# Pattern, Walls and Vortex: A micromagnetic excursion

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## Abstract

The micromagnetic model is a variational model for the description of ferromagnets, that includes small–scale quantum effects as well as large–scale effects of electromagnetic theory. As such, it is naturally a mesoscopic model. The object of interest is the magnetisation $m$, which can be viewed as a local average of spins, and which is subject to different interactions. The most prominent of these interactions are the exchange interaction, which is of quantum–mechanical origin, and the interaction of the magnetisation with the magnetic stray–field generated by $m$ itself. Other effects include the coupling to an external field via the Zeeman effect and the influence of crystalline anisotropy of the ferromagnetic sample. One important feature is the saturation condition: The magnetisation is assumed to be in a saturated state and by a suitable normalisation this yields the condition that $m$ be of unit length, or that $m : \Omega \mapsto \mathbb{S}^2$ for the given ferromagnetic sample $\Omega \subset \mathbb{R}^3$. Since the different interactions are active on different lengthscales, micromagnetism is an example for a multi–scale pattern–forming system.

A ferromagnetic pattern consists of domains and walls. Domains are regions of the ferromagnetic sample in which the magnetisation is almost constant. Walls are transition regions between such domains, usually extending over much smaller lengthscales than the typical domain. Such walls connect regions of different magnetisation directions by in–plane rotations — Néel walls — or out–of–plane rotations — Bloch walls —, though more complicated wall types exist. Another possible small structure in a ferromagnetic sample is a Bloch line, which comes in two variants. Circular Bloch lines are configurations in which the magnetisation forms a region of positive winding number or degree in a certain plane around a point. In this point, a topological singularity would form, which is avoided by the magnetisation turning out of the plane at that point. The two–dimensional image of such a situation in the plane is a vortex. A cross Bloch line is the same situation for negative winding number or degree around a point, its two–dimensional image is an antivortex.

In this thesis, we analyse two different settings in the micromagnetic model.

The first of these settings is an elongated thin–film strip. Our interest lies in the formation of a characteristic pattern, the concertina pattern, which is an almost periodic array of magnetic domains separated by Néel walls. We first perform a linear stability analysis for a uniform magnetisation along the long axis of the sample. As shown in (Cantero–Álvarez & Otto, 2006), we have four different scaling regimes for the onset of an instability due to an applied external field. We perform a linear stability analysis in the two intermediate regimes. This analysis is done by a Γ–convergence argument for the Rayleigh quotient of the Hessian of the energy functional, evaluated at the uniform magnetisation. The notion of Γ–convergence is a notion of convergence of functionals especially tailored for variational problems,

one of its main features being the fact that minimisers of the original variational problems converge to minimisers of the limit variational problem. Thus we can derive asymptotic properties of the system under consideration. The third, or second intermediate, regime is shown to feature an instability of nonzero oscillation period in direction of the long axis of the sample, which supports the conjecture that the concertina pattern may evolve out of this instability.

We then derive a normal form of the bifurcation in the third regime via a weakly nonlinear analysis. Again we resort to $\Gamma$–convergence for the derivation of this normal form. Due to possible charge cancellations on small scales — where "charge" means magnetic charges, which arise from the distributional divergence of $m$ and generate the stray–field —, the bifurcation is subcritical. This implies that the phase transition from the uniform magnetisation to whatever state in the third regime is discontinuous. Nevertheless due to a large scale coercivity, which is proved in (Cantero–Álvarez, Otto & Steiner, 2007), and by numerical simulations performed therein and in (Steiner, 2006) it can be shown that there exist concertina–type minimisers near the original uniform magnetisation.

As a result of the weakly nonlinear analysis, we obtain a model for an array of low–angle Néel walls, whose energy scaling we identify for large external fields, by matching upper and lower bounds. For the upper bound, we give a suitable construction, while the lower bound is calculated via suitable estimates.

The second setting we consider is a thin–film disk. For this setting, we investigate the stability of a central vortex. This problem is related to the Ginzburg–Landau problem without magnetic fields. We deviate from the latter by not prescribing Dirichlet data on the magnetisation, but by introducing a boundary penalty instead. The notion of stability for the central vortex is expressed by the positive definiteness of the Hessian of the energy functional evaluated at the central vortex solution, which in turn can be expressed as the positivity of the lowest eigenvalue of the corresponding Rayleigh quotient. We prove stability of the central vortex configuration for both tangential boundary data and a strong boundary penalty under a certain assumption on the lowest eigenvalue of the Rayleigh quotient for variations with zero boundary data.

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# Chapter 1

# Introduction and main results

In this thesis we study two different ferromagnetic settings, both of thin–film type, described by the micromagnetic model. The first setting is an elongated thin–film strip. In this case we are interested in the formation of a characteristic pattern, the concertina pattern. It is the almost periodic central pattern in Figure 1.1. The

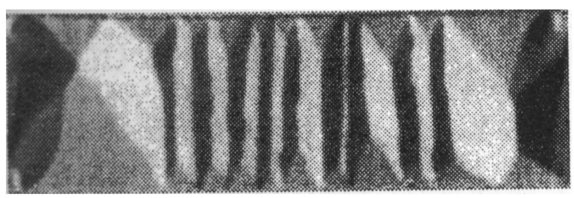

Figure 1.1: The concertina pattern, courtesy of R. Schäfer, IfW Dresden

pattern forming process is triggered by a bifurcation due to the influence of an applied external field.

The second setting is a thin–film disc. Here we study the stability of a central vortex, subject to a strong boundary penalty, cf. Figure 1.2.

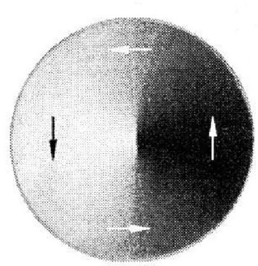

Figure 1.2: A circular Bloch line, magnetic vortex, courtesy of R. Schäfer, IfW Dresden

## 1.1 Micromagnetics

The micromagnetic model is a mesoscopic model for the description of ferromagnetic samples. We reminde the reader that in the microscopic approach to magnetism one analyses spin systems, while the macroscopic approach uses the Maxwell equations which are at the heart of the theory of electromagnetism. Micromagnetics combines elements of both extremes by considering the magnetisation $m$, which is known from electrodynamics of continuous media, as a local average of spins.

Micromagnetics is a variational model, typically set at temperatures below the Curie temperature, at which ferromagnetic behaviour is to be expected. In this case, the magnetisation is assumed to be in a state of saturation, i.e., it is assumed that in a ferromagnetic sample $\Omega$ the magnetisation is constrained to unit–length:

$$m : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{S}^2 \quad \Leftrightarrow \quad |m| \ = \ 1. \tag{1.1}$$

This constraint is nonconvex so that the direct method in the calculus of variations cannot be used to study minimisation problems.

The behaviour of $m$ is controlled by several energy contributions, two of which are always present:

- the exchange energy

$$E_{ex}(m) \ = \ d^2 \int_\Omega |\nabla m|^2 \, dx, \tag{1.2}$$

- the magnetostatic or stray–field energy

$$E_{stray}(m) \ = \ \int_{\mathbb{R}^3} |\nabla U|^2 \, dx.$$

We now comment on these two terms.

Microscopically, the magnetisation is a local average of atomic spins. The spins of different atoms interact with each other, based on symmetry requirements induced by the Pauli exclusion principle. In the ferromagnetic case we are interested in, this interaction favours a parallel alignment of the spins, which corresponds to the fact that a small Dirichlet integral (1.2) implies that $m$ is almost constant. Each of the energy contributions adding to the micromagnetic model is stated in nondimensional form, except for length. Thus, to make up for the gradient, the *exchange length* $d$ is introduced, which measures the effective reach of the exchange interaction.

The stray–field contribution is a consequence of Maxwell's equations for the special case of magnetostatics. In a medium, i.e., a ferromagnetic sample, a stray field $H$ is generated by the magnetisation $m$. Without currents, electric fields, or electric charges to be considered, the stray field field is curl–free and can be expressed as the gradient of a magnetostatic potential $U$. In distributional form, this yields the following characterisation of $U$:

$$-\int_{\mathbb{R}^3} H \cdot \nabla \zeta \, dx \ = \ \int_{\mathbb{R}^3} \nabla U \cdot \nabla \zeta \, dx \ = \ \int_\Omega m \cdot \nabla \zeta \, dx, \quad \forall \zeta \in C_0^\infty(\mathbb{R}^3).$$

Thus, $U$ is given by a Poisson equation in distributional sense

$$\Delta U = \nabla \cdot (m\chi_\Omega),$$

which can be interpreted in analogy to electrostatics. The potential is generated by charges, of which there exist two different types, (see Figure 1.3):

- volume charges: $-\nabla \cdot m$,
- boundary charges: $m \cdot \nu_\Omega$.

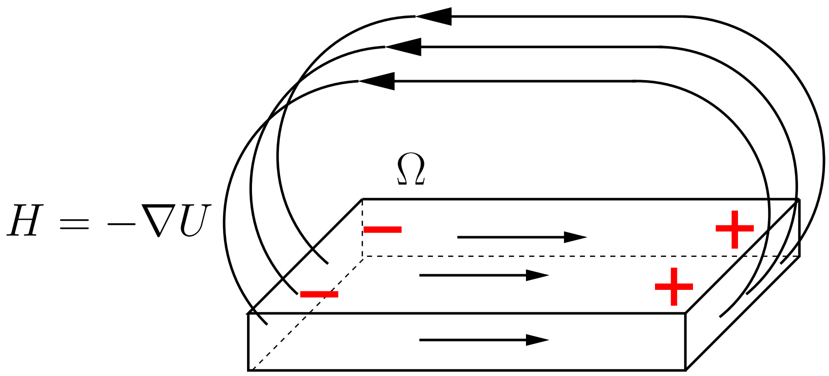


Figure 1.3: Stray–field generation by boundary charges

The stray–field energy acts as a penalisation of these charges, thus divergence–free magnetisations tangential to the boundary are favoured. Note that the stray–field contribution is nonlocal, as the Poisson equation is solved by a convolution of the charge density with the Newton potential.

Several other contributions may be added to the micromagnetic energy functional. The following two are the most common ones:

- the Zeeman or external field energy

$$E_{Zeeman}(m) = -2\int_\Omega H_{ext} \cdot m\, dx,$$

- the anisotropy energy

$$E_{aniso}(m) = Q\int_\Omega \varphi(m)\, dx.$$

The Zeeman energy is due to the Zeeman effect, by which spins couple to an external magnetic field and which favours a parallel alignment of the spins to that field.

For a special crystalline structure of the ferromagnetic material, the magnetisation may not be allowed to vary with the same ease in all directions. Such hindrances are encoded in the corresponding structural function $\varphi(m)$ and give rise to an anisotropy energy. The dimensionless prefactor $Q$ measures the effective size of this contribution compared to the stray–field energy. In this thesis we will focus on soft ferromagnetic samples, i.e., samples for which $Q \ll 1$, and thus set $Q = 0$.

For the energy contributions explained, the micromagnetic model is thus given by the energy functional

$$E(m) \;=\; d^2 \int_\Omega |\nabla m|^2 \, dx + \int_{\mathbb{R}^3} |\nabla U|^2 \, dx - 2 \int_\Omega H_{ext} \cdot m \, dx,$$

for a magnetisation

$$m : \Omega \mapsto \mathbb{S}^2$$

and the corresponding stray–field

$$\Delta U \;=\; \nabla \cdot (m \chi_\Omega)\,.$$

The main interest in the ferromagnetic samples or in micromagnetics which models them, stems from their ability to form patterns. Such patterns consist of a combination of *domains* and *walls*, with the occasional occurence of *vortices* and *antivortices*. We will now briefly comment on these structures:

- A *magnetic domain* is a region of a ferromagnetic sample in which the magnetisation is almost constant, see e.g. the uniformly coloured regions in Figure 1.1.

- *Walls* are transition layers that separate magnetic domains. The best known examples are Néel walls and Bloch walls. For Néel walls the magnetisation, which is a vector field, rotates in the plane, see the arrows in Figure 1.4. Bloch

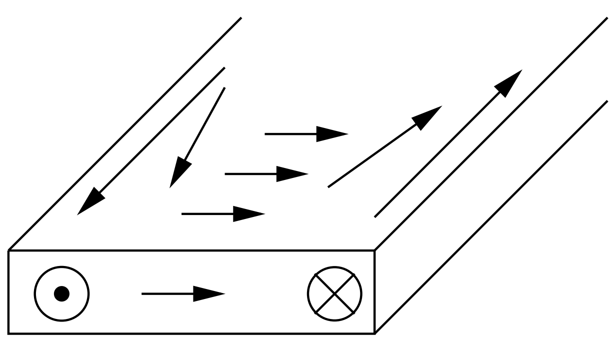

Figure 1.4: 180°–Néel wall, schematic view

  walls connect two domains of different magnetisation direction by an out–of–plane rotation. More complicated walls exist, for example the cross–tie wall, see Figure 1.5. For a mathematical treatment of the cross–tie wall, see [10].

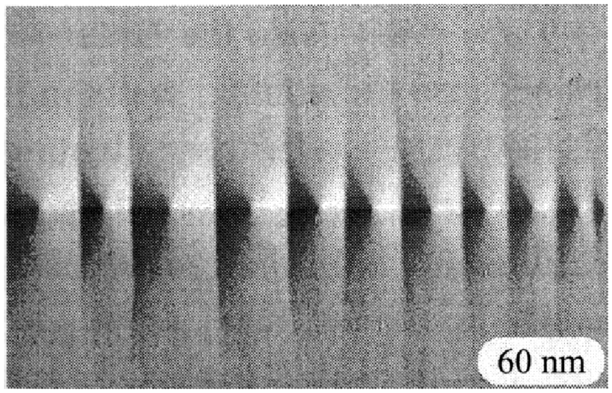


Figure 1.5: Cross–tie wall, experimental picture, courtesy of R. Schäfer, IfW Dresden

- *Vortices* and *antivortices* are configurations where the magnetisation rotates in–plane around a single point, which is the projection of a three–dimensional line, called a circular or a cross Bloch line. In a small circular region around the center of a vortex or antivortex, the magnetisation is forced to turn out–of–plane, to avoid a topological singularity. The cross–tie wall exhibits an array of alternating vortices and antivortices, see Figure 1.6. The cross Bloch lines are the black regions while the circular Bloch lines appear in the circled white regions.

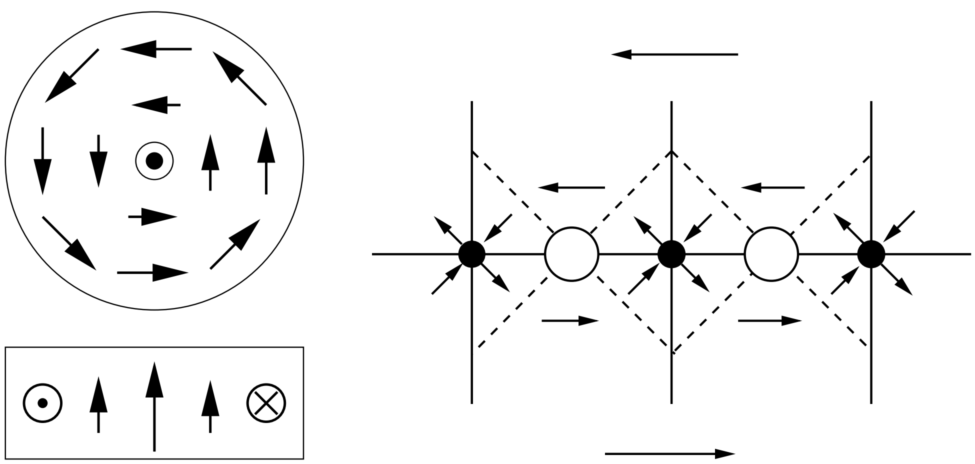

Figure 1.6: Circular Bloch line, top view and cross section; cross–tie wall

For more on the subject of pattern–formation in magnetism, with an introduction to experimental as well as analytical and numerical methods, we refer to the book by Hubert and Schäfer [17]. For the reader more inclined to the theory of ferromagnetism, we refer to Aharoni's book [1]. Several mathematical aspects of micromagnetism are summarised in [11].

## 1.2 Main results

### The concertina pattern: Linear stability analysis

Starting from a constant magnetisation in direction of the long axis of an elongated thin–film strip, consider an external field applied in the opposite direction to the magnetisation. When the applied field reaches a critical value, the constant magnetisation becomes unstable and a bifurcation occurs. The resulting instability depends upon the relationship between three lengthscales present in the problem, exchange length $d$, thickness $t$ and width $\ell$. There exist four regimes, which can be ordered by increasing thickness of the film, each of them with its own critical field and instability, cf. [6]. In a first step, we study the two intermediate regimes in the framework of $\Gamma$–convergence.

We set out from the variational problem of minimising the Rayleigh quotient of the Hessian of the energy functional, evaluated at the uniform magnetisation. Due to the infinite extension in $x_1$ of the sample under consideration, we factorise in $x_1$ and monitor the corresponding Fourier dual $k_1$ and the infinitesimal variations $\zeta$ from the uniform magnetisation. We first prove compactness for sequences that leave the Hessian bounded. As part of this study, we prove a weak form of edge–pinning, which means zero boundary data for the variations in the limit, such that the limit functions are in $H_0^1$. We then turn to the proof of $\Gamma$–convergence itself. The limit problems can be solved explicitly. In the second of the intermediate regimes, which is the overall third one, the asymptotically optimal variation — the unstable mode — is shown to exhibit oscillations in $x_1$–direction. In the other regime we analyse this is not the case. The oscillatory behaviour is of special interest, as it introduces a new lengthscale into the problem. We are therefore led to investigate the connection of this phenomenon to the concertina pattern. The oscillation wavelength is shown to be asymptotically equal to

$$w^* = \left(32\pi\frac{\ell^2 d^2}{t}\right)^{1/3}.$$

### The concertina pattern: A subcritical bifurcation

In order to investigate the connection of the oscillatory behaviour and the concertina pattern, a weakly nonlinear analysis is performed for the third regime. This kind of analysis can be used to identify the type of bifurcation occurring. In the case of a second–order phase transition — a supercritical bifurcation — the argument would be that the concertina pattern may evolve continuously out of the bifurcation. We derive a normal form for the bifurcation by identifying the first nonlinear term, which is a correction to the charge density in the stray–field contribution. The full charge density in the normal form is of the type of the Burgers operator

$$\hat\sigma = \hat\partial_2 \hat m_2 - \hat\partial_1\left(\tfrac12 \hat m_2^2\right),$$

if $\hat x_2$ is identified with "time" and $-\hat x_1$ with "space".

Again this analysis is done in the framework of Γ–convergence. The energy functional is renormalised according to the asymptotic results of the linear stability analysis. A new element is the rescaling of the amplitude of the magnetisation components perpendicular to the long axis, which corresponds to a magnification in configuration space. We also introduce a finite periodicity in $x_1$–direction, for the sake of working with a finite volume.

Due to the correction to the charge density, the proof of compactness is less straightforward than in the linear case and requires a compensated compactness argument. For the construction part in the Γ–convergence, we prove that admissible functions of finite energy can be approximated by smooth admissible functions in the energy topology. Again the main difficulty is the charge density. Due to the appearance of the Burgers operator, it seems natural to use concepts from the theory of conservation laws to prove the density result.

As the new charge density allows for dipolar charge cancellation, the bifurcation turns out to be subcritical. Results not included in this thesis show that the normal form is coercive in the large, i.e., that although the phase transition is discontinuous, there exist nontrivial minimisers close to the uniform magnetisation which can be found numerically via a path–following algorithm, cf. [8], [28].

**Néel walls: High energy scaling**

The scaling limit derived by the weakly nonlinear analysis has the property of connecting a linear regime — close to the Hessian of the original functional — to a highly nonlinear regime. The corresponding energy for the latter yields a one–dimensional model for an array of low–angle Néel walls perpendicular to the long axis of the elongated thin–film strip under consideration. We derive a scaling law for the energy in terms of the applied external field, which is valid for large external fields. More precisely, we have to leading order

$$E_{Neel} \;=\; -\frac{h_{ext}^2}{2\pi}\ln\left(\frac{h_{ext}}{2\pi^2}\right).$$

This result is established by deriving matching upper and lower bounds. We first analyse a singular model without exchange contribution, which yields a periodically divergent solution. For the lower bound, we use the regularising effect of the exchange energy to improve upon $L^\infty$–control given by the conditions on the solution, while for the upper bound we modify the Fourier coefficients obtained from the singular model to have an appropriate trial function. Both bounds match to leading order and yield the above result.

**Vortex stability**

For a thin–film disc, we analyse the stability of a central vortex. Starting from a reduced energy functional, which is closely related to the Ginzburg–Landau functional without magnetic fields, we deviate from the usual approach discussed in the literature in that we do not consider Dirichlet boundary conditions of unit length. Instead, we first assume tangential boundary conditions and later a boundary penalty, such that the energy is of the form

$$E(m) \;=\; \int_{B_1(0)} \left[\frac{1}{2}|\nabla m|^2 + \frac{1}{4\delta^2}(1-|m|^2)^2\right] dx + \frac{1}{2\varepsilon}\int_{\partial B_1(0)} (m\cdot\nu)^2 \, d\mathcal{H}^1.$$

The question of stability of the radial solution is answered by a spectral analysis of the Hessian of the energy functional at this solution. By a Fourier expansion in the polar angle, the analysis can be reduced to proving positivity of two eigenvalues, $\lambda_0$ and $\lambda_1$, corresponding to the Rayleigh quotients of two radial variational problems. One of these eigenvalues, $\lambda_0$, can be shown to be bounded below independently of the vortex core radius $\delta$.

We first consider tangential boundary data, which correspond to the value $\varepsilon = 0$ for the boundary penalty parameter. In this case, we use the result for the Dirichlet case and decompose the Hessian by an interior localisation into a part corresponding to Dirichlet boundary conditions, and error terms. Under the assumption that the Rayleigh quotient for perturbations with zero Dirichlet data scale as $|\ln\delta|^{-1}$, we derive positivity of $\lambda_1$ in this case, i.e., we bound the error terms and absorb them into the Dirichlet part.

In a second step, we consider a strong boundary penalty. More precisely, we require that $\varepsilon \ll |\ln\delta|^{-1}$. In this case, we use the result for the tangential case and decompose the Hessian by a localisation at the boundary into a part corresponding to tangential boundary conditions and error terms. We again bound the error terms and absorb them into the tangential part, to prove positivity of $\lambda_1$ in this case.

## 1.3 Notation

We will now introduce some notation which is used throughout this thesis.

Scaling symbols

We will use the following symbols for the purposes of determining the relative scaling of two expressions:

- "$f$ scales like $g$", or
$$f \;\sim\; g,$$
if there exists a positive constant $C$ such that
$$\frac{1}{C} f \;\le\; g \;\le\; C\, g.$$
- "$f$ is smaller in scaling than $g$", or
$$f \;\lesssim\; g,$$
if there exists a positive constant $C$ such that
$$f \;\le\; C\, g.$$

### Primes and Tildes

In Chapter 2, we will use the following notation:

$$x \;=\; (x_1, x_2, x_3), \quad x' \;=\; (x_1, x_2), \quad \tilde{x} \;=\; (x_2, x_3),$$

with the implicit understanding that these definitions of primes and tildes also apply to Fourier duals and gradients.

### Combined Fourier series and transform

Also in Chapter 2 it will be of use to work with the combined Fourier series in $x_1$ for periodicity $L$ and Fourier transform in $x_2$, i. e.,

$$\mathcal{F}(f)(k') \;=\; \frac{1}{\sqrt{2\pi L}} \int_{(0,L)\times\mathbb{R}} \exp(ik' \cdot x')\, f\, dx' \quad \text{for} \quad k' \;=\; (k_1, k_2) \in \frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}, \tag{1.3}$$

and to introduce the notation

$$\int_{\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}} dk' \;:=\; \sum_{k_1 \in \frac{2\pi}{L}\mathbb{Z}} \int_{\mathbb{R}} dk_2.$$

## 1.4 Outline

In Chapter 2 we first present the linear stability analysis for the two intermediate regimes. The asymptotic form of the instabilities is derived via a $\Gamma$–convergence result for the Hessian of the micromagnetic energy in both regimes. The corresponding limiting variational problems can be solved directly, yielding the desired asymptotic forms and also the wavelength of the instability in the third regime. These results have been published in *Journal of Nonlinear Science* [7]. We also present the derivation of the normal form of the bifurcation in the third regime, via $\Gamma$–convergence arguments. This result has been published in *Journal of Nonlinear Science* [8].

In Chapter 3 we present the analysis of a model for an array of Néel walls inspired by the normal form mentioned above. We first consider the problem without the regularising effect of the exchange energy and derive a Fourier series representation for the solution of the corresponding variational problem, which is periodically divergent. By a modification of the Fourier coefficients, we recover an upper bound for the original problem, which is complemented by the lower bound which incorporates the regularisation due to exchange.

Finally, Chapter 4 contains the stability analysis for a central vortex in a thin ferromagnetic disc. Considering the linearisation about the radial profile and its reexpression in terms of Fourier series in the polar angle, we have a sequence of quadratic forms which are ordered. The two lowest eigenvalues $\lambda_0$ and $\lambda_1$ of the corresponding Rayleigh quotients are of special interest, as stability corresponds to the positivity of both of them. In the case of tangential boundary conditions, we prove positivity of $\lambda_0$ by an ODE argument, while positivity of $\lambda_1$ requires an interior localisation argument. In the case of a boundary penalty, we prove positivity of $\lambda_0$ by a bound on the Rayleigh quotient while positivity of $\lambda_1$ requires a boundary localisation.

# Chapter 2

# The concertina pattern

In this chapter, we focus on the concertina pattern in soft ferromagnetic thin–film elements, i.e. thin samples of low crystalline anisotropy, cf. Figures 1.1 and 2.1.

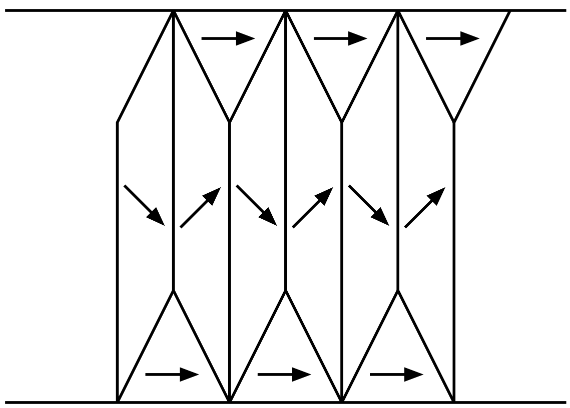

Figure 2.1: Schematic view of the concertina pattern

Experimentally, the pattern is generated by destabilising the groundstate magnetisation, uniform along the long axis, by applying a reverse external field, cf. Figure 2.2.

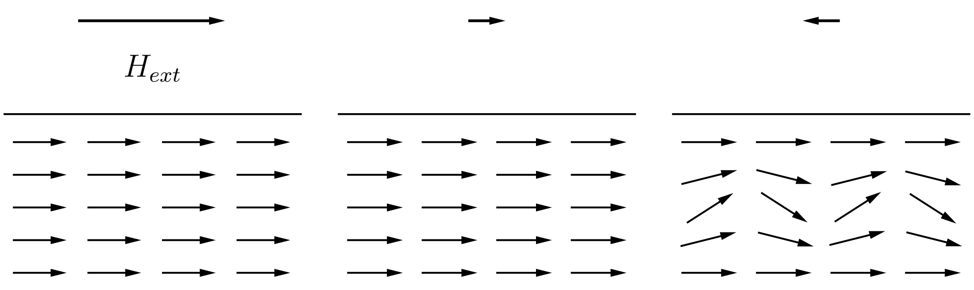


Figure 2.2: How the concertina pattern is generated

In order to understand this phenomenon, we first resort to nucleation theory for the micromagnetic model, which describes the onset of a bifurcation at a critical external field. The ferromagnetic sample is assumed to be saturated by a strong external field. As the field is reduced, an instability eventually occurs. The corresponding field is called the critical field and the first instability of the saturation branch is called nucleation. Such an instability may or may not be related to an irreversible event, see [27]. Whether this is the case or not depends on the type of bifurcation.

Mathematically speaking, the critical field is the value of the external field at which the Hessian of the micromagnetic energy ceases to be positive definite. The degenerate subspace consists of the "unstable modes". The related eigenvalue problem has been explicitly and completely solved for special geometries like ellipsoids of revolution [5], [12], [1]. As a consequence of the multiscale nature of the problem, there are different types of unstable modes, depending on and defining parameter regimes. For instance, for sufficiently small samples, the unstable mode corresponds to a coherent rotation of the magnetisation, as in the Stoner–Wohlfarth model [29]. For sufficiently large samples, the unstable mode corresponds to a curling of $m$ that does not generate a stray field [5]. A third mode, which corresponds to a buckling of the magnetisation, has been found numerically [12]. In [27], hysteresis simulations show the nucleation of a curling mode via a supercritical pitchfork bifurcation.

In [6], we studied the nucleation problem for a cylindrical geometry that mimics an elongated thin–film element. We identified four scaling regimes for nucleation in the two nondimensional parameters. One of these regimes displays an oscillatory buckling mode and is novel in the sense that the period of oscillation is determined by a subtle interaction of sample geometry and material length scales. It is therefore tempting to make the hypothesis that the concertina pattern evolves continuously out of this oscillatory instability. It is noteworthy that this thin–film buckling regime extends over a wide range in parameter space. This contradicts Aharoni's claim that buckling plays only a minor rôle [1, p. 202].

In order to investigate the hypothesis that the concertina pattern evolves continuously out of the oscillatory instability, it is first necessary to asymptotically identify the unstable mode in this regime and then to identify the type of bifurcation. This sets the scope for the present chapter. More precisely, in Section 2.2 we will asymptotically identify the unstable modes and critical fields in the two intermediate — buckling — regimes. In Section 2.3, we will concentrate on the second buckling regime, Regime III of [6]. We derive a normal form of the bifurcation in this regime and prove the bifurcation to be subcritical. Thus the phase transition ocurring at the bifurcation point is of first order, discontinuous. Nevertheless, a coercivity result proved in [8] and numerical simulations performed therein and in [28] show that concertina–type minimisers lie close in energy to the original uniform magnetisation.

# 2.1 Preliminaries

In this section we collect some preliminary information needed for the linear and nonlinear analysis to follow: We state the setting, quote the main result and two lemmata from [6], review the basic notions of nucleation and shed some light on the stray–field contribution.

## 2.1.1 The setting

We start by stating the setting. The micromagnetic energy in this case consists of exchange, stray–field and Zeeman energy:

$$E(m) \;=\; d^2 \int_\Omega |\nabla m|^2\,dx + \int_{\mathbb{R}^3} |\nabla U_m|^2\,dx - 2\int_\Omega H_{ext}\cdot m\,dx. \tag{2.1}$$

The sample $\Omega$ is a thin elongated strip

$$\Omega \;=\; \mathbb{R}\times(0,\ell)\times(0,t), \quad \text{with } \ell \ll t; \tag{2.2}$$

see Figure 2.3. We choose this geometry for two reasons:

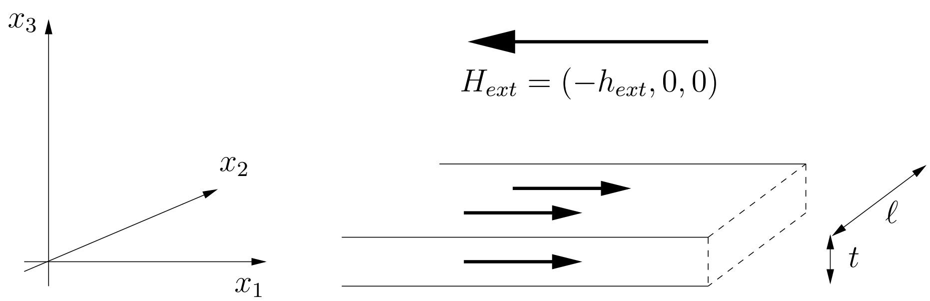


Figure 2.3: The geometry

- $\Omega$ mimics an elongated thin–film element of thickness $t$ and width $\ell$. Later we will impose a finite periodicity $L$ in $x_1$–direction.
- Translation invariance in $x_1$ implies that $m^* = (1,0,0)$ is a stationary point for all external fields of the form $H_{ext} = (-h_{ext},0,0)$, $h_{ext}\in\mathbb{R}$ (note the change of sign).

## 2.1.2 Hessian and nucleation

The local stability of the stationary point $m^*$ is described by the second variation of the energy $E$, its Hessian. Due to the nonconvex constraint $|m|^2 = 1$, infinitesimal perturbations of $m^* = (1,0,0)$ are of the form

$$\zeta \;=\; (0,\zeta_2,\zeta_3), \quad \zeta = \zeta(x_1,x_2,x_3).$$

The Hessian $E(m^*)$ in $m^*$ is given by

$$\tfrac{1}{2}\mathrm{Hess}E(m^*)(\zeta,\zeta) \;=\; \tfrac{1}{2}\mathrm{Hess}E^0(m^*)(\zeta,\zeta) - h_{ext}\int_\Omega |\zeta|^2, \tag{2.3}$$

where $\mathrm{Hess}E^0(m^*)$ denotes the Hessian without Zeeman term, i.e.,

$$\tfrac{1}{2}\mathrm{Hess}E^0(m^*)(\zeta,\zeta) \;=\; d^2\int_\Omega |\nabla\zeta|^2\,dx + \int_{\mathbb{R}^3} |\nabla u_\zeta|^2\,dx,$$

and the magnetostatic potential $u_\zeta$ is determined by $\zeta$ in the usual way. Anticipating the fact that we are interested in the Rayleigh quotient, we abbreviate

$$\mathcal{R}(\zeta) \;=\; \tfrac{1}{2}\mathrm{Hess}E^0(m^*)(\zeta,\zeta) \;=\; d^2\int_\Omega |\nabla\zeta|^2\,dx + \int_{\mathbb{R}^3} |\nabla u_\zeta|^2\,dx.$$

The critical field $h_{crit}$ is the value of $h_{ext}$ for which $\mathrm{Hess}E(m^*)$ ceases to be positive definite. The unstable modes are the elements of the degenerate subspace of $\mathrm{Hess}E(m^*)$ at $h_{ext} = h_{crit}$. The following variational characterisation of both can be inferred from (2.3): $h_{crit}$ and the (normalised) unstable modes are the minimum and the minimisers, respectively, of the variational problem

$$\mathcal{R}(\zeta) \quad \text{subject to} \quad \int_\Omega |\zeta|^2\,dx \;=\; 1. \tag{2.4}$$

We may make use of the translation invariance in $x_1$ by a partial Fourier transform in that variable. More precisely, we have a factorisation of $\mathcal{R}$ into $\{\mathcal{R}(k_1,\cdot)\}_{k_1\in\mathbb{R}}$, where

$$\begin{aligned}\mathcal{R}(k_1,\zeta) \;&=\; d^2\int_{(0,\ell)\times(0,t)} \left(k_1^2|\zeta|^2 + |\partial_2\zeta|^2 + |\partial_3\zeta|^2\right) dx_2dx_3 \\ &\quad + \int_{\mathbb{R}^2} \left(k_1^2|u_\zeta|^2 + |\partial_2 u_\zeta|^2 + |\partial_3 u_\zeta|^2\right) dx_2dx_3,\end{aligned} \tag{2.5}$$

$k_1$ denotes the Fourier dual to $x_1$ and $\zeta = \zeta(x_2,x_3)$. Hence we replace the variational problem (2.4) in $\zeta(x_1,x_2,x_3)$ by the variational problem in $\zeta(x_2,x_3)$ and $k_1$ of minimizing

$$\mathcal{R}(k_1,\zeta) \quad \text{subject to} \quad \int_{(0,\ell)\times(0,t)} |\zeta|^2\,dx_2dx_3 \;=\; 1. \tag{2.6}$$

Notice that unstable modes can also be seen as groundstates for the operator

$$\mathcal{L}\zeta \;=\; -d^2\Delta_{\mathrm{Neumann}} - \begin{pmatrix}\partial_2\\ \partial_3\end{pmatrix} u_\zeta. \tag{2.7}$$

A complete explicit diagonalisation of $\mathcal{L}$ beyond the obvious factorisation (2.5) seems not at hand. Indeed, the contribution by exchange is diagonal w.r.t. Fourier cosine series in $(x_2,x_3)$, while the stray–field contribution is diagonal w.r.t. the Fourier transform in $(x_2,x_3)$. This lack of compatibility reflects the fact that the exchange energy is local, i.e., confined to the sample, while the stray–field energy is nonlocal, i.e., extends into the ambient space.

### 2.1.3 The critical field: Four regimes

A rigorous analysis of the scaling of the critical field $h_{crit}$ was carried out in [6]. By dimensional analysis, $h_{crit}$ is a function of the nondimensional parameters $t/d$, $\ell/d$:

$$h_{crit} \; = \; h_{crit}(t/d, \ell/d). \tag{2.8}$$

In [6], all regimes for the scaling of this function are identified:

**Theorem.** *[6]. For $t \ll \ell$, there holds:*

$$h_{crit} \sim \left\{ \begin{array}{lllllll} \frac{t}{\ell}\ln\left(\frac{\ell}{t}\right) & for & & & t & \leq & \frac{d^2}{\ell}\ln^{-1}\left(\frac{\ell}{d}\right) & Regime\ I \\ \left(\frac{d}{\ell}\right)^2 & for & \frac{d^2}{\ell}\ln^{-1}\left(\frac{\ell}{d}\right) & \leq & t & \leq & \frac{d^2}{\ell} & Regime\ II \\ \left(\frac{dt}{\ell^2}\right)^{2/3} & for & \frac{d^2}{\ell} & \leq & t & \leq & (d\ell)^{1/2} & Regime\ III \\ \left(\frac{d}{t}\right)^2 & for & (d\ell)^{1/2} & \leq & t & & & Regime\ IV \end{array} \right\}. \tag{2.9}$$

This theorem can best be visualised in terms of a phase diagram for (2.8) in parameter space; see Figure 2.4.

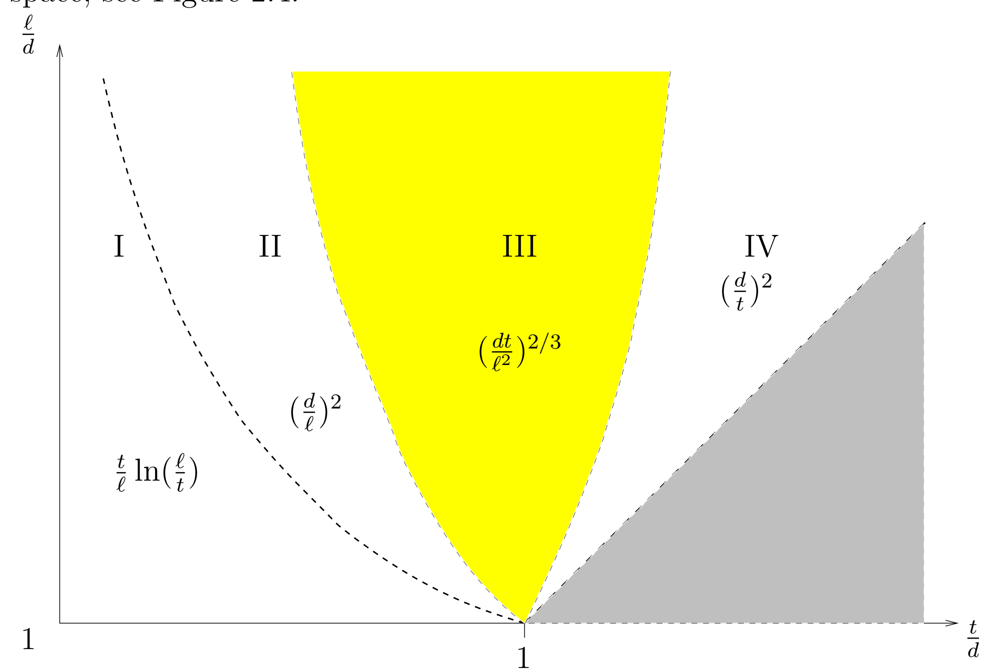


Figure 2.4: Phase diagram for $h_{crit}(t/d, \ell/d)$

### 2.1.4 Stray–field lemmata

For the purpose of the subsequent linear and nonlinear analysis, it is of use to acquire some knowledge on the properties of the stray–field contribution. This knowledge comes in the guise of variants of two basic lemmata. We will state the most general versions suited to our purposes and specify the peculiarities per case needed later.

Let us decompose the distributional divergence of the magnetisation into

$$\nabla \cdot m \;=\; \nabla' \cdot m' + \partial_3 m_3 \;=:\; \sigma + \partial_3 m_3,$$

where we think of $m$ as trivially extended outside of $\Omega$. Then we have the following result.

