HOMOTOPY THEORY OF S-BIMODULES, NAIVE RING SPECTRA AND STABLE MODEL CATEGORIES

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Contents

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Introduction

The basic subject in stable homotopy theory is the investigation of the stable homotopy category and its objects. This category can be thought of as the stabilization of the homotopy category of topological spaces. But it lies in the nature of the subject that working only inside the stable homotopy category is often not sufficient. The reason basically is that one often cannot perform constructions in the stable homotopy category itself since limits and colimits do not exist in general. Therefore it is useful, if not absolutely necessary, to consider a *model* for the stable homotopy category. By a model we mean a category in which limits and colimits exist and to which we can associate a homotopy category which is equivalent to the stable homotopy category. More precisely we mean a category equipped with a *model structure* in the sense of Quillen [Qui67] such that the associated homotopy category is equivalent to the stable homotopy category. A nowadays standard model for stable homotopy theory is the category of Bousfield-Friedlander spectra [BF78]. The advantage of Bousfield-Friedlander spectra is that the objects in that category are quite simple. A spectrum in this case is just a sequence of pointed spaces $X_0, X_1, \ldots$ together with maps

$$\Sigma X_n \to X_{n+1}.$$

A map of spectra is just a map in each degree compatible with the structure maps. The disadvantage of Bousfield-Friedlander spectra is that they do not admit a symmetric monoidal smash product which models the smash product on the homotopy category. As a consequence it is not possible to consider ring spectra on the level of the model which is necessary for most algebraic constructions. To circumvent this problem several models for multiplicative stable homotopy theory have been invented. The first published category of spectra with a symmetric monoidal product the category of S-modules due to Elmendorf, Kriz, Mandell and May [EKMM97]. At roughly the same time *symmetric spectra* were invented by Smith but published later by Hovey, Shipley and Smith in [HSS00]. There are also the simplicial functors due to Lydakis and the diagram spectra from Mandell, May, Schwede and Shipley [MMSS01].

Symmetric spectra are probably the model for spectra with smash product which is the closest to Bousfield-Friedlander spectra. A symmetric spectrum is a Bousfield-Friedlander spectrum X such that the nth symmetric group Σ_n acts on X_n and such that the structure maps are required to satisfy certain equivariance conditions with respect to symmetric group actions. Both symmetric and Bousfield-Friedlander spectra can be described as modules over the *sphere spectrum* S which consists in degree n of the n-sphere S^n. In the symmetric case S is a commutative monoid, but in the case of Bousfield-Friedlander S is only an associative monoid. As consequence, as in the case of modules over an ordinary ring, symmetric spectra can be considered as left and right modules over the commutative sphere spectrum and hence admit a "tensor product" over S whereas this is not possible for Bousfield-Friedlander spectra. Note that the crucial property to define a tensor or smash product over S is not the commutativity of S but to have a left and a right module over S. Therefore the category of bimodules over the associative sphere spectrum S has a monoidal, though not symmetric, smash product over S. In the first half of this text we study the homotopy theory of S-bimodules and its monoidal product. But before we start elaborating along these lines We recall the theory of model categories, Bousfield-Friedlander spectra and symmetric spectra in Section 1.

We discuss S-bimodules and their stable homotopy theory in Section 2. The two main results are that the category of S-bimodules admits a stable model structure with weak equivalences π_*-isomorphisms of underlying left S-modules (*i.e.* left

Bousfield-Friedlander spectrum), see Theorem 2.2.4. The other result is Theorem 2.3.3 where we show that S-bimodules are Quillen equivalent to symmetric spectra together with an endomorphism. This makes precise the way we actually think about S-bimodules: They are left S-modules together with a right action of S which up to homotopy correspond to an endomorphism of left S-bimodules. We make emphasize that we therefore do not get a new model for stable homotopy theory.

The monoidal structure on S-bimodules suffices to consider monoids in S-bimodules which we call *naive ring spectra*. The reason we call them "naive" is that it turns out that explicitly to have an naive ring spectrum means to have a graded space R_n, $n \in \mathbb{N}$, together with unital and associative multiplication maps

$$R_p \wedge R_q \to R_{p+q}$$

and a unit map

$$S^1 \to R_1$$

without any further assumptions.

