A}_t) \\ &= \int_M \widehat{L}(TM, \nabla^g) \wedge \mathrm{cs}(A_{t\varphi(u)}). \end{aligned}$$

Here, we have used Lemma A.2.1 in the second line to replace $\int_{N/M} \mathrm{ch}(\pi^* E, \widetilde{A}_t)$ with the transgression form of the Chern character computed with respect to the path $u \mapsto A_{t\varphi(u)}$. If we choose a different path, the result will differ by an exact form on M, see Proposition A.2.2 for a proof. This allows us to remove the function φ and use the path $u \mapsto A_{tu}$. Then

$$\mathrm{cs}(A_{t\varphi(u)}) = \mathrm{cs}(A_0, A_t) \mod d\Omega^{\mathrm{ev}}(M).$$

Since M is closed, this shows that

$$\xi(B_{A_t}) - \xi(B_{A_0}) = 2^{\frac{m+1}{2}} \int_M \widehat{L}(TM, \nabla^g) \wedge \mathrm{cs}(A_0, A_t) \mod \mathbb{Z},$$

which implies Proposition 1.5.1. □

Proof of Corollary 1.5.2. Part (i) is an immediate consequence of Proposition 1.5.1 and Proposition 1.3.14. Next, we note that

$$\xi(B_{A_1}) - \xi(B_{A_0}) = \eta(B^{\mathrm{ev}}_{A_1}) - \eta(B^{\mathrm{ev}}_{A_0}) + \dim\ker(B^{\mathrm{ev}}_{A_1}) - \dim\ker(B^{\mathrm{ev}}_{A_0}). \tag{1.51}$$

Using part (i) and reducing mod $\mathbb{Z}$ one finds that

$$\eta(B^{\mathrm{ev}}_{A_1}) - \eta(B^{\mathrm{ev}}_{A_0}) = 2^{\frac{m+1}{2}} \int_M \widehat{L}(TM, \nabla^g) \wedge \mathrm{cs}(A_0, A_1) \mod \mathbb{Z}.$$

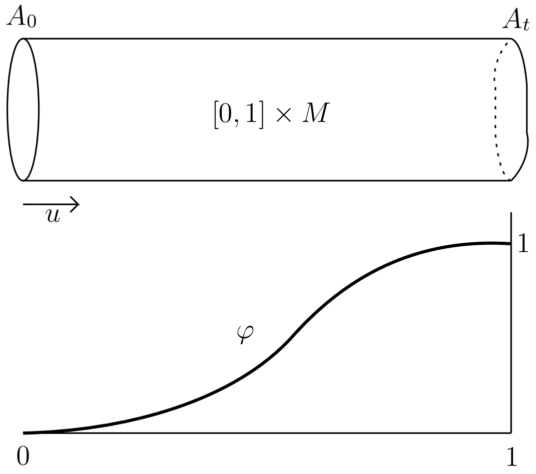

Figure 1.3: Cylinder N and the cutoff function φ

Now, if we assume that $m = 3$, the $\widehat{L}$-form equals 1, and we obtain part (ii). Concerning part (iii), we now consider the connection $\widetilde{A} := dt \wedge \frac{d}{dt} + A_t$ on the cylinder $[0,1] \times M$. Since we are assuming that A_t is a path of *flat* connections, we have

$$F_{\widetilde{A}} = dt \wedge \tfrac{d}{dt} A_t, \quad \text{and} \quad \exp\big(dt \wedge \tfrac{d}{dt} A_t\big) = 1 + dt \wedge \tfrac{d}{dt} A_t.$$

This implies

$$\begin{aligned}\int_M \widehat{L}(TM, \nabla^g) \wedge \mathrm{cs}(A_0, A_1) &= \int_{[0,1]\times M} \pi^* \widehat{L}(TM, \nabla^g) \wedge \mathrm{tr}_E \big[\exp(\tfrac{i}{2\pi} F_{\widetilde{A}})\big] \\ &= \int_M \widehat{L}(TM, \nabla^g) \wedge \tfrac{i}{2\pi} \mathrm{tr}_E[A_1 - A_0].\end{aligned}$$

If A_0 and A_1 are $\mathrm{SU}(k)$-connections, then $\mathrm{tr}_E[A_1 - A_0] = 0$, so that the integrand vanishes. On the other hand, if ($m \equiv 3 \mod 4$), the integrand has no degree m part so that the integral vanishes again. In both cases, part (iii) follows from part (i) and (1.51). ☐

Variation of the Metric. As already pointed out in Remark 1.5.3, an important tool for studying the Rho invariant is studying its variation under deformations of the flat connection. However, the moduli space of flat connections is often discrete or at least disconnected. Therefore, it is not always possible to find a path of flat connections which joins given endpoints. However, in many cases one can deform the geometry of the underlying manifold in such a way that twisted Eta invariants become computable. The main concern of this thesis are Rho invariants of fiber bundles, and as already explained in the introduction there are powerful methods to deform the geometry to a much simpler situation. One underlying result is the following, which is an analog of Proposition 1.5.1, see [8, Sec. 2].

Proposition 1.5.4. *Assume that g_t is a smooth path of Riemannian metrics on a closed, odd dimensional manifold M, and denote by ∇^{g_t} be the associated one-parameter family of Levi-Civita connections. Moreover, let A be a flat connection on a Hermitian vector bundle $E \to M$ of rank k. Then*

$$\eta(B_A^{\mathrm{ev}}, g_1) - \eta(B_A^{\mathrm{ev}}, g_0) = k \cdot \int_M TL(\nabla^{g_0}, \nabla^{g_1}),$$

where $TL(\nabla^{g_0}, \nabla^{g_1})$ is the transgression form of the L-class of TM, see Remark A.1.7.

Proof. As before, let $N := [0,1] \times M$ be the cylinder, endowed with the bundle π^*E and the connection $\widetilde{A} := dt \wedge \frac{d}{dt} + \pi^* A$. In contrast to the proof of Proposition 1.5.1, the connection $\widetilde{A}$ is flat, since A is independent of t. Let $\alpha : \pi_1(N) \to \mathrm{U}(k)$ denote the holonomy representation of $\widetilde{A}$. Recall that $\mathrm{Sign}_\alpha(N)$ is defined using the homomorphism

$$H^\bullet(N, M, E_\alpha) \to H^\bullet(N, E_\alpha).$$

However, in the case at hand, this map is trivial as M is a deformation retract of N, and α is compatible with the natural retraction. Therefore, $\mathrm{Sign}_\alpha(N) = 0$.

Now let φ be a cutoff function as in (1.50), and endow N with the metric

$$g_N := dt^2 + \pi^* g_{\varphi(t)}.$$

Let ∇^{g_N} be the Levi-Civita connection associated to g_N. Since g_N is in product form near the boundary and $\mathrm{Sign}_A(N) = 0$, Theorem 1.4.5 yields

$$\eta(B_A^{\mathrm{ev}}, g_1) - \eta(B_A^{\mathrm{ev}}, g_0) = k \cdot \int_N L(TN, \nabla^{g_N}).$$

Moreover, ∇^N is in temporal gauge on a collar of the boundary, so that we can deduce from Proposition A.2.4 and Remark A.2.5 that

$$\int_N L(TN, \nabla^{g_N}) = \int_M TL(\nabla^{g_0}, \nabla^{g_1}). \qquad \square$$

Remark 1.5.5. In the next chapter we will also encounter a one-parameter family of connections $(\nabla^t)_{t\in[0,1]}$ on TM which is not associated to a family of Riemannian metrics. Nevertheless, we can study the associated family of generalized odd signature operators

$$D_t := c^{\mathrm{ev}} \circ \nabla^t : \Omega^{\mathrm{ev}}(M) \to \Omega^{\mathrm{ev}}(M),$$

where c^{ev} is Clifford multiplication as defined in (1.49). Certainly, the operators D_t will in general not be formally self-adjoint, even if all ∇^t are compatible with the metric. Without going into detail, we note that a certain restriction on the torsion tensor of ∇^t guarantees that we get formally self-adjoint operators. Then we can use the variation formula of Proposition 1.3.14,

$$\xi(D_1) - \xi(D_0) = \mathrm{SF}(D_t)_{t\in[0,1]} + \int_0^1 \tfrac{d}{dt}\big[\xi(D_t)\big] dt.$$

However, since $(D_t)_{t\in[0,1]}$ is not a family of geometric Dirac operators, the local variation can not be identified using Theorem 1.4.5. Yet, if we are interested in Rho invariants, it follows from Corollary 1.3.12 that for any flat $\mathrm{U}(k)$-connection A, the local variations of $(D_t^{\oplus k})_{t\in[0,1]}$ and $(D_{A,t})_{t\in[0,1]}$ agree. In particular,

$$\rho(D_{A,1}) = \rho(D_{A,0}) + \mathrm{SF}(D_{A,t})_{t\in[0,1]} - k \cdot \mathrm{SF}(D_t)_{t\in[0,1]}. \tag{1.52}$$

1.5.2 The Variation Formula and Local Index Theory

In the remainder of this chapter we sketch a proof of Proposition 1.5.1 based on local index theory techniques. Mainly, this is because the underlying ideas will be helpful in the discussion of Rho invariants of fiber bundles in Chapter 3. Aside from that, a proof of the variation formula in Proposition 1.5.1, which does not rely on the index theorem for manifolds with boundary, underlines the intrinsic nature of Rho invariants.

The setup. Assume that M is a closed, oriented Riemannian manifold of odd dimension m, and let A_u be a smooth path of connections on a fixed Hermitian vector bundle $E \to M$. We want to have an explicit formula for the variation $\frac{d}{du}[\xi(B_{A_u})]$ of the reduced ξ-invariant, where

$$B_{A_u} = \tau\big(d_{A_u} + d^t_{A_u}\big) : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E).$$

Proposition D.2.5 shows that

$$\tfrac{d}{du}[\xi(B_{A_u})] = -\tfrac{1}{\sqrt{\pi}}\, a_m(B_{A_u}),$$

where $a_m(B_{A_u})$ is the constant term in the asymptotic expansion of

$$\sqrt{t}\ \mathrm{Tr}\,\big(\tfrac{dB_{A_u}}{du} e^{-tB^2_{A_u}}\big), \quad \text{as } t \to 0.$$

For brevity we will also use the common notation

$$a_m(B_{A_u}) = \underset{t\to 0}{\mathrm{LIM}}\, \sqrt{t}\ \mathrm{Tr}\,\big(\tfrac{dB_{A_u}}{du} e^{-tB^2_{A_u}}\big). \tag{1.53}$$

In the case at hand, it is immediate that

$$\tfrac{dB_{A_u}}{du} = \tau c\big(\tfrac{d}{du}A_u\big).$$

Now, to get a formula for $a_m(B_{A_u})$, we can fix u. To keep the notation short, we thus extract the following setup with which we will work

Definition 1.5.6. Let A be a connection on E, and let $\dot{A} \in \Omega^1\big(M, \mathrm{End}(E)\big)$. Define

$$D_A := d_A + d^t_A : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E),$$

and denote by

$$k_t(x,x) \in C^\infty\big(M, \mathrm{End}(\Lambda^\bullet T^*M \otimes E)\big)$$

the restriction to the diagonal of the kernel

$$k_t(x,y) := \Big(\sqrt{t}c(\dot{A})e^{-tD_A^2}\Big)(x,y).$$

With this notation our goal is now to compute

$$\underset{t\to 0}{\mathrm{LIM}}\ \mathrm{tr}_{\Lambda^\bullet T^*M\otimes E}\,\big[\tau k_t(x,x)\big].$$

According to (1.28) we can decompose

$$\mathrm{End}(\Lambda^\bullet T^*M \otimes E) = \mathrm{Cl}(T^*M) \otimes_s \mathrm{End}_{\mathrm{Cl}}\,\big(\Lambda^\bullet T^*M \otimes E\big),$$

where

$$\mathrm{End}_{\mathrm{Cl}}\left(\Lambda^\bullet T^*M \otimes E\right) = \widehat{\mathrm{Cl}}(T^*M) \otimes \mathrm{End}(E).$$

The local index theory proof of Proposition 1.5.1 will follow from the following odd dimensional version of [13, Thm. 4.1], see also Remark 1.2.15.

Theorem 1.5.7. *Let M be a Riemannian manifold of odd dimension m, and let $k_t(x,x)$ be as in Definition 1.5.6. There is an asymptotic expansion*

$$k_t(x,x) \sim (4\pi t)^{-\frac{m}{2}} \sum_{n=0}^{\infty} t^{\frac{n+1}{2}} k_n(x), \quad \text{as } t \to 0,$$

such that

$$k_n(x) \in C^\infty\Big(M, \mathrm{Cl}_{n+1}(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}\left(\Lambda^\bullet T^*M \otimes E\right)\Big).$$

With respect to the symbol map $\boldsymbol{\sigma} : \mathrm{Cl}(T^*M) \to \Lambda^\bullet T^*M$, one has

$$\sum_{n=0}^{m-1} \boldsymbol{\sigma}(k_n) = \det{}^{1/2}\left(\frac{R^g/2}{\sinh(R^g/2)}\right) \wedge \big(\dot{A} \wedge \exp(-F)\big),$$

where R^g is the Riemann curvature tensor, and F is the twisting curvature, defined in a local orthonormal frame $\{e_i\}$ for TM by

$$F := F_{\nabla^{A,g}} - \tfrac{1}{8} g\big(R^g(e_i,e_j)e_k, e_l\big) e^i \wedge e^j \otimes c^k c^l \in \Omega^2\big(M, \mathrm{End}_{\mathrm{Cl}}\left(\Lambda^\bullet T^*M \otimes E\right)\big).$$

Here, $\nabla^{A,g}$ is the connection on $\Lambda^\bullet T^*M \otimes E$ induced by the Levi-Civita connection and the connection A on E.

Before we sketch how Theorem 1.5.7 can be proved along parallel lines as in [13, Ch. 4], we deduce the following consequence which gives an alternative proof of Proposition 1.5.1.

Proposition 1.5.8. *Assume that M is a closed, oriented Riemannian manifold of odd dimension m. Let A_u be a smooth path of connections on a fixed Hermitian vector bundle $E \to M$. Then the reduced ξ-invariant associated to the family $B_{A_u} = \tau\big(d_{A_u} + d^t_{A_u}\big)$ satisfies*

$$\tfrac{d}{du}[\xi(B_{A_u})] = 2^{\frac{m+1}{2}} \int_M \widehat{L}(TM, \nabla^g) \wedge \tfrac{i}{2\pi} \operatorname{tr}_E \left[\tfrac{d}{du} A_u \wedge \exp(\tfrac{i}{2\pi} F_{A_u})\right],$$

where F_{A_u} is the curvature or A_u.

Proof. For fixed u we use the notation of Definition 1.5.6, and let $A := A_u$ and $\dot{A} := \frac{d}{du} A_u$. Then Proposition D.2.5 shows that with the kernel $k_t(x,x)$

$$\tfrac{d}{du}[\xi(B_{A_u})] = -\frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{t\to 0} \int_M \operatorname{tr}_{\Lambda^\bullet T^*M \otimes E}\left[\tau k_t(x,x)\right] \mathrm{vol}_M\,. \tag{1.54}$$

It follows from Theorem 1.5.7 that

$$\operatorname*{LIM}_{t\to 0} \int_M \operatorname{tr}_{\Lambda^\bullet T^*M \otimes E}\left[\tau k_t(x,x)\right] \mathrm{vol}_M = (4\pi)^{-m/2} \int_M \operatorname{tr}_{\Lambda^\bullet T^*M \otimes E}\left[\tau k_{m-1}\right] \mathrm{vol}_M\,.$$

As in Proposition 1.2.19 we can decompose

$$\operatorname{tr}_{\Lambda^\bullet T^*M\otimes E} = \operatorname{tr}_{\mathrm{Cl}} \otimes \widehat{\operatorname{tr}_{\mathrm{Cl}}} \otimes \operatorname{tr}_E\,.$$

Now, the definition of $\operatorname{tr}_{\mathrm{Cl}}$ is in such a way that for $a \in \mathrm{Cl}(T^*M)$

$$\operatorname{tr}_{\mathrm{Cl}}(\tau a) = \begin{cases} 0, & \text{if } a \in \mathrm{Cl}_{m-1}(T^*M), \\ 2^{m/2}, & \text{if } a = \tau. \end{cases}$$

This implies that for all $\kappa \in C^\infty\big(M, \operatorname{End}(T^*M \otimes E)\big)$

$$\int_M \operatorname{tr}_{\Lambda^\bullet T^*M\otimes E}\big[\tau\kappa\big]\operatorname{vol}_M = 2^{m/2}\int_M i^{-\frac{m+1}{2}}\big(\widehat{\operatorname{tr}_{\mathrm{Cl}}}\otimes \operatorname{tr}_E\big)\big[\boldsymbol{\sigma}(\kappa)\big],$$

where the factor $i^{-\frac{m+1}{2}}$ arises from the fact that $\boldsymbol{\sigma}(\tau) = i^{\frac{m+1}{2}}\operatorname{vol}_M$. Then, we can use Theorem 1.5.7 again to infer that

$$\begin{aligned}(4\pi)^{-m/2}\int_M &\operatorname{tr}_{\Lambda^\bullet T^*M\otimes E}\big[\tau k_{m-1}\big]\operatorname{vol}_M = \\ &\sqrt{2\pi}(2\pi i)^{-\frac{m+1}{2}}\int_M \det{}^{1/2}\left(\frac{R^g/2}{\sinh(R^g/2)}\right)\wedge\big(\widehat{\operatorname{tr}_{\mathrm{Cl}}}\otimes\operatorname{tr}_E\big)\big[\dot{A}\wedge\exp(-F)\big].\end{aligned}$$

Now, we decompose $F = \widehat{R}^S + F_A$, where $\widehat{R}^S$ is the twisting curvature of $\Lambda^\bullet T^*M$ as in Lemma 1.2.21, and $F_A \in \Omega^2\big(M, \operatorname{End}(E)\big)$ is the curvature of A. Then one computes

$$\begin{aligned}(2\pi i)^{-\frac{m+1}{2}}&\int_M \det{}^{1/2}\left(\frac{R^g/2}{\sinh(R^g/2)}\right)\wedge\big(\widehat{\operatorname{tr}_{\mathrm{Cl}}}\otimes\operatorname{tr}_E\big)\big[\dot{A}\wedge\exp(-F)\big] \\ &= -\int_M \widehat{A}(TM,\nabla^g)\wedge\big(\widehat{\operatorname{tr}_{\mathrm{Cl}}}\otimes\operatorname{tr}_E\big)\big[\tfrac{i}{2\pi}\dot{A}\wedge\exp(\tfrac{i}{2\pi}F)\big] \\ &= -\int_M \widehat{A}(TM,\nabla^g)\wedge\widehat{\operatorname{tr}_{\mathrm{Cl}}}\big[\exp\big(\tfrac{i}{2\pi}\widehat{R}^S\big)\big]\wedge\operatorname{tr}_E\big[\tfrac{i}{2\pi}\dot{A}\wedge\exp\big(\tfrac{i}{2\pi}F_A\big)\big] \\ &= -2^{m/2}\int_M \widehat{L}(TM,\nabla^g)\wedge\operatorname{tr}_E\big[\tfrac{i}{2\pi}\dot{A}\wedge\exp\big(\tfrac{i}{2\pi}F_A\big)\big],\end{aligned}$$

where we have used Proposition 1.2.22 in the last line. Using (1.54), the proof of Proposition 1.5.8 is finished. □

Getzler's Rescaling. We now want to motivate why Theorem 1.5.7 can be proved in the same way as the local index theorem [13, Thm. 4.1]. Since it is also basic for the considerations in the next chapters, we first want to extract one of the main ideas of Getzler's approach in [45]. This is to consider an appropriate rescaling of the Riemannian metric.

Let M be a closed manifold, and let g be a Riemannian metric. For $t > 0$ consider the rescaled metric $g_t := t^{-1}g$. We define a rescaled Clifford multiplication by

$$c_t : \big(T^*M, g_t\big) \to \big(\Lambda^\bullet T^*M, g\big), \quad c_t(\xi) := \sqrt{t}\big(\mathrm{e}(\xi) - \mathrm{i}(\xi)\big), \tag{1.55}$$

where $\mathrm{i}(\xi)$ denotes inner multiplication by ξ with respect to the fixed metric g on $\Lambda^\bullet T^*M$. One easily checks that

$$c_t(\xi)^2 = -t\cdot|\xi|_g^2 = -|\xi|_{g_t}^2, \quad\text{and}\quad c_t(\xi)^* = -c_t(\xi) \quad \text{w.r.t. } g. \tag{1.56}$$

This means that c_t defines a Clifford structure for (T^*M, g_t) on the bundle $\big(\Lambda^\bullet T^*M, g\big)$. Since the Levi-Civita connection ∇^g is invariant under rescaling with a constant parameter, it defines a Clifford connection for every t,

$$\big[\nabla^g, c_t(\xi)\big] = c_t\big(\nabla^g \xi\big), \quad \xi \in \Omega^1(M).$$

We thus get a family of de Rham operators D_t, defined in a local orthonormal frame $\{e_j\}$ for (TM, g) by

$$D_t := c_t(e^j)\nabla^g_{e_j} = \sqrt{t}(d + d^t) : \Omega^\bullet(M) \to \Omega^\bullet(M). \tag{1.57}$$

Here, d^t is the adjoint differential with respect to the fixed metric g.

Remark. It might be confusing that the de Rham differential d is also rescaled, although it is defined without using a metric. This is due to the fact that we are fixing g as a reference metric on $\Lambda^\bullet T^*M$ while varying the metric on T^*M.

We wish to make this more precise, and take g_t as a metric on $\Lambda^\bullet T^*M$ rather than the fixed metric g. Consider

$$\widetilde{c}_t : \big(T^*M, g_t\big) \to \big(\Lambda^\bullet T^*M, g_t\big), \quad \widetilde{c}_t(\xi) := \mathrm{e}(\xi) - \mathrm{i}_t(\xi), \tag{1.58}$$

where now, $\mathrm{i}_t(\xi)$ is inner multiplication with respect to g_t. Note that $\mathrm{i}_t(\xi)$ is related to inner multiplication with respect to g via $\mathrm{i}_t(\xi) = t \cdot \mathrm{i}(\xi)$. From this, one deduces that

$$\widetilde{c}_t(\xi)^2 = -t \cdot |\xi|_g^2 = -|\xi|_{g_t}^2, \quad \text{and} \quad c_t(\xi)^* = -c_t(\xi) \quad \text{w.r.t. } g_t. \tag{1.59}$$

Hence, $\widetilde{c}_t$ defines a Clifford structure for (T^*M, g_t) on the bundle $\big(\Lambda^\bullet T^*M, g_t\big)$. The relation between c_t and $\widetilde{c}_t$ is as follows.

Lemma 1.5.9. *There is an isometry of vector bundles given by*

$$\delta_t : \big(\Lambda^\bullet T^*M, g_t\big) \to \big(\Lambda^\bullet T^*M, g\big), \quad \delta_t(\alpha) := (\sqrt{t})^{|\alpha|}\alpha,$$

where α is homogenous of degree $|\alpha|$. Moreover, the Clifford structures c_t and $\widetilde{c}_t$ are related by

$$\widetilde{c}_t = \delta_t^{-1} \circ c_t \circ \delta_t.$$

The proof is straightforward and is left to the reader. We also note that if $d^t_{g_t}$ denotes the adjoint differential with respect to g_t, then

$$\sqrt{t}(d + d^t) = \delta_t \circ \big(d + d^t_{g_t}\big) \circ \delta_t^{-1}.$$

This explains in more detail why the de Rham differential in (1.57) is rescaled.

Now, one of the ideas underlying Getzler's approach in [45] can be expressed in the simple identity

$$e^{-tD^2} = e^{-sD_t^2}|_{s=1}, \tag{1.60}$$

where D is the de Rham operator with respect to g, and D_t is the rescaled de Rham operator (1.57). The deep insight behind (1.60) is that the asymptotic expansion as $t \to 0$ of the kernel of e^{-tD^2} can be related to the Euclidean heat kernel since—very roughly—the metric

g_t converges locally to a Euclidean metric as $t \to 0$.

Remarks on the Proof of Theorem 1.5.7. We shall make the above idea only a bit more precise, and refer to [13, Sec. 4.3] for more details. Let $x \in M$, and let $V := T_xM$. We choose a ball $U \subset V$ of radius less than the injectivity radius of M, so that $\exp_x : U \to M$ parametrizes a normal neighbourhood of x in M. Using parallel transport along geodesic rays with respect to the flat connection A, we can identify the bundle $E|_U$ with the trivial bundle $U \times E_x$. Hence, for $y \in U$ we can consider

$$h(t,y) := e^{-tD_A^2}\big(\exp_x(y), x\big) \in \operatorname{End}(\Lambda^\bullet V^* \otimes E_x), \tag{1.61}$$

and

$$k(t,y) := \Big(\sqrt{t}c(\dot{A})e^{-tD_A^2}\Big)\big(\exp_x(y), x\big) \in \operatorname{End}(\Lambda^\bullet V^* \otimes E_x). \tag{1.62}$$

Using the symbol map $\boldsymbol{\sigma} : \mathrm{Cl}(V^*) \to \Lambda^\bullet V^*$, we get sections

$$\boldsymbol{\sigma}(h), \boldsymbol{\sigma}(k) \in C^\infty\big(\mathbb{R}^+ \times U, \Lambda^\bullet V^* \otimes \operatorname{End}_{\mathrm{Cl}}(\Lambda^\bullet V^* \otimes E_x)\big).$$

Now Getzler's rescaling method can be described as follows: We fix U as the coordinate space, but replace the metric on M by $g_t := t^{-1}g$. This implies that the new system of normal coordinates is given by

$$\exp_x \circ \sqrt{t} : U \to M, \quad y \mapsto \exp(\sqrt{t}y).$$

On $\operatorname{End}_{\mathrm{Cl}}(\Lambda^\bullet V^* \otimes E_x)$, we fix the reference metric given by g and the metric on E, but we use the rescaled metric on $\mathrm{Cl}(V^*)$. According to Lemma 1.5.9 this means that the symbol map has to be replaced by

$$\boldsymbol{\sigma}_t := \delta_t^{-1} \circ \boldsymbol{\sigma} : \mathrm{Cl}(V^*) \to \Lambda^\bullet V^*.$$

One checks that if $\xi \in V^* \subset \mathrm{Cl}(V^*)$ and $a \in \mathrm{Cl}(V^*)$, then

$$\boldsymbol{\sigma}_t(\xi \cdot a) = \tfrac{1}{\sqrt{t}}\xi \wedge \boldsymbol{\sigma}_t(a) - \sqrt{t}\,\mathrm{i}(\xi)\boldsymbol{\sigma}_t(a), \tag{1.63}$$

where $\mathrm{i}(\xi)$ is interior multiplication with respect to the fixed metric g.

Definition 1.5.10. Let $s \in \mathbb{R}^+$ be an auxiliary parameter as in (1.60). Then the *rescaled heat kernel* is defined as

$$h(s,t,y) := t^{m/2}\boldsymbol{\sigma}_t\big(h(st, \sqrt{t}y)\big) \in \Lambda^\bullet V^* \otimes \operatorname{End}_{\mathrm{Cl}}(\Lambda^\bullet V^* \otimes E_x).$$

As remarked in [13, p. 155] the extra factor $t^{m/2}$ enters because $h(t,y)$ is a density in the y variable. Now Getzler's local index theorem can be reformulated as follows, see [13, Thm. 4.21].

Theorem 1.5.11. *The limit* $\lim_{t\to 0} h(s,t,y)$ *exists. For* $y = 0$ *and* $s = 1$,

$$\lim_{t\to 0} h(1,t,0) = (4\pi)^{-m/2}{\det}^{1/2}\left(\frac{R_x^g/2}{\sinh(R_x^g/2)}\right) \wedge \exp(-F_x),$$

where R_x^g *is the Riemann curvature tensor at* x *and* F *is the twisting curvature, defined in Theorem 1.5.7.*

Remark. In [13], Theorem 1.5.11 is proved under the assumption that m is even. However, a close examination of all intermediate steps shows that this restriction is not necessary. In particular, [13, Prop. 4.12] continues to hold for odd m as well. The argument is similar to (1.32) in the proof of Lemma 1.2.20.

Using (1.62) we now define as in Definition 1.5.10

$$k(s,t,y) := t^{m/2}\boldsymbol{\sigma}_t\big(k(st,\sqrt{t}y)\big) \in \Lambda^\bullet V^* \otimes \mathrm{End}_{\mathrm{Cl}}(\Lambda^\bullet V^* \otimes E_x).$$

For $y \in U$ let $\dot{A}_y \in \Lambda^1 V^* \otimes \mathrm{End}(E_x)$ denote the pullback of $\dot{A}$ to U at the point y. Then (1.63) shows that

$$k(s,t,y) = \dot{A}_{\sqrt{t}y} \wedge h(s,t,y) - t\,\mathrm{i}\,\big(\dot{A}_{\sqrt{t}y}\big)h(s,t,y),$$

and Theorem 1.5.11 implies

Corollary 1.5.12. *The limit* $\lim_{t\to 0} k(s,t,y)$ *exists. For* $y=0$ *and* $s=1$*,*

$$\lim_{t\to 0} k(1,t,0) = (4\pi)^{-m/2}\mathrm{det}^{1/2}\left(\frac{R^g_x/2}{\sinh(R^g_x/2)}\right) \wedge \big(\dot{A}_x \wedge \exp(-F_x)\big).$$

On the other hand—as in the even dimensional case—the rescaled kernel satisfies

$$\lim_{t\to 0} k(1,t,0) = (4\pi)^{-\frac{m}{2}} \sum_{n=0}^{m-1} \boldsymbol{\sigma}\big(k_n(x)\big),$$

where the k_n are the coefficients appearing in the asymptotic expansion in Theorem 1.5.7. This together with Corollary 1.5.12 finishes the outline of the proof of Theorem 1.5.7.

Chapter 2

Rho Invariants of Fiber Bundles, Basic Considerations

In this chapter we start to work on the main topic of this thesis. Our concern is to investigate how the structure of an oriented fiber bundle of closed manifolds can be used to analyze Rho invariants of its total space.

For this reason, we will first give a detailed summary of some geometric preliminaries as they appear in the theory of Riemannian submersions and in Bismut's local index theory for families. Since we are dealing with the odd signature operator, our emphasis is to understand the structure of the space of differential forms. In particular, we obtain descriptions of the exterior differential, the adjoint differential and the Levi-Civita connection, which account for the special situation arising in the context of the total space of a fiber bundle.

After having established how the odd signature operator can be expressed in terms of a submersion metric, we describe the idea of an *adiabatic metric* on a fiber bundle. This is the main tool on which our discussion of Rho invariants of fiber bundles relies. Very roughly, the idea is to rescale the metric on the base manifold in order to deform the geometry of the fiber bundle to an "almost product" situation. Using the variation formula, we shall see that the Eta invariant of the odd signature operator has a well-defined limit under this process, which is called the *adiabatic limit*. As the Rho invariant is independent of the underlying metric, we can then replace the Eta invariants in its definition by adiabatic limits.

With this idea in mind, we will analyze the first class of examples, namely principal circle bundles over closed surfaces. Thanks to the low dimensions of fiber and base as well as the enhanced symmetry provided by the principal bundle structure, one can compute Rho invariants without having to use the more advanced theory of Chapter 3. Nevertheless, adiabatic metrics will already play a prominent role in the discussion.

2.1 Fibered Calculus

2.1.1 Connections on Fiber Bundles.

Let $F \hookrightarrow M \xrightarrow{\pi} B$ be an oriented fiber bundle, where all manifolds are assumed to be closed, connected and oriented. Let $T^vM := \ker \pi_*$ be the vertical subbundle of TM. Then T^vM is *involutive*, i.e., if $[.,.]$ denotes the Lie bracket on $C^\infty(M, TM)$, we have

$$[U, V] \in T^vM, \quad U, V \in C^\infty(M, T^vM).$$

We now assume that M is endowed with a *connection*, i.e., a vertical projection

$$P^v : TM \to T^vM.$$

This induces a splitting

$$TM = T^hM \oplus T^vM, \quad T^hM := \ker P^v,$$

and $\pi_* : T^hM \to \pi^*TB$ is an isomorphism. In the following we usually identify T^hM and π^*TB via this isomorphism. Given a vector field $X \in C^\infty(B, TB)$ we can use the connection to lift X horizontally to a vector field $X^h \in C^\infty(M, T^hM)$. We will frequently use the following easy result, see [13, Lem. 10.7].

Lemma 2.1.1. *Let $V \in C^\infty(M, T^vM)$ be a vertical vector field on M, and let $X \in C^\infty(B, TB)$ be a vector field on B. Then*

$$[X^h, V] \in C^\infty(M, T^vM).$$

The horizontal distribution T^hM is in general not involutive. The following quantity measures the failure of being so.

Definition 2.1.2. The *curvature form* of the connection P^v is given by

$$\Omega \in C^\infty(M, \Lambda^2 T^hM^* \otimes T^vM), \quad \Omega(X, Y) := -P^v\big([X, Y]\big),$$

where $X, Y \in C^\infty(M, T^hM)$.

Riemannian Connections on Fiber Bundles. We now assume that the fiber bundle is equipped with a *submersion metric*. This means that with respect to the splitting $TM = \pi^*TB \oplus T^vM$ induced by the connection, we take a metric of the form

$$g = \pi^* g_B \oplus g_v, \tag{2.1}$$

where g_v is a Riemannian metric on T^vM, and g_B is a metric on B. We will frequently write $g = g_B \oplus g_v$, the pullback and the identification $TM = \pi^*TB \oplus T^vM$ being understood.

Remark. Note that our point of view is somewhat reversed to the situation encountered in differential geometry. Recall, see e.g. [81, pp. 212–214], that a *Riemannian submersion* is defined as a submersion $\pi : (M, g) \to (B, g_B)$ such that the push-forward $\pi_* : (\ker \pi_*)^\perp \to TB$ is an isometry. Then one deduces that M is a fiber bundle over B, endowed with a natural connection given by the orthogonal projection onto $\ker \pi_*$. Moreover, via the isomorphism $(\ker \pi_*)^\perp \cong \pi^*TB$ the metric g is of the form $g_B \oplus g_v$. In contrast to this point of view, we start with a fiber bundle, choose a connection and then endow the vertical and the horizontal bundles with metrics. This is because we often want to fix only a vertical metric.

Given a vertical metric g_v, the following result provides a natural connection ∇^v on T^vM, see [13, Prop. 10.2].

Proposition 2.1.3. *Let g_v be a metric on T^vM, and let P^v be a vertical projection. Then there is a natural connection ∇^v on T^vM defined by*

$$\nabla^v := P^v \circ \nabla^g \circ P^v,$$

where ∇^g is the Levi-Civita connection associated to a metric of the form (2.1)*. The connection ∇^v is independent of the choice of the metric g_B on B. It is compatible with g_v and torsion free when restricted to T^vM, i.e.,*

$$\nabla^v_U V - \nabla^v_V U = [U, V], \quad U, V \in C^\infty(M, T^vM).$$

Remark 2.1.4. The connection ∇^v can be thought of as a family of Levi-Civita connections parametrized by B, when we consider every fiber $F \subset M$ endowed with the metric induced by $g_v|_F$. This might give a good intuition why ∇^v is canonically associated to g_v and independent of g_B.

Definition 2.1.5. Let g_v be a vertical Riemannian metric. Then we define the *Weingarten map*

$$W \in C^\infty\big(M, T^hM^* \otimes \mathrm{End}(T^vM)\big), \quad W_X(V) := \nabla^v_X V - P^v\big([X, V]\big),$$

where $V \in C^\infty(M, T^vM)$ and $X \in C^\infty(M, T^hM)$. The *mean curvature* of ∇^v with respect to the vertical projection P^v is defined as

$$k_v = k_v(P^v, g_v) \in C^\infty(M, T^hM^*), \quad k_v(X) := \mathrm{tr}_v(W_X),$$

where $X \in C^\infty(M, T^hM)$ and $\mathrm{tr}_v : \mathrm{End}(T^vM) \to \mathbb{C}$ is the fiberwise vertical trace.

Remark.

(i) If $\{e_i\}$ is a local orthonormal frame for T^vM, we have

$$\mathrm{tr}_v(W_X) = \sum_i g_v\big(\nabla^v_X e_i - [X, e_i], e_i\big) = -\sum_i g_v\big([X, e_i], e_i\big). \tag{2.2}$$

This is because ∇^v is a metric connection so that

$$g_v(\nabla^v_X e_i, e_i) = X g_v(e_i, e_i) - g_v(e_i, \nabla^v_X e_i) = -g_v(\nabla^v_X e_i, e_i).$$

(ii) In the literature, one finds different conventions of how to define the Weingarten map. First of all in Riemannian geometry, one usually defines it as the negative of what we have defined. Moreover, one often normalizes the mean curvature by a factor of $(\dim F)^{-1}$. We chose to follow the conventions of [13, Sec. 10.1].

Associated to the metric g_B on B, there is the Levi-Civita connection ∇^B. Apart from the Levi-Civita connection ∇^g on TM we can thus form the direct sum connection

$$\nabla^\oplus := \pi^*\nabla^B \oplus \nabla^v \tag{2.3}$$

with respect to the splitting $TM = \pi^*TB \oplus T^vM$. Note that if X^h is a horizontal lift and if V is vertical, then $\nabla^\oplus_V X^h = 0$. Clearly, the connection $\nabla^\oplus$ preserves the metric $g = g_B \oplus g_v$. However, it is not necessarily torsion free. Hence, it does usually not coincide with the Levi-Civita connection of M. Following [14, Sec. I] we introduce the following natural tensors which measure the difference of $\nabla^\oplus$ and ∇^g.

Definition 2.1.6. Let $\nabla^{\oplus}$ be defined as in (2.3) and let ∇^g be the Levi-Civita connection associated to the metric (2.1). Define

$$S := \nabla^g - \nabla^{\oplus} \in C^\infty\big(M, \mathrm{End}(TM)\big),$$

and let θ be its metric contraction

$$\theta(X)(Y,Z) := g\big(S(X)Y, Z\big), \quad X, Y, Z \in C^\infty(M, TM).$$

Since both connections $\nabla^{\oplus}$ and ∇^g preserve the metric, the tensor θ is antisymmetric in Y and Z. Therefore, it is a section of $T^*M \otimes \Lambda^2 T^*M$. Also note that the above tensors are related to the *O'Neill tensors* of the Riemannian submersion, see [81, pp. 212–214]. The following result describes all non-trivial components of θ, see [16, Sec. 4 (a)].

Proposition 2.1.7. *The tensor θ is independent of the chosen metric g_B on B. Moreover, if X, Y are horizontal vector fields, and U, V are vertical vector fields, then*

$$\theta(X)(V,Y) = \theta(V)(X,Y) = \tfrac{1}{2} g\big(\Omega(X,Y), V\big), \quad \theta(U)(V,X) = g\big(\nabla^g_U V, X\big),$$

where Ω is the curvature of the fiber bundle, see Definition 2.1.2.

We also note the following formulæ for θ and the mean curvature k_v.

Lemma 2.1.8. *Let g_v be a vertical Riemannian metric. Then for U,V vertical and X horizontal*

$$\theta(U)(V,X) = -\tfrac{1}{2}\mathscr{L}_X(g_v)(U,V),$$

where $\mathscr{L}_X$ denotes the Lie derivative in the X direction. If $k_v \in C^\infty(M, T^hM^*)$ is the mean curvature of g_v, then

$$k_v(X) = \tfrac{1}{2}\,\mathrm{tr}_v\big[\mathscr{L}_X(g_v)\big] = -\sum_i \theta(e_i)(e_i, X),$$

where $\{e_i\}$ is an arbitrary local orthonormal frame for T^vM.

Proof. Let U, V be vertical and X horizontal. Then by definition of the Lie derivative

$$\begin{aligned}\mathscr{L}_X(g_v)(U,V) &= X g_v(U,V) - g_v\big([X,U],V\big) - g_v\big(U,[X,V]\big)\\ &= g_v\big(\nabla^v_X U - [X,U], V\big) + g_v\big(U, \nabla^v_X V - [X,V]\big),\end{aligned} \tag{2.4}$$

where we have used that ∇^v is a metric connection. This shows that

$$\mathrm{tr}_v\big[\mathscr{L}_X(g_v)\big] = 2\sum_i g_v\big(\nabla^v_X e_i - [X,e_i], e_i\big) = 2\,\mathrm{tr}_v(W_X) = 2\,k_v(X).$$

Now choose a metric g_B on B and endow M with the submersion metric $g = g_B \oplus g_v$. By definition of ∇^v, we can replace g_v and ∇^v in (2.4) with g and ∇^g, respectively. Since ∇^g is torsion free, we find

$$\begin{aligned}\mathscr{L}_X(g_v)(U,V) &= g\big(\nabla^g_X U - [X,U], V\big) + g\big(U, \nabla^g_X V - [X,V]\big)\\ &= g\big(\nabla^g_U X, V\big) + g\big(U, \nabla^g_V X\big)\\ &= g\big(X, -\nabla^g_U V - \nabla^g_V U\big) = -2g\big(X, \nabla^g_U V\big) + g\big(X, [U,V]\big).\end{aligned}$$

Now, $[U,V]$ is vertical, so that we find using Proposition 2.1.7 that

$$\mathscr{L}_X(g_v)(U,V) = -2g\big(X, \nabla^g_U V\big) = -2\theta(U)(V,X). \qquad \square$$

Convention. At this point it is convenient to introduce a convention regarding local computations. We will always denote by $\{e_i\}$ a local, oriented frame for T^vM and by $\{f_a\}$ a local, oriented frame for TB. The horizontal lifts to a frame for T^hM will be denoted with the same letters. Upper indices denote the dual frames and the summation convention will be understood. Indices $a, b, c, \dots$ always refer to horizontal directions and $i, j, k, \dots$ to vertical ones. If we have chosen metrics g_v and g_B, we will always choose local *orthonormal* frames. Whenever we do not want to distinguish horizontal and vertical directions, we use the notation $\{E_I\}$ for the frame $\{f_1, f_2, \dots, e_1, e_2, \dots\}$ with uppercase indices $I, J, K, \dots$. Moreover, if ∇ is a connection on a vector bundle over M, we will use the abbreviations ∇_a, ∇_i, ∇_I for ∇_{f_a}, ∇_{e_i}, ∇_{E_I}.

As an example regarding this convention, we write the tensor θ of Definition 2.1.6 as

$$\theta = \tfrac{1}{2}\theta_{IJK} E^I \otimes E^J \wedge E^K.$$

Then Proposition 2.1.7 shows that if we distinguish vertical and horizontal direction, the functions θ_{IJK} satisfy the relations

$$\begin{aligned} \theta_{aib} &= -\theta_{abi} = \theta_{iab} = \tfrac{1}{2} g\big(\Omega_{ab}, e_i\big), \\ \theta_{ija} &= -\theta_{iaj} = \theta_{jia} = g\big(\nabla^g_i e_j, f_a\big), \\ \theta_{ajk} &= \theta_{ijk} = \theta_{abc} = 0. \end{aligned} \tag{2.5}$$

Another Natural Connection. The connection ∇^v associated to a vertical Riemannian metric is not the only natural connection on T^vM. For $X \in C^\infty(M, T^hM)$ consider the vertical projection of the Lie derivative $\mathscr{L}_X$, i.e.,

$$\mathscr{L}^v_X : T^vM \to T^vM, \quad \mathscr{L}^v_X(V) := P^v[X, V].$$

Note that Lemma 2.1.1 yields that if X is a horizontal lift, then $[X, V]$ is automatically vertical. For general $X \in C^\infty(M, T^hM)$, $V \in C^\infty(M, T^vM)$ and $\varphi \in C^\infty(M)$ we have

$$\mathscr{L}^v_X(\varphi V) = P^v\big(\varphi[X, V] + (X\varphi)V\big) = \varphi\mathscr{L}^v_X(V) + (X\varphi)V$$

and, since X is horizontal,

$$\mathscr{L}^v_{\varphi X}(V) = P^v\big(\varphi[X, V] - (V\varphi)X\big) = \varphi\mathscr{L}^v_X(V).$$

Therefore, we can define a connection on T^vM as follows:

Definition 2.1.9. Let ∇^v be the natural connection associated to a vertical metric g_v. We define the connection $\widetilde{\nabla}^v$ on T^vM by

$$\widetilde{\nabla}^v_U := \nabla^v_U, \quad U \in C^\infty(M, T^vM), \quad \text{and} \quad \widetilde{\nabla}^v_X := \mathscr{L}^v_X, \quad X \in C^\infty(M, T^hM).$$

If g_B is a metric on B with Levi-Civita connection ∇^B, we also define a connection on $TM = \pi^*TB \oplus TM$ via

$$\widetilde{\nabla}^\oplus := \pi^*\nabla^B \oplus \widetilde{\nabla}^v.$$

Clearly, the definition of $\widetilde{\nabla}^v_X$ for horizontal X is independent of any choice of metric. However, for vertical U we cannot use the Lie derivative to define $\widetilde{\nabla}^v_U$ which is why the connection ∇^v_U enters the definition.

Lemma 2.1.10.

(i) *For* $X \in C^\infty(M, T^hM)$
$$\widetilde{\nabla}^v_X(g_v) = \mathscr{L}_X(g_v),$$
i.e., $\widetilde{\nabla}^v_X$ *does in general not preserve the metric* g_v. *Similarly* $\widetilde{\nabla}^\oplus$, *does in general not preserve the metric* $g = g_B \oplus g_v$.

(ii) *The torsion of* $\widetilde{\nabla}^\oplus$ *coincides with the curvature of the fiber bundle,*
$$T(\widetilde{\nabla}^\oplus) = \Omega \in C^\infty\big(M, \Lambda^2 T^hM^* \otimes T^vM\big).$$

Proof. Part (i) is clear by definition. For part (ii) we use local orthonormal frames $\{f_a\}$ and $\{e_i\}$ for TB and T^vM according to the convention on p. 53. Then one computes

$$\begin{aligned}
&\widetilde{\nabla}^\oplus_i e_j - \widetilde{\nabla}^\oplus_j e_i = \nabla^v_i e_j - \nabla^v_j e_i = P^v\big(\nabla^g_i e_j - \nabla^g_j e_i\big) = P^v[e_i, e_j] = [e_i, e_j],\\
&\widetilde{\nabla}^\oplus_a e_i - \widetilde{\nabla}^\oplus_i f_a = \widetilde{\nabla}^v_a e_i = P^v[f_a, e_i] = [f_a, e_i], \quad \text{and}\\
&\widetilde{\nabla}^\oplus_a f_b - \widetilde{\nabla}^\oplus_b f_a = \pi^*\big(\nabla^B_a f_b - \nabla^B_b f_a\big) = \pi^*[f_a, f_b] = [f_a, f_b]^h = [f_a, f_b] + \Omega(f_a, f_b).
\end{aligned}$$

Note that in the first and third row we have used that ∇^g respectively ∇^B are torsion free, and in the second row that $[f_a, e_i]$ is vertical as f_a is a horizontal lift, see Lemma 2.1.1. □

2.1.2 Differential Forms on a Fiber Bundle.

Let $F \hookrightarrow M \xrightarrow{\pi} B$ be an oriented fiber bundle as above. The natural exact sequence
$$T^vM \hookrightarrow TM \to \pi^*TB$$
translates to exterior bundles as
$$\pi^*\Lambda^p T^*B \hookrightarrow \Lambda^p T^*M \to \Lambda^p(T^vM)^*.$$
In terms of differential forms, the pullback of forms gives a natural inclusion
$$\pi^* : \Omega^p(B) \to \Omega^p(M).$$

Definition 2.1.11. Let $\omega \in \Omega^\bullet(M)$, and let i(.) denote inner multiplication with a vector field. Then ω is called

(i) *horizontal* if $\mathrm{i}(V)\omega = 0$ for all $V \in C^\infty(M, T^vM)$,

(ii) *basic* if $\omega = \pi^*\alpha$ for some $\alpha \in \Omega^\bullet(B)$, and

(iii) *vertical* if $\mathrm{i}(X)\omega = 0$ for all $X \in C^\infty(M, T^hM)$.

We denote by $\Omega^\bullet_v(M)$ the algebra of vertical differential forms, and by $\Omega^\bullet_h(M)$ the algebra of horizontal differential forms.

Remark.

(i) Note that the definition of vertical forms requires the choice of a vertical projection $P^v : TM \to T^vM$, whereas horizontal and basic forms are defined independently of such a choice. Moreover, P^v gives an identification

$$\Phi : C^\infty\big(M, \Lambda^q(T^vM)^*\big) \to \Omega_v^q(M), \quad \Phi(\omega)(X_1, \dots, X_q) := \omega(P^vX_1, \dots, P^vX_q).$$

We will usually suppress this isomorphism from the notation and identify a section of $(T^vM)^*$ with a vertical differential form.

(ii) A horizontal form ω can always be written (non-uniquely) as a sum

$$\omega = \sum_i \varphi_i(\pi^*\alpha_i), \quad \varphi_i \in C^\infty(M), \quad \alpha_i \in \Omega^\bullet(B).$$

Thus, one a somewhat formal level,

$$\Omega_h^\bullet(M) \cong \pi^*\Omega^\bullet(B) \otimes C^\infty(M),$$

where the tensor product is over $\pi^*C^\infty(B)$.

(iii) More generally, we can decompose every differential form ω on M as

$$\omega = \sum_i \alpha_i \wedge \beta_i, \quad \alpha_i \in \Omega_h^\bullet(M), \quad \beta_i \in \Omega_v^\bullet(M),$$

and so

$$\Omega^k(M) \cong \bigoplus_{p+q=k} \pi^*\Omega^p(B) \otimes_s \Omega_v^q(M) =: \bigoplus_{p+q=k} \Omega^{p,q}(M). \tag{2.6}$$

Here, $\otimes_s$ is a graded tensor product over $\pi^*C^\infty(B)$. We will often write $\otimes$ instead of $\otimes_s$, keeping in mind that there is a grading involved.

(iv) So far we have only implicitly remarked on our orientation convention. We use the *basis first* orientation which can be described as follows: In (2.6) we consider the case $k = \dim M$, $p = \dim B$ and $q = \dim F$. Let $\mathrm{vol}_B(g_B) \in \Omega^p(B)$ and $\mathrm{vol}_F(g_v) \in \Omega_v^q(M)$ be oriented volume forms associated to metrics g_B and g_v. Then we orient M according to the prescription

$$\mathrm{vol}_M(g) = \pi^*\big(\mathrm{vol}_B(g_B)\big) \wedge \mathrm{vol}_F(g_v). \tag{2.7}$$

Integration Along the Fiber. If the fiber bundle is endowed with a connection and a vertical metric, there is a natural right-inverse for the map $\pi^* : \Omega^p(B) \to \Omega^p(M)$. We recall the following well-known facts, see e.g. [22, Sec. 1.6].

Proposition 2.1.12. *There is a natural homomorphism of $C^\infty(B)$ modules,*

$$\int_{M/B} : \Omega^{\bullet,\dim F}(M) \to \Omega^\bullet(B),$$

called "integration along the fiber"*, which is uniquely defined by the property that*

$$\int_B \alpha \wedge \Big(\int_{M/B} \omega\Big) = \int_M \pi^*\alpha \wedge \omega, \quad \alpha \in \Omega^{\dim B - k}(B), \quad \omega \in \Omega^{k,\dim F}(M).$$

Moreover, for α and ω as above,

$$d_B \int_{M/B} \omega = \int_{M/B} d_M\omega \quad \textit{and} \quad \alpha \wedge \int_{M/B} \omega = \int_{M/B} \pi^*\alpha \wedge \omega.$$

Definition 2.1.13. Let $\mathrm{vol}_F(g_v) \in \Omega_v^{\dim F}(M)$ be the vertical volume form associated to a vertical metric, and let

$$v_F(g_v) := \int_{M/B} \mathrm{vol}_F(g_v) \in C^\infty(B)$$

be the function which associates to a point $y \in B$ the volume of the fiber over y. Then we define the *basic projection* on horizontal forms as

$$\Pi_B : \Omega_h^\bullet(M) \to \Omega^\bullet(B), \quad \Pi_B(\omega) := \frac{1}{v_F(g_v)} \int_{M/B} \omega \wedge \mathrm{vol}_F(g_v).$$

Here, the normalization factor enters since we want $\Pi_B(\pi^*\alpha) = \alpha$ for every $\alpha \in \Omega^\bullet(B)$. We also want to point out that if we allow conformal changes of the vertical metric, we can easily achieve that $v_F(g_v) = 1$. This is the content of the following simple result.

Lemma 2.1.14. *Let $n := \dim F$, and let g_v be a vertical metric. Define*

$$u := \tfrac{1}{n} \log\big(\pi^* v_F(g_v)\big) \in C^\infty(M).$$

Then the metric $\widetilde{g}_v := e^{-2u} g_v$ has unit volume along the fibers, i.e., $v_F(\widetilde{g}_v) = 1$.

Proof. Let $\{e_i\}$ be a local, oriented orthonormal frame for (T^vM, g_v). Let u be defined as above, and let $\widetilde{e}_i := \exp(u)e_i$. Then $\{\widetilde{e}_i\}$ is a local, oriented orthonormal frame with respect to the metric $\widetilde{g}_v$, and

$$\mathrm{vol}_F(\widetilde{g}_v) = \widetilde{e}^1 \wedge \ldots \wedge \widetilde{e}^n = (e^{-nu})e^1 \wedge \ldots \wedge e^n = \big(\pi^* v_F(g_v)\big)^{-1} \mathrm{vol}_F(g_v).$$

This yields that $v_F(\widetilde{g}_v) = 1$. □

One might expect, that there is a canonical description of the kernel of Π_B. This is indeed true. However, the corresponding result is not completely straightforward. We give a proof in Chapter 3, see Proposition 3.1.8. For the time being we need a better understanding of the calculus for differential forms on fiber bundles.

2.1.3 The Exterior Differential of a Fiber Bundle.

Let $F \hookrightarrow M \xrightarrow{\pi} B$ be an oriented fiber bundle of closed manifolds as before. Let $E \to M$ be a Hermitian vector bundle which admits a flat connection A. We denote by $\Omega_v^q(M,E)$ the space of vertical E-valued q-forms, i.e., the space of sections of $\Lambda^q T^vM^* \otimes E$. The canonical connection ∇^v in Proposition 2.1.3 together with A induces a natural connection

$$\nabla^{A,v} : \Omega_v^\bullet(M,E) \to \Omega^1(M) \otimes \Omega_v^\bullet(M,E).$$

Moreover, we get a vertical differential

$$d_{A,v} : \Omega_v^q(M,E) \to \Omega_v^{q+1}(M,E), \quad d_{A,v} = e^i \wedge \nabla_i^{A,v},$$

where $\{e_i\}$ is any local orthonormal frame for T^vM. As in (2.6), we can split the space of E-valued k-forms on M as

$$\Omega^k(M,E) = \bigoplus_{p+q=k} \Omega^{p,q}(M,E).$$

The vertical differential then extends to $\Omega^\bullet(M,E)$ by requiring that

$$d_{A,v}(\alpha \otimes \omega) = (-1)^p \alpha \otimes d_{A,v}\omega, \quad \alpha \otimes \omega \in \Omega^{p,q}(M,E).$$

On the other hand, the connection A defines a total exterior differential d_A on $\Omega^\bullet(M,E)$. It inherits a bigrading

$$d_A = \sum_{i+j=1} d_{ij}, \quad \text{where} \quad d_{ij} : \Omega^{p,q}(M,E) \to \Omega^{p+i,q+j}(M,E) \quad \text{for all } p,q.$$

We now want to describe this in terms of the data introduced in Section 2.1.1. Let $\widetilde{\nabla}^v$ be the connection on T^vM as in Definition 2.1.9. It induces a connection on vertical differential forms, which we denote by the same letter. Similarly, we obtain a connection $\widetilde{\nabla}^\oplus$ on $\Lambda^\bullet T^*M$, and using A, we define $\widetilde{\nabla}^{A,\oplus}$ on $\Lambda^\bullet T^*M \otimes E$.

Remark. There is a subtlety concerning the action on vertical differential forms of the connection $\widetilde{\nabla}^v_{X^h}$, if X^h is the horizontal lift of a vector field X on B. Since on T^vM, the action of $\widetilde{\nabla}^v_{X^h}$ is given by the Lie derivative, one might expect that the same is true for its action on forms. However, it is in general not true that $\mathscr{L}_{X^h}\omega$ is automatically vertical for a vertical differential form ω, compare with Lemma 2.1.1. For example, if $\omega \in \Omega^1_v(M)$, then the Cartan formula yields

$$\mathscr{L}_{X^h}(\omega)(Y^h) = (\mathrm{i}(X^h) \circ d\omega)(Y^h) = -\omega\big([X^h, Y^h]\big) = \omega\big(\Omega(X,Y)\big).$$

This is in general non-zero, so that $\mathscr{L}_{X^h}\omega$ is in general not a vertical form. Thus, $\widetilde{\nabla}^v_{X^h}$ agrees in general only with the vertical projection $\mathscr{L}^v_{X^h}$ of the Lie derivative.

As always let $\{f_a\}$ be a local frame for TB, and write $\Omega = \frac{1}{2} f^a \wedge f^b \otimes \Omega_{ab}$ for the curvature of the fiber bundle.

Proposition 2.1.15. *The total exterior differential d_A on $\Omega^\bullet(M,E)$ splits as*

$$d_A = d_{A,v} + d_{A,h} + \mathrm{i}(\Omega),$$

where for $\omega \in \Omega^\bullet(M,E)$

$$d_{A,h}\omega = f^a \wedge \widetilde{\nabla}^{A,\oplus}_a \omega, \quad \text{and} \quad \mathrm{i}(\Omega)\omega = \tfrac{1}{2} f^a \wedge f^b \wedge \mathrm{i}(\Omega_{ab})\omega.$$

Here, $\mathrm{i}(\Omega_{ab})$ denotes interior multiplication with $\Omega_{ab} \in C^\infty(M, T^vM)$.

For convenience we sketch a proof, although the result is well known, see [13, Prop. 10.1] or [19, Prop. 3.4]. Before we do so, let us point out that there is no $d_{-1,2}$ contribution to d_A, which is due to the fact that the vertical distribution is integrable, see [74, p. 58].

Proof of Proposition 2.1.15. We assume for simplicity that A is the trivial connection on the trivial line bundle. Recall, e.g. from [13, Prop. 1.22], that if ∇ is a torsion free connection on TM, then d can be expressed in terms of ∇ as the composition

$$\Omega^\bullet(M) \xrightarrow{\nabla} \Omega^1(M) \otimes \Omega^\bullet(M) \xrightarrow{e\,\circ} \Omega^{\bullet+1}(M). \tag{2.8}$$

Here, the second arrow means contraction with exterior multiplication. Let g be a metric of the form $g_B \oplus g_v$. It follows from Lemma 2.1.10, that we can define a torsion free[1] connection ∇ on TM by

$$\nabla_X := \widetilde{\nabla}^{\oplus}_X - \tfrac{1}{2}\Omega(X, .), \quad X \in C^\infty(M, TM).$$

Let $\{f_a\}$ and $\{e_i\}$ be local orthonormal frames for TB and T^vM respectively. Since d satisfies the Leibniz rule it suffices to compute de^i and df^a. From (2.8) and the definition of $\widetilde{\nabla}^{\oplus}$ we get

$$\begin{aligned} de^i &= e^j \wedge \nabla_j e^i + f^a \wedge \nabla_a e^i \\ &= e^j \wedge \nabla^v_j e^i + f^a \wedge \big(- e^i(\nabla_a e_k) e^k - e^i(\nabla_a f_b) f^b \big) \\ &= d_v e^i + f^a \wedge \big(- e^i([f_a, e_k]) e^k - e^i(-\tfrac{1}{2}\Omega_{ab}) f^b \big) \\ &= d_v e^i + f^a \wedge \widetilde{\nabla}^{\oplus}_a e^i + \tfrac{1}{2} e^i(\Omega_{ab}) f^a \wedge f^b. \end{aligned}$$

This is the required formula for de^i. On the other hand, since ∇_j acts trivially on basic forms, and $\Omega(f_b, f_c)$ has no horizontal component, one easily finds that

$$df^a = f^b \wedge \nabla_b f^a = f^b \wedge \widetilde{\nabla}^{\oplus}_b f^a. \qquad \square$$

Since A is flat we have $d_A^2 = 0$. This implies the following.

Corollary 2.1.16. *Let $\{.,.\}$ denote the anti-commutator of two operators in the ungraded sense. Then*

$$d^2_{A,v} = \mathrm{i}(\Omega)^2 = 0, \quad d^2_{A,h} + \big\{ d_{A,v}, \mathrm{i}(\Omega) \big\} = 0,$$
$$\big\{ d_{A,v}, d_{A,h} \big\} = \big\{ d_{A,h}, \mathrm{i}(\Omega) \big\} = 0.$$

More on the Mean Curvature. From Proposition 2.1.15, we can also deduce the following formula which relates the mean curvature with the differential of the vertical volume form, see [13, Lem. 10.4]. In the theory of foliations, this is known as *Rummler's formula*, see e.g. [95, p. 38].

Proposition 2.1.17. *Let $\mathrm{vol}_F(g_v)$ be the volume form associated to a vertical metric. Let $k_v \in \Omega^{1,0}(M)$ be the mean curvature form. Then*

$$d_M \,\mathrm{vol}_F(g_v) = k_v \wedge \mathrm{vol}_F(g_v) + \mathrm{i}(\Omega)\,\mathrm{vol}_F(g_v).$$

Proof. Since $\mathrm{vol}_F(g_v)$ has maximal vertical degree, Proposition 2.1.15 implies that

$$d_M \,\mathrm{vol}_F(g_v) = d_h \,\mathrm{vol}_F(g_v) + \mathrm{i}(\Omega)\,\mathrm{vol}_F(g_v).$$

[1] Clearly, we could also use the Levi-Civita connection associated to the metric g. However, its explicit formula is more complicated (see also Remark 2.1.20 below).

If $\{f_a\}$ and $\{e_j\}$ are local frames for TB and T^vM, we compute

$$\begin{aligned} d_h \operatorname{vol}_F(g_v) &= f^a \wedge \widetilde{\nabla}^v_a\big(\operatorname{vol}_F(g_v)\big) = f^a \wedge \widetilde{\nabla}^v_a(e^j) \wedge \mathrm{i}(e_j)\big(\operatorname{vol}_F(g_v)\big) \\ &= -f^a \wedge e^j\big([f_a, e_k]\big)e^k \wedge \mathrm{i}(e_j)\big(\operatorname{vol}_F(g_v)\big) \\ &= -f^a \wedge e^j\big([f_a, e_k]\big)\delta^k_j \wedge \operatorname{vol}_F(g_v) \\ &= -\sum_j g_v\big([f_a, e_j], e_j\big) f^a \wedge \operatorname{vol}_F(g_v). \end{aligned}$$

Now, (2.2) identifies the last line with $k_v \wedge \operatorname{vol}_F(g_v)$. □

Corollary 2.1.18. *Let* $\operatorname{vol}_F(g_v)$ *be the volume form associated to a vertical metric* g_v*, and let* v_F *be the volume of the fiber as in Definition 2.1.13. Then the basic projection of the mean curvature form is given by*

$$\Pi_B(k_v) = d_B \log(v_F) \in \Omega^1(B).$$

Proof. We differentiate v_F and use Proposition 2.1.17 to find that

$$d_B v_F = d_B \int_{M/B} \operatorname{vol}_F(g_v) = \int_{M/B} d_M \operatorname{vol}_F(g_v) = \int_{M/B} k_v \wedge \operatorname{vol}_F(g_v) = v_F\, \Pi_B(k_v). \quad \square$$

Corollary 2.1.18 shows that the basic projection of the mean curvature form gives a trivial element in the cohomology of the base. Moreover, it vanishes if the metric g_v has constant volume along the fiber. As we have seen in Lemma 2.1.14, this can be achieved by a conformal deformation of the vertical metric. In Section 3.1 we will transfer a result of [36] from the theory of foliations to the situation at hand and show that one can always deform the vertical projection and the vertical metric of the fiber bundle in such a way that not only the basic projection but the mean curvature form itself vanishes.

2.1.4 The Levi-Civita Connection on Forms and the Adjoint Differential

To study the de Rham operator on a fiber bundle $F \hookrightarrow M \xrightarrow{\pi} B$, we also need to understand the adjoint differential d^t_A, where A is a flat connection on a Hermitian vector bundle $E \to M$. For this we will use the local formula

$$d^t_A = -\,\mathrm{i}(E^I)\nabla^{A,g}_I,$$

where $\{E_I\}$ is a local orthonormal frame for TM, and $\nabla^{A,g}$ is the Levi-Civita connection on forms twisted by A. We want to use this formula to split d^t_A in terms of its bidegrees with respect to the decomposition (2.6). For this we need to relate the Levi-Civita connection $\nabla^{A,g}$ on forms with the connection $\nabla^{A,\oplus}$. Recall that in Definition 1.2.17 we have introduced a transposed Clifford as

$$\widehat{c}\colon T^*M \to \operatorname{End}(\Lambda^\bullet T^*M), \quad \widehat{c}(\xi) = \mathrm{e}(\xi) + \mathrm{i}(\xi).$$

Lemma 2.1.19. *Let* $E \to M$ *be a Hermitian bundle which admits a flat connection* A*. Then, on* $\Lambda^\bullet T^*M \otimes E$*, the difference of* $\nabla^{A,g}$ *and* $\nabla^{A,\oplus}$ *is given by*

$$\nabla^{A,g} = \nabla^{A,\oplus} + \tfrac{1}{2}\big(c(\theta) - \widehat{c}(\theta)\big),$$

where θ *is the tensor defined in Definition 2.1.6.*

Proof. Let $\{E_I\}$ be a local orthonormal frame of TM with dual coframe $\{E^I\}$. Then by definition of S and θ,

$$\begin{aligned}(\nabla_I^{A,g} - \nabla_I^{A,\oplus}) &= (\nabla_I^{A,g} - \nabla_I^{A,\oplus})(E^J) \wedge \mathrm{i}(E_J) = -E^J\big(S(E_I)E_K\big)\,\mathrm{e}(E^K)\,\mathrm{i}(E_J)\\ &= -\theta_{IKJ}\,\mathrm{e}(E^K)\,\mathrm{i}(E^J).\end{aligned}$$

On the other hand,

$$\begin{aligned}\tfrac{1}{2}\big(c(\theta_I) - \widehat{c}(\theta_I)\big) &= \tfrac{1}{4}\theta_{IJK}\big(c(E^J)c(E^K) - \widehat{c}(E^J)\widehat{c}(E^K)\big)\\ &= -\tfrac{1}{2}\theta_{IJK}\big(\,\mathrm{e}(E^J)\,\mathrm{i}(E^K) - \mathrm{i}(E^J)\,\mathrm{e}(E^K)\big)\\ &= -\tfrac{1}{2}\theta_{IKJ}\,\mathrm{e}(E^K)\,\mathrm{i}(E^J) + \tfrac{1}{2}\theta_{IJK}\,\mathrm{e}(E^K)\,\mathrm{i}(E^J) = -\theta_{IKJ}\,\mathrm{e}(E^K)\,\mathrm{i}(E^J),\end{aligned}$$

where in the last line we have first renamed J and K in the first summand and then used antisymmetry of θ_{IJK} in J and K for the second summand, see (2.5). □

Remark 2.1.20. One could use the above result to give a different proof of Proposition 2.1.15 by writing out locally

$$d_A = E^I \wedge \nabla_I^{A,g} = E^I \wedge \nabla_I^{A,\oplus} - \theta_{IKJ}E^I \wedge E^K \wedge \mathrm{i}(E^J)$$

Splitting this into horizontal and vertical contributions, and using (2.5), one verifies that

$$d_A = d_{A,v} + f^a \wedge \big(\nabla_a^{A,\oplus} - \theta_{kja}\, e^k \wedge \mathrm{i}(e^j)\big) + \mathrm{i}(\Omega).$$

Comparing this with Proposition 2.1.15 we see that in particular,

$$\widetilde{\nabla}_a^v = \nabla_a^v - \theta_{kja}\, e^k \wedge \mathrm{i}(e^j). \tag{2.9}$$

A more invariant description of the term occurring here is as follows: Define a tensor field $B \in C^\infty\big(M, T^hM^* \otimes \mathrm{End}(\Lambda^\bullet T^vM^*)\big)$ by requiring that

$$g_v\big(\alpha, B(X)\beta\big) = \mathscr{L}_X^v(g_v)(\alpha,\beta), \quad X \in C^\infty(M, T^hM), \quad \alpha,\beta \in \Omega_v^\bullet(M).$$

Then Lemma 2.1.8 (or a direct computation) easily implies that

$$B(X) = X^a\theta_{kja}\, e^k \wedge \mathrm{i}(e^j). \tag{2.10}$$

We now have the following analog of Proposition 2.1.15 for the adjoint differential.

Proposition 2.1.21. *Let $E \to M$ be a Hermitian vector bundle which admits a flat connection A. Then the twisted adjoint differential splits as*

$$d_A^t = d_{A,v}^t + d_{A,h}^t + \mathrm{i}(\Omega)^t,$$

where the terms are given in local orthonormal frames $\{e_i\}$ and $\{f_a\}$ by

$$\begin{aligned}d_{A,v}^t &= -\,\mathrm{i}(e^i)\circ\nabla_i^{A,\oplus} : \Omega^{p,q} \to \Omega^{p,q-1},\\ d_{A,h}^t &= -\,\mathrm{i}(f^a)\circ\big(\nabla_a^{A,\oplus} + B(f_a) + k_v(f_a)\big) : \Omega^{p,q} \to \Omega^{p-1,q},\\ \mathrm{i}(\Omega)^t &= -\tfrac{1}{2}\,\mathrm{i}(f^a)\,\mathrm{i}(f^b)\,\mathrm{e}(\Omega_{ab}) : \Omega^{p,q} \to \Omega^{p-2,q+1}.\end{aligned}$$

Here, $B(f_a)$ is defined as in (2.10), k_v is the mean curvature form, and $\mathrm{e}(\Omega_{ab})$ denotes exterior multiplication with the dual of $\Omega_{ab} \in C^\infty(M, T^vM)$.

Remark 2.1.22. The definition of the connection $\widetilde{\nabla}^v$ and the relation (2.9) between ∇^v and $\widetilde{\nabla}^v$ shows that we can write alternatively

$$\begin{aligned} d^t_{A,v} &= -\,\mathrm{i}(e^i) \circ \widetilde{\nabla}^{A,\oplus}_i : \Omega^{p,q} \to \Omega^{p,q-1}, \\ d^t_{A,h} &= -\,\mathrm{i}(f^a) \circ \big(\widetilde{\nabla}^{A,\oplus}_a + 2B(f_a) + k_v(f_a)\big) : \Omega^{p,q} \to \Omega^{p-1,q}. \end{aligned}$$

Proof of Proposition 2.1.21. For convenience we drop again the reference to the flat connection. According to Lemma 2.1.19 and the local formula for d^t we have

$$d^t = -\,\mathrm{i}(E^I)\nabla^{\oplus}_I + \mathrm{i}(E^I)\theta_{IJK}\,\mathrm{e}(E^J)\,\mathrm{i}(E^K).$$

Splitting this into horizontal and vertical parts and checking bidegrees one finds

$$d^t_v = -\,\mathrm{i}(e^j)\nabla^{\oplus}_j + \mathrm{i}(e^j)\theta_{jab}\,\mathrm{e}(f^a)\,\mathrm{i}(f^b) + \mathrm{i}(f^a)\theta_{abj}\,\mathrm{e}(f^b)\,\mathrm{i}(e^j) = -\,\mathrm{i}(e^j)\nabla^{\oplus}_j.$$

Here, we have used the relation $\theta_{abj} = -\theta_{jab}$, see (2.5). Similarly,

$$\begin{aligned} d^t_h &= -\,\mathrm{i}(f^a)\nabla^{\oplus}_a + \mathrm{i}(e^j)\theta_{jka}\,\mathrm{e}(e^k)\,\mathrm{i}(f^a) = -\,\mathrm{i}(f^a)\big(\nabla^{\oplus}_a - \theta_{jka}\,\mathrm{i}(e^j)\,\mathrm{e}(e^k)\big) \\ &= -\,\mathrm{i}(f^a)\Big(\nabla^{\oplus}_a + \theta_{kja}\big(\,\mathrm{e}(e^k)\,\mathrm{i}(e^j) - \delta^{jk}\big)\Big) = -\,\mathrm{i}(f^a)\big(\nabla^{A,\oplus}_a + B(f_a) + k_v(f_a)\big), \end{aligned}$$

where we have used symmetry of θ_{jka} in j and k, the definition of $B(f_a)$, and Lemma 2.1.8. Lastly, one has

$$\mathrm{i}(\Omega)^t = \mathrm{i}(f^a)\theta_{ajb}\,\mathrm{e}(e^j)\,\mathrm{i}(f^b) = -\tfrac{1}{2}\,\mathrm{i}(f^a)\,\mathrm{i}(f^b)\,\mathrm{e}(\Omega_{ab}).$$ □

Partial de Rham Operators. Having established the description of the adjoint differential in analogy to Proposition 2.1.15, we can now split the twisted de Rham operator on M.

Definition 2.1.23. We use the abbreviations

$$D_{A,v} := d_{A,v} + d^t_{A,v}, \quad D_{A,h} := d_{A,h} + d^t_{A,h}, \quad \text{and} \quad T := \mathrm{i}(\Omega) + \mathrm{i}(\Omega)^t.$$

The operators $D_{A,v}$ and $D_{A,h}$ are called the *vertical* respectively *horizontal* twisted de Rham operator on M.

Remark 2.1.24.

(i) Tautologically, the de Rham operator on M splits as

$$D_A = D_{A,v} + D_{A,h} + T : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E). \tag{2.11}$$

(ii) The vertical de Rham operator is a first order differential operator *acting fiberwise*, i.e., $[D_{A,v}, \pi^*\varphi] = 0$ for all $\varphi \in C^\infty(B)$. This means roughly, that it can be thought of as a smooth family of first order elliptic differential operators on $\Lambda^\bullet T^*F \otimes E|_F$ parametrized by B, see Definition 1.3.9. We refer to Section 3.1 for some more details. Clearly, an analogous statement cannot be formulated for the horizontal de Rham operator $D_{A,h}$, unless the horizontal distribution is integrable.

(iii) We want to add some remarks about the effect the splitting (2.11) has on the spectrum of D_A. Let us assume that A is the trivial connection. The appearance of the mean curvature form in the formula for d_h^t shows that in general, D_h will not restrict to an operator on basic forms, see Definition 2.1.11. Instead, if D_B denotes the de Rham operator on B,

$$D_h(\pi^*\alpha) = \pi^*(D_B\alpha) - \mathrm{i}(k_v)\pi^*\alpha, \quad \alpha \in \Omega^\bullet(B), \tag{2.12}$$

where $\mathrm{i}(k_v)$ denotes interior multiplication with the mean curvature form. However—as already pointed out—we will see in Section 3.1 that upon changing the vertical metric and the horizontal distribution, we can achieve that k_v vanishes. In this case (2.12) shows that eigenforms of D_B lift to eigenforms of D_h. Since D_v vanishes on basic forms, this produces eigenforms of $D_v + D_h$. In particular, $\mathrm{spec}(D_B) \subset \mathrm{spec}(D_v + D_h)$. However, the full de Rham operator on M is given by (2.11), and T will in general not act trivially on $\pi^*\Omega^\bullet(B)$. This should give a hint at why—even in the case that k_v vanishes—the relation between the spectrum of D_M and the spectra of D_h and D_v is non-trivial. We refer to [50, Ch.'s 3&4] for a detailed study of related questions.

Some Commutator Relations. The explicit description of the adjoint differential has the following consequence, see also [1, Prop. 3.1] for a generalization.

Proposition 2.1.25. *Let $\{f_a\}$ be local orthonormal frame for TB, and define a bundle endomorphism*

$$K := -\,\mathrm{i}(f^a) \circ \big(2B(f_a) + k_v(f_a)\big) : \Lambda^\bullet T^*M \to \Lambda^\bullet T^*M. \tag{2.13}$$

Then

$$d_{A,v} d_{A,h}^t + d_{A,h}^t d_{A,v} = d_{A,v} K + K d_{A,v}.$$

Proof. If $\{e_i\}$ is a local orthonormal frame for T^vM, one checks that

$$\widetilde{\nabla}_a^{A,\oplus} \circ d_{A,v} = d_{A,v} \circ \widetilde{\nabla}_a^{A,\oplus}.$$

Since $\widetilde{\nabla}_i^{A,\oplus} f^a = 0$, this implies

$$d_{A,v} \circ \big(\mathrm{i}(f^a) \circ \widetilde{\nabla}_a^{A,\oplus}\big) + \big(\mathrm{i}(f^a) \circ \widetilde{\nabla}_a^{A,\oplus}\big) \circ d_{A,v} = 0.$$

Now Remark 2.1.22 yields that

$$d_{A,h}^t = -\,\mathrm{i}(f^a) \circ \big(\widetilde{\nabla}_a^{A,\oplus} + 2B(f_a) + k_v(f_a)\big).$$

Then, with K as in (2.13), one easily verifies that indeed

$$d_{A,v} d_{A,h}^t + d_{A,h}^t d_{A,v} = d_{A,v} K + K d_{A,v}. \qquad \square$$

Corollary 2.1.26. *The anti-commutator $\{D_{A,v}, D_{A,h}\}$ is a first order differential operator acting fiberwise.*

Proof. According to Corollary 2.1.16 and the corresponding statement for the formal adjoints, we have

$$\{d_{A,h}, d_{A,v}\} = 0 \quad \text{and} \quad \{d_{A,h}^t, d_{A,v}^t\} = 0.$$

This implies

$$\{D_{A,v}, D_{A,h}\} = \{d_{A,v}, d_{A,h}^t\} + \{d_{A,v}^t, d_{A,h}\} = \{d_{A,v}, K\} + \{K^t, d_{A,v}^t\},$$

which is $C^\infty(B)$ linear and thus a first order differential operator acting fiberwise. $\square$

2.2 Rho Invariants and Adiabatic Metrics

2.2.1 The Odd Signature Operator

As before let $F \hookrightarrow M \xrightarrow{\pi} B$ be an oriented fiber bundle of closed manifolds, and let $E \to M$ be a Hermitian vector bundle endowed with a unitary flat connection A. We endow T^vM with a metric g_v and B with a Riemannian metric g_B, and consider the associated submersion metric $g := g_B \oplus g_v$.

If $\dim M$ is odd the odd signature operator on M twisted by A is given in terms of the partial de Rham operators introduced in Definition 2.1.23 as

$$B_A^{\mathrm{ev}} = \tau_M D_{A,v} + \tau_M D_{A,h} + \tau_M T : \Omega^{\mathrm{ev}}(M,E) \to \Omega^{\mathrm{ev}}(M,E), \tag{2.14}$$

where τ_M is the chirality operator on the total space M of the fiber bundle. It is useful to identify $\Omega^{\mathrm{ev}}(M,E)$ in terms of the splitting $TM = \pi^*TB \oplus T^vM$. From (2.6) we see that

$$\Omega^{\mathrm{ev}}(M,E) = \sum_{p+q\equiv 0\,(2)} \pi^*\Omega^p(B) \otimes \Omega_v^q(M,E).$$

Using this identification, we define

$$\Phi : \Omega^{\mathrm{ev}}(M,E) \to \pi^*\Omega^\bullet(B) \otimes \Omega_v^\bullet(M,E), \quad \Phi(\alpha \otimes \omega) = \alpha^e \otimes \omega^e + \tau_M(\alpha^o \otimes \omega^o),$$

where $\alpha^{e/o}$ and $\omega^{e/o}$ refer to the even/odd degree parts. Since it is straightforward, we skip the proof of the following result.

Lemma 2.2.1. *Assume that M is odd dimensional.*

(i) *If the fiber F is even dimensional, then Φ gives rise to an isometry*

$$\Phi : \Omega^{\mathrm{ev}}(M,E) \xrightarrow{\cong} \pi^*\Omega^{\mathrm{ev}}(B) \otimes \Omega_v^\bullet(M,E),$$

and the odd signature operator is equivalent to

$$\Phi \circ B_A^{\mathrm{ev}} \circ \Phi^{-1} = D_{A,v} + \tau_M D_{A,h} + T$$

(ii) *If F is odd dimensional, then*

$$\Phi : \Omega^{\mathrm{ev}}(M,E) \xrightarrow{\cong} \pi^*\Omega^\bullet(B) \otimes \Omega_v^{\mathrm{ev}}(M,E),$$

and

$$\Phi \circ B_A^{\mathrm{ev}} \circ \Phi^{-1} = \tau_M D_{A,v} + D_{A,h} + \tau_M T.$$

The Vertical Chirality Operator. For a more explicit formula for the odd signature one needs to understand how the chirality operator splits with respect to (2.6). In the general setting at hand we will not give a detail account but add some remarks which will be used in the examples below.

Definition 2.2.2. Let M be endowed with a vertical metric g_v, and let $n := \dim F$. Then the *vertical chirality operator* $\tau_v : \Omega_v^q(M) \to \Omega_v^{n-q}(M)$ is defined with respect to an oriented, orthonormal frame $\{e_i\}$ for T^vM by

$$\tau_v = i^{[\frac{n+1}{2}]} c_v(e^1) \cdot \ldots \cdot c_v(e^n).$$

Here, $c_v : T^vM^* \to \operatorname{End}\big(\Lambda^\bullet T^vM^*\big)$ is the vertical Clifford multiplication,

$$c_v(\xi) = \mathrm{e}(\xi) - \mathrm{i}(\xi), \quad \xi \in \Omega_v^1(M).$$

The vertical Clifford multiplication extends naturally to vertical differential forms, and up to the normalization factor, τ_v is Clifford multiplication with the vertical volume form. In particular, it is independent of the chosen frame. We also recall the convention (2.7) that if $g = g_B \oplus g_v$, we orient M using

$$\mathrm{vol}_M(g) = \pi^*\big(\mathrm{vol}_B(g_B)\big) \wedge \mathrm{vol}_F(g_v).$$

Lemma 2.2.3. *Let τ_B be the chirality operator on $\Omega^\bullet(B)$, and let $(\pi^*\alpha) \wedge \omega \in \Omega^{p,q}(M)$.*

(i) *Assume that F is even dimensional. Then*

$$\tau_M(\pi^*\alpha \wedge \omega) = \pi^*(\tau_B\alpha) \wedge \tau_v\omega.$$

(ii) *If F is odd dimensional, then*

$$\tau_M(\pi^*\alpha \wedge \omega) = (-1)^p \cdot \begin{cases} \pi^*(\tau_B\alpha) \wedge \tau_v\omega, & \text{if } B \text{ is even dimensional,} \\ -i \cdot \pi^*(\tau_B\alpha) \wedge \tau_v\omega, & \text{if } B \text{ is odd dimensional.} \end{cases}$$

The proof is a bit tedious but straightforward and shall be skipped.

Remark. In Proposition 2.1.21 we have described the adjoint differential d_A^t using the local formula $d_A^t = -\,\mathrm{i}(E^I) \circ \nabla_I^{A,g}$. However, as in (1.7), we also have the description

$$d_A^t = (-1)^{m+1} \tau_M \circ d_A \circ \tau_M,$$

where $m = \dim M$. Using this together with Lemma 2.2.3 and Proposition 2.1.15, one could give a different proof of Proposition 2.1.21. Clearly, the main point is then to compute $\tau_M d_{A,h} \tau_M$, which amounts to proof that

$$\tau_v\big[\widetilde{\nabla}_X^v, \tau_v\big] = 2B(X) + k_v(X), \quad X \in C^\infty(M, T^hM). \tag{2.15}$$

Conversely, (2.15) can be verified using Proposition 2.1.21 and (2.9).

2.2.2 Adiabatic Metrics on Fiber Bundles

In a similar way as in Section 1.5.2, we now want to rescale the metric on the fiber bundle $F \hookrightarrow M \xrightarrow{\pi} B$. Yet, the important difference is that we only rescale the metric on the base manifold. In order avoid square roots of ε, we are using ε^2 rather than ε to rescale the metric.

Definition 2.2.4. Let g_B be a metric on B and g_v be a vertical metric. For $\varepsilon > 0$ we define the *adiabatic metric*

$$g_\varepsilon := \tfrac{1}{\varepsilon^2} g_B \oplus g_v. \tag{2.16}$$

Associated to each g_ε, we have a Levi-Civita connection ∇^{g_ε}. Note that unlike in the case of a single manifold, the family ∇^{g_ε} is not independent of ε since we only scale the base metric. However, the direct sum connection $\nabla^{\oplus}$ is independent of ε since both, ∇^B and ∇^v are, see Proposition 2.1.3. Similarly, the tensor θ as in Definition 2.1.6 is independent of ε.

Adiabatic Families of Odd Signature Operators. Now let $E \to M$ be a flat unitary bundle with connection A, and let ∇^{A,g_ε} and $\nabla^{A,\oplus}$ be the induced connections on $\Lambda^\bullet T^*M \otimes E$. We can use Lemma 2.1.19 to write

$$\nabla^{A,g_\varepsilon} = \nabla^{A,\oplus} + \tfrac{1}{2}\big(c_\varepsilon(\theta) - \widehat{c}_\varepsilon(\theta)\big). \tag{2.17}$$

Here, Clifford multiplication is defined with respect to the fixed metric $g = g_B \oplus g_v$ on $\Lambda^\bullet T^*M$, i.e.,

$$c_\varepsilon(f^a) = \varepsilon c(f^a), \quad c_\varepsilon(e^i) = c(e^i), \quad \widehat{c}_\varepsilon(f^a) = \varepsilon \widehat{c}(f^a), \quad \widehat{c}_\varepsilon(e^i) = \widehat{c}(e^i),$$

compare with (1.55).

Lemma 2.2.5. *For each $\varepsilon > 0$, the connection ∇^{A,g_ε} on $\Lambda^\bullet T^*M \otimes E$ is compatible with the fixed metric $g = g_B \oplus g_v$. Moreover, it is a Clifford connection with respect to c_ε, i.e.,*

$$\big[\nabla^{A,g_\varepsilon}, c_\varepsilon(\xi)\big] = c_\varepsilon\big(\nabla^{g_\varepsilon}\xi\big), \quad \xi \in \Omega^1(M).$$

Sketch of proof. Lemma 2.2.5 is basically [13, Prop. 10.10]. The main observation there is that $\nabla^{A,\oplus}$ is compatible with g and satisfies

$$\big[\nabla^{A,\oplus}, c_\varepsilon(\xi)\big] = c_\varepsilon\big(\nabla^{\oplus}\xi\big), \quad \xi \in \Omega^1(M).$$

On the other hand, according to (2.17), we need to consider

$$c_\varepsilon\big(\theta(X)\big) - \widehat{c}_\varepsilon\big(\theta(X)\big) \in C^\infty\big(M, \mathrm{End}(\Lambda^\bullet T^*M \otimes E)\big), \quad X \in C^\infty(M, TM).$$

Since c_ε and $\widehat{c}_\varepsilon$ are defined with respect to the fixed metric g, and since $\theta(X)$ is a 2-form, one finds that $c_\varepsilon\big(\theta(X)\big)$ and $\widehat{c}_\varepsilon\big(\theta(X)\big)$ are self-adjoint with respect to g. This implies that ∇^{A,g_ε} is compatible with the metric. Lemma 1.2.18 (i) shows that for $\xi \in \Omega^1_v(M)$

$$\big[c_\varepsilon(\theta) - \widehat{c}_\varepsilon(\theta), c_\varepsilon(\xi)\big] = \big[c_\varepsilon(\theta), c_\varepsilon(\xi)\big]$$

As in [13, Prop. 10.10] one then finds that

$$\big[\nabla^{A,\oplus} + \tfrac{1}{2}c_\varepsilon(\theta), c_\varepsilon(\xi)\big] = c_\varepsilon\big(\nabla^{g_\varepsilon}\xi\big),$$

which proves that ∇^{A,g_ε} is indeed a Clifford connection. $\square$

Remark 2.2.6.