**Lemma 2.1.** *Let $m = m(x')$ be admissible and let $u_m$ be the solution of the distributional Poisson equation*

$$\Delta u_m \;=\; \sigma + \partial_3 m_3 \;=\; \sigma + m_3(x')\left(\delta_0(x_3) - \delta_t(x_3)\right). \tag{2.10}$$

*Then*

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla u_m|^2\, dx \;&=\; t^3 \int_{\mathbb{R}^2} \frac{\exp(-t|k'|) - 1 + t|k'|}{(t|k'|)^3} |\mathcal{F}(\sigma)|^2\, dk' \\ &+ t \int_{\mathbb{R}^2} \frac{1-\exp(-t|k'|)}{t|k'|} |\mathcal{F}(m_3)|^2\, dk'.\end{aligned} \tag{2.11}$$

Proof of Lemma 2.1.
For the purpose of this proof, we nondimensionalise length by $t$ and replace $(0,1)$ by $(-\frac{1}{2}, \frac{1}{2})$. We take the Fourier transform in $x'$ of (2.10):

$$-|k'|^2 \mathcal{F}(u_m) + \partial_3^2 \mathcal{F}(u_m) \;=\; 0, \qquad \text{for } x_3 \notin \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \tag{2.12}$$

$$-|k'|^2 \mathcal{F}(u_m) + \partial_3^2 \mathcal{F}(u_m) \;=\; \mathcal{F}(\sigma), \qquad \text{for } x_3 \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \tag{2.13}$$

$$\partial_3 \mathcal{F}(u_m)\left(k', \tfrac{1}{2}+\right) - \partial_3 \mathcal{F}(u_m)\left(k', \tfrac{1}{2}-\right) \;=\; -\mathcal{F}(m_3)(k'), \text{ for } x_3 \;=\; \tfrac{1}{2} \tag{2.14}$$

$$\partial_3 \mathcal{F}(u_m)\left(k', -\tfrac{1}{2}+\right) - \partial_3 \mathcal{F}(u_m)\left(k', -\tfrac{1}{2}-\right) \;=\; \mathcal{F}(m_3)(k'), \quad \text{for } x_3 \;=\; -\tfrac{1}{2}. \tag{2.15}$$

In view of (2.12) and (2.13), $\mathcal{F}(u_m)$ must be of the form

$$\begin{aligned}&\mathcal{F}(u_m)(k', x_3) \\ &=\; \bar{u}(k') \times \left\{\begin{array}{ll} \sinh\left(\frac{|k'|}{2}\right) \exp\left(\left(\frac{1}{2} - x_3\right)|k'|\right), & \frac{1}{2} \le x_3 \\ \exp\left(\frac{|k'|}{2}\right) - \cosh(|k'| x_3), & -\frac{1}{2} \le x_3 \le \frac{1}{2} \\ \sinh\left(\frac{|k'|}{2}\right) \exp\left(\left(\frac{1}{2} + x_3\right)|k'|\right), & x_3 \le -\frac{1}{2} \end{array}\right\} \\ &+\; \bar{v}(k') \times \left\{\begin{array}{ll} \sinh\left(\frac{|k'|}{2}\right) \exp\left(\left(\frac{1}{2} - x_3\right)|k'|\right), & \frac{1}{2} \le x_3 \\ \sinh(|k'| x_3), & -\frac{1}{2} \le x_3 \le \frac{1}{2} \\ -\sinh\left(\frac{|k'|}{2}\right) \exp\left(\left(\frac{1}{2} + x_3\right)|k'|\right), & x_3 \le -\frac{1}{2} \end{array}\right\}.\end{aligned} \tag{2.16}$$

Because of

$$
\begin{aligned}
&\partial_3\mathcal{F}(u_m)(k',x_3)\\
&= \bar{u}(k')\times\left\{\begin{array}{ll}
-|k'|\sinh\left(\frac{|k'|}{2}\right)\exp\left(\left(\frac{1}{2}-x_3\right)|k'|\right), & \frac{1}{2}\le x_3\\
-|k'|\sinh(|k'|x_3), & -\frac{1}{2}\le x_3\le \frac{1}{2}\\
|k'|\sinh\left(\frac{|k'|}{2}\right)\exp\left(\left(\frac{1}{2}+x_3\right)|k'|\right), & x_3\le -\frac{1}{2}
\end{array}\right\}\\
&+ \bar{v}(k')\times\left\{\begin{array}{ll}
-|k'|\sinh\left(\frac{|k'|}{2}\right)\exp\left(\left(\frac{1}{2}-x_3\right)|k'|\right), & \frac{1}{2}\le x_3\\
|k'|\cosh(|k'|x_3), & -\frac{1}{2}\le x_3\le \frac{1}{2}\\
-|k'|\sinh\left(\frac{|k'|}{2}\right)\exp\left(\left(\frac{1}{2}+x_3\right)|k'|\right), & x_3\le -\frac{1}{2}
\end{array}\right\}, \qquad (2.17)
\end{aligned}
$$

(2.14) and (2.15) both turn into

$$\mathcal{F}(m_3)(k') = |k'|\exp\left(\frac{|k'|}{2}\right)\bar{v}(k'), \qquad (2.18)$$

whereby

$$\bar{v}(k') = \frac{\exp\left(-\frac{|k'|}{2}\right)}{|k'|}\mathcal{F}(m_3)(k'). \qquad (2.19)$$

By (2.13),

$$\bar{u}(k') = -\frac{\exp\left(-\frac{|k'|}{2}\right)}{|k'|^2}\mathcal{F}(\sigma)(k'). \qquad (2.20)$$

Now by Plancherel

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla u_m|^2\,dx\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\left(|k'|^2|\mathcal{F}(u_m)|^2+|\partial_3\mathcal{F}(u_m)|^2\right)dx_3dk'\\
&\overset{(2.16),(2.17)}{=} \int_{\mathbb{R}^2}|\bar{u}|^2|k'|\{1+\exp(|k'|)(-1+|k'|)\}\,dk' + \int_{\mathbb{R}^2}|\bar{v}|^2|k'|(\exp(|k'|)-1)\,dk'\\
&\overset{(2.19),(2.20)}{=} \int_{\mathbb{R}^2}\frac{\exp(-|k'|)-1+|k'|}{|k'|^3}|\mathcal{F}(\sigma)|^2\,dk' + \int_{\mathbb{R}^2}\frac{1-\exp(-|k'|)}{|k'|}|\mathcal{F}(m_3)|^2\,dk'.
\end{aligned}
$$

□

For the second lemma, we now introduce vertically averaged quantities by the following definition

$$\langle f\rangle = \frac{1}{t}\int_0^t \zeta\,dx_3,$$

define

$$\sigma' := \partial_1\langle m_1\rangle + \partial_2\langle m_2\rangle$$

and state the follwing lemma:

**Lemma 2.2.**

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla u_m|^2 \, dx \\
&\geq \; t^3 \int_{\mathbb{R}^2} \frac{|\mathcal{F}(\sigma')|^2}{2t|k'| + (t|k'|)^2} \, dk' \\
&\quad + t \int_{\mathbb{R}^2} \frac{|\mathcal{F}(\langle m_3 \rangle) - ik' \cdot \mathcal{F}(\langle (x_3 - \frac{t}{2}) m' \rangle)|^2}{1 + \frac{1}{2} t|k'| + \frac{1}{12}(t|k'|)^2} \, dk'. \qquad (2.21)
\end{aligned}$$

Proof of Lemma 2.2.
Again, we nondimensionalise length by $t$ and replace $(0,1)$ by $(-\frac{1}{2}, \frac{1}{2})$. We will actually show that for any vector field $m$ supported in $\{-\frac{1}{2} < x_3 < \frac{1}{2}\}$ we have

$$\begin{aligned}
&\min_{m_3} \left\{ \int_{\mathbb{R}^3} |\nabla u_m|^2 \, dx \mid m' \text{ given}, \langle m_3 \rangle \text{ given} \right\} \\
&= \int_{\mathbb{R}^2} \frac{|\mathcal{F}(\sigma')|^2}{2|k'| + |k'|^2} \, dk' + \int_{\mathbb{R}^2} \frac{|\mathcal{F}(\langle m_3 \rangle) - ik' \cdot \mathcal{F}(\langle x_3 m' \rangle)|^2}{1 + \frac{1}{2}|k'| + \frac{1}{12}|k'|^2} \, dk'. \qquad (2.22)
\end{aligned}$$

Notice that this implies (2.21). We first note that (2.22) can be rewritten as a saddle point problem. This is a consequence of the following representation of the stray field energy:

$$\int_{\mathbb{R}^3} |\nabla u_m|^2 \, dx \;=\; \max_v \left\{ -\int_{\mathbb{R}^3} |\nabla v|^2 \, dx + 2 \int_{\mathbb{R}^3} \nabla v \cdot m \, dx \right\}.$$

The first variation w.r.t. $m_3$ yields

$$\partial_3^2 u_m \;=\; 0 \quad \text{for} \quad x_3 \in (-\tfrac{1}{2}, \tfrac{1}{2}). \qquad (2.23)$$

Note that since the average $\langle m_3 \rangle$ of $m_3$ in $x_3$ is prescribed, the second derivative $\partial_3^2 u_m$ w.r.t. $x_3$ vanishes and not the first. The first variation w.r.t. $v$ yields

$$\begin{aligned}
\Delta u_m &= 0, && \text{for } x_3 \notin \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \\
\Delta u_m &= \nabla \cdot m, && \text{for } x_3 \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \\
\partial_3 u_m \left(x', \tfrac{1}{2}+\right) - \partial_3 u_m \left(x', \tfrac{1}{2}-\right) &= -m_3(x', \tfrac{1}{2}), && \text{for } x_3 = \tfrac{1}{2} \\
\partial_3 u_m \left(x', -\tfrac{1}{2}+\right) - \partial_3 u_m \left(x', -\tfrac{1}{2}-\right) &= m_3(x', -\tfrac{1}{2}), && \text{for } x_3 = -\tfrac{1}{2}.
\end{aligned}$$

We take the Fourier transform in $x'$ and obtain, by (2.23),

$$\begin{aligned}
-|k'|^2 \mathcal{F}(u_m) + \partial_3^2 \mathcal{F}(u_m) &= 0, && \text{for } x_3 \notin \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \quad (2.24) \\
-|k'|^2 \mathcal{F}(u_m) &= \mathcal{F}(\nabla \cdot m), && \text{for } x_3 \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \quad (2.25) \\
\partial_3 \mathcal{F}(u_m) \left(k', \tfrac{1}{2}+\right) - \partial_3 \mathcal{F}(u_m) \left(k', \tfrac{1}{2}-\right) &= -\mathcal{F}(m_3)(k', \tfrac{1}{2}), && \text{for } x_3 = \tfrac{1}{2} \quad (2.26) \\
\partial_3 \mathcal{F}(u_m) \left(k', -\tfrac{1}{2}+\right) - \partial_3 \mathcal{F}(u_m) \left(k', -\tfrac{1}{2}-\right) &= \mathcal{F}(m_3)(k', -\tfrac{1}{2}), && \text{for } x_3 = -\tfrac{1}{2}. \quad (2.27)
\end{aligned}$$

In view of (2.23) and (2.24), $\mathcal{F}(u_m)$ must be of the form

$$\begin{aligned}
&\mathcal{F}(u_m)(k',x_3)\\
&= \bar u(k')\times\left\{\begin{array}{ll}\exp\left(\left(\frac12-x_3\right)|k'|\right), & \frac12\le x_3\\ 1, & -\frac12\le x_3\le \frac12\\ \exp\left(\left(\frac12+x_3\right)|k'|\right), & x_3\le -\frac12\end{array}\right\}\\
&+ \bar v(k')\times\left\{\begin{array}{ll}\frac12\exp\left(\left(\frac12-x_3\right)|k'|\right), & \frac12\le x_3\\ x_3, & -\frac12\le x_3\le \frac12\\ -\frac12\exp\left(\left(\frac12+x_3\right)|k'|\right), & x_3\le -\frac12\end{array}\right\}. \qquad (2.28)
\end{aligned}$$

Because of

$$\begin{aligned}
&\partial_3\mathcal{F}(u_m)(k',x_3)\\
&= \bar u(k')\times\left\{\begin{array}{ll}-|k'|\exp\left(\left(\frac12-x_3\right)|k'|\right), & \frac12\le x_3\\ 0, & -\frac12\le x_3\le \frac12\\ |k'|\exp\left(\left(\frac12+x_3\right)|k'|\right), & x_3\le -\frac12\end{array}\right\}\\
&+ \bar v(k')\times\left\{\begin{array}{ll}-\frac12|k'|\exp\left(\left(\frac12-x_3\right)|k'|\right), & \frac12\le x_3\\ 1, & -\frac12\le x_3\le \frac12\\ -\frac12|k'|\exp\left(\left(\frac12+x_3\right)|k'|\right), & x_3\le -\frac12\end{array}\right\}, \qquad (2.29)
\end{aligned}$$

(2.26) and (2.27) turn into

$$\begin{aligned}
\mathcal{F}(m_3)(k',\tfrac12) &= |k'|\bar u(k')+(1+\tfrac12|k'|)\bar v(k')\\
\mathcal{F}(m_3)(k',-\tfrac12) &= -|k'|\bar u(k')+(1+\tfrac12|k'|)\bar v(k'). \qquad (2.30)
\end{aligned}$$

We now determine $\bar u$ and $\bar v$. To this aim, we consider (2.25), which in view of (2.28) turns into

$$-|k'|^2(\bar u(k')+\bar v(k')\,x_3) \;=\; \mathcal{F}(\nabla\cdot m). \qquad (2.31)$$

For $\bar u$, we take the average in $x_3\in(-\frac12,\frac12)$ of (2.31):

$$\begin{aligned}
-|k'|^2\bar u(k') &= \mathcal{F}(\sigma')(k')+\big(\mathcal{F}(m_3)(k',\tfrac12)-\mathcal{F}(m_3)(k',-\tfrac12)\big)\\
&\overset{(2.30)}{=} \mathcal{F}(\sigma')(k')+2|k'|\bar u(k'),
\end{aligned}$$

yielding

$$\bar u(k') \;=\; -\frac{\mathcal{F}(\sigma')(k')}{2|k'|+|k'|^2}. \qquad (2.32)$$

For $\bar v$, we multiply (2.31) with $x_3$ and take the average in $x_3$:

$$\begin{aligned}
-\tfrac{1}{12}|k'|^2\bar v(k') &= -\langle x_3^2\rangle|k'|^2\bar v(k')\\
&= \mathcal{F}(\langle x_3\nabla'\cdot m'\rangle)(k')+\mathcal{F}(\langle x_3\partial_3 m_3\rangle)(k')\\
&= ik'\cdot\mathcal{F}(\langle x_3 m'\rangle)(k')-\mathcal{F}(\langle m_3\rangle)(k')\\
&\quad+\tfrac12(\mathcal{F}(m_3)(k',\tfrac12)+\mathcal{F}(m_3)(k',-\tfrac12))\\
&\overset{(2.30)}{=} ik'\cdot\mathcal{F}(\langle x_3 m'\rangle)(k')-\mathcal{F}(\langle m_3\rangle)(k')+(1+\tfrac12|k'|)\bar v(k'),
\end{aligned}$$

yielding

$$\bar{v}(k') \;=\; -\frac{ik'\cdot\mathcal{F}(\langle x_3 m'\rangle)(k') - \mathcal{F}(\langle m_3\rangle)(k')}{1+\frac{1}{2}|k'|+\frac{1}{12}|k'|^2}. \tag{2.33}$$

Hence, we obtain by Plancherel (mixed terms vanish because of different symmetry in $x_3$)

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla u_m|^2\,dx \;&=\; \int_{\mathbb{R}^2}\int_{\mathbb{R}} (|k'|^2|\mathcal{F}(u_m)|^2 + |\partial_3\mathcal{F}(u_m)|^2)\,dx_3\,dk' \\
&\overset{(2.28),(2.29)}{=} \int_{\mathbb{R}^2} \left(|\bar{u}|^2(|k'|^2+2|k'|) + |\bar{v}|^2(1+\tfrac{1}{2}|k'|+\tfrac{1}{12}|k'|^2\right)\,dk' \\
&\overset{(2.32),(2.33)}{=} \int_{\mathbb{R}^2} \frac{|\mathcal{F}(\sigma')|^2}{2|k'|+|k'|^2}\,dk' + \int_{\mathbb{R}^2} \frac{|\mathcal{F}(\langle m_3\rangle) - ik'\cdot\mathcal{F}(\langle x_3 m'\rangle)|^2}{1+\frac{1}{2}|k'|+\frac{1}{12}|k'|^2}.
\end{aligned}$$

□

### 2.1.5 Two Fourier lemmata

In this section, we quote two lemmata from [6] which will be needed and give the corresponding proofs. In the first lemma we consider only the $x_2$–dependence of $m_3$ for a one–dimensional estimate:

**Lemma 2.3.** *For a length scale* $\tau \ll \ell$ *and* $\operatorname{supp} m_3 \subset (0,\ell)$ *in* $x_2$, *we have*

$$\int_0^\ell |m_3|^2\,dx_2 \;\lesssim\; \int_{\{\tau|k_2|\le 1\}} |\mathcal{F}(m_3)|^2 + \tau^2\int_0^\ell |\partial_2 m_3|^2\,dx_2. \tag{2.34}$$

Proof of Lemma 2.3.
By Plancherel, it is sufficient to show that

$$\int_{\{\tau|k_2|\ge 1\}} |\mathcal{F}(m_3)|^2\,dk_2 \;\le\; \frac{1}{2}\int_0^\ell |m_3|^2\,dx_2 + C\tau^2\int_0^\ell |\partial_2 m_3|^2\,dx_2, \tag{2.35}$$

with some universal constant $C$.

We note that since $\partial_2|m_3|^2 = 2m_3\partial_2 m_3$, we have

$$\begin{aligned}
\sup_{x_2\in(0,\ell)} |m_3|^2 \;&\lesssim\; \frac{1}{\ell}\int_0^\ell |m_3|^2\,dx_2 + \int_0^\ell |m_3\partial_2 m_3|\,dx_2 \\
&\lesssim\; \frac{1}{\ell}\int_0^\ell |m_3|^2\,dx_2 + \left(\int_0^\ell |m_3|^2\,dx_2\int_0^\ell |\partial_2 m_3|^2\,dx_2\right)^{1/2}.
\end{aligned}$$

In particular,

$$|m_3(0)|^2 + |m_3(\ell)|^2 \;\lesssim\; \frac{1}{\ell}\int_0^\ell |m_3|^2\,dx_2 + \left(\int_0^\ell |m_3|^2\,dx_2\int_0^\ell |\partial_2 m_3|^2\,dx_2\right)^{1/2}. \tag{2.36}$$

We split $\mathcal{F}(m_3)(k_2)$ according to

$$\begin{aligned}\mathcal{F}(m_3)(k_2) &= \int_0^\ell e^{ik_2x_2} m_3\,dx_2 \\ &= \frac{i}{k_2}\int_0^\ell e^{ik_2x_2}\partial_2 m_3\,dx_2 - \frac{i}{k_2}\left[e^{ik_2x_2}m_3\right]_{x_2=0}^{x_2=\ell} \\ &= \mathcal{F}(m_3)^{(1)}(k_2) + \mathcal{F}(m_3)^{(2)}(k_2).\end{aligned}$$

We note that $\frac{k_2}{i}\mathcal{F}(m_3)^{(1)}$ is just the Fourier transform of $\partial_2 m_3$ (extended by zero on $\mathbb{R}$). Thus, one one hand, we have by Plancherel

$$\int_{\{\tau|k_2|\geq 1\}} |\mathcal{F}(m_3)^{(1)}|^2\,dk_2 \;\leq\; \tau^2\int_{\mathbb{R}} |k_2\,\mathcal{F}(m_3)^{(1)}|^2\,dk_2 \;=\; \tau^2\int_0^\ell |\partial_2 m_3|^2\,dx_2. \quad (2.37)$$

On the other hand, we have

$$\begin{aligned}&\int_{\{\tau|k_2|\geq 1\}} |\mathcal{F}(m_3)^{(2)}|^2\,dk_2 \\ &\lesssim \int_{\{\tau|k_2|\geq 1\}} \frac{1}{|k_2|^2}\,dk_2\left(|m_3(0)|^2 + |m_3(\ell)|^2\right) \\ &\overset{(2.36)}{\lesssim} \tau\left(\frac{1}{\ell}\int_0^\ell |m_3|^2\,dx_2 + \left(\int_0^\ell |m_3|^2\,dx_2\int_0^\ell |\partial_2 m_3|^2\,dx_2\right)^{1/2}\right). \quad (2.38)\end{aligned}$$

Combining (2.37) and (2.38), we obtain

$$\begin{aligned}&\int_{\{\tau|k_2|\geq 1\}} |\mathcal{F}(m_3)|^2\,dk_2 \\ &\lesssim \frac{\tau}{\ell}\int_0^\ell |m_3|^2\,dx_2 + \left(\tau^2\int_0^\ell |m_3|^2\,dx_2\int_0^\ell |\partial_2 m_3|^2\,dx_2\right)^{1/2} + \tau^2\int_0^\ell |\partial_2 m_3|^2\,dx_2.\end{aligned}$$

This implies (2.35) by Young's inequality and because of $\tau \ll \ell$. □

The second lemma is a simple Fourier multiplier estimate

**Lemma 2.4.** *For $t|k_2| \leq 1$, we have*

$$1 \;\lesssim\; \min\left\{1, \frac{1}{t^2|k'|^2}\right\} + t^2|k_1|^2.$$

Proof of Lemma 2.4.
The statement is proved as follows:

$$\begin{aligned}\min\left\{1, \frac{1}{t^2|k'|^2}\right\} + t^2|k_1|^2 &\sim \min\left\{1, \frac{1}{t^2|k_1|^2}, \frac{1}{t^2|k_2|^2}\right\} + t^2|k_1|^2 \\ &\overset{t|k_2|\leq 1}{=} \min\left\{1, \frac{1}{t^2|k_1|^2}\right\} + t^2|k_1|^2 \\ &\geq \min\left\{1, \frac{1}{t^2|k_1|^2} + t^2|k_1|^2\right\} \\ &\gtrsim 1.\end{aligned}$$

□

## 2.2 Linear stability analysis

We now concentrate on the identification of the asymptotic degenerate subspace in Regimes II and III via a Γ–convergence argument. This is in order to show that the regimes are different from each other. The similarity of both regimes lies in the edge–pinning effect, which restricts both unstable modes to have zero boundary values in $x_2$. Nevertheless, not only the difference in the scaling of the critical field but, more remarkably, in the $k_1$–dependence of the unstable mode clearly sets both regimes apart. On the level of the variational characterisation (2.6), the result of this section is formulated in the following two theorems.

**Theorem 2.1.** *Let* $(k_1^*, \zeta^*)$ *be a minimiser of*

$$\mathcal{R}(k_1, \zeta) \quad \textit{constrained by} \quad \int_{(0,\ell)\times(0,t)} |\zeta|^2 \, dx_2 dx_3 \;=\; 1.$$

*In the regime*

$$\frac{d^2}{\ell} \ln^{-1}\left(\frac{\ell}{d}\right) \;\ll\; t \;\ll\; \frac{d^2}{\ell}, \tag{2.39}$$

*we have*

$$\left(\frac{\ell}{d}\right)^2 \mathcal{R}(k_1^*, \zeta^*) \;\approx\; \pi^2, \tag{2.40}$$

$$|k_1^*| \;=\; 0, \tag{2.41}$$

$$\frac{1}{t\ell} \int_{(0,\ell)\times(0,t)} \left|\zeta^*(x_2, x_3) - \sqrt{2} c \sin(\pi x_2/\ell)\right|^2 dx_2 dx_3 \;\ll\; 1, \tag{2.42}$$

*for some* $c \in \mathbb{C}$ *with* $|c| = 1$.

We remark that (2.39) with $x := \frac{d^2}{\ell t}$ and $y := \frac{\ell}{d}$ reads as

$$x \;\ll\; 1 \;\ll\; x \ln(y).$$

This, in turn, yields

$$|\ln(x)| \;=\; -\ln(x) \;\ll\; \ln(\ln(y)) \;\ll\; \ln(y),$$

whereby

$$\ln(y) \;\sim\; \ln(xy^2),$$

which can be used to state that

$$\frac{d^2}{\ell} \ln^{-1}\left(\frac{\ell}{d}\right) \;\sim\; \frac{d^2}{\ell} \ln^{-1}\left(\frac{\ell}{t}\right). \tag{2.43}$$

This reformulation will replace the left–hand side term in (2.39) for all further purposes.

From (2.40), we infer that the critical field is given by

$$h_{crit} \approx \left(\pi \frac{d}{\ell}\right)^2,$$

while by (2.41) and (2.42) the unstable subspace asymptotically consists of all perturbations $\zeta$ of the form

$$\zeta_2^* = c \sin(\pi x_2/\ell), \ \zeta_3^* \equiv 0,$$

for some constant $c \in \mathbb{R}$.

**Theorem 2.2.** *Let $(k_1^*, \zeta^*)$ be a minimiser of*

$$\mathcal{R}(k_1, \zeta) \quad \textit{constrained by} \quad \int_{(0,\ell)\times(0,t)} |\zeta|^2 \, dx_2 dx_3 = 1.$$

*In the regime*

$$\frac{d^2}{\ell} \ll t \ll (d\ell)^{1/2}, \tag{2.44}$$

*we have*

$$\left(\frac{\ell^2}{dt}\right)^{2/3} \mathcal{R}(k_1^*, \zeta^*) \approx 3\left(\frac{\pi}{2}\right)^{4/3}, \tag{2.45}$$

$$\left(\frac{d^2\ell^2}{t}\right)^{1/3} |k_1^*| \approx \left(\frac{\pi}{2}\right)^{2/3}, \tag{2.46}$$

$$\frac{1}{t\ell}\int_{(0,\ell)\times(0,t)} \left|\zeta^*(x_2,x_3) - \sqrt{2}c\sin(\pi x_2/\ell)\right|^2 dx_2 dx_3 \ll 1, \tag{2.47}$$

*for some $c \in \mathbb{C}$ with $|c| = 1$.*

This implies that by (2.45) the critical field is given by

$$h_{crit} \approx 3\left(\frac{\pi^2}{4}\frac{dt}{\ell^2}\right)^{2/3},$$

while by (2.46) and (2.47) the unstable subspace asymptotically consists of all perturbations $\zeta$ of the form

$$\zeta_2^* = c \cos(2\pi(x_1+\xi)/w^*) \sin(\pi x_2/\ell), \ \zeta_3^* \equiv 0,$$

for some constants $c, \xi \in \mathbb{R}$, where the period $w^* = \frac{2\pi}{|k_1^*|}$ of oscillation in the infinite direction $x_1$ is given by

$$w^* = \left(32\pi \frac{d^2\ell^2}{t}\right)^{1/3}. \tag{2.48}$$

In order to be able to derive a Γ–limit, we first need to nondimensionalise according to the scaling result given in [6].

### 2.2.1 The scaling: Regime II

For Regime II, this means that we rescale length by:

$$x_1 \;=\; \ell\,\hat{x}_1, \quad x_2 \;=\; \ell\,\hat{x}_2, \quad x_3 \;=\; t\,\hat{x}_3. \tag{2.49}$$

with the understanding that this implies $k_1 = \ell^{-1}\hat{k}_1$, $\partial_2 = \ell^{-1}\hat{\partial}_2$, $dx_2 = \ell \hat{d}x_2$ and so on. Because of the constraint on $\zeta$, we rescale as follows

$$\zeta \;=\; (\ell t)^{-1/2}\widehat{\zeta}.$$

The Hessian — and thus $\mathcal{R}$ — itself has to be rescaled according to

$$\mathrm{Hess}E^0 \;=\; \left(\frac{d}{\ell}\right)^2 \widehat{\mathrm{Hess}E^0}.$$

Finally, we introduce three nondimensional parameters

$$\varepsilon \;:=\; \frac{d^2}{\ell t}\ln^{-1}\left(\frac{\ell}{t}\right), \quad \delta \;:=\; \frac{t\ell}{d^2}, \quad \text{and} \quad \alpha \;:=\; \frac{t}{\ell}, \tag{2.50}$$

where $\varepsilon$ and $\delta$ characterise Regime II:

$$\varepsilon \;\ll\; 1 \quad \text{and} \quad \delta \;\ll\; 1. \tag{2.51}$$

### 2.2.2 The scaling: Regime III

For Regime III, this means that we rescale length anisotropically as follows:

$$x_1 \;=\; \left(\frac{d^2\ell^2}{t}\right)^{1/3}\hat{x}_1, \quad x_2 \;=\; \ell\,\hat{x}_2, \quad x_3 \;=\; t\,\hat{x}_3, \tag{2.52}$$

with the understanding that this implies $k_1 = (t/d^2\ell^2)^{1/3}\hat{k}_1$, $\partial_2 = \ell^{-1}\hat{\partial}_2$, $dx_2 = \ell\hat{d}x_2$ and so on. Because of the constraint on $\zeta$, we rescale as follows

$$\zeta \;=\; (\ell t)^{-1/2}\widehat{\zeta}.$$

The Hessian — and thus $\mathcal{R}$ — itself has to be rescaled according to

$$\mathrm{Hess}E^0 \;=\; \left(\frac{dt}{\ell^2}\right)^{2/3} \widehat{\mathrm{Hess}E^0}.$$

Finally, we introduce two nondimensional parameters

$$\varepsilon \;:=\; \left(\frac{d^2}{\ell t}\right)^{2/3} \quad \text{and} \quad \delta \;:=\; \left(\frac{t^2}{d\ell}\right)^{2/3}, \tag{2.53}$$

which characterise Regime III:

$$\varepsilon \;\ll\; 1 \quad \text{and} \quad \delta \;\ll\; 1. \tag{2.54}$$

### 2.2.3 Compactness

In this subsection we state and prove the necessary compactness results for the Γ–convergence to follow. First we give the version of Lemma 2.2 suited to our purposes. We note that $\zeta = (0, \zeta_2, \zeta_3)$, and that we make use of the factorisation in $x_1$. Thus Lemma 2.2 takes the following form:

**Lemma 2.5.**

$$\begin{aligned}&\int_{\mathbb{R}^2} \left(k_1^2|u_\zeta|^2 + |\tilde\nabla u_\zeta|^2\right)\, d\tilde x\\ &\geq\ t\int_{\mathbb{R}} \frac{(tk_2)^2|\mathcal{F}(\zeta_2)|^2}{2t|k'| + (t|k'|)^2}\, dk_2 + t\int_{\mathbb{R}} \frac{|\mathcal{F}(\langle\zeta_3\rangle) - ik_2\mathcal{F}(\langle(x_3 - \frac{t}{2})\zeta_2\rangle)|^2}{1 + \frac{1}{2}t|k'| + \frac{1}{12}(t|k'|)^2}\, dk_2.\end{aligned}$$

We derive two corollaries, first for Regime II:

**Corollary 2.1.** *For any $\zeta \in L^2(\tilde\Omega)$ we have*

$$\hat{\mathcal{R}}(k_1,\zeta)\ \geq\ \int_0^1 \left(\hat k_1^2|\langle\hat\zeta_2\rangle|^2 + |\hat\partial_2\langle\hat\zeta_2\rangle|^2\right)\, d\hat x_2, \tag{2.55}$$

Proof of Corollary 2.1.
According to Lemma 2.5, we have

$$\mathcal{R}(k_1,\zeta)\ \geq\ d^2\int_{\tilde\Omega}\left(k_1^2|\zeta|^2 + |\tilde\nabla\zeta|^2\right)\, d\tilde x + t\int_{\mathbb{R}} \frac{1}{2t|k'| + (t|k'|)^2}(tk_2)^2|\mathcal{F}(\langle\zeta_2\rangle)|^2\, dk_2.$$

By Jensen's inequality

$$\int_{\tilde\Omega} |\zeta|^2\, d\tilde x\ \geq\ t\int_0^\ell |\langle\zeta\rangle|^2\, dx_2\ \geq\ t\int_0^\ell |\langle\zeta_2\rangle|^2\, dx_2$$

the above turns into

$$\mathcal{R}(k_1,\zeta)\ \geq\ d^2 t\int_0^\ell \left(k_1^2|\langle\zeta_2\rangle|^2 + |\partial_2\langle\zeta_2\rangle|^2\right)\, dx_2$$

which is the unrescaled version of (2.55). □

The corresponding corollary for Regime III is:

**Corollary 2.2.** *For any $\zeta \in L^2(\tilde\Omega)$ we have*

$$\hat{\mathcal{R}}(k_1,\zeta)\ \geq\ \hat k_1^2\int_0^1 |\langle\hat\zeta_2\rangle|^2\, d\hat x_2 + \int_{\mathbb{R}} \frac{\hat k_2^2|\mathcal{F}(\langle\hat\zeta_2\rangle)|^2}{2(\hat k_1^2 + \varepsilon\hat k_2^2)^{1/2} + \delta(\hat k_1^2 + \varepsilon\hat k_2^2)}\, d\hat k_2. \tag{2.56}$$

Proof of Corollary 2.2.
According to Lemma 2.5, we have

$$\mathcal{R}(k_1,\zeta)\ \geq\ d^2\int_{\tilde\Omega}\left(k_1^2|\zeta|^2 + |\tilde\nabla\zeta|^2\right)\, d\tilde x + t\int_{\mathbb{R}} \frac{1}{2t|k'| + (t|k'|)^2}(tk_2)^2|\mathcal{F}(\langle\zeta_2\rangle)|^2\, dk_2.$$

By Jensen's inequality

$$\int_{\tilde{\Omega}} |\zeta|^2 \, d\tilde{x} \;\geq\; t \int_0^\ell |\langle\zeta\rangle|^2 \, dx_2 \;\geq\; t \int_0^\ell |\langle\zeta_2\rangle|^2 \, dx_2$$

the above turns into

$$\mathcal{R}(k_1, \zeta) \;\geq\; d^2 t \, k_1^2 \int_0^\ell |\langle\zeta_2\rangle|^2 \, dx_2 + t^2 \int_{\mathbb{R}} \frac{k_2^2 |\mathcal{F}(\langle\zeta_2\rangle)|^2}{2(k_1^2+k_2^2)^{1/2} + t(k_1^2+k_2^2)}$$

which is the unrescaled version of (2.56). □

Now we show for both regimes that for bounded $\hat{\mathcal{R}}(k_1, \zeta)$

- the rescaled wave vector $\hat{k}_1$ is bounded,
- the rescaled perturbation $\hat{\zeta}$ is close to $\langle\hat{\zeta}_2\rangle$.

In Regime II this takes the form

**Lemma 2.6.** *For any $\zeta \in L^2(\tilde{\Omega})$ with $\int_{\tilde{\Omega}} |\zeta|^2 \, d\tilde{x} = 1$, we have*

$$\int_{\hat{\tilde{\Omega}}} |\hat{\zeta} - \langle\hat{\zeta}_2\rangle|^2 \, d\hat{\tilde{x}} \;\lesssim\; \left(\frac{\alpha}{\delta} + \alpha^2\right) \hat{\mathcal{R}}(k_1, \zeta), \tag{2.57}$$

$$|\hat{k}_1|^2 \;\leq\; \hat{\mathcal{R}}(k_1, \zeta) \tag{2.58}$$

*where we note that*

$$\frac{\alpha}{\delta} \;=\; \varepsilon\alpha \ln\left(\frac{1}{\alpha}\right).$$

In Regime III we prove

**Lemma 2.7.** *For any $\zeta \in L^2(\tilde{\Omega})$ with $\int_{\tilde{\Omega}} |\zeta|^2 \, d\tilde{x} = 1$, we have*

$$\int_{\hat{\tilde{\Omega}}} |\hat{\zeta} - \langle\hat{\zeta}_2\rangle|^2 \, d\hat{\tilde{x}} \;\lesssim\; \left(\delta^2 + \varepsilon\delta\right) \hat{\mathcal{R}}(k_1, \zeta), \tag{2.59}$$

$$|\hat{k}_1|^2 \;\leq\; \hat{\mathcal{R}}(k_1, \zeta). \tag{2.60}$$

Proof of Lemmata 2.6 and 2.7.
As the difference in both lemmata lies in the final rescaling, and their unrescaled version is identical, we may prove them together. We write

$$\int_{\tilde{\Omega}} |\zeta - \langle\zeta_2\rangle|^2 \, d\tilde{x} \;\leq\; \int_{\tilde{\Omega}} |\zeta - \langle\zeta\rangle|^2 \, d\tilde{x} + \int_{\tilde{\Omega}} |\langle\zeta_3\rangle|^2 \, d\tilde{x}. \tag{2.61}$$

The first term on the right–hand side in (2.61) is estimated by Poincaré's inequality

$$\int_{\tilde{\Omega}} |\zeta - \langle\zeta\rangle|^2 \, d\tilde{x} \;\lesssim\; t^2 \int_{\tilde{\Omega}} |\partial_3\zeta|^2 \, d\tilde{x} \;\leq\; \left(\frac{t}{d}\right)^2 \mathcal{R}(k_1, \zeta). \tag{2.62}$$

For the second term in (2.61) we observe that by Lemma 2.3 with $t \ll \ell$ there holds

$$t\int_0^\ell |\langle\zeta_3\rangle|^2\,dx_2 \;\lesssim\; t\int_{\{t|k_2|\le 1\}} |\mathcal{F}(\langle\zeta_3\rangle)|^2\,dk_2 + t^3\int_0^\ell |\partial_2\langle\zeta_3\rangle|^2\,dx_2. \tag{2.63}$$

The second term in (2.63) can be estimated by Jensen's inequality

$$t^3\int_0^\ell |\partial_2\langle\zeta_3\rangle|^2\,dx_2 \;\le\; t^2\int_{\tilde\Omega} |\partial_2\zeta_3|^2\,d\tilde x \;\le\; \left(\frac{t}{d}\right)^2 \mathcal{R}(k_1,\zeta). \tag{2.64}$$

For the first term in (2.63), we appeal to Lemma 2.4, whereby

$$\begin{aligned} &t\int_{\{t|k_2|\le 1\}} |\mathcal{F}(\langle\zeta_3\rangle)|^2\,dk_2 \\ &\lesssim\; t\int_{\mathbb{R}} \min\left\{1,\frac{1}{t^2|k'|^2}\right\} |\mathcal{F}(\langle\zeta_3\rangle)|^2\,dk_2 + t^3k_1^2\int_0^\ell |\langle\zeta_3\rangle|^2\,dx_2. \end{aligned} \tag{2.65}$$

For the last term in (2.65), we appeal once more to Jensen's inequality

$$t^3k_1^2\int_0^\ell |\langle\zeta_3\rangle|^2\,dx_2 \;\le\; t^2k_1^2\int_{\tilde\Omega} |\zeta_3|^2\,d\tilde x \;\le\; \left(\frac{t}{d}\right)^2 \mathcal{R}(k_1,\zeta). \tag{2.66}$$

We now turn to the first right–hand side term of (2.65). To this purpose, we appeal to Lemma 2.5, which yields in particular

$$t\int_{\mathbb{R}} \frac{|\mathcal{F}(\langle\zeta_3\rangle) - ik_2\mathcal{F}(\langle(x_3-\frac{t}{2})\zeta_2\rangle)|^2}{1+\frac12 t|k'|+\frac{1}{12}(t|k'|)^2}\,dk_2 \;\le\; \mathcal{R}(k_1,\zeta),$$

which we use in the form of

$$\begin{aligned} &t\int_{\mathbb{R}} \frac{|\mathcal{F}(\langle\zeta_3\rangle)|^2}{1+\frac12 t|k'|+\frac{1}{12}(t|k'|)^2}\,dk_2 \\ &\le\; 2\mathcal{R}(k_1,\zeta) + 2t\int_{\mathbb{R}} \frac{k_2^2|\mathcal{F}(\langle(x_3-\frac{t}{2})\zeta_2\rangle)|^2}{1+\frac12 t|k'|+\frac{1}{12}(t|k'|)^2}\,dk_2. \end{aligned}$$

Since

$$\frac{12}{t^2k_2^2} \;\ge\; \frac{1}{1+\frac12 t|k'|+\frac{1}{12}(t|k'|)^2} \;\gtrsim\; \min\left\{1,\frac{1}{t^2|k'|^2}\right\},$$

the above yields

$$t\int_{\mathbb{R}} \min\left\{1,\frac{1}{t^2|k'|^2}\right\} |\mathcal{F}(\langle\zeta_3\rangle)|^2 \;\lesssim\; \mathcal{R}(k_1,\zeta) + \frac{1}{t}\int_0^\ell |\langle(x_3-\tfrac{t}{2})\zeta_2\rangle|^2\,dx_2. \tag{2.67}$$

The last term in (2.67) can be estimated via Cauchy–Schwarz and Poincaré:

$$\begin{aligned} \frac{1}{t}\int_0^\ell |\langle (x_3-\tfrac{t}{2})\zeta_2\rangle|^2\,dx_2 &= \frac{1}{t}\int_0^\ell |\langle (x_3-\tfrac{t}{2})(\zeta_2-\langle\zeta_2\rangle)\rangle|^2\,dx_2 \\ &\le \frac{t}{12}\int_0^\ell \langle|\zeta_2-\langle\zeta_2\rangle|^2\rangle\,dx_2 \\ &= \frac{1}{12}\int_{\tilde\Omega} |\zeta_2-\langle\zeta_2\rangle|^2\,d\tilde x \\ &\lesssim t^2\int_{\tilde\Omega} |\partial_3\zeta_2|^2\,d\tilde x \\ &\le \left(\frac{t}{d}\right)^2 \mathcal{R}(k_1,\zeta). \end{aligned} \tag{2.68}$$

Inserting (2.68) into (2.67) yields

$$t\int_{\mathbb{R}} \min\left\{1,\frac{1}{t^2|k'|^2}\right\}|\mathcal{F}(\langle\zeta_3\rangle)|^2\,dk_2 \;\lesssim\; \left(1+\left(\frac{t}{d}\right)^2\right)\mathcal{R}(k_1,\zeta). \tag{2.69}$$

We now collect (2.62), (2.64), (2.66) and (2.69):

$$\int_{\tilde\Omega} |\zeta-\langle\zeta_2\rangle|^2\,d\tilde x \;\lesssim\; \left(1+\left(\frac{t}{d}\right)^2\right)\mathcal{R}(k_1,\zeta),$$

which is the unrescaled version of (2.57) and (2.59).

We finally address (2.58) and (2.60). We have

$$\mathcal{R}(k_1,\zeta) \;\ge\; d^2k_1^2\int_{\tilde\Omega}|\zeta|^2\,d\tilde x,$$

which by use of

$$\int_{\tilde\Omega}|\zeta|^2\,d\tilde x \;=\; 1,$$

yields

$$\mathcal{R}(k_1,\zeta) \;\ge\; d^2k_1^2.$$

This is an unrescaled version of (2.58) and (2.60). □

Before we continue with the statement and proof of the compactness results for Regimes II and III, we will make a short but necessary detour to consider edge–pinning. Edge–pinning implies that the unstable modes in both regimes have to be in $H^1_0((0,1))$ — after rescaling length. Here $H^1_0((0,1))$ denotes the Sobolev space of functions $\hat\zeta_2 \in L^2((0,1))$ whose distributional derivative $\hat\partial_2\hat\zeta_2$ is in $L^2((0,1))$ and which vanish at the boundary, i.e., $\hat\zeta_2(0)=\hat\zeta_2(1)=0$. The weak boundary conditions can be characterised as follows:

$$H^1_0((0,1)) \;=\; \left\{\hat\zeta_2|_{(0,1)}\;\middle|\;\hat\zeta_2\in L^2(\mathbb{R}),\ \hat\partial_2\hat\zeta_2\in L^2(\mathbb{R}),\ \operatorname{supp}\hat\zeta_2\subset[0,1]\right\}. \tag{2.70}$$

For Regime II we now show that we have zero boundary conditions in the limit:

**Lemma 2.8.** *For any $\zeta \in H^1(\tilde\Omega)$ with $\int_{\tilde\Omega} |\zeta|^2\, d\tilde x = 1$, we have*

$$|\langle\hat\zeta_2\rangle(0)|^2 + |\langle\hat\zeta_2\rangle(1)|^2 \lesssim \left(\varepsilon + \ln^{-1}\left(\frac{1}{\alpha}\right)\right)\hat{\mathcal{R}}(k_1,\zeta). \tag{2.71}$$

Proof of Lemma 2.8.
We have for $x \le 1$, where $x = \hat x_2$,

$$\begin{aligned} \langle\hat\zeta_2\rangle(0) &= \langle\hat\zeta_2\rangle(x) - \int_0^x \hat\partial_2\langle\hat\zeta_2\rangle\, dx \\ |\langle\hat\zeta_2\rangle(0)|^2 &\lesssim |\langle\hat\zeta_2\rangle(x)|^2 + x\int_0^1 |\hat\partial_2\langle\hat\zeta_2\rangle|^2\, dx. \end{aligned}$$

An averaged integration over the small interval $(0,h)$ yields

$$|\langle\hat\zeta_2\rangle(0)|^2 \lesssim \frac{1}{h}\int_0^h |\langle\hat\zeta_2\rangle|^2\, dx + h\int_0^1 |\hat\partial_2\langle\hat\zeta_2\rangle|^2\, dx.$$

Now by $\int_\alpha^1 \frac{1}{h}\cdot dh$,

$$\ln\left(\frac{1}{\alpha}\right)|\langle\hat\zeta_2\rangle(0)|^2 \lesssim \int_\alpha^1 \frac{1}{h^2}\int_0^h |\langle\hat\zeta_2\rangle|^2\, dx\, dh + \int_0^1 |\hat\partial_2\langle\hat\zeta_2\rangle|^2\, dx. \tag{2.72}$$

For the first part, we note that

$$\begin{aligned} \int_\alpha^1 \frac{1}{h^2}\int_0^h |\langle\hat\zeta_2\rangle|^2\, dx\, dh &\lesssim \int_\alpha^1 \frac{1}{h^2}\int_{\mathbb{R}} |\langle\hat\zeta_2\rangle(x+h) - \langle\hat\zeta_2\rangle(x)|^2\, dx\, dh \\ &= \int_\alpha^1 \frac{1}{h^2}\int_{\mathbb{R}} |e^{i\hat k_2 h} - 1|^2 |\mathcal{F}(\langle\hat\zeta_2\rangle)|^2\, d\hat k_2\, dh \\ &\le \int_{\mathbb{R}}\int_\alpha^1 \min\left\{\hat k_2^2, \frac{1}{h^2}\right\} dh\, |\mathcal{F}(\langle\hat\zeta_2\rangle)|^2\, d\hat k_2 \\ &\lesssim \int_{\mathbb{R}} \min\left\{\frac{1}{\alpha}, |\hat k_2|, \hat k_2^2\right\} |\mathcal{F}(\langle\hat\zeta_2\rangle)|^2\, d\hat k_2, \end{aligned} \tag{2.73}$$

where use was made of

$$\begin{aligned} \int_\alpha^1 \min\left\{\hat k_2^2, \frac{1}{h^2}\right\} dh &= \left\{ \begin{array}{lr} \int_\alpha^1 \hat k_2^2\, dh, & \text{for} \quad |\hat k_2| \le 1 \\ \int_\alpha^{\frac{1}{|\hat k_2|}} \hat k_2^2\, dh + \int_{\frac{1}{|\hat k_2|}}^1 \frac{1}{h^2}\, dh, & \text{for} \quad 1 \le |\hat k_2| \le \frac{1}{\alpha} \\ \int_\alpha^1 \frac{1}{h^2}\, dh, & \text{for} \quad |\hat k_2| \ge \frac{1}{\alpha} \end{array} \right\} \\ &\lesssim \left\{ \begin{array}{lr} \hat k_2^2 & \text{for} \quad |\hat k_2| \le 1 \\ |\hat k_2| & \text{for} \quad 1 \le |\hat k_2| \le \frac{1}{\alpha} \\ \frac{1}{\alpha}, & \text{for} \quad |\hat k_2| \ge \frac{1}{\alpha} \end{array} \right\} \\ &= \min\left\{\frac{1}{\alpha}, |\hat k_2|, \hat k_2^2\right\}. \end{aligned}$$

On the other hand, when considering the lower bound on the stray–field contribution, as given in Lemma 2.5 and after rescaling, we have that

$$\begin{aligned}\hat{\mathcal{R}}(k_1,\zeta) &\geq \delta \int_{\mathbb{R}} \frac{\hat{k}_2^2|\mathcal{F}(\langle\hat{\zeta}_2\rangle)|^2}{2|\hat{k}'|+\alpha|\hat{k}'|^2}\, d\hat{k}_2 \\ &\gtrsim \delta \int_{\mathbb{R}} \min\left\{\frac{1}{\alpha}, |\hat{k}_2|, \frac{\hat{k}_2^2}{|\hat{k}_1|}, \frac{\hat{k}_2^2}{\alpha\hat{k}_1^2}\right\} |\mathcal{F}(\langle\hat{\zeta}_2\rangle)|^2\, d\hat{k}_2. \end{aligned} \tag{2.74}$$

As $\mathcal{R}(k_1,\zeta)$ is considered to be bounded, we also have boundedness of $\hat{k}_1$ by Lemma 2.6 and

$$\begin{aligned}\int_\alpha^1 \frac{1}{h^2}\int_0^h |\langle\hat{\zeta}_2\rangle|^2\, dx\, dh &\overset{(2.73)}{\lesssim} \int_{\mathbb{R}} \min\left\{\frac{1}{\alpha}, |\hat{k}_2|, \hat{k}_2^2\right\} |\mathcal{F}(\langle\hat{\zeta}_2\rangle)|^2\, d\hat{k}_2 \\ &\lesssim \int_{\mathbb{R}} \min\left\{\frac{1}{\alpha}, |\hat{k}_2|, \frac{\hat{k}_2^2}{|\hat{k}_1|}, \frac{\hat{k}_2^2}{\alpha\hat{k}_1^2}\right\} |\mathcal{F}(\langle\hat{\zeta}_2\rangle)|^2\, d\hat{k}_2 \\ &\overset{(2.74)}{\lesssim} \frac{1}{\delta}\left(\delta \int_{\mathbb{R}} \frac{\hat{k}_2^2|\mathcal{F}(\langle\hat{\zeta}_2\rangle)|^2}{2|\hat{k}'|+\alpha|\hat{k}'|^2}\, d\hat{k}_2\right). \end{aligned} \tag{2.75}$$

By combining (2.72) and (2.75), then

$$\begin{aligned}\ln\left(\frac{1}{\alpha}\right)|\langle\hat{\zeta}_2\rangle(0)|^2 &\lesssim \frac{1}{\delta}\left(\delta \int_{\mathbb{R}} \frac{\hat{k}_2^2|\mathcal{F}(\langle\hat{\zeta}_2\rangle)|^2}{2|\hat{k}'|+\alpha|\hat{k}'|^2}\, d\hat{k}_2\right) + \int_0^1 |\hat{\partial}_2\langle\hat{\zeta}_2\rangle|^2\, dx \\ &\leq \left(\frac{1}{\delta}+1\right)\hat{\mathcal{R}}(k_1,\zeta),\end{aligned}$$

and finally

$$\begin{aligned}|\langle\hat{\zeta}\rangle(0)|^2 &\lesssim \ln^{-1}\left(\frac{1}{\alpha}\right)\left(\frac{1}{\delta}+1\right)\hat{\mathcal{R}}(k_1,\zeta) \\ &= \left(\varepsilon + \ln^{-1}\left(\frac{1}{\alpha}\right)\right)\hat{\mathcal{R}}(k_1,\zeta).\end{aligned}$$

This holds analogously for $|\langle\hat{\zeta}\rangle(1)|^2$. □

Now we can state and prove the two compactness results. For Regime II, we have

**Corollary 2.3.** *Let $(d_\nu, t_\nu, \ell_\nu)_{\nu\uparrow\infty}$ be such that*

$$\delta_\nu,\, \alpha_\nu \;\to\; 0 \quad (\Rightarrow\, \varepsilon_\nu \;\to\; 0). \tag{2.76}$$

*Let $(k_{1,\nu}, \zeta_\nu)_{\nu\uparrow\infty}$ be such that*

$$\int_{\tilde{\Omega}} |\zeta_\nu|^2\, d\tilde{x} \;=\; 1 \quad \textit{and} \quad \hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu) \quad \textit{is bounded.} \tag{2.77}$$

*Then $(\hat{k}_{1,\nu}, \hat{\zeta}_\nu)$ is relatively compact in $\mathbb{R}\times L^2(\hat{\tilde{\Omega}})$. Moreover, any limit $\hat{\zeta}$ is of the form*

$$\hat{\zeta} \;=\; \hat{\zeta}(\hat{x}_2), \quad \textit{and} \quad \hat{\zeta}_3 \;\equiv\; 0 \quad \textit{with} \quad \hat{\zeta}_2 \in H_0^1((0,1)). \tag{2.78}$$

Proof of Corollary 2.3.
The second estimate in Lemma 2.6 implies that

$$\{\hat{k}_{1,\nu}\}_{\nu\uparrow\infty} \quad \text{is bounded,} \tag{2.79}$$

and thus relatively compact in $\mathbb{R}$. The first estimate in Lemma 2.6 implies that any $L^2(\widehat{\tilde{\Omega}})$–limit $\hat{\zeta}$ of $\{\hat{\zeta}_\nu\}_{\nu\uparrow\infty}$ satisfies (2.78).