Having established the model structures on S-bimodules we would like to lift them to the category of monoids *i.e.* naive ring spectra. Unfortunately the model structures we consider are not monoidal in the sense of [Hov99, 4.2.6] and so we cannot apply the results from [SS00]. One could try to lift the model structure anyway but for our purposes it suffices to prove enough homotopical properties to replace a naive ring spectrum R by a symmetric ring spectrum R' whose underlying naive ring spectrum has the same stable homotopy type as R. This result appears as Theorem 3.3.15 in Section 3 where the homotopy theory of naive ring spectra is developed. There we also show that the category of modules over a naive ring spectrum R admits a stable model structure under the assumption that R is *right stable*, *i.e.* the right action of the underlying S-bimodule induces an isomorphism on homotopy groups (see Theorem 3.2.1). As expected it is then true that the category R-mod of modules over the naive ring spectrum R is Quillen equivalent to the category R'-mod$_\Sigma$ of modules over the symmetric ring spectrum R'. Therefore all homotopy theory of modules over a symmetric ring spectrum can also be done with modules over naive ring spectra. We should note that in Section 3 we also establish an additional model structure on S-bimodules in which the weak equivalences are the maps inducing isomorphisms on left and right homotopy groups.

We note again that we cannot consider commutative monoids in S-bimodules. But for many algebraic considerations it suffices to consider associative monoids. This is in particular true for Morita theory. The actual motivation to consider S-bimodules and naive ring spectra comes from the idea to generalize the following Morita like theorem due to Gabriel. Let $\mathcal{A}$ be an abelian category with a small projective generator P. Then the canonical functor

$$\mathrm{Hom}(P, -)\colon\ \mathcal{A} \to \mathrm{End}(P)\text{-mod}$$

sending an object X to the abelian group $\mathrm{Hom}(P, X)$ of morphisms from P to X in $\mathcal{A}$ which is a module over $\mathrm{End}(P) = \mathrm{Hom}(P, P)$, is an equivalence of categories. We prove the following analog to the result above. Let $\mathcal{C}$ be any stable model category such that the homotopy category $\mathrm{Ho}(\mathcal{C})$ has a compact generator X. Then there exists a (right stable) naive ring spectrum $\mathrm{End}(X)$ together with a Quillen equivalence

$$X \wedge_{\mathrm{End}(X)} (-)\colon\ \mathrm{End}(X)\text{-mod} \rightleftarrows \mathcal{C}\ :\mathrm{Hom}(X, -).$$

In particular there is an equivalence

$$\mathrm{Ho}(\mathcal{C}) \simeq \mathrm{Ho}(\mathrm{End}(X)\text{-mod})$$

of triangulated categories. This is all done in Section 4. First we prove an abstract Morita theorem, 4.1.4 which states that given an adjunction as above, then we already have an equivalence provided X is a compact generator. The major part of work is to ensure the existence of such Quillen adjunctions and in particular the existence of $\mathrm{End}(X)$.

We give constructions of $\mathrm{End}(X)$ and the adjoint functor pair in a axiomatic setting starting with a given so called *desuspension cospectrum* of X (see Definition 4.2.2 for details). Then we prove that the naive ring spectrum $\mathrm{End}(X)$ is right stable, or more precisely has an underlying S-module which is a left and right Ω-spectrum (Proposition 4.2.11). Afterwards we ensure the existence of desuspension cospectra from which $\mathrm{End}(X)$ and the adjoint pair are built. First we treat the case of a simplicial stable model category in Theorem 4.4.2 and afterwards the general case in Theorem 4.6.2 for where we need the theory of cosimplicial frames which is also developed there.

We would like to mention that, as off-spin of our theory, we obtain a new proof that Bousfield-Friedlander spectra are Quillen equivalent to symmetric spectra. This is a special case of Corollary 4.1.5.

