# On Modeling and Measuring Credit, Recovery and Liquidity Risks

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*to my parents*

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# Contents

# Introduction

Global financial markets are highly complex systems armed with a great variety of products, services, agents, institutions and authorities. Their functioning or malfunctioning depends greatly on the interacting mechanisms to which all market participants are subject. Mostly, financial crises arise from several failures of those mechanisms, which consist commonly of underestimation, misinterpretation, or even oversight of essential risks. Academic research tries to respond to demands of the financial industry by exploring some of the several aspects and expressions of risk. In line with academic responses, the purpose of the present dissertation is to provide an investigation on some praxis-relevant attributes of three major financial risks: credit, recovery and liquidity.

National and supranational regulation aims to ensure a stable basis for transactions of financial securities which are undertaken within financial markets. Nonetheless, legal frameworks are not sufficient to eliminate financial risk. Many multilateral financial contracts feature counterparty risk, which is the possibility that an involved party fails to accomplish some contractual agreement. One of the essential and basic types of financial contracts are bonds, which are debt obligations issued by a sponsoring institution, the issuer, who promises to pay in a timely manner some contractually agreed amount of capital to the counterparty, the bondholder. The failure of the issuer to fulfill the repayment schedule and eventually other protective clauses is known as default. *Credit risk* is the counterparty risk from the viewpoint of the bondholder or simply the uncertainty of the occurrence of default. Moreover, almost every financial obligation involving a creditor and a debtor embeds credit risk. The current financial crisis, originated in the underestimation of credit risk of mortgage borrowers in the United States, can be regarded as a collective and global failure in control and screening systems of financial institutions and authorities. Although financial entities and market regulators have access to state-of-the-art methods of risk management, the credit meltdown caused by the subprime crisis was unavoidable. Financial markets lacked of confidence, financial intermediaries reduced or abstained of taking credit risk exposure the following days, weeks and months after the crash of the investment banking industry. This lack of available capital extended over other business sectors and soon the global economy activity slowed down because credit risk skyrocketed.

*Recovery risk* is the immediate relative of credit risk. When payments are not paid as contractually arranged or some covenant is violated, default event occurs. The lender faces the uncertainty about the amount to be received in those cases when some legal clauses of the contract are infringed. This uncertainty is known as recovery risk and embeds two basic unknowns: date and amount of repayment. During the current financial crisis, when some of the largest investment companies were about to collapsed, the uncertainty of repayment amount increased and uncertainty about repayment date augmented even more. Increased credit and recovery risks depressed investors economic expectations. Because institutions reduced their risk acceptance, lending and borrowing were two functions of capital markets which were deficients. Despite of the combine efforts of central banks to reactivate the financial system, capital was a scarce resource. Capital markets suffered under liquidity problems, finance and refinance mechanisms evidenced serious traumas. Financial intermediaries not only reduced their exposure to credit and recovery risks but also to *liquidity risk* by maintaining large cash positions. Capital within financial markets dried up. Many economies entered into recession.

The three fundamental financial risks analyzed in this dissertation - credit, recovery and liquidity - played a decisive roll in the evolution of the current global crisis. In the following chapters these risks are investigated in three different scenarios, considering different aspects and problems. Although the present work is not directly addressed to unveil the origins and development of the financial crisis, results presented in this dissertation represent a small but further contribution in understanding those risks that shaped the current crisis.

Chapter 1 studies credit risk in an irreversible investment context. An investment project is considered to be irreversible if its initial implementation costs are lost in case of disinvestment, or if disinvestment is impossible at all. Individuals considering to invest in such a project, which generates uncertain cash flows after implementation, face the problem of finding an optimal time to initiate the project, known as investment time. McDonald and Siegel (McDonald & Siegel 1986) solve the investment problem by assuming that the implementation costs are paid in cash and that the value of the project is given by a geometric Brownian motion. In Chapter 1 we see how the solution of McDonald and Siegel changes if the irreversible investment project is financed by issuance of defaultable corporate debt. The investor is assumed to be a firm who borrows capital to finance the implementation costs of the project. As compensation, lenders are rewarded with coupon payments on the face value of the debt. However, the firm is empowered with the decision whether to continue or not paying coupons. If the firm opts to abstain of coupon payments, default event occurs and the project is turned over to bondholders. Accordingly, the firm faces an investment and default problem. By observing the evolution of the value of the project, the firm decides on optimal investment and default policies. Intuitively, the firm invests if cash flows are large and defaults if they are small. The analysis demonstrates that optimal investment time is influenced unambiguously by credit risk. A clear cut direction of this influence is not available since debt-financed investment time depends on the parameters governing the dynamics of the value of the project, taxes and default costs. If the firm can exploit tax shields and default costs, then a debt-financed investment occurs earlier than a cash-financed one. Contrary, if tax shields are low and default costs are large such that capital costs are high, then cash-financed investments occur earlier than debt-financed ones. Hence, the framework provides investment policies for debt-financed and cash-financed projects, which may differ because of tax shields and default costs, and allows firms to choose between these financing types in order to respond optimally to their corporate goals and interests.

Chapter 2 studies the risk structure of defaultable zero-coupon bonds which are financial obligations paying at expiry a fixed amount of capital, the principal or face value, in case default does not occur, and paying an unknown amount, the recovery payment, at some unknown time in case default event occurs. The uncertainty over the repayment amount and repayment date is defined as *recovery risk*. One of the main contributions of the analysis of Chapter 2 is the formulation of the riskiness of defaultable zero-coupon bonds. In particular, default time is modeled as a random variable, recovery time as a further random variable which is not necessarily identical to default time, and a random recovery payment. This approach has advantages with respect to main stream methods when trying to project real-world situations, because real-repayment dates of defaulted debt contracts are mostly unknown at default and in particular different than default time. The large literature on defaultable bonds neglects the differentiation of recovery and default times.

In academic research we find two main methodologies for valuing defaultable bonds: structural and reduced-form models. Merton (Merton 1974) is the pioneer of the structural valuation method. He regards the payment of a zero-coupon bond as a function of the value of the firm which is defined by a geometric Brownian motion, and assumes that payment date is always at expiry irrespectively of occurrence of default event. Later structural valuation research allows for recovery time to occur at any time before expiry, however, it is imposed to equal default time. A different approach to introduce default is the reduced-form methodology. While structural models consider default event to be determined as the earliest moment the value of the firm attains some pre-specified value, reduced-form models define default event through a stochastic process which is independent of the value of the firm. Reduced-form models are called intensity-based models when the default process is given by a (doubly stochastic) Poisson process. Similar as the structural framework, intensity-based valuation let default time to occur at any time before expiry and imposes no difference between recovery and default times as in Duffie et al. (Duffie et al. 1996) and Duffie and Singleton (Duffie & Singleton 1999).

In Chapter 2 we find valuations formulas for defaultable zero-coupon bonds which include parameters defining default and recovery times as well as recovery payment. Moreover, these valuations formulas are developed within a pure structural, a pure intensity-based and a mixed framework. By separating default and recovery times, we can combine the two main valuation methodologies, which is why the approach opens new frontiers of research. In addition, the valuation formulas permit a larger class of recovery payments than intensity-based models as the commonly used recovery rates in line with Duffie et al. (Duffie et al. 1996).

The last part of this dissertation is dedicated to the measurement of financial risks. In Chapter 3 we consider financial assets whose future prices are random. An investor holding a portfolio of those assets is interested in measuring the possibility of value losses of his positions in the future. The investor may measure such risk by using a conventional measure of risk as value-at-risk or expected shortfall. However, we acknowledge and warn investors for the shortcomings of conventional risk measures applied directly on portfolio values: value uncertainty originated by lack of marketability and by large volume trading is ignored. Contrary to the common no-transaction-costs assumption, real financial markets exhibit several types of imperfections. Among others, buy and sell prices are usually not equal, trading volume may induce an unambiguous impact on prices, and not all assets can be sold and bought at every time. These attributes are some typical traces of *liquidity risk*. Although liquidity risk represents a major source of risk in real financial markets, as evidenced in the current financial crisis, academic research has shown little interest for it. The prominent work of Çetin et al. (Çetin et al. 2002) introduces liquidity risk by differentiating asset prices during financial crisis and during normal trading periods. Nonetheless, other liquidity problems as lack of marketability remain excluded. The solid and innovative approach of Acerbi and Scandolo (Acerbi & Scandolo 2008) for measuring financial risk under consideration of liquidity risk tackles some ignored aspects of illiquidity mentioned previously. In particular, the researchers assume random demand and supply curves and, most important, a liquidity policy to be fulfilled by the investor. Acerbi and Scandolo put forward an adjustment to the conventional portfolio value consisting of a transaction that optimizes the value of the remaining positions while satisfying the required liquidity policy. The so called liquidity-adjusted risk measure and denoted by $\rho^{\mathcal{L}}$ is defined on the space of portfolio weights and implied from a coherent measure of risk applied to the liquidity-adjusted portfolio value $V^{\mathcal{L}}$. This risk measure is convex on the space of portfolio weights, as shown in (Acerbi & Scandolo 2008). However, this result holds only if large volume trading does not impact prices and full execution of transactions is always possible.

In Chapter 3 we use an extended Acerbi and Scandolo setup which includes abrupt price changes by large volume transactions and execution restrictions. Within this framework the liquidity-adjusted risk measure is not convex anymore. Moreover, large volume trading and partial execution increase the probability of large losses. Accordingly, we learn from these findings that ignoring and neglecting some aspects of liquidity risk leads us to wrong conjectures, which may us lead to take larger risks than wished.

All three chapters deal with different aspects of financial risk which is the most natural and essential component in the art of investment. Commonly, people accept risk only when there is some reward, a profit, justifying the risk exposure. If some aspects of risk are underestimated, misspecified or even neglected agents may take too much risk for too little reward. This dissertation provides three examples of the consequences of such misspecifications.

# Chapter 1

# Credit Risk and Optimal Investment

## 1.1 Introduction

The seminal work of McDonald and Siegel (McDonald & Siegel 1986) in 1986 shows an alternative approach in investment theory. The authors consider an *irreversible investment project* - disinvestment of the project is not possible once installed - which, once in operation, produces risky cash flows. An agent willing to invest in such a project will install it at that moment when expected cash flows are high enough. In other words, investors search for an optimal investment time. In order to determine this investment time, McDonald and Siegel realize that the opportunity to invest in an irreversible project can be regarded as an American call option with the value of the project as underlying asset and with the project's implementation cost as strike. By this observation and using techniques from financial engineering the authors provide the optimal investment time, which is fully defined by an investment threshold: if the current value of the project rises above this threshold, a firm should invest in the project at the earliest time this boundary is reached.
After the work of McDonald and Siegel a growing number of publications and articles on this subject has appeared, some of those relax certain assumptions, consider more complex scenarios and make further extensions. Some of these improvements are documented in the work of Dixit and Pindyck (Dixit & Pindyck 1994). Among

other frameworks these authors consider options to invest in an irreversible project and options to abandon an operating project. Nonetheless, they do not present a model where both options are enclosed in one investment problem. Additionally, none of the models in (Dixit & Pindyck 1994) are subjected to financial constraints. These shortcomings are addressed in studies of Trigeorgis (Trigeorgis 1993), Trigeorgis (Trigeorgis 1996) and Sabarwal (Sabarwal 2005). The first author identifies the advantages of having financial flexibility by considering debt financing. However, Trigeorgis' models are time-discrete and highly simplified. Sabarwal develops a more complex framework, where the firm finances project's costs with risky coupon bonds. Default occurs when revenue rate falls below coupon rate, forcing the firm to turn over current revenue to lenders. However, in case of default the firm is not obliged to surrender the project or any other asset to the lenders. Since the issued coupon bonds are perpetuities and given the nature of the dynamics of the value of the project, default occurs in several occasions. Despite the numerous defaults, lenders neither cancel nor sell the debt contract, instead they observe passively how default events occur. This feature makes Sabarwal framework unrealistic.

In the present work we analyze an irreversible investment problem where the project's implementation costs are financed by risky debt in form of defaultable corporate bonds. In other words, the investment project in our setup is *debt-financed*, different to *cash-financed* investments as those from McDonald and Siegel, and Dixit and Pindyck. In that context, firms have to find out not only the optimal investment time but also the optimal default time of interest payments at which the project is turned over to lenders. Similar as in the work of McDonald and Siegel (McDonald & Siegel 1986) we characterize an investment boundary and, additionally, a default boundary. Optimal investment is determined under these conditions.

In situations when the project is already operating and the value of the project reaches the default boundary, it is more convenient for the firm to default and turn over the project to debt holders than keeping the project alive. Thus, debt holders are exposed to default risk. As compensation for risk taking, debt holders demand higher coupon payments and lower prices of corporate bonds, lowering proceeds of the firm from bond sales. In some cases when coupon payments are high, available capital to cover the implementation costs is low, which delays investment. Hence, default risk of corporate bonds has a significant influence on the optimal investment strategy as well as for the value of the bonds. In order to provide - in some extent - analytical solutions of the investment problem, the defaultable bonds analyzed in this model are coupon bonds, without maturity and constant coupon rate, known as consols. Infinite maturity allows for an analytical valuation formula for the defaultable bonds. This in turn facilitates the valuation of the project which depends on the value of the bonds.

Recently, Sundaresan and Wang (Sundaresan & Wang 2007) present a model of irreversible investments where the equity holders of a firm can renegotiate the terms of debt. Equity holders choose the optimal investment time and coupon to maximize the value of the firm. Once the investment is operating, equity holders may threaten to default on debt. In this case debt holders may want to renegotiate the debt contract via a Nash bargaining game. In the present analysis equity holders choose a default time which maximizes the equity value, given the terms of debt. Hence, the present work and that of Sundaresan and Wang are complementary works, since Sundaresan and Wang do not provide an optimal default boundary. Morellec and Schürhoff in (Morellec & Schürhoff 2007) put forward an irreversible investment framework where investment and default time are endogenously determined. Furthermore, the firm decides on the coupon level of the issued debt to finance the investment project. The backbone of the their analysis relies in the personal tax advantages of the investors. The authors proceed similar as in the framework of this chapter when computing investment and default boundaries. Contrary to Morellec and Schürhoff, we use a verification theorem to prove optimality of our results.

Additionally, the present study analyzes the distribution probability of default as well as expected life (or expected default time) of corporate bonds . Defaultable bonds have been profoundly investigated in several works. The most relevant research for this framework is rooted in the works of Merton (Merton 1974) and Black and Cox (Black & Cox 1976). Here the authors price defaultable bonds with finite maturity, where default can only happen at maturity. Merton's study, where the default boundary equals the stock value at maturity, presents a closed-form solution for zero-coupon bonds with finite maturity as well as for coupon bonds with infinite maturity. Black and Cox specify the closed-form valuation formula for zero-coupon bonds when the default boundary evolves exponentially in time. Further research as in Leland (Leland 1994), Leland and Toft (Leland & Toft 1996) and Duffie and Lando (Duffie & D. Lando 2001) among other includes dividend and coupon payments, taxes and frictional default costs. These advances are incorporated in the present analysis of defaultable coupon bonds. By assuming existence of taxes and frictional default costs, investment strategy depends substantially on the financing form.

There are two main virtues of debt financing absent in cash financing which influence the investment problem unambiguously. First, debt financing allows firms to shift some risk to lenders because firms can default on debt. Second, tax and default costs represent important determinants of the investment boundary . In line with these characteristics, debt financing induce an earlier or later investment time than cash financing. Consequently, firms regard debt or cash financing as an additional alternative to their investment problem, and choose the financing type that matches their corporate goals the best.

The remainder of this chapter is organized as follows: Section 1.2 describes risky nature of the irreversible investment project. Section 1.3 presents optimal investment and default times. At the end of this section debt financed and cash financed projects are compared. Section 1.4 exhibits risk characteristics of corporate debt. Section 1.5 summaries and concludes this chapter.

## 1.2 Risky Investment Projects

Consider an investment project with risky future revenue and a firm which contemplates the possibility of investing in that project. In order to make an investment decision, the firm has to make conjectures about future performance of the project. Riskiness of the investment problem is captured in a time-continuous stochastic scenario embedded in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ fulfilling all usual conditions. Conjectures about project's performance are based on this probability structure. The project is an irreversible investment meaning that once the project is installed, disinvestment is not possible or at extremely high costs. The value of the irreversible investment project at time $t$ is denoted by $V_t$ and follows a geometric Brownian motion

$$dV_t = \mu V_t dt + \sigma V_t dW_t,$$

where $W_t$ is a standard Wiener process and parameters $\mu$ and $\sigma \geq 0$ represent constant and deterministic drift rate and volatility, respectively. The project generates after-tax cash flows, which are denoted by $D_t$ and are defined by

$$dD_t = \delta V_t dt,$$

where $\delta \geq 0$ represents the dividend rate that flows to equity holders. Assume that the firm is unlevered and is run by its equity holders. In this sense, equity holders' cash flow, i.e. dividend payments $D_t$, is the relevant quantity needed for investment decisions. Let implementation costs of the project $I$ be deterministic, constant and positive, and consider no operational costs.

Additionally, assume that the risk-free interest rate $r$ is deterministic and constant, and all market participants are risk neutral.

## 1.3 Optimal Investment Time and Endogenous Default Time

The investment problem for equity holders consists of two issues: (1) to find an optimal investment time for the project that is financed by defaultable corporate bonds, and (2) to choose the optimal time to default outstanding bonds, after the project has been installed. The solution proposed in this analysis is based on backward induction, solving first for the optimal default time, calculating the value of the investment, and finally determining the optimal investment time.

Following sections are organized as follows: Section 1.3.1 introduces the framework, in Section 1.3.2 optimal default time is determined as well as bonds's price and project's equity value. Section 1.3.3 is addressed to solve for optimal investment time. Finally, Section 1.3.4 provides a comparison between firms financing the project with defaultable bonds and firms financing the project with cash.

### 1.3.1 Corporate Financing

Before investment, the firm possesses own capital $K < I$. In order to invest in the project the firm needs to raise additional capital $I - K$ by selling firm's shares or by borrowing from a third party. The former case leads to the same investment problem proposed by McDonald and Siegel (McDonald & Siegel 1986). Hence, we draw our attention to the latter case.

The firm issues coupon bonds in order to raise funds. Coupon rate is $c \geq 0$ which is paid continuously over time up to infinity. Lenders are willing to accept the debt contract if bonds value equals the expected discounted future coupon payments. By debt financing the firm profits from tax shields on interest payments, because interests expenses are tax deductible. In other words and assuming a constant tax rate $\theta \in (0, 1)$, the firm tax payments are reduced by tax shields of $\theta c$. Under these circumstances, the after-tax cash flows for equity holders equal

$$\delta V_t - (1 - \theta)c.$$

Since the project's implementation costs are financed by selling corporate bonds, the firms chooses time $\tau$ which maximizes the sum of equity holder and bond values minus implementation costs, provided that the additional capital from bond sale is large enough. Let the sum of equity holder value and debt be given by a function $f$ of the project's value $V_t$, hence the firms investment problem is given by

$$\sup_t \mathbb{E}\left[ \mathrm{e}^{-rt} \left( f(V_t) - I \right) \middle| \mathcal{F}_0 \right],$$

provided that $d(V_\tau) \geq I - K$.

### 1.3.2 Corporate Bonds and Default Time

To derive optimal investment time $\tau$, we proceed with a backward analysis of the investment/default problem by assuming that the firm has already installed and operates the project with debt issuance. Once invested in the project, the firm cannot suspend it or abandon it because it is irreversible. Nonetheless, if the operating project is tangled in substantially adverse conditions the firm may opt to default on debt, in which case the project is turned over to debt holders with a frictional loss of value $\alpha \in (0, 1)$[1]. The firm opts to default, only if renouncing on the project is a better and more profitable option than keeping it operating. Hence, the firm chooses a default time $T_B$ on an operating project such that the expected payoff of the project is maximized at $T_B$. Optimal default time is the first-passage time $T(V_B) = \inf\{t \geq 0 : V_t \leq V_B\}$ that the project's value reaches a deterministic and constant default boundary $V_B$. The maximizing problem of the firm can be formally stated as follows. Let current time be $t$, then default time $T_B$ solves

$$\sup_T \mathbb{E}\left[\int_t^T e^{-r(s-t)}(\delta V_s - (1-\theta)c)ds \,\middle|\, \mathcal{F}_t\right]. \tag{1.1}$$

This problem has been studied in several works, the closest analysis are those in Duffie and Lando (Duffie & D. Lando 2001) and in Leland and Toft (Leland & Toft 1996). In order to use the results of these studies, consider following assumptions that are necessary for solving the investment problem.

**Assumption 1.3.1.**

- *Let parameters $\delta$, $c$, $r$, $I$, $\sigma$ and $\mu$ be deterministic, constant, positive and $r > \mu$.*
- *There exists no transaction costs and operational costs of the project.*

According to Duffie and Lando, optimal equity holders value denoted by $w(V_t)$ which equals the supremum of after-tax cash flows (1.1) solves the Hamilton-Jacobi-Bellman differential equation

$$w'(v)\mu v + \frac{1}{2}w''(v)\sigma^2 v^2 - rw(v) = (1-\theta)c - \delta v, \qquad v > V_B,$$

with boundary conditions

$$w(v) = 0, \qquad v \leq V_B,$$

and

$$w'(V_B) = 0.$$

[1]From the irreversible nature of the project one should expect that frictional loss $\alpha$ is close to 1.

By solving this differential equations with these conditions we have

$$V_B = \frac{\gamma(r-\mu)(1-\theta)c}{r(1+\gamma)\delta}, \tag{1.2}$$

where $\gamma = -\beta_2$ and $\beta_2$ is the negative solution[2] of the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0, \tag{1.3}$$

which is given by

$$\beta_2 = \frac{-m - \sqrt{m^2 + 2r\sigma^2}}{\sigma^2}. \tag{1.4}$$

Thus, whenever current project value $V_t$ falls below $V_B$ the firm defaults. Since maturity of issued bonds and project's life are infinite, expected value of cash flows depends only on current value.[3] Hence, the expected value of cash flows or equity holders value is given by

$$F(V_t) := w(V_t) = \frac{\delta V_t}{r-\mu} - \frac{\delta V_B}{r-\mu}\left(\frac{V_t}{V_B}\right)^{-\gamma} + (\theta-1)\frac{c}{r}\left(1 - \left(\frac{V_t}{V_B}\right)^{-\gamma}\right), \tag{1.5}$$

and the corresponding expected present value of the coupon payments before default is given by

$$d(V_t) = \frac{c}{r} + \left(\frac{\delta(1-\alpha)V_B}{r-\mu} - \frac{c}{r}\right)\left(\frac{V_t}{V_B}\right)^{-\gamma}. \tag{1.6}$$

Notice that the debt-financed value of the operating project, denoted by $f(V_t)$, is given by the sum of equity holders value and debt value as following

$$f(V_t) = F(V_t) + d(V_t) = \frac{\delta V_t}{r-\mu} + \theta\frac{c}{r} - \left(\frac{\alpha\delta V_B}{r-\mu} + \theta\frac{c}{r}\right)\left(\frac{V_t}{V_B}\right)^{-\gamma}, \tag{1.7}$$

which coincides with the value of a firm whose unique asset is the irreversible project and the only liabilities the outstanding coupon bonds. Further, notice that the value of a firm with the irreversible project as unique asset but without outstanding debt (and hence no possibility of default) denoted by $\tilde{f}(V_t)$ equals the expected present value of the dividend stream, i.e.

$$\tilde{f}(V_t) = \mathbb{E}\left[\int_t^\infty dD_s \,\middle|\, \mathcal{F}_t\right] = \delta \cdot \mathbb{E}\left[\int_t^\infty V_s ds \,\middle|\, \mathcal{F}_t\right] = \frac{\delta V_t}{r-\mu}.$$

We left the discussion and investigation on differences between project values $f$ and $\tilde{f}$ for further sections. In line with our arguments, we call projects financed by debt *levered projects* and projects financed with cash *unlevered projects*.

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[2]The solutions of the quadratic equation (1.3) are denoted with $\beta_1 > 1$ and $\beta_2 < 0$ as in Dixit and Pindyck (Dixit & Pindyck 1994).

[3]For more details consult (Leland & Toft 1996).

### 1.3.3 Investment Threshold

After debt and equity values are determined, it is possible to move to the next level in the backward analysis to infer optimal investment time. For this consider the problem of the firm already proposed in Section 1.3.1, where equity holders are interested in financing the implementation costs by debt issuance. At investment time $\tau$, the project yields an equity value of $F(V_t)$, bonds are issued and sold for $d(V_t)$ and implementation costs amount $I$. This implies that the firm's problem at current time $t = 0$ can be written as

$$\sup_t \mathbb{E}\left[\mathrm{e}^{-rt}\left(F(V_t) + d(V_t) - I\right)\middle|\, \mathcal{F}_0\right]. \tag{1.8}$$

Recall that the expected value of the project $f$ at time $t$ is the sum of debt and equity values. Consequently, the optimal reward function for the firm can be written as

$$g^*(V_0) = \operatorname*{ess\,sup}_t \mathbb{E}\left[\mathrm{e}^{-rt}\left(f(V_t) - I\right)\middle|\, \mathcal{F}_0\right].$$

Similar as before, the firm invests in the project at the first-passage time $\tau(V_I) = \inf\{t \geq 0 : V_t \geq V_I\}$ at which the value of the project reaches a deterministic, constant investment boundary $V_I$. Notice that at investment time $\tau(V_I)$ the revenue from bonds' sale plus initial own capital of the firm must be large enough to cover project's implementation costs, i.e. we must have $d(V_\tau) \geq I - K$. Otherwise the firm does not have enough capital to meet implementation expenses of the project and cannot invest. Closed-form solutions for $g^*(v)$ as for $V_I$ are not available. However, it is possible to characterize the investment threshold as follows.

**Proposition 1.3.2.** *Under Assumption 1.3.1, let $V_B$ be given by (1.2) and $V_I$ be the solution of*

$$-\left(\theta\frac{c}{r} + \frac{\alpha\delta V_B}{r-\mu}\right)\left(\frac{\beta_1+\gamma}{\beta_1}\right) V_B^{\gamma} V_I^{-\gamma} + \frac{\delta}{r-\mu}\left(\frac{\beta_1 - 1}{\beta_1}\right) V_I - I + \theta\frac{c}{r} = 0, \tag{1.9}$$

*where $\beta_1$ is the positive solution of (1.3). Provided that $d(V_I) \geq I - K$, then $V_I$ solves the investment problem in (1.8) if*

$$V_I \geq \frac{rI - \theta c}{\delta}. \tag{1.10}$$

*Proof.* Let $g(t, V_t) = \mathrm{e}^{-rt}(f(V_t) - I)$ be the reward function and $g^*(v)$ the optimal reward function with $g^*(v) = \operatorname{ess\,sup}_t \mathbb{E}[g(t, V_t)|\, v]$. Note that the project is not executed if the reward function is negative. Hence the investment problem can be regarded as

$$g^*(V_0) = \operatorname*{ess\,sup}_t \mathbb{E}[\tilde{g}(t, V_t)|\, \mathcal{F}_0],$$

where $\tilde{g}(t, v) := \max[g(t, v), 0]$ is continuous and non-negative. Consider the optimal investment time as the first-passage time $\tau(V_\tau) = \inf\{t \geq 0 : V_t \geq V_I\}$. The optimal reward function $g^*$, solves the Hamilton-Jacobi-Bellman differential equation

$$\mu v \frac{\partial g^*(v)}{\partial v} + \frac{1}{2}\sigma^2 v^2 \frac{\partial^2 g^*(v)}{\partial v^2} - r g^*(v) = 0 \qquad \text{for} \quad v < V_I, \tag{1.11}$$

with boundary condition

$$g^*(v) = f(v) - I \qquad \text{for} \quad v \geq V_I, \tag{1.12}$$

and smooth fit condition

$$\frac{\partial g^*(V_I)}{\partial v} = \frac{\delta}{r-\mu} + \gamma\left(\theta\frac{c}{r} + \frac{\alpha\delta V_B}{r-\mu}\right) V_B^{\gamma} V_I^{-\gamma-1}. \tag{1.13}$$

Condition (1.12) indicates that at any transgression of boundary $V_I$ it is optimal to exercise the option, i.e. to invest in the project. Expression (1.13) is the smooth-pasting condition derived from $f(V_t)$ in (1.7). Assume that optimal reward function has the following form $g^*(v) = A_1 v^{\beta_1} + A_2 v^{\beta_2}$ where $A_1$, $A_2$, $\beta_1$ and $\beta_2$ are constants. Coefficients $\beta_{1/2}$ can be calculated easily from (1.11), where

$$\beta_{1/2} = \frac{-m \pm \sqrt{m^2 + 2r\sigma^2}}{\sigma^2}, \tag{1.14}$$

with $m = \mu - \frac{\sigma^2}{2}$, $\beta_1 > 1$, $\beta_2 < 0$ and $\gamma = -\beta_2$ as in the previous subsection[4]. Since the value of the investment project $g^*(v)$ needs to be zero when $v = 0$, we have $A_2 = 0$. Hence

$$A_1 = \frac{\delta}{r-\mu}\frac{V_I^{1-\beta_1}}{\beta_1} + \frac{\gamma}{\beta_1}\left(\theta\frac{c}{r} + \frac{\alpha\delta V_B}{r-\mu}\right) V_B^{\gamma} V_I^{-\gamma-\beta_1}. \tag{1.15}$$

Equation (1.9) follows straightforwardly from this expression.

*Existence of $V_I$.* Let the left hand side of equation (1.9) be a function $h(x)$ with domain $\mathbb{R}_{++}$ given by

$$h(x) = -\eta x^{-\gamma} + \phi x + \chi, \tag{1.16}$$

[4]The coefficients $\beta_1$ and $\beta_2$ coincide with those in (1.3).

where

$$\eta = \left(\theta\frac{c}{r} + \frac{\alpha\delta V_B}{r-\mu}\right)\left(\frac{\beta_1+\gamma}{\beta_1}\right)V_B^{\gamma}, \qquad \phi = \frac{\delta}{r-\mu}\left(\frac{\beta_1-1}{\beta_1}\right),$$
$$\chi = \theta\frac{c}{r} - I.$$

Notice that $\eta, \phi > 0$ for $\alpha, \delta, \theta, r\ \sigma \in (0,1)$. This implies that $h'(x) > 0$ and $h''(x) < 0$ for $x > 0$, i.e. $h$ is strictly increasing and strictly concave. Furthermore, $h$ is continuous and $\lim_{x\to 0^+} h(x) = -\infty$ and $\lim_{x\to\infty} h(x) = \infty$. Thus, as a consequence of the intermediate value theorem, there exists a unique value $x^* > 0$ such that $h(x^*) = 0$.

Furthermore, in the exercise region $v \geq V_I$, the optimal regard function can never be less than the induced equity value $F(v)$ and the initial endowment $K$, i.e.

$$g^*(v) \geq F(v) - K \geq 0 \qquad \text{for} \quad v \geq V_I,$$

or equivalently

$$F(v) + d(v) - I \geq F(v) - K \qquad \Leftrightarrow \qquad d(v) \geq I - K,$$

for $v \geq V_I$. In particular, we must have $d(V_I) \geq I - K$.

*Optimality of* $\tau(V_I)$. This part of the proof is based on the verification Theorem 10.4.1 in Øksendal (Øksendal 2003). Consider the function

$$\psi(v) = \begin{cases} A_1 v^{\beta_1}; & 0 \leq v < V_I \\ f(v) - I & v \geq V_I, \end{cases} \tag{1.17}$$

defined on $\mathbb{R}_+$, where $A_1$ is given by equation (1.15) and $V_I$ from (1.9). Let $w(s,v) = e^{-rs}\psi(v)$. Most conditions (i)-(xi) of Theorem 10.4.1 in Øksendal are fulfilled by construction of investment time $\tau(V_I)$ and investment boundary $V_I$. Note that $w \in C^1$, $w \in C^2$ for $v \neq V_I$, and $Lw = 0$ for $v \leq V_I$ where $L$ is the second order partial differential operator. Thus, it remains to be shown:

(1) $w \geq g$ on $\mathbb{R}^2_+$, i.e. we need to verify that

$$A_1 v^{\beta_1} \geq f(v) - I.$$

Define $u(v) := A_1 v^{\beta_1} - (f(v) - I)$. Notice that $\lim_{v\to 0} u(v) = \infty$, $\lim_{v\to\infty} u(v) = \infty$, $u(V_I) = 0$, $u'(V_I) = 0$ and $u''(v) > 0$ for $v \in \mathbb{R}_+$. Hence, the global minimum of $u$ occurs at $v = V_I$, which implies $w \geq g$.

(2) $Lw \leq 0$ on $v > V_I$. Equivalently, it must be shown that for all $v > V_I$

$$\mu v \frac{\partial \psi(v)}{\partial v} + \frac{1}{2}\sigma^2 v^2 \frac{\partial^2 \psi(v)}{\partial v^2} - r\psi(v) \leq 0.$$

This expression yields the following inequality

$$-\delta v - \theta c + rI + \left[r + \gamma\left(\mu - \frac{1}{2}\sigma^2\right) - \frac{1}{2}\sigma^2\gamma^2\right]\left(\theta\frac{c}{r} + \frac{\alpha\delta V_B}{r-\mu}\right) V_B^{\gamma} v^{-\gamma} \leq 0. \quad (1.18)$$

Recall that $m = \mu - \frac{1}{2}\sigma^2$ and consider the following quadratic function

$$\mathcal{Q}(x) := -\frac{1}{2}\sigma^2 x^2 + \left(\mu - \frac{1}{2}\sigma^2\right)x + r = -\frac{1}{2}\sigma^2 x^2 + mx + r,$$

with roots

$$x_{1/2} = \frac{-m \pm \sqrt{m^2 + 2r\sigma^2}}{-\sigma^2}.$$

By definition of $\gamma$ and equation (1.4), we have $\mathcal{Q}(\gamma) = 0$. Further, note that the term in the brackets of inequality (1.18) is $\mathcal{Q}(\gamma)$. Thus, inequality (1.18) is equivalent to

$$-\delta v - \theta c + rI \leq 0.$$

Accordingly, last equation is satisfied for all $v > V_I$ if and only if

$$V_I \geq \frac{rI - \theta c}{\delta}. \quad (1.19)$$

Hence, $\tau$ is the optimal investment strategy and

$$g^* = w, \quad (1.20)$$

is the optimal reward function if condition (1.19) holds. This completes the proof.

□

For applicability of the proposed solution to the investment problem it is necessary to compare investment and default boundaries. For this purpose, suppose that conditions $d(V_I) \geq I - K$ and $V_I \geq \frac{rI-\theta c}{\delta}$ are satisfied and that $V_B > V_I$. In this case, if the project's value reaches the investment threshold at some time, the optimal strategy is to invest in the project and default on debt immediately thereafter. Although the strategy is optimal, it seems unrealistic. A firm may not want to incur in investment decisions and debt issuance if in the next second debt is going to be defaulted and the project is going to be turned over. In this model default threshold is never greater than the investment threshold, i.e. $V_I > V_B$, if initial investment costs are larger than present value of coupon payments multiplied by a certain factor $\xi$.

**Proposition 1.3.3.** *Investment threshold that solves equation (1.9) is never smaller than default threshold given in (1.2), i.e. $V_I > V_B$, if and only if*

$$I > \xi \frac{c}{r}, \tag{1.21}$$

*where*

$$\xi := \frac{\gamma}{(1+\gamma)\beta_1} \left[\beta_1 - 1 - (\beta_1 + \gamma)\left(\theta + \alpha(1-\theta)\right)\right]. \tag{1.22}$$

*Proof.* Let $h(x)$ by defined as in the proof of Proposition 1.3.2. Since $h$ is strictly increasing then $V_I > V_B$ if and only if $h(V_I) > h(V_B)$. Notice that $h(V_I) = 0$, hence it must hold $h(V_B) < 0$. After rearrangement of terms, the function $h(v)$ at $V_B$ can be written as

$$\begin{aligned} h(V_B) &= -\theta \frac{\gamma c}{\beta_1 r} - I + \frac{\delta}{r-\mu} \frac{V_B}{\beta_1} \left(\beta_1(1-\alpha) - 1 - \alpha\gamma\right) & (1.23) \\ &= \frac{\gamma c}{r(1+\gamma)\beta_1} \left[\beta_1 - 1 - (\beta_1 + \gamma)\left(\theta + \alpha(1-\theta)\right)\right] - I, & (1.24) \end{aligned}$$

where the last equality is obtained by plugging the expression for $V_B$ of (1.2) in the first equality and by redistributing terms. Thus, $h(V_B) < 0$ whenever

$$I > \frac{\gamma c}{r(1+\gamma)\beta_1} \left[\beta_1 - 1 - (\beta_1 + \gamma)\left(\theta + \alpha(1-\theta)\right)\right]. \tag{1.25}$$

□

Note that the coupon rate $c$ is an important determinant of the investment boundary $V_I$. When the coupon rate is *low enough*, the lower the coupon rate $c$, the lower financing costs of a levered project and the earlier the investment time. These consequences occur because default boundary is so low that capital collected from bond issuance is also remarkably low and defaulting seems to be very unlikely. Under these conditions, debt financing exhibits almost no differences to cash financing. On the other hand, if the coupon rate is *high enough*, then the greater the coupon rate $c$, the earlier the investment time. This implication follows from the increased tax shields the firm captures by debt financing. In this sense, the increased financing costs are compensated by the increased tax benefits which can be only received if the firm invests. Thus, as tax shields grow the firm has more incentives to invest earlier. The following lemma formalizes this discussion.

**Lemma 1.3.4.** *Investment threshold $V_I$ that solves equation (1.9) is inversely related to coupon rate if $c < M$, and positively related if $c > M$, i.e.*

$$V_I'(c) < 0 \qquad \text{if} \quad c < M \qquad \text{and} \qquad V_I'(c) > 0 \qquad \text{if} \quad c > M, \tag{1.26}$$

*where*

$$M = \left( \frac{\theta}{r\,(1+\gamma)\left(\frac{\alpha\delta}{r-\mu}\kappa + \frac{\theta}{r}\right)\left(\frac{\beta_1+\gamma}{\beta_1}\right)} \right)^{\frac{1}{\gamma}} \frac{v}{\kappa}, \tag{1.27}$$

*with $\kappa = \gamma(r-\mu)(1-\theta)/r(1+\gamma)\delta$.*

*Proof.* See Appendix 1.6.1. ∎

We examine next the difference between financing the project with debt and cash.

### 1.3.4 Unlevered and Levered Projects

Consider the case of a firm which finances the implementation costs without debt issuance. In this case the firm cannot default on coupon payments and henceforth, once invested, it has to keep operating the project ad infinitum. Formally, let $V_t$ be a geometric Brownian motion described in (1.2) and $I$ the project's costs. The equity holders' problem is to find an optimal reward function $g(v)$ such that

$$g(v) := \sup_t \mathbb{E}\left[ \mathrm{e}^{-rt} \left( \int_t^{\infty} \delta \mathrm{e}^{-r(s-t)} V_s ds - I \right) \middle| \mathcal{F}_0 \right], \tag{1.28}$$

where $v = V_0$. Let again $\tau(V^*) = \inf\{t \geq 0 \,|V_t \geq V^*\}$ be the optimal investment time given the low boundary $V^*$. Following Øksendal (Øksendal 2003), function $g(v)$ solves the Hamiltonian-Jacobi-Bellman differential equation

$$\frac{\partial g(v)}{\partial v}\mu v + \frac{1}{2}\sigma^2 v^2 \frac{\partial^2 g(v)}{\partial v^2}(v) - rg(v) = 0, \qquad \text{for} \quad v < V^*, \tag{1.29}$$

with the following boundary conditions:

$$g(V^*) = \frac{\delta}{r-\mu}V^* - I, \tag{1.30}$$

$$\frac{\partial g(V^*)}{\partial v} = \frac{\delta}{r-\mu}. \tag{1.31}$$

Let the optimal reward function be of the form $g(v) = Av^\beta$ for some constants $A$ and $\beta$. Then using differential equation (1.29) and boundary conditions (1.30) and (1.31) the investment threshold for unlevered project is given by

$$V^* = \frac{r-\mu}{\delta}\frac{\beta_1}{\beta_1 - 1}I, \tag{1.32}$$

where $\beta_1$ is again the positive solution of quadratic equation (1.3). This investment boundary corresponds to results in (Dixit & Pindyck 1994).

Consider now a firm which is planning to finance the project's costs with debt. The firm invests in the project whenever $v \geq V_I$, where $V_I$ is given in (1.9), and $d(v) \geq I - K$, where $d(v)$ corresponds to the value of debt given in (1.6). Assume a special case of no taxes and no liquidation costs in case of default, i.e. $\theta = \alpha = 0$. By equation (1.9), the optimal investment threshold for the levered project is given by

$$V_I = \frac{r-\mu}{\delta}\frac{\beta_1}{\beta_1 - 1}I,$$

and hence $V_I = V^*$. In a frictionless world without taxes the manner how the firm finances investment project is not relevant, although the firm can default on coupon payments in case of debt financing. The reason for this result is rooted in the Modigliani Miller Theorem in (Modigliani & Miller 1958). To see this, recall the interpretation of the project's present value made in previous sections as a company with only one asset. Accordingly, consider an unlevered company which value equals equity value since there is no debt. The investment project yields dividend cash flows $\delta V_t$ and where their expected value is give by

$$\tilde{f}(V_0) = \mathbb{E}\left[\int_0^\infty e^{-rs}\delta V_s ds \,\middle|\, \mathcal{F}_0\right] = \frac{\delta}{r-\mu}V_0,$$

which represents equity value. Now consider a levered company. The value of the firm at time zero is the sum of equity and debt value and is given by

$$f(V_0) = \frac{\delta V_0}{r-\mu} + \theta\frac{c}{r} - \left(\frac{\alpha\delta V_B}{r-\mu} + \theta\frac{c}{r}\right)\left(\frac{V_0}{V_B}\right)^{-\gamma}. \tag{1.33}$$

If there are no taxes and no default costs, $\theta = 0$ and $\alpha = 0$, then the value of the levered firm is

$$\frac{\delta}{r-\mu}V_0, \tag{1.34}$$

just the same as the unlevered firm. Hence, both companies face the same problem at investment time and henceforth their investment thresholds coincide.

Consequently, optimal investment thresholds of unlevered and levered projects differ because the existence of tax shields and default costs. Dependencies of the investment threshold on taxes and default costs are characterized as follows.

**Lemma 1.3.5.** *Let $\alpha$, $\theta \in (0,1)$. Optimal investment threshold $V_I$ is increasing in frictional default costs $\alpha$ and it is decreasing in tax rate $\theta$ whenever $V_I \geq \left(\frac{\beta_1+\gamma}{\beta_1}\right)^{1/\gamma} V_B$.*

*Proof.* See Appendix 1.6.2.

□

Moreover, because of tax benefits and default costs, investment threshold $V_I$ may differ from investment threshold $V^*$. In other words, when taxes and default costs exist, optimal investment time among financing types may differ. Firms invest in unlevered projects earlier than in levered ones when tax benefits are low and default costs are high. In this situation firms are not able to extract value from the issuance of defaultable bonds since their value is low. Similarly, if tax benefits are high and default costs are low firms will invest earlier in levered projects than in unlevered ones. This intuitive reflexions can be stated as follows.