It remains to argue that $\{\langle\hat{\zeta}_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ is relatively compact in $L^2((0,1))$. Actually by Corollary 2.1, for bounded

$$\hat{\mathcal{R}}(k_1,\zeta) \;\geq\; \int_0^1 \left(\hat{k}_1^2|\langle\hat{\zeta}_2\rangle|^2 + |\hat{\partial}_2\langle\hat{\zeta}_2\rangle|^2\right) d\hat{x}_2$$

both sequence and limit function are bounded in $H^1((0,1))$ and thus in $C^{0,\frac{1}{2}}([0,1])$, too. The Arzelà–Ascoli theorem asserts that there exists a $\langle\hat{\zeta}_2\rangle \in H^1((0,1)) \cap C^{0,\frac{1}{2}}([0,1])$ such that for a subsequence, which we do not distinguish in notation,

$$\langle\hat{\zeta}_{2,\nu}\rangle(x_2) \;\rightarrow\; \hat{\zeta}_2(x_2) \quad \text{uniformly in } x_2 \in [0,1]. \tag{2.80}$$

With the zero boundary values from Lemma 2.8, we thus have

$$\hat{\zeta}_2 \in H_0^1((0,1)).$$

□

The corresponding result for Regime III is

**Corollary 2.4.** *Let $(d_\nu, t_\nu, \ell_\nu)_{\nu\uparrow\infty}$ be such that*

$$\varepsilon_\nu,\, \delta_\nu \;\rightarrow\; 0. \tag{2.81}$$

*Let $(k_{1,\nu}, \zeta_\nu)_{\nu\uparrow\infty}$ be such that*

$$\int_{\tilde{\Omega}} |\zeta_\nu|^2\, d\tilde{x} \;=\; 1 \quad \textit{and} \quad \hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu) \quad \textit{is bounded.} \tag{2.82}$$

*Then $(\hat{k}_{1,\nu}, \hat{\zeta}_\nu)$ is relatively compact in $\mathbb{R}\times L^2(\widehat{\tilde{\Omega}})$. Moreover, any limit $\hat{\zeta}$ is of the form*

$$\hat{\zeta} \;=\; \hat{\zeta}(\hat{x}_2), \quad \textit{and} \quad \hat{\zeta}_3 \;\equiv\; 0 \quad \textit{with} \quad \hat{\zeta}_2 \in H_0^1((0,1)). \tag{2.83}$$

Proof of Corollary 2.4.
The second estimate in Lemma 2.7 implies that

$$\{\hat{k}_{1,\nu}\}_{\nu\uparrow\infty} \quad \text{is bounded,} \tag{2.84}$$

and thus relatively compact in $\mathbb{R}$. The first estimate in Lemma 2.7 implies that any $L^2(\widehat{\tilde{\Omega}})$–limit $\hat{\zeta}$ of $\{\hat{\zeta}_\nu\}_{\nu\uparrow\infty}$ satisfies (2.83).

It remains to argue that $\{\langle\hat\zeta_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ is relatively compact in $L^2((0,1))$. Since $(0,1)$ is bounded and since $\{\langle\hat\zeta_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ is bounded in $L^2((0,1))$, it remains to show that the high frequencies $|k_2|\gg 1$ are uniformly small, in the sense of

$$\lim_{M\uparrow\infty}\limsup_{\nu\uparrow\infty}\int_{|\hat k_2|\geq M}|\mathcal{F}(\langle\hat\zeta_{2,\nu}\rangle)|^2\,d\hat k_2 \;=\; 0. \tag{2.85}$$

We infer this from Corollary 2.2. Indeed,

$$\begin{aligned}
&\int_{|\hat k_2|\geq M}|\mathcal{F}(\langle\hat\zeta_{2,\nu}\rangle)|^2\,d\hat k_2\\
&\leq \left(\inf_{|\hat k_2|\geq M}\frac{\hat k_2^2}{2(\hat k_{1,\nu}^2+\varepsilon_\nu\hat k_2^2)^{1/2}+\delta_\nu(\hat k_{1,\nu}^2+\varepsilon_\nu\hat k_2^2)}\right)^{-1}\\
&\quad\times\int_{\mathbb{R}}\frac{\hat k_2^2|\mathcal{F}(\langle\hat\zeta_{2,\nu}\rangle)|^2}{2(\hat k_{1,\nu}^2+\varepsilon_\nu\hat k_2^2)^{1/2}+\delta_\nu(\hat k_{1,\nu}^2+\varepsilon_\nu\hat k_2^2)}\,d\hat k_2\\
&\leq \left(\frac{M^2}{2(\hat k_{1,\nu}^2+\varepsilon_\nu M^2)^{1/2}+\delta_\nu(\hat k_{1,\nu}^2+\varepsilon_\nu M^2)}\right)^{-1}\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu)\\
&= \left(\frac{1}{M}\left(\left(\frac{\hat k_{1,\nu}}{M}\right)^2+\varepsilon_\nu\right)^{1/2}+\delta_\nu\left(\left(\frac{\hat k_{1,\nu}}{M}\right)^2+\varepsilon_\nu\right)\right)\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu).
\end{aligned}$$

Hence we obtain from (2.81)

$$\limsup_{\nu\uparrow\infty}\int_{|\hat k_2|\geq M}|\mathcal{F}(\langle\hat\zeta_{2,\nu}\rangle)|^2\,d\hat k_2 \;\leq\; \frac{1}{M^2}\,\limsup_{\nu\uparrow\infty}|\hat k_{1,\nu}|\,\limsup_{\nu\uparrow\infty}\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu).$$

Thus (2.85) follows from (2.84) and (2.82). Finally, we show that the limit is in $H_0^1((0,1))$.

The notion (2.96) implies in particular

$$\langle\hat\zeta_{2,\nu}\rangle \;\to\; \hat\zeta_2 \quad\text{in } L^2((0,1)), \tag{2.86}$$

and thus by Plancherel

$$\mathcal{F}(\langle\hat\zeta_{2,\nu}\rangle) \;\to\; \mathcal{F}(\hat\zeta_2) \quad\text{in}\quad L^2((0,1)).$$

Hence for a subsequence, which we do not distinguish in notation,

$$\mathcal{F}(\langle\hat\zeta_{2,\nu}\rangle) \;\to\; \mathcal{F}(\hat\zeta_2) \quad\text{a.e. in } (0,1). \tag{2.87}$$

Again by Corollary 2.2, we have

$$\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu) \;\;\geq\;\; \hat k_{1,\nu}^2\int_0^1|\langle\hat\zeta_{2,\nu}\rangle|^2\,dx_2+\int_{\mathbb{R}}\frac{\hat k_2^2|\mathcal{F}(\langle\hat\zeta_{2,\nu}\rangle)|^2}{2(\hat k_{1,\nu}^2+\varepsilon_\nu\hat k_2^2)^{1/2}+\delta_\nu(\hat k_{1,\nu}^2+\varepsilon_\nu\hat k_2^2)}\,d\hat k_2.$$

We infer from $k_{1,\nu} \to k_1$, (2.86) and (2.87) (using Fatou's Lemma),

$$\liminf_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu) \;\geq\; \hat{k}_1^2 \int_0^1 |\langle\hat{\zeta}_2\rangle|^2\,dx_2 + \frac{1}{2|\hat{k}_1|}\int_{\mathbb{R}} \hat{k}_2^2|\mathcal{F}(\langle\hat{\zeta}_2\rangle)|^2\,d\hat{k}_2.$$

In particular

$$\int_{\mathbb{R}} \hat{k}_2^2|\mathcal{F}(\langle\hat{\zeta}_2\rangle)|^2\,d\hat{k}_2 \;<\; \infty.$$

In view of definition (2.70), this yields $\hat{\zeta}_2 \in H_0^1((0,1))$, as supp $\hat{\zeta}_2 \subset [0,1]$. □

### 2.2.4 Γ–convergence

After having proved the necessary compactness results, we now proceed to the Γ–convergence result. First, we will give an appropriate version of Lemma 2.1. We note that by factorisation in $k_1$ and $\zeta = (0,\zeta_2,\zeta_3)$, Lemma 2.1 takes the form

**Lemma 2.9.** *For $\zeta = \zeta(x_2)$ we have*

$$\begin{aligned}\int_{\mathbb{R}^2}\left(k_1^2|u_\zeta|^2 + |\tilde{\nabla} u_\zeta|^2\right)d\tilde{x} \;&=\; t\int_{\mathbb{R}} \frac{\exp(-t|k'|)-1+t|k'|}{(t|k'|)^3}(tk_2)^2|\mathcal{F}(\zeta_2)|^2\,dk_2 \\ &\quad + t\int_{\mathbb{R}} \frac{1-\exp(-t|k'|)}{t|k'|}|\mathcal{F}(\zeta_3)|^2\,dk_2.\end{aligned}$$

Now we prove two corollaries which are needed for the construction part in the Γ–convergence. For Regime II:

**Corollary 2.5.** *For any $\zeta \in L^2(\tilde{\Omega})$, such that $\zeta = \zeta(x_2)$, $\zeta_3 \equiv 0$ with $\hat{\zeta}_2 \in H_0^1((0,1))$, we have*

$$\hat{\mathcal{R}}(k_1,\zeta) \;\leq\; \int_0^1\left(\hat{k}_1^2|\hat{\zeta}_2|^2 + |\hat{\partial}_2\hat{\zeta}_2|^2\right)d\hat{x}_2 + \frac{\delta}{2}\int_{\mathbb{R}} |\hat{k}_2||\mathcal{F}(\hat{\zeta}_2)|^2\,d\hat{k}_2. \tag{2.88}$$

Proof of Corollary 2.5.
With $\zeta = \zeta(x_2)$ and $\zeta_3 \equiv 0$, we have by Lemma 2.9 that

$$\begin{aligned}\mathcal{R}(k_1,\zeta) \;&=\; d^2 t\int_0^\ell \left(k_1^2|\zeta_2|^2 + |\partial_2\zeta_2|^2\right)dx_2 \\ &\quad + t\int_{\mathbb{R}} \frac{\exp(-t|k'|)-1+t|k'|}{(t|k'|)^3}(tk_2)^2|\mathcal{F}(\zeta_2)|^2\,dk_2.\end{aligned}$$

We use the Fourier multiplier inequality

$$\frac{\exp(-t|k'|)-1+t|k'|}{(t|k'|)^3} \;\leq\; \frac{1}{2t|k'|} \;\leq\; \frac{1}{2t|k_2|},$$

and have

$$\mathcal{R}(k_1,\zeta) \;\leq\; d^2 t\int_0^\ell \left(k_1^2|\zeta_2|^2 + |\partial_2\zeta_2|^2\right)dx_2 + \frac{t^2}{2}\int_{\mathbb{R}} |k_2||\mathcal{F}(\zeta_2)|^2\,dk_2.$$

Upon the rescaling of Section 2.2.1, this turns into (2.88). □.

For Regime III, we have the following result:

**Corollary 2.6.** *For any $\zeta \in L^2(\tilde{\Omega})$, such that $\zeta = \zeta(x_2)$, $\zeta_3 \equiv 0$ with $\hat{\zeta}_2 \in H^1_0((0,1))$, we have*

$$\hat{\mathcal{R}}(k_1, \zeta) \;\le\; \hat{k}_1^2 \int_0^1 |\hat{\zeta}_2|^2 \, d\hat{x}_2 + \left( \frac{1}{2|\hat{k}_1|} + \varepsilon \right) \int_0^1 |\hat{\partial}_2 \hat{\zeta}_2|^2 \, d\hat{x}_2. \tag{2.89}$$

Proof of Corollary 2.6.
With $\zeta = \zeta(x_2)$ and $\zeta_3 \equiv 0$, we have by Lemma 2.9 that

$$\begin{aligned} \mathcal{R}(k_1, \zeta) \;&=\; d^2 t \int_0^\ell \left( k_1^2 |\zeta_2|^2 + |\partial_2 \zeta_2|^2 \right) dx_2 \\ &\quad + t \int_{\mathbb{R}} \frac{\exp(-t|k'|) - 1 + t|k'|}{(t|k'|)^3} (t k_2)^2 |\mathcal{F}(\zeta_2)|^2 \, dk_2. \end{aligned}$$

We use the Fourier multiplier inequality

$$\frac{\exp(-t|k'|) - 1 + t|k'|}{(t|k'|)^3} \;\le\; \frac{1}{2t|k'|} \;\le\; \frac{1}{2t|k_1|},$$

and have

$$\mathcal{R}(k_1, \zeta) \;\le\; d^2 t k_1^2 \int_0^\ell |\zeta_2|^2 \, dx_2 + d^2 t \int_0^\ell |\partial_2 \zeta_2|^2 \, dx_2 + \frac{t^2}{2|k_1|} \int_{\mathbb{R}} k_2^2 |\mathcal{F}(\zeta_2)|^2 \, dk_2.$$

Note that since $\zeta_2 \in H^1_0((0,\ell))$ (in the definition of (2.70)), we have

$$\int_{\mathbb{R}} k_2^2 |\mathcal{F}(\zeta_2)|^2 \, dk_2 \;=\; \int_0^\ell |\partial_2 \zeta_2|^2 \, dx_2,$$

such that the last inequality assumes the form

$$\mathcal{R}(k_1, \zeta) \;\le\; d^2 t k_1^2 \int_0^\ell |\zeta_2|^2 \, dx_2 + \left( d^2 t + \frac{t^2}{2|k_1|} \right) \int_0^\ell |\partial_2 \zeta_2|^2 \, dx_2.$$

Upon the rescaling of Section 2.2.2, this turns into (2.89). □

Now we are in the position to state and prove the Γ–convergence results. In Regime II, we have

**Proposition 2.1.** *Let $(d_\nu, t_\nu, \ell_\nu)_{\nu \uparrow \infty}$ be such that*

$$\delta_\nu,\, \alpha_\nu \;\to\; 0 \quad (\Rightarrow \varepsilon_\nu \;\to\; 0).$$

*Then the variational problem in $(k_1, \zeta) \in \mathbb{R} \times L^2(\tilde{\Omega})$ of minimising*

$$\hat{\mathcal{R}}(k_1, \zeta) \quad \text{constrained to} \quad \int_{\tilde{\Omega}} |\zeta|^2 \, d\tilde{x} \;=\; 1,$$

*Γ–converges to the variational problem in $(\hat{k}_1, \hat{\zeta}_2) \in \mathbb{R} \times H^1_0((0,1))$ of minimising*

$$\int_0^1 \left( \hat{k}_1^2 |\hat{\zeta}_2|^2 + |\hat{\partial}_2 \hat{\zeta}_2|^2 \right) d\hat{x}_2 \quad \text{constrained to} \quad \int_0^1 |\hat{\zeta}_2|^2 \, d\hat{x}_2 \;=\; 1, \tag{2.90}$$

*under the following notion of convergence:*

$$\hat{k}_{1,\nu} \;\to\; \hat{k}_1 \quad \text{and} \quad \int_{\widehat{\Omega}} |\hat{\zeta}_\nu(\widehat{x}) - \hat{\zeta}_2(\hat{x}_2)|^2 \, d\widehat{x} \;\to\; 0. \tag{2.91}$$

Proof of Proposition 2.1.
The proof of Γ–convergence consists of two parts:

- *Construction.* For any $(\hat{k}_1, \hat{\zeta}_2) \in \mathbb{R} \times H_0^1((0,1))$ with $\int_0^1 |\hat{\zeta}_2|^2 \, d\hat{x}_2 = 1$, there exists a sequence $(k_{1,\nu}, \zeta_\nu) \in \mathbb{R} \times L^2(\tilde{\Omega})$ with $\int_{\tilde{\Omega}} |\zeta_\nu|^2 \, d\tilde{x} = 1$ which converges in the sense of (2.91) such that

$$\limsup_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu) \;\le\; \int_0^1 \left( \hat{k}_1^2 |\hat{\zeta}_2|^2 + |\hat{\partial}_2 \hat{\zeta}_2|^2 \right) d\hat{x}_2. \tag{2.92}$$

- *Lower semicontinuity.* For any sequence $(k_{1,\nu}, \zeta_\nu) \in \mathbb{R} \times L^2(\tilde{\Omega})$ with bounded $\hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu)$ which converges to a $(\hat{k}_1, \hat{\zeta}_2) \in \mathbb{R} \times L^2((0,1))$ in the sense of (2.91), one has

$$\hat{\zeta}_2 \in H_0^1((0,1))$$

and

$$\liminf_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu) \;\ge\; \int_0^1 \left( \hat{k}_1^2 |\hat{\zeta}_2|^2 + |\hat{\partial}_2 \hat{\zeta}_2|^2 \right) d\hat{x}_2. \tag{2.93}$$

For the construction part, we define $(k_{1,\nu}, \zeta_\nu) \in \mathbb{R} \times L^2(\tilde{\Omega})$ according to Subsection 2.2.1:

$$\ell k_{1,\nu} \;=\; \hat{k}_1 \quad \text{and} \quad (t\ell)^{1/2} \zeta_\nu(\ell \hat{x}_2, t\hat{x}_3) \;=\; \hat{\zeta}_2(\hat{x}_2).$$

Then (2.91) is trivially fulfilled. Furthermore, by Corollary 2.5 we have

$$\hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu) \;\le\; \int_0^1 \left( \hat{k}_1^2 |\hat{\zeta}_2|^2 + |\hat{\partial}_2 \hat{\zeta}_2|^2 \right) d\hat{x}_2 + \frac{\delta_\nu}{2} \int_{\mathbb{R}} |\hat{k}_2| |\mathcal{F}(\hat{\zeta}_2)|^2 \, d\hat{k}_2.$$

This inequality establishes (2.92).

We now address the lower semicontinuity part. The notion (2.91) implies in particular

$$\langle \hat{\zeta}_{2,\nu} \rangle \;\to\; \hat{\zeta}_2 \quad \text{in} \quad L^2((0,1)). \tag{2.94}$$

Appealing to Corollary 2.1,

$$\hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu) \;\ge\; \int_0^1 \left( \hat{k}_{1,\nu}^2 |\langle \hat{\zeta}_{2,\nu} \rangle|^2 + |\hat{\partial}_2 \langle \hat{\zeta}_{2,\nu} \rangle|^2 \right) dx_2,$$

we infer from $k_{1,\nu} \to k_1$, (2.94) and the lower semicontinuity of the $H^1$–norm

$$\liminf_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu}, \zeta_\nu) \;\ge\; \int_0^1 \left( \hat{k}_1^2 |\hat{\zeta}_2|^2 + |\hat{\partial}_2 \hat{\zeta}_2|^2 \right) dx_2.$$

□

For Regime III, the corresponding result is

**Proposition 2.2.** *Let $(d_\nu, t_\nu, \ell_\nu)_{\nu\uparrow\infty}$ be such that*

$$\varepsilon_\nu,\ \delta_\nu \ \to\ 0.$$

*Then the variational problem in $(k_1,\zeta) \in \mathbb{R}\times L^2(\tilde\Omega)$ of minimising*

$$\hat{\mathcal{R}}(k_1,\zeta) \quad \text{constrained to} \quad \int_{\tilde\Omega} |\zeta|^2\, d\tilde{x} \ =\ 1,$$

*$\Gamma$–converges to the variational problem in $(\hat k_1,\hat\zeta_2)\in\mathbb{R}\times H_0^1((0,1))$ of minimising*

$$\hat k_1^2 \int_0^1 |\hat\zeta_2|^2 + \frac{1}{2|\hat k_1|}\int_0^1 |\hat\partial_2\hat\zeta_2|^2\, d\hat x_2 \quad \text{constrained to} \quad \int_0^1 |\hat\zeta_2|^2\, d\hat x_2 \ =\ 1, \tag{2.95}$$

*under the following notion of convergence:*

$$\hat k_{1,\nu} \ \to\ \hat k_1 \quad \text{and} \quad \int_{\hat{\tilde\Omega}} |\hat\zeta_\nu(\widehat{\tilde x}) - \hat\zeta_2(\hat x_2)|^2\, d\widehat{\tilde x} \ \to\ 0. \tag{2.96}$$

*For $k_1 = 0$, we assign the value $+\infty$ to the prefactor $\frac{1}{2|\hat k_1|}$ in (2.96).*

Proof of Proposition 2.2.
The proof of $\Gamma$–convergence consists of two parts:

- *Construction.* For any $(\hat k_1,\hat\zeta_2)\in\mathbb{R}\times H_0^1((0,1))$ with $\int_0^1|\hat\zeta_2|^2\,d\hat x_2 = 1$, there exists a sequence $(k_{1,\nu},\zeta_\nu)\in\mathbb{R}\times L^2(\tilde\Omega)$ with $\int_{\tilde\Omega}|\zeta_\nu|^2\,d\tilde x = 1$ which converges in the sense of (2.96) such that

$$\limsup_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu) \ \le\ \hat k_1^2\int_0^1 |\hat\zeta_2|^2\, d\hat x_2 + \frac{1}{2|\hat k_1|}\int_0^1 |\hat\partial_2\hat\zeta_2|^2\, d\hat x_2. \tag{2.97}$$

- *Lower semicontinuity.* For any sequence $(k_{1,\nu},\zeta_\nu)\in\mathbb{R}\times L^2(\tilde\Omega)$ with bounded $\hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu)$ which converges to a $(\hat k_1,\hat\zeta_2)\in\mathbb{R}\times L^2((0,1))$ in the sense of (2.96), one has

$$\hat\zeta_2 \in H_0^1((0,1))$$

  and

$$\liminf_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu) \ \ge\ \hat k_1^2\int_0^1 |\hat\zeta_2|^2\, d\hat x_2 + \frac{1}{2|\hat k_1|}\int_0^1 |\hat\partial_2\hat\zeta_2|^2\, d\hat x_2. \tag{2.98}$$

For the construction part, we define $(k_{1,\nu},\zeta_\nu)\in\mathbb{R}\times L^2(\tilde\Omega)$ as in Subsection 2.2.2:

$$\left(\frac{d_\nu^2\ell_\nu^2}{t_\nu}\right)^{1/3} k_{1,\nu} \ =\ \hat k_1 \quad \text{and} \quad (t\ell)^{1/2}\zeta_\nu(\ell\hat x_2, t\hat x_3) \ =\ \hat\zeta_2(\hat x_2).$$

Then (2.96) is trivially fulfilled. Furthermore, by Corollary 2.6:

$$\hat{\mathcal{R}}(k_1,\zeta) \ \le\ \hat k_1^2\int_0^1 |\hat\zeta_2|^2\, d\hat x_2 + \left(\frac{1}{2|\hat k_1|} + \varepsilon_\nu\right)\int_0^1 |\hat\partial_2\hat\zeta_2|^2\, d\hat x_2.$$

This inequality establishes (2.97).

We now address the lower semicontinuity part. As seen in the proof of Corollary 2.4,

$$\liminf_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu},\zeta_\nu) \ \geq\ \hat{k}_1^2\int_0^1 |\langle\hat{\zeta}_2\rangle|^2\,dx_2 + \frac{1}{2|\hat{k}_1|}\int_{\mathbb{R}} \hat{k}_2^2|\mathcal{F}(\langle\hat{\zeta}_2\rangle)|^2\,d\hat{k}_2.$$

which by $\hat{\zeta}_2 \in H_0^1((0,1))$ yields (2.98). □

### 2.2.5 Unstable modes: Proof of Theorems 2.1 and 2.2

Now that we have identified the Γ–limits, we may proceed by calculating the solutions to the limiting variational problems. We do so by the following two lemmata, first in Regime II:

**Lemma 2.10.** *Any minimiser* $(\hat{k}_1^*,\hat{\zeta}_2^*) \in \mathbb{R}\times H_0^1((0,1))$ *of*

$$\int_0^1 \left(\hat{k}_1^2|\hat{\zeta}_2|^2 + |\hat{\partial}_2\hat{\zeta}_2|^2\right)d\hat{x}_2 \quad \textit{constrained to} \quad \int_0^1 |\hat{\zeta}_2|^2\,d\hat{x}_2 = 1$$

*is of the form*

$$\hat{k}_1^* \ = \ 0 \tag{2.99}$$
$$\hat{\zeta}_2^*(\hat{x}_2) \ = \ \sqrt{2}c\,\sin(\pi\hat{x}_2) \quad \textit{for some} \quad |c| = 1. \tag{2.100}$$

*The minimum value is given by*

$$\pi^2.$$

Proof of Lemma 2.10.
The statement reduces to two observations:

- For any $\hat{\zeta}_2 \in H_0^1((0,1))$, we have

  $$\int_0^1 |\hat{\partial}_2\hat{\zeta}_2|^2\,d\hat{x}_2 \ \geq\ \pi^2\int_0^1 |\hat{\zeta}_2|^2\,d\hat{x}_2$$

  with equality if and only if $\hat{\zeta}_2$ is of the form (2.100) for some $c \in \mathbb{C}$.

- It holds that

  $$\hat{k}_1^2 + \pi^2 \ \geq\ \pi^2$$

  with equality if and only if $\hat{k}_1 = 0$. □

For the limit in Regime III we have

**Lemma 2.11.** *Any minimiser* $(\hat{k}_1^*,\hat{\zeta}_2^*) \in \mathbb{R}\times H_0^1((0,1))$ *of*

$$\hat{k}_1^2\int_0^1 |\hat{\zeta}_2|^2\,d\hat{x}_2 + \frac{1}{2|\hat{k}_1|}\int_0^1 |\hat{\partial}_2\hat{\zeta}_2|^2\,d\hat{x}_2 \quad \textit{constrained to} \quad \int_0^1 |\hat{\zeta}_2|^2\,d\hat{x}_2 = 1$$

*is of the form*

$$|\hat{k}_1^*| = \left(\frac{\pi}{2}\right)^{2/3} \tag{2.101}$$

$$\hat{\zeta}_2^*(\hat{x}_2) = \sqrt{2}c\,\sin(\pi\hat{x}_2) \quad \textit{for some} \quad |c| = 1. \tag{2.102}$$

*The minimum value is given by*

$$3\left(\frac{\pi}{2}\right)^{4/3}.$$

Proof of Lemma 2.11.
The statement reduces to two observations:

- For any $\hat{\zeta}_2 \in H_0^1((0,1))$, we have

$$\int_0^1 |\hat{\partial}_2\hat{\zeta}_2|^2\,d\hat{x}_2 \;\geq\; \pi^2\int_0^1 |\hat{\zeta}_2|^2\,d\hat{x}_2$$

  with equality if and only if $\hat{\zeta}_2$ is of the form (2.102) for some $c \in \mathbb{C}$.

- It holds that

$$\hat{k}_1^2 + \frac{\pi^2}{2|\hat{k}_1|} \;\geq\; 3\left(\frac{\pi}{2}\right)^{4/3}$$

  with equality if and only if $\hat{k}_1$ satisfies (2.101). □

With these lemmata, we are now in the position to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1.
The argument is indirect. Assume the statement were not true. Then there exists a sequence of numbers $(d_\nu, t_\nu, \ell_\nu)$ such that (2.76) holds, and a corresponding sequence $(k_{1,\nu}^*, \zeta_\nu^*)$ of minimisers such that one of the following conditions is violated:

$$|k_{1,\nu}^*| \;\rightarrow\; 0 \tag{2.103}$$

$$\int_{\hat{\Omega}} |\hat{\zeta}_\nu^*(\hat{\hat{x}}) - \sqrt{2}\,c\,\sin(\pi\hat{x}_2)|^2\,d\hat{\hat{x}} \;\rightarrow\; 0, \quad \text{for some } |c| \;=\; 1 \tag{2.104}$$

$$\hat{\mathcal{R}}(k_{1,\nu}^*, \zeta_\nu^*) \;\rightarrow\; \pi^2. \tag{2.105}$$

Lemma 2.10 implies in particular that the minimum of the Γ–limit (2.90) is finite. According to the construction part of Proposition 2.1, it follows that $\limsup_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu}^*, \zeta_\nu^*)$ is finite. According to Corollary 2.3, there thus exists $(\hat{k}_1^*, \hat{\zeta}_2^*) \in \mathbb{R} \times L^2((0,1))$ such that for a subsequence (which we do not distinguish in notation)

$$\hat{k}_{1,\nu}^* \;\rightarrow\; \hat{k}_1^*, \tag{2.106}$$

$$\int_{\hat{\Omega}} |\hat{\zeta}_\nu^*(\hat{\hat{x}}) - \hat{\zeta}_2^*(\hat{x}_2)|^2\,d\hat{\hat{x}} \;\rightarrow\; 0, \tag{2.107}$$

which is just the notion of convergence (2.91) in the Γ–convergence.

The Γ–convergence of Proposition 2.1 implies that, cf. [9],

$$(\hat{k}_1^*, \hat{\zeta}_2^*) \in \mathbb{R} \times H_0^1((0,1)) \quad \text{is a minimiser of (2.90)}, \tag{2.108}$$

and that

$$\hat{\mathcal{R}}(\hat{k}_{1,\nu}^*, \hat{\zeta}_\nu^*) \;\to\; |\hat{k}_1^*|^2 \int_0^1 |\hat{\zeta}_2^*|^2 \, d\hat{x}_2 + \int_0^1 |\hat{\partial}_2 \hat{\zeta}_2^*|^2 \, d\hat{x}_2. \tag{2.109}$$

Finally, we appeal to Lemma 2.10: (2.108) implies that

$$\begin{aligned} \hat{k}_1^* &= 0, \\ \hat{\zeta}_2(\hat{x}_2) &= \sqrt{2}\, c \sin(\pi \hat{x}_2), \quad \text{for some } |c| \;=\; 1, \\ |\hat{k}_1^*|^2 \int_0^1 |\hat{\zeta}_2|^2 \, d\hat{x}_2 + \int_0^1 |\partial_2 \hat{\zeta}_2|^2 \, d\hat{x}_2 &= \pi^2. \end{aligned}$$

In view of (2.106), (2.107) and (2.109), this is in contradiction to the assumption that one of the three properties (2.103), (2.104) or (2.105) is violated. □

Proof of Theorem 2.2.
The argument is indirect. Assume the statement were not true. Then there exists a sequence of numbers $(d_\nu, t_\nu, \ell_\nu)$ such that (2.81) holds, and a corresponding sequence $(k_{1,\nu}^*, \zeta_\nu^*)$ of minimisers such that one of the following conditions is violated:

$$|k_{1,\nu}^*| \;\to\; \left(\frac{\pi}{2}\right)^{2/3} \tag{2.110}$$

$$\int_{\hat{\Omega}} |\hat{\zeta}_\nu^*(\hat{\hat{x}}) - \sqrt{2}\, c \sin(\pi \hat{x}_2)|^2 \, d\hat{\hat{x}} \;\to\; 0, \quad \text{for some } |c| \;=\; 1 \tag{2.111}$$

$$\hat{\mathcal{R}}(k_{1,\nu}^*, \zeta_\nu^*) \;\to\; 3\left(\frac{\pi}{2}\right)^{4/3}. \tag{2.112}$$

Lemma 2.11 implies in particular that the minimum of the Γ–limit (2.95) is finite. According to the construction part of Proposition 2.2, it follows that $\limsup_{\nu\uparrow\infty} \hat{\mathcal{R}}(k_{1,\nu}^*, \zeta_\nu^*)$ is finite. According to Corollary 2.4, there thus exists $(\hat{k}_1^*, \hat{\zeta}_2^*) \in \mathbb{R} \times L^2((0,1))$ such that for a subsequence (which we do not distinguish in notation)

$$\hat{k}_{1,\nu}^* \;\to\; \hat{k}_1^* \tag{2.113}$$

$$\int_{\hat{\Omega}} |\hat{\zeta}_\nu^*(\hat{\hat{x}}) - \hat{\zeta}_2^*(\hat{x}_2)|^2 \, d\hat{\hat{x}} \;\to\; 0, \tag{2.114}$$

which is just the notion of convergence (2.96) in the Γ–convergence.

The Γ–convergence of Proposition 2.2 implies that, cf. [9],

$$(\hat{k}_1^*, \hat{\zeta}_2^*) \in \mathbb{R} \times H_0^1((0,1)) \quad \text{is a minimiser of (2.95)}, \tag{2.115}$$

and that

$$\hat{\mathcal{R}}(\hat{k}_{1,\nu}^*, \hat{\zeta}_\nu^*) \;\to\; |\hat{k}_1^*|^2 \int_0^1 |\hat{\zeta}_2^*|^2 \, d\hat{x}_2 + \frac{1}{2|\hat{k}_1^*|} \int_0^1 |\hat{\partial}_2 \hat{\zeta}_2^*|^2 \, d\hat{x}_2. \tag{2.116}$$

Finally, we appeal to Lemma 2.11: (2.115) implies that

$$\begin{aligned}
\hat{k}_1^* &= \left(\frac{\pi}{2}\right)^{2/3}, \\
\hat{\zeta}_2(\hat{x}_2) &= \sqrt{2}\, c\, \sin(\pi \hat{x}_2), \quad \text{for some } |c| \;=\; 1, \\
|\hat{k}_1^*|^2 \int_0^1 |\hat{\zeta}_2|^2 \, d\hat{x}_2 + \frac{1}{2|\hat{k}_1^*|} \int_0^1 |\partial_2 \hat{\zeta}_2|^2 \, d\hat{x}_2 &= 3 \left(\frac{\pi}{2}\right)^{4/3}.
\end{aligned}$$

In view of (2.113), (2.114) and (2.116), this is in contradiction to the assumption that one of the three properties (2.110), (2.111) or (2.112) is violated. □

### 2.2.6 Comparison with experiments

We return to the hypothesis that the concertina pattern evolves continuously out of the oscillatory instability. In that case, the observed period $w_{exp}$ of the concertina pattern would be the frozen–in length scale (2.48) of the unstable mode. First of all, expression (2.48) is consistent with the experimentally observed trends, see Figures 2.5 and 2.6: The width of the domains in the concertina pattern increases for increasing sample width $\ell$ while it decreases for increasing sample thickness $t$.

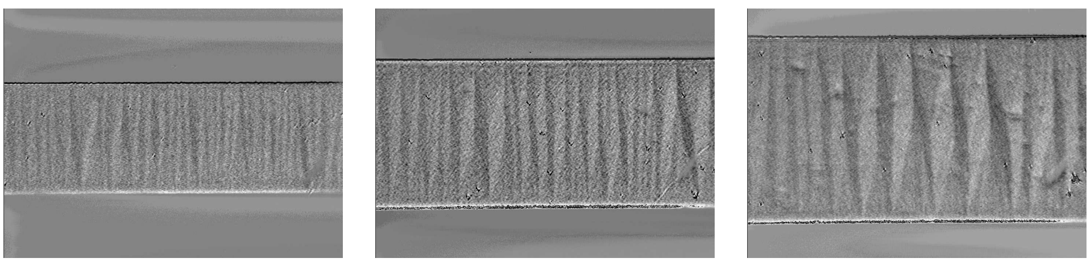

Figure 2.5: Concertina pattern for different widths, courtesy of R. Schäfer, IfW Dresden

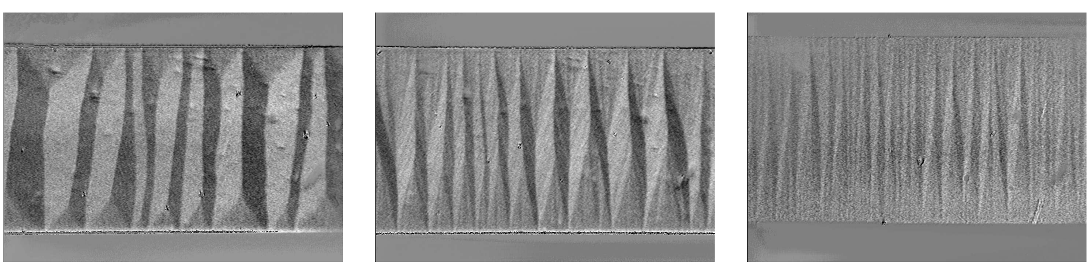

Figure 2.6: Concertina pattern for different thicknesses, courtesy of R. Schäfer, IfW Dresden

We have compared $w_{exp}$ to $w^*$ for eight experiments pictured in [17]. These experiments cover a substantial range of the non–dimensional parameters $t/d$ and $\ell/d$, see Table 2.2.6. We find a deviation by a factor $0.5 - 0.7$.

| $\frac{t}{d}$ | $\frac{\ell}{d}$ | $\frac{w_{exp}}{d}$ | $\frac{w^*}{d}$ | $\frac{w^*}{w_{exp}}$ |
|---|---|---|---|---|
| 8 | 4000 | 1200 | 586 | 0.49 |
| 48 | 8000 | 700 | 512 | 0.73 |
| 60 | 3600 | 540 | 279 | 0.52 |
| 10 | 2000 | 500 | 347 | 0.69 |
| 8 | 2800 | 800 | 462 | 0.56 |
| 8 | 7000 | 1700 | 851 | 0.50 |

It has to be emphasized that the concertina pattern is not a ground state but a metastable state and, as such, not a global minimizer of the energy functional, but merely a local one. Experimental evidence for this fact is given by coarsening phenomena, as seen in Figure 2.7. The concertina pattern for one fixed sample is shown in two stages of its development. In the second stage the domain structure has coarsened by fusion of different domains. This effect may explain the deviation between the period of oscillation of the unstable mode and the experimental results.

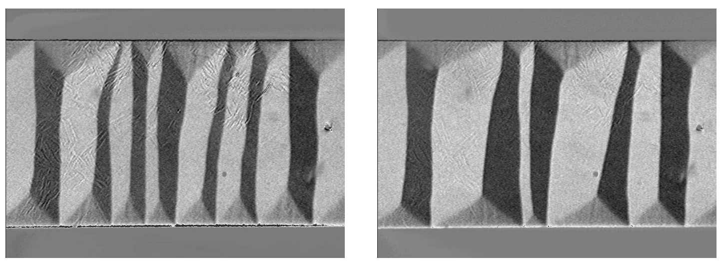

Figure 2.7: Coarsening of the concertina pattern, courtesy of R. Schäfer, IfW Dresden

## 2.3 A subcritical bifurcation

In this section, we renormalise the energy in Regime III and identify a scaling limit that includes the dominant nonlinearity. In order to preserve translation invariance while working with a finite volume, we impose a convenient periodicity $L$ in the infinite $x_1$–direction:

$$m(x_1 + L, x_2, x_3) \; = \; m(x_1, x_2, x_3).$$

This yields a well–defined energy functional $E$

$$\begin{aligned} E(m) \; &= \; d^2 \int_{(0,L)\times(0,\ell)\times(0,t)} |\nabla m|^2 \, dx + \int_{(0,L)\times\mathbb{R}^2} |\nabla u_m|^2 \, dx \\ &\quad + 2h_{ext} \int_{(0,L)\times(0,\ell)\times(0,t)} m_1 \, dx, \end{aligned} \tag{2.117}$$

where $u_m$ inherits the periodicity of $m$ in $x_1$.

In order to find the scaling limit, we will have to rescale not only length but also amplitudes $(m_2, m_3)$, thus combining a magnification around $m^*$ in the energy landscape with the asymptotic Regime III, all in the framework of $\Gamma$–convergence. We rescale length as in Subsection 2.2.2, while the new element of rescaling $(m_2, m_3)$ will be motivated now. For $m \approx m^* = (1, 0, 0)$, we have

$$m_1 \;=\; \sqrt{1 - m_2^2 - m_3^2} \;\approx\; 1 - \tfrac{1}{2}(m_2^2 + m_3^2).$$

As we expect the $m_3$–component to be strongly suppressed by the penalisation of surface charges, we have

$$m_1 \;\approx\; 1 - \tfrac{1}{2} m_2^2. \tag{2.118}$$

The magnetisation component $m_2$ will be rescaled in such a way as to balance the two contributions to the volume charge distribution $\nabla \cdot m$, i.e.,

$$\partial_1 m_1 \overset{(2.52),(2.118)}{\approx} -\left(\frac{t}{d^2\ell^2}\right)^{1/3} \hat{\partial}_1(\tfrac{1}{2}\hat{m}_2^2) \quad \text{and} \quad \partial_2 m_2 \overset{(2.52)}{=} \frac{1}{\ell}\hat{\partial}_2 m_2.$$

This leads to

$$(m_2, m_3) \;=\; \varepsilon^{1/2}(\hat{m}_2, \hat{m}_3), \quad \text{cf. (2.53)}. \tag{2.119}$$

The external field is measured in units of the critical field, and for the energy (2.117) we subtract $E(m^*)$ and normalise appropriately:

$$h_{ext} \;=\; \left(\frac{dt}{\ell^2}\right)^{2/3} \hat{h}_{ext}, \quad E - 2h_{ext}L\ell t \;=\; \left(\frac{d^8t^2}{\ell}\right)^{1/3} \hat{E}. \tag{2.120}$$

We are left with four nondimensional parameters, $\varepsilon$ and $\delta$ from (2.53) and also $\hat{L}$ and $\hat{h}_{ext}$. In order to end up with a finite volume in the limit, we are forced to assume

$$\hat{L} \;\sim\; 1 \quad \text{and likewise} \quad \hat{h}_{ext} \;\sim\; 1. \tag{2.121}$$

This means that $L$ is of the order of the oscillation period $w^*$; we think of $L$ as being a large, but $O(1)$ multiple of $w^*$, where according to (2.52), $\hat{L}$ is defined as

$$L \;=\; \left(\frac{d^2\ell^2}{t}\right)^{1/3} \hat{L}.$$

We note that in this section we make use of the combined Fourier series and transform as defined in (1.3).

We now state the main result of this section:

**Theorem 2.3.** *Fix an $\hat{M} \sim 1$. The variational problem of minimising*

$$\hat{E} \quad \textit{subject to} \quad |m|^2 \;=\; 1 \quad \textit{and} \quad \int_{(0,\hat{L})\times(0,1)\times(0,1)} |m - m^*|^2 \, d\hat{x} \;=\; \varepsilon \hat{M}^2 \tag{2.122}$$

*Γ–converges in the regime described by* (2.54) *and* (2.121) *under weak convergence for* $(\hat m_2, \hat m_3)$ *in* $L^2((0,\hat L)\times(0,1)\times(0,1))^2$ *to the variational problem of minimising*

$$\begin{aligned}\hat E_0 \;&:=\; \int_{(0,\hat L)\times(0,1)} (\hat\partial_1 \hat m_2)^2\, d\hat x' + \frac{1}{2}\int_{(0,\hat L)\times\mathbb{R}} \left| |\hat\partial_1|^{-1/2} \left(-\hat\partial_1(\tfrac{1}{2}\hat m_2^2) + \hat\partial_2 \hat m_2\right)\right|^2 d\hat x' \\ &\quad -\hat h_{ext} \int_{(0,\hat L)\times(0,1)} \hat m_2^2\, d\hat x', \quad \textit{subject to} \quad \int_{(0,\hat L)\times(0,1)} \hat m_2^2\, d\hat x' \;=\; \hat M^2, \qquad (2.123)\end{aligned}$$

*if* $(\hat m_2, \hat m_3)$ *is of the form* $(\hat m_2(\hat x'), 0)$ *and* $\hat E_0 = +\infty$ *if it is not of this form.*

Here $|\hat\partial_1|^{-1/2}$ denotes the operator with Fourier symbol $|\hat k_1|^{-1/2}$. The expression $-\hat\partial_1(\frac{1}{2}\hat m_2^2) + \hat\partial_2 \hat m_2$ is to be understood distributionally where $\hat m_2$ is zero outside of $\mathbb{R}\times(0,\ell)$. As in the linear stability analysis, this imposes edge–pinning, i.e., $\hat m_2(\hat x_2 = 1) = \hat m_2(\hat x_2 = 0) = 0$ in a weak sense.

We note that $\hat E_0$ interpolates between a linear regime and a highly nonlinear wall regime.

- For dominant $\hat\partial_2 \hat m_2$, $\hat E_0$ is close to the Γ–limit (2.95) of the Hessian, integrated over frequencies $\hat k_1$:

$$\hat E_0 \;\approx\; \int_{\frac{2\pi}{\hat L}\mathbb{Z}} \left( \hat k_1^2 \int_0^1 \hat m_2^2\, d\hat x_2 + \frac{1}{2|\hat k_1|}\int_0^1 (\hat\partial_2 \hat m_2)^2\, d\hat x_2 - \hat h_{ext}\int_0^1 \hat m_2^2\, d\hat x_2 \right) d\hat k_1,$$

  thus this is expected to be a good description near the bifurcation.