1. Preliminaries

In this section we recall standard results from model category theory and the theory of sequential and symmetric spectra. We state the most important results we will be using throughout this work.

1.1. Model categories.

In this section we recall the most important facts from the theory of model categories we need. For details we refer to the now standard text books [Hov99] and [Hir03]. For definitiveness we give the definition of a model category we use. Note that this definition differs from that given in [Hov99] in that we do not require functorial factorizations.

Definition 1.1.1. A *model category* is a bicomplete category $\mathcal{C}$ together with three subcategories

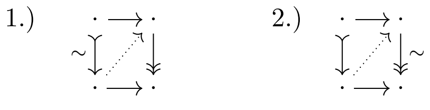
$$\mathrm{w}\,\mathcal{C}, \quad \mathrm{cof}\,\mathcal{C} \quad \text{and} \quad \mathrm{fib}\,\mathcal{C}$$

each containing all identities and whose morphisms are called *weak equivalences*, *cofibrations* and *fibrations* respectively and will be denoted by

$$\xrightarrow{\sim}\ , \qquad \rightarrowtail \qquad \text{and} \qquad \twoheadrightarrow$$

respectively, such that the following conditions hold:

MC1 If for two morphisms $f, g \in \mathcal{C}$ two of f, g, fg are in $\mathrm{w}\,\mathcal{C}$ then so is the third.
MC2 The three categories $\mathrm{w}\,\mathcal{C}$, $\mathrm{cof}\,\mathcal{C}$ and $\mathrm{fib}\,\mathcal{C}$ are closed under retracts as sub categories of $\mathrm{Ar}\mathcal{C}$ (the category of arrows of $\mathcal{C}$).
MC3 There exist lifts in the following two situations:



MC5 Every map f in $\mathcal{C}$ can be factored in two ways:

A model category is *pointed* if the initial and terminal object coincide.

For a pointed model category we can define the suspension ΣX of an object X by functorially factorizing the canonical map $X \vee X \to X$ into a cofibration followed by an acyclic fibration,

$$X \rightarrowtail CX \overset{\sim}{\twoheadrightarrow} X,$$

and taking the cofiber of the first map:

$$X \rightarrowtail CX \longrightarrow \Sigma X.$$

There is a dual construction yielding the *loop object* ΩX of X. One can check that these constructions give rise to an adjoint functor pair on the homotopy category $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$. Note that we do not have an adjoint pair on $\mathcal{C}$ itself nor are Σ and Ω functorial constructions.

Definition 1.1.2. A pointed model category $\mathcal{C}$ is *stable*, if the suspension functor on the homotopy category is an equivalence of categories.

Theorem 1.1.3. *The homotopy category of a stable model category is triangulated. A Quillen pair $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$ between stable model categories $\mathcal{C}$ and $\mathcal{D}$ induces an adjoint pair of exact functors of triangulated homotopy categories.*

Proof. The shift functor is given by the suspension functor given above. For a cofibration $i\colon A \to B$ between cofibrant objects in $\mathcal{C}$ define $B \to C(i)$ by the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ CA & \longrightarrow & C(i) \end{array}$$

where $A \to CA$ comes from a factorization of $* \to A$ into a cofibration followed by a weak equivalence. Taking the cofiber of $B \to C(i)$ yields a map $B \to \Sigma A$. We define the exact triangles to be those sequences isomorphic in the homotopy category the sequences

$$A \xrightarrow{i} B \to C(i) \to \Sigma A$$

as defined above. Dually one can define fiber sequences and it turns out that in the homotopy category cofiber and fiber sequences coincide up to a sign.