**Lemma 1.3.6.** *Firms invest earlier in unlevered projects than in levered projects, i.e. $\tau(V^*) \leq \tau(V_I)$, if*

$$V^* \leq \left[\left(1 + (1-\theta)\frac{\alpha\delta}{\theta(1+\gamma)}\left(\frac{\gamma+\beta_1}{\beta_1}\right)\right)\right]^{\frac{1}{\gamma}} V_B.$$

*Proof.* See Appendix 1.6.3.

□

Hence, firms can, with information about taxes and default costs, determine which financing option is of their convenience. The reader should be aware that this model does not aim to provide arguments which indicate why investing early (or later) can be convenient for a firm. However, these arguments can be found straightforwardly by glancing at R&D investment projects for example, where the first firm that develops a new product not only obtains the patent of the project but also monopolistic control over the product's market. In such a scenario, the firm is interested in investing optimally and as early as possible.

**Numerical Example**

Consider the following example which has similar parameter values as the illustration in the work of Duffie and Lando (Duffie & D. Lando 2001):

$$\mu = 0.0113, \quad r = 0.06, \quad \sigma = 0.05, \quad \delta = 0.05, \quad I = 100, \quad c = 8.$$

For these parameters, the quadratic equation previously introduced has solutions $\gamma = 12$ and $\beta_1 = 4$. Assuming current value $V_0 = 100$, taxes $\theta = 0.35$ and default costs $\alpha = 0.3$, the optimal thresholds are

$$V_B = 78, \quad V_I = 96.8150, \quad V^* = 130,$$

where $V^*$ is the optimal investment trigger for unlevered projects. Under this parameter constellation, firms financing the project with debt invest earlier than firms financing the project with cash.

As examined in Section 1.3.4, tax shields and default costs constitute a fundamental key for the determination of optimal investment and default strategies of levered firms. The next figures illustrate this importance.

Figure 1.1 presents default and investment thresholds $V_B$ and $V_I$ for levered projects when default costs are $\alpha = 0.3$. We observe that for this parameter constellation 1) the investment threshold is always greater than default threshold and 2) both boundaries decrease when taxes increase. Tax shields increase along with the tax rate. Since the firm benefits from tax shields only if it invests in the project, the opportunity costs of waiting increase if the tax rate increases. Hence, the firm waits less if taxes increase, i.e. the investment threshold falls as taxes grow. This explains the falling shape of the investment threshold. Similarly, once the project is installed, the firm exploits tax shields as long coupons are paid. Hence, cash flows increase along increments in the tax rate only if coupon payments are made on a timely manner. Therefore, the firm waits longer to default on debt if the tax rate increases. In this example, the value of the debt at investment time $d(V_I)$ is always larger than implementations costs and ranges between 128.93 and 125.19. Since investment threshold for unlevered projects $V^*$ equals 130, firms invest always earlier in levered than in unlevered projects.

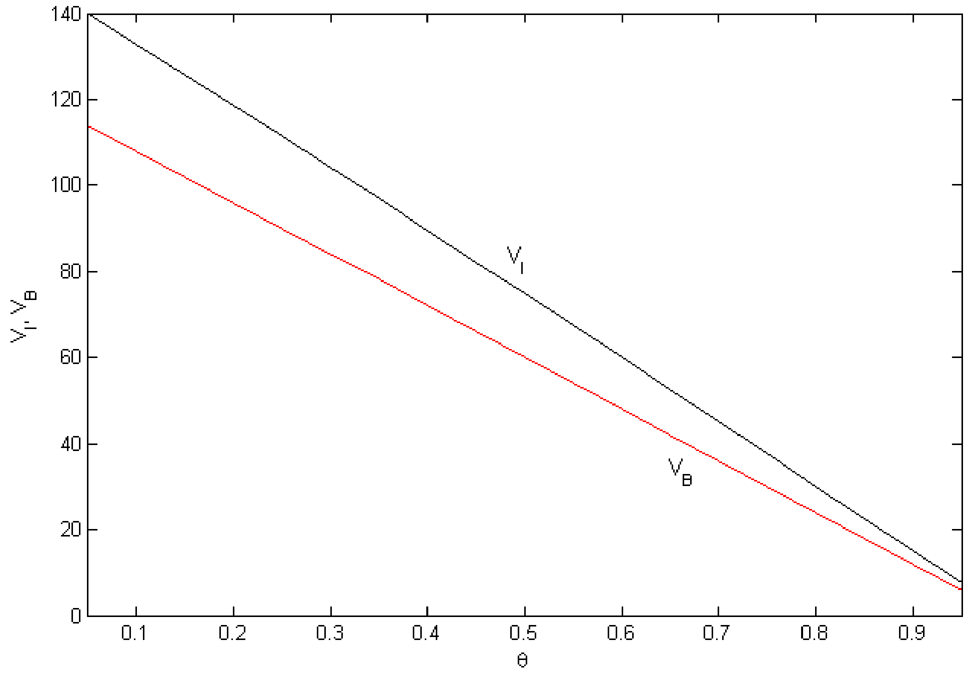


Figure 1.1: Investment $V_I$ and default $V_B$ thresholds for different tax rates $\theta$

Figure 1.2 shows the impact of default costs in investment threshold when taxes are $\theta = 0.35$. Default threshold $V_B$ stays constant at level 78 and investment threshold for unlevered projects at 130. As default costs $\alpha$ increase the value of the bonds at investment time $d(V_I)$ falls. Clearly, the greater $\alpha$, the less the bondholders recover at default and the less they are willing to pay for the bonds. Hence, for each $V_0$ the profitability of the project decreases as default costs $\alpha$ increase. Thus, the firm waits longer in order to attained a larger profit.

The last two figures present a more detailed illustration of Proposition 1.3.2. Figure 1.2 exhibits function $h(v)$ of the proof of that proposition. We can identify the strictly increasing and strictly concave shape of the function, which guaranties the existence of only one optimal investment threshold $V_I$.

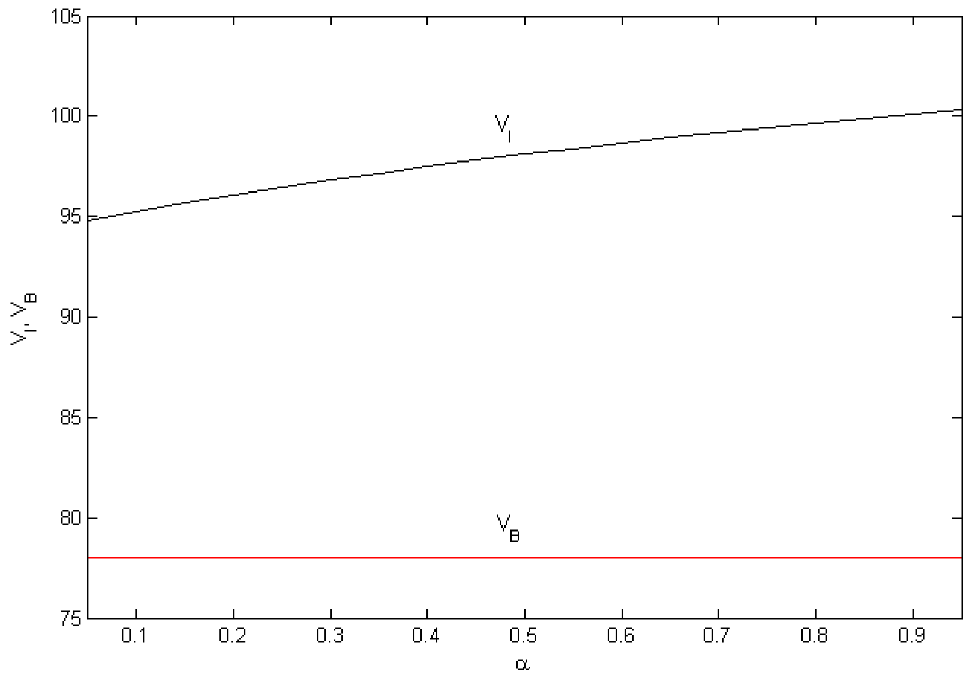


Figure 1.2: Investment $V_I$ and default $V_B$ thresholds for different default costs $\alpha$

Figure 1.4 displays two functions: $G(v)$ and $f(v) - I$. The former is given by

$$G(v) = A_1 v^{\beta_1},$$

and corresponds to the solution of the differential equation in the proof of Proposition 1.3.2. The function $f(v) - I$ is the sum of the equity holders value and bonds value minus implementation costs. As demonstrated in the mentioned proof, $G(v)$ is always greater than $f(v) - I$. The optimal reward function $g^*(v)$ consists of $G(v)$ for $v < V_I$ and of $f(v) - I$ for $v \geq V_I$.

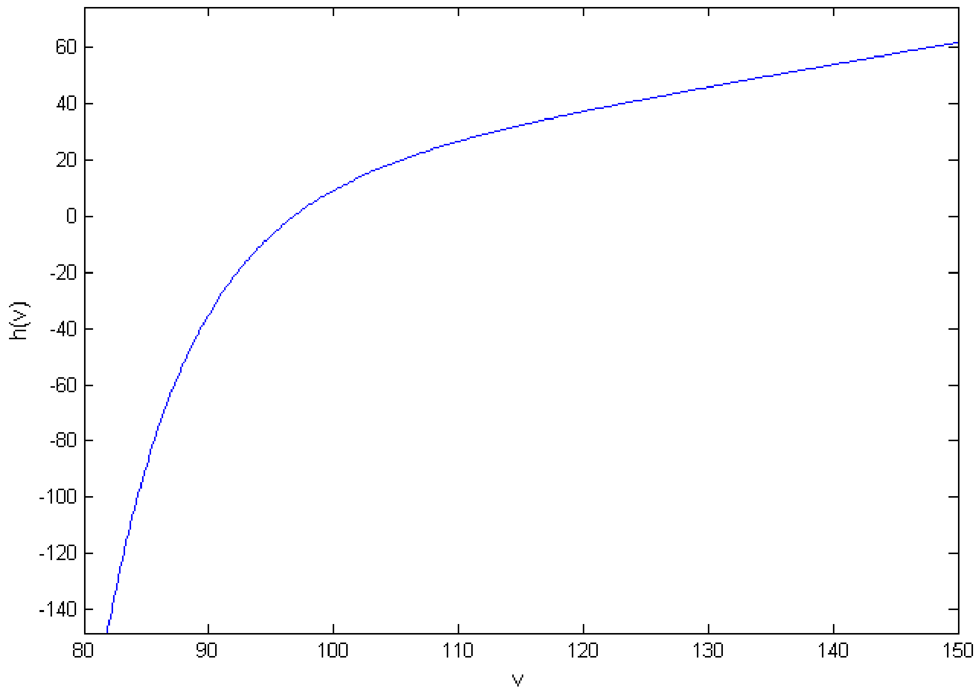


Figure 1.3: $h(v)$ function

## 1.4 Credit Risk

This section analyzes the risk exposure associated with defaultable bonds. Buyers of bonds are referred as lenders or debt holders and the firm issuing bonds is referred as borrower or issuer. Lenders are subjected to firm's default decision. Nonetheless, within this framework, the firm sticks to an optimal default strategy meaning that it is forced to default only when maintaining the project alive is suboptimal for equity holders. Hence, lenders are able to measure default risk studying default threshold given in equation (1.2) and the dynamics that govern the evolution of project's value $V_t$. Below we derive probability of default and expected default time.

### 1.4.1 Default Probability

We consider only situations when the project has been installed and is operating. Conditional default probability represents the likelihood of the project's value falling below default boundary at some future time given a current stock

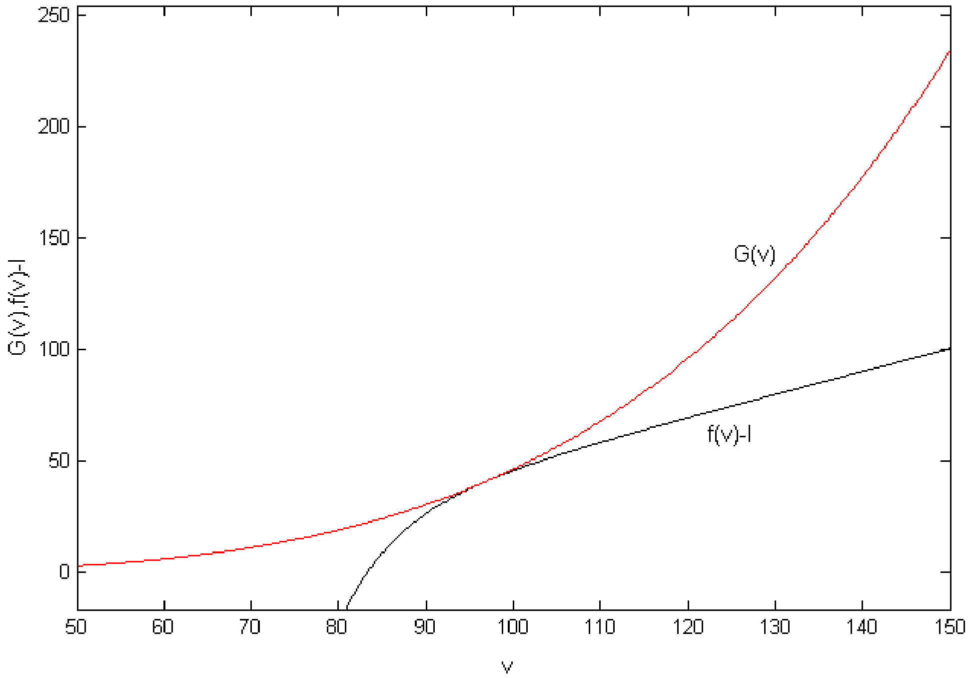


Figure 1.4: $G(v)$ and $f(v) - I$ functions

value. Formally, for an installed project with current value $v$ and default time $T_B := \inf\{\tau \geq t : V_\tau \leq V_B\}$ the conditional default probability is given by

$$\mathbb{P}_v\left(T_B < \infty\right) = 1 - \mathbb{P}_v\left(T_B = \infty\right) = \begin{cases} \left(\frac{V_B}{v}\right)^{|\nu|+\nu} & \text{for} \quad v > V_B \\ 1 & \text{otherwise,} \end{cases}$$

where $\nu := \mu/\sigma^2 - 1/2$. The last equality is obtained from the results in Borodin and Salminen (Borodin & Salminen 2002) because $V_t$ is a geometric Brownian motion and since $T_B = \inf\{\tau \leq t : V_\tau \leq V_B\} = \inf\{\tau \leq t : V_\tau = V_B\}$ for $V_t > V_B$.

Notice that default probability is always one in cases where $\nu \leq 0$, that is when the expected instantaneous mean return on the stock of the project lies below one half of the project instantaneous volatility, formally $\mu \leq \sigma^2/2$. Analogously, if the project's volatility is smaller than twice the expected mean return of the project's stocks, then default probability is less than one as long as $v > V_B$. Following these ideas, one can intuitively conjecture that high-volatility projects with poor rate of return are more likely to get into financial distress than low-volatility projects with high rate of return.

Moreover, denote $\mathbb{P}_v(T_B \in dt)$ as the density of default time $T_B$ and consider the probability of default occurring before time $T$ as follows[5]

$$\mathbb{P}_v(T_B < T) = \int_0^T \frac{|\ln(V_B/v)|}{\sigma\sqrt{2\pi t^3}} \left(\frac{V_B}{v}\right)^{\nu} \exp\left(-\frac{\nu^2\sigma^2 t}{2} - \frac{\ln^2(V_B/v)}{2\sigma^2 t}\right) dt. \qquad (1.35)$$

Consider now the $\alpha$-quantile function $q_\alpha(F_v)$ of the conditional distribution function $F_v(t) := \mathbb{P}_v(T_B \leq t)$ given by

$$q_\alpha(F_v) = \inf\{t \geq 0 \,|\, F_v(t) \geq \alpha\}.$$

This quantile gives the earliest time at which default occurs with probability of $\alpha$. Let

$$\hat{\tau} := q_\alpha(F_v),$$

and denote it as $\alpha$-default time. When $\alpha$ is small, e.g. 0.1 per cent, investors may use $\hat{\tau}$ as an approximation for calculating *maturity time* of a non-defaultable coupon bond with coupon rate $c$ and face value $(1-\alpha)\delta V_B/(r-\mu)$. Lenders may make this approximation in order to hedge their position in the bond or money market.

### 1.4.2 Expected Conditional Default Time

In addition, firm and lenders are interested in knowing the expected life of the bonds, which is equivalent to find the expected value of default time $T_B$. To determine this expected time recall the density function of $T_B$ associated to the expression in (1.35). Hence, expected default time conditional on current project value $v$ can be calculated by

$$\begin{aligned} \mathbb{E}_v[T_B |\; T_B < \infty] &= \int_0^\infty t\mathbb{P}_v(T_B \in dt) \\ &= \int_0^\infty \frac{\left|\ln\left(\frac{V_B}{v}\right)\right|}{\sigma\sqrt{2\pi t}} \left(\frac{V_B}{v}\right)^{\nu} \exp\left(-\frac{\nu^2\sigma^2 t}{2} - \frac{\ln^2\left(\frac{V_B}{v}\right)}{2\sigma^2 t}\right) dt. \end{aligned}$$

Notice that expected default time $T_B$ is conditioned on the set $\{T_B < \infty\}$. Rearranging terms and noting that $\ln^2(V_B/v) = \ln^2(v/V_B)$ the last equality can be written as

$$\left(\frac{V_B}{v}\right)^{\nu} \int_0^\infty \frac{|\ln(V_B/v)|}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\left(\frac{\ln^2(v/V_B) + \nu^2\sigma^4 t^2}{\sigma^2 t}\right)\right) dt.$$

[5]See Borodin and Salminen (Borodin & Salminen 2002) for details.

After some algebra this term can be expressed as

$$\left(\frac{V_B}{v}\right)^{\nu-|\nu|}\int_0^\infty \frac{|\ln(V_B/v)|}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{1}{2}\left(\frac{\ln(v/V_B)+|\nu|\,\sigma^2 t}{\sigma\sqrt{t}}\right)^2\right)dt.$$

Consider this integral as the following limit

$$\lim_{T\to\infty}\left(\frac{V_B}{v}\right)^{\nu-|\nu|}\int_0^T \frac{|\ln(V_B/v)|}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{1}{2}\left(\frac{\ln(v/V_B)+|\nu|\,\sigma^2 t}{\sigma\sqrt{t}}\right)^2\right)dt. \tag{1.36}$$

The next computational steps are based on the work of Leland and Toft (Leland & Toft 1996). Let $\bar{\sigma} = 2|\nu|\,\sigma$, $Y = v^{2|\nu|}$ and $Y_B = V_B^{2|\nu|}$. Substitute this terms in (1.36) which yields the following expression:

$$\lim_{T\to\infty}\frac{|\ln(V_B/v)|}{Y\sigma\bar{\sigma}}\left(\frac{V_B}{v}\right)^{\nu-|\nu|}\int_0^T \frac{\bar{\sigma}Y}{\sqrt{2\pi t}}\exp\left(-\frac{1}{2}\left(\frac{\ln(Y/Y_B)+\frac{1}{2}\bar{\sigma}^2 t}{\bar{\sigma}\sqrt{t}}\right)^2\right)dt.$$

Substituting $\epsilon = \frac{\bar{\sigma}\sqrt{t}}{\sqrt{T}}$ the last integral is given by

$$\lim_{T\to\infty}\frac{2\,|\ln(V_B/v)|}{Y\sigma\bar{\sigma}}\left(\frac{V_B}{v}\right)^{\nu-|\nu|}\int_0^{\bar{\sigma}} \frac{\sqrt{T}Y}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln(Y/Y_B)+\frac{1}{2}\epsilon^2 T}{\epsilon\sqrt{T}}\right)^2\right)d\epsilon. \tag{1.37}$$

The integral in this term is the integral over the partial derivative of a European call option with respect to volatility with underlying $Y$, strike $Y_B$, maturity $T$ and dividend yield as well as interest rate equaling zero. Assume $v > V_B$ as previously which implies $Y > Y_B$. By the fundamental theorem of calculus and the valuation formula for European options in the Black-Scholes model the expression in (1.37) is given by

$$\begin{aligned}
&\lim_{T\to\infty}\frac{2\,|\ln(V_B/v)|}{Y\sigma\bar{\sigma}}\left(\frac{V_B}{v}\right)^{\nu-|\nu|}\left[YN\left(\frac{\ln(Y/Y_B)+\frac{1}{2}\bar{\sigma}^2 T}{\bar{\sigma}\sqrt{T}}\right)\right.\\
&\left.-Y_BN\left(\frac{\ln(Y/Y_B)-\frac{1}{2}\bar{\sigma}^2 T}{\bar{\sigma}\sqrt{T}}\right)-(Y-Y_B)\right]\\
=\;&\lim_{T\to\infty}\frac{|\ln(V_B/v)|}{|\nu|\,\sigma^2}\left(\frac{V_B}{v}\right)^{\nu-|\nu|}\left[-\left(1-N\left(\frac{\ln(Y/Y_B)+\frac{1}{2}|\nu|\,\sigma^2 T}{\sigma\sqrt{T}}\right)\right)\right.\\
&\left.+\frac{Y_B}{Y}\left(1-N\left(\frac{\ln(Y/Y_B)-\frac{1}{2}|\nu|\,\sigma^2 T}{\sigma\sqrt{T}}\right)\right)\right],
\end{aligned}$$

where $N(\cdot)$ is the cumulative standard normal distribution. In the last equality one can substitute again $Y$ and use the attributes of the normal distribution which yields

$$\lim_{T\to\infty} \frac{|\ln (V_B/v)|}{|\nu|\,\sigma^2} \left(\frac{V_B}{v}\right)^{\nu-|\nu|} \left[-N\left(\frac{-\ln (v/V_B) - \frac{1}{2}|\nu|\,\sigma^2 T}{\sigma\sqrt{T}}\right)\right.$$
$$\left. + \left(\frac{V_B}{v}\right)^{2|\nu|} N\left(\frac{-\ln (v/V_B) + \frac{1}{2}|\nu|\,\sigma^2 T}{\sigma\sqrt{T}}\right)\right].$$

Notice that $|\ln (V_B/v)| = \ln (v/V_B)$ because $v > V_B > 0$. Hence, expected default time follows straightforwardly from the last expression.

**Proposition 1.4.1.** *Assume $v > V_B$. The expected default time conditional on current stock value $v$ is given by*

$$\mathbb{E}_v\left[T_B|\ T_B < \infty\right] = \frac{\ln (v/V_B)}{|\nu|\,\sigma^2} \left(\frac{V_B}{v}\right)^{\nu+|\nu|}.$$

Accordingly, the expected life of bonds is the longer the further away the project's value $v$ is from default boundary $V_B$. This result confirms economic intuition. Further, observe that expected default time is increasing in current stock value $v$ and decreasing in default boundary $V_B$.

## 1.5 Concluding Remarks

This chapter presents an optimal investment strategy for an irreversible project financed by defaultable bonds. In addition to investment threshold, the chapter shows an optimal default strategy for the issued bonds. Projects financed with defaultable debt and projects financed with cash are compared as well as their optimal investment strategy. In presence of taxes and default costs, optimal investment strategy depends on the financing method. Moreover, risk embedded in defaultable bonds allows those firms who choose this financing method to invest in the project earlier than firms financing the project's costs with cash, if the default threshold is small enough. Default probability and expected default time are calculated in order to quantify risk of the defaultable bonds.

Various extensions and modifications can be considered for the enrichment of the model. Regarding a stochastic interest rate as a second risk factor or a variable coupon rate are two fundamental extensions that can be made. Irreversible investment problems with stochastic interest rates are analyzed in the work of Schulmerich (Schulmerich 2005). However, the author does not consider issuance of defaultable coupon bonds as a financing alternative. Analytical solutions of a model including stochastic interest rates will be presumably not available and we will be likely forced to recur to numerical methods. Nonetheless, a model that incorporates these features would replicate real-world investment problems more accurately.

## 1.6 Appendix

### 1.6.1 Proof of Lemma 1.3.4.

The lemma follows from the implicit function theorem used for the function $h(V_I, c)$ presented in (1.16). Thus, consider

$$\begin{aligned}\frac{\partial h}{\partial c} &= \frac{\theta}{r} - (1+\gamma)\left(\frac{\theta}{r} + \frac{\alpha\delta\kappa}{r-\mu}\right)\left(\frac{\beta_1+\gamma}{\beta_1}\right)\left(\frac{V_I}{\kappa}\right)^{-\gamma} c^{\gamma}, \\ \frac{\partial h}{\partial V_I} &= \gamma\left(\frac{\theta}{r} + \frac{\alpha\delta\kappa}{r-\mu}\right)\left(\frac{\beta_1+\gamma}{\beta_1}\right)(V_I)^{-\gamma-1}\kappa^{\gamma}c^{\gamma-1} + \frac{\delta}{r-\mu}\left(\frac{\beta_1-1}{\beta_1}\right),\end{aligned}$$

where $\kappa = \gamma(r-\mu)(1-\theta)/r(1+\gamma)\delta$. The first order derivative of $V_I$ with respect to $c$ is given by

$$V_I'(c) = -\frac{\frac{\partial h}{\partial c}}{\frac{\partial h}{\partial V_I}},$$

by noticing that $\frac{\partial h}{\partial V_I} > 0$. Hence, $V_I'(c) < 0$ if and only if

$$\frac{\theta}{r} - (1+\gamma)\left(\frac{\theta}{r} + \frac{\alpha\delta\kappa}{r-\mu}\right)\left(\frac{\beta_1+\gamma}{\beta_1}\right)\left(\frac{V_I}{\kappa}\right)^{-\gamma} c^{\gamma} > 0,$$

which is equivalent to

$$c < \left(\frac{\theta}{r\,(1+\gamma)\left(\frac{\alpha\delta}{r-\mu}\kappa + \frac{\theta}{r}\right)\left(\frac{\beta_1+\gamma}{\beta_1}\right)}\right)^{\frac{1}{\gamma}} \frac{V_I}{\kappa}.$$

□

### 1.6.2 Proof of Lemma 1.3.5

To show the statement consider again the implicit function theorem for equation (1.9). Hence, assuming that the investment trigger is a function of $\alpha$ and $\theta$, one has to show that

$$\frac{\partial V_I}{\partial \alpha} \geq 0 \qquad \text{and} \qquad \frac{\partial V_I}{\partial \theta} \leq 0, \tag{1.38}$$

for $V_I \geq V_B$. Using the implicit function theorem for $h$ given in (1.16) with respect to $\alpha$ yields

$$\begin{aligned}\frac{\partial V_I}{\partial \alpha} &= \left[\gamma\left(\theta\frac{c}{r} + \frac{\alpha\delta V_B}{r-\mu}\right)\left(\frac{\gamma+\beta_1}{\beta_1}\right)V_B^{\gamma}V_I^{-\gamma-1} + \frac{\delta}{r-\mu}\left(\frac{\beta_1-1}{\beta_1}\right)\right]^{-1} \\ &\quad \cdot\frac{\delta V_B}{r-\mu}\left(\frac{\gamma+\beta_1}{\beta_1}\right)V_B^{\gamma}V_I^{-\gamma} \leq 0. \end{aligned}\tag{1.39}$$

The partial derivative of $h$ with respect to $\theta$ is given by

$$\begin{aligned}\frac{\partial h}{\partial \theta} &= -V_B'\left[\frac{\alpha\delta}{r-\mu}\left(\frac{\gamma+\beta_1}{\beta}\right)V_B^{\gamma}V_I^{-\gamma}+\gamma\left(\theta\frac{c}{r}+\frac{\alpha\delta V_B}{r-\mu}\right)\left(\frac{\gamma+\beta_1}{\beta_1}\right)V_B^{\gamma-1}V_I^{-\gamma}\right] \\ & \quad -\frac{c}{r}\left(\left(\frac{\beta_1+\gamma}{\beta_1}\right)V_B^{\gamma}V_I^{-\gamma}-1\right),\end{aligned} \tag{1.40}$$

Since $\frac{\partial h}{\partial V_I} > 0$,

$$\frac{\partial V_I}{\partial \theta} = -\frac{\frac{\partial h}{\partial \theta}}{\frac{\partial h}{\partial V_I}} \leq 0 \qquad \text{iff} \qquad -\frac{\partial h}{\partial \theta} \leq 0.$$

Because $\frac{\partial V_B}{\partial \theta} = -\frac{\gamma(r-\mu)c}{r(1-\gamma)\delta} < 0$ and the term in the brackets of (1.40) is positive, we have $\frac{\partial V_I}{\partial \theta} \leq 0$ if

$$\left(\frac{\beta_1+\gamma}{\beta_1}\right)V_B^{\gamma}V_I^{-\gamma}-1 \leq 0,$$

or equivalently

$$V_I \geq \left(\frac{\beta_1+\gamma}{\beta_1}\right)^{1/\gamma} V_B,$$

which completes the proof.

$\square$

### 1.6.3 Proof of Lemma 1.3.6.

Let $h(v)$ be the function in the left side of equation (1.9) as defined in the proof of Proposition 1.3.2. By inserting $V^*$ in $h$ one gets

$$h(V^*) = -\left(\theta\frac{c}{r}+\frac{\alpha\delta V_B}{r-\mu}\right)\left(\frac{\gamma+\beta_1}{\beta_1}\right)\left(\frac{V_B}{V^*}\right)^{\gamma}+\theta\frac{c}{r}. \tag{1.41}$$

Since $h$ is strictly increasing and $h(V_I)=0$, then $V_I \geq V^*$ is equivalent to $h(V^*) \leq 0$, i.e.

$$\left(\theta\frac{c}{r}+\frac{\alpha\delta V_B}{r-\mu}\right)\left(\frac{\gamma+\beta_1}{\beta_1}\right)\left(\frac{V_B}{V^*}\right)^{\gamma} \geq \theta\frac{c}{r}. \tag{1.42}$$

Solving this inequality for $V^*$ one gets the condition of the Corollary:

$$V^* \leq \left[\left(1+(1-\theta)\frac{\alpha\delta}{\theta(1+\gamma)}\left(\frac{\gamma+\beta_1}{\beta_1}\right)\right)\right]^{\frac{1}{\gamma}} V_B. \tag{1.43}$$

$\square$

# Chapter 2

# A Model on Default and Recovery Times of Defaultable Bonds

## 2.1 Introduction

In order to analyze and value defaultable securities, we firstly need to set up a framework describing default event. Issuers may default for different reasons, for example lack of operating earnings, failure of maintaining certain financial ratios,[1] bankruptcy, fraud, etc. The two most popular approaches for modeling default are the structural and the intensity-based frameworks. In the first setup, pioneered by Merton (Merton 1974), default occurs because the value of the issuer's assets falls below an acceptable level. In the second framework default is a random event described by a point process unobservable on the default-free market. In this case, default probability may be modeled in order to include some of the factors mentioned above. Intensity-based models are presented in Duffie, Schroeder and Skiadas (Duffie et al. 1996), Duffie and Singleton (Duffie & Singleton 1999), Jarrow and Yu (Jarrow & Yu 2001) and Collin-Dufresne et al. (Collin-Dufresne et al. 2004) among others. A further essential issue when modeling defaultable securities is the formulation of recovery payment in case of default. In structural models, recovery payment is defined as the remaining value of the issuer's assets after or at default time, while in intensity-based setups recovery payment is usually conceived as a fraction of a

[1] For example debt-to-equity and debt-to-assets ratio among others.

similar but riskless security, a fraction of a fixed money amount or as a fraction of its pre-default value. In case of default event, both valuation methodologies assume that recovery payment occurs either at maturity or at default time, which clearly is not always consistent with the real world. We drop the restrictive assumption that recovery time must match either default time or maturity.

In addition, one of the main challenges that arises from market observations is the need to incorporate correlation between default probability and recovery payment. The empirical analysis in Altman et al. (Altman et al. 2005) and Frye (Frye 2000) demonstrates that there exits a strong relationship between default and recovery rates. Furthermore, recovery payments of defaulted companies may also depend on economic-wide factors as documented in Acharya et al. (Acharya et al. 2007). Results of their study indicate that recovery payments are lower during economy-downturns and large during economy-upturns. Under the assumption of recovery payment at maturity or default, recovery rates reflect only the state of the economy at those points in time and will be inconsistent with empirical observations if real recovery differs from those points in time. By letting recovery time occur at any time after default, recovery rates will reflect the actual market conditions at the time of payment which is in accordance with Acharya et al. (Acharya et al. 2007). In line with this observation, we justify introducing a framework of defaultable bonds where recovery time occurs at any time after default. Moreover, in our model we can straightforwardly incorporate economic factors in default probabilities and in recovery payments. In this sense, our model allow for integration of real world aspects that neither intensity-based nor structural models can offer.

Aware of the weaknesses of intensity-based and structural models, Jarrow (Jarrow 2001) introduces a new approach of valuing defaultable bonds where recovery rates and default probabilities are correlated and depend on an economy-wide state variable. Within his approach, equity prices depend on default event which results in zero value of equity when default occurs. A zero equity value can only be accepted if the company is liquidated. Thus, in cases when firms are reorganized, the approach of Jarrow cannot be applied because equity value must not necessarily equal zero.

In this chapter we analyze an alternative valuation method defaultable for bonds based on stochastic default and recovery times. The setup considers three methodologies when defining default and recovery times: an *intensity-based approach* where both times are defined via point processes, a *structural approach* where default and recovery times occur at first-passage times of default-free market processes, and a *mixture approach* which is a combination of the previous setups. The separation of default and recovery times allow us to conceive and consider more realistic recovery payments, which can support empirical evidence. In this chapter we face two possible formulation of recovery payments: a *company-specific* and an *economy-wide* approach. For the company-specific approach the underlying determinant of recovery payment is the company's asset value[2] and for the economy-wide factor a market index or cycle-index is used.

By construction, the present model of defaultable bonds can combine aspects of structural and reduced-form models. Bond valuation formulas as in Merton (Merton 1974) and as in Duffie and Singleton (Duffie & Singleton 1999) can be derived from the following general specification. Moreover, the pricing formula derived in this chapter provides a pre-default bond value, as structural and intensity-based valuation models, and additionally a post-default-pre-recovery bond value. The latter is known as *distressed* value.

This chapter is organized as follows. Section 2 introduces default and recovery processes as well as the general market structure. In Section 3 we derive the main result of the paper which is the price of defaultable bonds with stochastic recovery and default times. The same section provides an analytical example of our valuation formula for the intensity-based approach. We discuss company-specific and an economy-wide specifications of recovery payments in Section 4. In Section 5 we cover examples and applications of those recovery payments. Section 6 concludes the chapter.

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[2]Current and liquid asset as well as marketable securities can be consider as underlying factors of recovery. Nonetheless, all company's assets may be considered when the company is forced to liquidation.

## 2.2 Framework

### 2.2.1 Financial Market

Consider a financial market embedded in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a complete and right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ representing arrival of overall information over time. Furthermore, assume the market is arbitrage-free and let $\mathbb{P}$ be an equivalent martingale measure.

Securities are discounted with the instantaneous, continuously compounded interest rate or short-rate $r_t$. For instance let the interest rate be a *càdlàg* $\mathcal{F}_t$-adapted process and let the bank account or money-market account at time $t \geq 0$ be

$$B_t = e^{\int_0^t r_u du},$$

and the value, at time $t$, of a default-free zero-coupon bond with face value 1 and maturity $T$ be given by

$$P(t,T) = \mathbb{E}^{\mathbb{P}}\left[ B_T^{-1} B_t \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}}\left[ e^{-\int_t^T r_u du} \middle| \mathcal{F}_t \right].$$

Additionally, consider two *càdlàg* $\mathcal{F}_t$-adapted stochastic processes $Y_t$ and $Z_t$ representing the value of the assets of the firm and the recovery payment, respectively. In Section 2.4 we provide some examples of the recovery process $Z_t$ which is defined as a function of the assets' value $Y_t$ which is seen as a *solvency proxy* of the firm. Since the functional dependency between $Z_t$ and $Y_t$ is not required to derive the pricing formulas below, we introduce these processes here without any further specification. Moreover, we consider an $\mathbb{R}^n$-valued, *càdlàg* $\mathcal{F}_t$-adapted stochastic process $V_t = (V_t^1, \ldots, V_t^n)$ which describes other relevant state variables observed in the financial market. In particular, under the structural and mixture approaches presented at the end of this section, some $V_t^i$ represent additional *solvency proxies* of the firm.[3] We assume that all processes of the default-free market are traded.[4]

On the default-free market information is generated by the interest rate, the value

[3]The case $V_t^i = Y_t$ for some $i = \{1, \ldots, n\}$ for all $t \in \mathbb{R}_+$ is not ruled out.

[4]Instead we can assume that the process $V_t$ is not traded and that there are traded processes $V_{i,t}$ with $i = 1, \ldots, k$, $k \in \mathbb{N}$, which replicate $V_t$.

of the assets, by recovery payment and by other state variables, i.e. information is generated by the real-valued vector $\Sigma_t = (r_t, Y_t, Z_t, V_t)$, the *state process*, such that for all[5] $t \geq 0$

$$\mathcal{G}_t := \sigma\left(\Sigma_u : 0 \leq u \leq t\right), \tag{2.1}$$

where $\mathcal{G}_t \subseteq \mathcal{F}_t$ for any $t \in \mathbb{R}_+$. Recovery payment is defined on the default-free market because it depends on solvency of the firm as well as refinancing options, which are all observed on the default-free market.

### 2.2.2 Default Time and Recovery Time

Before introducing the definitions of default and recovery times consider the following structure of the filtration $(\mathcal{F}_t)_{t\geq 0}$. Let the complete and right-continuous sub-filtrations[6] $(\mathcal{G}_t)_{t\geq 0}$, $(\mathcal{H}_t)_{t\geq 0}$ and $(\mathcal{H}^*_t)_{t\geq 0}$ be such that for any $t \in \mathbb{R}_+$,

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t \vee \mathcal{H}^*_t,$$

i.e. the filtration $\mathcal{F}_t$ coincides with the smallest $\sigma$-field containing $\mathcal{G}_t$, $\mathcal{H}_t$ and $\mathcal{H}^*_t$. We define below the sub-filtrations $\mathcal{H}_t$ and $\mathcal{H}^*_t$, which represent the information flow generated by default and recovery times, respectively. For the intensity-based approach we demand $\mathcal{H}_t \nsubseteq \mathcal{G}_t$ and $\mathcal{H}^*_t \nsubseteq \mathcal{G}_t$ for any $t \in \mathbb{R}_+$, for the structural approach $\mathcal{H}_t, \mathcal{H}^*_t \subseteq \mathcal{G}_t$ for all $t \in \mathbb{R}_+$, and for the mixture approach either $\mathcal{H}_t \nsubseteq \mathcal{G}_t$ or $\mathcal{H}^*_t \nsubseteq \mathcal{G}_t$ for any $t \in \mathbb{R}_+$.

#### Intensity-Based Approach

In an intensity-based context, default event is usually modeled by an $\mathcal{F}_t$-adapted point process $N_t$, in the sense, that default occurs at the first jump of $N_t$. Analogously, we define a process $N_t$ whose jump represents default event. For this and following Jarrow and Yu (Jarrow & Yu 2001) and Bielecki and Rutkowski (Bielecki & Rutkowski 2004) let $\lambda_t$ be a non-negative $\mathcal{G}_t$-progressively measurable process

[5] We call $\sigma(X)$ the $\sigma$-field generated by the random variable $X$.

[6] Where $(\mathcal{G}_t)_{t\geq 0}$ is defined in (2.1).

such that for all $t \geq 0$

$$\Lambda_t := \int_0^t \lambda_u du \quad < \quad \infty \qquad \mathbb{P} - a.s., \tag{2.2}$$

and

$$\Lambda_0 = 0 \quad and \quad \Lambda_\infty = \infty. \tag{2.3}$$

The process $\lambda_t$ is called the intensity process.

We assume that the underlying probability space on which the state process $\Sigma_t$ is defined is large enough in order to support a unit exponential random variable $\xi_1$ independent of the process[7] $\Sigma_t$. This setup is known as the *canonical construction* of the first jump of a point process, see for example Bielecki and Rutkowski (Bielecki & Rutkowski 2004). Accordingly, default time $\tau$ is given by

$$\tau := \inf \left\{ t \geq 0 : \int_0^t \lambda_u du \geq \xi_1 \right\}, \tag{2.4}$$

the process $N_t := \mathbb{1}_{\{\tau \leq t\}}$ represents the default process and

$$\mathcal{H}_t := \sigma(N_u : u \leq t),$$

the information flow generated by default time. By definition, the distribution function of default time $\tau$ conditioned on $\mathcal{G}_\infty$ is given by

$$\mathbb{P}\left(\{\tau > t\} \,|\mathcal{G}_\infty\right) = e^{-\Lambda_t},$$

where $\mathcal{G}_\infty = \sigma(\Sigma_u : \ u \in \mathbb{R}_+)$, and the unconditional distribution is given by

$$\mathbb{P}\left(\{\tau > t\}\right) = \mathbb{E}^{\mathbb{P}}\left[e^{-\Lambda_t}\right].$$

In addition, by construction we have

$$\mathbb{P}\left(\{\tau > t\} \,|\mathcal{G}_t\right) = \mathbb{E}^{\mathbb{P}}\left[\mathbb{P}\left(\{\tau > t\} \,|\mathcal{G}_\infty\right) |\mathcal{G}_t\right] = e^{-\Lambda_t},$$

and

$$\mathbb{P}\left(\{\tau \leq t\} \,|\mathcal{G}_t\right) = \mathbb{P}\left(\{\tau \leq t\} \,|\mathcal{G}_\infty\right), \tag{2.5}$$

[7]Alternatively, we could defined explicitly the structure of an enlarged probability space such that admits the existence of $\xi_1$ defined above. Since the assumption made above is equivalent to the explicit construction of the enlarged probability space, we opt to present the analysis in the short form and refer for technical details to Lando (Lando 1998) or Bielecki and Rutkowski (Bielecki & Rutkowski 2004).

i.e. $\Lambda_t$ represents the hazard process of $\tau$ with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$. As pointed out in Bielecki and Rutkowski (Bielecki & Rutkowski 2004), note that $\Lambda_t$ is not the hazard process of $\tau$ with respect to $(\mathcal{G}_t)_{t\geq 0} \vee (\mathcal{H}^*_t)_{t\geq 0}$.
Recovery time $\tau^*$ is similarly defined as default time. Let $\eta_t$ be a non-negative $\mathcal{G}_t$-progressively measurable process satisfying the conditions (2.2), (2.3) and consider the process

$$\lambda^*_t := \eta_t \mathbb{1}_{\{\tau \leq t\}}. \tag{2.6}$$

If default event has not occurred, recovery intensity is zero, i.e $\lambda^*_t = 0$ for $t < \tau$. After default $t > \tau$, we have $\lambda^*_t = \eta_t$. Moreover, we assume again that the probability space is large enough such that it admits an additional independent unit exponential random variable $\xi_2$, which is independent of the state process $\Sigma_t$ and default time $\tau$. Note that the definition of $\lambda^*_t$ is similar to the definition of equivalent processes in Jarrow and Yu (Jarrow & Yu 2001), where several random default times are model in order to illustrate counterparty risk.[8] Within our model, recovery time is defined as follows.