- For dominant $-\hat\partial_1(\frac{1}{2}\hat m_2^2)$, $\hat E_0$ is close to a 1–D model for small–angle thin–film Néel walls normal to $\hat x_1$, integrated over $\hat x_2$:

$$\hat E_0 \;\approx\; \int_0^1 \left( \int_0^{\hat L} (\hat\partial_1 \hat m_2)^2 d\hat x_1 + \frac{1}{2}\int_0^{\hat L} \left| |\hat\partial_1|^{1/2} (\tfrac{1}{2}\hat m_2^2)\right|^2 d\hat x_1 - \hat h_{ext}\int_0^{\hat L} \hat m_2^2\, d\hat x_1 \right) d\hat x_2. \qquad (2.124)$$

  We will analyse this model to some extent in the next chapter.

### 2.3.1 Compactness

In this subsection, we establish the compactness result which is necessary for the proof of the lower semicontinuity part of the Γ–convergence. First of all, we recall the definition of the vertical average of a quantity $f$:

$$\langle f\rangle \;=\; \frac{1}{t}\int_0^t f\, dx_3 \;=\; \int_0^1 f\, d\hat x_3.$$

**Proposition 2.3.** *Let the sequences* $\{\varepsilon_\nu\}_{\nu\uparrow\infty}$, $\{\delta_\nu\}_{\nu\uparrow\infty}$ *converge to zero. Let* $\hat L$, $\hat h_{ext}$ *be given. Consider a sequence* $\{m_\nu : \mathbb{R}^3 \mapsto \mathbb{S}^2\}_{\nu\uparrow\infty}$ *of vector fields that are L–periodic in* $x_1$*, supported in* $x_2 \in [0,\ell_\nu]$, $x_3 \in [0, t_\nu]$*, such that*

$$\frac{1}{\varepsilon_\nu}\int_{(0,\hat L)\times(0,1)\times(0,1)} |m_\nu - m^*|^2\, d\hat x, \quad \hat E(m_\nu) \quad \textit{bounded for } \nu\uparrow\infty. \qquad (2.125)$$

*Then*

$$\lim_{\nu\uparrow\infty}\frac{1}{\varepsilon_\nu}\int_{(0,\hat L)\times(0,1)\times(0,1)}|m_\nu-(1,\langle m_{\nu,2}\rangle,\langle m_{\nu,3}\rangle)|^2\,d\hat x \;=\; 0, \tag{2.126}$$

$$\lim_{\nu\uparrow\infty}\int_{(0,\hat L)\times(0,1)}\langle\hat m_{\nu,3}\rangle^2\,d\hat x' \;=\; 0, \tag{2.127}$$

$$\{\langle\hat m_{\nu,2}\rangle\}_{\nu\uparrow\infty} \text{ is compact in } L^2((0,\hat L)\times(0,1)). \tag{2.128}$$

The first lemma shows that the exchange energy is strong enough to force $m$ to stay close to $(1,\langle m_2\rangle,\langle m_3\rangle)$ in the $L^2$–topology. It proves part (2.126) of Proposition 2.3.

**Lemma 2.12.** *In regime* (2.54)*,*

$$\int_{(0,\hat L)\times(0,1)\times(0,1)}|m-m^*|^2\,d\hat x \;\lesssim\; \varepsilon \quad\text{and}\quad \hat E\;\lesssim\;1 \tag{2.129}$$

*imply*

$$\int_{(0,\hat L)\times(0,1)\times(0,1)}|m-(1,\langle m_2\rangle,\langle m_3\rangle)|^2\,d\hat x \;\lesssim\; \varepsilon^{3/2}+\delta^2\varepsilon. \tag{2.130}$$

Proof of Lemma 2.12.
For notational convenience, we write $\int d\hat x$ for the integral $\int_{(0,\hat L)\times(0,1)\times(0,1)} d\hat x$ and $\int d\hat x'$ for the integral $\int_{(0,\hat L)\times(0,1)} d\hat x'$. We start by observing that

$$\left|\hat h_{ext}\int\varepsilon^{-1}(m_1-1)d\hat x\right| \;\lesssim\; 1. \tag{2.131}$$

Indeed, since $\hat h_{ext}\sim 1$, cf. (2.121), it suffices to show $\int|m_1-1|\,d\hat x\lesssim\varepsilon$. To this purpose, we observe that due to $|m|^2=1$,

$$|m_1-1|\left\{\begin{array}{lll} =\;1-\sqrt{1-m_2^2-m_3^2}\;\le\;m_2^2+m_3^2 & \text{for } m_1\;\ge\;0 \\[2ex] \le\;(m_1-1)^2 & \text{for } m_1\;\le\;0\end{array}\right\},$$

and thus by assumption (2.129),

$$\int|m_1-1|\,d\hat x\;\le\;\int|m-m^*|^2\,d\hat x\;\lesssim\;\varepsilon.$$

This establishes (2.131).

We now argue that

$$\int|\hat\partial_1 m|^2\,d\hat x\;\lesssim\;\varepsilon,\quad \int|\hat\partial_2 m|^2\,d\hat x\;\lesssim\;1,\quad \int|\hat\partial_3 m|^2\,d\hat x\;\lesssim\;\delta^2\varepsilon. \tag{2.132}$$

Indeed, if we neglect the nonnegative stray–field contribution, we find after rescaling according to (2.52) and (2.120),

$$\hat E\;\ge\;\int\left(\varepsilon^{-1}|\hat\partial_1 m|^2+|\hat\partial_2 m|^2+\delta^{-2}\varepsilon^{-1}|\hat\partial_3 m|^2\right)d\hat x+2\hat h_{ext}\int\varepsilon^{-1}(m_1-1)\,d\hat x.$$

Now (2.132) follows from (2.131) and our assumption $\hat{E} \lesssim 1$, cf. (2.129).

We now establish that

$$\int \left(\langle m_2\rangle^4 + \langle m_3\rangle^4\right) \, d\hat{x}' \;\lesssim\; \varepsilon^{3/2}. \tag{2.133}$$

We treat only the $m_2$–component, since the argument for $m_3$ is identical. We remark that (2.133) follows from (2.132) in its $x_3$–averaged form, i.e.,

$$\begin{aligned}
\int (\hat{\partial}_1\langle m_2\rangle)^2 \, d\hat{x}' \;&\leq\; \int (\hat{\partial}_1 m_2)^2 \, d\hat{x} \;\lesssim\; \varepsilon, \\
\int (\hat{\partial}_2\langle m_2\rangle)^2 \, d\hat{x}' \;&\leq\; \int (\hat{\partial}_2 m_2)^2 \, d\hat{x} \;\lesssim\; 1,
\end{aligned}$$

and

$$\int \langle m_2\rangle^2 \, d\hat{x}' \;\leq\; \int m_2^2 \, d\hat{x} \;\overset{(2.129)}{\lesssim}\; \varepsilon,$$

via the interpolation estimate

$$\begin{aligned}
\int \langle m_2\rangle^4 \, d\hat{x}' \;\lesssim\; & \left(\int (\hat{\partial}_1\langle m_2\rangle)^2 \, d\hat{x}' + \int \langle m_2\rangle^2 \, d\hat{x}'\right) \\
& \times \left(\varepsilon^{1/2} \int (\hat{\partial}_2\langle m_2\rangle)^2 \, d\hat{x}' + \varepsilon^{-1/2} \int \langle m_2\rangle^2 \, d\hat{x}'\right).
\end{aligned} \tag{2.134}$$

The interpolation estimate (2.134) is easily established. We start with

$$\int \langle m_2\rangle^4 \, d\hat{x}' \;\leq\; \int_0^1 \sup_{\hat{x}_1} \langle m_2\rangle^2 \, d\hat{x}_2 \int_0^{\hat{L}} \sup_{\hat{x}_2} \langle m_2\rangle^2 \, d\hat{x}_1. \tag{2.135}$$

On the one hand for fixed $\hat{x}_1$, the follwing estimate on the unit length interval $\hat{x}_2 \in [0,1]$ holds for all $\varepsilon \lesssim 1$:

$$\sup_{\hat{x}_2} \langle m_2\rangle^2 \;\lesssim\; \varepsilon^{1/2} \int_0^1 (\hat{\partial}_2\langle m_2\rangle)^2 \, d\hat{x}_2 + \varepsilon^{-1/2} \int_0^1 \langle m_2\rangle^2 \, d\hat{x}_2,$$

and thus

$$\int_0^{\hat{L}} \sup_{\hat{x}_2} \langle m_2\rangle^2 \, d\hat{x}_1 \;\lesssim\; \varepsilon^{1/2} \int (\hat{\partial}_2\langle m_2\rangle)^2 \, d\hat{x}' + \varepsilon^{-1/2} \int \langle m_2\rangle^2 \, d\hat{x}'. \tag{2.136}$$

On the other hand, since $\hat{L} \sim 1$, cf. (2.121), we have for fixed $\hat{x}_2$

$$\sup_{\hat{x}_1} \langle m_2\rangle^2 \;\lesssim\; \int_0^{\hat{L}} (\hat{\partial}_1\langle m_2\rangle)^2 \, d\hat{x}_1 + \int_0^{\hat{L}} \langle m_2\rangle^2 \, d\hat{x}_1,$$

and thus

$$\int_0^1 \sup_{\hat{x}_1} \langle m_2\rangle^2 \, d\hat{x}_2 \;\lesssim\; \int (\hat{\partial}_1\langle m_2\rangle)^2 \, d\hat{x}' + \int \langle m_2\rangle^2 \, d\hat{x}'. \tag{2.137}$$

The interpolation inequality (2.134) now follows from inserting (2.136) and (2.137) into (2.135).

We now argue that (2.130) holds at least where $m_1$ is nonnegative:

$$\int_{\{m_1\geq 0\}}(m_1-1)^2\,d\hat{x}+\int\left((m_2-\langle m_2\rangle)^2+(m_3-\langle m_3\rangle)^2\right)\,d\hat{x}\;\lesssim\;\varepsilon^{3/2}+\delta^2\varepsilon. \tag{2.138}$$

Indeed, because of $|m|^2=1$, we have in the case of $m_1\geq 0$

$$(m_1-1)^2\;\leq\;(m_1+1)^2(m_1-1)^2\;=\;(1-m_1^2)^2\;=\;(m_2^2+m_3^2)^2\;\leq\;2(m_2^4+m_3^4).$$

Thus, (2.138) follows once we show

$$\int m_2^4+(m_2-\langle m_2\rangle)^2\,d\hat{x}\;\lesssim\;\varepsilon^{3/2}+\delta^2\varepsilon, \tag{2.139}$$

and a similar statement on $m_3$. Since $m_2^2\leq|m|^2=1$, we have

$$m_2^4\;\lesssim\;\langle m_2\rangle^4+(m_2-\langle m_2\rangle)^4\;\lesssim\;\langle m_2\rangle^4+(m_2-\langle m_2\rangle)^2,$$

so that (2.139) is a consequence of (2.133), i.e., $\int\langle m_2\rangle^4\,d\hat{x}'\lesssim\varepsilon^{3/2}$ and (2.132), via Poincaré's estimate in $\hat{x}_3\in[0,1]$,

$$\int(m_2-\langle m_2\rangle)^2\,d\hat{x}'\;\lesssim\;\int(\hat{\partial}_3 m_2)^2\,d\hat{x}\;\lesssim\;\delta^2\varepsilon.$$

Finally, we shall turn to the missing term in (2.138):

$$\int_{\{m_1\leq 0\}}(1-m_1)^2\,d\hat{x}\;\lesssim\;\varepsilon^{3/2}+\delta^2\varepsilon. \tag{2.140}$$

Obviously, (2.130) is just the sum of (2.138) and (2.140). To establish (2.140), we choose a smooth $\eta=\eta(\langle m_1\rangle)$ with the property

$$\eta\;=\;1\quad\text{for}\quad\langle m_1\rangle\;\leq\;\frac{1}{4}\quad\text{and}\quad\eta\;=\;0\quad\text{for}\quad\langle m_1\rangle\;\geq\;\frac{1}{2}. \tag{2.141}$$

Note that we have for the 2–D Lebesgue measure w.r.t. $\hat{x}'$

$$\begin{aligned}\mathcal{L}^2(\operatorname{supp}\eta(\langle m_1\rangle))\;&\overset{(2.141)}{\leq}\;\mathcal{L}^2(\{\langle m_1\rangle\;<\;\tfrac{1}{2}\})\\&\lesssim\;\int(1-\langle m_1\rangle)^2\,d\hat{x}'\\&\leq\;\int(1-m_1)^2\,d\hat{x}\\&\overset{(2.129)}{\lesssim}\;\varepsilon\;\ll\;1.\end{aligned}$$

Hence we have by the Sobolev–Poincaré inequality [13, Theorem 3.16],

$$\begin{aligned}\int \eta(\langle m_1\rangle)^2\,d\hat{x}' &\lesssim \left(\int |\hat{\nabla}'\eta(\langle m_1\rangle)|\,d\hat{x}'\right)^2 \\ &\le \int \eta'(\langle m_1\rangle)^2\,d\hat{x}' \int |\hat{\nabla}'\langle m_1\rangle|^2\,d\hat{x}',\end{aligned} \tag{2.142}$$

where $\eta'$ denotes the derivative of $\eta$ w.r.t. $\langle m_1\rangle$. According to (2.132),

$$\int |\hat{\nabla}'\langle m_1\rangle|^2\,d\hat{x}' \;\le\; \int \left((\hat{\partial}_1 m_1)^2 + (\hat{\partial}_2 m_1)^2\right)\,d\hat{x} \;\lesssim\; 1.$$

Furthermore, we have on the one hand,

$$\int \eta(\langle m_1\rangle)^2\,d\hat{x}' \overset{(2.141)}{\ge} \mathcal{L}^2(\{\langle m_1\rangle \;\le\; \tfrac{1}{4}\}),$$

and on the other hand,

$$\int \eta'(\langle m_1\rangle)^2\,d\hat{x}' \overset{(2.141)}{\lesssim} \mathcal{L}^2(\{\tfrac{1}{4} \;\le \langle m_1\rangle \;\le\; \tfrac{1}{2}\})$$

so that (2.142) turns into

$$\mathcal{L}^2(\{\langle m_1\rangle \;\le\; \tfrac{1}{4}\}) \;\lesssim\; \mathcal{L}^2(\{\tfrac{1}{4} \;\le \langle m_1\rangle \;\le\; \tfrac{1}{2}\}). \tag{2.143}$$

We estimate the right–hand side of (2.143) by Poincaré's estimate in $\hat{x}_3 \in [0,1]$

$$\begin{aligned}\mathcal{L}^2(\{\tfrac{1}{4} \;\le \langle m_1\rangle \;\le\; \tfrac{1}{2}\}) \quad &\lesssim \quad \mathcal{L}^3(\{0 \;\le m_1 \;\le\; \tfrac{3}{4}\}) + \textstyle\int (m_1 - \langle m_1\rangle)^2\,d\hat{x} \\ &\lesssim \quad \int_{\{m_1\ge 0\}} (m_1-1)^2\,d\hat{x} + \int (\hat{\partial}_3 m_1)^2\,d\hat{x} \\ &\overset{(2.138),(2.132)}{\lesssim} \quad \varepsilon^{3/2} + \delta^2\varepsilon.\end{aligned} \tag{2.144}$$

We now address the left–hand side of (2.143):

$$\begin{aligned}\int_{\{m_1\le 0\}} (1-m_1)^2\,d\hat{x} \quad &\lesssim \quad \mathcal{L}^3(\{m_1 \;\le\; 0\}) \\ &\lesssim \quad \mathcal{L}^2(\{\langle m_1\rangle \;\le\; \tfrac{1}{4}\}) + \textstyle\int (m_1 - \langle m_1\rangle)^2\,d\hat{x} \\ &\lesssim \quad \mathcal{L}^2(\{\langle m_1\rangle \;\le\; \tfrac{1}{4}\}) + \textstyle\int (\hat{\partial}_3 m_1)^2\,d\hat{x} \\ &\overset{(2.132)}{\lesssim} \quad \mathcal{L}^2(\{\langle m_1\rangle \;\le\; \tfrac{1}{4}\}) + \delta^2\varepsilon.\end{aligned} \tag{2.145}$$

Finally, (2.145), (2.143) and (2.144) yield (2.140). □

We now turn to the stray–field contribution and specify the version of Lemma 2.2 we need in this section, which simply differs by the introduction of $L$–periodicity in $x_1$–direction

**Lemma 2.13.**

$$\begin{aligned}
&\int_{(0,L)\times\mathbb{R}^2} |\nabla u_m|^2\,dx \\
&\geq\ t^3 \int_{\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}} \frac{|\mathcal{F}(\sigma')|^2}{2t|k'|+(t|k'|)^2}\,dk' \\
&\quad +t\int_{\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}} \frac{|\mathcal{F}(\langle m_3\rangle)-ik'\cdot\mathcal{F}(\langle(x_3-\frac{t}{2})m'\rangle)|^2}{1+\frac{1}{2}t|k'|+\frac{1}{12}(t|k'|)^2}\,dk'. \qquad (2.146)
\end{aligned}$$

The next lemma is a variant of Lemma 2.7, the proof of which we nevertheless give, and which proves part (2.127) of Proposition 2.3.

**Lemma 2.14.** *In regime* (2.54),

$$\int_{(0,\hat{L})\times(0,1)\times(0,1)} |m-m^*|^2\,d\hat{x}\ \lesssim\ \varepsilon \quad \textit{and} \quad \hat{E}\ \lesssim\ 1$$

*imply*

$$\int_{(0,\hat{L})\times(0,1)} \langle\hat{m}_3\rangle^2\,d\hat{x}'\ \lesssim\ \delta^2+\varepsilon\delta.$$

PROOF OF LEMMA 2.14.
We start by Lemma 2.3 for $t\ll\ell$, which yields

$$\begin{aligned}
&t\int_{(0,L)\times(0,\ell)} \langle m_3\rangle^2\,dx' \\
&\lesssim\ t\int_{\{t|k_2|\leq 1\}} |\mathcal{F}(\langle m_3\rangle)|^2\,dk' + t^3\int_{(0,L)\times(0,\ell)} (\partial_2\langle m_3\rangle)^2\,dx'. \qquad (2.147)
\end{aligned}$$

The second term in (2.147) can be estimated via Cauchy–Schwarz in $x_3$ by the exchange contribution:

$$t^3\int_{(0,L)\times(0,\ell)} (\partial_2\langle m_3\rangle)^2\,dx'\ \leq\ \left(\frac{t}{d}\right)^2 d^2\int_{(0,L)\times(0,\ell)\times(0,t)} |\nabla m|^2\,dx. \qquad (2.148)$$

For the first term in (2.147) we use Lemma 2.4 and have

$$\begin{aligned}
t\int_{\{t|k_2|\leq 1\}} |\mathcal{F}(\langle m_3\rangle)|^2\,dk'\ &\lesssim\ t\int_{\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}} \min\left\{1,\frac{1}{t^2|k'|^2}\right\}|\mathcal{F}(\langle m_3\rangle)|^2\,dk' \\
&\quad +t^3\int_{(0,L)\times(0,\ell)} (\partial_1\langle m_3\rangle)^2\,dx'. \qquad (2.149)
\end{aligned}$$

The last term in (2.149) can once more be estimated via Cauchy–Schwarz

$$t^3\int_{(0,L)\times(0,\ell)} (\partial_1\langle m_3\rangle)^2\,dx'\ \leq\ \left(\frac{t}{d}\right)^2 d^2\int_{(0,L)\times(0,\ell)\times(0,t)} |\nabla m|^2\,dx, \qquad (2.150)$$

while for the first term in (2.150) we make use of Lemma 2.13, which yields

$$t\int_{\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}} \frac{|\mathcal{F}(\langle m_3\rangle) - ik'\cdot\mathcal{F}(\langle(x_3-\frac{t}{2})m'\rangle)|^2}{1+\frac{1}{2}t|k'|+\frac{1}{12}(t|k'|)^2}\,dk' \;\le\; \int_{(0,L)\times\mathbb{R}^2} |\nabla u_m|^2\,dx. \tag{2.151}$$

We use (2.151) in the form

$$\begin{aligned} & t\int_{\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}} \frac{|\mathcal{F}(\langle m_3\rangle)|^2}{1+\frac{1}{2}t|k'|+\frac{1}{12}(t|k'|)^2}\,dk' \\ \le\;& 2\int_{(0,L)\times\mathbb{R}^2} |\nabla u_m|^2\,dx + 2t\int_{\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}} \frac{|k'|^2|\mathcal{F}(\langle(x_3-\frac{t}{2})m'\rangle)|^2}{1+\frac{1}{2}t|k'|+\frac{1}{12}(t|k'|)^2}\,dk'. \end{aligned} \tag{2.152}$$

Since

$$\frac{12}{t^2|k'|^2} \;\ge\; \frac{1}{1+\frac{1}{2}t|k'|+\frac{1}{12}(t|k'|)^2} \;\gtrsim\; \min\left\{1, \frac{1}{t^2|k'|^2}\right\},$$

(2.152) yields

$$\begin{aligned} & t\int_{\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}} \min\left\{1, \frac{1}{t^2|k'|^2}\right\} |\mathcal{F}(\langle m_3\rangle)|^2\,dk' \\ \lesssim\;& \int_{(0,L)\times\mathbb{R}^2} |\nabla u_m|^2\,dx + \frac{1}{t}\int_{(0,L)\times(0,\ell)} |\langle(x_3-\tfrac{t}{2})m'\rangle|^2\,dx'. \end{aligned} \tag{2.153}$$

The last term in (2.153) can be estimated via Cauchy–Schwarz and Poincaré:

$$\begin{aligned} \frac{1}{t}\int_{(0,L)\times(0,\ell)} |\langle(x_3-\tfrac{t}{2})m'\rangle|^2\,dx' \;&=\; \frac{1}{t}\int_{(0,L)\times(0,\ell)} |\langle(x_3-\tfrac{t}{2})(m'-\langle m'\rangle)\rangle|^2\,dx' \\ &\le\; \frac{t}{12}\int_{(0,L)\times(0,\ell)} \langle|m'-\langle m'\rangle|^2\rangle\,dx' \\ &=\; \frac{1}{12}\int_{(0,L)\times(0,\ell)\times(0,t)} |m'-\langle m'\rangle|^2\,dx \\ &\lesssim\; t^2\int_{(0,L)\times(0,\ell)\times(0,t)} |\partial_3 m'|^2\,dx \\ &\le\; \left(\frac{t}{d}\right)^2 d^2\int_{(0,L)\times(0,\ell)\times(0,t)} |\nabla m|^2\,dx. \end{aligned} \tag{2.154}$$

Inserting (2.154) into (2.153) yields

$$\begin{aligned} & t\int_{\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}} \min\left\{1, \frac{1}{t^2|k'|^2}\right\} |\mathcal{F}(\langle m_3\rangle)|^2\,dk' \\ \lesssim\;& \left(1+\left(\frac{t}{d}\right)^2\right)\left(d^2\int_{(0,L)\times(0,\ell)\times(0,t)} |\nabla m|^2\,dx + \int_{(0,L)\times\mathbb{R}^2} |\nabla u_m|^2\,dx\right). \end{aligned} \tag{2.155}$$

We now collect (2.148), (2.150) and (2.155):

$$
\begin{aligned}
& t \int_{(0,l)\times(0,\ell)} \langle m_3\rangle^2 \, dx' \\
& \quad \lesssim \quad \left(1+\left(\frac{t}{d}\right)^2\right) \left(d^2 \int_{(0,L)\times(0,\ell)\times(0,t)} |\nabla m|^2 \, dx + \int_{(0,L)\times\mathbb{R}^2} |\nabla u_m|^2 \, dx\right) \\
& \quad = \quad \left(1+\left(\frac{t}{d}\right)^2\right) \left(E - 2h_{ext} \int_{(0,L)\times(0,\ell)\times(0,t)} m_1 \, dx\right). \qquad (2.156)
\end{aligned}
$$

This is the unrescaled version of

$$
\begin{aligned}
& \int_{(0,\hat{L})\times(0,1)} \langle \hat{m}_3\rangle^2 \, d\hat{x}' \\
& \quad \lesssim \quad (\varepsilon\delta+\delta^2) \left(\hat{E} - 2\hat{h}_{ext} \int_{(0,\hat{L})\times(0,1)\times(0,1)} \varepsilon^{-1}(m_1-1) \, d\hat{x}\right).
\end{aligned}
$$

It remains to evoke (2.131). □

The following lemma shows that $\langle m_2\rangle$, which has been identified as the crucial quantity in Lemmata 2.12 and 2.14, is compact.

**Lemma 2.15.** *Let* $\{\varepsilon_\nu\}_{\nu\uparrow\infty}$, $\{\delta_\nu\}_{\nu\uparrow\infty} \subset (0,\infty)$ *and* $\{m_\nu : \mathbb{R}^3 \mapsto \mathbb{S}^2\}_{\nu\uparrow\infty}$ *be as in Proposition 2.3. Then* $\{\langle \hat{m}_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ *is compact in* $L^2((0,\hat{L})\times(0,1))$.

Proof of Lemma 2.15.

We will derive the compactness statement from the following ingredients:

$$
\begin{aligned}
\langle \hat{m}_{2,\nu}\rangle \quad & \text{is bounded in } L^2((0,\hat{L})\times(0,1)) && (2.157) \\
\hat{\partial}_1\langle \hat{m}_{2,\nu}\rangle \quad & \text{is bounded in } L^2((0,\hat{L})\times(0,1)) && (2.158) \\
\varepsilon_\nu^{-1}\hat{\partial}_1\langle m_{1,\nu}\rangle \quad & \text{is bounded in } L^1((0,\hat{L})\times(0,1)) && (2.159) \\
\varepsilon_\nu^{-1}\hat{\partial}_1\langle m_{1,\nu}\rangle + \hat{\partial}_2\langle \hat{m}_{2,\nu}\rangle \quad & \text{is compact in } H^{-1}((0,\hat{L})\times(0,1)). && (2.160)
\end{aligned}
$$

In view of $\hat{m}_{2,\nu} = \varepsilon_\nu^{-1/2} m_{2,\nu}$, (2.157) follows from the first item in (2.125) and Cauchy–Schwarz in $x_3$. Similarly, (2.158) is a consequence of the first item in (2.132) (which itself follows from (2.125)) and Cauchy–Schwarz in $x_3$. For (2.159) we need the following argument: Differentiating $|m_\nu|^2 = 1$ w.r.t. $x_1$, we obtain the identity

$$
\hat{\partial}_1 m_{1,\nu} \; = \; (1-m_{1,\nu})\,\hat{\partial}_1 m_{1,\nu} - m_{2,\nu}\,\hat{\partial}_1 m_{2,\nu} - m_{3,\nu}\,\hat{\partial}_1 m_{3,\nu},
$$

and thus by Cauchy–Schwarz,

$$
\int |\hat{\partial}_1 m_{1,\nu}| \, d\hat{x} \;\lesssim\; \left(\int |m_\nu - m^*|^2 \, d\hat{x} \int |\hat{\partial}_1 m_\nu|^2 \, d\hat{x}\right)^{1/2} \overset{(2.125),(2.132)}{\lesssim} \varepsilon_\nu.
$$

It remains to invoke $\int |\hat{\partial}_1\langle m_{1,\nu}\rangle| \, d\hat{x}' \le \int |\hat{\partial}_1 m_{1,\nu}| \, d\hat{x}$.

We finally address (2.160), that is, the compactness of the averaged in–plane divergence

$$\hat\sigma'_\nu \;=\; \varepsilon_\nu^{-1}\hat\partial_1\langle m_{1,\nu}\rangle + \hat\partial_2\langle \hat m_{2,\nu}\rangle. \tag{2.161}$$

in $H^{-1}$. For our purposes, the most suitable formulation of this statement is on the Fourier side. We recall that on the rescaled level, cf. (1.3),

$$\hat{\mathcal F}(f)(\hat k') \;=\; \frac{1}{\sqrt{2\pi\hat L}}\int_{(0,\hat L)\times\mathbb R}\exp(i\hat k'\cdot\hat x')\, f\, d\hat x' \quad \text{for } \hat k' \;=\; (\hat k_1,\hat k_2)\in\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R, \tag{2.162}$$

and

$$\int_{\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R} d\hat k' \;:=\; \sum_{\hat k_1\in\frac{2\pi}{\hat L}\mathbb Z}\int_{\mathbb R} d\hat k_2.$$

We shall show that there exists a function $\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R\ni\hat k'\mapsto\hat{\mathcal F}(\hat\sigma')(\hat k')$ such that for a subsequence

$$\lim_{\nu\uparrow\infty}\int_{\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R}\frac{1}{1+|\hat k'|^2}|\hat{\mathcal F}(\hat\sigma'_\nu)-\hat{\mathcal F}(\hat\sigma')|^2\,d\hat k' \;=\; 0. \tag{2.163}$$

Using the lower bound on the stray–field contribution from Lemma 2.13 and neglecting the nonnegative exchange contribution, we find after rescaling according to (2.52) and (2.120):

$$\begin{aligned}\hat E(m_\nu) \;\ge\; & \int_{\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R}\frac{|\hat{\mathcal F}(\hat\sigma'_\nu)|^2}{2\sqrt{\hat k_1^2+\varepsilon_\nu\hat k_2^2}+\delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)}\,d\hat k' \\ & +2\hat h_{ext}\int_{(0,\hat L)\times(0,1)\times(0,1)}\varepsilon_\nu^{-1}(m_{1,\nu}-1)\,d\hat x.\end{aligned}$$

In view of (2.125) and (2.131), this implies

$$\int_{\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R}\frac{|\hat{\mathcal F}(\hat\sigma'_\nu)|^2}{2\sqrt{\hat k_1^2+\varepsilon_\nu\hat k_2^2}+\delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)}\,d\hat k' \;\lesssim\; 1. \tag{2.164}$$

Furthermore we note that

$$\begin{aligned}\hat{\mathcal F}(\hat\sigma'_\nu)(\hat k') \;=\; & \frac{1}{\sqrt{2\pi\hat L}}\int_{(0,\hat L)\times(0,1)}\exp(i\hat k'\cdot\hat x')\varepsilon_\nu^{-1}\hat\partial_1\langle m_{1,\nu}\rangle\,d\hat x' \\ & -\frac{i\hat k_2}{\sqrt{2\pi\hat L}}\int_{(0,\hat L)\times(0,1)}\exp(i\hat k'\cdot\hat x')\langle\hat m_{2,\nu}\rangle\,d\hat x',\end{aligned}$$

and thus

$$\begin{aligned}\partial_{\hat k_2}\hat{\mathcal F}(\hat\sigma'_\nu)(\hat k') \;=\; & \frac{i}{\sqrt{2\pi\hat L}}\int_{(0,\hat L)\times(0,1)}\exp(i\hat k'\cdot\hat x')\,\hat x_2\,\varepsilon_\nu^{-1}\hat\partial_1\langle m_{1,\nu}\rangle\,d\hat x' \\ & +\frac{\hat k_2}{\sqrt{2\pi\hat L}}\int_{(0,\hat L)\times(0,1)}\exp(i\hat k'\cdot\hat x')\,\hat x_2\,\langle\hat m_{2,\nu}\rangle\,d\hat x'.\end{aligned}$$

We conclude, in view of (2.157) and (2.159),

$$
\begin{aligned}
&|\hat{\mathcal{F}}(\hat\sigma'_\nu)(\hat k')| + |\partial_{\hat k_2}\hat{\mathcal{F}}(\hat\sigma'_\nu)(\hat k')| \\
&\quad\le \frac{2}{\sqrt{2\pi\hat L}}\int_{(0,\hat L)\times(0,1)} |\varepsilon_\nu^{-1}\hat\partial_1\langle m_{1,\nu}\rangle|\,d\hat x' + 2|\hat k_2|\left(\int_{(0,\hat L)\times(0,1)}\langle \hat m_{2,\nu}\rangle^2\,d\hat x'\right)^{1/2}
\end{aligned}
$$

is bounded for $\nu\uparrow\infty$. Thus there exists a continuous $\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}\ni\hat k'\mapsto\hat{\mathcal{F}}(\hat\sigma')(\hat k')$ such that for a subsequence

$$
\lim_{\nu\uparrow\infty}\hat{\mathcal{F}}(\hat\sigma'_\nu)(\hat k') \;=\; \hat{\mathcal{F}}(\hat\sigma')(\hat k') \quad \text{locally uniformly in } \hat k'\in\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}. \tag{2.165}
$$

In order to establish (2.163), in view of (2.165), it remains to show that

$$
\limsup_{M\uparrow\infty,\,\nu\uparrow\infty}\int_{\{|k'|\ge M\}}\frac{1}{1+|\hat k'|^2}|\hat{\mathcal{F}}(\hat\sigma'_\nu)-\hat{\mathcal{F}}(\hat\sigma')|^2\,d\hat k' \;=\; 0. \tag{2.166}
$$

Since

$$
\frac{1}{2\sqrt{\hat k_1^2+\varepsilon_\nu\hat k_2^2}+\delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)} \;\uparrow\; \frac{1}{2|\hat k_1|} \quad \text{as } \nu\;\uparrow\;\infty,
$$

it follows from (2.164), (2.165), and Fatou's Lemma

$$
\int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}}\frac{|\hat{\mathcal{F}}(\hat\sigma'_\nu)|^2}{2\sqrt{\hat k_1^2+\varepsilon_\nu\hat k_2^2}+\delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)}\,d\hat k' \;\le\; \int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}}\frac{|\hat{\mathcal{F}}(\hat\sigma')|^2}{2|\hat k_1|}\,d\hat k' \;<\;\infty, \tag{2.167}
$$

so that

$$
\int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}}\frac{|\hat{\mathcal{F}}(\hat\sigma'_\nu)-\hat{\mathcal{F}}(\hat\sigma')|^2}{2\sqrt{\hat k_1^2+\varepsilon_\nu\hat k_2^2}+\delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)}\,d\hat k' \quad \text{is bounded as } \nu\uparrow\infty. \tag{2.168}
$$

We now observe that for $|\hat k'|\ge M$ and $\varepsilon_\nu\le 1$,

$$
\frac{1}{2\sqrt{\hat k_1^2+\varepsilon_\nu\hat k_2^2}+\delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)} \;\ge\; \frac{1}{\frac{2}{M}+\delta_\nu}\frac{1}{1+|\hat k'|^2},
$$

so that

$$
\begin{aligned}
&\int_{\{|\hat k'|\ge M\}}\frac{1}{1+|\hat k'|^2}|\hat{\mathcal{F}}(\hat\sigma'_\nu)-\hat{\mathcal{F}}(\hat\sigma')|^2\,d\hat k' \\
&\quad\le \left(\frac{2}{M}+\delta_\nu\right)\int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}}\frac{|\hat{\mathcal{F}}(\hat\sigma'_\nu)-\hat{\mathcal{F}}(\hat\sigma')|^2}{2\sqrt{\hat k_1^2+\varepsilon_\nu\hat k_2^2}+\delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)}\,d\hat k'.
\end{aligned}
$$

According to (2.168), this yields (2.166).

The argument that (2.157)–(2.160) yield the compactness of $\{\langle\hat m_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ is a classical compensated compactness result, in the sense that the strong equicontinuity properties of $\{\langle\hat m_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ in $\hat x_1$ compensate the weak equicontinuity properties in $\hat x_2$. In view of (2.157), we may assume that there exists $\hat m_2 \in L^2$ such that

$$\langle\hat m_{2,\nu}\rangle \;\rightharpoonup\; \hat m_2 \quad \text{weakly in } L^2. \tag{2.169}$$

Again, we pass to the Fourier side to express strong convergence: We have to show that $\hat{\mathcal F}(\langle\hat m_{2,\nu}\rangle)$ converges to $\hat{\mathcal F}(\hat m_2)$ in $L^2(d\hat k')$. Since $\langle\hat m_{2,\nu}\rangle$ is supported in $\hat x_2 \in [0,1]$, (2.169) automatically yields for all $\hat k'$

$$\begin{aligned} |\hat{\mathcal F}(\langle\hat m_{2,\nu}\rangle)(\hat k')|^2 &\le \int_{(0,\hat L)\times(0,1)} \langle\hat m_{2,\nu}\rangle^2\, d\hat x', \\ \hat{\mathcal F}(\langle\hat m_{2,\nu}\rangle)(\hat k') &\to \hat{\mathcal F}(\hat m_2)(\hat k'). \end{aligned}$$

Hence it remains to show that, uniformly in $\nu$, high frequencies do not carry much energy, i.e.,

$$\limsup_{\nu\uparrow\infty, M\uparrow\infty} \int_{\{|\hat k'|>M\}} |\hat{\mathcal F}(\langle\hat m_{2,\nu}\rangle)|^2\, d\hat k' \;=\; 0. \tag{2.170}$$

We note that in frequency space, (2.161) translates into

$$ik_2\, \hat{\mathcal F}(\langle\hat m_{2,\nu}\rangle) \;=\; \hat{\mathcal F}(f_\nu) + \hat{\mathcal F}(\hat\sigma_\nu'), \quad \text{where} \quad f_\nu \;:=\; -\varepsilon_\nu^{-1}\hat\partial_1\langle m_{1,\nu}\rangle, \tag{2.171}$$

and (2.158) and (2.159) turn into

$$\int_{\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R} \hat k_1^2 |\hat{\mathcal F}(\langle m_{2,\nu}\rangle)|^2\, d\hat k' \qquad \text{is bounded for } \nu\uparrow\infty, \tag{2.172}$$

$$\sup_{\hat k'} |\hat{\mathcal F}(f_\nu)|^2 \qquad \text{is bounded for } \nu\uparrow\infty. \tag{2.173}$$

For arbitrary

$$M_2 \;\gg\; M_1 \;\gg\; 1, \tag{2.174}$$

we note

$$\begin{aligned} &\int_{\{|\hat k_1|>M_1\}\cup\{|\hat k_2|>M_2\}} |\hat{\mathcal F}(\langle\hat m_{2,\nu}\rangle)|^2\, d\hat k' \\ &\quad\le\; \int_{\{|\hat k_1|>M_1\}} |\hat{\mathcal F}(\langle\hat m_{2,\nu}\rangle)|^2\, d\hat k' + \int_{\{|\hat k_1|\le M_1\}\cap\{|\hat k_2|>M_2\}} |\hat{\mathcal F}(\langle\hat m_{2,\nu}\rangle)|^2\, d\hat k' \\ &\quad\overset{(2.171)}{\lesssim}\; \frac{1}{M_1^2}\int_{\{|\hat k_1|>M_1\}} \hat k_1^2 |\hat{\mathcal F}(\langle\hat m_{2,\nu}\rangle)|^2\, d\hat k' \\ &\qquad + \int_{\{|\hat k_1|\le M_1\}\cap\{|\hat k_2|>M_2\}} \frac{1}{\hat k_2^2}|\hat{\mathcal F}(f_\nu)|^2\, d\hat k' + \int_{\{|\hat k_1|\le M_1\}\cap\{|\hat k_2|>M_2\}} \frac{1}{\hat k_2^2}|\hat{\mathcal F}(\hat\sigma_\nu')|^2\, d\hat k' \end{aligned}$$

$$\begin{aligned}
&\overset{(2.174)}{\lesssim} \frac{1}{M_1^2}\int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}} \hat k_1^2|\hat{\mathcal F}(\langle \hat m_{2,\nu}\rangle)|^2\,d\hat k' + \sup_{\hat k'}|\hat{\mathcal F}(f_\nu)|^2\int_{\{|\hat k_1|\le M_1\}\cap\{|\hat k_2|>M_2\}}\frac{1}{\hat k_2^2}\,d\hat k' \\
&\quad + \int_{\{|\hat k_1|\le M_1\}\cap\{|\hat k_2|>M_2\}}\frac{1}{1+|\hat k'|^2}|\hat{\mathcal F}(\hat\sigma_\nu')|^2\,d\hat k' \\
&\lesssim \frac{1}{M_1^2}\int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}} \hat k_1^2|\hat{\mathcal F}(\langle \hat m_{2,\nu}\rangle)|^2\,d\hat k' + \frac{M_1}{M_2}\sup_{\hat k'}|\hat{\mathcal F}(f_\nu)|^2 \\
&\quad + \int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}}\frac{1}{1+|\hat k'|^2}|\hat{\mathcal F}(\hat\sigma_\nu')-\hat{\mathcal F}(\hat\sigma')|^2\,d\hat k' + \int_{\{|\hat k_2|>M_2\}}\frac{1}{1+|\hat k'|^2}|\hat{\mathcal F}(\hat\sigma')|^2\,d\hat k'.
\end{aligned}$$

Choosing $M_2 = M \gg 1$ and $M_1 = M^{1/3}$ in the above, we obtain

$$\begin{aligned}
&\int_{\{|\hat k'|>M\}} |\hat{\mathcal F}(\langle \hat m_{2,\nu}\rangle)|^2\,d\hat k' \\
&\lesssim \frac{1}{M^{2/3}}\int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}} \hat k_1^2|\hat{\mathcal F}(\langle \hat m_{2,\nu}\rangle)|^2\,d\hat k' + \frac{1}{M^{2/3}}\sup_{\hat k'}|\hat{\mathcal F}(f_\nu)|^2 \\
&\quad + \int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}}\frac{1}{1+|\hat k'|^2}|\hat{\mathcal F}(\hat\sigma_\nu')-\hat{\mathcal F}(\hat\sigma')|^2\,d\hat k' + \int_{\{|\hat k'|>M\}}\frac{1}{1+|\hat k'|^2}|\hat{\mathcal F}(\hat\sigma')|^2\,d\hat k'.
\end{aligned} \tag{2.175}$$

Now (2.175) shows that (2.170) follows from (2.163), (2.167), (2.172) and (2.173). □

### 2.3.2 Lower–semicontinuity part in the Γ–convergence

In this subsection, we establish the lower semicontinuity part of the Γ–convergence to be proved. We formulate it in

**Proposition 2.4.** *Let $\{\varepsilon_\nu\}_{\nu\uparrow\infty}$, $\{\delta_\nu\}_{\nu\uparrow\infty} \subset (0,\infty)$ converge to zero. Let $\hat L$ and $\hat h_{ext}$ be given. Let $\{m_\nu : \mathbb{R}^3 \mapsto \mathbb{S}^2\}_{\nu\uparrow\infty}$ be a sequence of vector fields that are $L$–periodic in $x_1$ and supported in $x_2 \in [0,\ell_\nu]$, $x_3 \in [0,t_\nu]$ with*

$$\limsup_{\nu\uparrow\infty} \hat E(m_\nu) \;<\; \infty.$$

*Let $(\hat m_2,\hat m_3) : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be such that*

$$(\hat m_{2,\nu},\hat m_{3,\nu}) \rightharpoonup (\hat m_2,\hat m_3) \quad \textit{weakly in} \quad L^2((0,\hat L)\times(0,1)\times(0,1)). \tag{2.176}$$

*Then $(\hat m_2,\hat m_3)$ is of the form*

$$(\hat m_2,\hat m_3) = (\hat m_2(x'),0), \tag{2.177}$$

*and*

$$\begin{aligned}
\int_{(0,\hat L)\times(0,1)} \hat m_2^2\,d\hat x' &= \lim_{\nu\uparrow\infty}\frac{1}{\varepsilon_\nu}\int_{(0,\hat L)\times(0,1)\times(0,1)} |m_\nu - m^*|^2\,d\hat x, && (2.178)\\
\hat E_0(\hat m_2) &\le \liminf_{\nu\uparrow\infty} \hat E(m_\nu). && (2.179)
\end{aligned}$$

Proof of Proposition 2.4.
We continue to use the notation $\int d\hat{x}'$ and $\int d\hat{x}$. We start with the argument for (2.177). For this purpose we evoke part (2.126) of Proposition 2.3 in the form of

$$\lim_{\nu\uparrow\infty}\int\left((\hat{m}_{2,\nu}-\langle\hat{m}_{2,\nu}\rangle)^2+(\hat{m}_{3,\nu}-\langle\hat{m}_{3,\nu}\rangle)^2\right)\,d\hat{x}\;=\;0, \tag{2.180}$$

and part (2.127) of Proposition 2.3,

$$\lim_{\nu\uparrow\infty}\int\langle\hat{m}_{3,\nu}\rangle^2\,d\hat{x}'\;=\;0. \tag{2.181}$$

Now (2.180) and (2.181) combine into

$$\lim_{\nu\uparrow\infty}\int\left|(\hat{m}_{2,\nu},\hat{m}_{3,\nu})-(\langle\hat{m}_{2,\nu}\rangle,0)\right|^2\,d\hat{x}\;=\;0. \tag{2.182}$$

By the standard lower semicontinuity of this nonnegative quadratic expression under the weak convergence (2.176), (2.182) turns into

$$\int\left|(\hat{m}_2,\hat{m}_3)-(\langle\hat{m}_2\rangle,0)\right|^2\,d\hat{x}\;=\;0,$$

which is a reformulation of (2.177).

We now turn to (2.178). According to part (2.128) of Proposition 2.3, $\{\langle\hat{m}_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ is compact in $L^2((0,\hat{L}),(0,1))$. Hence the weak convergence of $\{(\hat{m}_{2,\nu},\hat{m}_{3,\nu})\}_{\nu\uparrow\infty}$ in $L^2((0,\hat{L})\times(0,1)\times(0,1))^2$, cf. (2.176), which yields weak convergence of $\{\langle\hat{m}_{2,\nu}\rangle\}_{\nu\uparrow\infty}$ in $L^2((0,\hat{L})\times(0,1))$, improves to strong convergence, i.e.,

$$\langle\hat{m}_{2,\nu}\rangle\;\to\;\hat{m}_2\quad\text{in } L^2((0,\hat{L})\times(0,1)). \tag{2.183}$$

For (2.178), we appeal to the triangle inequality

$$\begin{aligned}
&\left|\left(\varepsilon_\nu^{-1}\int|m_\nu-m^*|^2\,d\hat{x}\right)^{1/2}-\left(\int\hat{m}_2^2\,d\hat{x}'\right)^{1/2}\right|\\
&\le\;\left(\varepsilon_\nu^{-1}\int|m_\nu-(1,\langle m_{2,\nu}\rangle,\langle m_{3,\nu}\rangle)|^2\,d\hat{x}\right)^{1/2}\\
&\quad+\left(\int\langle\hat{m}_{3,\nu}\rangle^2 d\hat{x}'\right)^{1/2}+\left(\int(\langle\hat{m}_{2,\nu}\rangle-\hat{m}_2)^2\,d\hat{x}'\right)^{1/2}.
\end{aligned}$$

Hence, (2.178) follows from part (2.126) of Proposition 2.3, (2.181) and (2.183).