As a left Quillen functor (in particular left adjoint) F preserves all structure involved in the construction of cofiber sequences, so that F preserves suspension and cofiber sequences. For G the dual arguments applies to fiber sequences. The For more details we refer to [Hov99, Chapter 7]. □

Of course it is often desirable to have the suspension not only on the homotopy category but already on the model category. The most important class of model categories having this feature are the *simplicial model* categories. Recall that pointed simplicial sets together with the smash product of simplicial sets carry the structure of a symmetric monoidal model category (see *e.g.* [GJ99, I.11]). A *simplicial model category* is a pointed model category which is tensored, cotensored and enriched over pointed simplicial sets in homotopically compatible way. For a pointed simplicial set K and an objects X and Y in $\mathcal{C}$ we denote the tensor by $K \wedge X$, the cotensor by Y^K and the enrichment by $\mathrm{map}_{\mathcal{C}}(X, Y)$ which is a simplicial set and will be referred to as the *simplicial mapping space* of X and Y.

Lemma 1.1.4. *For a cofibrant object X in a simplicial model category $\mathcal{C}$ the object $S^1 \wedge X$ is a model for the suspension of X,* i.e. *there is an isomorphism*

$$\Sigma X \cong S^1 \wedge X$$

in the homotopy category of $\mathcal{C}$. □

Definition 1.1.5. Let $f\colon X \to Y$ be a map in a pointed simplicial model category $\mathcal{C}$ and $g\colon K \to L$ a map of pointed simplicial sets. The *pushout product* $f \square g$ is defined to be the map

$$f \square g\colon X \wedge L \cup_{X \wedge K} Y \wedge K \to Y \wedge L$$

The compatibility of the models structures is expressed in the following proposition we refer to as the *pushout product axiom.*

Proposition 1.1.6. *Let $f\colon X \to Y$ be a cofibration in a pointed simplicial model category $\mathcal{C}$ and $g\colon K \to L$ a cofibration of pointed simplicial sets. Then the pushout product $f \square g$ is a cofibration which is acyclic if f or g is.*

A proof can be found in [Hov99]. We finish this section with some useful results from simplicial model categories.

Lemma 1.1.7. *Let $A \to B$ be a map between cofibrant objects in a pointed simplicial model category. Then the inclusion $\iota\colon A \to \mathrm{Cyl}(f)$ of A into the mapping cylinder $\mathrm{Cyl}(f)$ of f is a cofibration*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} & & A \vee A & \xrightarrow{i_0 \vee i_1} & A \wedge \Delta^1_+ \\ & & \downarrow{\scriptstyle f \vee \mathrm{id}} & & \downarrow \\ A & \xrightarrow[i_1]{} & B \vee A & \longrightarrow & \mathrm{Cyl}(f) \end{array}$$

in which the square is a pushout. The top horizontal map is a cofibration since A is assumed to be cofibrant and the left bottom horizontal map is a cofibration since B is also assumed to be cofibrant. The right bottom horizontal map is a cofibration as the cobase change of the top horizontal map. Hence the bottom horizontal composition is a cofibration. But this is the map in question. □

Finally we mention *simplicial mapping spaces.* For A, B in a simplicial model category, we can define a simplicial set $\mathrm{map}_{\mathcal{C}}(A, B)$ given in degree n by

$$\mathrm{map}_{\mathcal{C}}(A, B)_n = \mathrm{map}_{\mathcal{C}}(A \wedge \Delta^n_+, B)$$

called the simplicial mapping space of maps from A to B. We have the following useful lemma.

Lemma 1.1.8. *If A is cofibrant and B fibrant, then $\mathrm{map}(A, B)$ is fibrant as simplicial set and there is a natural isomorphism*

$$\pi_0 \,\mathrm{map}(A, B) \cong [A, B]^{\mathcal{C}},$$

where $[-, -]^{\mathcal{C}}$ denotes morphisms in the homotopy category of $\mathcal{C}$.

1.2. **Spectra.** In this section we recall the most important results form Bousfield-Friedlander spectra which will we used throughout this text.

Definition 1.2.1. A *sequential spectrum* X is a sequence of (pointed) simplicial sets $X_n, n \geq 0$ in $\mathcal{C}$ together with maps $\sigma_n\colon S^1 \wedge X_n \to X_{n+1}$. A map $f\colon X \to Y$ of spectra is a sequence of maps $f_n\colon X_n \to Y_n$ such that the diagram

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\Sigma f_n} & \Sigma Y_n \\ \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_n} \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

commutes. We denote the category of sequential spectra by $\mathcal{S}p$.