**Definition 2.2.1.** *Recovery time $\tau^*$ is given by*

$$\tau^* := \inf\left\{t \geq 0 : \int_0^t \lambda^*_u du \geq \xi_2\right\},$$

*where $\lambda^*_t$ is given in (2.6), the process $N^*_t := \mathbb{1}_{\{\tau^* \leq t\}}$ represents the recovery process and*

$$\mathcal{H}^*_t := \sigma(N^*_u : u \leq t),$$

*the information flow generated by recovery time.*

Analogous to the arguments above and since $\xi_2$ is independent of $\Sigma_t$ and $\tau$, the conditional distribution of recovery time $\tau^*$ is given by[9]

$$\mathbb{P}\left(\{\tau^* > t\} \,|\mathcal{G}_\infty \vee \mathcal{H}_\infty\right) = e^{-\Lambda^*_t},$$

where $\Lambda^*_t = \int_0^t \lambda^*_u du$. As previous and by construction of recovery time we have

$$\mathbb{P}\left(\{\tau^* \leq t\} \,|\mathcal{G}_t \vee \mathcal{H}_t\right) = \mathbb{P}\left(\{\tau^* \leq t\} \,|\mathcal{G}_\infty \vee \mathcal{H}_\infty\right), \tag{2.7}$$

[8]Particularly, default time $\tau$ and recovery time $\tau^*$ correspond to default times of primary and secondary firms in Jarrow and Yu (Jarrow & Yu 2001), respectively.

[9]Again, we set $\mathcal{H}_\infty = \sigma(N_u : u \in \mathbb{R}_+)$.

i.e. $\Lambda_t^*$ represents the hazard process of $\tau^*$ with respect to the filtration $(\mathcal{G}_t)_{t\geq 0} \vee (\mathcal{H}_t)_{t\geq 0}$. In order to maintain the economic interpretation, recovery event should never occur before default event. For this regard the assumption below.

**Assumption 2.2.2.** *For all $t \geq 0$,*

$$\mathbb{P}\left(\{\tau > t\} \cap \{\tau^* \leq t\}\right) = 0 \qquad \textit{and} \qquad \mathbb{P}\left(\{\tau > t\} \cap \{\tau^* \leq t\} \,|\mathcal{G}_t \vee \mathcal{H}_t\right) = 0.$$

*Particularly, if the conditional probability of default time occurring after time $t$ given that recovery time has already occurred is well defined, then it must be zero, i.e.*

$$\mathbb{P}\left(\{\tau > t\} \,|\{\tau^* \leq t\}\right) = 0,$$

*for all $t \in \mathbb{R}_+$ with $\mathbb{P}(\{\tau^* \leq t\}) > 0$.*

Since we are interested in valuing defaultable bonds, the probability of default and recovery is supposed to be positive. We assume that neither default nor recovery occur at the origin but some time later. Formally, for $\nu \in \{\tau, \tau^*\}$

$$\mathbb{P}(\{\nu < +\infty\}) = 1, \qquad \mathbb{P}(\{\nu = 0\}) = 0 \qquad \text{and} \qquad \mathbb{P}(\{\nu > t\}) > 0 \quad \text{for all} \quad t \geq 0.$$

### Structural Approach

Alternatively, default and recovery times can be defined by some $\mathbb{R}^n$-valued, $\mathcal{G}_t$-adapted process $V_t$ which represents the solvency of the firm. Accordingly, let default time $\tau$ be defined as the first-passage time of the real-valued, *càdlàg* $\mathcal{G}_t$-adapted process $V_t^1$ such that default time is given by

$$\tau := \inf\left\{t \geq 0 : V_t^1 \leq \kappa^1\right\}, \tag{2.8}$$

where $\kappa^1 \in \mathbb{R}$. We denote the filtration generated by default event by

$$\mathcal{H}_t = \sigma(V_u^1 : 0 \leq u \leq t), \qquad \text{for any} \quad t \in \mathbb{R}_+.$$

Similarly, let recovery time $\tau^*$ be described by the real-valued, *càdlàg* $\mathcal{G}_t$-adapted processes $V_t^1$ and $V_t^2$ in the following

**Definition 2.2.3.** *Recovery time is defined by*

$$\tau^* := \inf\left\{t \geq 0 : \min_{0\leq u\leq t} V_u^1 \leq \kappa^1, \quad V_t^2 \geq \kappa^2\right\},$$

*where*[10] $\kappa^2 \in \mathbb{R}$. *As previous, the filtration generated by recovery event is denoted by*

$$\mathcal{H}^*_t = \sigma((V^1_u, V^2_u) : \ 0 \le u \le t), \qquad for \ \ any \ \ t \in \mathbb{R}_+.$$

Since $V^1_t$ and $V^2_t$ are $\mathcal{G}_t$-adapted processes, default and recovery times are $\mathcal{G}_t$-stopping times, which implies $\mathcal{H}_t \subseteq \mathcal{H}^*_t \subseteq \mathcal{G}_t = \mathcal{F}_t$ for any $t \in \mathbb{R}_+$. In this section we omit any further interpretation of the solvency proxies $V^1_t$ and $V^2_t$ in order to maintain the introduction of the model as general as possible.

### Mixture Approach

Clearly, we can conceive some defaultable bonds such that default event is governed by a process unobservable in the default-free market and recovery event (after default) by a process observable in the default-free market. In such a case, we are in the setup defined as follows.

**Definition 2.2.4.** *Let default time by defined as in (2.4) and recovery time by*

$$\tau^* := \inf \left\{ t \ge 0 : V^2_t \ge \kappa^2, \quad N_u > 0 \ \ for \ \ 0 \le u \le t \right\},$$

*where* $N_t = \mathbb{1}_{\{\tau \le t\}}$. *The information flow is given by*

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t,$$

*for all* $t \in \mathbb{R}_+$. *We denote this setup as Mixture Approach 1 or MA1.*

Of course, we can think of default event being described by a process observable in the default-free market and recovery event (after default) by a process unobservable in the default-free market. The following definition presents this case.

**Definition 2.2.5.** *Let default time be given by (2.8) and recovery time by Definition 2.2.1. The information flow is given by*

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}^*_t,$$

*for all* $t \in \mathbb{R}_+$. *This setup is denoted as Mixture Approach 2 or MA2.*

[10] Equivalently, we have $\tau^* = \inf \left\{ t \ge \tau : V^2_t \ge \kappa^2 \right\}$.

Note that under $MA2$, the definition of $\lambda_t^*$ has not change. In addition, $\lambda_t^*$ is a non-negative $\mathcal{G}_t$-progressively measurable process since $\mathbb{1}_{\{\tau \leq t\}}$ is a *càdlàg* $\mathcal{G}_t$-adapted process. Again, we postpone interpretation of the underlying stochastic processes $V_t^1$ and $V_t^2$ defining default and recovery times.

## 2.3 Financial Claims

Within this section we revise pricing rules of defaultable bonds considering default time $\tau$ and recovery time $\tau^*$ as previously introduced. Results of Sections 2.3.1 and 2.3.2 are general and are independent of the definition of default and recovery times. Thereafter, we use the general results for the different approaches presented above.

### 2.3.1 Dividend Price Process

By definition of martingale measures,[11] the price process of a dividend stream $D_t$ with settlement date $\theta$ is given by

$$S_t = B_t E^{\mathbb{P}} \left[ B_\theta^{-1} S_\theta + \int_{(t,\theta]} B_u^{-1} dD_u \middle| \mathcal{F}_t \right], \qquad \forall t \leq \theta. \tag{2.9}$$

By assuming that the price process reflects only future dividends, it must necessarily be zero at settlement date, i.e. $S_\theta = 0$. This treatment is usually known as *ex-dividend price process*[12] because past and present dividends are omitted from the price of the analyzed claims. The ex-dividend price process $S_t$ of a financial claim that pays dividends $D_t$, with settlement date $\theta$ is given by

$$S_t = B_t \mathbb{E}^{\mathbb{P}} \left[ \int_{(t,\theta]} B_u^{-1} dD_u \middle| \mathcal{F}_t \right], \qquad \forall t < \theta. \tag{2.10}$$

The dividend process $D_t$ of a defaultable claim consists of three elements: payoff at maturity, payoff during the life of the claim and payment at recovery time. At

[11] Equation (2.9) can be derived from a self-financing strategy and by some martingale arguments as presented in Appendix 2.7.1.

[12] Following the definition in Duffie et al. (Duffie et al. 1996), Duffie and Singleton (Duffie & Singleton 1999), Bielecki and Rutkowski (Bielecki & Rutkowski 2004) and Collin-Dufresne et al.(Collin-Dufresne et al. 2004) among others.

maturity $T$ a claim with notional value $X$ pays its face value if no default has occurred. If default takes place during the life of the claim, payment at maturity is $R$. So payoff at maturity is given by

$$X_T^d = X \cdot \mathbb{1}_{\{\tau > T\}} + R \cdot \mathbb{1}_{\{\tau \leq T\}},$$

where $R$ is a $\mathcal{G}_T$-measurable and bounded random variable. A defaultable claim may offer a stream of payments $A_t$ before maturity and default, which equals

$$\int \mathbb{1}_{\{\tau > u\}} dA_u.$$

Finally, if default occurs before maturity the claim pays at recovery time $\tau^*$ an amount $Z_{\tau^*}$ which is a $\mathcal{G}_t$-adapted, non-negative bounded process. Hence payment at recovery time can be written as

$$\int Z_u d\mathbb{1}_{\{\tau^* \leq u\}}.$$

Combining these elements together we derive the dividend process $D_t$ which is given by

$$D_t = X_T^d \cdot \mathbb{1}_{\{t \geq T\}} + \int_{(0,t]} \mathbb{1}_{\{\tau > u\}} dA_u + \int_{(0,t]} Z_u d\mathbb{1}_{\{\tau^* \leq u\}}, \qquad t > 0. \tag{2.11}$$

### 2.3.2 Zero-Coupon Bonds Valuation

In case of zero-coupon bonds there is no payment stream before maturity and so $A_t = 0$ for all $t \geq 0$. Furthermore, if default occurs recovery payment is made at recovery time and not necessarily at maturity, thus $R = 0$. A defaultable claim will be priced and traded until maturity $T$ in case of no default previous maturity or until recovery time $\tau^*$ if default occurs before maturity. In this sense, the random settlement date of a defaultable claim is given by

$$\theta := T \cdot \mathbb{1}_{\{\tau > T\}} + \tau^* \cdot \mathbb{1}_{\{\tau \leq T\}}.$$

Note that there are only two possible repayment dates for a defaultable zero-coupon bond either $\theta$ or $\tau^*$. Since $S_t$ is the ex-dividend price process of a defaultable claim that pays at maturity or recovery time, $S_t$ given in (2.10) is the *pre-repayment* price, i.e. $S_t$ is only given for $t < \tau^* \wedge \theta$. Hence, combining definitions (2.10) and (2.11) the price process[13] of a zero-coupon bond with face value $X$, maturity $T$ and random settlement[14] $\theta$ is given by

$$S_t = B_t \mathbb{E}^{\mathbb{P}} \left[ \int_{(t,\theta]} B_u^{-1} X_T^d d\mathbb{1}_{\{u \geq T\}} + \int_{(t,\theta]} B_u^{-1} Z_u d\mathbb{1}_{\{\tau^* \leq u\}} \middle| \mathcal{F}_t \right], \qquad \forall t < \tau^* \wedge \theta.$$

Notice that on $\{\theta = \tau^*\}$ payoff at maturity is zero, $X_T^d = 0$, because $R = 0$ and so inducing

$$S_t = B_t \mathbb{E}^{\mathbb{P}} \left[ \mathbb{1}_{\{\tau > T\}} \cdot \int_{(t,T]} B_u^{-1} X_T^d d\mathbb{1}_{\{u \geq T\}} + \int_{(t,\theta]} B_u^{-1} Z_u d\mathbb{1}_{\{\tau^* \leq u\}} \middle| \mathcal{F}_t \right],$$

for all $t < \tau^* \wedge \theta$. Integrating the first term the expression is equivalent to

$$S_t = B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X_T^d \cdot \mathbb{1}_{\{\tau > T\}} + \int_{(t,\theta]} B_u^{-1} Z_u d\mathbb{1}_{\{\tau^* \leq u\}} \middle| \mathcal{F}_t \right], \quad \forall t < \tau^* \wedge \theta. \qquad (2.12)$$

Notice that the integral in (2.12) is zero on $\{\tau > T\}$. Hence, it suffices to consider the price process

$$S_t = B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X_T^d \cdot \mathbb{1}_{\{\tau > T\}} + \mathbb{1}_{\{\tau \leq T\}} \cdot \int_{(t,\tau^*]} B_u^{-1} Z_u d\mathbb{1}_{\{\tau^* \leq u\}} \middle| \mathcal{F}_t \right], \quad \forall t < \tau^* \wedge \theta.$$

By adopting the notation $B^d(t,T) := S_t$ it follows

$$B^d(t,T) = B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X_T^d \cdot \mathbb{1}_{\{\tau > T\}} + \mathbb{1}_{\{\tau \leq T\}} \cdot B_{\tau^*}^{-1} Z_{\tau^*} \middle| \mathcal{F}_t \right], \qquad \forall t < \tau^* \wedge \theta.$$

On the set $\{\tau^* \leq t\}$ the price process is zero. Therefore the price of a defaultable zero-bond results after plugging the definition of $X_T^d$ as follows.

**Proposition 2.3.1.** *The ex-dividend price process $B^d(t,T)$ of a defaultable zero-coupon bond with maturity $T$ and notional value $X$ for $0 < t < T$ is given by*

$$B^d(t,T) = \mathbb{1}_{\{\tau^* > t\}} \cdot B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \cdot \mathbb{1}_{\{\tau > T\}} + B_{\tau^*}^{-1} Z_{\tau^*} \cdot \mathbb{1}_{\{\tau \leq T\}} \middle| \mathcal{F}_t \right].$$

**Remark 2.3.2.** *A special case of this framework is the model introduced in Duffie, Schroeder and Skiadas (Duffie et al. 1996) and Duffie and Singleton (Duffie & Singleton 1999). By letting recovery time equal default time, $\tau = \tau^*$, the intensities of jump processes describing default and recovery must be identical, i.e. $\lambda_t = \lambda_t^*$ for*

[13]From now on all prices of defaultable bonds are ex-dividend.

[14]For a discussion of the validity of the pricing rule $S_t$ see Appendix 2.7.1 and 2.7.2. An alternative derivation of Proposition 2.3.1 can be also found in Appendix 2.7.2.

*all $t \geq 0$. Furthermore, since recovery can never occur after maturity, settlement date must equal maturity, $\theta = T$. So the price process corresponding to definition (2.10) reads*

$$S_t = B_t \mathbb{E}^{\mathbb{P}} \left[ \int_{(t,T]} B_u^{-1} dD_u \middle| \mathcal{F}_t \right], \qquad \forall t < T.$$

*Moreover, the integral of recovery payment in (2.12) is*

$$\int_{(t,T]} B_u^{-1} Z_u d\mathbb{1}_{\{\tau \leq u\}}.$$

*By noticing that $M_t := \mathbb{1}_{\{\tau > t\}} - \Lambda_{t \wedge \tau}$ follows an $\mathcal{F}_t$-martingale under $\mathbb{P}$, the last expression can be divided in two integrals*

$$\int_{(t,T]} B_u^{-1} Z_u \lambda_u \cdot \mathbb{1}_{\{\tau > u\}} du + \int_{(t,T]} B_u^{-1} Z_u dM_u, \qquad t \in [0, T].$$

*If $Z_t$ is a $\mathcal{G}_t$-predictable process, the second integral is a local martingale and so the price process examined by Duffie et al. follows from (2.12)*

$$S_t = B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \cdot \mathbb{1}_{\{\tau > T\}} + \int_{(t,T]} B_u^{-1} Z_u \lambda_u \cdot \mathbb{1}_{\{\tau > u\}} du \middle| \mathcal{F}_t \right], \qquad t < T.$$

*Obviously, information arrival of default and recovery time coincide, that is $\mathcal{H}_t = \mathcal{H}_t^*$ for all $t \geq 0$, implying $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$.*

### 2.3.3 Integral Representation for the Intensity-Based Approach

In order to introduce our results consider the next assumption.

**Assumption 2.3.3.** *Let the expectations*

$$\mathbb{E}^{\mathbb{P}} \left[ e^{-\int_t^T (r_s + \lambda_s) ds} |X| \right],$$
$$\mathbb{E}^{\mathbb{P}} \left[ \int_{(t,\infty)} |Z_u \eta_u| \, e^{-\int_t^u (r_s + \eta_s) ds} du \right],$$

*and*

$$\mathbb{E}^{\mathbb{P}} \left[ \int_{(t,T]} \int_{(t,T]} \left| Z_u \eta_u \mathbb{1}_{\{q \leq u\}} \lambda_q \right| e^{-\int_t^u (r_s + \eta_s \mathbb{1}_{\{q \leq s\}}) ds - \int_t^q \lambda_v dv} du dq \right],$$

*be finite.*

In view of Proposition 2.3.1, the value of a defaultable bond under the intensity-based approach can be expressed via the processes $\lambda_t$ and $\eta_t$ as shown in the following result.

**Proposition 2.3.4.** *Under Assumption 2.3.3 the ex-dividend price process $B^d(t,T)$ of a defaultable zero-coupon bond with maturity $T$ and notional value $X$ for any $0 < t < T$ is given by*

$$B^d(t,T) = \mathbb{1}_{\{\tau>t\}} \cdot (I_t^1 + I_t^2) + \mathbb{1}_{\{\tau\leq t\}}\mathbb{1}_{\{\tau^*>t\}} \cdot I_t^3, \tag{2.13}$$

*where*

$$\begin{aligned} I_t^1 &= \mathbb{E}^{\mathbb{P}}\left[ e^{-\int_t^T (r_s+\lambda_s)ds} X \middle| \mathcal{G}_t \right], \\ I_t^2 &= \mathbb{E}^{\mathbb{P}}\left[ \int_{(t,T]} \int_{(t,T]} Z_u \eta_u \mathbb{1}_{\{q\leq u\}} \lambda_q e^{-\int_t^u (r_s+\eta_s \mathbb{1}_{\{q\leq s\}})ds - \int_t^q \lambda_v dv} du dq \middle| \mathcal{G}_t \right], \end{aligned}$$

*and*

$$I_t^3 = \mathbb{E}^{\mathbb{P}}\left[ \int_{(t,\infty)} Z_u \eta_u e^{-\int_t^u (r_s+\eta_s)ds} du \middle| \mathcal{G}_t \right].$$

*Proof.* See Appendix 2.7.3.

□

The price of defaultable bonds is divided into pre-default and post-default-pre-recovery prices. The pre-default value is represented by $I^1 + I^2$ and it is influenced by default and recovery parameters $\lambda_t$ and $\eta_t$, respectively. In the case default event occurs, then the term $\mathbb{1}_{\{\tau>t\}}(I^1 + I^2)$ vanishes and the price of the financial claim equals $I^3$, which is commonly called *distressed* price (post-default bonds are usually denoted as *distressed debt*). In the expectation $I^3$ only the recovery parameter $\eta_t$ is present and the upper limit of the integral is $\infty$ reflecting the randomness of recovery time $\tau^*$. Furthermore, note that nothing has been said about the joint density of recovery payment $Z_t$, default intensity $\lambda_t$ and the process $\eta_t$ conditioned on $\mathcal{G}_t$. Hence, correlation between default event and recovery payment can be introduced in order to be consistent with empirical evidence.[15]

[15] See Altman et al. (Altman et al. 2005) and Frye (Frye 2000).

**Example**

Regard now a simple application for the valuation rule of Proposition 2.3.4 of defaultable zero-coupon bonds under the intensity-based approach. Let default intensity $\lambda_t$ and the process $\eta_t$ be deterministic and constant such that $\lambda_t = \lambda > 0$ and $\eta_t = \eta > 0$ for any $t \geq 0$. Furthermore, suppose that current time $t = 0$ and assume the $\sigma$-field $\mathcal{G}_0$ is trivial. Recovery payment is a deterministic fraction of par value, i.e. $Z_t = \phi$ for all $t \geq 0$ where $0 \leq \phi \leq X$ (we implicitly suppose that par value is deterministic $X > 0$). In view of Cox, Ingersoll and Ross (Cox et al. 1985), the dynamics of the instantaneous short rate process are

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t,$$

under $\mathbb{P}$ where $r_0, k, \theta, \sigma > 0$ and $2k\theta > \sigma^2$. Since the Cox-Ingersoll-Ross setup corresponds to the class of affine term-structure models[16], the value at time $t$ of a default-free zero-coupon bond with maturity $T$ is given by

$$P(t,T) = A(t,T)e^{-C(t,T)r_t},$$

where

$$\begin{aligned} A(t,T) &= \left(\frac{2h\exp\{(k+h)(T-t)/2\}}{2h + (k+h)(\exp\{(T-t)h\} - 1)}\right)^{2k\theta/\sigma^2}, \\ C(t,T) &= \frac{2(\exp\{(T-t)h\} - 1)}{2h + (k+h)(\exp\{(T-t)h\} - 1)}, \\ h &= \sqrt{k^2 + 2\sigma^2}. \end{aligned}$$

Notice, that we can still apply Proposition 2.3.4 for $t = 0$, which consists of

$$B^d(0,T) = I_0^1 + I_0^2,$$

since the third summand in (2.13) vanishes[17]. Hence, the first term is given by

$$I_0^1 = \mathbb{E}^{\mathbb{P}}\left[B_T^{-1}e^{-\lambda T}X\right] = e^{-\lambda T}XP(0,T). \tag{2.14}$$

By applying Fubini-Tonelli's theorem repeatedly, Appendix 2.7.4 shows that for $\lambda \neq \eta$ the second term is given by

$$I_0^2 = \frac{\phi\eta\lambda}{\eta - \lambda}\int_{(0,T]} \left(e^{-\lambda u} - e^{-\eta u}\right) A(0,u)e^{-C(0,u)r_0}du.$$

[16]See Brigo and Mercurio (Brigo & Mercurio 2006) for details.

[17]Recall that $\mathbb{P}(\{\tau = 0\}) = 0$ is assumed.

In order to appreciate the insights of modeling recovery time as a random time that may differ from default time, we consider the value of the same bond under the assumption that recovery and default events occur simultaneously, i.e. $\tau = \tau^*$ $\mathbb{P}$-a.s., and we denote that bond's price by $\tilde{B}^d(t,T)$. By assuming $\tau = \tau^*$, default and recovery processes coincide, as well as the corresponding $\sigma$-fields. In particular, we have $\mathcal{H}_t = \mathcal{H}_t^*$ for all $t \geq 0$, which implies $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$ for all $t \geq 0$. Hence, the ex-dividend price process of a defaultable zero-coupon bond with par value $X$ and maturity $T$ under the assumption $\tau = \tau^*$ $\mathbb{P}$-a.s. is given by

$$\tilde{B}^d(t,T) = B_t \mathbb{E}^{\mathbb{P}} \left[ \int_{(t,T]} B_u^{-1} dD_u \middle| \mathcal{F}_t \right], \qquad \forall t < \tau,$$

where

$$D_t = X \cdot \mathbb{1}_{\{t \geq T\}} + \int_{(0,t]} Z_u d\mathbb{1}_{\{\tau \leq u\}}, \qquad t > 0.$$

Following the same arguments of Section 2.3.2, the price of the defaultable security under $\tau = \tau^*$ $\mathbb{P}$-a.s for $t < T$ is given by

$$\tilde{B}^d(t,T) = \mathbb{1}_{\{\tau > t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \cdot \mathbb{1}_{\{\tau > T\}} + B_\tau^{-1} Z_\tau \mathbb{1}_{\{\tau \leq T\}} \middle| \mathcal{F}_t \right],$$

which is well known[18] to be equivalent to

$$\tilde{B}^d(t,T) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}} \left[ e^{-\int_t^T (r_s + \lambda_s) ds} X + \int_{(t,T]} \lambda_u Z_u e^{-\int_t^u (r_s + \lambda_s) ds} du \middle| \mathcal{G}_t \right].$$

In the present example this formula yields the following expression

$$\tilde{B}^d(0,T) = I_0^1 + \tilde{I}_0^2,$$

where $I_0^1$ is given in (2.14) and

$$\tilde{I}_0^2 = \lambda \phi \int_{(0,T]} A(0,u) e^{-C(0,u) r_0 - \lambda u} du,$$

whose derivation is shown in Appendix 2.7.4. Differences between $B^d(t,T)$ and $\tilde{B}^d(t,T)$ are explained by differences between $I_0^2$ and $\tilde{I}_0^2$, which are generated by the existence of the recovery parameter $\eta$. In order to highlight these, we present below a numerical illustration for which we additionally consider the credit spread $s(t,T)$ at time $t = 0$ of a defaultable bond $B^d(0,T)$ with face value $X$ defined by

$$s(0,T) = -\frac{1}{T} \ln \left( \frac{B^d(0,T)}{P_X(0,T)} \right),$$

[18] See Bielecki and Rutkowski (Bielecki & Rutkowski 2004) for example.

where $P_X(0,T) = P(0,T)X$.

For the numerical example, we assume the following values of the Cox-Ingersoll-Ross term structure parameters corresponding to empirical observations[19]

$$k = 0.0373, \qquad \theta = 0.0697, \qquad \sigma = 0.0283.$$

Furthermore, we set the remaining parameters of the model as follows.

$$X = 1000, \qquad T = 5, \qquad \phi = 0.5, \qquad r_0 = 0.0295, \qquad \lambda = 0.04, \qquad \eta = 0.064.$$

Table 2.1 shows the differences in the components of the defaultable bonds $B^d(0,T)$ and $\tilde{B}^d(0,T)$. Recall that the latter price excludes the possibility of $\tau^* \neq \tau$. Hence, we abbreviate this description and refer to $\tilde{B}^d(0,T)$ as the price ignoring $\tau^* \neq \tau$ and to $B^d(0,T)$ as the price including $\tau^* \neq \tau$.

For the given parameters, the value of a default-free zero coupon bond with par

| | Bond Price | $I_0^1$ | $I_0^2$ or $\tilde{I}_0^2$ | Spread |
|---|---|---|---|---|
| Including $\tau^* \neq \tau$ | 706.5763 | 694.4216 | 12.1547 | 365.2960 |
| Excluding $\tau^* \neq \tau$ | 778.4423 | 694.4216 | 84.0207 | 171.5687 |

Table 2.1: Price Differences Depending on Recovery Time Risk

value $X$ and maturity $T$ is $P_X(0,T) = 848.1684$. Figures 3.1 and 2.2 display some interesting comparative statics.

### 2.3.4 Valuation under the Structural Approach

Recall pricing formula of Proposition 2.3.1

$$B^d(t,T) = \mathbb{1}_{\{\tau^*>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \mathbb{1}_{\{\tau>T\}} + B_{\tau^*}^{-1} Z_{\tau^*} \mathbb{1}_{\{\tau\leq T\}} \middle| \mathcal{F}_t \right], \qquad \text{for } \ t < T,$$

or equivalently,

$$B^d(t,T) = \mathbb{1}_{\{\tau^*>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \mathbb{1}_{\{\tau>T\}} \middle| \mathcal{F}_t \right] + \mathbb{1}_{\{\tau^*>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_{\tau^*}^{-1} Z_{\tau^*} \mathbb{1}_{\{\tau\leq T\}} \middle| \mathcal{F}_t \right].$$

The determinants of default and recovery events are the processes $V_t^1$ and $V_t^2$. In general, these solvency proxies cannot be assumed to be independent of the default-free interest rate $r_t$. For example, let $V_t^1$ be the market value of the stock of the issuer

[19]See Nowman (Nowman 1997).

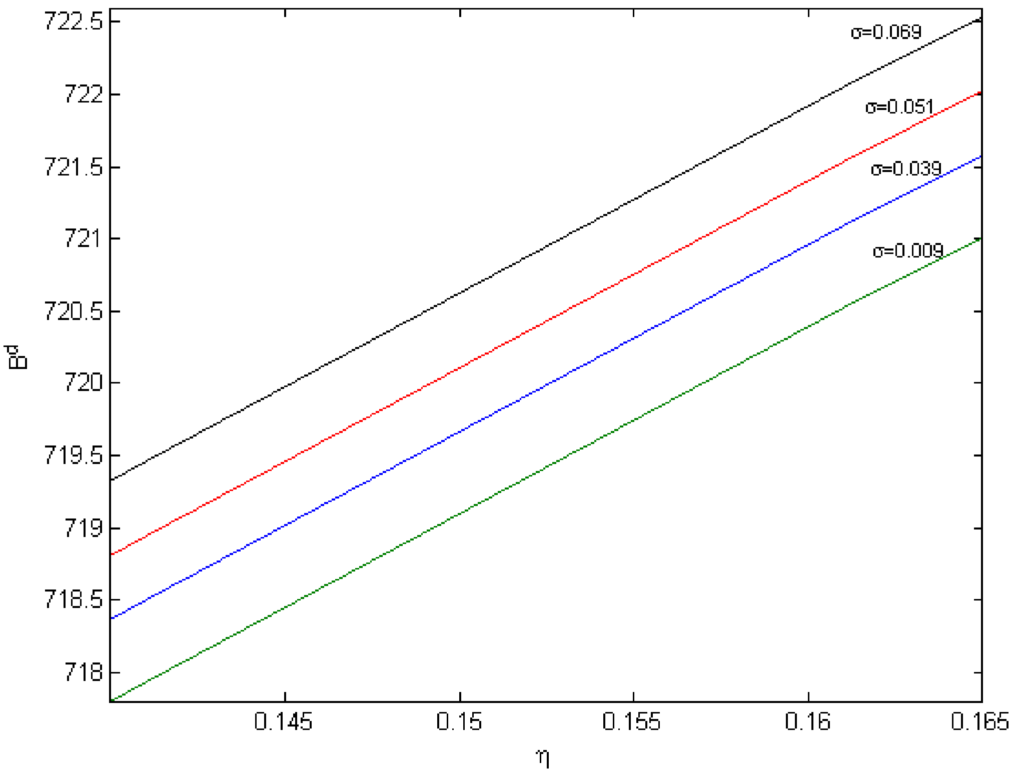


Figure 2.1: Defaultable Bond Price $B^d(0,T)$ Including $\tau^* \neq \tau$

firm. In addition, let the firm have variable-coupon debentures of higher seniority with a coupon formula based on the default-free interest rate $r_t$. If these debentures are protected by covenants, then $V_t^1$ cannot be supposed to be independent of $r_t$ because the analyzed zero-coupon bond has a lower priority. However, in cases where default time and bank account are independent, we find the following pricing formula.

**Corollary 2.3.5.** *Assume default time $\tau$ and default-free interest rate $r_t$ are independent. The value of a zero-coupon bond with maturity $T$ and par value $X$ for $0 < t < T$ is given by*

$$B^d(t,T) = \mathbb{1}_{\{\tau^*>t\}} P_X(t,T) \mathbb{P}\left(\{\tau > T\} \,|\mathcal{G}_t\right) + \mathbb{1}_{\{\tau^*>t\}} B_t \mathbb{E}^{\mathbb{P}}\left[ B_{\tau^*}^{-1} Z_{\tau^*} \mathbb{1}_{\{\tau \leq T\}} \middle| \, \mathcal{G}_t \right].$$

*Proof.* Follows straightforwardly from the independence of $\tau$ and $r_t$ and from the property of the structural approach of $\mathcal{G}_t = \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.

□

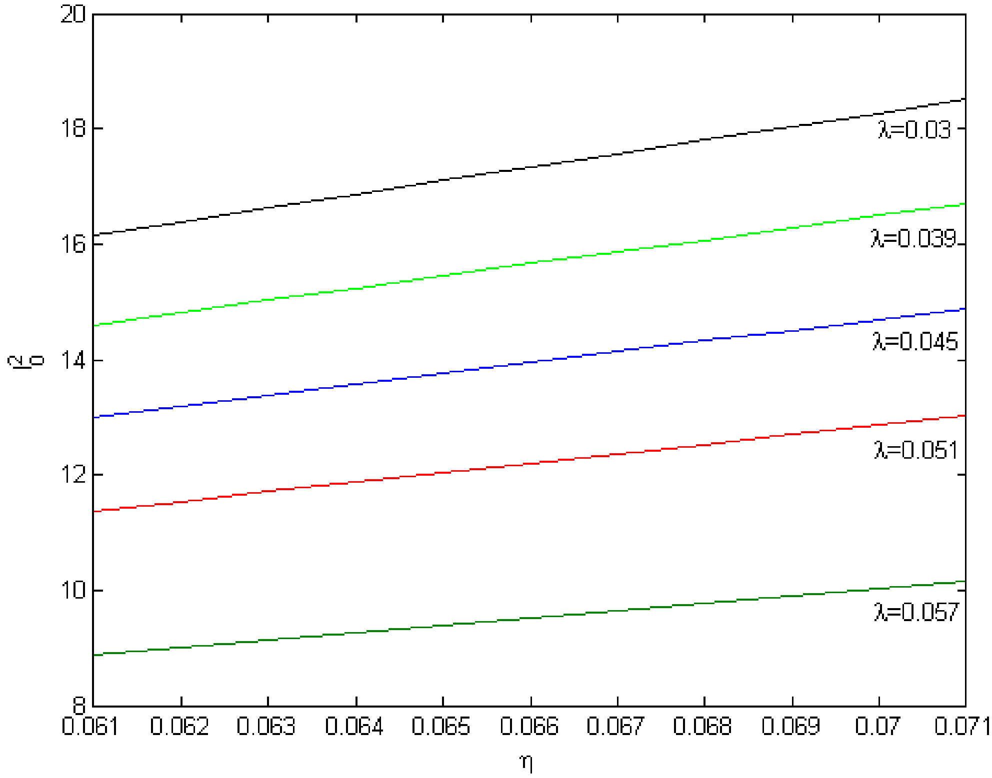


Figure 2.2: $I_0^2$ Including $\tau^* \neq \tau$

In general, the computation of the second expectation in Corollary 2.3.5 is a complex task. Black and Cox (Black & Cox 1976), Kim et al. (Kim et al. 1993), Longstaff and Schwartz (Longstaff & Schwartz 1995) and Briys and de Varenne (Briys & de Varenne 1997) among others examine defaultable bonds within a structural framework considering recovery payment at default time. In the spirit of those analysis, let recovery process be given by $Z_t := B_t\phi$ where $\phi$ is a constant with $0 \leq \phi \leq X$. Consequently, for $t < T$ we have

$$B_t \mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1} Z_{\tau^*} \mathbb{1}_{\{\tau \leq T\}} \middle| \mathcal{G}_t\right] = B_t \phi \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau \leq T\}} \middle| \mathcal{G}_t\right] = B_t \phi \mathbb{P}\left(\{\tau \leq T\} \middle| \mathcal{G}_t\right).$$

Hence, for $t < T$ the price of a defaultable bond is given by

$$B^d(t,T) = \mathbb{1}_{\{\tau^* > t\}} P_X(t,T) \mathbb{P}\left(\{\tau > T\} | \mathcal{G}_t\right) + \mathbb{1}_{\{\tau^* > t\}} B_t \phi \mathbb{P}\left(\{\tau \leq T\} \middle| \mathcal{G}_t\right).$$

Note that we must take great caution when choosing the term structure model otherwise the value of the bond may explode on $\{\tau \leq T\}$.

### 2.3.5 Pricing under the Mixture Approach

From previous comments in the derivation of a pricing rule under the structural approach, the reader may already have noted that in most cases we have to recur to the joint density function of default and recovery times in order to compute the bonds value. Within the mixture approach this is also true. However, there are some cases where we can avoid finding a joint density. We assume again the recovery payment process $Z_t = B_t\phi$, where $0 \leq \phi \leq X$. Using the same assumption of independence between default time and default-free interest rate as before, the value of a zero-coupon bond with maturity $T$ and par value $X$ under both $MA1$ and $MA2$ is given by

$$B^d(t,T) = \mathbb{1}_{\{\tau^*>t\}} P_X(t,T)\mathbb{P}\left(\{\tau > T\} \left|\mathcal{F}_t\right.\right) + \mathbb{1}_{\{\tau^*>t\}} B_t \phi \mathbb{P}\left(\{\tau \leq T\} \middle| \mathcal{F}_t\right),$$

for $t < T$. Note that the expectations are conditioned on $\mathcal{F}_t$ and not on $\mathcal{G}_t$. For $MA1$ observe that for $t < T$

$$\begin{aligned}
\mathbb{1}_{\{\tau^*>t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau>T\}}\middle|\mathcal{F}_t\right] &= \mathbb{1}_{\{\tau^*>t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau>T\}}\middle|\mathcal{G}_t \vee \mathcal{H}_t\right] \\
&= \mathbb{1}_{\{\tau^*>t\}}\frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau>T\}}\middle|\mathcal{G}_t\right]}{\mathbb{P}\left(\{\tau>t\}\left|\mathcal{G}_t\right.\right)} \\
&= \mathbb{1}_{\{\tau^*>t\}}\frac{\mathbb{P}\left(\{\tau>T\}\left|\mathcal{G}_t\right.\right)}{\mathbb{P}\left(\{\tau>t\}\left|\mathcal{G}_t\right.\right)} = \mathbb{1}_{\{\tau^*>t\}} e^{-\int_t^T \lambda_s ds},
\end{aligned}$$

since default process $N_t$ is a doubly stochastic Poisson process with intensity $\lambda_t$. Similarly, for $t < T$ we have

$$\begin{aligned}
&\mathbb{1}_{\{\tau^*>t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau\leq T\}}\middle|\mathcal{F}_t\right] \\
&= \mathbb{1}_{\{\tau^*>t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{t<\tau\leq T\}}\middle|\mathcal{F}_t\right] + \mathbb{1}_{\{\tau^*>t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau\leq t\leq T\}}\middle|\mathcal{F}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau\leq T\}}\middle|\mathcal{F}_t\right] + \mathbb{1}_{\{\tau\leq t\}}\mathbb{1}_{\{\tau^*>t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau\leq T\}}\middle|\mathcal{F}_t\right].
\end{aligned} \tag{2.15}$$

By previous arguments the first summand in (2.15) equals

$$\mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau\leq T\}}\middle|\mathcal{G}_t\right]}{\mathbb{P}\left(\{\tau>t\}\left|\mathcal{G}_t\right.\right)} = \mathbb{1}_{\{\tau>t\}}\left(1 - e^{-\int_t^T \lambda_s ds}\right)$$

Hence, the pre-default value of zero-coupon bonds is given by

$$B^d(t,T) = P_X(t,T)e^{-\int_t^T \lambda_s ds} + \phi B_t\left(1 - e^{-\int_t^T \lambda_s ds}\right), \qquad t < T,$$

and the post-default-pre-recovery value of distressed debt is given by

$$B^d(t, T) = \phi,$$

since the second summand in (2.15) is one on $\{\tau \leq t\}$. Hence, after default the value of the bond stays constant. If we neglect the possibility of $\tau^* \neq \tau$, i.e. recovery occurring at default, then at default time we receive $\phi$ amount of money which we can invest in the bank account and earn the default-free interest rate on it. When we allow for $\tau^* \neq \tau$, we receive $\phi$ at recovery and we forgo interests for the time period $\tau^* - \tau$. Although in this simple example we avoid modeling the joint density function of recovery and default time, the consequences of letting recovery time differ from default time still have an impact in the value of defaultable bonds.

## 2.4 Recovery Modeling

Once a company defaults on its outstanding debt, recovery payment is most likely lower than the original contractual arrangement. Recovery payment depends on the causes of default. In case of a technical default, i.e. when a protective covenant has been violated, recovery payment may be close to the face value. If default is originated by insolvency or bankruptcy recovery depends on whether the company is reorganized or liquidated. In case the company is reorganized, its capital structure is modified, its debt refinanced and the company keeps its operations. In a liquidation the company ceases to exist and its assets are sold in order to repay creditors. Hence, recovery depends on the *solvency* of the issuer company at repayment date.
*Firm-specific* solvency proxies can be regarded as cash flows, net income, market value of current assets, impairment-adjusted total assets and financial statement liquidity ratios among others. Refinancing possibilities are also relevant for covering outstanding debt in case of default. Additional to company-specific factors it is also necessary to consider an *economy-wide* proxy reflecting company solvency within the current economic situation. A company in financial distress is most likely to have more problems reorganizing its capital structure during market-wide financial crisis or industry-wide crashes than during normal economic conditions.

In the present analysis we discuss the process governing recovery rates both from a firm-specific and from an economy-wide point of view.

### 2.4.1 Firm-specific Solvency Proxy

Suppose that liquid assets of a company, which could effortlessly be sold to repay debt, are represented by a càdlàg $\mathcal{G}_t$-adapted process $\tilde{Y}_t$. Let $\tilde{Y}_t$ be the portion of assets designated to repayment of the face value $X$ of a certain bond. Furthermore, suppose that a company may obtain capital from refinancing which is represented by a *càdlàg* $\mathcal{G}_t$-adapted process $RF_t$. Thus, in case of default the firm's available capital to redeem its obligations at recovery time equals $Y_{\tau^*} = \tilde{Y}_{\tau^*} + RF_{\tau^*}$. Of course, we can assumed that sale of assets and refinancing occur at an earlier point of time $t'$ than at recovery time $\tau^*$. In this case, we can model $Y_t$ as a constant after $t'$ or as the value of assets' sale and refinancing at time $t'$ and the corresponding accrued interest of the bank account for the time elapse $\tau^* - t'$, i.e. $Y_{\tau^*} = B_{\tau^*} B_{t'}^{-1} Y_{t'}$ by abuse of notation. If $Y_t$ represents the value of regularly traded financial instruments, we can also make the accurate assumption that recovery payment at recovery time is defined as some function of the market price of those financial instruments. Independently of the modeling and interpretation of $Y_t$, consider the following definition of recovery payment $Z_t$.

**Specification 2.4.1** (MR)**.** *Once defaulted, repayment is a fraction $\delta$, with $\delta \in [0, 1]$ of the remaining of face value and current assets, i.e.*

$$Z_{\tau^*} = \delta(X - \max(X - Y_{\tau^*}, 0)).$$

The fraction $\delta$ may be specified stochastically at costs of rising complexity for estimation methods. For practical means, $\delta$ can be fixed as the maximum recovery rate observed from historical data within that economy and industry or equal to one in order to reduce assumptions and calibrations. Following these ideas we denote this specification as *maximum recovery* (MR).

### 2.4.2 Economy-wide Solvency Proxy

During financial crisis defaults are more common than during financial stability, recovery rates tend to decrease and their volatility augments.[20] In an economy-wide crisis, reorganization of companies in financial distress is more difficult since there are fewer institutions willing to provide capital for reorganization. Additional, defaulted companies may produce a contagion effect[21] on their creditors and other companies, increasing defaults and lowering recoveries.

In order to capture the market-wide business condition, let the process $Y_t$ represent the "state" of the economy. One can regard this process as an index reflecting financial-economic cycles or a benchmark of economic climate. Further, define economy-wide distress whenever $Y_t < K$ for some constant and deterministic $K$. Hence, the specification of an economy-wide recovery rate can take the following form.