We now address (2.179) and start with the observation that, due to Lemma 2.13,

$$\begin{aligned}
\hat{E}(m_\nu) &\geq \int \Big(\varepsilon_\nu^{-1}|\hat\partial_1 m_\nu|^2 + |\hat\partial_2 m_\nu|^2 + \varepsilon_\nu^{-1}\delta_\nu^{-2}|\hat\partial_3 m_\nu|^2\Big)\, d\hat x \\
&\quad + \int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}} \frac{|\hat{\mathcal{F}}(\hat\sigma_\nu')|^2}{2\sqrt{\hat k_1^2+\varepsilon_\nu \hat k_2^2} + \delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)}\, d\hat k' \\
&\quad + 2\hat h_{ext}\int \varepsilon_\nu^{-1}(m_{1,\nu}-1)\, d\hat x \\
&\geq \int (\hat\partial_1\langle \hat m_{2,\nu}\rangle)^2\, d\hat x' \\
&\quad + \int_{\frac{2\pi}{\hat L}\mathbb{Z}\times\mathbb{R}} \frac{|\hat{\mathcal{F}}(\hat\sigma_\nu')|^2}{2\sqrt{\hat k_1^2+\varepsilon_\nu \hat k_2^2} + \delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)}\, d\hat k' \\
&\quad + 2\hat h_{ext}\int \varepsilon_\nu^{-1}(\langle m_{1,\nu}\rangle-1)\, d\hat x', \qquad (2.184)
\end{aligned}$$

where

$$\hat\sigma_\nu' \;:=\; \varepsilon_\nu^{-1}\hat\partial_1\langle m_{1,\nu}\rangle + \hat\partial_2\langle \hat m_{2,\nu}\rangle \;=\; \hat\partial_1\left(\varepsilon_\nu^{-1}(\langle m_{1,\nu}\rangle - 1)\right) + \hat\partial_2\langle \hat m_{2,\nu}\rangle.$$

Hence, the main nonlinear ingredient is

$$\varepsilon_\nu^{-1}(\langle m_{1,\nu}\rangle - 1) \;\to -\tfrac{1}{2}\hat m_2^2 \quad \text{in } L^1((0,\hat L)\times(0,1)). \qquad (2.185)$$

We start by noticing that

$$\begin{aligned}
&\int \left|\hat m_{2,\nu}^2 + \hat m_{3,\nu}^2 - \hat m_2^2\right|\, d\hat x \\
&\leq \int |(\hat m_{2,\nu}-\hat m_2)(\hat m_{2,\nu}+\hat m_2)|\, d\hat x + \int \hat m_{3,\nu}^2\, d\hat x \\
&\leq \left[\left(\int (\hat m_{2,\nu} - \langle \hat m_{2,\nu}\rangle)^2\, d\hat x\right)^{1/2} + \left(\int (\langle \hat m_{2,\nu}\rangle - \hat m_2)^2\, d\hat x'\right)^{1/2}\right] \\
&\quad \times \left[\left(\int \hat m_{2,\nu}^2\, d\hat x\right)^{1/2} + \left(\int \hat m_2^2\, d\hat x'\right)^{1/2}\right] + \int \hat m_{3,\nu}^2\, d\hat x.
\end{aligned}$$

Hence, (2.182) and (2.183) imply

$$\lim_{\nu\uparrow\infty}\int \left|\hat m_{2,\nu}^2 + \hat m_{3,\nu}^2 - \hat m_2^2\right|\, d\hat x \;=\; 0.$$

Thus, for (2.185), it suffices to show that

$$\lim_{\nu\uparrow\infty}\int \left|\varepsilon_\nu^{-1}(m_{1,\nu}-1) + \tfrac{1}{2}(\hat m_{2,\nu}^2 + \hat m_{3,\nu}^2)\right|\, d\hat x \;=\; 0,$$

which can be reformulated as

$$\lim_{\nu\uparrow\infty}\varepsilon_\nu^{-1}\int_{(0,\hat L)\times(0,1)\times(0,1)}\left|(m_{1,\nu}-1)+\tfrac12(m_{2,\nu}^2+m_{3,\nu}^2)\right|\,d\hat x. \tag{2.186}$$

We notice that, due to $|m_\nu|^2=1$,

$$(m_{1,\nu}-1)+\tfrac12(m_{2,\nu}^2+m_{3,\nu}^2) \;=\; (m_{1,\nu}-1)+\tfrac12(1-m_{1,\nu}^2) \;=\; -\tfrac12(m_{1,\nu}-1)^2,$$

so that (2.186) follows from part (2.126) of Proposition 2.3. This establishes (2.185).

Setting $\hat\sigma' := -\hat\partial_1(\frac12\hat m_2^2)+\hat\partial_2\hat m_2$, we have

$$\lim_{\nu\uparrow\infty}\hat{\mathcal F}(\hat\sigma'_\nu)(\hat k') \;=\; \hat{\mathcal F}(\hat\sigma')(\hat k') \quad \text{for all } \hat k'\in\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R. \tag{2.187}$$

Indeed, this follows from the representation

$$\begin{aligned}\hat{\mathcal F}(\hat\sigma'_\nu)(\hat k') \;&=\; \frac{1}{\sqrt{2\pi\hat L}}\int_{(0,\hat L)\times\mathbb R}\exp(i\hat k'\cdot\hat x')\left(\hat\partial_1\varepsilon_\nu^{-1}(\langle m_{1,\nu}\rangle-1)+\hat\partial_2\langle\hat m_{2,\nu}\rangle\right)d\hat x'\\ &=\; -\frac{i\hat k_1}{\sqrt{2\pi\hat L}}\int_{(0,\hat L)\times\mathbb R}\exp(i\hat k'\cdot\hat x')\varepsilon_\nu^{-1}(\langle m_{1,\nu}\rangle-1)\,d\hat x'\\ &\quad\; -\frac{i\hat k_2}{\sqrt{2\pi\hat L}}\int_{(0,\hat L)\times\mathbb R}\exp(i\hat k'\cdot\hat x')\langle\hat m_{2,\nu}\rangle\,d\hat x',\end{aligned}$$

and the convergences stated in (2.185) and (2.183).

We are now ready to conclude on (2.179) based on the lower bound (2.184). From (2.183) and standard lower semicontinuity, we obtain

$$\int_{(0,\hat L)\times(0,1)}(\hat\partial_1\hat m_2)^2\,d\hat x' \;\le\; \liminf_{\nu\uparrow\infty}\int_{(0,\hat L)\times(0,1)}(\hat\partial_1\langle\hat m_{2,\nu}\rangle)^2\,d\hat x'.$$

Furthermore, (2.185) yields

$$-\hat h_{ext}\int_{(0,\hat L)\times(0,1)}\hat m_2^2\,d\hat x' \;=\; \lim_{\nu\uparrow\infty}2\hat h_{ext}\int_{(0,\hat L)\times(0,1)}\varepsilon_\nu^{-1}(\langle m_{1,\nu}\rangle-1)\,d\hat x'.$$

Finally, since

$$\lim_{\nu\uparrow\infty}\frac{1}{2\sqrt{\hat k_1^2+\varepsilon_\nu\hat k_2^2}+\delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)} \;=\; \frac{1}{2|\hat k_1|} \quad \text{for all } \hat k'\in\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R,$$

(2.187) yields by Fatou's Lemma

$$\int_{\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R}\frac{1}{2|\hat k_1|}|\hat{\mathcal F}(\hat\sigma')|^2\,d\hat k' \;\le\; \liminf_{\nu\uparrow\infty}\int_{\frac{2\pi}{\hat L}\mathbb Z\times\mathbb R}\frac{|\hat{\mathcal F}(\hat\sigma'_\nu)|^2}{2\sqrt{\hat k_1^2+\varepsilon_\nu\hat k_2^2}+\delta_\nu(\hat k_1^2+\varepsilon_\nu\hat k_2^2)}\,d\hat k'.$$

□

### 2.3.3 Approximation by smooth functions

In this subsection we prove that admissible functions $\hat{m}_2$ of finite energy $\hat{E}_0$ can be approximated by smooth admissible functions $\{\hat{m}_{2,\alpha}\}_{\alpha\downarrow 0}$ in the energy topology, i.e., such that $\hat{E}_0(\hat{m}_{2,\alpha}) \to \hat{E}_0(\hat{m}_2)$. This is an important ingredient for the construction of the recovery sequence in the proof of Γ–convergence. Because of the nonlinear term $\hat{\partial}_1(\frac{1}{2}\hat{m}_2^2)$, the approximation argument is not obvious.

Before stating the approximation result in Proposition 2.5, we now make precise what exactly we understand by admissible functions with finite energy $\hat{E}_0$. Consider $\hat{m}_2 \in L^2((0,\hat{L})\times\mathbb{R})$, $\hat{L}$–periodic in $\hat{x}_1$ and with bounded support in $x_2$, such that (2.193) is finite. This means in particular that $\hat{\partial}_1\hat{m}_2$ (which always exists as a distribution, since $\hat{m}_2$ is in particular locally integrable) is actually in $L^2$. Finiteness of $\hat{E}_0$ also means that

$$\hat{\sigma} \;:=\; \hat{\partial}_2\hat{m}_2 - \hat{\partial}_1(\tfrac{1}{2}\hat{m}_2^2), \tag{2.188}$$

which is well–defined as a distribution due to $\hat{m}_2 \in L^2$, has the property that $|\hat{\partial}_1|^{-1/2}\hat{\sigma}$ (which is suitably defined with the help of Fourier series in $\hat{x}_1$) is also in $L^2$. This implies in particular that in a distributional sense

$$\int_0^{\hat{L}} \hat{\sigma}\, d\hat{x}_1 \;=\; 0 \quad \text{for all } \hat{x}_2 \in \mathbb{R}. \tag{2.189}$$

In view of (2.188) and the periodicity of $\hat{m}_2^2$ in $\hat{x}_1$, this yields in a distributional sense

$$\hat{\partial}_2 \int_0^{\hat{L}} \hat{m}_2\, d\hat{x}_1 \;=\; 0.$$

Since $\hat{m}_2$ has finite support in $\hat{x}_2$, the latter implies

$$\int_0^{\hat{L}} \hat{m}_2\, d\hat{x}_1 \;=\; 0 \quad \text{for a.e. } \hat{x}_2 \in \mathbb{R}. \tag{2.190}$$

We now formulate the main result of this subsection:

**Proposition 2.5.** *Let $\hat{m}_2 : \mathbb{R}^2 \mapsto \mathbb{R}$ be $\hat{L}$–periodic in $\hat{x}_1$ with vanishing mean in $\hat{x}_1$ and supported in $\hat{x}_2 \in [0,1]$ with $\hat{m}_2 \in L^2((0,\hat{L})\times(0,1))$. Let $\hat{m}_{2,\alpha}$ denote the convolution of $\hat{m}_2$ with a Dirac sequence in $\alpha \downarrow 0$. Then*

$$\begin{aligned}\limsup_{\alpha\downarrow 0}\Bigg(\int_{(0,\hat{L})\times\mathbb{R}} (\hat{\partial}_1\hat{m}_{2,\alpha})^2\, d\hat{x} + \frac{1}{2}\int_{(0,\hat{L})\times\mathbb{R}} \big||\hat{\partial}_1|^{-1/2}\big(-\hat{\partial}_1(\tfrac{1}{2}\hat{m}_{2,\alpha}^2) + \hat{\partial}_2\hat{m}_{2,\alpha}\big)\big|^2\, d\hat{x}\Bigg) \\ \le \int_{(0,\hat{L})\times\mathbb{R}} (\hat{\partial}_1\hat{m}_2)^2\, d\hat{x} + \frac{1}{2}\int_{(0,\hat{L})\times\mathbb{R}} \big||\hat{\partial}_1|^{-1/2}\big(-\hat{\partial}_1(\tfrac{1}{2}\hat{m}_2^2) + \hat{\partial}_2\hat{m}_2\big)\big|^2\, d\hat{x}.\end{aligned}$$

Since the nonlinearity involves Burgers' operator $-\hat{\partial}_1(\frac{1}{2}\hat{m}_2^2) + \hat{\partial}_2\hat{m}_2$ (see also below), one might think of a vanishing viscosity ansatz,

$$-\hat{\partial}_1(\tfrac{1}{2}\hat{m}_{2,\alpha}^2) + \hat{\partial}_2\hat{m}_{2,\alpha} - \alpha\,\hat{\partial}_1^2\hat{m}_{2,\alpha} \;=\; \hat{\sigma}$$

to construct an approximation $\{\hat{m}_{2,\alpha}\}_{\alpha\downarrow 0}$ of $\hat{m}_2$. However, it is not clear to us how to control the difference $\hat{m}_{2,\alpha} - \hat{m}_2$ for this approach.

The following corollary will be helpful for the construction of a recovery sequence in the next subsection.

**Corollary 2.7.** *Let $\hat{m}_2 : \mathbb{R}^2 \mapsto \mathbb{R}$ be $\hat{L}$–periodic in $\hat{x}_1$ with vanishing mean in $\hat{x}_1$ and supported in $\hat{x}_2 \in [0,1]$ with $\hat{m}_2 \in L^2((0,\hat{L}) \times (0,1))$ and $\hat{E}_0(\hat{m}_2) < \infty$. Then there exists a sequence $\{\hat{m}_{2,\nu} : \mathbb{R}^2 \mapsto \mathbb{R}\}_{\nu\uparrow\infty}$ of smooth functions, $\hat{L}$–periodic in $\hat{x}_1$, with vanishing mean in $\hat{x}_1$ and supported in $\hat{x}_2 \in [0,1]$, such that*

$$\begin{aligned} \hat{m}_{2,\nu} \quad &\to \quad \hat{m}_2 \quad \text{in} \quad L^2((0,\hat{L}) \times (0,1)), && (2.191)\\ \limsup_{\nu\uparrow\infty} \hat{E}_0(\hat{m}_{2,\nu}) \quad &\le \quad \hat{E}_0(\hat{m}_2). && (2.192) \end{aligned}$$

We start with some notational simplifications: For convenience we omit the hats and the primes. We also notice that the main part of $E_0$, i.e.,

$$\int_{(0,L)\times\mathbb{R}} (\partial_1 m_2)^2\,dx + \frac{1}{2}\int_{(0,L)\times\mathbb{R}} \big||\partial_1|^{-1/2}\big(-\partial_1(\tfrac{1}{2}m_2^2) + \partial_2 m_2\big)\big|^2\,dx, \qquad (2.193)$$

scales under the change of variables

$$x_1 \;=\; \lambda\,\hat{x}_1, \quad x_2 \;=\; \lambda^{3/2}\,\hat{x}_2 \quad \text{and} \quad m_2 \;=\; \lambda^{-1/2}\,\hat{m}_2.$$

Hence we may assume that $L = 1$. More precisely, the admissible $m_2$'s are 1–periodic in $x_1$ and have bounded support in $x_2$. Also, we write $\int dx$ for $\int_{(0,1)\times\mathbb{R}} dx$.

We start with the following remark: If one identifies $x_2$ with "time" and $-x_1$ with "space" (2.188), i.e.,

$$\partial_2 m_2 - \partial_1(\tfrac{1}{2}m_2^2) \;=\; \sigma \qquad (2.194)$$

can be read as the inviscid Burgers' equation with a right hand side $\sigma$. This point of view will motivate most of the subsequent analysis. We start by deriving what are called entropy equations. For an introduction to the notion of entropy for scalar conservation laws, see [24, Chapter 2.3]. Consider an "entropy" $\eta = \eta(m_2)$ and its "entropy flux" $q = q(m_2)$ related by

$$q'(m_2) \;=\; -m_2\,\eta'(m_2). \qquad (2.195)$$

The entropy flux is defined such that for a smooth solution $m_2$ of (2.194) one has

$$\partial_2\eta(m_2) + \partial_1 q(m_2) \;=\; \eta'(m_2)\,\sigma.$$

The following lemma shows that this remains true for our class of solutions under appropriate growth conditions for $\eta$. It shows that the chain rule remains valid for the class of considered functions.

**Lemma 2.16.** *Let the "entropy" $\eta = \eta(m_2)$ be smooth with*

$$\sup_{m_2}(|\eta'| + |\eta''|) \;<\; \infty \tag{2.196}$$

*and $q$ defined as in* (2.195). *Then we have for an $m_2$ as in Corollary 2.7*

$$\partial_2\eta(m_2) + \partial_1 q(m_2) \;=\; \eta'(m_2)\sigma \quad \textit{distributionally.} \tag{2.197}$$

Proof of Lemma 2.16.
Notice that by $m_2 \in L^2$, the growth condition (2.196) on $\eta$, and thus on $q$ in view of (2.195), ensure that $\eta(m_2)$ and $q(m_2)$ are locally integrable. Hence the left–hand side of (2.197) is well–defined distributionally. For an arbitrary test function $\zeta$, the right–hand side of (2.197) is to be read as $\langle\sigma, \eta'(m_2)\zeta\rangle$. This expression is well–defined, since on the one hand $\int ||\partial_1|^{-1/2}\sigma|^2\,dx < \infty$ and thus a fortiori $\int ||\partial_1|^{-1}\sigma|^2\,dx < \infty$, due to the periodicity in $x_1$ and on the other hand $m_2$, $\partial_1 m_2 \in L^2$ together with (2.196) imply that $\partial_1(\eta'(m_2)\zeta) \in L^2$.

As $\eta$ can be recovered as the limit of linear combinations of convex $\eta$'s, we may assume

$$\eta \quad \text{is convex.} \tag{2.198}$$

(In fact we will apply Lemma 2.16 only for convex $\eta$'s.) By the symmetry $x_2 \rightsquigarrow -x_2$, it is enough to establish the distributional inequality

$$\partial_2\eta(m_2) + \partial_1 q(m_2) \;\geq\; \eta'(m_2)\sigma.$$

So let a nonnegative test function $\zeta \in C_0^\infty(\mathbb{R}^2)$ be given. We would like to test (2.194) with $\eta'(m_2)\zeta$. In order to carry this out, we need to regularize $\eta'(m_2)\zeta$ in $x_2$, a technique we learned from [2]. Hence, we test (2.194) with

$$\frac{1}{h}\int_{x_2-h}^{x_2} \eta'(m_2)\zeta\,dy_2 \quad \text{for } h > 0.$$

It remains to investigate the three terms

$$\left\langle \partial_2 m_2, \frac{1}{h}\int_{x_2-h}^{x_2} \eta'(m_2)\zeta\,dy_2 \right\rangle, \tag{2.199}$$

$$\left\langle \partial_1(\tfrac{1}{2}m_2^2), \frac{1}{h}\int_{x_2-h}^{x_2} \eta'(m_2)\zeta\,dy_2 \right\rangle, \tag{2.200}$$

$$\left\langle \sigma, \frac{1}{h}\int_{x_2-h}^{x_2} \eta'(m_2)\zeta\,dy_2 \right\rangle, \tag{2.201}$$

in the limit $h \downarrow 0$. For (2.199), we note

$$\begin{aligned}
&\left\langle \partial_2 m_2, \frac{1}{h}\int_{x_2-h}^{x_2} \eta'(m_2)\zeta\,dy_2 \right\rangle \\
&= \; -\int m_2(x_2)\frac{1}{h}(\eta'(m_2(x_2))\zeta(x_2) - \eta'(m_2(x_2-h))\zeta(x_2-h))\,dx \\
&= \; \int \frac{1}{h}(m_2(x_2+h) - m_2(x_2))\eta'(m_2(x_2))\zeta(x_2)\,dx.
\end{aligned}$$

We observe that due to (2.198) and the nonnegativity of $\zeta$,

$$\frac{1}{h}(m_2(x_2+h)-m_2(x_2))\eta'(m_2(x_2))\zeta(x_2) \;\le\; \frac{1}{h}(\eta(m_2(x_2+h))-\eta(m_2(x_2)))\zeta(x_2),$$

so that the above turns into

$$\begin{aligned}
&\left\langle \partial_2 m_2, \frac{1}{h}\int_{x_2-h}^{x_2} \eta'(m_2)\zeta\,dy_2 \right\rangle \\
&\le\; \int \frac{1}{h}(\eta(m_2(x_2+h))-\eta(m_2(x_2)))\zeta(x_2)\,dx \\
&=\; -\int \eta(m_2(x_2))\frac{1}{h}(\zeta(x_2+h)-\zeta(x_2))\,dx.
\end{aligned}$$

Hence we obtain for the first term, as desired,

$$\limsup_{h\downarrow 0}\left\langle \partial_2 m_2, \frac{1}{h}\int_{x_2-h}^{x_2} \eta'(m_2)\zeta\,dy_2 \right\rangle \;\le\; -\int \eta(m_2)\partial_2\zeta\,dx \;=\; \langle \partial_2\eta(m_2),\zeta\rangle.$$

We now address the second term, (2.200). Because of $m_2 \in L^2$ and $\partial_1 m_2 \in L^2$, we have

$$\partial_1(-\tfrac{1}{2}m_2^2) \;=\; -m_2\partial_1 m_2 \in L^1,$$

so that

$$\langle \partial_1(-\tfrac{1}{2}m_2^2),\zeta\rangle \;=\; \int(-m_2\partial_1 m_2)\zeta\,dx. \tag{2.202}$$

Since $\eta'$ is bounded, we have by definition (2.195) of $q$

$$\partial_1 q(m_2) \;=\; -m_2\eta'(m_2)\partial_1 m_2 \in L^1,$$

so that

$$\langle \partial_1 q(m_2),\zeta\rangle \;=\; \int(-m_2\eta'(m_2)\partial_1 m_2)\zeta\,dx. \tag{2.203}$$

We further note that (modulo subsequence)

$$\lim_{h\downarrow 0}\frac{1}{h}\int_{x_2-h}^{x_2}\eta'(m_2)\zeta\,dy_2 \;=\; \eta'(m_2)\zeta \quad \text{a.e.},$$

so that by dominated convergence

$$\begin{aligned}
&\lim_{h\downarrow 0}\left\langle \partial_1(-\tfrac{1}{2}m_2^2), \frac{1}{h}\int_{x_2-h}^{x_2}\eta'(m_2)\zeta\,dy_2\right\rangle \\
&\overset{(2.202)}{=}\; \lim_{h\downarrow 0}\int(-m_2\partial_1 m_2)\frac{1}{h}\int_{x_2-h}^{x_2}\eta'(m_2)\zeta\,dy_2\,dx \\
&=\; \int(-m_2\partial_1 m_2)\eta'(m_2)\zeta\,dx \\
&\overset{(2.203)}{=}\; \langle \partial_1 q(m_2),\zeta\rangle.
\end{aligned}$$

Finally, we turn to the last term, (2.201). We note that because of $m_2$, $\partial_1 m_2 \in L^2$, we have

$$\lim_{h\downarrow 0}\int\left|\partial_1\left(\frac{1}{h}\int_{x_2-h}^{x_2}\eta'(m_2)\zeta dy_2-\eta'(m_2)\zeta\right)\right|^2 dx \;=\; 0,$$

which by $\int||\partial_1|^{-1}\sigma|^2dx<\infty$ immediately implies

$$\lim_{h\downarrow 0}\left\langle\sigma,\frac{1}{h}\int_{x_2-h}^{x_2}\eta'(m_2)\zeta\,dy_2\right\rangle \;=\; \langle\sigma,\eta'(m_2)\zeta\rangle.$$

□

The main consequence of Lemma 2.16 is the following estimate of $m_2$:

**Lemma 2.17.** *We have for any $m_2$ as in Corollary 2.7*

$$\mathrm{esssup}_{x_2}\int_0^1 m_2^2\,dx_1 \;\lesssim\; \left(\int(\partial_1 m_2)^2\,dx\int||\partial_1|^{-1/2}\sigma|^2\,dx\right)^{1/2}, \tag{2.204}$$

$$\int m_2^4\,dx \;\lesssim\; \left(\int(\partial_1 m_2)^2\,dx\right)^{3/2}\left(\int||\partial_1|^{-1/2}\sigma|^2\,dx\right)^{1/2}. \tag{2.205}$$

Proof of Lemma 2.17.

We first argue in favour of (2.204). Formally, (2.204) follows from choosing $\eta(m_2)=\frac{1}{2}m_2^2$ in Lemma 2.16 and integrating over $x_1$ and $x_2$. Since this $\eta$ violates the growth condition (2.196), we need an approximation argument. For any $M<\infty$, the entropy

$$\eta_M(m_2) \;=\; \left\{\begin{array}{cc}\frac{1}{2}m_2^2 & \text{for } |m_2|\le M\\ Mm_2-\frac{1}{2}M^2 & \text{for } |m_2|\ge M\end{array}\right\}$$

is admissible in Lemma 2.16 (after appropriate smoothing near $|m_2|=M$). Integrating (2.197) over $x_1$ yields, due to the periodicity of $\frac{1}{2}m_2^2$ in $x_1$,

$$\frac{d}{dx_2}\int_0^1\eta_M(m_2)dx_1 \;=\; \int_0^1\eta_M'(m_2)\sigma\,dx_1 \quad\text{distributionally in } x_2. \tag{2.206}$$

We note that the right–hand side of (2.206) even makes sense in a pointwise way, as

$$\begin{aligned}&\int_{\mathbb{R}}\left|\int_0^1\eta_M'(m_2)\sigma\,dx_1\right|dx_2\\ &\overset{(2.206)}{\le}\quad\left(\int|\partial_1(\eta_M'(m_2))|^2dx\int||\partial_1|^{-1}\sigma|^2dx\right)^{1/2}\\ &\lesssim\quad\mathrm{esssup}_{m_2}|\eta_M''|\left(\int|\partial_1 m_2|^2dx\int||\partial_1|^{-1/2}\sigma|^2\,dx\right)^{1/2}\\ &\le\quad\left(\int|\partial_1 m_2|^2dx\int||\partial_1|^{-1/2}\sigma|^2\,dx\right)^{1/2}\end{aligned} \tag{2.207}$$

Since $m_2$ has compact support in $x_2$, we conclude from (2.206) and (2.207) that

$$\begin{aligned}\mathrm{esssup}_{x_2}\int_0^1 \eta_M(m_2)dx_1 &\le \int_{\mathbb{R}}\left|\frac{d}{dx_2}\int_0^1 \eta_M(m_2)dx_1\right| dx_2\\ &\lesssim \left(\int |\partial_1 m_2|^2 dx \int ||\partial_1|^{-1/2}\sigma|^2\,dx\right)^{1/2}.\end{aligned}$$

We now evoke the monotone convergence principle in order to obtain (2.204) in the limit $M\uparrow\infty$.

In view of (2.190), we have for a.e. $x_2\in\mathbb{R}$,

$$\mathrm{esssup}_{x_1} m_2^2 \lesssim \int_0^1 (\partial_1 m_2)^2 dx_1,$$

and thus

$$\int_{\mathbb{R}} \mathrm{esssup}_{x_1} m_2^2\,dx_2 \lesssim \int (\partial_1 m_2)^2\,dx. \tag{2.208}$$

In order to obtain (2.205), it remains to combine (2.204) with (2.208):

$$\begin{aligned}\int m_2^4\,dx &\le \int_{\mathbb{R}}\left(\int_0^1 m_2^2\,dx_1\right)\mathrm{esssup}_{x_1} m_2^2\,dx_2\\ &\le \left(\mathrm{esssup}_{x_2}\int_0^1 m_2^2\,dx_1\right)\left(\int_{\mathbb{R}}\mathrm{esssup}_{x_1} m_2^2\,dx_2\right)\\ &\overset{(2.204),(2.208)}{\lesssim} \left(\int (\partial_1 m_2)^2\,dx\right)^{3/2}\left(\int ||\partial_1|^{-1/2}\sigma|^2\,dx\right)^{1/2}.\end{aligned}$$

□

Proof of Proposition 2.5.

Let the subscript $\alpha$ denote convolution with the Dirac sequence under consideration. Because of the standard inequalities

$$\begin{aligned}\int (\partial_1 m_{2,\alpha})^2\,dx &\le \int (\partial_1 m_2)^2\,dx,\\ \int ||\partial_1|^{-1/2}(-\partial_1(\tfrac12 m_2^2)_\alpha + \partial_2 m_{2,\alpha})|^2\,dx &\le \int ||\partial_1|^{-1/2}(-\partial_1(\tfrac12 m_2^2)+\partial_2 m_2)|^2\,dx,\end{aligned}$$

the statement of Proposition 2.5 follows from

$$\begin{aligned}\limsup_{\alpha\downarrow 0}\Big\{\int ||\partial_1|^{-1/2}(-\partial_1(\tfrac12 m_{2,\alpha}^2)+\partial_2 m_{2,\alpha})|^2\,dx\\ -\int ||\partial_1|^{-1/2}(-\partial_1(\tfrac12 m_2^2)_\alpha+\partial_2 m_{2,\alpha})|^2\,dx\Big\} \le 0.\end{aligned} \tag{2.209}$$

Hence we need to estimate the commutator of mollification and quadratic nonlinearity. Indeed, by the triangle inequality, (2.209) follows from

$$\lim_{\alpha\downarrow 0}\int ||\partial_1|^{-1/2}(-\partial_1(\tfrac12 m_{2,\alpha}^2)+\partial_1(\tfrac12 m_2^2)_\alpha)|^2\,dx = 0. \tag{2.210}$$

Again, by the triangle inequality, we may split (2.210) into

$$\lim_{\alpha\downarrow 0}\int ||\partial_1|^{1/2}((\tfrac{1}{2}m_2^2)_\alpha-\tfrac{1}{2}m_2^2)|^2\,dx = 0 \tag{2.211}$$

$$\lim_{\alpha\downarrow 0}\int ||\partial_1|^{1/2}(\tfrac{1}{2}m_{2,\alpha}^2-\tfrac{1}{2}m_2^2)|^2\,dx = 0. \tag{2.212}$$

A crucial ingredient in the sequel is the following estimate for two one–periodic functions $g$ and $h$ of $x_1$ with mean zero:

$$\begin{aligned}&\int_0^1 ||\partial_1|^{1/2}(hg)|^2\,dx_1\\ &\lesssim \left(\int_0^1 h^2\,dx_1\int_0^1(\partial_1 h)^2\,dx_1\int_0^1 g^2\,dx_1\int_0^1(\partial_1 g)^2\,dx_1\right)^{1/2}.\end{aligned} \tag{2.213}$$

We shall start by arguing that

$$\begin{aligned}\int_0^1 ||\partial_1|^{1/2}(hg)|^2\,dx_1 \lesssim\ & \mathrm{esssup}_{x_1} g^2\int_0^1 ||\partial_1|^{1/2}h|^2 dx_1\\ &+\mathrm{esssup}_{x_1} h^2\int_0^1 ||\partial_1|^{1/2}g|^2 dx_1.\end{aligned} \tag{2.214}$$

For this purpose we recall the characterisation of the homogeneous $H^{1/2}$–norm as trace–norm:

$$\int_0^1 ||\partial_1|^{1/2}h|^2\,dx_1 = \int_0^\infty\int_0^1(\partial_1\bar h)^2+(\partial_3\bar h)^2\,dx_1\,dx_3, \tag{2.215}$$

where $\bar h(x_1,x_3)$ is the unique decaying harmonic extension of $h(x_1)$ in the upper half–plane $\{x_3>0\}$. Then $\bar h\bar g$ is a (nonharmonic) extension of $hg$ and we have by the variational characterisation of harmonic functions and the maximum principle

$$\begin{aligned}&\int_0^1 ||\partial_1|^{1/2}(hg)|^2\,dx_1\\ &\overset{(2.215)}{=} \int_0^\infty\int_0^1(\partial_1(\overline{hg}))^2+(\partial_3(\overline{hg}))^2\,dx_1\,dx_3\\ &\le \int_0^\infty\int_0^1(\partial_1(\bar h\bar g))^2+(\partial_3(\bar h\bar g))^2\,dx_1\,dx_3\\ &\lesssim \mathrm{esssup}_{x_1,x_3}\bar g^2\int_0^\infty\int_0^1(\partial_1\bar h)^2+(\partial_3\bar h)^2\,dx_1\,dx_3\\ &+ \mathrm{esssup}_{x_1,x_3}\bar h^2\int_0^\infty\int_0^1(\partial_1\bar g)^2+(\partial_3\bar g)^2\,dx_1\,dx_3\\ &\overset{(2.215)}{=} \mathrm{esssup}_{x_1} g^2\int_0^1||\partial_1|^{1/2}h|^2\,dx_1+\mathrm{esssup}_{x_1}h^2\int_0^1||\partial_1|^{1/2}g|^2\,dx_1.\end{aligned}$$

This establishes (2.214).

We now turn to (2.213). It is a consequence of (2.214) and the following two observations (which hold for both $h$ and $g$). Notice that $\int_0^1 h\,dx_1 = 0$ implies that there exists an $x_1$ with $h(x_1) = 0$ and thus $h^2(x_1) = 0$, so that

$$\begin{aligned} \operatorname{esssup}_{x_1} h^2 &\leq \int_0^1 |\partial_1(h^2)|\,dx_1 \\ &= 2\int_0^1 |h\,\partial_1 h|\,dx_1 \\ &\lesssim \left(\int_0^1 h^2\,dx_1 \int_0^1 (\partial_1 h)^2\,dx_1\right)^{1/2}. \end{aligned} \tag{2.216}$$

Observe further that by Cauchy–Schwarz in Fourier space,

$$\int_0^1 ||\partial_1|^{1/2} h|^2\,dx_1 \;\leq\; \left(\int_0^1 h^2\,dx_1 \int_0^1 (\partial_1 h)^2\,dx_1\right)^{1/2}. \tag{2.217}$$

Now (2.213) follows from inserting (2.216) and (2.217) into (2.214).

We now turn to (2.211). By the standard convolution argument, this follows from

$$\int ||\partial_1|^{1/2}(\tfrac{1}{2}m_2^2)|^2\,dx \;<\; \infty. \tag{2.218}$$

The finiteness of (2.218) is a consequence of the estimate

$$\int ||\partial_1|^{1/2}(\tfrac{1}{2}m_2^2)|^2\,dx \;\lesssim\; \operatorname{esssup}_{x_2} \int_0^1 m_2^2\,dx_1 \int (\partial_1 m_2)^2\,dx \tag{2.219}$$

and (2.204) in Lemma 2.17. The argument for (2.219) is at hand: After integration in $x_2$, it follows from

$$\int_0^1 ||\partial_1|^{1/2}(\tfrac{1}{2}m_2^2)|^2\,dx_1 \;\lesssim\; \int_0^1 m_2^2\,dx_1 \int_0^1 (\partial_1 m_2)^2\,dx_1,$$

which itself is a consequence of (2.213) for $g = h = m_2$.

We finally turn to (2.212). In fact, we shall show that

$$\begin{aligned} &\int ||\partial_1|^{1/2}(\tfrac{1}{2}m_{2,\alpha}^2 - \tfrac{1}{2}m_2^2)|^2\,dx \\ &\lesssim\; \operatorname{esssup}_{x_2} \int_0^1 m_2^2\,dx_1 \left(\int (\partial_1 m_2)^2\,dx\right)^{1/2} \left(\int (\partial_1 m_{2,\alpha} - \partial_1 m_2)^2\,dx\right)^{1/2} \end{aligned} \tag{2.220}$$

invoke (2.204) and the standard property of the convolution

$$\lim_{\alpha\downarrow 0} \int (\partial_1 m_{2,\alpha} - \partial_1 m_2)^2\,dx \;=\; 0.$$

For (2.220), we apply (2.213) to $g = \frac{1}{2}(m_\alpha + m)$ and $h = m_\alpha - m$, which yields

$$\begin{aligned}
&\int_0^1 ||\partial_1|^{1/2}(\tfrac{1}{2}m_{2,\alpha}^2 - \tfrac{1}{2}m_2^2)|^2\,dx_1 \\
&\quad\lesssim \Big(\int_0^1 (m_{2,\alpha}+m_2)^2\,dx_1 \int_0^1 (\partial_1 m_{2,\alpha}+\partial_1 m_2)^2\,dx_1 \\
&\qquad\times \int_0^1 (m_{2,\alpha}-m_2)^2\,dx_1 \int_0^1 (\partial_1 m_{2,\alpha}-\partial_1 m_2)^2\,dx_1\Big)^{1/2},
\end{aligned}$$

and thus by Cauchy–Schwarz in $x_2$ and the triangle inequality,

$$\begin{aligned}
&\int ||\partial_1|^{1/2}(\tfrac{1}{2}m_{2,\alpha}^2 - \tfrac{1}{2}m_2^2)|^2\,dx \\
&\lesssim \Big(\operatorname{esssup}_{x_2}\int_0^1 (m_{2,\alpha}+m_2)^2\,dx_1 \int (\partial_1 m_{2,\alpha}+\partial_1 m_2)^2\,dx \\
&\quad\times \operatorname{esssup}_{x_2}\int_0^1 (m_{2,\alpha}-m_2)^2\,dx_1 \int (\partial_1 m_{2,\alpha}-\partial_1 m_2)^2\,dx\Big)^{1/2} \\
&\lesssim \Big(\operatorname{esssup}_{x_2}\int_0^1 m_{2,\alpha}^2\,dx_1 + \operatorname{esssup}_{x_2}\int_0^1 m_2^2\,dx_1\Big) \\
&\quad\times \Big(\int (\partial_1 m_{2,\alpha})^2\,dx + \int (\partial_1 m_2)^2\,dx\Big)^{1/2} \Big(\int (\partial_1 m_{2,\alpha}-\partial_1 m_2)^2\,dx\Big)^{1/2} \\
&\lesssim \operatorname{esssup}_{x_2}\int_0^1 m_2^2\,dx_1 \Big(\int (\partial_1 m_2)^2\,dx\Big)^{1/2} \Big(\int (\partial_1 m_{2,\alpha}-\partial_1 m_2)^2\,dx\Big)^{1/2}.
\end{aligned}$$

□

Proof of Corollary 2.7.

According to Proposition 2.5, the given function $m_2$ can be smoothly approximated by a convolution with a Dirac sequence. The corresponding approximations can be called $\{\tilde m_{2,\alpha}\}_\alpha$. The support of the original function in $x_2$ is changed to $[-\alpha, 1+\alpha]$. This can be amended by rescaling with an affine function:

$$m_{2,\nu}(x_1,x_2) := (1+2\alpha)\tilde m_{2,\alpha}(x_1,(1+2\alpha)x_2-\alpha),$$

where $\alpha = \nu^{-1}$. The prefactor preserves a uniform scaling of the charge density:

$$\begin{aligned}
\sigma_\nu &= -\partial_1(\tfrac{1}{2}m_{2,\nu}^2) + \partial_2 m_{2,\nu} \\
&= -(1+2\alpha)^2\partial_1(\tfrac{1}{2}\tilde m_{2,\alpha}^2) + (1+2\alpha)^2\partial_2\tilde m_{2,\alpha} \\
&=: (1+2\alpha)^2\tilde\sigma_\alpha.
\end{aligned}$$

Likewise we obtain

$$\begin{aligned}
(\partial_1 m_{2,\nu})^2 &= (1+2\alpha)^2(\partial_1\tilde m_{2,\alpha})^2 \\
m_{2,\nu}^2 &= (1+2\alpha)^2\tilde m_{2,\alpha}^2.
\end{aligned}$$

Thus we have

$$E_0(m_{2,\nu}) = (1+2\alpha)^3 E_0(\tilde m_{2,\alpha}).$$

The standard convolution argument for (2.191) and Proposition 2.5 for (2.192) conclude the proof. □

### 2.3.4 Recovery sequence in Γ–convergence

In this subsection, we construct the recovery sequence for the Γ–convergence. The construction of a recovery sequence can be carried out starting from a smooth function $\hat{m}_2$, preceded by the approximation argument of the last section. The crucial statement is given in the following proposition:

**Proposition 2.6.** *Let $\hat{m}_2 : \mathbb{R}^2 \mapsto \mathbb{R}$ be smooth, $\hat{L}$–periodic in $\hat{x}_1$, with vanishing mean in $\hat{x}_1$ and supported in $\hat{x}_2 \in [0,1]$. Then there exists a sequence $\{m_\nu : \mathbb{R}^3 \mapsto \mathbb{S}^2\}_{\nu\uparrow\infty}$ of smooth vector fields that are $L$–periodic in $x_1$ and supported in $x_2 \in [0,\ell_\nu]$, $x_3 \in [0,t_\nu]$, such that in the limit $\nu \uparrow \infty$,*

$$(\hat{m}_{2,\nu}, \hat{m}_{3,\nu}) \;\rightarrow\; (\hat{m}_2, 0) \quad in \quad L^2((0,\hat{L})\times(0,1)\times(0,1)), \tag{2.221}$$

$$\int_{(0,\hat{L})\times(0,1)} \hat{m}_2^2 \, d\hat{x}' \;=\; \lim_{\nu\uparrow\infty} \frac{1}{\varepsilon_\nu} \int_{(0,\hat{L})\times(0,1)\times(0,1)} |m_\nu - m^*|^2 \, d\hat{x}, \tag{2.222}$$

$$\hat{E}_0(\hat{m}_2) \;\geq\; \limsup_{\nu\uparrow\infty} \hat{E}(m_\nu). \tag{2.223}$$

The construction of the recovery sequence can be generalised to nonsmooth functions via the following corollary:

**Corollary 2.8.** *Let $\hat{m}_2 : \mathbb{R}^2 \mapsto \mathbb{R}$ be $\hat{L}$–periodic in $\hat{x}_1$ and supported in $\hat{x}_2 \in [0,1]$ with $\hat{m}_2 \in L^2((0,\hat{L})\times(0,1))$ and $\hat{E}_0(\hat{m}_2) < \infty$. Then there exists a sequence $\{m_\nu : \mathbb{R}^3 \mapsto \mathbb{S}^2\}_{\nu\uparrow\infty}$, $L$–periodic in $x_1$ and supported in $x_2 \in [0,\ell_\nu]$, $x_3 \in [0,t_\nu]$, such that* (2.221), (2.222) *and* (2.223) *are fulfilled.*

Proof of Proposition 2.6.
In order to have $|m_\nu|^2 = 1$ and $m_\nu \approx m^*$, and in view of (2.119), we set

$$m_{1,\nu}(x) \;=\; \sqrt{1-\varepsilon_\nu\,\hat{m}_2^2(\hat{x}')}, \quad m_{2,\nu}(x) \;=\; \varepsilon_\nu^{1/2}\hat{m}_2(\hat{x}') \quad \text{and} \quad m_{3,\nu}(x) \;\equiv\; 0, \tag{2.224}$$

where $x$ and $\hat{x}$ are related via (2.52). With this definition, statement (2.221) is obvious. Statement (2.222) follows from

$$\frac{1}{\varepsilon_\nu}\int_{(0,\hat{L})\times(0,1)\times(0,1)} |m_\nu - m^*|^2 \, d\hat{x}$$
$$\overset{(2.224)}{=} \frac{1}{\varepsilon_\nu}\int_{(0,\hat{L})\times(0,1)\times(0,1)} (m_{1,\nu}-1)^2 \, d\hat{x} + \int_{(0,\hat{L})\times(0,1)} \hat{m}_2^2 \, d\hat{x}',$$

and

$$\begin{aligned}\frac{1}{\varepsilon_\nu}\int_{(0,\hat{L})\times(0,1)\times(0,1)} (m_{1,\nu}-1)^2 \, d\hat{x} \;&\overset{(2.224)}{=}\; \frac{1}{\varepsilon_\nu}\int_{(0,\hat{L})\times(0,1)} \left(\sqrt{1-\varepsilon_\nu\,\hat{m}_2^2} - 1\right)^2 d\hat{x}' \\ &\leq\; \frac{\varepsilon_\nu}{4}\int_{(0,\hat{L})\times(0,1)} \hat{m}_2^4 \, d\hat{x}' \\ &\rightarrow\; 0.\end{aligned}$$

We now turn to (2.223). First of all, recall the definition of $\hat{E}(m_\nu)$ in (2.120), and proceed separately for each energy contribution. The exchange contribution turns into

$$\begin{aligned}\hat{E}_{ex}(m_\nu) &\overset{(2.120)}{=} \left(\frac{\ell_\nu}{d_\nu^8 t_\nu^2}\right)^{1/3} d_\nu^2 \int_{(0,L)\times(0,\ell_\nu)\times(0,t_\nu)} |\nabla m_\nu|^2\, dx \\ &\overset{(2.224)}{=} \left(\frac{\ell_\nu}{d_\nu^8 t_\nu^2}\right)^{1/3} d_\nu^2 t_\nu \int_{(0,L)\times(0,\ell_\nu)} \frac{\varepsilon_\nu}{1-\varepsilon_\nu\,\hat{m}_2^2} |\nabla \hat{m}_2|^2\, dx' \\ &\overset{(2.52),(2.53)}{=} \int_{(0,\hat{L})\times(0,1)} \frac{1}{1-\varepsilon_\nu\,\hat{m}_2^2}\left((\hat{\partial}_1\hat{m}_2)^2 + \varepsilon_\nu(\hat{\partial}_2\hat{m}_2)^2\right) d\hat{x}'.\end{aligned}$$

Therefore, by dominated convergence we have

$$\lim_{\nu\uparrow\infty} \hat{E}_{ex}(m_\nu) = \int_{(0,\hat{L})\times(0,1)} (\hat{\partial}_1\hat{m}_2)^2\, d\hat{x}'.$$

We combine the additive constant $E(m^*) = 2h_{ext}L\ell_\nu t_\nu$ with the Zeeman term, cf. (2.120):

$$\begin{aligned}\hat{E}_{Zeeman}(m_\nu) &\overset{(2.120)}{=} 2h_{ext}\left(\frac{\ell_\nu}{d_\nu^8 t_\nu^2}\right)^{1/3} \int_{(0,L)\times(0,\ell_\nu)\times(0,t_\nu)} (m_{1,\nu}-1)\, dx \\ &\overset{(2.224)}{=} 2h_{ext}\left(\frac{\ell_\nu}{d_\nu^8 t_\nu^2}\right)^{1/3} t_\nu \int_{(0,L)\times(0,\ell_\nu)} \left(\sqrt{1-\varepsilon_\nu\,\hat{m}_2^2} - 1\right) dx' \\ &\leq -h_{ext}\left(\frac{\ell_\nu}{d_\nu^8 t_\nu^2}\right)^{1/3} t_\nu \int_{(0,L)\times(0,\ell_\nu)} \varepsilon_\nu\,\hat{m}_2^2\, dx' \\ &\overset{(2.52),(2.53),(2.120)}{=} -\hat{h}_{ext}\int_{(0,\hat{L})\times(0,1)} \hat{m}_2^2\, d\hat{x}'.\end{aligned}$$