(i) Again, it might be confusing that all connections ∇^{A,g_ε} are compatible with the fixed metric g on $\Lambda^\bullet T^*M$. This is due to the fact that we have defined ∇^{A,g_ε} in such a way, that it already incorporates the isometry of Lemma 1.5.9, which in the case at hand takes the form

$$\delta_\varepsilon : \big(\Omega^\bullet(M,E), g_\varepsilon\big) \to \big(\Omega^\bullet(M,E), g\big), \quad \delta_\varepsilon(\pi^*\alpha \wedge \omega) := \varepsilon^{|\alpha|}\pi^*\alpha \wedge \omega.$$

(ii) We also want to point out that the chirality operator τ_M on $\big(\Omega^\bullet(M,E), g\big)$ does not change with ε. Indeed, it is immediate that $\mathrm{vol}_M(g_\varepsilon) = \varepsilon^{-\dim B}\,\mathrm{vol}_M(g)$ from which it follows that

$$c_\varepsilon\big(\mathrm{vol}_M(g_\varepsilon)\big) = c\big(\mathrm{vol}_M(g)\big)$$

Definition 2.2.7. Let g_ε be an adiabatic metric on M, and assume that $m = \dim M$ is odd. We define the *adiabatic family* of odd signature operators as

$$B^{\mathrm{ev}}_{A,\varepsilon} := \tau_M D_{A,v} + \varepsilon \cdot \tau_M D_{A,h} + \varepsilon^2 \cdot \tau_M T : \Omega^{\mathrm{ev}}(M,E) \to \Omega^{\mathrm{ev}}(M,E). \tag{2.18}$$

The definition is in such a way that $B^{\mathrm{ev}}_{A,\varepsilon}$ is given by Clifford contraction of ∇^{A,g_ε} with respect to the Clifford multiplication $\tau_M \circ c_\varepsilon$. Thus, all $B^{\mathrm{ev}}_{A,\varepsilon}$ are geometric Dirac operators on $\Omega^{\mathrm{ev}}(M,E)$ which are formally self-adjoint with respect to the L^2-structure induced by the fixed reference metric g. The ε factors occur since each horizontal Clifford variable is scaled with ε.

Adiabatic Limit of the Eta Invariant. The family of operators $B^{\mathrm{ev}}_{A,\varepsilon}$ converges pointwise to $\tau_M D_{A,v}$, which is not an elliptic operator. Therefore, the following result is remarkable, see also [16, Prop. 4.3].

Proposition 2.2.8. *Let g_ε be an adiabatic metric on the total space of a fiber bundle $F \hookrightarrow M \xrightarrow{\pi} B$, and assume that $m = \dim M$ is odd. Let A be a flat $\mathrm{U}(k)$-connection, and let $\eta(B^{\mathrm{ev}}_{A,\varepsilon})$ be the family of Eta invariants associated to the adiabatic family of odd signature operators $B^{\mathrm{ev}}_{A,\varepsilon}$. Then the* "adiabatic limit of the Eta invariant" *exists in $\mathbb{R}$. More precisely,*

$$\lim_{\varepsilon\to 0}\eta(B^{\mathrm{ev}}_{A,\varepsilon}) = \eta(B^{\mathrm{ev}}_{A}) + k \cdot \int_M TL(\nabla^g, \nabla^\oplus),$$

where $TL(\nabla^g, \nabla^\oplus)$ is the transgression form of the L-class with respect to the connection ∇^g and $\nabla^\oplus$ on TM.

Proof. Fix $\varepsilon \in (0,1)$. Then Proposition 1.5.4 shows that

$$\eta(B^{\mathrm{ev}}_{A,\varepsilon}) = \eta(B^{\mathrm{ev}}_{A}) + k \cdot \int_M TL(\nabla^g, \nabla^{g_\varepsilon}). \tag{2.19}$$

Moreover, we deduce from Proposition A.2.4 that

$$\int_M TL(\nabla^g, \nabla^{g_\varepsilon}) = \int_M TL(\nabla^g, \nabla^\oplus) + \int_M TL(\nabla^\oplus, \nabla^{g_\varepsilon}).$$

As in Definition 2.1.6 consider

$$S = \nabla^g - \nabla^{\oplus} \quad \text{and} \quad S^\varepsilon = \nabla^{g_\varepsilon} - \nabla^{\oplus}.$$

It is straightforward to check that

$$P^v S^\varepsilon = P^v S \quad \text{and} \quad P^h S^\varepsilon = \varepsilon^2 P^h S,$$

where $P^{v/h} : TM \to T^{v/h}M$ is the vertical, respectively horizontal, projection of the fiber bundle. Hence,

$$\lim_{\varepsilon \to 0} \nabla^{g_\varepsilon} = \nabla^{\oplus} + P^v S,$$

and the limit is uniform in ε. Therefore,

$$\lim_{\varepsilon \to 0} \int_M TL(\nabla^{\oplus}, \nabla^{g_\varepsilon}) = \int_M TL(\nabla^{\oplus}, \nabla^{\oplus} + P^v S).$$

Now, a consequence of Proposition 2.1.7 is that for fixed $X \in C^\infty(M, TM)$, the only non-trivial component of $P^v S(X)$ is

$$P^v S(X) : T^h M \to T^v M.$$

In particular, $P^v S(X)$ and all its powers are trace-free which implies that

$$TL(\nabla^{\oplus}, \nabla^{\oplus} + P^v S) = 0.$$

Hence, we can take the limit in (2.19) and get

$$\lim_{\varepsilon \to 0} \eta(B^{\text{ev}}_{A,\varepsilon}) = \eta(B^{\text{ev}}_A) + k \cdot \int_M TL(\nabla^g, \nabla^{\oplus}). \qquad \square$$

Remark. So far Proposition 2.2.8 is not of particular value for explicit computations of $\eta(B^{\text{ev}}_A)$. First of all, we do not yet know anything about the adiabatic limit $\lim_{\varepsilon \to 0} \eta(B^{\text{ev}}_{A,\varepsilon})$. However, in Chapter 3 we will describe how powerful methods of local families index theory give an alternative expression of the adiabatic limit in more topological terms. Another aspect worth mentioning is that the Chern-Simons term $\int_M TL(\nabla^g, \nabla^{\oplus})$ can be very difficult to compute, see e.g. [79] for very explicit computations in the case of circle bundles over surfaces.

Concerning Rho invariants we already know at this point that the transgression term does not play a role. This is because the transgression term is the same for B^{ev}_A and the untwisted odd signature operator B^{ev}, since A is flat. Moreover, according to Proposition 1.4.7 the Rho invariant is independent of the metric, so that we have the freedom of choosing particular well-suited vertical and horizontal metrics. Summarizing these observations, we obtain the following result, which is the underlying idea for our discussion of Rho invariants of fiber bundles.

Corollary 2.2.9. *With respect to all adiabatic metrics g_ε on M we have*

$$\rho_A(M) = \lim_{\varepsilon \to 0} \eta(B^{\text{ev}}_{A,\varepsilon}) - k \cdot \lim_{\varepsilon \to 0} \eta(B^{\text{ev}}_\varepsilon)$$

2.3 The U(1)-Rho Invariant for Principal S^1-Bundles over Surfaces

In this section we will see how the idea of Corollary 2.2.9 is already helpful for explicit computations, even without employing more abstract theory we will encounter in Chapter 3. We give an elementary computation of Rho invariants for the simple but already non-trivial example of a principal S^1-bundle over a closed surface. We content ourselves with the U(1)-Rho invariant since all phenomena related to adiabatic limits appear. Some parts of our discussion are borrowed from [79]. The setup there is the Spin^c Dirac operator which is, however, closely related to a twisted odd signature operator.

Before we can start with the discussion of the odd signature operator on a principal S^1-bundle over a Riemannian surface, we need an explicit description of flat U(1)-connections.

2.3.1 The U(1)-Moduli Space

Let Σ be a closed, oriented surface of genus g, and let $S^1 \hookrightarrow M \xrightarrow{\pi} \Sigma$ be an oriented principal circle bundle. Since $H^2(\Sigma,\mathbb{Z}) = \mathbb{Z}$, such a bundle is classified up to isomorphism by its degree $l \in \mathbb{Z}$. Given that, there is a very explicit construction, which we describe now.

Topological Description. Let $\mathbb{D} \subset \Sigma$ be an embedded disc, and let $\Sigma_0 := \Sigma \setminus \mathbb{D}$. Clearly, $H^2(\mathbb{D},\mathbb{Z}) = \{0\}$, and the long exact cohomology sequence of the pair $(\Sigma_0, \partial\Sigma_0)$ implies that $H^2(\Sigma_0,\mathbb{Z}) = \{0\}$ as well. Since principal S^1-bundles are classified by their first Chern class, this shows that the restriction of $S^1 \hookrightarrow M \xrightarrow{\pi} \Sigma$ to $\mathbb{D}$ and Σ_0 is trivializable. Fixing an identification $\partial\mathbb{D} = -\partial\Sigma_0 = S^1$ as oriented manifolds, we conclude that—up to isomorphism—the bundle $\pi : M \to \Sigma$ is given by a glueing function of the form

$$\varphi : \partial(\mathbb{D} \times S^1) \to \partial(\Sigma_0 \times S^1), \quad \varphi(z,\lambda) = (z, z^{-l}\lambda), \tag{2.20}$$

where $z \in S^1 = \partial\mathbb{D} = -\partial\Sigma_0$. We want to use this description to determine the fundamental group of M. For elements $a, b \in \pi_1(\Sigma)$ we write $[a,b] = b^{-1}a^{-1}ba$, which according to our convention means to first follow the path a, then b and then the same again with the orientations reversed.

Lemma 2.3.1. *Let $S^1 \hookrightarrow M \xrightarrow{\pi} \Sigma$ be an oriented principal circle bundle of degree $l \in \mathbb{Z}$. Then the fundamental group of M has the presentation*

$$\pi_1(M) = \Big\langle a_1, b_1, \ldots, a_g, b_g, \gamma \Big| \prod_{i=1}^{g} [a_i, b_i] = \gamma^l,\ \gamma \text{ central} \Big\rangle,$$

where $a_1, b_1, \ldots, a_g, b_g$ are lifts to M of the standard generators of $\pi_1\Sigma$ and γ is the homotopy class of the S^1-fiber.

Proof. Let c be the homotopy class of $\partial\Sigma_0$. Then the canonical generators of $\pi_1(\Sigma_0)$ are the ones indicated in Figure 2.1. It is well known, see e.g. [40, Sec. III.3.5], that

$$\pi_1(\Sigma_0) = \Big\langle a_1, b_1, \ldots, a_g, b_g, c \Big| \prod_{i=1}^{g} [a_i, b_i] = c^{-1} \Big\rangle.$$

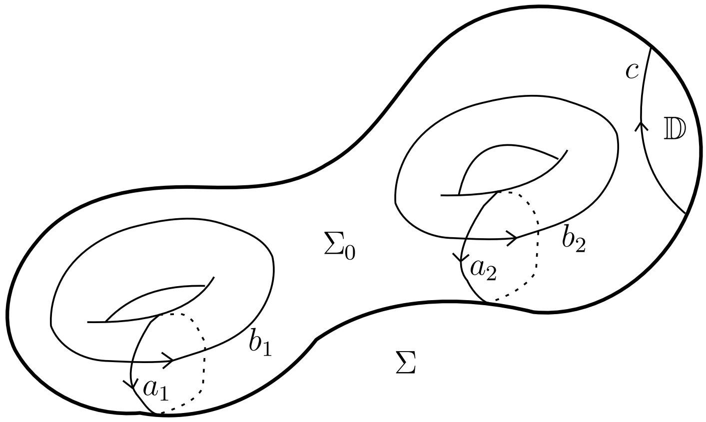

Figure 2.1: Generators of $\pi_1(\Sigma_0)$

Write

$$\pi_1\big(\partial(\mathbb{D}\times S^1)\big) = \big\langle \tilde{c},\tilde{\gamma}\,\big|[\tilde{\gamma},\tilde{c}]=1\big\rangle, \quad \pi_1\big(\partial(\Sigma_0\times S^1)\big) = \big\langle c,\gamma\,\big|[\gamma,c]=1\big\rangle.$$

Note that $\tilde{c}$ is annihilated under the inclusion $\partial\mathbb{D}\hookrightarrow\mathbb{D}$. Moreover, the map (2.20) induces a map on fundamental groups

$$\varphi_*:\pi_1\big(\partial(\mathbb{D}\times S^1)\big)\to\pi_1\big(\partial(\Sigma_0\times S^1)\big), \quad \varphi_*(\tilde{\gamma})=\gamma, \quad \varphi_*(\tilde{c})=\gamma^{-l}c^{-1}.$$

Van Kampen's Theorem now shows that

$$\pi_1(M) = \Big\langle a_1,b_1,\ldots,a_g,b_g,c,\gamma\,\Big|\prod_{i=1}^{g}[a_i,b_i]=c^{-1},\ \forall_i\ [a_i,\gamma]=[b_i,\gamma]=1, c^{-1}=\gamma^l\Big\rangle$$

which by cancelling c coincides with the claimed presentation. □

Since $H_1=\pi_1/[\pi_1,\pi_1]$, it follows immediately from the above Lemma that

$$H_1(M,\mathbb{Z}) = H_1(\Sigma,\mathbb{Z})\oplus\mathbb{Z}_l, \tag{2.21}$$

where we set $\mathbb{Z}_l=\mathbb{Z}$ if $l=0$. As the first homology group $H_1(\Sigma,\mathbb{Z})$ is equal to $\mathbb{Z}^{2g}$, we deduce that

$$\operatorname{Hom}\big(H_1(\Sigma,\mathbb{Z}),\mathrm{U}(1)\big) = \mathrm{U}(1)^{2g}.$$

The long exact coefficient sequence shows that this $2g$-dimensional torus can be identified with $H^1(\Sigma,\mathbb{R})/H^1(\Sigma,\mathbb{Z})$. From (2.21) one can now determine the moduli space of U(1)-representations, which is the topological version of moduli space of flat Hermitian line bundles, see Proposition B.1.8.

Lemma 2.3.2. *Let $M\to\Sigma$ be an oriented principal circle bundle of degree l. Then the moduli space of flat Hermitian line bundles on M is given by*

$$\mathcal{M}\big(M,\mathrm{U}(1)\big)\cong\begin{cases}\mathrm{U}(1)^{2g}\times\mathbb{Z}_l, & \text{if } l\neq 0,\\ \mathrm{U}(1)^{2g+1}, & \text{if } l=0.\end{cases}$$

Remark 2.3.3. Note that in the case $l \neq 0$ it follows from Poincaré duality $H_1(M,\mathbb{Z}) \cong H^2(M,\mathbb{Z})$ and (2.21) that $\operatorname{Tor} H^2(M,\mathbb{Z}) = \mathbb{Z}_l$. Hence there are flat line bundles which are topologically non-trivial. This corresponds to the above decomposition of $\mathcal{M}\big(M,\mathrm{U}(1)\big)$ into l different components. We have included some more details on flat line bundles which are topologically non-trivial in Appendix B.1, see in particular Lemma B.1.10.

Flat Line Bundles over M. We now need a description of $\mathcal{M}\big(M,\mathrm{U}(1)\big)$ in terms of flat line bundles. We will see that every flat line bundle over the total space M arises as the pullback of a line bundle on the base Σ. Since $M \to \Sigma$ is a principal S^1-bundle, there exists an associated Hermitian line bundle $L \to \Sigma$ which clearly has to play a particular role. Much of the discussion to follow is inspired by [78, Sec. 3.3], although we include some more details and put more emphasis on the explicit description of the U(1)-moduli space.

We will work with respect to a fixed connection on the principal S^1-bundle. For this we first endow Σ with a Riemannian metric g_Σ of unit volume. As noted in Appendix B.3, this amounts to fixing a complex structure on Σ. We identify the Lie algebra of S^1 with $i\mathbb{R}$. Let e be the vector field on M associated to the S^1-action,

$$e|_p = \tfrac{d}{dt}\big|_{t=0} p \cdot e^{it}, \quad p \in M.$$

A connection on the principal bundle $\pi : M \to \Sigma$ is a 1-form $i\omega \in \Omega^1(M, i\mathbb{R})$ such that

$$\omega(e) = 1 \quad \text{and} \quad R^*_{e^{it}}\omega = \omega,$$

where $R_{e^{it}}$ denotes right-multiplication, compare with (B.1). Let $F_\omega \in \Omega^2(\Sigma, i\mathbb{R})$ be the curvature of $i\omega$. Since the cohomology class of $\frac{i}{2\pi}F_\omega$ represents the rational first Chern class of the bundle $\pi : M \to \Sigma$ we can choose ω in such a way that

$$-\tfrac{1}{2\pi}d\omega = \tfrac{i}{2\pi}\pi^* F_\omega = l \cdot \pi^* \operatorname{vol}_\Sigma . \tag{2.22}$$

Let $L \to \Sigma$ be the line bundle associated to the principal bundle structure. The connection ω induces a natural connection A_ω on L. We write L_ω for the line bundle L endowed with this particular connection A_ω. As explained in Appendix B.3, this is the same as fixing a holomorphic structure on L. The following simple observation relies only on the principal bundle structure and not on the particular structure group U(1) or the dimension of the base.

Lemma 2.3.4. *The pullback $\pi^* L_\omega \to M$ is canonically trivial and the pullback connection $\pi^* A_\omega$ satisfies*

$$\pi^* A_\omega = d_M + i\omega.$$

Proof. Recall that the associated bundle $L_\omega \to \Sigma$ is defined by the pullback diagram

$$\begin{array}{ccc} M \times \mathbb{C} & \longrightarrow & M \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M \times \mathbb{C}/\sim & \longrightarrow & \Sigma \end{array}$$

where $(p,v) \sim (pz, z^{-1}v)$ for all $z \in S^1$. Since $\pi^* L_\omega$ is given by the same pullback diagram, we tautologically get $M \times \mathbb{C} = \pi^* L_\omega$. Under this identification,

$$\pi^* C^\infty(\Sigma, L_\omega) = \big\{\varphi : M \to \mathbb{C} \;\big|\; \varphi(pz) = z^{-1}\varphi(p)\big\}.$$

The pullback connection $\pi^* A_\omega$ acts on equivariant functions φ as

$$(\pi^* A_\omega)_X \varphi = X^h \varphi = d_M \varphi(X) - X^v \varphi,$$

where $X^{h/v}$ denotes the horizontal/vertical projection of X with respect to the connection $i\omega$. The latter is explicitly given by

$$X_p^v = \tfrac{d}{dt}\big|_0 p \cdot \exp\big(ti\omega(X_p)\big) = \omega(X)\, e|_p.$$

If φ is an equivariant function, then $e\varphi = -i\varphi$. Therefore,

$$X_p^v \varphi = -i\omega(X_p)\varphi(p).$$

Extending by the Leibniz rule to all functions on M, we get

$$\pi^* A_\omega = d_M + i\omega. \qquad \square$$

Now let $L_A \to \Sigma$ be an arbitrary Hermitian line bundle of degree k with a holomorphic structure given by a unitary connection A, see Appendix B.3. It follows from Proposition B.3.5 that upon transforming A with a complex gauge transformation $f \in \mathcal{G}^c$ we may—and will—assume in the following that

$$\tfrac{i}{2\pi} F_A = k \operatorname{vol}_\Sigma . \tag{2.23}$$

In general, to achieve this, we really need to transform with a complexified gauge transformation and not just a unitary one, see Proposition B.3.5.

Lemma 2.3.5. *Assume that $l \neq 0$, and let $q := k/l$. Then the connection*

$$A_q := \pi^* A - q\, i\omega \quad \text{on } \pi^* L_A$$

is flat. Moreover, the holonomy of A_q along the S^1-fiber γ is given by

$$\operatorname{hol}_{A_q}(\gamma) = \exp(2\pi i q).$$

Proof. By functoriality, we have $F_{\pi^* A} = \pi^* F_A$. Thus, it follows from assumptions (2.22) and (2.23) that the curvature of A_q satisfies

$$F_{A_q} = \pi^* F_A - q\, i d\omega = -2\pi i \big(k - ql\big) \pi^* \operatorname{vol}_\Sigma = 0.$$

To compute the holonomy, let $p \in M$ be arbitrary and let $\gamma(t) = p \cdot \exp(it)$ with $t \in [0, 2\pi]$ parametrize the fiber containing p. Clearly, $\operatorname{hol}_{\pi^* A}(\gamma) = 0$ and $\omega_{\gamma(t)}(\dot\gamma(t)) = 1$. Therefore,

$$\operatorname{hol}_{A_q}(\gamma) = \exp\Big(- \int_\gamma -q\, i\omega \Big) = \exp\Big(q\, i \int_0^{2\pi} \omega_{\gamma(t)}(\dot\gamma(t)) dt \Big) = \exp(2\pi i q). \qquad \square$$

The Moduli Space of Flat Line Bundles. After this preparation, we can now give the geometric description of $\mathcal{M}\big(M, \mathrm{U}(1)\big)$. Recall that $\operatorname{Pic}(\Sigma)$ denotes the Picard group of holomorphic line bundles over Σ, see Definition B.3.4.

Proposition 2.3.6. *Let $l \neq 0$ and let $\mathcal{M}\big(M, \mathrm{U}(1)\big)$ be the moduli space of flat line bundles over M. Then ω induces a natural surjection*

$$\pi^* : \mathrm{Pic}(\Sigma) \to \mathcal{M}\big(M, \mathrm{U}(1)\big), \quad [L_A] \mapsto \big[\pi^* L_A, A_q\big],$$

where A_q is defined as in Lemma 2.3.5. There is a natural $\mathbb{Z}$-action on $\mathrm{Pic}(\Sigma)$ *given by $(L_A, k) \mapsto L_A \otimes L_\omega^{\otimes k}$, and with respect to this,*

$$\mathrm{Pic}(\Sigma)/\mathbb{Z} \cong \mathcal{M}\big(M, \mathrm{U}(1)\big).$$

Proof. Let $L_A \to \Sigma$ be a holomorphic line bundle. Assume that B is another unitary connection on L, satisfying condition (2.23) and inducing an equivalent holomorphic structure, i.e.,

$$B = A + u^{-1} du, \quad \text{for some } u \in \mathcal{G}^c.$$

Condition (2.23) means in particular that $F_A = F_B$, which implies that in fact $u \in \mathcal{G}$. From this we obtain that

$$B_q = A_q + \pi^*(u^{-1} du), \quad \text{for some } u \in \mathcal{G}.$$

i.e., the flat connections B_q and A_q on $\pi^* L_A$ are equivalent. This shows that the map in Proposition 2.3.6 is well-defined.

To verify that it is surjective, let $L \to M$ be a flat Hermitian line bundle with connection $\tilde{A}$. Let γ denote the generator of the S^1-fiber in $\pi_1(M)$. Then Lemma 2.3.1 shows that γ^l is a commutator. It follows that for some $k \in \mathbb{Z}$,

$$\mathrm{hol}_{\tilde{A}}(\gamma) = \exp(2\pi i k/l). \tag{2.24}$$

Now let $L_A \to \Sigma$ be an arbitrary holomorphic line bundle of degree k. We infer from (2.24) and Lemma 2.3.5 that $\pi^* L_A \otimes L^{-1}$, endowed with the connection $A_q \otimes 1 - 1 \otimes \tilde{A}$, is a flat line bundle on M with trivial holonomy along the fiber γ. This easily implies that it is equivalent to the pullback $\pi^* \mathbb{C}_B$ of the trivial line bundle over Σ endowed with a flat connection B. Thus, as line bundles with connection,

$$L = \pi^*(L_A \otimes \mathbb{C}_B),$$

which proves surjectivity of the map in Proposition 2.3.6.

As we have seen in Lemma 2.3.4, the pullback $\pi^* L_\omega$ with connection $\pi^* A_\omega - i\omega$ is the trivial flat line bundle. Using this one observes that the map π^* is invariant under the natural $\mathbb{Z}$-action on $\mathrm{Pic}(\Sigma)$. Assume now that $\pi^* L_A = \pi^* L_B$ for two holomorphic line bundles over Σ of degree k and m respectively. Since their holonomies along γ agree, it follows that $m - k = nl$ for some $n \in \mathbb{Z}$. Thus,

$$\pi^*(L_B \otimes L_A^{-1}) = \pi^* L_\omega^{\otimes n} \quad \text{and} \quad \pi^* B = \pi^* A + n \cdot i\omega.$$

We deduce that $L_B = L_A \otimes L_\omega^{\otimes n}$ as holomorphic line bundles, which is what we needed to prove. □

Remark 2.3.7. Recall that the S^1-bundle $\pi : M \to \Sigma$ gives rise to the Gysin sequence

$$\ldots \to H^0(\Sigma) \xrightarrow{\cup c} H^2(\Sigma) \xrightarrow{\pi^*} H^2(M) \xrightarrow{\pi_*} H^1(\Sigma) \to 0,$$

see [22, Prop. 14.33]. Here, $c = c(M) \in H^2(\Sigma)$ is the first Chern class (or Euler class) of the oriented S^1-bundle. If we are assuming that $l \neq 0$, the map $H^0(\Sigma) \xrightarrow{\cup c} H^2(\Sigma)$ gives an isomorphism in de Rham cohomology. This implies that for cohomology with integer coefficients, the map $\pi^* : H^2(\Sigma, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ appearing in the Gysin sequence surjects onto the torsion subgroup of $H^2(M, \mathbb{Z})$. It is related to the map π^* of Proposition 2.3.6 by the following diagram

$$\begin{array}{ccc} \mathrm{Pic}(\Sigma) & \xrightarrow[c_1]{} & H^2(M,\mathbb{Z}) \\ {\scriptstyle \pi^*}\big\downarrow & & {\scriptstyle \pi^*}\big\downarrow \\ \mathcal{M}\big(M, \mathrm{U}(1)\big) & \xrightarrow[c_1]{} & \mathrm{Tor}\,\big(H^2(M,\mathbb{Z})\big). \end{array}$$

Note also that the first Chern class c_1 is equivariant with respect to the $\mathbb{Z}$-action on $\mathrm{Pic}(\Sigma)$,

$$c_1(L_A \otimes L_\omega^{\otimes k}) = c_1(L_A) + k \cdot c(M).$$

Using Proposition B.3.5 one can now interpret the above diagram as the geometric version of Lemma 2.3.2 in the case $l \neq 0$.

The structure result Proposition 2.3.6 excludes the case that the circle bundle is of degree $l = 0$, i.e., isomorphic to $\Sigma \times S^1$. However, a geometric description in this case is easy to find directly. As in Remark 1.4.8 (iii), a flat line bundle L_q over S^1 is the trivial line bundle endowed with the connection $d - qz^{-1}dz$ for some $q \in \mathbb{R}$. Here, we view S^1 as a subset of $\mathbb{C}$, and $z^{-1}dz$ expresses the Maurer-Cartan form of S^1. Clearly, L_q and $L_{q'}$ are unitarily equivalent if and only if $q - q' \in 2\pi i\mathbb{Z}$. Without effort one verifies the following result.

Lemma 2.3.8. *If $M = \Sigma \times S^1$ is the trivial circle bundle over Σ, then*

$$\mathcal{M}\big(\Sigma, \mathrm{U}(1)\big) \times \mathcal{M}\big(S^1, \mathrm{U}(1)\big) \cong \mathcal{M}\big(M, \mathrm{U}(1)\big).$$

Here, the isomorphism is given by

$$\big([L_A], [L_q]\big) \mapsto \big[L_A \boxtimes L_q\big],$$

where $L_A \boxtimes L_q$ is the fiber product defined in (1.15)*, endowed with its natural connection.*

2.3.2 The Odd Signature Operator.

We now want to identify the odd signature operator on the total space of a principal circle bundle over a closed, oriented surface. Certainly, the underlying principal bundle structure will play an important role, and many features generalize to arbitrary principal bundles with compact structure group. However, we will not give many comments about these generalizations.

Fibered Calculus on M. To start, we need to identify some of the quantities defined in Section 2.1 in the case at hand. Let $i\omega$ be a connection on the principal S^1-bundle $\pi : M \to \Sigma$. Since the vector field e associated to the S^1-action gives a trivialization of the vertical tangent bundle, we get a vertical projection

$$P^v : TM \to T^vM, \quad X \mapsto \omega(X)e.$$

With respect to this, the curvature Ω in the sense of Definition 2.1.2 is related to the curvature of $i\omega$ by

$$\Omega(X^h, Y^h) = -\omega\big([X^h, Y^h]\big)e = d\omega(X^h, Y^h)e = -iF_\omega(X, Y)e,$$

where X^h and Y^h are horizontal lifts of vector fields X, Y on Σ. In particular, when we fix a metric g_Σ of unit volume and require that ω satisfies (2.22), we have

$$\Omega(X^h, Y^h) = -2\pi l \operatorname{vol}_\Sigma(X, Y)e. \tag{2.25}$$

We now endow T^vM with the vertical metric $g_v := \omega \otimes \omega$, and consider the submersion metric $g = g_\Sigma \oplus g_v$.

Lemma 2.3.9.

(i) *With respect to the trivialization given by e, the canonical connection ∇^v on T^vM is the trivial connection, i.e.,*

$$\nabla^v_X e = 0, \quad X \in C^\infty(M, TM).$$

(ii) *If $X \in C^\infty(\Sigma, T\Sigma)$, we have*

$$\mathscr{L}_{X^h}(e) = [X^h, e] = 0, \quad \text{and} \quad \mathscr{L}^v_{X^h}(g_v) = 0.$$

In particular, the connection $\widetilde{\nabla}^v$ from Definition 2.1.9 agrees with ∇^v, and the mean curvature k_v as well as the tensor B in (2.10) *vanish.*

Proof. The connection ∇^v is compatible with g_v. Hence,

$$0 = g_v(\nabla^v_X e, e) + g_v(e, \nabla^v_X e),$$

which yields $\nabla^v_X e = 0$. This proves (i). Since $i\omega$ is a connection, we have $R^*_{e^{it}}\omega = \omega$. This implies that the metric $g_v = \omega \otimes \omega$ is invariant under the flow associated to the vector field e. Since $\pi^* g_\Sigma$ is constant along the fiber, we find that for all vector fields X on Σ

$$0 = \mathscr{L}_e(g)(X^h, e) = g\big([e, X^h], e\big) + g\big(X^h, [e, e]\big).$$

As $[e, e] = 0$ we conclude that $g\big([e, X^h], e\big) = 0$. This implies that $[e, X^h] = 0$, because Lemma 2.1.1 ensures that the vector field $[e, X^h]$ is vertical. In particular, since

$$\mathscr{L}^v_{X^h} g_v(e, e) = X^h\big(g_v(e, e)\big) - 2g_v\big([X^h, e], e\big),$$

we deduce that the vertical Lie derivative of g_v vanishes. Using its very definition, we see that the tensor B is indeed trivial. Moreover, we know from Lemma 2.1.8 that the mean curvature is given by the trace of $\mathscr{L}^v_{X^h}(g_v)$. Thus, it is also is zero. Moreover, using part (i) we find that the derivations $\mathscr{L}_{X^h}$ and $\nabla^v_{X^h}$ agree on e. Since both satisfy the Leibniz rule they are necessarily equal. Hence, by definition, the connection $\widetilde{\nabla}^v$ agrees with ∇^v. $\square$

Rho Invariants for Trivial Circle Bundles. Before we continue with the general discussion, we assume that $l = 0$ so that $M = \Sigma \times S^1$. We endow M with the natural connection $i\omega = z^{-1}dz$ given by the Maurer-Cartan form on S^1. Choose a flat line bundle $L \to M$, i.e.,

$$L = L_A \boxtimes L_q \to \Sigma \times S^1,$$

where L_A and L_q are flat line bundles over Σ respectively S^1, see Lemma 2.3.8. We identify

$$\Omega^{\mathrm{ev}}(\Sigma \times S^1, L) = \Omega^\bullet(\Sigma, L_A) \otimes C^\infty(S^1).$$

Using Lemma 2.2.1 and Lemma 2.2.3, we can write the odd signature operator as

$$B_{A,q} := B^{\mathrm{ev}}_{A,q} = \tau_\Sigma \otimes B_q + D_A \otimes 1,$$

where D_A is the twisted de Rham operator on Σ and B_q is the odd signature operator on S^1,

$$B_q = -i(\mathscr{L}_e - iq) : C^\infty(S^1) \to C^\infty(S^1).$$

Hence, $B_{A,q}$ is of the form considered in Lemma 1.3.6 (iii). According to the Hirzebruch Signature Theorem, the index of D_A^+ vanishes for all flat connections A on Σ and so

$$\eta(B_{A,q}) = \mathrm{ind}(D_A^+) \cdot \eta(B_q) = 0. \tag{2.26}$$

Therefore, all Rho invariants for the trivial circle bundle $\Sigma \times S^1$ vanish.

The Structure of $B_{A,q}$ in the General Case. We now assume that $l \neq 0$. Let $L_A \to \Sigma$ be a line bundle of degree k endowed with a Hermitian connection A which satisfies the condition of (2.23),

$$\tfrac{i}{2\pi} F_A = k \cdot \mathrm{vol}_\Sigma\,.$$

We endow the pullback $L := \pi^* L_A \to M$ with the flat connection of Lemma 2.3.5, i.e.,

$$A_q = \pi^* A - iq\,\omega, \quad q := k/l.$$

Since L is the pullback of L_A, we alter the identification (2.6) slightly to

$$\Omega^\bullet(M, L) = \pi^*\Omega^\bullet(\Sigma, L_A) \otimes \Omega^\bullet_v(M).$$

As in Lemma 2.1.15 we write the twisted de Rham operator as

$$d_{A_q} = d_{q,v} + d_{A,h} + \mathrm{i}(\Omega),$$

where

$$d_{q,v} = d_v - iq\ \mathrm{e}(\omega), \quad d_{A,h} = \mathrm{e}(f^a)\widetilde{\nabla}^{A_q,\oplus}_a = (\pi^* d_A) \otimes 1 + \mathrm{e}(f^a) \otimes \widetilde{\nabla}^v_a. \tag{2.27}$$

As always $\mathrm{e}(.)$ denotes exterior multiplication and $\{f_1, f_2\}$ is a local orthonormal frame for $T\Sigma$.

To describe the odd signature operator, we split the space of L-valued differential forms of even degree as in Lemma 2.2.1,

$$\Omega^{\mathrm{ev}}(M, L) = \pi^*\Omega^\bullet(\Sigma, L_A) \otimes C^\infty(M).$$

Proposition 2.3.10. *With respect to the above identification, the odd signature operator is given by*

$$B_{A,q} = \tau_\Sigma \otimes B_{q,v} + D_{A,h} + \tau_M T,$$

where the individual terms are

$$B_{q,v} = -i(\mathscr{L}_e - iq), \quad D_{A,h} = D_A \otimes 1 + c(f^a) \otimes \mathscr{L}^v_{f_a},$$

and

$$\tau_M T = \begin{cases} 0 & \text{on } \Omega^{0,0} \oplus \Omega^{1,0}, \\ -2\pi l & \text{on } \Omega^{2,0}. \end{cases}$$

Moreover, we have the (anti-)commutator relations

$$\big[1 \otimes B_{q,v}, D_{A,h}\big] = 0 \quad \textit{and} \quad \big\{\tau_\Sigma \otimes B_{q,v}, D_{A,h}\big\} = 0. \tag{2.28}$$

Proof. Let $\alpha \in \pi^*\Omega^p(\Sigma, L_A)$ and $\varphi \in C^\infty(M)$. Then,

$$(\tau_M D_{q,v})(\alpha \wedge \varphi) = (-1)^p \tau_M\big(\alpha \wedge (D_{q,v}\varphi)\big) = (\tau_\Sigma \alpha) \wedge (\tau_v D_{q,v}\varphi),$$

where we have used Lemma 2.2.3 in the last equality. Now, checking the factors of i in Definition 2.2.2 one finds that $\tau_v(\omega) = -i$. Thus,

$$(\tau_v D_{q,v})\varphi = (\tau_v d_{q,v})\varphi = \tau_v\big(d_v - iq\omega\big)\varphi = \tau_v(\omega)\big(\mathscr{L}_e - iq\big)\varphi = -i\big(\mathscr{L}_e - iq\big)\varphi.$$

According to Lemma 2.2.1 the horizontal part of $B_{A,q}$ coincides with the horizontal de Rham operator

$$D_{A,h} = d_{A,h} + d^t_{A,h} = \mathrm{e}(f^a)\widetilde{\nabla}^{A,\oplus}_a - \mathrm{i}(f^a)\big(\widetilde{\nabla}^{A,\oplus}_a + 2B(f_a) + k_v(f_a)\big).$$

Here, we have used Proposition 2.1.21 and (2.9). Hence, we deduce from Lemma 2.3.9 that

$$D_{A,h} = c(f^a)\widetilde{\nabla}^{A,\oplus}_a = D_A \otimes 1 + c(f^a) \otimes \widetilde{\nabla}^v_a.$$

For the last term appearing in the formula for $B_{A,q}$ note that

$$T(\alpha \otimes \varphi) = \mathrm{e}(f^1)\,\mathrm{e}(f^2)\,\mathrm{i}(\Omega_{12}) - \mathrm{i}(f^1)\,\mathrm{i}(f^2)\,\mathrm{e}(\Omega_{12}),$$

where (2.25) shows that $\Omega_{12} = -2\pi l e$. Therefore, T is non-zero only on $\Omega^{2,0}$, and

$$\begin{aligned}(\tau_M T)(\mathrm{vol}_\Sigma \wedge \varphi) &= \tau_M\big(-2\pi l\,\mathrm{i}(f^1)\,\mathrm{i}(f^2)\,\mathrm{vol}_\Sigma \wedge \varphi\omega\big) \\ &= -2\pi l\big(\tau_\Sigma(1) \wedge \varphi\tau_v(\omega)\big) = -2\pi l\,\mathrm{vol}_\Sigma \wedge \varphi.\end{aligned}$$

In this computation we have used Lemma 2.2.3 to express τ_M in terms of τ_Σ and τ_v. Also note that $\tau_\Sigma(1) = i\,\mathrm{vol}_\Sigma$ and $\tau_v(\omega) = -i$.

Lemma 2.3.9 (ii) implies that the bundle endomorphism K as defined in (2.13) vanishes. Thus, we deduce from Proposition 2.1.25 that

$$d_{q,v}d^t_{A,h} + d^t_{A,h}d_{q,v} = 0, \quad \text{and} \quad D_{q,v}D_{A,h} + D_{A,h}D_{q,v} = 0. \tag{2.29}$$

Also D_A and $c(f^a)$ anti-commute with τ_Σ, since Σ is even dimensional. This yields the relations in (2.28). □

The Spectrum of $B_{q,v}$. The vertical odd signature operator $B_{q,v}$ is not elliptic, since its principal symbol vanishes in all directions orthogonal to the fiber. Thus, we do not know much about the spectrum of $B_{q,v}$ by employing the general theory. However, due to the S^1-symmetry, we can determine its eigenvalues by hand.

Remark. Before we state the next result, recall that $L_\omega \to \Sigma$ denotes the line bundle associated to M endowed with the connection A_ω induced by ω. As we have seen in Lemma 2.3.4, the pullback $\pi^* L_\omega \to M$ is isomorphic to the trivial line bundle. Under this identification, a function $\varphi \in C^\infty(M)$ is the pullback of a section $s_\varphi \in C^\infty(\Sigma, L_\omega)$ if and only if

$$\varphi(p \cdot z) = z^{-1}\varphi(p), \quad p \in M, \quad z \in S^1. \tag{2.30}$$

Moreover, the Lie derivative is related to the connection A_ω via

$$s_{(X^h\varphi)} = A_\omega(X) s_\varphi. \tag{2.31}$$

We refer to the proof of Lemma 2.3.4 for more details.

Lemma 2.3.11. *Assume that $l \neq 0$, and let L_A be a holomorphic line bundle over Σ of degree k. Moreover, let $q := k/l$, and let $L = \pi^* L_A$ be the associated flat line bundle over M. Then*

$$\ker\big(B_{q,v} - \lambda\big) \neq \{0\} \quad \textit{if and only if} \quad \lambda + q \in \mathbb{Z}.$$

Moreover, if $\lambda + q \in \mathbb{Z}$, then

$$\ker\big(B_{q,v} - \lambda\big) \cong \pi^* C^\infty(\Sigma, L_{B_\lambda}), \quad \textit{where} \quad L_{B_\lambda} := L_A \otimes L_\omega^{-(\lambda+q)}.$$

The operator $D_{A,h}$ restricted to $\Omega^\bullet(\Sigma) \otimes \ker\big(B_{q,v} - \lambda\big)$ corresponds under the above isomorphism to

$$D_{B_\lambda} : \Omega^\bullet(\Sigma, L_{B_\lambda}) \to \Omega^\bullet(\Sigma, L_{B_\lambda}),$$

where $B_\lambda = A \otimes 1 + 1 \otimes -(\lambda + q)A_\omega$ is the natural connection on L_{B_λ}.

Proof. Let $\varphi \in \ker(B_{q,v} - \lambda)$. According to Proposition 2.3.10 this means that

$$\mathscr{L}_e \varphi = i\big(q + \lambda\big)\varphi.$$

For $t \in \mathbb{R}$ and $p \in M$ let $\widehat{\varphi}_t(p) := \varphi(p \cdot e^{it})$. Then, since e is the vector field generated by the S^1-action,

$$\tfrac{d}{dt}\widehat{\varphi}_t = \mathscr{L}_e(\widehat{\varphi}_t) = i(q + \lambda) \cdot \widehat{\varphi}_t, \quad \text{i.e.,} \quad \widehat{\varphi}_t = e^{i(q+\lambda)t}\varphi.$$

This implies that $q + \lambda \in \mathbb{Z}$ or $\varphi = 0$. Rewriting the result in terms of $z = e^{it}$ we see that

$$\varphi(p \cdot z) = z^{q+\lambda} \cdot \varphi(p), \quad z \in S^1.$$

As in (2.30) this means that we can identify φ with a section

$$\varphi \in \pi^* C^\infty\big(\Sigma, L_A \otimes L_\omega^{-q-\lambda}\big) = \pi^* C^\infty(\Sigma, L_{B_\lambda}).$$

Tracing the proof backwards shows that conversely every such element gives an eigenvector of $B_{q,v}$. The assertion about $D_{A,h}$ easily follows from (2.31) and Proposition 2.3.10. □

Remark 2.3.12. Without going into details, we want to mention that we have actually determined the full spectrum of $B_{q,v}$. We note without proof that $B_{q,v}$ is essentially self-adjoint in $\pi^*\Omega^\bullet(\Sigma, L_A)\otimes C^\infty(M)$ and that (2.28) implies that it commutes with the formally self-adjoint elliptic operator $D_h + B_{q,v}$. This suffices to guarantee that $\operatorname{spec}(B_{q,v})$ consists only of eigenvalues—though, with infinite multiplicities. Then Lemma 2.3.11 implies that

$$\operatorname{spec}(B_{q,v}) = \{\lambda \mid \lambda + q \in \mathbb{Z}\}.$$

Moreover, as in the case of eigenvalues with finite multiplicity, we can decompose

$$\pi^*\Omega^\bullet(\Sigma, L_A)\otimes C^\infty(M) = \bigoplus_{\lambda\in\operatorname{spec}(B_{q,v})} \Omega^\bullet(\Sigma)\otimes\ker\big(B_{q,v}-\lambda\big) \cong \bigoplus_{\lambda\in\operatorname{spec}(B_{q,v})} \Omega^\bullet(\Sigma, L_{B_\lambda}).$$

We also want to note that this decomposition is essentially the decomposition of the infinite dimensional S^1-module $\pi^*\Omega^\bullet(\Sigma, L_A)\otimes C^\infty(M)$ into its irreducible components. A similar situation should occur for general Lie groups.

2.3.3 The Eta Invariant of the Truncated Odd Signature Operator.

The fact that the commutators in (2.28) in Proposition 2.3.10 are zero allows us to give an elementary computation of Eta invariants, see [79, App. C] for a related treatment. However, the Eta invariant of the full signature operator is not directly tractable. Therefore, we introduce the following.

Definition 2.3.13. Let $L_A \to \Sigma$ be a line bundle of degree k, and let $L := \pi^*L_A$ be the corresponding flat line bundle over M. We call the operator

$$B^{\oplus}_{A,q} := \tau_\Sigma \otimes B_{q,v} + D_{A,h} \quad \text{on} \quad \pi^*\Omega^\bullet(\Sigma, L_A)\otimes C^\infty(M)$$

the *truncated odd signature operator* twisted by L.

Remark 2.3.14. The connection $\nabla^\oplus$ from (2.3) together with A_q induces a connection $\nabla^{A_q,\oplus}$ on $\Lambda^\bullet T^*M\otimes L$. Then the truncated odd signature operator is given by Clifford contraction of $\nabla^{A_q,\oplus}$. Therefore, it is almost as good as a geometric Dirac operator. However, $\nabla^{A_q,\oplus}$ is not a Clifford connection since it is compatible with $\nabla^\oplus$ and not with the Levi-Civita connection ∇^g. As remarked earlier an operator of this type is in general not formally self-adjoint. However, in the situation at hand, $B^{\oplus}_{A,q}$ is clearly formally self-adjoint, since $B_{q,v}$ and $D_{A,h}$ are.

Since $B^{\oplus}_{A,q}$ is an formally self-adjoint elliptic differential operator on a closed manifold, its Eta function is well-defined and for $\operatorname{Re}(s)$ large,

$$\eta(B^{\oplus}_{A,q}, s) = \frac{1}{\Gamma\big(\frac{s+1}{2}\big)}\int_0^\infty \operatorname{Tr}\big[B^{\oplus}_{A,q}\exp\big(-t(B^{\oplus}_{A,q})^2\big)\big]t^{\frac{s-1}{2}}\,dt.$$

Moreover, Theorem 1.3.3 implies that the meromorphic extension of $\eta(B^{\oplus}_{A,q}, s)$ has no pole in 0. Our strategy is now to compute the Eta invariant of the truncated odd signature operator explicitly and determine its kernel, see Proposition 2.3.15 and Proposition 2.3.16. Then in Section 2.3.4 we will use these results to determine the Rho invariant of the full odd signature operator $B_{A,q}$. For this we want to use the variation formula of Proposition 1.3.14 for Eta invariants so that we will need to understand the spectral flow between $B^{\oplus}_{A,q}$ and $B_{A,q}$. However, the difference $B_{A,q} - B^{\oplus}_{A,q}$ might be "too large" compared to $B^{\oplus}_{A,q}$ to get a good control of the spectrum near zero. As we will see the solution to this problem is to study an adiabatic metric. After this short digression on our strategy let us now investigate the truncated odd signature operator.

Proposition 2.3.15. *Let $S^1 \hookrightarrow M \xrightarrow{\pi} \Sigma$ be a principal circle bundle of degree $l \neq 0$. Let $L_A \to \Sigma$ be a line bundle over Σ of degree k, and let $L := \pi^* L_A$ be the corresponding flat line bundle over M. Then, with $q = k/l$,*

$$\eta(B^{\oplus}_{A,q}) = 2l P_2(q),$$

where P_2 is the second periodic Bernoulli function, i.e., if $q - [q] = q_0$ with $q_0 \in [0,1)$ and $[q] \in \mathbb{Z}$, then

$$P_2(q) = q_0^2 - q_0 + \tfrac{1}{6},$$

see Definition C.1.1. In particular, $\eta(B^{\oplus}_{A,q})$ is independent of the metric g_Σ, and the connection A involved in its definition.

Proof. Formula (2.28) in Proposition 2.3.10 shows that $\tau_\Sigma \otimes B_{q,v}$ anti-commutes with $D_{A,h}$. Hence, we can split

$$B^{\oplus}_{A,q} e^{-t(B^{\oplus}_{A,q})^2} = D_{A,h} e^{-tD^2_{A,h} - t(\tau_\Sigma \otimes B_{q,v})^2} + \big(\tau_\Sigma \otimes B_{q,v}\big) e^{-tD^2_{A,h} - t(\tau_\Sigma \otimes B_{q,v})^2}.$$

Since $D_{A,h}$ anti-commutes with τ_Σ one finds as in the proof of Lemma 1.3.6 that

$$\operatorname{Tr}\big[D_{A,h} e^{-tD^2_{A,h} - t(\tau_\Sigma \otimes B_{q,v})^2}\big] = 0.$$

Now let $\lambda \in \operatorname{spec}(B_{q,v})$. Then according to Lemma 2.3.11, the operator

$$\big(\tau_\Sigma \otimes B_{q,v}\big) e^{-tD^2_{A,h} - t(\tau_\Sigma \otimes B_{q,v})^2} \quad \text{on} \quad \Omega^\bullet(\Sigma) \otimes \ker\big(B_{q,v} - \lambda\big)$$

is unitarily equivalent to

$$\big(\tau_\Sigma e^{-tD^2_{B_\lambda}}\big) \lambda e^{-t\lambda^2} : \Omega^\bullet(\Sigma, L_{B_\lambda}) \to \Omega^\bullet(\Sigma, L_{B_\lambda}),$$

where $B_\lambda = A - (\lambda + q) A_\omega$. It follows from the McKean-Singer formula and the index theorem for the signature operator that

$$\operatorname{Tr}\big[\tau_\Sigma e^{-t(D_{B_\lambda})^2}\big] = \operatorname{ind}\big(D^+_{B_\lambda}\big) = 2\big(k - l(\lambda + q)\big) = -2l\lambda.$$

Hence, using the decomposition from Remark 2.3.12 we find

$$\begin{aligned}
\operatorname{Tr}\big[B^{\oplus}_{A,q} \exp\big(-t(B^{\oplus}_{A,q})^2\big)\big] &= \sum_{\lambda \in \operatorname{spec}(B_{q,v})} \operatorname{Tr}\big[\tau_\Sigma e^{-t(D_{B_\lambda})^2}\big] \lambda e^{-t\lambda^2} \\
&= -2l \sum_{\lambda \in \operatorname{spec}(B_{q,v})} \lambda^2 e^{-t\lambda^2},
\end{aligned}$$

Hence, we find that for $\operatorname{Re}(s)$ large,

$$\begin{aligned}\eta(B^{\oplus}_{A,q},s) &= \frac{1}{\Gamma\big(\frac{s+1}{2}\big)}\int_0^\infty \operatorname{Tr}\big[B^{\oplus}_{A,q}\exp\big(-t(B^{\oplus}_{A,q})^2\big)\big]t^{\frac{s-1}{2}}\,dt\\ &= \frac{1}{\Gamma\big(\frac{s+1}{2}\big)}\int_0^\infty -2l\sum_{\lambda\in\operatorname{spec}(B_{q,v})}\lambda^2 e^{-t\lambda^2}t^{\frac{s-1}{2}}\,dt\\ &= -2l\sum_{\lambda\in\operatorname{spec}(B_{q,v})}|\lambda|^{1-s}\frac{1}{\Gamma\big(\frac{s+1}{2}\big)}\int_0^\infty e^{-x}x^{\frac{s-1}{2}}\,dx\\ &= -2l\sum_{\lambda\in\operatorname{spec}(B_{q,v})}|\lambda|^{1-s},\end{aligned}$$

where we have substituted $x = t\lambda^2$. Also note that interchanging summation and integration can be justified by the large and small time estimates on $\sum_{\lambda\in\operatorname{spec}(B_{q,v})}\lambda^2e^{-t\lambda^2}$, see Proposition 1.2.4 and Theorem 1.2.7. Now, since

$$\operatorname{spec}(B_{q,v}) = \big\{\lambda\in\mathbb{R}\ \big|\ \lambda+q\in\mathbb{Z}\big\},$$

we find that

$$\eta(B^{\oplus}_{A,q},s) = -2l\sum_{\substack{n\in\mathbb{Z}\\ n\neq q}}|n-q|^{1-s},\quad \operatorname{Re}(s)>1.$$

We have included a computation of the value at $s=0$ in Proposition C.1.2 (ii). The result is the claimed formula

$$\eta(B^{\oplus}_{A,q}) = 2lP_2(q).$$

□

Proposition 2.3.16. *The kernel of the truncated odd signature operator is given by*

$$\ker(B^{\oplus}_{A,q}) = \ker(1\otimes B_{q,v})\cap\ker(D_{A,h}) \cong \begin{cases}\{0\}, & \text{if } q\notin\mathbb{Z},\\ H^\bullet(\Sigma,L_B), & \text{if } q\in\mathbb{Z}.\end{cases}$$

Here, L_B is the trivial line bundle endowed with the flat connection $B = A - qA_\omega$. Moreover, if $q\in\mathbb{Z}$, and if g denotes the genus of Σ, then

$$H^\bullet(\Sigma,L_B)\cong\begin{cases}\mathbb{C}\oplus\mathbb{C}^{2g}\oplus\mathbb{C}, & \textit{if } B \textit{ is the trivial connection},\\ \{0\}\oplus\mathbb{C}^{2g-2}\oplus\{0\}, & \textit{otherwise}.\end{cases}$$

Proof. Since $B^{\oplus}_{A,q}$ is formally self adjoint, we have

$$\ker(B^{\oplus}_{A,q}) = \ker(B^{\oplus}_{A,q})^2 = \ker\big(1\otimes B^2_{q,v} + D^2_{A,h}\big),$$

where we have used that $\tau_\Sigma\otimes B_{q,v}$ anti-commutes with $D_{A,h}$. Since both, $1\otimes B_{q,v}$ and $D_{A,h}$, are formally self-adjoint we get

$$\ker\big(1\otimes B^2_{q,v} + D^2_{A,h}\big) = \ker(1\otimes B_{q,v})\cap\ker(D_{A,h}).$$

Now Lemma 2.3.11 shows that

$$\ker(1\otimes B_{q,v})\cong\begin{cases}\{0\}, & \text{if } q\notin\mathbb{Z},\\ \Omega^\bullet(\Sigma,L_B), & \text{if } q\in\mathbb{Z},\end{cases}\tag{2.32}$$

where $L_B = L_A \otimes L_\omega^{-q}$, endowed with the connection $B = A \otimes 1 + 1 \otimes qA_\omega$. If $q \notin \mathbb{Z}$ the proof is finished, so that we assume from now on that $q \in \mathbb{Z}$. Since L_A is of degree k and L_ω of degree l, we find that $L_A \cong L_\omega^{k/l} = L_\omega^q$. Thus, L_B is isomorphic to the trivial line bundle and B is flat. Moreover, Lemma 2.3.11 identifies the restriction of the operator $D_{A,h}$ to $\ker(1 \otimes B_{q,v})$ with the de Rham operator D_B on $\Omega^\bullet(\Sigma, L_B)$. Using this and (2.32) we deduce from the Hodge-de-Rham isomorphism that

$$\ker(1 \otimes B_{q,v}) \cap \ker(D_{A,h}) \cong \ker(D_B) \cong H^\bullet(\Sigma, L_B).$$

Now if B is isomorphic to the trivial connection, we have the well-known cohomology groups of a surface

$$H^\bullet(\Sigma) \cong \mathbb{C} \oplus \mathbb{C}^{2g} \oplus \mathbb{C},$$

where g is the genus of Σ. In the case that B is non-trivial, the index theorem for the twisted de Rham operator shows that $\operatorname{ind}(D_B)$ is independent of B as long as B is flat. Hence,

$$\sum_p (-1)^p \dim H^p(\Sigma, L_B) = \operatorname{ind}(D_B) = \operatorname{ind}(D) = \chi(\Sigma) = 2 - 2g.$$

Moreover, Poincaré duality shows that

$$\dim H^0(\Sigma, L_B) = \dim H^2(\Sigma, L_B).$$

However, if B is a non-trivial flat connection, then

$$H^0(\Sigma, L_B) = \{0\}.$$

One way to see this is as follows[2]: Let $\beta : \pi_1(\Sigma) \to \mathrm{U}(1)$ be the holonomy representation of B. Then according to Proposition 1.1.5 and [33, Prop. 5.14],

$$H^0(\Sigma, L_B) = \big\{ z \in \mathbb{C} \;\big|\; \beta(c)z = z \text{ for all } c \in \pi_1(\Sigma) \big\}.$$

Therefore, for $H^0(\Sigma, L_B)$ is trivial unless $\beta \equiv 1$. Putting these observations together, we find that for non-trivial B we have indeed

$$H^\bullet(\Sigma, L_B) \cong \{0\} \oplus \mathbb{C}^{2g-2} \oplus \{0\}. \qquad \square$$

2.3.4 Adiabatic Metrics and the Spectral Flow

After having calculated the Eta invariant for the truncated signature operator $B_{A,q}^{\oplus}$, we turn our attention to the Rho invariant of the odd signature operator $B_{A,q}$. As before, let $S^1 \hookrightarrow M \xrightarrow{\pi} \Sigma$ be a circle bundle of degree $l \neq 0$ over a closed surface Σ. We let g_Σ be a metric on Σ of unit volume, and g_v be the metric on T^vM such that the vector field e has length 1. For $\varepsilon > 0$ we consider the adiabatic metric (2.16),

$$g_\varepsilon := \tfrac{1}{\varepsilon^2} g_\Sigma \oplus g_v,$$

[2] See also Corollary 4.3.6 below for a different proof of this fact.

and let ∇^{g_ε} be the Levi-Civita connection associated to g_ε. For each $t \in [0,1]$ and $\varepsilon > 0$ we define a connection on TM by

$$\nabla^{\varepsilon,t} := (1-t)\nabla^{\oplus} + t\nabla^{g_\varepsilon},$$

where $\nabla^{\oplus}$ is the direct sum connection of (2.3), which is independent of the scaling parameter ε.

Now let $L_A \to \Sigma$ be a holomorphic line bundle of degree k, and let $L := \pi^* L_A$ be the corresponding flat line bundle over M. Contracting the connection

$$\nabla^{\varepsilon,t} \otimes 1 + 1 \otimes A_q \quad \text{on} \quad \pi^*\Lambda^\bullet(T^*\Sigma) \otimes \pi^* L_A$$

with the natural Clifford multiplication, we obtain a 2-parameter family of formally self-adjoint elliptic operators

$$B_{A,q}^{\varepsilon,t} := \tau_\Sigma \otimes B_{q,v} + \varepsilon D_{A,h} + t\varepsilon^2 \tau_M T \quad \text{on} \quad \pi^*\Omega^\bullet(\Sigma, L_A) \otimes C^\infty(M),$$

which connects the truncated odd signature operator with the full odd signature operator associated to g_ε.

Proposition 2.3.17. *There exists ε_0 such that for all $\varepsilon < \varepsilon_0$ the following holds.*

(i) *If $q \notin \mathbb{Z}$, then for all $t \in [0,1]$*

$$\ker\left(B_{A,q}^{\varepsilon,t}\right) = \{0\}, \quad \textit{and} \quad \operatorname{SF}\left(B_{A,q}^{\varepsilon,t}\right)_{t\in[0,1]} = 0.$$

(ii) *If $q \in \mathbb{Z}$ and A_q is non-trivial, then for all $t \in [0,1]$*

$$\ker\left(B_{A,q}^{\varepsilon,t}\right) \cong \mathbb{C}^{2g-2}, \quad \textit{and} \quad \operatorname{SF}\left(B_{A,q}^{\varepsilon,t}\right)_{t\in[0,1]} = 0.$$

(iii) *For the trivial connection we have*

$$\ker\left(B^{\varepsilon,t}\right) \cong \begin{cases} \mathbb{C}^{2g+1}, & \text{if } t \neq 0, \\ \mathbb{C}^{2g+2}, & \text{if } t = 0. \end{cases} \quad \textit{and} \quad \operatorname{SF}\left(B^{\varepsilon,t}\right)_{t\in[0,1]} = \begin{cases} 0, & \text{if } l < 0, \\ -1, & \text{if } l > 0. \end{cases}$$

Proof. To keep the notation simple we abbreviate

$$B_{A,q}^{\varepsilon,t} = \tau_\Sigma \otimes B_{q,v} + \varepsilon D_{A,h} + t\varepsilon^2 \tau_M T =: B + \varepsilon D + t\varepsilon^2 S.$$

According to (2.28) we have $\{B, D\} = 0$, and so

$$\left(B_{A,q}^{\varepsilon,t}\right)^2 = B^2 + \varepsilon^2 D^2 + \varepsilon^4 t^2 S^2 + \varepsilon^2 t\{B,S\} + \varepsilon^3 t\{D,S\}. \tag{2.33}$$

By definition the operators B, D and S are formally self-adjoint. Thus,

$$(\varepsilon^{1/2} B + \varepsilon^{3/2} tS)^2 \geq 0 \quad \text{and} \quad (\varepsilon^{3/2} D + \varepsilon^{3/2} tS)^2 \geq 0.$$

This implies

$$\varepsilon^2 t\{B,S\} \geq -\varepsilon B^2 - \varepsilon^3 t^2 S^2 \quad \text{and} \quad \varepsilon^3 t\{D,S\} \geq -\varepsilon^3 D^2 - \varepsilon^3 t^2 S^2.$$

Using this in (2.33) we can estimate that for $\varepsilon < 1/2$

$$\left(B^{\varepsilon,t}_{A,q}\right)^2 \geq \tfrac{\varepsilon^2}{2}\left(B^2 + D^2 - 4\varepsilon t^2 S^2\right).$$

Now $B^2 + D^2$ is an elliptic operator, and since B and D are both formally self-adjoint, it has non-negative spectrum. Hence, its non-zero eigenvalues are bounded from below by some $\lambda > 0$. Moreover, S is an operator of order 0 so that S^2 is a bounded operator. Letting $\varepsilon_0 < \min\{\frac{1}{2}, \frac{\lambda}{4\|S^2\|}\}$ we find that for all $\varepsilon < \varepsilon_0$ and $t \neq 0$

$$\ker\left(B^2 + D^2 - 4\varepsilon t^2 S^2\right) = \ker B \cap \ker D \cap \ker S,$$

where we have used that $\ker(B^2 + D^2) = \ker B \cap \ker D$.

We now switch back to the usual notation. Using Proposition 2.3.16 we can reformulate what we have observed so far:

(i) If $q \notin \mathbb{Z}$ and $\varepsilon < \varepsilon_0$, then
$$\ker(B^{\varepsilon,t}_{A,q}) = \{0\}.$$

(ii) If $q \in \mathbb{Z}$ and $\varepsilon < \varepsilon_0$, then for all $t \neq 0$
$$\ker(B^{\varepsilon,t}_{A,q}) = \ker(\tau_M T) \cap H^\bullet(\Sigma, L_B), \quad L_B = L_A \otimes L_\omega^{-q}.$$

Thus, in the case $q \notin \mathbb{Z}$ the proof is finished and we assume henceforth that $q \in \mathbb{Z}$. From Proposition 2.3.10 we know that

$$\tau_M T = \begin{cases} 0 & \text{on } \Omega^{0,0} \oplus \Omega^{1,0}, \\ -2\pi l & \text{on } \Omega^{2,0}. \end{cases}$$

On the other hand, we have seen in Proposition 2.3.16 that

$$H^\bullet(\Sigma, L_B) \cong \begin{cases} \mathbb{C} \oplus \mathbb{C}^{2g} \oplus \mathbb{C}, & \text{if } B \text{ is the trivial connection}, \\ \{0\} \oplus \mathbb{C}^{2g-2} \oplus \{0\}, & \text{otherwise}. \end{cases}$$

Therefore, since $l \neq 0$,

$$\ker(\tau_M T) \cap H^\bullet(\Sigma, L_B) \cong \begin{cases} \mathbb{C} \oplus \mathbb{C}^{2g} \oplus \{0\}, & \text{if } B \text{ is the trivial connection}, \\ \{0\} \oplus \mathbb{C}^{2g-2} \oplus \{0\}, & \text{otherwise}. \end{cases}$$

This implies part (ii) and the first assertion of (iii). To finish the proof we still have to compute the spectral flow in the case that B is the trivial connection. Since the kernels of $B^{\varepsilon,t}$ are of constant dimension for $t \neq 0$ the only possible spectral flow contribution is at $t = 0$. As we have seen $\ker(B^{\varepsilon,0})$ contains $H^2(\Sigma)$ as a summand, whereas $\ker(B^{\varepsilon,t})$ for $t \neq 0$ does not. Now $H^2(\Sigma)$ is spanned by the cohomology class of vol_Σ and we can explicitly compute that

$$B^{\varepsilon,t}(\mathrm{vol}_\Sigma) = \left(\tau_\Sigma B_v + \varepsilon D_h + t\varepsilon^2 \tau_M T\right)(\mathrm{vol}_\Sigma) = -2\pi l \varepsilon^2 t \,\mathrm{vol}_\Sigma\,. \tag{2.34}$$

According to the convention in Definition 1.3.13 of how to count eigenvalues at the endpoints we find that

$$\mathrm{SF}\left(B^{\varepsilon,t}\right)_{t\in[0,1]} = \begin{cases} 0, & \text{if } l < 0, \\ -1, & \text{if } l > 0. \end{cases} \qquad \square$$

Remark. Note that (2.34) also shows that the odd signature operator associated to the metric g_ε has a non-trivial eigenvalue $-2\pi l\varepsilon^2$ which is of order ε^2. Eigenvalues of this type play a special role in the general adiabatic limit formula of [30]. We will discuss this in more detail in Sections 3.3.2 and 3.3.3, see in particular Proposition 3.4.3.

We now have collected all ingredients to compute the U(1)-Rho invariant for circle bundles over surfaces.

Theorem 2.3.18. *Let $S^1 \hookrightarrow M \xrightarrow{\pi} \Sigma$ be a principal circle bundle of degree $l \neq 0$. Let $L_A \to \Sigma$ be a line bundle over Σ of degree k, and let $L := \pi^* L_A$ be the corresponding flat line bundle over M. Write $q := k/l$ and assume that the flat connection $A_q = \pi^* A - iq\omega$ is not the trivial connection. Then*

$$\rho_{A_q}(M) = 2l\big(P_2(q) - \tfrac{1}{6}\big) + \operatorname{sgn}(l).$$

If $M = \Sigma \times S^1$ is the trivial circle bundle, then all Rho invariants vanish.

Proof. The Rho invariant associated to the odd signature operator is independent of the metric. In particular,

$$\rho_{A_q}(M) = \eta(B^{\varepsilon,1}_{A,q}) - \eta(B^{\varepsilon,1}), \quad \varepsilon > 0.$$

Hence, the variation formula (1.52) implies that for all $\varepsilon > 0$

$$\begin{aligned}\rho_{A_q}(M) = \eta(B^{\varepsilon,0}_{A,q}) - \eta(B^{\varepsilon,0}) + 2\,\mathrm{SF}\,\big(B^{\varepsilon,t}_{A,q}\big)_{t\in[0,1]} - 2\,\mathrm{SF}\,\big(B^{\varepsilon,t}\big)_{t\in[0,1]}\\ - \dim(\ker B^{\varepsilon,1}_{A,q}) + \dim(\ker B^{\varepsilon,1}) + \dim(\ker B^{\varepsilon,0}_{A,q}) - \dim(\ker B^{\varepsilon,0}).\end{aligned}$$

As we have seen in Proposition 2.3.15, the Eta invariant associated to the truncated odd signature operator does not change if the metric on the base is rescaled. Thus,

$$\eta(B^{\varepsilon,0}_{A,q}) - \eta(B^{\varepsilon,0}) = 2l\big(P_2(q) - \tfrac{1}{6}\big).$$

From Propositions 2.3.16 and 2.3.17 we see that if A_q is non-trivial

$$\dim(\ker B^{\varepsilon,1}_{A,q}) = \dim(\ker B^{\varepsilon,0}_{A,q}).$$

On the other hand, in the untwisted case

$$\dim(\ker B^{\varepsilon,1}) - \dim(\ker B^{\varepsilon,0}) = (2g+1) - (2g+2) = -1.$$

Lastly, we have seen in Proposition 2.3.17 that for ε small enough

$$\mathrm{SF}\,\big(B^{\varepsilon,t}_{A,q}\big)_{t\in[0,1]} = 0 \quad \text{and} \quad \mathrm{SF}\,\big(B^{\varepsilon,t}\big)_{t\in[0,1]} = \begin{cases} 0, & \text{if } l < 0,\\ -1, & \text{if } l > 0.\end{cases}$$

Putting all pieces together we find that

$$\rho_{A_q}(M) = 2l\big(P_2(q) - \tfrac{1}{6}\big) + \operatorname{sgn}(l).$$

The triviality of Rho invariants for $\Sigma \times S^1$ follows from (2.26). $\square$

Chapter 3

Rho Invariants of Fiber Bundles, Abstract Theory

This chapter forms the main theoretical part of the thesis. After having introduced the idea of adiabatic metrics on fiber bundles and seen their effect in the computation of the Rho invariant for principal circle bundles, we now want to describe how powerful tools of local index theory lead to a general formula for the adiabatic limit of Eta invariants. Since there exists a wide range of literature on this subject, the ideas presented here are not new. Nevertheless, we give a detailed account, including some proofs if feasible.

The treatment starts with the bundle of vertical cohomology groups over the base of the fiber bundle. To relate it to the kernel of the vertical de Rham operator, we discuss a fibered version of the Hodge decomposition theorem. As a byproduct of this we can prove a result about how to achieve that the mean curvature of a fiber bundle vanishes. Continuing with the main line of argument, we give a detailed discussion of the natural flat connection that exists on the bundle of vertical cohomology groups. It is precisely this topological nature of the kernel of the vertical de Rham operator which will make the adiabatic limit formula accessible for computational purposes. In particular, we will need to discuss a version of the odd signature operator on the base twisted by the bundle of vertical cohomology groups.

As the formulation of the general adiabatic limit formula relies on Bismut's local index theory for families, we continue with a brief survey of the main constructions and necessary results. In particular, we include a short discussion of superconnections and associated Dirac operators. Returning to the context of fiber bundles, we introduce the Bismut superconnection and recall how it appears in the local index theorem for families.

With these notions at hand, we will give a heuristic derivation of the adiabatic limit formula for families of odd signature operators. Then, referring to the literature for rigorous proofs, we finally state the general adiabatic limit formula for the Eta invariant due to Dai. One of the terms appearing there has a topological interpretation in terms of the Leray-Serre spectral sequence, and we discuss this briefly.

We finish this chapter using the adiabatic limit formula to derive again the formula for the Rho invariant of a principal S^1-bundle over a closed surface. Although we have obtained the formula already in the last chapter, it is illuminating to observe how the abstract theory leads to a shorter a more conceptual proof.

3.1 The Bundle of Vertical Cohomology Groups

3.1.1 The Vertical de Rham Operator

In Remark 2.1.24 we have pointed out without further comments that there is a relationship between differential operators acting fiberwise and families of differential operators in the sense of Definition 1.3.9. We want to make this a bit more precise now. Let $F \hookrightarrow M \xrightarrow{\pi} B$ be an oriented fiber bundle, where as before all manifolds are assumed to be closed, connected and oriented, and let $E \to M$ be a Hermitian over M.

Definition 3.1.1. Let $D : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$ be a differential operator. Then we call D a *fiberwise* differential operator, if

$$[D, \pi^*\varphi] = 0, \quad \text{for all } \varphi \in C^\infty(B).$$

We call D *fiberwise elliptic* if in addition its principal symbol

$$\sigma(D)(x, \xi) : E_x \to E_x, \quad x \in M,$$

is invertible for every non-vanishing $\xi \in T^v_x M^*$.

Certainly, if T^vM is endowed with a metric, and A is a flat connection on E, the vertical de Rham operator $D_{A,v}$ as in Definition 2.1.23 is a fiberwise elliptic operator in the sense of this definition.

Local Trivializations and Families. To relate fiberwise differential operators with families of differential operators as in Definition 1.3.9, we describe a particular way to construct local trivializations of the fiber bundle, see [50, Lem. 1.3.3].

Lemma 3.1.2. *Let $g = g_B \oplus g_v$ be a submersion metric as in Section 2.1. Let $y \in B$ and $F := \pi^{-1}(y)$. Then for every sufficiently small geodesic neighbourhood U around y there exists an isomorphism of fiber bundles*

$$\Phi : U \times F \to \pi^{-1}(U),$$

such that for all $(u, x) \in U \times F$ and every vector $v \in T_yU \subset T_{(y,x)}U \times F$,

$$\Phi(y, x) = x, \quad \pi \circ \Phi(u, x) = u, \quad \textit{and} \quad \Phi_* v = v^h,$$

where v^h refers to the horizontal lift of v.

Proof. Let $b = \dim B$, and let $U \subset B$ be a geodesic ball centered in y. We identify U with an open ball in $\mathbb{R}^b$ in such a way that $y = 0$. We can use the horizontal projection $P^h : TM \to T^hM$ to lift the coordinate vector fields ∂_a to horizontal vector fields ∂^h_a on $\pi^{-1}(U)$. Identifying a point $u \in U$ with the vector field $u^a\partial_a$ we get a vector field $u^h = u^a\partial^h_a$ on $\pi^{-1}(U)$, and hence a flow

$$\Phi_t(u, .) : M \to M, \quad u \in U.$$

It follows from the construction that for small t, the flow $\Phi_t(u,.)$ maps the fiber F diffeomorphically onto the fiber $\pi^{-1}(tu)$. Moreover, for all $x \in F$ we have $\Phi_{st}(u,x) = \Phi_t(su,x)$, so that we can choose U small enough to define

$$\Phi : U \times F \to \pi^{-1}(U), \quad (u,x) \mapsto \Phi_1(u,x).$$

The claimed properties all follow immediately from this definition. □

Using a fiber bundle chart $\Phi : U \times F \to \pi^{-1}(U)$ of the form just described, we can transfer all geometric structures on $\pi^{-1}(U)$ to $U \times F$: First of all, it is straightforward to check that

$$\Phi^* T^v M = U \times TF.$$

Therefore, the pullback $\Phi^*(g_v)$ of the vertical metric is the same as a family of Riemannian metrics $g_{F,u}$ on F. In the same way the restriction of ∇^v to $T^v M$ induces a family $\nabla^{F,u}$ of covariant derivatives on TF, and Proposition 2.1.3 shows that each $\nabla^{F,u}$ is the Levi-Civita connection on F with respect to the metric $g_{F,u}$, compare with Remark 2.1.4. Similarly, we pull the horizontal distribution $T^h M$ back to $U \times F$ and use this to identify

$$\Omega_v^\bullet\big(\pi^{-1}(U)\big) \cong C^\infty\big(U, \Omega^\bullet(F)\big), \quad \text{and} \quad \Omega^{p,q}\big(\pi^{-1}(U)\big) \cong \Omega^p\big(U, \Omega^q(F)\big). \tag{3.1}$$

Note, however, that $\Phi^* T^h M$ will in general not coincide with $TU \times F$, unless the curvature Ω of the fiber bundle is trivial.

Remark. We want to give a note about the definition of $\Omega^p\big(U, \Omega^q(F)\big)$. A naive way—which is sufficient for our purposes—is to define elements of $C^\infty\big(U, \Omega^q(F)\big)$ to be locally of the form

$$\sum_{|I|=q} f_I(y,x) dx^I, \quad y \in U,$$

with local coordinates x_i for F, multi indices I, and smooth functions $f_I(y,x)$ satisfying the appropriate transformation laws with respect to changes of the coordinate chart. Similarly one treats elements of $\Omega^p\big(U, \Omega^q(F)\big)$. From a more invariant perspective, one could endow $\Omega^q(F)$ with its natural Fréchét topology and consider smooth maps with respect to this.

Under the identification of (3.1), a vertical differential operator D on $\Omega^\bullet(M)$ can be written over $U \times F$ as

$$D = \sum_j K_j(u) \otimes D_j(u),$$

where each $K_j(u)$ is a bundle endomorphism of $\Lambda^\bullet T^* U$ and each $D_j(u)$ is a smooth b-parameter family of differential operators on $\Omega^\bullet(F)$ in the sense of Definition 1.3.9.

Vertical de Rham Operators. We use the above digression to give a description of the vertical de Rham operator. For this we first need to incorporate a bundle E over M, endowed with a flat connection A. Let $\pi_F : U \times F \to F$ be the projection onto the second factor, and denote by $E|_F$ be the restriction of E to the fiber F.

Lemma 3.1.3. *There exists a natural lift of the bundle isomorphism*

$$\Phi : U \times F \to \pi^{-1}(U)$$

to an isomorphism of flat Hermitian vector bundles

$$\Phi_E : \pi_F^*(E|_F) \to E|_{\pi^{-1}(U)}.$$

Proof. Let $u \in U$, and let F_u be the fiber over u. Then $\Phi(u, .)$ maps F diffeomorphically to F_u. Using parallel transport with respect to A along the flow lines of $\Phi_t(u, .)$ we can lift this to an isomorphism

$$\Phi_E(u, .) : E|_F \to E|_{F_u}.$$

Now, since A is a flat Hermitian connection, the bundles $E|_F$ and $E|_{F_u}$ are naturally endowed with flat Hermitian connections induced by A. Since we are using parallel transport with respect to A, a locally constant, unitary frame for $E|_F$ will be mapped by Φ_E to a locally constant, unitary frame for $E|_{F_u}$. This implies that Φ_E is, in fact, an isomorphism of flat Hermitian bundles. □

In a similar way as in (3.1), we can use Lemma 3.1.3 to identify

$$\Omega^{p,q}\big(\pi^{-1}(U), E|_{\pi^{-1}(U)}\big) \cong \Omega^p\big(U, \Omega^q(F, E|_F)\big), \tag{3.2}$$

where $E|_F$ is endowed with a fixed flat Hermitian connection A_F. We then let $D_{A_F,u}$ be the family of de Rham operators on $\Omega^\bullet(F, E|_F)$ associated to the metric $g_{F,u}$ and the flat connection A_F. Then under the identification (3.2) we can write

$$D_{A,v} = (-1)^p \otimes D_{A_F,u} \quad \text{on } \Omega^p\big(U, \Omega^\bullet(F, E|_F)\big). \tag{3.3}$$

Proposition 3.1.4. *The $C^\infty(B)$-module $\ker(D_{A,v}) \cap \Omega_v^\bullet(M, E)$ is isomorphic to the space of smooth sections of a vector bundle, which we denote by $\mathscr{H}^\bullet_{A,v}(M) \to B$. Moreover,*

$$\ker(D_{A,v}) \cong \Omega^\bullet\big(B, \mathscr{H}^\bullet_{A,v}(M)\big).$$

Sketch of proof. It suffices to show that the assertion is true locally, i.e., that for sufficiently small open subsets $U \subset M$

$$\ker(D_{A,v}) \cap \Omega^\bullet\big(\pi^{-1}(U), E\big)$$

is isomorphic to the space of differential forms over U with values in a vector bundle. For this let $U \subset B$ be as in Lemma 3.1.2 such that $\pi^{-1}(U) \cong U \times F$, and write $D_{A,v}$ as in (3.3). Since $D_{A,v}$ acts as $\pm\,\mathrm{Id}$ on $\Omega^\bullet(U)$, it suffices to consider $D_{A_F,u}$ acting on

$$\Omega_v^\bullet\big(\pi^{-1}(U), E\big) \cong C^\infty\big(U, \Omega^\bullet(F, E|_F)\big).$$

For fixed u, the Hodge-de-Rham theorem for $D_{A_F,u}$ implies that $\ker(D_{A_F,u})$ is isomorphic to $H^\bullet(F, E_A|_F)$, where $E_A|_F$ is short for $E|_F$ endowed with the flat connection A_F. Since we know from Lemma 3.1.3 that A_F does not vary with u, we infer that $\dim\ker(D_{A_F,u})$ is constant for $u \in U$. Hence, we are precisely in the situation of Proposition D.1.8—respectively Remark D.1.9. Therefore, the family of projections

$$P_u : \Omega^\bullet(F, E|_F) \to \ker\big(D_{A_F,u}\big), \quad u \in U,$$

is a smooth family of finite rank smoothing operators. Using this, it is straightforward to check that the collection

$$\mathscr{H}^\bullet_{A,v}(U) := \bigcup_{u\in U} \ker\left(D_{A_F,u}\right) \to U$$

forms a smooth vector bundle over U, see [13, Lem. 9.9] for a detailed proof. Then the assertion of Proposition 3.1.4 easily follows. $\square$

3.1.2 Vertical Hodge Decomposition

We now want to use Proposition 3.1.4 to prove the following fibered version of the de Hodge decomposition theorem.

Theorem 3.1.5. *Let $E \to M$ be a Hermitian vector bundle over the total space of an oriented fiber bundle of closed manifolds $F \hookrightarrow M \xrightarrow{\pi} B$. Assume that E admits a flat connection A. With respect to every submersion metric, there is an L^2-orthogonal splitting of smooth forms*

$$\begin{aligned}\Omega^\bullet(M,E) &= \ker(D_{A,v}) \oplus \operatorname{im}(D_{A,v})\\ &= \left(\ker d_{A,v} \cap \ker d^t_{A,v}\right) \oplus \operatorname{im}(d_{A,v}) \oplus \operatorname{im}(d^t_{A,v}).\end{aligned}$$

Moreover, the splitting is independent of the chosen metric g_B on B.

Proof. We start with a local consideration. With the same notation as in the proof of Proposition 3.1.4 we consider the family of de Rham operators $D_{A_F,u}$ on $\Omega^\bullet(F, E|_F)$. We know from the proof of Proposition 3.1.4 that $\dim\ker(D_{A_F,u})$ is constant for $u \in U$, so that the family of projections P_u onto the kernels depends smoothly on u. According to Proposition D.1.8 (ii) the same is true for the family of Green's operators,

$$G_u : \Omega^\bullet(F,E|_F) \to \Omega^\bullet(F,E|_F).$$

Let $\omega_u \in C^\infty\big(U, \Omega^\bullet(F,E|_F)\big)$. For fixed u we can decompose

$$\omega_u = P_u\omega_u + D_u \circ G_u \circ (\operatorname{Id} - P_u)\omega_u,$$

and both summands depend smoothly on u as P_u and G_u do so. Writing $D_{A,v}|_{\pi^{-1}(U)}$ for the restriction of $D_{A,v}$ to $\Omega^\bullet\big(\pi^{-1}(U), E|_{\pi^{-1}(U)}\big)$, one readily concludes that

$$\Omega^\bullet\big(\pi^{-1}(U), E|_{\pi^{-1}(U)}\big) = \ker\big(D_{A,v}|_{\pi^{-1}(U)}\big) \oplus \operatorname{im}\big(D_{A,v}|_{\pi^{-1}(U)}\big). \tag{3.4}$$

Now let $\{\varphi_i\}$ be a partition of unity on B, subordinate to a finite covering $B = \bigcup_i U_i$, such that (3.4) holds for every $\Omega^\bullet\big(\pi^{-1}(U_i), E|_{\pi^{-1}(U_i)}\big)$. For $\omega \in \Omega^\bullet(M,E)$ and every i we can decompose

$$(\pi^*\varphi_i)\omega = \alpha_i + D_{A,v}\beta_i, \quad \alpha_i \in \ker\big(D_{A,v}|_{\pi^{-1}(U_i)}\big), \quad \beta_i \in \Omega^\bullet\big(\pi^{-1}(U_i), E|_{\pi^{-1}(U_i)}\big).$$

Then, since $D_{A,v}$ is $C^\infty(B)$ linear,

$$\omega = \sum_i (\pi^*\varphi_i)\alpha_i + \sum_i (\pi^*\varphi_i) D_{A,v}\beta_i = \sum_i (\pi^*\varphi_i)\alpha_i + D_{A,v}\Big(\sum_i (\pi^*\varphi_i)\beta_i\Big),$$

which is a decomposition of ω in terms of $\ker(D_{A,v}) \oplus \operatorname{im}(D_{A,v})$. Clearly, the decomposition is L^2-orthogonal, since $D_{A,v}$ is formally self-adjoint. The equalities

$$\ker(D_{A,v}) = \big(\ker d_{A,v} \cap \ker d_{A,v}^t\big) \quad \text{and} \quad \operatorname{im}(D_{A,v}) = \operatorname{im}(d_{A,v}) \oplus \operatorname{im}(d_{A,v}^t)$$

follow as in the unparametrized case. Finally, the assertion that the vertical Hodge decomposition is independent of the metric g_B on B is immediate from the fact that $D_{A,v}$ is independent of g_B. □

3.1.3 Vanishing Mean Curvature

Before we continue the discussion of the bundle $\mathscr{H}^\bullet_{A,v}(M) \to B$, we want to give an interesting application of the fibered Hodge decomposition theorem. The corresponding result for foliations is [36, Thm. 4.18]. However, in the case of fiber bundles, the proof can be simplified considerably, and the author of this thesis is not aware of a corresponding treatment in the literature. This subsection is not essential for the line of thoughts in the later sections. Yet, it might be helpful in more complicated examples.

Theorem 3.1.6. *Let g_v be a metric on T^vM of unit volume. Then there exists a vertical projection $P^v : TM \to T^vM$ such that the associated mean curvature form $k_v(g_v, P^v)$ vanishes.*

Since we have seen in Lemma 2.1.14 that we can deform a vertical metric conformally to a metric of unit volume, Theorem 3.1.6 implies

Corollary 3.1.7. *Every oriented fiber bundle of closed manifolds admits a connection and a vertical metric such that the mean curvature form vanishes.*

Before we give the proof of Theorem 3.1.6, we extract the part where we will use the vertical Hodge decomposition of Theorem 3.1.5. Recall that we have introduced the basic projection Π_B in Definition 2.1.13.

Proposition 3.1.8. *There is an L^2-orthogonal splitting of smooth horizontal forms,*

$$\Omega_h^\bullet(M) = \pi^*\Omega^\bullet(B) \oplus d_v^t\big(\Omega^{\bullet,1}(M)\big).$$

Moreover, the kernel of the basic projection is given by

$$\ker \Pi_B = d_v^t\big(\Omega^{\bullet,1}(M)\big).$$

Proof. First of all, as the 0th cohomology group of the fiber consists only of constant functions, one deduces from Proposition 3.1.4 that

$$\ker D_v \cap \Omega_h^\bullet(M) = \pi^*\Omega^\bullet(B).$$

Since $\Omega_h^\bullet(M) \perp \operatorname{im}(d_v)$, the vertical Hodge decomposition in Theorem 3.1.5 yields

$$\Omega_h^\bullet(M) = \pi^*\Omega^\bullet(B) \oplus d_v^t\big(\Omega^{\bullet,1}(M)\big).$$

For the second assertion we note that for all $\alpha \in \Omega^\bullet(B)$ and $\omega \in \Omega_h^\bullet(M)$

$$\big\langle \alpha\,,\, \Pi_B(\omega)\big\rangle = v_F^{-1} \int_{M/B} \langle \pi^*\alpha, \omega\rangle \operatorname{vol}_F,$$

where v_F is the function which associates to a point $y \in B$ the volume of the fiber over y, see Definition 2.1.13. This implies that $\ker \Pi_B \perp \pi^*\Omega^\bullet(B)$. On the other hand, as in the unparametrized case, one finds that for $\omega \in \Omega^{\bullet,1}(M)$,

$$\Pi_B(d_v^t\omega) = v_F^{-1} \int_{M/B} d_v^t(\omega) \wedge \mathrm{vol}_F = 0.$$

Therefore,

$$d_v^t\big(\Omega^{\bullet,1}(M)\big) \subset \ker \Pi_B,$$

which finishes the proof. □

Proof of Theorem 3.1.6. Let g_v be a vertical metric such that $v_F(g_v) = 1$. We choose an arbitrary vertical projection $P^v : TM \to T^vM$, and let g be a submersion metric on M satisfying

$$T^vM^\perp = \ker P^v \quad \text{and} \quad g\big|_{T^vM\times T^vM} = g_v.$$

According to Corollary 2.1.18, the assumption that $v_F(g_v) = 1$ implies that the basic projection of the mean curvature k_v vanishes. Form Proposition 3.1.8 we deduce that there exists $\eta \in \Omega^{1,1}(M)$ such that

$$d_v^t\eta = k_v \in \Omega^{1,0}(M).$$

Define $h \in C^\infty(M, T^*M \otimes T^*M)$ by

$$h(X,Y) := \eta(P^hX, P^vY) + \eta(P^hY, P^vX), \quad X, Y \in C^\infty(M, TM).$$

Here, $P^h = \mathrm{Id} - P^v$ is the horizontal projection. Then h is a symmetric 2-tensor, and $h(X,X) = 0$ for all $X \in C^\infty(M,TM)$. Thus, we can define a new metric on TM by letting

$$\widetilde{g} := g + h.$$

Note that the restriction of $\widetilde{g}$ to T^vM still coincides with g_v. Let $\widetilde{P}^{v/h}$ denote the vertical respectively horizontal projection associated to $\widetilde{g}$. Then, if $\{e_i\}$ is any local orthonormal frame for T^vM with respect to g_v, we have

$$\widetilde{P}^v(X) = P^v(X) + \sum_i \eta(X, e_i)e_i, \quad \widetilde{P}^h(X) = P^h(X) - \sum_i \eta(X, e_i)e_i.$$

Let $\widetilde{k}_v$ denote the mean curvature form associated to g_v and $\widetilde{P}^v$, and let $\{f_a\}$ be a local orthonormal frame for T^hM with respect to the original metric g. Then, according to formula (2.2) for the mean curvature,

$$\widetilde{k}_v(f_a) = -\sum_i g_v\big([\widetilde{P}^h f_a, e_i], e_i\big) = k_v(f_a) + \sum_{ij} g_v\big([\eta(f_a, e_j)e_j, e_i], e_i\big).$$

Now, using standard arguments involving the Lie bracket and the fact that ∇^v is metric and torsion-free as a connection on T^vM, one gets

$$\begin{aligned} g_v\big([\eta(f_a, e_j)e_j, e_i], e_i\big) &= -e_i\big[\eta(f_a, e_j)\big] g_v(e_j, e_i) + \eta(f_a, e_j) g_v\big([e_j, e_i], e_i\big) \\ &= -e_i\big[\eta(f_a, e_i)\big] + \eta(f_a, e_j) g_v(e_j, \nabla^v_{e_i} e_i). \end{aligned}$$

On the other hand, according to Proposition 2.1.21,

$$d_v^t\eta(f_a) = \sum_i (\nabla^{\oplus}_{e_i}\eta)(f_a, e_i) = \sum_i \Big(e_i\big[\eta(f_a, e_i)\big] - \eta(\nabla^{\oplus}_{e_i} f_a, e_i) - \eta(f_a, \nabla^{\oplus}_{e_i} e_i)\Big).$$

Since $\nabla^{\oplus}_{e_i} f_a = 0$ and $\nabla^{\oplus}_{e_i} e_i = \nabla^{v}_{e_i} e_i$ we conclude that

$$\sum_{ij} g_v\big([\eta(f_a, e_j)e_j, e_i], e_i\big) = -d_v^t\eta(f_a).$$

Employing the definition of η we have thus achieved that

$$\widetilde{k}_v = k_v - d_v^t\eta = 0.$$

This shows that the mean curvature associated to $\widetilde{P}^v$ and g_v vanishes. $\square$

Remark. The statement of Theorem 3.1.6 for foliations is not true without changes. The underlying reason is that Corollary 2.1.18 does not generalize, i.e., the basic projection of the mean curvature does not necessarily give a trivial cohomology class. Note that the definition of cohomology requires extra work for the possibly singular leaf space of a foliation. But even if one uses the *basic cohomology* as the correct substitute, Corollary 2.1.18 does not carry over, and one finds topological obstructions to the vanishing of the mean curvature form. In the language of foliation theory, Corollary 3.1.7 asserts that the vertical distribution of a fiber bundles is a *taut* foliation. For a detailed discussion of the aspects mentioned here, in particular the difference between *tense* and taut foliations, we refer to [36] and references given therein.