**Specification 2.4.2** (TSR). *Let $\delta > \gamma$ be two recovery rates with $\delta, \gamma \in [0,1]$ for different scenarios:*

$$Z_{\tau^*} = \left(\delta \mathbb{1}_{\{Y_{\tau^*} \geq K\}} + \gamma \mathbb{1}_{\{Y_{\tau^*} < K\}}\right) X,$$

We denote this specification as *two-scenario recovery* (TSR). A generalization can be achieved by differentiating between several scenarios.

**Remark 2.4.3.** *Notice that the firm-specific framework MR can be adjusted to reflect economy-wide effects. This can be done by letting $\delta$ be a function of macroeconomic variables. Similarly, the economy-wide specification, TSR, can be defined such that $\delta$ and $\gamma$ are recovery rates determined by firm-specific factors.*

## 2.5 A Special Case: Intensity-Based Approach with Deterministic Recovery Time $\tau^* = T$

Stochastic recovery time, which may differ from default time, introduces additional uncertainty in models of defaultable claims, which leads to changes in pricing rules

[20] See Altman et al. (Altman 2006)

[21] See Jarrow and Yu (Jarrow & Yu 2001).

as can be seen comparing Proposition 2.3.4 and results of conventional models, e.g. Bielecki and Rutkowski (Bielecki & Rutkowski 2004). However, within our intensity-based approach and by fixing recovery time equal to maturity, differences of bond prices between conventional models and the present study remain. Traditionally, intensity-based models consider recovery processes corresponding to fractions of face, treasury or pre-default value. However, in case of liquidation recovery payment depends mainly on solvency of the firm while in case of reorganization depends on refinancing alternatives. In the present framework recovery process reflects the firms solvency and not necessarily the value of other bonds.

When setting $\tau^* = T$, it becomes clear that settlement date in definition (2.10) equals maturity, that is $\theta = T$. Moreover, information arrival of recovery time is neglected. Accordingly, arrival of all information available on the market is driven by default-free processes and default time, that is $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. Hence, the value of a defaultable zero-coupon bond is given by

**Proposition 2.5.1.** *The arbitrage-free price at $t < T$ of a defaultable zero-coupon bond with face value $X$ and recovery payment at maturity $R$ is given by*

$$B^d(t,T) = \mathbb{E}^{\mathbb{P}}\left[ e^{\int_t^T r_s ds} R \middle| \mathcal{G}_t \right] + \mathbb{1}_{\{\tau>t\}} \cdot \mathbb{E}^{\mathbb{P}}\left[ e^{\int_t^T (r_s+\lambda_s)ds}(X-R) \middle| \mathcal{G}_t \right],$$

*where $R$ is a $\mathcal{G}_T$-measurable random variable.*

*Proof.* Recall that $\theta = T$, set $Z_t = A_t = 0$ for all $t \geq 0$ in definition (2.11) and combine it with equation (2.9) to obtain

$$\begin{aligned} B^d(t,T) &= \mathbb{E}^{\mathbb{P}}\left[ B_t B_T^{-t} X_T^d \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{P}}\left[ B_t B_T^{-t} (\mathbb{1}_{\{\tau>T\}} X + R(1-\mathbb{1}_{\{\tau>T\}})) \middle| \mathcal{F}_t \right] \\ &= B_t \mathbb{E}^{\mathbb{P}}\left[ B_T^{-t} R \middle| \mathcal{F}_t \right] + B_t \mathbb{E}^{\mathbb{P}}\left[ \mathbb{1}_{\{\tau>t\}} \mathbb{1}_{\{\tau>T\}} B_T^{-t}(X-R) \middle| \mathcal{F}_t \right]. \end{aligned}$$

For the first conditional expectation information contained in $\mathcal{H}_t$ is irrelevant. Hence, the last equality can be written as

$$B^d(t,T) = B_t \mathbb{E}^{\mathbb{P}}\left[ B_T^{-t} R \middle| \mathcal{G}_t \right] + \mathbb{1}_{\{\tau>t\}} B_t \frac{\mathbb{E}^{\mathbb{P}}\left[ \mathbb{1}_{\{\tau>T\}} B_T^{-t}(X-R) \middle| \mathcal{G}_t \right]}{\mathbb{P}(\{\tau>t\}|\mathcal{G}_t)}.$$

Since $R$ is $\mathcal{G}_T$-measurable and iterating expectations we obtain after substituting $\mathbb{P}(\{\tau > t\}|\mathcal{G}_t)$ for $e^{-\int_0^t \lambda_s ds}$

$$B^d(t,T) = \mathbb{E}^{\mathbb{P}}\left[ B_T^{-t} B_t R \middle| \mathcal{G}_t \right] + \mathbb{1}_{\{\tau>t\}} \mathbb{E}^{\mathbb{P}}\left[ B_T^{-t} B_t e^{-\int_t^T \lambda_s ds}(X-R) \middle| \mathcal{G}_t \right].$$

$\square$

The following examples of recovery payment specifications are based on the assumption of the special case of recovery time $\tau^* = T$.

### 2.5.1 Valuation under the MR Specification

Below we present an application of Proposition 2.5.1 for Specification 2.4.1 by setting $R = Z_{\tau^*}$, where $Z_t = \delta(X - [X - Y_t]^+)$ and $\delta$ is constant.

**Corollary 2.5.2.** *The value of defaultable zero-bond under MR specification for $t < T$ is given by*

$$\begin{aligned}
B^d(t,T) &= \mathbb{E}^{\mathbb{P}}\left[ e^{-\int_t^T r_s ds} \delta X \middle| \mathcal{G}_t \right] + \mathbb{1}_{\{\tau > t\}} \cdot \mathbb{E}^{\mathbb{P}}\left[ e^{-\int_t^T (r_s + \lambda_s) ds} (1-\delta) X \middle| \mathcal{G}_t \right] \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[ e^{-\int_t^T r_s ds} \delta [X - Y_T]^+ \middle| \mathcal{G}_t \right] \\
&\quad + \mathbb{1}_{\{\tau > t\}} \cdot \mathbb{E}^{\mathbb{P}}\left[ e^{-\int_t^T (r_s + \lambda_s) ds} \delta [X - Y_T]^+ \middle| \mathcal{G}_t \right]. \qquad (2.16)
\end{aligned}$$

If default has not occurred until time $t$, the price of a defaultable zero-bond is a combination of default-free zero-bonds and European put options on the value of current assets of the company $Y_t$. The first term in (2.16) represents the value of a default-free zero-bond with face value $\delta X$, the second term can be interpreted as a synthetic default-free zero-bond whose face value $(1-\delta)X$ is discounted by $r_t + \lambda_t$. The third term is a short position in $\delta$ put options with respect to the assets of the company $Y_t$ and strike $X$, while using $r_t$ to discount. Finally, the last term represents a $\delta$ long position in an identical put option using $r_t + \lambda_t$ for discounting. Let $Put[Y_t, X, t, T, r_t; \delta]$ be the expected value under the equivalent martingale measure $\mathbb{P}$ at time $t$ of $\delta$ units of the payoff of a European put option with underlying $Y_t$, strike $X$, maturity $T$ and interest rate $r_t$. Hence the price of a defaultable zero-bond is given by

$$\begin{aligned}
B^d(t,T) &= \mathbb{E}^{\mathbb{P}}\left[ e^{-\int_t^T r_s ds} \delta X \middle| \mathcal{G}_t \right] + \mathbb{1}_{\{\tau > t\}} \cdot \mathbb{E}^{\mathbb{P}}\left[ e^{-\int_t^T (r_s + \lambda_s) ds} (1-\delta) X \middle| \mathcal{G}_t \right] \\
&\quad - Put[Y_t, X, t, T, r_t; \delta] + \mathbb{1}_{\{\tau > t\}} \cdot Put[Y_t, X, t, T, r_t + \lambda_t; \delta].
\end{aligned}$$

Denote the difference of put values in the last equation as *put differential* given by

$$\Delta Put(t, \delta) := \mathbb{1}_{\{\tau > t\}} \cdot Put[Y_t, X, t, T, r_t + \lambda_t; \delta] - Put[Y_t, X, t, T, r_t; \delta].$$

By recalling that $\lambda_t$ is non-negative for all $t \geq 0$ and assuming $r_t$ is also non-negative for any $t \in \mathbb{R}_+$, we have $\Delta Put(t, \delta) \leq 0$. Moreover, notice

$$P(t,T) = \mathbb{E}^{\mathbb{P}}\left[B_T^{-1}B_t \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[B_T^{-1}B_t \middle| \mathcal{G}_t\right],$$

because there is no relevant information in $\mathcal{H}_t$ for $B_t$. Assuming that $\delta$ is deterministic and using the introduced notation[22], the value of a defaultable zero bond can be expressed as

$$B^d(t,T) = P_{\delta X}(t,T) + \mathbb{1}_{\{\tau>t\}} \cdot (1-\delta)X\mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T (r_s+\lambda_s)ds} \middle| \mathcal{G}_t\right] + \Delta Put(t,\delta). \quad (2.17)$$

For the special case $\delta = 1$, the price of a defaultable bond is given by

$$B^d(t,T) = P_X(t,T) + \Delta Put(t,1),$$

which consists of a default-free zero-bond with face value $X$ and the difference of put options on the assets of the company. Note that defaultable bonds pay either the face value or the sale value of the company's asset, i.e. $B^d(T,T) = X$ on $\{\tau > T\}$ and $B^d(T,T) = \min\{Y_T, X\}$ on $\{\tau \leq T\}$, respectively. The same bonds' payoff profile can be found in the structural model of Merton (Merton 1974). However, the present framework does not impose a default trigger given by some predefined asset value as in Merton. Instead, we let default event be governed by a point process which is not observable in the default-free market. Evidently, even in this simplified illustration our framework offers more modeling flexibility than the model of Merton. Depending on the underlying assumptions on the processes $r_t$, $\lambda_t$ and $Y_t$, it can be advantageous to formulate Corollary 2.5.2 under the T-forward measure $\mathbb{Q}^T$.

**Corollary 2.5.3.** *The value of a defaultable zero-coupon bond under the MR specification for $t < T$ is given by*

$$\begin{aligned} B^d(t,T) \;&=\; P(t,T)\Big(\mathbb{E}^{\mathbb{Q}^T}\left[\delta X \middle| \mathcal{G}_t\right] + \mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_t^T \lambda_s ds}(1-\delta)X \middle| \mathcal{G}_t\right] \\ &\quad -\mathbb{E}^{\mathbb{Q}^T}\left[\delta[X-Y_T]^+ \middle| \mathcal{G}_t\right] + \mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_t^T \lambda_s ds}\delta[X-Y_T]^+ \middle| \mathcal{G}_t\right]\Big), \end{aligned}$$

*where $\mathbb{Q}^T$ is defined by the Radon-Nikodym derivative*

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{P}}\right|_{\mathcal{G}_T} := \frac{P(T,T)B_0}{P(0,T)B_T} = \frac{e^{-\int_0^T r_s ds}P(T,T)}{P(0,T)}, \quad (2.18)$$

*where $P(t,T)$ is the price of default-free zero-bond with face value* 1.

[22]Particularly, $P(0,T) \cdot X = P_X(0,T)$.

*Proof.* Let the Radon-Nikodym derivative defining $\mathbb{Q}^T$ be given by (2.18). The price of a default-free zero-coupon bond is given by

$$P(t,T) = \mathbb{E}^{\mathbb{P}}\left[ B_T^{-1} B_t \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}}\left[ B_T^{-1} B_t \middle| \mathcal{G}_t \right],$$

by previous arguments. Since $P(0,T) > 0$ is a martingale under $\mathbb{P}$, the expression in (2.18) represents a density function and $\mathbb{Q}^T$ is equivalent to $\mathbb{P}$. For a $\mathcal{G}_T$-measurable payoff $H_T$, its present value under the risk neutral measure at time $t$ is determined by

$$\mathbb{E}^{\mathbb{P}}\left[ \frac{B_t}{B_T} H_T \middle| \mathcal{G}_t \right] = P(t,T)\mathbb{E}^{\mathbb{P}}\left[ \frac{P(T,T)B_t}{P(t,T)B_T} \frac{H_T}{P(T,T)} \middle| \mathcal{G}_t \right] = P(t,T)\mathbb{E}^{\mathbb{Q}^T}\left[ H_T \middle| \mathcal{G}_t \right],$$

because $P(T,T) = 1$. Applying this procedure to Corollary 2.5.2 the proof is completed.

□

**Example 2.5.4.** *Assume $\delta = 1$, $t = 0$ and the following conditions:*

- *Asset values $Y_t$ are lognormally distributed under $\mathbb{Q}^T$ with deterministic volatility $\sigma$,*

$$dY_t = \sigma Y_t dW_t^T.$$

- *Default intensity and asset values are stochastically independent.*

*Hence, we obtain*

$$Y_T = Y_t \exp\left\{ -\frac{\sigma^2}{2}(T-t) + \sigma(W_T^T - W_t^T) \right\},$$

*where $W_T^T - W_t^T \sim N(0, T-t)$ and*

$$Y_T = Y_t \exp\left\{ -\frac{\sigma^2}{2}(T-t) + \sigma\sqrt{T-t} \cdot z \right\},$$

*with $z$ as a standard normally distributed random variable. Thus, $X \geq Y_T$ for*

$$z \leq \frac{\ln(X/Y_t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} =: d_{1,t}.$$

*The value of a simple put option is given by*

$$\begin{aligned}
Put[Y_t, X, 0, T, r_t; 1] &= P(0,T)\mathbb{E}^{\mathbb{Q}^T}\left[[X - Y_T]^+\right] \\
&= P(0,T)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \left[X - Y_0 e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}\cdot z}\right]^+ dz \\
&= P(0,T)\int_{-\infty}^{d_{1,0}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \left(X - Y_0 e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}\cdot z}\right) dz \\
&= P_X(0,T)\cdot N(d_{1,0}) - P_{Y_0}(0,T)\cdot \int_{-\infty}^{d_{1,0}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma\sqrt{T})^2}{2}} dz \\
&= P_X(0,T)\cdot N(d_{1,0}) - P_{Y_0}(0,T)\cdot N(d_{1,0} - \sigma\sqrt{T}).
\end{aligned}$$

*By defining*

$$d_{2,0} = d_{1,0} - \sigma\sqrt{T},$$

*the put option value is*

$$Put[Y_t, X, 0, T, r_t; 1] = P(0,T)(X\cdot N(d_{1,0}) - Y_0\cdot N(d_{2,0})).$$

*Similarly, $Put[Y_t, X, 0, T, r_t + \lambda_t; 1]$ is determined using the assumption of independence between $Y_t$ and $\lambda_t$, which yields*

$$\begin{aligned}
Put[Y_t, X, 0, T, r_t + \lambda_t; 1] &= P(0,T)\cdot \mathbb{E}^{\mathbb{Q}^T}\left[ e^{-\int_t^T \lambda_s ds}\delta[X - Y_T]^+ \middle| \mathcal{G}_0\right] \\
&= \mathbb{E}^{\mathbb{Q}^T}\left[ e^{-\int_t^T \lambda_s ds} \middle| \mathcal{G}_0\right]\cdot Put[Y_t, X, 0, T, r_t; 1] \\
&= \mathbb{Q}_0^T(\{\tau > T\})\cdot Put[Y_t, X, 0, T, r_t; 1],
\end{aligned}$$

*where $\mathbb{Q}_0^T(\{\tau > T\})$ is the survival probability under the T-forward measure conditional on information in $\mathcal{G}_0$. The value of a defaultable zero-coupon bond is given by*

$$\begin{aligned}
B^d(0,T) &= P_X(0,T) - (1 - \mathbb{Q}_0^T(\{\tau > T\})\cdot Put[Y_t, X, 0, T, r_0; 1] \\
&= P(0,T)\left[X - (1 - \mathbb{Q}_0^T(\{\tau > T\})(X\cdot N(d_{1,0}) - Y_0\cdot N(d_{2,0}))\right].
\end{aligned}$$

*The credit spread, which is the difference of the continuously compounded yield to maturity of a defaultable and a default-free zero-coupon bond, can be computed using the last equation. For this note that we need the yield of a default-free bond with face value $X$. Accordingly, the credit spread $s(0,T)$ is given by*

$$\begin{aligned}
s(0,T) &= -\frac{1}{T}\ln\left(\frac{B^d(0,T)}{P_X(0,T)}\right) \\
&= -\frac{1}{T}\ln\left(1 - (1 - \mathbb{Q}_0^T(\{\tau > T\}))\left(N(d_{1,0}) - \frac{Y_0}{X}\cdot N(d_{2,0})\right)\right).
\end{aligned}$$

*By analyzing the credit spread, one can see that default risk (captured in the survival probability) and recovery risk (illustrated in the expression for the value of the put) are components of credit risk.*

Further, the special case of zero recovery in the MR specification is given when $\delta = 0$. Here the company can neither sell its assets nor reorganize its capital structure when default occurs. Thus, the value of a defaultable zero-coupon bond corresponds to

$$B^d(t,T) = \mathbb{1}_{\{\tau>t\}} \cdot X\mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T (r_s+\lambda_s)ds}\middle|\, \mathcal{G}_t\right].$$

As mentioned previously specification TSR can be used to model recovery rates when the underlying process represents current assets or the state of the economy/industry. This specification is analyzed in the next section.

### 2.5.2 Valuation under the TSR Specification

During economic/industry crisis reorganization and liquidation of financially distressed companies are more difficult to conduct which induces lower recovery payments of defaulted bonds. We differentiate between a recovery rate $\delta$ during for stable phases and a recovery rate $\gamma$ during crisis. Let business cycles of the economy/industry be represented by the process $Y_t$ and the critical crisis value be given by some constant $K$. This framework is the *two-scenario recovery* rate specification discussed previously, where $R = (\delta\mathbb{1}_{\{Y_t\geq K\}} + \gamma\mathbb{1}_{\{Y_t<K\}})X$. By Proposition 2.5.1 the value of a defaultable bond under TSR is given as follows.

**Corollary 2.5.5.** *The value of a defaultable zero-coupon bond under TSR for $t < T$ is given by*

$$\begin{aligned} B^d(t,T) &= \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r_s ds}\gamma X\middle|\, \mathcal{G}_t\right] + \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r_s ds}(\delta-\gamma)X\cdot\mathbb{1}_{\{Y_T\geq K\}}\middle|\, \mathcal{G}_t\right] \\ &\quad +\mathbb{1}_{\{\tau>t\}}\cdot\mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T (r_s+\lambda_s)ds}(1-\gamma)X\middle|\, \mathcal{G}_t\right] \\ &\quad -\mathbb{1}_{\{\tau>t\}}\cdot\mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T (r_s+\lambda_s)ds}(\delta-\gamma)X\cdot\mathbb{1}_{\{Y_T\geq K\}}\middle|\, \mathcal{G}_t\right]. \end{aligned}$$

*Proof.* The corollary is an immediate consequence of Proposition 2.5.1, Specification 2.4.2 and of

$$\begin{aligned} X - R &= \left(1-\delta\cdot\mathbb{1}_{\{Y_T\geq K\}} - \gamma\cdot\mathbb{1}_{\{Y_T<K\}}\right)X \\ &= \left(1-\gamma-(\delta-\gamma)\cdot\mathbb{1}_{\{Y_T\geq K\}}\right)X. \end{aligned}$$

$\square$

The first two expectations in Corollary 2.5.5 represent the value of the bond when default has occurred. It is given by the sum of conditionally expected, discounted values of the minimum recovery payment $\gamma X$ and the difference of loss given defaults $\delta - \gamma$ only if at maturity there is no financial crisis. If default has not occurred, the value of the claim must be adjusted for possible future default. This adjustment is expressed in the two last expectations of the corollary. The first of these adjusts the minimum recovery payment by the maximum loss given default $(1-\gamma)X$. Note that $r_t + \lambda_t$ is used for discounting. The last term corrects the loss given default differential using $r_t + \lambda_t$ as discount rate. Hence, the second and fourth expectations in the corollary do not cancel out. Instead a small positive value remains. So the expectation of the discounted value of the smallest recovery payment is adjusted by possible outcomes of no-default and default when no financial crisis takes place.
Changing numeraire in Corollary 2.5.5 by the T-forward measure and using the same techniques as explained in the previous section, we obtain the following result.

**Corollary 2.5.6.** *The value of a defaultable zero-coupon bond under TSR for $t < T$ is given by*

$$\begin{aligned} B^d(t,T) &= P(t,T)\Big(\mathbb{E}^{\mathbb{Q}^T}\left[\gamma X \middle| \mathcal{G}_t\right] \\ &+\mathbb{E}^{\mathbb{Q}^T}\left[(\delta-\gamma)X \cdot \mathbb{1}_{\{Y_T \geq K\}} \middle| \mathcal{G}_t\right] \\ &+\mathbb{1}_{\{\tau > t\}} \cdot \mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_t^T \lambda_s ds}(1-\gamma)X \middle| \mathcal{G}_t\right] \\ &-\mathbb{1}_{\{\tau > t\}} \cdot \mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_t^T \lambda_s ds}(\delta-\gamma)X \cdot \mathbb{1}_{\{Y_T \geq K\}} \middle| \mathcal{G}_t\right]\Big). \end{aligned} \tag{2.19}$$

**Example 2.5.7.** *Assume that the maximum recovery rate in both scenarios can be identified with certainty. Therefore, let $\delta$ and $\gamma$ be deterministic. Additionally, consider the same assumptions from the previous example.*

- *The state process $Y_t$ is lognormally distributed under $\mathbb{Q}^T$ with deterministic volatility $\sigma$,*

$$dY_t = \sigma Y_t dW_t^T.$$

- *Default intensity and state process are stochastically independent.*

*Notice that*

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^T}\left[\mathbb{1}_{\{Y_T \geq K\}} \middle| \mathcal{G}_0\right] &= \mathbb{Q}_0^T\left(Y_T \geq K\right) \\
&= \mathbb{Q}_0^T\left(\frac{\ln(Y_T) - (\ln(Y_0) - \frac{\sigma^2}{2}T)}{\sigma\sqrt{T}} \geq -\hat{d}_{2,0}\right) \\
&= \mathbb{Q}_0^T\left(Z \leq \hat{d}_{2,0}\right) \\
&= N(\hat{d}_{2,0}),
\end{aligned}$$

*where* $\hat{d}_{2,0} = \frac{\ln(Y_0/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$*. From Corollary 2.5.6 we have*

$$\begin{aligned}
B^d(0,T) &= P(0,T)\Big[\gamma X + (\delta - \gamma)X \cdot N(\hat{d}_{2,0}) + (1-\gamma)X \cdot \mathbb{Q}_0^T(\{\tau > T\}) \\
&\quad -(\delta - \gamma)X \cdot N(\hat{d}_{2,0}) \cdot \mathbb{Q}_0^T(\{\tau > T\})\Big].
\end{aligned}$$

*By rearranging terms we obtain*

$$B^d(0,T) = P_X(0,T)\Big[1 + \big(1 - \mathbb{Q}_0^T(\{\tau > T\})\big)\Big((\delta - \gamma)N(\hat{d}_{2,0}) - (1-\gamma)\Big)\Big].$$

*Notice that the term in the brackets is never larger than 1, since* $((\delta - \gamma)N(\hat{d}_{2,0}) - (1-\gamma)) < 0$*. Again, the credit spread* $s(0,T)$ *embeds default and recovery risks*

$$s(0,T) = -\frac{1}{T}\ln\Big(1 + \big(1 - \mathbb{Q}_0^T(\{\tau > T\})\big)\Big[(\delta - \gamma)N(\hat{d}_{2,0}) - (1-\gamma)\Big]\Big).$$

## 2.6 Concluding Remarks

The model on defaultable bonds of this chapter incorporates stochastic default and recovery times which may differ from each other. Observing real-world situations where recovery payment may neither occur at default time nor at maturity, the separation of default and recovery times is well founded. Moreover, recovery payment is modeled considering company's solvency and governing economic conditions. The resulting models which gathers all these concepts combines virtues of intensity-based and structural approaches.

In this chapter we regard default and recovery times without three setups: intensity-based, structural and mixture approaches. Under these specifications we derived a general pricing rule for defaultable zero-coupon bonds. Furthermore, we discussed a model for recovery processes that can take firm-specific and economy-wide settings.

Finally, we analyzed different examples in the special case when recovery time equals maturity.

Empirical evidence has uncovered relationships between default events and recovery payments. Theoretical work has started to react on this findings. This chapter contributes to that cause by providing a more flexible framework capable to consider recovery risk as the uncertainty about recovery payment and the uncertainty about the time of this payment. Within this framework correlation of recovery rates and default have a straight economic foundation. We do not need to make unrealistic assumptions about the bond price at default (value of distressed debt at default time) because it is derived from the model.

Further research should concentrate in 1) empirical verification of the pricing rule proposed and 2) hedging strategies that can be used. Additionally, other types of defaultable securities should be investigated. A more extensive analysis on correlation between default and recovery rates should be also undertaken.

## 2.7 Appendix

### 2.7.1 Self-financing strategy

Note that $S_t$ given in (2.9) is defined for an arbitrary dividend process $D_t$. We rephrase the borrowed derivation of the pricing rule for some non-defaultable security $S_t$ from Bielecki and Rutkowski (Bielecki & Rutkowski 2004). Consider an admissible self-financing, buy-and-hold trading strategy $\phi_t = (\phi_t^1, \phi_t^2) = (1, \phi_t^2)$, where we buy one unit of asset $S_t$ and hold it until settlement date $\theta$, and deposit all gains in the bank account. Accordingly, the wealth process $U_t$ of $\phi_t$ is given by

$$U_t = S_t + \phi_t^2 B_t, \qquad t \in [0, \theta], \tag{2.20}$$

with initial capital $U_0 = S_0 + \phi_0^2 B_0$. Since $\phi_t$ is self-financing, we obtain

$$U_t - U_0 = S_t - S_0 + D_t + \int_{(0,t]} \phi_u^2 dB_u, \qquad t \in [0, \theta], \tag{2.21}$$

and by defining $\tilde{U}_t = B_t^{-1} U_t$ it follows

$$\tilde{U}_t - \tilde{U}_0 = \tilde{S}_t - \tilde{S}_0 + \int_{(0,t]} B_u^{-1} dD_u, \qquad t \in [0, \theta], \tag{2.22}$$

which corresponds to Lemma 2.1.1 in Bielecki and Rutkowski (Bielecki & Rutkowski 2004). Because the discounted wealth process is an $\mathcal{F}_t$-martingale under $\mathbb{P}$, we have for all $t \leq \theta$

$$\mathbb{E}^{\mathbb{P}} \left[ \tilde{U}_\theta - \tilde{U}_t \middle| \mathcal{F}_t \right] = 0, \tag{2.23}$$

implying

$$\tilde{S}_t = \mathbb{E}^{\mathbb{P}} \left[ \tilde{S}_\theta + \int_{(t,\theta]} B_u^{-1} dD_u \middle| \mathcal{F}_t \right]. \tag{2.24}$$

By assuming $\tilde{S}_\theta = S_\theta = 0$ equation (2.9) results.

In Appendix 2.7.2 we provide arguments for the validity of the pricing formula for $S_t$ of Section 2.3.2 when it represents the price of a defaultable zero-coupon bond when settlement is random and recovery time may differ from default time.

### 2.7.2 Random $\theta$

Equation (2.24) is not altered when $\theta$ is random, i.e. the ex-dividend pricing rule

$$S_t = \mathbb{E}^{\mathbb{P}}\left[\int_{(t,\theta]} B_u^{-1} dD_u \middle| \mathcal{F}_t\right] \qquad \text{for} \quad t < \tau^* \wedge \theta, \tag{2.25}$$

is valid. For this consider the following ideas.

As in Duffie and Singleton (Duffie & Singleton 1999), let any traded contingent claim be characterized by a pair $(\mathcal{X}, \nu)$, where $\nu$ is a stopping time and $\mathcal{X}$ an $\mathcal{F}_\nu$-measurable random variable. Given the equivalent martingale measure $\mathbb{P}$, the *ex-dividend* price of this contingent claim is denoted by $C_t$ and given by

$$C_t = B_t \mathbb{E}^{\mathbb{P}}\left[B_\nu^{-1}\mathcal{X} \middle| \mathcal{F}_t\right],$$

for all $t < \nu$. In our case, a defaultable claim is a combination of two contingent claims $(\mathcal{X}^1, \nu^1)$ and $(\mathcal{X}^2, \nu^2)$, where

$$\nu^1 := \tau \wedge T, \qquad \nu^2 := \tau^*,$$

and

$$\mathcal{X}^1 := X\mathbb{1}_{\{\tau > T\}} + \mathbb{1}_{\{\tau \le T\}} C_\tau^2, \qquad \mathcal{X}^2 := Z_{\tau^*},$$

where $C_\tau^2$ represents the value of the second contingent claim at time $\tau$. Formally, the value of our defaultable claim at $t < \nu^1$ is given by

$$\begin{aligned} S_t &= B_t \mathbb{E}^{\mathbb{P}}\left[B_{\nu^1}^{-1}\left(X\mathbb{1}_{\{\tau > T\}} + X\mathbb{1}_{\{\tau \le T\}} C_\tau^2\right) \middle| \mathcal{F}_t\right] \\ &= B_t \mathbb{E}^{\mathbb{P}}\left[B_{\nu^1}^{-1}\left(X\mathbb{1}_{\{\tau > T\}} + \mathbb{1}_{\{\tau \le T\}} B_\tau \mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1} Z_{\tau^*} \middle| \mathcal{F}_\tau\right]\right) \middle| \mathcal{F}_t\right]. \end{aligned}$$

This implies

$$S_t = B_t \mathbb{E}^{\mathbb{P}}\left[B_T^{-1} X\mathbb{1}_{\{\tau > T\}} + \mathbb{1}_{\{\tau \le T\}} B_{\tau^*}^{-1} Z_{\tau^*} \middle| \mathcal{F}_t\right], \qquad \text{for} \quad t < \nu^1.$$

We assume that at default time $\tau$ on $\{\tau \le T\}$ there is no exchange of money. Instead the investor holding the contingent claim $(\mathcal{X}^1, \nu^1)$ becomes owner of the second contingent claim $(\mathcal{X}^2, \nu^2)$. Consequently, the payment of the combination of contingent claims occur only at $T$ or $\tau^*$. Thus, last equation is equivalent to

$$S_t = \mathbb{1}_{\{\tau^* > T\}} B_t \mathbb{E}^{\mathbb{P}}\left[B_T^{-1} X\mathbb{1}_{\{\tau > T\}} + \mathbb{1}_{\{\tau \le T\}} B_{\tau^*}^{-1} Z_{\tau^*} \middle| \mathcal{F}_t\right], \qquad \text{for} \quad t < T,$$

which matches the results of Proposition 2.3.1 and is equivalent to (2.25) for $t < T$ with $A_t = 0$ for all $t \in \mathbb{R}_+$ and $R = 0$. Hence, the definition of the ex-dividend price process can be applied even if settlement $\theta$ is a random time.

### 2.7.3 Proof of Proposition 2.3.4

By Proposition 2.3.1 the price of a defaultable zero-coupon bond is given by written as

$$B^d(t,T) = \mathbb{1}_{\{\tau^*>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \mathbb{1}_{\{\tau>T\}} \middle| \mathcal{F}_t \right] + \mathbb{1}_{\{\tau^*>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_{\tau^*}^{-1} Z_{\tau^*} \mathbb{1}_{\{\tau\leq T\}} \middle| \mathcal{F}_t \right]. \tag{2.26}$$

Consider the first summand

$$\begin{aligned} &\mathbb{1}_{\{\tau^*>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \mathbb{1}_{\{\tau>T\}} \middle| \mathcal{F}_t \right] \\ &\quad = \mathbb{1}_{\{\tau^*>t\}} \frac{B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \mathbb{1}_{\{\tau^*>t\}} \mathbb{1}_{\{\tau>T\}} \middle| \mathcal{G}_t \vee \mathcal{H}_t \right]}{\mathbb{P}(\{\tau^*>t\} | \mathcal{G}_t \vee \mathcal{H}_t)}. \end{aligned} \tag{2.27}$$

Equality holds because the $\mathcal{F}_t$-conditional expectation is zero on the set $\{\tau^* \leq t\}$, and so the only relevant information of $\mathcal{H}_t^*$ is on the set $\{\tau^* > t\}$. Since $\mathbb{1}_{\{\tau^*>t\}} \mathbb{1}_{\{\tau>t\}} = \mathbb{1}_{\{\tau>t\}}$ and by noticing that the expression (2.27) is zero on $\{\tau \leq t\}$, it is equivalent to

$$\begin{aligned} &\mathbb{1}_{\{\tau>t\}} \frac{B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \mathbb{1}_{\{\tau^*>t\}} \mathbb{1}_{\{\tau>T\}} \middle| \mathcal{G}_t \vee \mathcal{H}_t \right]}{\mathbb{P}(\{\tau^*>t\} | \mathcal{G}_t \vee \mathcal{H}_t)} \\ &\quad = \mathbb{1}_{\{\tau>t\}} \frac{B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \mathbb{E}^{\mathbb{P}} \left[ \mathbb{1}_{\{\tau^*>t\}} | \mathcal{G}_\infty \vee \mathcal{H}_\infty \right] \mathbb{1}_{\{\tau>T\}} \middle| \mathcal{G}_t \vee \mathcal{H}_t \right]}{\mathbb{P}(\{\tau^*>t\} | \mathcal{G}_t \vee \mathcal{H}_t)}, \end{aligned}$$

which follows because the term $B_T^{-1} X \mathbb{1}_{\{\tau>T\}}$ is $\mathcal{G}_\infty \vee \mathcal{H}_\infty$. By the properties of the hazard process of $\tau^*$ pointed out in (2.7) the last argument is given by

$$\begin{aligned} &\mathbb{1}_{\{\tau>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \mathbb{1}_{\{\tau>T\}} \middle| \mathcal{G}_t \vee \mathcal{H}_t \right] \\ &\quad = \mathbb{1}_{\{\tau>t\}} \frac{B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \mathbb{1}_{\{\tau>t\}} \mathbb{1}_{\{\tau>T\}} \middle| \mathcal{G}_t \right]}{\mathbb{P}(\{\tau>t\} | \mathcal{G}_t)} \\ &\quad = \mathbb{1}_{\{\tau>t\}} \frac{B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} X \mathbb{P}(\{\tau>T\} | \mathcal{G}_\infty) \middle| \mathcal{G}_t \right]}{\mathbb{P}(\{\tau>t\} | \mathcal{G}_t)} \\ &\quad = \mathbb{1}_{\{\tau>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_T^{-1} e^{\Lambda_t - \Lambda_T} X \middle| \mathcal{G}_t \right], \end{aligned} \tag{2.28}$$

where the third equality follows from the properties of the hazard process of $\tau$ presented in (2.5). Now consider the second term in (2.26)

$$\mathbb{1}_{\{\tau^*>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_{\tau^*}^{-1} Z_{\tau^*} \mathbb{1}_{\{\tau\leq T\}} \mathbb{1}_{\{\tau^*>t\}} \middle| \mathcal{F}_t \right]. \tag{2.29}$$

Equivalently, for $t < T$ this expression is given by

$$\mathbb{1}_{\{\tau^*>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_{\tau^*}^{-1} Z_{\tau^*} \mathbb{1}_{\{t<\tau\leq T\}} + B_{\tau^*}^{-1} Z_{\tau^*} \mathbb{1}_{\{\tau\leq t\leq T\}} \middle| \mathcal{F}_t \right]. \tag{2.30}$$

Consider the first summand of (2.30)

$$\begin{aligned}\mathbb{1}_{\{\tau^*>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\mathbb{1}_{\{t<\tau\le T\}}\middle|\mathcal{F}_t\right] &= \mathbb{1}_{\{\tau^*>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau\le T\}}\middle|\mathcal{F}_t\right]\\ &= \mathbb{1}_{\{\tau>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\mathbb{1}_{\{\tau\le T\}}\middle|\mathcal{F}_t\right].\end{aligned}$$

Similarly, the second summand of (2.30) can be reformulated as follows.

$$\begin{aligned}\mathbb{1}_{\{\tau^*>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\mathbb{1}_{\{\tau\le t\le T\}}\middle|\mathcal{F}_t\right] &= \mathbb{1}_{\{\tau^*>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\mathbb{1}_{\{\tau\le t\}}\middle|\mathcal{F}_t\right]\\ &= \mathbb{1}_{\{\tau^*>t\}}\mathbb{1}_{\{\tau\le t\}}B_t\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\middle|\mathcal{F}_t\right].\end{aligned}$$

Hence, for $t<T$ the conditional expectation (2.29) is given by

$$\mathbb{1}_{\{\tau>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\mathbb{1}_{\{\tau\le T\}}\middle|\mathcal{F}_t\right]+\mathbb{1}_{\{\tau^*>t\}}\mathbb{1}_{\{\tau\le t\}}B_t\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\middle|\mathcal{F}_t\right]. \tag{2.31}$$

Consider the first conditional expectation above

$$\begin{aligned}&\mathbb{1}_{\{\tau>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\mathbb{1}_{\{\tau\le T\}}\middle|\mathcal{F}_t\right]\\ &\quad=\mathbb{1}_{\{\tau>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\mathbb{1}_{\{\tau\le T\}}\left|\mathcal{G}_\infty\vee\mathcal{H}_\infty\vee\mathcal{H}_t^*\right.\right]\middle|\mathcal{F}_t\right]\\ &\quad=\mathbb{1}_{\{\tau>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[B_{\tau^*}^{-1}Z_{\tau^*}\left|\mathcal{G}_\infty\vee\mathcal{H}_\infty\vee\mathcal{H}_t^*\right.\right]\mathbb{1}_{\{\tau\le T\}}\middle|\mathcal{F}_t\right].\end{aligned}$$

Since the density of recovery time conditionally on $\mathcal{G}_\infty\vee\mathcal{H}_\infty$ for $\tau^*>t$ is given by

$$\frac{\partial\mathbb{P}}{\partial s}\left(\{\tau^*\le s\}\left|\tau^*>t;\mathcal{G}_\infty\vee\mathcal{H}_\infty\right.\right)=\lambda_s^*e^{-\int_t^s\lambda_u^*du}=\lambda_s^*e^{\Lambda_t^*-\Lambda_s^*}\qquad\text{for}\quad s>t, \tag{2.32}$$

the last expression can be written as

$$\begin{aligned}&\mathbb{1}_{\{\tau>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau^*>t\}}\int_{(t,\infty)}B_u^{-1}Z_u\lambda_u^*e^{\Lambda_t^*-\Lambda_u^*}du\,\mathbb{1}_{\{\tau\le T\}}\middle|\mathcal{F}_t\right]\\ &\quad=\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau^*>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[\int_{(t,\infty)}B_u^{-1}Z_u\eta_u\mathbb{1}_{\{\tau\le u\}}e^{\Lambda_t^*-\Lambda_u^*}du\,\mathbb{1}_{\{\tau\le T\}}\middle|\mathcal{F}_t\right]\\ &\quad=\mathbb{1}_{\{\tau>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[\int_{(t,\infty)}B_u^{-1}Z_u\eta_u\mathbb{1}_{\{\tau\le u\le T\}}e^{\Lambda_t^*-\Lambda_u^*}du\middle|\mathcal{F}_t\right]\\ &\quad=\mathbb{1}_{\{\tau>t\}}B_t\mathbb{E}^{\mathbb{P}}\left[\int_{(t,T]}B_u^{-1}Z_u\eta_u\mathbb{1}_{\{\tau\le u\}}e^{\Lambda_t^*-\Lambda_u^*}du\middle|\mathcal{F}_t\right].\end{aligned}$$

We use the definition of $\lambda_t^*$ for the first equality. Note that the conditional expectation above is zero on the set $\{\tau^*\le t\}$. Thus, the last expression results in

$$\mathbb{1}_{\{\tau>t\}}\frac{B_t\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau^*>t\}}\int_{(t,T]}B_u^{-1}Z_u\eta_u\mathbb{1}_{\{\tau\le u\}}e^{\Lambda_t^*-\Lambda_u^*}du\middle|\mathcal{G}_t\vee\mathcal{H}_t\right]}{\mathbb{P}\left(\{\tau^*>t\}\left|\mathcal{G}_t\vee\mathcal{H}_t\right.\right)}.$$

Using the exact previous arguments of the hazard process of $\tau^*$ and while noticing that the integral in the conditional expectation is $\mathcal{G}_\infty \vee \mathcal{H}_\infty$-measurable, the last term is equivalent to

$$\mathbb{1}_{\{\tau>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ \int_{(t,T]} B_u^{-1} Z_u \eta_u \mathbb{1}_{\{\tau \le u\}} e^{\Lambda_t^* - \Lambda_u^*} du \middle| \mathcal{G}_t \vee \mathcal{H}_t \right]$$

$$= \mathbb{1}_{\{\tau>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ \mathbb{E}^{\mathbb{P}} \left[ \int_{(t,T]} B_u^{-1} Z_u \eta_u \mathbb{1}_{\{\tau \le u\}} e^{-\int_t^u \eta_s \mathbb{1}_{\{\tau \le s\}} ds} du \middle| \mathcal{G}_\infty \vee \mathcal{H}_t \right] \middle| \mathcal{G}_t \vee \mathcal{H}_t \right].$$

Note that the density of default conditionally on $\mathcal{G}_\infty$ for $\tau > t$ is given by

$$\frac{\partial \mathbb{P}}{\partial s} \left( \{\tau \le s\} \, | \tau > t; \mathcal{G}_\infty \right) = \lambda_s e^{-\int_t^s \lambda_u du} \qquad \text{for} \quad s > t.$$

Thus, last expression is given by

$$\mathbb{1}_{\{\tau>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ \int_{(t,\infty)} \int_{(t,T]} B_u^{-1} Z_u \eta_u \mathbb{1}_{\{q \le u\}} e^{-\int_t^u \eta_s \mathbb{1}_{\{q \le s\}} ds} du \lambda_q e^{-\int_t^q \lambda_v dv} dq \middle| \mathcal{G}_t \vee \mathcal{H}_t \right],$$

or, equivalently,

$$\mathbb{1}_{\{\tau>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ \int_{(t,T]} \int_{(t,T]} B_u^{-1} Z_u \eta_u \mathbb{1}_{\{q \le u\}} \lambda_q e^{-\int_t^u \eta_s \mathbb{1}_{\{q \le s\}} ds - \int_t^q \lambda_v dv} du dq \middle| \mathcal{G}_t \vee \mathcal{H}_t \right],$$

since $u$ cannot be greater than $T$. By noting that the expression is zero on $\{\tau \le t\}$, we have

$$\mathbb{1}_{\{\tau>t\}} \frac{B_t \mathbb{E}^{\mathbb{P}} \left[ \mathbb{1}_{\{\tau>t\}} \int_{(t,T]} \int_{(t,T]} B_u^{-1} Z_u \eta_u \mathbb{1}_{\{q \le u\}} \lambda_q e^{-\int_t^u \eta_s \mathbb{1}_{\{q \le s\}} ds - \int_t^q \lambda_v dv} du dq \middle| \mathcal{G}_t \right]}{\mathbb{P} \left( \{\tau > t\} \, | \mathcal{G}_t \right)}.$$

Since the double integral conditionally is $\mathcal{G}_\infty$-measurable, we use the properties of the hazard process of $\tau$ to derive the equivalent expression

$$\mathbb{1}_{\{\tau>t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ \int_{(t,T]} \int_{(t,T]} B_u^{-1} Z_u \eta_u \mathbb{1}_{\{q \le u\}} \lambda_q e^{-\int_t^u \eta_s \mathbb{1}_{\{q \le s\}} ds - \int_t^q \lambda_v dv} du dq \middle| \mathcal{G}_t \right]. \tag{2.33}$$

Finally, we proceed with the second term of (2.31)

$$\mathbb{1}_{\{\tau^*>t\}} \mathbb{1}_{\{\tau \le t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ B_{\tau^*}^{-1} Z_{\tau^*} \middle| \mathcal{F}_t \right]$$

$$= \mathbb{1}_{\{\tau^*>t\}} \mathbb{1}_{\{\tau \le t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ \mathbb{E}^{\mathbb{P}} \left[ B_{\tau^*}^{-1} Z_{\tau^*} \middle| \mathcal{G}_\infty \vee \mathcal{H}_\infty \vee \mathcal{H}_t^* \right] \middle| \mathcal{F}_t \right].$$

By the density of $\tau^*$ conditioned on $\mathcal{G}_\infty \vee \mathcal{F}_\infty$ given in (2.32), we can write last expression as

$$\mathbb{1}_{\{\tau^*>t\}} \mathbb{1}_{\{\tau \le t\}} B_t \mathbb{E}^{\mathbb{P}} \left[ \int_{(t,\infty)} B_u^{-1} Z_u \eta_u \mathbb{1}_{\{\tau \le u\}} e^{-\int_t^u \eta_s \mathbb{1}_{\{\tau \le s\}} ds} du \middle| \mathcal{F}_t \right]$$

$$= \mathbb{1}_{\{\tau^*>t\}} \mathbb{1}_{\{\tau \le t\}} \frac{B_t \mathbb{E}^{\mathbb{P}} \left[ \mathbb{1}_{\{\tau^*>t\}} \int_{(t,\infty)} B_u^{-1} Z_u \eta_u \mathbb{1}_{\{\tau \le u\}} e^{-\int_t^u \eta_s \mathbb{1}_{\{\tau \le s\}} ds} du \middle| \mathcal{G}_t \vee \mathcal{H}_t \right]}{\mathbb{P} \left( \{\tau^* > t\} \, | \mathcal{G}_t \vee \mathcal{H}_t \right)}.$$

By the same procedure as before, the last term equals

$$\mathbb{1}_{\{\tau^*>t\}}\mathbb{1}_{\{\tau\le t\}}B_t\mathbb{E}^{\mathbb{P}}\left[\int_{(t,\infty)} B_u^{-1}Z_u\eta_u\mathbb{1}_{\{\tau\le u\}}e^{-\int_t^u \eta_s\mathbb{1}_{\{\tau\le s\}}ds}du\,\middle|\,\mathcal{G}_t\vee\mathcal{H}_t\right]$$

Since the conditional expectation is zero on $\{\tau > t\}$, then the last expression is given by

$$\begin{aligned}
&\mathbb{1}_{\{\tau^*>t\}}\mathbb{1}_{\{\tau\le t\}}B_t\mathbb{E}^{\mathbb{P}}\left[\int_{(t,\infty)} B_u^{-1}Z_u\eta_u e^{-\int_t^u \eta_s ds}du\,\middle|\,\mathcal{G}_t\vee\mathcal{H}_t\right]\\
&= \mathbb{1}_{\{\tau^*>t\}}\mathbb{1}_{\{\tau\le t\}}\frac{B_t\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau\le t\}}\int_{(t,\infty)} B_u^{-1}Z_u\eta_u e^{-\int_t^u \eta_s ds}du\,\middle|\,\mathcal{G}_t\right]}{\mathbb{P}\left(\{\tau\le t\}\,|\mathcal{G}_t\right)}\\
&= \mathbb{1}_{\{\tau^*>t\}}\mathbb{1}_{\{\tau\le t\}}B_t\mathbb{E}^{\mathbb{P}}\left[\int_{(t,\infty)} B_u^{-1}Z_u\eta_u e^{-\int_t^u \eta_s ds}du\,\middle|\,\mathcal{G}_t\right], \qquad (2.34)
\end{aligned}$$

where the second equality follows from the properties of the hazard process of $\tau$ conditioned on $\mathcal{G}_\infty$. The value of a defaultable zero-coupon bond is the sum of (2.28), (2.33), and (2.34).