In order to treat the stray–field contribution, we use a variant of Lemma 2.1 suited to our purposes:

**Lemma 2.18.** *Let $\sigma = \sigma(x')$ be $L$–periodic in $x_1$ with vanishing mean, and of bounded support in $x_2$. Let $u_\sigma = u_\sigma(x)$ be $L$–periodic in $x_1$, decaying in $x_2$ and $x_3$, the continuously differentiable solution of*

$$\Delta u_\sigma(x', x_3) = \left\{ \begin{array}{ll} \sigma(x') & \text{for } x_3 \in (0,t) \\ 0 & \text{else} \end{array} \right\}. \tag{2.225}$$

*Then*

$$\begin{aligned}\int_{(0,L)\times\mathbb{R}^2} |\nabla u_\sigma|^2\, dx &= t\int_{\frac{2\pi}{L}\mathbb{Z}\times\mathbb{R}} \frac{|\mathcal{F}(\sigma)|^2}{|k'|^2}\left(\frac{e^{-t|k'|} - 1 + t|k'|}{t|k'|}\right) dk' \\ &\leq \frac{t^2}{2}\int_{\left(\frac{2\pi}{L}\mathbb{Z}\setminus\{0\}\right)\times\mathbb{R}} \frac{|\mathcal{F}(\sigma)|^2}{|k_1|}\, dk'.\end{aligned} \tag{2.226}$$

The inequality in (2.226) follows from

$$\frac{e^{-|k'|}-1+|k'|}{|k'|^3} \;\le\; \frac{1}{2|k'|} \;\le\; \frac{1}{2|k_1|}.$$

Note that the wave numbers $k_1 = 0$ do not contribute to the Fourier integral because of our assumption on the mean of $\sigma$. □

We continue with the proof of Proposition 2.6. The charge density $\sigma_\nu = \sigma_\nu(x')$ scales as follows:

$$\begin{aligned}
\sigma_\nu \quad &= \quad \partial_1 m_{1,\nu} + \partial_2 m_{2,\nu} \\
&\overset{(2.224)}{=} \quad \varepsilon_\nu \left( -\frac{\partial_1(\frac{1}{2}\hat m_2^2)}{\sqrt{1-\varepsilon_\nu \hat m_2^2}} \right) + \sqrt{\varepsilon_\nu}\,\partial_2 \hat m_2 \\
&\overset{(2.52),(2.53)}{=} \quad \left(\frac{d_\nu^2}{\ell_\nu^4 t_\nu}\right)^{1/3} \left[ -\frac{\hat\partial_1(\frac{1}{2}\hat m_2^2)}{\sqrt{1-\varepsilon_\nu \hat m_2^2}} + \hat\partial_2 \hat m_2 \right] \\
&=: \quad \left(\frac{d_\nu^2}{\ell_\nu^4 t_\nu}\right)^{1/3} \hat\sigma_\nu. 
\end{aligned} \tag{2.227}$$

Note that

$$\hat\sigma_\nu \;\to\; \hat\sigma \;=\; -\hat\partial_1(\tfrac{1}{2}\hat m_2^2) + \hat\partial_2 \hat m_2, \tag{2.228}$$

uniformly with all derivatives. Hence we gather from (2.52):

$$|\mathcal{F}(\sigma_\nu)(k')|^2 \;=\; \left(\frac{d_\nu^4 \ell_\nu}{t_\nu^2}\right)^{2/3} |\hat{\mathcal{F}}(\hat\sigma_\nu)(\hat k')|^2, \tag{2.229}$$

cf. (2.162) for the definition of the rescaled Fourier transform. Thus we obtain from Lemma 2.18:

$$\begin{aligned}
\hat E_{stray}(m_\nu) \quad &= \quad \left(\frac{\ell_\nu}{d_\nu^8 t_\nu^2}\right)^{1/3} \int_{(0,L)\times\mathbb{R}^2} |\nabla u_\nu|^2\, dx \\
&\le \quad \left(\frac{\ell_\nu}{d_\nu^8 t_\nu^2}\right)^{1/3} \frac{t_\nu^2}{2} \int_{(\frac{2\pi}{L}\mathbb{Z}\setminus\{0\})\times\mathbb{R}} \frac{|\mathcal{F}(\sigma_\nu)|^2}{|k_1|}\, dk' \\
&\overset{(2.229),(2.52),(2.53)}{=} \quad \frac{1}{2} \int_{(\frac{2\pi}{\hat L}\mathbb{Z}\setminus\{0\})\times\mathbb{R}} \frac{|\hat{\mathcal{F}}(\hat\sigma_\nu)|^2}{|\hat k_1|}\, d\hat k'.
\end{aligned}$$

It remains to argue that

$$\begin{aligned}
\lim_{\nu\uparrow\infty} \int_{(\frac{2\pi}{\hat L}\mathbb{Z}\setminus\{0\})\times\mathbb{R}} \frac{|\hat{\mathcal{F}}(\hat\sigma_\nu)|^2}{|\hat k_1|}\, d\hat k' \;&=\; \int_{(\frac{2\pi}{\hat L}\mathbb{Z}\setminus\{0\})\times\mathbb{R}} \frac{|\hat{\mathcal{F}}(\hat\sigma)|^2}{|\hat k_1|}\, d\hat k' \\
&=\; \int_{(0,\hat L)\times\mathbb{R}} ||\hat\partial_1|^{-1/2}\hat\sigma|^2\, d\hat x'.
\end{aligned} \tag{2.230}$$

Indeed, by the triangle inequality and Plancherel, we have

$$\begin{aligned}
&\left|\left(\int_{(\frac{2\pi}{\hat{L}}\mathbb{Z}\setminus\{0\})\times\mathbb{R}} \frac{|\hat{\mathcal{F}}(\hat{\sigma}_\nu)|^2}{|\hat{k}_1|}\, d\hat{k}'\right)^{1/2} - \left(\int_{(\frac{2\pi}{\hat{L}}\mathbb{Z}\setminus\{0\})\times\mathbb{R}} \frac{|\hat{\mathcal{F}}(\hat{\sigma})|^2}{|\hat{k}_1|}\, d\hat{k}'\right)^{1/2}\right| \\
\leq\ & \left(\int_{(\frac{2\pi}{\hat{L}}\mathbb{Z}\setminus\{0\})\times\mathbb{R}} \frac{|\hat{\mathcal{F}}(\hat{\sigma}_\nu - \hat{\sigma})|^2}{|\hat{k}_1|}\, d\hat{k}'\right)^{1/2} \\
\leq\ & \left(\frac{\hat{L}}{2\pi}\int_{\frac{2\pi}{\hat{L}}\mathbb{Z}\times\mathbb{R}} |\hat{\mathcal{F}}(\hat{\sigma}_\nu - \hat{\sigma})|^2\, d\hat{k}'\right)^{1/2} \\
=\ & \left(\frac{\hat{L}}{2\pi}\int_{(0,\hat{L})\times(0,1)} (\hat{\sigma}_\nu - \hat{\sigma})^2\, d\hat{x}'\right)^{1/2}.
\end{aligned}$$

Hence (2.230) follows from (2.228). □

Proof of Corollary 2.8.
The idea is to approximate $\hat{m}_2$ by smooth functions and apply Proposition 2.5. If the smoothing process of Corollary 2.7 is marked by the index $\nu$ and the approximation procedure of Proposition 2.5 by the index $\mu$, we have

$$\begin{aligned}
\limsup_{\mu\uparrow\infty} \hat{E}(m_{\nu,\mu}) &\leq \hat{E}_0(\hat{m}_{2,\nu}) \quad \text{for fixed } \nu \\
\limsup_{\mu\uparrow\infty} \hat{E}_0(\hat{m}_{2,\nu}) &\leq \hat{E}_0(\hat{m}_2).
\end{aligned}$$

It remains to choose an appropriate diagonal sequence in $\mu = \mu_\nu$. □

### 2.3.5 Subcriticality of the bifurcation

We shall now argue on the level of the scaling limit $\hat{E}_0$ that the bifurcation ocurring at $h_{ext} = h_{crit}$ is subcritical. Hence we cannot ascertain the existence of minimisers by standard bifurcation theory.

Neglecting the hats, the scaling limit can be rewritten as

$$\begin{aligned}
E_0(m_2) &= \frac{1}{2}\langle m_2, \mathcal{L}m_2\rangle + \mathcal{N}_3(m_2,m_2,m_2) + \mathcal{N}_4(m_2,m_2,m_2,m_2) \\
&\quad -(h_{ext} - h_{crit})\int_0^1\int_0^L m_2^2\, dx_1\, dx_2,
\end{aligned}$$

where

$$\frac{1}{2}\langle u, \mathcal{L}v\rangle = \int_0^1\int_0^L \left[\partial_1 u \partial_1 v + \tfrac{1}{2}|\partial_1|^{-1/2}\partial_2 u |\partial_1|^{-1/2}\partial_2 v - h_{crit} uv\right]\, dx_1\, dx_2 \tag{2.231}$$

$$\begin{aligned}
\mathcal{N}_3(u,v,w) &= -\int_0^1\int_0^L |\partial_1|^{-1/2}\partial_2 u |\partial_1|^{-1/2}\partial_1(\tfrac{1}{2}vw)\, dx_1\, dx_2 \\
\mathcal{N}_4(r,u,v,w) &= \frac{1}{2}\int_0^1\int_0^L |\partial_1|^{-1/2}\partial_1(\tfrac{1}{2}ru)|\partial_1|^{-1/2}\partial_1(\tfrac{1}{2}vw)\, dx_1\, dx_2.
\end{aligned}$$

The linear operator $\mathcal{L}$ defined by the bilinear form (2.231) corresponds to the $\Gamma$–limit of the Hessian for the critical external field, integrated in $x_1$. Thus, it has the following properties:

$$\mathcal{L} \;\geq\; 0 \quad \text{and} \quad \ker\mathcal{L} \;=\; \mathrm{span}\{\cos(k_1^* x_1)\,\sin(\pi x_2)\} \;=:\; \mathrm{span}\{m_2^+\};$$

cf. Lemma 2.11. For an asymptotic expansion, the contributions of lowest order to $E_0$ restricted onto the one–dimensional kernel of $\mathcal{L}$, $\{t m_2^+ \,|\, t \in \mathbb{R}\}$, have to be calculated. It turns out that by symmetry

$$\mathcal{N}_3(m_2^+, m_2^+, m_2^+) \;=\; 0.$$

As the cubic term degenerates on $\ker\mathcal{L}$, one has to include a perturbation of higher order in the asymptotic analysis; thus the ansatz is to restrict $E_0$ onto $\{t m_2^+ + t^2 m_2^{++} \,|\, t \in \mathbb{R}\}$, to neglect all terms $o(t^4)$ and to minimise in $m_2^{++}$. This leads to an Euler–Lagrange equation for $m_2^{++}$, by variation

$$\forall \zeta : \quad 0 \;=\; \langle \zeta, \mathcal{L} m_2^{++}\rangle + \big(\mathcal{N}_3(\zeta, m_2^+, m_2^+) + \mathcal{N}_3(m_2^+, \zeta, m_2^+) + \mathcal{N}_3(m_2^+, m_2^+, \zeta)\big)\,.$$

Integration by parts yields

$$\begin{aligned} 0 \;&=\; -2\partial_1^2 m_2^{++} - |\partial_1|^{-1}\partial_2^2 m_2^{++} - 2h_{crit} m_2^{++} \\ &\quad +|\partial_1|^{-1}\partial_2\partial_1(\tfrac{1}{2}(m_2^+)^2) + (\partial_1\partial_2|\partial_1|^{-1} m_2^+) m_2^+ \\ &=\; -2\partial_1^2 m_2^{++} - |\partial_1|^{-1}\partial_2^2 m_2^{++} - 2h_{crit} m_2^{++} - \frac{\pi}{2}\sin(2k_1^* x_1)\sin(2\pi x_2). \end{aligned}$$

The solution is given by

$$m_2^{++} \;=\; \frac{1}{10}\left(\frac{2}{\pi}\right)^{1/3} \sin(2k_1^* x_1)\sin(2\pi x_2).$$

Thus, the behaviour of $\hat{E}_0$ at the critical field in fourth order is given by

$$\begin{aligned} \hat{E}_0(t m_2^+ + t^2 m_2^{++}) \;&=\; \left(-\frac{1}{2}\langle m_2^{++}, \mathcal{L} m_2^{++}\rangle + \mathcal{N}_4(m_2^+, m_2^+, m_2^+, m_2^+)\right) t^4 + O(t^5) \\ &=\; \left(-\frac{\pi}{40} + \frac{3\pi}{128}\right) t^4 + O(t^5) \\ &=\; -\frac{\pi}{640} t^4 + O(t^5). \end{aligned} \tag{2.232}$$

Because of the negative quartic term in the energy, the bifurcation is subcritical. This is due to cancellations of the charge density $\sigma$ in the small. In fact, the secondary mode $m^{++}$ lowers the energy; see Figure 2.8. Though the exchange energy rises by the new spectral component, opposite charges are moved closer together, s.t. the stray–field contribution is lowered. The second effect is stronger than the first one, yielding a lower energy.

In spite of the subcriticality of the bifurcation, nontrivial local minimisers of concertina type exist near the former minimiser $m_0$. This is due to the fact that the

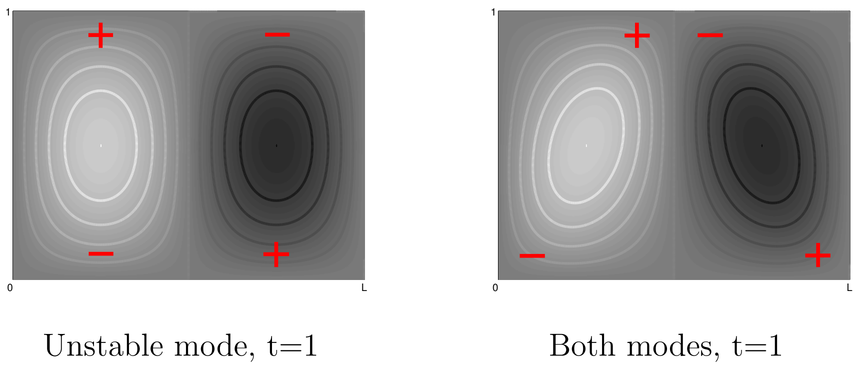


Figure 2.8: Improved dipolar charge screening

scaling limit $\hat{E}_0$ is coercive such that charge cancellations in the small have no consequences in the large, cf. [8]. Numerical simulations, cf. [28], [8], show that the bifurcating subcritical branch exhibits a turning point and thus the energy landscape and the bifurcation are of the form as seen in Figure 2.9, where the continuous branch of local minimisers consists of concertina–type configurations.

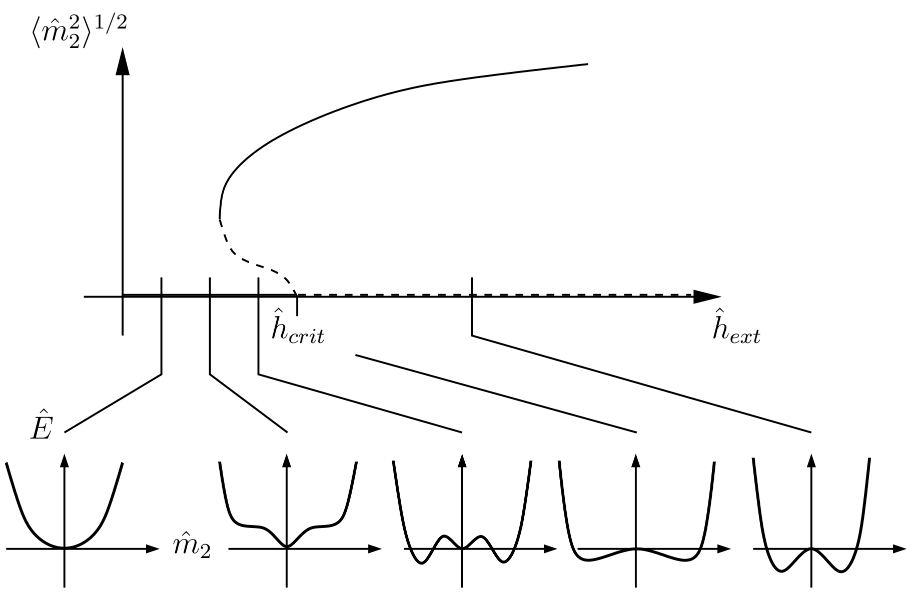


Figure 2.9: Shape of the bifurcation

# Chapter 3

# Néel walls: High energy scaling

In this chapter, we focus on one of the two limiting behaviours of $E_0$, as noted in (2.124), namely a 1-D model for small angle Néel walls in thin films under the influence of an external field. We assume the walls to form an array of periodicity one in $x_1$, neglect any dependence on $x_2$ and arrive at

$$E_0(m) \;=\; \int_{-\frac{1}{2}}^{\frac{1}{2}} |m'|^2 \, dx + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \left|\frac{d}{dx}\right|^{1/2} \left(\tfrac{1}{2}\, m^2\right)\right|^2 dx - h_{ext} \int_{-\frac{1}{2}}^{\frac{1}{2}} m^2 \, dx.$$

We replace $u := \frac{1}{2}m^2$ and have

$$E_0(u) \;=\; \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u'^2}{u}\, dx + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\left|\frac{d}{dx}\right|^{1/2} u\right|^2 dx - 2\, h_{ext} \int_{-\frac{1}{2}}^{\frac{1}{2}} u \, dx.$$

The conditions on $u$ are as follows:

$$u(0) \;=\; 0, \quad u(-x) \;=\; u(x), \quad \text{and} \quad u\left(x + \tfrac{1}{2}\right) \;=\; u(x), \tag{3.1}$$

thus

$$E_0(u) \;=\; \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{u'^2}{u}\, dx + \int_{-\frac{1}{4}}^{\frac{1}{4}} \left|\left|\frac{d}{dx}\right|^{1/2} u\right|^2 dx - 4\, h_{ext} \int_{-\frac{1}{4}}^{\frac{1}{4}} u \, dx.$$

Because of periodicity and symmetry we make the ansatz

$$u(x) \;=\; u_0 + \sum_{n\in\mathbb{N}} u_n \cos(4\pi n x) \;=\; \sum_{n\in\mathbb{N}_0} u_n \cos(4\pi n x),$$

while the condition at $x = 0$ yields

$$0 \;=\; u_0 + \sum_{n\in\mathbb{N}} u_n. \tag{3.2}$$

Note that the coefficients $u_n$ are real-valued.

We now state the main result of this chapter.

**Theorem 3.1.** *For $h_{ext} > 8\pi^2$, we have the following estimate on the energy for an array of small–angle Néel walls:*

$$E_0(m) \;\sim\; -\frac{h_{ext}^2}{2\pi}\ln\left(\frac{h_{ext}}{2\pi^2}\right).$$

This result will follow immediately from a comparison of the lower bound given in Theorem 3.2 and the upper bound given in Theorem 3.3

## 3.1 Lower bound

For a lower bound, it is necessary to gain a certain insight into the effect of each energy contribution as well as the constraint. By means of the Fourier cosine representation, we obtain immediate information about the last two energy contributions: The Zeeman term favors a large mean value of $u$, while the stray–field contribution holds oscillations at bay. The only communication between both energy contributions happens through the constraint (3.2). This communication is too weak, in that it does not prevent singularities of solutions without the influence of the exchange contribution, while the latter has a regularising effect on the problem.

### 3.1.1 Unboundedness without exchange

To illustrate the singular behaviour mentioned above, we consider the sum of stray–field and Zeeman contributions only, neglect the fact that $u$ has to be nonnegative by construction and have in Fourier space

$$\begin{aligned}\tilde{E}_0(u) \;&:=\; \int_{-\frac{1}{4}}^{\frac{1}{4}}\left|\left|\frac{d}{dx}\right|^{1/2}u\right|^2 dx - 4\,h_{ext}\int_{-\frac{1}{4}}^{\frac{1}{4}}u\,dx\\ &=\; \pi\sum_{n\in\mathbb{N}} n\,u_n^2 - 2h_{ext}\,u_0.\end{aligned}\tag{3.3}$$

Now we have

**Lemma 3.1.** *The variational problem of minimising* (3.3) *under the constraint* (3.2) *admits no solution in $L^\infty$.*

PROOF OF LEMMA 3.1:
If we take into account the constraint (3.2), we arrive at

$$\begin{aligned}2\pi\,n\,u_n \;&=\; \lambda\\ -2h_{ext} \;&=\; \lambda\\ \Rightarrow\quad u_n \;&=\; -\frac{h_{ext}}{\pi\,n}.\end{aligned}$$

Neglecting the constant component $u_0$, the function $u(x)$ is then calculated to be

$$\begin{aligned} u(x) &= -\frac{h_{ext}}{\pi} \sum_{n\in\mathbb{N}} \frac{\cos(4\pi n x)}{n} \\ &= \frac{h_{ext}}{\pi} \ln(2\sin(2\pi x)), \end{aligned}$$

for $0 < x \leq \frac{1}{4}$, which has to be extended periodically to

$$u(x) = \frac{h_{ext}}{\pi} \ln(2|\sin(2\pi x)|).$$

The function $u$ diverges for $x = \frac{k}{2}$, $k \in \mathbb{Z}$, cf Figure 3.1, and the requirement that $u(0)$ vanish cannot be met for $u \in L^\infty$. □

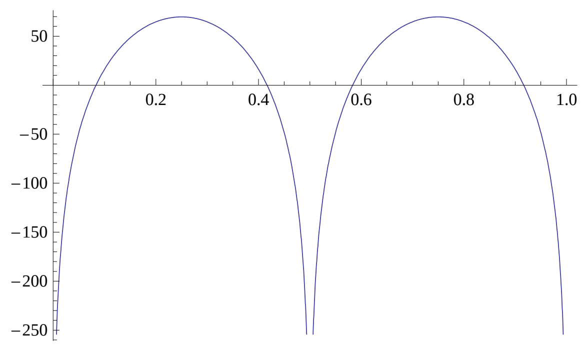


Figure 3.1: Solution to the singular problem, $h_{ext} = 32\pi^2$

### 3.1.2 Introducing the exchange energy

Instead of turning our attention to the full problem, we will try to improve from $L^\infty$–control by means of a different constraint:

$$0 = \int_{-\delta}^{\delta} u\,dx. \tag{3.4}$$

This constraint is much stronger than (3.2), in that it immediately connects the $L^1$–norm to the $H^{1/2}$–norm, as we prove in the following lemma.

**Lemma 3.2.** *For $\frac{1}{2}$–periodic and symmetric functions $u$ fulfilling (3.4) we have*

$$2\pi\langle u\rangle^2 \leq \left(3 + 2\ln\left(\frac{1}{4\pi\delta}\right)\right) ||u||^2_{\dot{H}^{1/2}(I)},$$

*where $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$.*

Proof of Lemma 3.2:
As before, we work with the Fourier cosine representation, whereby (3.4) yields

$$0 \;=\; 2\delta\left[u_0+\sum_{n\in\mathbb{N}}u_n\,\frac{\sin(4\pi n\delta)}{4\pi n\delta}\right],$$

and we have

$$\begin{aligned}\langle u\rangle^2 &= u_0^2\\ &= \left(\sum_{n\in\mathbb{N}}u_n\,\frac{\sin(4\pi n\delta)}{4\pi n\delta}\right)^2\\ &\le \left(\pi\sum_{n\in\mathbb{N}}n\,u_n^2\right)\left(\sum_{n\in\mathbb{N}}\frac{1}{\pi n}\,\frac{\sin^2(4\pi n\delta)}{(4\pi n\delta)^2}\right)\\ &= \left(\sum_{n\in\mathbb{N}}\frac{1}{\pi n}\,\frac{\sin^2(4\pi n\delta)}{(4\pi n\delta)^2}\right)||u||^2_{\dot H^{1/2}(I)}.\end{aligned}$$

The prefactor is estimated as follows:

$$\begin{aligned}\sum_{n\in\mathbb{N}}\frac{1}{\pi n}\,\frac{\sin^2(4\pi n\delta)}{(4\pi n\delta)^2} &\le \sum_{n\in\mathbb{N}}\frac{1}{\pi n}\,\frac{\min\{1,\,(4\pi n\delta)^2\}}{(4\pi n\delta)^2}\\ &\le \frac{1}{\pi}+\frac{1}{\pi}\int_1^\infty\frac{\min\{1,\,(4\pi\delta)^2x^2\}}{(4\pi\delta)^2x^3}\\ &= \frac{1}{\pi}+\frac{1}{\pi}\int_1^{(4\pi\delta)^{-1}}\frac{dx}{x}+\frac{1}{\pi}\int_{(4\pi\delta)^{-1}}^\infty\frac{dx}{(4\pi\delta)^2x^3}\\ &= \frac{3}{2\pi}+\frac{1}{\pi}\ln\left(\frac{1}{4\pi\delta}\right).\end{aligned}$$

□

Now we incorporate our improvised constraint (3.4) into the above estimate.

**Corollary 3.1.** *For $\frac{1}{2}$–periodic and symmetric functions $u$ we have*

$$\frac{2\pi}{\kappa_\delta}\left(\langle u\rangle-\int_{-\delta}^{\delta}u\,dx\right)^2 \;\le\; ||u||^2_{\dot H^{1/2}(I)}$$

*where $I=\left(-\frac{1}{4},\,\frac{1}{4}\right)$ and $\kappa_\delta=3+2\ln\left(\frac{1}{4\pi\delta}\right)$.*

Proof of Corollary 3.1:
With the following notation:

$$u^0 \;=\; u-\int_{-\delta}^{\delta}u\,dx,$$

we have

$$\langle u^0 \rangle \;=\; \langle u \rangle - \int_{-\delta}^{\delta} u\,dx$$

and

$$||u^0||^2_{\dot{H}^{1/2}(I)} \;=\; ||u||^2_{\dot{H}^{1/2}(I)}.$$

□

In a third step, we replace the constraint by the exchange contribution.

**Lemma 3.3.** *For a nonnegative function $u$ satisfying $u(0) = 0$ we have*

$$\int_{-\delta}^{\delta} u\,dx \;\leq\; 4\delta^2 \int_I \frac{|u'|^2}{u}\,dx,$$

*where $(-\delta, \delta) \subset I$.*

Proof of Lemma 3.3:
We have

$$\begin{aligned}\int_{-\delta}^{\delta} u\,dx &= \int_{-\delta}^{\delta}\int_0^x u'(y)\,dy\,dx \\ &\leq \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |u'(y)|\,dy\,dx \\ &= 2\delta \int_{-\delta}^{\delta} |u'|\,dx \\ &\leq c\delta \int_{-\delta}^{\delta} \frac{|u'|^2}{u}\,dx + \frac{\delta}{c}\int_{-\delta}^{\delta} u\,dx,\end{aligned}$$

for arbitrary positive $c$. Thus

$$\int_{-\delta}^{\delta} u\,dx \;\leq\; \frac{c^2\delta}{c-\delta}\int_I \frac{|u'|^2}{u}\,dx.$$

The sharpest bound is realized for $c = 2\delta$, yielding

$$\int_{-\delta}^{\delta} u\,dx \;\leq\; 4\delta^2 \int_I \frac{|u'|^2}{u}\,dx.$$

□

Now we are in the position to give a first estimate for $E_0$.

**Lemma 3.4.** *For a nonnegative $u$ as in (3.1) we have the following lower bound:*

$$E_0(u) \;\geq\; -h_{ext}^2\,\frac{\kappa_\delta}{2\pi} + \left(\frac{1}{4\delta^2} - 2h_{ext}\right)\int_{-\delta}^{\delta} u\,dx,$$

*where $\kappa_\delta = 3 + 2\ln\left(\frac{1}{4\pi\delta}\right)$.*

PROOF OF LEMMA 3.4:
By Lemma 3.3 and Corollary 3.1 we have for $E_0$ that

$$\begin{aligned}
E_0(u) &= \int_I \frac{|u'|^2}{u}\,dx + ||u||^2_{\dot{H}^{1/2}(I)} - 2h_{ext}\langle u\rangle \\
&\geq \frac{1}{4\delta^2}\int_{-\delta}^{\delta} u\,dx + \frac{2\pi}{\kappa_\delta}\left(\langle u\rangle - \int_{-\delta}^{\delta} u\,dx\right)^2 - 2h_{ext}\langle u\rangle \\
&= \frac{1}{4\delta^2}\int_{-\delta}^{\delta} u\,dx + \frac{2\pi}{\kappa_\delta}\left(\langle u\rangle^2 - 2\langle u\rangle\left[\int_{-\delta}^{\delta} u\,dx + h_{ext}\frac{\kappa_\delta}{2\pi}\right] + \left[\int_{-\delta}^{\delta} u\,dx\right]^2\right) \\
&\geq \frac{1}{4\delta^2}\int_{-\delta}^{\delta} u\,dx - h_{ext}^2\frac{\kappa_\delta}{2\pi} - 2h_{ext}\int_{-\delta}^{\delta} u\,dx.
\end{aligned}$$

The last estimate follows by minimisation in $\langle u\rangle$.

□

This finally enables us to give a lower bound.

**Theorem 3.2.** *For a nonnegative $u$ as in (3.1) and $2 < h_{ext}$, we have:*

$$E_0(u) \geq -\frac{h_{ext}^2}{2\pi}\ln\left(\frac{h_{ext}}{2\pi^2}\right) - \frac{3h_{ext}^2}{2\pi}.$$

PROOF OF THEOREM 3.2:
By choice of

$$\delta = \frac{1}{\sqrt{8h_{ext}}}$$

in Lemma 3.4, we have

$$\begin{aligned}
E_0(u) &\geq -\frac{3h_{ext}^2}{2\pi} - \frac{h_{ext}^2}{\pi}\ln\left(\frac{\sqrt{8\,h_{ext}}}{4\pi}\right) \\
&= -\frac{h_{ext}^2}{2\pi}\ln\left(\frac{h_{ext}}{2\pi^2}\right) - \frac{3h_{ext}^2}{2\pi}.
\end{aligned}$$

The condition $2 < h_{ext}$ ensures that $\delta < \frac{1}{4}$, as required in Lemma 3.3. □

## 3.2 Upper Bound

For an upper bound on the energy, we will make use of the insights gained in Subsection 3.1.1. As magnetostatics and the Zeeman contribution have nice Fourier representations, it seems in order to perform a modification on the level of the Fourier coefficients of Subsection 3.1.1, which ensures that the corresponding function is nonnegative. This can be done as follows:

**Lemma 3.5.** *Assume $u^\delta$ has the following Fourier coefficients:*

$$u_n^\delta = -\frac{h_{ext}}{\pi}\frac{1}{n}\frac{1}{(1+\delta)^n}, \quad n \in \mathbb{N}.$$

*Then*

$$u^\delta(x) = \frac{h_{ext}}{2\pi} \ln\left(1 + \frac{4(1+\delta)}{\delta^2}\sin^2(2\pi x)\right).$$

Note that then $u^\delta(0) = 0$, which by (3.2) implies

$$u_0^\delta = -\sum_{n\in\mathbb{N}} u_n^\delta = \frac{h_{ext}}{\pi}\sum_{n\in\mathbb{N}}\frac{1}{n}\frac{1}{(1+\delta)^n} = \frac{h_{ext}}{\pi}\ln\left(1+\frac{1}{\delta}\right). \tag{3.5}$$

Proof of Lemma 3.5:
We have, cf. [14]:

$$\begin{aligned}
-\frac{\pi}{n}\frac{1}{(1+\delta)^n} &= \int_0^\pi \ln(1-2(1+\delta)\cos(x)+(1+\delta)^2)\,\cos(nx)\,dx \\
&= 4\pi\int_0^{\frac14} \ln(1-2(1+\delta)\cos(4\pi x)+(1+\delta)^2)\,\cos(4\pi nx)\,dx \\
&= 2\pi\int_{-\frac14}^{\frac14} \ln(\delta^2+4(1+\delta)\sin^2(2\pi x))\,\cos(4\pi nx)\,dx \\
&= 2\pi\int_{-\frac14}^{\frac14} \ln\left(1+\frac{4(1+\delta)}{\delta^2}\sin^2(2\pi x)\right)\cos(4\pi nx)\,dx,
\end{aligned}$$

and thus

$$u_n^\delta = \frac{2h_{ext}}{\pi}\int_{-\frac14}^{\frac14} \ln\left(1+\frac{4(1+\delta)}{\delta^2}\sin^2(2\pi x)\right)\cos(4\pi nx)\,dx.$$

Note that we have for $n \in \mathbb{N}$:

$$\begin{aligned}
u_n^\delta &= 4\,u_n^\delta\int_{-\frac14}^{\frac14}\cos^2(4\pi nx)\,dx \\
&= 4\sum_{m\in\mathbb{N}_0} u_m^\delta\int_{-\frac14}^{\frac14}\cos(4\pi nx)\,\cos(4\pi mx)\,dx \\
&= 4\int_{-\frac14}^{\frac14}\cos(4\pi nx)\left[\sum_{m\in\mathbb{N}_0} u_m^\delta\cos(4\pi mx)\right]dx \\
&= 4\int_{-\frac14}^{\frac14} u^\delta(x)\,\cos(4\pi nx)\,dx.
\end{aligned}$$

□

For a connection between $u$ and $u^\delta$ in real space, note that

$$u^\delta(x) \;=\; \frac{h_{ext}}{2\pi}\,\left[\ln\left(\delta^2+4(1+\delta)\sin^2(2\pi x)\right)-\ln\left(\delta^2\right)\right]$$

and that

$$\lim_{\delta\downarrow 0}\frac{h_{ext}}{2\pi}\,\ln\left(\delta^2+4(1+\delta)\sin^2(2\pi x)\right) \;=\; \frac{h_{ext}}{\pi}\,\ln(2|\sin(2\pi x)|),$$

which is the solution for of the variational problem without exchange, not in $L^\infty$, as calculated in Subsection 3.1.1. The second part of $u^\delta$ is necessary for the condition $u^\delta(0)=0$ and diverges as $\delta\to 0$.

With the help of the Fourier coefficients, we can calculate both the stray–field and the Zeeman contributions with some ease.

**Lemma 3.6.** *The sum of stray–field and Zeeman contributions for $u^\delta$ is*

$$E_{SZ}(u^\delta) \;=\; \frac{h_{ext}^2}{\pi}\,\ln\left(\frac{\delta}{2+\delta}\right).$$

Proof of Lemma 3.6:
For the Zeeman energy we have with (3.5):

$$-4h_{ext}\int_{-\frac14}^{\frac14}u^\delta(x)\,dx \;=\; -2h_{ext}\,u_0^\delta \;=\; -\frac{2\,h_{ext}^2}{\pi}\,\ln\left(1+\frac{1}{\delta}\right).$$

The stray–field contribution amounts to:

$$\begin{aligned}\int_{-\frac14}^{\frac14}\left||\partial_1|^{-1/2}u^\delta\right|\,dx \;&=\; \pi\sum_{n\in\mathbb{N}}n\,|u_n^\delta|^2\\ &=\; \frac{h_{ext}^2}{\pi^2}\sum_{n\in\mathbb{N}}\frac{\pi}{n}\frac{1}{(1+\delta)^{2n}}\\ &=\; -\frac{h_{ext}^2}{\pi}\ln\left(\frac{\delta(2+\delta)}{(1+\delta)^2}\right).\end{aligned}$$

The sum of both contributions yields the desired result:

$$E_{SZ}(u^\delta) \;=\; -\frac{h_{ext}^2}{\pi}\ln\left(\frac{(1+\delta)^2}{\delta^2}\,\frac{\delta(2+\delta)}{(1+\delta)^2}\right) \;=\; \frac{h_{ext}^2}{\pi}\,\ln\left(\frac{\delta}{2+\delta}\right)$$

□

The exchange energy cannot be calculated in such a straightforward manner, yet as we expect it to be of little effect, we only aim for an appropriate $L^\infty$ estimate. To simplify notation, we adopt the following convention:

$$C_\delta \;=\; \frac{4(1+\delta)}{\delta^2},$$

such that

$$u^\delta(x) \;=\; \frac{h_{ext}}{2\pi}\,\ln\left(1 + C_\delta\,\sin^2(2\pi x)\right).$$

Then

$$\begin{aligned}
(u^\delta)'(x) &= 2h_{ext}\,\frac{C_\delta\,\sin(2\pi x)\,\cos(2\pi x)}{1 + C_\delta\,\sin^2(2\pi x)}\\
\frac{|(u^\delta)'(x)|^2}{|u^\delta(x)|} &= 8\pi\,h_{ext}\,\frac{C_\delta^2\,\sin^2(2\pi x)\,\cos^2(2\pi x)}{[1 + C_\delta\,\sin^2(2\pi x)]^2\,\ln\left(1 + C_\delta\,\sin^2(2\pi x)\right)}\\
\lim_{x\to 0}\frac{|(u^\delta)'(x)|^2}{|u^\delta(x)|} &= 8\pi\,h_{ext}\,C_\delta.
\end{aligned}$$

Now we go for the estimate:

**Lemma 3.7.**

$$\max_{[0,\frac{1}{4}]}\frac{|(u^\delta)'(x)|^2}{u^\delta(x)} \;=\; 8\pi\,h_{ext}\,C_\delta.$$

Proof of Lemma 3.7:
With the redefinition:

$$y \;:=\; C_\delta\,\sin^2(2\pi x) \in [0,\,C_\delta],$$

we have to ascertain the nonnegativity of the difference:

$$\begin{aligned}
\Delta(y) &= 8\pi\,h_{ext}\,C_\delta - 8\pi\,h_{ext}\,\frac{y\,(C_\delta - y)}{(1+y)^2\,\ln(1+y)}\\
&= 8\pi\,h_{ext}\frac{C_\delta(1+y)^2\ln(1+y) - y\,(C_\delta - y)}{(1+y)^2\,\ln(1+y)}
\end{aligned}$$

This is equivalent to showing nonnegativity of the numerator

$$f(y) \;=\; C_\delta\,(1+y)^2\ln(1+y) - y(C_\delta - y),$$

as the denominator is positive on $(0,\,C_\delta]$, while $\Delta(0) = 0$. For $f(y)$ we have

$$\begin{aligned}
f(0) &= 0\\
f'(y) &= 2C_\delta\,(1+y)\ln(1+y) + y(C_\delta + 2) \;\geq\; 0 \;\text{ on }\; (0,\,C_\delta],
\end{aligned}$$

which completes the proof. □

We are now in the position to give an upper bound.

**Theorem 3.3.**

$$E_0(u^\delta) \;\leq\; -\frac{h_{ext}^2}{2\pi}\ln\left(\frac{h_{ext}}{2\pi^2}\right) + \frac{h_{ext}^2}{2\pi}(1 + \ln 4) - 4\pi h_{ext}$$

Proof of Theorem 3.3.

$$
\begin{aligned}
E_0^\delta(u) &= E_0\left(u^\delta\right) \\
&= 2\int_0^{\frac{1}{4}} \frac{|(u^\delta)'(x)|^2}{|u^\delta(x)|}\,dx + \frac{h_{ext}^2}{\pi}\ \ln\left(\frac{\delta}{2+\delta}\right) \\
&\leq 4\pi\, h_{ext}\, C_\delta + \frac{h_{ext}^2}{\pi}\ \ln\left(\frac{\delta}{2+\delta}\right) \\
&= 16\pi\, h_{ext}\,\frac{1+\delta}{\delta^2} + \frac{h_{ext}^2}{\pi}\ \ln\left(\frac{\delta}{2+\delta}\right),
\end{aligned}
$$

A minimisation of this upper bound in $\delta$ yields:

$$
\begin{aligned}
0 &\overset{!}{=} -16\pi\, h_{ext}\frac{\delta+2}{\delta^3} + \frac{2h_{ext}^2}{\pi}\frac{1}{\delta(\delta+2)} \\
\delta &= 2\frac{\sqrt{8\pi^2}}{\sqrt{h_{ext}} - \sqrt{8\pi^2}},
\end{aligned}
$$

valid for $h_{ext} > 8\pi^2$, the other solution being negative and thus not appropriate. The corresponding upper bound on $E_\delta$ is

$$
\begin{aligned}
E_0^\delta(u) &\leq 16\pi h_{ext}\frac{h_{ext}-8\pi^2}{32\pi^2} - \frac{h_{ext}^2}{2\pi}\ln\left(\frac{h_{ext}}{8\pi^2}\right) \\
&= -\frac{h_{ext}^2}{2\pi}\ln\left(\frac{h_{ext}}{2\pi^2}\right) + \frac{h_{ext}^2}{2\pi}(1+\ln 4) - 4\pi h_{ext}.
\end{aligned}
$$

□

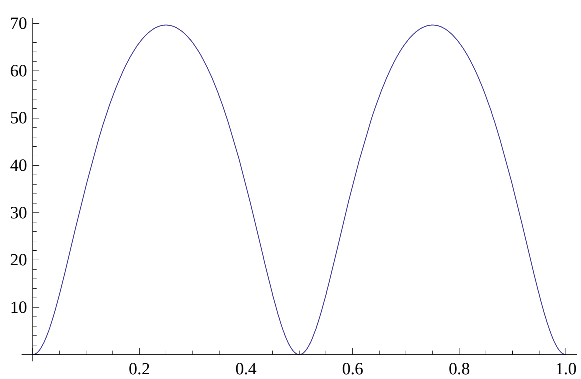


Figure 3.2: Trial function $u^\delta$ for $\delta = 2$, $h_{ext} = 32\pi^2$

# Chapter 4

# Vortex stability

In this chapter, we will investigate a variant of the Ginzburg–Landau energy without magnetic fields for a magnetisation $m : B_1(0) \mapsto B_1(0)$, where we include a boundary penalty as used in [20]

$$E(m) \;=\; \int_{B_1(0)} \left[\frac{1}{2}|\nabla m|^2 + \frac{1}{4\delta^2}(1-|m|^2)^2\right] dx + \frac{1}{2\varepsilon}\int_{\partial B_1(0)} (m\cdot\nu)^2 \, d\mathcal{H}^1. \quad (4.1)$$

Thus we impose no Dirichlet boundary condition on $m$, in which we deviate from the vast number of articles that follow the book by Bethuel, Brezis and Hélein, [4], in which the boundary condition is that

$$m|_{\partial B_1(0)} \;=\; g, \quad g : \partial B_1(0) \mapsto \mathbb{S}^1. \quad (4.2)$$

We will be interested in the case where $\delta \ll 1$ and $\varepsilon \ll \frac{1}{|\ln \delta|}$. In fact, we will start by studying

$$E^0(m) \;=\; \int_{B_1(0)} \left[\frac{1}{2}|\nabla m|^2 + \frac{1}{4\delta^2}(1-|m|^2)^2\right] dx, \quad m\cdot\nu \;=\; 0, \quad (4.3)$$

which corresponds to the limit $\varepsilon \to 0$ and then consider the more general case (4.1), for $\varepsilon > 0$. Our goal is to investigate the stability of the radially symmetric one–vortex solution corresponding to this functional, first for the case (4.3), then for (4.1).

For $E^0(m)$ with Dirichlet data $g(\theta) = e^{id\theta}$ of degree $d \in \mathbb{N}$ on $\partial B_1(0)$, it is known that radially symmetric solutions exist, see [4]. We call such solutions central $d$–vortices. They are unique, see [16]. The question of stability for such central $d$–vortices has been addressed by several authors, and we briefly summarise the main results. Lieb and Loss, [21], show that the radially symmetric solution is weakly stable for $d = 1$, i.e., that the spectrum of the Hessian at the radial solution is bounded below by zero for solutions of degree one. Mironescu, [25], shows that the spectrum of the Hessian is strictly positive in the case of radial solutions of degree one, while it has negative components for degree larger than one, if $\delta$ is small. In [22], Lin considers the same problem for degree one and shows that that of the two relevant eigenvalues identified by Mironescu, see Section 4.3, one is bounded below by a constant independent of the parameter $\delta$, while the other one converges to zero with $\delta$. Finally, in [23], Lin considers the problem for perturbations of the radial solution of degree one on $\mathbb{R}^2$ in $H^1_0(B_1(0), \mathbb{C})$, and proves that the smallest eigenvalue converges to zero with $\delta$ as $O(|\ln\delta|^{-1})$. Moreover, he shows that the eigenfunctions corresponding to this eigenvalue are close in $L^2$ to the generators of translations of the radial solution.

We now motivate our version of the Ginzburg–Landau functional by a reduction of the original micromagnetic energy

$$E(m) \;=\; d^2 \int_\Omega |\nabla m|^2\,dx + \int_{\mathbb{R}^3} |\nabla U|^2\,dx \tag{4.4}$$

to our case, in which the boundary penalty appears as a natural term.