3.1.4 A Flat Connection on the Bundle of Vertical Cohomology Groups

Let $E \to M$ be a Hermitian vector bundle over the total space of the fiber bundle $F \hookrightarrow M \xrightarrow{\pi} B$, and assume that E admits a flat connection A. As we have seen in Corollary 2.1.16 there is a vertical differential

$$d_{A,v} : \Omega_v^{\bullet}(M, E) \to \Omega_v^{\bullet+1}(M, E), \quad d_{A,v}^2 = 0.$$

Hence, we can form the quotient $\ker d_{A,v}/\operatorname{im} d_{A,v}$. If M is endowed with a vertical metric, it is an immediate consequence of the fibered Hodge decomposition theorem that

$$\ker d_{A,v}/\operatorname{im} d_{A,v} \cong \ker(D_{A,v}) \cap \Omega_v^{\bullet}(M, E), \tag{3.5}$$

which is the space of sections of the bundle $\mathscr{H}^{\bullet}_{A,v}(M) \to B$ of Proposition 3.1.4. This is not surprising, since if $F \subset M$ is a fiber of $\pi : M \to B$ we can restrict $d_{A,v}$ as in Section 3.1.1 to an operator

$$d_{A,v}|_F : \Omega^{\bullet}(F, E|_F) \to \Omega^{\bullet+1}(F, E|_F).$$

Then $\ker(d_{A,v}|_F)/\operatorname{im}(d_{A,v}|_F)$ is just the de Rham cohomology of F with values in the flat bundle $E|_F$, so that $\ker d_{A,v}/\operatorname{im} d_{A,v}$ is roughly the union of the cohomology groups of all fibers. These observations and Proposition 3.1.4 imply

Proposition 3.1.9. *The space* $\ker d_{A,v}/\operatorname{im} d_{A,v}$ *is isomorphic to the space of sections of a finite rank vector bundle* $H^\bullet_{A,v}(M) \to B$*, which we call the* "bundle of vertical cohomology groups". *Its fiber over a point* $y \in B$ *is isomorphic to the de Rham cohomology of* $\pi^{-1}(y)$ *with values in the flat bundle* $E|_{\pi^{-1}(y)}$.

Remark.

(i) Although the bundles $H^\bullet_{A,v}(M)$ and $\mathscr{H}^\bullet_{A,v}(M)$ are isomorphic, we will usually not identify them, since the latter is only defined when we have chosen a vertical metric. This is why we have introduced the term "bundle of vertical cohomology groups" only here rather than already in Proposition 3.1.4.

(ii) As for the bundle $\mathscr{H}^\bullet_{A,v}(M)$ we consider also differential forms on B with values in the bundle of vertical cohomology groups. Then it is immediate that

$$\Omega^p\big(B, H^q_{A,v}(M)\big) \cong \frac{\ker\big(d_{A,v} : \Omega^{p,q}(M,E) \to \Omega^{p,q+1}(M,E)\big)}{\operatorname{im}\big(d_{A,v} : \Omega^{p,q-1}(M,E) \to \Omega^{p,q}(M,E)\big)}. \tag{3.6}$$

The Natural Flat Connection. Recall from Corollary 2.1.16, that on $\Omega^\bullet(M,E)$,

$$d^2_{A,h} + \big\{d_{A,v}, \mathrm{i}(\Omega)\big\} = 0, \quad \text{and} \quad \big\{d_{A,v}, d_{A,h}\big\} = \big\{d_{A,h}, \mathrm{i}(\Omega)\big\} = 0,$$

where Ω is the curvature of the fiber bundle. Now the fact that $d_{A,h}$ anti-commutes with $d_{A,v}$ implies that $d_{A,h}$ descends to a well-defined map

$$\bar{d}_{A,h} : \Omega^p\big(B, H^\bullet_{A,v}(M)\big) \to \Omega^{p+1}\big(B, H^\bullet_{A,v}(M)\big).$$

Moreover, if $\omega \in \Omega^{p,q}(M,E)$ satisfies $d_{A,v}\omega = 0$, then it follows from the relation

$$d^2_{A,h}\omega = -d_{A,v} \circ \mathrm{i}(\Omega)\omega$$

that $d^2_{A,h}\omega$ is a $d_{A,v}$-exact element of $\ker\big(d_{A,v} : \Omega^{p+2,q}(M,E) \to \Omega^{p+2,q+1}(M,E)\big)$. This implies that $(\bar{d}_{A,h})^2 = 0$. In other words, we have found a natural flat connection on the bundle of vertical cohomology groups which is induced by $d_{A,h}$.

Definition 3.1.10. We denote by

$$\nabla^{H_{A,v}} : C^\infty\big(B, H^\bullet_{A,v}(M)\big) \to \Omega^1\big(B, H^\bullet_{A,v}(M)\big)$$

the flat connection defined by $\bar{d}_{A,h}$. More precisely, for all $X \in C^\infty(B,TB)$ and $\omega \in \Omega^\bullet_v(M,E)$ with $d_{A,v}\omega = 0$ we define

$$\nabla^{H_{A,v}}_X[\omega] := \big[\mathrm{i}(X^h)d_{A,h}\omega\big] \in C^\infty\big(B, H^\bullet_{A,v}(M)\big).$$

Relation to the Leray-Serre Spectral Sequence. We want to point out that we have just constructed the term $(E_1^{\bullet,\bullet}, d_1)$ of the spectral sequence associated to the complex $\big(\Omega^\bullet(M,E), d_A\big)$. To explain this—and also for later use—we make a short digression on the Leray-Serre spectral sequence.

Recall, e.g. from [67, Sec 2.2], that a complex endowed with a decreasing filtration gives rise to a spectral sequence. The appropriate filtration in the case at hand is the *Serre filtration* given by

$$F^k\Omega^\bullet := \sum_{p\geq k} \Omega^{p,\bullet}(M,E). \tag{3.7}$$

Then, if $b = \dim B$ and $d = d_A$, we have

$$\{0\} = F^{b+1}\Omega^\bullet \subset F^b\Omega^\bullet \subset \ldots \subset F^0\Omega^\bullet = \Omega^\bullet, \quad d\big(F^k\Omega^\bullet\big) \subset F^k\Omega^\bullet.$$

Note that the latter follows from Proposition 2.1.15, since each of the terms appearing in d_A,

$$d_A = d_{A,v} + d_{A,h} + \mathrm{i}(\Omega),$$

preserves $F^k\Omega^\bullet$. In the same way one verifies that the de Rham cohomology $H^\bullet(M,E_A)$ inherits a filtration defined by

$$F^kH^n := \operatorname{im}\big(H^n(F^k\Omega^\bullet,d) \to H^n(M,E_A)\big). \tag{3.8}$$

One now constructs a spectral sequence as follows, see the proof of [67, Thm. 2.6]. For $r\in\mathbb{N}$ define

$$\begin{aligned} Z_r^{p,q} &:= F^p\Omega^{p+q} \cap d^{-1}\big(F^{p+r}\Omega^{p+q+1}\big),\\ B_r^{p,q} &:= F^p\Omega^{p+q} \cap d\big(F^{p-r}\Omega^{p+q-1}\big),\\ E_r^{p,q} &:= \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}}, \quad E_0^{p,q} := \frac{F^p\Omega^{p+q}}{F^{p+1}\Omega^{p+q}} \cong \Omega^{p,q}(M,E). \end{aligned} \tag{3.9}$$

Then the differential d naturally defines on each bigraded module $E_r^{\bullet,\bullet}$ a differential d_r of bidegree $(r,1-r)$ in such a way that

$$E_{r+1}^{p,q} \cong \frac{\ker\big(d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}\big)}{\operatorname{im}\big(d_r : E_r^{p-r,q+r-1} \to E_r^{p+r,q+1-r}\big)}.$$

The general theory of spectral sequences now implies the following, see again [67, Thm. 2.6].

Theorem 3.1.11. *The spectral sequence $\big(E_r^{\bullet,\bullet}, d_r\big)$ collapses for $r = b+1$ and converges to $H^\bullet(M,E_A)$. More precisely, for all $r \geq b+1$*

$$E_r^{p,q} \cong \frac{F^pH^{p+q}}{F^{p+1}H^{p+q}},$$

where $F^pH^\bullet$ is the filtration of $H^\bullet(M,E)$ given by (3.8).

Now we can interpret the bundle of vertical cohomology groups in terms of the Leray-Serre spectral sequence. Certailnly, the term $E_0^{\bullet,\bullet}$ in (3.9) coincides with $\Omega^{\bullet,\bullet}(M,E)$. Moreover, one easily verifies that the natural construction of the differential in the proof of [67, Thm. 2.6] coincides with $d_{A,v}$. Thus,

$$\big(E_0^{\bullet,\bullet}, d_0\big) = \big(\Omega^{\bullet,\bullet}(M,E), d_{A,v}\big).$$

In the discussion following (3.6) we have constructed a natural differential $\bar{d}_{A,h}$ on the cohomology of $\big(E_0^{\bullet,\bullet}, d_0\big)$, and again, one can easily check that it coincides with the differential on $E_1^{\bullet,\bullet}$ abstractly constructed from (3.9). Without giving more details we summarize that

Lemma 3.1.12. *The Lerray-Serre spectral sequence satisfies*

$$E_1^{p,q} \cong \big(\Omega^p\big(B, H^q_{A,v}(M)\big), \bar{d}_{A,h}\big) \quad \textit{and} \quad E_2^{p,q} \cong H^p\big(\Omega^\bullet\big(B, H^q_{A,v}(M)\big)\big).$$

3.1.5 Twisting with the Bundle of Vertical Cohomology Groups

We can also use the vertical Hodge decomposition of Theorem 3.1.5 to give a Hodge theoretic description of the flat connection $\nabla^{H_{A,v}}$. We fix a vertical metric g_v and use this to identify $H^\bullet_{A,v}(M)$ with $\mathscr{H}^\bullet_{A,v}(M)$ using the vertical Hodge-de-Rham isomorphism in (3.5). From Section 2.1.3 we know that $\Omega^\bullet_v(M,E)$ is endowed with the natural connection $\widetilde{\nabla}^{A,v}$, induced by the vertical Lie derivative and the connection A. Then we have the following, see also [19, Prop 3.14].

Proposition 3.1.13. *Under the vertical Hodge-de-Rham isomorphism the flat connection $\nabla^{H_{A,v}}$ coincides with the connection defined by*

$$\nabla^{\mathscr{H}_{A,v}}_X := P_{\ker(D_{A,v})} \circ \widetilde{\nabla}^{A,v}_{X^h}, \quad X \in C^\infty(B, TB).$$

Proof. For convenience we drop the reference to the flat connection A. Denote by

$$\Psi : \ker d_v \cap \Omega^\bullet_v(M) \to C^\infty\big(B, H^\bullet_v(M)\big)$$

the quotient map. Then, according to Definition 3.1.10,

$$\nabla^{H_v}_X\big(\Psi(\omega)\big) = \Psi \circ \mathrm{i}(X^h) \circ d_h(\omega), \quad \omega \in \ker d_v \cap \Omega^\bullet_v(M), \quad X \in C^\infty(B, TB).$$

Using Proposition 3.1.9 and Theorem 3.1.5, we can explicitly describe the isomorphism $\mathscr{H}^\bullet_v(M) \cong H^\bullet_v(M)$ in terms of sections by the composition

$$C^\infty\big(B, \mathscr{H}^\bullet_v(M)\big) = \ker D_v \cap \Omega^\bullet_v(M) \hookrightarrow \ker d_v \cap \Omega^\bullet_v(M) \xrightarrow{\Psi} C^\infty\big(B, H^\bullet_v(M)\big).$$

This implies that $\nabla^{H_v}_X \Psi(\omega) \in C^\infty\big(B, H^\bullet_v(M)\big)$ corresponds to

$$P_{\ker D_v} \circ \mathrm{i}(X^h) \circ d_h\big(P_{\ker D_v}\omega\big) \in C^\infty\big(B, \mathscr{H}^\bullet_v(M)\big).$$

Finally, Proposition 2.1.15 shows that on $\Omega^\bullet_v(M)$

$$\mathrm{i}(X^h) \circ d_h = \widetilde{\nabla}^v_{X^h},$$

from which we obtain the claimed formula. □

Metrics on the Bundle of Vertical Cohomology Groups. The $C^\infty(B)$-module $\Omega^\bullet_v(M,E)$ is endowed with the pairing

$$(\omega, \eta)_{M/B} := \int_{M/B} \langle \omega, \eta \rangle \, \mathrm{vol}_F(g_v) \in C^\infty(B), \quad \omega, \eta \in \Omega^\bullet_v(M,E), \tag{3.10}$$

where the scalar product in the integrand is induced by g_v together with the Hermitian metric on E.

Definition 3.1.14. Let g_v be a vertical metric. We define

$$\langle \omega, \eta \rangle_{\mathscr{H}_{A,v}} := (\omega, \eta)_{M/B}, \quad \omega, \eta \in C^\infty\big(B, \mathscr{H}^\bullet_{A,v}(M)\big),$$

and, if τ_v is the vertical chirality operator,

$$Q_{A,v}(\omega, \eta) := \langle \omega, \tau_v \eta \rangle_{\mathscr{H}_{A,v}}.$$

We also use $\langle .,. \rangle_{\mathscr{H}_{A,v}}$ and $Q_{A,v}$ to the corresponding objects induced by the vertical Hodge-de-Rham isomorphism on $H^\bullet_{A,v}(M)$.

Clearly, $\langle.,.\rangle_{\mathscr{H}_{A,v}}$ is a Hermitian metric on the vector bundle $H^\bullet_{A,v}(M) \to B$, which through the vertical Hodge-de-Rham isomorphism depends on the vertical metric g_v. In contrast, $Q_{A,v}$ is independent of g_v since it is related to the vertical intersection form via

$$Q_{A,v}\big([\alpha],[\beta]\big) = i^k \int_{M/B} \langle \alpha \wedge \beta \rangle, \quad [\alpha],[\beta] \in H^\bullet_{A,v}(M), \tag{3.11}$$

where k depends only on the degrees of α and β. Furthermore, one easily checks that $Q_{A,v}$ is an indefinite Hermitian form with signature

$$\operatorname{Sign}(Q_{A,v}) = \operatorname{rk}\big(\mathscr{H}^+_{A,v}(M)\big) - \operatorname{rk}\big(\mathscr{H}^-_{A,v}(M)\big).$$

Here, $\mathscr{H}^\pm_{A,v}(M)$ denotes the ± 1 eigenbundle of τ_v. This implies that $Q_{A,v}$ has signature 0 unless the dimension of the fiber is divisible by 4 in which case $\operatorname{Sign}(Q_{A,v}) = \operatorname{Sign}(F)$.

Proposition 3.1.15. *The flat connection $\nabla^{H_{A,v}}$ is compatible with the indefinite Hermitian metric $Q_{A,v}$. It is compatible with the Hermitian metric $\langle.,.\rangle_{\mathscr{H}_{A,v}}$ if and only if for all $X \in C^\infty(M, T^hM)$*

$$2P_{\ker D_v} \circ B(X) \circ P_{\ker D_v} + k_v(X) = 0, \tag{3.12}$$

where $B(X)$ is the tensor as in (2.10)*, and k_v is the mean curvature form.*

Proof. For the first part we use the description (3.11) for $Q_{A,v}$. Let $\alpha, \beta \in \Omega^\bullet_v(M,E)$ be chosen in such a way that $\langle \alpha \wedge \beta \rangle$ is of maximal vertical degree. Then

$$d_B \int_{M/B} \langle \alpha \wedge \beta \rangle = \int_{M/B} d_h \langle \alpha \wedge \beta \rangle = \int_{M/B} \langle d_{A,h}\alpha \wedge \beta \rangle + (-1)^{|\alpha|} \langle \alpha \wedge d_{A,h}\beta \rangle.$$

Hence, if $d_{A,v}\alpha = d_{A,v}\beta = 0$ we have

$$d_B Q_{A,v}\big([\alpha],[\beta]\big) = Q_{A,v}\big([d_{A,h}\alpha],[\beta]\big) + (-1)^{|\alpha|} Q_{A,v}\big([\alpha],[d_{A,h}\beta]\big),$$

so that, according to Definition 3.1.10,

$$X Q_{A,v}\big([\alpha],[\beta]\big) = Q_{A,v}\big(\nabla^{H_{A,v}}_X [\alpha],[\beta]\big) + Q_{A,v}\big([\alpha],\nabla^{H_{A,v}}_X [\beta]\big), \quad X \in C^\infty(B,TB).$$

This shows that $\nabla^{H_{A,v}}$ is indeed compatible with $Q_{A,v}$. Now let g_v be a vertical metric, and let $\omega, \eta \in C^\infty\big(B, \mathscr{H}^\bullet_{A,v}(M)\big)$. Then

$$d_B \int_{M/B} \langle \omega, \eta \rangle \operatorname{vol}_F(g_v) = \int_{M/B} d_h\big(\langle \omega, \eta \rangle\big) \wedge \operatorname{vol}_F(g_v) + \int_{M/B} \langle \omega, \eta \rangle \, d_h \operatorname{vol}_F(g_v).$$

It follows from Proposition 2.1.17 that

$$d_h \operatorname{vol}_F(g_v) = k_v \wedge \operatorname{vol}_F(g_v).$$

Since the connection $\nabla^{A,v}$ is compatible with the metric on $\Omega^\bullet_v(M,E)$ we know that for all $X \in C^\infty(B,TB)$

$$X^h \langle \omega, \eta \rangle = \big\langle \nabla^{A,v}_{X^h} \omega \,,\, \eta \big\rangle + \big\langle \omega \,,\, \nabla^{A,v}_{X^h} \eta \big\rangle.$$

From (2.9) and (2.10) we then deduce

$$X^h\langle\omega,\eta\rangle = \big\langle\widetilde{\nabla}^{A,v}_{X^h}\omega\,,\,\eta\big\rangle + \big\langle\omega\,,\,\widetilde{\nabla}^{A,v}_{X^h}\eta\big\rangle + \big\langle B(X^h)\omega\,,\,\eta\big\rangle + \big\langle\omega\,,\,B(X^h)\eta\big\rangle.$$

Now, $B(X^h)$ is easily seen to be self-adjoint with respect to the metric $\langle .,.\rangle$ on $\Omega^\bullet_v(M,E)$. Putting all pieces together, we find that for the connection $\nabla^{\mathscr{H}_{A,v}}_X$ of Proposition 3.1.13

$$\begin{aligned} X\langle\omega,\eta\rangle_{\mathscr{H}_{A,v}} = {} & \big\langle\nabla^{\mathscr{H}_{A,v}}_X\omega\,,\,\eta\big\rangle_{\mathscr{H}_{A,v}} + \big\langle\omega\,,\,\nabla^{\mathscr{H}_{A,v}}_X\eta\big\rangle_{\mathscr{H}_{A,v}} \\ & + \big\langle\omega\,,\,2B(X^h)\eta\big\rangle_{\mathscr{H}_{A,v}} + \big\langle\omega\,,\,k_v(X^h)\eta\big\rangle_{\mathscr{H}_{A,v}}, \end{aligned}$$

which proves that $\nabla^{\mathscr{H}_{A,v}}$ is compatible with $\langle .,.\rangle_{\mathscr{H}_{A,v}}$ if and only if (3.12) holds. Since the metrics as well as the connections $\nabla^{H_{A,v}}$ and $\nabla^{\mathscr{H}_{A,v}}$ on $H^\bullet_{A,v}(M)$ and $\mathscr{H}^\bullet_{A,v}(M)$ coincide under the vertical Hodge-de-Rham isomorphism, the proof of Proposition 3.1.15 is finished. □

Remark. If we denote by p and q the maximal ranks of subbundles of $H^\bullet_{A,v}(M)$ on which $Q_{A,v}$ is positive respectively negative definite, we can rephrase the first part of Proposition 3.1.15 by saying that $\nabla^{H_{A,v}}$ is a flat $\mathrm{U}(p,q)$-connection.[1] The choice of a vertical metric reduces the structure group of $H^\bullet_{A,v}(M)$ to the subgroup $\mathrm{U}(p)\times\mathrm{U}(q)$. However, the connection does not necessarily reduce to a flat $\mathrm{U}(p)\times\mathrm{U}(q)$-connection, the geometric obstruction being (3.12). As we have seen in Theorem 3.1.6 we can always arrange that the mean curvature form vanishes. For arbitrary fiber bundles, the tensor $B(X)$ is, however, a non-trivial obstruction. It would be interesting to find a topological condition which guarantees that there exists a vertical metric such that (3.12) holds.

Definition 3.1.16. Let $D_{A,v}$ and $D_{A,h}$ be the vertical and horizontal de Rham operators as in Definition 2.1.23. If $\dim M$ is odd, we define the odd signature operator on B with *values in the bundle of vertical cohomology groups*,

$$D_B\otimes\nabla^{\mathscr{H}_{A,v}} : \Omega^\bullet\big(B,\mathscr{H}^\bullet_{A,v}(M)\big)\to\Omega^\bullet\big(B,\mathscr{H}^\bullet_{A,v}(M)\big),$$

by

$$D_B\otimes\nabla^{\mathscr{H}_{A,v}} := P_{\ker D_{A,v}}\circ\tau_M D_{A,h}\circ P_{\ker D_{A,v}}.$$

Here, τ_M is the chirality operator associated to a fixed submersion metric on M.

Remark 3.1.17.

(i) Certainly, $D_B\otimes\nabla^{\mathscr{H}_{A,v}}$ is a formally self-adjoint elliptic differential operator and thus has a well-defined Eta invariant. This will play an important role in Dai's general adiabatic limit formula for the Eta in Section 3.3.

(ii) We note that if $\dim B$ is odd, and (3.12) is satisfied, then $D_B\otimes\nabla^{\mathscr{H}_{A,v}}$ is actually isometric to two copies of the odd signature operator on B twisted by $\nabla^{\mathscr{H}_{A,v}}$. This is because we have not restricted to forms on the base of even degree, compare with Remark 1.4.4 (i).

[1] Recall that $\mathrm{U}(p,q)$ denotes the isometry group of the quadratic from

$$\sum_{j=1}^{p}|z_j|^2-\sum_{j=p+1}^{p+q}|z_j|^2.$$

3.1.6 Eta Invariants of $\mathrm{U}(p,q)$-Connections

Before we continue with the general discussion, we briefly want to digress on the Eta invariant of the operator $D_B \otimes \nabla^{\mathscr{H}_{A,v}}$ introduced above. Without any effort, we can treat the more general case that $E \to B$ is a complex vector bundle, endowed with an indefinite Hermitian metric Q and a connection ∇, not necessarily flat, but compatible with Q. We choose a splitting $E = E^+ \oplus E^-$ into subbundles where Q is positive respectively negative definite, and define τ_E to be $\pm\,\mathrm{id}_E$ on $E^\pm$. Then we can define a Hermitian metric on E via

$$h(e,f) := Q(e,\tau f), \quad e,f \in E,$$

compare with Definition 3.1.14. Note that the splitting $E = E^+ \oplus E^-$ is orthogonal with respect to h. We now define an $\mathrm{End}(E)$-valued 1-form on B

$$\omega^{\nabla,\tau_E}(X) := \tau_E\big[\nabla_X, \tau_E\big] = \tau_E \circ \nabla_X \circ \tau_E - \nabla_X, \quad X \in C^\infty(B,TB).$$

Then we have the following simple result.

Lemma 3.1.18. *For all $X \in C^\infty(B,TB)$, the endomorphism $\omega^{\nabla,\tau_E}(X)$ is self-adjoint with respect to h. It interchanges the subbundles E^+ and E^-. Moreover, the connection*

$$\nabla^u := \nabla + \tfrac{1}{2}\omega^{\nabla,\tau_E},$$

is unitary with respect to h.

Proof. Let $e,f \in C^\infty(B,E)$. Since ∇_X is compatible with Q, one verifies—using in particular that $\tau_E^2 = \mathrm{id}_E$ and that $Q \circ \tau_E = Q$,

$$\begin{aligned} h\big(\omega^{\nabla,\tau_E}(X)e, f\big) &= Q\big(\tau_E\big[\nabla_X,\tau_E\big]e, \tau_E f\big) = Q\big(\nabla_X(\tau_E e), f\big) - Q\big(\nabla_X e, \tau_E f\big) \\ &= -Q\big(\tau_E e, \nabla_X f\big) + Q\big(e, \nabla_X(\tau_E f)\big) = Q\big(e, \big[\nabla_X,\tau_E\big]f\big) \\ &= h\big(e, \omega^{\nabla,\tau_E}(X)f\big). \end{aligned}$$

Hence, $\omega^{\nabla,\tau_E}(X)$ is self-adjoint with respect to h. Now, let $P_{E^\pm} := \frac{1}{2}(\mathrm{id}_E \pm \tau_E)$ denote the projection onto $E^\pm$. Then one easily obtains that

$$P_{E^+} \circ \nabla_X \circ P_{E^-} + P_{E^-} \circ \nabla_X \circ P_{E^+} = -\tfrac{1}{2}\omega^{\nabla,\tau_E}(X).$$

On the one hand, this implies that $\omega^{\nabla,\tau_E}(X)$ interchanges the subbundles E^+ and E^-. By definition of ∇^u, we can deduce on the other hand, that ∇^u preserves E^+ and E^- from which it easily follows that ∇^u is unitary with respect to h. $\square$

Remark. In the case that $E = \mathscr{H}^\bullet_{A,v}(M)$ is the bundle of vertical cohomology groups, $Q = Q_v$ is the vertical intersection form and $\nabla = \nabla^{H_{A,v}}$ is the natural flat connection, the 1-form ω^{∇,τ_E} is precisely the 1-form appearing in (3.12), compare also with (2.15). This gives a more abstract explanation of Proposition 3.1.15.

The Odd Signature Operator with values in *E*. To define the analog of $D_B \otimes \nabla^{\mathscr{H}_{A,v}}$ in the case at hand, we choose a metric g_B on B, and let τ_B the associated chirality operator on $\Omega^\bullet(B,E)$. Let $b := \dim B$, and extend τ_E to $\Omega^\bullet(B,E)$ by requiring that

$$\tau_E(\alpha \otimes e) = (-1)^{p(b+1)}\alpha \otimes \tau_E e, \quad \alpha \in \Omega^p(B), \quad e \in C^\infty(B,E). \tag{3.13}$$

Checking signs one finds that $\tau_B\tau_E = \tau_E\tau_B$. We then define

$$\tau := \tau_B\tau_E : \Omega^\bullet(B,E) \to \Omega^{b-\bullet}(B,E),$$

which takes the place of the total chirality operator τ_M, compare with Lemma 2.2.3. Note that more explicitly, if $\alpha \otimes e \in \Omega^p(B,E)$, then

$$\tau(\alpha\otimes e) = \tau_B\big((-1)^{p(b+1)}\alpha\otimes\tau_E e\big) = (-1)^{p(b+1)}(\tau_B\alpha)\otimes\tau_E e.$$

Then the analog of $D_B\otimes\nabla^{\mathscr{H}_{A,v}}$ is given by

$$D_B\otimes\nabla := \tau d_\nabla + d_\nabla\tau : \Omega^\bullet(B,E)\to\Omega^\bullet(B,E), \tag{3.14}$$

where d_∇ is the exterior differential on B twisted by the connection ∇ on E. We also define

$$D_B\otimes\nabla^u := \tau d_{\nabla^u} + d_{\nabla^u}\tau,$$

and denote by $\nabla^{u,\pm}$ the restriction of ∇^u to $E^\pm$.

Lemma 3.1.19. *With respect to the splitting $E = E^+\oplus E^-$, the operator $D_B\otimes\nabla^u$ is of the form*

$$D_B\otimes\nabla^u = \begin{pmatrix} D_B^{\nabla^{u,+}} & 0 \\ 0 & -D_B^{\nabla^{u,-}}\end{pmatrix},$$

where

$$D_B^{\nabla^{u,\pm}} := \tau_B(-1)^{b+1}d_{\nabla^{u,\pm}} + d_{\nabla^{u,\pm}}\tau_B, \quad b := \dim B.$$

Moreover, if we define

$$V := D_B\otimes\nabla - D_B\otimes\nabla^u,$$

then V is a self-adjoint operator on $\Omega^\bullet(B,E)$ of order 0 which interchanges $\Omega^\bullet(B,E^+)$ and $\Omega^\bullet(B,E^-)$. In particular, $D_B\otimes\nabla$ and $D_B\otimes\nabla^u$ are formally self-adjoint.

Proof. It follows from Lemma 3.1.18 that τ_E commutes with ∇^u. Using the sign convention in (3.13) it is immediate that

$$\tau_E d_{\nabla^u} = (-1)^{b+1}d_{\nabla^u}\tau_E.$$

Hence,

$$\tau d_{\nabla^u} + d_{\nabla^u}\tau = \big((-1)^{b+1}\tau_B d_{\nabla^u} + d_{\nabla^u}\tau_B\big)\tau_E,$$

which proves the first assertion. The other assertions are a simple consequence of the corresponding properties of ω^{∇,τ_E} in Lemma 3.1.18, since by definition

$$d_{\nabla^u} = d_\nabla + \tfrac{1}{2}\,\mathrm{e}\left(\omega^{\nabla,\tau_E}\right),$$

where $\mathrm{e}(.)$ is exterior multiplication. $\square$

Difference of Eta Invariants. Roughly, Lemma 3.1.19 asserts that $D\otimes\nabla$ is the direct sum of two geometric Dirac operator plus a lower order perturbation which interchanges the twisting bundles. This leads to a simple relation between the Eta invariants of $D\otimes\nabla$ and $D\otimes\nabla^u$. The following result is a reformulation of [16, Thm.'s 2.7 & 2.35]. We formulate it in terms of the ξ-invariant, see Definition 1.3.4.

Theorem 3.1.20. *As before, let* $V = D_B \otimes \nabla - D_B \otimes \nabla^u$*. Then*

$$\xi(D_B \otimes \nabla) - \xi(D_B \otimes \nabla^u) = \mathrm{SF}\left(D_B \otimes \nabla^u + xV\right)_{x\in[0,1]}.$$

Remark. By comparison with the variation formula of Corollary D.2.6, we see that Theorem 3.1.20 asserts that the contribution coming from the variation of the reduced ξ-invariant vanishes, i.e.,

$$\int_0^1 \tfrac{d}{dx}[\xi(D_x)]dx = 0.$$

In fact, this is precisely what Bismut and Cheeger prove, see [16, Lem. 2.11]. Recall from Proposition D.2.5 that

$$\tfrac{d}{dt}[\xi(D_x)]dx = -\tfrac{1}{\sqrt{\pi}}\, a_n(V, D_x^2),$$

where $a_n(V, D_x^2)$ is the constant term in the asymptotic expansion of

$$\sqrt{t}\ \mathrm{Tr}\left(Ve^{-tD_x^2}\right), \quad \text{as } t \to 0.$$

In the case that $\dim B$ is even, Theorem 1.2.7 shows that there are no half integer powers of t in the asymptotic expansion of $\mathrm{Tr}\left(Ve^{-tD_x^2}\right)$, so that Theorem 3.1.20 is a consequence of the general theory for elliptic operators. However, the odd dimensional case requires considerably more work. In [16, Lem. 2.11], Bismut and Cheeger prove the corresponding result for operators of the form we are considering here. Their proof uses Getzler's local index theory techniques for twisted Dirac operators, adapted to odd dimensional base spaces, in a similar way as we have described in Section 1.5.2.

3.2 Elements of Bismut's Local Index Theory for Families

To discuss Dai's adiabatic limit formula, we need to recall some aspects of Bismut's local index theory for families. We will be rather sketchy and refer to the original article [14] and the treatment in [13, Ch.'s 9 & 10] for more details. The survey article [15] is also recommended. For convenience and since we will not need a greater generality, we restrict to the case of the signature operator.

3.2.1 The Index Theorem for Families

The predecessor of local index theorem for families is the K-theoretic version by Atiyah and Singer [10], which we briefly recall. As announced we consider only the case of the signature operator. Let $F \hookrightarrow M \xrightarrow{\pi} B$ be an oriented fiber bundle of closed manifolds, where F is assumed to be even dimensional. We choose a vertical projection and a vertical metric g_v. Let ∇^v be the associated connection on T^vM, and let D_v^+ be the vertical signature operator defined by the vertical chirality operator. As in Proposition 3.1.4 we can view $\ker D_v^+$ and $\operatorname{coker} D_v^+$ as (spaces of sections of) finite dimensional vector bundles over B.

Definition 3.2.1. The *index bundle* associated to D_v^+ is defined by

$$\operatorname{Ind} D_v^+ := [\ker D_v^+] - [\operatorname{coker} D_v^+] \in K^0(B).$$

Note that as we are considering only families of signature operators we do not need the beautiful construction for the case of varying dimensions as in [10]. The Chern character defines a map in K-theory,

$$\mathrm{ch} : K^0(B) \otimes \mathbb{C} \to H^{\mathrm{ev}}(B).$$

Then cohomological version of the families index theorem as in [10, Thm. 5.1] is

Theorem 3.2.2 (Atiyah-Singer)**.** *The Chern character of the index bundle associated to the signature operator is given by*

$$\mathrm{ch}(\mathrm{Ind}\, D_v^+) = \Big[\int_{M/B} L(T^v M, \nabla^v) \Big] \in H^{\mathrm{ev}}(B),$$

where $L(T^v M, \nabla^v)$ is the Hirzebruch L-form of $T^v M$ defined via Chern-Weil theory as in (A.4) *in terms of ∇^v.*

3.2.2 Superconnections and Associated Dirac Operators

Quillen [84] introduced superconnections to study Chern-Weil theory for the Chern character of a difference bundle. We briefly recall the basic definitions, and refer to [13, Sec. 1.4] and [84] for details. Let B be a closed, oriented manifold, and let $E \to B$ be a complex vector bundle.

Definition 3.2.3. A differential operator $\mathbb{A}$ on $\Omega^\bullet(B, E)$ is called a *generalized connection* on E if it satisfies the Leibniz rule

$$\mathbb{A}(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \mathbb{A}\beta,$$

where $\alpha \in \Omega^\bullet(B)$ and $\beta \in \Omega^\bullet(B, E)$. The *curvature* of $\mathbb{A}$ is defined as

$$\mathbb{A}^2 \in \Omega^\bullet\big(B, \mathrm{End}(E)\big).$$

If E is $\mathbb{Z}_2$-graded and $\mathbb{A}$ is of odd parity with respect to the total grading on $\Omega^\bullet(B, E)$, then $\mathbb{A}$ is called a *superconnection.*

Remark.

(i) If $E = E^+ \oplus E^-$ is $\mathbb{Z}_2$-graded, the total grading of $\Omega^\bullet(B, E)$ referred to above is defined by

$$\Omega(B, E)^\pm := \Omega^{\mathrm{ev}}(B, E^\pm) + \Omega^{\mathrm{odd}}(B, E^\mp).$$

This should not be confused with the grading $\Omega^\pm(B, E)$, induced by the chirality operator τ_B.

(ii) The fact that the curvature $\mathbb{A}^2$ is indeed given by the action of an element in $\Omega^\bullet\big(B, \mathrm{End}(E)\big)$ works as in the case of a usual connection, see [13, Prop. 1.38].

(iii) A generalized connection $\mathbb{A}$ is determined by its homogeneous components

$$\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \mathbb{A}_{[2]} + \dots,$$

where $\mathbb{A}_{[p]} \in \Omega^p\big(B, \mathrm{End}(E)\big)$ for $p \neq 1$ and $\mathbb{A}_{[1]}$ is a connection on E. In the case that $\mathbb{A}$ is a superconnection, the connection part $\mathbb{A}_{[1]}$ preserves the splitting $E = E^+ \oplus E^-$.

Definition 3.2.4. Let E be $\mathbb{Z}_2$-graded, and let $\mathbb{A}$ be a superconnection on E. Then we define the *Chern character form* of $\mathbb{A}$ as

$$\mathrm{ch}_s(E,\mathbb{A}) := \mathrm{str}_E\bigl(\gamma\exp(-\mathbb{A}^2)\bigr).$$

Here, $\gamma : \Omega^\bullet(B) \to \Omega^\bullet(B)$ is a normalization function, defined for forms of homogeneous degree by

$$\gamma(\alpha) := \bigl(\tfrac{1}{\sqrt{2\pi i}}\bigr)^{|\alpha|}\alpha, \quad \text{where } \sqrt{i} = e^{\frac{i\pi}{4}}. \tag{3.15}$$

The discussion in Appendix A generalizes to the case of superconnections. In particular, $\mathrm{ch}_s(E,\mathbb{A})$ is a closed differential from on B whose cohomology class is independent of the superconnection $\mathbb{A}$. Since the supertrace vanishes on endomorphisms of odd parity, one can also check that $\mathrm{ch}_s(E,\mathbb{A}) \in \Omega^{\mathrm{ev}}(B,E)$. We also note that if $\mathbb{A} = \nabla$ is a connection in the usual sense, which decomposes with respect to the splitting $E = E^+ \oplus E^-$ into $\nabla = \nabla^+ \oplus \nabla^-$, then

$$\mathrm{ch}_s(E,\nabla) = \mathrm{ch}(E^+,\nabla^+) - \mathrm{ch}(E^-,\nabla^-), \tag{3.16}$$

where the right hand side is as in Definition A.1.3. Then the main idea of [84] can be summarized as

Theorem 3.2.5 (Quillen)**.** *Let $E \to B$ be a $\mathbb{Z}_2$-graded Hermitian bundle over B. Let $[E^+] - [E^-]$ be the induced element in $K^0(B)$, and let $\mathbb{A}$ be a superconnection on E. Then*

$$\mathrm{ch}\bigl([E^+] - [E^-]\bigr) = \bigl[\mathrm{ch}_s(E,\mathbb{A})\bigr] \in H^{\mathrm{ev}}(B).$$

Generalized Clifford Connections and Dirac Operators. Whenever E is endowed with the structure of a Clifford module, one can associate a Dirac operator to a generalized connection $\mathbb{A}$. Let

$$\boldsymbol{\sigma}^{-1} : \Lambda^\bullet T^*B \to \mathrm{Cl}(T^*B)$$

be the quantization map. If $c : \mathrm{Cl}(T^*B) \to \mathrm{End}(E)$ denotes Clifford multiplication, we get a natural Clifford contraction

$$\Omega^\bullet(B,E) \xrightarrow{\boldsymbol{\sigma}^{-1}} C^\infty\bigl(B, \mathrm{Cl}(T^*B)\otimes E\bigr) \xrightarrow{c} C^\infty(B,E).$$

Since $\mathbb{A}$ maps $C^\infty(B,E)$ to $\Omega^\bullet(B,E)$, we can define

$$D_{\mathbb{A}} := c \circ \boldsymbol{\sigma}^{-1} \circ \mathbb{A} : C^\infty(B,E) \to C^\infty(B,E). \tag{3.17}$$

Clearly, this defines an elliptic operator of first order. In order for $D_{\mathbb{A}}$ to be formally self-adjoint, we certainly have to require that the connection part $\mathbb{A}_{[1]}$ of $\mathbb{A}$ is a Clifford connection in sense of Definition 1.2.11. Furthermore, some condition has to be imposed on the other homogeneous components of $\mathbb{A}$, which we derive now, see also [13, p. 117]. For $p \neq 1$ and a local orthonormal frame $\{f_a\}$ for TB we can write locally

$$A_{[p]} = \tfrac{1}{p!} f^{a_1} \wedge \ldots \wedge f^{a_p} \wedge T_{a_1 \ldots a_p}, \quad \text{with} \quad T_{a_1\ldots a_p} \in C^\infty\bigl(B, \mathrm{End}(E)\bigr).$$

The contribution to $D_{\mathbb{A}}$ is then given by

$$c \circ \boldsymbol{\sigma}^{-1} \circ A_{[p]} = \tfrac{1}{p!} c^{a_1} \ldots c^{a_p} T_{a_1 \ldots a_p},$$

where c^{a_j} is short for Clifford multiplication with f^{a_j}. For $e, \widetilde{e} \in C^\infty(B,E)$ one computes that

$$\begin{aligned}\left\langle c^{a_1}\dots c^{a_p} T_{a_1\dots a_p} e\,,\,\widetilde{e}\right\rangle &= (-1)^p\left\langle e\,,\,T^*_{a_1\dots a_p} c^{a_p}\dots c^{a_1}\widetilde{e}\right\rangle \\ &= (-1)^{\frac{p(p+1)}{2}}\left\langle e\,,\,T^*_{a_1\dots a_p} c^{a_1}\dots c^{a_p}\widetilde{e}\right\rangle.\end{aligned}$$

This motivates the following

Definition 3.2.6. Let E be a Hermitian vector bundle. A generalized connection $\mathbb{A}$ is called *unitary* if its connection part is a unitary connection and if for $p \neq 1$

$$\mathbb{A}^*_{[p]} = (-1)^{\frac{p(p+1)}{2}}\mathbb{A}_{[p]}.$$

Here, taking the adjoint is meant with respect to the endomorphism part only. $\mathbb{A}$ is is called a *generalized Clifford connection*, if in addition, its connection part $\mathbb{A}_{[1]}$ is a Clifford connection, and if for $p \neq 1$ and $\xi \in \Omega^1(B)$

$$\mathbb{A}_{[p]}c(\xi) = (-1)^p c(\xi)\mathbb{A}_{[p]},$$

where again the product is to be understood in the endomorphism part.

The essential part of the following result is a consequence of the discussion preceding Definition 3.2.6. The other claims are immediate.

Proposition 3.2.7. *Let $E \to B$ be a Hermitian vector bundle endowed with a Clifford structure, and let $\mathbb{A}$ be a generalized Clifford connection. Then $D_{\mathbb{A}}$ is formally self-adjoint and $D^2_{\mathbb{A}}$ is a generalized Laplacian. The symbol of $\mathbb{A}$ is given by Clifford multiplication $c : T^*B \otimes E \to E$. If in addition E is $\mathbb{Z}_2$-graded and $\mathbb{A}$ is a superconnection, then $D_{\mathbb{A}}$ is $\mathbb{Z}_2$-graded.*

Remark. As we have pointed out in Remark 1.2.12, not every Dirac operator arises as a geometric Dirac operator associated to a Clifford connection. However, it is shown in [13, Prop. 3.42] that there is a 1-1 correspondence between Clifford superconnections and $\mathbb{Z}_2$-graded Dirac operators with symbol being the given Clifford structure. Going through the proof of loc.cit. one sees that the same statement is true for ungraded Dirac operators and generalized Clifford connections as defined above. Note, however, that in contrast to [13], we require Dirac operators to be formally self-adjoint.

3.2.3 The Families Index Theorem for the Signature Operator

Generalizing Quillen's construction to infinite dimensional bundles, Bismut [14] found a heat equation formula for the Chern character form of the index bundle. We describe the setup briefly, again restricting to the case of the untwisted signature operator.

Bismut's Superconnection. Let $F \hookrightarrow M \xrightarrow{\pi} B$ be an oriented fiber bundle of closed manifolds. We choose a vertical projection, and let $\Omega^\bullet_v(M)$ be the $C^\infty(B)$-module of vertical differential forms. We formally interpret this as the space of sections of an infinite dimensional bundle $\mathscr{E}$ over B, where the fiber $\mathscr{E}_y$ over $y \in B$ is given by the space of differential forms over $\pi^{-1}(y)$. Since this picture has only motivational character, we do not give any details of how this bundle of Fréchét spaces is defined rigorously. We can then view the space of all differential forms $\Omega^\bullet(M)$ as

$$\Omega^\bullet(M) \cong \Omega^\bullet(B, \mathscr{E}),$$

compare with (2.6). Proposition 2.1.15 shows that the total exterior differential d_M on $\Omega^\bullet(M)$ splits as

$$d_M = d_v + d_h + \mathrm{i}(\Omega), \tag{3.18}$$

where with respect to any choice of metric g_B in a local orthonormal frame $\{f_a\}$ for TB

$$d_h = f^a \wedge \widetilde{\nabla}^\oplus_a, \quad \mathrm{i}(\Omega) = \tfrac{1}{2} f^a \wedge f^b \wedge \mathrm{i}(\Omega_{ab}). \tag{3.19}$$

Recall that when restricted to $\Omega^\bullet_v(M)$, the connection $\widetilde{\nabla}^\oplus_a$ is defined as $\widetilde{\nabla}^v_a$, see Definition 2.1.9. We view the latter as a natural connection $\nabla^{\mathscr{E}}$ on the infinite dimensional bundle $\mathscr{E}$. Then (3.18) and (3.19) express d_M as a generalized connection on $\mathscr{E}$ with connection part $\nabla^{\mathscr{E}}$. It is a superconnection with respect to the even/odd grading on $\Omega^\bullet_v(M)$. In this interpretation, the property $d_M^2 = 0$ states that d_M is a *flat* superconnection on the bundle of vertical differential forms, an observation which is due to [19, Sec. III (b)]. For this reason, d_M together with its interpretation as a superconnection is sometimes called the *Bismut-Lott superconnection.*

Definition 3.2.8. Assume that $F \hookrightarrow M \xrightarrow{\pi} B$ is endowed with a vertical metric g_v and a vertical projection. Let ∇^v be the associated canonical connection, and let D_v be the vertical de Rham operator. With respect to any choice of g_B and a local orthonormal frame $\{f_a\}$ on B define

$$\nabla^{\mathscr{E},u} := f^a \wedge \big(\nabla^v_a + \tfrac{1}{2} k_v(f_a)\big) : \Omega^\bullet_v(M) \to \Omega^{1,\bullet}(M), \tag{3.20}$$

where k_v is the mean curvature form. Then, the *Bismut superconnection* is defined as

$$\mathbb{B} := \tfrac{1}{2} D_v + \nabla^{\mathscr{E},u} - \tfrac{1}{2} c_v(\Omega) : \Omega^\bullet_v(M) \to \Omega^\bullet(M),$$

where c_v denotes the vertical Clifford multiplication on $\Omega^\bullet(M)$, and locally

$$c_v(\Omega) = \tfrac{1}{2} f^a \wedge f^b \wedge c_v(\Omega_{ab}).$$

The Bismut superconnection naturally extends to an operator $\Omega^\bullet(M) \to \Omega^\bullet(M)$, if we replace ∇^v with $\nabla^\oplus$ in (3.20). For this note that for $\alpha \in \Omega^p(B)$ and $\omega \in \Omega^\bullet_v(M)$,

$$f^a \wedge \nabla^\oplus_a\big((\pi^*\alpha) \wedge \omega)\big) = \pi^*(d_B\alpha) \wedge \omega + (-1)^p \pi^*\alpha \wedge f^a \wedge \nabla^v_a \omega.$$

This relation also shows that replacing ∇^v with $\nabla^\oplus$ is the same as extending $\mathbb{B}$ from $\Omega^\bullet_v(M) = C^\infty(B, \mathscr{E})$ to $\Omega^\bullet(M) = \Omega^\bullet(B, \mathscr{E})$ by requiring the Leibniz rule. Moreover, we find that the term $f^a \wedge \nabla^\oplus_a$ is independent of the chosen metric g_B, since this is true for the connection ∇^v, see Proposition 2.1.3.

Remark 3.2.9. We want to point out that the definition of $\mathbb{B}$ can be motivated by an infinite dimensional version of Lemma 3.1.18. If we choose a vertical metric g_v on T^vM, we can view the pairing $(.,.)_{M/B}$ in (3.10) as a metric on the bundle $\mathscr{E}$. As in Proposition 3.1.15, the connection part $\nabla^{\mathscr{E}} = f^a \wedge \widetilde{\nabla}^v_a$ of d_M is compatible with the vertical intersection pairing, but not necessarily with $(.,.)_{M/B}$. If we proceed as in Lemma 3.1.18—using in particular (2.9) and (2.15)—we see that the unitary connection associated to $\nabla^{\mathscr{E}}$ is given by

$$\begin{aligned} f^a \wedge \widetilde{\nabla}^v_a + \tfrac{1}{2} f^a \wedge \tau_v \big[\widetilde{\nabla}^v_a, \tau_v\big] &= f^a \wedge \big(\widetilde{\nabla}^v_a + B(f_a) + \tfrac{1}{2} k_v(f_a)\big) \\ &= f^a \wedge \big(\nabla^v_a + \tfrac{1}{2} k_v(f_a)\big). \end{aligned}$$

This is precisely the connection part $\nabla^{\mathscr{E},u}$ of the Bismut superconnection. In order to get the unitary superconnection associated to d_M we proceed as in Definition 3.2.6 and replace the other homogeneous components d_v and $\mathrm{i}(\Omega)$ of d_M with

$$\tfrac{1}{2}(d_v + d_v^t) = \tfrac{1}{2} D_v \quad \text{and} \quad \tfrac{1}{2}(\mathrm{i}(\Omega) - \mathrm{i}(\Omega)^t) = -\tfrac{1}{2} c_v(\Omega),$$

which explain the remaining terms in the definition of $\mathbb{B}$.

Since Remark 3.2.9 is the underlying motivation for large parts of the treatment in this section, we extract the following result, adding some observations which are immediate.

Proposition 3.2.10. *The Bismut superconnection is the generalized unitary connection associated to the flat superconnection d_M. It is a superconnection with respect to the even/odd grading on $\Omega^\bullet_v(M)$. If the fiber is even dimensional, it is also a superconnection with respect to the grading induced by the vertical chirality operator τ_v.*

Remark 3.2.11. In the context of the signature operator we are interested in the grading given by the vertical chirality operator. However, we get a superconnection with respect to this grading only if the fiber is even dimensional. In the case that the fiber is odd dimensional, one can turn $\mathbb{B}$ into a superconnection by adding an auxiliary Grassmann variable, see [18, Sec.'s II (b) & (f)]. This is based on Quillen's ideas in the finite dimensional case as in [84, Sec. 5].

The Chern Character of the Bismut Superconnection. As in Definition 3.2.3, the curvature of the Bismut superconnection is defined as the differential operator

$$\mathbb{B}^2 : \Omega^\bullet(M) \to \Omega^\bullet(M).$$

In analogy with the finite dimensional situation in Section 3.1.6, we cannot expect that the Bismut superconnection is flat.

Proposition 3.2.12. *The curvature of $\mathbb{B}$ is a fiberwise elliptic operator. It is of second order with leading term given by the vertical Laplacian*

$$D_v^2 : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E).$$

Proof. According to Definition 3.1.1, we have to check first that $\mathbb{B}^2$ is $C^\infty(B)$ linear. For all $\varphi \in C^\infty(B)$,

$$[\mathbb{B}, \pi^* \varphi] = \mathrm{e}(\pi^* d_B \varphi),$$

and thus,

$$\big[\mathbb{B}^2, \pi^* \varphi\big] = \mathbb{B} \circ \mathrm{e}(\pi^* d_B \varphi) + \mathrm{e}(\pi^* d_B \varphi) \circ \mathbb{B}.$$

Now, the operators D_v, $c_v(\Omega)$ and $f^a \wedge k_v(f_a)$ all anti-commute with exterior multiplication with a horizontal 1-form. Hence,

$$\big[\mathbb{B}^2, \pi^*\varphi\big] = f^a \wedge \nabla_a^{\oplus}(\pi^* d_B\varphi) = \pi^* d_B^2\varphi = 0,$$

so that $\mathbb{B}^2$ is indeed a fiberwise differential operator. Now, $\mathbb{B}^2$ contains D_v^2 as term of second order but a priori there might be other contributions coming from $(\nabla^{\mathscr{E},u})^2$ and the anti-commutator of D_v and $\nabla^{\mathscr{E},u}$. To see that this is not the case we note that $\nabla^{\mathscr{E},u}$ agrees with d_h up to a term of order 0. Moreover, we know from Corollary 2.1.16 that

$$d_h^2 = -\big\{d_v, \mathrm{i}(\Omega)\big\} \quad \text{and} \quad \big\{D_v, d_h\big\} = \big\{d_v^t, d_h\big\},$$

where both terms are of order ≤ 1, see Proposition 2.1.25 for the second term. This implies that $(\nabla^{\mathscr{E},u})^2$ and $\{D_v, \nabla^{\mathscr{E},u}\}$ are also of order ≤ 1. $\square$

Since fiberwise elliptic operators are intimately related to families of elliptic operators, the following version of Theorem D.1.7 should be plausible. We do not give a proof, referring to [13, Thm.'s 9.48 & 9.51] for more details.

Theorem 3.2.13. *The operator $e^{-\mathbb{B}^2} : \Omega^\bullet(M) \to \Omega^\bullet(M)$ is a well-defined, fiberwise smoothing operators with coefficients in $\Omega^\bullet(B)$.*

Using the fiberwise supertrace, this result allows us to study the Chern character of the superconnection $\mathbb{B}$. To keep the motivational part of this section reasonably short we skip the discussion of the odd dimensional case and assume henceforth that the fiber is even dimensional.

Definition 3.2.14. Let $F \hookrightarrow M \xrightarrow{\pi} B$ have even dimensional fiber F, and let g_v be a vertical metric. We define the *Chern character of the Bismut superconnection* as

$$\mathrm{ch}_s(\mathscr{E}, \mathbb{B}) := \mathrm{Str}_v\left(\gamma e^{-\mathbb{B}^2}\right) \in \Omega^\bullet(B),$$

where γ is the normalization function as in (3.15).

Transgression and the Rescaled Superconnection. One of Bismut's main observations in [14] is that $\mathrm{ch}_s(\mathscr{E}, \mathbb{B})$ is the right candidate for the Chern character form for the index bundle as in Definition 3.2.1. This becomes apparent when rescaling the vertical metric, see [14, Sec. III (c)].

For $t \in (0, \infty)$ we rescale the vertical metric with a factor of t^{-1}. As in (1.55) this means that Clifford multiplication with a vertical 1-form ξ has to be replaced with

$$c_{v,t}(\xi) = \sqrt{t}\big(\,\mathrm{e}(\xi) - \mathrm{i}(\xi)\big),$$

where inner multiplication is with respect to the fixed metric g_v. Also, if $\{e_i\}$ is a local orthonormal frame for T^vM with respect to g_v, then $\{\sqrt{t}e_i\}$ is orthonormal with respect to the rescaled metric. Since ∇^v and k_v are independent of t, this motivates the following

Definition 3.2.15. For $t \in (0, \infty)$ define the *rescaled Bismut superconnection* by

$$\mathbb{B}_t := \tfrac{\sqrt{t}}{2} D_v + \nabla^{\mathscr{E},u} - \tfrac{1}{2\sqrt{t}} c_v(\Omega).$$

Now, the infinite dimensional version of the transgression formula for the Chern character is the following, see [13, Thm. 9.17].

Theorem 3.2.16. *For all $t \in (0, \infty)$, the differential form $\mathrm{ch}_s(\mathscr{E}, \mathbb{B}_t)$ is closed and satisfies the transgression formula*

$$\frac{d}{dt}\,\mathrm{ch}_s(\mathscr{E}, \mathbb{B}_t) = -\gamma\, d_B \,\mathrm{Str}_v\Big(\frac{d\mathbb{B}_t}{dt} e^{-\mathbb{B}_t^2}\Big).$$

Since this transgression form will be of importance in the next section, we make the following abbreviation.

Definition 3.2.17. The *transgression form* associated to the rescaled Bismut superconnection is given by

$$\alpha(\mathbb{B}_t) := \frac{\gamma}{\sqrt{2\pi i}}\,\mathrm{Str}_v\Big(\frac{d\mathbb{B}_t}{dt} e^{-\mathbb{B}_t^2}\Big) \in \Omega^{\mathrm{odd}}(B).$$

Bismut's Local Index Theorem for Families. With the above ingredients, we can now summarize Bismut's main results [14, Thm.'s 3.4, 4.12 & 4.16] in the case of the signature operator, see also [13, Thm.'s 10.21, 10.23 & 10.32].

Theorem 3.2.18 (Bismut)**.** *Let $F \hookrightarrow M \xrightarrow{\pi} B$ be endowed with a vertical metric g_v and a vertical projection, and assume that F is even dimensional. Then the rescaled superconnection $\mathbb{B}_t$ satisfies*

$$\big[\,\mathrm{ch}_s(\mathscr{E}, \mathbb{B}_t)\big] = \mathrm{ch}(\mathrm{Ind}\, D_v^+) \in H^{ev}(B). \tag{3.21}$$

Moreover, with respect to the C^∞-topology on $\Omega^\bullet(B)$ and for $t \to 0$

$$\lim_{t\to 0} \mathrm{ch}_s(\mathscr{E}, \mathbb{B}_t) = \int_{M/B} L(T^vM, \nabla^v) \quad \text{and} \quad \alpha(\mathbb{B}_t) = O(1), \tag{3.22}$$

where $L(T^vM, \nabla^v)$ is the Hirzebruch L-form of T^vM with respect to the connection ∇^v. In particular,

$$\mathrm{ch}_s(\mathscr{E}, \mathbb{B}_t) = \int_{M/B} L(T^vM, \nabla^v) - d\int_0^t \alpha(\mathbb{B}_s)ds. \tag{3.23}$$

Remark.

(i) An immediate consequence of Bismut's local index theorem for families is Atiyah and Singer's earlier result as stated in Theorem 3.2.2. This is because (3.21) and (3.23) imply that

$$\mathrm{ch}(\mathrm{Ind}\, D_v^+) = \Big[\int_{M/B} L(T^vM, \nabla^v)\Big] \in H^{\mathrm{ev}}(B).$$

(ii) We also want to remark that (3.21) is a generalization of the McKean-Singer formula in Theorem 1.2.6. In the case that B is a point, the term $\mathrm{ch}(\mathrm{Ind}\, D_v^+)$ coincides with the numerical index. On the other hand, $e^{-\mathbb{B}_t^2} = e^{-tD_v^2}$, since the higher degree terms in the Bismut superconnection vanish. Hence, if B is a point, the equation in (3.21) is equivalent to the usual McKean-Singer formula.

To understand why (3.21) is related to the limit $t \to \infty$ also in higher dimensions, we summarize some results from [13, Sec. 9.3]. Let $P_{\ker(D_v)}$ be the projection $\mathscr{E} \to \ker D_v$, and define

$$\nabla^{\mathscr{H}_v,u} := P_{\ker(D_v)} \circ \nabla^{\mathscr{E},u} \circ P_{\ker(D_v)}. \tag{3.24}$$

According to Lemma 3.1.18 and Proposition 3.2.10, the connection $\nabla^{\mathscr{H}_v,u}$ is a unitary connection on the bundle $\mathscr{H}_v^\bullet(M) \to B$, compatible with the grading given by τ_v. We can thus define the associated Chern character form $\mathrm{ch}_s\left(\mathscr{H}_v^\bullet(M), \nabla^{\mathscr{H}_v,u}\right)$ as in (3.16). Then there is the following result, see [13, Thm.'s 9.19 & 9.23].

Theorem 3.2.19 (Berline-Getzler-Vergne)**.** *With respect to the C^∞-topology on $\Omega^\bullet(B)$ and for $t \to \infty$*

$$\lim_{t\to\infty} \mathrm{ch}_s(\mathscr{E}, \mathbb{B}_t) = \mathrm{ch}_s\left(\mathscr{H}_v^\bullet(M), \nabla^{\mathscr{H}_v,u}\right), \quad \textit{and} \quad \alpha(\mathbb{B}_t) = O(t^{-3/2}).$$

In particular,

$$\mathrm{ch}_s\left(\mathscr{H}_v^\bullet(M), \nabla^{\mathscr{H}_v,u}\right) = \mathrm{ch}_s(\mathscr{E}, \mathbb{B}_t) - d\int_t^\infty \alpha(\mathbb{B}_s)ds. \tag{3.25}$$

It is immediate from Definition 3.2.1, the definition of $\nabla^{\mathscr{H}_v,u}$ and (3.16) that

$$\left[\,\mathrm{ch}_s\left(\mathscr{H}_v^\bullet(M), \nabla^{\mathscr{H}_v,u}\right)\right] = \mathrm{ch}(\mathrm{Ind}\, D^+).$$

so that (3.25) is an extension to differential forms of (3.21), and explains why the latter is related to taking the limit $t \to \infty$.

Remark. We want to point out that Theorem 3.2.19 holds for more general superconnections but relies on the fact that the dimensions of the kernels of its homogeneous component of degree 0 do not jump. In contrast, Theorem 3.2.18 continues to hold without any assumption on the kernels but only for the Bismut superconnection associated to a family of Dirac operators.

3.3 A General Formula for Rho Invariants

3.3.1 Transgression and Adiabatic Limits

We now want to relate the discussion in the last section to adiabatic limits of Eta invariants and give a motivation for Dai's general adiabatic limit formula. A large part of this subsection will be heuristic without rigorous arguments. We hope that nevertheless, our discussion helps to give some intuition underlying the sophisticated theory.

Statement of the Reduced Adiabatic Limit Formula. Let $F \hookrightarrow M \xrightarrow{\pi} B$ be an oriented fiber bundle of closed manifolds, where M is odd dimensional. Since this section has only motivational character, we assume for simplicity that F is even dimensional and that B is odd dimensional. We endow the fiber bundle with a vertical projection and a submersion metric $g = g_B \oplus g_v$. Let $\alpha(\mathbb{B}_t)$ be the transgression form associated to the rescaled Bismut superconnection, see Definition 3.2.17. Theorem 3.2.18 and Theorem 3.2.19 show that $\alpha(\mathbb{B}_t) = O(1)$ as $t \to 0$ and $\alpha(\mathbb{B}_t) = O(t^{-3/2})$ as $t \to \infty$. This implies that we can make the following definition, which goes back to [16].

Definition 3.3.1. If F is even dimensional and B is odd dimensional, we define the *Bismut-Cheeger Eta form*

$$\widehat{\eta} := \int_0^\infty \alpha(\mathbb{B}_t)dt.$$

Now let g_ε be the adiabatic metric (2.16) on M, and consider the associated adiabatic family of de Rham operators,

$$D_{M,\varepsilon} = D_v + \varepsilon \cdot D_h + \varepsilon^2 \cdot T : \Omega^\bullet(M) \to \Omega^\bullet(M).$$

Moreover, recall that in Definition 3.1.16 we have introduced the odd signature operator on B with values in the bundle of vertical cohomology groups,

$$D_B \otimes \nabla^{\mathscr{H}_v} : \Omega^\bullet\big(B, \mathscr{H}_v^\bullet(M)\big) \to \Omega^\bullet\big(B, \mathscr{H}_v^\bullet(M)\big).$$

Then we have the following version of [30, Thm.'s 0.1 & 4.4].

Theorem 3.3.2 (Dai)**.** *Assume that* $b := \dim B$ *is odd and that* $\dim F$ *is even. Then the adiabatic limit of the reduced* ξ*-invariant is given by*

$$\lim_{\varepsilon\to 0} \xi\big(\tau_M D_{M,\varepsilon}\big) = \xi\big(D_B \otimes \nabla^{\mathscr{H}_v}\big) + 2^{\frac{b+1}{2}} \int_B \widehat{L}(TB, \nabla^B) \wedge \widehat{\eta} \mod \mathbb{Z}.$$

This formula has a long history which started with Witten's famous holonomy formula in [97]. The first mathematically rigorous treatments were given by Bismut and Freed in [18] as well as Cheeger in [26], both for the case that $B = S^1$. Bismut and Freed emphasize the relation to Bismut's local index theory, whereas Cheeger gives an independent proof based on Duhamel's formula and finite propagation speed methods. Later in [16], Bismut and Cheeger generalized the adiabatic limit formula to higher dimensional base spaces under the assumption that the vertical Dirac operator is invertible. Note that in this case the twisted Eta invariant on the base does not appear. Dai's remarkable generalization in [30] allows the vertical Dirac operators to have non-trivial kernels—which, however, need to form a vector bundle over B. This is why his result applies in particular to the odd signature operator. We will state Dai's result in its full generality below, see Theorem 3.3.10.

The Total Dirac Operator. For the rest of this subsection we want to give a heuristic derivation of Theorem 3.3.2. Recall that we have seen in Theorem 3.2.16 that the form $\alpha(\mathbb{B}_t)$ plays the role of the transgression form associated to the Chern character of the superconnection $\mathbb{B}_t$. The rough idea is now that the adiabatic limit formula of Theorem 3.3.2 is an infinite dimensional analog of the variation formula for the Eta invariant as in Proposition 1.5.1, respectively Corollary 1.5.2.

This idea does not apply directly to the adiabatic family of odd signature operators, and as a tool we have to introduce another natural Dirac operator acting on $\Omega^\bullet(M)$. Let $c : T^*M \to \mathrm{End}\big(\Lambda^\bullet T^*M\big)$ denote the natural Clifford structure, and define a connection on $\Lambda^\bullet T^*M$ via

$$\nabla^S := \nabla^\oplus + \tfrac{1}{2}c(\theta),$$

where θ is the tensor as in Definition 2.1.6. One easily checks that ∇^S is a Clifford connection as in Definition 1.2.11, see e.g. [13, Prop. 10.10].

Definition 3.3.3. The *total Dirac operator* D_S on M is the geometric Dirac operator associated to ∇^S,

$$D_S = c \circ \nabla^S : \Omega^\bullet(M) \to \Omega^\bullet(M).$$

Remark. We recall from Lemma 2.1.19 that on $\Lambda^\bullet T^*M$, the difference of the Levi-Civita connection ∇^g and the connection $\nabla^\oplus$ is given by

$$\nabla^g = \nabla^\oplus + \tfrac{1}{2}\big(c(\theta) - \widehat{c}(\theta)\big),$$

where $\widehat{c}$ is the transposed Clifford structure. Therefore, the total Dirac operator D_S does in general not coincide with the de Rham operator D_M.

Relation to the Bismut Superconnection. As in Section 3.2.3, we interpret $\Omega^\bullet(M)$ as the space of differential forms on B with values in the infinite dimensional vector bundle $\mathscr{E} \to B$ of vertical differential forms. Then the total Dirac operator is a differential operator

$$D_S : \Omega^\bullet(B, \mathscr{E}) \to \Omega^\bullet(B, \mathscr{E}).$$

Using the local description (2.5) for the tensor θ, one then checks that with respect to a local orthonormal frame $\{f_a\}$ for TB,

$$D_S = D_v + c^a \nabla_a^{\mathscr{E},u} - \tfrac{1}{8} c^a c^b c_v(\Omega_{ab}), \tag{3.26}$$

where $\nabla^{\mathscr{E},u}$ is defined as in (3.20). The fact that the terms appearing in this formula are reminiscent of the terms appearing in the definition of the Bismut superconnection seems to be one of the main ideas which lead Bismut to the definition of $\mathbb{B}$, compare with [15, Thm. 2.24] and [14, Sec. V]. We want to make this relation more precise now.

In the interpretation of Proposition 3.2.10, the Bismut superconnection is a superconnection on $\Lambda^\bullet T^*B \otimes \mathscr{E}$. Comparing with Definition 3.2.6, one finds that it is a Clifford connection with respect to the Clifford structure induced by $c : T^*B \to \Lambda^\bullet T^*B$. Now formally using (3.17), we can associate a Dirac operator to this, which is locally given by

$$\tfrac{1}{2} D_v + c^a \nabla_a^{\mathscr{E},u} - \tfrac{1}{4} c^a c^b c_v(\Omega_{ab}).$$

Up to factors of $\frac{1}{2}$, this is coincides with (3.26). Hence, the total Dirac operator is essentially the Dirac operator on $\Omega^\bullet(B, \mathscr{E})$ associated to the Bismut superconnection. To state the relation more precisely, consider a metric of the form $g_B \oplus \frac{1}{t} g_v$, where $t \in (0, \infty)$, and let $D_{S,t}$ be the associated total Dirac operator. As in the discussion preceding Definition 3.2.15 one infers that

$$D_{S,t} = \sqrt{t} D_v + c^a \nabla_a^{\mathscr{E},u} - \frac{1}{8\sqrt{t}}\, c^a c^b c_v(\Omega_{ab}).$$

Then we have as in [13, Thm. 10.19]

Proposition 3.3.4. *Consider the Dirac operator on $\Omega^\bullet(B, \mathscr{E})$ associated to the rescaled Bismut superconnection $\mathbb{B}_t$ on $\Lambda^\bullet T^*B \otimes \mathscr{E}$, i.e.,*

$$D_{\mathbb{B}_t} := c \circ \boldsymbol{\sigma}^{-1} \circ \mathbb{B}_t : \Omega^\bullet(B, \mathscr{E}) \to \Omega^\bullet(B, \mathscr{E})$$

Then, under the identification $\Omega^\bullet(M) = \Omega^\bullet(B, \mathscr{E})$, we have $D_{\mathbb{B}_{4t}} = D_{S,t}$.

Remark.

(i) The factor 4 occurs because we have defined the Bismut superconnection in a slightly different way compared to [14]. This is because we wanted Proposition 3.2.10 to hold without any constants appearing in $\mathbb{B}$, see also [19, Rem. 3.10].

(ii) We also want to stress that Proposition 3.3.4—in the same way as Proposition 3.2.10 earlier—is more a formal interpretation than a mathematical statement. Nevertheless, this helps in understanding some of the ideas underlying the technicalities of local families index theory.

Variation of the Eta Invariant of the Total Dirac Operator. After having introduced the total Dirac operator and its interpretation in term of the rescaled Bismut superconnection our aim is now to give a heuristic explanation of Theorem 3.3.2, in the case that the adiabatic family of de Rham operators is replaced with a corresponding family of total Dirac operators.

We first note that since $D_{S,t}$ is a path of formally self-adjoint elliptic operators of first order on $\Omega^\bullet(M)$, the general variation formula for the ξ-invariant in Proposition D.2.5 shows that

$$\xi\big(\tau_M D_{S,t_0}\big) - \xi\big(\tau_M D_{S,t_1}\big) = \int_{t_0}^{t_1} \frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{u\to 0} \sqrt{u}\ \mathrm{Tr}\left[\tau_M \tfrac{dD_{S,t}}{dt} e^{-uD_{S,t}^2}\right] dt \mod \mathbb{Z}, \tag{3.27}$$

where $\mathrm{LIM}_{u\to 0}$ means taking the constant term in the asymptotic expansion as $u \to 0$. Given the interpretation of Proposition 3.3.4, we now formally apply Proposition 1.5.8 to the case at hand and get

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{u\to 0} \sqrt{u}\ \mathrm{Tr}\left[\tau_M \tfrac{dD_{S,t}}{dt} e^{-uD_{S,t}^2}\right] \\
&\qquad = -2^{\frac{b+1}{2}} \int_B \widehat{L}(TB,\nabla^B) \wedge \mathrm{Tr}_v\left[\tau_v \tfrac{\gamma}{\sqrt{2\pi i}} \big(\tfrac{d\mathbb{B}_{4t}}{dt} e^{-\mathbb{B}_{4t}^2}\big)\right] \\
&\qquad = 2^{\frac{b+1}{2}} \int_B \widehat{L}(TB,\nabla^B) \wedge 4\alpha(\mathbb{B}_{4t}).
\end{aligned} \tag{3.28}$$

Remark. Note that in contrast to Proposition 1.5.8 we are not only in the situation that the twisting bundle $\mathscr{E}$ is infinite dimensional, but are also dealing with a path of superconnections rather than usual connections. This requires special considerations already in the case of a finite dimensional twisting bundle, see [46, Sec. 2].

Despite this apparent lack of mathematical rigor, we assume for the rest of this motivational part that the formula in (3.28) is valid.

The Limit as $t \to \infty$. Note that for the total Dirac operator, the component $c^a \nabla_a^{\mathscr{E},u}$ is the analog of the horizontal de Rham operator D_h in Definition 2.1.23. In analogy to Definition 3.1.16 we thus define an operator

$$\begin{aligned}
D_B \otimes \nabla^{\mathscr{H}_v,u} &: \Omega^\bullet\big(B, \mathscr{H}_v^\bullet(M)\big) \to \Omega^\bullet\big(B, \mathscr{H}_v^\bullet(M)\big), \\
D_B \otimes \nabla^{\mathscr{H}_v,u} &:= P_{\ker D_v} \circ \tau_M\big(c^a \nabla_a^{\mathscr{E},u}\big) \circ P_{\ker D_v},
\end{aligned} \tag{3.29}$$

compare with (3.24). Then the remarkable result [30, Thm. 1.6] guarantees the following

Theorem 3.3.5 (Dai)**.** *Abbreviate* $D_{S,\infty} := D_B \otimes \nabla^{\mathscr{H}_v,u}$*. Then, for* N *large enough, there exist positive constants* C_1 *and* C_2 *such for all* $s > 0$ *as* $t \to \infty$

$$\left| \operatorname{Tr}' \left(\tau_M D_{S,t} e^{-sD_{S,t}^2} \right) - \operatorname{Tr} \left(D_{S,\infty} e^{-sD_{S,\infty}^2} \right) \right| \le \frac{C_1}{\sqrt{t} s^N} e^{-C_2 s}.$$

Here, Tr' *indicates taking the trace over those eigenvalues of* $D_{S,t}$ *which are bounded away from 0 as* $t \to \infty$*.*

It is shown in [30, Thm. 1.5] that there are only finitely many eigenvalues of $D_{S,t}$ which converge to zero as $t \to \infty$. We also note that the formulation in [30] is slightly different from what is stated here. This is due to the fact that we are using a different scaling and a different parameter, see Remark 3.3.8 (iii) below. For the time being, we only use that—very roughly—Theorem 3.3.5 means that

$$\lim_{t\to\infty} \xi\big(\tau_M D_{S,t}\big) = \xi\big(D_B \otimes \nabla^{\mathscr{H}_v,u}\big) \mod \mathbb{Z}.$$

Together with (3.27) and (3.28) we arrive at the following heuristic formula

$$\xi\big(\tau_M D_S\big) = \xi\big(D_B \otimes \nabla^{\mathscr{H}_v,u}\big) + 2^{\frac{b+1}{2}} \int_B \widehat{L}(TB, \nabla^B) \wedge \int_1^\infty 4\alpha(\mathbb{B}_{4t}) dt \mod \mathbb{Z}. \tag{3.30}$$

The Adiabatic Limit Formula for D_S. To understand how an adiabatic limit comes into play, we consider a metric of the form

$$g_{\varepsilon,t} = \tfrac{1}{\varepsilon^2} g_B \oplus \tfrac{1}{t} g_v.$$

Let $D_{S,\varepsilon,t}$ be the associated total Dirac operator. Since multiplying $g_{\varepsilon,t}$ by ε^{-1} does not change the ξ-invariant, we have

$$\xi\big(D_{S,\varepsilon,t}\big) = \xi\big(\sqrt{\varepsilon} D_{S,\varepsilon,t}\big).$$

We note that explicitly,

$$\sqrt{\varepsilon} D_{S,\varepsilon,t} = \sqrt{\varepsilon t} D_v + \sqrt{\varepsilon}^3 c^a \nabla_a^{\mathscr{E},u} - \tfrac{\varepsilon^3}{8\sqrt{\varepsilon t}}\, c^a c^b c_v(\Omega_{ab}).$$

As we have seen in Section 1.5, the proof of Proposition 1.5.8 uses Getzler's rescaling by $\sqrt{u}$. Since (3.28) is a formal transition to the case at hand, it is reasonable to expect that the same rescaling is involved there. Without going into detail, we note that using the rescaling $\sqrt{u\varepsilon^3}$ instead, one formally obtains

$$\frac{1}{\sqrt{\pi}} \operatorname*{LIM}_{u\to 0} \sqrt{u} \operatorname{Tr}\left[\tau_M \tfrac{dD_{S,\varepsilon,t}}{dt} e^{-uD_{S,\varepsilon,t}^2}\right] = 2^{\frac{b+1}{2}} \int_B \widehat{L}(TB, \nabla^B) \wedge 4\varepsilon\alpha(\mathbb{B}_{4\varepsilon t}).$$

Inserting this in (3.30) and substituting $s = 4\varepsilon t$, we arrive at

$$\xi\big(\tau_M D_{S,\varepsilon}\big) = \xi\big(D_B \otimes \nabla^{\mathscr{H}_v,u}\big) + 2^{\frac{b+1}{2}} \int_B \widehat{L}(TB, \nabla^B) \wedge \int_{4\varepsilon}^\infty \alpha(\mathbb{B}_s) ds \mod \mathbb{Z}, \tag{3.31}$$

where we have used that the term $\xi\big(D_B \otimes \nabla^{\mathscr{H}_v,u}\big)$ is independent of ε. This is because $D_B \otimes \nabla^{\mathscr{H}_v,u}$ is an operator over B and rescaling the full metric does not change the ξ-invariant. Now letting $\varepsilon \to 0$ yields the analog of Theorem 3.3.2 for the total Dirac operator $D_{S,\varepsilon}$.

Remark. We want to point out again, that our considerations leading to (3.31) are heuristic and have no rigorous mathematical foundation. In fact, we do not even expect (3.27), (3.30) and (3.31) to be correct without any changes. However, it would be interesting to find estimates on the correction term in (3.31) along the line of thoughts we have presented to give an alternative proof of Theorem 3.3.2 for the total Dirac operator.

Relation between $D_{S,\varepsilon}$ and $D_{M,\varepsilon}$. What is still missing in our heuristic explanation of Theorem 3.3.2 is why we can replace $D_{S,\varepsilon}$ with $D_{M,\varepsilon}$ and $D_B \otimes \nabla^{\mathscr{H}_v,u}$ with $D_B \otimes \nabla^{\mathscr{H}_v}$. First of all, the relation between the latter two operators fits exactly into the framework of Section 3.1.6. In particular, Theorem 3.1.20 yields that

$$\xi\big(D_B \otimes \nabla^{\mathscr{H}_v,u}\big) = \xi\big(D_B \otimes \nabla^{\mathscr{H}_v}\big) \mod \mathbb{Z}.$$

Formally, the relation between $D_{S,\varepsilon}$ and $D_{M,\varepsilon}$ is an infinite dimensional analog of the same situation, where the odd signature operator $\tau_M D_{M,\varepsilon}$ plays the role of $D_B \otimes \nabla$, whereas $\tau_M D_{S,\varepsilon}$ is an analog of the operator $D_B \otimes \nabla^u$, compare with Lemma 3.1.19 and Proposition 3.2.10. Then the heuristic analog of Theorem 3.1.20 is that we can make the required substitution in Theorem 3.3.2. For a technically precise explanation, see [30, p. 304] and [17, p. 374].

3.3.2 Dai's Adiabatic Limit Formula

After this heuristic digression, we now want to state Dai's general adiabatic limit formula for the Eta invariant of the odd signature operator in its full generality. From now on we will also include the case that $\dim B$ is even and $\dim F$ is odd. Hence, we first need the analog of the Bismut-Cheeger Eta form in Definition 3.3.1 for the case of odd dimensional fibers, see [16, Def. 4.93 & Rem. 4.100].

Definition 3.3.6. Assume that $\dim F$ is odd and $\dim B$ is even, and let $\mathbb{B}_t$ be the rescaled Bismut superconnection as in Definition 3.2.15. Then we define

$$\widehat{\eta} := \frac{1}{\sqrt{\pi}} \int_0^\infty \gamma \operatorname{Tr}_v^{\mathrm{ev}} \big[\tau_v \tfrac{d\mathbb{B}_t}{dt} e^{-\mathbb{B}_t^2}\big] dt \in \Omega^{\mathrm{ev}}(B),$$

where γ is the normalization function as in (3.15), and $\operatorname{Tr}_v^{\mathrm{ev}}$ indicates taking the even form part of Tr_v.

Remark.

(i) The vertical chirality operator enters in Definition 3.3.6, since in contrast to [16] we are dealing with the operator $\tau_M D_M$ rather than the spin Dirac operator. Nevertheless, $\operatorname{Tr}_v \circ \tau_v$ should not be viewed as a supertrace, since in the case that $\dim F$ is odd, τ_v commutes with vertical Clifford multiplication.

(ii) The fact that the integral defining $\widehat{\eta}$ is indeed convergent relies on the odd dimensional analogs of the small and large time behaviour of $\alpha(\mathbb{B}_t)$ in Theorem 3.2.18 and Theorem 3.2.19, i.e.,

$$\operatorname{Tr}_v^{\mathrm{ev}} \big[\tau_v \tfrac{d\mathbb{B}_t}{dt} e^{-\mathbb{B}_t^2}\big] = O(1) \quad \text{as } t \to 0, \quad \operatorname{Tr}_v^{\mathrm{ev}} \big[\tau_v \tfrac{d\mathbb{B}_t}{dt} e^{-\mathbb{B}_t^2}\big] = O(t^{-3/2}) \quad \text{as } t \to \infty.$$

It is difficult to find an explicit proof in the literature, since the treatment is usually in the superconnection formalism which does not apply to the operator in question. However, as pointed out in Remark 3.2.11 one can overcome this difficulty by introducing an extra Grassmann variable to turn $\mathbb{B}_t$ into a superconnection of the required form, see again [18, Sec.'s II (b) & (f)], and also [25, Sec. 5.2.2] for additional remarks and references. The reader who feels uncomfortable with this can equally well consider

$$\widehat{\eta}(s) := \frac{1}{\sqrt{\pi}} \int_0^\infty \gamma \operatorname{Tr}_v^{\mathrm{ev}} \left[\tau_v \tfrac{d\mathbb{B}_t}{dt} e^{-\mathbb{B}_t^2}\right] t^s dt, \quad \operatorname{Re}(s) \text{ large},$$

and define the Eta form as the constant term in the Laurent series around $s = 0$ of the meromorphic continuation. Then the whole discussion to follow goes through with only minor changes—the sole problem being a more awkward notation.

(iii) In contrast to the case that F is even dimensional, the Bismut-Cheeger Eta form is now a differential form on B of *even* degree. One easily checks that the degree 0 term is given by

$$\widehat{\eta}_{[0]} = \frac{1}{2\sqrt{\pi}} \int_0^\infty u^{-1/2} \operatorname{Tr}_v \left[\tau_v D_v e^{-uD_v^2}\right] ds,$$

where we have substituted $4u = t$. Hence, if $B_v^{\mathrm{ev}} = \tau_v D_v|_{\Omega_v^{\mathrm{ev}}(M)}$ is the family of vertical odd signature operators, we see that

$$\widehat{\eta}_{[0]} = \eta(B_v^{\mathrm{ev}}) \in C^\infty(B), \tag{3.32}$$

which is the function that associates to each point $y \in B$ the Eta invariant of the fiber $\pi^{-1}(y)$. Note that we have used (1.38) which is possible since B_v^{ev} is a family of geometric Dirac operators so that the Eta function can be defined without making use of a meromorphic extension.

Preliminary Adiabatic Limit Formula. The starting point for the rigorous treatment of the adiabatic limit formula is [30, Prop. 1.4], which in the case of the odd signature operator reads

Proposition 3.3.7 (Dai). *Let D_ε be the family of de Rham operator associated to the adiabatic metric* (2.16). *Then there exists a small positive number α such that*

$$\lim_{\varepsilon\to 0} \eta(\tau_M D_\varepsilon) = 2^{[\frac{b+1}{2}]+1} \int_B \widehat{L}(TB, \nabla^B) \wedge \widehat{\eta} + \lim_{\varepsilon\to 0} \frac{1}{\sqrt{\pi}} \int_{\varepsilon^{-\alpha}}^\infty u^{-1/2} \operatorname{Tr}\left[\tau_M D_\varepsilon e^{-uD_\varepsilon^2}\right] du,$$

provided either one of the limits exists.

Remark 3.3.8.

(i) Dai deduces Proposition 3.3.7 from the main result in [16] which—translated to the situation at hand—gives an explicit formula for

$$\lim_{\varepsilon\to 0} \operatorname{Tr}\left[\tau_M D_\varepsilon e^{-uD_\varepsilon^2}\right],$$

together with remainder estimates, which are uniform in ε for compact u-intervals, see [16, (4.40)]. We want to point out, that Bismut's and Cheeger's proof is rather involved, one difficulty being the presence of the term $\operatorname{Tr}\left[\tau_M \varepsilon D_h e^{-uD_\varepsilon^2}\right]$, which behaves "singular" with respect to Getzler's rescaling, see [16, Rem. 3.4]. This part should simplify if one could find a proof along the lines of the heuristic discussion in Section 3.3.1.

(ii) Note that for the odd signature operator, we already know from Proposition 2.2.8 that the limit on the left hand side in Proposition 3.3.7 exists. By comparison with Theorem 3.3.2 we see that the second term on the right hand side will produce the twisted Eta invariant of the base as well as the integer contribution which we omitted so far.

(iii) To relate this term to the discussion in Section 3.3.1, we substitute $s = \varepsilon^2 u$, which corresponds to rescaling the metric g_ε to $\varepsilon^2 g_\varepsilon$. Then

$$\int_{\varepsilon^{-\alpha}}^{\infty} u^{-1/2} \operatorname{Tr}\left[\tau_M D_\varepsilon e^{-uD_\varepsilon^2}\right] du = \int_{\varepsilon^{2-\alpha}}^{\infty} s^{-1/2} \operatorname{Tr}\left[\tau_M (\tfrac{1}{\varepsilon} D_\varepsilon) e^{-s(\frac{1}{\varepsilon}D_\varepsilon)^2}\right] ds.$$

We now note that if we rename $\sqrt{t} = \varepsilon^{-1}$, then $\frac{1}{\varepsilon} D_\varepsilon$ becomes

$$\sqrt{t} D_{(\sqrt{t})^{-1}} = \sqrt{t} D_v + D_h + \tfrac{1}{\sqrt{t}} T.$$

where the individual terms are defined as in Definition 2.1.23. This means that the operator $\frac{1}{\varepsilon} D_\varepsilon$ plays essentially the role of the operator $D_{S,t}$ in the discussion of Section 3.3.1, and that taking the limit $\varepsilon \to 0$ on the right hand side in Proposition 3.3.7 is related to the limit $t \to \infty$ in our heuristic explanation of Theorem 3.3.2. We also note without giving the details that this also explains why Theorem 3.3.5 as we have stated it is indeed a reformulation of [30, Thm. 1.6].

Behaviour of Small Eigenvalues. The limit on the right hand side of the formula in Proposition 3.3.7 is closely related to eigenvalues of $\tau_M D_\varepsilon$ which approach zero as $\varepsilon \to 0$. This is roughly because of their presence, there is no uniform bound of the form Ce^{-cu} for the term $\operatorname{Tr}\left[\tau_M D_\varepsilon e^{-uD_\varepsilon^2}\right]$ as $\varepsilon \to 0$ for large u, compare with Lemma D.2.1. The following result provides the essential analysis of the spectrum of D_ε as $\varepsilon \to 0$, see [30, Thm. 1.5].

Theorem 3.3.9 (Dai). *For $\varepsilon > 0$ the eigenvalues of $\tau_M D_\varepsilon$ depend analytically on ε. There exist analytic functions $\{\lambda_\varepsilon^i \mid i \in \mathbb{Z}\}$ such that $\operatorname{spec}(\tau_M D_\varepsilon) = \bigcup_{i\in\mathbb{Z}} \lambda_\varepsilon^i$ for all $\varepsilon > 0$. Moreover, the functions λ_ε^i have the following properties.*

(i) *There exists a positive constant λ_0 such that either for some ε_0*

$$|\lambda_\varepsilon^i| \geq \lambda_0 > 0, \quad \textit{for } \varepsilon \leq \varepsilon_0,$$

or λ_ε^i has a complete asymptotic expansion

$$\lambda_\varepsilon^i \sim \sum_{k\geq 1} \mu_k^i \varepsilon^k \quad \textit{as } \varepsilon \to 0,$$

where $\mu_1^i \in \operatorname{spec}\left(D_B \otimes \nabla^{\mathscr{H}_v}\right)$, see Definition 3.1.16. The latter gives a bijective correspondence

$$\left\{ \lambda_\varepsilon \in \operatorname{spec}(\tau_M D_\varepsilon) \;\middle|\; \lambda_\varepsilon = O(\varepsilon) \textit{ as } \varepsilon \to 0 \right\} \xleftrightarrow{1:1} \operatorname{spec}\left(D_B \otimes \nabla^{\mathscr{H}_v}\right).$$

(ii) *Assume that* $\lambda_\varepsilon^i = O(\varepsilon)$ *as* $\varepsilon \to 0$*, and that* $\mu_1^i \neq 0$*. Then there is a uniform remainder estimate of the form*

$$\lambda_\varepsilon^i = \mu_1^i \varepsilon + \varepsilon^2 C(\varepsilon)(\mu_1^i)^2, \quad |C(\varepsilon)| \leq \text{const}\,.$$

(iii) *For every* $K > 0$*,*

$$\#\Big\{ i \in \mathbb{Z} \;\Big|\; \lambda_\varepsilon^i = O(\varepsilon) \text{ as } \varepsilon \to 0, \text{ and } |\mu_1^i| \leq K \Big\} < \infty.$$

Remark. As one might expect, the proof of Theorem 3.3.9 in [30] relies on standard perturbation theory as in [56], and this part is in fact not very difficult. However, to prove that the eigenvalues have a complete asymptotic expansion, Dai makes use of Melrose's theory of degenerate elliptic problems as used in [66]. We refer to [30, Sec. 2] for more details and references.