For a simplicial set K we can define the homotopy groups of K to be either the homotopy groups $\pi_n|K|$ of the geometric realization or as the simplicial homotopy groups $\pi_n K^f$ of an fibrant replacement in the category of simplicial sets (see *e.g.* [Hov99, Chapter 3] for a discussion of the model structure on simplicial sets). We remark that in the model category of simplicial set there is a functorial fibrant replacement available. We will just write $\pi_n K$ for the rest in this work.

Definition 1.2.2. Let X be a sequential spectrum. For any integer q define the qth homotopy group $\pi_q X$ of X to be the colimit of the sequence

$$\cdots \to \pi_{q+k}X_k \xrightarrow{S^1\wedge(-)} \pi_{q+k+1}(S^1 \wedge X_k) \xrightarrow{(\sigma_k)_*} \pi_{q+k+1}X_{q+k+1} \to \cdots .$$

Definition 1.2.3. Let X be a spectrum. We define the *suspension spectrum* ΣX by $(\Sigma X)_n = X_n \wedge S^1$ with structure maps by applying $(-) \wedge S^1$ to the structure maps of X.

Further, we define the m-th *shift spectrum* $\operatorname{sh}_m X$ by $(\operatorname{sh}_m X)_n = X_{n+m}$ and structure maps $\sigma_n^{\operatorname{sh}_m X} = \sigma_{n+m}^X$.

Finally, denote by F_m the left adjoint to the evaluation at n functor which sends a spectrum X to its mth space. In degree n we have $(F_m K)_n = S^{n-m} \wedge K$ for $n \geq m$ and $*$ else.

Before we state the stable model structure, we give the *level model* structure. Define a map between spectra to be a weak equivalence respectively a fibration if it is one of simplicial sets in each degree and a cofibration if it has the left lifting property with respect to all maps which are acyclic fibrations of simplicial sets in each degree. This equips the category of sequential spectra with a model structure we refer to as the *level model structure.* We call weak equivalences and fibrations in this model structure *level equivalences* and *level fibrations* respectively. Cofibrations in the level structure are referred to as *projective cofibrations.* Note that the level structure is cofibrantly generated with generating acyclic cofibrations the maps

$$F_m\Lambda^n_{k+} \to F_m\Delta^n_+$$

and generating cofibrations

$$F_m\partial\Delta^n_+ \to F_m\Delta^n_+$$

with $n, m \geq 0$, $n \geq k \geq 0$ and where $\Lambda^n_{k+} \to \Delta^n_+$ and $\partial\Delta^n_+ \to \Delta^n_+$ are the usual inclusions of simplicial sets.

A proof of the following fibration criterion can be found in [GJ99, Chapter X.4].

Theorem 1.2.4. *The category of sequential spectra admits a proper stable simplicial model structure in which a map $f\colon X \to Y$ is weak equivalences if it is a π_*-isomorphisms, a cofibration if is one in the level structure and a fibration if it is a fibration in each degree and for each $n \geq 0$ the diagram*

$$\begin{array}{ccc} X_n & \longrightarrow & \Omega X_{n+1} \\ \Big\downarrow{\scriptstyle f_n} & & \Big\downarrow{\scriptstyle \Omega f_n} \\ Y_n & \longrightarrow & \Omega Y_{n+1} \end{array}$$

is a homotopy pullback diagram.

Since we have a stable model category the homotopy category is triangulated and we have cofiber sequences inducing long exact sequences on homotopy groups. But the situation is actually a little better in that we do not require a cofibration but just a map which is injective in each degree.

Lemma 1.2.5. *Let $f\colon A \to B$ be a map of spectra which is injective in each degree. Then there is a long exact sequence*

$$\cdots \to \pi_n(A) \xrightarrow{f_*} \pi_n(B) \xrightarrow{p_*} \pi_n(B/A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots,$$

where p is the projection onto the quotient.