## 4.1 Motivation and main result

In order to reduce the micromagnetic functional (4.4) to the one we intend to work with, we will start from a thin–film cylinder

$$\Omega \;=\; B_1(0)\times[0,t], \quad \text{with } t \ll 1,$$

where we nondimensionalise all lengths by the radius of the cylinder. We assume that $m$ does not vary in the thickness direction, i.e.,

$$m(x', x_3) \;=\; m(x'),$$

which implies that the stray–field contribution falls into three parts, due to in–plane volume charges, lateral boundary charges and vertical boundary charges, respectively. With $m = (m', m_3)$, we have formally, cf. [19], that

$$\begin{aligned} E(m', m_3) \;=\; & d^2 t \int_{B_1(0)} (|\nabla' m'|^2 + |\nabla' m_3|^2)\,dx' + t^2 ||\nabla'\cdot m'||^2_{H^{-1/2}} \\ & + \frac{t^2|\ln t|}{2\pi} \int_{\partial B_1(0)} (m'\cdot\nu')^2\,d\mathcal{H}^1 + t\int_{B_1(0)} m_3^2\,dx'. \end{aligned}$$

We now simplify the functional by neglecting the Dirichlet integral of $m_3$, which changes the local structure of a vortex solution but is of little importance far from the vortex location, see [15]. Now we use the fact that $|m| = 1$, thus $|m'|^2 + m_3^2 = 1$, and drop the primes:

$$\begin{aligned} E(m) \;=\; & d^2 t \int_{B_1(0)} |\nabla m|^2\,dx + t^2 ||\nabla\cdot m||^2_{H^{-1/2}} \\ & + \frac{t^2|\ln t|}{2\pi} \int_{\partial B_1(0)} (m\cdot\nu)^2\,d\mathcal{H}^1 + t\int_{B_1(0)} (1-|m|^2)\,dx. \end{aligned}$$

We note that the divergence in the $H^{-1/2}$–norm is the regular part of the distributional divergence, continued by zero outside of $B_1(0)$, i.e. $(\nabla \cdot m)\chi_{B_1(0)}$, while the singular part is given by the boundary term. We rename

$$\delta = \frac{d}{\sqrt{2}},$$

and thus we may start from

$$\begin{aligned} E(m) &= 2\delta^2 t \int_{B_1(0)} |\nabla m|^2 \, dx + t^2 ||\nabla \cdot m||^2_{H^{-1/2}} \\ &\quad + \frac{t^2 |\ln t|}{2\pi} \int_{\partial B_1(0)} (m \cdot \nu)^2 \, d\mathcal{H}^1 + t \int_{B_1(0)} (1 - |m|^2) \, dx, \end{aligned}$$

divide by $4\delta^2 t$ and arrive at

$$\begin{aligned} E(m) &= \frac{1}{2} \int_{B_1(0)} |\nabla m|^2 \, dx + \frac{t}{4\delta^2} ||\nabla \cdot m||^2_{H^{-1/2}} \\ &\quad + \frac{t |\ln t|}{8\pi\delta^2} \int_{\partial B_1(0)} (m \cdot \nu)^2 \, d\mathcal{H}^1 + \frac{1}{4\delta^2} \int_{B_1(0)} (1 - |m|^2) \, dx. \end{aligned}$$

For the purpose of the stability analysis of the central vortex solution, we neglect the divergence contribution. This can be done, as the central vortex solution is divergence–free and thus still a stationary point of the functional with divergence term, while the divergence contribution only has the effect to improve stability of the central vortex solution. Then we rename

$$\varepsilon = \frac{4\pi\delta^2}{t|\ln t|}$$

and have

$$E(m) = \int_{B_1(0)} \left[ \frac{1}{2} |\nabla m|^2 + \frac{1}{4\delta^2} (1 - |m|^2) \right] dx + \frac{1}{2\varepsilon} \int_{\partial B_1(0)} (m \cdot \nu)^2 \, d\mathcal{H}^1. \quad (4.5)$$

Note that by the unit–length constraint on the original magnetisation, we are limited to in–plane magnetisations with

$$m : B_1(0) \mapsto B_1(0).$$

In order to incorporate this requirement without dealing with an extra constraint, we modify the functional further by squaring the volume potential term without the prefactor:

$$E(m) = \int_{B_1(0)} \left[ \frac{1}{2} |\nabla m|^2 + \frac{1}{4\delta^2} (1 - |m|^2)^2 \right] dx + \frac{1}{2\varepsilon} \int_{\partial B_1(0)} (m \cdot \nu)^2 \, d\mathcal{H}^1. \quad (4.6)$$

This is a variant of a model problem introduced by Moser, [26], where the difference lies in the fact that the parameters $\delta$ and $\varepsilon$ are independent of each other. Both $\delta$

and $\varepsilon$ will stay fixed, be positive and not converge to zero. The case of tangential boundary conditions, though, which we will also treat, corresponds to the limit $\varepsilon \to 0$.

The main result of this chapter is the following theorem.

**Theorem 4.1.** *Assume that the Rayleigh quotient of the Hessian of $E(m)$ as given in (4.6) evaluated at the central vortex solution $m = m_0$ is bounded below as $\frac{1}{|\ln\delta|}$ for variations with zero boundary data. Furthermore, assume that $\varepsilon \ll \frac{1}{|\ln\delta|}$. Then the central one–vortex solution is stable, in the sense that the Hessian of $E(m)$ evaluated at $m = m_0$ is positive definite.*

This theorem will follow immediately from Theorem 4.3. The assumption on the lower bound for the Hessian in the case of zero boundary data will turn into assumption (4.28). We emphasise the fact that this theorem differs from the aforementioned results by including a boundary penalty which naturally results from a reduction of the micromagnetic energy functional, and is thus more meaningful than a Dirichlet boundary condition of unit length.

## 4.2 The central vortex solution

Before analysing the stability of the central vortex solution $m_0$, we will collect some qualitative properties of $m_0$ itself. We start with the Euler–Lagrange equation for (4.1):

$$\Delta m + \frac{1}{\delta^2} m(1 - |m|^2) = 0 \quad \text{in } B_1(0) \tag{4.7}$$

$$\partial_\nu m + \frac{1}{\varepsilon}(m \cdot \nu)\nu = 0 \quad \text{on } \partial B_1(0), \tag{4.8}$$

and make the following ansatz for a central vortex

$$m_0(r, \varphi) = i\varrho(r)e^{i\varphi}, \quad \varrho(0) = 0, \tag{4.9}$$

which, when substituted into (4.7), yields

$$\varrho''(r) + \frac{1}{r}\varrho'(r) - \frac{1}{r^2}\varrho(r) + \frac{1}{\delta^2}\varrho(r)(1 - \varrho^2(r)) = 0 \tag{4.10}$$

and by (4.8), the boundary conditions

$$\varrho(0) = 0 \quad \text{and} \quad \varrho'(1) = 0. \tag{4.11}$$

By the results of [16] there exists a monotonic increasing profile $\varrho_\delta(r)$, solving

$$\varrho_\delta'' + \frac{1}{r}\varrho_\delta' - \frac{1}{r^2}\varrho_\delta + \varrho_\delta(1 - \varrho_\delta^2) = 0 \quad \text{on } \left(0, \frac{1}{\delta}\right) \tag{4.12}$$

with the corresponding boundary conditions

$$\varrho_\delta(0) \;=\; 0 \qquad \text{and} \qquad \varrho_\delta'\left(\frac{1}{\delta}\right) \;=\; 0. \tag{4.13}$$

We will show that such a profile can only exist for $\delta < 1$. By continuation of the profile $\varrho_\delta$ beyond $r = \frac{1}{\delta}$, we know it to be one of the oscillating solutions lying below the unique monotonic increasing solution of the problem

$$f'' + \frac{1}{r} f' - \frac{1}{r^2} f + f\,(1 - f^2), \quad f(0) \;=\; 0, \quad \lim_{r\to\infty} f(r) \;=\; 1.$$

Moreover we know it to assume its first local maximum at $r = \frac{1}{\delta}$, which characterises it completely, according to [16]. The function $\varrho$ is related to $\varrho_\delta$ by rescaling of the argument, i.e. $\varrho_\delta\left(\frac{r}{\delta}\right) = \varrho(r)$.

We continue with an upper bound on the energy of $m_0$, which is well–known in the literature, but for which we give a proof for the commodity of the reader. Constants $C$ appearing in any of the proofs in this chapter are independent of $\delta$, for $\delta$ small enough.

**Lemma 4.1.** *Let $\delta < 1$. Then we have the following upper bound on the energy:*

$$E(m_0) \;\leq\; \pi|\ln\delta| + C. \tag{4.14}$$

Proof of Lemma 4.1.
We start with rewriting $E(m_0)$ in terms of $\varrho$:

$$\begin{aligned} E(m_0) \;&=\; \int_{B_1(0)} \left[\frac{1}{2}|\nabla m_0|^2 + \frac{1}{4\delta^2}(1-|m_0|^2)^2\right] dx + \frac{1}{2\varepsilon}\int_{\partial B_1(0)} (m_0\cdot\nu)^2\, d\mathcal{H}^1 \\ &=\; \pi\int_0^1 \left[\varrho'^2 + \frac{1}{r^2}\varrho^2 + \frac{1}{2\delta^2}(1-\varrho^2)^2\right] r\,dr \;=:\; E(\varrho). \end{aligned} \tag{4.15}$$

Then we insert the following profile

$$\varrho^+(r) \;=\; \min\left\{\frac{r}{\delta}, 1\right\}.$$

As $\varrho$ minimises $E(\varrho)$, we obtain an upper bound by insertion of $\varrho^+$.

$$\begin{aligned} E(\varrho) \;\leq\; E(\varrho^+) \;&=\; \pi\int_0^\delta \left[\frac{2}{\delta^2} + \frac{1}{2\delta^2}\left(1 - \frac{r^2}{\delta^2}\right)^2\right] r\,dr + \pi\ln\frac{1}{\delta} \\ &=\; \pi\int_0^1 \left[2 + \frac{1}{2}(1-x^2)^2\right] x\,dx + \pi\ln\frac{1}{\delta} \\ &=\; C + \pi|\ln\delta|. \end{aligned}$$

□

We are now in the position to prove bounds on $\varrho^2(1)$:

**Lemma 4.2.** *Consider a monontonic increasing positive profile $\varrho$.*

*(i) The profile can only exist for $\delta < 1$, and*

$$\varrho^2(1) \;\leq\; 1 - \delta^2.$$

*(ii) The profile satisfies the lower bound*

$$\varrho^2(1) \;\geq\; 1 - 2\delta\sqrt{C + |\ln\delta|}.$$

Proof of Lemma 4.2:

(i) For the upper bound, we recall that $\varrho$ has a maximum at $r = 1$, which implies

$$\begin{aligned} 0 &= \varrho''(1) + \varrho'(1) - \varrho(1) + \frac{1}{\delta^2}\,\varrho(1)\,(1 - \varrho^2(1)) \\ &\leq -\varrho(1) + \frac{1}{\delta^2}\,\varrho(1)\,(1 - \varrho^2(1)) \\ &= \frac{1}{\delta^2}\,\varrho(1)\,\left(1 - \delta^2 - \varrho^2(1)\right). \end{aligned} \tag{4.16}$$

Positivity of $\varrho(1)$ in (4.16) implies the upper bound. Again, by positivity, we deduce that $\delta < 1$ is necessary.

(ii) Since $\varrho(r)$ is monotonic increasing, we have that

$$\frac{\pi}{4\delta^2}(1 - \varrho^2(1))^2 \;\leq\; \frac{1}{4\delta^2}\int_{B_1(0)} (1 - |m_0|^2)^2\,dx \;\leq\; \pi|\ln\delta| + C.$$

Solving for $\varrho^2(1)$ yields the result. □

We will improve on the lower bound in Corollary 4.1, after first collecting some more information on $\varrho$. The next lemma provides us with a comparison function.

**Lemma 4.3.** *The radial profile $\varrho$ lies above the linear function*

$$g(r) \;=\; \varrho(1)\,r.$$

Proof of Lemma 4.3:
We note that $g(r)$ fulfils the differential equation

$$g'' + \frac{1}{r}g' - \frac{1}{r^2}g \;=\; 0,$$

while $\varrho$ has the same boundary data, yet fulfils

$$\varrho'' + \frac{1}{r}\varrho' - \frac{1}{r^2}\varrho \;=\; -\frac{2}{\delta^2}\varrho(1 - \varrho^2) \;<\; 0.$$

Thus we have by the maximum principle that $g(r) \;\leq\; \varrho(r)$. □

The last two lemmata enable us to find sets on which the profile $\varrho(r)$ is larger than a given value, provided $\delta$ is small enough, a fact which we will need in the analysis to follow.

Due to the upper bound from Lemma 4.1, we have energy concentration on a small ball around the vortex location, i.e., the origin, see e.g., [18], which in turn yields for some $r_0 > 0$, which we may set to $r_0 = \frac{1}{4}$ for $\delta$ small enough:

$$\int_{B_1(0)\setminus B_{r_0}(0)} |\nabla m_0|^2 \, dx \;\leq\; C.$$

With this bound we can derive a pointwise bound on the squared gradient, due to the fact that it is radially decreasing.

**Lemma 4.4.**

$$\partial_r |\nabla m_0|^2 \;<\; 0.$$

Proof of Lemma 4.4.
We differentiate and have

$$\begin{aligned}
\partial_r |\nabla m_0|^2 \quad &= \quad \left(\varrho'^2 + \frac{1}{r^2}\varrho^2\right)' \\
&= \quad 2\varrho'\varrho'' + \frac{2}{r^2}\varrho\varrho' - \frac{2}{r^3}\varrho^2 \\
&\overset{(4.10)}{=} \quad -\frac{2}{r}\varrho'^2 + \frac{4}{r^2}\varrho\varrho' - \frac{2}{r^3}\varrho^2 - \frac{2}{\delta^2}\varrho\varrho'(1-\varrho^2) \\
&= \quad -\frac{2}{r}\left(\varrho' - \frac{1}{r}\varrho\right)^2 - \frac{2}{\delta^2}\varrho\varrho'(1-\varrho^2) \\
&< \quad 0,
\end{aligned}$$

by monotonicity of $\varrho$ and $\varrho \leq 1$. □

With this and the above bound, we conclude that

$$\begin{aligned}
|\nabla m_0|^2(r)|B_r(0) \setminus B_{\frac{1}{4}}(0)| \quad &\leq \quad \int_{B_r(0)\setminus B_{\frac{1}{4}}(0)} |\nabla m_0|^2 \, dx \;\leq\; C \\
\Rightarrow \quad |\nabla m_0|^2(r) \quad &\leq \quad \frac{C}{|B_r(0) \setminus B_{\frac{1}{4}}(0)|}
\end{aligned} \tag{4.17}$$

This enables us to give bounds on the quantity

$$\frac{1}{\delta^2}(1 - \varrho^2(r)).$$

We will need these bounds for the proof of interior estimates and estimates up to the boundary in Subsections 4.4.2 and 4.5.2. In order to achieve such bounds, we prove the following lemma.

**Lemma 4.5.** *Let the following problem be given:*

$$\left\{ \begin{array}{rcl} -\delta^2\Delta w + w &=& 0 \quad on \quad B_1(0)\setminus B_R(0) \\ w &=& 1 \quad on \quad \partial B_R(0) \\ \partial_r w &=& 0 \quad on \quad \partial B_1(0) \end{array} \right\}$$

*Then a supersolution to this problem is given by*

$$w^+(r) \;=\; e^{\frac{1}{2\delta}(r^2-2r+2R-R^2)} \;=\; e^{\frac{1}{2\delta}((r-1)^2-(R-1)^2)}$$

*for* $\delta < 2R-1$.

Proof of Lemma 4.5.
First of all, the boundary data are fulfilled. Moreover, to show that $w^+$ is a supersolution, we note that

$$\begin{aligned} -\delta^2\Delta w^+ + w^+ &= -\delta^2\left((w^+)'' + \frac{1}{r}(w^+)'\right) + w^+ \\ &= -\delta^2\left(\frac{(r-1)^2}{\delta^2} + \frac{1}{\delta} + \frac{r-1}{r\delta}\right) w^+ + w^+ \\ &= w^+\left(-(r-1)^2 - 2\delta + \frac{\delta}{r} + 1\right) \\ &= w^+\left(2r - r^2 - 2\delta + \frac{\delta}{r}\right) \\ &\geq w^+(2R-1-\delta) \end{aligned}$$

which is positive for $\delta < 2R-1$. □

We note that we need to have $R > \frac{1}{2}$ for Lemma 4.5 to be meaningful. Thus we restrict the following argument to radii in $[\frac{2}{3}, 1]$. Now using Lemma 4.5, we can bound $\frac{1}{\delta^2}(1-\varrho^2(r))$ from above.

**Lemma 4.6.** *On the set* $\{r \mid \varrho(r) > \frac{1}{\sqrt{2}}\} \cap \left(\frac{2}{3}, 1\right]$, *we have for* $\delta$ *small enough that*

$$\frac{1}{\delta^2}(1-\varrho^2(r)) \;\leq\; C.$$

Proof of Lemma 4.6.
We define $v := 1 - |m_0|^2 = 1 - \varrho^2(r)$, and have

$$\begin{aligned} -\delta^2\Delta v &= -\delta^2\Delta(1-|m_0|^2) \\ &= 2\delta^2\nabla\cdot(m_0\nabla m_0) \\ &= 2\delta^2 m_0\Delta m_0 - 2\delta^2|\nabla m_0|^2 \\ &\overset{(4.7)}{=} -2|m_0|^2(1-|m_0|^2) + 2\delta^2|\nabla m_0|^2 \\ &= -2\varrho^2 v + 2\delta^2|\nabla m_0|^2 \end{aligned}$$

which yields the equation

$$-\delta^2 \Delta v + 2\varrho^2 v \;=\; 2\delta^2 |\nabla m_0|^2,$$

with $v \leq 1$ and

$$\partial_r v(1) \;=\; \partial_r (1 - |m_0|^2)(1) \;=\; (1-\varrho^2)'(1) \;=\; -2\varrho(1)\,\varrho'(1) \;=\; 0.$$

If we restrict our attention to the set of radii for which $\varrho(r) \geq \frac{1}{\sqrt{2}}$ and use the estimate on the gradient, we have that

$$-\delta^2 \Delta v + v \;\leq\; C\delta^2.$$

We now consider the function $w_- = v - C\delta^2$ and note that

$$-\delta^2 \Delta w_- + w_- \;\leq\; 0, \quad \text{with} \quad w_-(r) \;\leq\; 1 \quad \text{and} \quad \partial_r w_-(1) \;=\; 0.$$

Thus we can refer to Lemma 4.5 and choose $R$ as the maximum of the radius for which $\varrho(r) = \frac{1}{\sqrt{2}}$ and $\frac{2}{3}$. We now use our supersolution $w^+$ for $\delta$ small enough and $r > R$ with a strictly negative exponent $-\mu(r) := (r-1)^2 - (R-1)^2$. By Lemma 4.5 we have that

$$\begin{aligned} w_-(r) \;&\leq\; w^+(r) \\ v(r) \;&\leq\; C\delta^2 + w^+(r) \\ \frac{1}{\delta^2}(1-\varrho^2(r)) \;&\leq\; C + \frac{1}{\delta^2} e^{-\frac{\mu(r)}{2\delta}}, \end{aligned}$$

which is bounded from above by a constant as long as we stay away from $r = R$, by maximising the right–hand side in $\delta$. Thus we have that

$$\frac{1}{\delta^2}(1-\varrho^2(r)) \;\leq\; C. \tag{4.18}$$

□

Now we can give an improved lower bound on $\varrho^2(1)$.

**Corollary 4.1.** *For the profile $\varrho$, we have the following improved lower bound:*

$$\varrho^2(1) \;\geq\; 1 - C\,\delta^2.$$

Lemma 4.3 and Corollary 4.1 enable us to prove two further corollaries. The first one is needed for the construction of suitable localisation functions in Section 4.4.3.

**Corollary 4.2.** *For $\delta$ small enough it is possible to choose radii $a < b < 1$ such that*

$$a \;<\; 1 - 2\delta \quad \text{and} \quad 3\varrho^2 - 1 \;\geq\; \frac{3}{2} \quad \text{for } r \;\geq\; a,$$

*while $b - a$ and $1 - b$ are bounded below by constants not depending on $\delta$.*

Proof of Corollary 4.2.
To achieve the bound on $3\varrho^2-1$, we combine Lemma 4.3 with Corollary 4.1. Thereby, it suffices to have

$$\begin{aligned} 3r^2(1-C\delta^2)-1 &\geq \frac{3}{2} \\ \Rightarrow \quad r &\geq \sqrt{\frac{5}{6(1-C\delta^2)}} \end{aligned}$$

which is smaller than $1-2\delta$ for $\delta$ small enough.

Finally, we note that the radius $a$ stays away macroscopically from $r=1$ for small $\delta$ (e.g., for $C\delta^2 < \frac{1}{12}$ we have the bound $a < \sqrt{\frac{10}{11}}$ ), thus it is indeed possible to find a radius $b<1$ for which $b-a$ and $1-b$ are bounded below by constants independent of $\delta$. □

The second corollary makes the set on which Lemma 4.6 is active more accessible:

**Corollary 4.3.** *The interval* $[\frac{3}{4},1]$ *is contained in the set* $\{\varrho(r) > \frac{1}{\sqrt{2}}\} \cap (\frac{2}{3},1]$ *for* $\delta$ *small enough.*

Proof of Corollary 4.3:
We note that by Lemma 4.3 and Corollary 4.1, it suffices to have

$$\begin{aligned} r^2(1-C\delta^2) &> \frac{1}{2} \\ \Rightarrow \quad r &> \sqrt{\frac{1}{2(1-C\delta^2)}} \end{aligned}$$

which is smaller than $\frac{3}{4}$ for $\delta$ small enough. □

Thus we may consider the interval $[\frac{3}{4},1]$ for all future purposes in which bounds are needed on $\frac{1}{\delta^2}(1-\varrho^2)$.

## 4.3 Linearisation about the radial profile

In order to analyse the stability of the central vortex solution, we linearise the Ginzburg–Landau energy (4.1) about this solution, i.e., we calculate the Hessian

$$\begin{aligned} D^2E(m_0)(\zeta,\zeta) &= \int_{B_1(0)} \left[ |\nabla\zeta|^2 - \frac{1}{\delta^2}|\zeta|^2(1-|m_0|^2) + \frac{2}{\delta^2}(m_0\cdot\zeta)^2 \right] dx \\ &\quad + \frac{1}{\varepsilon}\int_{\partial B_1(0)} (\zeta\cdot\nu)^2 \, d\mathcal{H}^1, \end{aligned} \tag{4.19}$$

which is the quadratic form corresponding to the linearisation $L$ of the Ginzburg–Landau operator

$$\mathcal{F}m = \Delta m + \frac{1}{\delta^2}(1-|m|^2)m$$

about the central vortex solution

$$L\,\zeta \;:=\; \Delta\zeta + \frac{1}{\delta^2}(1-|m_0|^2)\zeta - \frac{2}{\delta^2}(m_0\cdot\zeta)\zeta, \tag{4.20}$$

with boundary conditions

$$0 \;=\; \partial_\nu\zeta + \frac{1}{\varepsilon}(\zeta\cdot\nu)\,\nu. \tag{4.21}$$

We now examine the behaviour of the Hessian more closely, by a Fourier expansion in the polar angle. We have

$$\zeta \;=\; \sum_{k\in\mathbb{Z}} \zeta_k(r)\, e^{ik\varphi}, \tag{4.22}$$

and note that

$$\begin{aligned}
\zeta\cdot\nu &= \frac{1}{2}\left(\bar\zeta\nu + \zeta\bar\nu\right)\\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\bar\zeta_k e^{-ik\varphi}e^{i\varphi} + \zeta_k e^{ik\varphi}e^{-i\varphi}\right)\\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\bar\zeta_k e^{i(1-k)\varphi} + \zeta_k e^{i(k-1)\varphi}\right)\\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\bar\zeta_{1-k} + \zeta_{k+1}\right)e^{ik\varphi}\\
\int_0^{2\pi}(\zeta\cdot\nu)^2\,d\varphi &= \frac{\pi}{2}\sum_{k\in\mathbb{Z}}|\bar\zeta_{1-k}+\zeta_{k+1}|^2 &(4.23)\\
\zeta\cdot ie^{i\varphi} &= \frac{i}{2}\left(\bar\zeta e^{i\varphi} - \zeta e^{-i\varphi}\right)\\
&= \frac{i}{2}\sum_{k\in\mathbb{Z}}\left(\bar\zeta_{1-k} - \zeta_{k+1}\right)e^{ik\varphi}\\
\int_0^{2\pi}(\zeta\cdot ie^{i\varphi})^2\,d\varphi &= \frac{\pi}{2}\sum_{k\in\mathbb{Z}}|\bar\zeta_{1-k}-\zeta_{k+1}|^2. &(4.24)
\end{aligned}$$

Substituting (4.22), (4.23) and (4.24) into (4.19) yields

$$\begin{aligned}
Q(\zeta) &:= \frac{1}{2\pi}\,D^2E(m_0)(\zeta,\zeta)\\
&= \sum_{k\in\mathbb{Z}}\int_0^1 r\left[|\zeta_k'|^2 + \frac{k^2}{r^2}|\zeta_k|^2 - \frac{1}{\delta^2}|\zeta_k|^2(1-\varrho^2) + \frac{\varrho^2}{2\delta^2}|\bar\zeta_{1-k}-\zeta_{k+1}|^2\right]dr\\
&\qquad + \frac{1}{4\varepsilon}\sum_{k\in\mathbb{Z}}|\bar\zeta_{1-k}(1)+\zeta_{k+1}(1)|^2.
\end{aligned}$$

We note that the terms for $k+1$ and $1-k$ are coupled, whereby we have the following decomposition of $Q(\zeta)$ into disjoint quadratic forms:

$$Q(\zeta) \;=\; Q(\zeta_1) + \sum_{k\in\mathbb{N}} Q(\zeta_{k+1},\zeta_{1-k}) \;=:\; Q_0 + \sum_{k\in\mathbb{N}} Q_k,$$

with

$$Q_0 = \int_0^1 r \left[ |\zeta_1'|^2 + \frac{1}{r^2}|\zeta_1|^2 - \frac{1}{\delta^2}|\zeta_1|^2(1-\varrho^2) + \frac{\varrho^2}{2\delta^2}|\zeta_1 - \bar{\zeta}_1|^2 \right] dr + \frac{1}{4\varepsilon}|\bar{\zeta}_1(1) + \zeta_1(1)|^2$$

and

$$\begin{aligned} Q_k = & \int_0^1 r \Big[ |\zeta_{k+1}'|^2 + |\zeta_{1-k}'|^2 + \frac{(k+1)^2}{r^2}|\zeta_{k+1}|^2 + \frac{(1-k)^2}{r^2}|\zeta_{1-k}|^2 \\ & -\frac{1}{\delta^2}(|\zeta_{k+1}|^2 + |\zeta_{1-k}|^2)(1-\varrho^2) + \frac{\varrho^2}{\delta^2}|\bar{\zeta}_{1-k} - \zeta_{k+1}|^2 \Big] dr \\ & + \frac{1}{2\varepsilon}|\bar{\zeta}_{1-k}(1) + \zeta_{k+1}(1)|^2. \end{aligned}$$

We now replace the $\zeta_k$ by

$$A_k = \bar{\zeta}_{1-k} - \zeta_{k+1} \quad \text{and} \quad B_k = \bar{\zeta}_{1-k} + \zeta_{k+1},$$

whereby

$$\begin{aligned} Q_0 = & \int_0^1 r \Big[ \frac{1}{4}(|A_0'|^2 + |B_0'|^2) + \frac{1}{4r^2}(|A_0|^2 + |B_0|^2) \\ & -\frac{1}{4\delta^2}(|A_0|^2 + |B_0|^2)(1-\varrho^2) + \frac{\varrho^2}{2\delta^2}|A_0|^2 \Big] dr + \frac{1}{4\varepsilon}|B_0(1)|^2 \end{aligned}$$

and

$$\begin{aligned} Q_k = & \int_0^1 r \Big[ \frac{1}{2}(|A_k'|^2 + |B_k'|^2) + \frac{k^2+1}{2r^2}(|A_k|^2 + |B_k|^2) - \frac{2k}{r^2}\mathcal{R}e(A_k B_k) \\ & -\frac{1}{2\delta^2}(|A_k|^2 + |B_k|^2)(1-\varrho^2) + \frac{\varrho^2}{\delta^2}|A_k|^2 \Big] dr + \frac{1}{2\varepsilon}|B_k|^2(1) \\ \geq & \int_0^1 r \Big[ \frac{1}{2}(|A_k'|^2 + |B_k'|^2) + \frac{(k-1)^2}{2r^2}(|A_k|^2 + |B_k|^2) \\ & -\frac{1}{2\delta^2}(|A_k|^2 + |B_k|^2)(1-\varrho^2) + \frac{\varrho^2}{\delta^2}|A_k|^2 \Big] dr + \frac{1}{2\varepsilon}|B_k(1)|^2 \\ =: & \tilde{Q}_k. \end{aligned}$$

Note that $\tilde{Q}_2 = 2Q_0$. Therefore, positivity of $Q_0$ implies positivity of $\tilde{Q}_2$. Furthermore, note that $\tilde{Q}_k \leq \tilde{Q}_{k+1}$. Thus positivity of $Q_0$ implies positivity of $\tilde{Q}_k$, $k \geq 2$. Finally, this implies positivity of $Q_k$, $k \geq 2$. Following T.-C. Lin, [22], we can relate the eigenvalues $\lambda_k$ of $Q$ to the different $Q_k$, by minimisation over

$$V_0 = \left\{ (A_0, B_0) \;\middle|\; (B_0 - A_0)(r)e^{i\varphi} \in H^1(B_1(0), \mathbb{C}), \int_0^1 r(|A_0|^2 + |B_0|^2)\, dr = 1 \right\}$$

in the case of $Q_0$ and

$$\begin{aligned} V_k = \Big\{ (A_k, B_k) \;\Big|\; & (B_k - A_k)(r)e^{i(k+1)\varphi} + (\bar{A}_k + \bar{B}_k)e^{i(1-k)\varphi} \in H^1(B_1(0), \mathbb{C}), \\ & \int_0^1 r(|A_k|^2 + |B_k|^2)\, dr = 1 \Big\} \end{aligned}$$

in the case of $Q_k$. By direct comparison of the functionals in their $\zeta$–version, we have at once that

$$\lambda_1 \;<\; \lambda_2 \;<\; \lambda_3 \;<\; \ldots \tag{4.25}$$

Thus our interest lies in $\lambda_0$ and $\lambda_1$, as $\lambda_0$ and $\lambda_1$ bound the $\lambda_k$, $k \geq 2$ from below. One of the two eigenvalues $\lambda_0$ and $\lambda_1$ is the lowest eigenvalue of $Q$.

We note at this point that for purposes of minimisation we can assume both $A_k$ and $B_k$ to be real–valued and nonnegative, else they can be replaced by $|A_k|$ and $|B_k|$. We further note that the difference between tangential boundary conditions, i.e., $\varepsilon = 0$, and boundary penalty, i.e., $\varepsilon > 0$, lies in the fact that the boundary conditions for the variations $A_k$ and $B_k$ are different. We always have Neumann boundary conditions on the $A_k$:

- For $\varepsilon = 0$, the boundary conditions (4.3) translate into

$$A_k'(1) \;=\; 0 \quad \text{and} \quad B_k(1) \;=\; 0. \tag{4.26}$$

- For $\varepsilon > 0$, the boundary conditions are

$$A_k'(1) \;=\; 0 \quad \text{and} \quad B_k'(1) + \frac{1}{\varepsilon} B_k(1) \;=\; 0. \tag{4.27}$$

As a last remark, we will formulate a main assumption, which will be needed for the proof of stability later on:

**Assumption.** *Given the eigenvalue $\lambda_1^0$ of $Q_1$ in the case of full Dirichlet boundary conditions $(A_1, B_1)(1) = 0$, we assume*

$$\lambda_1^0 \;\sim\; \frac{1}{|\ln \delta|}. \tag{4.28}$$

We note that though upper bounds for this eigenvalue have been proved by Lin in [23] for the case of variations of the radial solution of degree one on $\mathbb{R}^2$, no lower bound is known to us. Nevertheless, numerical simulations, [3], seem to support our assumption.

We note at this point that an assumption of the form $\lambda_1^0 \gg \delta$ would suffice in a qualitative way for the results we prove, yet we follow the numerical lead and assume the logarithmic behaviour as stated above.

## 4.4 Tangential boundary conditions

In this section, we will exclusively treat the situation in which $\varepsilon = 0$, i.e., (4.3) holds. This implies that the variations $\zeta$ are restricted to fulfil this boundary condition, too, which in terms of $(A_k, B_k)$ by (4.26) implies that

$$B_k(1) \;=\; 0. \tag{4.29}$$

We will start by giving a lower bound on the eigenvalue $\lambda_0$ and then proceed by giving one for $\lambda_1$.

### 4.4.1 Lower bound on $\lambda_0$

We follow T.–C. Lin, [22], who proves positivity of $\lambda_0$ for a profile $\varrho$ which fulfils boundary data $\varrho(1) = 1$. The proof we give for the case of boundary conditions $\varrho'(1) = 0$ is similar. The only requirement on $\varrho(r)$ which is needed is that the profile be positive and monotonic increasing, which we have. We now state and prove the corresponding lemma.

**Lemma 4.7.**

$$\lambda_0 \ \geq \ C_0 \ > 0, \quad \textit{for } C_0 \textit{ independent of } \delta.$$

Proof of Lemma 4.7.
In order to show that $\lambda_0$ is bounded from below by a constant $C_0$ it suffices to consider the Euler–Lagrange equation for $B_0$. The Euler–Lagrange equation is

$$0 \ = \ -(rB_0')' + \frac{1}{r}B_0 - \frac{r}{\delta^2}B_0(1-\varrho^2) - 4r\lambda_0 B_0. \tag{4.30}$$

We set

$$\Psi(s) \ = \ B_0(e^{-s}), \quad \text{whereby} \quad \Psi(0) \ = \ 0.$$

Equation (4.30) then becomes

$$\Psi''(s) \ = \ \left\{1 + e^{-2s}\left[\frac{1}{\delta^2}(\varrho^2(e^{-s}) - 1) - 4\lambda_0\right]\right\}\Psi(s), \tag{4.31}$$

and with $\xi := \Psi'$, we have the first order system

$$\begin{pmatrix} \Psi \\ \xi \end{pmatrix}' \ = \ \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A}\begin{pmatrix} \Psi \\ \xi \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b(s) & 0 \end{pmatrix}\begin{pmatrix} \Psi \\ \xi \end{pmatrix} \tag{4.32}$$

with

$$b(s) = e^{-2s}\left[\frac{1}{\delta^2}(\varrho^2(e^{-s}) - 1) - 4\lambda_0\right].$$

The asymptotic behaviour of (4.32) is given by the matrix $A$, with eigenvalues and eigenvectors

$$\mu_1 \ = \ 1 \ \Rightarrow \ v_1 \ = \ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mu_2 \ = \ -1 \ \Rightarrow \ v_2 \ = \ \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

resulting in two asymptotic solutions

$$x(s) \ \sim \ e^{s}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad y(s) \ \sim \ e^{-s}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{4.33}$$

For the system with $\lambda_0 = 0$

$$\begin{pmatrix} \Xi \\ \eta \end{pmatrix}' \ = \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Xi \\ \eta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \tilde{b}(s) & 0 \end{pmatrix}\begin{pmatrix} \Xi \\ \eta \end{pmatrix}, \tag{4.34}$$

where

$$\tilde{b}(s) = e^{-2s}\frac{1}{\delta^2}(\varrho^2(e^{-s}) - 1),$$

we have the fundamental matrix and its inverse

$$Y(s) = \begin{pmatrix} \varrho(e^{-s}) & z_1(s) \\ -\varrho'(e^{-s})e^{-s} & z_2(s) \end{pmatrix}, \qquad Y^{-1}(s) = \begin{pmatrix} z_2(s) & -z_1(s) \\ -\varrho'(e^{-s})e^{-s} & \varrho(e^{-s}) \end{pmatrix}$$

where by reduction of order

$$\begin{aligned} z_1(s) &= \varrho(e^{-s})\int_0^s \varrho^{-2}(e^{-t})\,dt \\ z_2(s) &= -\varrho'(e^{-s})e^{-s}\int_0^s \varrho^{-2}(e^{-t})\,dt + \varrho^{-1}(e^{-s}). \end{aligned}$$

We use the fundamental matrix and the two projectors $P_1 = e_1 \otimes e_1$, $P_2 = e_2 \otimes e_2$, to give the solution to (4.32) as

$$\begin{aligned} x(t) &= Y(t)P_1x(0) - \int_0^t Y(t)P_1Y^{-1}(s)\begin{pmatrix} 0 \\ 4\lambda_0 e^{-2s}x_1(s) \end{pmatrix} ds \\ &\quad + \int_t^\infty Y(t)P_2Y^{-1}(s)\begin{pmatrix} 0 \\ 4\lambda_0 e^{-2s}x_1(s) \end{pmatrix} ds \\ &= \begin{pmatrix} \varrho(e^{-t}) \\ -\varrho'(e^{-t})e^{-t} \end{pmatrix} x_1(0) + 4\lambda_0 \begin{pmatrix} \varrho(e^{-t}) \\ -\varrho'(e^{-t})e^{-t} \end{pmatrix} \int_0^t e^{-2s}z_1(s)x_1(s)\,ds \\ &\quad + 4\lambda_0 \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \int_t^\infty e^{-2s}\varrho(e^{-s})x_1(s)\,ds. \end{aligned}$$

The estimate for $z_1(s)$ requires positivity and monotonicity of $\varrho(r)$, which we have, thus

$$z_1(s) = \varrho(e^{-s})\int_0^s \varrho^{-2}(e^{-t})\,dt \le \varrho(e^{-s})\int_0^s \varrho^{-2}(e^{-s})\,dt = s\varrho^{-1}(e^{-s})$$

and in consequence

$$\begin{aligned} \left|\int_0^t e^{-2s}z_1(s)x_1(s)\,ds\right| &\le ||x_1(t)||_\infty \int_0^t e^{-2s}z_1(s)\,ds \\ &\le ||x_1(t)||_\infty \varrho^{-1}(e^{-t})\int_0^t se^{-2s}\,ds \\ &= \frac{1}{4}\varrho^{-1}(e^{-t})||x_1(t)||_\infty \end{aligned} \tag{4.35}$$

as well as

$$\begin{aligned} z_1(t)\left|\int_t^\infty e^{-2s}\varrho(e^{-s})x_1(s)\,ds\right| &\le z_1(t)||x_1(t)||_\infty \int_t^\infty e^{-2s}\varrho(e^{-s})\,ds \\ &\le z_1(t)||x_1(t)||_\infty \varrho(e^{-t})\int_t^\infty e^{-2s}\,ds \\ &= \frac{1}{2}te^{-2t}||x_1(t)||_\infty \\ &\le \frac{1}{4e}||x_1(t)||_\infty. \end{aligned} \tag{4.36}$$

We use the formula for $x(t)$ and have for the first component:

$$||x_1(t)||_\infty \;\le\; |x_1(0)|\varrho(1) + 4\lambda_0\left(\frac{1}{4} + \frac{1}{4e}\right)||x_1(t)||_\infty.$$

Now $\Psi(s) = x_1(s)$, whereby $x_1(0) = 0$ and thus

$$||x_1(t)||_\infty \;\le\; \lambda_0\left(1 + e^{-1}\right)||x_1(t)||_\infty.$$

Unless $x_1 \equiv 0$, which is impossible by the asymptotic behaviour (4.33), we have that

$$\lambda_0 \;\ge\; (1 + e^{-1})^{-1}.$$

□

We note that this proof relied on $B_0(1) = 0$ which we will not have in the case of a boundary penalty. There we will use a different proof.

### 4.4.2 Interior estimates

The positivity of $\lambda_0$ leaves a dichotomy for $\lambda_1$. Either $\lambda_1$ is larger than $\lambda_0$ and we are done, or $\lambda_1$ is the smallest eigenvalue. Henceforth we shall assume the latter. We also note that the constants $a$ and $b$ which appear in this and the following subsection are fixed by Corollary 4.2. As we will be concerned from now on with $(A_1, B_1)$, we drop the subscripts and prove the following Caccioppoli estimate:

**Lemma 4.8.** *For an interval $(a, b) \Subset [\frac{3}{4}, 1]$ as in Corollary 4.2, we have*

$$\int_a^b (A'^2 + B'^2)\, r\, dr \;\le\; C\int_0^1 (A^2 + B^2)\, r\, dr. \tag{4.37}$$

PROOF OF LEMMA 4.8.
We start from the eigenvalue equation

$$-\Delta\zeta - \frac{1}{\delta^2}(1 - |m_0|^2)\zeta + \frac{2}{\delta^2}(m_0\cdot\zeta)m_0 \;=\; \lambda_1\zeta, \tag{4.38}$$

and test it with $\eta^2\zeta$, where $\eta(r)$ is a smooth cutoff function with support in $(\frac{3}{4}, 1)$ such that $\eta(r) = 1$ on $(a, b)$:

$$\int\left[\eta^2\zeta\cdot(-\Delta\zeta) - \frac{1}{\delta^2}(1 - |m_0|^2)\eta^2|\zeta|^2 + \frac{2}{\delta^2}\eta^2(m_0\cdot\zeta)^2\right]dx \;=\; \lambda_1\int \eta^2|\zeta|^2\, dx.$$

An integration by parts yields

$$\int\left[\nabla(\eta^2\zeta) : \nabla\zeta - \frac{1}{\delta^2}(1 - |m_0|^2)\eta^2|\zeta|^2 + \frac{2}{\delta^2}\eta^2(m_0\cdot\zeta)^2\right]dx \;=\; \lambda_1\int \eta^2|\zeta|^2\, dx.$$

We may now execute the derivative

$$\int \left[\eta^2|\nabla\zeta|^2 + 2\eta\eta'\zeta\cdot\partial_r\zeta - \frac{1}{\delta^2}(1-|m_0|^2)\eta^2|\zeta|^2\right] dx \;\le\; \lambda_1 \int \eta^2|\zeta|^2\, dx.$$

and use Young's inequality

$$2\eta\eta'\zeta\cdot\partial_r\zeta \;\ge\; -\frac{1}{2}\eta^2|\nabla\zeta|^2 - 2\eta'^2|\zeta|^2,$$

to arrive at

$$\int \left[\frac{1}{2}\eta^2|\nabla\zeta|^2 - 2\eta'^2|\zeta|^2 - \frac{1}{\delta^2}(1-|m_0|^2)\eta^2|\zeta|^2\right] dx \;\le\; \lambda_1 \int \eta^2|\zeta|^2\, dx$$

or

$$\int \eta^2|\nabla\zeta|^2\, dx \;\le\; \int \left[\left(2\lambda_1 + \frac{2}{\delta^2}(1-\varrho(r)^2)\right)\eta^2 + 4\eta'^2\right]|\zeta|^2\, dx.$$

Now we use Lemma 4.6 for a bound $C_1$ on the term $\frac{1}{\delta^2}(1-\varrho(r)^2)$. For $\lambda_1$ we note that an upper bound can be given by suitable test functions, e.g. with zero boundary data as in [22]. Thus we have

$$\begin{aligned}\int_{B_b(0)\setminus B_a(0)} |\nabla\zeta|^2\, dx \;&\le\; \left((2\lambda_1 + 2C_1) + 4\sup_{(a_0,b_0)}|\eta'|^2\right)\int_{B_{b_0}(0)\setminus B_{a_0}(0)} |\zeta|^2\, dx\\ &\le\; C\int_{B_1(0)} |\zeta|^2\, dx.\end{aligned}$$

As we know the eigenfunction to be of the form

$$\begin{aligned}\zeta(r,\varphi) \;&=\; \frac{A+B}{2} + \frac{B-A}{2}\,e^{2i\varphi}\\ &=\; -A(r)\,\sin\varphi\, i\, e^{i\varphi} + B(r)\,\cos\varphi\, e^{i\varphi},\end{aligned}$$

we have that

$$\int_a^b (A'^2 + B'^2)\, r\, dr \;\le\; C\int_0^1 (A^2+B^2)\, r\, dr.$$

□

We combine this Caccioppoli estimate with the following result.

**Lemma 4.9.** *There holds*

$$\sup_{[a,b]} A^2(r) \;\le\; \left(\frac{2}{b^2-a^2} + \ln\frac{b}{a}\right)\left[\int_a^b A^2(r) r\, dr + \int_a^b A'^2(r) r\, dr\right]. \qquad (4.39)$$

Proof of Lemma 4.9:
We have by the fundamental theorem of calculus and Young's inequality, for $d \in \mathbb{R}^+$:

$$\begin{aligned}
A(r) &= A(s) + \int_s^r A'(t)\,dt \\
A^2(r) &\leq (1+d)A^2(s) + (1+d^{-1})\left(\int_s^r A'(t)dt\right)^2 \\
&\leq (1+d)A^2(s) + (1+d^{-1})\int_s^r A'^2(t)t\,dt \int_s^r \frac{dt}{t} \\
&\leq (1+d)A^2(s) + (1+d^{-1})\ln\frac{r}{s}\int_s^r A'^2(t)t\,dt \\
&\leq (1+d)A^2(s) + (1+d^{-1})\ln\frac{b}{a}\int_a^b A'^2(r)r\,dr.
\end{aligned}$$

Upon integration $\int_a^b \cdot s\,ds$ and division by $\frac{b^2-a^2}{2}$ — an integral average —, this yields

$$A^2(r) \leq \frac{2(1+d)}{b^2-a^2}\int_a^b A^2(s)s\,ds + (1+d^{-1})\ln\frac{b}{a}\int_a^b A'^2(r)r\,dr.$$

In order for the two terms to be of equal weight, we require that

$$\frac{2(1+d)}{b^2-a^2} = (1+d^{-1})\ln\frac{b}{a} \quad \Rightarrow \quad d = \frac{b^2-a^2}{2}\ln\frac{b}{a},$$

thus we have that

$$\sup_{[a,b]} A^2(r) \leq \left(\frac{2}{b^2-a^2} + \ln\frac{b}{a}\right)\left[\int_a^b A^2(r)r\,dr + \int_a^b A'^2(r)r\,dr\right].$$

□

The combination of Lemma 4.8 and Lemma 4.9 yields:

**Corollary 4.4.** *There holds for $\delta < \frac{b-a}{2}$:*

$$\int_a^{a+2\delta} A^2(r)\,r\,dr \leq C\delta\int_0^1 (A^2+B^2)\,r\,dr.$$

Proof of Corollary 4.4.
We start with the left–hand side:

$$\begin{aligned}
&\int_a^{a+2\delta} A^2(r)r\,dr \\
&\leq 2\delta \sup_{[a,a+2\delta]} A^2(r) \leq 2\delta \sup_{[a,b]} A^2(r) \\
&\overset{(4.39)}{\leq} 2\delta\left(\frac{2}{b^2-a^2} + \ln\frac{b}{a}\right)\left[\int_a^b A^2(r)r\,dr + \int_a^b A'^2(r)r\,dr\right] \\
&\overset{(4.37)}{\leq} 2\delta\left(\frac{2}{b^2-a^2} + \ln\frac{b}{a}\right)\left[\int_a^b A^2(r)r\,dr + \int_0^1 (A^2+B^2)r\,dr\right] \\
&\leq C\delta\int_0^1 (A^2+B^2)r\,dr.
\end{aligned}$$

□

### 4.4.3 Proof by localisation

In this subsection, we prove the positive definiteness of $Q_1$, which we label $Q_1(A, B)$, for tangential boundary data, i.e., $B(1) = 0$.