□

### 2.7.4 Computation of $I_0^2$ and $\tilde{I}_0^2$ of Section 2.3.3

First, we calculate $I_0^2$.

$$\begin{aligned}
I_0^2 &= \mathbb{E}^{\mathbb{P}}\left[\int_{(0,T]}\int_{(0,T]} B_T^{-1}\phi\eta\mathbb{1}_{\{q\le u\}}\lambda e^{-\int_0^u \eta\mathbb{1}_{\{q\le s\}}ds-\lambda q}dudq\right]\\
&= \phi\eta\lambda\mathbb{E}^{\mathbb{P}}\left[\int_{(0,T]}\int_{(0,T]} B_T^{-1}\mathbb{1}_{\{q\le u\}}e^{-\int_0^u \eta\mathbb{1}_{\{q\le s\}}ds-\lambda q}dqdu\right]\\
&= \phi\eta\lambda\mathbb{E}^{\mathbb{P}}\left[\int_{(0,T]} B_T^{-1}\int_{(0,u]} e^{-\int_q^u \eta ds-\lambda q}dqdu\right]\\
&= \phi\eta\lambda\mathbb{E}^{\mathbb{P}}\left[\int_{(0,T]} B_T^{-1}\int_{(0,u]} e^{q(\eta-\lambda)-\eta u}dqdu\right].
\end{aligned}$$

For $\mu \neq \eta$, we have

$$\begin{aligned}
I_0^2 &= \phi\eta\lambda\mathbb{E}^{\mathbb{P}}\left[\int_{(0,T]} B_T^{-1}\frac{1}{\eta-\lambda}\left(e^{-\lambda u}-e^{-\eta u}\right)du\right] \\
&= \frac{\phi\eta\lambda}{\eta-\lambda}\int_\Omega\int_{(0,T]} B_T^{-1}\left(e^{-\lambda u}-e^{-\eta u}\right)dud\mathbb{P} \\
&= \frac{\phi\eta\lambda}{\eta-\lambda}\int_{(0,T]}\left(e^{-\lambda u}-e^{-\eta u}\right)\int_\Omega B_T^{-1}d\mathbb{P}du \\
&= \frac{\phi\eta\lambda}{\eta-\lambda}\int_{(0,T]}\left(e^{-\lambda u}-e^{-\eta u}\right)P(0,u)du \\
&= \frac{\phi\eta\lambda}{\eta-\lambda}\int_{(0,T]}\left(e^{-\lambda u}-e^{-\eta u}\right)A(0,u)e^{-B(0,u)r_0}du,
\end{aligned}$$

where $A(0,u)$ and $B(0,u)$ are define in Section 2.3.3. Consider $\tilde{I}_0^2$,

$$\begin{aligned}
\tilde{I}_0^2 &= \lambda\phi\mathbb{E}^{\mathbb{P}}\left[\int_{(0,T]} B_u^{-1}e^{-\lambda u}du\right] \\
&= \lambda\phi\int_\Omega\int_{(0,T]} B_u^{-1}e^{-\lambda u}dud\mathbb{P} \\
&= \lambda\phi\int_{(0,T]} e^{-\lambda u}\int_\Omega B_u^{-1}d\mathbb{P}du \\
&= \lambda\phi\int_{(0,T)} e^{-\lambda u}P(0,u)du \\
&= \lambda\phi\int_{(0,T]} A(0,u)e^{-B(0,u)r_0-\lambda u}du.
\end{aligned}$$

# Chapter 3

# Liquidity Risk Under Partial Execution and Block Trading

## 3.1 Introduction

The urge of a comprehensive, tractable and computationally light model to measure liquidity risk has increased since the last financial crisis. During the credit crunch we have observed how liquidity risk can manifest itself in a wide variety of forms, as for example the uncertainty about the width of the bid-ask spread, the uncertainty about the marketability of assets, the uncertainty about the price of infrequently traded assets, etc. Most researchers and practitioners identify mainly two types of liquidity risks: solvency and market-price. Solvency risk is the uncertainty about capital flows needed to cover operating costs of a firm, such as interest expenses and debenture repayments. Exposure to solvency risk is of great concern of banks and other financial intermediaries since their operations are based on daily lending and borrowing of large amounts of capital. Financial institutions are also exposed to market liquidity risk, which is the uncertainty about 1) the asset prices at which they can be bought and sold and 2) the volume which can be traded. Generally, highly traded assets qualify as liquid assets. Infrequently traded securities which exhibit large price movements when medium and large volumes are traded may be regarded as illiquid assets. One of the common approaches to measure liquidity risk is the analysis of transaction costs represented by the bid-ask spread. Market participants surely agree that the bid-ask spread, as a measure of liquidity risk, explains the problem of illiquidity only partially. Empirical models have analyzed other relevant factors which help to explain the liquidity conundrum of assets and portfolios. Two of the most important of these factors are trading volume and transaction price-impact, as presented in Holthausen et al. (Holthausen et al. 1987) and (Holthausen et al. 1990), Gallant et al. (Gallant et al. 1992) and Keim and Madhavan (Keim & Madhavan 1996). These studies indicate that trading large volumes, specially trading large block orders, have consequences on asset prices. In particular, trading *large blocks* has a great effect on the price and marketability of infrequently traded assets. The present chapter presents a framework which includes not only the uncertainty about the bid and ask prices but also includes price-impact originated from block trading and the uncertainty of marketability.

This chapter analyzes market price liquidity risk on asset portfolios. According to empirical work, the present framework formulates *block trading effects* as a change in the future best bid and ask quotes after fulfillment of a large block transaction. Additionally, marketability is modeled here as a set of executable trades, leaving all trades outside of this set as non-executable. Hence, execution of sell or buy orders may be partial. In addition, the framework imposes a solvency restriction on the asset portfolio, later denoted as liquidity policy, which reflects solvency risk. Hence, the framework introduced in the following sections covers both market price liquidity and solvency risks.

Jarrow and Protter (Jarrow & Protter 2005) present a framework of liquidity risk. Their contribution is based on the celebrated model of Çetin et al. (Çetin et al. 2002) of liquidity risk. Concretely, they consider different states of the financial market: business as usual and liquidity crisis. Under a liquidity crisis, asset values deteriorate implying a decline in portfolio values. Liquidity risk captured by their model consists of only of a fraction of the market price liquidity risk, because the authors neglect the importance of trading volume as a factor of risk.

Recently, Acerbi and Scandolo (Acerbi & Scandolo 2008) put forward an alternative adjustment to portfolio values which relies on an optimization problem of portfolio and transaction values. The authors model liquidity risk through the uncertainty concerning future bid and ask quotes. Under a specified liquidity policy/restriction, the value of the asset portfolio is corrected. They denote the liquidity-adjusted portfolio value by $V^{\mathcal{L}}$. This approach tackles issues with respect to solving risk and up to some extent market price uncertainty .However, the authors neglect the existence of block trading and partial execution effects. Furthermore, Acerbi and Scandolo define a liquidity-adjusted risk measure $\rho^{\mathcal{L}}$ on the space of portfolio weights $\mathcal{P}$ using the liquidity-adjusted portfolio value $V^{\mathcal{L}}$. In their setup $V^{\mathcal{L}}$ has the necessary properties such that $\rho^{\mathcal{L}}$ is a convex risk measure on $\mathcal{P}$.

We borrow the setup of Acerbi and Scandolo and undertake the same liquidity adjustment on portfolio values but we admit the existence of block trading and partial execution effects. Under this treatment the following *main contributions* arises.

*The liquidity-adjusted risk measure $\rho^{\mathcal{L}}$ fails to be convex on $\mathcal{P}$ whenever block trading and partial execution effects are present,...*

*...and the probability distribution of $V^{\mathcal{L}}$ shifts to lower values, the expected value falls and the Value-at-Risk rises.*

In other words, the fact that $\rho^{\mathcal{L}}$ is not convex on $\mathcal{P}$ indicates that the risk of the average portfolio of a collection of portfolios is not necessarily lower than the average risk of the collection of the portfolios. The liquidity adjustment of any portfolio consists in a correction of the mark-to-market portfolio value by answering the question: what is the price of those positions after executing a necessary transaction in order to maintain some liquidity restriction, while considering that this transaction - if execution is possible - may cause an impact on the bid-ask spread? It is evident, that this type of correction will adjust the mark-to-market portfolio downwards, which explains why $V^{\mathcal{L}}$ and its whole probability function shift to lower values than the liquidity-adjusted value of Acerbi and Scandolo.

The remainder is organized as follows. The model and some helpful preliminary results are introduced in Section 3.2. We advise the reader to skip Section 3.2.2 and advance to Section 3.3, where we define and investigate the properties of the liquidity-adjusted portfolio value. Section 3.4 presents the analysis on the liquidity-adjusted measure of risk and the consequences of introducing block trading and partial execution effects. In Section 3.5 we find a numerical example that illustrates the impact of those effects on $\rho^{\mathcal{L}}$. We conclude in Section 3.6.

## 3.2 Framework

### 3.2.1 Setup, Partial Execution and Block Trading

The following setup is a one-period financial model. Hence, fix some arbitrary point in time and consider $N+1$ traded assets in the financial market, $N$ securities denoted by $A_i$ for $i \in I := \{1, \ldots, N\} := I \subset \mathbb{N}$ and the bank account denoted by $A_0$. Trading takes place in a market which functions under a continuous matching mechanism, i.e. buy and sell orders are processed immediately by their arrival, in particular there are no auctions. Our market is order driven. Investors have access to information of the order book, where sell orders and buy orders are listed and ordered using a price-time priority. In other words, an asset is traded with prices corresponding to the prices given[1] in the order book. Following Acerbi and Scandolo (Acerbi & Scandolo 2008) the order book implies a 'Marginal Supply-Demand Curve' or MSDC. Formally, the MSDC describes the prices at which asset $i \in I$ can be traded is given by a map $m_i : X \to \mathbb{R}$ with $X := \mathbb{R} \setminus \{0\}$ which satisfies

1. If $x_1 < x_2$, then $m_i(x_1) \geq m_i(x_2)$.

2. $m_i$ is càdlàg for $x < 0$ and làdcàg for $x > 0$.

As in (Acerbi & Scandolo 2008), cash has an special MSDC: $m_0(x) = 1$ for all $x \in X$. An order $x < 0$ ($x > 0$) in the order book represents a buy (sell) order. Thus, the price of the $dx$-th share is given by the bid (ask) price $m_i(x)$ for $x < 0$ ($x > 0$).

**Remark 3.2.1.** *Since our purpose is to model partial order execution, the set $X$ represents the set of orders and not the set of transactions as in (Acerbi & Scandolo 2008).*

Each asset traded in financial markets has a finite trading volume, e.g. the stock shares of a company, the number of futures contracts, the number of bonds, the number of asset-backed securities. The largest possible trading volume equals the sum of the posted orders and we call it the *maximal executable transaction.*

[1] Or implied, in case of market orders or other special cases.

Obviously, for each asset there are a maximal executable buy transaction and a maximal executable sell transaction, which we define as follows.

**Definition 3.2.2.** *The maximum executable sell and buy transactions are given by the sum of all sell and buy orders of the order book and are denoted by* $\underline{y}_i \leq 0$ *and* $\bar{y}_i \geq 0$ *for* $i \in \{0, 1, \ldots, N\}$*, respectively. The set of executable transactions is denoted by* $\mathcal{Y} := Y_0 \times Y_1 \times \ldots \times Y_N$*, where*[2] $Y_i := [\underline{y}_i, \bar{y}_i] \subset \mathbb{R}$*. Additionally, we denote the upper- and lower-bound vectors by* $\bar{y} := (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_N)$ *and* $\underline{y} := (\underline{y}_0, \underline{y}_1, \ldots, \underline{y}_N)$*.*

Note that $\mathcal{Y}$ is a set of transactions. Moreover, observe that the set of orders $\mathbb{R}^{N+1}$ is larger than the set of executable transactions, i.e. $\mathcal{Y} \subset \mathbb{R}^{N+1}$. In the present analysis we are interested in the value of portfolios after execution of some specific executable transactions. In this view, we focus on transactions and not on orders. We find below the definition of a function that translates any order into an executable transaction.

**Definition 3.2.3.** *For any order* $\theta_i \in \mathbb{R}$ *the corresponding executable transaction is given by a mapping* $f_i : \mathbb{R} \to Y_i$*,*

$$f_i(\theta_i) = \mathbb{1}_{\{\theta_i \in Y_i\}} \theta_i + \mathbb{1}_{\{\theta_i < \underline{y}_i\}} \underline{y}_i + \mathbb{1}_{\{\theta_i > \bar{y}_i\}} \bar{y}_i.$$

*We denote* $\hat{\theta}_i := f(\theta_i)$*.*

An investor trying to sell (buy) $\theta_i$ shares places a sell (buy) order $\theta_i$ which will be matched against buy (sell) orders of the order book and receives (pays) all bid (ask) prices $m_i(x)$ until the transaction $\hat{\theta}_i$ is completed. Hence, trading proceeds of asset $A_i$ from transaction $\hat{\theta}_i \in Y_i$ is given by the function $p_i : Y_i \to \mathbb{R}$,

$$p_i(\hat{\theta}_i) = \int_0^{\hat{\theta}_i} m_i(x) dx.$$

In this sense $p_i(\hat{\theta}_i)$ is the cash amount the investor receives when *selling* $\hat{\theta}_i > 0$ shares and the cash amount he pays when *buying* $|\hat{\theta}_i|$ shares when $\hat{\theta}_i < 0$. The proceeds from submitting an order of $\theta_i \in \mathbb{R}$ shares are represented by the function $P_i(\theta_i)$ where $P_i := p_i \circ f_i$. Recall that submitting order $\theta_i$ does not necessary

[2]We impose cash bounds, which can represent all current money amount on the market, just for technical reasons.

imply a transaction amounting $\theta_i$, because of the existence of maximal executable transactions it is possible to have $\theta_i \neq \hat{\theta}_i$. Since our interest is the analysis of transactions, we focus on $p_i$.

Based on the definition in (Acerbi & Scandolo 2008), a portfolio is a vector $\psi = (\psi_0, \psi_1, \ldots, \psi_N)$ in $\mathbb{R}^{N+1} =: \mathcal{P}$. A long, short and flat position in asset $A_i$ for $i \in \{0, 1, \ldots, N\}$ is denoted by $\psi_i > 0$, $\psi_i < 0$ and $\psi_i = 0$, respectively. Market participants hold portfolios $\psi \in \mathcal{P}$ and execute transactions $\hat{\theta} \in \mathcal{Y}$ and are interested to correctly identify corresponding *transaction proceeds* and *mark-to-market values* of asset positions. Following (Acerbi & Scandolo 2008), the *transaction proceeds* of an arbitrary transaction $\hat{\theta} \in \mathcal{Y}$ are given by the map $L : \mathcal{Y} \to \mathbb{R}$,

$$L(\hat{\theta}) = \sum_{i=0}^{N} p_i(\hat{\theta}_i) = \hat{\theta}_0 + \sum_{i=1}^{N} \int_0^{\hat{\theta}_i} m_i(x)dx.$$

In order to introduce the mark-to-market value of a portfolio, consider the best bid and the best ask which are denoted by $m_i^+ := m_i(0^+)$ and $m_i^- := m_i(0^-)$, respectively.[3] In practice is common to use prevailing bid and ask prices for valuing long and short portfolio positions, respectively. This usage is known as *mark-to-market* and is given for portfolio $\psi \in \mathcal{P}$ by

$$\tilde{U}(\psi) = \psi_0 + \sum_{i=1}^{N} \left( m_i^+ \psi_i \mathbb{1}_{\{\psi_i > 0\}} + m_i^- \psi_i \mathbb{1}_{\{\psi_i < 0\}} \right). \tag{3.1}$$

The mark-to-market value $\tilde{U}$ reflects only prevailing best bid and ask prices. As mentioned before, the bid-ask spread is a limited measure of liquidity risk, which is why we opt to consider the liquidity adjustment of portfolio positions $\psi$ put forward by Acerbi and Scandolo (Acerbi & Scandolo 2008). Tis adjustment is based on a specific transaction $\hat{\theta}^*$ which is introduced later in the analysis. For now, it is enough to keep in mind that by adjusting portfolio $\psi$ we end up with the mark-to-market value $\tilde{U}(\psi - \hat{\theta})$. We should note, however, that $\tilde{U}$ is not capable to internalize any possible liquidity effects such as block trading produced by transaction $\hat{\theta}$. The mentioned disadvantage is evident since best bids $m_i^+$ and best asks $m_i^-$ are independent of the execution of any transaction $\hat{\theta} \in \mathcal{Y}$. Accordingly, we call $\tilde{U}$ the *pre-execution mark-to-market portfolio value*. Aware of the shortcomings of $\tilde{U}$ we consider an

[3]Hence, the bid-ask spread is $m_i^- - m_i^+$.

extended version of the conventional mark-to-market value for which we define the *post-execution* best bid and ask prices.

**Definition 3.2.4.** *The post-execution best bid and ask prices after transaction $\hat{\theta}_i \in Y_i$ are given by functions $m_i^{+,\hat{\theta}_i}$ with $m_i^{+,\cdot} : Y_i \to \mathbb{R}_+$ and $m_i^{-,\hat{\theta}_i}$ with $m_i^{-,\cdot} : Y_i \to \mathbb{R}_+$, respectively.*

Thus, if no transaction is undertaken post-execution best bids and asks coincide with pre-execution best bids and asks, i.e. $m_i^- = m_i^{-,\hat{\theta}_i}$ and $m_i^+ = m_i^{+,\hat{\theta}_i}$ for all $i \in I$. However, from empirical observations presented in Klein and Madhavan (Keim & Madhavan 1996) there are transactions that shift temporarily or permanently best bids and asks. These transactions usually consist of large volume trades executed in one piece (if possible), which are commonly called *large block trades*. Execution of large trades may take a longer time than small transactions and may carry unveiled essential information which is reflected in an abrupt price change. Hence, the post-execution best bid and ask prices of a block transaction may not coincide with the pre-execution best bid and ask prices, respectively. The possible difference between pre- and post-execution prices represents a liquidity effect that is missing in Acerbi and Scandolo (Acerbi & Scandolo 2008).
In order to incorporate this additional liquidity effect originated by block trading we recur to the following simple but effective approach. Assume lower and upper bounds $\underline{b}_i$ and $\bar{b}_i$ such that large block trades are those transactions outside these bounds.

**Definition 3.2.5.** *For any non-cash asset $A_i$ for $i \in I$ suppose there exits lower and upper bounds $\underline{b}_i, \bar{b}_i \in \mathbb{R} \setminus \{0\}$, which define the set*

$$B_i = \left\{ x \in \mathbb{R} \,\middle|\, x \notin \left[\underline{b}_i, \bar{b}_i\right] \right\}.$$

*Denote $\bar{b} := (\bar{b}_1, \ldots, \bar{b}_N)$ and $\underline{b} := (\underline{b}_1, \ldots, \underline{b}_N)$. A transaction $\hat{\theta}_i \in Y_i$ for $i \in I$ is called block trade or block transaction if $\hat{\theta}_i \in B_i$. Additionally, denote the set of large block transactions by*

$$\mathcal{B} := \{\psi \in \mathcal{P} \mid \exists\, i \in I \quad \textit{with} \quad \psi_i \in B_i\}.$$

**Remark 3.2.6.** *Note that the cash positions $\psi_0$ and cash transactions $\hat{\theta}_0$ are irrelevant for the definition of block transactions. Furthermore, notice that*

$$\mathcal{B}^c = \{\psi \in \mathcal{P} \mid \forall i \in I, \quad \psi_i \notin B_i\}.$$

By previous arguments, we expect that a block transaction $\hat{\theta} \in \mathcal{B}$ induce a difference between pre-execution and post-execution best bids and best asks. When executing a block buy trade $\hat{\theta}_i < \underline{b}_i$ for asset $A_i$ market participants interpret the large volume of that single transaction as relevant information. For example, some agents will foresee a future imminent increase of the price of $A_i$ and will try to make a trading gain by buying that asset. This will overflow momentarily the market for asset $A_i$ with buy orders, which will induce an increase of the best ask, i.e. $m_i^{-,\hat{\theta}_i} > m_i^-$ for $\hat{\theta}_i < \underline{b}_i$. Similarly, during the execution of a block sell trade $\hat{\theta}_i > \bar{b}_i$ for asset $A_i$ market participants will expect a drop in the price of $A_i$ and try to 'dump' their positions as fast as possible inducing a fall in the best bid, i.e. $m_i^{+,\hat{\theta}_i} < m_i^+$ for $\hat{\theta}_i > \bar{b}_i$. We assume that large buy trades affect only best asks while large sell trades affect only best bids. The following assumption summarize these ideas.

**Assumption 3.2.7.** *For any $\hat{\theta}_i \in Y_i$, $i \in I$, it holds $m_i^- \leq m_i^{-,\hat{\theta}_i}$ and $m_i^+ \geq m_i^{+,\hat{\theta}_i}$. In particular, we have $m_i^{-,\hat{\theta}_i} > m_i^-$ for any $\hat{\theta}_i < \underline{b}_i$ and $m_i^{+,\hat{\theta}_i} < m_i^+$ for $\hat{\theta}_i > \bar{b}_i$.*

Note that only the best bids and asks shift after execution of block transactions. The previously introduced MSDCs are those price curves prevailing during the execution of transactions. We do not model a post-execution MSDC. In addition, we impose a nontrivial bid-ask spread, which is a common assumption when handling liquidity risk.

**Assumption 3.2.8.** *The bid-ask spread is always positive, i.e. for all $i \in I$ we have $m_i^- - m_i^+ > 0$ and $m_i^{-,\hat{\theta}_i} - m_i^{+,\hat{\theta}_i} > 0$ for any $\hat{\theta}_i \in Y_i$.*

In order to present the analysis as simple as possible without losing generality, we introduce the following assumption which is consistent with Assumption 3.2.7.

**Assumption 3.2.9.** *Consider the post-execution best bid and ask prices of asset $A_i$ after some transaction $\hat{\theta}_i \in Y_i$*

$$m_i^{+,\hat{\theta}_i} = \begin{cases} m_i^{+,\bar{b}_i} & for \quad \hat{\theta}_i > 0,\ \hat{\theta}_i \in B_i \\ m_i^+ & else \end{cases},$$

*and*

$$m_i^{-,\hat{\theta}_i} = \begin{cases} m_i^{-,\underline{b}_i} & for \quad \hat{\theta}_i < 0,\ \hat{\theta}_i \in B_i \\ m_i^- & else \end{cases},$$

*respectively, where* $m_i^+ > m_i^{+,\bar{b}_i} \geq m_i(\bar{b}_i) \in \mathbb{R}_{++}$*,* $m_i^- < m_i^{-,\underline{b}_i} \leq m_i(\underline{b}_i) \in \mathbb{R}_{++}$*, whenever* $\bar{b}_i, \underline{b}_i \in Y_i$*, and* $m_i^{+,\bar{b}_i} \geq 1$*.*

Assuming the existence of the set of block trades $\mathcal{B}$ is not sufficient to guarantee the existence of block trading effects. We define *block trading effects* as the price change in either a best bid or a best ask caused by execution of some block transaction. In other words, block trading effects exist only if block trades are executable. If none executable transaction produces a change in best bid or ask prices, then there is no block trading effect.[4] Formally, consider the following

**Definition 3.2.10.** *There are no block trading effects if for all executable transactions all post-execution best bid and ask prices match the pre-execution best bid and ask prices, respectively, i.e. there are no block trading effects if for all* $\hat{\theta} \in \mathcal{Y}$ *we have* $m_i^{+,\hat{\theta}_i} = m_i^+$ *and* $m_i^{-,\hat{\theta}_i} = m_i^-$ *for all* $i \in I$*.*

Equivalently, we rephrase this definition by a more handy statement, which is of great help for the presentation of the results of the next sections.

**Proposition 3.2.11.** *There are no block trading effects if and only if* $\mathcal{Y} \subseteq \mathcal{B}^c$*.*

*Proof.* It follows straightforwardly from Assumption 3.2.9 and Definition 3.2.10.

□

Because the execution of a block buy (ask) order induces a rise (fall) in the best ask (best bid) price, any investor undertaking a block transaction must acknowledge the impact of this transaction in the value of its portfolio. For example, if an investor holding a large long position in asset $A_i$ executes a block sell trade, the remaining long position after execution will be priced with the post-execution best bid $m_i^{+,\bar{b}_i}$ which is strictly smaller than the pre-execution best bid $m_i^+$. Hence, the investor's portfolio experience a loss in its value due to block trading effects. Consequently, we propose to use a concept of mark-to-market value that embeds block trading

[4]Usually in reality execution of small and large transactions will induce changes in best bids and asks, which are mostly small and momentarily. We refer to block trading effects to *large* changes in best bids and ask, which may be*permanently* or last a long period of time.

effects. Accordingly, we introduce the notion of *post-execution mark-to-market value* as follows.

**Definition 3.2.12.** *The post-execution mark-to-market value (MtM) value of portfolio $\psi \in \mathcal{P}$ after execution of transaction $\hat{\theta} \in \mathcal{Y}$ is given by the mapping $U : \mathcal{P} \times \mathcal{Y} \to \mathbb{R}$,*

$$U(\psi, \hat{\theta}) = \psi_0 + \sum_{i=1}^{N} \left( m_i^{+,\hat{\theta}_i} \psi_i \mathbb{1}_{\{\psi_i > 0\}} + m_i^{-,\hat{\theta}_i} \psi_i \mathbb{1}_{\{\psi_i < 0\}} \right).$$

As pointed out in Acerbi and Scandolo (Acerbi & Scandolo 2008), a mark-to-market valuation approach based only on best bids and best asks fails to capture the depth of the market, i.e. we miss the information contained in the whole MSDCs because we regard only best bids and best asks. Alternatively, Acerbi and Scandolo propose to consider the sum of the mark-to-market value of a portfolio after the execution of some transaction and the value of the proceeds of that transaction as a more informative valuation method. Formally, for some portfolio $\psi \in \mathcal{P}$ and by choosing $\theta$ optimally they consider

$$\tilde{U}(\psi - \theta) + L(\theta), \tag{3.2}$$

as a valuation method that reflects the overall market situation more accurately than just $\tilde{U}$. Valuation method (3.2) gives an answer to the question: how much cash can an investor holding an initial portfolio $\psi$ collect by executing transaction $\theta$, and what is the mark-to-market value of the resulting portfolio? Note that valuation approach (3.2) evidences some weak points concerning the incorporation of market-liquidity conditions since it neglects block trading and partial execution effects because Acerbi and Scandolo assume that any order can be traded. In order to avoid these shortcomings we consider the following valuation of some portfolio $\psi \in \mathcal{P}$ by choosing optimally some $\hat{\theta} \in \mathcal{Y}$,

$$U(\psi - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}). \tag{3.3}$$

As mentioned previously, in Section 3.3.1 we introduce the notion of liquidity-adjusted portfolio value, which is based on (3.3) for some special transaction $\hat{\theta}^* \in \mathcal{Y}$. We denote expression (3.3) as the *post-execution portfolio value* of $\psi \in \mathcal{P}$ given transaction $\hat{\theta} \in \mathcal{Y}$.

In order to provide a further argument for the validity of post-execution portfolio values as a valuation approach reflecting liquidity effects, we discuss next *portfolio liquidation*. An investor trying to liquidate an entire portfolio will face execution restrictions if the portfolio positions are too large for the governing market situation. In cases when the market depth does not permit full liquidation, the investor receives the liquidation proceeds from the executable portion and hold the unexecuted portion of his portfolio. The executable portion of a portfolio $\psi \in \mathcal{P}$ is given by $\hat{\psi} = f(\psi) \in \mathcal{Y}$ and the unexecutable portion by $\psi - \hat{\psi}$. Hence, differently as Acerbi and Scandolo, the *liquidation value* of portfolio $\psi \in \mathcal{P}$ is given by

$$U(\psi - \hat{\psi}, \hat{\psi}) + L(\hat{\psi}),$$

which is the post-execution value of $\psi$ given transaction $\hat{\psi}$. Liquidation value of portfolio $\psi$ reflects the underlying market depth involving MSDCs, executable transactions and price impacts caused by block trading, all of which are important components of liquidity risk.

*Liquidity costs* of any transaction $\hat{\theta} \in \mathcal{Y}$ are the difference of the post-execution portfolio value given no transaction and the post-execution portfolio value given transaction $\hat{\theta}$. Formally, the liquidity costs of transaction $\hat{\theta} \in \mathcal{Y}$ for portfolio $\psi \in \mathcal{P}$ is given by the function $C : \mathcal{P} \times \mathcal{Y} \to \mathbb{R}$

$$C(\psi, \hat{\theta}) = U(\psi, 0) - (L(\hat{\theta}) + U(\psi - \hat{\theta}, \hat{\theta})). \tag{3.4}$$

Consequently, the liquidation costs of any portfolio $\psi \in \mathcal{P}$ are given by $C(\psi, \hat{\psi}) = U(\psi, 0) - (L(\hat{\psi}) + U(\psi - \hat{\psi}, \hat{\psi}))$. Consider the following characterization of portfolios from (Acerbi & Scandolo 2008). Let $\psi, \xi \in \mathcal{P}$ be two portfolios.

1. They are concordant, $\psi \upuparrows \xi$ , if $\psi_i \xi_i \geq 0$ for any $i \in \{0, 1, \ldots, N\}$.
2. They are discordant, $\psi \downdownarrows \xi$ , if $\psi_i \xi_i \leq 0$ for any $i \in \{0, 1, \ldots, N\}$.
3. $\psi \geq \xi$, if $\psi_i \geq \xi_i$ for all $i \in \{0, 1, \ldots, N\}$.

As mentioned earlier, the liquidity-adjusted portfolio value is based on the post-execution portfolio value given some *specific* transaction. Since we are modeling liquidity risk, we are interested only in transactions that fulfill some liquidity constraint or *liquidity policy.* By (Acerbi & Scandolo 2008), a liquidity policy $\mathcal{L}$ is any closed convex subset $\mathcal{L} \subseteq \mathcal{P}$ satisfying the following conditions.

1. If $\psi \in \mathcal{L}$, then $\psi + (a, 0, \ldots, 0) \in \mathcal{L}$ for all $a > 0$.

2. If $(\psi_0, \psi_1, \ldots, \psi_N) \in \mathcal{L}$, then $(\psi_0, 0, \ldots, 0) \in \mathcal{L}$.

Accordingly, the liquidity-adjusted portfolio value of $\psi \in \mathcal{P}$ is given by $U(\psi - \hat{\theta}^*, \hat{\theta}^*) + L(\hat{\theta}^*)$, where the *resulting portfolio* $\psi - \hat{\theta}^*$ is in $\mathcal{L}$ for transaction $\hat{\theta}^* \in \mathcal{Y}$ which is defined in Section 3.3.1.

### 3.2.2 Preliminary Results

In this section we present some useful insights and results concerning post-execution MtM value $U$, transactions proceeds $L$, post-execution portfolio value $U + L$ and liquidity costs $C$, which are needed later in the analysis. However, we advice the reader to skip this section and proceed with Section 3.3.1. Results there cite propositions, lemmas and corollaries of this section, which can be read on timely demand. In Acerbi and Scandolo (Acerbi & Scandolo 2008), we find that $U$, $L$, $U + L$ and $C$ are continuous functions, the latter is convex and the rest concave. However, following our setup and in presence of block trading and partial execution effects, we learn from this section that continuity of $U$, $L$, $U + L$ and $C$ is destroyed. Furthermore, when these effects exist $U$, $L$ and $U + L$ are not concave and $C$ is not convex. We develop these results meticulously as follows.

**Proposition 3.2.13.** *Consider the function $L : \mathcal{Y} \longrightarrow \mathbb{R}$ and for every $\hat{\theta} \in \mathcal{Y}$ the function $U(\cdot, \hat{\theta}) : \mathcal{P} \longrightarrow \mathbb{R}$.*

1. *$L$ is increasing[5], continuous and concave on $\mathcal{Y}$, subadditive on concordant portfolios and superadditive on discordant portfolios.*

2. *$U(\cdot, \hat{\theta})$ is increasing, continuous and concave on $\mathcal{P}$, additive on concordant portfolios and superadditive on $\mathcal{P}$.*

[5] A function $f : \mathcal{X} \longrightarrow \mathbb{R}$, $\mathcal{X} \subseteq \mathcal{P}$, is increasing if $f(\psi) \geq f(\xi)$ for $\psi, \xi \in \mathcal{P}$ with $\psi \geq \xi$.

*Proof.* Continuity of $L$ and $U(\cdot, \hat{\theta})$ follows from their definitions. By definition of MSDC, for each $i \in I$ the proceeds function $p_i(\hat{\theta}_i)$, $\hat{\theta}_i \in Y_i$, is increasing. Hence, $L(\hat{\theta}) = \sum_{i=0}^{N} p_i(\hat{\theta}_i)$ is increasing on $\mathcal{Y}$. Similarly, because of the definition of MSDC and by construction, $U$ is an increasing function on $\mathcal{P}$. The rest of the proof can be found in Acerbi and Scandolo (Acerbi & Scandolo 2008) by noticing that the functions $L(\cdot)$ and $U(\cdot, \hat{\theta})$ are decomposable.

□

Last proposition almost coincides with results in (Acerbi & Scandolo 2008). There are two differences: transactions proceeds $L$ is defined on the set of executable transactions $\mathcal{Y}$ and we use the post-execution MtM value $U$ given some transaction $\hat{\theta} \in \mathcal{Y}$ instead of the pre-execution mark-to-market value $\tilde{U}$ as Acerbi and Scandolo. Note that the function $U$ has the same characteristics as $\tilde{U}$ if we fix transaction $\hat{\theta}$. Let us next draw our attention to the post-execution value $U$ for the resulting portfolio $\psi - \hat{\theta} \in \mathcal{P}$ after transaction $\hat{\theta} \in \mathcal{Y}$,

$$U(\psi - \hat{\theta}, \hat{\theta}),$$

where $\psi \in \mathcal{P}$ is the initial portfolio. For each $\psi \in \mathcal{P}$ denote the function $z^{\psi} : \mathcal{Y} \to \mathbb{R}$ defined by

$$z^{\psi}(\hat{\theta}) := U(\psi - \hat{\theta}, \hat{\theta}) \qquad \text{for} \qquad \hat{\theta} \in \mathcal{Y}. \tag{3.5}$$

The proposition below states that $z^{\psi}$ is continuous and concave when the underlying portfolio is not too large and on the set of small executable transactions $\mathcal{B}^c \cap \mathcal{Y}$. If however, the portfolio is large and block trades are executable, then $z^{\psi}$ looses those characteristics.

**Proposition 3.2.14.** *Given $\psi \in \mathcal{P}$, $\bar{y}, \underline{y} \in \mathcal{P}$ and $\bar{b}, \underline{b} \in \mathbb{R}^N$,*

1. *$z^{\psi}$ is continuous and concave*

   *(a) if $\psi \in \mathcal{B}^c$.*

   *(b) on[6] $\mathcal{B}^c \cap \mathcal{Y}$.*

[6]By definition $\mathcal{B}^c \cap \mathcal{Y} \neq \emptyset$.

*2. $z^\psi$ is neither concave nor continuous if there exists some $i \in I$ such that*

*(a) $\psi_i < \underline{b}_i$ and $\underline{y}_i < \underline{b}_i$, or*

*(b) $\psi_i > \bar{b}_i$ and $\bar{y}_i > \bar{b}_i$.*

*Proof.* For any $\psi \in \mathcal{P}$, $z^\psi(x)$ is a decomposable function, i.e. for $x \in \mathcal{Y}$

$$z^\psi(x) = \sum_{i=0}^{N} f_i^\psi(x_i), \qquad with \qquad f_i^\psi(x_i) = \int_0^{\psi_i - x_i} g_i^\psi(u)du$$

where

$$g_i^\psi(x_i) = m_i^{+,x_i} \cdot \mathbb{1}_{\{\psi_i - x_i > 0\}} + m_i^{-,x_i} \cdot \mathbb{1}_{\{\psi_i - x_i < 0\}},$$

for $i \in I$, with $g_i^\psi(\psi_i) = 0$ and $g_0^\psi(x_i) = 1$ for all $x_i \in Y_i$.

*1. (a) Continuity.* By Assumption 3.2.9, we have for $\bar{b}_i \geq \psi_i \geq 0$ and $\underline{b}_i \leq \psi_i \leq 0$

$$f_i^\psi(x_i) = \begin{cases} m_i^+(\psi_i - x_i) & for \quad x_i < \psi_i \\ 0 & for \quad x_i = \psi_i \\ m_i^-(\psi_i - x_i) & for \quad x_i > \psi_i. \end{cases}$$

For the cases $x_i < \psi_i$ and $x_i > \psi_i$, the function $f_i^\psi$ is clearly continuous. For $x_i = \psi_i$, consider a sequence $(\xi_i^n)$, with $\underline{y}_i \leq \xi_i^n < \psi_i$ for all $n \in \mathbb{N}$ and $\xi_i^n \to \psi_i$ as $n \to \infty$, and another sequence $(\eta_i^n)$, with $\psi_i < \eta_i^n \leq \bar{y}_i$ for all $n \in \mathbb{N}$ and $\eta_i^n \to \psi_i$ as $n \to \infty$. Since $f_i^\psi(\xi_i^n) \to 0$ and $f_i^\psi(\eta_i^n) \to 0$ as $n \to \infty$ and $f_i^\psi(\psi_i) = 0$, the function $f_i^\psi$ is continuous. Because all components $f_i$ of $f$ are continuous, $f$ is also continuous.

*1. (a) Concavity.* Note that

$$f_i^\psi(x_i) = \begin{cases} \int_0^{\psi_i - x_i} g_i^\psi(u)du & if \quad \psi_i - x_i > 0 \\ \\ -\int_{\psi_i - x_i}^0 g_i^\psi(u)du & if \quad \psi_i - x_i < 0. \end{cases}$$

Furthermore, for $\psi_i - x_i > 0$ and $\psi_i - x_i < 0$ the derivative of $f_i^\psi$ is given by

$$\frac{df_i^\psi(x_i)}{dx_i} = -g_i^\psi(\psi_i - x_i).$$

Additionally, for $\bar{b}_i \geq \psi_i \geq 0$ and $\underline{b}_i \leq \psi_i \leq 0$ we have

$$-g_i^\psi(\psi_i - x_i) = \begin{cases} -m_i^+ & for \quad x_i < \psi_i \\ -m_i^- & for \quad x_i > \psi_i. \end{cases}$$

For $\bar{b}_i \geq \psi_i \geq 0$ and $\underline{b}_i \leq \psi_i \leq 0$ and $\psi_i = x_i$ observe the left and right derivatives are given by

$$\frac{d^+ f_i^\psi(\psi_i)}{dx_i} = \lim_{\epsilon \to 0} \frac{f_i^\psi(\psi_i + \epsilon) - f_i^\psi(\psi_i)}{\epsilon} = \lim_{\epsilon \to 0} \frac{-m_i^- \epsilon}{\epsilon} = -m_i^-,$$

and

$$\frac{d^- f_i^\psi(\psi_i)}{dx_i} = \lim_{\epsilon \to 0} \frac{f_i^\psi(\psi_i - \epsilon) - f_i^\psi(\psi_i)}{\epsilon} = \lim_{\epsilon \to 0} \frac{-m_i^+ \epsilon}{\epsilon} = -m_i^+,$$

where $\epsilon > 0$. Thus, the slope of any tangent on $f^\psi$ at $x_i = \psi_i$ takes values in $[-m_i^-, -m_i^+]$, i.e. $-m_i^- \leq \frac{df_i^\psi(\psi_i)}{dx_i} \leq -m_i^+$. Since $-m_i^+ > -m_i^-$, $\frac{df_i^\psi(x_i)}{dx_i}$ is decreasing in $x_i$. Thus, $f_i^\psi$ is concave, hence $z^\psi$ is also concave[7].