The General Adiabatic Limit Formula. For $r \in \mathbb{N}$ define

$$\Lambda(\varepsilon^r) := \big\{ \lambda_\varepsilon \in \operatorname{spec}(\tau_M D_\varepsilon) \;\big|\; \lambda_\varepsilon = O(\varepsilon^r), \text{ as } \varepsilon \to 0 \big\}. \tag{3.33}$$

Note that part (iii) of Theorem 3.3.9 shows that $\#\Lambda(\varepsilon^r) < \infty$ for all $r \geq 2$. Then Dai's main results can be stated in the following way, see [30, Cor. 1.6 & Prop. 1.8]. The formal but illuminating outline of the proof in [30, p. 275] is also recommended.

Theorem 3.3.10 (Dai). *With the notation of Proposition 3.3.7,*

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi}} \int_{\varepsilon^{-\alpha}}^{\infty} u^{-1/2} \operatorname{Tr}\big[\tau_M D_\varepsilon e^{-u D_\varepsilon^2}\big] du = \eta\big(D_B \otimes \nabla^{\mathscr{H}_v}\big) + \lim_{\varepsilon \to 0} \sum_{\lambda_\varepsilon \in \Lambda(\varepsilon^2)} \operatorname{sgn}(\lambda_\varepsilon).$$

In particular,

$$\begin{aligned}\lim_{\varepsilon \to 0} \eta(\tau_M D_\varepsilon) = 2^{[\frac{b+1}{2}]+1} &\int_B \widehat{L}(TB, \nabla^B) \wedge \widehat{\eta} \\ &+ \eta\big(D_B \otimes \nabla^{\mathscr{H}_v}\big) + \lim_{\varepsilon \to 0} \sum_{\lambda_\varepsilon \in \Lambda(\varepsilon^2)} \operatorname{sgn}(\lambda_\varepsilon).\end{aligned}$$

This separates the computation of the adiabatic limit of the Eta invariant on the total space of the fiber bundle into three terms, which are all of a very different nature. Intuitively, the first term contains global information about the fiber, but is local on B. The second term is global on the base and contains cohomological information about the fiber. The third term is global on both, the base and the fiber. It fits into the heuristic discussion of Section 3.3.1 as an analog of the spectral flow term in Corollary 1.5.2. Following again [30], we will see in the next subsection that for the odd signature operator, this term is expressible in completely topological terms.

3.3.3 Small Eigenvalues and the Leray Spectral Sequence

Let Δ_ε be the Laplace operator associated to an adiabatic metric g_ε on an oriented fiber bundle $F \hookrightarrow M \xrightarrow{\pi} B$ of closed manifolds. In [66], Mazzeo and Melrose analyze the space of harmonic forms $\mathscr{H}^\bullet_\varepsilon(M) = \ker(\Delta_\varepsilon)$ as $\varepsilon \to 0$, and show that it has a basis which extends smoothly to $\varepsilon = 0$. Using a Taylor series analysis to determine which forms lie in this limiting space, they find a Hodge theoretic version of the Leray-Serre spectral sequence. We the result briefly, and refer to [66] as well as [41] and [30, Sec. 4.2] for details. As in the latter reference, we restrict to the odd dimensional case and use the formulation in terms of the odd signature operator.

The Hodge Theoretic Spectral Sequence. Assume that $\dim M$ is odd. For $\Lambda(\varepsilon^r)$ as in (3.33) we define

$$G_{\Lambda(\varepsilon^r)} := \sum_{\lambda_\varepsilon \in \Lambda(\varepsilon^r)} \ker\big(\tau_M D_\varepsilon - \lambda_\varepsilon\big).$$

We view this as a family of subspaces of $\Omega^\bullet(M)$, parametrized by $\varepsilon \in (0,\infty)$. Note that Theorem 3.3.9 implies that for $r \geq 2$ each $G_{\Lambda(\varepsilon^r)}$ is finite dimensional. Now the analysis of [66] adapted to the case at hand yields the following, see [30, Thm. 0.2 & Prop. 4.2].

Theorem 3.3.11 (Mazzeo-Melrose, Dai)**.** *For $r \geq 2$ the family $G_{\Lambda(\varepsilon^r)}$ depends smoothly on ε down to $\varepsilon = 0$, i.e., there exists a smooth family of orthonormal bases for $G_{\Lambda(\varepsilon^r)}$, parametrized by $\varepsilon \in (0,\infty)$, that extends smoothly to $\varepsilon = 0$. One can then define*

$$G_r := \lim_{\varepsilon\to 0} G_{\Lambda(\varepsilon^r)}, \quad d_r := \lim_{\varepsilon \to 0} \varepsilon^{-r} d_\varepsilon : G_r \to G_r, \tag{3.34}$$

where $d_\varepsilon = d_v + \varepsilon d_h + \varepsilon^2\,\mathrm{i}(\Omega)$. This defines a spectral sequence that is isomorphic to the Leray-Serre spectral sequence, i.e.,

$$(G_r, d_r) \cong (E_r^{\bullet,\bullet}, d_r), \quad r \geq 2.$$

Remark.

(i) Note that the spectral sequence (G_r, d_r) is not defined as a spectral sequence associated to a filtered complex as in (3.9). Nevertheless, as subspaces of $\Omega^\bullet(M) = \bigoplus_{p,q} \Omega^{p,q}(M)$, the spaces G_r are naturally bigraded.

(ii) Since we are working on $\Omega^\bullet(M)$ with the fixed reference metric g we are using the modified de Rham differential d_ε, compare with Remark 2.2.6.

(iii) It is immediate that each d_ε maps $G_{\Lambda(\varepsilon^r)}$ to itself, since d_ε commutes with $\tau_M D_\varepsilon$. However, note that for an element ω_ε of a basis of $G_{\Lambda(\varepsilon^r)}$ as in Theorem 3.3.11, one has

$$(\tau_M D_\varepsilon)\omega_\varepsilon = \lambda_\varepsilon \omega_\varepsilon = O(\varepsilon^r), \quad \text{as } \varepsilon \to 0.$$

Since $\tau_M D_\varepsilon = \tau_M d_\varepsilon + d_\varepsilon \tau_M$, this implies that $d_\varepsilon \omega_\varepsilon = O(\varepsilon^r)$ as well. This should serve as a motivation for the factor ε^{-r} in (3.34).

The case $r = 1$. Because of the extra difficulty which arises from the fact that $G_{\Lambda(\varepsilon)}$ is not finite dimensional the case $r = 1$ is excluded from Theorem 3.3.11. Nevertheless, for motivational purposes, we now want to make a formal digression on this case, since it might give an idea of the mechanism lying behind Theorem 3.3.11. For a rigorous treatment, we refer to [66, Sec.'s 3 & 4].

Let λ_ε be a family of eigenvalues of $\tau_M D_\varepsilon$, which is of order ε as $\varepsilon \to 0$. Since for $\varepsilon \in (0,\infty)$, the family $\tau_M D_\varepsilon$ depends analytically on ε, standard perturbation theory ensures that λ_ε depends analytically on ε and that there exists an analytic family of eigenforms $\omega_\varepsilon \in \Omega^\bullet(M)$ with eigenvalue λ_ε. We now assume without justification that λ_ε and ω_ε extend analytically to $[0,\infty)$ so that we can write

$$\lambda_\varepsilon = \sum_{k\geq 1} \lambda_k \varepsilon^k, \quad \text{and} \quad \omega_\varepsilon = \sum_{k\geq 0} \omega_k \varepsilon^k, \quad \text{as } \varepsilon \to 0.$$

From Definition 2.2.7 we know that

$$\tau_M D_\varepsilon = \tau_M D_v + \varepsilon \tau_M D_h + \varepsilon^2 \tau_M T : \Omega^\bullet(M) \to \Omega^\bullet(M).$$

Then comparing ε-powers in the identity $(\tau_M D_\varepsilon)\omega_\varepsilon = \lambda_\varepsilon \omega_\varepsilon$ one finds that

$$D_v \omega_0 = 0, \quad \tau_M\big(D_v\omega_1 + D_h\omega_0\big) = \lambda_1 \omega_0.$$

In particular, $\omega_0 \in \ker(D_v) = \Omega^\bullet\big(B, \mathscr{H}_v^\bullet(M)\big)$ and thus also $d_v\omega_0 = 0$. Then the second equation yields

$$P_{\ker(D_v)^\perp}\big(\tau_M\big(D_v\omega_1 + D_h\omega_0\big)\big) = 0. \tag{3.35}$$

On the other hand, when we compare ε-powers in the identity $\big(\tau_M d_\varepsilon \omega_\varepsilon\,,\, d_\varepsilon \tau_M \omega_\varepsilon\big)_{L^2} = 0$, we can deduce that

$$\big(\tau_M d_v \omega_1 + \tau_M d_h \omega_0\,,\, d_v \tau_M \omega_1 + d_h \tau_M \omega_0\big)_{L^2} = 0.$$

Using this and (3.35) one infers that

$$P_{\ker(D_v)^\perp}\big(d_v\omega_1 + d_h\omega_0\big) = 0.$$

Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0} \varepsilon^{-1} d_\varepsilon \omega_\varepsilon &= \lim_{\varepsilon\to 0} \varepsilon^{-1}\big(d_v\omega_0 + \varepsilon(d_v\omega_1 + d_h\omega_0) + \varepsilon^2(\ldots)\big) = d_v\omega_1 + d_h\omega_0 \\ &= P_{\ker(D_v)}(d_h\omega_0).\end{aligned}$$

This shows at least formally that the construction of Theorem 3.3.11 extends to the case $r = 1$ and gives

$$G_1 = \Omega^\bullet\big(B, \mathscr{H}_v^\bullet(M)\big), \quad d_1 = P_{\ker(D_v)} \circ d_h.$$

Now Proposition 3.1.13 shows that via the Hodge-de-Rham isomorphism,

$$(G_1, d_1) \cong \big(\Omega^p\big(B, H_v^\bullet(M)\big), \bar{d}_h\big),$$

with the differential $\bar{d}_h$ on $\Omega^p\big(B, H_v^\bullet(M)\big)$ associated to the flat connection ∇^{H_v}, see Definition 3.1.10. According to Lemma 3.1.12 this means that (G_1, d_1) is isomorphic to the E_1-term of the Leray-Serre spectral sequence of the fiber bundle.

Remark. In principle, one could now continue and give a formal derivation of Theorem 3.3.11 along the lines just presented. However, already in the case $r = 2$, the Hodge theoretical expression of the differential becomes very unpleasant. Therefore, we end the digression and refer to [66, 30, 41] for more details.

We also want to point out that Theorem 3.3.11 will not be explicitly used later on. Yet, we have included the above discussion to motivate why it is reasonable to expect that the term

$$\lim_{\varepsilon\to 0} \sum_{\lambda_\varepsilon \in \Lambda(\varepsilon^2)} \operatorname{sgn}(\lambda_\varepsilon)$$

appearing in Dai's general adiabatic limit formula has a topological interpretation in terms of the Leray-Serre spectral sequence. We shall make this more precise now.

Multiplicative Structure on the Leray-Serre Spectral Sequence. From now on we also incorporate a flat twisting bundle E over M with connection A. The entire treatment of Section 3.3 carries over verbatim; we have omitted it so far only for notational convenience. However, in the discussion to follow, there are some small distinctions to be made. We use the notation $(E_{A,r}^{\bullet,\bullet}, d_{A,r})$ for the spectral sequence associated to the flat bundle, and reserve the notation $(E_r^{\bullet,\bullet}, d_r)$ from (3.9) for the spectral sequence associated to the trivial connection.

Recall that there is a multiplicative structure of the form

$$\cdot : E_{A,r}^{p,q} \times E_{A,r}^{s,t} \to E_r^{p+s,q+t},$$

which is canonically induced by the wedge-product and the metric $h : E \otimes E \to \mathbb{C}$ on the bundle E, see [22, pp. 174–177], [67, Thm. 5.2] or [33, Thm. 9.24]. For brevity, we introduce the notation

$$E_{A,r}^k := \bigoplus_{p+q=k} E_{A,r}^{p,q}, \quad \cdot : E_{A,r}^k \times E_{A,r}^l \to E_r^{k+l}.$$

Then the differential and the multiplicative structure satisfy the relation

$$d_r(\omega \cdot \eta) = d_{A,r}(\omega) \cdot \eta + (-1)^k \omega \cdot d_{A,r}(\eta), \quad \omega \in E_{A,r}^k, \quad \eta \in E_{A,r}^l. \tag{3.36}$$

Denote $m = \dim M$ and $b = \dim B$. Since we are assuming that the fiber bundle is oriented, the bundle $H_v^{m-b}(M) \to B$ is trivializable, where each vertical volume form gives a canonical trivialization. We then have

$$E_2^m = E_2^{b,m-b} = H^b\big(B, H_v^{m-b}(M)\big) \cong \mathbb{C}, \quad ,$$

where a natural basis $\xi_2 \in E_2^m$ is induced by any volume form on M. Since M is closed $H^m(M) \cong \mathbb{C}$, which implies that E_r^m for all $r \geq 2$. Moreover, as the isomporphism $H^m(M) \cong \mathbb{C}$ is also canonically induced by any volume form on M, we obtain natural bases ξ_r for all E_r^m with $r \geq 2$. Using the multiplicative structure on $E_{A,r}^{\bullet,\bullet}$, we can thus define a natural pairing

$$Q_{A,r} : E_{A,r}^k \times E_{A,r}^{m-k} \to \mathbb{C}, \tag{3.37}$$

by requiring that

$$Q_{A,r}(\omega, \eta)\, \xi_r = \omega \cdot \eta, \quad \text{for all} \quad (\omega, \eta) \in E_{A,r}^k \times E_{A,r}^{m-k}.$$

Remark. Due to the presence of the pairing $E \times E \to \mathbb{C}$ in its definition, the pairing (3.37) is complex anti-linear in the first variable. For even dimensional manifolds, the above pairing has been analyzed already by [27] in the context of the signature of fiber bundles.

As in [30, Sec. 4.3] we now define the analog for the odd dimensional case,

$$P_{A,r} : E^k_{A,r} \times E^{m-k-1}_{A,r} \to \mathbb{C}, \quad P_{A,r}(\omega, \eta) := Q_{A,r}\big(\omega, d_{A,r}\eta\big). \tag{3.38}$$

Lemma 3.3.12. *Let $\omega \in E^k_{A,r}$ and $\eta \in E^{m-k-1}_{A,r}$. Then*

$$P_{A,r}(\omega, \eta) = (-1)^{(k+1)(m-k)}\overline{P_{A,r}(\eta, \omega)}.$$

In particular, if $k = \frac{m-1}{2}$, then $P_{A,r}$ is Hermitian if $m = 4n-1$ and skew Hermitian if $m = 4n-3$.

Proof. Since $E^m_r \cong \mathbb{C}$ for each $r \geq 2$ it follows that $d_r|_{E^{m-1}_r} \equiv 0$. Then (3.36) implies that

$$\begin{aligned} Q_{A,r}\big(\omega, d_{A,r}\eta\big)\xi_r &= \omega \cdot d_{A,r}\eta = (-1)^{k+1}(d_{A,r}\omega) \cdot \eta \\ &= (-1)^{(k+1)(m-k)}\overline{\eta \cdot d_{A,r}\omega} \\ &= (-1)^{(k+1)(m-k)}\overline{Q_{A,r}\big(\eta, d_{A,r}\omega\big)}\xi_r. \end{aligned}$$

This implies the first assertion. Setting $k = \frac{m-1}{2}$, the exponent becomes $(\frac{m+1}{2})^2$, which yields the second assertion. ☐

Dai's Correction Term. Using the pairing $P_{A,r}$ introduced in (3.38), we now come to the topological interpretation of the third term in the general adiabatic limit formula of Theorem 3.3.10.

Definition 3.3.13. For each $r \geq 2$ we define

$$\sigma_{A,r} := \operatorname{Sign}\big(P_{A,r} : E^k_{A,r} \times E^k_{A,r} \to \mathbb{C}\big), \quad \text{with} \quad k := \tfrac{m-1}{2},$$

where as before the signature of a skew form is defined as the number of positive imaginary eigenvalues minus the number of negative imaginary ones. Moreover, we write

$$\sigma_A := \sum_{r \geq 2} \sigma_{A,r}.$$

Note that the sum defining σ_A is finite since the spectral sequence collapses after finitely many steps so that for large r the number $\sigma_{A,r}$ is always 0. This is due to the presence of the differential $d_{A,r}$ in the definition of $Q_{A,r}$. Now we can state [30, Thm. 4.4].

Theorem 3.3.14 (Dai). *Let E be a flat Hermitian bundle with connection A over the odd dimensional total space of an oriented fiber bundle $F \hookrightarrow M \xrightarrow{\pi} B$ of closed manifolds. For any adiabatic metric on M, let $\Lambda_A(\varepsilon^2)$ be defined in analogy to (3.33) with respect to the operator $\tau_M D_{A,\varepsilon}$. Then*

$$\lim_{\varepsilon \to 0} \sum_{\lambda_\varepsilon \in \Lambda_A(\varepsilon^2)} \operatorname{sgn}(\lambda_\varepsilon) = 2\sigma_A.$$

Consequently, the following adiabatic limit formula holds

$$\lim_{\varepsilon\to 0}\eta(B^{\mathrm{ev}}_{A,\varepsilon}) = 2^{[\frac{b+1}{2}]}\int_B \widehat{L}(TB,\nabla^B)\wedge\widehat{\eta}_A + \tfrac{1}{2}\eta\big(D_B\otimes\nabla^{\mathscr{H}_{A,v}}\big) + \sigma_A.$$

Here, $\widehat{\eta}_A \in \Omega^\bullet(B)$ is defined in analogy to the untwisted case in Definition 3.3.1, respectively Definition 3.3.6.

Note that we have divided the adiabatic limit formula in Theorem 3.3.10 by a factor of 2 to get the corresponding formula for the odd signature operator, see Remark 1.4.4.

An Adiabatic Limit Formula for the Rho Invariant. An immediate consequence of Theorem 3.3.14 is the result we were aiming at in this section. In analogy to Definition 1.4.6 we first make the following

Definition 3.3.15. Let E be a flat bundle of rank k with connection A over the odd dimensional total space of an oriented fiber bundle of closed manifolds. For every submersion metric, we define the *Bismut-Cheeger Rho form* as

$$\widehat{\rho}_A := \widehat{\eta}_A - k\cdot\widehat{\eta} \in \Omega^\bullet(B),$$

and the *Rho invariant of the bundle of vertical cohomology groups* as

$$\rho_{\mathscr{H}_{A,v}}(B) := \tfrac{1}{2}\eta\big(D_B\otimes\nabla^{\mathscr{H}_{A,v}}\big) - k\cdot\tfrac{1}{2}\eta\big(D_B\otimes\nabla^{\mathscr{H}_v}\big) \in \mathbb{R}.$$

We include the factor $\frac{1}{2}$ in the definition of $\rho_{\mathscr{H}_{A,v}}(B)$ since the operator $D_B\otimes\nabla^{\mathscr{H}_{A,v}}$ is essentially two copies of a usual odd signature operator, see Remark 3.1.17 (ii). Now an immediate—yet interesting—consequence of Theorem 3.3.14 and Corollary 2.2.9 is

Theorem 3.3.16. *Let E be a flat Hermitian bundle with connection A over the odd dimensional total space of an oriented fiber bundle $F\hookrightarrow M\xrightarrow{\pi}B$ of closed manifolds. Then with respect to every submersion metric*

$$\rho_A(M) = 2^{[\frac{b+1}{2}]}\int_B \widehat{L}(TB,\nabla^B)\wedge\widehat{\rho}_A + \rho_{\mathscr{H}_{A,v}}(B) + \sigma_A - k\cdot\sigma,$$

where σ_A and σ refer to Dai's correction term as in Definition 3.3.13.

3.4 Circle Bundles Revisited

We now want to use Theorem 3.3.16, to compute the U(1)-Rho invariant for principal circle bundles over Riemann surfaces thus giving a different proof of Theorem 2.3.18. Related results are due to Zhang [98] and Dai-Zhang [32]. In the first reference, the case of the untwisted spin Dirac operator for circle bundles over general even dimensional spin manifolds is studied, see also [51, Sec. 3 a)]. Dai and Zhang use the main result of [98] and a computation of the Eta invariant for the untwisted odd signature operator to determine the Kreck-Stolz invariant for circle bundles. We want to point out that the strategy in [32] for the untwisted odd signature operator is to apply the signature theorem for manifolds with boundary to the disk bundle associated to the given circle bundle. Related discussions can be found in [11, 63]. However, this approach does not carry over to non-trivial flat twisting bundles, since one would need an extension to a *flat* bundle over the disk bundle, see also Remark 1.4.8 (iv). In this respect our strategy is of a purely intrinsic nature.

The Bismut-Cheeger Rho Form. As in Section 2.3 let Σ be a closed, oriented Riemann surface of unit volume, and let $S^1 \hookrightarrow M \xrightarrow{\pi} \Sigma$ a principal S^1-bundle over Σ of degree $0 \neq l \in \mathbb{Z}$. Choose a connection $i\omega \in \Omega^1(M, i\mathbb{R})$ on M, and use this and the metric on Σ to equip M with a submersion metric. Let $L_A \to \Sigma$ be a holomorphic line bundle of degree k, the holomorphic structure being induced by a unitary connection A. As before, we assume that

$$\tfrac{i}{2\pi} F_\omega = l \cdot \mathrm{vol}_\Sigma, \quad \text{and} \quad \tfrac{i}{2\pi} F_A = k \, \mathrm{vol}_\Sigma \, .$$

We endow $L := \pi^* L_A \to M$ with the flat connection of Lemma 2.3.5, i.e.,

$$A_q = \pi^* A - iq\,\omega, \quad q := k/l.$$

Proposition 3.4.1. *The Bismut-Cheeger Rho form associated of the flat line bundle $L = \pi^* L_A$ is given by*

$$\widehat{\rho}_{A_q} = 2P_1(q) + l\big(P_2(q) - \tfrac{1}{6}\big)\,\mathrm{vol}_\Sigma \in \Omega^{\mathrm{ev}}(\Sigma).$$

Here, P_1 and P_2 are the first and second periodic Bernoulli functions of Definition C.1.1.

Proof. First, we infer from (3.32) that

$$\big(\widehat{\rho}_{A_q}\big)_{[0]} = \eta(B_{q,v}) - \eta(B_{0,v}),$$

where $B_{q,v}$ is the vertical odd signature operator

$$B_{q,v} = -i(\mathscr{L}_e - iq) : C^\infty(M, L) \to C^\infty(M, L),$$

see Proposition 2.3.10. Hence, we can use Remark 1.4.8 (iii) to deduce that

$$\big(\widehat{\rho}_{A_q}\big)_{[0]} = 2P_1(q).$$

To identify the 2-form part of the Rho form, we recall from Definition 3.2.15 that the rescaled Bismut superconnection is given by

$$\mathbb{B}_t = \tfrac{\sqrt{t}}{2} D_{q,v} + \nabla^{\mathscr{E},u} - \tfrac{1}{2\sqrt{t}} c_v(\Omega).$$

First, we need to identify the terms appearing here. In the same way as in Proposition 2.3.10 one finds that the vertical de Rham operator associated to the connection A_q is given by

$$D_{q,v} = -i\tau_v(\mathscr{L}_e - iq) : \Omega_v^\bullet(M, L) \to \Omega_v^\bullet(M, L).$$

Moreover, Lemma 2.3.9 shows that the mean curvature of the fiber bundle vanishes, and that the connections ∇^v and $\widetilde{\nabla}^v$ coincide. This implies that the connection part of $\mathbb{B}$ coincides with the horizontal part of the exterior differential,

$$\nabla^{\mathscr{E},u} = d_{A_q,h} : \Omega_v^\bullet(M, L) \to \Omega^{1,\bullet}(M, L).$$

Lastly, one easily checks—again as in Proposition 2.3.10—that

$$c_v(\Omega) = 2\pi il \operatorname{vol}_\Sigma \wedge \tau_v : \Omega_v^\bullet(M, L) \to \Omega^{2,\bullet}(M, L). \tag{3.39}$$

We now claim that in the case at hand

$$\mathbb{B}_t^2 = \tfrac{1}{4} t D_{q,v}^2 + \tfrac{1}{2} c_v(\Omega) D_{q,v}. \tag{3.40}$$

Proof of (3.40). From (2.29) and Corollary 2.1.16 we have the anti-commutator relations

$$\big\{\nabla^{\mathscr{E},u}, D_{q,v}\big\} = \big\{\nabla^{\mathscr{E},u}, c_v(\Omega)\big\} = 0.$$

Moreover, the explicit formulæ above easily yield

$$c_v(\Omega) D_{q,v} = D_{q,v} c_v(\Omega) = 2\pi l \operatorname{vol}_\Sigma \wedge (\mathscr{L}_e - iq).$$

Since $\nabla^{\mathscr{E},u}$ agrees with $d_{A_q,h}$ we infer from Corollary 2.1.16 that

$$(\nabla^{\mathscr{E},u})^2 = -\big\{d_{q,v}, \mathrm{i}(\Omega)\big\},$$

and one verifies that in the case at hand,

$$\big\{d_{q,v}, \mathrm{i}(\Omega)\big\} = -2\pi l \operatorname{vol}_\Sigma \wedge (\mathscr{L}_e - iq) = -c_v(\Omega) D_{q,v}.$$

Lastly as Σ is 2-dimensional, we have $c_v(\Omega)^2 = 0$. Putting all pieces together, we obtain

$$\mathbb{B}_t^2 = \tfrac{1}{4} t D_{q,v}^2 - \tfrac{1}{2} c_v(\Omega) D_{q,v} - \big\{d_{q,v}, \mathrm{i}(\Omega)\big\} = \tfrac{1}{4} t D_{q,v}^2 + \tfrac{1}{2} c_v(\Omega) D_{q,v}. \qquad \square$$

Having established the formula in (3.40), we continue with the proof of Proposition 3.4.1. Since $D_{q,v}^2$ and $c_v(\Omega) D_{q,v}$ commute, one can use (3.40) and Duhamel's formula to show that

$$\operatorname{Tr}_v\Big[\tau_v \tfrac{d\mathbb{B}_t}{dt} e^{-\mathbb{B}_t^2}\Big] = \operatorname{Tr}_v\Big[\tfrac{1}{4\sqrt{t}} \tau_v \big(D_{q,v} + \tfrac{c_v(\Omega)}{t}\big) e^{-\frac{1}{4} t D_{q,v}^2}\big(1 - \tfrac{1}{2} c_v(\Omega) D_{q,v}\big)\Big].$$

According to Definition 3.3.6, one thus finds that the 2-form part of $\widehat{\eta}_{A_q}$ is given by

$$\begin{aligned}
(\widehat{\eta}_{A_q})_{[2]} &= \frac{1}{2\pi i} \frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{Tr}_v\Big[\tfrac{1}{4\sqrt{t}} \tau_v c_v(\Omega)\big(-\tfrac{1}{2} D_{q,v}^2 + \tfrac{1}{t}\big) e^{-\frac{1}{4} t D_{q,v}^2}\Big] dt \\
&= \frac{l \operatorname{vol}_\Sigma}{4\sqrt{\pi}} \int_0^\infty \Big(-u^{-1/2} \operatorname{Tr}_v\big[D_{q,v}^2 e^{-u D_{q,v}^2}\big] + \tfrac{1}{2} u^{-3/2} \operatorname{Tr}_v\big[e^{-u D_{q,v}^2}\big]\Big) du.
\end{aligned} \tag{3.41}$$

Note that in the second equality we have used (3.39) to replace $\tau_v c_v(\Omega)$ with $2\pi il \operatorname{vol}_\Sigma$, and then made the substitution $t = 4u$. We now introduce a complex parameter s with $\operatorname{Re}(s) \geq 0$ and define

$$(\widehat{\eta}_{A_q})_{[2]}(s) = \frac{l \operatorname{vol}_\Sigma}{4\Gamma\big(\frac{s+1}{2}\big)} \int_0^\infty \Big(-u^{\frac{s-1}{2}} \operatorname{Tr}_v\big[D_{q,v}^2 e^{-u D_{q,v}^2}\big] + \tfrac{1}{2} u^{\frac{s-3}{2}} \operatorname{Tr}_v\big[e^{-u D_{q,v}^2}\big]\Big) du,$$

see Remark 3.4.2 (i) below. Now for $\operatorname{Re}(s)$ large enough, we can split up the integral and compute using the Mellin transform that

$$\frac{1}{\Gamma\big(\frac{s+1}{2}\big)} \int_0^\infty u^{\frac{s-1}{2}} \operatorname{Tr}_v\big[D_{q,v}^2 e^{-u D_{q,v}^2}\big] du = \sum_{\lambda \in \operatorname{spec}(D_{q,v}^2)} \lambda^{-\frac{s-1}{2}},$$

and

$$\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty u^{\frac{s-3}{2}}\operatorname{Tr}_v\left[e^{-uD_{q,v}^2}\right]du = \sum_{\lambda\in\operatorname{spec}(D_{q,v}^2)}\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty u^{\frac{s-3}{2}}e^{-u\lambda}du$$
$$= \sum_{\lambda\in\operatorname{spec}(D_{q,v}^2)}\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\lambda^{-\frac{s-1}{2}} = \sum_{\lambda\in\operatorname{spec}(D_{q,v}^2)}\tfrac{2}{s-1}\lambda^{-\frac{s-1}{2}}.$$

Now we can use the computation of the spectrum of the vertical odd signature operator in Lemma 2.3.11. Note, however, that here we are using $\tau_v D_{q,v}$ which corresponds to two copies of the operator considered there. We find, again for $\operatorname{Re}(s)$ large enough, that

$$\sum_{\lambda\in\operatorname{spec}(D_{q,v}^2)}\lambda^{-\frac{s-1}{2}} = \sum_{\lambda\in\operatorname{spec}(\tau_v D_{q,v})}|\lambda|^{1-s} = 2\sum_{\substack{n\in\mathbb{Z}\\ n\neq q}}|n-q|^{1-s} = 2\widetilde{\zeta}_q(s-1),$$

where $\widetilde{\zeta}_q$ is the Zeta function in Lemma C.1.2. Hence,

$$(\widehat{\eta}_{A_q})_{[2]}(s) = \frac{l\operatorname{vol}_\Sigma}{2}\frac{2-s}{s-1}\widetilde{\zeta}_q(s-1). \tag{3.42}$$

We know the value of the meromorphic continuation $\widetilde{\zeta}_q(s)$ to $s=-1$ from Lemma C.1.2. Using this we arrive at

$$(\widehat{\eta}_{A_q})_{[2]} = lP_2(q)\operatorname{vol}_\Sigma,$$

so that indeed

$$(\widehat{\rho}_{A_q})_{[2]} = (\widehat{\eta}_{A_q})_{[2]} - \widehat{\eta}_{[2]} = l\big(P_2(q)-\tfrac{1}{6}\big)\operatorname{vol}_\Sigma. \qquad \square$$

Remark 3.4.2.

(i) We want to point out that introducing a complex parameter to split up the sum in (3.41) is necessary. For this note that the individual terms do not give functions which are holomorphic for $\operatorname{Re}(s)\geq 0$. However—at least in the case that $q\notin\mathbb{Z}$—their sum is holomorphic for $\operatorname{Re}(s)\geq 0$, since Lemma C.1.2 implies that the poles and zeros of $\frac{2-s}{s-1}$ and $\widetilde{\zeta}_q(s-1)$ in (3.42) cancel each other out.

(ii) Even though the above computations are very similar to the ones in the proof of Proposition 2.3.15, we want to point out that there is a conceptual difference. Before, we had to incorporate the operator $D_{A_q,h}$ and work on $\Omega^\bullet(M,L)$, whereas now, we work only on $\Omega_v^\bullet(M,L)$.

Using Proposition 3.4.1 we can identify the first term in the general formula of Theorem 3.3.16. Since Σ is 2-dimensional, we have $\widehat{L}(T\Sigma,\nabla^\Sigma)=1$ so that

$$2\int_\Sigma \widehat{L}(T\Sigma,\nabla^\Sigma)\wedge\widehat{\rho}_{A_q} = 2l\big(P_2(q)-\tfrac{1}{6}\big). \tag{3.43}$$

Dai's Correction Term. We now want to understand the remaining terms appearing in Theorem 3.3.16 for the example at hand. Using (3.43) it is then immediate that our second proof of Theorem 2.3.18 will follow from the following result.

Proposition 3.4.3. *Let $l \neq 0$ be the degree of the principal circle bundle, and let A_q be the flat connection on $L = \pi^* L_A$ as before. Then $\rho_{\mathscr{H}_{A_q,v}}(\Sigma) = 0$, and*

$$\sigma_{A_q} = \begin{cases} -\operatorname{sgn}(l), & \text{if } A_q \text{ is the trivial connection,} \\ \quad 0, & \text{if } A_q \text{ is non-trivial.} \end{cases}$$

Proof. Recall that $L_\omega \to \Sigma$ denotes the line bundle associated to the principal bundle $\pi : M \to \Sigma$, endowed with the connection A_ω naturally induced by ω. As in (2.32) we can identify

$$\ker(D_{q,v}) \cong \begin{cases} \{0\}, & \text{if } q \notin \mathbb{Z}, \\ \Omega^\bullet(\Sigma, L_B) \oplus \big(\Omega^\bullet(\Sigma, L_B) \otimes \mathbb{C}\,[\omega]\big), & \text{if } q \in \mathbb{Z}. \end{cases} \tag{3.44}$$

Here, $L_B = L_A \otimes L_\omega^{-q}$, which is endowed with the connection $B = A \otimes 1 + 1 \otimes qA_\omega$. Note that in addition to (2.32), the term $\Omega^\bullet(\Sigma, L_B) \otimes \mathbb{C}\,[\omega]$ appears since we do not restrict $D_{q,v}$ to $\Omega^{\bullet,0}(M, L)$.

If $q \notin \mathbb{Z}$, we deduce from (3.44) that the bundle of vertical cohomology groups vanishes, which certainly implies that $\eta\big(D_\Sigma \otimes \nabla^{\mathscr{H}_{A,v}}\big)$ and σ_{A_q} are both zero. If $q \in \mathbb{Z}$, we conclude as in the proof of Proposition 2.3.16 that L_B is isomorphic to the trivial line bundle and that the connection B is flat. This implies that the operator $D_\Sigma \otimes \nabla^{\mathscr{H}_{A,v}}$ is unitarily equivalent to several copies of a twisted odd signature operator on Σ, see Remark 3.1.17 (ii) and Lemma 3.1.19. Since Σ is even dimensional, we thus infer that $\eta\big(D_\Sigma \otimes \nabla^{\mathscr{H}_{A,v}}\big) = 0$. To finish the proof, it remains to compute σ_{A_q} for $q \in \mathbb{Z}$.

First of all, we explicitly describe the E_2-term of the Leray-Serre spectral sequence. Using Lemma 3.1.12 we proceed as in Proposition 2.3.16 to conclude from (3.44) that for $q \in \mathbb{Z}$,

$$E^{\bullet,0}_{A_q,2} \cong H^\bullet(\Sigma, L_B), \quad E^{\bullet,1}_{A_q,2} \cong H^\bullet(\Sigma, L_B) \otimes \mathbb{C}[\omega]. \tag{3.45}$$

Now, according to Definition 3.3.13, we have to compute the signature of the pairing

$$P_{A_q,2} = Q_{A,2}\big(\,.\,, d_{A_q,2}(\,.\,)\big) : E^{0,1}_{A_q,2} \times E^{0,1}_{A_q,2} \to \mathbb{C}, \tag{3.46}$$

see (3.38). Here, we have used that concerning the signature of $P_{A_q,2}$ we can neglect the spaces $E^{1,0}_{A_q,2}$, since the differential $d_{A_q,2}$ is of bidegree $(2,-1)$ and thus zero on $E^{1,0}_{A_q,2}$, see Figure 3.1.

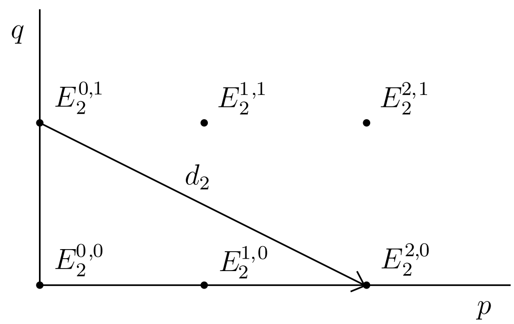

Figure 3.1: The E_2-term of the spectral sequence

Now, if A_q is a non-trivial connection, the line bundle $L_B = L_A \otimes L_\omega^{-q}$ with its natural connection is not isomorphic to the trivial flat line bundle. According to (3.45) this implies that $E_{A_q,2}^{0,1} = \{0\}$, see also Proposition 2.3.16. Hence, in this case $\sigma_{A_q} = 0$. Now we assume that A_q is the trivial connection, and drop the subscripts A_q from the notation. Using (3.44), one checks directly that

$$E_2^{0,1} = \mathbb{C}\,[\omega] \quad \text{and} \quad E_2^{2,0} = \mathbb{C}\,[\mathrm{vol}_\Sigma].$$

According to its definition in (3.37), the pairing Q_2 in (3.46) is induced by the wedge product, followed by evaluation on the fundamental class. Using this, one easily verifies that Q_2 satisfies

$$Q_2 : E_2^{0,1} \times E_2^{2,0} \mapsto \mathbb{C}, \quad Q_2\big([\omega], [\mathrm{vol}_\Sigma]\big) = 1.$$

Moreover, since the differential d_2 is naturally induced by exterior differentiation, one obtains from Proposition 2.1.15 that

$$d_2 : E_2^{0,1} \to E_2^{2,0}, \quad d_2[\omega] = [\mathrm{i}(\Omega)\omega] = -2\pi l[\mathrm{vol}_\Sigma].$$

According to (3.46), this means

$$P_2\big([\omega], [\omega]\big) = Q_2\big([\omega], d_2[\omega]\big) = -2\pi l.$$

This yields that $\mathrm{Sign}(P_2) = -\operatorname{sgn}(l)$. Since the spectral sequence collapses at the E_3-stage, there are no higher signatures. This finishes the computation of σ_{A_q} in the case that A_q is the trivial connection. $\square$

Chapter 4

3-dimensional Mapping Tori

After having described the abstract theory leading to a general formula for Rho invariants of a fiber bundle, we now want to use this to discuss a second class of examples in detail. In contrast to the previous class considered, we now reverse the role of fiber and base, and consider fiber bundles over S^1 with fiber a closed, oriented surface.

We start this chapter to describe a particular way of constructing submersion metrics. Here, it is convenient to use a formulation in terms of symplectic forms and almost complex structures. We then identify the geometric objects in this setting, most notably the bundle of vertical cohomology groups and the transgression form of the Bismut superconnection.

Under the assumption that the mapping torus is of finite order, we derive a formula which expresses the Rho invariant in terms of Hodge-de-Rham cohomology. Here, we can treat the case of a higher dimensional gauge group and arbitrary genus of the surface fiber without effort.

From then on we restrict to the case that the fiber is a 2-dimensional torus. We describe in some detail the geometric setup, including a discussion of the spectrum of the Laplace operator on a torus twisted by a flat U(1)-connection. Using this, we obtain a formula for the Rho form. We then employ ideas related to the classical Kronecker limit formula, to cast this expression into a different form which is more accessible for direct computations.

To obtain explicit formulæ for the Rho invariant, we now have to distinguish between the cases that the monodromy of the mapping torus is elliptic, parabolic or hyperbolic. In the elliptic case, we can specialize the previous result about finite order mapping tori to obtain a simple formula for the Rho invariant. The parabolic case turns out to be more involved. Most notably, the Eta invariant of the odd signature operator with values in the bundle of vertical cohomology groups has to be analyzed carefully. Yet, we shall arrive again at a very explicit formula for U(1)-Rho invariants.

The last part of our discussion will be concerned with the case of a hyperbolic mapping torus. Here, the difficulty lies in identifying the Rho form. Generalizing considerations by Atiyah [3], we relate this term to the logarithm of a generalized Dedekind Eta function. We will find that the transformation property of this logarithm under the action of the modular group determines the value of the Rho invariant of a hyperbolic mapping torus. Using a result due to Dieter [35], we can then express the Rho invariant as the difference of certain Dedekind sums. Simplifying this expression we finally arrive at a general formula for U(1)-Rho invariants in the hyperbolic case as well.

4.1 Geometric Preliminaries

Let Σ be a closed, oriented surface, and let $f \in \mathrm{Diff}^+(\Sigma)$ be an orientation preserving diffeomorphism of Σ. Consider the mapping torus

$$\Sigma_f := \big(\Sigma \times \mathbb{R}\big)/\mathbb{Z}, \tag{4.1}$$

where $\mathbb{Z}$ acts on $\Sigma \times \mathbb{R}$ via

$$k \cdot (x,t) = \big(f^{-k}(x), t+k\big), \quad (x,t) \in \Sigma \times \mathbb{R}, \quad k \in \mathbb{Z}. \tag{4.2}$$

Then Σ_f is naturally the total space of a fiber bundle over S^1, see Appendix B.2. Moreover, according to Lemma B.2.1, the diffeomorphism type of the mapping torus depends only on the isotopy class of f, i.e., on the image of f in the mapping class group

$$\mathrm{Diff}^+(\Sigma)/\,\mathrm{Diff}_0(\Sigma).$$

We now describe the geometric structure we need on a mapping torus in some detail. Related material can be found in the context of Seiberg-Witten equations in [90, Sec. 8].

Moser's Trick. The freedom of varying f in its isotopy class allows us to fix particularly convenient choices for the monodromy map. To exhibit such a choice we need the following result of [76] which is sometimes called *Moser's trick*, see also [68, Sec. 3.2].

Proposition 4.1.1. *Let $\omega \in \Omega^2(\Sigma)$ be a symplectic form on the closed, oriented surface Σ, and let $f \in \mathrm{Diff}^+(\Sigma)$. Then there exists $\widetilde{f} \in \mathrm{Diff}^+(\Sigma)$, isotopic to f, with $\widetilde{f}^*\omega = \omega$.*

Sketch of proof. We first note that since Σ is of dimension 2, a symplectic form ω on Σ is simply a 2-form which is non-degenerate in the sense that $\int_\Sigma \omega \neq 0$. Moreover, as f is orientation preserving, we have

$$[f^*\omega] = [\omega] \in H^2(\Sigma, \mathbb{R}).$$

Thus, there exists $\alpha \in \Omega^1(\Sigma)$ with

$$f^*\omega = \omega + d\alpha.$$

We now define a time dependent vector field $X : \mathbb{R} \to C^\infty(\Sigma, T\Sigma)$ by requiring that

$$\mathrm{i}(X_t)(\omega + td\alpha) = -\alpha, \quad t \in \mathbb{R}.$$

This is well-defined because ω is non-degenerate. Let $\Phi : \mathbb{R} \to \mathrm{Diff}^+(\Sigma)$ be the flow uniquely defined by the initial value problem

$$\frac{d}{dt}\Phi_t = X_t \circ \Phi_t, \quad \Phi_0 = \mathrm{id}_\Sigma .$$

Then one checks using Cartan's formula that

$$\begin{aligned}\frac{d}{dt}\Phi_t^*(\omega + td\alpha) &= \Phi_t^* d\alpha + \Phi_t^*\big(\mathscr{L}_{X_t}(\omega + td\alpha)\big)\\ &= \Phi_t^* d\alpha + \Phi_t^*\big(d \circ \mathrm{i}(X_t)(\omega + td\alpha)\big)\\ &= \Phi_t^* d\alpha - \Phi_t^* d\alpha = 0,\end{aligned}$$

where we have used the definition of X_t in the last line. Hence, $\Phi_t^*(\omega + td\alpha)$ is independent of t and so

$$\omega = \Phi_0^*\omega = \Phi_1^*(\omega + d\alpha) = \Phi_1^*(f^*\omega).$$

Now the claim follows with $\widetilde{f} := f \circ \Phi_1$. □

Metrics on Mapping Tori. We can use Proposition 4.1.1 to define particular Riemannian metrics on a mapping torus Σ_f. As remarked before, a symplectic form $\omega \in \Omega^2(\Sigma)$ is the same as a volume form on Σ. The space of metrics on Σ with ω as a volume form has the following description, see e.g. [68, Sec. 4.1].

Recall that an almost complex structure is an endomorphism $J \in C^\infty\big(\Sigma, \mathrm{End}(T\Sigma)\big)$ with $J^2 = -\mathrm{Id}$. It is called *ω-compatible* if for all $v, w \in T\Sigma$ with $v \neq 0$,

$$\omega(Jv, Jw) = \omega(v, w), \quad \text{and} \quad \omega(v, Jv) > 0.$$

Then we get a Riemannian metric with volume form ω by letting

$$g_J(v, w) := \omega(v, Jw), \quad v, w \in T\Sigma. \tag{4.3}$$

Moreover, the space of metrics with volume form ω is naturally isomorphic to

$$\mathscr{J}_\omega := \big\{ J \in C^\infty\big(\Sigma, \mathrm{End}(T\Sigma)\big) \;\big|\; J \text{ is an } \omega\text{-compatible almost complex structure} \big\}.$$

Given a mapping torus Σ_f, we now fix a symplectic form $\omega \in \Omega^2(\Sigma)$ of unit volume. Invoking Proposition 4.1.1 and Lemma B.2.1, we may assume that $f^*\omega = \omega$. Note that this implies that $\mathscr{J}_\omega$ is invariant under conjugation with f_*. Since $\mathscr{J}_\omega$ is easily seen to be path connected—in fact, even contractible—we can choose a path $J_t : \mathbb{R} \to \mathscr{J}_\omega$ of ω-compatible almost complex structures on Σ satisfying

$$J_{t+1} = f_*^{-1} \circ J_t \circ f_*.$$

Let g_t be the path of Riemannian metrics defined by J_t and ω as in (4.3), and define a metric on $\Sigma \times \mathbb{R}$ by

$$g := dt \otimes dt + g_t. \tag{4.4}$$

It is immediate from the fact that $f^*\omega = \omega$ and the convention (4.2) of how to define the mapping torus as a quotient of $\Sigma \times \mathbb{R}$ that g descends to a Riemannian metric on the mapping torus Σ_f.

Calculus on Mapping Tori. We now want to identify the various quantities introduced in Section 2.1. Since the base of the fiber bundle is 1-dimensional, the curvature form Ω vanishes. Clearly, each vertical vector field on Σ_f is induced by a path

$$V : \mathbb{R} \to C^\infty(\Sigma, T\Sigma), \quad V_{t+1} = f_* V_t, \tag{4.5}$$

and each horizontal vector field X can be identified with

$$X = \varphi_t \partial_t, \quad \text{with} \quad \varphi : \mathbb{R} \to C^\infty(\Sigma), \quad \varphi_{t+1} = \varphi_t \circ f.$$

Lemma 4.1.2. *Let U and V be vertical vector fields on Σ. With respect to a metric g as in* (4.4)*, the Levi-Civita connection ∇^g on Σ_f is given by*

$$\nabla^g_{\partial_t} V\big|_t = \partial_t V_t + \tfrac{1}{2}\dot{J}_t J_t V_t, \quad \nabla^g_U V\big|_t = \nabla^{g_t}_{U_t} V_t - \tfrac{1}{2}\omega(U_t, \dot{J}_t V_t)\partial_t, \quad \nabla^g_{\partial_t}\partial_t = 0, \tag{4.6}$$

where ∇^{g_t} is the Levi-Civita connection on Σ with respect to the metric g_t and $\dot{J}_t = [\partial_t, J_t]$. Moreover, the natural vertical connection ∇^v is given by

$$\nabla^v_U V\big|_t = \nabla^{g_t}_{U_t} V_t, \quad \nabla^v_{\partial_t} V\big|_t = \partial_t V_t + \tfrac{1}{2}\dot{J}_t J_t V_t. \tag{4.7}$$

In particular, the difference tensor S of Definition (2.1.6) *is given by*

$$S(U,V)\big|_t = -\tfrac{1}{2}\omega(U_t, \dot{J}_t V_t)\partial_t,$$

and the mean curvature form k_v vanishes.

Sketch of proof. The description of the Levi-Civita connection easily follows from the explicit formula, see e.g. [13, Sec. 1.2],

$$\begin{aligned} 2g\big(\nabla^g_X Y, Z\big) &= g\big([X,Y],Z\big) - g\big([Y,Z],X\big) + g\big([Z,X],Y\big) \\ &\quad + Xg(Y,Z) + Yg(Z,X) - Zg(X,Y). \end{aligned}$$

For example,

$$\begin{aligned} 2g\big(\nabla^g_{\partial_t} V, U\big) &= g\big([\partial_t, V], U\big) + g\big([U,\partial_t], V\big) + \partial_t g(V,U) \\ &= g\big(\partial_t V, U\big) - g\big(V, \partial_t U\big) + \partial_t \omega(V, JU) \\ &= 2g\big(\partial_t V, U\big) + \omega(V, \dot{J}U) \\ &= 2g\big(\partial_t V, U\big) + g(JV, \dot{J}U) = 2g\big(\partial_t V + \tfrac{1}{2}\dot{J}JV,\, U\big), \end{aligned}$$

where we have used that $\dot{J}_t$ is self-adjoint with respect to g_t for each t. The second equation in (4.6) is proven similarly, while the third is clear. Then (4.7) follows by taking the vertical projection of ∇^g. Taking differences yields the formula for S. If $\{e_1, e_2\}$ is a vertical local orthonormal frame, Lemma 2.1.8 implies that

$$k_v(\partial_t) = \tfrac{1}{2}g\big(\dot{J}Je_1, e_1\big) + \tfrac{1}{2}g\big(\dot{J}Je_2, e_2\big),$$

which is zero since $\dot{J}_t J_t$ is easily seen to be skew-adjoint with respect to g_t. □

Flat Connections and the Bundle of Vertical Cohomology Groups. In Appendix B.2 we have included a detailed description of the moduli space of flat $\mathrm{U}(k)$-connections over mapping tori. According to Proposition B.2.12, a flat Hermitian vector bundle over Σ_f is given—up to isomorphism—by a pair (a,u), where $u \in C^\infty\big(\Sigma, \mathrm{U}(k)\big)$ is a gauge transformation, and a is a flat $\mathrm{U}(k)$-connection over Σ satisfying

$$a = u^{-1}(f^*a)u + u^{-1}du. \tag{4.8}$$

We briefly recall from Section B.2 how this data defines a flat vector bundle over Σ_f. First, let

$$\widehat{f}_u : \Sigma \times \mathbb{C}^k \to \Sigma \times \mathbb{C}^k, \quad \widehat{f}_u(x,z) = \big(f(x), u(x)z\big),$$

be the automorphism of the trivial bundle over Σ defined by u, see Remark B.2.5. Then we define a vector bundle $E_u \to \Sigma_f$ as the mapping torus

$$\big((\Sigma \times \mathbb{C}^k) \times \mathbb{R}\big)/\sim, \quad \big((x,z), t+1\big) \sim \big(\widehat{f}_u(x,z), t\big).$$

Viewing a as a constant path of Lie algebra valued 1-form, $a \in C^\infty\big(\mathbb{R}, \Omega^1(\Sigma, \mathfrak{u}(k)\big)$, condition (4.8) ensures that we can define a connection A on E_u. Since a is flat, the same is true for A, see (B.23) and (B.24).

Lemma 4.1.3. *Let $u \in C^\infty\big(\Sigma, \mathrm{U}(k)\big)$ be a gauge transformation defining a bundle $E_u \to \Sigma_f$, and let A be a flat $\mathrm{U}(k)$-connection over T^2_M defined by a pair (a,u) satisfying (4.8).*

(i) *The bundle automorphism $\widehat{f}_u : \Sigma \times \mathbb{C}^k \to \Sigma \times \mathbb{C}^k$ induces an isomorphism*

$$\widehat{f}_u^* : H^\bullet(\Sigma, E_a) \to H^\bullet(\Sigma, E_a).$$

(ii) *Let $\nabla^{H_{A,v}}$ be the natural flat connection on the bundle $H^\bullet_{A,v}(\Sigma_f) \to S^1$ of vertical cohomology groups, see Definition 3.1.10. Then its holonomy representation is given by*

$$\mathrm{hol}_{\nabla^{H_{A,v}}} : \pi_1(S^1) \to \mathrm{GL}\left(H^\bullet(\Sigma, E_a)\right), \quad \mathrm{hol}_{\nabla^{H_{A,v}}}(\gamma) = \widehat{f}_u^*,$$

where $\gamma \in \pi_1(S^1)$ is the canonical generator.

Proof. The map $\widehat{f}_u$ acts on $\Omega^\bullet(\Sigma, \mathbb{C}^k)$ via

$$\widehat{f}_u^* \alpha = u^{-1} f^* \alpha, \quad \alpha \in \Omega^\bullet(\Sigma, \mathbb{C}^k).$$

Condition (4.8) is easily seen to be equivalent to $\widehat{f}_u^* \circ d_a = d_a \circ \widehat{f}_u^*$. This implies that $\widehat{f}_u^*$ descends to an isomorphism on cohomology, which proves part (i). As in (B.22), we have the following identification of the space of vertical, E_u-valued differential forms

$$\Omega^\bullet_v(\Sigma_f, E_u) = \big\{\alpha_t : \mathbb{R} \to \Omega^\bullet(\Sigma, \mathbb{C}^k) \;\big|\; \alpha_{t+1} = \widehat{f}_u^* \alpha_t\big\}. \tag{4.9}$$

With respect to this identification, the vertical differential $d_{A,v}$ coincides with d_a applied pointwise for each t. Moreover, since the connection A has no dt component, the horizontal differential

$$d_{A,h} : \Omega^\bullet_v(\Sigma_f, E_u) \to \Omega^{1,\bullet}(\Sigma_f, E_u)$$

is given with respect to (4.9) by

$$d_{A,h} \alpha_t = dt \wedge \partial_t \alpha_t.$$

From these observations one finds that the space of sections of $H^\bullet_{A,v}(\Sigma_f) \to S^1$ can be described as

$$C^\infty\big(S^1, H_{A,v}(\Sigma_f)\big) = \big\{[\alpha]_t : \mathbb{R} \to H^\bullet(\Sigma, E_a) \;\big|\; [\alpha]_{t+1} = f_u^*[\alpha]_t\big\}.$$

Now it is immediate from the definition of the connection $\nabla^{H_{A,v}}$ that the holonomy representation has the claimed form. $\square$

The Bismut Superconnection. To identify the terms appearing in Dai's adiabatic limit formula, we first need to understand the connection $\nabla^{A,v}$ acting on forms, and then the Bismut superconnection in the setting at hand.

Lemma 4.1.4. *With respect to the identification* (4.9), *the connection* $\nabla^{A,v}$ *acting on* $\Omega_v^\bullet(\Sigma_f, E_u)$ *is given by*

$$\nabla^{A,v}_{\partial_t} = \partial_t - \tfrac{1}{2}\dot{\tau}_v\tau_v, \quad \nabla^{A,v}_V\big|_t = \nabla^{a,g_t}_{V_t},$$

where V *is a vertical vector field, and* ∇^{a,g_t} *denotes the connection on* $\Omega^\bullet(\Sigma, \mathbb{C}^k)$ *induced by* g_t *and* a*, acting pointwise for each* t*.*

Proof. The assertion about $\nabla^{A,v}_V$ is true by definition of $\nabla^{A,v}$ and the fact that A is independent of t. Concerning $\nabla^{A,v}_{\partial_t}$, we observe that $d_{A,h} = dt \wedge \partial_t$, which implies that

$$\widetilde{\nabla}^{A,v}_{\partial_t} = \partial_t. \tag{4.10}$$

Since the mean curvature form vanishes, we can use (2.9) and (2.15) to deduce that

$$\nabla^{A,v}_{\partial_t} = \partial_t + \tfrac{1}{2}\tau_v[\partial_t, \tau_v] = \partial_t - \tfrac{1}{2}\dot{\tau}_v\tau_v,$$

where we have used that $\dot{\tau}_v = [\partial_t, \tau_v]$ and $\dot{\tau}_v\tau_v = -\tau_v\dot{\tau}_v$. □

Using Lemma 4.1.4 we can now find a more explicit expression for the Bismut superconnection and its transgression form.

Proposition 4.1.5. *The rescaled Bismut superconnection associated to* A *is given for* $s \in (0, \infty)$ *by*

$$\mathbb{B}_s = \tfrac{\sqrt{s}}{2} D_{A,v} + dt \wedge \big(\partial_t - \tfrac{1}{2}\dot{\tau}_v\tau_v\big) : \Omega_v^\bullet(\Sigma_f, E_u) \to \Omega^\bullet(\Sigma_f, E_u),$$

where $D_{A,v}$ *is the vertical de Rham operator. Moreover,*

$$\alpha(\mathbb{B}_s) = \frac{i}{16\pi} dt \wedge \operatorname{Tr}_v \left(\dot{\tau}_v\big(d^t_{A,v}d_{A,v} - d_{A,v}d^t_{A,v}\big)e^{-\frac{s}{4}D^2_{A,v}}\right) \in \Omega^1(S^1).$$

Proof. The formula for $\mathbb{B}_s$ follows immediately from Lemma 4.1.4 and the fact that k_v and Ω are zero, see (3.20) and Definition 3.2.15. For the second assertion, we drop the connection A from the notation. Since the base is 1-dimensional, and dt anti-commutes with D_v, one finds that

$$\mathbb{B}_s^2\big|_{s=4r} = rD_v^2 + \sqrt{r}dt \wedge \big[\nabla^v_{\partial_t}, D_v\big].$$

As in [13, Lem. 9.42] and [18, Thm. 3.3] an application of Duhamel's formula then yields that

$$e^{-rD_v^2 - \sqrt{r}dt\wedge[\nabla^v_{\partial_t}, D_v]} = e^{-rD_v^2} - \sqrt{r}\, dt \wedge \int_0^1 e^{-r'rD_v^2}\big[\nabla^v_{\partial_t}, D_v\big]e^{-(1-r')rD_v^2}dr'.$$

where the higher correction terms vanish, again since the base is 1-dimensional. Therefore,

$$\begin{aligned}\operatorname{Str}_v\left(\tfrac{d\mathbb{B}_s}{ds}e^{-\mathbb{B}_s^2}\right)\Big|_{s=4r} &= \tfrac{1}{8\sqrt{r}}\operatorname{Str}_v\left(D_v e^{-rD_v^2 - \sqrt{r}dt\wedge[\nabla^v_{\partial_t}, D_v]}\right)\\ &= \tfrac{1}{8\sqrt{r}}\operatorname{Str}_v\left(D_v e^{-rD_v^2}\right) + \tfrac{1}{8}dt \wedge \operatorname{Str}_v\left(D_v[\nabla^v_{\partial_t}, D_v]e^{-rD_v^2}\right).\end{aligned}$$

Note that a factor of -1 enters in front of the second term since we have interchanged dt and D_v. Since τ_v anti-commutes with D_v, the first term vanishes. To simplify the second term, we also drop the sub- and subscripts v from the notation. Then one verifies using Lemma 4.1.4 and the relations $D = d - \tau d\tau$, $[\partial_t, \tau] = \dot{\tau}$ as well as $\dot{\tau}\tau = -\tau\dot{\tau}$ that

$$\tau[\nabla_{\partial_t}, D] = -\tfrac{1}{2}\big(\tau[\dot{\tau}\tau, d] - [\dot{\tau}\tau, d]\tau\big) = \tfrac{1}{2}\big(\dot{\tau}d + \tau d\dot{\tau}\tau + \dot{\tau}\tau d\tau - d\dot{\tau}\big)$$

Now, since $\tau D = -D\tau$, we can use the trace property and the fact that e^{-rD^2} is a semi-group of smoothing operators to find that

$$\operatorname{Tr}\big(\tau d\dot{\tau}\tau De^{-rD^2}\big) = -\operatorname{Tr}\big(d\dot{\tau}De^{-rD^2}\big), \quad \operatorname{Tr}\big(\dot{\tau}\tau d\tau De^{-rD^2}\big) = \operatorname{Tr}\big(\dot{\tau}dDe^{-rD^2}\big).$$

Thus, by repeatedly making use of the trace property, one obtains

$$\begin{aligned}\operatorname{Str}\big(D[\nabla_{\partial_t}, D]e^{-rD^2}\big) &= \operatorname{Str}\big([\nabla_{\partial_t}, D]De^{-rD^2}\big) \\ &= -\operatorname{Tr}\big((d\dot{\tau} - \dot{\tau}d)De^{-rD^2}\big) = -\operatorname{Tr}\big(\dot{\tau}(d^t d - dd^t)e^{-rD^2}\big).\end{aligned}$$

Recalling the normalization factor in Definition 3.2.17, we arrive at the claimed formula for the transgression form. □

4.2 Finite Order Mapping Tori

Proposition 4.1.5 shows that if we can achieve that $\dot{\tau}_v \equiv 0$, then the Bismut-Cheeger Eta form associated to a flat connection over Σ_f vanishes. From the discussion in Section 4.1 we know that $\dot{\tau}_v \equiv 0$ is equivalent to finding an f-invariant metric g_Σ on Σ. Clearly, such a metric will not exist for arbitrary $f \in \operatorname{Diff}^+(\Sigma)$.

Lemma 4.2.1. *If $f \in \operatorname{Diff}^+(\Sigma)$ is of finite order n, there exists a metric g_Σ of unit volume with $f^*g_\Sigma = g_\Sigma$.*

Proof. Choose an arbitrary metric g_Σ on Σ of unit volume and define

$$\widetilde{g}_\Sigma := \tfrac{1}{n}\sum_{j=0}^{n-1}(f^j)^*g_\Sigma.$$

Then $\widetilde{g}_\Sigma$ is again a metric of unit volume. Moreover, since $f^n = \mathrm{id}_\Sigma$, one finds that indeed $f^*\widetilde{g}_\Sigma = \widetilde{g}_\Sigma$. □

Remark 4.2.2.

(i) A metric g_Σ defines an almost complex structure on Σ which is integrable, see Section B.3. If g_Σ is f-invariant for some $f \in \operatorname{Diff}^+(\Sigma)$, then f is holomorphic with respect to the complex structure defined by g_Σ.

(ii) It can be shown that if there exists an f-invariant metric g_Σ, then the mapping class $[f] \in \operatorname{Diff}^+(\Sigma)/\operatorname{Diff}_0(\Sigma)$ is necessarily of finite order. If the genus of Σ is 0 or 1, this can be checked directly, see Proposition 4.4.3 below for $\Sigma = T^2$. For higher genera, one can use for example the holomorphic description, and invoke the Riemann-Hurwitz formula, see [40, Ch. V].

Rho Invariants of a Finite Order Mapping Torus. Assume from now on that $f \in \operatorname{Diff}^+(\Sigma)$ is of finite order, and that an f-invariant metric g_Σ has been chosen. We also fix a flat connection A on a Hermitian vector bundle $E_u \to \Sigma_f$ defined by a pair (a, u) satisfying $\widehat{f}_u^* a = a$ as in (4.8). In Lemma 4.1.3 we have given a description of the bundle of vertical cohomology groups in terms of de Rham cohomology. However, the Rho invariant of the bundle of vertical cohomology groups in Definition 3.3.15—which appears in the formula for the Rho invariant in Theorem 3.3.16—is defined using the Hodge theoretic description. In the case of a finite order mapping torus, we have the following extension of Lemma 4.1.3.

Lemma 4.2.3.

(i) *With respect to the induced metric on $\mathscr{H}^\bullet(\Sigma, E_a)$, the bundle map $\widehat{f}_u$ defines an isometry*

$$\widehat{f}_u^* : \mathscr{H}^\bullet(\Sigma, E_a) \to \mathscr{H}^\bullet(\Sigma, E_a), \quad \mathscr{H}^\bullet(\Sigma, E_a) = \ker(d_a + d_a^t) \subset \Omega^\bullet(\Sigma, \mathbb{C}^k).$$

The splitting into ± 1-eigenspaces of τ_Σ,

$$\mathscr{H}^\bullet(\Sigma, E_a) = \mathscr{H}^+(\Sigma, E_a) \oplus \mathscr{H}^-(\Sigma, E_a),$$

is invariant with respect to $\widehat{f}_u^*$.

(ii) *The flat connection $\nabla^{H_{A,v}}$ on the bundle of vertical cohomology groups is compatible with the metric $\langle .,. \rangle_{\mathscr{H}_{A,v}}$ of Definition 3.1.14, and its holonomy representation is given by*

$$\operatorname{hol}_{\nabla^{H_{A,v}}}(\gamma) = \widehat{f}_u^* \in \operatorname{GL}\big(\mathscr{H}^\bullet(\Sigma, E_a)\big),$$

where $\gamma \in \pi_1(S^1)$ is the canonical generator.

Proof. Since f is an isometry with respect to the metric g_Σ, the pullback

$$f^* : \Omega^\bullet(\Sigma) \to \Omega^\bullet(\Sigma)$$

commutes with the chirality operator τ_Σ. Moreover, as $\widehat{f}_u^* = u^{-1} f^*$, the same is true for $\widehat{f}_u^*$. This, and the fact that $f_u^* \circ d_a = d_a \circ \widehat{f}_u^*$ implies part (i). Since $\dot{\tau} \equiv 0$, we can deduce from (2.15) and Proposition 3.1.15 that the connection $\nabla^{H_{A,v}}$ is indeed compatible with the metric $\langle .,. \rangle_{\mathscr{H}_{A,v}}$. Moreover, if

$$\Psi : \mathscr{H}^\bullet(\Sigma, E_a) \to H^\bullet(\Sigma, E_a)$$

denotes the Hodge-de-Rham isomorphism, the diagram

$$\begin{array}{ccc} \mathscr{H}^\bullet(\Sigma, E_a) & \xrightarrow{\widehat{f}_u^*} & \mathscr{H}^\bullet(\Sigma, E_a) \\ {\scriptstyle\Psi}\downarrow & & {\scriptstyle\Psi}\downarrow \\ H^\bullet(\Sigma, E_a) & \xrightarrow{\widehat{f}_u^*} & H^\bullet(\Sigma, E_a), \end{array}$$

is commutative. This is because it is naturally induced by $\widehat{f}_u^* : \Omega^\bullet(\Sigma, \mathbb{C}^k) \to \Omega^\bullet(\Sigma, \mathbb{C}^k)$. Hence, the description of the holonomy representation follows from Lemma 4.1.3. $\square$

Theorem 4.2.4. *Let $f \in \mathrm{Diff}^+(\Sigma)$ be of finite order. Let A be a flat $\mathrm{U}(k)$-connection over Σ_f, defined by a pair (a, u) of flat connection and gauge transformation over Σ satisfying $\widehat{f}_u^* a = a$. Then with respect to every f-invariant metric g_Σ on Σ*

$$\begin{aligned}\rho_A(\Sigma_f) =& 2\operatorname{tr}\log\big[\widehat{f}_u^*|_{\mathscr{H}^+(\Sigma,E_a)\cap\Omega^1}\big] - \operatorname{rk}\big[(\widehat{f}_u^* - \mathrm{Id})|_{\mathscr{H}^+(\Sigma,E_a)\cap\Omega^1}\big]\\ &- 2\operatorname{tr}\log\big[\widehat{f}_u^*|_{\mathscr{H}^-(\Sigma,E_a)\cap\Omega^1}\big] + \operatorname{rk}\big[(\widehat{f}_u^* - \mathrm{Id})|_{\mathscr{H}^-(\Sigma,E_a)\cap\Omega^1}\big]\\ &- 4k\operatorname{tr}\log\big[f^*|_{\mathscr{H}^+(\Sigma)\cap\Omega^1}\big] + 2k\operatorname{rk}\big[(f^* - \mathrm{Id})|_{\mathscr{H}^+(\Sigma)\cap\Omega^1}\big].\end{aligned}$$

Here, "$\operatorname{tr}\log$" is defined for a unitary map $T \in \mathrm{U}(n)$ as

$$\operatorname{tr}\log T := \sum_{j=1}^n \theta_j \in \mathbb{R},$$

where $e^{2\pi i\theta_j}$ are the eigenvalues of T, and where we require $\theta_j \in [0, 1)$.

Proof. As we have already noted before, Proposition 4.1.5 implies that the Bismut-Cheeger Rho form vanishes, because g_Σ is f-invariant. Moreover, the Leray-Serre spectral sequence associated to A—respectively the trivial connection—collapses at the E_2-stage, since the base is 1-dimensional, see Theorem 3.1.11. This implies that Dai's correction term in Definition 3.3.13 vanishes. Hence, Theorem 3.3.16 yields

$$\rho_A(\Sigma_f) = \rho_{\mathscr{H}_{A,v}}(S^1) = \tfrac{1}{2}\eta\big(D_{S^1} \otimes \nabla^{\mathscr{H}_{A,v}}\big) - k \cdot \tfrac{1}{2}\eta\big(D_{S^1} \otimes \nabla^{\mathscr{H}_v}\big), \tag{4.11}$$

where $D_{S^1} \otimes \nabla^{\mathscr{H}_{A,v}}$ and $D_{S^1} \otimes \nabla^{\mathscr{H}_v}$ are as in Definition 3.1.16. We have seen in Lemma 4.2.3 that the flat connection $\nabla^{\mathscr{H}_v}$ is unitary with respect to the metric $\langle .,. \rangle_{\mathscr{H}_v}$. Hence, we deduce from Lemma 3.1.19 that

$$D_{S^1} \otimes \nabla^{\mathscr{H}_v} = \begin{pmatrix} D_{S^1} \otimes \nabla^{\mathscr{H}_v,+} & 0 \\ 0 & -D_{S^1} \otimes \nabla^{\mathscr{H}_v,-} \end{pmatrix}, \tag{4.12}$$

where

$$D_{S^1} \otimes \nabla^{\mathscr{H}_v,\pm} = \tau_{S^1} d_{\nabla^{\mathscr{H}_v,\pm}} + d_{\nabla^{\mathscr{H}_v,\pm}} \tau_{S^1}, \tag{4.13}$$

and $\nabla^{\mathscr{H}_v,\pm}$ denotes the restriction of $\nabla^{\mathscr{H}_v}$ to $\mathscr{H}_v^\pm(\Sigma_f)$. The same formula holds for $\mathscr{H}_v^\bullet(\Sigma_f)$ replaced with $\mathscr{H}_{A,v}^\bullet(\Sigma_f)$. Let us abbreviate

$$B_\pm := \tau_{S^1} d_{\nabla^{\mathscr{H}_v,\pm}} : C^\infty\big(S^1, \mathscr{H}_v^\pm(\Sigma_f)\big) \to C^\infty\big(S^1, \mathscr{H}_v^\pm(\Sigma_f)\big), \tag{4.14}$$

and define $B_{A,\pm}$ correspondingly. Then (4.11) and (4.12) show that

$$\rho_A(\Sigma_f) = \eta(B_{A,+}) - \eta(B_{A,-}) - k \cdot \eta(B_+) + k \cdot \eta(B_-). \tag{4.15}$$

Note that the factors $\frac{1}{2}$ disappear as the operators in (4.13) are equivalent to two copies of the operators in (4.14), see Remark 3.1.17 (ii). We deduce from Lemma 4.2.3 (i) that $\widehat{f}_u^*$ restricts to a unitary map on $\mathscr{H}^\pm(\Sigma, E_a)$. Hence, we can find a basis of $\mathscr{H}^\pm(\Sigma, E_a)$ such that

$$\widehat{f}_u^*|_{\mathscr{H}^\pm(\Sigma,E_a)} = \operatorname{diag}\big(e^{2\pi i\theta_1^\pm}, \ldots, e^{2\pi i\theta_n^\pm}\big),$$

where $\theta_j^\pm \in [0,1)$, and $n = \dim \mathscr{H}^+(\Sigma, E_a) = \dim \mathscr{H}^-(\Sigma, E_a)$. Note that the equality of dimensions follows from the fact that $\mathrm{Sign}_a(\Sigma) = 0$, which in turn is a consequence of the signature formula, see Theorem 1.2.23. Now Lemma 4.2.3 (ii) yields that the restriction $\widehat{f}_u^*|_{\mathscr{H}^\pm(\Sigma,E_a)}$ defines the holonomy representation of the connection $\nabla^{\mathscr{H}_{A,v},\pm}$. From this we deduce as in Remark 1.4.8 (iii) that

$$\begin{aligned}\eta(B_{A,+}) - \eta(B_{A,-}) = &\sum_{\theta_j^+\neq 0} (2\theta_j^+ - 1) - \sum_{\theta_j^-\neq 0} (2\theta_j^- - 1)\\ =&2\operatorname{tr}\log\left[\widehat{f}_u^*|_{\mathscr{H}^+}\right] - \operatorname{rk}\left[(\widehat{f}_u^* - \mathrm{Id})|_{\mathscr{H}^+}\right]\\ &- 2\operatorname{tr}\log\left[\widehat{f}_u^*|_{\mathscr{H}^-}\right] + \operatorname{rk}\left[(\widehat{f}_u^* - \mathrm{Id})|_{\mathscr{H}^-}\right],\end{aligned}$$

where for convenience we have abbreviated $\mathscr{H}^\pm = \mathscr{H}^\pm(\Sigma, E_a)$ in the last equality. Continuing with obvious abbreviations, we now decompose

$$\mathscr{H}^\pm = \mathscr{H}^\pm \cap (\Omega^0 \oplus \Omega^2) \oplus \mathscr{H}^\pm \cap \Omega^1.$$

Each element of $\mathscr{H}^\pm \cap \left(\Omega^0 \oplus \Omega^2\right)$ is of the form $\varphi \pm \tau_\Sigma\varphi$ with $\varphi \in \mathscr{H}^0$. Hence, we have a natural isomorphism (compare also with the proof of Proposition 1.1.8),

$$\mathscr{H}^+ \cap \left(\Omega^0 \oplus \Omega^2\right) \xrightarrow{\cong} \mathscr{H}^- \cap \left(\Omega^0 \oplus \Omega^2\right), \quad \varphi + \tau_\Sigma\varphi \mapsto \varphi - \tau_\Sigma\varphi.$$

Since $\widehat{f}_u^*$ commutes with τ_Σ, we can conclude that

$$\begin{aligned}&2\operatorname{tr}\log\left[\widehat{f}_u^*|_{\mathscr{H}^+\cap(\Omega^0\oplus\Omega^2)}\right] - \operatorname{rk}\left[(\widehat{f}_u^* - \mathrm{Id})|_{\mathscr{H}^+\cap(\Omega^0\oplus\Omega^2)}\right]\\ &\qquad = 2\operatorname{tr}\log\left[\widehat{f}_u^*|_{\mathscr{H}^-\cap(\Omega^0\oplus\Omega^2)}\right] - \operatorname{rk}\left[(\widehat{f}_u^* - \mathrm{Id})|_{\mathscr{H}^-\cap(\Omega^0\oplus\Omega^2)}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\eta(B_{A,+}) - \eta(B_{A,-}) =&2\operatorname{tr}\log\left[\widehat{f}_u^*|_{\mathscr{H}^+\cap\Omega^1}\right] - \operatorname{rk}\left[(\widehat{f}_u^* - \mathrm{Id})|_{\mathscr{H}^+\cap\Omega^1}\right]\\ &- 2\operatorname{tr}\log\left[\widehat{f}_u^*|_{\mathscr{H}^-\cap\Omega^1}\right] + \operatorname{rk}\left[(\widehat{f}_u^* - \mathrm{Id})|_{\mathscr{H}^-\cap\Omega^1}\right].\end{aligned}$$

This identifies the twisted terms appearing in the formula of Theorem 4.2.4. In the case that a is the trivial connection and $u \equiv 1$, we can simplify this further. Since we are considering complex valued forms, we have a conjugation

$$\mathscr{H}^1(\Sigma) \to \mathscr{H}^1(\Sigma), \quad \alpha \mapsto \bar{\alpha}.$$

The chirality operator τ_Σ is readily seen to anti-commute with conjugation. This yields an anti-linear isomorphism

$$\mathscr{H}^+(\Sigma) \cap \Omega^1(\Sigma) \xrightarrow{\cong} \mathscr{H}^-(\Sigma) \cap \Omega^1(\Sigma).$$

Since f^* is the complex linear extension of a real automorphism, it commutes with conjugation. From this one readily deduces that the eigenvalues of $f^*|_{\mathscr{H}^-\cap\Omega^1}$ are complex conjugate to the eigenvalues of $f^*|_{\mathscr{H}^+\cap\Omega^1}$. By checking the definition of "tr log" carefully, one conlcudes

$$\operatorname{tr}\log\left[f^*|_{\mathscr{H}^-\cap\Omega^1}\right] = \operatorname{rk}\left[(f^* - \mathrm{Id})|_{\mathscr{H}^+\cap\Omega^1}\right] - \operatorname{tr}\log\left[f^*|_{\mathscr{H}^+\cap\Omega^1}\right].$$

Now, as $\operatorname{rk}\left[(f^* - \mathrm{Id})|_{\mathscr{H}^-\cap\Omega^1}\right] = \operatorname{rk}\left[(f^* - \mathrm{Id})|_{\mathscr{H}^+\cap\Omega^1}\right]$, we finally get

$$\eta(B_+) - \eta(B_-) = 4\operatorname{tr}\log\left[f^*|_{\mathscr{H}^+\cap\Omega^1}\right] - 2\operatorname{rk}\left[(f^* - \mathrm{Id})|_{\mathscr{H}^+\cap\Omega^1}\right],$$

which is precisely the untwisted term in the claimed formula. □

Remark.

(i) The last step of the above proof is essentially equivalent to observing that f^* acting on $\mathscr{H}^1(\Sigma)$ is the complexification of a symplectic map. This explains why the eigenvalues come in conjugate pairs. Clearly, this is no longer true if we consider $u^{-1}f^*$ with $u \in \mathrm{U}(1)$, which also defines a flat connection over Σ_f, see also Theorem 4.4.4 below.

(ii) In a similar direction, if the connection a is non-trivial, complex conjugation gives rise to an anti-linear isomorphism

$$\mathscr{H}^+(\Sigma, E_a) \xrightarrow{\cong} \mathscr{H}^-(\Sigma, E_{\bar{a}}).$$

From this, one can relate the eigenvalues of $\widehat{f}_u^*$ acting on $\mathscr{H}^+(\Sigma, E_a)$ with the eigenvalues of $\widehat{f}_{\bar{u}}^*$ acting on $\mathscr{H}^-(\Sigma, E_{\bar{a}})$. However, this only simplifies the formula of Theorem 4.2.4 in the case that a and u are real, in the sense that they arise from an $\mathrm{O}(k)$-structure.

(iii) Although Theorem 4.2.4 expresses the Rho invariants of Σ_f in terms of Hodge-de-Rham cohomology of Σ, it is only an intermediate step to an expression in completely topological terms. The next step would be to use the ideas of the Atiyah-Bott fixed point formula—see [13, Sec. 6.2]—to relate the traces appearing in Theorem 4.2.4 to the fixed point data of f. This, in turn, can be expressed in terms of the Seifert invariants of the finite order mapping torus. We refer to [2, Sec. 5] and [43, Sec. 2.2] for a discussion of these ideas in the context of the determinant line bundle over the moduli space of flat connections associated to a finite order mapping torus.

(iv) In [75] an interpretation of the untwisted Eta invariant for finite order mapping tori is given in terms of Meyer's cocycle for the mapping class group, see [72]. It follows from the proof of Theorem 4.2.4 that the adiabatic limit of the untwisted Eta invariant is given by

$$4 \operatorname{tr} \log \left[f^*|_{\mathscr{H}^+(\Sigma) \cap \Omega^1(\Sigma)} \right] - 2 \operatorname{rk} \left[(f^* - \mathrm{Id})|_{\mathscr{H}^+(\Sigma) \cap \Omega^1(\Sigma)} \right]. \tag{4.16}$$

It would be interesting to relate this to the main result of [75].

4.3 Torus Bundles over S^1, General Setup

We now consider the case that $\Sigma = T^2$ is the 2-dimensional torus. In the same setting, Atiyah [3] studies a rich interplay between the untwisted Eta invariant and other topological invariants, the Dedekind Eta function and also number theoretical L-series. As a tool Atiyah also makes intensive use of the idea of adiabatic limits, and much of our discussion is influenced by the treatment in [3]. We shall restrict to the case of $\mathrm{U}(1)$-connections, which already contains many important ideas. However, in view of the computations of Chern-Simons invariants for torus bundles in [54, 57] the generalization to higher gauge groups would be extremely interesting.

4.3.1 Geometry Torus Bundles over S^1

Complex Structures on T^2. We fix the standard torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, endowed with the volume form induced by $\omega = dx \wedge dy$. As in Section 4.1 we are interested in the space of all metrics which have ω as a volume form. Equivalently, we need to understand the space $\mathscr{J}_\omega$ of all ω-compatible almost complex structures. It is well known that $\mathscr{J}_\omega$ is the Teichmüller space of T^2, i.e., the upper half plane[1]

$$\mathbb{H} := \big\{\sigma = \sigma_1 + i\sigma_2 \in \mathbb{C} \;\big|\; v > 0\big\},$$

see [55, Thm. 2.7.2]. For definiteness, we will use the following explicit isomorphism. Note that each almost complex structure $J \in \mathscr{J}_\omega$ can be identified with a matrix in $\mathrm{M}_2(\mathbb{R})$ because the tangent space of T^2 is canonically isomorphic to $\mathbb{R}^2$.

Lemma 4.3.1. *The map*

$$\Phi : \mathbb{H} \to \mathscr{J}_\omega, \quad \Phi(\sigma) = \frac{1}{\sigma_2}\begin{pmatrix} -\sigma_1 & -|\sigma|^2 \\ 1 & \sigma_1 \end{pmatrix}, \quad \sigma = \sigma_1 + i\sigma_2,$$

is a bijection. The metric on T^2 defined by $\Phi(\sigma)$ as in (4.3) *is given with respect to the standard coordinate basis as*

$$g_\sigma = \frac{1}{\sigma_2}\Big(dx \otimes dx + \sigma_1(dx \otimes dy + dy \otimes dx) + |\sigma|^2 dy \otimes dy\Big).$$

Proof. If we identify $J \in \mathscr{J}_\omega$ with an element in $\mathrm{M}_2(\mathbb{R})$, one easily checks that $J^2 = -\mathrm{Id}$ is equivalent to $\det(J) = 1$ and $\mathrm{tr}(J) = 0$. Hence there exist $r, s, t \in \mathbb{R}$ such that

$$J = \begin{pmatrix} -r & t \\ s & r \end{pmatrix}, \quad r^2 + st = -1. \tag{4.17}$$

Let J_0 be the almost complex structure which induces the standard scalar product on $\mathbb{R}^2$, i.e.,

$$J_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad g_0 = \omega(., J_0.) = dx \otimes dx + dy \otimes dy.$$

Then one verifies that J is ω-compatible if and only if $-J_0 J$ is positive definite and symmetric. Now

$$-J_0 J = \begin{pmatrix} s & r \\ r & -t \end{pmatrix}, \tag{4.18}$$

and so (4.17) implies that $-J_0 J$ is positive definite if and only if $s > 0$. Using this one finds that Φ is well-defined with inverse given by

$$\Phi^{-1}(J) = \frac{1}{s}(r + i).$$

Moreover, (4.18) relates $\Phi(\sigma)$ and the associated metric g_σ and easily yields the second claim. □

[1] We are using the letter σ for elements in $\mathbb{H}$ rather than the more common letter τ to avoid confusion with the chirality operator.

Remark 4.3.2. Viewed from a complex analytic perspective, the above identification might seem a bit cumbersome. As in [55, Sec. 2.7], any $\sigma \in \mathbb{H}$ defines a lattice

$$\Lambda(\sigma) := \{m + n\sigma \mid (m,n) \in \mathbb{Z}^2\} \subset \mathbb{C},$$

and the quotient torus $\mathbb{C}/\Lambda(\sigma)$ is naturally a complex manifold, with complex structure induced by the one of the complex plane, and metric induced by $\frac{1}{\sigma_2}(dx^2 + dy^2)$. Note that one has to divide by σ_2 to get a metric of unit volume. It is easy to check that

$$\psi_\sigma : \mathbb{R}^2 \to \mathbb{C}, \quad (x,y) \mapsto x + \sigma y,$$

descends to an isometry

$$\psi_\sigma : (T^2, g_\sigma) \to \mathbb{C}/\Lambda(\sigma), \tag{4.19}$$

where the metric g_σ is defined as in Lemma 4.3.1. The complex analytic description is better suited for explicit computations if σ is fixed. However, if σ varies the underlying manifold varies as well. This is sometimes inconvenient in the study of families. In the following, we will use both descriptions. To avoid confusion we will reserve T^2 for the standard 2-torus and use the notation $\mathbb{C}/\Lambda(\sigma)$ whenever we prefer to think in complex analytic terms.

Working on $\mathbb{C}/\Lambda(\sigma)$ has the advantage that the metric is up to a constant factor induced by the standard metric. In particular, we will consider the 1-forms $dz = dx + idy$ and $d\bar{z} = dx - idy$. Note that

$$\big(dz\,,\, d\bar{z}\big)_{L^2} = 0, \quad \|dz\|_{L^2} = \|d\bar{z}\|_L^2 = \sqrt{2\sigma_2},$$

and, with the chirality operator τ,

$$\tau dz = dz, \quad \tau d\bar{z} = -d\bar{z}, \quad \text{and} \quad \tau(dz \wedge d\bar{z}) = -2\sigma_2.$$

Using the isometry ψ_σ of (4.19), one translates this easily to (T^2, g_σ). More precisely, we define

$$\omega_\sigma := \psi_\sigma^*(dz) = dx + \sigma dy, \quad \text{and} \quad \omega_{\bar{\sigma}} := \psi_\sigma^*(d\bar{z}) = dx + \bar{\sigma} dy. \tag{4.20}$$

Then we have a natural basis $(\omega_\sigma, \omega_{\bar{\sigma}})$ for the $C^\infty(T^2)$-module $\Omega^1(T^2)$ satisfying

$$\begin{gathered}(\omega_\sigma, \omega_{\bar{\sigma}})_{L^2} = 0, \quad \|\omega_\sigma\|_{L^2} = \|\omega_{\bar{\sigma}}\|_{L^2} = \sqrt{2\sigma_2}, \\ \tau\omega_\sigma = \omega_\sigma, \quad \text{and} \quad \tau\omega_{\bar{\sigma}} = -\omega_{\bar{\sigma}}.\end{gathered} \tag{4.21}$$

Flat Connections and Dolbeault Operators. The moduli space of flat U(1)-connections over T^2 has a simple structure. Since T^2 is the quotient of $\mathbb{R}^2$ by the standard lattice $\mathbb{Z}^2$, we have

$$\pi_1(T^2) \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \mathbb{R}^2,$$

where (e_1, e_2) is the standard basis of $\mathbb{R}^2$. Then, since U(1) is abelian,

$$\mathcal{M}\big(T^2, \mathrm{U}(1)\big) \cong \mathrm{Hom}\,\big(\pi_1(T^2), \mathrm{U}(1)\big) \cong \mathrm{U}(1) \times \mathrm{U}(1).$$

As we are usually working with connections rather than representations of the fundamental group, we summarize how the above isomorphism works explicitly.

Lemma 4.3.3.