**Theorem 4.2.** *Under the assumption* (4.28) *and for* $B(1) = 0$, *we have*

$$Q_1(A, B) \;\geq\; \frac{\lambda_1^0}{4}\int_0^1 (A^2 + B^2)\, r\, dr.$$

PROOF OF THEOREM 4.2.
We will need to incorporate assumption (4.28) into the analysis and use the Dirichlet case $(A, B)(1) = 0$. To this aim we localise $A$.

We choose cutoff functions, for $a < b < 1$ as in Corollary 4.2:

$$\begin{aligned}\alpha(r) &= \chi_{[0,a]}(r) + \cos\left(\frac{\pi}{2}\frac{r-a}{b-a}\right)\chi_{[a,b]}(r)\\ \beta(r) &= \sin\left(\frac{\pi}{2}\frac{r-a}{b-a}\right)\chi_{[a,b]}(r) + \chi_{[b,1]}(r),\end{aligned}$$

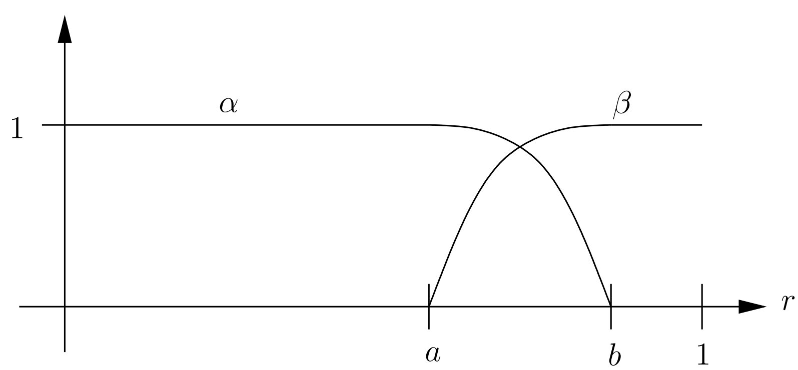


Figure 4.1: Cutoff functions $\alpha$ and $\beta$

whereby $\alpha^2 + \beta^2 = 1$, and set $A_{int} = \alpha A$ and $A_{ext} = \beta A$. Thus

$$\begin{aligned}A_{int}'^2 + A_{ext}'^2 &= (\alpha'^2 + \beta'^2)A^2 + \underbrace{(2\alpha\alpha' + 2\beta\beta')}_{=0}AA' + (\alpha^2 + \beta^2)A'^2\\ &= (\alpha'^2 + \beta'^2)A^2 + A'^2 \;=\; \frac{\pi^2}{4(b-a)^2}\chi_{(a,b)}(r)\, A^2 + A'^2. \qquad (4.40)\end{aligned}$$

With this knowledge we have for $Q_1(A, B)$:

$$\begin{aligned}&Q_1(A, B)\\ &= \frac{1}{2}\int_0^1 \left[(A'^2 + B'^2) + \frac{2}{r^2}(B-A)^2 + \frac{1}{\delta^2}A^2(3\varrho^2 - 1) + \frac{1}{\delta^2}B^2(\varrho^2 - 1)\right] r\, dr\\ &= \frac{1}{2}\int_0^1 \Big[(A_{int}'^2 + A_{ext}'^2 + B'^2 - (\alpha'^2 + \beta'^2)A^2) + \frac{2}{r^2}(B - A_{int} + A_{int} - A)^2\\ &\qquad + \frac{1}{\delta^2}(A_{int}^2 + A_{ext}^2)(3\varrho^2 - 1) + \frac{1}{\delta^2}B^2(\varrho^2 - 1)\Big]\, r\, dr\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2}\int_0^1 \left[(A_{int}'^2 + B'^2) + \frac{2}{r^2}(B - A_{int})^2 + \frac{1}{\delta^2}A_{int}^2(3\varrho^2 - 1) + \frac{1}{\delta^2}B^2(\varrho^2 - 1)\right] r\,dr \\
&\quad + \frac{1}{2}\int_0^1 \left[A_{ext}'^2 + \frac{1}{\delta^2}A_{ext}^2(3\varrho^2 - 1)\right] r\,dr - \int_0^1 \frac{2}{r^2}(B - A_{int})(A - A_{int})\, r\,dr \\
&\quad - \frac{1}{2}\int_0^1 (\alpha'^2 + \beta'^2)A^2\, r\,dr \\
&= Q_1(A_{int}, B) + \frac{1}{2}\int_a^1 \left[A_{ext}'^2 + \frac{1}{\delta^2}A_{ext}^2(3\varrho^2 - 1)\right] r\,dr \\
&\quad - \int_a^1 \frac{2}{r^2}(B - A_{int})(1 - \alpha)A\, r\,dr - \frac{\pi^2}{8(b-a)^2}\int_a^b A^2\, r\,dr,
\end{aligned}$$

where we have changed integration limits according to the support of the integrands. We recall that $a$ and $b$ are fixed by Corollary 4.2 and that $(b - a)$ is bounded below independently of $\delta$ for $\delta$ small enough. Thus, by Corollary 4.2:

$$\begin{aligned}
Q_1(A, B) \;\geq\; & Q_1(A_{int}, B) + \frac{3}{4\delta^2}\int_a^1 A_{ext}^2\, r\,dr \\
& - \int_a^1 \frac{2}{r^2}(B - A_{int})(1 - \alpha)A\, r\,dr - \frac{\pi^2}{8(b-a)^2}\int_a^b A^2\, r\,dr.
\end{aligned}$$

For the second to last term, we apply Young's inequality with $c \in \mathbb{R}^+$, noting that $(1 - \alpha)^2 \leq 1 - \alpha^2 = \beta^2$:

$$\begin{aligned}
\int_a^1 \frac{2}{r^2}(B - A_{int})(1 - \alpha)A\, r\,dr \;&\leq\; \frac{c}{a^2}\int_a^1 (B - A_{int})^2\, r\,dr + \frac{1}{ca^2}\int_a^1 (1 - \alpha)^2 A^2\, r\,dr \\
&\leq\; \frac{2c}{a^2}\int_a^1 (A_{int}^2 + B^2)\, r\,dr + \frac{1}{ca^2}\int_a^1 (1 - \alpha)^2 A^2\, r\,dr \\
&\leq\; \frac{2c}{a^2}\int_a^1 (A_{int}^2 + B^2)\, r\,dr + \frac{1}{ca^2}\int_a^1 (1 - \alpha^2) A^2\, r\,dr \\
&=\; \frac{2c}{a^2}\int_a^1 (A_{int}^2 + B^2)\, r\,dr + \frac{1}{ca^2}\int_a^1 \beta^2 A^2\, r\,dr \\
&=\; \frac{2c}{a^2}\int_a^1 (A_{int}^2 + B^2)\, r\,dr + \frac{1}{ca^2}\int_a^1 A_{ext}^2\, r\,dr.
\end{aligned}$$

In the last step we used that $\beta A = A_{ext}$.

For the last term, we split $[a, b]$ into $[a, a + 2\delta)$ and $[a + 2\delta, b]$, noting that

$$\beta(r) \;=\; \sin\left(\frac{\pi}{2}\frac{r - a}{b - a}\right) \text{ on } [a, b] \quad \Rightarrow \quad \frac{r - a}{b - a} \;\leq\; \beta(r) \text{ on } [a, b].$$

We now use this inequality to estimate the first part of the last term against an integral over $A^2_{ext}$:

$$\begin{aligned}
\int_{a+2\delta}^{b} A^2\, r\, dr &\leq \frac{(b-a)^2}{4\delta^2}\int_{a+2\delta}^{b}\frac{(r-a)^2}{(b-a)^2}A^2\, r\, dr\\
&\leq \frac{(b-a)^2}{4\delta^2}\int_{a+2\delta}^{b}\beta^2 A^2\, r\, dr\\
&= \frac{(b-a)^2}{4\delta^2}\int_{a+2\delta}^{b} A^2_{ext}\, r\, dr\\
\Rightarrow \quad -\frac{\pi^2}{8(b-a)^2}\int_{a+2\delta}^{b} A^2\, r\, dr &\geq -\frac{\pi^2}{32\delta^2}\int_{a+2\delta}^{b} A^2_{ext}\, r\, dr.
\end{aligned}$$

For the other contribution we refer to Corollary 4.4:

$$\int_a^{a+2\delta} A^2\, r\, dr \;\leq\; C\,\delta\int_0^1 (A^2+B^2)\, r\, dr.$$

With this result we can estimate the second part of the last term:

$$-\frac{\pi^2}{8(b-a)^2}\int_a^{a+2\delta} A^2\, r\, dr \;\geq\; -C\,\delta\int_0^1 (A^2+B^2)\, r\, dr,$$

and thus we arrive at

$$-\frac{\pi^2}{8(b-a)^2}\int_a^{b} A^2\, r\, dr \;\geq\; -\frac{\pi^2}{32\delta^2}\int_a^1 A^2_{ext}\, r\, dr - C\,\delta\int_0^1 (A^2+B^2)\, r\, dr.$$

For $Q_1$ this implies

$$\begin{aligned}
Q_1(A,B) \;\geq\; & Q_1(A_{int},B) + \left(\frac{3}{4\delta^2} - \frac{\pi^2}{32\delta^2} - C\,\delta - \frac{1}{ca^2}\right)\int_a^1 A^2_{ext}\, r\, dr\\
& -\frac{2c}{a^2}\int_a^1 (A^2_{int}+B^2)\, r\, dr - C\,\delta\int_0^1 (A^2_{int}+B^2)\, r\, dr,
\end{aligned}$$

where we modified the limits of integration according to the support $(a,1)$ for the contribution by $A_{ext}$. We now use the lower bound assumed in (4.28) for $Q_1(A_{int},B)$ with $(A_{int},B)(1)=0$ and have

$$\begin{aligned}
Q_1(A,B) \;\geq\; & \left(\lambda_1^0 - C\,\delta - \frac{2c}{a^2}\right)\int_0^1 (A^2_{int}+B^2)\, r\, dr\\
& + \left(\frac{1}{4\delta^2} - C\,\delta - \frac{1}{ca^2}\right)\int_a^1 A^2_{ext}\, r\, dr.
\end{aligned}$$

We choose $c$ as follows

$$\frac{8\delta^2}{a^2} \;\leq\; c \;\leq\; \lambda_1^0\,\frac{a^2}{4}.$$

This choice requires $\lambda_1^0 \gtrsim \delta^2$ and is thus possible by (4.28). We now have

$$Q_1(A,B) \;\geq\; \left(\frac{\lambda_1^0}{2} - C\,\delta\right)\int_0^1 (A^2_{int}+B^2)\, r\, dr + \left(\frac{1}{8\delta^2} - C\,\delta\right)\int_a^1 A^2_{ext}\, r\, dr.$$

Finally, we again use (4.28) to arrive at

$$Q_1(A,B) \;\geq\; \frac{\lambda_1^0}{4}\int_0^1 (A_{int}^2 + A_{ext}^2 + B^2)\, r\, dr \;=\; \frac{\lambda_1^0}{4}\int_0^1 (A^2+B^2)\, r\, dr$$

for $\delta$ small enough. □

## 4.5 Boundary penalty

Now we extend our result for tangential boundary conditions to the case of a strong boundary penalty, i.e., $\varepsilon > 0$, which implies the appearance of a boundary term, or, in the minimisation context in terms of $(A_k, B_k)$, cf. (4.27), the boundary condition

$$B_k'(1) + \frac{1}{\varepsilon} B_k(1) \;=\; 0. \tag{4.41}$$

We will proceed as in the previous section by giving a lower bound on $\lambda_0$ and then one on $\lambda_1$. Any eigenvalue apart from $\lambda_1^0$ appearing in this section is related to the $\varepsilon$–problem, however, we will not mark the dependence of the eigenvalues on $\varepsilon$.

### 4.5.1 Lower bound on $\lambda_0$

We will need to give a proof different to that in the previous section, as $B_0$ does not have zero boundary data. Nevertheless, we still concentrate on $B_0$, as the variational problem in $(A_0, B_0)$ decomposes into one for $A_0$ and one for $B_0$. We first show a formula for the partial Rayleigh quotient in $B_0$, which we multiply by four to keep notation simpler.

**Lemma 4.10.** *For $B_0$ we have:*

$$\int_0^1 \left[ B_0'^2 + \frac{1}{r^2} B_0^2 + \frac{1}{\delta^2} B_0^2(\varrho^2 - 1) \right] r\, dr \;=\; \int_0^1 \varrho^2 \left[ \left( \frac{B_0}{\varrho} \right)' \right]^2 r\, dr. \tag{4.42}$$

Proof of Lemma 4.10:
We start with the right–hand side and calculate

$$\begin{aligned}
\int_0^1 \varrho^2 \left[ \left( \frac{B_0}{\varrho} \right)' \right]^2 r\, dr \;&=\; \int_0^1 \varrho^2 \frac{B_0'^2 \varrho^2 - 2B_0 B_0' \varrho \varrho' + B_0^2 \varrho'^2}{\varrho^4}\, r\, dr \\
&=\; \int_0^1 \left[ B_0'^2 - 2B_0 B_0' \frac{\varrho'}{\varrho} + B_0^2 \frac{\varrho'^2}{\varrho^2} \right] r\, dr.
\end{aligned} \tag{4.43}$$

Now we take the middle term and perform an integration by parts:

$$\begin{aligned}
-\int_0^1 2B_0 B_0' \frac{\varrho'}{\varrho}\, r\, dr \;&=\; \int_0^1 \left( 2rB_0 \frac{\varrho'}{\varrho} \right)' B_0\, dr - \left[ 2rB_0^2 \frac{\varrho'}{\varrho} \right]_0^1 \\
&=\; \int_0^1 \left[ 2B_0^2 \frac{\varrho'}{\varrho} + 2rB_0 B_0' \frac{\varrho'}{\varrho} + 2rB_0^2 \frac{\varrho''}{\varrho} - 2rB_0^2 \frac{\varrho'^2}{\varrho^2} \right] dr + 0 \\
&=\; \int_0^1 \left[ B_0^2 \frac{\varrho'}{\varrho} + rB_0^2 \frac{\varrho''}{\varrho} - rB_0^2 \frac{\varrho'^2}{\varrho^2} \right] dr.
\end{aligned} \tag{4.44}$$

The boundary conditions are zero, as $B_0(0) = 0$ by integrability, and as $\varrho'(1) = 0$. With equation (4.10) we have

$$\frac{r\varrho'' + \varrho'}{\varrho} = \frac{1}{r} + \frac{r}{\delta^2}(\varrho^2 - 1). \tag{4.45}$$

If we insert (4.45) into (4.44), we have

$$-\int_0^1 2B_0 B_0' \frac{\varrho'}{\varrho} r\, dr = \int_0^1 \left[\frac{1}{r}B_0^2 + \frac{r}{\delta^2}B_0^2(\varrho^2 - 1) - rB_0^2\frac{\varrho'^2}{\varrho^2}\right] dr,$$

which we insert into (4.43) to arrive at

$$\int_0^1 \varrho^2 \left[\left(\frac{B_0}{\varrho}\right)'\right]^2 r\, dr = \int_0^1 \left[B_0'^2 + \frac{1}{r^2}B_0^2 + \frac{1}{\delta^2}B_0^2(\varrho^2 - 1)\right] r\, dr.$$

□

The next lemma gives the bound on the Rayleigh quotient:

**Lemma 4.11.** *For $B_0$ with boundary condition $B_0(0) = 0$ there holds*

$$\frac{1}{2-\varepsilon}\int_0^1 \varrho^2 \left[\left(\frac{B_0}{\varrho}\right)'\right]^2 r\, dr + \frac{1}{\varepsilon}B_0^2(1) \geq \int_0^1 B_0^2\, r\, dr.$$

Proof of Lemma 4.11:
By the fundamental theorem of calculus,

$$\begin{aligned}\left(\frac{B_0(r)}{\varrho(r)}\right) &= \left(\frac{B_0(1)}{\varrho(1)}\right) - \int_r^1 \left(\frac{B_0(s)}{\varrho(s)}\right)' ds \\ B_0(r) &= B_0(1)\frac{\varrho(r)}{\varrho(1)} - \varrho(r)\int_r^1 \left(\frac{B_0(s)}{\varrho(s)}\right)' ds.\end{aligned}$$

Now we use Young's inequality with $\mu \in \mathbb{R}^+$

$$\begin{aligned}B_0^2(r) \leq\ & (1+\mu)B_0^2(1)\frac{\varrho^2(r)}{\varrho^2(1)} \\ & +(1+\mu^{-1})\varrho^2(r)\int_0^1 \varrho^2\left[\left(\frac{B_0}{\varrho}\right)'\right]^2 r\, dr \int_r^1 \frac{ds}{s\varrho^2(s)}.\end{aligned}$$

By monotonicity of $\varrho(r)$, this yields

$$\begin{aligned}B_0^2(r)r \leq\ & (1+\mu)rB_0^2(1)\underbrace{\frac{\varrho^2(r)}{\varrho^2(1)}}_{\leq 1} \\ & +(1+\mu^{-1})\int_0^1 \varrho^2\left[\left(\frac{B_0}{\varrho}\right)'\right]^2 r\, dr \int_r^1 \underbrace{\frac{r\varrho^2(r)}{s\varrho^2(s)}}_{\leq 1} ds \\ \leq\ & (1+\mu)rB_0^2(1) + (1+\mu^{-1})(1-r)\int_0^1 \varrho^2\left[\left(\frac{B_0}{\varrho}\right)'\right]^2 r\, dr,\end{aligned}$$

and upon integration in $r$ we have

$$\int_0^1 B_0^2(r) r\,dr \;\le\; \frac{1+\mu}{2} B_0^2(1) + \frac{1+\mu^{-1}}{2} \int_0^1 \varrho^2 \left[\left(\frac{B_0}{\varrho}\right)'\right]^2 r\,dr.$$

If we choose

$$\mu \;=\; \frac{2}{\varepsilon} - 1,$$

we arrive at

$$\int_0^1 B_0^2 r\,dr \;\le\; \frac{1}{\varepsilon} B_0^2(1) + \frac{1}{2-\varepsilon} \int_0^1 \varrho^2 \left[\left(\frac{B_0}{\varrho}\right)'\right]^2 r\,dr.$$

□

As a result, we obtain an estimate on the partial Rayleigh quotient for $B_0$, which is equivalent to a lower bound on $\lambda_0$.

**Corollary 4.5.** *For $B_0$ with zero boundary data in 0 and $\varepsilon \le 1$, there holds*

$$\int_0^1 B_0^2(r) r\,dr \;\le\; \int_0^1 \left[B_0'^2 + \frac{1}{r^2} B_0^2 + \frac{1}{\delta^2} B_0^2(\varrho^2 - 1)\right] r\,dr + \frac{1}{\varepsilon} B_0^2(1).$$

*Thus we have the lower bound $\lambda_0 \ge \frac{1}{4}$.*

### 4.5.2 Estimates up to the boundary

As in Subsection 4.4.2, we assume w.l.o.g. that $\lambda_1 \le \lambda_0$, and we will again use a localisation argument, this time for $B$. This will be effectuated by cutting off on a small annulus at the boundary of the disc. In order to have the necessary estimates, we will have to prove some inequalities up to the boundary.

We begin with a Caccioppoli inequality:

**Lemma 4.12.** *For $(1-\gamma, 1] \subset [\frac{3}{4}, 1]$, where $\gamma \ll 1$ will be chosen later, we have*

$$\int_{1-\gamma}^1 (A'^2 + B'^2)\, r\,dr + \frac{2}{\varepsilon} B^2(1) \;\le\; C \int_0^1 (A^2 + B^2)\, r\,dr. \tag{4.46}$$

Proof of Lemma 4.12.
As in Lemma 4.8, we start from the eigenvalue equation

$$-\Delta\zeta - \frac{1}{\delta^2}(1 - |m_0|^2)\zeta + \frac{2}{\delta^2}(m_0 \cdot \zeta) m_0 \;=\; \lambda_1 \zeta$$

and test with $\eta^2\zeta$, where $\eta$ has support in $(\frac{3}{4}, 1]$ and $\eta(r) = 1$ on $(1-\gamma, 1]$. Then we have

$$\int \left[\eta^2 \zeta \cdot (-\Delta\zeta) - \frac{1}{\delta^2}(1 - |m_0|^2)\eta^2 |\zeta|^2 + \frac{2}{\delta^2}\eta^2 (m_0 \cdot \zeta)^2\right] dx \;=\; \lambda_1 \int \eta^2 |\zeta|^2\, dx.$$

We integrate by parts

$$\int \left[\nabla(\eta^2\zeta) : \nabla\zeta - \frac{1}{\delta^2}(1-|m_0|^2)\eta^2|\zeta|^2\right] dx - \int_{\partial B_1(0)} \zeta\cdot\partial_\nu\zeta\, d\mathcal{H}^1 \;\leq\; \lambda_1 \int \eta^2|\zeta|^2\, dx,$$

and insert the boundary data (4.21)

$$\int \left[\nabla(\eta^2\zeta) : \nabla\zeta - \frac{1}{\delta^2}(1-\varrho^2)\eta^2|\zeta|^2\right] dx + \frac{1}{\varepsilon}\int_{\partial B_1(0)} (\zeta\cdot\nu)^2\, d\mathcal{H}^1 \;\leq\; \lambda_1 \int \eta^2|\zeta|^2\, dx.$$

As in Lemma 4.8 we use $L^\infty$–bounds on $\frac{1}{\delta^2}(1-\varrho^2)$ up to the boundary, cf. Lemma 4.6. We also note again that we may bound $\lambda_1$ by a suitable choice of test functions, e.g. with zero boundary data, as in [22]. Thus we have

$$\int_{B_1(0)\setminus B_{1-\gamma}(0)} |\nabla\zeta|^2\, dx + \frac{2}{\varepsilon}\int_{\partial B_1(0)} (\zeta\cdot\nu)^2\, d\mathcal{H}^1 \;\leq\; C\int_{B_1(0)} |\zeta|^2\, dx.$$

Because of the special form of $\zeta$, namely

$$\begin{aligned} \zeta(r,\varphi) &= \frac{A+B}{2} + \frac{B-A}{2}\, e^{2i\varphi} \\ &= -A(r)\,\sin\varphi\, ie^{i\varphi} + B(r)\,\cos\varphi\, e^{i\varphi}, \end{aligned}$$

we therefore have that

$$\int_{1-\gamma}^1 (A'^2 + B'^2)\, r\, dr + \frac{2}{\varepsilon}B^2(1) \;\leq\; C\int_0^1 (A^2+B^2)\, r\, dr.$$

□

Now we consider the equation for $B$ itself. To derive it, we note that $Q_1(A,B)$ has the form

$$\begin{aligned} &Q_1(A,B) \\ &= \frac{1}{2}\int_0^1 \left[A'^2 + B'^2 + \frac{2}{r^2}(B-A)^2 + \frac{1}{\delta^2}A^2(3\varrho^2-1) + \frac{1}{\delta^2}B^2(\varrho^2-1)\right] r\, dr \\ &\qquad + \frac{1}{2\varepsilon}B^2(1). \end{aligned}$$

The equation for the Rayleigh quotient is then given by

$$-B'' - \frac{1}{r}B' + \frac{2}{r^2}(B-A) + \frac{1}{\delta^2}B(\varrho^2-1) \;=\; 2\lambda_1 B. \tag{4.47}$$

Using this equation, we can now prove a lemma on the second derivatives.

**Lemma 4.13.** *For the interval $(1-\gamma,1]$ of Lemma 4.12 we have*

$$\int_{1-\gamma}^1 (B'')^2\, r\, dr \;\leq\; C\int_0^1 (A^2+B^2)\, r\, dr. \tag{4.48}$$

PROOF OF LEMMA 4.13.
We rewrite (4.47)

$$\begin{aligned} B'' &= -\frac{1}{r}B' + \frac{2}{r^2}(B-A) - \frac{1}{\delta^2}(1-\varrho^2)B - 2\lambda_1 B \\ &= -\frac{1}{r}B' + \left(\frac{2}{r^2} - \frac{1}{\delta^2}(1-\varrho^2) - 2\lambda_1\right)B - \frac{2}{r^2}A. \end{aligned}$$

Then we have by integrating the square of this expression and using Young's inequality several times

$$\begin{aligned} \int_{1-\gamma}^1 (B'')^2 r\,dr &= \int_{1-\gamma}^1 \left[-\frac{1}{r}B' + \left(\frac{2}{r^2} - \frac{1}{\delta^2}(1-\varrho^2) - 2\lambda_1\right)B - \frac{2}{r^2}A\right]^2 r\,dr \\ &= \int_{1-\gamma}^1 \Big[\frac{1}{r^2}B'^2 + \left(\frac{2}{r^2} - \frac{1}{\delta^2}(1-\varrho^2) - 2\lambda_1\right)^2 B^2 + \frac{4}{r^4}A^2 \\ &\qquad -\frac{2}{r}\left(\frac{2}{r^2} - \frac{1}{\delta^2}(1-\varrho^2) - 2\lambda_1\right)BB' + \frac{4}{r^3}AB' \\ &\qquad -\frac{4}{r^2}\left(\frac{2}{r^2} - \frac{1}{\delta^2}(1-\varrho^2) - 2\lambda_1\right)AB\Big]\, r\,dr \\ &\leq \int_{1-\gamma}^1 \left[\frac{4}{r^2}B'^2 + 4\left(\frac{2}{r^2} - \frac{1}{\delta^2}(1-\varrho^2) - 2\lambda_1\right)^2 B^2 + \frac{8}{r^4}A^2\right] r\,dr. \end{aligned}$$

We use the $L^\infty$–bound on $\frac{1}{\delta^2}(1-\varrho^2)$ from Lemma 4.6 — calling it $C_1$ — to arrive at

$$\int_{1-\gamma}^1 (B'')^2 r\,dr \leq \int_{1-\gamma}^1 \left[\frac{4}{r^2}B'^2 + 4\left(\frac{2}{r^2} + C_1 + 2\lambda_1\right)^2 B^2 + \frac{8}{r^4}A^2\right] r\,dr.$$

Since $r \geq 1-\gamma$, and since $\lambda_1$ can be bounded as before, we have that

$$\begin{aligned} \int_{1-\gamma}^1 (B'')^2 r\,dr &\leq C\int_{1-\gamma}^1 B'^2\, r\,dr + C\int_{1-\gamma}^1 (A^2+B^2)\, r\,dr \\ &\overset{(4.46)}{\leq} C\int_0^1 (A^2+B^2)\, r\,dr. \end{aligned}$$

□

We can now derive estimates on the derivative at the boundary.

**Lemma 4.14.** *For the boundary value $B'(1)$ there holds*

$$B'^2(1) \leq C\int_0^1 (A^2+B^2)\, r\,dr. \tag{4.49}$$

PROOF OF LEMMA 4.14.
We proceed as in the proof of Lemma 4.9, where $1-\gamma$ replaces $a$ and 1 replaces $b$. Thus we arrive at

$$B'^2(1) \leq \left(\frac{2}{1-(1-\gamma)^2} + \ln\frac{1}{1-\gamma}\right)\left[\int_{1-\gamma}^1 B'^2(r)\, r\,dr + \int_{1-\gamma}^1 (B'')^2(r)\, r\,dr\right].$$

From Lemma 4.12 and Lemma 4.13 we conclude the result. □

Substituting (4.41) into (4.49) yields

**Corollary 4.6.** *For the boundary value $B(1) = \varepsilon B'(1)$ there holds*

$$B^2(1) \;\le\; C\,\varepsilon^2 \int_0^1 (A^2 + B^2)\, r\, dr. \tag{4.50}$$

With this corollary, we can now prove the final lemma of this subsection, which is the crucial inclusion of the boundary penalty in the problem.

**Lemma 4.15.** *For $\gamma \ll 1$, we have*

$$\sup_{(1-\gamma,1)} B^2(r) \;\le\; C(\varepsilon^2 + \gamma^2) \int_0^1 (A^2 + B^2)\, r\, dr. \tag{4.51}$$

Proof of Lemma 4.15.
By the fundamental theorem of calculus and Young's inequality, we have

$$\begin{aligned} B'(r) \quad &= \quad B'(1) - \int_r^1 B''(s)\, ds \\ B'^2(r) \quad &\le \quad 2B'^2(1) + 2\ln\frac{1}{1-\gamma}\int_{1-\gamma}^1 (B'')^2(r)\, r\, dr \\ &\overset{(4.48),(4.49)}{\le} \quad C\int_0^1 (A^2+B^2)\, r\, dr \end{aligned} \tag{4.52}$$

Similarly

$$\begin{aligned} B(r) \quad &= \quad B(1) - \int_r^1 B'(s)\, ds \\ B^2(r) \quad &\le \quad 2B^2(1) + 2\ln\frac{1}{1-\gamma}\int_{1-\gamma}^1 B'^2(r)\, r\, dr \\ &\overset{(4.52),(4.50)}{\le} \quad C\,\varepsilon^2\int_0^1 (A^2+B^2)\, r\, dr + 2C\gamma\ln\frac{1}{1-\gamma}\int_0^1 (A^2+B^2)\, r\, dr \\ &\overset{\gamma\ll 1}{\le} \quad C(\varepsilon^2+\gamma^2)\int_0^1 (A^2+B^2)\, r\, dr. \end{aligned}$$

□

### 4.5.3 Proof by localisation

In this subsection, we prove the positive definiteness of $Q_1(A, B)$ for nontangential boundary data, i.e. $B'(1) + \frac{1}{\varepsilon}B(1) = 0$. We will use our knowledge on the case of tangential boundary conditions and to this aim we localise $B$. We also note that our result is only valid for a strong boundary penalty, i.e., for

$$\varepsilon \;\ll\; \frac{1}{|\ln\delta|}. \tag{4.53}$$

For the localisation of $B$ we take a radial cutoff function $\eta_\gamma$ with $\eta_\gamma(r) = 1$ for $r \in [0, 1-\gamma]$, smoothly decaying to zero in $(1-\gamma, 1)$, with $\gamma \ll 1$ to be determined later.

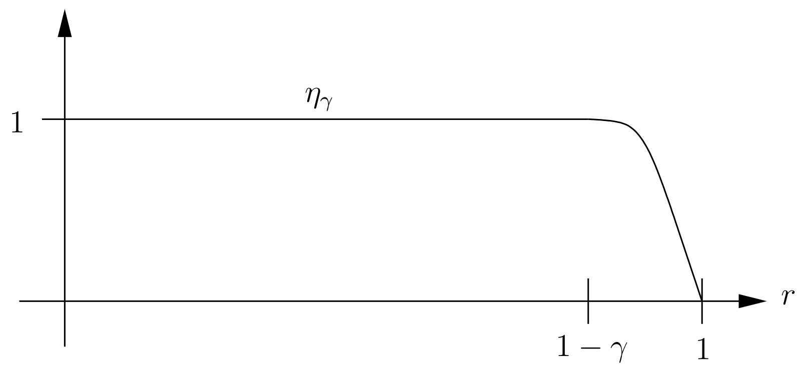


Figure 4.2: Cutoff function $\eta_\gamma$

As we know by Theorem 4.2 that

$$Q_1(A, \eta_\gamma B) \;\geq\; \frac{\lambda_1^0}{4} \int_0^1 (A^2 + (\eta_\gamma B)^2)\, r\, dr,$$

we are interested in the behaviour of the quantity

$$\Delta Q_1 \;:=\; Q_1(A, B) - Q_1(A, \eta_\gamma B).$$

Now we prove

**Theorem 4.3.** *Under the assumption* (4.28)*, and for* $\varepsilon \ll \frac{1}{|\ln \delta|}$ *there holds*

$$Q_1(A, B) \;\geq\; \frac{\lambda_1^0}{8} \int_0^1 (A^2 + B^2)\, r\, dr. \tag{4.54}$$

Proof of Theorem 4.3.
We start with an analysis of $\Delta Q_1$. First we note that

$$\begin{aligned}
\Delta Q_1 \;&=\; \int_{1-\gamma}^1 \Big[B'^2 - (\eta_\gamma B)'^2 + \frac{2}{r^2}(B-A)^2 - \frac{2}{r^2}(\eta_\gamma B - A)^2 \\
&\qquad + \frac{1}{\delta^2}(1-\eta_\gamma^2)(\varrho^2 - 1)B^2\Big]\, r\, dr + \frac{1}{\varepsilon} B^2(1) \\
&=\; \int_{1-\gamma}^1 \Big[(1-\eta_\gamma^2)B'^2 - 2\eta_\gamma \eta_\gamma' B B' - \eta_\gamma'^2 B^2 + \frac{2}{r^2} B^2 (1-\eta_\gamma^2) - \frac{4}{r^2}(1-\eta_\gamma) A B \\
&\qquad + \frac{1}{\delta^2}(1-\eta_\gamma^2)(\varrho^2 - 1)B^2\Big]\, r\, dr + \frac{1}{\varepsilon} B^2(1).
\end{aligned}$$

We now integrate the mixed term by parts and use Young's inequality with $c \in \mathbb{R}^+$:

$$\begin{aligned}\Delta Q_1 &= \int_{1-\gamma}^1 \Big[(1-\eta_\gamma^2)B'^2 + \eta_\gamma\left(\eta_\gamma'' + \frac{1}{r}\eta_\gamma'\right)B^2 + \frac{2}{r^2}B^2(1-\eta_\gamma^2) - \frac{4}{r^2}(1-\eta_\gamma)AB \\ &\qquad + \frac{1}{\delta^2}(1-\eta_\gamma^2)(\varrho^2-1)B^2\Big]\, r\,dr + \frac{1}{\varepsilon}B^2(1) \\ &\geq \int_{1-\gamma}^1 \Big[(1-\eta_\gamma^2)B'^2 + \eta_\gamma\left(\eta_\gamma'' + \frac{1}{r}\eta_\gamma'\right)B^2 + \frac{2}{r^2}B^2(1-\eta_\gamma^2) - \frac{2c}{r^2}A^2 - \frac{2}{cr^2}B^2 \\ &\qquad + \frac{1}{\delta^2}(1-\eta_\gamma^2)(\varrho^2-1)B^2\Big]\, r\,dr + \frac{1}{\varepsilon}B^2(1).\end{aligned}$$

The three relevant contributions are then, noting that $1-\gamma > \frac{1}{2}$:

$$\begin{aligned}\Delta Q_1 &\geq \underbrace{-8c\int_{1-\gamma}^1 A^2\, r\,dr}_{\Delta_I} + \underbrace{\int_{1-\gamma}^1 \left[\eta_\gamma\left(\eta_\gamma'' + \frac{1}{r}\eta_\gamma'\right) - \frac{2}{cr^2}\right]B^2\, r\,dr}_{\Delta_{II}} \\ &\quad \underbrace{+\int_{1-\gamma}^1 \left[(1-\eta_\gamma^2)B'^2 + \left(\frac{2}{r^2} - \frac{1}{\delta^2}(1-\varrho^2)\right)(1-\eta_\gamma^2)B^2\right] r\,dr + \frac{1}{\varepsilon}B^2(1)}_{\Delta_{III}}.\end{aligned}$$

For term $\Delta_I$, we choose

$$c = \frac{\lambda_1^0}{128},$$

thus

$$\begin{aligned}\Delta Q_1 &\geq -\frac{\lambda_1^0}{16}\int_{1-\gamma}^1 A^2\, r\,dr + \int_{1-\gamma}^1 \left[\eta_\gamma\left(\eta_\gamma'' + \frac{1}{r}\eta_\gamma'\right) - \frac{256}{\lambda_1^0 r^2}\right]B^2\, r\,dr \\ &\quad + \int_{1-\gamma}^1 \left[(1-\eta_\gamma^2)B'^2 + \left(\frac{2}{r^2} - \frac{1}{\delta^2}(1-\varrho^2)\right)(1-\eta_\gamma^2)B^2\right] r\,dr + \frac{1}{\varepsilon}B^2(1).\end{aligned}$$

Concerning term $\Delta_{III}$, we first ignore the positive contributions and give a lower bound for the negative term. By the $L^\infty$–bounds on $\frac{1}{\delta^2}(1-\varrho^2)$ from Lemma 4.6, we have

$$\begin{aligned}-\int_{1-\gamma}^1 \frac{(1-\varrho^2)}{\delta^2}(1-\eta_\gamma^2)B^2\, r\,dr &\geq -C\,\gamma \sup_{(1-\gamma,1)} B^2 \\ &\overset{(4.51)}{\geq} -C\,\gamma\,(\varepsilon^2+\gamma^2)\int_0^1 (A^2+B^2)\, r\,dr. \qquad (4.55)\end{aligned}$$

For term $\Delta_{II}$, we proceed in a similar way. We note that we may estimate the derivatives of $\eta_\gamma$ by inverse powers of $\gamma$, whereby due to $1-\gamma > \frac{1}{2}$

$$\begin{aligned}&\int_{1-\gamma}^1 \left[\eta_\gamma\left(\eta_\gamma'' + \frac{1}{r}\eta_\gamma'\right) - \frac{256}{\lambda_1^0 r^2}\right]B^2\, r\,dr \\ &\quad\geq -C\,\gamma\left(\frac{1}{\gamma^2} + \frac{1}{\gamma} + \frac{1}{\lambda_1^0}\right)\sup_{(1-\gamma,1)} B^2 \\ &\quad\overset{(4.51)}{\geq} -C\,(\varepsilon^2+\gamma^2)\left(\frac{1}{\gamma} + 1 + \frac{\gamma}{\lambda_1^0}\right)\int_0^1 (A^2+B^2)\, r\,dr. \qquad (4.56)\end{aligned}$$

We now collect (4.55) and (4.56). In order to later absorb both terms, we want to choose $\gamma$ in such a way as to have

$$C\left(\varepsilon^2+\gamma^2\right)\max\left\{\gamma,\ \frac{1}{\gamma},\ 1,\ \frac{\gamma}{\lambda_1^0}\right\}\ \leq\ \frac{\lambda_1^0}{64}. \tag{4.57}$$

If $\gamma^2 \leq \lambda_1^0$, the maximum is achieved for $\frac{1}{\gamma}$. Hence (4.57) becomes

$$\gamma^2-\frac{\lambda_1^0}{64C}\gamma+\varepsilon^2\ \leq\ 0. \tag{4.58}$$

The requirement (4.53) on the boundary penalty parameter $\varepsilon$ stems from this inequality, as the latter is only solvable for $\gamma$ if

$$\varepsilon\ \leq\ \frac{\lambda_1^0}{128C}\quad\overset{(4.28)}{\Rightarrow}\quad\varepsilon\ \ll\ \frac{1}{|\ln\delta|}.$$

Solving (4.58) for $\gamma$ yields $\gamma\in\left[\frac{\lambda_1^0}{128C},\frac{\lambda_1^0}{64C}\right]\ll 1$, for $\delta$ small, so that indeed $\gamma^2\leq\lambda_1^0$. Now, if we sum all contributions, including the positive contributions to $\Delta_{III}$, we have

$$\begin{aligned}
\Delta Q_1 \ \geq\ & -\frac{\lambda_1^0}{16}\int_0^1 A^2\,r\,dr-\frac{\lambda_1^0}{16}\int_0^1(A^2+B^2)\,r\,dr+\frac{1}{\varepsilon}B^2(1)\\
&+\int_{1-\gamma}^1\left[B'^2+\frac{2}{r^2}B^2\right](1-\eta_\gamma^2)\,r\,dr\\
\geq\ & -\frac{\lambda_1^0}{8}\int_0^1 A^2\,r\,dr-\frac{\lambda_1^0}{16}\int_0^1 B^2\,r\,dr+2\int_{1-\gamma}^1(1-\eta_\gamma^2)B^2\,r\,dr\\
\geq\ & -\frac{\lambda_1^0}{8}\int_0^1(A^2+(\eta_\gamma B)^2)\,r\,dr+\left(2-\frac{\lambda_1^0}{16}\right)\int_{1-\gamma}^1(1-\eta_\gamma^2)B^2\,r\,dr\\
\geq\ & -\frac{\lambda_1^0}{8}\int_0^1(A^2+(\eta_\gamma B)^2)\,r\,dr+\frac{\lambda_1^0}{8}\int_{1-\gamma}^1(1-\eta_\gamma^2)B^2\,r\,dr,
\end{aligned}$$

for $\delta$ small enough. Thus we arrive at

$$\begin{aligned}
Q_1(A,B)\ =\ & Q_1(A,\eta_\gamma B)+\Delta Q_1\\
\geq\ & \frac{\lambda_1^0}{4}\int_0^1(A^2+(\eta_\gamma B)^2)\,r\,dr\\
&-\frac{\lambda_1^0}{8}\int_0^1(A^2+(\eta_\gamma B)^2)\,r\,dr+\frac{\lambda_1^0}{8}\int_{1-\gamma}^1(1-\eta_\gamma^2)B^2\,r\,dr\\
=\ & \frac{\lambda_1^0}{8}\int_0^1(A^2+(\eta_\gamma B)^2)\,r\,dr+\frac{\lambda_1^0}{8}\int_{1-\gamma}^1(1-\eta_\gamma^2)B^2\,r\,dr\\
=\ & \frac{\lambda_1^0}{8}\int_0^1(A^2+B^2)\,r\,dr.
\end{aligned}$$

□

# Bibliography


[1] A. Aharoni, *Introduction to the Theory of Ferromagnetism*, 2nd edition, Oxford Science Publications, Oxford University Press, 2000.

[2] H.–W. Alt, S. Luckhaus, Quasilinear elliptic–parabolic differential equations, *Math. Z.*, **183**(3), 311–341, 1983.

[3] S. Bartels, R. Müller, private communication.

[4] F. Bethuel, H. Brezis, F. Hélein, *Ginzburg–Landau vortices*, Birkhäuser, Basel, 1994.

[5] W. F. Brown, Criterion for uniform micromagnetization, *Phys. Rev.* **105**, 1497–1482, 1957.

[6] R. Cantero–Álvarez, F. Otto, Critical fields in ferromagnetic thin films: Identification of four regimes, *J. Nonlinear Sci.*, **16**(4), 351–383, 2006.

[7] R. Cantero–Álvarez, F. Otto, Oscillatory buckling mode in thin-film nucleation, *J. Nonlinear Sci.*, **16**(4), 385–413, 2006.

[8] R. Cantero–Álvarez, F. Otto and J. Steiner, The concertina pattern: A bifurcation in ferromagnetic thin films, *J. Nonlinear Sci.*, **17**(3), 221–281, 2007.

[9] G. Dal Maso, *An introduction to* $\Gamma$*–convergence*, Birkhäuser, Boston, 1993.

[10] A. DeSimone, R. V. Kohn, S. Müller and F. Otto, Repulsive interaction of Néel walls, and the internal length scale of the cross-tie wall, *Multiscale Model. Simul.*, **1**(1), 57–104, 2003

[11] A. DeSimone, R. V. Kohn, S. Müller and F. Otto, Recent analytical developments in micromagnetics, in: *Science of Hysteresis*, Elsevier, G. Bertotti and I. Magyergyoz, Eds., 2005.

[12] E. H. Frei, S. Shtrikman and D. Treves, Critical size and nucleation field of ideal ferromagnetic particles, *Phys. Rev.* **107**, 446–455, 1957.

[13] E. Giusti, *Direct methods in the calculus of variations*, World Scientific, 2003.

[14] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Corrected and Enlarged Edition, Academic Press, 1980.

[15] F. B. Hang, F. H. Lin, Static theory for planar ferromagnets and antiferromagnets, *Acta Math. Sinica, Engl. Series*, **17**(4), 541–580, 2001.

[16] R.–M. Hervé, M. Hervé, Étude qualitative des solutions réelles d'une équation différentielle liée à l'équation de Ginzburg–Landau, *Ann. Inst. Henri Poincaré*, **11**(4), 427–440, 1994.

[17] A. Hubert, R. Schäfer, *Magnetic domains*, Springer, 1998.

[18] R. L. Jerrard, Lower bounds for generalized Ginzburg–Landau functionals, *SIAM J. Math. Anal.*, **30**(4), 721–746, 1999.

[19] R. V. Kohn and V. V. Slastikov, Another thin-film limit of micromagnetics, *ARMA* **178**, 227–245, 2005.

[20] M. Kurzke, Boundary vortices in thin magnetic films, *Calc. Var.* **26**, 1–28, 2006.

[21] E. H. Lieb and M. Loss, Symmetry of the Ginzburg Landau minimizer in a disc, *Math. Res. Lett.*, **1**(6), 701–715, 1994.

[22] T.–C. Lin, The stability of the radial solution to the Ginzburg–Landau equation, *Comm. in PDE*, **22**(3), 619–632, 1997.

[23] T.–C. Lin, Spectrum of the linearized operator for the Ginzburg–Landau equation, *Electronic Journal of Differential Equations*, **42**, 1–25, 2000.

[24] J. Málek, J. Nečas, M. Rokyta and M. Růžička, *Weak and Measure–valued Solutions to Evolutionary PDEs*, Chapman & Hall, 1996.

[25] P. Mironescu, On the stability of radial solutions to the Ginzburg–Landau equation, *J. Funct. Anal.*, **130**(2), 334–344, 1995.

[26] R. Moser, Ginzburg–Landau vortices for thin ferromagnetic films, *Appl. Math. Res. Express*, **1**, 1–32, 2003.

[27] A. J. Newell and R. T. Merrill, Nucleation and stability of ferromagnetic states, *J. Geophys. Res.–Sol. Ea.* **105**(B8), 19,377–19,391, 2000.

[28] J. Steiner, *Reduzierte Modelle für dünne ferromagnetische Filme: Analysis und Numerik*, Diploma thesis, 2006.

[29] E. C. Stoner and E. P. Wohlfarth, A mechanism of magnetic hysteresis in heterogeneous media, *Phil. Trans. Roy. Soc. London A* **240**, 599–642, 1948.