*1. (b) Continuity and Concavity.* Since $x_i \in B_i^c \cap Y_i$ we have as before

$$f_i^\psi(x_i) = \begin{cases} m_i^+(\psi_i - x_i) & if \quad x_i < \psi_i \\ 0 & if \quad x_i = \psi_i \\ m_i^-(\psi_i - x_i) & if \quad x_i > \psi_i, \end{cases}$$

and

$$\frac{df_i^\psi(x_i)}{dx_i} = \begin{cases} -m_i^+ & for \quad x_i < \psi_i \\ -m_i^- & for \quad x_i > \psi_i, \end{cases}$$

for any $\psi_i \in \mathbb{R}$. As observed previously $\frac{df_i^\psi(\psi_i)}{dx_i}$ is decreasing. Hence, $f_i^\psi$ is continuous and concave, which implies that $z^\psi$ is continuous and concave on $\mathcal{B}^c \cap \mathcal{Y}$.

*2. Non-Concavity.* As previously found the derivative of $f_i^\psi$ with respect to $x_i$ equals $-g_i^\psi(\psi_i - x_i)$.

Let $\bar{y}_i < \bar{b}_i$. For $\psi_i > \bar{b}_i$ we have

$$-g_i^\psi(\psi_i - x_i) = \begin{cases} -m_i^+ & for \quad x_i \leq \bar{b}_i \\ -m_i^{+,\bar{b}_i} & for \quad \bar{b}_i < x_i < \psi_i \\ -m_i^- & for \quad x_i > \psi_i. \end{cases}$$

Since $-m_i^+ < -m_i^{+,\bar{b}_i} > -m_i^-$, then $-g_i^\psi$ is not a decreasing function. Hence, $f_i^\psi$ is not concave for $\psi_i > \bar{b}_i$ and $\bar{y}_i > \bar{b}_i$. Now let $\underline{y}_i < \underline{b}_i$. Similarly, for $\psi_i < \underline{b}_i$

$$-g_i^\psi(\psi_i - x_i) = \begin{cases} -m_i^+ & for \quad x_i < \psi_i \\ -m_i^{-,\underline{b}_i} & for \quad \psi_i < x_i < \underline{b}_i \\ -m_i^- & for \quad x_i > \underline{b}_i. \end{cases}$$

[7] A decomposable function is concave if and only if all of its components are concave.

Since $-m_i^+ > -m_i^{-,\underline{b}_i} < -m_i^-$, $-g_i^\psi$ is not a decreasing function. Hence, $f_i^\psi$ is not concave for $\psi_i < \underline{b}_i$ and $\underline{y}_i < \underline{b}_i$. Thus, $f^\psi$ is not concave.

*2. Non-Continuity.* Consider $\psi$, $\psi' \in \mathcal{P}$ with $\psi_j < \underline{b}_j$, $\psi'_k > \bar{b}_k$ for some $j, k \in I$. Additionally, regard two sequences $(\xi^n), (\eta^n) \subset \mathcal{Y}$, with

$$\xi_i^n = \begin{cases} \psi_i & for \quad i \neq j \\ \xi_j^n & for \quad i = j \end{cases}, \qquad \text{and} \qquad \eta_i^n = \begin{cases} \psi'_i & for \quad i \neq k \\ \eta_k^n & for \quad i = k \end{cases},$$

where $\underline{y}_j < \xi_j^n < \underline{b}_j$, $\psi_j < \xi_j^n$ and $\bar{y}_k > \eta_k^n > \bar{b}_k$, $\psi'_k > \eta_k^n$ for all $n \in \mathbb{N}$. Assume $\xi^n \to \xi$ and $\eta^n \to \eta$ with $\xi_j^n \to \underline{b}_j$ and $\eta_k^n \to \bar{b}_k$ as $n \to \infty$. Hence,

$$f_j^\psi(\xi^n) = m_j^{-,\underline{b}_j}(\psi_j - \xi_j^n) \overset{n\to\infty}{\longrightarrow} m_j^{-,\underline{b}_j}(\psi_j - \underline{b}_j) < m_j^-(\psi_j - \underline{b}_j) = f_j^\psi(\xi),$$

and

$$f_k^{\psi'}(\eta^n) = m_k^{+,\bar{b}_k}(\psi'_k - \eta_k^n) \overset{n\to\infty}{\longrightarrow} m_k^{+,\bar{b}_k}(\psi'_k - \bar{b}_k) < m_k^+(\psi'_k - \bar{b}_k) = f_k^{\psi'}(\eta).$$

Thus, $f_i^\psi$ and $z^\psi$ are not continuous if either 1) $\psi_i < \underline{b}_i$ and $\underline{y}_i < \underline{b}_i$ or 2) $\psi_i > \bar{b}_i$ and $\bar{y}_i > \bar{b}_i$.

$\square$

By this proposition, the function $z^\psi$ is continuous and concave whenever block trading effects are not present.[8]

The same statements of the previous proposition hold true for the post-execution portfolio value $U(\psi - \hat{\theta}, \hat{\theta}) + L(\hat{\theta})$ and for the liquidity costs $C(\psi, \hat{\theta})$. For this we consider the next convenient representations. For every $\psi \in \mathcal{P}$ consider the function $v^\psi : \mathcal{Y} \to \mathbb{R}$ given by

$$v^\psi(\hat{\theta}) := U(\psi - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}), \tag{3.6}$$

for $\hat{\theta} \in \mathcal{Y}$, and the function $C(\psi, \cdot) : \mathcal{Y} \to \mathbb{R}$ defined in (3.4). The auxiliary function $v^\psi$ represents the post-execution portfolio value when portfolio $\psi$ is fixed and transactions $\hat{\theta}$ vary.

**Corollary 3.2.15.** *Given $\psi \in \mathcal{P}$, $\bar{y}, \underline{y} \in \mathcal{P}$ and $\bar{b}, \underline{b} \in \mathbb{R}^N$,*

*1. $v^\psi$ and $C(\psi, \cdot)$ are continuous, $v^\psi$ is concave and $C(\psi, \cdot)$ is convex*

[8]Recall that these effects are not present if $\mathcal{B}^c \subseteq \mathcal{Y}$.

(a) *if* $\psi \in \mathcal{B}^c$.

(b) *on* $\mathcal{B}^c \cap \mathcal{Y}$.

2. $v^\psi$ *and* $C(\psi, \cdot)$ *are not continuous,* $v^\psi$ *is not concave and* $C(\psi, \cdot)$ *is not convex if there exists some* $i \in I$ *such that*

   (a) $\psi_i < \underline{b}_i$ *and* $\underline{y}_i < \underline{b}_i$*, or*

   (b) $\psi_i > \bar{b}_i$ *and* $\bar{y}_i > \bar{b}_i$.

*Proof.* By Proposition 3.2.13 and Proposition 3.2.14 statement 1. holds true. For point 2., observe that

$$v^\psi(x) = \sum_{i=0}^{N} F_i^\psi(x_i),$$

where

$$F_i^\psi(x_i) = \int_0^{\psi_i - x_i} g_i^\psi(u)du + \int_0^{x_i} m_i(u)du,$$

with $g_i^\psi$ defined in the proof of Proposition 3.2.14. Hence and by Proposition 3.2.14, $v^\psi$ is not continuous if either conditions (a) or (b) holds true, because the derivative $\frac{dF_i^\psi}{dx_i} = -g_i^\psi(\psi_i - x_i) + m_i(x_i)$ is not decreasing in cases (a) and (b). Thus, the remaining statements hold also true by the definition of $C(\psi, \cdot)$ in (3.4).

□

Last corollary and following lemma indicate that block trading effects induce discontinuity and non-concavity in the post-execution portfolio value $v^\psi$. This result suggests that valuation approaches based on $v^\psi$ are expected to be discontinuous and non-concave in presence of block trading effects. In the next section we introduced a liquidity-adjusted portfolio value based on $v^\psi$, which we prove to obey this rule.

**Lemma 3.2.16.** *Given some portfolio* $\psi \in \mathcal{P}$*,* $\bar{y}, \underline{y} \in \mathcal{P}$ *and* $\bar{b}, \underline{b} \in \mathbb{R}^N$*, the function* $v^\psi$ *is not continuous if and only if there exists some nonempty set* $J \subseteq I$ *such that* $\psi_j > \bar{b}_j$ *with* $\bar{y}_j > \bar{b}_j$ *for some* $j \in J$ *or* $\psi_k < \underline{b}_k$ *with* $\underline{y}_k < \underline{b}_k$ *for some* $k \in J$.

*Proof.* By Corollary 3.2.15 the existence of $J$ is sufficient in order to have a discontinuous function $v^\psi$. We show here that its existence is a necessary condition.

Consider the function $v^\psi$ defined in Corollary 3.2.15 as a decomposable function $v^\psi(\hat{\theta}) = \sum_{i=0}^N v_i^\psi(\hat{\theta}_i)$, where $\hat{\theta}_i \in Y_i$, with

$$v_i^\psi(\hat{\theta}_i) = m^{+,\hat{\theta}_i}(\psi_i - \hat{\theta}_i)\mathbb{1}_{\{\psi_i - \hat{\theta}_i > 0\}} + m^{-,\hat{\theta}_i}(\psi_i - \hat{\theta}_i)\mathbb{1}_{\{\psi_i - \hat{\theta}_i < 0\}} + \int_0^{\hat{\theta}_i} m_i(u)du,$$

for $i \in I$ and $v_0^\psi(\hat{\theta}_0) = \psi_0$. By Corollary 3.2.15, the function $v^\psi(\hat{\theta})$ is continuous for the following cases: (i) $\psi_i \in [\underline{b}_i, \bar{b}_i]$ for all $i \in I$, and (ii) $\psi_i \notin [\underline{b}_i, \bar{b}_i]$, (a) $\underline{b}_i \leq \underline{y}_i \leq \bar{y}_i \leq \bar{b}_i$ for all $i \in I$. We have the following cases left:
(b) $\underline{b}_i > \underline{y}_i$ and (c) $\bar{b}_i < \bar{y}_i$. Additionally, for (b) we have the possibilities (1) $\psi_i < \underline{b}_i$ and (2) $\psi_i > \bar{b}_i$. For (ii.b.1), we obtain

$$v_i^\psi(\hat{\theta}_i) = \begin{cases} m_i^-(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u)du & \hat{\theta}_i \geq \underline{b}_i \\ m_i^{-,\underline{b}_i}(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u)du & \psi_i \leq \hat{\theta}_i < \underline{b}_i \\ m_i^+(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u)du & \hat{\theta}_i > \psi_i, \end{cases}$$

and for (ii.b.2)

$$v_i^\psi(\hat{\theta}_i) = \begin{cases} m_i^-(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u)du & \hat{\theta}_i \geq \psi_i \\ m_i^+(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u)du & \hat{\theta}_i < \psi_i. \end{cases}$$

By Proposition 3.2.14 in case (ii.b.1) the function $v_i^\psi$ is not continuous and in case (ii.b.1) it is continuous. Analogously, non continuity of the function $v_i^\psi$ follows for case (c) for the possibilities (3) $\psi < \bar{b}_i$ and (4) $\psi > \bar{b}_i$.

□

From Corollary 3.2.15 and Lemma 3.2.16 is clear that the post-execution portfolio value $v^\psi$ presents discontinuities and non-concavity when block trading effects exist. Hence, regard the following corollary whose proof is the collection of last results.

**Corollary 3.2.17.** *In general the function $v^\psi$ is neither concave nor continuous.*

In Acerbi and Scandolo's setup we consider the pre-execution portfolio value $\tilde{v}^\psi : \mathcal{Y} \to \mathbb{R}$, given by

$$\tilde{v}^\psi(\hat{\theta}) := \tilde{U}(\psi - \hat{\theta}) + L(\hat{\theta}), \tag{3.7}$$

for $\hat{\theta} \in \mathcal{Y}$. The authors show that $\tilde{v}^\psi$ is a continuous concave function. The liquidity-adjusted portfolio value proposed by Acerbi and Scandolo turns out to be also continuous and concave because it is based on the pre-execution portfolio value $\tilde{v}^\psi$. In our framework, the liquidity-adjusted portfolio value introduced in the following section is based on the post-execution portfolio value $v^\psi$, inheriting the discontinuity and non-concavity from $v^\psi$.

However, if block trading effects are not present, i.e. $\mathcal{Y} \subseteq \mathcal{B}^c$, the post-execution mark-to-market value $U$ coincides with the pre-execution value $\tilde{U}$. Formally, if $\mathcal{Y} \subseteq \mathcal{B}^c$, we have by construction

$$U(\psi, \hat{\theta}) = \tilde{U}(\psi),$$

for all $\psi \in \mathcal{P}$ and any $\hat{\theta} \in \mathcal{Y}$. Following this observation, we obtain the next evident result.

**Corollary 3.2.18.** *If $\mathcal{Y} \subseteq \mathcal{B}^c$, for all $\psi \in \mathcal{P}$ and any $\hat{\theta} \in \mathcal{Y}$,*

$$v^\psi(\hat{\theta}) = \tilde{v}^\psi(\hat{\theta}).$$

Hence, in absence of block trading effects, i.e. $\mathcal{Y} \subseteq \mathcal{B}^c$, and by borrowing results from (Acerbi & Scandolo 2008), $v^\psi$ is continuous, concave and for $\lambda > 1$

$$v^{\lambda\psi}(\lambda\hat{\theta}) \leq \lambda v^\psi(\hat{\theta}). \tag{3.8}$$

Additionally, the function $C$ is continuous, convex and for $\lambda > 1$

$$C(\lambda\psi, \lambda\hat{\theta}) \geq \lambda C(\psi, \hat{\theta}).$$

Block trading effects cause not only discontinuities, non-concavity and non-convexity, they also induce lower mark-to-market values. To see this, recall that Acerbi and Scandolo's framework neglects block trading effects. Relevant mark-to-market values of their framework are $\tilde{U}$ and $\tilde{v}^\psi$. Our model conceives block trading effects as possible. Our relevant mark-to-market values are $U$ and $v^\psi$. Hence, consider the following

**Proposition 3.2.19.** *For any $\psi \in \mathcal{P}$ and $\hat{\theta} \in \mathcal{Y}$*

$$\tilde{U}(\psi) = U(\psi, 0) \geq U(\psi, \hat{\theta}),$$

*and*

$$\tilde{v}^\psi(\hat{\theta}) \geq v^\psi(\hat{\theta}).$$

*Proof.* The results follow from observing that $m_i^{+,\hat{\theta}_i} \leq m_i^+$ and $m_i^{-,\hat{\theta}_i} \geq m_i^-$ for any $\hat{\theta}_i \in Y_i$ for $i \in I$ and from the definition of $U$, $\tilde{U}$, $v^\psi$ and $\tilde{v}^\psi$.

□

By this proposition we see in Section 3.3.3 that block trading effects reduce the liquidity-adjusted portfolio value, which is introduced next.

## 3.3 Liquidity-Adjusted Portfolio Value

In this section we present the liquidity-adjusted portfolio value $V^{\mathcal{L}}$ and prove that it is well defined for every portfolio $\psi$. Then we show that $V^{\mathcal{L}}$ is not concave in general. Last, we identify the effects of block trading and partial execution on $V^{\mathcal{L}}$.

### 3.3.1 Liquidity Adjustment

Previously we introduced the pre-execution mark-to-market value for portfolio $\psi \in \mathcal{P}$, which is given by $\tilde{U}(\hat{\theta})$. As pointed out, this valuation approach can not reflect liquidity conditions of the market. Moreover, $\tilde{U}(\hat{\theta})$ can neither capture the specific liquidity needs or constraints of the investor holding that position.

A more appropriate value that considers market-liquidity conditions as MSDCs, executable transactions and block trading is the post-execution value $U(\psi - \hat{\theta}, \hat{\theta}) + L(\hat{\theta})$ of portfolio $\psi$ for some transaction $\hat{\theta} \in \mathcal{Y}$. In order to adequate this valuation approach to reflect liquidity restrictions of the investor, we need to impose that the resulting portfolio from the transaction must meet the requirements of some given liquidity policy $\mathcal{L}$ to which the investor is subjected. Starting from a portfolio $\psi \in \mathcal{P}$ and executing transaction $\hat{\theta} \in \mathcal{Y}$, the resulting portfolio including cash proceeds of the transaction is given by

$$\psi - \hat{\theta} + (L(\hat{\theta}), 0, \dots, 0),$$

where $(L(\hat{\theta}), 0, \dots, 0) \in \mathcal{P}$ represents a cash portfolio with a cash position which equals the proceeds from transaction $\hat{\theta}$. Hence, given some liquidity policy $\mathcal{L}$ we are interested for liquidity adjustments $\hat{\theta}$ to portfolio $\psi$ whose adjusted value is given by

$$v^{\psi}(\hat{\theta}) = U(\psi - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}), \tag{3.9}$$

such that $\hat{\theta} \in \mathcal{Y}$, and $\psi - \hat{\theta} + (L(\hat{\theta}), 0, \ldots, 0) \in \mathcal{L}$. Note that there may be several transactions $\hat{\theta}$ that fulfill last conditions but with different $v^{\psi}(\hat{\theta})$. Following Acerbi and Scandolo, we impose a further condition for transaction $\hat{\theta}$: it must be chosen optimally, i.e. it must maximize $v^{\psi}(\hat{\theta})$. Formally, we define the *liquidity-adjusted portfolio value* of $\psi \in \mathcal{P}$ under liquidity policy $\mathcal{L}$ as a map $V^{\mathcal{L}} : \mathcal{P} \to \mathbb{R}$ given by

$$V^{\mathcal{L}}(\psi) = \sup \left\{ U(\psi - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}) \middle| \, \hat{\theta} \in \mathcal{C}^{\mathcal{L}}(\psi) \right\}, \tag{3.10}$$

with

$$\mathcal{C}^{\mathcal{L}}(\psi) = \left\{ \hat{\theta} \in \mathcal{Y} \;\middle|\; \psi - \hat{\theta} + (L(\hat{\theta}), 0, \ldots, 0) \in \mathcal{L} \right\}.$$

Additionally, if $\mathcal{C}^{\mathcal{L}}(\psi) = \emptyset$, then $V^{\mathcal{L}}(\psi) = -\infty$.
Although the function $U(\psi - \hat{\theta}, \hat{\theta}) + L(\hat{\theta})$ of optimization problem (3.10) is in general non-concave in $\mathcal{Y}$ as shown in the previous section, the problem is solvable, i.e. the liquidity-adjusted portfolio value $V^{\mathcal{L}}$ is well defined. We show this in the following two propositions.

**Proposition 3.3.1.** *For all $\psi \in \mathcal{P}$ with $\mathcal{C}^{\mathcal{L}}(\psi) \neq \emptyset$, the set $\mathcal{C}^{\mathcal{L}}(\psi)$ is compact and $v^{\psi}(\hat{\theta})$ is bounded.*

*Proof.* Firstly, we verify the compactness of the set $\mathcal{C}^{\mathcal{L}}(\psi) \subseteq \mathcal{Y} \cap \mathcal{L}$ under the assumption $\mathcal{C}^{\mathcal{L}}(\psi) \neq \emptyset$. Consider a sequence $(\xi^n)$ in $\mathcal{C}^{\mathcal{L}}(\psi)$ with $\xi^n \to \xi$. Hence, $\psi - \xi^n + (L(\xi^n), 0, \ldots, 0) \in \mathcal{L}$. Due to the continuity of $L$ and the fact that $\mathcal{L}$ is closed we have

$$\psi - \xi^n + (L(\xi^n), 0, \ldots, 0) \longrightarrow \psi - \xi + (L(\xi), 0, \ldots, 0) \in \mathcal{L}.$$

Thus the set $\mathcal{C}^{\mathcal{L}}(\psi)$ is closed. Since $\mathcal{Y}$ is a bounded set, $\mathcal{C}^{\mathcal{L}}(\psi)$ is compact.
Secondly, we show that for every $\psi \in \mathcal{P}$ the function $v^{\psi}(\hat{\theta})$ is bounded. For this fix

some $\psi \in \mathcal{P}$ and note that for all $\hat{\theta} \in \mathcal{Y}$ follows

$$
\begin{aligned}
&\bar{K}(\psi, \bar{y}, \underline{y}) := \psi_0 + \sum_{i \in I} \left( m_i^+(\psi_i - \underline{y}_i) + \int_0^{\bar{y}_i} m_i(u) du \right) \\
&\geq \psi_0 + \sum_{i \in I} \left( \mathbb{1}_{\{\psi_i - \hat{\theta}_i > 0\}} m_i^{+,\hat{\theta}_i}(\psi_i - \hat{\theta}_i) + \mathbb{1}_{\{\psi_i - \hat{\theta}_i < 0\}} m_i^{-,\hat{\theta}_i}(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u) du \right) \\
&\geq \psi_0 + \sum_{i \in I} \left( m_i^{-,\underline{b}}(\psi_i - \bar{y}_i) + \sum_{i=1}^{N} \int_0^{\underline{y}_i} m_i(u) du \right) =: \underline{K}(\psi, \bar{y}, \underline{y}). \qquad (3.11)
\end{aligned}
$$

Notice that the second expression after the first inequality in (3.11) is $v^{\psi}(\hat{\theta})$. Furthermore, note that $|\bar{K}(\psi, \bar{y}, \underline{y})|, |\underline{K}(\psi, \bar{y}, \underline{y})| < \infty$ for any $\psi \in \mathcal{P}$, and any $\bar{y}, \underline{y} \in \mathcal{Y}$. Hence, $|v^{\psi}(\hat{\theta})| < \infty$ for any $\hat{\theta} \in \mathcal{Y}$.

□

**Remark 3.3.2.** *If $\psi \in \mathcal{B}^c$, problem (3.10) is a convex optimization problem due to the concavity of $g^{\psi}$ by Corollary 3.2.15 and to the convexity of $\mathcal{C}^{\mathcal{L}}(\psi)$.*

**Remark 3.3.3.** *The function $U(\psi - \hat{\theta}, \hat{\theta}) + L(\hat{\theta})$ does not depend on $\hat{\theta}_0$. Hence, for any trade $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1, \ldots, \hat{\theta}_N) \in \mathcal{Y}$ we set $\hat{\theta}_0 = 0$ for the remainder of the paper.*

**Proposition 3.3.4.** *For every $\psi \in \mathcal{P}$ with $\mathcal{C}^{\mathcal{L}}(\psi) \neq \emptyset$, the supremum of optimization problem (3.10) is attained, i.e. there is some $\hat{\theta}^* \in \mathcal{C}^{\mathcal{L}}(\psi)$ such that*

$$
V^{\mathcal{L}}(\psi) = U(\psi - \hat{\theta}^*, \hat{\theta}^*) + L(\hat{\theta}^*). \qquad (3.12)
$$

*Proof.* Assume $\mathcal{C}^{\mathcal{L}}(\psi) \neq \emptyset$ and there is no $\hat{\theta}^* \in \mathcal{C}^{\mathcal{L}}(\psi)$ that satisfies (3.12). Because $\mathcal{C}^{\mathcal{L}}(\psi)$ is compact and $v^{\psi}$ is bounded, the assumption is equivalent to the existence of some sequence $(\hat{\theta}^n) \subset \mathcal{C}^{\mathcal{L}}(\psi)$ with $\hat{\theta}^n \to \tilde{\theta}$ as $n \to \infty$ such that

$$
V^{\mathcal{L}}(\psi) = \lim_{n \to \infty} U(\psi - \hat{\theta}^n, \hat{\theta}^n) + L(\hat{\theta}^n) > U(\psi - \tilde{\theta}, \tilde{\theta}) + L(\tilde{\theta}). \qquad (3.13)
$$

Since $\mathcal{C}^{\mathcal{L}}(\psi)$ is closed, we have $\tilde{\theta} \in \mathcal{C}^{\mathcal{L}}(\psi)$. Clearly, such cases can only occur on those regions where $v^{\psi}(\hat{\theta}) = U(\psi - \hat{\theta}) + L(\hat{\theta})$ is discontinuous. We prove that inequality (3.13) does not hold true.

By Lemma 3.2.16, there are only two cases at which the function $v^{\psi}$ is discontinuous: (1) if there exists some $j \in I$ with $\psi_j > \bar{b}_j$ and $\bar{y}_j > \bar{b}_j$, or (2) if there exists some $k \in I$ with $\psi_k < \underline{b}_k$ and $\underline{y}_k < \underline{b}_k$. Since the function $v_i^{\psi}$ is discontinuous only at

$\hat{\theta}_j = \bar{b}_j$ if condition (1) holds, or at $\hat{\theta}_k = \underline{b}_k$ if condition (2) holds, any sequence $(\xi^n)$ in $\mathcal{C}^{\mathcal{L}}(\psi)$ with $\xi^n \to \xi$ as $n \to \infty$ such that

$$v^{\psi}(\xi) \neq \lim_{n\to\infty} v^{\psi}(\xi^n). \tag{3.14}$$

must be of the form $\xi^n = (\xi_0^n, \xi_1^n, \ldots, \xi_N^n) \in \mathcal{C}^{\mathcal{L}}(\psi)$ with at least one $i \in I$ such that

$$\xi_i^n \overset{n\to\infty}{\longrightarrow} \bar{b}_i \quad \text{and} \quad \xi_i^n > \bar{b}_i, \quad \forall n \in \mathbb{N},$$

if $i$ satisfies condition (1), or

$$\xi_i^n \overset{n\to\infty}{\longrightarrow} \underline{b}_i \quad \text{and} \quad \xi_i^n < \underline{b}_i, \quad \forall n \in \mathbb{N},$$

if $i$ satisfies condition (2). By Lemma 3.7.1 of the Appendix we have

$$v_i^{\psi}(\bar{b}_i) \geq v_i^{\psi}(\hat{\theta}_i) \qquad \text{for} \ \ \hat{\theta}_i \in (\bar{b}_i, \bar{y}_i],$$

if $i$ satisfies condition (1), and

$$v_i^{\psi}(\underline{b}_i) \geq v_i^{\psi}(\hat{\theta}_i) \qquad \text{for} \ \ \hat{\theta}_i \in [\bar{y}_i, \bar{b}_i),$$

if $i$ satisfies condition (2). Hence, for any sequence $(\xi^n)$ which converges to a discontinuous point of $v^{\psi}$ and satisfies expression (3.14), we have

$$v^{\psi}(\xi) > \lim_{n\to\infty} v^{\psi}(\xi^n),$$

which contradicts (3.13).

□

### 3.3.2 General Properties of $V^{\mathcal{L}}$

We begin this section by presenting the basis of the main result of this chapter. As mentioned earlier, the liquidity-adjusted portfolio value $V^{\mathcal{L}}$ is non-concave in general because $v^{\psi}$ is not concave.

**Proposition 3.3.5.** *The liquidity-adjusted portfolio value $V^{\mathcal{L}}$ is not necessarily concave.*

The proof of this proposition is undertaken via counterexample, for which we need to introduce some concrete liquidity policies. Following Acerbi and Scandolo, the *cash liquidity policy* $\mathcal{L}(a)$ is given by

$$\mathcal{L}(a) = \{ (\psi_0, \psi_1, \ldots, \psi_N) \in \mathcal{P} |\, \psi_0 \geq a \}, \qquad a \in \mathbb{R},$$

the *total liquidation policy* $\mathcal{L}^L$ is defined by

$$\mathcal{L}^L = \{ (\psi_0, 0, \ldots, 0) \in \mathcal{P} |\, \psi_0 \in \mathbb{R} \},$$

and the *unrestricted liquidation policy* $\mathcal{L}^U$ is given by

$$\mathcal{L}^U = \mathcal{P}.$$

Additionally, we introduce the *cash liquidity policy without buy transactions* $\mathcal{L}^S(a)$ which is given by[9]

$$\mathcal{L}^S(a) = \{ (\psi_0, \psi_1, \ldots, \psi_N) \in \mathcal{P} |\, \psi_0 \geq a \quad \textit{and} \quad \psi_i \geq 0, \quad \forall i \in I \}, \qquad a \in \mathbb{R}.$$

Now we provide a simple counterexample in order to prove that $V^{\mathcal{L}}$ is not concave in general. For this we consider $\mathcal{L}$ as a cash liquidity policy or as a cash liquidity policy without buy transactions.

*Proof of Proposition 3.3.5.* For sake of simplicity we impose $N = 1$. Consider $a > 0$, $\mathcal{L} \in \{\mathcal{L}(a), \mathcal{L}^S(a)\}$, $0 < \bar{b}_1 < \bar{y}_1 \in \mathbb{R}_{++}$. Let the MSDC satisfy the following assumption

$$m_1(x) = \begin{cases} m & \text{for } x \in (0, \bar{b}_1] \\ m' & \text{for } x \in (\bar{b}_1, \bar{y}_1], \end{cases}$$

where $m, m' \in \mathbb{R}_{++}$. Further assume

$$\int_0^{\bar{b}_1} m_1(u)du < a \quad \text{and} \quad \int_0^{\bar{y}_1} m_1(u) > a, \tag{3.15}$$

which is equivalent to assuming $\bar{b}_1 m < a$ and $(\bar{y}_1 - \bar{b}_1)m' - \bar{b}_1 m > a$, respectively. Set $\beta := a - \int_0^{\bar{b}_1} m_1(u)du = a - \bar{b}_1 m$ and note that $\beta > 0$. Choose some large $\gamma > 0$ such that $\gamma - \bar{y}_1 > 0$ and consider the portfolios $\psi^1 = (\beta, \gamma)$ and $\psi^2 = (0, \gamma)$. By construction we have $\mathcal{C}^{\mathcal{L}}(\psi^1) \neq \emptyset$ and $\mathcal{C}^{\mathcal{L}}(\psi^2) \neq \emptyset$. Denote the corresponding

[9] $\mathcal{L}^S(a)$ is clearly convex and closed.

solutions of optimization problem (3.10) for $\psi^1$ and $\psi^2$ by $\hat{\theta}^1$ and $\hat{\theta}^2$, respectively. By Lemma 3.7.2 of the Appendix we demand that optimal transactions $\hat{\theta}^1$ and $\hat{\theta}^2$ satisfy $\beta + L(\hat{\theta}^1) = a$ and $L(\hat{\theta}^2) = a$. Clearly, $\hat{\theta}^1 = (0, \bar{b}_1)$. Optimal transaction $\hat{\theta}^2 = (0, \hat{\theta}^2_1)$ must satisfy the following

$$(\hat{\theta}^2_1 - \bar{b}_1)m' - \bar{b}_1 m = a,$$

or, equivalently,

$$\hat{\theta}^2_1 = \frac{\beta}{m'} + \bar{b}_1.$$

Denote $\psi^\lambda := \lambda\psi^1 + (1-\lambda)\psi^2 = (\lambda\beta, \gamma)$ for $\lambda \in [0,1]$ and note that $\mathcal{C}^{\mathcal{L}}(\psi^\lambda) \neq \emptyset$ by condition 3.15. Hence, let the solution of problem (3.10) for $\psi^\lambda$ be given by $\hat{\theta}^* \in \mathcal{Y}$, which satisfies

$$\lambda\beta + L(\hat{\theta}^*) = a, \tag{3.16}$$

which is equal to

$$(\lambda\beta + \hat{\theta}^*_1 - \bar{b}_1)m' - \bar{b}_1 m = a,$$

implying

$$\hat{\theta}^*_1 = (1-\lambda)\frac{\beta}{m'} + \bar{b}_1.$$

Further, consider the mapping $F : [0,1] \to \mathbb{R}$ given by

$$F(\lambda) := V^{\mathcal{L}}(\psi^\lambda) = U(\psi^\lambda - \hat{\theta}^*, \hat{\theta}^*) + L(\hat{\theta}^*) = a + m^{+,\bar{b}_1}\left(\gamma - (1-\lambda)\frac{\beta}{m'} - \bar{b}_1\right),$$

for $\lambda \in [0,1]$, and the mapping $G : [0,1] \to \mathbb{R}$ given by

$$\begin{aligned} G(\lambda) &:= \lambda V^{\mathcal{L}}(\psi^1) + (1-\lambda)V^{\mathcal{L}}(\psi^2) \\ &= \lambda\left(U(\psi^1 - \hat{\theta}^1, \hat{\theta}^1) + L(\hat{\theta}^1)\right) + (1-\lambda)\left(U(\psi^2 - \hat{\theta}^2, \hat{\theta}^2) + L(\hat{\theta}^2)\right) \\ &= a + \lambda m_1^+(\gamma - \bar{b}_1) + (1-\lambda)m_1^{+,\bar{b}_1}\left(\gamma - \frac{\beta}{m'} - \bar{b}_1\right), \end{aligned}$$

for $\lambda \in [0,1]$. Last, note that the difference of $G$ and $F$

$$\begin{aligned} G(\lambda) - F(\lambda) &= \lambda m_1^+(\gamma - \bar{b}_1) + (1-\lambda)m_1^{+,\bar{b}_1}\left(\gamma - \frac{\beta}{m'} - \bar{b}_1\right) \\ &- m^{+,\bar{b}_1}\left(\gamma - (1-\lambda)\frac{\beta}{m'} - \bar{b}_1\right) \\ &= \left(\lambda m_1^+ + (1-\lambda)m_1^{+,\bar{b}_1} - m^{+,\bar{b}_1}\right)(\gamma - \bar{b}_1) \\ &> \left(\lambda m_1^{+,\bar{b}_1} + (1-\lambda)m_1^{+,\bar{b}_1} - m^{+,\bar{b}_1}\right)(\gamma - \bar{b}_1) = 0 \end{aligned}$$

is positive. Hence, we have $\lambda V^{\mathcal{L}}(\psi^1) + (1-\lambda)V^{\mathcal{L}}(\psi^2) > V^{\mathcal{L}}(\psi^\lambda)$.

$\square$

A further general property of the liquidity-adjusted portfolio value valid for any liquidity policy is its *translation supervariance*, introduced in (Acerbi & Scandolo 2008). This concept states that augmenting a cash position to a portfolio induces a larger liquidity-adjusted portfolio value than the sum of the liquidity-adjusted value of the initial portfolio and the cash amount.

**Proposition 3.3.6.** *Given a liquidity policy $\mathcal{L}$ the liquidity-adjusted portfolio value is translation supervariant, i.e. for $\psi \in \mathcal{P}$, $e \geq 0$*

$$V^{\mathcal{L}}(\psi + (e, 0, \dots, 0)) \geq V^{\mathcal{L}}(\psi) + e.$$

*Proof.* By noting that the MtM value $U$ of a portfolio is additive on cash portfolios, i.e. for $\psi \in \mathcal{P}$, $e \geq 0$ and $\theta \in \mathcal{Y}$ we obtain $U(\psi + (e, 0, \dots, 0), \theta) = U(\psi, \theta) + e$, the proof follows by the same arguments used in (Acerbi & Scandolo 2008).

$\square$

When valuing portfolios under liquidity constraints, translation supervariance is a more natural concept than translation invariance. The liquidity adjustment defined at the beginning of the section takes into account multiple liquidity issues such as MSDCs, partial execution, block trading and liquidity policy. These market imperfections influence mark-to-market portfolio values $U$ and $v^{\psi}$ in a non-linear manner. Accordingly, it is intuitive that the adjustment to the value of a portfolio with a cash position is less severe than the adjustment to the value of the same portfolio without the cash position. The severity of the liquidity adjustment is such that adding the cash position after the adjustment to the latter portfolio is not enough to equate the liquidity-adjustment of the former portfolio.

In addition to translation supervariance, the liquidity-adjusted portfolio value is monotonic on $\mathcal{P}$ for unrestricted liquidation and cash liquidity policies.

**Proposition 3.3.7.** *Consider some liquidity policy $\mathcal{L} \in \left\{\mathcal{L}(a), \mathcal{L}^S(a), \mathcal{L}^U\right\}$. The function $V^{\mathcal{L}}$ is monotone increasing on $\mathcal{P}$, i.e. for any $\psi^1, \psi^2 \in \mathcal{P}$ with $\psi^1 \geq \psi^2$,*

$$V^{\mathcal{L}}(\psi^1) \geq V^{\mathcal{L}}(\psi^2).$$

*Proof.* Let $\psi^1, \psi^2 \in \mathcal{P}$ with $\psi^1 \geq \psi^2$ and $\mathcal{L} \in \left\{\mathcal{L}(a), \mathcal{L}^S(a), \mathcal{L}^U\right\}$ be given. First note that

$$\mathcal{C}^{\mathcal{L}}(\psi^2) \subseteq \mathcal{C}^{\mathcal{L}}(\psi^1). \tag{3.17}$$

First, we verify this inclusion for the case $\mathcal{C}^{\mathcal{L}}(\psi^2) \neq \emptyset$. Consider some $\hat{\theta} \in \mathcal{Y}$ satisfying

$$\psi^2 - \hat{\theta} + (L(\hat{\theta}), 0, \ldots, 0) \in \mathcal{L}, \tag{3.18}$$

i.e. $\hat{\theta} \in \mathcal{C}^{\mathcal{L}}(\psi^2)$. Because $\mathcal{L}^U = \mathcal{P}$ and $\psi^1_0 - \psi^2_0 \geq 0$, every $\hat{\theta} \in \mathcal{Y}$ satisfying (3.18) also fulfills

$$(\psi^1 - \psi^2) + \psi^2 - \hat{\theta} + (L(\hat{\theta}), 0, \ldots, 0) \in \mathcal{L},$$

for $\mathcal{L} \in \left\{\mathcal{L}(a), \mathcal{L}^S(a), \mathcal{L}^U,\right\}$. Hence, $\hat{\theta} \in \mathcal{C}^{\mathcal{L}}(\psi^1)$. Further,

$$\begin{aligned}
V^{\mathcal{L}}(\psi^1) &= \sup\left\{U(\psi^1 - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}) \,\middle|\, \hat{\theta} \in \mathcal{Y};\ \psi^1 - \hat{\theta} + (L(\hat{\theta}), 0, \ldots, 0) \in \mathcal{L}\right\} \\
&= \sup\Big\{U((\psi^1 - \psi^2) + \psi^2 - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}) \,\Big|\, \hat{\theta} \in \mathcal{Y}; \\
&\qquad\qquad \psi^1 - \hat{\theta} + (L(\hat{\theta}), 0, \ldots, 0) \in \mathcal{L}\Big\} \\
&= \sup\Big\{U(\psi^1 - \psi^2, \hat{\theta}) + U(\psi^2 - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}) \,\Big|\, \hat{\theta} \in \mathcal{Y}; \\
&\qquad\qquad \psi^1 - \hat{\theta} + (L(\hat{\theta}), 0, \ldots, 0) \in \mathcal{L}\Big\} \\
&\geq U(\psi^1 - \psi^2, b) + \sup\Big\{U(\psi^2 - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}) \,\Big|\, \hat{\theta} \in \mathcal{Y}; \\
&\qquad\qquad \psi^1 - \hat{\theta} + (L(\hat{\theta}), 0, \ldots, 0) \in \mathcal{L}\Big\} \\
&\geq U(\psi^1 - \psi^2, b) + \sup\Big\{U(\psi^2 - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}) \,\Big|\, \hat{\theta} \in \mathcal{Y}; \\
&\qquad\qquad \psi^2 - \hat{\theta} + (L(\hat{\theta}), 0, \ldots, 0) \in \mathcal{L}\Big\} \\
&= U(\psi^1 - \psi^2, b) + V^{\mathcal{L}}(\psi^2) \geq V^{\mathcal{L}}(\psi^2),
\end{aligned}$$

where $b \in \mathcal{P}$ has elements $b_i = \underline{b}_i$ if $\psi^1_i - \psi^2_i < 0$ and $b_i = \bar{b}_i$ if $\psi^1_i - \psi^2_i > 0$ for $i \in I$. The first inequality follows from the definition of the post-execution best bid and ask prices, the second follows from (3.17) and the third from definition of $U$ and from $\psi^1 - \psi^2 \geq 0$.

For the case $\mathcal{C}^{\mathcal{L}}(\psi^2) = \emptyset$, we have either $\mathcal{C}^{\mathcal{L}}(\psi^1) \neq \emptyset$ or $\mathcal{C}^{\mathcal{L}}(\psi^1) = \emptyset$. In the first case, it holds $V^{\mathcal{L}}(\psi^1) > -\infty$ and in the latter $V^{\mathcal{L}}(\psi^1) = -\infty$. Since $V^{\mathcal{L}}(\psi^2) = -\infty$ whenever $\mathcal{C}^{\mathcal{L}}(\psi^2) = \emptyset$, inclusion 3.17 is satisfied and $V^{\mathcal{L}}(\psi^1) \geq V^{\mathcal{L}}(\psi^2)$ is also fulfilled.

□

Monotonicity of $V^{\mathcal{L}}$ is consistent with economic intuition. For this, regard two portfolios $\psi^1, \psi^2 \in \mathcal{P}$ with $\psi^1 > \psi^2$. Consequently, consider all short positions of portfolio $\psi^2$ and note that for the same assets those positions must be either long, flat or smaller (in absolute value) short positions in portfolio $\psi^1$. Analogously, we find larger long positions in $\psi^1$ for those assets with long and flat positions in $\psi^2$. Hence, holding portfolio $\psi^1$ is more convenient than holding portfolio $\psi^2$ if we need to execute some transaction $\hat{\theta} \in \mathcal{Y}$ in order to fulfill some liquidity constraint. It seems that, any reasonable liquidity-adjusted portfolio value for $\psi^1$ can never be smaller than the liquidity-adjusted portfolio value for $\psi^2$, as stated in Proposition 3.3.7. This is, however, not true if the liquidity-adjusted portfolio is based on the liquidity policy $\mathcal{L}^L$. Under this policy the optimal adjusting transaction closes all non cash portfolio positions, which is shown in Proposition 3.3.8 below. The nature of this adjustments is essential for the non-monotonicity under $\mathcal{L}^L$. To see this, consider again portfolios $\psi^1 > \psi^2 \in \mathcal{P}$. Under partial execution, it is possible that only positions $\psi^2$ can be closed but positions $\psi^1$ not, which yield $V^{\mathcal{L}^L}(\psi^2) > V^{\mathcal{L}^L}(\psi^1) = -\infty$.