(i) *Up to gauge equivalence, a flat* U(1)*-connection over* T^2 *is induced by a* $\mathbb{Z}^2$*-invariant 1-form with constant coefficients,*

$$a_\nu = -2\pi i(\nu_1 dx + \nu_2 dy) \in \Omega^1(\mathbb{R}^2, i\mathbb{R}), \quad \nu = (\nu_1, \nu_2) \in \mathbb{R}^2. \tag{4.22}$$

Two connections a_ν *and* $a_{\nu'}$ *are gauge equivalent if and only if* $\nu - \nu' \in \mathbb{Z}^2$*.*

(ii) *In terms of the generators* e_1, e_2 *of* $\pi_1(T^2)$*, the holonomy representation of* a_ν *is given by*

$$\mathrm{hol}_{a_\nu} : \pi_1(T^2) \to \mathrm{U}(1), \quad \mathrm{hol}_{a_\nu}(e_j) = e^{2\pi i \nu_j}.$$

Proof. A flat connection a over T^2 is a $\mathbb{Z}^2$-invariant 1-form, satisfying $da = 0$. Therefore, it gives an element in de Rham cohomology. Since

$$H^1(T^2, \mathbb{R}) = \mathbb{R}^2,$$

we can find a $\mathbb{Z}^2$-invariant function $f : \mathbb{R}^2 \to \mathbb{R}$ and $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ such that

$$a - idf = -2\pi i(\nu_1 dx + \nu_2 dy).$$

Defining $u := \exp(if)$, we get a gauge transformation on T^2 which brings a into the claimed form. If a_ν and $a_{\nu'}$ are gauge equivalent, there exists a $\mathbb{Z}^2$-invariant function $u : \mathbb{R}^2 \to \mathrm{U}(1)$ satisfying

$$2\pi i\big((\nu_1 - \nu_1')dx + (\nu_2 - \nu_2')dy)\big) = u^{-1}du.$$

This easily implies that u is of the form

$$u = C \cdot \exp\big(2\pi i \langle \nu - \nu' \,, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \rangle\big), \quad C \in \mathrm{U}(1),$$

and this is $\mathbb{Z}^2$-invariant precisely if $\nu - \nu' \in \mathbb{Z}^2$. This proves part (i). Concerning (ii), we compute using Definition B.1.2

$$\mathrm{hol}_{a_\nu}(e_j) = \exp\Big(-\int_0^1 a_\nu\big|_{se_i}(e_i)ds\Big) = \exp\big(2\pi i \nu_i\big). \qquad \square$$

As pointed out in Remark 4.3.2 it is often convenient to work on $\mathbb{C}/\Lambda(\sigma)$ rather than T^2. We collect the following formulæ, for definitions see Appendix B.3, in particular (B.28).

Proposition 4.3.4. *Let* a_ν *be a flat* U(1)*-connection over* T^2 *as in Lemma 4.3.3, and let* $\sigma = \sigma_1 + i\sigma_2 \in \mathbb{H}$*.*

(i) *The pullback of* a_ν *to* $\mathbb{C}/\Lambda(\sigma)$ *is given by*

$$a = (\psi_\sigma^{-1})^* a_\nu = -\bar{w}_\nu dz + w_\nu d\bar{z}, \quad \textit{where} \quad w_\nu := \tfrac{\pi}{\sigma_2}(\nu_2 - \sigma\nu_1).$$

(ii) *Let* ∂_a *and* $\bar{\partial}_a$ *be the Dolbeault operators associated to* a*. Then the twisted Laplace operator on* $C^\infty\big(\mathbb{C}/\Lambda(\sigma)\big)$ *is given by*

$$\Delta_a = 2\partial_a^t \partial_a = 2\bar{\partial}_a^t \bar{\partial}_a = -4\sigma_2\Big(\frac{\partial^2}{\partial z \partial \bar{z}} - \bar{w}_\nu \frac{\partial}{\partial \bar{z}} + w_\nu \frac{\partial}{\partial z} - |w_\nu|^2\Big).$$

(iii) *If $\varphi \in C^\infty(\mathbb{C})$ is $\Lambda(\sigma)$-invariant, then*

$$\partial_a \partial_a^t(\varphi dz) = \bar\partial_a^t \bar\partial_a(\varphi dz) = (\tfrac{1}{2}\Delta_a \varphi)dz,$$
$$\bar\partial_a \bar\partial_a^t(\varphi d\bar z) = \partial_a^t \partial_a(\varphi d\bar z) = (\tfrac{1}{2}\Delta_a \varphi)d\bar z,$$

and

$$\bar\partial_a \partial_a^t(\varphi dz) = -\partial_a^t \bar\partial_a(\varphi dz) = -2\sigma_2\Big(\frac{\partial^2\varphi}{\partial \bar z^2} + 2w_\nu \frac{\partial \varphi}{\partial \bar z} + w_\nu^2 \varphi\Big)d\bar z,$$
$$\partial_a \bar\partial_a^t(\varphi d\bar z) = -\bar\partial_a^t \partial_a(\varphi d\bar z) = -2\sigma_2\Big(\frac{\partial^2\varphi}{\partial z^2} - 2\bar w_\nu \frac{\partial \varphi}{\partial z} + \bar w_\nu^2 \varphi\Big)dz.$$

In particular,

$$\Delta_a(\varphi dz) = (\Delta_a\varphi)dz \quad \textit{and} \quad \Delta_a(\varphi d\bar z) = (\Delta_a \varphi)d\bar z.$$

Sketch of proof. Although Proposition 4.3.4 is a standard exercise in complex analysis, we want to give some remarks on the proof to clarify the sign conventions we are using. For part (i), we use the 1-forms ω_σ and $\omega_{\bar\sigma}$ of (4.20) to express

$$dy = \tfrac{1}{2i\sigma_2}(\omega_\sigma - \omega_{\bar\sigma}), \quad dx = \tfrac{1}{2i\sigma_2}(\sigma\omega_{\bar\sigma} - \bar\sigma\omega_\sigma).$$

Since $\omega_\sigma = \psi_\sigma^*(dz)$ and $\omega_{\bar\sigma} = \psi_\sigma^*(d\bar z)$ we deduce that

$$(\psi_\sigma^{-1})^* a_\nu = -\tfrac{\pi}{\sigma_2}\big(\nu_1(\sigma d\bar z - \bar\sigma dz) + \nu_2(dz - d\bar z)\big) = -\bar w_\nu dz + w_\nu d\bar z,$$

with $w_\nu = \frac{\pi}{\sigma_2}(\nu_2 - \sigma\nu_1)$ as claimed. For part (ii) and (iii) we note that

$$\partial_a = \partial - \mathrm{e}(\bar w_\nu dz), \quad \bar\partial_a = \bar\partial + \mathrm{e}(w_\nu d\bar z),$$

and, since we are using the complex linear chirality operator,

$$\partial_a^t = -\tau \circ \bar\partial_a \circ \tau, \quad \bar\partial_a^t = -\tau \circ \partial_a \circ \tau.$$

Then one computes that for every $\varphi \in C^\infty(\mathbb{C})$,

$$\partial_a^t \partial_a \varphi = -\tau \bar\partial_a \tau\Big(\frac{\partial\varphi}{\partial z} - \bar w_\nu \varphi\Big)dz = -\tau\Big(\frac{\partial^2 \varphi}{\partial z \partial \bar z} - \bar w_\nu \frac{\partial\varphi}{\partial \bar z} + w_\nu \frac{\partial \varphi}{\partial z} - |w_\nu|^2\varphi\Big) d\bar z \wedge dz,$$

where we have used that $\tau dz = dz$. Since $\tau(d\bar z \wedge dz) = 2\sigma_2$, this yields the claimed formula for $\partial_a^t\partial_a\varphi$. In a similar way one computes $\bar\partial_a^t\bar\partial_a\varphi$. Then part (ii) follows since

$$\Delta_a\varphi = (\partial_a + \bar\partial_a)^t(\partial_a + \bar\partial_a)\varphi = \partial_a^t\partial_a\varphi + \bar\partial_a^t\bar\partial_a\varphi.$$

Using the same ideas one easily verifies part (iii). □

Using Proposition 4.3.4 it is straightforward to determine the spectrum of Δ_a.

Proposition 4.3.5. *Let $\sigma \in \mathbb{H}$. For $n = (n_1, n_2) \in \mathbb{Z}^2$ define*

$$\varphi_n := e^{\bar w_n z - w_n \bar z} : \mathbb{C} \to \mathrm{U}(1), \quad w_n = \tfrac{\pi}{\sigma_2}(n_2 - \sigma n_1).$$

Then φ_n is $\Lambda(\sigma)$-invariant, and $\{\varphi_n \,|\, n \in \mathbb{Z}^2\}$ is a orthonormal basis for $L^2\big(\mathbb{C}/\Lambda(\sigma)\big)$. If $a = -\bar{w}_\nu dz + w_\nu d\bar{z}$ is a flat U(1)*-connection as before, then*

$$\Delta_a \varphi_n = \lambda_{n,\nu}\varphi_n, \quad \textit{where} \quad \lambda_{n,\nu} = 4\sigma_2 |w_{n-\nu}|^2 \varphi_n.$$

Moreover,

$$\Big\{ \frac{\varphi_n dz}{\sqrt{2\sigma_2}}, \frac{\varphi_n d\bar{z}}{\sqrt{2\sigma_2}} \;\Big|\; n \in \mathbb{Z}^2 \Big\}$$

gives an orthonormal basis for the space of 1-forms consisting of eigenforms for Δ_a with respect to the same eigenvalues $\lambda_{n,\nu}$.

Since Proposition 4.3.5 is well known, we shall proceed without further comments on the proof. However, we want to note that we can use Proposition 4.3.5 and the Hodge-de-Rham isomorphism to determine the twisted cohomology groups of T^2. Recall that in the proof of Proposition 2.3.16 we have already done this using topological methods.

Corollary 4.3.6. *Let a_ν be a flat* U(1)*-connection over T^2, and let $\sigma \in \mathbb{H}$ determine the metric g_σ. Then the cohomology of T^2 with values in the line bundle L_{a_ν} is given in terms of harmonic forms by*

$$\mathscr{H}^\bullet(T^2, L_{a_\nu}) = \begin{cases} \mathbb{C} \oplus (\mathbb{C}\omega_\sigma \oplus \mathbb{C}\omega_{\bar{\sigma}}) \oplus \mathbb{C}dx \wedge dy, & \text{if } \nu \in \mathbb{Z}^2, \\ \{0\}, & \text{if } \nu \notin \mathbb{Z}^2, \end{cases}$$

where ω_σ and $\omega_{\bar{\sigma}}$ are as in (4.20)*.*

Mapping Tori with Fiber T^2. It is well known that the mapping class group of T^2 is isomorphic $\mathrm{SL}_2(\mathbb{Z})$, see [53, Sec. 2.9]. Here, the action of an element $M \in \mathrm{SL}_2(\mathbb{Z})$ on T^2 is the one induced by matrix multiplication

$$\mathbb{R}^2 \to \mathbb{R}^2, \quad \big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big) \mapsto M\big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big).$$

As every $M \in \mathrm{SL}_2(\mathbb{Z})$ has determinant 1, it preserves the volume form $dx \wedge dy$ and we do not have to invoke Proposition 4.1.1. On the other hand, $\mathrm{SL}_2(\mathbb{Z})$ acts on the Riemann sphere $\widehat{\mathbb{C}}$ by fractional linear transformations, and this restricts to an action on $\mathbb{H}$,

$$M : \mathbb{H} \to \mathbb{H}, \quad M\sigma := \frac{a\sigma + b}{c\sigma + d}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{4.23}$$

see for example [93, p. 6]. Unfortunately, the isometry (4.19) does not behave equivariantly with respect to these two $\mathrm{SL}_2(\mathbb{Z})$-actions. We can remedy this, using the following involution on $\mathrm{SL}_2(\mathbb{Z})$,

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto M^{\mathrm{op}} := \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Note that $(M_1M_2)^{\mathrm{op}} = M_2^{\mathrm{op}} M_1^{\mathrm{op}}$, so that we can use the involution to turn a left action of $\mathrm{SL}_2(\mathbb{Z})$ into a right action.

Lemma 4.3.7. *Let $\Phi : \mathbb{H} \to \mathscr{J}_\omega$ be the map of Lemma 4.3.1. Then for $M \in \mathrm{SL}_2(\mathbb{Z})$*

$$\Phi(M\sigma) = (M^{\mathrm{op}})^{-1}\Phi(\sigma)M^{\mathrm{op}}.$$

Sketch of proof. The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elements

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S^2 = (ST)^3, \quad S^4 = \mathrm{Id},$$

see [93, pp. 16–17]. As fractional linear transformations they act as

$$S(\sigma) = -\frac{1}{\sigma}, \quad T(\sigma) = \sigma + 1.$$

One then computes that for $\sigma = \sigma_1 + i\sigma_2 \in \mathbb{H}$,

$$\begin{aligned} \Phi(S\sigma) = \Phi\Big(-\frac{\bar{\sigma}}{|\sigma|^2}\Big) &= \frac{1}{\sigma_2}\begin{pmatrix} \sigma_1 & -1 \\ |\sigma|^2 & -\sigma_1 \end{pmatrix} \\ &= \frac{1}{\sigma_2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\sigma_1 & -|\sigma|^2 \\ 1 & \sigma_1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S^{-1}\Phi(\sigma)S, \end{aligned}$$

and

$$\Phi(T\sigma) = \Phi(\sigma + 1) = \frac{1}{\sigma_2}\begin{pmatrix} -\sigma_1 - 1 & -|\sigma_1 + 1|^2 \\ 1 & \sigma_1 + 1 \end{pmatrix} = \ldots = T^{-1}\Phi(\sigma)T.$$

Now one has to verify that the formula of Lemma 4.3.7 holds for all words in S and T. This is almost tautologically true. For example, if we consider $M = ST$, then

$$\Phi(M\sigma) = S^{-1}\Phi(T\sigma)S = (TS)^{-1}\Phi(\sigma)TS = (M^{\mathrm{op}})^{-1}\Phi(\sigma)M^{\mathrm{op}}. \qquad \square$$

The above lemma suggests that using the involution $M \mapsto M^{\mathrm{op}}$ we should either redefine the action of $\mathrm{SL}_2(\mathbb{Z})$ as the mapping class group or turn the natural left action (4.23) of $\mathrm{SL}_2(\mathbb{Z})$ into a right action. We opt for the latter, although this leads to an unfortunate difference in notation compared to the literature. However, redefining the action of $\mathrm{SL}_2(\mathbb{Z})$ as the mapping class group seems more unnatural.

Definition 4.3.8. Let $M \in SL_2(\mathbb{Z})$, and let $\sigma(t) : \mathbb{R} \to \mathbb{H}$ be M-invariant in the sense that $M^{\mathrm{op}}\sigma(t) = \sigma(t+1)$. Then we define (T^2_M, g_σ) to be the mapping torus

$$\big(T^2 \times \mathbb{R}\big)/\sim, \quad \big(M\big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big), t\big) \sim \big(\big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big), t+1\big), \quad \big(\big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big), t\big) \in T^2 \times \mathbb{R},$$

endowed with the metric induced by

$$g_\sigma := dt \otimes dt + g_{\sigma(t)}.$$

Here, for each $t \in \mathbb{R}$ the metric $g_{\sigma(t)}$ on T^2 is defined as in Lemma 4.3.1.

Flat U(1)-connections over $\boldsymbol{T^2_M}$. We now give an explicit description of flat $\mathrm{U}(1)$-connections over T^2_M up to gauge equivalence.

Proposition 4.3.9. *Let $M \in SL_2(\mathbb{Z})$. Every flat $\mathrm{U}(1)$-connection A over the mapping torus T^2_M is equivalent to one induced by a flat connection a_ν over T^2 as in Lemma 4.3.3 and a gauge transformation $u \in C^\infty\big(T^2, \mathrm{U}(1)\big)$ satisfying*

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} := (\mathrm{Id} - M^t)\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \in \mathbb{Z}^2, \tag{4.24}$$

and

$$u = \exp\big[-2\pi i\big(m_1 x + m_2 y + \lambda\big)\big]$$

for some $\lambda \in [0, 1)$.

Proof. Let $a_\nu = -2\pi i(\nu_1 dx + \nu_2 dy)$ be a connection over T^2 as in Lemma 4.3.3. Since M acts by matrix multiplication on $\mathbb{R}^2$, the pullback of a_ν by M is given by

$$M^* a_\nu = -2\pi i(\mu_1 dx + \mu_2 dy), \quad \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = M^t \begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}.$$

Now the condition $\widehat{M}_u^* a_\nu = a_\nu$ of (4.8) means that connection a_ν is the restriction of a connection over T^2_M if and only if there exists a $\mathbb{Z}^2$-invariant function $u : \mathbb{R}^2 \to \mathrm{U}(1)$ such that

$$\begin{pmatrix}m_1\\ m_2\end{pmatrix} = \big(\mathrm{Id} - M^t\big)\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \frac{i}{2\pi}\begin{pmatrix}u^{-1}\partial_x u\\ u^{-1}\partial_y u\end{pmatrix}. \tag{4.25}$$

Clearly, a function $u : \mathbb{R}^2 \to \mathrm{U}(1)$ satisfying (4.25) is necessarily of the form

$$u = \exp\big[-2\pi i\big(m_1 x + m_2 y + \lambda\big)\big], \quad \lambda \in [0,1),$$

and this is $\mathbb{Z}^2$-invariant precisely if $(m_1, m_2) \in \mathbb{Z}^2$. □

Remark 4.3.10.

(i) Note that the gauge transformation u is—up to the number $\lambda \in [0,1)$—determined by (4.24). For simplicity we will sometimes consider only the case $\lambda = 0$ and neglect the gauge transformation from the notation.

(ii) In Remark B.2.10 (ii) we have pointed out that the bundle $L \to T^2_M$ on which the flat connection A is defined is not necessarily trivializable. In the case at hand this topological data is encoded in (4.24). One verifies—using for example Proposition B.2.9—that L is trivializable if and only if

$$\begin{pmatrix}m_1\\ m_2\end{pmatrix} \in \mathrm{im}\,\big(\mathrm{Id} - M^t : \mathbb{Z}^2 \to \mathbb{Z}^2\big).$$

For example, if $M = \left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right)$, then $\mathrm{Id} - M^t = \left(\begin{smallmatrix}0 & 0\\ 2 & 0\end{smallmatrix}\right)$ and so

$$\begin{pmatrix}0\\ 1\end{pmatrix} \notin \mathrm{im}\,\big(\mathrm{Id} - M^t : \mathbb{Z}^2 \to \mathbb{Z}^2\big).$$

This implies that in this case, the gauge transformation $u = \exp(-2\pi i y)$ defines a flat bundle $L \to T^2_M$ which is not trivializable.

4.3.2 The Bismut-Cheeger Eta Form

We now want to use Proposition 4.1.5 to express the Eta form in terms of the data introduced in the last paragraphs. First of all, we need to understand the vertical chirality operator and its variation.

Let $\sigma(t) = \sigma_1(t) + i\sigma_2(t)$ be an M-invariant path in $\mathbb{H}$ in the sense of Definition 4.3.8, and let $\omega_{\sigma(t)}$ and $\omega_{\bar\sigma(t)}$ be the associated path of 1-forms as defined in (4.20). Differentiating with respect to t easily yields that

$$\dot\omega_{\sigma(t)} = \tfrac{\dot\sigma(t)}{2i\sigma_2(t)}\big(\omega_{\sigma(t)} - \omega_{\bar\sigma(t)}\big) \quad \text{and} \quad \dot\omega_{\bar\sigma(t)} = \tfrac{\dot{\bar\sigma}(t)}{2i\sigma_2(t)}\big(\omega_{\sigma(t)} - \omega_{\bar\sigma(t)}\big).$$

Let τ_t be the associated path of chirality operators on $\Omega^\bullet(T^2)$. Since the volume form on T^2 does not vary with t the action of τ_t on $\Omega^0(T^2)$ and $\Omega^2(T^2)$ is independent of t. On $\Omega^1(T^2)$ it is determined by $\tau_t\omega_{\sigma(t)} = \omega_{\sigma(t)}$ and $\tau_t\omega_{\bar\sigma(t)} = -\omega_{\bar\sigma(t)}$, see (4.21). This readily implies that the derivative of τ_t with respect to t is given by

$$\dot\tau_t|_{\Omega^0(T^2)} = 0, \quad \dot\tau_t|_{\Omega^2(T^2)} = 0, \quad \dot\tau_t\omega_{\sigma(t)} = \tfrac{i\dot\sigma(t)}{\sigma_2(t)}\omega_{\bar\sigma(t)}, \quad \dot\tau_t\omega_{\bar\sigma(t)} = \tfrac{i\dot{\bar\sigma}(t)}{\sigma_2(t)}\omega_{\sigma(t)}. \tag{4.26}$$

Proposition 4.3.11. *Let a_ν be a flat connection over T^2 as in Proposition 4.3.9. Denote by A be the associated flat connection on the line bundle $L \to T^2_M$, and let $\sigma(t)$ be an M-invariant path in $\mathbb{H}$. Then the Bismut-Cheeger Eta form is given by*

$$\widehat{\eta}_A = \frac{1}{2\pi}\operatorname{Re}\Big(\dot\sigma(t)\int_0^\infty F_\nu\big(\sigma(t),u\big)du\Big)dt,$$

where for every $\sigma = \sigma_1 + i\sigma_2 \in \mathbb{H}$ and $u \in \mathbb{R}^+$

$$F_\nu(\sigma,u) = \sum_{n\in\mathbb{Z}^2}\frac{\pi^2}{\sigma_2^2}\big(n_2-\nu_2-\bar\sigma(n_1-\nu_1)\big)^2 e^{-u\frac{\pi^2}{\sigma_2}|n_2-\nu_2-\sigma(n_1-\nu_1)|^2}. \tag{4.27}$$

The sum in (4.27) *converges absolutely and there are estimates, locally uniform in σ,*

$$|F_\nu(\sigma,u)| \le Ce^{-cu} \quad \text{as } u\to\infty, \qquad |F_\nu(\sigma,u)| \le Ce^{-\frac{c}{u}} \quad \text{as } u \to 0.$$

Proof. According to Proposition 4.1.5, the Eta form associated to the connection A and the path $\sigma(t)$ is given by

$$\widehat{\eta}_A = \frac{i}{16\pi}dt\wedge\int_0^\infty \operatorname{Tr}_v\Big(\dot\tau_v\big(d^t_{A,v}d_{A,v} - d_{A,v}d^t_{A,v}\big)e^{-\frac{u}{4}D^2_{A,v}}\Big)du.$$

It follows from Lemma 4.1.4 that under the identification

$$\Omega^\bullet_v(T^2_M, L) = \big\{\alpha_t : \mathbb{R}\to\Omega^\bullet(T^2) \;\big|\; \alpha_{t+1} = \widehat{M}^*\alpha_t\big\},$$

the operator $d_{A,v}$ coincides with d_{a_ν} applied pointwise for each t. The same is true for $d^t_{A,v}$ and $d^t_{a_\nu}$, where the transpose has to be taken pointwise for each t with respect to the metric induced by $\sigma(t)$. Now, the operators $d_{a_\nu}d_{a_\nu}$, $d_{a_\nu}d^t_{a_\nu}$ and $e^{-\frac{u}{4}D^2_{a_\nu}}$ all preserve the decomposition

$$\Omega^\bullet(T^2) = \Omega^0(T^2)\oplus\Omega^1(T^2)\oplus\Omega^2(T^2).$$

Moreover, we know from (4.26) that the operator $\dot\tau_v$ acts trivially on $\Omega^0\oplus\Omega^2$. Therefore,

$$\operatorname{Tr}_v\Big(\dot\tau_v\big(d^t_{A,v}d_{A,v} - d_{A,v}d^t_{A,v}\big)e^{-\frac{u}{4}D^2_{A,v}}\Big) = \operatorname{Tr}_t\Big(\dot\tau_t\big(d^t_{a_\nu}d_{a_\nu} - d_a d^t_{a_\nu}\big)e^{-\frac{u}{4}D^2_{a_\nu}}|_{\Omega^1(T^2)}\Big),$$

where the subscripts t indicate that we consider the right hand side as a function of t. Instead of working over (T^2, g_σ) we now switch to $\mathbb{C}/\Lambda(\sigma)$ to be able to use Proposition 4.3.4 and Proposition 4.3.5. Using the notation introduced used there, we write

$$a = (\psi_\sigma^{-1})^* a_\nu, \quad w_n = \tfrac{\pi}{\sigma_2}(n_2 - \sigma n_1), \quad \varphi_n = e^{\bar w_n z - w_n\bar z} : \mathbb{C}\to \mathrm{U}(1),$$

where $n = (n_1, n_2) \in \mathbb{Z}^2$. Note that for notational convenience, we have now dropped the reference to the t dependence. Then Proposition 4.3.4 yields that

$$\begin{aligned}\big(d_a^t d_a - d_a d_a^t\big)(\varphi_n dz) &= 2\bar{\partial}_a \partial_a^t(\varphi_n dz) = -4\sigma_2(w_n^2 - 2w_\nu w_n + w_\nu^2)\varphi_n d\bar{z}\\ &= -4\sigma_2(w_{n-\nu})^2\varphi_n d\bar{z}.\end{aligned}$$

Under the isometry $\psi_\sigma : (T^2, \sigma) \to \mathbb{C}/\Lambda(\sigma)$, the pair $(dz, d\bar{z})$ pulls back to $(\omega_\sigma, \omega_{\bar{\sigma}})$. Hence, we deduce from (4.26) that

$$\dot{\tau}\big(d_a^t d_a - d_a d_a^t\big)(\varphi_n dz) = -4i\dot{\bar{\sigma}}(w_{n-\nu})^2\varphi_n dz.$$

Similarly, one finds that

$$\dot{\tau}\big(d_a^t d_a - d_a d_a^t\big)(\varphi_n d\bar{z}) = -4i\dot{\sigma}(\bar{w}_{n-\nu})^2\varphi_n d\bar{z}.$$

The Laplace operators Δ_a and $D_{a_\nu}^2$ coincide via $\psi_\sigma : (T^2, \sigma) \to \mathbb{C}/\Lambda(\sigma)$. Using Proposition 4.3.5 and the fact that $e^{-\frac{u}{4}\Delta_a^2}$ is an operator with smooth kernel, one then concludes that

$$\begin{aligned}\frac{i}{16\pi}\operatorname{Tr}\Big(\dot{\tau}\big(d_a^t d_a - d_a d_a^t\big)e^{-\frac{u}{4}\Delta_a^2}\Big) &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}^2}\operatorname{Re}\big(\dot{\sigma}(\bar{w}_{n-\nu})^2\big)e^{-u\sigma_2|w_{n-\nu}|^2}\\ &= \frac{1}{2\pi}\operatorname{Re}\big(\dot{\sigma}F_\nu(\sigma, u)\big),\end{aligned}$$

where in the last step we have simply used the definition of $F_\nu(\sigma, u)$. Concerning the absolute convergence of $F_\nu(\sigma, u)$ and the estimate as $u \to \infty$ we assume for simplicity that $\nu = 0$. The general case requires only minor changes. Define

$$\begin{aligned}r_\sigma &:= \min\big\{|x_2 - \sigma x_1| \;\big|\; (x_1, x_2) \in \mathbb{R}^2,\ |x_1| + |x_2| = 1\big\},\\ R_\sigma &:= \max\big\{|x_2 - \sigma x_1| \;\big|\; (x_1, x_2) \in \mathbb{R}^2,\ |x_1| + |x_2| = 1\big\}.\end{aligned}$$

Clearly, r_σ and R_σ depend continuously on σ. For $n \in \mathbb{Z}^2$ and some constants c and C, not depending on n and σ, we have

$$|w_n|^2 \geq c r_\sigma |n|^2, \quad |w_n|^2 \leq C R_\sigma |n|^2,$$

so that

$$\big|\bar{w}_n^2 e^{-u\sigma_2|w_n|^2}\big| \leq C|n|^2 e^{-uc|n|^2},$$

where the constants c and C now depend continuously on σ. This implies absolute convergence of the series in (4.27). Concerning the estimate as $u \to \infty$ one now proceeds exactly as in the proof of Lemma 1.2.3 and we will skip the details. However, the estimate for $u \to 0$ cannot be read off in the same way, and the general theory only yields that $F_\nu(\sigma, u)$ is bounded as $u \to 0$, see Theorem 3.2.18. To obtain the required estimate, we proceed as in [17, Thm. 2.15]. Recall that the Poisson summation formula states that for any rapidly decreasing function f on $\mathbb{R}^d$

$$\sum_{n\in\mathbb{Z}^d} f(n) = \sum_{n\in\mathbb{Z}^d}\widehat{f}(n), \quad \widehat{f}(n) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle n, x\rangle}dx, \tag{4.28}$$

see e.g., [64, Sec. 20.1]. We now let $\nu \in \mathbb{R}^2$ be arbitrary again and use the Poisson summation formula to bring $F_\nu(\sigma, u)$ into a different form. Since we will need the formula only to obtain the estimate as $u \to 0$, we give only some intermediate steps

$$\begin{aligned} F_\nu(\sigma, u) &= \sum_{n\in\mathbb{Z}^2} (\bar{w}_{n-\nu})^2 e^{-u\sigma_2|w_{n-\nu}|^2} = \sum_{n\in\mathbb{Z}^2} \int_{\mathbb{R}^2} (\bar{w}_{x-\nu})^2 e^{-u\sigma_2|w_{x-\nu}|^2} e^{-2\pi i\langle x,n\rangle} dx \\ &= \sum_{n\in\mathbb{Z}^2} e^{-2\pi i\langle \nu,n\rangle} \frac{1}{\sigma_2 u^2} \int_{\mathbb{R}^2} (x_1 + ix_2)^2 e^{-\pi|x|^2} e^{-2\pi i\langle x,\xi_n\rangle} dx\Big|_{\xi_n = \frac{1}{\sqrt{\pi u\sigma_2}}\left(\begin{smallmatrix}\sigma_2 n_2 \\ n_1+\sigma_1 n_2\end{smallmatrix}\right)}, \end{aligned}$$

where the last line follows from a suitable substitution. For arbitrary $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ one computes that

$$\int_{\mathbb{R}^2} (x_1 + ix_2)^2 e^{-\pi|x|^2} e^{-2\pi i\langle x,\xi_n\rangle} dx = \big(-i\xi_1 + \xi_2\big)^2 e^{-\pi|\xi|^2}.$$

Moreover, with $\xi = \frac{1}{\sqrt{\pi u\sigma_2}}\left(\begin{smallmatrix}\sigma_2 n_2 \\ n_1+\sigma_1 n_2\end{smallmatrix}\right)$ as above,

$$-i\xi_1 + \xi_2 = \frac{1}{\sqrt{\pi u\sigma_2}}(n_1 + \bar{\sigma} n_2),$$

and thus,

$$F_\nu(\sigma, u) = \frac{1}{\pi\sigma_2^2} \sum_{n\in\mathbb{Z}^2} e^{-2\pi i\langle\nu,n\rangle} (\bar{w}_n^*)^2 u^{-3} e^{-\frac{|w_n^*|^2}{u\sigma_2}}, \quad w_n^* := n_1 + \sigma n_2. \tag{4.29}$$

Since $\bar{w}_n = 0$ for $n = (0, 0)$, we see that the only possible term which might fail to decrease exponentially as $u \to 0$ drops out. Hence, we can proceed again as in Lemma 1.2.3 to deduce that $|F_\nu(\sigma, u)| \le Ce^{-\frac{c}{u}}$ as $u \to 0$, with constants c and C depending continuously on σ. $\square$

We now want to simplify the expression for $\widehat{\eta}_A$ further and get an expression for the integral of the Eta form $\widehat{\eta}_A$ over the base. We note that the function F_ν in (4.27) depends on $\nu \in \mathbb{R}^2$ only modulo $\mathbb{Z}^2$, see also (4.29). Hence, we will often assume in the following that $0 \le \nu_1 < 1$ or even that $\nu \in [0, 1)^2$.

Theorem 4.3.12. *For $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ with $0 \le \nu_1 < 1$ and $\sigma \in \mathbb{H}$ write $z = \nu_1\sigma - \nu_2$.*

(i) *Employing the notation*

$$q_\sigma = e^{2\pi i\sigma}, \quad \textit{and} \quad q_z = e^{2\pi iz},$$

we define

$$E_\nu(\sigma) := \sum_{n_1>0}\sum_{n_2>0} \tfrac{1}{n_2}(q_z + q_z^{-1})^{n_2} q_\sigma^{n_1 n_2}, \quad \textit{if } \nu_1 = 0,$$

and

$$E_\nu(\sigma) := \sum_{n_2>0} \tfrac{1}{n_2} q_z^{n_2} + \sum_{n_1>0}\sum_{n_2>0} \tfrac{1}{n_2}(q_z + q_z^{-1})^{n_2} q_\sigma^{n_1 n_2}, \quad \textit{if } \nu_1 \ne 0.$$

Then the sum defining $E_\nu(\sigma)$ converges absolutely to a function which is holomorphic on $\mathbb{H}$.

(ii) *Let $F_\nu(\sigma, u)$ be as in (4.27). Then*

$$\frac{1}{2\pi}\int_0^\infty F_0(\sigma,u)du = \frac{1}{6} - \frac{1}{2\pi\sigma_2} + \frac{i}{\pi}\frac{\partial}{\partial\sigma}E_0(\sigma),$$

and for $\nu \notin \mathbb{Z}$

$$\frac{1}{2\pi}\int_0^\infty F_\nu(\sigma,u)du = P_2(\nu_1) + \frac{i}{\pi}\frac{\partial}{\partial\sigma}E_\nu(\sigma),$$

where P_2 is the second periodic Bernoulli function, see Definition C.1.1.

Remark 4.3.13.

(i) The function $E_\nu(\sigma)$ is related to the logarithm of a generalized Dedekind Eta function, see (4.61) and Lemma 4.4.18 below. As such it appears in the constant term of the Laurent series at $s = 1$ of certain Eisenstein series. Without going into details about the exact relation, we recall that the determination of this constant term is classically referred to as a *Kronecker limit formula*, see [64, Ch. 20] and [88, Sec. 4]. In the proof of Theorem 4.3.12 below, we mimic a combined proof of the *first* and the *second* Kronecker limit formula as in [64].

(ii) Since the sum defining $E_\nu(\sigma)$ converges absolutely, we can interchange the summation over n_1 and n_2. Since $|q_\sigma^{n_2}| < 1$ for $n_2 > 0$ we find

$$\sum_{n_1>0} q_\sigma^{n_1 n_2} = \frac{q_\sigma^{n_2}}{1-q_\sigma^{n_2}} = \frac{q_\sigma^{n_2/2}}{q_\sigma^{-n_2/2} - q_\sigma^{n_2/2}} = \frac{i}{2}\big(\cot(\pi n_2\sigma) + i\big).$$

Hence, for $\nu_1 = 0$

$$E_\nu(\sigma) = i\sum_{n_2>0} \tfrac{1}{n_2}\cos(2\pi z n_2)\big(\cot(\pi n_2 \sigma) + i\big).$$

In the case that $\nu_1 \neq 0$ one can combine the two sums over n_2. Then

$$E_\nu(\sigma) = i\sum_{n_2\geq 1} \tfrac{1}{n_2}\big(\cos(2\pi n_2 z)\cot(\pi n_2\sigma) + \sin(2\pi n_2 z)\big).$$

Proof of Theorem 4.3.12. We can assume for simplicity that $0 \leq \nu_2 < 1$ as well, since the claims of Theorem 4.3.12 depend on ν_2 only modulo $\mathbb{Z}$. As an auxiliary tool we define

$$G_\nu(\sigma, s) := \frac{1}{2\pi\Gamma(s)}\int_0^\infty u^{s-1}F_\nu(\sigma,u)du. \tag{4.30}$$

The estimates in Proposition 4.3.11 ensure that $G_\nu(\sigma, s)$ is a holomorphic function for all $s \in \mathbb{C}$. Clearly,

$$G_\nu(\sigma, 1) = \frac{1}{2\pi}\int_0^\infty F_\nu(\sigma,u)du.$$

Since the sum over $n = (n_1, n_2) \in \mathbb{Z}^2$ defining $F_\nu(\sigma, u)$ converges absolutely, we can first extract possible terms with $n_1 = \nu_1 = 0$, and then sum up the remaining terms. More precisely, define

$$F_\nu^0(\sigma,u) := \begin{cases} \frac{\pi^2}{\sigma_2^2}\sum_{n_2\in\mathbb{Z}}|n_2-\nu_2|^2 e^{-u\frac{\pi^2}{\sigma_2}|n_2-\nu_2|^2}, & \text{if } \nu_1 = 0,\\ 0, & \text{if } \nu_1 \neq 0,\end{cases}$$

and

$$F_\nu^1(\sigma,u) := \frac{\pi^2}{\sigma_2^2}\sum_{n_1\neq\nu_1}\sum_{n_2\in\mathbb{Z}}\big(n_2-\bar\sigma n_1+\bar z\big)^2 e^{-u\frac{\pi^2}{\sigma_2}|n_2-\sigma n_1+z|^2},$$

where $z=\nu_1\sigma-\nu_2$. Accordingly, we can split $G_\nu(\sigma,s)$ for $\mathrm{Re}(s)$ large enough as

$$\begin{aligned}G_\nu(\sigma,s) &= \frac{1}{2\pi\Gamma(s)}\int_0^\infty u^{s-1}F_\nu^0(\sigma,u)du + \frac{1}{2\pi\Gamma(s)}\int_0^\infty u^{s-1}F_\nu^1(\sigma,u)du\\ &=: G_\nu^0(\sigma,s)+G_\nu^1(\sigma,s).\end{aligned}$$

Now, again for $\mathrm{Re}(s)$ large enough, we can interchange summation and integration, so that for $\nu_1=0$ the substitution $u\mapsto\frac{\pi^2}{\sigma_2}|n_2-\nu_2|^2u$ yields

$$\begin{aligned}G_\nu^0(\sigma,s) &= \frac{\pi}{2\sigma_2^2}\sum_{n_2\in\mathbb{Z}}|n_2-\nu_2|^2\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-u\frac{\pi^2}{\sigma_2}|n_2-\nu_2|^2}du\\ &= \frac{\sigma_2^{s-2}}{2\pi^{2s-1}}\sum_{n_2\in\mathbb{Z}}|n_2-\nu_2|^{2-2s} = \frac{\sigma_2^{s-2}}{2\pi^{2s-1}}\widetilde{\zeta}_{\nu_2}(s-1),\end{aligned}$$

where $\widetilde{\zeta}_{\nu_2}(s-1)$ is the periodic Zeta function in Proposition C.1.2. This implies that $G_\nu^0(\sigma,s)$ admits a meromorphic continuation to the whole s-plane, and

$$G_\nu^0(\sigma,1) = \begin{cases} 0, & \text{if } \nu_2\neq 0,\\ -(2\pi\sigma_2)^{-1}, & \text{if } \nu_1=\nu_2=0.\end{cases} \tag{4.31}$$

To identify $G_\nu^1(\sigma,s)$ we assume again that $\mathrm{Re}(s)$ is large enough, so that we can freely interchange summation and integration. Then, with the substitution $u\mapsto\frac{\pi^2}{\sigma_2}u$, we get

$$G_\nu^1(\sigma,s) = \frac{\sigma_2^{s-2}}{2\pi^{2s-1}}\sum_{n_1\neq\nu_1}\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\sum_{n_2\in\mathbb{Z}}\big(n_2-\bar\sigma n_1+\bar z\big)^2e^{-u|n_2-\sigma n_1+z|^2}du.$$

We now apply the Poisson summation formula (4.28) to the inner sum and compute that

$$\begin{aligned}&\sum_{n_2\in\mathbb{Z}}\big(n_2-\bar\sigma n_1+\bar z\big)^2e^{-u|n_2-\sigma n_1+z|^2}\\ &\quad=\sum_{n_2\in\mathbb{Z}}\int_{\mathbb{R}}\big(x-\bar\sigma n_1+\bar z\big)^2e^{-u|x-\sigma n_1+z|^2}e^{-2\pi i x n_2}dx\\ &\quad=\sum_{n_2\in\mathbb{Z}}e^{-2\pi i\,\mathrm{Re}(\sigma n_1-z)n_2}e^{-u(\mathrm{Im}(\sigma n_1-z))^2}\int_{\mathbb{R}}\big(x+i\,\mathrm{Im}(n_1\sigma-z)\big)^2e^{-ux^2}e^{-2\pi i x n_2}dx,\end{aligned}$$

where we have separated the real and imaginary parts and then made the substitution $x\mapsto x-\mathrm{Re}\big(\sigma n_1-z\big)$. Clearly, the integral in the last expression decays exponentially as $|n_2|\to\infty$. Hence, the sum converges absolutely, and we can rearrange the order of summation again. Write

$$G_\nu^1(\sigma,s) = G_\nu^{10}(\sigma,s)+G_\nu^{11}(\sigma,s), \tag{4.32}$$

where $G_\nu^{10}(\sigma,s)$ is the contribution coming from $n_2=0$, i.e.,

$$G_\nu^{10}(\sigma,s)=\frac{\sigma_2^{s-2}}{2\pi^{2s-1}}\frac{1}{\Gamma(s)}\sum_{n_1\neq\nu_1}\int_0^\infty u^{s-1}e^{-u(\mathrm{Im}(\sigma n_1-z))^2}\int_{\mathbb{R}}\big(x+i\,\mathrm{Im}(n_1\sigma-z)\big)^2e^{-ux^2}dx.$$

Setting $a:=\mathrm{Im}(n_1\sigma-z)$ we have

$$\int_{\mathbb{R}}(x+ia)^2e^{-ux^2}dx=\sqrt{\pi}\big(\tfrac12u^{-3/2}-a^2u^{-1/2}\big).$$

Therefore, standard manipulations involving the Gamma function yield

$$\begin{aligned}\int_0^\infty u^{s-1}e^{-ua^2}\int_{\mathbb{R}}(x+ia)^2e^{-ux^2}dx&=\sqrt{\pi}|a|^{3-2s}\big(\tfrac12\Gamma(s-\tfrac32)-\Gamma(s-\tfrac12)\big)\\&=\sqrt{\pi}|a|^{3-2s}\Gamma(s-\tfrac12)\frac{4-2s}{2s-3}.\end{aligned}$$

Recalling that $a=\mathrm{Im}(n_1\sigma-z)$ we find that for $\mathrm{Re}(s)$ large enough,

$$\begin{aligned}G_\nu^{10}(\sigma,s)&=\sqrt{\pi}\frac{\sigma_2^{s-2}}{2\pi^{2s-1}}\frac{\Gamma(s-\frac12)}{\Gamma(s)}\frac{4-2s}{2s-3}\sum_{n_1\neq\nu_1}\big|\,\mathrm{Im}(n_1\sigma-z)\big|^{3-2s}\\&=\sqrt{\pi}\frac{\sigma_2^{1-s}}{2\pi^{2s-1}}\frac{\Gamma(s-\frac12)}{\Gamma(s)}\frac{4-2s}{2s-3}\tilde{\zeta}_{\nu_1}(2s-3),\end{aligned}$$

where we have used that $\mathrm{Im}(n_1\sigma-z)=\sigma_2(n_1-\nu_1)$. Hence we have found an expression for $G_\nu^{10}(\sigma,s)$ which can be extended to a meromorphic function on the whole s-plane. It follows from Proposition C.1.2 that $s=1$ is not a pole, and that

$$G_\nu^{10}(\sigma,1)=P_2(\nu_1)=\nu_1^2-\nu_1+\tfrac16. \tag{4.33}$$

Now we have to consider the general term $G_\nu^{11}(\sigma,s)$ in (4.32). Write $a_n=|\,\mathrm{Im}(n_1\sigma-z)|$ and $b_n=\pi|n_2|$. Note that $a_n,b_n>0$ if $n_1\neq\nu_1$ and $n_2\neq0$. Then, for $\mathrm{Re}(s)$ large,

$$\begin{aligned}G_\nu^{11}(\sigma,s)=\frac{\sigma_2^{s-2}}{2\pi^{2s-1}}\frac{1}{\Gamma(s)}\sum_{n_1\neq\nu_1}\sum_{n_2\neq0}e^{-2\pi i\,\mathrm{Re}(\sigma n_1-z)n_2}\int_0^\infty u^{s-1}e^{-ua_n^2}\\\int_{\mathbb{R}}\Big(x+\frac{ia_n}{\mathrm{sgn}(n_1-\nu_1)}\Big)^2e^{-ux^2}e^{-2ix\,\mathrm{sgn}(n_2)b_n}dxdu.\end{aligned} \tag{4.34}$$

To compute the integrals in the sum, we replace a_n and b_n by real parameters $a,b>0$. Then

$$\begin{aligned}&\int_{\mathbb{R}}\Big(x+\frac{ia}{\mathrm{sgn}(n_1-\nu_1)}\Big)^2e^{-ux^2}e^{-2ix\,\mathrm{sgn}(n_2)b}dx\\&\qquad=-\Big(\frac{\partial_b}{2\,\mathrm{sgn}(n_2)}+\frac{a}{\mathrm{sgn}(n_1-\nu_1)}\Big)^2\int_{\mathbb{R}}e^{-ux^2}e^{-2ix\,\mathrm{sgn}(n_2)b}dx\\&\qquad=-\Big(\frac{\partial_b}{2\,\mathrm{sgn}(n_2)}+\frac{a}{\mathrm{sgn}(n_1-\nu_1)}\Big)^2\sqrt{\frac{\pi}{u}}e^{-b^2/u}.\end{aligned}$$

Therefore, the u-integral in (4.34) in terms of the parameters a and b is given by

$$-\sqrt{\pi}\Big(\frac{\partial_b}{2\,\mathrm{sgn}(n_2)}+\frac{a}{\mathrm{sgn}(n_1-\nu_1)}\Big)^2K_{s-\frac12}(a,b), \tag{4.35}$$

where $K_s(a,b)$ is the *Bessel K-function* [64, Sec. 20.3]

$$K_s(a,b) = \int_0^\infty u^{s-1} e^{-(a^2u+b^2/u)} du.$$

Moreover, for fixed s, one has

$$\partial_b K_s(a,b) = -2bK_{s-1}(a,b),$$

so that (4.35) is actually a sum of Bessel K-functions for different s-parameters. We also collect from [64, Sec. 20.3] that $K_s(a,b)$ is holomorphic on the whole s-plane and satisfies estimates, locally uniform in s, of the form

$$|K_s(a,b)| \le C\Big(\frac{b}{a}\Big)^s e^{-2ab}, \quad ab \to \infty.$$

This implies that the summand in (4.34) decays exponentially as $|(n_1,n_2)| \to \infty$, locally uniform in s. From this one deduces that $G_\nu^{11}(\sigma,s)$ can be extended holomorphically to the whole s-plane, and that we can simply put $s=1$ to find the value we are interested in. Now, $K_{\frac{1}{2}}(a,b) = \frac{\sqrt{\pi}}{a} e^{-2ab}$, see [64, p. 271]. Using this, one verifies without effort that the value of (4.35) at $s=1$ is equal to

$$-2\pi a\big(1 - \operatorname{sgn}(n_2(n_1-\nu_1))\big)e^{-2ab}.$$

Using this one finds that

$$G_\nu^{11}(\sigma,1) = -\sum_{n_1\neq\nu_1}\sum_{n_2\neq 0} |n_1-\nu_1|\big(1-\operatorname{sgn}(n_2(n_1-\nu_1))\big)e^{-2\pi|n_2||\operatorname{Im}(\sigma n_1 - z)|}e^{-2\pi i \operatorname{Re}(\sigma n_1 - z)n_2}.$$

Since $n_2 \neq 0$ and $n_1 \neq \nu_1$, the above sum converges absolutely. Moreover, all the terms with $\operatorname{sign}(n_2) = \operatorname{sign}(n_1-\nu_1)$ drop out. Using the notation $q_\sigma = e^{2\pi i\sigma}$ and $q_z = e^{2\pi i z}$, one obtains

$$G_\nu^{11}(\sigma,1) = -2\sum_{n_2>0}\nu_1 q_z^{n_2} - 2\sum_{n_1>0}\sum_{n_2>0}\Big[(n_1+\nu_1)(q_z q_\sigma^{n_1})^{n_2} + (n_1-\nu_1)(q_z^{-1}q_\sigma^{n_1})^{n_2}\Big]. \quad (4.36)$$

Note that for $\nu_1 = 0$ the first term is equal to zero. Now, q_z and q_σ are holomorphic as functions of σ, and

$$\nu_1 q_z^{n_2} = \frac{1}{2\pi i n_2}\frac{\partial}{\partial\sigma}q_z^{n_2}, \quad (n_1 \pm \nu_1)(q_z q_\sigma^{n_1})^{n_2} = \frac{1}{2\pi i n_2}\frac{\partial}{\partial\sigma}(q_z^{\pm 1}q_\sigma^{n_1})^{n_2},$$

In the case that $\nu_1 \neq 0$ we have $z \in \mathbb{H}$. This yields that $\frac{1}{n_2}q_z^{n_2}$ decays exponentially as $n_2 \to \infty$. For arbitrary ν_1 and $n_1 > 0$ the term $\frac{1}{n_2}(q_z^{\pm 1}q_\sigma^{n_1})^{n_2}$ decays exponentially in both, n_1 and n_2. Moreover, this decay is certainly locally uniform in σ. This implies that in (4.36) we can interchange summation and differentiation to find that

$$G_\nu^{11}(\sigma,1) = \frac{i}{\pi}\frac{\partial}{\partial\sigma}E_\nu(\sigma), \quad (4.37)$$

where $E_\nu(\sigma)$ is defined as in part (i) of the theorem. Since the sums converge absolutely and locally uniform in σ, we conclude that $E_\nu(\sigma)$ defines a holomorphic function on $\mathbb{H}$, which proves part (i). Moreover, we have split the auxiliary function in (4.30) for $\operatorname{Re}(s)$ large as

$$G_\nu(\sigma,s) = G_\nu^0(\sigma,s) + G_\nu^{10}(\sigma,s) + G_\nu^{11}(\sigma,s).$$

As we have seen, the terms on the right hand side extend to meromorphic function on the s-plane, and thus, the above equality continues to hold for all s.[2] Therefore, we can insert the values at $s = 1$, which we have computed in (4.31), (4.33) and (4.37), and deduce that

$$G_\nu(\sigma, 1) = P_2(\nu_1) + \frac{i}{\pi}\frac{\partial}{\partial\sigma} E_\nu(\sigma) + \begin{cases} 0, & \text{if } \nu_2 \neq 0, \\ -(2\pi\sigma_2)^{-1}, & \text{if } \nu_1 = \nu_2 = 0, \end{cases}$$

which proves part (ii) of Theorem 4.3.12. □

As a consequence of Theorem 4.3.12, we obtain the expression for the integral of Bismut-Cheeger Eta form we were aiming at. We know from Proposition 4.3.11 that

$$\widehat{\eta}_A = \frac{1}{2\pi} \operatorname{Re}\Big(\dot\sigma(t) \int_0^\infty F_\nu\big(\sigma(t), u\big) du\Big) dt$$

Hence, integrating the formula in Theorem 4.3.12 (ii) with respect to t one easily arrives at

Theorem 4.3.14. *Let $M \in \mathrm{SL}_2(\mathbb{Z})$, let $\sigma(t)$ be an M-invariant path in $\mathbb{H}$, and use $\sigma(t)$ to endow the mapping torus T^2_M with a metric.*

(i) *The untwisted Eta form $\widehat{\eta}$ satisfies*

$$\int_0^1 \widehat{\eta} = \frac{1}{\pi} \operatorname{Re}\Big[\pi\sigma P_2(0) + iE_\nu(\sigma)\Big]_{\sigma(0)}^{\sigma(1)} - \frac{1}{2\pi}\int_0^1 \frac{\dot\sigma_1(t)}{\sigma_2(t)} dt,$$

where we use the abbreviation $[f(\sigma)]_{\sigma(0)}^{\sigma(1)} = f(\sigma(1)) - f(\sigma(0))$.

(ii) *Let $\nu \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ with $0 \leq \nu_1 < 1$ satisfy $(\mathrm{Id} - M^t)\nu \in \mathbb{Z}^2$, and let A be the corresponding flat $\mathrm{U}(1)$-connection over the mapping torus T^2_M. Then*

$$\int_0^1 \widehat{\eta}_A = \frac{1}{\pi} \operatorname{Re}\Big[\pi\sigma P_2(\nu_1) + iE_\nu(\sigma)\Big]_{\sigma(0)}^{\sigma(1)}.$$

In particular, the Rho form $\widehat{\rho}_A$ satisfies

$$\int_0^1 \widehat{\rho}_A = \frac{1}{\pi} \operatorname{Re}\Big[\pi\sigma\big(P_2(\nu_1) - \tfrac{1}{6}\big) + i\big(E_\nu(\sigma) - E_0(\sigma)\big)\Big]_{\sigma(0)}^{\sigma(1)} + \frac{1}{2\pi}\int_0^1 \frac{\dot\sigma_1(t)}{\sigma_2(t)} dt.$$

Remark 4.3.15. The forms $\widehat{\eta}_A$ and $\widehat{\rho}_A$ depend only on ν modulo $\mathbb{Z}^2$. Also, by its very definition, $P_2(\nu_1)$ depends on ν_1 only modulo $\mathbb{Z}$. Therefore, it is reasonable—and convenient—to extend the definition of $E_\nu(\sigma)$ to arbitrary $(\nu_1, \nu_2) \in \mathbb{R}^2$ by letting

$$E_{(\nu_1,\nu_2)}(\sigma) := E_{(\nu_1 - [\nu_1], \nu_2)}(\sigma),$$

where $[\nu_1]$ is the largest integer less or equal than ν_1. Then Theorem 4.3.14 (ii) continues to hold without the assumption on ν_1.

[2] Here, we encounter a similar situation as in Remark 3.4.2 (i). The terms $G^0_\nu(\sigma, s)$ and $G^{10}_\nu(\sigma, s)$ are not necessarily holomorphic, whereas $G_\nu(\sigma, s)$ and $G^{11}_\nu(\sigma, s)$ are. This implies that the poles of $G^0_\nu(\sigma, s)$ and $G^{10}_\nu(\sigma, s)$ have to cancel each other out. With some effort one can check this directly. We will not go into further details, since the value we are interested in is $s = 1$, which is no pole for any of the summands.

4.4 Torus Bundles over S^1, Explicit Computations

In this section, we want to give a more explicit formula for the Rho invariants of a mapping torus T^2_M with $M \in \mathrm{SL}_2(\mathbb{Z})$. The result depends considerably on whether M is elliptic, parabolic or hyperbolic—see Definition 4.4.1 below—and we have to treat all three cases separately. Explicit formulæ for the untwisted Eta invariant have been obtained in [3] and [26, App. 3]. Both references make use of adiabatic limits, and much of our treatment parallels their discussion. In [26] the focus is on the hyperbolic case, and the Eta invariant is identified with the value of certain number theoretical L-series, see also [6, 17, 77]. This has its origin in Hirzebruch's work [52], where a topological interpretation of the aforementioned L-series was conjectured. A similar relation can also be found for twisted Eta invariants. However, our aim is to get a simple formula for the Rho invariant, and values of L-series are certainly not easy to compute. Fortunately, Atiyah [3] found a number of very different ways to express the untwisted Eta invariant of T^2_M, and we shall derive a formula for the Rho invariant along those lines.

Rough Classification of Elements in $\mathbf{SL_2}(\mathbb{Z})$. For explicit computations we now have to find M-invariant paths in $\mathbb{H}$. For this we will use that elements in $\mathrm{SL}_2(\mathbb{Z})$ split into three natural classes.

Definition 4.4.1. Let $M \in \mathrm{SL}_2(\mathbb{Z})$, and let $\Delta := (\operatorname{tr} M)^2 - 4$. Then M is called

(i) *elliptic*, if $\Delta < 0$,

(ii) *parabolic*, if $\Delta = 0$, and

(iii) *hyperbolic*, if $\Delta > 0$.

Remark. Recall that according to Lemma B.2.1, the diffeomorphism type of T^2_M depends only on the conjugacy class of M in $\mathrm{SL}_2(\mathbb{Z})$. Moreover, $T^2_{M^{-1}}$ and T^2_M are related by an orientation reversing diffeomorphism. In addition, one verifies that for all $M \in \mathrm{SL}_2(\mathbb{Z})$

$$S^{-1} M^t S = M^{-1}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This implies that in the case at hand, there also exists an orientation reversing diffeomorphism $T^2_M \cong T^2_{M^t}$. Since Rho invariants depends only on the oriented diffeomorphism type of the mapping torus T^2_M, and the relation among Rho invariants for different orientations is determined by Lemma 1.3.6 (ii), we are interested in elements of $\mathrm{SL}_2(\mathbb{Z})$ only up to conjugation, taking inverses and transposes. Note that Δ in Definition 4.4.1 is invariant under these operations so that M, M^{-1} and M^t all belong to the same class.

We first collect some well-known facts, see for example [93, Sec. 1.4].

Proposition 4.4.2. *Let $M \in \mathrm{SL}_2(\mathbb{Z})$.*

(i) *M is parabolic if and only if M is conjugate in $\mathrm{SL}_2(\mathbb{Z})$ to $\pm\left(\begin{smallmatrix} 1 & l \\ 0 & 1 \end{smallmatrix}\right)$ with $l \in \mathbb{Z}$.*

(ii) *M is elliptic if and only if M it is of finite order with $M \neq \pm \mathrm{Id}$. In this case, M is of order 3,4 or 6, and conjugate in* $\mathrm{SL}_2(\mathbb{Z})$ *to an element of the form*

$$\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Sketch of proof. The eigenvalues of M are easily seen to be

$$\kappa = \tfrac{1}{2}\big(\operatorname{tr} M + \sqrt{\Delta}\big), \quad \kappa^{-1} = \tfrac{1}{2}\big(\operatorname{tr} M - \sqrt{\Delta}\big), \tag{4.38}$$

where we fix the complex square root with $\sqrt{-1} = i$. By definition, M is parabolic if and only if $\kappa = \kappa^{-1} = \pm 1$, so that the "if" part of (i) is clear. To prove the "only if" part, write $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and assume that M is parabolic with $M \neq \pm \mathrm{Id}$. We can then assume—modulo conjugation by S—that $c \neq 0$. Replacing M with $-M$ if necessary we can also achieve that $\operatorname{tr} M = a + d = 2$. Define

$$g := \gcd\big(a - d, 2c\big), \quad p := \frac{a-d}{g}, \quad q := \frac{2c}{g}.$$

It follows from $a + d = 2$ and $ad - bc = 1$ that

$$ap + bq = p, \quad cp + dq = q.$$

Moreover, $\gcd(p, q) = 1$ so that we can find $r, s \in \mathbb{Z}$ with $pr - qs = 1$. Then $\left(\begin{smallmatrix} p & s \\ q & r \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and one verifies that

$$\begin{pmatrix} p & s \\ q & r \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & s \\ q & r \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix},$$

for some $l \in \mathbb{Z}$. This proves part (i).

Concerning part (ii), we first note that part (i) implies that the only parabolic elements of finite order are $\pm \mathrm{Id}$. Hence, we can assume that $\kappa \neq \kappa^{-1}$. Then M is conjugate in $\mathrm{GL}_2(\mathbb{C})$ to $\left(\begin{smallmatrix} \kappa & 0 \\ 0 & \kappa^{-1} \end{smallmatrix}\right)$. Hence, M is of finite order if and only if κ is a root of unity. Then $\kappa^{-1} = \bar{\kappa}$, and M is elliptic because

$$|\operatorname{tr} M| = 2|\operatorname{Re}(\kappa)| < 2, \quad \text{since} \quad \kappa \neq \bar{\kappa}.$$

For the reverse direction, we only note that if M is elliptic, then $\operatorname{tr} M \in \{-1, 0, 1\}$ and one easily checks by hand that κ is a root of unity—in fact, $\kappa = e^{i\frac{2\pi}{3}}, e^{i\frac{\pi}{2}}$ or $e^{i\frac{\pi}{3}}$. From this it is not difficult to determine explicitly all conjugacy classes of elliptic elements. We refer to [93, pp. 14–15]. □

We now analyze the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in some more detail. Recall from Definition 4.3.8 that we let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{H}$ by the restriction of the fractional linear transformation

$$M^{\mathrm{op}} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \quad M^{\mathrm{op}} z = \frac{dz + b}{cz + a}.$$

The following results are well-known but to fix notation we sketch the proof.

Proposition 4.4.3. *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ with $M \neq \pm\mathrm{Id}$.*

(i) *If M is parabolic of the form $M = \pm\left(\begin{smallmatrix}1 & l\\ 0 & 1\end{smallmatrix}\right)$ with $l \neq 0$, then M^{op} has no fixed points in $\mathbb{C}$, and horizontal lines $\{\sigma \in \mathbb{H} \mid \mathrm{Im}(\sigma) = \mathrm{const}\}$ are invariant under the action of M^{op}.*

(ii) *If M is elliptic, the fractional linear transformation given by M^{op} has exactly one fixed point in $\mathbb{H}$.*

(iii) *M is hyperbolic, if and only if the fractional transformation given by M^{op} has two distinct fixed points $\alpha, \beta \in \mathbb{R} \subset \mathbb{C}$, and the circle*

$$\left\{\sigma \in \mathbb{H} \;\middle|\; \left|\sigma - \tfrac{\alpha+\beta}{2}\right| = \left|\tfrac{\alpha-\beta}{2}\right|\right\}$$

is invariant under the action of M^{op}.

Sketch of proof. Part (i) is immediate since for all $z \in \mathbb{C}$ we have $M^{\mathrm{op}}z = z + l$. If $M = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ is not parabolic, one easily verfies that the eigenvalues κ and κ^{-1} as in (4.38) cannot be integers. Hence, M is not in diagonal or triangular form, which implies that $b, c \neq 0$. Then the fixed points of M^{op} acting on $\widehat{\mathbb{C}}$ are easily seen to be

$$\alpha = \frac{\kappa - a}{c} \quad \text{and} \quad \beta = \frac{\kappa^{-1} - a}{c}.$$

If M is elliptic, then $\mathrm{Im}(\kappa) > 0$ and $\bar{\alpha} = \beta$. Thus, the unique fixed point of M^{op} as claimed in part (ii) is given by $\alpha \in \mathbb{H}$ if $c > 0$, and by $\beta \in \mathbb{H}$ if $c < 0$. Let us now assume that M is hyperbolic. Then the eigenvalues are real and $\kappa > \kappa^{-1}$. If we also assume for simplicity that $c > 0$, we get $\beta < \alpha$. Then one verifies using elementary linear algebra that for all $\sigma \in \mathbb{H}$

$$\left|\sigma - \frac{\alpha+\beta}{2}\right| = \frac{\alpha-\beta}{2} \quad \Longleftrightarrow \quad \mathrm{Re}\left(\frac{\sigma-\alpha}{\sigma-\beta}\right) = 0. \tag{4.39}$$

On the other hand, $M^{\mathrm{op}}\sigma$ is uniquely defined by the normal form of the fractional linear transformation

$$\frac{M^{\mathrm{op}}\sigma - \alpha}{M^{\mathrm{op}}\sigma - \beta} = \kappa^{-2}\frac{\sigma - \alpha}{\sigma - \beta}. \tag{4.40}$$

Since M is hyperbolic we have $\kappa \in \mathbb{R}$. Hence, one finds from (4.39) and (4.40) that the circle

$$\left\{\sigma \in \mathbb{H} \;\middle|\; \left|\sigma - \tfrac{\alpha+\beta}{2}\right| = \tfrac{\alpha-\beta}{2}\right\}$$

is indeed invariant under the action of M^{op}. This proves part (iii). □

4.4.1 The Elliptic Case

Assume that $M = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is elliptic. Then, according to Proposition 4.4.2, M is of finite order, so that we are in the situation considered in Section 4.2. As in Proposition 4.4.3 there are precisely two fixed points for M^{op} acting on $\mathbb{C}$, one of which lies in $\mathbb{H}$, explicitly given by

$$\sigma = \frac{\kappa - a}{c}, \quad \bar{\sigma} = \frac{\bar{\kappa} - a}{c},$$

where,

$$\kappa = \tfrac{1}{2}\big(\operatorname{tr} M + i\sqrt{4-(\operatorname{tr} M)^2}\big) = e^{2\pi i\theta}, \quad \theta \in (0, \tfrac{1}{2}). \tag{4.41}$$

Actually, one easily checks that $\theta \in \{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}\}$.

Theorem 4.4.4. *Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ be elliptic, and let A be a flat connection over T^2_M defined by a pair (a_ν, u) as in Proposition 4.3.9.*

(i) *If $\nu \notin \mathbb{Z}^2$, and θ is as in (4.41), then*

$$\rho_A(T^2_M) = (2 - 4\theta)\operatorname{sgn}(c).$$

(ii) *If $a_\nu = 0$ is the trivial connection, so that $u \equiv e^{-2\pi i\lambda} \in \mathrm{U}(1)$, then*

$$\rho_A(T^2_M) = \begin{cases} 0, & \text{if } \operatorname{Re}(u) < \operatorname{Re}(\kappa), \\ \operatorname{sgn}(c), & \text{if } \operatorname{Re}(u) = \operatorname{Re}(\kappa), \\ 2\operatorname{sgn}(c), & \text{if } \operatorname{Re}(u) > \operatorname{Re}(\kappa). \end{cases}$$

Proof. If $\nu \notin \mathbb{Z}^2$, it follows from Lemma 4.3.3 (iii) that $H^\bullet(T^2, L_{a_\nu}) = \{0\}$. Hence, Theorem 4.2.4 yields that in this case the only contribution to the Rho invariant comes from the Eta invariant of the trivial connection. More precisely,

$$\rho_A(T^2_M) = -4\operatorname{tr}\log\big[M^*|_{\mathscr{H}^+(T^2)\cap\Omega^1}\big] + 2\operatorname{rk}\big[(M^* - \mathrm{Id})|_{\mathscr{H}^+(T^2)\cap\Omega^1}\big].$$

Let σ respectively $\bar\sigma$ be the fixed point of M^{op} in $\mathbb{H}$ as above, depending on whether $c > 0$ or $c < 0$. Use this to define an M-invariant metric on T^2 as in Lemma 4.3.1. With the notation of (4.20), it follows from Corollary 4.3.6 that if $c > 0$, then

$$\mathscr{H}^+(T^2) \cap \Omega^1 = \mathbb{C}\omega_\sigma, \quad \mathscr{H}^-(T^2) \cap \Omega^1 = \mathbb{C}\omega_{\bar\sigma},$$

and, if $c < 0$, then

$$\mathscr{H}^+(T^2) \cap \Omega^1 = \mathbb{C}\omega_{\bar\sigma}, \quad \mathscr{H}^-(T^2) \cap \Omega^1 = \mathbb{C}\omega_\sigma.$$

Moreover, it is immediate that

$$M^*\omega_\sigma = \kappa \cdot \omega_\sigma, \quad \text{and} \quad M^*\omega_{\bar\sigma} = \bar\kappa \cdot \omega_{\bar\sigma}.$$

This implies that

$$M^*|_{\mathscr{H}^\pm(T^2)\cap\Omega^1} = \kappa, \quad M^*|_{\mathscr{H}^\mp(T^2)\cap\Omega^1} = \bar\kappa, \quad \text{if } \pm c > 0. \tag{4.42}$$

Hence, with θ as in (4.41) and the definition of "tr log" in Theorem 4.2.4, one finds that if $\nu \notin \mathbb{Z}^2$,

$$\rho_A(T^2_M) = \begin{cases} -4\theta + 2, & \text{if } c > 0, \\ -4(1-\theta) + 2, & \text{if } c < 0. \end{cases}$$

This proves part (i) of Theorem 4.4.4. Now assume that a_ν is the trivial connection. Lemma 4.3.3 and Proposition 4.3.9 then imply that we can choose $\nu = 0$ and u to be the constant gauge transformation $e^{-2\pi i\lambda}$, with $\lambda \in [0,1)$. Then part (i) and Theorem 4.2.4 show that the Rho invariant of A is given by

$$\begin{aligned}2\operatorname{tr}\log\left[u^{-1}M^*|_{\mathscr{H}^+(T^2)\cap\Omega^1}\right] - \operatorname{rk}\left[(u^{-1}M^* - \mathrm{Id})|_{\mathscr{H}^+(T^2)\cap\Omega^1}\right]\\ - 2\operatorname{tr}\log\left[u^{-1}M^*|_{\mathscr{H}^-(T^2)\cap\Omega^1}\right] + \operatorname{rk}\left[(u^{-1}M^* - \mathrm{Id})|_{\mathscr{H}^-(T^2)\cap\Omega^1}\right] + (2-4\theta)\operatorname{sgn}(c).\end{aligned}$$

To compute the above quantities, we have to replace M^* in (4.42) with $u^{-1}M^*$. We assume for simplicity that $c > 0$; the other case works analogously. Then

$$u^{-1}M^*|_{\mathscr{H}^+(T^2)\cap\Omega^1} = u^{-1}\kappa, \quad u^{-1}M^*|_{\mathscr{H}^-(T^2)\cap\Omega^1} = u^{-1}\bar\kappa.$$

Now if $\operatorname{Re}(u) < \operatorname{Re}(\kappa)$, then $\lambda \in [0,\theta)$ or $\lambda \in (1-\theta, 1)$. One readily verifies that in this case,

$$\operatorname{rk}\left[(u^{-1}M^* - \mathrm{Id})|_{\mathscr{H}^+(T^2)\cap\Omega^1}\right] = \operatorname{rk}\left[(u^{-1}M^* - \mathrm{Id})|_{\mathscr{H}^-(T^2)\cap\Omega^1}\right],$$

and

$$2\operatorname{tr}\log\left[u^{-1}M^*|_{\mathscr{H}^+(T^2)\cap\Omega^1}\right] - 2\operatorname{tr}\log\left[u^{-1}M^*|_{\mathscr{H}^-(T^2)\cap\Omega^1}\right] = 4\theta - 2.$$

This implies that if $\operatorname{Re}(u) < \operatorname{Re}(\kappa)$, then $\rho_A(T^2_M) = 0$. Similarly, if $\operatorname{Re}(u) > \operatorname{Re}(\kappa)$, one computes that

$$\operatorname{rk}\left[(u^{-1}M^* - \mathrm{Id})|_{\mathscr{H}^+(T^2)\cap\Omega^1}\right] = \operatorname{rk}\left[(u^{-1}M^* - \mathrm{Id})|_{\mathscr{H}^-(T^2)\cap\Omega^1}\right],$$

and

$$2\operatorname{tr}\log\left[u^{-1}M^*|_{\mathscr{H}^+(T^2)\cap\Omega^1}\right] - 2\operatorname{tr}\log\left[u^{-1}M^*|_{\mathscr{H}^-(T^2)\cap\Omega^1}\right] = 4\theta,$$

so that $\rho_A(T^2_M) = 2$. Lastly, if $\operatorname{Re}(u) = \operatorname{Re}(\kappa)$, then either $\lambda = \theta$ or $\lambda = 1-\theta$. In the first case,

$$\operatorname{rk}\left[(u^{-1}M^* - \mathrm{Id})|_{\mathscr{H}^+(T^2)\cap\Omega^1}\right] = 1, \quad \operatorname{rk}\left[(u^{-1}M^* - \mathrm{Id})|_{\mathscr{H}^-(T^2)\cap\Omega^1}\right] = 0,$$

and

$$2\operatorname{tr}\log\left[u^{-1}M^*|_{\mathscr{H}^+(T^2)\cap\Omega^1}\right] = 4\theta, \quad 2\operatorname{tr}\log\left[u^{-1}M^*|_{\mathscr{H}^-(T^2)\cap\Omega^1}\right] = 0.$$

This yields $\rho_A(T^2_M) = 1$. In a similar way one deals with the case $\lambda = 1-\theta$. If $c < 0$, one has to replace κ with $\bar\kappa$ in the above computations, and one easily verifies that the result is the negative of what we computed in the case $c > 0$. □

In part (i) of Theorem 4.4.4 the twisting connection does not contribute. Also recall that through Theorem 4.2.4 we have used Theorem 3.3.16, which expresses the Rho invariant as the difference of adiabatic limits of Eta invariants. Hence, part (i) of Theorem 4.4.4 can be rephrased as

$$\lim_{\varepsilon\to 0}\eta(B^{\mathrm{ev}}_\varepsilon) = (4\theta - 2)\operatorname{sgn}(c).$$

Here, $B^{\mathrm{ev}}_\varepsilon$ is the adiabatic family of untwisted odd signature operators associated to the M-invariant metric on T^2 induced by σ respectively $\bar\sigma$. Now in the case at hand, the family $\eta(B^{\mathrm{ev}}_\varepsilon)$ is independent of ε, see [3, p. 360]. Hence, we arrive at the following

Corollary 4.4.5. *Let $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ be elliptic, and endow T^2_M with the metric induced by an M-invariant metric on T^2. Then, with θ is as in (4.41),*

$$\eta(B^{\mathrm{ev}}) = (4\theta - 2)\operatorname{sgn}(c).$$

To check consistency with previous results, we now use Corollary 4.4.5 to compute the Eta invariant for the examples considered in [3, p. 372], respectively [75, p. 48].

(i) $M = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Then $\kappa = e^{\frac{\pi i}{3}}$, so that $\theta = \frac{1}{6}$. Hence, $\eta(B^{\mathrm{ev}}) = \frac{4}{6} - 2 = -\frac{4}{3}$.

(ii) $M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\kappa = e^{\frac{2\pi i}{3}}$ and $\eta(B^{\mathrm{ev}}) = \frac{4}{3} - 2 = -\frac{2}{3}$.

(iii) $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\kappa = e^{\frac{\pi i}{2}}$, so that $\eta(B^{\mathrm{ev}}) = \frac{4}{4} - 2 = -1$.

Therefore, we obtain the same values as in [3] and [75]. Yet, the underlying abstract ideas we have used are very different from what is considered there.

4.4.2 The Parabolic Case

If M is parabolic, we know from Proposition 4.4.2 that M is conjugate to an element of the form $\pm\left(\begin{smallmatrix} 1 & l \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, we can always choose $\sigma(t) := tl + i$ as an M-invariant path in $\mathbb{H}$. We first compute the integral of the Rho form using Theorem 4.3.14.

Proposition 4.4.6. *Let $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ with $\nu \notin \mathbb{Z}^2$ satisfy $(M^t - \mathrm{Id})\nu \in \mathbb{Z}^2$. Let A be the corresponding flat connection over the mapping torus T^2_M. Then the integral of the Rho form with respect to the metric induced by $\sigma(t) := tl + i$ is given by*

$$\int_0^1 \widehat{\rho}_A = l\big(P_2(\nu_1) - \tfrac{1}{6}\big) + \frac{l}{2\pi},$$

where P_2 is the second periodic Bernoulli function.

Proof. Since both sides of the equation in Proposition 4.4.6 depend on ν only modulo $\mathbb{Z}^2$, we can assume that $\nu \in [0,1)^2$. Since $\sigma(0) = i$ and $\sigma(1) = l + i$, we have

$$\cot(\pi\sigma(1)n) = \cot(\pi\sigma(0)n) \quad \text{for all } n \in \mathbb{N}.$$

This implies—using the notation of Theorem 4.3.12 and Remark 4.3.13 (ii)—that

$$E_0\big(\sigma(1)\big) = E_0\big(\sigma(0)\big).$$

We now claim that also

$$E_\nu\big(\sigma(1)\big) = E_\nu\big(\sigma(0)\big). \tag{4.43}$$

Indeed, if $M = \left(\begin{smallmatrix} 1 & l \\ 0 & 1 \end{smallmatrix}\right)$, the condition $(M^t - \mathrm{Id})\nu \in \mathbb{Z}^2$ guarantees that $l\nu_1 \in \mathbb{Z}$, whereas ν_2 is arbitrary. Thus, with $z(t) = \nu_1\sigma(t) - \nu_2$ as in Theorem 4.3.12, we get

$$z(1) = z(0) + l\nu_1 \in z(0) + \mathbb{Z}.$$

This easily yields (4.43) in the case at hand. If $M = -\left(\begin{smallmatrix} 1 & l \\ 0 & 1 \end{smallmatrix}\right)$, then $(M^t - \mathrm{Id})\nu \in \mathbb{Z}^2$ means that

$$2\nu_1 \in \mathbb{Z}, \quad l\nu_1 + 2\nu_2 \in \mathbb{Z}.$$

Since we are assuming that $\nu \in [0,1)^2$, there are only a few possible values for ν. First of all, if $\nu_1 = 0$, then $z(1) = z(0)$, so that (4.43) holds again. If $\nu_1 = \frac{1}{2}$, then

$$\nu_2 \in \begin{cases} \{0, \frac{1}{2}\}, & \text{if } l \text{ is even,} \\ \{\frac{1}{4}, \frac{3}{4}\}, & \text{if } l \text{ is odd.} \end{cases}$$

Moreover, we have $z(1) = z(0) + \frac{l}{2}$ and so

$$\begin{aligned} \cos\big(2\pi z(1)n\big) &= (-1)^{nl} \cos\big(2\pi z(0)n\big), \\ \sin\big(2\pi z(1)n\big) &= (-1)^{nl} \sin\big(2\pi z(0)n\big). \end{aligned} \tag{4.44}$$

This implies that only summands such nl is odd contribute to $E_\nu\big(\sigma(1)\big) - E_\nu\big(\sigma(0)\big)$. Hence, if l is even we are done. Let us thus assume that l is odd and—for definiteness—that $\nu_2 = \frac{1}{4}$. The case $\nu_2 = \frac{3}{4}$ is analogous. Then, for odd $n \in \mathbb{N}$,

$$\cos\big(2\pi z(0)n\big) = \cos\big(\pi i n - \tfrac{\pi}{2} n\big) = i^{n-1} \sin(\pi i n), \quad \sin\big(2\pi z(0)n\big) = -i^{n-1} \cos(\pi i n),$$

and so

$$\cos\big(2\pi n z(0)\big) \cot(\pi n i) + \sin\big(2\pi n z(0)\big) = 0.$$

This, together with (4.44), implies (4.43) in this last case as well. We can now use (4.43) and Theorem 4.3.14 to deduce that in all cases

$$\int_0^1 \widehat{\rho}_A = l\big(P_2(\nu_1) - \tfrac{1}{6}\big) + \frac{1}{2\pi} \int_0^1 l dt = l\big(P_2(\nu_1) - \tfrac{1}{6}\big) + \frac{l}{2\pi}. \qquad \square$$

To conclude the computation of the Rho invariants of T^2_M for parabolic $M \in \mathrm{SL}_2(\mathbb{Z})$, we still have to determine the Rho invariant of the bundle of vertical cohomology groups $\rho_{\mathscr{H}_{A,v}}(S^1)$ appearing in Theorem 3.3.16. Recall that Dai's correction term vanishes because the base is 1-dimensional, see the proof of Theorem 4.2.4.

Proposition 4.4.7. *For $\varepsilon = \pm 1$ and $l \in \mathbb{Z}$ let T^2_M be the mapping torus of $M = \varepsilon\left(\begin{smallmatrix} 1 & l \\ 0 & 1 \end{smallmatrix}\right)$. Endow T^2_M with the metric given by $\sigma(t) = tl + i$. Then, for all connections A as in Proposition 4.4.6,*

$$\rho_{\mathscr{H}_{A,v}}(S^1) = \begin{cases} 0, & \text{if } l = 0, \\ -\frac{l}{\pi} + \operatorname{sgn}(l), & \text{if } \varepsilon = 1,\, l \neq 0, \\ -\frac{l}{\pi}, & \text{if } \varepsilon = -1. \end{cases}$$

The proof turns out to be somewhat involved, and we sketch the strategy first. We know from Corollary 4.3.6 that the twisted cohomology groups of T^2 vanish except for the trivial connection. Hence, for A as in Proposition 4.4.6 we can argue as in the proof of part (i) of Theorem 4.4.4 that

$$\rho_{\mathscr{H}_{A,v}}(S^1) = -\tfrac{1}{2}\eta\big(D_{S^1} \otimes \nabla^{\mathscr{H}_v}\big), \tag{4.45}$$

where $D_{S^1} \otimes \nabla^{\mathscr{H}_v}$ is as in Definition 3.1.16. However, unlike in the case of elliptic elements, the connection $\nabla^{\mathscr{H}_v}$ on the bundle of vertical cohomology groups is not unitary, so that it is difficult to compute the above Eta invariant directly. The idea of our proof is to study the difference between $D_{S^1} \otimes \nabla^{\mathscr{H}_v}$ and the odd signature operator associated to the unitary connection $\nabla^{\mathscr{H}_v,u}$, see (3.24) and (3.29). More precisely, we will compute $\eta\big(D_{S^1} \otimes \nabla^{\mathscr{H}_v,u}\big)$ and then use the variation formula of Proposition 1.3.14 to obtain $\eta\big(D_{S^1} \otimes \nabla^{\mathscr{H}_v}\big)$. Here, the considerations of Section 3.1.6 will play a role.

Proof of Proposition 4.4.7. We split the bundle of vertical cohomology groups as

$$\mathscr{H}_v^\bullet(T_M^2) = \mathscr{H}_v^0(T_M^2) \oplus \mathscr{H}_v^1(T_M^2) \oplus \mathscr{H}_v^2(T_M^2).$$

It follows from Corollary 4.3.6, that $\mathscr{H}_v^0(T_M^2)$ and $\mathscr{H}_v^2(T_M^2)$ can be trivialized by the constant sections 1 respectively $dx \wedge dy$. With respect to this trivialization, the connection $\nabla^{\mathscr{H}_v}$ is the trivial connection, see (4.10). According to Remark 1.4.8 (iii) the Eta invariant of the untwisted odd signature operator over S^1 vanishes, so that we only have to compute the contribution to $\eta\big(D_{S^1} \otimes \nabla^{\mathscr{H}_v}\big)$ coming from $\mathscr{H}_v^1(T_M^2)$.

Let $\omega_{\sigma(t)}$ and $\omega_{\bar\sigma(t)}$ be as in (4.20) with respect to $\sigma(t) = tl + i$, and define

$$\alpha_t := \omega_{\sigma(t)} + \omega_{\bar\sigma(t)}, \quad \beta_t := \omega_{\sigma(t)} - \omega_{\bar\sigma(t)}.$$

It is immediate from Corollary 4.3.6 that for each t the pair (α_t, β_t) forms an orthogonal basis of $\mathscr{H}^1(T_M^2, g_{\sigma(t)})$. However, it is not necessarily a trivialization of the bundle of vertical cohomology groups. For this note that, with $\varepsilon = \pm 1$ as in the statement of the proposition,

$$M^*\omega_{\sigma(t)} = \varepsilon\omega_{\sigma(t+1)}, \quad M^*\omega_{\bar\sigma(t)} = \varepsilon\omega_{\bar\sigma(t+1)},$$

so that also

$$M^*\alpha_t = \varepsilon\alpha_{t+1}, \quad M^*\beta_t = \varepsilon\beta_{t+1}.$$

Nevertheless, this means that we can write every section of $\mathscr{H}_v^1(T_M^2) \to S^1$ as

$$\varphi_\alpha(t)\alpha_t + \varphi_\beta(t)\beta_t,$$

with functions φ_α and φ_β on $\mathbb{R}$ satisfying the condition

$$\varphi_\alpha(t+1) = \varepsilon\varphi_\alpha(t), \quad \varphi_\beta(t+1) = \varepsilon\varphi_\beta(t). \tag{4.46}$$

Now, note that

$$\partial_t\omega_{\sigma(t)} = \partial_t\omega_{\bar\sigma(t)} = l\,dy = \frac{l}{2i}\big(\omega_{\sigma(t)} - \omega_{\bar\sigma(t)}\big).$$

Using this we see that the flat connection $\nabla^{\mathscr{H}_v}$ on $\mathscr{H}_v^1(T_M^2)$ is given by

$$\nabla^{\mathscr{H}_v}_{\partial_t}\alpha_t = \partial_t\alpha_t = -il\beta_t, \quad \nabla^{\mathscr{H}_v}_{\partial_t}\beta_t = \partial_t\beta_t = 0. \tag{4.47}$$

Moreover, one verifies using (4.21) and (4.26) that

$$\tau_t\alpha_t = \beta_t, \quad \tau_t\beta_t = \alpha_t, \quad \dot\tau_t\alpha_t = il\alpha_t, \quad \dot\tau_t\beta_t = -il\beta_t. \tag{4.48}$$

According to Lemma 4.1.4, this means that the unitary connection $\nabla^{\mathscr{H}_v,u}$ of (3.24) on $\mathscr{H}_v^1(T_M^2)$ is given by

$$\nabla^{\mathscr{H}_v,u}_{\partial_t}\alpha_t = -\frac{il}{2}\beta_t, \quad \nabla^{\mathscr{H}_v,u}_{\partial_t}\beta_t = -\frac{il}{2}\alpha_t. \tag{4.49}$$

With the operators of Definition 3.1.16 and (3.29) we introduce the abbreviations

$$D := D_{S^1} \otimes \nabla^{\mathscr{H}_v}|_{C^\infty(S^1,\mathscr{H}_v^1(T_M^2))}, \quad D^u := D_{S^1} \otimes \nabla^{\mathscr{H}_v,u}|_{C^\infty(S^1,\mathscr{H}_v^1(T_M^2))}.$$

Then, using the splitting of the total chirality operator as in Lemma 2.2.3 one verifies that

$$D = -i\tau_t\nabla^{\mathscr{H}_v}_{\partial_t}, \quad D^u = -i\tau_t\nabla^{\mathscr{H}_v,u}_{\partial_t}.$$

From the considerations at the beginning of this proof we deduce that

$$\eta(D) = \tfrac{1}{2}\eta\big(D_{S^1}\otimes\nabla^{\mathscr{H}_v}\big), \tag{4.50}$$

where the factor enters for the same reason as in Remark 3.1.17 (ii) and Remark 1.4.4 (i). Similarly,

$$\eta(D^u) = \tfrac{1}{2}\eta\big(D_{S^1}\otimes\nabla^{\mathscr{H}_v,u}\big).$$

Now let

$$D_s := D^u + s(D-D^u), \quad s\in[0,1].$$

Then D_s is a family of self-adjoint operators, which is precisely of the form considered in Theorem 3.1.20. Hence, the "local variation" of the Eta invariant vanishes , so that the general variation formula of Proposition 1.3.14 reduces to

$$\eta(D) = \eta(D^u) + \dim(\ker D^u) - \dim(\ker D) + 2\,\mathrm{SF}(D_s)_{s\in[0,1]}. \tag{4.51}$$

To determine the terms appearing in (4.51), we now explicitly compute $\operatorname{spec}(D_s)$. If $\varphi_\alpha\alpha_t + \varphi_\beta\beta_t$ is a section of $\mathscr{H}^1_v(T^2_M)$, then (4.47), (4.48) and (4.49) imply that D_s acts in terms of the coordinate functions $(\varphi_\alpha,\varphi_\beta)$ as

$$D_s\begin{pmatrix}\varphi_\alpha\\ \varphi_\beta\end{pmatrix} = -i\left[\begin{pmatrix}0 & \partial_t\\ \partial_t & 0\end{pmatrix} - \frac{il}{2}\begin{pmatrix}1+s & 0\\ 0 & 1-s\end{pmatrix}\right]\begin{pmatrix}\varphi_\alpha\\ \varphi_\beta\end{pmatrix}.$$

Hence, if $\varphi_\alpha\alpha_t + \varphi_\beta\beta_t$ is an eigenvector with eigenvalue $\lambda(s)\in\mathbb{R}$, then

$$\partial_t\begin{pmatrix}\varphi_\alpha\\ \varphi_\beta\end{pmatrix} = i\begin{pmatrix}0 & \lambda(s)+(1-s)\frac{l}{2}\\ \lambda(s)+(1+s)\frac{l}{2} & 0\end{pmatrix}\begin{pmatrix}\varphi_\alpha\\ \varphi_\beta\end{pmatrix} =: iT_{\lambda(s)}\begin{pmatrix}\varphi_\alpha\\ \varphi_\beta\end{pmatrix}, \tag{4.52}$$

which is an ordinary linear differential equation with constant coefficients. Let us assume from now on that $l\neq 0$. The case $l=0$ will be dealt with separately at the end. The characteristic equation for the eigenvalues κ of $T_{\lambda(s)}$ is

$$\kappa^2 = \big(\lambda(s)+\tfrac{l}{2}\big)^2 - s^2\big(\tfrac{l}{2}\big)^2.$$

Therefore, unless $\big(\lambda(s)+\frac{l}{2}\big)^2 = s^2\big(\frac{l}{2}\big)^2$, the matrix $T_{\lambda(s)}$ has two distinct eigenvalues κ and $-\kappa$, and can be brought into diagonal form. Now a solution to (4.52) satisfies the condition (4.46) if and only if $e^{i\kappa}=\varepsilon$. Write $\varepsilon = e^{i\theta}$ with $\theta\in\{0,\pi\}$. Then the condition $e^{i\kappa}=\varepsilon$ is equivalent to $\kappa = 2\pi n+\theta$ with $n\in\mathbb{N}$. Hence, $\lambda(s)$ is an eigenvalue of D_s if and only if

$$\lambda(s) = \lambda_n^\pm(s) := \pm\sqrt{\kappa_n^2 + s^2\big(\tfrac{l}{2}\big)^2} - \tfrac{l}{2}, \quad \kappa_n = 2\pi n+\theta \text{ for some } n\in\mathbb{N}.$$

Moreover, unless $n=0$ and $\theta=0$, the standard procedure for solving (4.52) gives us two linearly independent solutions for both eigenvalues $\lambda_n^+(s)$ and $\lambda_n^-(s)$. In the special case $n=0$ and $\theta=0$ we have

$$\lambda_0^\pm(s) = \tfrac{1}{2}(\pm s|l| - l), \quad \text{and so} \quad T_{\lambda_0^\pm(s)} = \frac{s}{2}\begin{pmatrix}0 & \pm|l|-l\\ \pm|l|+l & 0\end{pmatrix}.$$

If $s = 0$, this matrix vanishes so that the eigenvalues $\lambda_0^\pm(0)$ have multiplicity 2. If $s \neq 0$, this matrix is in triangular form. This implies that the eigenvalues $\lambda_0^\pm(s)$ have multiplicity 1. To see this, consider for example the case $l > 0$, and $T = T_{\lambda_0^+(s)}$. Then

$$e^{itT} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + it \begin{pmatrix} 0 & 0 \\ sl & 0 \end{pmatrix},$$

so that the only solutions of (4.52) satisfying the condition (4.46) are constant multiples of $(\varphi_\alpha, \varphi_\beta) = (0, 1)$.