Proof. The claim follows from [HSS00, Lemma 3.1.13]. There it is proved for symmetric spectra, but the proof only depends on the underlying sequential spectra. □

Note that X is a fibrant spectrum if and only if X is fibrant in each degree and the adjoints of the structure maps

$$X_n \to \Omega X_{n+1}$$

are weak equivalences.

Definition 1.2.6. A spectrum X is an Ω-*spectrum* if it is fibrant in each degree and the adjoints of the strut rue maps are weak equivalences.

Note that in case of an Ω-spectrum X there is a natural isomorphism

$$\pi_{n-m} X \cong \pi_n X_m$$

for all $m, n \geq 0$. From this we have the

Lemma 1.2.7. *A map between Ω-spectra is a π_*-isomorphism if and only if it is a level equivalence.*

We finish this section with a discussion of generating (acyclic) cofibrations for the stable model structure on spectra.

Definition 1.2.8. Define $\lambda_n\colon F_{n+1}S^1 \to F_n S^0$ to be the adjoint to the identity $S^1 \to (F_n S^0)_{n+1}$. Factor this map $\lambda_n = r_n \circ c_n$ into an inclusion followed by a simplicial homotopy equivalence using the simplicial mapping cylinder construction.

Denote by F_n the left adjoint of the functor which takes a spectrum to its nth space.

Proposition 1.2.9. *The stable model structure on $\mathcal{S}p$ is cofibrantly generated with generating cofibrations given by the maps*

$$F_n \partial \Delta^m_+ \to F_n \Delta^m_+$$

and acyclic cofibrations the maps

$$F_n \Lambda^m_{k+} \to F_n \Delta^m_+ \quad \text{and} \quad c_n \,\square\, \iota_m,$$

where ι_m denotes the inclusion $\partial\Delta^m \to \Delta^m$.

For a proof see [Sch01, Lemma A.3].

1.3. **Symmetric Spectra.** In this section we quickly recall the most important things from the theory of symmetric spectra. We emphasize the construction of the symmetric monoidal smash product since this is conceptually most important for our understanding of the differences between sequential spectra, S-bimodules and symmetric spectra.

Definition 1.3.1. Let Σ be the category with objects finite sets $\overline{n} = \{1, \ldots, n\}$ and morphisms $\Sigma(\overline{n}, \overline{n}) = \Sigma_n$ and $\Sigma(\overline{m}, \overline{n}) = \emptyset$, if $m \neq n$. Here Σ_n denotes the symmetric group on n letters. If $\mathcal{C}$ is any category then $\mathcal{S}_*^\Sigma$ denotes the category of functors $\Sigma \to \mathcal{S}_*$ with morphisms natural transformations of functors. We call the objects in $\mathcal{S}_*^\Sigma$ *symmetric sequences*. By abuse of language we will write n instead of $\overline{n}$ and X_n instead of $X(\overline{n})$ for $X \in \mathcal{S}_*^\Sigma$.

Lemma 1.3.2. *The category Σ is symmetric monoidal.*

Proof. For objects $\overline{n}$ and $\overline{m}$ in Σ define their product $\overline{n} \square \overline{m}$ to be $\overline{m+n}$. The unit is $\overline{0}$ and all structure isomorphisms are given by identities except the twist isomorphism $\tau \colon \overline{p} \square \overline{q} \to \overline{q} \square \overline{p}$ which is given by block sum permutation $\chi_{p,q}$, *i.e.* $\chi_{p,q}(i) = i + q$ for $1 \leq i \leq p$ and $\chi_{p,q}(i) = i - p$ for $p < i \leq q$. $\square$

Now the point is, that $\mathcal{S}_*^\Sigma$ inherits a closed symmetric monoidal structure from Σ and $\mathcal{S}_*$.

Definition 1.3.3. For two symmetric sequences X and Y define their product $X \otimes Y$ by

$$(X \otimes Y)_n = \bigvee_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)$$

with the obvious action of Σ_n in degree n.