**Proposition 3.3.8.** *Let $\psi \in \mathcal{P}$ and consider the total liquidation policy $\mathcal{L}^L$. The solution of optimization problem (3.10) is given by $\hat{\theta}^* = (0, \psi_1, \dots, \psi_N)$ and*

$$V^{\mathcal{L}^L}(\psi) = \psi_0 + L(\hat{\theta}^*),$$

*if $\hat{\theta}^* \in \mathcal{Y}$. Otherwise, $V^{\mathcal{L}^L}(\psi) = -\infty$.*

*Proof.* By definition of $\mathcal{L}^L$, we have

$$\mathcal{C}^{\mathcal{L}^L}(\psi) = \left\{ \hat{\theta} \in \mathcal{Y} \,\middle|\, \psi - \hat{\theta} + (L(\hat{\theta}), 0, \dots, 0) = (x, 0, \dots, 0) \in \mathcal{P} \quad for\ x \in \mathbb{R} \right\}.$$

Clearly, if $\hat{\theta}^* := (0, \psi_1, \dots, \psi_N) \in \mathcal{Y}$, then $\mathcal{C}^{\mathcal{L}^L}(\psi) = \{\hat{\theta}^*\}$ and $V^{\mathcal{L}^L}(\psi) = \psi_0 + L(\hat{\theta}^*)$. If $\hat{\theta}^* \notin \mathcal{Y}$, then $\mathcal{C}^{\mathcal{L}^L}(\psi) = \emptyset$ and $V^{\mathcal{L}^L}(\psi) = -\infty$.

□

We discuss now some additional properties of $\mathcal{L}^U$,$\mathcal{L}(a)$ and $\mathcal{L}^S(a)$. If all MSDCs are strictly decreasing, the optimal transaction $\hat{\theta}^* \in \mathcal{Y}$ that solves problem (3.10) under the unrestricted liquidation policy $\mathcal{L}^U$ is a non-trade transaction, i.e. $\hat{\theta}^* = 0$, independently of block trading and partial execution effects.

**Proposition 3.3.9.** *Consider some $\psi \in \mathcal{P}$, the unrestricted liquidation policy $\mathcal{L}^U$ and assume that for all $i \in I$ the MSDC is strictly decreasing, i.e.*

$$m_i(x) < m_i(x'),$$

*for $x, x' \in \mathbb{R} \setminus \{0\}$ with $x > x'$. Then,*

$$V^{\mathcal{L}^U}(\psi) = U(\psi, 0),$$

*in other words, $\hat{\theta}^* = 0$ solves problem (3.10) for $\psi$ under $\mathcal{L}^U$.*

*Proof.* Let $\psi \in \mathcal{P}$ and $\mathcal{L}^U$. Assume optimal transaction $\hat{\theta}^* \neq 0$ where $\hat{\theta}^* \in \mathcal{Y}$. Hence,

$$V^{\mathcal{L}}(\psi) = v^{\psi}(\hat{\theta}^*) = \sum_{i \in I^+} v_i^{\psi}(\hat{\theta}_i^*) + \sum_{i \in I^-} v_i^{\psi}(\hat{\theta}_i^*) + \sum_{i \in I^0} v_i^{\psi}(\hat{\theta}_i^*),$$

where $I^+, I^-, I^0 \subseteq I$ such that for all $i \in I^+$ we have $\hat{\theta}_i^* > 0$, for all $i \in I^-$ we have $\hat{\theta}_i^* < 0$, and for all $i \in I^0$ we have $\hat{\theta}_i^* = 0$. Since we assume $\hat{\theta}^* \neq 0$, either $I^+ \neq \emptyset$, $I^- \neq \emptyset$, or both subsets are not empty. From Lemma 3.7.3 of the Appendix we know that $v_i^{\psi}$ is strictly increasing on $[\underline{y}_i, 0)$ and strictly decreasing on $(0, \bar{y}_i]$. Hence,

$$v^{\psi}(0) = \sum_{i \in I} v_i^{\psi}(0) > \sum_{i \in I^+} v_i^{\psi}(\hat{\theta}_i^*) + \sum_{i \in I^-} v_i^{\psi}(\hat{\theta}_i^*) + \sum_{i \in I^0} v_i^{\psi}(0) = v^{\psi}(\hat{\theta}^*),$$

which contradicts optimality of $\hat{\theta}^*$. Since last inequality holds for all $\hat{\theta}^* \neq 0$ and $\mathcal{C}^{\mathcal{L}^U}(\psi) \neq \emptyset$, then the non-trade transaction, i.e. $\hat{\theta}^* = 0$, must be the solution of optimization problem (3.10).

$\square$

In addition to the cash policy $\mathcal{L}(a)$ of Acerbi and Scandolo we introduced previously a similar cash policy $\mathcal{L}^S(a)$ which admits only sell transactions. By analyzing closer $\mathcal{L}(a)$ we see that it allows buy trades as optimal solution of optimization problem (3.10). To illustrate this possibility regard the following example.

**Example 3.3.10.** *Assume $N = 2$ and consider the following bounds*

$$\underline{y}_1 = \underline{b}_1 < 0, \qquad \bar{b}_1 = 2 \cdot 10^3, \qquad \bar{y}_1 = 6 \cdot 10^3, \qquad \underline{y}_2 = \underline{b}_2 = 3 \cdot 10^3, \qquad \bar{y}_2 = 0.$$

*Let the portfolio $\psi \in \mathcal{P}$ consist of a flat cash position $\psi_0 = 0$, of a long position $\psi_1 > \bar{y}_1$ in asset $A_1$ which is liquid and of a long position $\psi_2 > |\underline{y}_2|$ in asset $A_2$ which is sale-illiquid. Accordingly, consider the MSDCs for the assets as follows*

$$m_1(x) = \begin{cases} -10^{-2}x + 10^2 & for\ x \leq \bar{b}_1 \\ 8 \cdot 10 & else \end{cases}, \qquad m_2(x) = 10^2 \quad for\ x < 0.$$

*Recall that $m_i(0)$ is not defined. However, observe that $m_1^+ = m_1(0^+) = 10^2 = m_1(0^-) = m_1^-$, i.e. the bid-ask spread is zero, $\delta_1 := m_1^- - m_1^+ = 0$. For asset $A_2$ we also suppose a trivial bid-ask spread $\delta_2 = 0$. Furthermore, let $m^{+,\bar{b}_1} = 8 \cdot 10$. Regard the cash liquidity policy $\mathcal{L}(a)$ where $a = 2 \cdot 10^5$. Because the proceeds of selling $\bar{b}_1$ units of asset $A_1$ are not sufficient to cover the liquidity requirement $a$, i.e.*

$$p_1(\bar{b}_1) = \int_0^{\bar{b}_1} m_1(u)du = 1,8 \cdot 10^5 < 2 \cdot 10^5 = a,$$

*the optimal transaction $\hat{\theta}_1^*$ solving problem (3.10) must be larger than $\bar{b}_1$. Particularly, the liquidity constraint $\psi_0 + L(\hat{\theta}^*) \geq a$ for an optimal $\hat{\theta}^* \in \mathcal{Y}$ is given by*

$$8 \cdot 10\hat{\theta}_1^* + 10^2\hat{\theta}_2^* - 1,8 \cdot 10^5 \geq 0, \tag{3.19}$$

*where $\hat{\theta}_1^* \geq 2,25 \cdot 10^3$, since $p_1(2,25 \cdot 10^3) = a$, and $\hat{\theta}_2^* \in [\underline{y}_2, 0)$. By Lemma 3.7.2 it suffices to regard (3.19) as an equality, which leads to*

$$\hat{\theta}_2^* = -8 \cdot 10^{-1}\hat{\theta}_1^* + 1,8 \cdot 10^3.$$

*Hence, the liquidity-adjusted portfolio value is given by*

$$\begin{aligned} V^{\mathcal{L}}(\psi) &= \sup\Big\{U(\psi - \hat{\theta}^*, \hat{\theta}^*) + L(\hat{\theta}^*) \,\Big|\, \hat{\theta}_1^* \in \big[2,25 \cdot 10^3, 6 \cdot 10^3\big], \\ &\qquad \hat{\theta}_2^* = -8 \cdot 10^{-1}\hat{\theta}_1^* + 1,8 \cdot 10^3 \Big\} \\ &= \sup\Big\{8 \cdot 10(\psi_1 - \hat{\theta}_1^*) + 10^2(\psi_2 - \hat{\theta}_2^*) \\ &\qquad +a \,\Big|\, \hat{\theta}_1^* \in \big[2,25 \cdot 10^3, 6 \cdot 10^3\big],\ \hat{\theta}_2^* = -8 \cdot 10^{-1}\hat{\theta}_1^* + 1,8 \cdot 10^3 \Big\}, \end{aligned}$$

*because $\psi_1 - \hat{\theta}_1^* > 0$ for any $\hat{\theta}_1 \in (0, \bar{y}_1]$ and $\psi_2 - \hat{\theta}_2^* > 0$ for any $\hat{\theta}_2^* \in [\underline{y}_2, 0]$. Equivalently,*

$$\begin{aligned} V^{\mathcal{L}}(\psi) &= \sup\Big\{8 \cdot 10(\psi_1 - \hat{\theta}_1^*) + 10^2(\psi_2 + 8 \cdot 10^{-1}\hat{\theta}_1^* - 1,8 \cdot 10^3) \\ &\qquad +a \,\Big|\, \hat{\theta}_1^* \in \big[2,25 \cdot 10^3, 6 \cdot 10^3\big] \Big\} \\ &= \sup\Big\{8 \cdot 10\psi_1 + 10^2\psi_2 - 1,8 \cdot 10^5 + a \,\Big|\, \hat{\theta}_1^* \in \big[2,25 \cdot 10^3, 6 \cdot 10^3\big] \Big\}. \end{aligned}$$

*Thus, the solution set of problem (3.10) satisfying $\psi_0 + L(\hat{\theta}^*) = a$ is given by*

$$\Sigma = \left\{ \hat{\theta}^* \in \mathcal{Y} \middle| \hat{\theta}_1^* \in [2,25 \cdot 10^3, 6 \cdot 10^3],\ \hat{\theta}_2^* = -8 \cdot 10^{-1} \hat{\theta}_1^* + 1,8 \cdot 10^3 \right\}.$$

*Note that $\hat{\theta}_2^* < 0$ whenever $\hat{\theta}_1^* \in (2,25 \cdot 10^3, 6 \cdot 10^3]$. Furthermore, consider some $c > 0$ and $m_2(x) = c \cdot 10^2$. Thus, optimal transaction for $A_2$ is given by*

$$\hat{\theta}_2^* = -\frac{8}{c} \cdot 10^{-1} \hat{\theta}_1^* + \frac{1,8}{c} \cdot 10^3,$$

*and the liquidity-adjusted portfolio value has not changed*

$$V^{\mathcal{L}}(\psi) = \sup \left\{ 8 \cdot 10 \psi_1 + 10^2 \psi_2 - 1,8 \cdot 10^5 + a \,\middle|\, \hat{\theta}_1^* \in \left[2,25 \cdot 10^3, 6 \cdot 10^3\right] \right\}.$$

*In view of Assumption 3.2.8 we introduce a nontrivial bid-ask spread for $A_2$, such that*

$$m_2^+ = 9,5 \cdot 10, \qquad m_2^- = 10^2,$$

*the liquidity-adjusted portfolio value becomes*

$$\begin{aligned} V^{\mathcal{L}}(\psi) \;&=\; \sup \Big\{ -4\hat{\theta}_1^* + 8 \cdot 10 \psi_1 + 9,5 \cdot 10 \psi_2 - 17,1 \cdot 10^4 \\ &\qquad\qquad + a \,\Big|\, \hat{\theta}_1^* \in [2,25 \cdot 10^3, 6 \cdot 10^3] \Big\}. \end{aligned}$$

*Hence, the solution set is a singleton given by*

$$\Sigma = \left\{ (2,25 \cdot 10^3, 0) \right\}.$$

*Note that this result holds for any $\delta_2 > 0$ such that $m_2^- = 10^2$ and $m_2^+ = 10 - \delta_2$, since*

$$\begin{aligned} V^{\mathcal{L}}(\psi) \;&=\; \sup \Big\{ -8 \cdot 10 \delta_2 \hat{\theta}_1^* + 8 \cdot 10 \psi_1 + (10^2 - \delta_2) \psi_2 - (1,8 \cdot 10^2 - \delta_2) 10^3 \\ &\qquad\qquad + a \,\Big|\, \hat{\theta}_1^* \in [2,25 \cdot 10^3, 6 \cdot 10^3] \Big\}. \end{aligned}$$

Under cash liquidity policies, optimal transaction $\hat{\theta}^*$ indicates which trades must be undertaken in order to obtain at least $a - \psi_0$ money units. If the purpose of the liquidity adjustment to portfolio values is aimed for risk management and not for active portfolio management, optimal buy transactions $\hat{\theta}_i^* < 0$ fail the purpose of the adjustment. Alternatively, a cash liquidity policy without buy transactions $\mathcal{L}^S(a)$, contemplates only those sell transactions[10] $\hat{\theta}_i^* > 0$ that must be undertaken

[10] And, of course, non-trade transactions $\hat{\theta}_i^* = 0$.

in order to satisfy the capital needs. Hence, in a risk management context the liquidity policy $\mathcal{L}^S(a)$ should be prefer over $\mathcal{L}(a)$.

However, if the MSDCs are strictly decreasing for buy trades and the capital requirement is positive, $a > 0$, then any optimal transaction $\hat{\theta}^*$ under cash liquidity $\mathcal{L}(a)$ consists of sales or no trades.

**Proposition 3.3.11.** *Assume strictly decreasing MSDCs for buy trades, i.e. for all $i \in I$ let $m_i(x) > m_i(x')$ for any $x < x' < 0$. Fix some cash liquidity policy $\mathcal{L}(a)$ with $a > 0$, some portfolio $\psi \in \mathcal{P}$ and denote $\hat{\theta}^* \in \mathcal{Y}$ a solution of optimization problem (3.10) for $\psi$. If $\mathcal{C}^{\mathcal{L}(a)}(\psi) \neq \emptyset$, then $\hat{\theta}^* \geq 0$.*

*Proof.* Let $a > 0$, $\psi \in \mathcal{P}$ and denote $\hat{\theta}^* \in \mathcal{Y}$ a solution of optimization problem (3.10) for $\psi$, i.e. $\hat{\theta}^* \in \mathcal{C}^{\mathcal{L}(a)}(\psi) \neq \emptyset$. Firstly, consider the case $a - \psi_0 > 0$. Since optimal transaction $\hat{\theta}^*$ must satisfy $\psi_0 + L(\hat{\theta}^*) \geq a$ or, equivalently, $L(\hat{\theta}^*) \geq a - \psi_0 > 0$, there exists a nonempty set $I^+ \subseteq I$ such that $\hat{\theta}_i^* > 0$ for all $i \in I^+$. Assume that $\hat{\theta}^*$ contains buy transactions, also. Hence, $I^+ \subset I$ and there exists a nonempty set $I^- \subset I$ such that $\hat{\theta}_i^* < 0$ for all $i \in I^-$. Denote the complement of the union of $I^+$ and $I^-$ by $I^0$ and observe that $\hat{\theta}_i^* = 0$ for all $i \in I^0$. Notice that

$$\sum_{i \in I^+} p_i(\hat{\theta}_i^*) = \sum_{i \in I^+} \int_0^{\hat{\theta}_i^*} m_i(u) du > a - \psi_0. \tag{3.20}$$

Regard the liquidity-adjusted portfolio value of $\psi$ given by

$$V^{\mathcal{L}}(\psi) = U(\psi - \hat{\theta}^*, \hat{\theta}^*) + L(\hat{\theta}^*) = \sum_{i \in I^+} v_i^{\psi}(\hat{\theta}_i^*) + \sum_{i \in I^-} v_i^{\psi}(\hat{\theta}_i^*) + \sum_{i \in I^0} v_i^{\psi}(0).$$

From Lemma 3.7.3 of the Appendix we know that $v_i^{\psi}$ is strictly increasing on $[\underline{u}_i, 0)$. Consider the transaction $\hat{\theta}^{**} \in \mathcal{Y}$ with components

$$\hat{\theta}_i^{**} = \begin{cases} \hat{\theta}_i^* & \text{if } i \in I^+ \\ 0 & \text{else} \end{cases}. \tag{3.21}$$

By (3.20), we have $\hat{\theta}^{**} \in \mathcal{C}^{\mathcal{L}}(\psi)$. Furthermore,

$$\begin{aligned} v^{\psi}(\hat{\theta}^*) &= \sum_{i \in I^+} v_i^{\psi}(\hat{\theta}_i^*) + \sum_{i \in I^-} v_i^{\psi}(\hat{\theta}_i^*) + \sum_{i \in I^0} v_i^{\psi}(0) \\ &< \sum_{i \in I^+} v_i^{\psi}(\hat{\theta}_i^*) + \sum_{i \in I^-} v_i^{\psi}(0) + \sum_{i \in I^0} v_i^{\psi}(0) \\ &= v^{\psi}(\hat{\theta}^{**}), \end{aligned} \tag{3.22}$$

which contradicts the optimality of $\hat{\theta}^*$. Now consider the case $\psi_0 \geq a$ and $\mathcal{C}^{\mathcal{L}(a)}(\psi) \neq \emptyset$. Because the function $v_i^\psi$ is strictly increasing on $[\underline{y}_i, 0)$, it is evident that for any transaction $\hat{\theta}^* \in \mathcal{Y}$ that solves problem (3.10) the set $I^-$ must be empty. Hence, any optimal transaction $\hat{\theta}^* \in \mathcal{C}^{\mathcal{L}(a)}(\psi)$ is nonnegative, i.e. $\hat{\theta}_i^* \geq 0$ for all $i \in I$.

□

### 3.3.3 Block Trading and Partial Execution Effects

By Corollary 3.2.18 we know that in absence of block trading effects the post-execution portfolio value $v^\psi$ coincides with the pre-execution portfolio value $\tilde{v}^\psi$. In addition, by Proposition 3.3.4 there exists a transaction $\hat{\theta}^* \in \mathcal{Y}$ such that $V^{\mathcal{L}}(\psi) = v^\psi(\hat{\theta}^*)$ for every $\psi \in \mathcal{P}$. Hence, in absence of block trading effects, for every $\psi \in \mathcal{P}$ we have

$$V^{\mathcal{L}}(\psi) = \tilde{v}^\psi(\hat{\theta}^*).$$

Consequently, if there are no block trading effects, the results encountered in (Acerbi & Scandolo 2008) apply also for the present framework's liquidity-adjusted portfolio value $V^{\mathcal{L}}$. In particular, the liquidity-adjusted portfolio value $V^{\mathcal{L}}$ is concave on $\mathcal{P}$ for any liquidity policy. Additionally, for $\psi \in \mathcal{P}$ with $\psi_0 \geq 0$, $\mathcal{L} \in \{\mathcal{L}(a), \mathcal{L}^S(a)\}$ and any $\lambda > 1$, we have

$$V^{\mathcal{L}}(\lambda\psi) \geq \lambda V^{\mathcal{L}}(\psi).$$

However, these observations do not explain how the liquidity-adjusted value $V^{\mathcal{L}}$ behaves in presence of block trading and partial execution effects. We address this question by defining *governing market conditions* as follows.

**Definition 3.3.12.** *Consider the sets $\mathcal{Y}, \mathcal{B}^c, \mathcal{X} \subseteq \mathcal{P}$ with $\mathcal{Y} \subseteq \mathcal{X}$, where $\mathcal{Y}$ is the set of executable transactions and $\mathcal{B}^c$ the set of non-block transactions. The governing market conditions reflecting block trading and partial execution is given by the pair $\mathcal{MC} := \{\mathcal{Y}, \mathcal{B}^c\}$.*
*The pair*

- *$\mathcal{MC}^{\neg PEBT} := \{\mathcal{P}, \mathcal{P}\}$ excludes block trading and partial execution effects from governing market conditions,*

- $\mathcal{MC}^{\neg BT} := \{\mathcal{Y}, \mathcal{X}\}$ *excludes block trading effects from governing market conditions,*
- $\mathcal{MC}^{\neg PE} := \{\mathcal{P}, \mathcal{B}^c\}$ *excludes partial execution effects from governing market conditions.*

**Remark 3.3.13.** *Note that $\mathcal{MC}^{\neg B}$ satisfies the condition in Proposition 3.2.11.*

In order to conduct an efficient comparison between market conditions consider the following extension of the notation.

**Definition 3.3.14.** *Let $\psi \in \mathcal{P}$ and some liquidity policy $\mathcal{L}$ be given. The liquidity-adjusted portfolio value for $\psi$ subjected to $\mathcal{L}$ under market conditions $\mathcal{MC} = \{\mathcal{X}, \mathcal{X}'\}$ where $\mathcal{X}, \mathcal{X}' \subseteq \mathcal{P}$ is denoted by*

$$V^{\mathcal{L}}(\psi\,|\mathcal{MC}) = \sup\left\{U(\psi - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}) \middle| \hat{\theta} \in \mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC})\right\},$$

*with*

$$\mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}) = \left\{\hat{\theta} \in \mathcal{X} \,\middle|\, \psi - \hat{\theta} + (L(\hat{\theta}), 0, \ldots, 0) \in \mathcal{L}\right\},$$

*and where $\mathcal{X}'$ is the set of non-block transactions.*

The presence of either block trading or partial execution effects represents further restrictions to which we must optimally adjust the post-execution portfolio value $v^{\psi}$. Consequently and consistent with economic intuition, these effects impair the liquidity-adjusted portfolio value $V^{\mathcal{L}}$.

**Proposition 3.3.15.** *Assume some governing market conditions $\mathcal{MC} = \{\mathcal{Y}, \mathcal{B}^c\}$ where $\mathcal{Y}, \mathcal{B}^c \subseteq \mathcal{P}$. Consider some portfolio $\psi \in \mathcal{P}$ and some liquidity policy $\mathcal{L}$. Then*

$$V^{\mathcal{L}}(\psi\,|\mathcal{MC}) \leq V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PE}\right) \leq V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\right), \tag{3.23}$$

*and*

$$V^{\mathcal{L}}(\psi\,|\mathcal{MC}) \leq V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg BT}\right) \leq V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\right). \tag{3.24}$$

*Proof.* By the definition of $L$, $\mathcal{L}$, Definition 3.3.14 and since $\mathcal{Y} \subseteq \mathcal{P}$ we have

$$\mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}) \subseteq \mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}^{\neg PE}) = \mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}^{\neg PEBT}), \tag{3.25}$$

and

$$\mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}) = \mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}^{\neg BT}) \subseteq \mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}^{\neg PEBT}). \tag{3.26}$$

First, consider the case $\mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}\,) \neq \emptyset$. The first inequality in (3.23) follows straightforwardly by the first inclusion in (3.25). By Proposition 3.2.19 and $\mathcal{B}^c \subseteq \mathcal{P}$ we obtain

$$\begin{aligned} V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PE}\right) &= \sup\left\{U(\psi-\hat{\theta},\hat{\theta})+L(\hat{\theta})\middle|\,\hat{\theta}\in\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg PE}\,)\right\}\\ &\leq \sup\left\{U(\psi-\hat{\theta},0)+L(\hat{\theta})\middle|\,\hat{\theta}\in\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg PE}\,)\right\}\\ &= \sup\left\{U(\psi-\hat{\theta})+L(\hat{\theta})\middle|\,\hat{\theta}\in\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\,)\right\}\\ &= V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\right). \end{aligned} \tag{3.27}$$

Similarly, the second inequality in (3.24) follows straightforwardly from the second inclusion in (3.26) and the first inequality from

$$\begin{aligned} V^{\mathcal{L}}\left(\psi\,|\mathcal{MC}\,\right) &= \sup\left\{U(\psi-\hat{\theta},\hat{\theta})+L(\hat{\theta})\middle|\,\hat{\theta}\in\mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}\,)\right\}\\ &\leq \sup\left\{U(\psi-\hat{\theta},0)+L(\hat{\theta})\middle|\,\hat{\theta}\in\mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}\,)\right\}\\ &= \sup\left\{U(\psi-\hat{\theta})+L(\hat{\theta})\middle|\,\hat{\theta}\in\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg BT}\,)\right\}\\ &= V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg BT}\right). \end{aligned}$$

Finally, consider the case $\mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}\,) = \emptyset$. Notice that $V^{\mathcal{L}}\left(\psi\,|\mathcal{MC}\,\right) = -\infty$. Moreover, we have either $\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg PE}\,) \neq \emptyset$ or $\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg PE}\,) = \emptyset$. For the former case

$$\left|V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PE}\right)\right| < \infty,$$

which holds by Proposition 3.3.1 and for the latter we have $V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PE}\right) = -\infty$. Whenever $\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg PE}\,) \neq \emptyset$ we also have $\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\,) \neq \emptyset$ and hence $V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PE}\right) \leq V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\right)$. If $\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg PE}\,) = \emptyset$, then $\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\,) = \emptyset$ and thus

$$V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\right) = -\infty.$$

Furthermore, note that in the case $\mathcal{C}^{\mathcal{L}}(\psi\,|\mathcal{MC}\,) = \emptyset$, we have $\mathcal{C}^{\mathcal{L}}(\psi\,\middle|\mathcal{MC}^{\neg BT}\,) = \emptyset$ by definition of $\mathcal{MC}^{\neg BT}$. Hence,

$$V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg BT}\right) = -\infty,$$

and the first inequality in (3.24) holds with equality. Analogous to previous arguments, we have either

$$\left|V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\right)\right| < \infty \qquad \text{or} \qquad V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\right) = -\infty.$$

$\square$

**Remark 3.3.16.** *Notice that the liquidity-adjusted portfolio value of the Acerbi and Scandolo's setup in (Acerbi & Scandolo 2008) is equivalent to* $V^{\mathcal{L}}\left(\psi\,\middle|\mathcal{MC}^{\neg PEBT}\right)$*.*

## 3.4 Measures of Risk under Liquidity Adjustments

In this section we introduce the *liquidity-adjusted risk measure* $\rho^{\mathcal{L}}$ put forward in Acerbi and Scandolo (Acerbi & Scandolo 2008), which is a risk measure implied by a coherent measure of risk and defined on the set of portfolio weights $\mathcal{P}$. Also in this section we find one of the main contributions of the chapter, which indicates that in presence of block trading and partial execution effects liquidity-adjusted risk measures are not convex on $\mathcal{P}$. In addition, we also examine other properties of liquidity-adjusted risk measures $\rho^{\mathcal{L}}$ under consideration of those effects. At the end of the section, we investigate the probability distribution of liquidity-adjusted portfolio values and show that block trading and partial execution effects increase the probability of large losses.

### 3.4.1 Coherent Measures of Risk

As mentioned at the beginning, we study a one-period model. Hence, given some future point in time we model MSDCs, post-execution best bids and post-execution best asks to be random, while we assume that all other parameters are deterministic. In particular, we suppose that the set of executable transactions $\mathcal{Y} \subseteq \mathcal{P}$ and the set of block trades $\mathcal{B} \subseteq \mathcal{P}$ are deterministic and given. Assume that some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ governs the uncertainty on the market. Accordingly, consider the following definition of MSDCs, post-execution best bids and post-execution best asks for non-cash assets.

**Definition 3.4.1.** *For every non-cash asset the MSDC and the post-execution best bid and ask prices are real-valued random variables, i.e. for each* $i \in I$*,* $x \in \mathbb{R}\setminus\{0\}$

*and* $\hat{\theta}_i \in Y_i$

$$\begin{aligned} m_i(x) : \Omega &\rightarrow \mathbb{R}, \\ \omega &\mapsto m_i(x)(\omega) \qquad \text{for} \qquad \omega \in \Omega, \end{aligned}$$

*and*

$$\begin{aligned} m_i^{*,\hat{\theta}_i} : \Omega &\rightarrow \mathbb{R}, \\ \omega &\mapsto m_i^{*,\hat{\theta}_i}(\omega) \qquad \text{for} \qquad \omega \in \Omega, \end{aligned}$$

*where* $* \in \{+,-\}$.

Hence, the liquidity-adjusted portfolio value is a random variable as defined below.

**Definition 3.4.2.** *Consider governing market conditions* $\mathcal{MC} = \{\mathcal{Y}, \mathcal{B}^c\}$ *where* $\mathcal{Y}, \mathcal{B}^c \subseteq \mathcal{P}$*, a liquidity policy* $\mathcal{L}$ *and some portfolio* $\psi \in \mathcal{P}$*. Under* $\mathcal{MC}$ *and policy* $\mathcal{L}$*, the liquidity-adjusted portfolio value* $V^{\mathcal{L}}(\psi \,|\mathcal{MC})$ *for* $\psi$ *is a real-valued random variable, i.e.*

$$\begin{aligned} V^{\mathcal{L}}(\psi \,|\mathcal{MC}) : \Omega &\rightarrow \mathbb{R}, \\ \omega &\mapsto V^{\mathcal{L}}(\psi \,|\mathcal{MC})(\omega) \qquad \text{for} \qquad \omega \in \Omega. \end{aligned}$$

*We set* $V^{\mathcal{L}}(\psi) = V^{\mathcal{L}}(\psi \,|\mathcal{MC})$ *whenever there is no chance of confusion.*

Furthermore we assume that whenever $V^{\mathcal{L}}(\psi) \neq -\infty$ $\mathbb{P}$-a.s., then $V^{\mathcal{L}}(\psi) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Following Acerbi and Scandolo's concept, we define the liquidity-adjusted risk measure $\rho^{\mathcal{L}}$, which, differently from Acerbi and Scandolo, represents a measure of risk reflecting block trading and partial execution effects as part of liquidity risks. Formally, consider a measure of risk $\rho : \mathcal{M} \rightarrow \mathbb{R}$ where $\mathcal{M} \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$. Given a liquidity policy $\mathcal{L}$ and a risk measure $\rho$, the implied liquidity-adjusted measure of risk $\rho^{\mathcal{L}} : \mathcal{P} \rightarrow \mathbb{R}$ is defined by

$$\rho^{\mathcal{L}}(\psi) = \rho(V^{\mathcal{L}}(\psi)), \qquad \psi \in \mathcal{P}.$$

In case $V^{\mathcal{L}}(\psi) = -\infty$ we set $\rho(-\infty) = \infty$. The liquidity-adjusted risk measure introduced in (Acerbi & Scandolo 2008) uses a coherent risk measure $\rho$. Coherent measures of risk are risk measures satisfying the following four axioms presented by

Artzner et al. in (Artzner et al. 1999), which we call ADEH axioms.
*(M) Monotonicity*: For all $x, x' \in \mathcal{M}$ with $x \leq x'$ a.s., we have

$$\rho(x) \geq \rho(x'). \tag{3.28}$$

*(TI) Translation Invariance*: For all $x \in \mathcal{M}$ and $e \in \mathbb{R}$ we obtain

$$\rho(x + e) = \rho(x) - e., \tag{3.29}$$

*(PH) Positive Homogeneity*: For all $x \in \mathcal{M}$ and $\lambda > 0$

$$\rho(\lambda x) = \lambda \rho(x). \tag{3.30}$$

*(S) Subadditivity*: For all $x, x' \in \mathcal{M}$ we have

$$\rho(x + x') \leq \rho(x) + \rho(x'). \tag{3.31}$$

A less restrictive risk measure class is the convex risk measure family. A measure of risk $\rho : \mathcal{M} \to \mathbb{R}$ satisfying properties (M), (TI) and

$$\rho(\lambda x + (1 - \lambda)x') \leq \lambda \rho(x) + (1 - \lambda)\rho(x'),$$

for $x, x' \in \mathcal{M}$ and $\lambda \in [0, 1]$, is called convex measure of risk. It is evident that a coherent measure of risk is a convex measure. However, a convex measure of risk is not necessarily coherent.
Furthermore, note that the liquidity-adjusted risk measure $\rho^{\mathcal{L}}$ is defined on the set of portfolios $\mathcal{P}$ instead on the set of random variables $\mathcal{M}$, for which the ADEH and convexity axioms are conceived. A natural question that arises with this remark is if $\rho^{\mathcal{L}}$ presents similar properties on $\mathcal{P}$. Acerbi and Scandolo show that the liquidity-adjusted risk measure is *always* convex. They also characterize the situations when the ADEH axioms are fulfilled. Within our setup, we show that the convexity found by Acerbi and Scandolo does not hold, which is the main contribution of this chapter. In particular, under block trading and partial execution effects the liquidity-adjusted risk measure is not convex in general.

**Proposition 3.4.3.** *Consider some coherent risk measure $\rho$ and liquidity policy $\mathcal{L}$. In general, the liquidity-adjusted risk measure $\rho^{\mathcal{L}}$ is not convex on $\mathcal{P}$.*

*Proof.* See Appendix 3.7.4.

□

By noting that block trading and partial execution effects produce non-concavity on the liquidity-adjusted portfolio value $V^{\mathcal{L}}$ shown in Proposition 3.3.5, this result should not surprise the reader. In the appendix we use the non-concavity of $V^{\mathcal{L}}$ and the expected shortfall as a coherent risk measure to show that coherence of $\rho$ is not sufficient to even out the non-concavity of $V^{\mathcal{L}}$ in order to have a convex liquidity-adjusted risk measure $\rho^{\mathcal{L}}$ on $\mathcal{P}$. Recall from the proof of Proposition 3.3.5 non-concavity of $V^{\mathcal{L}}$ can arise when considering a cash liquidity policy and two portfolios where one of them has a positive cash position and the other no cash.

Now we examine some interesting properties of $\rho^{\mathcal{L}}$. The following properties in the proposition below of the liquidity-adjusted risk measure coincide with the results of Acerbi and Scandolo.

**Proposition 3.4.4.** *Let $\mathcal{L}$ be some liquidity policy and $\rho$ a coherent measure of risk.*

1. *For every $\psi, \xi \in \mathcal{P}$ with $V^{\mathcal{L}}(\psi) \geq V^{\mathcal{L}}(\xi)$ $\mathbb{P}$-a.s., then $\rho^{\mathcal{L}}(\xi) \geq \rho^{\mathcal{L}}(\psi)$.*
2. *The liquidity-adjusted risk measure $\rho^{\mathcal{L}}$ is translation subvariant, i.e. for any $e \geq 0$ and $\psi \in \mathcal{P}$*

$$\rho^{\mathcal{L}}(\psi + (e, 0, \ldots, 0)) \leq \rho^{\mathcal{L}}(\psi) - e \leq \rho^{\mathcal{L}}(\psi) + e.$$

*Proof.* 1. Follows from axiom (M) of $\rho$ and 2. follows from Proposition 3.3.6, (M) and (TI) of $\rho$.

□

Both properties have a natural interpretation. The first states that if the liquidity-adjusted portfolio value for $\psi$ is almost surely larger than the liquidity-adjusted portfolio value for $\xi$, then the liquidity-adjusted risk for the former must be lower than the risk for the latter, which is intuitive. The second property embeds the fact that the liquidity adjustments to portfolio values are non-linear, which makes a portfolio with a positive cash position less risky than the same portfolio without the cash position.

Furthermore, liquidity-adjusted risk measures $\rho^{\mathcal{L}}$ for cash liquidity, total liquidation and unrestricted liquidation policies are monotonic on the set of portfolios $\mathcal{P}$.

**Proposition 3.4.5.** *Let $\mathcal{L} \in \left\{\mathcal{L}(a), \mathcal{L}^S(a), \mathcal{L}^U, \mathcal{L}^L\right\}$ and $\rho$ some coherent measure of risk. For any $\psi^1, \psi^2 \in \mathcal{P}$ with $\psi^1 \geq \psi^2$,*

$$\rho^{\mathcal{L}}(\psi^1) \leq \rho^{\mathcal{L}}(\psi^2).$$

*Proof.* From Proposition 3.3.7 we have

$$V^{\mathcal{L}}(\psi^1) \geq V^{\mathcal{L}}(\psi^2), \qquad \mathbb{P} - a.s.$$

The results follows from the (M) of $\rho$ rephrased in Proposition 3.4.4.

□

As shown previously, the total liquidation policy $\mathcal{L}^L$ implies that the optimal adjustment transaction - solution of optimization problem (3.10) - consists of closing position of the initial portfolio. Consequently, the liquidity-adjusted portfolio can be easily be characterized, which also facilitates the characterization of $\rho^{\mathcal{L}^L}$ presented below.

**Proposition 3.4.6.** *Let $\psi \in \mathcal{P}$ and the total liquidation policy $\mathcal{L}^L$. Then*

1. *For $\lambda > 1$,*
$$\rho^{\mathcal{L}^L}(\lambda\psi) \geq \lambda\rho^{\mathcal{L}^L}(\psi).$$
2. *$\rho^{\mathcal{L}^L}$ is subadditive on discordant portfolios, i.e. for $\psi^1, \psi^2 \in \mathcal{P}$ with $\psi^1 \downarrow\downarrow \psi^2$*
$$\rho^{\mathcal{L}^L}(\psi^1 + \psi^2) \leq \rho^{\mathcal{L}^L}(\psi^1) + \rho^{\mathcal{L}^L}(\psi^2).$$

*Proof.* By Proposition 3.3.8 we have

$$V^{\mathcal{L}^L}(\psi) = \psi_0 + L(\hat{\theta}^*) \qquad \mathbb{P} - a.s.,$$

where $\hat{\theta}^* = (0, \psi_1, \ldots, \psi_N)$. Since $L$ is a concave function, we have for $\lambda > 1$

$$L(\lambda\hat{\theta}^*) \leq \lambda L(\hat{\theta}^*), \tag{3.32}$$

which is proved in (Acerbi & Scandolo 2008). Thus, inequality (3.32) holds $\mathbb{P}$-almost surely. Result 1. follows from (M) of $\rho$. Subadditivity follows from Proposition 3.2.13, because $L$ is superadditive on discordant portfolios, and from (M) and from (S) of $\rho$.

$\square$

Last, consider the unrestricted liquidation policy $\mathcal{L}^U$ and assume strictly decreasing MSDCs. Recall that in this case the optimal liquidity adjustment solving (3.10) is a non-trade transaction, i.e. $\hat{\theta}^* = 0$. Hence, the liquidity-adjusted portfolio value matches the pre-execution mark-to-market value, which is linear in $\psi$. Because of this linearity, a coherent risk measure applied on the liquidity-adjusted portfolio value preserves its properties on the set $\mathcal{P}$. In other words, $\rho^{\mathcal{L}^U}$ is coherent on $\mathcal{P}$. To see this, note that $\psi > \xi$ implies $V^{\mathcal{L}^U}(\psi) > V^{\mathcal{L}^U}(\xi)$, which guaranties monotonicity of $\rho^{\mathcal{L}^U}$. Positive homogeneity follows from the linearity of the liquidity-adjusted portfolio value because it equals $\tilde{U}(\psi)$. The remaining properties are shown below.

**Proposition 3.4.7.** *Assume strictly decreasing MSDCs for all non-cash assets, consider $\mathcal{L}^U$ and any coherent risk measure $\rho$. The liquidity-adjusted measure of risk $\rho^{\mathcal{L}^U}$ is subadditive and translation invariant for positive cash positions, i.e.*

1. *For $\psi^1, \psi^2 \in \mathcal{P}$,*
$$\rho^{\mathcal{L}^U}(\psi^1 + \psi^2) \leq \rho^{\mathcal{L}^U}(\psi^1) + \rho^{\mathcal{L}^U}(\psi^2).$$
2. *For $\psi \in \mathcal{P}$ and $e \geq 0$,*
$$\rho^{\mathcal{L}^U}(\psi + (e, 0, \dots, 0)) = \rho^{\mathcal{L}^U}(\psi) - e.$$

*Proof. Subadditivity.* By Proposition 3.3.9 and Proposition 3.2.13 we have for $\psi^1, \psi^2 \in \mathcal{P}$

$$V^{\mathcal{L}^U}(\psi^1 + \psi^2) = U(\psi^1 + \psi^2, 0) \geq U(\psi^1, 0) + U(\psi^2, 0) = V^{\mathcal{L}^U}(\psi^1) + V^{\mathcal{L}^U}(\psi^2) \quad \mathbb{P}-a.s.$$

The result follows from (M) of $\rho$.

*Translation Invariance.* Follows from (TI) of $\rho$, from Proposition 3.3.9 and by noting that for $\psi \in \mathcal{P}$ and $e \geq 0$

$$V^{\mathcal{L}^U}(\psi + (e, 0, \dots, 0)) = U(\psi + (e, 0, \dots, 0), 0) = U(\psi, 0) + e = V^{\mathcal{L}^U}(\psi) + e, \quad \mathbb{P}-a.s.$$

$\square$

For the unrestricted liquidation policy $\mathcal{L}^U$ and strictly decreasing MSDCs, the liquidity-adjusted risk measure of risk is subadditive and translation invariant because block trading does not affect the liquidity-adjusted portfolio value.
As shown earlier, whenever block trading effects are not present, the liquidity-adjusted portfolio value $\mathcal{L}$ preserves the properties put forward in (Acerbi & Scandolo 2008). Accordingly, the liquidity-adjusted measure of risk $\rho^{\mathcal{L}}$ also preserves all the properties presented in (Acerbi & Scandolo 2008). However, partial execution and block trading effects lead to greater risk. This issue is discussed next.

### 3.4.2 Probability Distribution of $V^{\mathcal{L}}(\psi)$

In this section we analyze the probability distribution of the liquidity-adjusted portfolio value under governing market conditions $\mathcal{MC}$ and under market conditions excluding block trading $\mathcal{MC}^{\neg BT}$, partial execution $\mathcal{MC}^{\neg PE}$ and both $\mathcal{MC}^{\neg PEBT}$. We find that block trading and partial execution produces a shift of the probability distribution to lower values, i.e. the probability of large losses increases under these market imperfections.
Let governing market conditions be $\mathcal{MC} = \{\mathcal{Y}, \mathcal{B}^c\}$ where $\mathcal{Y}, \mathcal{B}^c \subseteq \mathcal{P}$, consider some liquidity policy $\mathcal{L}$ and some portfolio $\psi \in \mathcal{P}$. The cumulative probability distribution function $F$ of the random variable $V^{\mathcal{L}}(\psi \,|\mathcal{MC})$ is denoted by

$$\mathbb{P}\left(V^{\mathcal{L}}(\psi \,|\mathcal{MC}) \leq x\right) = F\left(V^{\mathcal{L}}(\psi \,|\mathcal{MC}) \leq x\right) = F_{V^{\mathcal{L}}(\psi|\mathcal{MC})}(x),$$

for $x \in \mathbb{R}$. Regard two real-valued random variables $X, Y$. We say $X$ dominates stochastically $Y$ in the first order and denote $X \succ_1 Y$, if for each $x \in \mathbb{R}$ we have[11]

$$\mathbb{P}(X \leq x) \leq \mathbb{P}(Y \leq x).$$

If the random variables are integrable, then we have the following well known result for which we abstain to present the proof.

**Lemma 3.4.8.** *Observe two real-valued random variables $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. If $X \succ_1 Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.*

[11] Following Föllmer and Schied (Föllmer & Schied 2004).

Previously we have examined the impact of block trading and partial execution on the liquidity-adjusted portfolio value $V^{\mathcal{L}}$. Under stochastic MSDCs and post-execution best bids and asks, liquidity-adjusted portfolio value $V^{\mathcal{L}}$ is a random variable and the results in Sections 3.3.2 and 3.3.3 must be understood as almost surely statements under probability measure $\mathbb{P}$. By this consideration and the following convention, the next proposition follows straightforwardly.