Since the operator D^u coincides with D_0 we see in particular that

$$\operatorname{spec}(D^u) = \left\{ \pm(2\pi n + \theta) - \tfrac{l}{2} \;\middle|\; n \in \mathbb{N} \right\},$$

where all eigenvalues have multiplicity 2. Hence, for $\operatorname{Re}(z) > 1$

$$\eta(D^u, z) = 2 \sum_{n \in \mathbb{Z}} \frac{\operatorname{sgn}(2\pi n + \theta - \frac{l}{2})}{|2\pi n + \theta - \frac{l}{2}|^z} = 2(2\pi)^z \sum_{n \in \mathbb{Z}} \frac{\operatorname{sgn}(n - \frac{l - 2\theta}{4\pi})}{|n - \frac{l-2\theta}{4\pi}|^z},$$

which up to a factor is the Eta function considered in Proposition C.1.2. Hence, the value at $z = 0$ of the meromorphic continuation of $\eta(D^u, z)$ is given as follows: Let $m \in \mathbb{Z}$ be such that

$$\frac{l}{4\pi} - \frac{\theta}{2\pi} - m \in (0, 1). \quad \text{Then} \quad \eta(D^u) = \frac{l}{\pi} - 2\frac{\theta}{\pi} - 4m - 2. \tag{4.53}$$

This identifies the first term in (4.51).

To compute the spectral flow term, we first assume that $l > 0$. Then the zero eigenvalues of D_s for $s \in (0, 1)$ are given by those $\lambda_n^+(s)$ for which s and n are related by

$$\kappa_n^2 = (1 - s^2)(\tfrac{l}{2})^2.$$

The family $\lambda_n^+(s)$ is strictly increasing with s and all eigenvalues have multiplicity 2. For the latter note that since we are assuming that $s \neq 1$, the eigenvalues of multiplicity 1, which we have found in the case $\theta = 0$, are never zero. Therefore, each zero eigenvalue will contribute $+2$ to $\operatorname{SF}(D_s)_{s \in [0,1]}$. Since $1 - s^2$ maps $(0, 1)$ bijectively onto itself, we have to count the number of $n \in \mathbb{N}$ for which $0 < \kappa_n < \frac{l}{2}$, or—equivalently—for which $-\frac{\theta}{2\pi} < n < \frac{l}{4\pi} - \frac{\theta}{2\pi}$. Now, with m as in (4.53), it is immediate to check that

$$\#\left\{ n \in \mathbb{N} \;\middle|\; -\tfrac{\theta}{2\pi} < n < \tfrac{l}{4\pi} - \tfrac{\theta}{2\pi} \right\} = \begin{cases} m, & \text{if } \theta = 0, \\ m + 1, & \text{if } \theta = \pi. \end{cases}$$

Note that since we are assuming that $l > 0$, we certainly have $m \geq 0$ if $\theta = 0$ and $m \geq -1$ if $\theta = \pi$. Concerning the endpoints of the path, there are no zero eigenvalues for $s = 0$. For $s = 1$ we only have one if $\theta = 0$, and this is the eigenvalue $\lambda_0^+(1)$ of multiplicity 1. Putting all information together, we find that for $l > 0$,

$$\dim(\ker D^u) - \dim(\ker D) + 2 \operatorname{SF}(D_s)_{s \in [0,1]} = \begin{cases} -1 + 2(2m + 1), & \text{if } \theta = 0, \\ 4(m + 1), & \text{if } \theta = \pi. \end{cases}$$

Together with (4.53) and (4.51), we conclude that in the case that $l > 0$,

$$\eta(D) = \begin{cases} \frac{l}{\pi} - 1, & \text{if } \theta = 0, \\ \frac{l}{\pi}, & \text{if } \theta = \pi. \end{cases}$$

Let us now assume that $l < 0$. Then the role of $\lambda_n^+(s)$ in the preceding discussion is replaced by $\lambda_n^-(s)$, which strictly decreases with s. Hence, the contribution to the spectral flow is -2 for each zero eigenvalue. With m as in (4.53) we now have $m \leq -1$ and for both values of θ

$$\#\big\{n \in \mathbb{N} \;\big|\; -\tfrac{\theta}{2\pi} < n < -\tfrac{l}{4\pi} - \tfrac{\theta}{2\pi}\big\} = -m - 1.$$

For $l < 0$ and $\theta = 0$, the zero eigenvalue $\lambda_0^-(1)$ of multiplicity 1 does not contribute to the spectral flow. Therefore, we arrive at

$$\dim(\ker D^u) - \dim(\ker D) + 2\,\mathrm{SF}(D_s)_{s\in[0,1]} = \begin{cases} -1 + 4(m+1), & \text{if } \theta = 0, \\ 4(m+1), & \text{if } \theta = \pi. \end{cases}$$

Hence, we conclude that for $l < 0$

$$\eta(D) = \begin{cases} \frac{l}{\pi} + 1, & \text{if } \theta = 0, \\ \frac{l}{\pi}, & \text{if } \theta = \pi. \end{cases}$$

Hence, using (4.45) and (4.50) we have proved Proposition 4.4.7 in the case that $l \neq 0$. If $l = 0$, one easily checks that

$$\mathrm{spec}(D) = \big\{2\pi n + \theta \;\big|\; n \in \mathbb{Z}\big\},$$

where all eigenvalues have multiplicity 2. This implies that $\mathrm{spec}(D)$ is symmetric, so that $\eta(D) = 0$. $\square$

We can now combine Propositions 4.4.6 and 4.4.7 to obtain the formula for $U(1)$-Rho invariants for mapping tori with parabolic monodromy. According to Theorem 3.3.16, we have

$$\rho_A(T_M^2) = 2\int_0^1 \widehat{\rho}_A + \rho_{\mathscr{H}_{A,v}}(S^1).$$

Therefore, we arrive at the following

Theorem 4.4.8. *Let $\varepsilon = \pm 1$ and $l \in \mathbb{Z}$, and let T_M^2 be the mapping torus of the parabolic element $M = \varepsilon\left(\begin{smallmatrix} 1 & l \\ 0 & 1 \end{smallmatrix}\right)$. Let A be a flat connection over the mapping torus T_M^2, defined by $\nu \in \mathbb{R}^2$ with $\nu \notin \mathbb{Z}^2$, satisfying $(M^t - \mathrm{Id})\nu \in \mathbb{Z}^2$. If $l = 0$, the Rho invariant $\rho_A(T_M^2)$ vanishes. For $l \neq 0$ we have*

$$\rho_A(T_M^2) = 2l\big(P_2(\nu_1) - \tfrac{1}{6}\big) + \begin{cases} \mathrm{sgn}(l), & \text{if } \varepsilon = 1 \\ 0, & \text{if } \varepsilon = -1. \end{cases}$$

Remark 4.4.9.

(i) We want to point out that the assumption that $\nu \notin \mathbb{Z}^2$ excludes possibly non-trivial flat connections on T^2_M which restrict to the trivial connection over T^2. Note that for elliptic elements in Theorem 4.4.4 (ii) we included a discussion. However, in the case of parabolic elements—and also in the hyperbolic case below—the case $\nu \notin \mathbb{Z}^2$ is much more interesting and a parallel treatment of the remaining case would lead to more notational inconvenience and a tedious distinction between all cases. Since the insight gained seemed not worth the effort, we opted to work under the assumption that $\nu \notin \mathbb{Z}^2$ only.

(ii) Note that if $\varepsilon = 1$ in Theorem 4.4.8, then $\nu_1 l \in \mathbb{Z}$, so that $\nu_1 = k/l$ for some $k \in \mathbb{Z}$. Hence, the formula for the Rho invariant is the same as the formula for the Rho invariant for a principal circle bundle of degree l over T^2 in Theorem 2.3.18. The underlying reason is that for $\varepsilon = 1$, the mapping torus T^2_M is at the same time a principal S^1-bundle of degree l over T^2, see [91, p. 470].

4.4.3 The Hyperbolic Case

Now we turn to the generic—and most interesting—case that M is hyperbolic. This section is less self-contained than the previous sections, since we will deduce the main result from a well-known transformation formula for certain generalized Dedekind Eta functions. Since this would lead to far afield, we shall not attempt to give a detailed treatment but refer to the literature for proofs.

***M*-invariant Paths in the upper half plane.** Assume that $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is hyperbolic. As in the proof of Proposition 4.4.3, we know that $b, c \neq 0$, and that the fixed points of M^{op} acting on $\widehat{\mathbb{C}}$ are given by

$$\alpha = \frac{\kappa - a}{c}, \quad \beta = \frac{\kappa^{-1} - a}{c}, \quad \text{where} \quad \kappa = \tfrac{1}{2}\big(a + d + \sqrt{\Delta}\big). \tag{4.54}$$

For $t \in \mathbb{R}$ define

$$\sigma(t) := \frac{1}{|\kappa|^{2t} + |\kappa|^{-2t}}\big(\alpha|\kappa|^{2t} + \beta|\kappa|^{-2t} + i|\alpha - \beta|\big). \tag{4.55}$$

Lemma 4.4.10. *The path $\sigma(t)$ in $\mathbb{H}$ lies on the circle*

$$\Big\{\sigma \in \mathbb{H} \;\Big|\; \big|\sigma - \tfrac{\alpha+\beta}{2}\big| = \big|\tfrac{\alpha-\beta}{2}\big|\Big\}$$

and satisfies $M^{\mathrm{op}}\sigma(t) = \sigma(t+1)$.

Proof. We proof the second assertion first. By comparison with (4.40), we thus have to show that

$$\frac{\sigma(t+1) - \alpha}{\sigma(t+1) - \beta} = \kappa^{-2}\frac{\sigma(t) - \alpha}{\sigma(t) - \beta} \tag{4.56}$$

Now,

$$\begin{aligned}
\sigma(t) - \alpha &= \frac{1}{|\kappa|^{2t} + |\kappa|^{-2t}}\big(\alpha|\kappa|^{2t} + \beta|\kappa|^{-2t} - \alpha|\kappa|^{2t} - \alpha|\kappa|^{-2t} + i|\alpha - \beta|\big) \\
&= \frac{1}{|\kappa|^{2t} + |\kappa|^{-2t}}\big((\beta - \alpha)|\kappa|^{-2t} + i|\alpha - \beta|\big),
\end{aligned}$$

and, similarly,

$$\sigma(t) - \beta = \frac{1}{|\kappa|^{2t} + |\kappa|^{2t}}\big((\alpha-\beta)|\kappa|^{2t} + i|\alpha-\beta|\big).$$

Let $t, t' \in \mathbb{R}$, and abbreviate $\varepsilon := \operatorname{sgn}(\alpha-\beta) = \operatorname{sgn}(c)$. Then

$$\begin{aligned}\frac{\big(\sigma(t')-\alpha\big)\big(\sigma(t)-\beta\big)}{\big(\sigma(t')-\beta\big)\big(\sigma(t)-\alpha\big)} &= \frac{\big(-|\kappa|^{-2t'}+i\varepsilon\big)\big(|\kappa|^{2t}+i\varepsilon\big)}{\big(|\kappa|^{2t'}+i\varepsilon\big)\big(-|\kappa|^{-2t}+i\varepsilon\big)}\\ &= \frac{-|\kappa|^{-2(t'-t)} - 1 + i\varepsilon\big(|\kappa|^{2t} - |\kappa|^{-2t'}\big)}{-|\kappa|^{2(t'-t)} - 1 + i\varepsilon\big(|\kappa|^{2t'} - |\kappa|^{-2t}\big)} = |\kappa|^{-2(t'-t)}.\end{aligned}$$

In particular, for $t' = t+1$ we obtain the formula in (4.56), which in turn shows that $\sigma(t)$ is M-invariant. Moreover,

$$\frac{\sigma(t)-\alpha}{\sigma(t)-\beta} = \kappa^{-2t}\frac{\sigma(0)-\alpha}{\sigma(0)-\beta}.$$

Now, as

$$\sigma(0) = \frac{\alpha+\beta}{2} + i\frac{|\alpha-\beta|}{2},$$

this implies as in the proof of Proposition 4.4.3 that all points $\sigma(t)$ lie on the circle

$$\Big\{\sigma\in\mathbb{H} \;\Big|\; \big|\sigma - \tfrac{\alpha+\beta}{2}\big| = \big|\tfrac{\alpha-\beta}{2}\big|\Big\}. \qquad \square$$

The Rho invariant of the bundle of vertical cohomology groups. Having found an M-invariant path in $\mathbb{H}$ we now need to compute the integral over the Rho form and the Rho invariant of the bundle of vertical cohomology groups. We start with the latter, which is more straightforward than in the parabolic case.

Proposition 4.4.11. *Let T^2_M be the mapping torus of a hyperbolic element $M \in \mathrm{SL}_2(\mathbb{Z})$. Endow T^2_M with the metric given by (4.55), and let A be a flat connection determined by $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ with $\nu \notin \mathbb{Z}^2$ and $(M^t - \mathrm{Id})\nu \in \mathbb{Z}^2$. Then*

$$\rho_{\mathscr{H}_{A,v}}(S^1) = 0.$$

Proof. Again we know from Corollary 4.3.6 that the twisted cohomology groups of T^2 vanish except for the case that the underlying connection is the trivial one. Thus, as in (4.45)

$$\rho_{\mathscr{H}_{A,v}}(S^1) = -\tfrac{1}{2}\eta\big(D_{S^1}\otimes\nabla^{\mathscr{H}_v}\big).$$

Moreover, as explained in the proof of Proposition 4.4.7, we only need to study the restriction of $D_{S^1}\otimes\nabla^{\mathscr{H}_v}$ to $C^\infty\big(S^1, \mathscr{H}^1_v(T^2_M)\big)$.

In view of the rather complicated formula for $\sigma(t)$ it is inconvenient to work directly with the basis $(\omega_{\sigma(t)}, \omega_{\bar\sigma(t)})$ of $\mathscr{H}^1(T^2_M, g_{\sigma(t)})$, given by Corollary 4.3.6. Instead we define

$$\omega_\alpha(t) := \frac{|\kappa|^t - i\varepsilon|\kappa|^{-t}}{2}\,\omega_{\sigma(t)} + \frac{|\kappa|^t + i\varepsilon|\kappa|^{-t}}{2}\,\omega_{\bar\sigma(t)},$$

and

$$\omega_\beta(t) := \frac{|\kappa|^{-t} + i\varepsilon|\kappa|^{t}}{2}\,\omega_{\sigma(t)} + \frac{|\kappa|^{-t} - i\varepsilon|\kappa|^{t}}{2}\,\omega_{\bar\sigma(t)},$$

where as before $\varepsilon = \operatorname{sgn}(c)$. Then $\big(\omega_\alpha(t), \omega_\beta(t)\big)$ is a clearly basis of $\mathscr{H}^1(T^2_M, g_{\sigma(t)})$ for each t. Moreover, (4.21) implies that

$$\tau_t \omega_\alpha(t) = -i\omega_\beta(t), \quad \tau_t \omega_\beta(t) = i\omega_\alpha(t), \tag{4.57}$$

where τ_t is the chirality operator defined by $\sigma(t)$. Now, a straightforward calculation—which we skip—shows that

$$\omega_\alpha(t) = |\kappa|^t (dx + \alpha dy), \quad \omega_\beta(t) = |\kappa|^{-t}(dx + \beta dy). \tag{4.58}$$

Thanks to this identity, we obtain—without having to compute the derivatives of $\omega_{\sigma(t)}$ and $\omega_{\bar{\sigma}(t)}$ explicitly—that

$$\partial_t \omega_\alpha(t) = \log|\kappa| \cdot \omega_\alpha(t), \quad \partial_t \omega_\beta(t) = -\log|\kappa| \cdot \omega_\beta(t). \tag{4.59}$$

From the definition of α and β in (4.54) we immediately see that $c\alpha + a = \kappa$ and $c\beta + a = \kappa^{-1}$. Moreover, using that $ad - bc = 1$ and that $\kappa + \kappa^{-1} = a + d$, one computes

$$d\alpha + b = \frac{d\kappa - ad}{c} + b = \kappa \frac{d - \kappa^{-1}}{c} = \kappa\alpha, \quad d\beta + b = \ldots = \kappa^{-1}\beta.$$

This means that $(1, \alpha)$ and $(1, \beta)$ are eigenvectors of $M^t = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ with eigenvalues κ and κ^{-1}, respectively. Therefore, it follows from (4.58) that

$$M^*\omega_\alpha(t) = |\kappa|^t \big((a + c\alpha)dx + (b + d\alpha)dy\big) = |\kappa|^t \big(\kappa dx + \alpha\kappa dy\big) = \operatorname{sgn}(\kappa)\omega_\alpha(t+1),$$

and similarly,

$$M^*\omega_\beta(t) = \operatorname{sgn}(\kappa)\omega_\beta(t+1).$$

Hence, any section of $\mathscr{H}^1_v(T^2_M) \to S^1$ can be written as

$$\varphi_\alpha(t)\omega_\alpha(t) + \varphi_\beta(t)\omega_\beta(t),$$

where $\varphi_\alpha, \varphi_\beta \in C^\infty(\mathbb{R})$ satisfy

$$\varphi_\alpha(t+1) = \operatorname{sgn}(\kappa)\varphi_\alpha(t), \quad \varphi_\beta(t+1) = \operatorname{sgn}(\kappa)\varphi_\beta(t)$$

We deduce from (4.57) and (4.59) that the operator $D := -i\tau_t\partial_t$ on $C^\infty\big(S^1, \mathscr{H}^1_v(T^2_M)\big)$ acts in terms of the coordinate functions $(\varphi_\alpha, \varphi_\beta)$ as

$$D \begin{pmatrix} \varphi_\alpha \\ \varphi_\beta \end{pmatrix} = \begin{pmatrix} 0 & \partial_t - \log|\kappa| \\ -\partial_t - \log|\kappa| & 0 \end{pmatrix} \begin{pmatrix} \varphi_\alpha \\ \varphi_\beta \end{pmatrix}$$

Hence, it becomes clear that if $(\varphi_\alpha, \varphi_\beta)$ defines an eigenvector of D with eigenvalue λ, then $(\varphi_\alpha, -\varphi_\beta)$ gives rise to an eigenvector with eigenvalue $-\lambda$. This means that $\operatorname{spec}(D)$ is symmetric, so that $\eta(D) = 0$. Since the operator D is precisely the restriction of $D_{S^1} \otimes \nabla^{\mathscr{H}_v}$ to $C^\infty\big(S^1, \mathscr{H}^1_v(T^2_M)\big)$, we obtain the desired result. $\square$

The Logarithm of the Dedekind Eta Function. The discussion of the Rho form is more transparent, if we consider the twisted and the untwisted Eta forms separately. We start with the untwisted case. This case has already received a far-reaching treatment in the beautiful article [3], from which we borrow the main ideas.

Recall that the classical *Dedekind Eta function* is defined as

$$\boldsymbol{\eta}(\sigma) := q_\sigma^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q_\sigma^n), \quad \sigma \in \mathbb{H}, \quad q_\sigma := e^{2\pi i \sigma},$$

see [64, Sec. 18.5]. As in [3] we use the bold symbol $\boldsymbol{\eta}$ to avoid confusion with an Eta function in the sense of Definition 1.3.1. Using the power series expansion

$$\log(1-z) = -\sum_{m=1}^{\infty} \frac{z^m}{m}, \quad |z| < 1,$$

one can define a logarithm of $\boldsymbol{\eta}(\sigma)$ by

$$\log \boldsymbol{\eta}(\sigma) := \frac{\pi i \sigma}{12} - \sum_{n>0} \sum_{m>0} \frac{q_\sigma^{mn}}{n}. \tag{4.60}$$

The sum in (4.60) is of the same form as the one in defining $E_0(\sigma)$ in Theorem 4.3.12 (i). Hence, we can make Remark 4.3.13 (i) more precise and note that

$$\log \boldsymbol{\eta}(\sigma) = \frac{\pi i \sigma}{12} - \frac{E_0(\sigma)}{2} = \frac{1}{2}\big(\pi i \sigma P_2(0) - E_0(\sigma)\big), \tag{4.61}$$

where as always, P_2 is the second periodic Bernoulli function. Therefore, we can reformulate Theorem 4.3.14 (i) as

$$\int_0^1 \widehat{\eta} = \frac{2}{\pi} \operatorname{Im}\big[\log \boldsymbol{\eta}\big(M^{\mathrm{op}}\sigma(0)\big) - \log \boldsymbol{\eta}\big(\sigma(0)\big)\big] - \frac{1}{2\pi} \int_0^1 \frac{\dot{\sigma}_1(t)}{\sigma_2(t)} dt, \tag{4.62}$$

where $\sigma(t) = \sigma_1(t) + i\sigma_2(t)$ is an M-invariant path in $\mathbb{H}$. Hence, the Eta invariant of T^2_M is related to the transformation property of $\log \boldsymbol{\eta}$ under modular transformations.

The study of this has a long history, starting with Dedekind's work [34]. There are several different proofs of the following theorem, see for example [92] for references and a beautifully simple proof. A short discussion of the Dedekind sums appearing below is included in Appendix C.2.

Theorem 4.4.12 (Dedekind)**.** *Let $\sigma \in \mathbb{H}$, and let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$. Then*

$$\log \boldsymbol{\eta}(M^{\mathrm{op}}\sigma) - \log \boldsymbol{\eta}(\sigma) = \frac{1}{2} \log\Big(\frac{c\sigma + a}{\operatorname{sgn}(c) i}\Big) + \pi i \Big(\frac{a+d}{12c} - \operatorname{sgn}(c) s(a,c)\Big),$$

where the logarithm on the right hand side is the standard branch on $\mathbb{C} \setminus \mathbb{R}^-$, and $s(a,c)$ is the classical Dedekind sum, see (C.21)*,*

$$s(a,c) = \sum_{k=1}^{|c|-1} P_1\big(\tfrac{ak}{c}\big) P_1\big(\tfrac{k}{c}\big).$$

Remark. Note that since we have defined the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ using the involution $M \mapsto M^{\mathrm{op}}$ as in Lemma 4.3.7, we have to interchange a and d in the classical formula. However, $s(a,c)$ is not affected by this, see (C.22).

The Untwisted Eta Invariant. From Theorem 4.4.12 we can deduce the formula for the Eta invariant of T^2_M for hyperbolic M. The formula we shall obtain appears as a signature cocycle for the mapping class group the formula already in [72], and as a signature defect in [52]. However, its derivation using Theorem 4.4.12 and the adiabatic limit formula as well as an explanation of the relation among these different invariants are due to Atiyah [3].

Theorem 4.4.13 (Atiyah, Hirzebruch, Meyer)**.** *Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ by hyperbolic. Let g be the metric on T^2_M defined by by $\sigma(t)$ as in (4.55), and let B^{ev} be the associated odd signature operator on T^2_M. Then*

$$\eta(B^{\mathrm{ev}}) = \frac{a+d}{3c} - 4\,\mathrm{sgn}(c)s(a,c) - \mathrm{sgn}\big(c(a+d)\big).$$

Proof. Let g_ε be the adiabatic metric associated to g, and denote by $B^{\mathrm{ev}}_\varepsilon$ the corresponding adiabatic family of odd signature operators. It follows from Proposition 4.4.11 and its proof that the Eta invariant of the bundle of vertical cohomology groups vanishes. We thus deduce from Theorem 3.3.14 that

$$\lim_{\varepsilon\to 0} \eta(B^{\mathrm{ev}}_\varepsilon) = 2\int_0^1 \widehat{\eta} = \frac{4}{\pi}\,\mathrm{Im}\,\big[\log \boldsymbol{\eta}\big(M^{\mathrm{op}}\sigma(0)\big) - \log \boldsymbol{\eta}\big(\sigma(0)\big)\big] - \frac{1}{\pi}\int_0^1 \frac{\dot\sigma_1(t)}{\sigma_2(t)}dt,$$

where we have used (4.62) for the last equality. Hence, Theorem 4.4.12 implies that

$$\lim_{\varepsilon\to 0} \eta(B^{\mathrm{ev}}_\varepsilon) = \frac{a+d}{3c} - 4\,\mathrm{sgn}(c)s(a,c) + \frac{1}{\pi}\Big[2\,\mathrm{Im}\log\Big(\frac{c\sigma(0)+a}{\mathrm{sgn}(c)i}\Big) - \int_0^1 \frac{\dot\sigma_1(t)}{\sigma_2(t)}dt\Big]. \tag{4.63}$$

We now note that it follows from Lemma 4.4.10 that

$$\sigma(t) = \tfrac{\alpha+\beta}{2} + \tfrac{|\alpha-\beta|}{2}e^{i\varphi(t)}, \quad \varphi(t) = \arg\big(\sigma(t) - \tfrac{\alpha+\beta}{2}\big).$$

Here, the argument function is such that for $z \in \mathbb{C}\setminus\mathbb{R}^-$, one has $\arg(z) \in (-\pi,\pi)$. We obtain

$$\int_0^1 \frac{\dot\sigma_1(t)}{\sigma_2(t)}dt = -\arg\big(\sigma(1) - \tfrac{\alpha+\beta}{2}\big) + \arg\big(\sigma(0) - \tfrac{\alpha+\beta}{2}\big).$$

Using the explicit formula (4.55) for $\sigma(t)$, one finds that $\arg\big(\sigma(0) - \tfrac{\alpha+\beta}{2}\big) = \tfrac{\pi}{2}$, and

$$\sigma(1) - \frac{\alpha+\beta}{2} = \frac{1}{2}\frac{(\alpha-\beta)(\kappa^2-\kappa^{-2})}{\kappa^2+\kappa^{-2}} + i\frac{|\alpha-\beta|}{\kappa^2+\kappa^{-2}}.$$

Hence, using the abbreviations

$$x := \tfrac{1}{2}\,\mathrm{sgn}(c)(\kappa+\kappa^{-1}), \quad y := \tfrac{1}{2}(\kappa-\kappa^{-1}), \tag{4.64}$$

we find that

$$\arg\big(\sigma(1) - \tfrac{\alpha+\beta}{2}\big) = \arg\big(2xy + i\big).$$

Note that $y^2 = x^2 - 1$, so that $2xy + i = i(x - iy)^2$. Moreover, $y > 0$ and $y < |x|$ so that

$$\arg(x + iy) \in \begin{cases} \big(0, \frac{\pi}{4}\big), & \text{if } x > 0, \\ \big(\frac{3\pi}{4}, 2\pi\big), & \text{if } x < 0. \end{cases}$$

Thus, we obtain

$$\int_0^1 \frac{\dot\sigma_1(t)}{\sigma_2(t)} dt = \arg\big(-i(x+iy)^2\big) + \tfrac{\pi}{2} = 2\arg(x+iy) - \begin{cases} 0, & \text{if } x > 0, \\ 2\pi, & \text{if } x < 0. \end{cases} \tag{4.65}$$

On the other hand, it follows from the definition of α and β that

$$c\sigma(0) + a = c\big(\tfrac{\alpha+\beta}{2} + i\tfrac{|\alpha-\beta|}{2}\big) + a = \tfrac{1}{2}\big(\kappa + \kappa^{-1} + i\,\mathrm{sgn}(c)(\kappa - \kappa^{-1})\big) = \mathrm{sgn}(c)(x+iy),$$

with x and y as in (4.64). Since we are using the standard branch of the logarithm, we have

$$\operatorname{Im}\log\Big(\frac{c\sigma(0)+a}{\mathrm{sgn}(c)i}\Big) = \arg\big(-i(x+iy)\big) = \arg(x+iy) - \tfrac{\pi}{2}.$$

Combining this with (4.65) we find that

$$2\operatorname{Im}\log\Big(\frac{c\sigma(0)+a}{\mathrm{sgn}(c)i}\Big) - \int_0^1 \frac{\dot\sigma_1(t)}{\sigma_2(t)} dt = -\,\mathrm{sgn}(x)\pi.$$

As $\kappa + \kappa^{-1} = a + d$ we have $\mathrm{sgn}(x) = \mathrm{sgn}\,\big(c(a+d)\big)$ so that using (4.63), we finally arrive at

$$\lim_{\varepsilon\to 0} \eta(B_\varepsilon^{\mathrm{ev}}) = \frac{a+d}{3c} - 4\,\mathrm{sgn}(c)s(a,c) - \mathrm{sgn}\,\big(c(a+d)\big).$$

Hence, it remains to argue that in the case at hand,

$$\eta(B^{\mathrm{ev}}) = \lim_{\varepsilon\to 0} \eta(B_\varepsilon^{\mathrm{ev}}),$$

This is precisely [3, Lem. 5.56] and we will not repeat the argument here. □

Remark. The proof of Theorem 4.4.13 in [3] is along different lines than our discussion. In [3, Thm. 5.60], the Eta invariant of T_M^2 is seen to be equal to a large number of quantities, including a signature defect. Then in [3, Sec. 6], the formula for $\eta(B^{\mathrm{ev}})$ is obtained by explicitly constructing a bounding manifold and a computation of the signature defect. In particular, the transformation formula of the Dedekind Eta function in Theorem 4.4.12 is not used. However, since our focus is the application of the adiabatic limit formula to compute Eta respectively Rho invariants, we have to use Theorem 4.4.12 in some form.

The Generalized Dedekind Eta Function. To obtain the formula for U(1)-Rho invariants of T_M^2 in the spirit of the discussion of the untwisted case, we now need a twisted version of $\boldsymbol{\eta}(\sigma)$ and a transformation formula for its logarithm. Fortunately, a corresponding treatment can be found in [35]. As in [35, p. 38] we make the following

Definition 4.4.14. For $g, h \in \mathbb{R}$ and $\sigma \in \mathbb{H}$ let $z := g\sigma + h$, $q_z = e^{2\pi i z}$ and $q_\sigma = e^{2\pi i \sigma}$. Define

$$\boldsymbol{\eta}_{g,h}(\sigma) := \xi(g,h)\, q_\sigma^{\frac{P_2(g)}{2}} (1 - q_z) \prod_{n=1}^{\infty} (1 - q_z q_\sigma^m)(1 - q_z^{-1} q_\sigma^m),$$

where

$$\xi(g,h) := \begin{cases} e^{2\pi i (g - \frac{1}{2}) P_1(h)}, & \text{if } g \in \mathbb{Z} \\ e^{2\pi i [g] P_1(h)}, & \text{if } g \notin \mathbb{Z}. \end{cases}$$

As always, P_1 is the first periodic Bernoulli function, and $[g]$ is the largest integer less or equal than g.

Remark 4.4.15.

(i) Since $\sigma \in \mathbb{H}$, the term q_σ^m decays exponentially with m. This implies that $\boldsymbol{\eta}_{g,h}(\sigma)$ is well-defined.

(ii) Definition 4.4.14 might look slightly different than the formula in [35]. Yet, writing g and h as $\tilde{g}/f$ and $\tilde{h}/f$ with integers $\tilde{g}$, $\tilde{h}$ and f, the function $\boldsymbol{\eta}_{g,h}(\sigma)$ is easily seen to be equal to what is denoted $\eta_{\tilde{g},\tilde{h}}(\sigma)$ in loc.cit.

(iii) The reason for the factor $\xi(g,h)$ is to achieve that $\boldsymbol{\eta}_{g,h}(\sigma)$ depends on g and h only modulo $\mathbb{Z}$, see [35, p. 39].

(iv) The Dedekind Eta function $\boldsymbol{\eta}(\sigma)$ is not equal to $\boldsymbol{\eta}_{0,0}(\sigma)$, since the latter obviously vanishes. Dropping the factor $1 - q_z$ from the definition of $\boldsymbol{\eta}_{g,h}(\sigma)$ one would get a direct generalization of $\boldsymbol{\eta}(\sigma)^2$. However, Definition 4.4.14 allows us to use the results of [35] without too many changes.

One defines $\log \boldsymbol{\eta}_{g,h}(\sigma)$ in analogy to (4.60), see [35, p. 40].

Definition 4.4.16. Let $(g,h) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$. If $0 \leq g < 1$, we define

$$\log \boldsymbol{\eta}_{g,h}(\sigma) := \pi i \big(\varphi(g,h) + P_2(g)\big) - \sum_{n>0} \tfrac{1}{n} q_z^n - \sum_{m>0} \sum_{n>0} \tfrac{1}{n} (q_z + q_z^{-1})^n q_\sigma^{mn},$$

where

$$\varphi(g,h) = \begin{cases} -P_1(h), & \text{if } g = 0, \\ 0, & \text{if } g \neq 0. \end{cases}$$

For general g we define

$$\log \boldsymbol{\eta}_{g,h}(\sigma) := \log \boldsymbol{\eta}_{g-[g],h}(\sigma).$$

The transformation formula of $\log \boldsymbol{\eta}_{g,h}(\sigma)$ is then given by [35, Thm. 1],

Theorem 4.4.17 (Dieter)**.** *Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$, let $(g,h) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, and define*

$$\begin{pmatrix} g' \\ h' \end{pmatrix} := \begin{pmatrix} a & -c \\ -b & d \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix}.$$

Then for all $\sigma \in \mathbb{H}$

$$\log \boldsymbol{\eta}_{g',h'}(M^{\mathrm{op}}\sigma) - \log \boldsymbol{\eta}_{g,h}(\sigma) = \pi i \Big(\frac{a}{c} P_2(g) + \frac{d}{c} P_2(g') - 2\,\mathrm{sgn}(c) s_{g',h'}(d,c) \Big),$$

where $s_{g',h'}(d,c)$ is the generalized Dedekind sum, see Definition C.2.5,

$$s_{g',h'}(d,c) = \sum_{k=0}^{|c|-1} P_1\big(d\tfrac{k+g'}{c} + h'\big) P_1\big(\tfrac{k+g'}{c}\big).$$

Remark.

(i) As for the transformation formula for $\log \boldsymbol{\eta}(\sigma)$, we have formulated Theorem 4.4.17 in terms of M^{op} acting on $\mathbb{H}$, which means that a and d have been interchanged in comparison to [35, Thm. 1].

(ii) The generalized Dedekind sums appeared first in [70]. A brief discussion of the aspects we need is contained in C.2.

(iii) We want to point out that the proof of Theorem 4.4.17 in [35] is rather involved. The simple proof in [92] of the transformation formula for $\log \boldsymbol{\eta}(\sigma)$ carries over with minor changes in the case that $g \in \mathbb{Z}$. It would be interesting to know if there is a proof for the general case of Theorem 4.4.17 along the lines of [92].

Application to the Rho Form. As in the untwisted case, the structure of the formula in Definition 4.4.16 resembles what we have encountered in Theorem 4.3.14 (ii). In fact,

Lemma 4.4.18. *With the notation of Theorem 4.3.12 and Remark 4.3.15, we have for all $\nu \in \mathbb{Q}^2 \setminus \mathbb{Z}$.*

$$\operatorname{Im}\big(\log \boldsymbol{\eta}_{\nu_1,-\nu_2}(\sigma)\big) = \operatorname{Im}\big(\pi i \sigma P_2(\nu_1) - E_\nu(\sigma)\big).$$

Proof. Both sides of the equation are defined in terms of $\nu_1 - [\nu_1]$ and are $\mathbb{Z}$-periodic in ν_2 Hence, we can assume that $\nu \in [0,1)^2$. Then, if $\nu_1 \neq 0$, the relation is immediate—and clearly holds for the real parts as well. If $\nu_1 = 0$, one observes that

$$\operatorname{Im}\Big(\pi i P_1(\nu_2) - \sum_{n>0} \tfrac{1}{n} e^{-2\pi i \nu_2}\Big) = -\operatorname{Im}\Big(\sum_{n>0} \tfrac{1}{n}\cos(2\pi\nu_2)\Big) = 0, \tag{4.66}$$

where the first equality follows from the Fourier series expansion

$$P_1(\nu_2) = \nu_2 - \tfrac{1}{2} = -\frac{1}{2\pi i}\sum_{n>0} \tfrac{1}{n}\big(e^{2\pi i \nu_2} - e^{-2\pi i \nu_2}\big), \quad \nu_2 \notin \mathbb{Z},$$

whose proof is a standard exercise. Then (4.66) implies that the imaginary parts of the extra terms in Definition 4.4.16 cancel each other out so that the result is indeed the right hand side of the formula in Lemma 4.4.18. □

Proposition 4.4.19. *Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic, and $\nu \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ with $(\mathrm{Id} - M^t)\nu \in \mathbb{Z}^2$. Let A be the corresponding flat* U(1)*-connection over the mapping torus T^2_M, and use* (4.55) *to define a metric. Then*

$$\int_0^1 \widehat{\eta}_A = \tfrac{a+d}{c} P_2(\nu_1) - 2\,\mathrm{sgn}(c) s_{\nu_1,\nu_2}(a,c).$$

Proof. Certainly, we want to use Lemma 4.4.18 to apply Theorem 4.4.17 to part (ii) of Theorem 4.3.14. We first note that $(\mathrm{Id} - M^t)\nu \in \mathbb{Z}^2$ implies that $\nu \in \mathbb{Q}^2$, so that we are precisely in the situation of Lemma 4.4.18. Abbreviate $\nu' := M^t\nu$, so that by assumption $\nu - \nu' \in \mathbb{Z}$. Thus, according to Definition 4.4.16,

$$\log \boldsymbol{\eta}_{\nu_1', -\nu_2'}(\sigma) = \log \boldsymbol{\eta}_{\nu_1, -\nu_2}(\sigma), \quad \text{for all } \sigma \in \mathbb{H}.$$

Moreover, $P_1(\nu_1') = P_1(\nu_1)$, and

$$\begin{pmatrix} \nu_1' \\ -\nu_2' \end{pmatrix} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix} \begin{pmatrix} \nu_1 \\ -\nu_2 \end{pmatrix}.$$

Hence, Lemma 4.4.18, Theorem 4.4.17 and Theorem 4.3.14 (ii) imply that

$$\int_0^1 \widehat{\eta}_A = \tfrac{a+d}{c} P_2(\nu_1) - 2\,\mathrm{sgn}(c) s_{\nu_1', -\nu_2'}(d, c).$$

Now,

$$\begin{aligned} s_{\nu_1', -\nu_2'}(d, c) &= \sum_{k=0}^{|c|-1} P_1\big(d\tfrac{k+\nu_1'}{c} - \nu_2'\big) P_1\big(\tfrac{k+\nu_1'}{c}\big) = \sum_{k=0}^{|c|-1} P_1\big(\tfrac{dk + d\nu_1' - c\nu_2'}{c}\big) P_1\big(\tfrac{k + a\nu_1 + c\nu_2}{c}\big) \\ &= \sum_{k=0}^{|c|-1} P_1\big(\tfrac{k+\nu_1}{c}\big) P_1\big(a\tfrac{k+\nu_1}{c} + \nu_2\big) = s_{\nu_1, \nu_2}(a, c), \end{aligned}$$

where we have rewritten ν' in terms of ν, and then used that $\{ak \,|\, k = 0, \ldots, |c| - 1\}$ is a representation system of $\mathbb{Z}$ modulo c, see Appendix C.2 for more details. This implies the desired result. $\square$

Rho Invariants of Hyperbolic Mapping Tori. After this preparation, we finally arrive at the main result of this section.

Theorem 4.4.20. *Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic, and let $\nu \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ satisfy*

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = (\mathrm{Id} - M^t) \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \in \mathbb{Z}^2.$$

Let A be the corresponding flat $\mathrm{U}(1)$-connection over the mapping torus T^2_M, and define $r \in \{0, \ldots |c| - 1\}$ by requiring that $m_1 \equiv r\,(c)$. Then

$$\begin{aligned} \rho_A(T^2_M) = \tfrac{2(a+d)-4}{c}\big(P_2(\nu_1) - \tfrac{1}{6}\big) - 4 \sum_{k=1}^{|c|-r} P_1\big(\tfrac{dk}{c}\big) + \mathrm{sgn}\,\big(c(a+d)\big) - \mathrm{sgn}(c)\delta(\nu_1)\big(1 - \delta(\tfrac{m_1}{c})\big) \\ - 2P_1\big(\tfrac{dm_1}{c}\big) - 2\delta(\nu_1)\Big(P_1\big(\tfrac{m_1}{c}\big) - P_1\big(\tfrac{dm_1}{c}\big)\Big), \end{aligned}$$

where δ is the characteristic function of $\mathbb{R} \setminus \mathbb{Z}$.

Proof. According to Proposition 4.4.11, the Rho invariant of the bundle of vertical cohomology groups vanishes. Also, since the base is 1-dimensional, Dai's correction term is zero. Hence, we can use the general formula for Rho invariants in Theorem 3.3.16, together with the formulæ of Theorem 4.4.13 and Proposition 4.4.19, to deduce that

$$\rho_A(T^2_M) = \tfrac{2(a+d)}{c}\big(P_2(\nu_1) - \tfrac{1}{6}\big) - 4\,\mathrm{sgn}(c)\big(s_{\nu_1,\nu_2}(a,c) - s(a,c)\big) + \mathrm{sgn}\,\big(c(a+d)\big). \tag{4.67}$$

A formula for the difference of $s_{\nu_1,\nu_2}(a,c)$ and $s(a,c)$ is given in Proposition C.2.7. With $r \in \{0, \dots |c|-1\}$ such that $m_1 \equiv r\ (c)$, we have

$$\begin{aligned} s_{\nu_1,\nu_2}(a,c) - s(a,c) = \tfrac{1}{|c|}\big(P_2(\nu_1) - \tfrac{1}{6}\big) + \sum_{k=1}^{|c|-r} P_1\big(\tfrac{dk}{|c|}\big) + \tfrac{1}{2}P_1\big(\tfrac{dm_1}{|c|}\big) \\ + \tfrac{1}{2}\delta(\nu_1)\Big(P_1\big(\tfrac{m_1}{|c|}\big) - P_1\big(\tfrac{dm_1}{|c|}\big)\Big) + \tfrac{1}{4}\delta(\nu_1)\big(1 - \delta(\tfrac{m_1}{c})\big). \end{aligned}$$

We now insert this into (4.67). Since P_1 is odd, the factor $\mathrm{sgn}(c)$ in front of $s_{\nu_1,\nu_2}(a,c) - s(a,c)$ cancels the norms in the denominators. Then we arrive at the formula of Theorem 4.4.20. ☐

Immediate Applications. The main formula in Theorem 4.4.20 might look more complicated than the intermediate formula (4.67). Yet, it is more satisfactory from a computational point of view, since the sum $\sum_{k=1}^{|c|-r} P_1\big(\frac{dk}{c}\big)$ is much easier to compute than the individual Dedekind sums. For concreteness, let us use Theorem 4.4.20 for some explicit computations.

Example.

(i) Consider

$$M = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{so that} \quad \mathrm{Id} - M^t = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}.$$

Since $\det(\mathrm{Id} - M^t) = 5$, a pair $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ with $m = (\mathrm{Id} - M^t)\nu \in \mathbb{Z}^2$ has to consist of rational numbers with denominator 5. Recall that we exclude the case $\nu \in \mathbb{Z}^2$ and may restrict to $\nu \in [0,1)^2$. One then verifies that to obtain a full set of representatives for the flat connections on T^2_M we are interested in, we need to consider pairs ν and m with

$$\begin{array}{rcllll} \nu & = & (\frac{1}{5}, \frac{3}{5}) & (\frac{2}{5}, \frac{1}{5}) & (\frac{3}{5}, \frac{4}{5}) & (\frac{4}{5}, \frac{2}{5}) \\ m & = & (0,1) & (1,0) & (1,1) & (2,0). \end{array}$$

As $c = 1$ and $\nu_1 \neq 0$, the formula of Theorem 4.4.20 reduces to

$$\rho_A(T^2_M) = 2\big((a+d) - 2\big)(\nu_1^2 - \nu_1) - \mathrm{sgn}\,\big(c(a+d)\big) - \mathrm{sgn}(c) = -10(\nu_1^2 - \nu_1).$$

Hence, one computes

$$\begin{array}{rcllll} \nu_1 & = & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5}, \\ \rho_A(T^2_M) & = & \frac{8}{5} & \frac{12}{5} & \frac{12}{5} & \frac{8}{5}. \end{array}$$

(ii) As a further example, let us consider

$$M = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \quad \text{so that} \quad \mathrm{Id} - M^t = \begin{pmatrix} -2 & -4 \\ -2 & -2 \end{pmatrix}.$$

Now, one easily verifies that we can represent the conjugacy classes of flat connections of interest by

$$\begin{array}{rcccc} \nu &=& (0,\frac{1}{2}) & (\frac{1}{2},0) & (\frac{1}{2},\frac{1}{2}), \\ m &=& (-2,-1) & (-1,-1) & (-3,-2), \\ r &=& 2 & 3 & 1. \end{array}$$

Then

$$\rho_A(T_M^2) = 2(\nu_1^2 - \nu_1) - 4\sum_{k=1}^{4-r} P_1\big(\tfrac{3k}{4}\big) + 1 - 2P_1\big(\tfrac{3r}{4}\big) - 2\delta(\nu_1)\Big(P_1\big(\tfrac{r}{4}\big) - P_1\big(\tfrac{3r}{4}\big)\Big).$$

For $\nu = (0, \frac{1}{2})$, we have $r = 2$, and so

$$\rho_A(T_M^2) = -4\big(P_1(\tfrac{3}{4}) + P_1(\tfrac{1}{2})\big) + 1 - 2P_1(\tfrac{1}{2}) = 0,$$

for $\nu = (\frac{1}{2}, 0)$ with $r = 3$,

$$\rho_A(T_M^2) = 2\big(\tfrac{1}{4} - \tfrac{1}{2}\big) - 4P_1(\tfrac{3}{4}) + 1 - 2P_1(\tfrac{3}{4}) = 0,$$

and lastly, for $\nu = (\frac{1}{2}, \frac{1}{2})$ with $r = 1$,

$$\rho_A(T_M^2) = 2\big(\tfrac{1}{4} - \tfrac{1}{2}\big) + 1 - 2P_1(\tfrac{1}{4}) = 1,$$

where we have used that $\sum_{k=1}^{3} P_1\big(\frac{3k}{4}\big) = 0$, see (C.18).

Recall from Corollary 1.5.2 (ii) that the non-integer part of the Rho invariant on a 3-dimensional manifold is essentially the Chern-Simons invariant associated to the Chern character. More precisely,

$$\rho_A(T_M^2) \equiv 4\,\mathrm{CS}(A) \mod \mathbb{Z}. \tag{4.68}$$

In the case of torus bundles over surfaces, computations for Chern-Simons invariants are contained in [42, 54, 57]. For the case of U(1)-connections, see for example [42, Thm. 7.22].

Corollary 4.4.21. *Under the assumptions of Theorem 4.4.20 we have*

$$\rho_A(T_M^2) \equiv 2\big(\nu_2 m_1 - \nu_1 m_2\big) \mod \mathbb{Z}.$$

Proof. First note that for all $k \in \mathbb{Z}$

$$2P_1\big(\tfrac{k}{c}\big) \equiv 2\tfrac{k}{c} \mod \mathbb{Z}.$$

In particular,

$$4\sum_{k=1}^{|c|-r} P_1\big(\tfrac{dk}{c}\big) \equiv 4\tfrac{d}{c}\tfrac{(|c|-r)(|c|-r+1)}{2} \equiv 2\big(\tfrac{dm_1^2}{c} - \tfrac{dm_1}{c}\big) \mod \mathbb{Z}.$$

Here, we have used that by definition $r \equiv m_1\,(c)$. We also note that $(\mathrm{Id} - M^t)\nu = m$ means explicitly that

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1-a & -c \\ -b & 1-d \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \quad \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \tfrac{1}{a+d-2}\begin{pmatrix} d-1 & -c \\ -b & a-1 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \tag{4.69}$$

Let us assume now that $\nu_1 \in \mathbb{Z}$. Then

$$\rho_A(T^2_M) \equiv -2\left(\tfrac{dm_1^2}{c} - \tfrac{dm_1}{c}\right) - 2\tfrac{dm_1}{c} \equiv -2\tfrac{dm_1^2}{c} \mod \mathbb{Z}.$$

It follows from (4.69) that $\frac{m_1}{c} \equiv -\nu_2$ modulo $\mathbb{Z}$, and $dm_1 \equiv m_1$ modulo $\mathbb{Z}$. Therefore,

$$\rho_A(T^2_M) \equiv 2\nu_2 m_1 \mod \mathbb{Z},$$

which is the claim of Corollary 4.4.21 in the case that $\nu_1 \in \mathbb{Z}$. If $\nu \notin \mathbb{Z}$, we have

$$\rho_A(T^2_M) \equiv \tfrac{2(a+d)-4}{c}(\nu_1^2 - \nu_1) - 2\left(\tfrac{dm_1^2}{c} - \tfrac{dm_1}{c}\right) - 2\tfrac{m_1}{c} \mod \mathbb{Z}. \tag{4.70}$$

From (4.69) we know that

$$\tfrac{(a+d)-2}{c}\nu_1 = \tfrac{(d-1)m_1}{c} - m_2.$$

Inserting this into (4.70), one finds that

$$\begin{aligned}\rho_A(T^2_M) &\equiv 2\Big(\big(\tfrac{(d-1)m_1}{c} - m_2\big)(\nu_1 - 1) + \tfrac{(d-1)m_1}{c} - \tfrac{dm_1^2}{c}\Big) \mod \mathbb{Z}\\ &\equiv 2\big(-\nu_1 m_2 + \tfrac{dm_1}{c}(\nu_1 - m_1) - \tfrac{m_1}{c}\nu_1\big) \mod \mathbb{Z}\\ &\equiv 2\big(-\nu_1 m_2 + dm_1\nu_2 + \tfrac{(ad-1)m_1}{c}\nu_1\big) \mod \mathbb{Z},\end{aligned}$$

where in the last line we have used that $\nu_1 - m_1 = a\nu_1 + c\nu_1$, see (4.69). Now using that $ad - 1 = bc$ and observing that $b\nu_1 + d\nu_2 \equiv \nu_2$ modulo $\mathbb{Z}$, we arrive at

$$\rho_A(T^2_M) \equiv 2\big(-\nu_1 m_2 + m_1\nu_2\big) \mod \mathbb{Z}. \qquad \square$$

Remark. The formula of Corollary 4.4.21 also holds in the parabolic case: Let $\varepsilon\left(\begin{smallmatrix}1 & l\\ 0 & 1\end{smallmatrix}\right)$ with $l \in \mathbb{Z}$ and $\varepsilon = \pm 1$, and let $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ satisfy $m = (\mathrm{Id} - M^t)\nu \in \mathbb{Z}^2$. Then, if $\varepsilon = 1$,

$$-l\nu_1 = m_2 \in \mathbb{Z}, \quad m_1 = 0, \quad \text{so that} \quad 2\big(-\nu_1 m_2 + m_1\nu_2\big) = 2l\nu_1^2.$$

According to Theorem 4.4.8, this is congruent to $\rho_A(T^2_M)$ modulo $\mathbb{Z}$. If $\varepsilon = -1$, then

$$2\nu_1 = m_1 \in \mathbb{Z}, \quad -l\nu_1 + 2\nu_2 = m_2 \in \mathbb{Z},$$

so that again

$$2\big(-\nu_1 m_2 + m_1\nu_2\big) = 2l\nu_1^2 - 2\nu_1\nu_2 + 2\nu_1\nu_2 = 2l\nu_1^2.$$

Jeffrey's Conjecture. We end the main discussion of this thesis with a remark concerning a possible perspective for further research. According to (4.68), Corollary 4.4.21 identifies the Chern-Simons invariant only modulo $\frac{1}{4}\mathbb{Z}$, which might seem a bit disappointing. Moreover, the methods of [42, 54, 57] to obtain the formula for the Chern-Simons invariant are much less involved than what we have presented. However, this is precisely the real strength of Theorem 4.4.20. It can be used to compute the difference $\rho_A(T^2_M) - 4\,\mathrm{CS}(A) \in \mathbb{Z}$.

Recall from Corollary 1.5.2 (i) that this is essentially the spectral flow of the odd signature operator between the trivial connection and A. For this reason it is promising that a generalization of Theorem 4.4.20 to higher gauge groups might be a way to prove Jeffrey's conjecture about the mod 4 reduction of this spectral flow term, see [54, Conj. 5.8].

Appendix A

Characteristic Classes and Chern-Simons Forms

Although we assume that the reader is familiar with the theory of characteristic classes, we include a short survey of Chern-Weil theory in the way we will use it. We closely follow [13, Sec. 1.5] and [99, Ch. 1] to which we also refer for more details. We place some emphasis on transgression forms and formulate the results about Chern-Simons invariants, which we use in Section 1.5.

A.1 Chern-Weil Theory

A.1.1 Connections and Characteristic Forms

We start with a short algebraic preliminary. Let V be a complex vector space. For $m \in \mathbb{N}$ consider $(\Lambda^{\mathrm{ev}}\mathbb{C}^m) \otimes V$ as module over the commutative algebra $\Lambda^{\mathrm{ev}}\mathbb{C}^m$. Then any element $T \in \Lambda^{\mathrm{ev}}\mathbb{C}^m \otimes \mathrm{End}(V)$ may be viewed as a module endomorphism. Upon choosing a basis for V, this is a matrix with entries in $\Lambda^{\mathrm{ev}}\mathbb{C}^m$. In this way we can define expressions like T^n and $\det T$. We extend the trace $\mathrm{tr}_V : \mathrm{End}(V) \to \mathbb{C}$ on V in the natural way to a trace

$$\mathrm{tr}_V : (\Lambda^{\mathrm{ev}}\mathbb{C}^m) \otimes \mathrm{End}(V) \to \Lambda^{\mathrm{ev}}\mathbb{C}^m.$$

Let $f(z) = \sum_{n \geq 0} a_n z^n$ be a formal power series with coefficients a_n in $\mathbb{C}$, and assume that $T \in (\Lambda^{2\bullet+2}\mathbb{C}^m) \otimes \mathrm{End}(V)$. An endomorphism T of this form is nilpotent, so that we can define

$$f(T) = \sum_{n \geq 0} a_n T^n \in (\Lambda^{\mathrm{ev}}\mathbb{C}^m) \otimes \mathrm{End}(V).$$

The following algebraic result is the main tool we use for defining the characteristic forms we need. We skip the easy proof.

Lemma A.1.1. *For every $T \in (\Lambda^{2\bullet+2}\mathbb{C}^m) \otimes \mathrm{End}(V)$,*

$$\det\big(1 + T\big) = \exp\big(\mathrm{tr}_V\big[\log(1 + T)\big]\big),$$

where the $\exp(.)$ *is taken in the algebra* $\Lambda^{\mathrm{ev}}\mathbb{C}^m$ *and the logarithm is defined using the formal power series*

$$\log(1+z) = \sum_{n\geq 0} \frac{(-1)^n}{n+1} z^{n+1}.$$

It follows from Lemma A.1.1 that if $f(z) = 1 + \sum_{n\geq 1} a_n z^n$ is a normalized formal power series, then for all $T \in (\Lambda^{2\bullet+2}\mathbb{C}^m) \otimes \mathrm{End}(V)$

$$\det\big(f(T)\big) = \exp\big(\mathrm{tr}_V\left[\log f(T)\right]\big).$$

Motivated by this we also define

$$\det{}^{1/2}\big(f(T)\big) := \exp\big(\tfrac{1}{2}\,\mathrm{tr}_V\left[\log f(T)\right]\big). \tag{A.1}$$

Characteristic Forms of Complex Vector Bundles. Now let $E \to M$ be a complex vector bundle over an m-dimensional manifold M. Let ∇ be a connection on E with curvature $F_\nabla \in \Omega^2\big(M, \mathrm{End}(E)\big)$. In this context $\Omega^{\mathrm{ev}}\big(M, \mathrm{End}(E)\big)$ plays the role of $\Lambda^{\mathrm{ev}}\mathbb{C}^m \otimes \mathrm{End}(V)$ in the above considerations.

If $T \in C^\infty\big(M, \mathrm{End}(E)\big)$, then the commutator $[\nabla, T]$ is an element of $\Omega^1\big(M, \mathrm{End}(E)\big)$. We can extend this to a derivation

$$[\nabla, \cdot] : \Omega^\bullet\big(M, \mathrm{End}(E)\big) \to \Omega^{\bullet+1}\big(M, \mathrm{End}(E)\big),$$

by requiring that for $\alpha \in \Omega^\bullet(M)$ of pure degree $|\alpha|$, and $T \in \Omega^\bullet\big(M, \mathrm{End}(E)\big)$,

$$[\nabla, \alpha \wedge T] = (d\alpha) \wedge T + (-1)^{|\alpha|} \alpha \wedge [\nabla, T].$$

Then it is easy to check that for every such T,

$$\mathrm{tr}_E[\nabla, T] = d\big(\mathrm{tr}_E\, T\big). \tag{A.2}$$

Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series. We define

$$f(\nabla) := \sum_{n\geq 0} a_n \big(\tfrac{i}{2\pi} F_\nabla\big)^n \in \Omega^{\mathrm{ev}}\big(M, \mathrm{End}(E)\big).$$

From (A.2) and the fact that $[\nabla, F_\nabla] = 0$, we obtain

$$d\,\mathrm{tr}_E\big(f(\nabla)\big) = \mathrm{tr}_E\left[\nabla, f(\nabla)\right] = 0. \tag{A.3}$$

Definition A.1.2. Let E be a complex vector bundle over M with connection ∇, and let $f(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series. Then we define the *characteristic form* of ∇ associated to f by

$$\mathrm{tr}_E\left[f(\nabla)\right] = \mathrm{tr}_E\Big[\sum_{n\geq 0} a_n \big(\tfrac{i}{2\pi} F_\nabla\big)^n\Big] \in \Omega^{\mathrm{ev}}(M).$$

Definition A.1.3. Let E be a complex vector bundle over M with connection ∇.

(i) The characteristic form associated to $\exp(z)$ is called the *Chern character form*

$$\mathrm{ch}(E, \nabla) := \mathrm{tr}_E\left[\exp\big(\tfrac{i}{2\pi} F_\nabla\big)\right] \in \Omega^{\mathrm{ev}}(M).$$

(ii) The characteristic form

$$c(E,\nabla) := \det\left(1 + \tfrac{i}{2\pi} F_\nabla\right) = \exp\left(\operatorname{tr}_E\left[\log(1 + \tfrac{i}{2\pi} F_\nabla)\right]\right) \in \Omega^{\mathrm{ev}}(M)$$

is called the *total Chern form*.

(iii) The *j-th Chern form*

$$c_j(E,\nabla) \in \Omega^{2j}(M)$$

is defined as the component of degree $2j$ of the total Chern form, i.e.,

$$c(E,\nabla) = \sum_{j=0}^{[m/2]} c_j(E,\nabla) = 1 + c_1(E,\nabla) + c_2(E,\nabla) + \dots .$$

Remark.

(i) Note that it follows form Lemma A.1.1 that the total Chern form fits into the framework of Definition A.1.2 if we take $f(z) = \log(1+z)$ and exponentiate in $\Omega^{\mathrm{ev}}(M)$ after taking $\operatorname{tr}_E\left[f(\nabla)\right]$. Since

$$d\big(\exp\circ\operatorname{tr}_E\left[f(\nabla)\right]\big) = d\big(\operatorname{tr}_E\left[f(\nabla)\right]\big) \wedge \big(\exp\circ\operatorname{tr}_E\left[f(\nabla)\right]\big),$$

it follows from (A.3) that this construction also gives closed forms.

(ii) When we decompose the Chern character form into its homogeneous components

$$\operatorname{ch}(E,\nabla) = \sum_{j}^{[m/2]} \operatorname{ch}_j(E,\nabla),$$

then one easily finds relations between ch_j and the Chern forms c_j for small j. Here, we are dropping the reference to E and ∇ for the moment. For example,

$$\operatorname{ch}_0 = \operatorname{rk} E, \quad \operatorname{ch}_1 = c_1, \quad \operatorname{ch}_2 = \tfrac{1}{2}c_1^2 - c_2, \quad \operatorname{ch}_3 = \tfrac{1}{6}(3c_3 - 3c_2c_1 + c_1^3), \quad \dots .$$

(iii) If E_1 and E_2 are two complex vector bundles over M endowed with connections ∇_1 and ∇_2, the following relations are immediate.

a) The Chern character form satisfies

$$\operatorname{ch}(E_1 \oplus E_2, \nabla_1 \oplus \nabla_2) = \operatorname{ch}(E_1,\nabla_1) + \operatorname{ch}(E_2,\nabla_2)$$

and

$$\operatorname{ch}(E_1 \otimes E_2, \nabla_1 \otimes 1 + 1 \otimes \nabla_2) = \operatorname{ch}(E_1,\nabla_1) \wedge \operatorname{ch}(E_2,\nabla_2)$$

b) The total Chern form satisfies

$$c(E_1 \oplus E_2, \nabla_1 \oplus \nabla_2) = c(E_1,\nabla_1) \wedge c(E_2,\nabla_2)$$

(iv) If E is equipped with a metric and ∇ is a compatible connection, then the associated Chern forms and the Chern character form are $\mathbb{R}$ valued forms. Moreover, assume that E is of rank k and admits an $\mathrm{SU}(k)$-structure. The latter means that the determinant line $\det(E) = \Lambda^k E$ is trivial. Then for every compatible connection ∇

$$\mathrm{ch}_{2j+1}(E, \nabla) = 0,$$

which is due to the fact that the trace of elements in the Lie algebra $\mathfrak{su}(k)$ vanishes. In particular, the first Chern form $c_1(E, \nabla)$ is trivial for $\mathrm{SU}(k)$-bundles.

Characteristic Forms of Real Vector Bundles. For our purposes it is enough to define the characteristic forms which are obtained by complexifying the bundle and the connection. In particular, we need not restrict to orthogonal connections as one would need to define the Euler class.

Definition A.1.4. Let M be an m-dimensional manifold, and let ∇ be a connection on a real vector bundle $E \to M$. Let $E^{\mathbb{C}} := E \otimes \mathbb{C}$ be endowed with the induced connection $\nabla^{\mathbb{C}}$.

(i) We call the characteristic form

$$p(E, \nabla) := \det{}^{1/2}\big(1 + (\tfrac{i}{2\pi} F_{\nabla^{\mathbb{C}}})^2\big) = \exp\Big(\tfrac{1}{2}\,\mathrm{tr}_V \Big[\log\big(1 + (\tfrac{i}{2\pi} F_{\nabla^{\mathbb{C}}})^2\big)\Big]\Big)$$

the *total Pontrjagin form* of ∇.

(ii) The *j-th Pontrjagin form*

$$p_j(E, \nabla) \in \Omega^{4j}(M)$$

is defined as the component of degree $4j$ of the total Pontrjagin form, i.e.,

$$p(E, \nabla) = \sum_{j=0}^{[m/4]} p_j(E, \nabla) = 1 + p_1(E, \nabla) + p_2(E, \nabla) + \dots .$$

(iii) We define the *Hirzebruch $\widehat{L}$-form* as

$$\widehat{L}(E, \nabla) := \det{}^{1/2}\left(\frac{\frac{i}{4\pi} F_{\nabla^{\mathbb{C}}}}{\tanh\big(\frac{i}{4\pi} F_{\nabla^{\mathbb{C}}}\big)}\right) \in \Omega^{4\bullet}(M).$$

(iv) Moreover, the *$\widehat{A}$-form* is defined as

$$\widehat{A}(E, \nabla) := \det{}^{1/2}\left(\frac{\frac{i}{4\pi} F_{\nabla^{\mathbb{C}}}}{\sinh\big(\frac{i}{4\pi} F_{\nabla^{\mathbb{C}}}\big)}\right) \in \Omega^{4\bullet}(M).$$

Remark.

(i) The definition of the characteristic forms above varies in the literature. First of all, some authors, e.g. [13], drop the normalizing constants $\frac{i}{2\pi}$ from the definition. We include them to get integer valued characteristic classes. Moreover, the $\widehat{L}$-form is related to the classical Hirzebruch L-form via

$$2^{2n} \cdot \widehat{L}(E, \nabla)_{[4n]} = L(E, \nabla)_{[4n]}, \tag{A.4}$$

where $(\dots)_{[n]}$ means taking the n-form component of a differential form.

(ii) We have not yet remarked, why the $\widehat{L}$-form and the $\widehat{A}$-form are well-defined. We give some brief remarks and refer to [73, App. B] for more details. Recall that the Bernoulli numbers B_n can be defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi, \tag{A.5}$$

see [29, Sec. 9.1]. With respect to this sign convention, the first non-trivial B_n are given by

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30}, \quad B_6 = \tfrac{1}{42}, \quad \ldots$$

Using (A.5), one finds that for $|z| < \pi$

$$\frac{z/2}{\tanh(z/2)} = 1 + \sum_{n \geq 1} \frac{1}{(2n)!} B_{2n} z^{2n} = 1 + \tfrac{1}{12} z^2 - \tfrac{1}{720} z^4 + \ldots$$

and

$$\frac{z/2}{\sinh(z/2)} = 1 + \sum_{n \geq 1} \frac{2^{2n-1} - 1}{2^{2n-1}(2n)!} B_{2n} z^{2n} = 1 - \tfrac{1}{24} z^2 + \tfrac{1}{5760} z^4 + \ldots.$$

This shows that both are normalized power series, which implies that the Hirzebruch $\widehat{L}$-form and the $\widehat{A}$-form are well-defined. Moreover, if $\dim M = 4$, then

$$\widehat{L}(E, \nabla) = 1 + \tfrac{1}{12} p_1(E, \nabla), \quad \widehat{A}(E, \nabla) = 1 - \tfrac{1}{24} p_1(E, \nabla). \tag{A.6}$$

(iii) Note that if E is endowed with a bundle metric, and compatible connection ∇, then $\nabla^{\mathbb{C}}$ on $E^{\mathbb{C}}$ satisfies $F^t_{\nabla^{\mathbb{C}}} = -F_{\nabla^{\mathbb{C}}}$. Thus,

$$\left(1 + \tfrac{i}{2\pi} F_{\nabla^{\mathbb{C}}}\right)^t = 1 - \tfrac{i}{2\pi} F_{\nabla^{\mathbb{C}}}.$$

Hence also,

$$\det{}^{1/2}\left(1 + \tfrac{i}{2\pi} F_{\nabla^{\mathbb{C}}}\right) = \det{}^{1/2}\left(1 - \tfrac{i}{2\pi} F_{\nabla^{\mathbb{C}}}\right),$$

and so

$$\begin{aligned}\sum_{l=0}^{[m/2]} c_l(E^{\mathbb{C}}, \nabla^{\mathbb{C}}) &= \det\left(1 + \tfrac{i}{2\pi} F_{\nabla^{\mathbb{C}}}\right) = \det{}^{1/2}\left(1 + \tfrac{i}{2\pi} F_{\nabla^{\mathbb{C}}}\right)\det{}^{1/2}\left(1 + \tfrac{i}{2\pi} F_{\nabla^{\mathbb{C}}}\right) \\ &= \det{}^{1/2}\left(1 - (\tfrac{i}{2\pi} F_{\nabla^{\mathbb{C}}})^2\right) = \sum_{j=0}^{[m/4]} (-1)^j p_j(E, \nabla).\end{aligned}$$

From this we deduce that

$$c_{2j}(E^{\mathbb{C}}, \nabla^{\mathbb{C}}) = (-1)^j p_j(E, \nabla), \quad \text{and} \quad c_{2j-1}(E^{\mathbb{C}}, \nabla^{\mathbb{C}}) = 0.$$

(iv) If E can be written as $E = E_1 \oplus E_2$, and ∇ decomposes as $\nabla = \nabla_1 \oplus \nabla_2$, the total Pontrjagin form p and the forms introduced in Definition A.1.4 satisfy

$$p(E) = p(E_1) \wedge p(E_2), \quad \widehat{L}(E) = \widehat{L}(E_1) \wedge \widehat{L}(E_2), \quad \widehat{A}(E) = \widehat{A}(E_1) \wedge \widehat{A}(E_2),$$

where we are dropping the references to the connections.

A.1.2 Transgression and Characteristic Classes

As we have seen in (A.3), characteristic forms associated to a formal power series f as in Definition A.1.2 are closed. Therefore, they define de Rham cohomology classes. The famous Chern-Weil theorem states that the difference

$$\mathrm{tr}_E\left[f(\nabla^1)\right] - \mathrm{tr}_E\left[f(\nabla^0)\right]$$

for two connections ∇^0 and ∇^1 on E is an exact form. Therefore, the cohomology class is independent of the connection. We refer to [13, Prop. 1.41] for a proof of the following result.

Theorem A.1.5. *Let $E \to M$ be a complex vector bundle over a manifold M, and let $f(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series. If ∇^t is a smooth path of connections on E, then*

$$\tfrac{d}{dt}\,\mathrm{tr}_E\left[f(\nabla^t)\right] = d\,\mathrm{tr}_E\left[\tfrac{i}{2\pi}(\tfrac{d}{dt}\nabla^t)\wedge f'(\nabla^t)\right].$$

In particular, if $a := \nabla^1 - \nabla^0 \in \Omega^1\big(M, \mathrm{End}(E)\big)$ is the difference of two connections, then

$$\mathrm{tr}_E\left[f(\nabla^1)\right] - \mathrm{tr}_E\left[f(\nabla^0)\right] = d\int_0^1 \tfrac{i}{2\pi}\,\mathrm{tr}_E\left[a\wedge f'(\nabla^0+ta)\right]dt$$

Therefore, we have an equality of cohomology classes

$$\Big[\mathrm{tr}_E\left[f(\nabla^1)\right]\Big] = \Big[\mathrm{tr}_E\left[f(\nabla^0)\right]\Big] \in H^{\mathrm{ev}}(M).$$

Definition A.1.6. Let $E \to M$ be a complex vector bundle over a manifold M, and let $f(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series.

(i) Let ∇ be an arbitrary connection on E. Then the cohomology class

$$c_f(E) := \Big[\mathrm{tr}_E\left[f(\nabla)\right]\Big] \in H^{\mathrm{ev}}(M)$$

is called the *f-class* of E or the *characteristic class* of E associated to f.

(ii) If M is closed and oriented, the number

$$\big\langle c_f(E), [M]\big\rangle = \int_M c_f(E) \in \mathbb{R}$$

is called the *characteristic number* of E associated to f. If all characteristic numbers associated to f are integers, the f-class is called *integer valued.*

(iii) If ∇^t is a path of connections, we call

$$Tc_f(\nabla^t) := \int_0^1 \tfrac{i}{2\pi}\,\mathrm{tr}_E\left[(\tfrac{d}{dt}\nabla^t)\wedge f'(\nabla^t)\right]dt \in \Omega^{\mathrm{odd}}(M)$$

the *transgression form of the f-class* associated to ∇^t. If $\nabla^t = \nabla^0 + ta$ we also use the notation

$$Tc_f(\nabla^0,\nabla^1) := \int_0^1 \tfrac{i}{2\pi}\,\mathrm{tr}_E\left[a\wedge f'(\nabla^0+ta)\right]dt \in \Omega^{\mathrm{odd}}(M).$$

(iv) The transgression form of the Chern character is called the *Chern-Simons form* of ∇^1 with respect to ∇^0,

$$\mathrm{cs}(\nabla^0, \nabla^1) = \int_0^1 \tfrac{i}{2\pi} \,\mathrm{tr}_E \left[a \wedge \exp(\nabla^0 + ta)\right] dt \in \Omega^{\mathrm{odd}}(M).$$

Remark A.1.7.

(i) Theorem A.1.5 also applies to characteristic forms of the form $\exp\big(\mathrm{tr}_E\left[f(\nabla)\right]\big)$. For this note that

$$\begin{aligned}\tfrac{d}{dt} \exp\big(\mathrm{tr}_E\left[f(\nabla^t)\right]\big) &= \tfrac{d}{dt}\big(\mathrm{tr}_E\left[f(\nabla^t)\right]\big) \wedge \exp\big(\mathrm{tr}_E\left[f(\nabla^t)\right]\big) \\ &= d\big(\tfrac{i}{2\pi} \,\mathrm{tr}_E\left[(\tfrac{d}{dt}\nabla^t) \wedge f'(\nabla^t)\right]\big) \wedge \exp\big(\mathrm{tr}_E\left[f(\nabla^t)\right]\big).\end{aligned}$$

This form is exact, since $\exp\big(\mathrm{tr}_E\left[f(\nabla^t)\right]\big)$ is closed. Hence, the transgression form in this case is

$$\int_0^1 \tfrac{i}{2\pi} \,\mathrm{tr}_E\left[(\tfrac{d}{dt}\nabla^t) \wedge f'(\nabla^t)\right] \wedge \exp\big(\mathrm{tr}_E\left[f(\nabla^t)\right]\big) dt$$

(ii) When considering the cohomology class of one of the particular characteristic forms introduced in the last section, we will call them *Chern character*, *Chern class*, $\widehat{L}$*-class*, etc. The distinction between forms and classes is done by incorporating the connection in the notation. For example,

$$\mathrm{ch}(E, \nabla) \in \Omega^{\mathrm{ev}}(M), \quad \text{but} \quad \mathrm{ch}(E) \in H^{\mathrm{ev}}(M).$$

(iii) The Chern and Pontrjagin classes are integer valued due to the normalization factor of $\frac{i}{2\pi}$, see [73, App. C]. The other characteristic classes we have defined are in general only $\mathbb{Q}$ valued.

(iv) Often the term Chern-Simons form is reserved for the degree 3 part of what we have called the Chern-Simons form. Due to its importance in 3-manifold topology, we want to derive an explicit formula for it. We abbreviate $\nabla := \nabla^0$ and let F_t denote the curvature of $\nabla^t := \nabla + ta$. Then

$$F_t = F_\nabla + t(\nabla a) + t^2 a \wedge a.$$

For the component of degree 4 of the Chern character form we have $f(z) = z^2/2$, so that $f'(z) = z$. According to Definition A.1.6,

$$\mathrm{cs}(\nabla^0, \nabla^1)_{[3]} = -\frac{1}{4\pi^2} \int_0^1 \mathrm{tr}_E \left[a \wedge \big(F_\nabla + t\nabla a + t^2 a \wedge a\big)\right] dt.$$

Integrating this expression we get

$$\mathrm{cs}(\nabla^0, \nabla^1)_{[3]} = -\frac{1}{4\pi^2} \,\mathrm{tr}_E \left[a \wedge F_\nabla + \tfrac{1}{2} a \wedge \nabla a + \tfrac{1}{3} a \wedge a \wedge a\right]. \tag{A.7}$$

In particular, if ∇ is a flat connection we get the well-known expression

$$\mathrm{cs}(\nabla^0, \nabla^1)_{[3]} = -\frac{1}{8\pi^2} \,\mathrm{tr}_E \left[a \wedge \nabla a + \tfrac{2}{3} a \wedge a \wedge a\right].$$

A.2 Chern-Simons Invariants

There is also a different description of transgression forms, which we shall describe now. Let $E \to M$ be a complex vector bundle over a manifold M, endowed with a path ∇^t of connections. Over the cylinder $N := [0,1] \times M$, we consider the vector bundle $\pi^*E \to N$, where $\pi : N \to M$ is the natural projection. The path ∇^t defines a connection on π^*E via

$$\widetilde{\nabla} := dt \wedge \tfrac{d}{dt} + \pi^*\nabla^t. \tag{A.8}$$

Its curvature is easily seen to be given by

$$F_{\widetilde{\nabla}} = dt \wedge \left(\tfrac{d}{dt}\pi^*\nabla^t\right) + \pi^* F_{\nabla^t}.$$

Since $dt \wedge dt = 0$, one deduces from the trace property that for all $n \geq 1$,

$$\mathrm{tr}_{\pi^*E}\left[F^n_{\widetilde{\nabla}}\right] = dt \wedge \pi^* \mathrm{tr}_E\left[n\left(\tfrac{d}{dt}\nabla^t\right) \wedge F^{n-1}_{\nabla^t}\right] + \pi^* \mathrm{tr}_E\left[F^n_{\nabla^t}\right].$$

This implies that for any formal power series $f(z) = \sum_n a_n z^n$,

$$\mathrm{tr}_{\pi^*E}\left[f(\widetilde{\nabla})\right] = \tfrac{i}{2\pi} dt \wedge \pi^* \mathrm{tr}_E\left[\left(\tfrac{d}{dt}\nabla^t\right) \wedge f'(\nabla^t)\right] + \pi^* \mathrm{tr}_E\left[f(\nabla^t)\right]. \tag{A.9}$$

Now, consider integration along the fiber as in Proposition 2.1.12,

$$\int_{N/M} : \Omega^\bullet(N) \to \Omega^{\bullet-1}(M).$$

Then, comparing (A.9) and the definition of the transgression form in Definition A.1.6, one readily obtains

Lemma A.2.1. *If ∇^t is a path of connections over M, and $\widetilde{\nabla}$ denotes the associated connection* (A.8) *over the cylinder $N := [0,1] \times M$, then*

$$Tc_f(\nabla^t) = \int_{N/M} \mathrm{tr}_{\pi^*E}\left[f(\widetilde{\nabla})\right] \in \Omega^{\mathrm{odd}}(M).$$

Using this result we can now derive the following important property of transgression forms, see [28, Sec. 3].

Proposition A.2.2. *Let $E \to M$ be a complex vector bundle over a manifold M. If ∇^t is a closed path of connections on E, then*

$$Tc_f(\nabla^t) \in d\Omega^{\mathrm{ev}}(M).$$

Proof. Since the space of connections on E is contractible, we can find a smooth two-parameter family $\nabla^{s,t}$ of connections which gives a homotopy relative endpoints from ∇^t to the constant path. On the cylinder N we consider the one-parameter family

$$\widetilde{\nabla}^s := dt \wedge \tfrac{d}{dt} + \nabla^{s,t},$$

where we are dropping the pullback with π from the notation. Using Theorem A.1.5 and (A.9) one finds that

$$\begin{aligned}\tfrac{d}{ds}\operatorname{tr}_E\big[f(\widetilde{\nabla}^s)\big] &= -\frac{1}{4\pi^2}\,d_N\operatorname{tr}_{\pi^*E}\Big[\big(\tfrac{d}{ds}\nabla^{s,t}\big)\wedge dt\wedge\big(\tfrac{d}{dt}\nabla^{s,t}\big)\wedge f''(\nabla^{s,t})\Big]\\ &\qquad + d_N\operatorname{tr}_E\Big[\tfrac{i}{2\pi}\big(\tfrac{d}{ds}\nabla^{s,t}\big)\wedge f'(\nabla^{s,t})\Big]\\ &=: d_N\big(dt\wedge\alpha(s,t)\big)+d_N\beta(s,t),\end{aligned}$$

where $\alpha(s,t)$ and $\beta(s,t)$ are two-parameter families of differential forms on M. Then

$$\begin{aligned}\frac{d}{ds}\int_{N/M}\operatorname{tr}_E\big[f(\widetilde{\nabla}^s)\big] &= \int_{N/M} d_N\big(dt\wedge\alpha(s,t)\big)+\int_{N/M}d_N\beta(s,t)\\ &= d_M\int_{N/M}dt\wedge\alpha(s,t)+\int_{N/M}dt\wedge\big(\tfrac{d}{dt}\beta(s,t)\big)\\ &= d_M\int_{N/M}dt\wedge\alpha(s,t)+\beta(s,1)-\beta(s,0).\end{aligned}$$

By assumption, $\nabla^{s,1}=\nabla^{s,0}$ is constant for all s. Checking the explicit formula for $\beta(s,t)$ one finds that $\beta(s,1)=\beta(s,0)$. According to Lemma A.2.1, this shows that

$$\tfrac{d}{ds}Tc_f(\nabla^{s,t})=\frac{d}{ds}\int_{N/M}\operatorname{tr}_E\big[f(\widetilde{\nabla}^s)\big]\in d\Omega^{\mathrm{ev}}(M),$$

from which the result follows. □

Chern-Simons Invariants. The last result shows that transgression forms can be used to define numerical invariants associated to pairs of connections on odd dimensional manifolds. In this respect they are odd analogues of characteristic numbers.

Definition A.2.3. Let M be a closed manifold, and let $f(z)=\sum_{n\geq 0}a_n z^n$ be a formal power series. If ∇^0 and ∇^1 are two connections on a complex vector bundle $E\to M$ we define the *Chern-Simons invariant* of ∇^1 with respect to ∇^0 associated to f as

$$\mathrm{CS}_f(\nabla^0,\nabla^1):=\int_M Tc_f(\nabla^0,\nabla^1).$$

Proposition A.2.4. *Let M be a closed manifold, and let f be a formal power series. Consider two connections ∇^0 and ∇^1 on a complex vector bundle $E\to M$.*

(i) *If ∇^t is any path connecting ∇^0 and ∇^1, then*

$$\mathrm{CS}_f(\nabla^0,\nabla^1)=\int_M Tc_f(\nabla^t).$$

(ii) *Let ∇^2 be a third connection on E, then*

$$\mathrm{CS}_f(\nabla^0,\nabla^2)=\mathrm{CS}_f(\nabla^0,\nabla^1)+\mathrm{CS}_f(\nabla^1,\nabla^2).$$

(iii) *Assume that* ∇^N *is a connection over the cylinder* $N = [0,1] \times M$ *such that on a collar of the boundary it is of the form* (A.8)*. Then*

$$\mathrm{CS}_f(\nabla^0, \nabla^1) = \int_N \mathrm{tr}_E \big[f(\nabla^N) \big]. \tag{A.10}$$

(iv) *If* W *is a compact manifold with boundary* M*, and* E*,* ∇^0 *and* ∇^1 *extend to* E_W*,* $\hat{\nabla}^0$ *and* $\hat{\nabla}^1$*, then*

$$\mathrm{CS}_f(\nabla^0, \nabla^1) = \int_W \mathrm{tr}_E \big[f(\hat{\nabla}^1) \big] - \int_W \mathrm{tr}_E \big[f(\hat{\nabla}^0) \big].$$

(v) *Assume that* f *gives an integer valued characteristic class, and let* $\Phi : E \to E$ *be a bundle isomorphism. Then for every connection* ∇ *on* E*,*

$$\mathrm{CS}_f(\nabla, \Phi^*\nabla) \in \mathbb{Z}.$$

Sketch of proof. Since M is assumed to be closed, part (i) follows from Proposition A.2.2. Part (ii) is an immediate consequence of (i). For (iii) let ∇^t be a path connecting ∇^0 and ∇^1 such that on a collar of the boundary, ∇^N and $\widetilde{\nabla}$ as in (A.8) agree. Theorem A.1.5 implies that $\nabla^N - \widetilde{\nabla}$ is the differential of a form on N with compact support away from the boundary. Then Stokes' Theorem readily yeilds (iii). Part (iv) also follows from Theorem A.1.5 and Stokes' Theorem.[1] For part (v) denote by φ the map covered by Φ. Then the mapping torus

$$E_\Phi := \big([0,1] \times E\big)/\sim\,, \quad (1,x) \sim (0, \Phi(x))$$

is a Hermitian vector bundle over the mapping torus M_φ. Endow E_Φ with a connection ∇^Φ, induced by connecting ∇ and $\Phi^*\nabla$ over M. Then one easily finds that

$$\mathrm{CS}_f(\nabla, \Phi^*\nabla) = \int_{M_\varphi} \mathrm{tr}_{E_\Phi} \big[f(\nabla^\Phi) \big].$$

The right hand side is integer valued as M_φ is closed. ☐

Remark A.2.5. For a characteristic class of the form $\exp\big(\mathrm{tr}_E \big[f(\nabla)\big]\big)$ we have seen in Remark A.1.7 that the transgression form is given by

$$\int_0^1 \tfrac{i}{2\pi} \mathrm{tr}_E \big[(\tfrac{d}{dt}\nabla^t) \wedge f'(\nabla^t) \big] \wedge \exp\big(\mathrm{tr}_E \big[f(\nabla^t)\big]\big) dt.$$

Lemma A.2.1 extends to this context: Let $\widetilde{\nabla}$ be the connection over $N = [0,1] \times M$, and write

$$\mathrm{tr}_E \big[f(\widetilde{\nabla}) \big] = dt \wedge \alpha(t) + \beta(t),$$

where $\alpha(t)$ and $\beta(t)$ contain no dt-factor. Then

$$\exp\big(\mathrm{tr}_E \big[f(\widetilde{\nabla})\big]\big) = \exp\big(dt \wedge \alpha(t)\big) \wedge \exp\big(\beta(t)\big) = \big(1 + dt \wedge \alpha(t)\big) \wedge \exp\big(\beta(t)\big).$$

[1] Alternatively, one could glue a cylinder to the boundary of W to interpolate between $\hat{\nabla}^0$ and $\hat{\nabla}^1$, and then use Lemma A.2.1 as well as the additivity under cutting and pasting of characteristic numbers.

Therefore,

$$\begin{aligned}
\int_{N/M} \exp\big(\operatorname{tr}_E\big[f(\widetilde{\nabla})\big]\big) &= \int_{N/M} dt \wedge \alpha(t) \wedge \exp\big(\beta(t)\big) \\
&= \int_{N/M} \operatorname{tr}_E\big[f(\widetilde{\nabla})\big] \wedge \exp\big(\operatorname{tr}_E\big[f(\nabla^t)\big]\big) \\
&= \int_0^1 \tfrac{i}{2\pi} \operatorname{tr}_E\big[(\tfrac{d}{dt}\nabla^t) \wedge f'(\nabla^t)\big] \wedge \exp\big(\operatorname{tr}_E\big[f(\nabla^t)\big]\big) dt
\end{aligned}$$

where we have used (A.9) in the last line. Similarly, one checks that Proposition A.2.2 and Proposition A.2.4 continue to hold in this context.

Appendix B

Remarks on Moduli Spaces

In this appendix we include some details concerning the moduli space of flat connections and the moduli space of holomorphic line bundles over a Riemann surface. Since the Rho invariant depends only on the gauge equivalence class of the underlying flat connection, understanding the moduli space of flat connections is a prerequisite for the computation of Rho invariants. Moreover, the interplay between flat connections and representations of the fundamental group is often used in the main body of this thesis. Therefore, we start with a detailed discussion of these topics, in particular including some remarks on the question of whether a given flat bundle is trivializable or not.

We proceed with a discussion of the moduli space of flat connections associated to a mapping torus. Here the objective is to prove the facts we have used in Chapter 4. After this, we add some remarks about the moduli space of holomorphic line bundles over a Riemann surface and its relation to the moduli space of (flat) connections. This will establish some facts we have freely used in Section 2.3.

B.1 The Moduli Space of Flat Connections

B.1.1 Flat Connections and Representations of the Fundamental Group

Since many features become more transparent in a more general setup, we start working with a principal G-bundles, where G is an arbitrary connected matrix Lie group. Ultimately we are interested in flat Hermitian vector bundles and restrict to $G = \mathrm{U}(k)$. A general reference for the contents of this section are [37, Sec. 2.1] and [62, Ch. II].

Denote by $\mathfrak{g}$ the Lie algebra of G. Since we are assuming that G is a matrix Lie group, $\mathfrak{g}$ is a matrix Lie algebra. We use the notation "Ad" for the adjoint action of G on itself and "ad" to denote the adjoint action of G on $\mathfrak{g}$.

Connections and Curvature. Let M be a connected manifold, and let $P \xrightarrow{\pi} M$ be a principal G-bundle. Let R_g denote the right-action of $g \in G$ on P. Recall that a G-*connection* on P is a Lie algebra valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ satisfying

$$R_g^* A = g^{-1} A g, \quad \text{and} \quad A\big(\tfrac{d}{dt}\big|_{t=0} p \cdot \exp(tX)\big) = X, \quad p \in P, \quad X \in \mathfrak{g}. \qquad \text{(B.1)}$$

We denote the space of all G-connections on P by $\mathcal{A}(P)$. The *curvature* of A is defined as

$$F_A = dA + A \wedge A \in \Omega^2(P, \mathfrak{g}),$$

where $A \wedge A$ stands for taking the exterior product in the form part and matrix multiplication in the Lie algebra part. The curvature is easily seen to be ad-equivariant and horizontal, i.e.,

$$R_g^* F_A = g^{-1} F_A g, \quad \mathrm{i}\left(\tfrac{d}{dt}\big|_{t=0} p \cdot \exp(tX)\right) F_A = 0.$$

This implies that F_A can also be viewed as a 2-form on M with values in the bundle $\mathrm{ad}(\mathfrak{g}) = P \times_{\mathrm{ad}} \mathfrak{g}$. A connection A is called *flat* if $F_A = 0$, and we denote by

$$\mathcal{F}(P) = \{A \in \mathcal{A}(P) \,|\, F_A = 0\}$$

the space of flat G-connections on P.