**Notation 3.4.9.** *For the remainder, let $\Lambda$ be either $PE$, $BT$ or $PEBT$.*

**Proposition 3.4.10.** *Consider some given governing market conditions $\mathcal{MC} = \{\mathcal{Y}, \mathcal{B}^c\}$ where $\mathcal{Y}, \mathcal{B}^c \subseteq \mathcal{P}$, some liquidity policy $\mathcal{L}$ and some portfolio $\psi \in \mathcal{P}$. Then*

$$V^{\mathcal{L}}\left(\psi \left| \mathcal{MC}^{\neg\Lambda}\right.\right) \succ_1 V^{\mathcal{L}}\left(\psi \left| \mathcal{MC}\right.\right).$$

*If $V^{\mathcal{L}}\left(\psi \left| \mathcal{MC}\right.\right) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for any $\mathcal{MC}$, then*

$$\mathbb{E}\left[V^{\mathcal{L}}\left(\psi \left| \mathcal{MC}^{\neg\Lambda}\right.\right)\right] \geq \mathbb{E}\left[V^{\mathcal{L}}\left(\psi \left| \mathcal{MC}\right.\right)\right].$$

*Proof.* Since inequalities (3.23) and (3.24) of Proposition 3.3.15 hold $\mathbb{P}$-almost surely we have

$$F_{V^{\mathcal{L}}(\psi|\mathcal{MC})}(x) \geq F_{V^{\mathcal{L}}\left(\psi\left|\mathcal{MC}^{\neg\Lambda}\right.\right)}(x),$$

for all $x \in \mathbb{R}$. The second part follows from Corollary 3.4.8.

□

This result points out that block trading and partial execution effects shift the probability distribution of the liquidity-adjusted portfolio value towards left to lower values. Furthermore, by considering these effects the expected value drops to lower values also.

To conclude this section, we throw a glance to the quantiles of the probability distribution of $V^{\mathcal{L}}$. In particular, we focus on the most popular quantile used in practice: Value-at-Risk (VaR). Following McNeil et al. (McNeil et al. 2005), let the probability distribution function of the random variable $X$ be denoted by $F_X$. For $\alpha \in (0, 1)$ the right $\alpha$-quantile of $F_X$ is given by

$$q_\alpha(F_X) = \inf\left\{x \in \mathbb{R} \left| F_X(x) > \alpha\right.\right\}.$$

Additionally, the VaR at level $\alpha \in (0,1)$ is given by

$$VaR_\alpha\left(X\right) = -q_\alpha(F_X).$$

Although VaR is not a coherent risk measure, its usage has a wide range for which we make it part of our analysis. According to results presented above, the quantile of the liquidity-adjusted portfolio value is lower when block trading and partial execution effects exist. Hence, VaR is larger under block trading and partial execution.

**Proposition 3.4.11.** *Consider $\mathcal{MC} = \{\mathcal{Y}, \mathcal{B}^c\}$ where $\mathcal{Y}, \mathcal{B}^c \subseteq \mathcal{P}$, any liquidity policy $\mathcal{L}$ and $\psi \in \mathcal{P}$. For $\alpha \in (0,1)$*

$$q_\alpha\left(F_{V^{\mathcal{L}}(\psi|\mathcal{MC})}\right) \leq q_\alpha\left(F_{V^{\mathcal{L}}\left(\psi\left|\mathcal{MC}^{\neg\Lambda}\right.\right)}\right),$$

*and*

$$VaR_\alpha\left(V^{\mathcal{L}}\left(\psi\left|\mathcal{MC}\right.\right)\right) \geq VaR_\alpha\left(V^{\mathcal{L}}\left(\psi\left|\mathcal{MC}^{\neg\Lambda}\right.\right)\right).$$

*Proof.* Since $F_{V^{\mathcal{L}}(\psi|\mathcal{MC})}(x) \geq F_{V^{\mathcal{L}}\left(\psi\left|\mathcal{MC}^{\neg\Lambda}\right.\right)}(x)$ for each $x \in \mathbb{R}$, we obtain for each $\alpha \in (0,1)$

$$\begin{aligned} q_\alpha\left(F_{V^{\mathcal{L}}(\psi|\mathcal{MC})}\right) &= \inf\left\{x \in \mathbb{R}\left|F_{V^{\mathcal{L}}(x)(\psi|\mathcal{MC})} > \alpha\right.\right\} \\ &\leq \inf\left\{x \in \mathbb{R}\left|F_{V^{\mathcal{L}}(x)\left(\psi\left|\mathcal{MC}^{\neg\Lambda}\right.\right)} > \alpha\right.\right\} = q_\alpha\left(F_{V^{\mathcal{L}}\left(\psi\left|\mathcal{MC}^{\neg\Lambda}\right.\right)}\right). \end{aligned}$$

Since $VaR_\alpha(X) = -q_\alpha(F_X)$ the proof is complete.

□

## 3.5 Numerical Example

In order to illustrate our results by comparing them with (1) the framework of Acerbi and Scandolo and (2) a framework without liquidity adjustments, we present a numerical example in line with the analytically solvable class of $V^{\mathcal{L}}$ exhibit in (Acerbi & Scandolo 2008). Accordingly, the MSDCs are strictly decreasing of the form $m_i(x) = A_i e^{-k_i x}$ with $A_i, k_i > 0$ for $x \in \mathbb{R}$, $i \in I$. Hence, the transaction proceeds are given by

$$L(\hat{\theta}) = \sum_{i=1}^{N} \frac{A_i}{k_i}\left(1 - e^{-k_i\hat{\theta}_i}\right), \qquad \hat{\theta}_i \in Y_i.$$

We consider the cash liquidity policy without buy trades $\mathcal{L}^S(a)$ for some $a > 0$. Whenever block trading is not present, $\mathcal{Y} \subseteq \mathcal{B}^c$, the optimal transaction $\hat{\theta}^*$ that solves problem (3.10) is given by[12]

$$\hat{\theta}_i^* = \frac{1}{k_i} \log \left( \frac{\sum_{i=1}^N \frac{A_i}{k_i}}{\sum_{i=1}^N \frac{A_i}{k_i} - a} \right),$$

for $i \in I$, and the liquidity-adjusted portfolio value is given by

$$V^{\mathcal{L}^S(a)}(\psi) = \sum_{i=1}^N A_i(\psi_i - \hat{\theta}_i^*) + a, \qquad \psi \in \mathcal{P}.$$

For the Monte Carlo simulation we consider the cases $\mathcal{Y} \subseteq \mathcal{B}^c$ and $\mathcal{B}^c \subset \mathcal{Y}$ where the vectors $A := (A_1, \ldots, A_N)$ and $k := (k_1, \ldots, k_N)$ are randomly generated from a lognormal distribution. We impose post-execution best bid and ask prices that conform with Assumption 3.2.9. Additionally, we also compute the portfolio value without any liquidity adjustment. Figure 3.1 exhibits the histogram resulting from the simulation of (i) the liquidity-adjusted portfolio value with block trading and partial execution effects, (ii) the liquidity-adjusted portfolio value without those effects corresponding to the framework of Acerbi and Scandolo and (iii) the portfolio value without any liquidity risk adjustment.

Table 3.1 displays the means and quantiles for all three cases. According to Proposition 3.4.10 and Proposition 3.4.11 we observe lower values for case (i) than cases (ii) and (iii), which indicates greater risk for (i).

Table 3.1: Quantiles from Simulation

| Quantiles | 1% | 5% | 10% | 25% | 50% | 75% | Mean |
|---|---|---|---|---|---|---|---|
| (i) Block Trading and Partial Execution | 8719.2 | 8848 | 8957.6 | 9159.2 | 9323.5 | 9522.4 | 9342.5 |
| (ii) Without Block Trading and Partial Execution | 9311.1 | 9442.5 | 9568.2 | 9775.8 | 9969.4 | 10159.6 | 9975.5 |
| (iii) Without Liquidity Risk Adjustment | 9335.6 | 9467.1 | 9590.7 | 9798 | 9990.7 | 10181.6 | 9997.2 |

## 3.6 Concluding Remarks

In the present chapter we have investigated the consequences of block trading and partial execution on the setup from Acerbi and Scandolo (Acerbi & Scandolo 2008).

[12]For a formal proof see (Acerbi & Scandolo 2008).

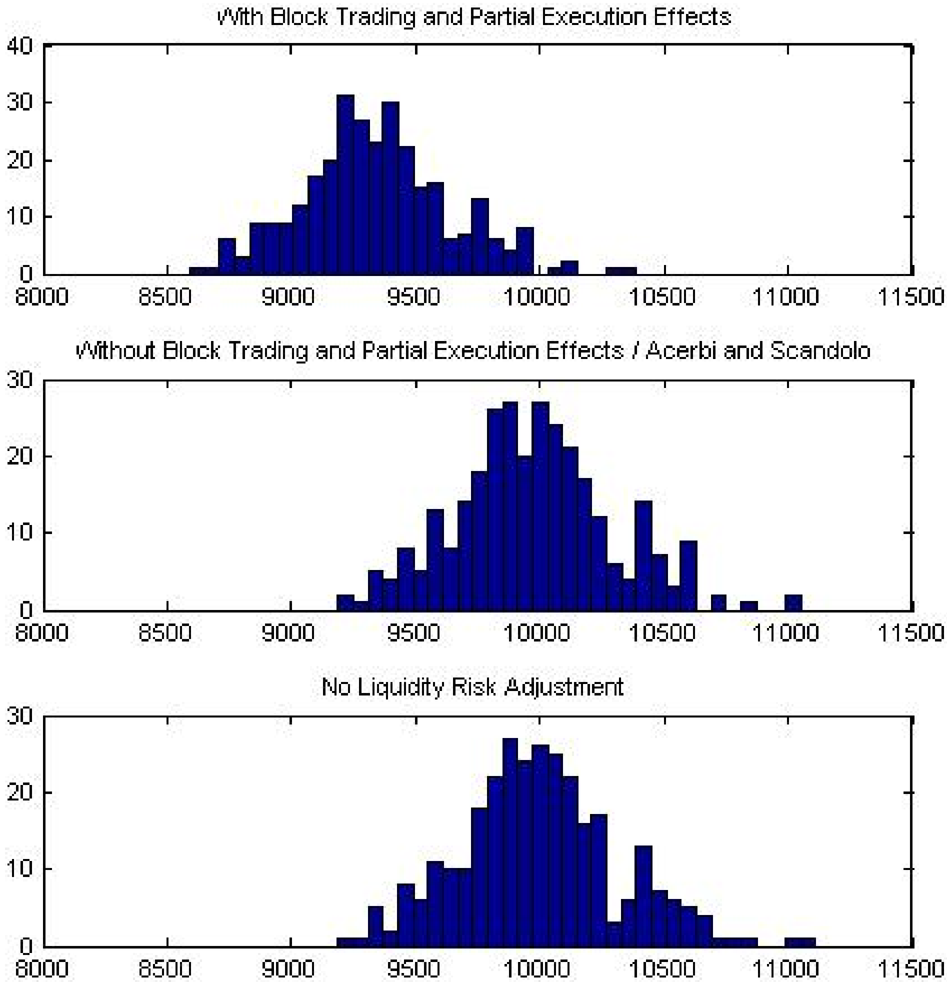


Figure 3.1: Histograms from Simulation

We undertake the same liquidity adjustment to portfolio values as in (Acerbi & Scandolo 2008), which produces the liquidity-adjusted portfolio value $V^{\mathcal{L}}$. Via a coherent risk measure we define the liquidity-adjusted risk measures $\rho^{\mathcal{L}}$ put forward in (Acerbi & Scandolo 2008). Acerbi and Scandolo show that $V^{\mathcal{L}}$ is concave on $\mathcal{P}$ and $\rho^{\mathcal{L}}$ is convex on $\mathcal{P}$. By introducing block trading and partial execution effects, we show that, under these circumstances, $V^{\mathcal{L}}$ is not concave on $\mathcal{P}$ and $\rho^{\mathcal{L}}$ is not convex on $\mathcal{P}$. In addition, we show that the existence of block trading and partial execution effects induces a shift of the probability function of $V^{\mathcal{L}}$ to lower values, i.e. the probability of large losses increases.

Measuring liquidity risk is evidently not a simple task. This may rely on the fact that liquidity risk expresses itself through several channels. The lack of a unique definition of liquidity risk or, more precisely, the ample variety of definitions, effects and consequences of liquidity risk represents a cumbersome issue that researchers may need to solve first. As this analysis points out, by ignoring some aspects of liquidity risk, we arrive at wrong conclusions, which may cause catastrophic damages.

Our model does not reflect all forms of liquidity risk, and those which are included are introduced in the most simple manner. In this sense, researchers may find worthy augmenting more forms of liquidity risk to our model or to handle more elaborated concepts.

## 3.7 Appendix

### 3.7.1 Additional Lemma in Proof of Proposition 3.3.4

**Lemma 3.7.1.** *Consider the function*

$$v_i^\psi(\hat{\theta}_i) = m^{+,\hat{\theta}_i}(\psi_i - \hat{\theta}_i)\mathbb{1}_{\{\psi_i - \hat{\theta}_i > 0\}} + m^{-,\hat{\theta}_i}(\psi_i - \hat{\theta}_i)\mathbb{1}_{\{\psi_i - \hat{\theta}_i < 0\}} + \int_0^{\hat{\theta}_i} m_i(u)du,$$

*for $i \in I$.*

1. *If $\bar{y}_i \geq \psi_i > \bar{b}_i$ or $\psi_i > \bar{y}_i > \bar{b}_i$, then*

$$v_i^\psi(\bar{b}_i) \geq v_i^\psi(\hat{\theta}_i) \qquad for \quad \hat{\theta}_i \in (\bar{b}_i, \bar{y}_i].$$

2. *If $\underline{y}_i \leq \psi_i < \underline{b}_i$ or $\psi_i < \underline{y}_i < \underline{b}_i$, then*

$$v_i^\psi(\underline{b}_i) \geq v_i^\psi(\hat{\theta}_i) \qquad for \quad \hat{\theta}_i \in [\underline{y}_i, \underline{b}_i).$$

*Proof. 1.* For $\bar{y}_i \geq \psi_i > \bar{b}_i$,

$$v_i^\psi(\bar{b}_i) = m_i^+(\psi_i - \bar{b}_i) + \int_0^{\bar{b}_i} m_i(u)du,$$

and

$$v_i^\psi(\hat{\theta}_i) = \begin{cases} m_i^{+,\bar{b}_i}(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u)du & \text{for} \quad \hat{\theta}_i \in (\bar{b}_i, \psi_i] \\ m_i^-(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u)du & \text{for} \quad \hat{\theta}_i \in (\psi_i, \bar{y}_i]. \end{cases}$$

Thus, for $\hat{\theta}_i \in (\bar{b}_i, \psi_i]$ we have

$$\begin{aligned} v_i^\psi(\bar{b}_i) - v_i^\psi(\hat{\theta}_i) &= m_i^+(\psi_i - \bar{b}_i) - m_i^{+,\bar{b}_i}(\psi_i - \hat{\theta}_i) - \int_{\bar{b}_i}^{\hat{\theta}_i} m_i(u)du \\ &\geq m_i^+(\psi_i - \bar{b}_i) - m_i^+(\psi_i - \hat{\theta}_i) - \int_{\bar{b}_i}^{\hat{\theta}_i} m_i(u)du \\ &= \int_{\bar{b}_i}^{\hat{\theta}_i} \left(m_i^+ - m_i(u)\right) du \geq 0, \end{aligned} \tag{3.33}$$

and for $\hat{\theta}_i \in (\psi_i, \bar{y}_i]$,

$$\begin{aligned} v_i^\psi(\bar{b}_i) - v_i^\psi(\hat{\theta}_i) &= m_i^+(\psi_i - \bar{b}_i) - m_i^-(\psi_i - \hat{\theta}_i) - \int_{\bar{b}_i}^{\hat{\theta}_i} m_i(u)du \\ &\geq m_i^+(\hat{\theta}_i - \bar{b}_i) - \int_{\bar{b}_i}^{\hat{\theta}_i} m_i(u)du \geq 0. \end{aligned}$$

*1.* For $\psi_i > \bar{y}_i > \bar{b}_i$, we have (3.33) for $\hat{\theta}_i \in (\bar{b}_i, \bar{y}_i]$.

*2.* For $\underline{y}_i \leq \psi_i < \underline{b}_i$, we obtain

$$v_i^{\psi}(\underline{b}_i) = m_i^-(\psi_i - \underline{b}_i) - \int_{\underline{b}_i}^{0} m_i(u)du,$$

and

$$v_i^{\psi}(\hat{\theta}_i) = \begin{cases} m_i^{-,\underline{b}_i}(\psi_i - \hat{\theta}_i) - \int_{\hat{\theta}_i}^{0} m_i(u)du & \text{for} \quad \hat{\theta}_i \in [\psi_i, \underline{b}_i) \\ m_i^+(\psi_i - \hat{\theta}_i) - \int_{\hat{\theta}_i}^{0} m_i(u)du & \text{for} \quad \hat{\theta}_i \in [\underline{b}_i, \psi_i). \end{cases}$$

Hence, for $\hat{\theta}_i \in [\psi_i, \underline{b}_i)$,

$$\begin{aligned} v_i^{\psi}(\underline{b}_i) - v_i^{\psi}(\hat{\theta}_i) &= m_i^-(\psi_i - \underline{b}_i) - m_i^{-,\underline{b}_i}(\psi_i - \hat{\theta}_i) + \int_{\hat{\theta}_i}^{\underline{b}_i} m_i(u)du \\ &\geq m_i^-(\hat{\theta}_i - \underline{b}_i) + \int_{\hat{\theta}_i}^{\underline{b}_i} m_i(u)du \\ &= \int_{\hat{\theta}_i}^{\underline{b}_i} \left(m_i(u) - m_i^-\right) du \geq 0, \end{aligned} \tag{3.34}$$

and for $\hat{\theta}_i \in [\underline{y}_i, \psi_i)$,

$$\begin{aligned} v_i^{\psi}(\underline{b}_i) - v_i^{\psi}(\hat{\theta}_i) &= m_i^-(\psi_i - \underline{b}_i) - m_i^+(\psi_i - \hat{\theta}_i) + \int_{\hat{\theta}_i}^{\underline{b}_i} m_i(u)du \\ &\geq m_i^-(\hat{\theta}_i - \underline{b}_i) + \int_{\hat{\theta}_i}^{\underline{b}_i} m_i(u)du \\ &= \int_{\hat{\theta}_i}^{\underline{b}_i} \left(m_i(u) - m_i^-\right) du \geq 0. \end{aligned}$$

*2.* For $\psi_i < \underline{y}_i < \underline{b}_i$ we have (3.34) for all $\hat{\theta}_i \in [\underline{y}_i, \underline{b}_i)$.

□

### 3.7.2 Liquidity Restriction

**Lemma 3.7.2.** *Consider some $a > 0$ and $\psi \in \mathcal{P}$ with $a - \psi_0 > 0$. For every $\hat{\theta}^* \in \mathcal{Y}$ that solves optimization problem (3.10) for $\mathcal{L} \in \{\mathcal{L}(a), \mathcal{L}^S(a)\}$, there exists some $\hat{\theta}^{**} \in \mathcal{Y}$ that also solves problem (3.10) for $\mathcal{L}$ with*[13]

$$\psi_0 + L(\hat{\theta}^{**}) = a.$$

[13]Note that if there is some $\hat{\theta}^* \in \mathcal{Y}$ that solves (3.10), then $V^{\mathcal{L}}(\hat{\theta}^*) \neq -\infty$.

*Proof.* Let $a > 0$, $\mathcal{L} \in \{\mathcal{L}(a), \mathcal{L}^S(a)\}$ and $\psi \in \mathcal{P}$ with $a - \psi_0 > 0$. Assume $\hat{\theta}^* \in \mathcal{C}^{\mathcal{L}}(\psi) \neq \emptyset$ solves problem (3.10), i.e.

$$U(\psi - \hat{\theta}^*, \hat{\theta}^*) + L(\hat{\theta}^*) \geq U(\psi - \hat{\theta}, \hat{\theta}) + L(\hat{\theta}) \qquad \forall\, \hat{\theta} \in \mathcal{C}^{\mathcal{L}}(\psi) \neq \emptyset, \tag{3.35}$$

such that

$$\psi_0 + L(\hat{\theta}^*) > a. \tag{3.36}$$

Hence, there exits $K \subseteq I$ such that $\hat{\theta}^*_i > 0$ for all $i \in K$. This implies the existence of a some nonempty subset $J \subseteq K$ such that transaction $\hat{\theta}^{**} \in \mathcal{Y}$ is given by

$$\hat{\theta}^{**}_i = \begin{cases} \hat{\theta}^*_i & for\ \ i \notin J \\ \hat{\theta}^*_i - \hat{x}_i & for\ \ i \in J \end{cases},$$

where $0 < \hat{x}_i \leq \hat{\theta}^*_i$ such that $\psi_0 + L(\hat{\theta}^{**}) = a$. Clearly, $J \neq \emptyset$ because of condition (3.36). Thus,

$$\begin{aligned}
& U(\psi - \hat{\theta}^{**}, \hat{\theta}^{**}) + L(\hat{\theta}^{**}) \\
= \;& \psi_0 + \sum_{i \notin J} \Big( m_i^{+,\hat{\theta}^*_i}(\psi_i - \hat{\theta}^*_i) \mathbb{1}_{\{\psi_i - \hat{\theta}^*_i > 0\}} + m_i^{-,\hat{\theta}^*_i}(\psi_i - \hat{\theta}^*_i) \mathbb{1}_{\{\psi_i - \hat{\theta}^*_i < 0\}} \Big) \\
+ \;& \sum_{i \in J} \Big( m_i^{+,\hat{\theta}^*_i - \hat{x}_i}(\psi_i - \hat{\theta}^*_i + \hat{x}_i) \mathbb{1}_{\{\psi_i - \hat{\theta}^*_i + \hat{x}_i > 0\}} \\
+ \;& m_i^{-,\hat{\theta}^*_i - \hat{x}_i}(\psi_i - \hat{\theta}^*_i + \hat{x}_i) \mathbb{1}_{\{\psi_i - \hat{\theta}^*_i + \hat{x}_i < 0\}} \Big) + \sum_{i \notin J} \int_0^{\hat{\theta}^*_i} m_i(u) du \\
+ \;& \sum_{i \in J} \int_0^{\hat{\theta}^*_i - \hat{x}_i} m_i(u) du \\
\geq \;& \psi_0 + \sum_{i \notin J} \Big( m_i^{+,\hat{\theta}^*_i}(\psi_i - \hat{\theta}^*_i) \mathbb{1}_{\{\psi_i - \hat{\theta}^*_i > 0\}} + m_i^{-,\hat{\theta}^*_i}(\psi_i - \hat{\theta}^*_i) \mathbb{1}_{\{\psi_i - \hat{\theta}^*_i < 0\}} \Big) \\
+ \;& \sum_{i \in J} \Big( m_i^{+,\hat{\theta}^*_i - \hat{x}_i}(\psi_i - \hat{\theta}^*_i) \mathbb{1}_{\{\psi_i - \hat{\theta}^*_i + \hat{x}_i \geq 0\}} + m_i^{-,\hat{\theta}^*_i - \hat{x}_i}(\psi_i - \hat{\theta}^*_i) \mathbb{1}_{\{\psi_i - \hat{\theta}^*_i + \hat{x}_i < 0\}} \Big) \\
+ \;& \sum_{i=1}^{N} \int_0^{\hat{\theta}^*_i} m_i(u) du \\
\geq \;& \psi_0 + \sum_{i=0}^{N} \Big( m_i^{+,\hat{\theta}^*_i}(\psi_i - \hat{\theta}^*_i) \mathbb{1}_{\{\psi_i - \hat{\theta}^*_i > 0\}} + m_i^{-,\hat{\theta}^*_i}(\psi_i - \hat{\theta}^*_i) \mathbb{1}_{\{\psi_i - \hat{\theta}^*_i < 0\}} \Big) \\
+ \;& \sum_{i=1}^{N} \int_0^{\hat{\theta}^*_i} m_i(u) du \\
= \;& U(\psi - \hat{\theta}^*, \hat{\theta}^*) + L(\hat{\theta}^*).
\end{aligned}$$

The first inequality follows from the decreasing shape of MSDC and condition $m^{+,\bar{b}_i} \geq m_i(x)$ for $x \geq \bar{b}_i$ of Assumption 3.2.9, since

$$\sum_{i\in J}\left(m_i^{+,\hat{\theta}_i^*-\hat{x}_i}\cdot\hat{x}_i\cdot\mathbb{1}_{\{\psi_i-\hat{\theta}_i^*+\hat{x}_i\geq 0\}}+m_i^{-,\hat{\theta}_i^*-\hat{x}_i}\cdot\hat{x}_i\cdot\mathbb{1}_{\{\psi_i-\hat{\theta}_i^*+\hat{x}_i<0\}}\right)$$
$$\geq\sum_{i\in J}\int_{\hat{\theta}_i^*-\hat{x}_i}^{\hat{\theta}_i^*}m_i(u)du.$$

For the second inequality we must consider the following three cases for $i \in J$: (i) $\psi_i - \hat{\theta}_i^* > 0$ which implies $\psi_i - \hat{\theta}_i^* + \hat{x}_i \geq 0$, (ii) $\psi_i - \hat{\theta}_i^* < 0$, $\psi_i - \hat{\theta}_i^* + \hat{x}_i < 0$ and (iii) $\psi_i - \hat{\theta}_i^* < 0$, $\psi_i - \hat{\theta}_i^* + \hat{x}_i \geq 0$. Considering Assumption 3.2.9, for those $i \in J$ satisfying cases (i) or (ii) the second inequality becomes an equality and for those $i \in J$ satisfying case (iii) it is a strict inequality. By construction we have $\hat{\theta}^{**} \in \mathcal{C}^{\mathcal{L}}(\psi)$. Hence, it must hold

$$U(\psi-\hat{\theta}^{**},\hat{\theta}^{**})+L(\hat{\theta}^{**})=U(\psi-\hat{\theta}^*,\hat{\theta}^*)+L(\hat{\theta}^*),$$

by condition (3.35) and optimality of $\hat{\theta}^*$. In other words, $\hat{\theta}^{**}$ solves problem (3.10) and satisfies $\psi_0 + L(\hat{\theta}^{**}) = a$.

□

### 3.7.3 Additional Lemma in Proof of Proposition 3.3.9

**Lemma 3.7.3.** *Assume that for all $i \in I$ the MSDC is strictly decreasing, i.e.*

$$m_i(x) < m_i(x'),$$

*for $x, x' \in \mathbb{R}\setminus\{0\}$ with $x > x'$. Then, for any $\psi \in \mathcal{P}$ and every $i \in I$ the function $v_i^\psi$ is strictly increasing on $[\underline{y}_i, 0)$ and strictly decreasing on $(0, \bar{y}_i]$.*

*Proof.* First, we analyze the function $v_i^\psi$ on $[\underline{y}_i, 0)$. Note that $v_i^\psi$ can take only the following values

$$v_i^\psi(\hat{\theta}_i)=\begin{cases}m_i^-(\psi_i-\hat{\theta}_i)+\int_0^{\hat{\theta}_i}m_i(u)du & \text{if }\hat{\theta}_i\geq\underline{b}_i\text{ and }\hat{\theta}_i>\psi_i\\ m_i^{-,\underline{b}_i}(\psi_i-\hat{\theta}_i)+\int_0^{\hat{\theta}_i}m_i(u)du & \text{if }\hat{\theta}_i<\underline{b}_i\text{ and }\hat{\theta}_i>\psi_i\\ \int_0^{\hat{\theta}_i}m_i(u)du & \text{if }\hat{\theta}_i=\psi_i\\ m_i^+(\psi_i-\hat{\theta}_i)+\int_0^{\hat{\theta}_i}m_i(u)du & \text{else.}\end{cases}$$

For the two first cases we have the following derivative

$$\frac{dv_i^\psi(\hat{\theta}_i)}{d\hat{\theta}_i} = \begin{cases} -m_i^- + m_i(\hat{\theta}_i) & \text{if } \hat{\theta}_i > \underline{b}_i \text{ and } \hat{\theta}_i > \psi_i \\ -m_i^{-,\underline{b}_i} + m_i(\hat{\theta}_i) & \text{if } \hat{\theta}_i < \underline{b}_i \text{ and } \hat{\theta}_i > \psi_i, \end{cases}$$

and for the fourth case the derivative is given by

$$\frac{dv_i^\psi(\hat{\theta}_i)}{d\hat{\theta}_i} = -m_i^+ + m_i(\hat{\theta}_i).$$

Since $m_i(x)$ is strictly decreasing and by Assumption 3.2.9, we obtain $\frac{dv_i^\psi(\hat{\theta}_i)}{d\hat{\theta}_i} > 0$ for these three cases. Furthermore, from Lemma 3.7.1 we know that $v_i^\psi$ increases at discontinuity points $\hat{\theta}_i = \underline{b}_i$. Because we assume strictly decreasing MSDCs, $v_i^\psi$ strictly increases at discontinuity points. For the case $\hat{\theta}_i = \psi_i$, consider some $\epsilon > 0$ and observe that by strictly decreasing MSDCs we have

$$m_i^{-,\psi_i-\epsilon} \cdot \epsilon + \int_0^{\psi_i-\epsilon} m_i(u)du < \int_0^{\psi_i} m_i(u)du < -m_i^{-,\psi_i+\epsilon} \cdot \epsilon + \int_0^{\psi_i+\epsilon} m_i(u)du,$$

or equivalently,

$$v_i^\psi(\hat{\theta}_i - \epsilon) < v_i^\psi(\hat{\theta}_i) < v_i^\psi(\hat{\theta}_i + \epsilon).$$

Hence, $v_i^\psi$ is strictly increasing on $[\underline{y}_i, 0)$ for any $\psi \in \mathcal{P}$. We find symmetrical arguments of the function $v_i^\psi$ for $\hat{\theta}_i \in (0, \bar{y}_i]$:

$$v_i^\psi(\hat{\theta}_i^*) = \begin{cases} m_i^+(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u)du & \text{if } \hat{\theta}_i \leq \bar{b}_i \text{ and } \hat{\theta}_i < \psi_i \\ m_i^{+,\bar{b}_i}(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u)du & \text{if } \hat{\theta}_i > \bar{b}_i \text{ and } \hat{\theta}_i < \psi_i \\ \int_0^{\hat{\theta}_i} m_i(u)du & \text{if } \hat{\theta}_i = \psi_i \\ m_i^-(\psi_i - \hat{\theta}_i) + \int_0^{\hat{\theta}_i} m_i(u)du & \text{else.} \end{cases}$$

Following the same steps as before, we find

$$\frac{dv_i^\psi(\hat{\theta}_i)}{d\hat{\theta}_i} < 0,$$

and for the first two cases $\hat{\theta}_i = \bar{b}_i$ and $\hat{\theta}_i = \psi_i$

$$v_i^\psi(\hat{\theta}_i - \epsilon) > v_i^\psi(\hat{\theta}_i) > v_i^\psi(\hat{\theta}_i + \epsilon),$$

for some $\epsilon > 0$. Hence, $v_i^\psi$ is strictly decreasing on $[\underline{y}_i, 0)$ for any $\psi \in \mathcal{P}$.

□

### 3.7.4 Proof of Proposition 3.4.3

Consider the same scenario as in the proof of Proposition 3.3.5 for the portfolios $\psi^1, \psi^2 \in \mathcal{P}$ and choose some $\lambda \in (0,1)$ such that

$$\lambda V^{\mathcal{L}(a)}(\psi^1) + (1-\lambda)V^{\mathcal{L}(a)}(\psi^2) - V^{\mathcal{L}(a)}(\psi^\lambda) > 0, \tag{3.37}$$

as shown in the mentioned proof.
Assume there are only two states of the world. Hence, for any portfolio $\psi \in \mathcal{P}$ its liquidity-adjusted value equals $V^{\mathcal{L}(a)}(\psi; p)$ with probability $p < 1/2$ and $V^{\mathcal{L}(a)}(\psi; 1-p)$ with probability $1-p$. Assume that the MSDC is so distributed that for the chosen inequality (3.37) holds in both states of the world $\lambda \in (0,1)$. Let

$$V^{\mathcal{L}(a)}(\psi; p) < V^{\mathcal{L}(a)}(\psi; 1-p),$$

and

$$V^{\mathcal{L}(a)}(\psi; p) \neq -\infty,$$

for $\psi \in \left\{\psi^1, \psi^2, \psi^\lambda\right\}$. Furthermore, choose $\alpha \in (0,1)$ such that $p < \alpha$ and consider the following lower $\alpha$-quantile

$$q_\alpha\left(V^{\mathcal{L}(a)}(\psi)\right) := \inf\left\{x \in \mathbb{R} | \, \mathbb{P}\left(V^{\mathcal{L}}(a) \leq x\right) \geq \alpha\right\},$$

and set $q_\alpha(\psi) := q_\alpha\left(V^{\mathcal{L}(a)}(\psi)\right)$. Hence,

$$q_\alpha\left(\psi^1\right) = V^{\mathcal{L}(a)}(\psi^1; 1-p), \qquad q_\alpha\left(\psi^2\right) = V^{\mathcal{L}(a)}(\psi^2; 1-p),$$
$$q_\alpha\left(\psi^\lambda\right) = V^{\mathcal{L}(a)}(\psi^\lambda; 1-p).$$

Consider the expected shortfall $ES$ which is a coherent risk measure and the representation of Acerbi and Tasche (Acerbi & Tasche 2002). Accordingly, the expected shortfall at level $\alpha \in (0,1)$ for a discrete random variable $X$ is given by

$$ES_\alpha(X) = -\frac{1}{\alpha}\mathbb{E}\left[X\mathbb{1}_{\{X \leq q_\alpha(X)\}} + X\frac{\alpha - \mathbb{P}(X \leq q_\alpha(X))}{\mathbb{P}(X = q_\alpha(X))}\mathbb{1}_{\{X = q_\alpha(X)\}}\right].$$

In our case, the expected shortfall is given for portfolios $\psi \in \left\{\psi^1, \psi^2, \psi^\lambda\right\}$ by

$$\begin{aligned}
ES_\alpha(\psi) &= -\frac{1}{\alpha}\mathbb{E}\left[V^{\mathcal{L}(a)}(\psi)\right] - \frac{1}{\alpha}\mathbb{E}\left[V^{\mathcal{L}(a)}(\psi) \cdot \frac{\alpha - 1}{1-p} \cdot \mathbb{1}_{\left\{V^{\mathcal{L}(a)}(\psi) = q_\alpha(\psi)\right\}}\right] \\
&= -\frac{1}{\alpha}\left[pV^{\mathcal{L}(a)}(\psi; p) + (1-p)V^{\mathcal{L}(a)}(\psi; 1-p)\right] \\
&- \frac{1}{\alpha}(\alpha - 1)V^{\mathcal{L}(a)}(\psi; 1-p) \\
&= -\frac{1}{\alpha}pV^{\mathcal{L}(a)}(\psi; p) - \frac{1}{\alpha}(\alpha - p)V^{\mathcal{L}(a)}(\psi; 1-p).
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\alpha\left[\lambda ES_\alpha\left(\psi^1\right)+(1-\lambda)ES_\alpha\left(\psi^2\right)-ES_\alpha\left(\psi^\lambda\right)\right]\\
&=pV^{\mathcal{L}(a)}(\psi^\lambda;p)+(\alpha-p)V^{\mathcal{L}(a)}(\psi^\lambda;1-p)\\
&-\lambda\left[pV^{\mathcal{L}(a)}(\psi^1;p)+(\alpha-p)V^{\mathcal{L}(a)}(\psi^1;1-p)\right]\\
&-(1-\lambda)\left[pV^{\mathcal{L}(a)}(\psi^2;p)+(\alpha-p)V^{\mathcal{L}(a)}(\psi^2;1-p)\right]\\
&=-p\left[\lambda V^{\mathcal{L}(a)}(\psi^1;p)+(1-\lambda)V^{\mathcal{L}(a)}(\psi^2;p)-V^{\mathcal{L}(a)}(\psi^\lambda;p)\right]\\
&-(\alpha-p)\left[\lambda V^{\mathcal{L}(a)}(\psi^1;1-p)+(1-\lambda)V^{\mathcal{L}(a)}(\psi^2;1-p)-V^{\mathcal{L}(a)}(\psi^\lambda;1-p)\right]<0,
\end{aligned}
$$

which follows from inequality (3.37) and $\alpha>p$. Hence,

$$\lambda ES_\alpha\left(\psi^1\right)+(1-\lambda)ES_\alpha\left(\psi^2\right)-ES_\alpha\left(\psi^\lambda\right)<0,$$

or, equivalently,

$$\rho^{\mathcal{L}}\left(\lambda\psi^1+(1-\lambda)\psi^2\right)>\lambda\rho^{\mathcal{L}}\left(\psi^1\right)+(1-\lambda)\rho^{\mathcal{L}}\left(\psi^2\right).$$

□

# Bibliography


Acerbi, C. & Scandolo, G. (2008), 'Liquidity and coherent risk measures', *Quantitative Finance* **8**(7), 681–692.

Acerbi, C. & Tasche, D. (2002), 'On the coherence of expected shortfall', *Journal of Banking and Finance* **26**(7), 1487–1503.

Acharya, V. V., Bharath, S. T. & Srinivasan, A. (2007), 'Does industry-wide distress affect defaulted firms? evidence from creditor recoveries', *The Journal of Financial Economics* **85**, 787–821.

Altman, E., Brady, B., Resti, A. & Sironi, A. (2005), 'The link between default and recovery rates: Theory, empirical evidence, and implications', *Journal of Business* **78**(6), 2203–2227.

Altman, E. I. (2006), Default recovery rates and lgd in credit risk modeling and practice: An updated review of the literature and empirical evidence. Working Paper.

Artzner, P., Delbaen, F., Eber, J. & D., H. (1999), 'Coherent measures of risk', *Mathematical Finance* **9**(3), 203–228.

Bielecki, T. R. & Rutkowski, M. (2004), *Credit Risk: Modeling, Valuation and Hedging*, Springer.

Black, F. & Cox, J. (1976), 'Valuing corporate securities: Some effects of bond indenture provisions', *The Journal of Finance* **31**, 351–367.

Borodin, A. N. & Salminen, P. (2002), *Handbook of Brownian motion. Facts and formulae*, Birkhäuser Verlag.

Brigo, D. & Mercurio, F. (2006), *Interest Rate Models-Theory and Practice*, Springer.

Briys, E. & de Varenne, F. (1997), 'Valuing risky fixed rate debt: An extension', *Journal of Financial and Quantitative Analysis* **32**, 239–248.

Çetin, U., Jarrow, R. & Protter, P. (2002), 'Liquidity risk and arbitrage pricing theory', *Finance and Stochastics* **3**(8), 311–341.

Collin-Dufresne, P., Goldstein, R. & Hugonnier, J. (2004), 'A general formula for valuing defaultable securities', *Econometrica* **72**(5), 1377–1407.

Cox, J., Ingersoll, J. & Ross, S. (1985), 'A theory of the term structure of interest rates', *Econometrica* **53**, 385–407.

Dixit, A. K. & Pindyck, R. (1994), *Investment Under Uncertainty*, Princeton University Press.

Duffie, D. & D. Lando, D. (2001), 'Term structures of credit spread with incomplete accounting information', *Econometrica* **69**(3), 633–664.

Duffie, D., Schroeder, M. & Skiadas, C. (1996), 'Recursive valuation of defaultable securities and the timing of resolution of uncertainty', *The Annals of Applied Probability* **6**(4), 1075–1090.

Duffie, D. & Singleton, K. (1999), 'Modelling term structures of defaultable bonds', *The Review of Financial Studies* **12**(4), 687–720.

Föllmer, H. & Schied, A. (2004), *Stochastic Finance*, Walter de Gruyter.

Frye, J. (2000), 'Collateral damage', *Risk* **13**(4), 91–94.

Gallant, A. R., Rossi, P. E. & Tauchen, G. (1992), 'Stock prices and volume', *The Review of Financial Studies* **5**(2), 199–242.

Holthausen, R. W., Leftwich, R. W. & Mayers, D. (1987), 'The effect of large block trading transactions on security prices, a cross-sectional analysis', *Journal of Financial Economics* **19**, 237–267.

Holthausen, R. W., Leftwich, R. W. & Mayers, D. (1990), 'Large-block transactions, the speed of response, and temporary and permanent stock-price effects', *Journal of Financial Economics* **26**, 71–95.

Jarrow, R. A. (2001), 'Default parameter estimation using market prices', *Financial Analysts Journal* **57**(5), 75–92.

Jarrow, R. A. & Yu, F. (2001), ‘Counterparty risk and the pricing of defaultable security’, *The Journal of Finance* **56**(5), 1765–1799.

Jarrow, R. & Protter, P. (2005), ‘Liquidity risk and risk measure computation’, *Review of Futures Markets* **11**(1), 27–39.

Keim, D. B. & Madhavan, A. (1996), ‘The upstairs market for large-block transactions: Analysis and measurement of price effects’, *The Review of Financial Studies* **9**(1), 1–36.

Kim, I., Ramaswamy, K. & Sundaresan, S. (1993), The valuation of corporate fixed income securities. Working Paper, Wharton School, University of Pennsylvania.

Lando, D. (1998), ‘On cox processes and credit risky securities’, *Review of Derivative Research* **2**, 99–120.

Leland, H. E. (1994), ‘Corporate debt value, bond covenants, and optimal capital structure’, *The Journal of Finance* **49**(4), 1213–1252.

Leland, H. E. & Toft, F. B. (1996), ‘Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads’, *The Journal of Finance* **51**(3), 987–1019.

Longstaff, F. & Schwartz, E. (1995), ‘A simple approach to valuing risky fixed and floating rate debt’, *The Journal of Finance* **50**, 789–819.

McDonald, R. & Siegel, D. (1986), ‘The value of waiting to invest’, *Quarterly Journal of Economics* **101**(4), 707–728.

McNeil, A., Frey, R. & Embrechts, P. (2005), *Quantitative Risk Management, Concepts, Technoques and Tools*, Princeton University Press.

Merton, R. (1974), ‘On the pricing of corporate debt: The risk structure of interest rates’, *The Journal of Finance* **29**(2), 449–470.

Modigliani, F. & Miller, M. H. (1958), ‘The cost of capital, corporate finance and the theory of investment’, *American Economic Review* **48**, 261–297.

Morellec, E. & Schürhoff, N. (2007), Personal taxes, leverage, and real investment.

Nowman, K. (1997), ‘Gaussian estimation of single-factor continuous time models of the term structure of interest rates’, *The Journal of Finance* **52**(4), 1695–1706.

Øksendal, B. (2003), *Stochastic Differential Equations. An Introduction with Applications*, Springer.

Sabarwal, T. (2005), 'The non-neutrality of debt in investment timing: A new npv rule', *Annals of Finance* **4**(1), 433–445.

Schulmerich, M. (2005), *Real Options Valuation. The Importance of interest Rate Modelling in Theory and Practice*, Springer.

Sundaresan, S. & Wang, N. (2007), 'Investment under uncertainty with strategic debt service', *American Economic Review* **97**(2), 256–261.

Trigeorgis, L. (1993), 'Real options and interactions with financial flexibility', *Financial Management* **22**(3), 202–224.

Trigeorgis, L. (1996), *Real Options: Managerial Flexibility and Strategy in Resource Allocation*, Cambridge, MA: MIT Press.