Gauge Transformations and the Moduli Space. We also recall that a *gauge transformation* is a G-equivariant bundle isomorphism,

$$\Phi : P \to P, \quad \Phi(p \cdot g) = \Phi(p) \cdot g.$$

If one defines $u : P \to G$ by requiring that $\Phi(p) = p \cdot u(p)$, then u is Ad-equivariant,

$$u : P \to G, \quad u(p \cdot g) = g^{-1} u(p) g.$$

Conversely, it is easy to see that every gauge transformation arises this way. Hence, one of several equivalent ways to define the group of gauge transformations is

$$\mathcal{G}(P) := C^\infty\big(M, \mathrm{Ad}(P)\big), \quad \text{where} \quad \mathrm{Ad}(P) = P \times_{\mathrm{Ad}} G.$$

The pullback of a connection by a gauge transformation gives a natural action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$. In terms of an Ad-equivariant map $u : P \to G$ this takes the form

$$A \cdot u = u^{-1} A u + u^{-1} du, \quad A \in \mathcal{A}(P), \quad u \in \mathcal{G}(P).$$

We point out that $u^{-1}du$ is the pullback of the Maurer-Cartan form on G via u. The curvature behaves equivariantly with respect to this action,

$$F_{A \cdot u} = u^{-1} F_A u, \quad A \in \mathcal{A}(P), \quad u \in \mathcal{G}(P).$$

In particular, the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ leaves the space $\mathcal{F}(P)$ of flat connections invariant. One can thus define the *moduli space* of flat connections on P as

$$\mathcal{M}(P) := \mathcal{F}(P)/\mathcal{G}(P).$$

Remark B.1.1. We will often encounter the situation that P is trivializable. If we fix a trivialization $P \cong M \times G$, we can identify

$$\mathcal{A}(M \times G) \cong \Omega^1(M, \mathfrak{g}), \quad \mathcal{G}(M \times G) \cong C^\infty(M, G).$$

The Holonomy Representation. Fix a base point $p_0 \in P$, and let $x_0 := \pi(p_0)$. Consider a closed loop based at x_0, i.e.,

$$c : I \to M, \quad I = [0,1], \quad \text{with} \quad c(0) = c(1) = x_0.$$

Since I is contractible, the pullback $c^*P \to I$ is trivializable. As we are assuming that G is connected, we can fix a lift $\widehat{c} : I \to P$ of c such that $\widehat{c}(0) = \widehat{c}(1) = p_0$. Now let A be a G-connection—not necessarily flat for the moment—and let

$$A_t := (\widehat{c}^*A)(\partial_t) \in C^\infty(I, \mathfrak{g}). \tag{B.2}$$

Definition B.1.2. Let $g_t : I \to G$ be the unique solution of the ordinary differential equation

$$\partial_t g_t = -A_t g_t, \quad g_0 = e,$$

where $e \in G$ is the identity element. Then the *holonomy* of A along c with respect to the base point p_0 is defined by

$$\mathrm{hol}_A(c, p_0) := g_1 \in G.$$

Note that the definition gives no reference to the lift $\widehat{c}$ we have fixed. The reason why we are allowed to do so is one of the contents of the following result.

Lemma B.1.3. *Let A be a connection on P, and let $c : I \to M$ be a closed loop, based at x_0.*

(i) *If $\varphi : I \to I$ is an orientation preserving reparametrization, then*

$$\mathrm{hol}_A(c \circ \varphi, p_0) = \mathrm{hol}_A(c, p_0).$$

(ii) *For every gauge transformation $u \in \mathcal{G}(P)$,*

$$\mathrm{hol}_{A\cdot u}(c, p_0) = u(p_0)^{-1}\,\mathrm{hol}_A(c, p_0)\,u(p_0).$$

In particular, $\mathrm{hol}_A(c, p_0)$ is independent of the lift $\widehat{c}$ chosen in its definition.

(iii) *Assume that $\widetilde{c}$ is another loop, based at x_0, and denote by $c * \widetilde{c}$ the loop defined by first running along c and then along $\widetilde{c}$. Then*

$$\mathrm{hol}_A(c * \widetilde{c}, p_0) = \mathrm{hol}_A(\widetilde{c}, p_0)\,\mathrm{hol}_A(c, p_0).$$

(iv) *Let $p_1 \in P$ be a different base point, and $\widehat{c}_0 : I \to P$ be a path connecting p_0 with p_1. Then there exists $g \in G$ such that*

$$\mathrm{hol}_A(c_0^{-1} * c * c_0, p_1) = g\,\mathrm{hol}_A(c, p_0)\,g^{-1}, \quad \textit{where } c_0 := \pi \circ \widehat{c}_0.$$

Proof. [1] To prove part (i) let g_t be as in Definition B.1.2. Then

$$\partial_s g_{\varphi(s)} = \varphi'(s)(\partial_t g_t)|_{t=\varphi(s)} = -\varphi'(s)A_{\varphi(s)}g_{\varphi(s)} = -(\widehat{c} \circ \varphi)^*A(\partial_s)g_{\varphi(s)}.$$

[1] Although the assertions in Lemma B.1.3 is standard, we include a proof as the discussion to follow relies on similar arguments.

Moreover, we have $g_{\varphi(0)} = g_0 = e$ and $g_{\varphi(1)} = g_1$. Then part (i) is true by definition. To prove (ii) define $u_t := u \circ \widehat{c} \in C^\infty(I, G)$, and $\widehat{c}_u := \widehat{c} \cdot u_t$. Then $\widehat{c}_u$ is a lift of c with

$$\widehat{c}_u(0) = \widehat{c}_u(1) = p_0 \cdot u(p_0), \quad \text{and} \quad \widehat{c}^*(A \cdot u)(\partial_t) = \widehat{c}_u^* A(\partial_t) = A_t \cdot u_t.$$

If g_t is as in Definition B.1.2, then

$$\begin{aligned}\partial_t(u_t^{-1} g_t u_0) &= (\partial_t u_t^{-1}) g_t u_0 + u^{-1}(\partial_t g_t) u_0 \\ &= -u_t^{-1}(\partial_t u_t)(u_t^{-1} g_t u_0) - (u_t^{-1} A_t u_t)(u_t^{-1} g_t u_0) \\ &= -(A_t \cdot u_t)(u_t^{-1} g_t u_0).\end{aligned}$$

Since $u_0^{-1} g_0 u_0 = e$, this implies that

$$\mathrm{hol}_{A \cdot u_t}(c, p_0 \cdot g) = u_1^{-1} g_1 u_0 = u^{-1}(p_0) \, \mathrm{hol}_A(c, p_0) \, u(p_0).$$

The second assertion of (ii) follows from the fact that every lift of c with base point p_0 is of the form $\widehat{c} \cdot u$ with $u(p) = e$. Concerning part (iii), we define $\widetilde{A}_t$ as in (B.2) with respect to a lift of $\widetilde{c}$ and note without going into detail that $\mathrm{hol}_A(c * \widetilde{c}, p_0)$ is given by $\widetilde{g}_1$, where $\widetilde{g}_t$ is the unique solution to

$$\partial_t \widetilde{g}_t = -\widetilde{A}_t \widetilde{g}_t, \quad \widetilde{g}_0 = \mathrm{hol}_A(c, p_0).$$

This readily yields $\widetilde{g}_1 = \mathrm{hol}_A(\widetilde{c}, p_0) \, \mathrm{hol}_A(c, p_0)$. To prove (iv) we can solve the initial value problem

$$\partial_t g_t = -(\widehat{c}_0^* A)(\partial_t) g_t, \quad g_0 = e.$$

Then one verifies without effort that the assertion holds with $g := g_1$. □

Let $\Omega(M, x_0)$ be the *based loop group* of M. This is the set of all loops, based at x_0, modulo orientation preserving reparametrization. For reasons of functoriality we endow $\Omega(M, x_0)$ with the product $c \cdot \widetilde{c} := \widetilde{c} * c$, where $\widetilde{c} * c$ is as in Lemma B.1.3. Let $p_0 \in P$ with $x_0 = \pi(p_0)$. Using the above results, one obtains a well-defined homomorphism

$$\mathrm{hol}_A : \Omega(M, x_0) \to G, \quad c \mapsto \mathrm{hol}_A(c, p_0) \tag{B.3}$$

Definition B.1.4. Let A be a connection on P. Then the homomorphism (B.3) is called the *holonomy representation* of A with respect to the base point p_0. We also define the *holonomy group* of A with respect to p_0 as

$$G_A(p_0) := \mathrm{im}\left(\mathrm{hol}_A : \Omega(M, x_0) \to G\right)$$

A connection A is called *irreducible*, if $G_A(p_0) = G$. Otherwise, it is called *reducible*. Moreover, the *isotropy group* of A is defined as

$$I(A) := \left\{ u \in \mathcal{G}(P) \;\middle|\; A \cdot u = A \right\}.$$

Lemma B.1.5. *The conjugacy class of $G_A(p_0)$ is independent of p_0 and the gauge equivalence class of A. Moreover, for fixed $p_0 \in P$, the map*

$$I(A) \to G, \quad u \mapsto u(p_0),$$

maps $I(A)$ isomorphically to the centralizer of $G_A(p_0)$ in G.

Proof. The first assertion is immediate from Lemma B.1.3. Now fix p_0, and assume that $u \in I(A)$. Lemma B.1.3 implies that for every loop $c : I \to M$, based at $x_0 = \pi(p_0)$, we have

$$\mathrm{hol}_A(c, p_0) = \mathrm{hol}_{A\cdot u}(c, p_0) = u(p_0)^{-1}\,\mathrm{hol}_A(c, p_0)\,u(p_0).$$

Thus, $u(p_0)$ lies in the centralizer of $G_A(p_0)$. Now let P_0 be the set of all $p \in P$ such that there exists a horizontal path $\widehat{c} : I \to P$ with $\widehat{c}(0) = p_0$ and $\widehat{c}(1) = p$. Then P_0 intersects every fiber of $\pi : P \to M$. This is because if $\widehat{c} : I \to P$ is an arbitrary path with $\widehat{c}(0) = p_0$, we can solve

$$\partial_t g_t = -A_t g_t, \quad g_0 = e,$$

with A_t as in (B.2), to get a horizontal path $\widehat{c} \cdot g_t : I \to P$ whose endpoint lies in the same fiber as the endpoint of $\widehat{c}$.

To prove injectivity of the map $I(A) \to G$, $u \mapsto u(p_0)$, let $u \in I(A)$ with $u(p_0) = e$. Since u is Ad-equivariant and P_0 intersects every fiber of $\pi : P \to M$, it suffices to show that $u|_{P_0} \equiv e$. Let $p \in P_0$, and let $\widehat{c}$ a horizontal path connecting p_0 with p. Then $A \cdot u = A$ implies that

$$(u^{-1} \circ \widehat{c})\partial_t(u \circ \widehat{c})(t) = u^{-1}du|_{\widehat{c}(t)}(\tfrac{d}{dt}\widehat{c}) = (A - u^{-1}Au)|_{\widehat{c}(t)}(\tfrac{d}{dt}\widehat{c}) = 0,$$

where we have used that $\frac{d}{dt}\widehat{c}$ is horizontal. Hence u is constant along $\widehat{c}$ and thus, $u(p) = e$. Next, assume that $g_0 \in G$ lies in the centralizer of $G_A(p_0)$. We need to define $u \in I(A)$ such that $u(p_0) = g_0$. We first define $u|_{P_0}$ to be the constant map g_0. To see that this defines a gauge transformation, we need to check that $u(p \cdot g) = g^{-1}u(p)g$, whenever p and $p \cdot g$ both lie in P_0. Let $\widehat{c}$ and $\widehat{c}_g$ be horizontal paths connecting p_0 with p, respectively with $p \cdot g$. Then $\widehat{c}_g * (\widehat{c}^{-1} \cdot g)$ is a horizontal path which connects p_0 with $p_0 \cdot g$. This implies that $g \in G_A(p_0)$. Since we have assumed that g_0 lies in the centralizer of $G_A(p_0)$, we obtain

$$u(p \cdot g) = g^{-1}u(p)g = g^{-1}g_0 g = g_0.$$

Hence, $u|_{P_0}$ is Ad-equivariant and can be extended to a gauge transformation on P. To prove that $u \in I(A)$ first note that the values of the 1-forms $A \cdot u$ and A on vertical vectors are both prescribed by (B.1). To see that $A \cdot u$ and A also agree on horizontal vectors, it suffices to consider $A|_p$ and $A \cdot u|_p$ for $p \in P_0$ since both, $A \cdot u$ and A, are ad-equivariant. Now if $v \in T_pP$ is horizontal, there exists a horizontal path $\widehat{c} : (-\varepsilon, \varepsilon) \to P$ with $\widehat{c}(0) = p$ and $\frac{d}{dt}\widehat{c}(0) = v$. By definition of P_0 one easily checks that $\mathrm{im}(\widehat{c}) \subset P_0$. Thus, u is constant along $\widehat{c}$ so that

$$A \cdot u|_p(v) - A|_p(v) = u^{-1}du|_p(v) = u^{-1}(p)\tfrac{d}{dt}\big|_{t=0} u \circ \widehat{c} = 0. \qquad \square$$

Flat Connections and the Fundamental Group. After this technical preparation, we turn our attention to flat connections. The following result shows that they are of a topological nature.

Proposition B.1.6. *If A is flat, then the holonomy $\mathrm{hol}_A(c, p_0)$ depends only on the homotopy class $[c] \in \pi_1(M) = \pi_1(M, x_0)$. In particular, the holonomy representation defines a homomorphism*

$$\mathrm{hol}_A \in \mathrm{Hom}\big(\pi_1(M), G\big).$$

Moreover, the assignment $\mathcal{F}(P) \to \mathrm{Hom}\big(\pi_1(M), G\big)$, $A \mapsto \mathrm{hol}_A$ gives well-defined map

$$\mathcal{M}(P) \to \mathrm{Hom}\big(\pi_1(M), G\big)/G.$$

Proof. Consider a homotopy

$$c : I \times I \to M, \quad c(s,0) = c(s,1) = x_0.$$

Since any fiber bundle has the homotopy lifting property, we can choose a lift

$$\widehat{c} : I \times I \to P, \quad \widehat{c}(s,0) = \widehat{c}(s,1) = p_0$$

Abusing notation we use the letter A also to denote the 1-form $\widehat{c}^*A \in \Omega^1(I \times I, \mathfrak{g})$. The flatness condition $dA + A \wedge A = 0$ written out with respect to the coordinates $(s,t) \in I \times I$ is

$$\partial_s A(\partial_t) - \partial_t A(\partial_s) + A(\partial_s)A(\partial_t) - A(\partial_t)A(\partial_s) = 0. \tag{B.4}$$

For fixed s let $g_s = g_s(t) : I \to G$ denote the solution to

$$\partial_t g_s = -A(\partial_t)g_s, \quad g_s(0) = e. \tag{B.5}$$

Since A depends smoothly on s and t, it follows from the standard theory of ordinary differential equations that g_s depends smoothly on s. We then compute

$$\begin{aligned}\partial_t\big(\partial_s g_s + A(\partial_s)g_s\big) &= \partial_s(\partial_t g_s) + \big(\partial_t A(\partial_s)\big)g_s + A(\partial_s)(\partial_t g_s)\\ &= -\partial_s\big(A(\partial_t)g_s\big) + \big(\partial_t A(\partial_s)\big)g_s - A(\partial_s)A(\partial_t)g_s,\end{aligned}$$

where we have used (B.5). Then (B.4) implies that

$$\begin{aligned}\partial_t\big(\partial_s g_s + A(\partial_s)g_s\big) &= \big(A(\partial_s)A(\partial_t) - A(\partial_t)A(\partial_s)\big)g_s - A(\partial_t)\partial_s g_s - A(\partial_s)A(\partial_t)g_s\\ &= -A(\partial_t)\big(\partial_s g_s + A(\partial_s)g_s\big).\end{aligned}$$

The initial condition in (B.5) and the fact that $\widehat{c}(s,0)$ is constant for $s \in I$ implies that $\partial_s g_s|_{t=0} = 0$ and $A(\partial_s)|_{t=0} = 0$. Hence,

$$\partial_s g_s = -A(\partial_s)g_s, \quad \text{for all } t \in I.$$

Moreover, we have $A(\partial_s)|_{t=1} = 0$ so that $g_s(1)$ is independent of s. By definition of the holonomy and (B.5) this proves the first assertion of Proposition B.1.6. The other assertions are immediate from Lemma B.1.3. □

The Moduli Space of Representations. Let $\mathcal{M}(M,G)$ be the moduli space of flat principal G-bundles, i.e., the space of isomorphism classes of pairs (P,A) where P is a principal G-bundle and A is a flat connection on P. Our next goal is to show that the map in Proposition B.1.6 induces an isomorphism

$$\mathcal{M}(M,G) \cong \mathrm{Hom}\big(\pi_1(M), G\big)/G.$$

Remark B.1.7. Note that in general $\mathcal{M}(M,G)$ will be strictly larger than the moduli space $\mathcal{M}(P)$ for one fixed flat bundle P. This is because there might exist flat bundles such that the underlying principal G-bundles are not isomorphic. In the case that $G = \mathrm{U}(k)$ we will say more about this in Section B.1.2 below.

To associate a flat G-bundle to any representation $\alpha : \pi_1(M) \to G$, let $\widetilde{M}$ be the universal cover of M. For definiteness we fix a base point $x_0 \in M$ and identify $\widetilde{M}$ with the space of homotopy classes of paths in M starting at x_0. Then $\pi_1(M, x_0)$ naturally acts on $\widetilde{M}$ from the right. For $\alpha : \pi_1(M, x_0) \to G$ we define the principal G-bundle

$$P_\alpha := \widetilde{M} \times_\alpha G = (\widetilde{M} \times G)/\sim$$

where

$$(\widetilde{x}, g) \sim (\widetilde{x} \cdot c, \alpha(c)^{-1} g), \quad (\widetilde{x}, g) \in \widetilde{M} \times G, \quad c \in \pi_1(M). \tag{B.6}$$

Pulling back the Maurer-Cartan form $g^{-1}dg \in \Omega^1(G, \mathfrak{g})$ to $\widetilde{M} \times G$ defines a natural flat connection on $\widetilde{M} \times G$, which is invariant under the action (B.6) of $\pi_1(M)$. In this way we get an induced flat connection A_α on P_α. It is straightforward to check that with respect to the base point $p_0 := [x_0, e] \in P_\alpha$,

$$\mathrm{hol}_{A_\alpha}\big(c, p_0\big) = \alpha(c), \quad c \in \pi_1(M, x_0).$$

More generally, we have

Proposition B.1.8. *Let P be a principal G-bundle with flat connection A, and let $\alpha : \pi_1(M) \to G$ be a representation of the fundamental group. Then (P, A) is isomorphic to (P_α, A_α) if and only if there exists $g \in G$ with $\mathrm{hol}_A = g^{-1}\alpha g$. In particular, we have a bijection*

$$\mathcal{M}(M, G) \xrightarrow{\cong} \mathrm{Hom}\big(\pi_1(M), G\big)/G, \quad [P, A] \mapsto [\mathrm{hol}_A].$$

Sketch of proof. The assertion that $(P, A) \cong (P_\alpha, A_\alpha)$ implies $\mathrm{hol}_A = g^{-1}\alpha g$ is an immediate generalization of Lemma B.1.3 (ii). For the reverse direction first consider a representation α and let $\widetilde{\alpha} := g_0^{-1}\alpha g_0$ for some $g_0 \in G$. Define

$$\widetilde{M} \times G \to \widetilde{M} \times G, \quad (\widetilde{x}, g) \mapsto (\widetilde{x}, g_0 g).$$

This descend to a bundle map $P_\alpha \to P_{\widetilde{\alpha}}$ since

$$\big(\widetilde{x} \cdot c, \widetilde{\alpha}(c)^{-1}(g_0 g)\big) = \big(\widetilde{x} \cdot c, g_0(\alpha(c)^{-1} g)\big).$$

One verifies that this gives an isomorphism of flat bundles. Now assume that $\mathrm{hol}_A = \alpha$, and fix a base point $p_0 \in P$. For $\widetilde{x} \in \widetilde{M}$, let $c_{\widetilde{x}} : I \to M$ be a path in M representing $\widetilde{x}$ and starting at $x_0 = \pi(p_0)$. Let $\widehat{c}_{\widetilde{x}} : I \to P$ be the horizontal lift to P, starting at p_0. Using the same ideas as in Proposition B.1.6 one finds that $\widehat{c}_{\widetilde{x}}(1)$ depends only on the homotopy class of $c_{\widetilde{x}}$. Hence, we get a well-defined map

$$\Phi : \widetilde{M} \times G \to P, \quad (\widetilde{x}, g) \mapsto \widehat{c}_{\widetilde{x}}(1) \cdot g.$$

The construction is in such a way that Φ is G-equivariant and surjective. Moreover, if $c \in \pi_1(M, x_0)$, then a straightforward calculation shows that

$$\widehat{c}_{(\widetilde{x} \cdot c)}(1) = \widehat{c}_{\widetilde{x}}(1) \cdot \mathrm{hol}_A(c, p_0),$$

and

$$\Phi^{-1}\big(\widehat{c}_{\widetilde{x}}(1)\big) = \big\{\big(\widetilde{x} \cdot c, \mathrm{hol}_A(c, p_0)^{-1}\big) \;\big|\; c \in \pi_1(M, x_0)\big\}.$$

Since we are assuming that $\mathrm{hol}_A(c, p_0) = \alpha(c)$, this implies that Φ descends to a bundle isomorphism $P_\alpha \to P$. Concerning the relation between the flat connections A and A_α we remark without further comments that Φ is defined in such a way that it maps the horizontal distribution on the trivial bundle $\widetilde{M} \times G$ to the horizontal distribution on P given by A. This implies that via the isomorphism $P_\alpha \cong P$ the connections A_α and A agree. $\square$

B.1.2 Flatness and Triviality of U(k)-Bundles

As mentioned in Remark B.1.7, a flat principal G-bundle is not necessarily trivializable. In this section we give some details for the case $G = \mathrm{U}(k)$. Via the standard representation of $\mathrm{U}(k)$ on $\mathbb{C}^k$, a principal $\mathrm{U}(k)$-bundle P defines a Hermitian vector bundle $E \to M$ and vice versa. We will freely switch between the two equivalent notions.

Flat Line Bundles. We first recall that the space of all Hermitian line bundles over a given manifold can be described in terms of Čech cohomology, see [96, Sec. III.4] for details. Let M be a compact, connected manifold. For a Lie group G we denote by $\underline{G}$ the sheaf of locally smooth functions on M with values in G. Then $H^1\big(M, \underline{\mathrm{U}}(1)\big)$ is isomorphic to the set of Hermitian line bundles over M up to isomorphism. Note that the group structure on the latter is given by the tensor products of line bundles. There is an exact sequence of sheaves.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \underline{\mathbb{R}} \xrightarrow{e^{2\pi i x}} \underline{\mathrm{U}}(1) \longrightarrow 0. \tag{B.7}$$

Since the sheaf $\underline{\mathbb{R}}$ is fine (i.e., admits partitions of unity), the cohomology $H^\bullet(M, \underline{\mathbb{R}})$ vanishes away from degree 0. The long exact sequence in cohomology then produces natural isomorphisms

$$H^p\big(M, \underline{\mathrm{U}}(1)\big) \cong H^{p+1}(M, \mathbb{Z}), \quad p \geq 1.$$

For $p = 1$, this isomorphism coincides with the integral first Chern class

$$c_1 : H^1\big(M, \underline{\mathrm{U}}(1)\big) \xrightarrow{\cong} H^2(M, \mathbb{Z}). \tag{B.8}$$

On the other hand, we have seen in Proposition B.1.8 that *flat* Hermitian line bundles are classified by representations of $\pi_1(M)$ in $\mathrm{U}(1)$. Here, conjugation does not play a role here since $\mathrm{U}(1)$ is abelian. Therefore, the moduli space of flat Hermitian line bundles has the cohomological description

$$\mathrm{Hom}\,\big(\pi_1(M), \mathrm{U}(1)\big) = H^1\big(M, \mathrm{U}(1)\big).$$

Note that in terms of Čech cohomology, $H^p\big(M, \mathrm{U}(1)\big)$ refers to the sheaf of locally constant (rather than C^∞) functions with values in $\mathrm{U}(1)$. Similarly, we have to distinguish between $H^\bullet(M, \mathbb{R})$ and $H^\bullet(M, \underline{\mathbb{R}})$. We have a long exact coefficient sequence

$$\ldots \longrightarrow H^p(M, \mathbb{Z}) \longrightarrow H^p(M, \mathbb{R}) \longrightarrow H^p\big(M, \mathrm{U}(1)\big) \longrightarrow H^{p+1}(M, \mathbb{Z}) \longrightarrow \ldots$$

Here, the integral first Chern class appears again as the map

$$c_1 : H^1\big(M, \mathrm{U}(1)\big) \longrightarrow H^2(M, \mathbb{Z}).$$

Moreover, the universal coefficient theorem shows that

$$\ker\big(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})\big) = \mathrm{Tor}\,\big(H^2(M, \mathbb{Z})\big).$$

Together with (B.8) this easily yields

Lemma B.1.9. *A line bundle $L \to M$ admits a flat connection if and only if its integral first Chern class satisfies $c_1(L) \in \operatorname{Tor}\big(H^2(M,\mathbb{Z})\big)$.*

Remark. The above result also follows from Chern-Weil theory: The representative of the first Chern class in de Rham cohomology is given by $\big[\operatorname{tr}(\frac{i}{2\pi}F_A)\big]$, where A is any U(1)-connection on L. Hence, L admits a flat connection if and only if $c_1(L)$ vanishes in $H^2(M,\mathbb{R})$. This is precisely the case if the integral first Chern class is a torsion class.

There is also a topological condition for triviality of a flat Hermitian line bundle. It is straightforward to check that the natural map of sheaves $\mathrm{U}(1) \to \underline{\mathrm{U}}(1)$ relates the two exact coefficient sequences via

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1\big(M,\underline{U}(1)\big) & \xrightarrow{c_1} & H^2(M,\mathbb{Z}) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \| & & \uparrow \\
H^1(M,\mathbb{R}) & \longrightarrow & H^1\big(M,U(1)\big) & \xrightarrow{c_1} & H^2(M,\mathbb{Z}) & \longrightarrow & H^2(M,\mathbb{R}).
\end{array}$$

Lemma B.1.10. *Let $L_\alpha \to M$ be a flat Hermitian line bundle on M with holonomy $\alpha : H_1(M,\mathbb{Z}) \to \mathrm{U}(1)$. Then L_α is trivializable if and only if the restriction of α to the torsion subgroup $\operatorname{Tor}\big(H_1(M,\mathbb{Z})\big)$ is trivial.*

Proof. The line bundle L_α is trivializable if and only if $c_1(L_\alpha) = 0$ in $H^2(M,\mathbb{Z})$. The above diagram shows that this is precisely if

$$L_\alpha \in \operatorname{im}\big[H^1(M,\mathbb{R}) \to H^1\big(M,\mathrm{U}(1)\big)\big].$$

The latter means that $\alpha = \exp(2\pi i\widehat{\alpha})$ with $\widehat{\alpha} : H_1(M,\mathbb{Z}) \to \mathbb{R}$. Since $(\mathbb{R},+)$ is a torsion free abelian group, the restriction of $\widehat{\alpha}$ to $\operatorname{Tor}\big(H_1(M,\mathbb{Z})\big)$ vanishes. Conversely, if α is trivial on $\operatorname{Tor}\big(H_1(M,\mathbb{Z})\big)$, it can be lifted to a homomorphism $\widehat{\alpha}$ as above. □

Flat U(k)-undles. If we now consider $\mathrm{U}(k)$ for $k > 1$, Lemma B.1.9 and Lemma B.1.10 do not generalize immediately, since we do not have a simple cohomological description of the space of (flat) $\mathrm{U}(k)$-bundles. This is mainly due to the fact that $H^1\big(M,\underline{\mathrm{U}}(k)\big)$ is not a group since $\mathrm{U}(k)$ is non-abelian. However, in the case that $\dim M \leq 3$ the results about flat U(1)-bundles generalize to $\mathrm{U}(k)$. The underlying reason is the following

Proposition B.1.11. *Let M be a manifold of dimension ≤ 3. Then every principal $\mathrm{SU}(k)$-bundle $P \to M$ is trivializable.*

Idea of proof. Let $E\,\mathrm{SU}(k) \to B\,\mathrm{SU}(k)$ be the universal $\mathrm{SU}(k)$-bundle, see [33, Sec. 8.6]. It has the property that $E\,\mathrm{SU}(k)$ is contractible, and the set of isomorphism classes of $\mathrm{SU}(k)$-bundles is isomorphic to the set of homotopy classes of maps $M \to B\,\mathrm{SU}(k)$. Since the total space of the universal $\mathrm{SU}(k)$-bundle is contractible, the long exact homotopy sequence yields that

$$\pi_n\big(B\,\mathrm{SU}(k)\big) = \pi_{n-1}\big(\mathrm{SU}(k)\big), \quad n \geq 1. \tag{B.9}$$

It is well known that $\mathrm{SU}(k)$ is 2-connected, see [33, Sec. 6.14]. Hence, (B.9) implies that $B\,\mathrm{SU}(k)$ is 3-connected. Since we are assuming that $\dim M \leq 3$ it follows that every map $M \to B\,\mathrm{SU}(k)$ is homotopic to a constant map. Therefore, every $\mathrm{SU}(k)$-bundle over M is trivializable. □

We then have the following generalization of Lemma B.1.9 and Lemma B.1.10.

Corollary B.1.12. *Let M be a compact, connected manifold of dimension ≤ 3, and let $E \to M$ be a Hermitian vector bundle over M of rank k.*

(i) *The bundle E admits a flat connection if and only if its integral first Chern class satisfies $c_1(E) \in \mathrm{Tor}\big(H^2(M,\mathbb{Z})\big)$.*

(ii) *Assume that E is flat with holonomy $\alpha : \pi_1(M) \to \mathrm{U}(k)$. Then E is trivializable if and only if the restriction of $\det(\alpha) : H^1(M,\mathbb{Z}) \to \mathrm{U}(1)$ to $\mathrm{Tor}\big(H_1(M,\mathbb{Z})\big)$ is trivial.*

Proof. Let $\det(E) := \Lambda^k E \to M$ denote the determinant line bundle of E. One concludes from the exact sequence

$$0 \longrightarrow \mathrm{SU}(k) \longrightarrow \mathrm{U}(k) \xrightarrow{\det} \mathrm{U}(1) \longrightarrow 0,$$

that $E \otimes \det(E)^{-1}$ admits an $\mathrm{SU}(k)$-structure. As we are assuming $\dim M \leq 3$, it follows from Proposition B.1.11 that $E \otimes \det(E)^{-1}$ is isomorphic to $M \times \mathbb{C}^k$. This implies that E is flat/trivializable if and only if $\det(E)$ is flat/trivializable. Since

$$c_1(E) = c_1\big(\det(E)\big) \in H^2(M,\mathbb{Z}),$$

part (i) readily follows from Lemma B.1.9, whereas (ii) is immediate from Lemma B.1.10. $\square$

Remark B.1.13. If Σ is a closed, oriented surface, then $H^2(\Sigma,\mathbb{Z}) \cong \mathbb{Z}$. Therefore, there are no torsion elements and every flat Hermitian vector bundle over Σ is isomorphic to the trivial bundle. For 3-manifolds there will in general exist non-trivial flat Hermitian bundles, see for example Remark B.2.10 below or Section 2.3.

B.2 Flat Connections over Mapping Tori

We let M be a closed, oriented manifold, and let $f \in \mathrm{Diff}^+(M)$ be an orientation preserving diffeomorphism. Then we define the *mapping torus* M_f of f as

$$M_f := \big(M \times \mathbb{R}\big)/\mathbb{Z},$$

where $\mathbb{Z}$ acts on $M \times \mathbb{R}$ via

$$k \cdot (x,t) = \big(f^{-k}(x), t+k\big), \quad (x,t) \in M \times \mathbb{R}, \quad k \in \mathbb{Z}. \tag{B.10}$$

We use this definition rather than defining M_f as a quotient of $M \times [0,1]$, since it is convenient to work $\mathbb{Z}$-equivariantly on $M \times \mathbb{R}$. Since we assume that f is orientation preserving, the product orientation of $M \times \mathbb{R}$ defines an orientation on M_f. The map

$$M \times \mathbb{R} \to S^1, \quad (x,t) \mapsto \exp(2\pi i t)$$

is invariant with respect to the action (B.10), and gives rise to a fiber bundle

$$M \hookrightarrow M_f \xrightarrow{\pi} S^1.$$

Let $\mathrm{Diff}_0(M)$ be the component of the identity in $\mathrm{Diff}^+(M)$ with respect to the C^∞-topology. We collect the following well-known material.

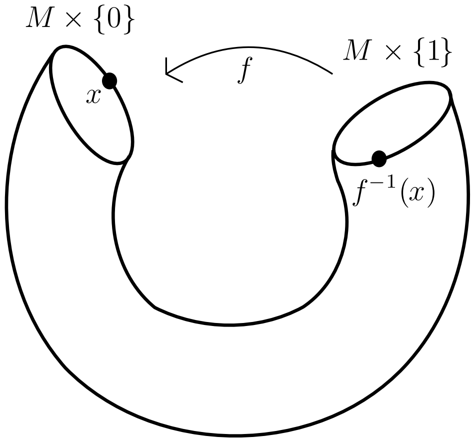

Figure B.1: The mapping torus of f

Lemma B.2.1. *The diffeomorphism class of M_f depends only on the conjugacy class of f inside the mapping class group*

$$\mathrm{Diff}^+(M)/\,\mathrm{Diff}_0(M).$$

Moreover, there exists a diffeomorphism $-M_f \cong M_{f^{-1}}$, where $-M_f$ carries the reversed orientation.

Proof. To show that the diffeomorphism class of M_f depends only on the isotopy class of f, let $f_t : [0,1] \to \mathrm{Diff}^+(M)$ be an isotopy. Possibly using a reparametrization of $[0,1]$ we may assume that f_t is constant near 0 and 1. Define $\varphi_t := f_t^{-1} \circ f_0$ and extend φ_t to a path $\varphi_t : \mathbb{R} \to \mathrm{Diff}^+(M)$ by requiring that $\varphi_{t+1} = f_1^{-1} \circ \varphi_t \circ f_0$. Then one easily checks that

$$\Phi : M \times \mathbb{R} \to M \times \mathbb{R}, \quad \Phi(x,t) := \big(\varphi_t(x), t\big),$$

is $\mathbb{Z}$-equivariant with respect to (B.10). Hence, it descends to a diffeomorphism $\Phi : M_{f_0} \to M_{f_1}$. Similarly, if $g \in \mathrm{Diff}^+(M)$, then the map

$$\Psi : M \times \mathbb{R} \to M \times \mathbb{R}, \quad \Psi(x,t) := \big(g(x), t\big)$$

defines a diffeomorphism $\Psi : M_f \to M_{gfg^{-1}}$. Thus, the diffeomorphism class of M_f depends only on the conjugacy class of f in $\mathrm{Diff}^+(M)/\,\mathrm{Diff}_0(M)$. In a similar way one verifies that an orientation reversing diffeomorphism $M_f \cong M_{f^{-1}}$ is induced by

$$M \times \mathbb{R} \to M \times \mathbb{R}, \quad (x,t) \mapsto (x,-t). \qquad \square$$

Remark B.2.2. We also want to point out that every oriented fiber bundle $M \hookrightarrow \widetilde{M} \xrightarrow{\pi} S^1$ arises in the way just described: Identify M with the fiber $\pi^{-1}(1)$, and endow M with a vertical projection $P^v : TM \to T^vM$, see Section 2.1.1. For $x \in M$ denote by $c_x : [0,1] \to M$ the horizontal lift of the path

$$[0,1] \to S^1, \quad t \mapsto e^{2\pi i t}$$

which starts at x. Then $c_x(1) \in M$, so that we can define

$$f : M \to M, \quad f(x) := c_x(1).$$

It follows from the standard theory of ordinary differential equations that f is a diffeomorphism. Moreover, since we have assumed that $M \hookrightarrow \widetilde{M} \xrightarrow{\pi} S^1$ is an oriented fiber bundle, it follows that f is orientation preserving, i.e., $f \in \mathrm{Diff}^+(M)$. One then verifies that the fiber bundle $\pi : \widetilde{M} \to S^1$ is indeed isomorphic to the fiber bundle given by the mapping torus M_f. This identification might seem to depend on the choice of P^v and the identification of M as a fiber. However, since all vertical projections are homotopic, one can check that the conjugacy class of f in the mapping class group $\mathrm{Diff}^+(M)/\mathrm{Diff}_0(M)$ does not change, so that by Lemma B.2.1 the isomorphism class of the mapping torus is unambiguously defined.

B.2.1 Algebraic Description of the Moduli Space

Let G be a connected matrix Lie group. Ultimately G will be $\mathrm{U}(k)$. We use Proposition B.1.8 to identify

$$\mathcal{M}(M_f, G) = \mathrm{Hom}\big(\pi_1(M_f), G\big)/G,$$

where G acts by conjugation. We fix a base point $x_0 \in M$ assume for simplicity that $f(x_0) = x_0$. This is possible as we can always find an element in the isotopy class of $f \in \mathrm{Diff}^+(M)$ which fixes x_0. Then the path

$$\gamma : \mathbb{R} \to M \times \mathbb{R}, \quad t \mapsto (x_0, t),$$

descends to a closed path in M_f, whose homotopy class we also denote by $\gamma \in \pi_1(M_f)$. Without including it in the notation, we are using $[x_0, 0]$ as a base point for M_f. On the other hand, the inclusion $M = M \times \{0\} \subset M \times \mathbb{R}$ induces a map

$$i_* : \pi_1(M) \to \pi_1(M_f).$$

Then the following is easily verified.[2]

Lemma B.2.3. *The fundamental group of M_f with respect to the base point $[x_0, 0]$ is given by*

$$\pi_1(M_f) = \big\langle \pi_1(M), \gamma \,\big|\, \gamma^{-1} c \gamma = f_* c,\ c \in \pi_1(M) \big\rangle,$$

where $f_* : \pi_1(M) \to \pi_1(M)$ is the induced map on the fundamental group.

The map $i_* : \pi_1(M) \to \pi_1(M_f)$ gives rise to a natural homomorphism

$$i^* : \mathrm{Hom}\big(\pi_1(M_f), G\big) \to \mathrm{Hom}\big(\pi_1(M), G\big), \quad \alpha \mapsto \alpha \circ i_*.$$

This map is G-equivariant and we get an induced map

$$[i^*] : \mathcal{M}(M_f, G) \to \mathcal{M}(M, G), \quad [\alpha] \mapsto [\alpha \circ i_*]. \tag{B.11}$$

Similarly, $f^* : \mathrm{Hom}\big(\pi_1(M), G\big) \to \mathrm{Hom}\big(\pi_1(M), G\big)$ is G-equivariant so that it descends to a map

$$[f^*] : \mathcal{M}(M, G) \to \mathcal{M}(M, G), \quad [\alpha] \mapsto [\alpha \circ f_*].$$

For the following result see also [2, Sec. 8].

[2] Recall that for fundamental groups we are using the product $[c] \cdot [\widetilde{c}] := [\widetilde{c} * c]$, where $\widetilde{c} * c$ means first $\widetilde{c}$ and then c.

Proposition B.2.4. *The natural map* (B.11) *defines a surjection*

$$[i^*] : \mathcal{M}(M_f, G) \to \mathrm{Fix}[f^*] \subset \mathcal{M}(M, G).$$

Moreover, if $\alpha \in \mathrm{Hom}\big(\pi_1(M), G\big)$ *is such that* $[\alpha] \in \mathrm{Fix}[f^*]$, *then*

$$[i^*]^{-1}[\alpha] \cong \big\{g \in G \;\big|\; g^{-1}\alpha g = f^*\alpha\big\}/Z(\alpha) \tag{B.12}$$

where $Z(\alpha) := \{h \in G \,|\, h^{-1}\alpha h = \alpha\}$ *is the centralizer of* α, *and acts on* G *by conjugation.*

Proof. With γ as in Lemma B.2.3 define

$$\Phi : \mathrm{Hom}\big(\pi_1(M_f), G\big) \to \mathrm{Hom}\big(\pi_1(M), G\big) \times G, \quad \alpha \mapsto \big(\alpha \circ i_*, \alpha(\gamma)\big).$$

Then Lemma B.2.3 implies that Φ induces an isomorphism

$$\Phi : \mathrm{Hom}\big(\pi_1(M_f), G\big) \xrightarrow{\cong} \big\{(\alpha, g) \in \mathrm{Hom}\big(\pi_1(M), G\big) \times G \;\big|\; g^{-1}\alpha g = f^*\alpha\big\}. \tag{B.13}$$

The action of G by conjugation on $\mathrm{Hom}\big(\pi_1(M_f), G\big)$ translates to to the right hand side as G acting diagonally by conjugation, i.e., $h \in G$ acts on an element (α, g) via

$$(\alpha, g) \cdot h := \big(h^{-1}\alpha h, h^{-1}gh\big).$$

Then

$$\mathcal{M}(M_f, G) \cong \big\{(\alpha, g) \in \mathrm{Hom}\big(\pi_1(M), G\big) \times G \;\big|\; g^{-1}\alpha g = f^*\alpha\big\}/G.$$

Under this identification, the map $[i^*]$ in (B.11) is given by the projection onto the first factor. Now $[\alpha] = [f^*\alpha]$ if and only if there exists $g \in G$ with $g^{-1}\alpha g = f^*\alpha$, which guarantees that

$$[i^*] : \mathcal{M}(M_f, G) \to \mathrm{Fix}[f^*] \subset \mathcal{M}(M, G), \quad [\alpha, g] \mapsto [\alpha],$$

is well-defined and surjective. The inverse image of $[\alpha] \in \mathrm{Fix}[f^*]$ is given by

$$[i^*]^{-1}[\alpha] = \big\{(\widetilde{\alpha}, g) \in \mathrm{Hom}\big(\pi_1(M), G\big) \times G \;\big|\; g^{-1}\widetilde{\alpha}g = f^*\widetilde{\alpha},\, [\widetilde{\alpha}] = [\alpha]\big\}/G.$$

Hence, if we fix a representative α, then

$$[i^*]^{-1}[\alpha] \cong \big\{g \in G \;\big|\; g^{-1}\alpha g = f^*\alpha\big\}/Z(\alpha). \qquad \square$$

Remark.

(i) If we fix a different representative $\widetilde{\alpha} = h^{-1}\alpha h$ in (B.12), then one easily checks that

$$\big\{\widetilde{g} \in G \;\big|\; \widetilde{g}^{-1}\widetilde{\alpha}\widetilde{g} = f^*\widetilde{\alpha}\big\} = \big\{h^{-1}gh \in G \;\big|\; g^{-1}\alpha g = f^*\alpha\big\}.$$

Moreover,

$$Z(\widetilde{\alpha}) = h^{-1}Z(\alpha)h.$$

This describes the relation among the isomorphisms in (B.12) for different choices of representatives for $[\alpha] \in \mathrm{Fix}[f^*]$.

(ii) We also want to point out that if we fix g with $g^{-1}\alpha g = f^*\alpha$, then

$$\big\{g \in G \;\big|\; g^{-1}\alpha g = f^*\alpha\big\} = Z(\alpha)g.$$

Furthermore, note that if $\alpha \in \mathrm{Hom}\big(\pi_1(M), G\big)$ is irreducible, then $Z(\alpha)$ coincides with the center of G,

$$Z(\alpha) = Z(G) = \big\{g \in G \;\big|\; gh = hg \quad \text{for all } h \in G\big\}.$$

B.2.2 Geometric Description of the Moduli Space

We now turn to a more geometric description of $\mathcal{M}(M_f, G)$. A related discussion, yet in a different context, can be found in [38, Sec. 5] and [90, Sec. 7]. Since we prefer to work with Hermitian vector bundles over M_f, we assume from now on that $G = \mathrm{U}(k)$.

Vector Bundles over Mapping Tori. Using the ideas of Remark B.2.2 one finds that every Hermitian vector bundle over M_f is isomorphic to a mapping torus

$$E_{\widehat{f}} := \big(E \times \mathbb{R}\big)/\mathbb{Z} \to \big(M \times \mathbb{R}\big)/\mathbb{Z} = M_f. \tag{B.14}$$

Here, $\widehat{f}$ is a bundle isomorphism covering f, and $\mathbb{Z}$ acts on $E \times \mathbb{R}$ via

$$k \cdot (e,t) = \big(\widehat{f}^{-k}(e), t+k\big), \quad (e,t) \in E \times \mathbb{R}, \quad k \in \mathbb{Z}.$$

To keep the notation simple, we are identifying $E \times \mathbb{R}$ with the pullback bundle $\pi_M^* E \to M \times \mathbb{R}$, where $\pi_M : M \times \mathbb{R} \to M$. For a fixed Hermitian vector bundle $E \to M$, we denote by $\mathcal{G}(E)$ the group of gauge transformations and by $\mathcal{G}_f(E)$ the set of isomorphism classes of bundle isomorphisms covering f. Note that there is a free and transitive action of $\mathcal{G}(E)$ on $\mathcal{G}_f(E)$, given by

$$\mathcal{G}_f(E) \times \mathcal{G}(E) \to \mathcal{G}_f(E), \quad (\widehat{f}, u) \mapsto \widehat{f} \circ u.$$

Hence, upon fixing one particular $\widehat{f} \in \mathcal{G}_f(E)$, the space $\mathcal{G}_f(E)$ is isomorphic to the group of gauge transformations $\mathcal{G}(E)$.

Remark B.2.5. If $E = M \times \mathbb{C}^k$ is the trivial bundle, the group $\mathcal{G}(E)$ coincides with $C^\infty\big(M, \mathrm{U}(k)\big)$, see Remark B.1.1. Then for $u \in C^\infty\big(M, \mathrm{U}(k)\big)$, we define

$$\widehat{f}_u : M \times \mathbb{C}^k \to M \times \mathbb{C}^k, \quad \widehat{f}_u(x,z) := \big(f(x), u(x)z\big).$$

This gives a canonical identification

$$C^\infty\big(M, \mathrm{U}(k)\big) \xrightarrow{\cong} \mathcal{G}_f(M \times \mathbb{C}^k), \quad u \mapsto \widehat{f}_u.$$

We then write E_u for the bundle defined by $\widehat{f}_u$.

Lemma B.2.6. *If $\widehat{f}_1, \widehat{f}_2 : E \to E$ are two bundle isomorphisms covering f, then $E_{\widehat{f}_1} \cong E_{\widehat{f}_2}$ if and only if there exists $\varphi_t \in C^\infty\big(\mathbb{R}, \mathcal{G}(E)\big)$ such that*

$$\varphi_{t+1} = \widehat{f}_2^{-1} \circ \varphi_t \circ \widehat{f}_1. \tag{B.15}$$

Proof. If we define

$$\Phi : E \times \mathbb{R} \to E \times \mathbb{R}, \quad \Phi(e,t) := \big(\varphi_t(e), t\big),$$

then

$$\Phi\big(\widehat{f}_1^{-1}(e), t+1\big) = \big(\varphi_{t+1} \circ \widehat{f}_1^{-1}(e), t+1\big) = \big(\widehat{f}_2^{-1} \circ \varphi_t(e), t+1\big).$$

This implies that Φ is a $\mathbb{Z}$-equivariant bundle isomorphism and so $E_{\widehat{f}_1}$ and $E_{\widehat{f}_2}$ are isomorphic. Conversely, we can lift a bundle isomorphism $E_{\widehat{f}_1} \cong E_{\widehat{f}_2}$ to a $\mathbb{Z}$-equivariant map $E \times \mathbb{R} \to E \times \mathbb{R}$ and use this to define φ_t with the required property. $\square$

Remark B.2.7. Assume that $E = M \times \mathbb{C}^k$ is a trivial bundle, and let $u, v \in C^\infty\big(M, \mathrm{U}(k)\big)$ define vector bundles E_u and E_v as in Remark B.2.5. Then (B.15) takes the form

$$\varphi_{t+1} = \widehat{f}_v^{-1} \circ \varphi_t \circ \widehat{f}_u = v^{-1}(\varphi_t \circ f)u.$$

Additional Remarks for Line Bundles. In the case that $E = L$ is a line bundle, we can also give a more topological interpretation of (B.15). Since $\mathrm{U}(1)$ is abelian, the group of gauge transformations of L is given by $\mathcal{G}(L) = C^\infty\big(M, \mathrm{U}(1)\big)$, irrespectively of whether L is trivializable or not. From the long exact sequence associated to the coefficient sequence (B.7), we get an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow C^\infty(M, \mathbb{R}) \longrightarrow C^\infty\big(M, \mathrm{U}(1)\big) \longrightarrow H^1(M, \mathbb{Z}) \longrightarrow 0. \tag{B.16}$$

Here, the map $C^\infty\big(M, \mathrm{U}(1)\big) \to H^1(M, \mathbb{Z})$ is given by

$$u \mapsto u_* \in \mathrm{Hom}\,\big(\pi_1(M), \mathbb{Z}\big) = H^1(M, \mathbb{Z}).$$

Remark B.2.8. The de Rham theorem defines a map

$$\Omega^p_{cl}(M, \mathbb{R}) \to H^p(M, \mathbb{R}),$$

where the left hand side denotes the space of closed p-forms. Since $H^1(M, \mathbb{Z})$ has no torsion, we can use this to identify

$$H^1(M, \mathbb{Z}) \cong \frac{\Omega^1_{cl}(M, \mathbb{Z})}{dC^\infty(M, \mathbb{R})}.$$

Here, $\Omega^p_{cl}(M, \mathbb{Z})$ denotes the space of closed p-forms with *integral periods*, i.e., the kernel of the projection

$$\Omega^p_{cl}(M, \mathbb{R}) \to H^p(M, \mathbb{R})/H^p(M, \mathbb{Z}).$$

Using this, the last map in (B.16) can be expressed as

$$C^\infty\big(M, \mathrm{U}(1)\big) \to H^1(M, \mathbb{Z}), \quad u \mapsto \Big[\frac{u^{-1}du}{2\pi i}\Big]. \tag{B.17}$$

Now, $u \in C^\infty\big(M, \mathrm{U}(1)\big)$ is mapped to 0 if and only if $u = \exp(2\pi i g)$ for some $g \in C^\infty(M, \mathbb{R})$, because then

$$u^{-1}du = 2\pi i dg.$$

Also note that there exists g with $u = \exp(2\pi i g)$ precisely if u is homotopic to a constant map. Then we have the following topological interpretation of (B.15).

Proposition B.2.9. *Let $L \to M$ be a Hermitian line bundle, and let $\widehat{f}_1, \widehat{f}_2 \in \mathcal{G}_f(L)$. Define $u \in \mathcal{G}(L)$ by requiring that $\widehat{f}_2 = \widehat{f}_1 \circ u$. Then the line bundles $L_{\widehat{f}_1}$ and $L_{\widehat{f}_2}$ are isomorphic if and only if*

$$\Big[\frac{u^{-1}du}{2\pi i}\Big] \in \mathrm{im}\,\big(\mathrm{Id} - f^*\big) \subset H^1(M, \mathbb{Z}). \tag{B.18}$$

Proof. Assume first that $L_{\widehat{f}_1} \cong L_{\widehat{f}_2}$. Then there exists $\varphi_t \in C^\infty\big(\mathbb{R}, \mathcal{G}(L)\big)$ as in Lemma B.2.6 such that $\varphi_{t+1} = \widehat{f}_2^{-1} \circ \varphi_t \circ \widehat{f}_1$. Now, since U(1) is abelian, $\widehat{f}_1^{-1} \circ \varphi_t \circ \widehat{f}_1 = \varphi_t \circ f$. Hence,

$$\varphi_{t+1} = u^{-1} \circ \widehat{f}_1^{-1} \circ \varphi_t \circ \widehat{f}_1 = u^{-1}(\varphi_t \circ f).$$

This yields

$$\varphi_1^{-1} d\varphi_1 = f^*(\varphi_0^{-1} d\varphi_0) - u^{-1} du \tag{B.19}$$

Moreover, since φ_0 is homotopic to φ_1 we can find $g : M \to \mathbb{R}$ such that

$$\varphi_0^{-1} d\varphi_0 = \varphi_1^{-1} d\varphi_1 + 2\pi i dg.$$

From this and (B.19) we see that (B.18) is valid. Conversely, let us assume that (B.18) holds. This means that there exist $\varphi \in C^\infty\big(M, \mathrm{U}(1)\big)$ and $g \in C^\infty(M, \mathbb{R})$ such that

$$u^{-1} du = f^*(\varphi^{-1} d\varphi) - \varphi^{-1} d\varphi + 2\pi i dg.$$

For $t \in [0,1]$ define $\varphi_t := \varphi \exp\big(-2\pi i t g\big)$. Then, upon adding a constant to g, we conclude that

$$u = \varphi_1^{-1}(\varphi_0 \circ f).$$

This allows us to extend φ_t for all $t \in \mathbb{R}$ in such a way that

$$\varphi_{t+1} = u^{-1}(\varphi_t \circ f) = \widehat{f}_2^{-1} \circ \varphi_t \circ \widehat{f}_1,$$

and we get the required isomorphism. □

Remark B.2.10.

(i) As we have mentioned in Section B.1.2, line bundles over M_f are classified by the group $H^2(M_f, \mathbb{Z})$. This group fits into an exact sequence associated to the fiber bundle $M \hookrightarrow M_f \to S^1$. One can use for example the five-term exact sequence induced by the Leray-Serre spectral sequence to obtain an exact sequence

$$\begin{aligned} \ldots \longrightarrow H^1(M) \xrightarrow{\mathrm{Id}-f^*} H^1(M) \longrightarrow H^2(M_f) \\ \xrightarrow{i^*} H^2(M) \xrightarrow{\mathrm{Id}-f^*} H^2(M) \longrightarrow \ldots \end{aligned} \tag{B.20}$$

For higher dimensional spheres as the base this is usually referred as the *Wang sequence*, see [33, p. 254].

Using the discussion preceding Proposition B.2.9 we can give a geometric interpretation of the map $H^1(M) \to H^2(M_f)$ in (B.20). First we use (B.17) to represent an element of $H^1(M, \mathbb{Z})$ by a gauge transformation $u \in C^\infty\big(M, \mathrm{U}(1)\big)$, and let $L_u \to M_f$ be the line bundle defined by u as in Remark B.2.5. Since the isomorphism class of L_u is independent of the homotopy class of u we get a well-defined map

$$H^1(M, \mathbb{Z}) \to H^2(M_f, \mathbb{Z}), \quad \left[\frac{u^{-1} du}{2\pi i}\right] \mapsto c_1(L_u).$$

Now Proposition B.2.9 identifies the kernel of this map and gives a geometric explanation for the exactness of the (B.20) at $H^1(M)$. Also, from a geometric point of view, exactness at $H^2(M_f, \mathbb{Z})$ is immediate; this simply means that a line bundle $L \to M_f$ is of the form $L = L_u$ if and only if it restricts to the trivial line bundle over M. Concerning exactness at $H^2(M)$ we observe that the restriction $L|_M$ of a line bundle $L \to M_f$ has to satisfy $f^* c_1(L|_M) = c_1(L|_M)$. Conversely, this is precisely the condition which enables us to define a line bundle over M_f.

(ii) The sequence (B.20) also shows that there will in general be flat line bundles over M_f which are topologically non-trivial. A gauge transformation $u \in C^\infty\big(M, \mathrm{U}(1)\big)$ gives a flat bundle L_u if and only if $c_1(L_u)$ is a torsion element in $H^2(M_f, \mathbb{Z})$. According to Proposition B.2.9 and part (i) of this remark, this is precisely the case if there exists $N \in \mathbb{N}$ such that

$$N \cdot \left[\frac{u^{-1} du}{2\pi i}\right] = \left[\frac{u^{-N} du^N}{2\pi i}\right] \in \operatorname{im}(\mathrm{Id} - f^*) \subset H^1(M, \mathbb{Z}).$$

Already in the case that M is the 2-torus T^2 one can easily find a diffeomorphism $f : T^2 \to T^2$ such that

$$\operatorname{coker}\big(\,\mathrm{Id} - f^* : H^1(M, \mathbb{Z}) \to H^1(M, \mathbb{Z})\big)$$

contains torsion elements, see Remark 4.3.10 in Section 4.3.

Flat Connections over Mapping Tori. For convenience we assume for the rest of this section that a flat $\mathrm{U}(k)$-bundle over M is necessarily trivializable. According to Remark B.1.13 this is satisfied for example if $M = \Sigma$ is a closed surface, which is the case we are considering in Chapter 4.

Under this assumption a flat bundle $E \to M_f$ restricts to the trivial bundle over the fiber, so that it is of the form considered in Remark B.2.5, i.e.,

$$E = E_u \quad u \in C^\infty\big(M, \mathrm{U}(k)\big),$$

where E_u is the mapping torus of the bundle isomorphism $\widehat{f}_u : M \times \mathbb{C}^k \to M \times \mathbb{C}^k$. We can then identify the space of sections of E_u as

$$C^\infty(M_f, E_u) = \big\{\varphi_t : \mathbb{R} \to C^\infty(M, \mathbb{C}^k) \;\big|\; \varphi_{t+1} = u^{-1}(\varphi_t \circ f)\big\}.$$

More generally, $\widehat{f}_u$ induces a pullback

$$\widehat{f}_u^* \alpha = u^{-1}(f^* \alpha), \quad \alpha \in \Omega^\bullet(M, \mathbb{C}^k), \tag{B.21}$$

and

$$\Omega^\bullet(M_f, E_u) = \big\{\alpha_t \in \Omega^1(M \times \mathbb{R}, \mathbb{C}^k) \;\big|\; \alpha_{t+1} = \widehat{f}_u^* \alpha_t\big\}. \tag{B.22}$$

In a similar way, a $\mathrm{U}(k)$-connection A on E_u can equivalently be described as a Lie algebra valued 1-form

$$A = a_t + b_t dt, \quad a_t \in C^\infty\big(\mathbb{R}, \Omega^1(M, \mathfrak{u}(k)\big), \quad b_t \in C^\infty\big(\mathbb{R}, C^\infty(M, \mathfrak{u}(k)\big),$$

which is $\mathbb{Z}$-equivariant in the sense that

$$a_{t+1} = \widehat{f}_u^* a_t = u^{-1}(f^* a_t) u + u^{-1} du, \quad b_{t+1} = \widehat{f}_u^* b_t = u^{-1}(b_t \circ f) u. \tag{B.23}$$

The curvature of A is given by

$$F_A = d_{M\times\mathbb{R}}A + A\wedge A = d_M a_t + a_t\wedge a_t + \big(d_M b_t - \partial_t a_t + [a_t, b_t]\big)\wedge dt.$$

Hence, if $F_{a_t}\in C^\infty\big(\mathbb{R}, \Omega^2(M,\mathfrak{u}(k)\big)$ denotes the curvature associated to the path of connections a_t over M, we have

$$F_A = F_{a_t} + \big(d_{a_t}b_t - \partial_t a_t\big)\wedge dt.$$

Therefore, A is flat if and only if

$$F_{a_t} = 0, \quad \text{and} \quad \partial_t a_t = d_{a_t}b_t. \tag{B.24}$$

Before we can describe the structure of the moduli space of flat $\mathrm{U}(k)$-bundles on M_f we need the following technical result.

Lemma B.2.11. *Let $A = a_t + b_t dt$ be a flat connection on E_u. Then there exists a gauge transformation $v\in C^\infty\big(M,\mathrm{U}(k)\big)$ such that the constant path a_0 defines a connection A_0 on E_v, and there exists an isomorphism*

$$\Phi: E_v\to E_u, \quad \textit{with} \quad \Phi^*A = A_0.$$

Proof. Let $\varphi_t:\mathbb{R}\to C^\infty\big(M,\mathrm{U}(k)\big)$ be the unique solution to

$$\partial_t\varphi_t = -b_t\varphi_t, \quad \varphi_0\equiv e,$$

where $e\in\mathrm{U}(k)$ is the identity matrix. Define $v := u\varphi_1\in C^\infty\big(M,\mathrm{U}(k)\big)$. We claim that

$$\varphi_{t+1} = u^{-1}(\varphi_t\circ f)v \quad \text{and} \quad \partial_t(a_t\cdot\varphi_t) = 0. \tag{B.25}$$

Proof of (B.25). For the first assertion we recall from (B.23) that b_t satisfies

$$b_{t+1} = u^{-1}(b_t\circ f)u.$$

From this and the definition of φ_t, one easily checks that both sides of the claimed equality are solutions of the initial value problem

$$\partial_t\widetilde{\varphi}_t = -u^{-1}(b_t\circ f)u\,\widetilde{\varphi}_t, \quad \widetilde{\varphi}_0 = u^{-1}v,$$

and hence agree. Now, using the identity $\partial_t\varphi_t^{-1} = -\varphi_t^{-1}(\partial_t\varphi_t)\varphi_t^{-1}$, we compute that

$$\begin{aligned}
\partial_t(a_t\cdot\varphi_t) &= \partial_t(\varphi_t^{-1}a_t\varphi_t + \varphi_t^{-1}d_M\varphi_t)\\
&= -\varphi_t^{-1}(\partial_t\varphi_t)\varphi_t^{-1}a_t\varphi_t + \varphi_t^{-1}(\partial_t a_t)\varphi_t + \varphi_t^{-1}a_t(\partial_t\varphi_t)\\
&\qquad - \varphi_t^{-1}(\partial_t\varphi_t)\varphi_t^{-1}d_M\varphi_t + \varphi_t^{-1}d_M(\partial_t\varphi_t)\\
&= \varphi_t^{-1}(\partial_t a_t)\varphi_t + \varphi_t^{-1}(b_t a_t)\varphi_t - \varphi_t^{-1}(a_t b_t)\varphi_t\\
&\qquad + \varphi_t^{-1}b_t d_M\varphi_t - \varphi_t^{-1}d_M(b_t\varphi_t)\\
&= \varphi_t^{-1}\big(\partial_t a_t - d_M b_t - [a_t, b_t]\big)\varphi_t = \varphi_t^{-1}\big(\partial_t a_t - d_{a_t}b_t\big)\varphi_t.
\end{aligned}$$

Since we are assuming that A is flat, condition (B.24) yields the second part of (B.25). $\square$

Having established (B.25) we continue with the proof of Lemma B.2.11. First of all we can use the first formula in (B.25) and Remark B.2.7, to deduce that the family φ_t defines a bundle isomorphism

$$\Phi : E_v \xrightarrow{\cong} E_u.$$

On the other hand, we can use (B.23) and the definition of φ_1 to compute that

$$f^* a_0 = a_1 \cdot u^{-1} = (a_1 \cdot \varphi_1) \cdot v^{-1} = a_0 \cdot v^{-1},$$

where in the last step we have used (B.25) to see that $a_t \cdot \varphi_t$ is constant. Also note that $a_0 \cdot \varphi_0 = a_0 \cdot e = a_0$. This implies that the constant path a_0 defines a connection A_0 on E_v. Moreover, $\Phi^* A$ is given by

$$\Phi^* A = a_t \cdot \varphi_t + \big(\varphi_t^{-1} b_t \varphi_t + \varphi_t^{-1} \partial_t \varphi_t\big) dt, \tag{B.26}$$

and thus, $\Phi^* A = a_t \cdot \varphi_t \equiv A_0$. □

The Moduli Space of Flat Connections. We can now state geometric version of Proposition B.2.4. Recall that we are assuming that a flat bundle over M is necessarily trivial. We let $\mathcal{F}_M$ denote the space of flat $\mathrm{U}(k)$-connections over M, and define

$$\widehat{\mathcal{M}}(M_f) := \big\{(a, v) \in \mathcal{F}_M \times C^\infty\big(M, \mathrm{U}(k)\big) \;\big|\; \widehat{f}_v^* a = a\big\}.$$

Here, $\widehat{f}_v^* a$ is defined as in (B.21). Note the similarity of the definition of $\widehat{\mathcal{M}}(M_f)$ and

$$\big\{(\alpha, g) \in \mathrm{Hom}\,\big(\pi_1(M), \mathrm{U}(k)\big) \times \mathrm{U}(k) \;\big|\; g^{-1} \alpha g = f^* \alpha\big\},$$

which we considered in (B.13) in the context of the algebraic description of $\mathcal{M}\big(M_f, \mathrm{U}(k)\big)$. There is a natural action of $C^\infty\big(M, \mathrm{U}(k)\big)$ on $\widehat{\mathcal{M}}(M_f)$, given by

$$(a, v) \cdot \varphi := \big(a \cdot \varphi, (\varphi^{-1} \circ f) v \varphi\big), \quad (a, v) \in \widehat{\mathcal{M}}(M_f), \quad \varphi \in C^\infty\big(M, \mathrm{U}(k)\big).$$

This action is well-defined since for $u := (\varphi^{-1} \circ f) v \varphi$ we have

$$\widehat{f}_u^* (a \cdot \varphi) = (\widehat{f}_v^* a) \cdot v^{-1} u (\varphi \circ f) = a \cdot \varphi.$$

For notational brevity we use the following abbreviations for the moduli spaces of flat bundles

$$\mathcal{M}(M) := \mathcal{M}\big(M, \mathrm{U}(k)\big), \quad \mathcal{M}(M_f) := \mathcal{M}\big(M_f, \mathrm{U}(k)\big).$$

We then have the following analog of of Proposition B.2.4.

Proposition B.2.12. *With respect to the natural action of $C^\infty\big(M, \mathrm{U}(k)\big)$ on $\widehat{\mathcal{M}}(M_f)$ we have*

$$\widehat{\mathcal{M}}(M_f) / C^\infty\big(M, \mathrm{U}(k)\big) \cong \mathcal{M}(M_f).$$

Moreover, the projection $\widehat{\mathcal{M}}(M_f) \to \mathcal{F}_M$ onto the first factor induces a surjection

$$[i^*] : \mathcal{M}(M_f) \to \mathrm{Fix}(f^*) \subset \mathcal{M}(M).$$

If we represent $[a] \in \mathrm{Fix}(f^*) \subset \mathcal{M}(M)$ by $a \in \mathcal{F}_M$, then

$$[i^*]^{-1}[a] \cong \big\{v \in C^\infty\big(M, \mathrm{U}(k)\big) \;\big|\; \widehat{f}_v^* a = a\big\} / I(a),$$

where $I(a) = \{g \in \mathrm{U}(k) \,|\, g^{-1} a g = a\}$ denotes the isotropy group of a in $\mathrm{U}(k)$.

Proof. Every element $(a,v) \in \mathcal{M}(M_f)$ defines a flat $\mathrm{U}(k)$-bundle (E_v, A_0) over M_f. Here, A_0 is used as in Lemma B.2.11 to denote the flat connection on E_v induced by the constant path $a : \mathbb{R} \to \Omega^1\big(M, \mathfrak{u}(k)\big)$. We define

$$\Psi : \widehat{\mathcal{M}}(M_f) \to \mathcal{M}(M_f), \quad (a,v) \mapsto [E_v, A_0].$$

If $(a,v) \in \widehat{\mathcal{M}}(M_f)$ and $\varphi \in C^\infty\big(M, \mathrm{U}(k)\big)$, then

$$\varphi = v^{-1}(\varphi \circ f)u, \quad \text{where} \quad u := (\varphi^{-1} \circ f)v\varphi.$$

According to Lemma B.2.6 this implies that $E_u \cong E_v$. Moreover, the connection A_0 on E_v pulls back to the connection $A_0 \cdot \varphi$ on E_u, see (B.26). Thus,

$$\Psi\big((a,v)\cdot\varphi\big) = \Psi(a,v),$$

and we get a well-defined map

$$\overline{\Psi} : \widehat{\mathcal{M}}(M_f)/C^\infty\big(M, \mathrm{U}(k)\big) \to \mathcal{M}(M_f).$$

To see that $\overline{\Psi}$ is surjective, let $[E_u, A] \in \mathcal{M}(M_f)$. Then Lemma B.2.11 implies that there exist $a \in \mathcal{F}_M$ and $v \in C^\infty\big(M, \mathrm{U}(k)\big)$ in such a way that E_u is isomorphic to E_v, and the connection A pulls back to the connection A_0 defined by a. By definition, this gives an element $(a,v) \in \widehat{\mathcal{M}}(M_f)$ such that $\Psi(a,v) = [E_u, A]$. To check injectivity, assume that $\Psi(a,v) = \Psi(\widetilde{a}, \widetilde{v})$. According to Lemma B.2.6 and (B.26) this means that there exists $\varphi_t : \mathbb{R} \to C^\infty\big(M, \mathrm{U}(k)\big)$ such that

$$\varphi_{t+1} = v^{-1}(\varphi_t \circ f)\widetilde{v}, \quad a \cdot \varphi_t = \widetilde{a}, \quad \text{and} \quad \varphi_t^{-1}\partial_t\varphi_t = 0.$$

In particular, $\varphi_t \equiv \varphi$ is independent of t, and

$$\widetilde{a} = a \cdot \varphi, \quad \text{and} \quad \widetilde{v} = (\varphi^{-1} \circ f)v\varphi.$$

Hence, $(\widetilde{a}, \widetilde{v}) = (a,v)\cdot\varphi$, which establishes injectivity. The rest of the proof is formally the same as the proof of Proposition B.2.4 and shall be omitted. □

B.3 Holomorphic Line Bundles over Riemann Surfaces.

In this section we discuss some aspects of closed surfaces related to complex geometry. A general reference is [96, Ch.'s I–III]. Moreover, a concise introduction can be found in [80, Sec. 1.4].

Complex Structures on Closed Surfaces. Let Σ be a closed, oriented surface. Endow Σ with a Riemannian metric g_Σ, with volume form vol_Σ of unit volume. The metric g_Σ defines the structure of a complex manifold on Σ in the following way: Let $*$ be the Hodge star operator on Σ. On $\Omega^1(\Sigma)$ it satisfies $*^2 = -1$ and thus gives an almost complex structure on Σ such that

$$\Omega^{1,0} = \{\alpha \in \Omega^1 \mid *\,\alpha = -i\alpha\} \quad \text{and} \quad \Omega^{0,1} = \{\alpha \in \Omega^1 \mid *\,\alpha = i\alpha\}. \tag{B.27}$$

Denote by $P^{1,0}$ and $P^{0,1}$ the associated projections and define the Dolbeault operators

$$\partial := P^{1,0} \circ d, \quad \bar{\partial} := P^{0,1} \circ d.$$

Using the Leibniz rule they extend to $\Omega^\bullet \to \Omega^\bullet$ and satisfy

$$\partial^2 = \bar{\partial}^2 = 0.$$

This is because $\Omega^{2,0}$ and $\Omega^{0,2}$ are trivial on 2-dimensional almost complex manifolds. Therefore, the almost complex structure is *integrable*. This means that we can find local coordinates $z = x + iy$ with holomorphic transition functions such that $*dx = dy$. In these coordinates, the metric g_Σ is conformal to the standard metric on $\mathbb{C}$, i.e.,

$$g_\Sigma = e^{2u(z,\bar{z})} dz \otimes d\bar{z}, \quad u : U \subset \Sigma \to \mathbb{R}.$$

For a proof see [94, Sec. 5.10] or [68, Thm 4.16]. The sheaf of holomorphic functions on Σ is given by

$$\mathcal{O}_\Sigma(U) = \ker \bar{\partial}|_U \subset C^\infty(U), \quad U \subset \Sigma \text{ open}.$$

Holomorphic Line Bundles. Now let $L \to \Sigma$ be a Hermitian line bundle on Σ, and let A be a unitary connection on L with associated covariant derivative

$$d_A : \Omega^0(\Sigma, L) \to \Omega^1(\Sigma, L).$$

Define the twisted Dolbeault operators by

$$\begin{aligned} \partial_A &:= P^{1,0} \circ d_A : \Omega^0(\Sigma, L) \to \Omega^{1,0}(\Sigma, L), \\ \bar{\partial}_A &:= P^{0,1} \circ d_A : \Omega^0(\Sigma, L) \to \Omega^{0,1}(\Sigma, L). \end{aligned} \tag{B.28}$$

As in the untwisted case the extension of $\bar{\partial}_A$ to $\Omega^{0,\bullet}(\Sigma, L)$ satisfies $\bar{\partial}_A^2 = 0$. We can then define a holomorphic structure on L by requiring that its sheaf of holomorphic sections is given by

$$\mathcal{O}(U, L_A) := \ker \bar{\partial}_A|_U.$$

Since $\bar{\partial}_A(fs) = (\bar{\partial}f)s + f\bar{\partial}_A s$, it is clear that $\mathcal{O}(L_A)$ is a sheaf of $\mathcal{O}_\Sigma$-modules. Moreover, it follows from elliptic theory that the space of global sections $\mathcal{O}(\Sigma, L_A) := \ker \bar{\partial}_A$ is finite dimensional.

Remark. According to the above definition, every unitary connection on a line bundle defines a holomorphic structure and one might ask whether this is a suitable definition. Indeed, one can construct a complex structure on the total space of L such that the projection $L \to \Sigma$ is holomorphic. A section $s \in C^\infty(\Sigma, L)$ is then holomorphic as a map between Σ and L if and only if $\bar{\partial}_A s = 0$, see [37, Thm. 2.1.53 & Sec. 2.2.2].

The Riemann-Roch Theorem. Although we will not need it in the main body of the thesis, we digress briefly on the famous Riemann-Roch Theorem in its version for line bundles, see [55, Thm 5.4.1] and [55, Sec. 5.6]. Let $K := T\Sigma \to \Sigma$ be the tangent bundle of Σ viewed as a complex line bundle. The metric g_Σ and the Levi-Civita connection endow K with a Hermitian metric, respectively, a holomorphic structure. K together with this structure is called the *canonical line bundle* of Σ.

Theorem B.3.1 (Riemann-Roch). *Let Σ be a closed Riemann surface of genus g. Let $L_A \to \Sigma$ be a Hermitian line bundle endowed with the holomorphic structure given by a unitary connection A. Then*

$$\dim \mathcal{O}(\Sigma, L_A) - \dim \mathcal{O}(\Sigma, K \otimes L_A^{-1}) = \deg L_A - g + 1.$$

Remark. The left hand side of the Riemann-Roch Theorem is the index of the operator

$$\bar{\partial}_A : \Omega^0(\Sigma, L_A) \to \Omega^{0,1}(\Sigma, L_A).$$

To see this, note first that by definition $\ker \bar{\partial}_A = \mathcal{O}(\Sigma, L_A)$. To identify the cokernel, we note that $\bar{\partial}_A^t = - * \partial_A *$. Recall that we are using the complex *linear* $*$ operator. Therefore, the $*$ operator maps the kernel of $\bar{\partial}_A^t$ to

$$\ker \big(\partial_A : \Omega^{0,1}(\Sigma, L_A) \to \Omega^2(\Sigma, L_A)\big).$$

Observing that $K^{-1} = (T^*\Sigma)^{0,1}$, we can interpret ∂_A as an operator

$$\partial_A : \Omega^0(\Sigma, K^{-1} \otimes L_A) \to \Omega^{1,0}(\Sigma, K^{-1} \otimes L_A).$$

Since anti-holomorphic sections of a line bundle are in 1-1 correspondence to holomorphic sections of the dual bundle we get that

$$\dim(\ker \bar{\partial}_A^t) = \dim \mathcal{O}(\Sigma, K \otimes L_A^{-1})$$

which identifies the left hand side of the Riemann-Roch Formula as an index. The right hand side is then the integral over the corresponding index density as in the Atiyah-Singer Index Theorem 1.2.16, see [13, Sec. 4.1]. □

Relation to the Signature Operator. Since we are usually dealing with the signature operator rather than the Dolbeault operator we also want to mention how they can be related. It follows from the definition (B.27) of the almost complex structure on Σ that

$$\Omega^{1,0} = \Omega^+ \cap \Omega^1 \quad \text{and} \quad \Omega^{0,1} = \Omega^- \cap \Omega^1,$$

We thus have isometries

$$\Phi_+ : \Omega^+ \to \Omega^0 \oplus \Omega^{1,0}, \quad \alpha \mapsto \sqrt{2}\alpha_{[0]} + \alpha_{[1]},$$

and

$$\Phi_- : \Omega^- \to \Omega^{0,1} \oplus \Omega^2, \quad \alpha \mapsto \alpha_{[1]} + \sqrt{2}\alpha_{[2]}.$$

Let $\bar{\partial}_A$ be the Dolbeault operator on $\Omega^\bullet(\Sigma, L_A)$. One checks that

$$\Phi_- \circ (d_A + d_A^t) \circ \Phi_+^{-1} = \sqrt{2}\bar{\partial}_A : \Omega^0 \oplus \Omega^{1,0} \to \Omega^{0,1} \oplus \Omega^2$$

and

$$\Phi_+ \circ (d + d^t) \circ \Phi_-^{-1} = \sqrt{2}\bar{\partial}_A^t : \Omega^{0,1} \oplus \Omega^2 \to \Omega^0 \oplus \Omega^{1,0}.$$

This implies

Lemma B.3.2. *The de Rham operator $d_A + d_A^t$ on Σ with values in the line bundle L_A is unitary equivalent to*

$$\sqrt{2}(\bar\partial_A + \bar\partial_A^t) : \Omega^\bullet(\Sigma, L_A) \to \Omega^\bullet(\Sigma, L_A).$$

Under this equivalence, the signature operator D_A^+ corresponds to

$$\sqrt{2}\bar\partial_A : \Omega^0(\Sigma, L_A) \oplus \Omega^{1,0}(\Sigma, L_A) \to \Omega^{0,1}(\Sigma, L_A) \oplus \Omega^2(\Sigma, L_A)$$

Remark. With Lemma B.3.2 at hand one could derive the Riemann-Roch Theorem from the Hirzebruch Signature Theorem and the Gauss-Bonnet Theorem, or vice versa. Certainly, the relation among these results becomes more complicated in higher dimensions.

The Moduli Space of Holomorphic Line Bundles. We now want to give some remarks on the notion of equivalence of holomorphic line bundles. First, recall from Section B.1.1 that the group of gauge transformations $\mathcal{G} = C^\infty\big(\Sigma, \mathrm{U}(1)\big)$ acts on the space of Hermitian connections $\mathcal{A}(L)$ on L via

$$A \cdot u = A + u^{-1}du, \quad u \in \mathcal{G}, \quad A \in \mathcal{A}(L)$$

One easily checks that the associated twisted Dolbeault operators satisfy

$$\bar\partial_{A\cdot u} = \bar\partial_A + u^{-1}\bar\partial u = u^{-1}\bar\partial_A u.$$

Therefore, for every $U \subset \Sigma$ we get an isomorphism of $\mathcal{O}_\Sigma(U)$-modules

$$\mathcal{O}(U, L_A) \to \mathcal{O}(U, L_{A\cdot u}), \quad s \mapsto u^{-1}s. \tag{B.29}$$

Because of this, gauge equivalent connections on L give rise to equivalent holomorphic structures. The converse is certainly not true. For this note that we can use any $f \in C^\infty(\Sigma, \mathbb{C}^*)$ to define a holomorphic structure on L via

$$(\bar\partial_A)_f := f^{-1}\bar\partial_A f = \bar\partial_A + f^{-1}\bar\partial f. \tag{B.30}$$

Via an isomorphism of the form (B.29), this holomorphic structure is equivalent to the one induced by A. However, the underlying connection $A + f^{-1}df$ is in general not unitary.

The relation between the moduli space of complex line bundles and the moduli space of line bundles with connection can be made very explicit. Since we want to avoid dealing with isomorphic Hermitian line bundles which are not equal, we fix one Hermitian line bundle $L \to \Sigma$ of degree 1 and let $L_k := L^{\otimes k}$, where a negative exponent means taking the tensor product of the dual. We then let $\mathcal{A}_k$ be the space of Hermitian connections on L_k and $\mathcal{A} = \bigcup_k \mathcal{A}_k$.

We now consider the complexified group $\mathcal{G}^c := C^\infty(\Sigma, \mathbb{C}^*)$ of gauge transformations. It acts on the set of holomorphic structures on L_k via (B.30). We want to lift this to an action on $\mathcal{A}_k$. The underlying idea why this should be possible is a theorem of Chern that associates to every holomorphic structure and Hermitian metric on a vector bundle a unique compatible Hermitian connection, see [96, Thm. III.2.1]. In the simple case at hand we have the following:

Lemma B.3.3. *Let $A \in \mathcal{A}_k$ with associated Dolbeault operator $\bar{\partial}_A$, and let $f \in \mathcal{G}^c$. Define*

$$A \cdot f := A + f^{-1}\bar{\partial} f - \bar{f}^{-1}\partial \bar{f}.$$

The $A \cdot f$ is a Hermitian connection satisfying

$$\bar{\partial}_{A_f} = f^{-1}\bar{\partial}_A f.$$

This defines an action of $\mathcal{G}^c$ on $\mathcal{A}_k$ which for $\mathcal{G} \subset \mathcal{G}^c$ coincides with the standard action.

Proof. It is straightforward to check that $f^{-1}\bar{\partial} f - \bar{f}^{-1}\partial \bar{f}$ is an imaginary valued 1-form on Σ. Thus, the connection $A \cdot f$ is Hermitian. Since $P^{0,1}\big(\bar{f}^{-1}\partial \bar{f}\big) = 0$, the second assertion also follows. The Leibniz rule applied to $\bar{\partial}$ and ∂ shows that $A \cdot (fg) = (A \cdot f) \cdot g$ so that we indeed get an action of $\mathcal{G}^c$ on $\mathcal{A}_k$. Moreover, a short computations shows that

$$f^{-1}\bar{\partial} f - \bar{f}^{-1}\partial \bar{f} = f^{-1}df - \partial \log |f|^2 \tag{B.31}$$

from which one readily finds that if f takes values in $\mathrm{U}(1)$, the action coincides with the usual one. $\square$

Definition B.3.4. The moduli space of *holomorphic structures* on L_k is defined as the quotient $\mathcal{A}_k/\mathcal{G}^c$. Moreover, we define the *Picard group* as

$$\mathrm{Pic}(\Sigma) := \bigcup_{k \in \mathbb{Z}} \mathcal{A}_k/\mathcal{G}^c = \mathcal{A}/\mathcal{G}^c,$$

where the group structure is induced by the tensor product of line bundles with connection.

Remark. We note without further details that we have defined $\mathcal{A}_k/\mathcal{G}^c$ and thus $\mathrm{Pic}(\Sigma)$ purely in differential geometric language. The proof that the objects we obtain coincide with the ones defined in holomorphic terms requires more work than sketched here.

Relation to the Moduli Space of Unitary Connections. We now have the ingredients to relate the moduli space $\mathcal{A}/\mathcal{G}$ of line bundles with connections and the moduli space $\mathrm{Pic}(\Sigma)$ of holomorphic line bundles. We start by recalling a structure result for $\mathcal{A}/\mathcal{G}$, see [37, Sec. 2.2.1]. Consider the map sending a connection to its Chern-Weil representative,

$$CW : \mathcal{A} \to \Omega^2(\Sigma, \mathbb{R}), \quad CW(A) := \frac{i}{2\pi} F_A.$$

Note that no trace is involved since we are dealing with line bundles. The image of CW is easily seen to coincide with the space $\Omega^2_{cl}(\Sigma, \mathbb{Z})$ of closed 2-forms with integral periods, see Remark B.2.8. Moreover, $CW(A + ia) = CW(A)$ for every closed 1-form a. These considerations produce a short exact sequence

$$0 \to \Omega^1_{cl}(\Sigma, \mathbb{R}) \to \mathcal{A} \xrightarrow{CW} \Omega^2_{cl}(\Sigma, \mathbb{Z}) \to 0.$$

We also note that the action of $\mathcal{G}$ on $\mathcal{A}$ changes a connection A by a closed 1-form with integral periods. In particular, the map CW is invariant under the action of $\mathcal{G}$. Moreover, in the case at hand

$$\Omega^1_{cl}(\Sigma, \mathbb{R})/\Omega^1_{cl}(\Sigma, \mathbb{Z}) = H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z}) = H^1\big(\Sigma, \mathrm{U}(1)\big). \tag{B.32}$$

Thus, taking quotients, we get the following exact sequence of groups

$$0 \to H^1\big(\Sigma, \mathrm{U}(1)\big) \to \mathcal{A}/\mathcal{G} \xrightarrow{CW} \Omega^2_{cl}(\Sigma, \mathbb{Z}) \to 0.$$

Remark. This exact sequence generalizes to the case of an arbitrary closed manifold M. However, the proof we sketched does not generalize immediately. For this note that in the case of a surface Σ, there are no line bundles which give torsion elements. Moreover, the first homology group $H_1(\Sigma, \mathbb{Z})$ is torsion-free, a fact we used in the second equality of (B.32). In the general case, the Chern-Weil map does not capture possible torsion. However, the universal coefficient theorem shows that

$$\mathrm{Tor}\,\big(H^2(M, \mathbb{Z})\big) = \mathrm{Tor}\,\big(H_1(M, \mathbb{Z})\big).$$

Therefore, the information about torsion is contained in

$$H^1\big(M, \mathrm{U}(1)\big) = \mathrm{Hom}\,\big(H_1(M, \mathbb{Z}), \mathrm{U}(1)\big)$$

which appears on the left hand side of the above sequence.

Returning now to holomorphic line bundles over surfaces we state the following structure result.

Proposition B.3.5. *The natural projections*

$$\mathcal{A}/\mathcal{G} \to \mathrm{Pic}(\Sigma) \quad \textit{and} \quad \Omega^2_{cl}(\Sigma, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z})$$

fit into the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1\big(\Sigma, \mathrm{U}(1)\big) & \longrightarrow & \mathcal{A}/\mathcal{G} & \xrightarrow{CW} & \Omega^2_{cl}(\Sigma, \mathbb{Z}) & \longrightarrow & 0 \\ & & \Big\| & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & H^1\big(\Sigma, \mathrm{U}(1)\big) & \longrightarrow & \mathrm{Pic}(\Sigma) & \xrightarrow{c_1} & H^2(\Sigma, \mathbb{Z}) & \longrightarrow & 0. \end{array}$$

Sketch of proof. We can write every element $f \in \mathcal{G}^c$ uniquely as

$$f = \exp(\varphi)u, \quad u \in \mathcal{G}, \quad \varphi \in C^\infty(\Sigma, \mathbb{R}).$$

It follows from (B.31) that for every $A \in \mathcal{A}$

$$A \cdot f = A + u^{-1}du + d\varphi - 2\partial\varphi.$$

Therefore, the moduli space $\mathrm{Pic}(\Sigma) = \mathcal{A}/\mathcal{G}^c$ can alternatively be described as a quotient of $\mathcal{A}/\mathcal{G}$ by the action

$$[A] \cdot \varphi := \big[A + d\varphi - 2\partial\varphi\big], \quad [A] \in \mathcal{A}/\mathcal{G}, \quad \varphi \in C^\infty(\Sigma, \mathbb{R}).$$

With respect to this, the map CW has the following equivariance property:

$$CW\big([A] \cdot \varphi\big) = CW([A]) + \frac{1}{\pi i}\bar{\partial}\partial\varphi = CW([A]) + \frac{1}{2\pi}(\Delta\varphi)\,\mathrm{vol}_\Sigma\,.$$

The latter equality is a consequence of the Kähler identities but can also be checked directly in this simple case. Therefore, we can equip the image of CW, i.e., the space $\Omega^2_{cl}(\Sigma, \mathbb{Z})$, with a natural action of $C^\infty(\Sigma, \mathbb{R})$ by defining

$$\omega \cdot \varphi := \omega + \frac{1}{2\pi}(\Delta\varphi)\,\mathrm{vol}_\Sigma, \quad \varphi \in C^\infty(\Sigma, \mathbb{R}), \quad \omega \in \Omega^2_{cl}(\Sigma, \mathbb{Z}).$$

It follows from the Hodge decomposition theorem that the quotient of this action coincides with $H^2(\Sigma, \mathbb{Z})$. The stabilizers consist of the constant functions and thus agree with the stabilizers of the action of $C^\infty(\Sigma, \mathbb{R})$ on $\mathcal{A}/\mathcal{G}$. Moreover, the action of $C^\infty(\Sigma, \mathbb{R})$ on the fiber $H^1\big(\Sigma, \mathrm{U}(1)\big)$ is trivial. Therefore, taking quotients in the equivariant exact sequence

$$0 \to H^1\big(\Sigma, \mathrm{U}(1)\big) \to \mathcal{A}/\mathcal{G} \xrightarrow{CW} \Omega^2_{cl}(\Sigma, \mathbb{Z}) \to 0$$

gives the exact sequence

$$0 \to H^1\big(\Sigma, \mathrm{U}(1)\big) \to \mathrm{Pic}(\Sigma) \xrightarrow{c_1} H^2(\Sigma, \mathbb{Z}) \to 0$$

and thus the requested diagram. □

Remark. As in the case of Hermitian line bundles, there is a description of $\mathrm{Pic}(\Sigma)$ in terms of Čech cohomology. Let $\mathcal{O}^*_\Sigma$ be the sheaf of nowhere vanishing holomorphic functions on Σ. Then

$$\mathrm{Pic}(\Sigma) = H^1(\Sigma, \mathcal{O}^*_\Sigma),$$

see [96, Lem. III.4.4]. Moreover, the exponential sequence $\mathbb{Z} \to \mathcal{O}_\Sigma \to \mathcal{O}^*_\Sigma$ gives rise to a long exact sequence in cohomology

$$... \to H^p(\Sigma, \mathbb{Z}) \to H^p(\Sigma, \mathcal{O}_\Sigma) \to H^p(\Sigma, \mathcal{O}^*_\Sigma) \to H^{p+1}(\Sigma, \mathbb{Z}) \to ...$$

The sheaf $\mathcal{O}_\Sigma$ is not fine so that the above sequence contains much more information than its smooth version discussed in Section B.1.2. In the case at hand, this sequence is essentially the second line of the diagram in Proposition B.3.5. To see this, note that as Σ is complex 1-dimensional, we have $H^2(\Sigma, \mathcal{O}_\Sigma) = 0$. Thus, the above sequence reads

$$H^1(\Sigma, \mathbb{Z}) \to H^1(\Sigma, \mathcal{O}_\Sigma) \to \mathrm{Pic}(\Sigma) \to H^2(\Sigma, \mathbb{Z}) \to 0.$$

Moreover, the space $H^1(\Sigma, \mathcal{O}_\Sigma)$ is isomorphic to the Dolbeault cohomology group $H^{0,1}(\Sigma)$. Via Hodge theory, the latter can be identified with $H^1(\Sigma, \mathbb{R})$. Using (B.32) we arrive at the sequence of Proposition B.3.5.

Appendix C

Some Computations

Here, we include a computational discussion which will be used in the main body of this thesis. We introduce basic Eta and Zeta functions, and derive some of their values. In the second part of this appendix, we briefly discuss the Dedekind sums and their generalization which we use in Section 4.4. In particular, we establish the relation among them which we need to prove Theorem 4.4.20.

C.1 Values of Zeta and Eta Functions

C.1.1 The Gamma and the Hurwitz Zeta Function

We will need some standard facts concerning the Gamma function and the generalization by Hurwitz of the Riemann Zeta function, see for example [29, Ch. 9] as a general reference. Recall that the integral representation of the Gamma function is

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \quad \mathrm{Re}(s) > 0.$$

It satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s), \tag{C.1}$$

which can be used to extend $\Gamma(s)$ meromorphically to the whole plane. Then $\Gamma(s)$ has no zeros and only simple poles at $s = 0, -1, -2, \ldots$ The residues are given by

$$\mathrm{Res}\,\Gamma(s)\big|_{s=-n} = \frac{(-1)^n}{n!}, \quad n \in \mathbb{N}. \tag{C.2}$$

The *Hurwitz Zeta function* is defined for $q \in \mathbb{R}^+$ as

$$\zeta_q(s) := \sum_{n=0}^\infty \frac{1}{(n+q)^s}, \quad \mathrm{Re}(s) > 1. \tag{C.3}$$

In particular, $\zeta_1(s)$ is the Riemann Zeta function. Using the Mellin transform one can derive basic properties of $\zeta_q(s)$. First, we note that

$$\frac{1}{(n+q)^s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t(n+q)} dt.$$

The formula for the geometric series shows that for $t > 0$,

$$\sum_{n=0}^{\infty} e^{-t(n+q)} = \frac{e^{-tq}}{1-e^{-t}}, \tag{C.4}$$

which clearly decays exponentially with t as $t \to \infty$.

Bernoulli Polynomials. The behaviour as $t \to 0$ of (C.4) is determined by the *Bernoulli polynomials* $B_n(x)$. Recall, e.g. from [29, Sec. 9.1], that they can be defined via the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi, \quad x \in \mathbb{R}. \tag{C.5}$$

Comparing this with (A.5), we note that the Bernoulli numbers with respect to the normalization we are using are given by $B_n = B_n(0)$. Moreover, expanding e^{xt} in (C.5) and comparing coefficients of t^n yields that

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

In particular,

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}. \tag{C.6}$$

From the definition (C.5), one easily finds that the Bernoulli polynomials have the symmetry property

$$B_n(1-x) = (-1)^n B_n(x). \tag{C.7}$$

Values of the Hurwitz Zeta Function. It now follows from (C.5) and (C.7) applied to (C.4) that for $t \in (0, 2\pi)$

$$\sum_{n=0}^{\infty} e^{-t(n+q)} = \frac{e^{(1-q)t}}{e^t - 1} = \sum_{n=0}^{\infty} (-1)^n B_n(q) \frac{t^{n-1}}{n!}.$$

In particular, $\sum_{n=0}^{\infty} e^{-t(n+q)} = O(t^{-1})$ as $t \to 0$. This implies that we can apply the Mellin transform to (C.3) and interchange summation and integration. Then splitting the integral into $\int_0^1 + \int_1^\infty$ one easily obtains that for $\mathrm{Re}(s) > 1$

$$\Gamma(s)\zeta_q(s) = h(s) + \sum_{n=0}^{\infty} \frac{(-1)^n B_n(q)}{(s+n-1)n!},$$

where $h(s)$ can be extended to a holomorphic function of $s \in \mathbb{C}$. Therefore, $\Gamma(s)\zeta_q(s)$ extends to a meromorphic function on the whole plane. It has simple poles at the points $s = 1, 0, -1, -2, \dots$ with residues

$$\mathrm{Res}\left(\Gamma(s)\zeta_q(s)\right)\big|_{s=-n+1} = (-1)^n \frac{B_n(q)}{n!}, \quad n \in \mathbb{N}. \tag{C.8}$$

Now, since $\Gamma(s)$ has no zeros, we can deduce that $\zeta_q(s)$ extends to a meromorphic function on the whole plane which can have only simple poles. Using (C.2) and (C.8) one finds that $\zeta_q(s)$ has a simple pole at $s = 1$ with residue 1. The other poles are cancelled out by the zeroes of $\Gamma(s)^{-1}$ and

$$\zeta_q(-n) = -\frac{B_{n+1}(q)}{n+1}, \quad n \in \mathbb{N}. \tag{C.9}$$

C.1.2 An Eta Function and a "Periodic" Zeta Function

Definition C.1.1. For $x \in \mathbb{R}$ let $[x]$ denote the largest integer less or equal than x. We define the *n-th periodic Bernoulli function* as

$$P_n(x) := \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ B_n\big(x - [x]\big), & \text{if } x \notin \mathbb{Z}, \end{cases} \quad \text{for } n \text{ odd},$$

and

$$P_n(x) := B_n\big(x - [x]\big), \quad \text{for } n \text{ even}.$$

Remark. The definition of $P_n(x)$ for odd n is a bit artificial. Note that (C.7) implies that $B_n(1) = (-1)^n B_n(0)$, so that for n even, we have $B_n(1) = B_n(0)$. We note without proof, that for odd n with $n \neq 1$, one has $B_n(1) = B_n(0) = 0$ so that a distinction is unnecessary. However, when working with $P_1(x)$, the above convention is convenient. Using (C.6) we note that explicitly,

$$P_1(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ x - [x] - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \end{cases} \quad P_2(x) = \big(x - [x]\big)^2 - \big(x - [x]\big) + \tfrac{1}{6}. \tag{C.10}$$

Most of our computations of Eta invariants in the main body of this thesis will be based on the following result.

Proposition C.1.2. *Let $q \in \mathbb{R}$, and define for* $\mathrm{Re}(s) > 1$

$$\eta_q(s) := \sum_{\substack{n \in \mathbb{Z} \\ n \neq q}} \frac{\operatorname{sgn}(n-q)}{|n-q|^s}, \quad \textit{and} \quad \widetilde{\zeta}_q(s) := \sum_{\substack{n \in \mathbb{Z} \\ n \neq q}} \frac{1}{|n-q|^s}.$$

(i) *The function $\eta_q(s)$ extends to a holomorphic function for all $s \in \mathbb{C}$, and*

$$\eta_q(0) = 2P_1(q).$$

(ii) *The function $\widetilde{\zeta}_q(s)$ extends to a meromorphic function with only one simple pole at $s = 1$. Moreover,*

$$\widetilde{\zeta}_q(0) = \begin{cases} 0, & \text{if } q \notin \mathbb{Z}, \\ -1, & \text{if } q \in \mathbb{Z}, \end{cases} \qquad \zeta_q(-1) = -P_2(q).$$

Proof. Write $q_0 := q - [q] \in [0, 1)$. Since the sums defining $\eta_q(s)$ and $\widetilde{\zeta}_q(s)$ converge absolutely for $\mathrm{Re}(s) > 1$, we can change the order of summation. Then, if $q \in \mathbb{Z}$ so that $q_0 = 0$, we find that

$$\eta_q(s) = \zeta_1(s) - \zeta_1(s) = 0, \quad \text{and} \quad \widetilde{\zeta}_q(s) = 2\zeta_1(s).$$

If $q \notin \mathbb{Z}$ we get

$$\eta_q(s) = \sum_{n=1}^{\infty} \left(\tfrac{1}{n-q_0}\right)^s - \sum_{n=0}^{\infty} \left(\tfrac{1}{n+q_0}\right)^s = \zeta_{1-q_0}(s) - \zeta_{q_0}(s),$$

and

$$\widetilde{\zeta}_q(s) = \sum_{n=1}^{\infty} \left(\tfrac{1}{n-q_0}\right)^s + \sum_{n=0}^{\infty} \left(\tfrac{1}{n+q_0}\right)^s = \zeta_{1-q_0}(s) + \zeta_{q_0}(s).$$

Since $\zeta_1(s)$, $\zeta_{1-q_0}(s)$ and $\zeta_{q_0}(s)$ extend to meromorphic functions on the whole plane, with only one simple pole at $s = 1$ of residue 1, we can extend $\eta_q(s)$ and $\widetilde{\zeta}_q(s)$ meromorphically. One sees that $\eta_q(s)$ has no pole, whereas $\widetilde{\zeta}_q(s)$ has a simple pole at $s = 1$. Moreover, $\eta_q(s)$ vanishes if $q \in \mathbb{Z}$. Otherwise, we deduce from (C.9), (C.6) and (C.7) that

$$\eta_q(0) = \zeta_{1-q_0}(s) - \zeta_{q_0}(s) = -B_1(1-q_0) + B_1(q_0) = 2B_1(q_0).$$

From the definition of $P_1(q)$ and q_0, part (i) follows. Concerning part (ii), we first assume that $q_0 = 0$. Then

$$\widetilde{\zeta}_q(0) = -2B_1(1) = -1, \quad \text{and} \quad \widetilde{\zeta}_q(-1) = -2\tfrac{B_2(1)}{2} = -B_2(0).$$

For $q_0 \neq 0$, one finds that

$$\widetilde{\zeta}_q(0) = -B_1(1-q_0) - B_1(q_0) = B_1(q_0) - B_1(q_0) = 0,$$

and

$$\widetilde{\zeta}_q(-1) = -\tfrac{1}{2}\big(B_2(1-q_0) + B_2(q_0)\big) = -B_2(q_0). \qquad \square$$

Remark. Clearly, one could go on without difficulty, and determine more values of $\eta_q(s)$ and $\widetilde{\zeta}_q(s)$ in terms of the periodic Bernoulli functions. However, Proposition C.1.2 covers all the cases we are interested in.

C.2 Generalized Dedekind Sums

When studying the Eta invariant for 2-dimensional torus bundles over the circle, one naturally encounters versions of the *Dedekind sums*. In this section, we include some relevant definitions and computations. We start to collect some facts about finite Fourier series, see [12, Ch. 7].

C.2.1 Some Finite Fourier Analysis

Let $c \in \mathbb{Z}$ with $c \neq 0$. In this section we will always use the c-th root of unity $\xi := \exp\left(\frac{2\pi i}{c}\right)$.

Definition C.2.1. Assume that $f : \mathbb{Z} \to \mathbb{C}$ is *c-periodic*, i.e., $f(k+c) = f(k)$ for all $k \in \mathbb{Z}$. The *Fourier transform* $\widehat{f} : \mathbb{Z} \to \mathbb{C}$ is defined as

$$\widehat{f}(p) = \sum_{k=0}^{|c|-1} f(k)\xi^{-kp}, \quad p \in \mathbb{Z}.$$

Remark C.2.2. Since we allow c to be negative, one has to be a bit careful concerning signs. Let $\varepsilon = \operatorname{sgn}(c)$ and denote by $\widehat{f}^{\varepsilon}$ the Fourier transform of f with respect to the c-th root of unity $\xi^{\varepsilon} = \exp\left(\frac{2\pi i}{|c|}\right)$. Then for all $p \in \mathbb{Z}$

$$\widehat{f^{\varepsilon}}(p) = \sum_{k=0}^{|c|-1} f(k)(\xi^{\varepsilon})^{-kp} = \widehat{f}(\varepsilon p).$$

Since ξ^{-kp} is c-periodic in p, the Fourier transform is again c-periodic. Moreover, we can shift the sum by any $m \in \mathbb{Z}$, i.e.,

$$\widehat{f}(p) = \sum_{k=m}^{|c|+m-1} f(k)\xi^{-kp}.$$

This implies that if $g(k) := f(k+m)$, then

$$\widehat{g}(p) = \sum_{k=0}^{|c|-1} f(k+m)\xi^{-kp} = \sum_{k=0}^{|c|-1} f(k)\xi^{-(k-m)p} = \xi^{mp}\widehat{f}(p). \tag{C.11}$$

Furthermore, if $a \in \mathbb{Z}$ with $\gcd(a,c) = 1$, then $\{ak \mid k = 0, \dots, |c|-1\}$ is a representation system of $\mathbb{Z}$ modulo c, so that

$$\widehat{f}(p) = \sum_{k=0}^{|c|-1} f(ak)\xi^{-akp}, \quad \gcd(a,c) = 1.$$

Hence, if $d \in Z$ is an inverse of a modulo c, i.e., $ad \equiv 1\,(c)$, then $g(k) := f(ak)$ satisfies

$$\widehat{g}(p) = \sum_{k=0}^{|c|-1} f(ak)\xi^{-kp} = \sum_{k=0}^{|c|-1} f(k)\xi^{-kdp} = \widehat{f}(dp). \tag{C.12}$$

The finite geometric series yields that

$$\sum_{k=0}^{|c|-1} \xi^{kp} = \begin{cases} |c|, & \text{if } p \equiv 0\,(c), \\ 0 & \text{otherwise.} \end{cases}$$

From this one easily deduces the *Fourier inversion formula*

$$f(k) = \tfrac{1}{|c|} \sum_{p=0}^{|c|-1} \widehat{f}(p)\xi^{pk}, \tag{C.13}$$

see [12, Thm. 7.2]. Moreover, if g is another c-periodic function, one has the convolution formulæ

$$(f * g)(k) := \sum_{l=0}^{|c|-1} f(l)g(k-l) = \tfrac{1}{|c|} \sum_{p=0}^{|c|-1} \widehat{f}(p)\widehat{g}(p)\xi^{pk}, \tag{C.14}$$

and

$$(\widehat{f} * \widehat{g})(p) = |c| \sum_{k=0}^{|c|-1} f(k)g(k)\xi^{-pk}, \tag{C.15}$$

see [12, Thm 7.10].

The facts we have collected so far are sufficient for the application to generalized Dedekind sums in Section C.2.2 below. Yet, we need to compute the Fourier transform for one particular class of functions, which form the building blocks of generalized Dedekind sums. First, we introduce some notation. Note that fixing a pair (a, c) with $\gcd(a, c) = 1$, and an inverse d of a modulo c is the same as fixing a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad c \neq 0.$$

Here, b is uniquely determined by requiring that $ad - bc = 1$. Moreover, for $x, y \in \mathbb{R}$ we define

$$\begin{pmatrix} x' \\ y' \end{pmatrix} := M^t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}. \tag{C.16}$$

Proposition C.2.3. *Let P_1 be the first periodic Bernoulli function, see Definition C.1.1. Fix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$, let $x, y \in \mathbb{R}$ and x' as in* (C.16)*. Define a c-periodic function by*

$$f_{x,y,a,c}(k) := P_1\left(a\frac{k + x}{c} + y\right), \quad k \in \mathbb{Z}.$$

Then,

$$\widehat{f}_{x,y,a,c}(p) = \operatorname{sgn}(c) \begin{cases} P_1(x'), & \text{if } p \equiv 0\,(c), \\ \frac{1}{2}\Big(i \cot\left(\frac{\pi d p}{c}\right) - \delta(x')\Big)\xi^{d[x']p}, & \text{otherwise.} \end{cases}$$

Here, δ is the characteristic function of $\mathbb{R} \setminus \mathbb{Z}$, i.e., $\delta(x') = 0$ if $x' \in \mathbb{Z}$ and $\delta(x') = 1$ if $x' \notin \mathbb{Z}$. Moreover, as in Definition C.1.1, the expression $[x']$ refers to the largest integer less or equal than x'.

Before we give the proof of Proposition C.2.3 let us collect some special cases.

Corollary C.2.4.

(i) *If $f(k) := f_{0,0,1,c}(k) = P_1\left(\frac{k}{c}\right)$, then*

$$\widehat{f}(p) = \begin{cases} 0, & \text{if } p \equiv 0\,(c), \\ \frac{i}{2} \cot\left(\frac{\pi p}{|c|}\right), & \text{otherwise.} \end{cases} \tag{C.17}$$

In particular,

$$\sum_{k=1}^{|c|-1} P_1\left(\tfrac{k}{c}\right) = 0. \tag{C.18}$$

(ii) *Let $x \in \mathbb{R}$, and $f_x(k) := f_{x,0,1,c}(k) = P_1\left(\frac{k+x}{c}\right)$. Then*

$$\widehat{f}(p) = \operatorname{sgn}(c) \begin{cases} P_1(x), & \text{if } p \equiv 0\,(c), \\ \frac{1}{2}\Big(i \cot\left(\frac{\pi p}{c}\right) - \delta(x)\Big)\xi^{[x]p}, & \text{otherwise.} \end{cases} \tag{C.19}$$

Proof of Proposition C.2.3. The proof consists of proving the special cases (C.17) and (C.19) first. The general case then follows using (C.11) and (C.12). Formula (C.17) is standard, see [12, Lem 7.3]. Nevertheless, we sketch a proof, since most computations we encounter are deduced from this formula. We assume first that $c > 0$. For $k \in \{0, \ldots, c - 1\}$ we have

$$f(k) = P_1\left(\tfrac{k}{c}\right) = \begin{cases} 0, & \text{if } k = 0, \\ \frac{k}{c} - \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Therefore, for $p \equiv 0\,(c)$,

$$\widehat{f}(p) = \sum_{k=1}^{c-1} \left(\tfrac{k}{c} - \tfrac{1}{2}\right) = \tfrac{1}{c}\tfrac{c(c-1)}{2} - \tfrac{c-1}{2} = 0,$$

which is also the claim in (C.18). Now, for p not divisible by c, we have

$$\widehat{f}(p) = \sum_{k=1}^{c-1} \left(\tfrac{k}{c} - \tfrac{1}{2}\right)\xi^{-kp} = -\tfrac{1}{2}\sum_{k=1}^{c-1} \xi^{-kp} + \frac{\partial}{\partial x}\Big|_{x=-2\pi i p} \sum_{k=1}^{c-1} \exp\left(\tfrac{kx}{c}\right).$$

For $x \notin 2\pi i c\mathbb{Z}$, the formula for the finite geometric series states that

$$\sum_{k=1}^{c-1} \exp\left(\tfrac{kx}{c}\right) = \frac{1 - e^{x}}{1 - e^{x/c}} - 1.$$

Using this one verifies that

$$\frac{\partial}{\partial x}\Big|_{x=-2\pi i p} \sum_{k=1}^{c-1} \exp\left(\tfrac{kx}{c}\right) = -\frac{1}{1-\xi^{-p}} = -\frac{\xi^{p/2}}{\xi^{p/2} - \xi^{-p/2}} = \tfrac{1}{2}\left(i \cot\left(\tfrac{\pi p}{c}\right) - 1\right).$$

Moreover, for p not divisible by c, one has $\sum_{k=1}^{c-1} \xi^{-kp} = -1$. Therefore, in this case

$$\widehat{f}(p) = \tfrac{1}{2} + \tfrac{1}{2}\left(i \cot\left(\tfrac{\pi p}{c}\right) - 1\right) = \tfrac{i}{2}\cot\left(\tfrac{\pi p}{c}\right),$$

which proves (C.17) for $c > 0$. For general $c \neq 0$, let $\varepsilon = \operatorname{sgn}(c)$. Then $P_1\left(\tfrac{k}{c}\right) = \varepsilon P_1\left(\tfrac{k}{|c|}\right)$, so that we deduce from the case $c > 0$ and Remark C.2.2 that

$$\widehat{f}(p) = \begin{cases} 0, & \text{if } p \equiv 0\,(c), \\ \tfrac{i}{2}\varepsilon\cot\left(\tfrac{\pi \varepsilon p}{|c|}\right), & \text{otherwise.} \end{cases}$$

Since the cotangent is an odd function, we obtain (C.17) for $c < 0$ as well.

Concerning (C.19), we assume first that $x \in [0, 1)$, and again that $c > 0$. Then

$$f_x(k) = \begin{cases} P_1\left(\tfrac{x}{c}\right), & \text{if } k \equiv 0\,(c), \\ P_1\left(\tfrac{k}{c}\right) + \tfrac{x}{c}, & \text{otherwise.} \end{cases} \tag{C.20}$$

Therefore, for $p \equiv 0\,(c)$,

$$\widehat{f}_x(p) = P_1\left(\tfrac{x}{c}\right) + \sum_{k=1}^{c-1} \tfrac{x}{c} + \sum_{k=1}^{c-1} P_1\left(\tfrac{k}{c}\right) = P_1\left(\tfrac{x}{c}\right) + \tfrac{c-1}{c}x,$$

where we have used (C.18). Now, one easily verifies that for $x \in [0, 1)$

$$P_1\left(\tfrac{x}{c}\right) + \tfrac{c-1}{c}x = P_1(x),$$

which implies (C.19) for the case $p \equiv 0\,(c)$. If p is not divisible by c we use (C.20) and (C.17) to deduce that

$$\widehat{f}_x(p) = P_1\left(\tfrac{x}{c}\right) + \tfrac{x}{c}\sum_{k=1}^{c-1} \xi^{-kp} + \sum_{k=1}^{c-1} P_1\left(\tfrac{k}{c}\right)\xi^{-kp} = P_1\left(\tfrac{x}{c}\right) - \tfrac{x}{c} + \tfrac{i}{2}\cot\left(\tfrac{\pi p}{c}\right).$$

Since $P_1\big(\frac{x}{c}\big) - \frac{x}{c} = -\frac{1}{2}\delta(x)$, formula (C.19) follows for $x \in [0,1)$. For arbitrary $x \in \mathbb{R}$ we write $x = [x] + x_0$, so that

$$f_x(k) = f_{x_0}\big(k + [x]\big), \quad x_0 \in [0,1).$$

Since we have already proved (C.19) for f_{x_0}, we can use (C.11) to get the required formula for f_x. For this note that in the case $p \equiv 0\,(c)$ we have $\xi^{[x]p} = 1$. The case $c < 0$ follows as before, using Remark C.2.2 and the fact that f_x is odd.

Now for the general formula of Proposition C.2.3 observe that

$$f_{x,y,a,c}(k) = P_1\big(a\tfrac{k+x}{c} + y\big) = P_1\big(\tfrac{ak+ax+cy}{c}\big) = P_1\big(\tfrac{ak+x'}{c}\big) = f_{x'}(ak),$$

where $x' = ax + cy$. Then (C.12) implies that

$$\widehat{f}_{x,y,a,c}(p) = \widehat{f}_{x'}(dp),$$

so that Proposition C.2.3 follows immediately from (C.19). □

C.2.2 Relation Among Some Dedekind Sums

Let $M = \big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) \in \mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$. Recall that the classical *Dedekind sums* are defined by

$$s(a,c) := \sum_{k=1}^{|c|-1} P_1\big(\tfrac{ak}{c}\big) P_1\big(\tfrac{k}{c}\big), \tag{C.21}$$

see e.g., [12, p. 128]. Since the first periodic Bernoulli function is odd, we can replace c with $|c|$ in both denominators of (C.21). Moreover, using that $\{ap \,|\, p = 0, \ldots, |c|-1\}$ is a representation system for $\mathbb{Z}$ modulo c, and that $ad \equiv 1\,(c)$, one can replace a with d, so that

$$s(a,c) = s(a,|c|) = s(d,c) = s(d,|c|). \tag{C.22}$$

There are many more relations among different Dedekind sums but their discussion would lead to far afield. We only mention that (C.17) and (C.15) easily imply the classical *cotangent formula*

$$s(a,c) = \frac{1}{|c|} \sum_{p=1}^{|c|-1} \tfrac{i}{2} \cot\big(\tfrac{dp}{|c|}\big) \tfrac{i}{2} \cot\big(-\tfrac{p}{|c|}\big) = \frac{1}{4|c|} \sum_{p=1}^{|c|-1} \cot\big(\tfrac{dp}{c}\big) \cot\big(\tfrac{p}{c}\big),$$

The Dedekind sums $s(a,c)$ were generalized in several ways. The generalization we are interested in was considered in [35, 70, 85].

Definition C.2.5. For $x, y \in \mathbb{R}$ define

$$s_{x,y}(a,c) := \sum_{k=0}^{|c|-1} P_1\big(a\tfrac{k+x}{c} + y\big) P_1\big(\tfrac{k+x}{c}\big).$$

Again, there are several relations among generalized Dedekind sums for different values of (x, y) and (a, c). For example, $s_{x,y}(a, c)$ depends on (x, y) only modulo $\mathbb{Z}^2$. For y this is immediate and for x this is because for $m \in \mathbb{Z}$, one has

$$s_{x+m,y}(a,c) = \sum_{k=-m}^{|c|-m-1} P_1\big(a\tfrac{k+x}{c}+y\big)P_1\big(\tfrac{k+x}{c}\big) = \sum_{k=0}^{|c|-1} P_1\big(a\tfrac{k+x}{c}+y\big)P_1\big(\tfrac{k+x}{c}\big). \tag{C.23}$$

We will not include a separate treatment of other relations but give an ad hoc explanation whenever we are using them. However, what we want to single out is the following straightforward consequence of Proposition C.2.3 and (C.15):

Lemma C.2.6. *Let x' be as in* (C.16)*, and let δ be the characteristic function of $\mathbb{R} \setminus \mathbb{Z}$. Then*

$$s_{x,y}(a,c) = \tfrac{1}{|c|}P_1(x)P_1(x') - \tfrac{1}{4|c|}\sum_{p=1}^{|c|-1}\Big(i\cot\big(\tfrac{\pi p}{c}\big) - \delta(x)\Big) \times \Big(i\cot\big(\tfrac{\pi dp}{c}\big) + \delta(x')\Big)\xi^{([x]-d[x'])p}.$$

We will now make a simplifying assumption, which renders the following formulæ a bit more transparent.

Assumption. From now on, $(x, y) \in \mathbb{R}^2$ will always be chosen in such a way that $x \in [0, 1)$ and

$$\begin{pmatrix} m \\ n \end{pmatrix} := (\mathrm{Id} - M^t)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - x' \\ y - y' \end{pmatrix} \in \mathbb{Z}^2. \tag{C.24}$$

Note that under this assumption we have $x - x' = m \in \mathbb{Z}$ and $[x] = 0$, which yields that $[x'] = -m$ and $x' - [x'] = x$. Thus, the cotangent formula of Lemma C.2.6 simplifies to

$$s_{x,y}(a,c) = \tfrac{1}{|c|}P_1(x)^2 - \tfrac{1}{4|c|}\sum_{p=1}^{|c|-1}\Big(i\cot\big(\tfrac{\pi p}{c}\big) - \delta(x)\Big)\Big(i\cot\big(\tfrac{\pi dp}{c}\big) + \delta(x)\Big)\xi^{dmp}. \tag{C.25}$$

We then have the following relation between the generalized Dedekind sums and the classical Dedekind sums, see also [70, Sec. 8] and [71, Sec. 4].

Proposition C.2.7. *Under the above assumption, let $r \in \{0, \ldots |c| - 1\}$ with $m \equiv r\,(c)$. Then*

$$\begin{aligned} s_{x,y}(a,c) - s(a,c) = \tfrac{1}{|c|}\big(P_2(x) - \tfrac{1}{6}\big) + \sum_{k=1}^{|c|-r} P_1\big(\tfrac{dk}{|c|}\big) + \tfrac{1}{2}P_1\big(\tfrac{dm}{|c|}\big) \\ + \tfrac{1}{2}\delta(x)\Big(P_1\big(\tfrac{m}{|c|}\big) - P_1\big(\tfrac{dm}{|c|}\big)\Big) + \tfrac{1}{4}\delta(x)\big(1 - \delta\big(\tfrac{m}{c}\big)\big). \end{aligned}$$

Proof. The cotangent formula (C.25) shows that

$$\begin{aligned} s_{x,y}(a,c) = \tfrac{1}{|c|}P_1(x)^2 + \tfrac{1}{4|c|}\sum_{p=1}^{|c|-1}\cot\big(\tfrac{\pi p}{c}\big)\cot\big(\tfrac{\pi dp}{c}\big)\xi^{dmp} + \tfrac{1}{4|c|}\delta(x)\sum_{p=1}^{|c|-1}\xi^{dmp} \\ + \tfrac{1}{2|c|}\delta(x)\sum_{p=1}^{|c|-1}\tfrac{i}{2}\cot\big(\tfrac{\pi dp}{c}\big)\xi^{dmp} - \tfrac{1}{2|c|}\delta(x)\sum_{p=1}^{|c|-1}\tfrac{i}{2}\cot\big(\tfrac{\pi p}{c}\big)\xi^{dmp}. \end{aligned} \tag{C.26}$$

Let $\tilde{x} = 0$ and $\tilde{y} = -\frac{m}{c}$, so that in the notation of (C.16) we have $\tilde{x}' = -m$, and $[\tilde{x}] - d[\tilde{x}'] = dm$. Applying Lemma C.2.6 to $s_{\tilde{x},\tilde{y}}(a,c)$ yields

$$s_{\tilde{x},\tilde{y}}(a,c) = \tfrac{1}{4|c|} \sum_{p=1}^{|c|-1} \cot\left(\tfrac{\pi p}{c}\right) \cot\left(\tfrac{\pi dp}{c}\right) \xi^{dmp},$$

which is exactly the first sum in (C.26). On the other hand, by definition,

$$s_{\tilde{x},\tilde{y}}(a,c) = \sum_{k=1}^{|c|-1} P_1\left(\tfrac{ak-m}{c}\right) P_1\left(\tfrac{k}{c}\right) = \sum_{k=1}^{|c|-1} P_1\left(\tfrac{ak+m}{c}\right) P_1\left(\tfrac{k}{c}\right) = \sum_{k=1}^{c-1} P_1\left(\tfrac{k+m}{c}\right) P_1\left(\tfrac{dk}{c}\right),$$

where we have first used that P_1 is odd and summed over $-k$, and then summed over ak instead of k. Comparing the two expressions for $s_{\tilde{x},\tilde{y}}(a,c)$ we find that

$$\tfrac{1}{4|c|} \sum_{p=1}^{|c|-1} \cot\left(\tfrac{\pi p}{c}\right) \cot\left(\tfrac{\pi dp}{c}\right) \xi^{dmp} = \sum_{k=1}^{|c|-1} P_1\left(\tfrac{k+m}{c}\right) P_1\left(\tfrac{dk}{c}\right) \tag{C.27}$$

Using (C.17) and the Fourier inversion formula (C.13) we can immediately identify two other terms in (C.26), namely

$$\tfrac{1}{|c|} \sum_{p=1}^{|c|-1} \tfrac{i}{2} \cot\left(\tfrac{\pi dp}{c}\right) \xi^{dmp} = P_1\left(\tfrac{m}{|c|}\right), \quad \tfrac{1}{|c|} \sum_{p=1}^{|c|-1} \tfrac{i}{2} \cot\left(\tfrac{\pi p}{c}\right) \xi^{dmp} = P_1\left(\tfrac{dm}{|c|}\right). \tag{C.28}$$

Moreover,

$$\sum_{p=1}^{|c|-1} \xi^{dmp} = |c|\big(1 - \delta(\tfrac{dm}{c})\big) - 1 = |c|\big(1 - \delta(\tfrac{m}{c})\big) - 1,$$

where in the last equality we have used that $\gcd(d,c) = 1$. Employing this together with (C.27) and (C.28), we can rewrite (C.26) as

$$\begin{aligned} s_{x,y}(a,c) = &\tfrac{1}{|c|} P_1(x)^2 + \sum_{k=1}^{c-1} P_1\left(\tfrac{k+m}{c}\right) P_1\left(\tfrac{dk}{c}\right) \\ &+ \tfrac{1}{4}\delta(x)\Big(1 - \delta(\tfrac{m}{c}) - \tfrac{1}{|c|}\Big) + \tfrac{1}{2}\delta(x)\Big(P_1\left(\tfrac{m}{|c|}\right) - P_1\left(\tfrac{dm}{|c|}\right)\Big). \end{aligned} \tag{C.29}$$

To find the claimed formula for $s_{x,y}(a,c) - s(a,c)$, let us first study the difference

$$\sum_{k=1}^{c-1} P_1\left(\tfrac{k+m}{c}\right) P_1\left(\tfrac{dk}{c}\right) - s(a,c) = \sum_{k=1}^{c-1} P_1\left(\tfrac{dk}{|c|}\right) \Big(P_1\left(\tfrac{k+m}{|c|}\right) - P_1\left(\tfrac{k}{|c|}\right)\Big), \tag{C.30}$$

where we have used that we can replace c with $|c|$ in the first term and that according to (C.22) we have $s(a,c) = s(d,|c|)$. Now, with $r \in \{0, \ldots, |c|-1\}$ satisfying $m \equiv r\,(c)$, we get

$$P_1\left(\tfrac{k+m}{|c|}\right) - P_1\left(\tfrac{k}{|c|}\right) = \begin{cases} \tfrac{r}{|c|}, & \text{if } k < |c| - r, \\ P_1\left(\tfrac{r}{|c|}\right), & \text{if } k = |c| - r, \\ \tfrac{r}{|c|} - 1, & \text{if } k > |c| - r. \end{cases}$$

Hence, straightforward manipulations yield that (C.30) equals

$$\begin{aligned}\left(\tfrac{r}{|c|}-1\right)\sum_{k=1}^{|c|-1}P_1\left(\tfrac{dk}{|c|}\right)+\sum_{k=1}^{|c|-r-1}P_1\left(\tfrac{dk}{|c|}\right)-\left(\tfrac{r}{|c|}-1\right)P_1\left(\tfrac{d(|c|-r)}{|c|}\right)+P_1\left(\tfrac{d(|c|-r)}{|c|}\right)P_1\left(\tfrac{r}{|c|}\right)\\ =\sum_{k=1}^{|c|-r}P_1\left(\tfrac{dk}{|c|}\right)+P_1\left(\tfrac{d(|c|-r)}{|c|}\right)\Big(P_1\left(\tfrac{r}{|c|}\right)-\tfrac{r}{|c|}\Big)\\ =\sum_{k=1}^{|c|-r}P_1\left(\tfrac{dk}{|c|}\right)+\tfrac{1}{2}P_1\left(\tfrac{dm}{|c|}\right).\end{aligned}$$

Note that we have used (C.18) to see that $\sum_{k=1}^{|c|-1}P_1\left(\frac{dk}{|c|}\right)=0$. Combining this computation with (C.29), we arrive at

$$\begin{aligned}s_{x,y}(a,c)-s(a,c)=&\tfrac{1}{|c|}P_1(x)^2-\tfrac{1}{4|c|}\delta(x)+\sum_{k=1}^{|c|-r}P_1\left(\tfrac{dk}{|c|}\right)+\tfrac{1}{2}P_1\left(\tfrac{dm}{|c|}\right)\\ &+\tfrac{1}{4}\delta(x)\left(1-\delta\left(\tfrac{m}{c}\right)\right)+\tfrac{1}{2}\delta(x)\Big(P_1\left(\tfrac{m}{|c|}\right)-P_1\left(\tfrac{dm}{|c|}\right)\Big),\end{aligned}$$

which is the claimed formula, since $P_1(x)^2-\frac{1}{4}\delta(x)=x^2-x=P_2(x)-\frac{1}{6}$. $\square$

Appendix D

Local Variation of the Eta Invariant

We include some more analytical details concerning the heat operator. Specifically, we will give some remarks concerning families of heat operators, and use this to give a proof of Proposition 1.3.14.

D.1 More Results on the Heat Operator

In Chapter 1 we have defined the heat operator e^{-tH} for a formally self-adjoint elliptic differential operator H of order 2 with positive definite leading symbol. We have done this by explicitly writing it as the limit of integral operators with smooth kernels, which were defined in terms of a spectral decomposition of H. In Lemma 1.2.3, we have derived a basic estimate for the heat kernel for large times. Concerning the small time behaviour of the heat operator, it is useful to have a description of e^{-tH} in terms of the resolvent $(H-z)^{-1}$ for $z \notin \operatorname{spec}(H)$. In particular, the proof of the asymptotic expansion in Theorem 1.2.7 as in [49] uses this description.

D.1.1 Expression via the Resolvent

Let E be a vector bundle over a closed manifold M of dimension m, and let H be a formally self-adjoint elliptic differential operator of order 2 with positive definite leading symbol. As before, for $s, s' \in \mathbb{R}$, let $\mathscr{B}(L^2_s, L^2_{s'})$ denote the space of bounded operators from L^2_s to $L^2_{s'}$ endowed with the operator norm $\|.\|_{s,s'}$. Although this is not really necessary, we assume for simplicity that H is non-negative, i.e., $\operatorname{spec}(H) \subset [0, \infty)$, which is certainly true for an operator of the form $H = D^2$. Then the elliptic estimate (1.11) implies that for every $s \geq 0$, there exists a constant C such that

$$C^{-1}\big\|(\mathrm{Id}+H)^{s/2}\varphi\big\|_{L^2} \leq \|\varphi\|_{L^2_s} \leq C\big\|(\mathrm{Id}+H)^{s/2}\varphi\big\|_{L^2}, \quad \varphi \in C^\infty(M,E). \tag{D.1}$$

Consider the region

$$\Lambda := \big\{z \in \mathbb{C} \bigm| \operatorname{Re}(z) + 1 \leq |\operatorname{Im}(z)|\big\}. \tag{D.2}$$

Then there exists a constant $C > 0$ such that

$$\operatorname{dist}(z, \operatorname{spec}(H)) \geq C|z| \quad \text{for all } z \in \Lambda.$$

This, together with (D.1) and the spectral theorem, implies that for $s \geq 0$, $z \in \Lambda$, and $\varphi \in C^\infty(M, E)$

$$\begin{aligned}\|(H-z)^{-1}\varphi\|_{L^2_s} &\leq C_1 \|(\mathrm{Id} + H)^{s/2}(H-z)^{-1}\varphi\|_{L^2}\\ &\leq C_2|z|^{-1}\,\|(\mathrm{Id}+H)^{s/2}\varphi\|_{L^2} \leq C_3|z|^{-1}\,\|\varphi\|_{L^2_s}\end{aligned}$$

where we have used that $(\mathrm{Id} + H)$ and $(H-z)^{-1}$ commute. By duality, this also holds for $s < 0$. Thus, for every $s \in \mathbb{R}$, there exists a constant C such that

$$\|(H-z)^{-1}\|_{s,s} \leq C|z|^{-1}, \quad z \in \Lambda. \tag{D.3}$$

We now consider the contour $\Gamma := \partial\Lambda$, oriented as the boundary, i.e., in such a way that $[0, \infty)$ lies in the interior, see Figure D.1. Then the Cauchy formula implies that

$$e^{-tH} = \frac{1}{2\pi i}\int_\Gamma e^{-tz}(H-z)^{-1}dz. \tag{D.4}$$

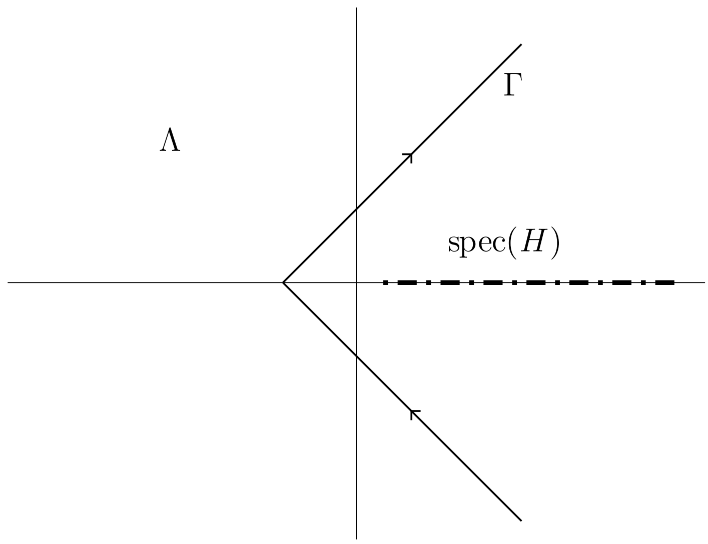

Figure D.1: The contour Γ

Because of (D.3), this expression converges in the operator norm in $L^2_s(M, E)$ for every $s \in \mathbb{R}$. To get norm estimates in $\mathscr{B}(L^2_s, L^2_{s+l})$ for $l > 0$ we need the following basic resolvent estimate.

Proposition D.1.1. *Let $s \in \mathbb{R}$, $k \in \mathbb{N}$, and $0 \leq l \leq 2k$. Then there exists a constant C such that for all $z \in \Lambda$*

$$\|(H-z)^{-k}\|_{s,s+l} \leq C|z|^{l/2-k}. \tag{D.5}$$

Proof. For $z \in \Lambda$, the spectrum of the operator $(H-z)^{-1}$ is not contained in the negative real line, so that $(H-z)^{-r}$ can be defined for every $r \geq 0$ by the spectral theorem. As in (D.3) one then finds

$$\|(H-z)^{-r}\|_{s,s} \leq C|z|^{-r}.$$

Then,

$$\begin{aligned}\|(H-z)^{-k}\|_{s,s+l} &\leq C_1\big\|(\mathrm{Id}+H)^{l/2}(H-z)^{-k}\big\|_{s,s}\\ &\leq C_1\big\|\big((\mathrm{Id}+H)(H-z)^{-1}\big)^{l/2}\big\|_{s,s}\big\|(H-z)^{l/2-k}\big\|_{s,s}\\ &\leq C_2|z|^{l/2-k}\Big(1+\big\|(\mathrm{Id}+z)(H-z)^{-1}\big\|_{s,s}^{k}\Big)\\ &\leq C_2|z|^{l/2-k}\Big(1+C_3\tfrac{|z+1|}{|z|}\Big)^{k}.\end{aligned}$$

The last factor can be bounded, independently of $z \in \Lambda$. This proves (D.5). □

An important consequence of (D.5) is that e^{-tH} does indeed solve the heat equation. We give the following summary of well-known facts, see in particular [49, Lem. 1.7.5].

Proposition D.1.2. *Let H be a non-negative operator in $\mathscr{P}^2_{s,e}(M,E)$ and let $s \in \mathbb{R}$.*

(i) *The one-parameter family $(e^{-tH})_{t\in(0,\infty)}$ is a smooth family*[1] *of smoothing operators. If $\varphi \in C^\infty(M,E)$, then*

$$(\tfrac{d}{dt}+H)e^{-tH}\varphi = 0, \quad \textit{and} \quad \lim_{t\to 0}\|e^{-tH}\varphi-\varphi\|_{L^2_s} = 0.$$

(ii) *The collection e^{-tH} forms a semi-group, i.e.,*

$$e^{-(t+t')H} = e^{-tH}e^{-t'H}, \quad t,t' > 0.$$

(iii) *For $l \geq 0$, there exists a constant $C > 0$ such that for $t \in (0,1)$*

$$\|e^{-tH}\|_{s,s+l} \leq Ct^{-l/2}. \tag{D.6}$$

(iv) *Let $c > 0$ be smaller than the smallest non-zero eigenvalue of H and let $t_0 > 0$. Then there exists a constant $C > 0$ such that for all $t \geq t_0$*

$$\|e^{-tH}-P_0\|_{s,s} \leq Ce^{-ct}, \tag{D.7}$$

where P_0 is the projection onto $\ker H$.

Sketch of proof. The assertion (i) is [49, Lem. 1.7.5]. We skip the proof since it uses the same ideas as the proof of part (iii). Part (ii) is also standard: Let $\varepsilon > 0$, and consider the contour $\Gamma' := \Gamma - \varepsilon$. Then, using Cauchy's formula and the resolvent equation, one finds that

$$\begin{aligned}e^{-t'H}e^{-tH} &= -\frac{1}{4\pi^2}\int_{\Gamma'}\int_{\Gamma} e^{-t'z'}e^{-tz}(H-z)^{-1}(H-z')^{-1}dzdz'\\ &= -\frac{1}{4\pi^2}\int_{\Gamma'}\int_{\Gamma}\frac{e^{-tz-t'z'}}{z-z'}\big[(H-z)^{-1}-(H-z')^{-1}\big]dzdz'.\end{aligned}$$

[1] See Definition D.1.6 below.

It now follows from standard "$\mathbb{C}$-valued" complex analysis, that first term is equal to $e^{-(t'+t)H}$, while the second term vanishes. To prove (D.6) choose $k \in \mathbb{N}$ with $l \leq 2k$. We integrate (D.4) by parts to find that

$$e^{-tH} = \frac{1}{2\pi i}\frac{(k-1)!}{t^{k-1}}\int_\Gamma e^{-tz}(H-z)^{-k}dz. \tag{D.8}$$

Now let $t \in (0,1)$ and substitute $\zeta = tz$ in (D.8). Then we can use Cauchy's theorem to change integration over $t\Gamma$ back to integration over Γ. This shows that for $t \in (0,1)$

$$e^{-tH} = \frac{1}{2\pi i}\frac{(k-1)!}{t^{k}}\int_\Gamma e^{-\zeta}(H-\zeta/t)^{-k}d\zeta.$$

Now (D.5) applied to $z = \zeta/t \in \Lambda$ shows that

$$\|e^{-tH}\|_{s,s+l} \leq C_1 t^{-k}\Big(\int_\Gamma |e^{-\zeta}|\,|\zeta/t|^{l/2-k}d\zeta\Big) \leq C_2|t|^{-l/2}.$$

This proves (D.6). Concerning (iii) we only note that the large time estimate can be easily deduced from Lemma 1.2.3. Alternatively it can be proved using (D.8) by considering the contour

$$\Gamma_c := \big\{z \in \mathbb{C} \;\big|\; \mathrm{Re}(z) - c/2 = |\,\mathrm{Im}(z)|\big\},$$

and employing estimates corresponding to the ones in (D.5), see Figure D.2 $\square$

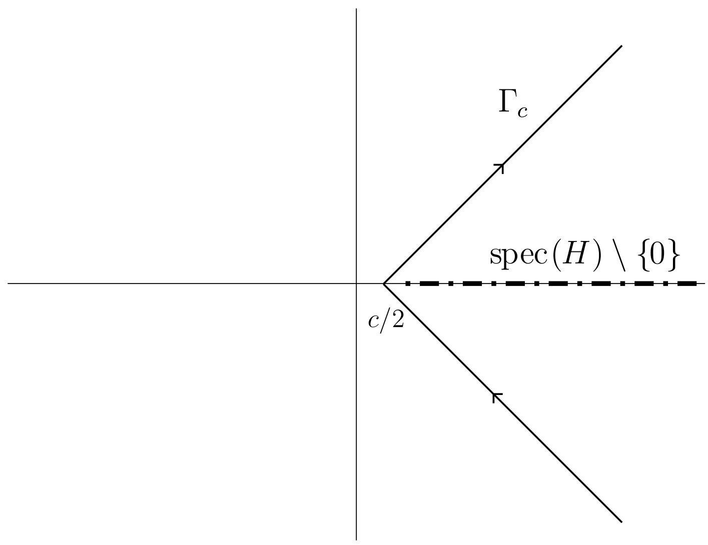

Figure D.2: The contour Γ_c

D.1.2 Perturbed Operators

Let E be a vector bundle over a closed manifold M of dimension m, and let K be an integral operator on $L^2(M,E)$ with smooth kernel $k(x,y) \in C^\infty(M \times M, E \boxtimes E^*)$. Then K is a smoothing operator and thus,

$$K \in \mathscr{B}\big(L^2_s, L^2_{s'}\big) \quad \text{for all } s, s' \in \mathbb{R}.$$

Moreover, the C^k-norms of $k(x,y)$ can be controlled by the operator norms of K. More precisely, for each $k \in \mathbb{N}$, we can find a constant C as in [49, Lem. 1.2.7] such that

$$\|k(x,y)\|_{C^k} \le C\,\|K\|_{-l,l}, \quad l > k + m/2. \tag{D.9}$$

Remark. Conversely, the Schwartz Kernel Theorem (see e.g. [94, Sec. 4.6]) ensures that an operator K on $L^2(M,E)$, which satisfies $\|K\|_{-l,l} < \infty$ for $l > k + m/2$, has a kernel $k(x,y)$ of class C^k such that (D.9) holds.

As before, assume that $H \in \mathscr{P}^2_{s,e}(M,E)$ is non-negative. If K is a symmetric smoothing operator on $C^\infty(M,E)$, the operator $H+K$ will in general not be an elliptic differential operator. Nevertheless, it follows from standard perturbation theory that $H+K$ has all the properties, we have obtained in Theorem 1.2.2 for the unperturbed case. In particular, we get a well-defined heat operator $e^{-t(H+K)}$. We need to have a control on the difference $e^{-t(H+K)} - e^{-tH}$. The following result is basically [13, Prop. 9.46], only that we have changed it into a statement about operators rather than kernels.

Proposition D.1.3. *Let K be a symmetric smoothing operator on $C^\infty(M,E)$, and let $H \in \mathscr{P}^2_{s,e}(M,E)$ be non-negative. For $k \ge 1$ and $t > 0$ define inductively*

$$K_0(t) := e^{-tH}, \quad K_k(t) := \int_0^t e^{-(t-s)H} K K_{k-1}(s) ds.$$

Then, if $l \in \mathbb{N}$, there exists a constant $C > 0$ such that for all $k \ge 1$

$$\|K_k(t)\|_{-l,l} \le \frac{C^k t^k}{k!}\,\|K\|^k_{-l,l},$$

Moreover, for each $N \ge 1$ there exists $C > 0$ such that for all $t \in (0,1)$

$$\Big\| e^{-t(H+K)} - e^{-tH} - \sum_{k=1}^N (-1)^k K_k(t) \Big\|_{-l,l} \le C t^{N+1}.$$

Remark. Before we sketch the proof we want to point out that for $k \ge 1$ the operator $K_k(t)$ can also be described as follows. Let

$$\Delta_k := \big\{ (s_1,\dots,s_k) \in \mathbb{R}^k \;\big|\; 0 \le s_1 \le \ldots \le s_k = 1 \big\}$$

be the standard k-simplex. Then

$$K_k(t) = t^k \int_{\Delta_k} e^{-t(1-s_k)H} K e^{-t(s_k - s_{k-1})H} \ldots K e^{-t(s_2-s_1)H} K e^{-ts_1 H} ds. \tag{D.10}$$

Sketch of proof. We use (D.10) to prove the estimate on $K_k(t)$. First it follows from (D.6) and (D.7) that $\|e^{-tH}\|_{s,s}$ can be bounded, independently of t. Thus,

$$\|K e^{-ts_1 H}\|_{-l,l} \le \|K\|_{-l,l} \|e^{-ts_1 H}\|_{-l,-l} \le C \|K\|_{-l,l},$$

On the other hand,

$$\big\| e^{-t(1-s_k)H} K e^{-t(s_k - s_{k-1})} \ldots K e^{-t(s_2 - s_1)H} \big\|_{l,l} \le C^k \|K\|^{k-1}_{l,l},$$

Since $\|K\|_{l,l} \leq C\|K\|_{-l,l}$ we deduce,

$$\|K_k(t)\|_{-l,l} \leq t^k \operatorname{vol}(\Delta_k) C^k \|K\|_{-l,l}^k \leq \frac{C^k t^k}{k!} \|K\|_{-l,l}^k,$$

where we have used that $\operatorname{vol}(\Delta_k) = \frac{1}{k!}$. Now the definition of $K_k(t)$ shows that

$$\tfrac{d}{dt} K_k(t) = K K_{k-1}(t) - H K_k(t).$$

Thus, for all $N \geq 1$,

$$\big(\tfrac{d}{dt} + H + K\big) \sum_{k=0}^{N} (-1)^k K_k(t) = (-1)^N K K_N(t).$$

Using the estimate on $\|K_k(t)\|_{-l,l}$ we see that

$$\Big\| \big(\tfrac{d}{dt} + H + K\big) \sum_{k=0}^{N} (-1)^k K_k(t) \Big\|_{-l,l} \leq C_N t^N$$

Thus, $\sum_{k=0}^{N} (-1)^k K_k(t)$ is an approximate solution to the heat equation in terms of $H + K$ in the sense of [13, Sec. 2.4]. This implies that

$$\Big\| e^{-t(H+K)} - e^{-tH} - \sum_{k=1}^{N} (-1)^k K_k(t) \Big\|_{-l,l} \leq C t^{N+1}. \qquad \square$$

We also need a result on the heat trace asymptotics when we perturb the operator H by a smoothing operator K. We first borrow the following from [13, Prop. 2.47].

Proposition D.1.4. *Let K be a smoothing operator on $L^2(M, E)$. Then there exists an asymptotic expansion of the form*

$$\operatorname{Tr}(e^{-tH} K) \sim \operatorname{Tr}(K) + \sum_{n=0}^{\infty} a_n t^n, \quad \text{as } t \to 0.$$

Remark. The proof in [13] relies on the explicit description of the heat kernel by geometrically constructed approximations. This method to obtain the heat trace asymptotics is different from the one presented in [49], which is the one we are following in our presentation. However, a rough idea to prove Proposition D.1.4 with the methods already established is the following: For each $N \in \mathbb{N}$ let

$$K_N(t) := \sum_{n=0}^{N} \frac{(-t)^n}{n!} H^n K.$$

Since K is a smoothing operator, $K_N(t)$ is a smoothing operator as well. Applying the heat equation to $K_N(t)$ yields

$$\big(\tfrac{d}{dt} + H\big) K_N(t) = \frac{(-t)^N}{N!} H^{N+1} K.$$

Since we can bound $\|H^{N+1}K\|_{-l,l}$ for fixed N, this shows that $K_N(t)$ is an approximate solution to the heat equation. For $t \to 0$ it converges strongly to K which implies that

$$\left\|e^{-tH}K - K_N(t)\right\|_{-l,l} \leq Ct^{N+1}.$$

Using (D.9) and the expression of the trace in terms of kernels, we can then estimate

$$\left|\operatorname{Tr}\left(e^{-tH}K - K_N(t)\right)\right| \leq Ct^{N+1}.$$

From this one finds that the assertion holds with

$$a_n := \frac{(-1)^n}{n!}\operatorname{Tr}(H^n K).$$

The following result is what we were aiming at in this section. We will give a version which suffices for our considerations, although a more general statement should be possible.

Proposition D.1.5. *Let H and K be as before, and assume in addition that K and H commute. Let D be an auxiliary formally self-adjoint differential operator of order d, and let K' be a symmetric smoothing operator on $C^\infty(M,E)$. Then there exists an asymptotic expansion*

$$\operatorname{Tr}\left((D+K')e^{-t(H+K)}\right) \sim \sum_{n=0}^{\infty} t^{\frac{n-m-d}{2}} a_n, \quad \text{as } t \to 0.$$

Moreover, if we denote by $a_n(D,H)$ the coefficients of the asymptotic expansion of $\operatorname{Tr}(De^{-tH})$*, we have*

$$a_n = a_n(D,H), \quad \text{for } n \leq m.$$

Sketch of proof. According to Theorem 1.2.7 it suffices to check that

$$\operatorname{Tr}\left((D+K')e^{-t(H+K)}\right) - \operatorname{Tr}\left(De^{-tH}\right)$$

has an asymptotic expansion as $t \to 0$ as a power series in t. The assumption that K and H commute simplifies the situation considerably, since then

$$e^{-t(H+K)} = e^{-tH}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}K^n,$$

where the series converges in every $L^2_{-l,l}$, because K is smoothing and thus bounded. This reduces the claim to the study of terms of the form

$$\operatorname{Tr}\left(De^{-tH}K^n\right) \quad \text{and} \quad \operatorname{Tr}(K'e^{-tH}K^n), \tag{D.11}$$

where $n \geq 1$. Now the trace property shows that both are of the form $\operatorname{Tr}\left(e^{-tH}\tilde{K}\right)$ for some smoothing operator $\tilde{K}$. We can thus apply Proposition D.1.4 to deduce that the terms in (D.11) are indeed asymptotic to power series in t. □

D.1.3 Variation of the Heat Operator

Now let $(K_u)_{u\in U}$ be a p-parameter family of smoothing operators. The constant C in the estimate (D.9) is independent of u. This implies that if K_u depends smoothly on u with respect to $\|.\|_{-l,l}$ for each $l \in \mathbb{N}$, then the family of kernels $k_u(x,y)$ will depend smoothly on u with respect to all C^k-norms, where $l > k + m/2$. This motivates the following definition.

Definition D.1.6. Let K_u be a one-parameter family of operators with smooth kernels. Then K_u is called a *smooth family of smoothing operators*, if for all $l \in \mathbb{N}$, the assignment $u \mapsto K_u$ is smooth in $\mathscr{B}\big(L^2_{-l}, L^2_l\big)$.

We can now formulate the following version of Duhamel's formula, compare with [13, Thm. 2.48].

Theorem D.1.7. *Consider a one-parameter family H_u in $\mathscr{P}^2_{s,e}(M,E)$ of non-negative operators, and assume that H_u is smooth in the sense of Definition 1.3.9. Then e^{-tH_u} is a smooth family of smoothing operators, and*

$$\tfrac{d}{du}e^{-tH_u} = -\int_0^t e^{-(t-s)H_u}\big(\tfrac{d}{du}H_u\big)e^{-sH_u}ds. \tag{D.12}$$

Remark. Note that in contrast to the discussion in [13, Sec. 2.7] the smoothness in u does not follow from our description (1.14) of the heat kernel since in general the eigenvalues and eigenvectors of H_u will not depend smoothly on u. However, we can use the description of e^{-tH_u} in terms of the resolvent to obtain the result.

Proof. First note that the basic resolvent estimates (D.3) and (D.5) can be made uniform in u since the constants appearing there depend on u only through the elliptic estimate, which can be made locally uniform in u.

We now want to prove that $(H_u-z)^{-1}$ varies smoothly with u. Without loss of generality we consider an interval around 0. First of all, let Γ be the contour defined as the boundary of (D.2). Then for all $z \in \Gamma$,

$$H_u - z = (H_0 - z)\big(\operatorname{Id} + (H_0 - z)^{-1}(H_u - H_0)\big) =: (H_0 - z)(\operatorname{Id} + T_u). \tag{D.13}$$

It follows from our smoothness assumption that T_u is a bounded operator on each L^2_l for every choice of $l \in \mathbb{Z}$, and that the assignment

$$\mathbb{R} \to \mathscr{B}(L^2_l, L^2_l), \quad u \mapsto T_u$$

is smooth. Moreover, we can choose $\delta > 0$ such that for all $u \in (-\delta, \delta)$

$$\|T_u\|_{n,n} \leq \tfrac{1}{2} \quad \text{for } n \in \{-l, \dots, l\}.$$

Using the Neumann series, this ensures that $\operatorname{Id} + T_u$ is invertible in each $\mathscr{B}(L^2_n, L^2_n)$. Furthermore, the assignment $u \mapsto (\operatorname{Id} + T_u)^{-1}$ is differentiable with

$$\tfrac{d}{du}(\operatorname{Id} + T_u)^{-1} = (\operatorname{Id} + T_u)^{-1}\big(\tfrac{d}{du}T_u\big)(\operatorname{Id} + T_u)^{-1}.$$

Inductively, one finds that $(\operatorname{Id} + T_u)^{-1}$ is smooth in u. Now (D.13) shows that for all $z \in \Gamma$

$$(H_u - z)^{-1} = (\operatorname{Id} + T_u)^{-1}(H_0 - z)^{-1}.$$

This yields that the family $(H_u - z)^{-1}$ depends smoothly on u as a map to $\mathscr{B}(L^2_{n-2}, L^2_n)$, where as before $n \in \{-l, \dots, l\}$. Hence, for k large enough, the assignment

$$(-\delta, \delta) \to \mathscr{B}(L^2_{-l}, L^2_l), \quad u \mapsto (H_u - z)^{-k}$$

is well-defined and smooth in u. Moreover, one easily finds that

$$\tfrac{d}{du}(H_u - z)^{-k} = -\sum_{n=1}^{k} (H_u - z)^{-n} \big(\tfrac{d}{du} H_u\big)(H_u - z)^{n-k-1}. \tag{D.14}$$

Since the basic resolvent estimate (D.5) can be made uniform in u, we deduce from (D.8) that e^{-tH_u} is differentiable and that we can differentiate under the integral using (D.14). Inductively, one finds that e^{-tH_u} is a smooth map to $\mathscr{B}(L^2_{-l}, L^2_l)$.

Having established the smoothness of e^{-tH_u} in u, we can prove Duhamel's formula (D.12) along the same lines as in [13, Sec. 2.7]: First, the heat equation implies that

$$(\tfrac{d}{dt} + H_u)\tfrac{d}{du} e^{-tH_u} = -(\tfrac{d}{du} H_u) e^{-tH_u}.$$

On the other hand, one finds that

$$(\tfrac{d}{dt} + H_u) \int_0^t e^{-(t-s)H_u} (\tfrac{d}{du} H_u) e^{-sH_u} ds = (\tfrac{d}{du} H_u) e^{-tH_u}.$$

Thus,

$$\tfrac{d}{du} e^{-tH_u} + \int_0^t e^{-(t-s)H_u} (\tfrac{d}{du} H_u) e^{-sH_u} ds \tag{D.15}$$

solves the heat equation. We are thus left to show that (D.15) converges to 0 in $L^2(M, E)$ as $t \to 0$. Concerning the first term, we use

$$\tfrac{d}{du} e^{-tH_u} = -\frac{1}{2\pi i} \int_\Gamma e^{-tz} (H_u - z)^{-1} \big(\tfrac{d}{du} H_u\big)(H_u - z)^{-1} dz$$

and the resolvent estimate (D.3). Then, as in the proof of (D.6), one easily finds small time estimates

$$\|\tfrac{d}{du} e^{-tH_u} \varphi\|_{L^2} \le Ct \|\varphi\|_{L^2}, \quad \varphi \in L^2(M, E),$$

which are uniform in u and $t \in (0, 1)$. This shows that $\frac{d}{du} e^{-tH_u}$ converges to 0 as $t \to 0$ in $\mathscr{B}(L^2, L^2)$. For the second term in (D.15), we can argue as in the proof of Proposition D.1.3 to get uniform bounds on the integrand as $t \to 0$. This implies our assertion. □

In a similar way one can show the following result.

Proposition D.1.8. *Let D_u be a smooth family in $\mathscr{P}^d_{s,e}(M, E)$ and assume further that* $\dim(\ker D_u)$ *is constant.*

(i) *The projection P_u onto* $\ker(D_u)$ *is a smooth family of smoothing operators.*

(ii) *Denote by G_u the family of Green's operators of D_u^2, defined by*

$$G_u|_{\ker(P_u)} = \big(D_u|_{\ker(P_u)}\big)^{-1}, \quad G_u|_{\mathrm{im}(P_u)} = 0.$$

Then, for every $s \in \mathbb{R}$, the family G_u is smooth in u as a map to $\mathscr{B}(L^2_s, L^2_{s+d})$.

Sketch of proof. Since D_u varies smoothly with u, one can verify—using the smooth dependence of the resolvent on u—that for compact u-intervals, the non-zero eigenvalues are uniformly bonded away from 0. Then, according to the spectral theorem, the projection P_u onto $\ker D_u$ is given by

$$P_u = \frac{1}{2\pi i}\int_\Gamma (D_u - z)^{-1} dz, \tag{D.16}$$

where Γ is the clockwise oriented boundary of a small disk B such that

$$B \cap \operatorname{spec}(D_u) = \{0\}, \quad \text{for all } u.$$

Integrating by parts, we get

$$P_u = \frac{1}{2\pi i}\int_\Gamma (-z)^{k-1}(D_u - z)^{-k} dz.$$

Resolvent estimates as in (D.3) and (D.5) and the formula (D.14) for the derivative of the resolvent with respect to u then yield that P_u is a smooth family of smoothing operators, see [13, Prop. 9.10] for a related discussion.

Concerning the family of Green's operators, we can take the same small disk B and orient the boundary Γ counter-clockwise. Then

$$G_u = \frac{1}{2\pi i}\int_\Gamma z^{-1}(D_u - z)^{-1} dz. \tag{D.17}$$

Now the uniform bound $\|(D_u - z)^{-1}\|_{s,s+d} \le C$ and the formula for the derivative of the resolvent show that G_u is a smooth family of operators in $\mathscr{B}(L^2_s, L^2_{s+d})$. □

Remark D.1.9.

(i) Note that in (D.17) we cannot integrate by parts to increase the regularity of G_u. This is, of course, already clear in the case of a single operator.

(ii) If there exists a constant c such that $c \notin \operatorname{spec}(|D_u|)$ for all u in some interval $[-\delta, \delta]$, we can use (D.16)—with Γ being a circle of radius c around 0—to deduce that the spectral projection onto all eigenvalues of norm less than c is smooth u, see also [13, Prop. 9.10].

(iii) Theorem D.1.7 and Proposition D.1.8 carry over verbatim, if we consider a smooth p-parameter families of formally self-adjoint elliptic operators. This is important for the discussion of fiber bundles.

We also need the following consequence of Theorem D.1.7

Lemma D.1.10. *Let H_u be a smooth one-parameter family of non-negative operators in $\mathscr{P}^2_{s,e}(M, E)$, and let D_u be an auxiliary smooth one-parameter family of formally self-adjoint differential operators of order d. Then $D_u e^{-tH_u}$ is a differentiable family of trace-class operators and, if D_u commutes with H_u,*

$$\tfrac{d}{du}\operatorname{Tr}\left(D_u e^{-tH_u}\right) = \operatorname{Tr}\left(\tfrac{dD_u}{du} e^{-tH_u}\right) - t\operatorname{Tr}\left(D_u \tfrac{dH_u}{du} e^{-tH_u}\right). \tag{D.18}$$

Proof. Let $K_u := D_u e^{-tH_u/2}$ and $L_u := e^{-tH_u/2}$. Then Theorem D.1.7 implies that L_u is a smooth family of smoothing operators, and the same is true for K_u, since by Lemma 1.3.10 the operators $D_u \in \mathscr{B}(L^2_{l+d}, L^2_l)$ depend smoothly on u for every $l \in \mathbb{Z}$. We now observe the following

(i) K_u and L_u are smooth families of operators in $\mathscr{B}(L^2, L^2)$, and

$$\tfrac{d}{du}(K_u L_u) = \tfrac{dD_u}{du} e^{-tH_u} + D_u \tfrac{d}{du} e^{-tH_u},$$

where we are using the pairing

$$\mathscr{B}(L^2_d, L^2) \times \mathscr{B}(L^2, L^2_d) \to \mathscr{B}(L^2, L^2), \quad (S,T) \mapsto ST,$$

to differentiate $D_u e^{-tH_u/2}$.

(ii) K_u and L_u are trace-class operators. Expressing the trace in terms of the kernels it follows from (D.9) that they depend continuously on u with respect to the trace norm.

Using the Hölder inequality $|\operatorname{Tr}(ST)| \leq \operatorname{Tr}|S|\,\|T\|_{0,0}$, these observations imply that $\operatorname{Tr}\big(D_u e^{-tH_u}\big)$ is differentiable. If D_u and H_u commute, we can use (D.12) to compute that

$$\begin{aligned}
\tfrac{d}{du}\operatorname{Tr}\big(D_u e^{-tH_u}\big) &= \operatorname{Tr}\big(\tfrac{dD_u}{du} e^{-tH_u}\big) + \operatorname{Tr}\big(D_u \tfrac{d}{du} e^{-tH_u}\big)\\
&= \operatorname{Tr}\big(\tfrac{dD_u}{du} e^{-tH_u}\big) - \int_0^t \operatorname{Tr}\big(D_u e^{-(t-s)H_u}(\tfrac{d}{du}H_u)e^{-sH_u}\big)ds\\
&= \operatorname{Tr}\big(\tfrac{dD_u}{du} e^{-tH_u}\big) - \int_0^t \operatorname{Tr}\big(D_u(\tfrac{d}{du}H_u)e^{-tH_u}\big)ds.\\
&= \operatorname{Tr}\big(\tfrac{dD_u}{du} e^{-tH_u}\big) - t\operatorname{Tr}\big(D_u \tfrac{dH_u}{du} e^{-tH_u}\big).
\end{aligned}$$

□

D.2 Parameter Dependent Eta Invariants

As a consequence of the above discussion, we can now describe a proof of the variation formula for the Eta invariant.

D.2.1 Large Time Behaviour

To study parameter dependent Eta invariants we let $H_u = D_u^2$, where D_u is a smooth family in $\mathscr{P}^1_{s,e}(M,E)$. We can then state the following parameter dependent version of Proposition 1.2.4, see also [31, Lem. A.14].

Lemma D.2.1. *Under the assumptions of Theorem D.1.7, the one-parameter family $D_u e^{-tD_u^2}$ is a differentiable family of trace-class operators, and*

$$\tfrac{d}{du}\operatorname{Tr}(D_u e^{-tD_u^2}) = \big(1 + 2t\tfrac{d}{dt}\big)\operatorname{Tr}\big(\tfrac{dD_u}{du} e^{-tD_u^2}\big). \tag{D.19}$$

Moreover, if $\dim(\ker D_u)$ *is constant, then for* $t_0 > 0$, *there exist constants* c *and* C *such that for* $t \geq t_0$

$$\big|\operatorname{Tr}(D_u e^{-tD_u^2})\big| \leq Ce^{-ct}, \tag{D.20}$$

locally uniform in u.

Proof. By (D.18) we have

$$\begin{aligned}
\tfrac{d}{du}\operatorname{Tr}(D_u e^{-tD_u^2}) &= \operatorname{Tr}\left(\tfrac{dD_u}{du}e^{-tD_u^2}\right) - t\operatorname{Tr}\left(D_u\tfrac{dD_u^2}{du}e^{-tD_u^2}\right)\\
&= \operatorname{Tr}\left(\tfrac{dD_u}{du}e^{-tD_u^2}\right) - 2t\operatorname{Tr}\left(\tfrac{dD_u}{du}D_u^2 e^{-tD_u^2}\right)\\
&= \operatorname{Tr}\left(\tfrac{dD_u}{du}e^{-tD_u^2}\right) + 2t\tfrac{d}{dt}\operatorname{Tr}\left(\tfrac{dD_u}{du}e^{-tD_u^2}\right).
\end{aligned}$$

Proposition 1.2.4 shows that for $t_0 > 0$, we can find constants $C(u)$ such that for $t \geq t_0$

$$\left|\operatorname{Tr}(D_u e^{-tD_u^2})\right| \leq C(u)e^{-t\lambda_0(u)/2},$$

where $\lambda_0(u)$ is the smallest non-zero eigenvalue of D_u^2. The proof of Proposition 1.2.4 shows that $C(u)$ depends continuously on u so that we can find C with $C(u) \leq C$ for compact u intervals. Moreover, the eigenvalues of D_u^2 vary continuously. Hence if $\dim(\ker D_u)$ is constant, the non-zero eigenvalues of D_u^2 have a uniform positive lower bound on compact u intervals. This proves the second assertion. □

The above lemma has the following consequence

Corollary D.2.2. *If* $\dim(\ker D_u)$ *is constant, then*

$$\frac{d}{du}\int_1^\infty \operatorname{Tr}\left(D_u e^{-tD_u^2}\right)t^{\frac{s-1}{2}}\,dt = -2\operatorname{Tr}\left(\tfrac{dD_u}{du}e^{-D_u^2}\right) - s\int_1^\infty \operatorname{Tr}\left(\tfrac{dD_u}{du}e^{-tD_u^2}\right)t^{\frac{s-1}{2}}\,dt,$$

and both sides are holomorphic for all $s \in \mathbb{C}$.

Proof. Let P_u be the orthogonal projection onto $\ker D_u$. Since $\dim(\ker D_u)$ is constant we know from Proposition D.1.8 that P_u depends smoothly on u. Moreover,

$$\begin{aligned}
\tfrac{d}{du}D_u &= \tfrac{d}{du}\big((\operatorname{Id}-P_u)D_u(\operatorname{Id}-P_u)\big)\\
&= -\tfrac{dPu}{du}D_u(\operatorname{Id}-P_u) + (\operatorname{Id}-P_u)\tfrac{dD_u}{du}(\operatorname{Id}-P_u) - (\operatorname{Id}-P_u)D_u\tfrac{dPu}{du}
\end{aligned}$$

From this one deduces that $\operatorname{Tr}\left(\frac{dD_u}{du}e^{-tD_u^2}\right)$ satisfies an estimate of the form (D.20). Thus, for fixed $T > 1$ we can use (D.19) to differentiate under the integral. Integrating by parts, we find that

$$\begin{aligned}
\frac{d}{du}\int_1^T t^{\frac{s-1}{2}}\operatorname{Tr}\left(D_u e^{-tD_u^2}\right)dt &= \int_1^T t^{\frac{s-1}{2}}\left(1+2t\tfrac{d}{dt}\right)\operatorname{Tr}\left(\tfrac{dD_u}{du}e^{-tD_u^2}\right)dt\\
&= 2\Big[T^{\frac{s+1}{2}}\operatorname{Tr}\left(\tfrac{dD_u}{du}e^{-TD_u^2}\right) - \operatorname{Tr}\left(\tfrac{dD_u}{du}e^{-D_u^2}\right)\Big]\\
&\qquad - s\int_1^T \operatorname{Tr}\left(\tfrac{dD_u}{du}e^{-tD_u^2}\right)t^{\frac{s-1}{2}}\,dt.
\end{aligned}$$

The uniform estimates on $\operatorname{Tr}\left(D_u e^{-tD_u^2}\right)$ and $\operatorname{Tr}\left(\frac{dD_u}{du}e^{-tD_u^2}\right)$ show that both sides are holomorphic for $s \in \mathbb{C}$ and allow us to take $T \to \infty$. This proves the result. □

D.2.2 Small Times and Meromorphic Extension

To extend Corollary D.2.2 to small times $t \in [0,1]$, we need the following parameter dependent version of Theorem 1.2.7, see [49, Lem. 1.9.3]. A detailed proof can also be found in [23, Thm.'s 3.3 & 3.4].

Theorem D.2.3. *Let A_u be an auxiliary smooth family of formally self-adjoint operators of order a. There is an asymptotic expansion, locally uniform in u, such that*

$$\operatorname{Tr}(A_u e^{-tD_u^2}) \sim \sum_{n=0}^{\infty} t^{\frac{n-m-a}{2}} a_n(A_u, D_u), \quad \text{as } t \to 0. \tag{D.21}$$

The $a_n(A_u, D_u)$ are integrals over quantities locally computable from the total symbols of A_u and D_u. Moreover, the functions $a_n(A_u, D_u)$ are smooth in u and (D.21) *can be differentiated term by term, i.e.,*

$$\tfrac{d}{du} \operatorname{Tr}(A_u e^{-tD_u^2}) \sim \sum_{n=0}^{\infty} t^{\frac{n-m-a}{2}} \tfrac{d}{du} a_n(A_u, D_u), \quad \text{as } t \to 0.$$

Proposition D.2.4. *If $\dim(\ker D_u)$ is constant, the meromorphic extension of $\eta(D_u, s)$ is continuously differentiable in u, and*

$$\tfrac{d}{du}\eta(D_u) = -\tfrac{2}{\sqrt{\pi}}\, a_m(\tfrac{dD_u}{du}, D_u^2),$$

where $a_m(\frac{dD_u}{du}, D_u^2)$ is the constant term in the asymptotic expansion of

$$\sqrt{t}\ \operatorname{Tr}\left(\tfrac{dD_u}{du} e^{-tD_u^2}\right), \quad \text{as } t \to 0.$$

Proof. According to Theorem D.2.3 we have asymptotic expansions, which can be differentiated in u,

$$\operatorname{Tr}\left(D_u e^{-tD_u^2}\right) \sim \sum_{n=0}^{\infty} t^{\frac{n-m-1}{2}} a_n(u), \quad \text{as } t \to 0,$$

and

$$\operatorname{Tr}\left(\tfrac{dD_u}{du} e^{-tD_u^2}\right) \sim \sum_{n=0}^{\infty} t^{\frac{n-m-1}{2}} a_n'(u), \quad \text{as } t \to 0.$$

As remarked in Theorem 1.2.7 they can be differentiated in t as well so that (D.19) implies

$$\tfrac{d}{du} a_n(u) = (n-m)\, a_n'(u). \tag{D.22}$$

Let $N \in \mathbb{N}$ be fixed, and let

$$r_N(t,u) := \operatorname{Tr}\left(D_u e^{-tD_u^2}\right) - \sum_{n=0}^{N} t^{\frac{n-m-1}{2}} a_n(u),$$

and

$$r_N'(t,u) := \operatorname{Tr}\left(\tfrac{dD_u}{du} e^{-tD_u^2}\right) - \sum_{n=0}^{N} t^{\frac{n-m-1}{2}} a_n'(u).$$

Then r_N and r'_N satisfy estimates, locally uniform in u,

$$|r_N(t,u)| \le Ct^N, \quad |r'_N(t,u)| \le Ct^N, \quad \text{as } t \to 0. \tag{D.23}$$

Moreover, (D.19) and (D.22) imply that

$$\tfrac{d}{du} r_N(t,u) = (1 + 2t\tfrac{d}{dt}) r'_N(t,u). \tag{D.24}$$

Now for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > m - (N+1)$ and $s \notin \{m - n \,|\, n \in \mathbb{N}\}$

$$\int_0^1 t^{\frac{s-1}{2}} \operatorname{Tr}\big(D_u e^{-tD_u^2}\big) dt = \sum_{n=0}^N \frac{2a_n(u)}{s+n-m} + \int_0^1 t^{\frac{s-1}{2}} r_N(t,u) dt.$$

Using (D.23) we can differentiate under the integral to deduce from (D.22) and (D.24) that

$$\begin{aligned}
\frac{d}{du}\int_0^1 t^{\frac{s-1}{2}} \operatorname{Tr}\big(D_u e^{-tD_u^2}\big) dt &= \sum_{n=0}^N \frac{2(n-m)\,a'_n(u)}{s+n-m} + \int_0^1 t^{\frac{s-1}{2}} (1 + 2t\tfrac{d}{dt}) r'_N(t,u) dt \\
&= \sum_{n=0}^N \frac{2(n-m)\,a'_n(u)}{s+n-m} + 2r'_N(1,u) - s\int_0^1 t^{\frac{s-1}{2}} r'_N(t,u) dt \\
&= 2\operatorname{Tr}\big(\tfrac{dD_u}{du} e^{-D_u^2}\big) + \sum_{n=0}^N \frac{-2s\,a'_n(u)}{s+n-m} - s\int_0^1 t^{\frac{s-1}{2}} r'_N(t,u) dt,
\end{aligned}$$

since

$$\frac{2(n-m)\,a'_n(u)}{s+n-m} - 2a'_n(u) = \frac{-2s\,a'_n(u)}{s+n-m}.$$

On the other hand,

$$\int_0^1 t^{\frac{s-1}{2}} \operatorname{Tr}\big(\tfrac{dD_u}{du} e^{-D_u^2}\big) dt = \sum_{n=0}^N \frac{2\,a'_n(u)}{s+n-m} + \int_0^1 t^{\frac{s-1}{2}} r'_N(t,u) dt.$$

This shows that the meromorphic extension to $\mathbb{C}$ of $\int_0^1 t^{\frac{s-1}{2}} \operatorname{Tr}\big(D_u e^{-tD_u^2}\big) dt$ is continuously differentiable in u away from the poles, with derivative given by the meromorphic extension of

$$2\operatorname{Tr}\big(\tfrac{dD_u}{du} e^{-D_u^2}\big) - s\int_0^1 t^{\frac{s-1}{2}} \operatorname{Tr}\big(\tfrac{dD_u}{du} e^{-D_u^2}\big) dt.$$

Since $\dim(\ker D_u)$ is constant we can apply Corollary D.2.2 to deduce that the meromorphic extension of the Eta function $\eta(D_u, s)$ is continuously differentiable in u. Moreover, for $N > m$ and a suitable function $h_N(u,s)$, holomorphic for $\mathrm{Re}(s) > m - (N+1)$,

$$\tfrac{d}{du}\eta(D_u, s) = \frac{-s}{\Gamma(\frac{s+1}{2})}\Big(\sum_{n=0}^N \frac{2\,a'_n(u)}{s+n-m} + h_N(u,s)\Big).$$

In particular,

$$\tfrac{d}{du}\eta(D_u) = -2\,\Gamma(\tfrac{1}{2})^{-1} a'_m(u) = -\tfrac{2}{\sqrt{\pi}}\, a_m(\tfrac{dD_u}{du}, D_u^2). \qquad \square$$

D.2.3 The Case of Varying Kernel Dimension

If we want to drop the assumption on $\dim(\ker D_u)$, we have to study the reduced ξ-invariant $[\xi(D_u)] \in \mathbb{R}/\mathbb{Z}$ as in Definition 1.3.4.

Proposition D.2.5. *Let D_u be a smooth family in $\mathscr{P}^1_{s,e}(M,E)$. Then the reduced ξ-function $[\xi(D_u)] \in \mathbb{R}/\mathbb{Z}$ is continuously differentiable in u, and*

$$\tfrac{d}{du}[\xi(D_u)] = -\tfrac{1}{\sqrt{\pi}}\, a_m\big(\tfrac{dD_u}{du}, D_u^2\big),$$

where $a_m(\frac{dD_u}{du}, D_u^2)$ is the constant term in the asymptotic expansion of

$$\sqrt{t}\ \mathrm{Tr}\left(\tfrac{dD_u}{du} e^{-tD_u^2}\right), \quad \textit{as } t \to 0.$$

The result is the same as [31, Prop. A.17]. We include a proof for completeness and sketch a few more details.

Proof. Choose δ small enough so that there exists $c \in (0,1)$ with $c \notin \mathrm{spec}(|D_u|)$ for all $u \in (-\delta, \delta)$. Denote by λ_u^i with $i = 1, \ldots, i_0$ the finite number of eigenvalues of D_u with $|\lambda_u^i| < c$, and let

$$E_u(c) := \bigoplus_{i=1}^{i_0} \ker\left(D_u - \lambda_u^i\right).$$

Let $P_u(c)$ be the projection onto $E_u(c)$. According to Remark D.1.9 $P_u(c)$ is a smooth[2] family of finite rank operators with smooth kernel. Thus

$$D_u(c) := D_u\big(\mathrm{Id} - P_u(c)\big) + P_u(c)$$

is a smooth perturbation of D_u by finite rank operators with smooth kernel. Note that for fixed u we replace with 1 the finite number of eigenvalues of D_u which are of norm smaller than c. Thus, the large eigenvalues of $D_u(c)$ are the same as those of D_u. This implies that the Eta function is well-defined for $\mathrm{Re}(s) > m$ and satisfies

$$\eta\big(D_u(c), s\big) = \eta(D_u, s) + \dim\ker(D_u) + \sum_{\lambda_u^i \neq 0} \Big[1 - \tfrac{\mathrm{sgn}(\lambda_u^i)}{|\lambda_u^i|^s}\Big], \quad \mathrm{Re}(s) > m. \tag{D.25}$$

Since the right hand side admits a meromorphic continuation to $\mathbb{C}$, the same holds for $\eta\big(D_u(c), s\big)$. Moreover, $s = 0$ is no pole, and the reduced ξ-invariant satisfies

$$\big[\xi\big(D_u(c)\big)\big] = \big[\xi(D_u)\big]. \tag{D.26}$$

We now need to understand the variation of $\xi\big(D_u(c)\big)$. Since $D_u(c)$ is invertible we will study $\eta\big(D_u(c)\big)$ instead. Note that we cannot directly apply Proposition D.2.4 since $D_u(c)$ is in general not a family of differential operators. However, we have already done the major work and indicate the changes to be made:

[2] Kato's selection theorem ([56, Sec. II.6]), ensures that the eigenvalues λ_u^i can be ordered in such a way that they are parametrized by continuously differentiable functions. Nevertheless, the total projection onto all eigenspaces spanned by the collection λ_u^i depends smoothly on u.

(i) Since $D_u(c)$ is a smooth perturbation of D_u by symmetric smoothing operators, the proof of Theorem D.1.7 goes through verbatim, showing that $e^{-tD_u(c)^2}$ is a smooth family of smoothing operators which satisfies (D.12).

(ii) Then, as in Lemma D.2.1, one finds that the one-parameter family $D_u(c)e^{-tD_u(c)^2}$ is a differentiable family of trace-class operators satisfying (D.19). Clearly, the uniform large-time estimate (D.20) and Corollary D.2.2 also continue to hold.

(iii) Since $D_u(c)^2$ is of the form $D_u^2 + K_u$, where K_u is a smooth family of smoothing operators which commutes with D_u^2, we can apply Proposition D.1.5 for fixed u to get asymptotic expansions

$$\operatorname{Tr}\left(D_u(c)e^{-tD_u(c)^2}\right) \sim \sum_{n=0}^{\infty} t^{\frac{n-m-1}{2}} a_n(u), \quad \text{as } t \to 0, \tag{D.27}$$

and

$$\operatorname{Tr}\left(\left(\tfrac{d}{du}D_u(c)\right)e^{-tD_u(c)^2}\right) \sim \sum_{n=0}^{\infty} t^{\frac{n-m-1}{2}} a_n'(u), \quad \text{as } t \to 0, \tag{D.28}$$

where $a_m'(u)$ is equal to the constant term in the asymptotic expansion as $t \to 0$ of $\sqrt{t}\,\operatorname{Tr}\left(\frac{dD_u}{du}e^{-tD_u^2}\right)$. Moreover, one deduces from Theorem D.2.3 and the proof of Proposition D.1.5 that (D.27) and (D.28) are locally uniform in u with coefficients depending smoothly on u.

Now the proof of Proposition D.2.4 carries over to the situation at hand yielding that $\eta\big(D_u(c)\big)$ is continuously differentiable in u with

$$\tfrac{d}{du}\eta\big(D_u(c)\big) = -\tfrac{2}{\sqrt{\pi}}\, a_m\left(\tfrac{dD_u}{du}, D_u^2\right).$$

Since $\big[\xi(D_u)\big] = \frac{1}{2}\eta\big(D_u(c)\big)$, the proposition follows. □

From the proof of Proposition D.2.5 we can also easily deduce the variation formula for the ξ-invariant, as stated in Proposition 1.3.14.

Corollary D.2.6. *Let D_u with $u \in [a,b]$ be a smooth one-parameter family of operators in $\mathscr{P}^1_{s,e}(M,E)$. Then*

$$\xi(D_b) - \xi(D_a) = \mathrm{SF}(D_u)_{u\in[a,b]} + \int_a^b \tfrac{d}{du}[\xi(D_u)]du.$$

Proof. Without loss of generality we may assume that $[a,b] = [-\delta,\delta]$ for δ as in the proof of Proposition D.2.5. Moreover, let $D_u(c)$ be the family of perturbed operators defined there. Then the fundamental theorem of calculus shows that

$$\eta(D_\delta(c)) - \eta(D_{-\delta}(c)) = \int_{-\delta}^{\delta} \tfrac{d}{du}\eta(D_u(c))du.$$

We deduce from analytic continuation of (D.25) to $s = 0$ that

$$\begin{aligned}\xi(D_\delta) - \xi(D_{-\delta}) &= \int_{-\delta}^{\delta} \tfrac{d}{du}\xi(D_u(c))du - \#\big\{i \in \{1, \ldots, i_0\} \;\big|\; \lambda^i_\delta < 0\big\} \\ &\qquad\qquad + \#\big\{i \in \{1, \ldots, i_0\} \;\big|\; \lambda^i_{-\delta} < 0\big\} \\ &= \int_{-\delta}^{\delta} \tfrac{d}{du}\xi(D_u(c))du + \mathrm{SF}(D_u)_{u\in[-\delta,\delta]}.\end{aligned}$$

Since $\xi(D_u(c))$ is differentiable,

$$\tfrac{d}{du}\xi(D_u(c)) = \tfrac{d}{du}\big[\xi(D_u(c))\big].$$

Now the equality $\big[\xi\big(D_u(c)\big)\big] = \big[\xi(D_u)\big]$ from (D.26) proves the result. $\square$

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