Definition 1.3.4. For $n \geq 0$ the functor $\mathrm{Ev}_n \colon \mathcal{S}_*^\Sigma \to \mathcal{S}_*$ evaluation at n is the functor which takes a symmetric sequence X to X_n. The evaluation functor has an adjoint $G_n \colon \mathcal{S}_* \to \mathcal{S}_*^\Sigma$ given by $G_n A = \Sigma(n, -)_+ \wedge A$, *i.e.* $(G_n A)_k = \Sigma_{n+} \wedge A$ for $k = n$ and $*$ else.

Now we can give the definition of a symmetric spectrum. First note that the free commutative monoid $\mathrm{Sym}(K)$ (*i.e.* the symmetric algebra) generated by the symmetric sequence $(*, K, *, \dots)$ in $\mathcal{S}_*^\Sigma$ for any object $K \in \mathcal{S}_*$ is given by $(S, K, K^{\wedge 2}, \dots, K^{\wedge n}, \dots)$, where Σ_n acts by permutation of factors off $K^{\wedge n}$. Now specialize to the case $K = S^1$ and denote $\mathrm{Sym}(S^1)$ by $\mathbb{S}$. Note that we have $\mathbb{S}_n = S^n$.

Definition 1.3.5. A *symmetric spectrum* X is a (left) module in $\mathcal{S}_*^\Sigma$ over the free commutative monoid $\mathbb{S} = \mathrm{Sym}(S^1)$ in $\mathcal{S}_*^\Sigma$. The category of symmetric spectra (*i.e.* $\mathbb{S}$-modules) is denoted by Sp^Σ.

Theorem 1.3.6. *The category of symmetric spectra is a closed symmetric monoidal category.*

Proof. This is completely formal and works as in case of the tensor product of modules over a commutative ring. Suppose $\mathcal{M}$ is the category of (left) modules over a commutative monoid in a closed symmetric monoidal category $\mathcal{C}$ with product $\wedge$ and unit S. For modules M and N in $\mathcal{M}$ with products μ and λ define their product $M \wedge_R N$ by the coequalizer displayed in the diagram

$$M \wedge R \wedge N \underset{\mu\tau \wedge N}{\overset{M \wedge \lambda}{\rightrightarrows}} M \wedge N \longrightarrow M \wedge_R N.$$

Dually define the function R-module $\mathrm{Hom}_R(M,N)$ by the equalizer displayed in the diagram

$$\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M,N) \underset{\lambda_*}{\overset{\mu^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{C}}(R \wedge M, N),$$

where the bottom map is actually given by the composite

$$\mathrm{Hom}_{\mathcal{C}}(M,N) \xrightarrow{R \wedge (-)} \mathrm{Hom}_{\mathcal{C}}(R \wedge M, R \wedge N) \xrightarrow{\lambda_*} \mathrm{Hom}_{\mathcal{C}}(R \wedge M, N)$$

rather than by λ_* itself. We omit checking the remaining details for a closed symmetric monoidal category. $\square$

Explicitly a symmetric spectrum X is a sequential spectrum such that Σ_n acts on X_n and the "composed structure maps"

$$S^p \wedge X_q \to X_{p+q}$$

are $\Sigma_p \times \Sigma_q$-equivariant. So by forgetting the symmetric group actions we obtain a sequential spectrum. For example the underlying sequential spectrum of $\mathbb{S}$ is S.

We define the *level model structure* on symmetric spectra analogous to the case of sequential spectra. So a map is a weak equivalence, fibration if it is one of underlying sequential spectra. Similarly a symmetric spectrum is an Ω-spectrum if the underlying sequential spectrum is.

We would like to establish a stable model structure for symmetric spectra. But it turns out, that we cannot take π_*-isomorphisms as weak equivalences for the following reason. Let F_n denote the left adjoint to the functor which takes a symmetric spectrum to its nth space. We can then take the the adjoint $F_1S^1 \to F_0S^0$ of the identity $S^1 \to (F_0S^0)_1$ corresponding to the maps of Definition 1.2.8. But the symmetric group actions which have to be build into F_1S^1 "blows up" the spectrum and as a result the map $F_1S^1 \to \mathbb{S}$ is not a π_*-isomorphism (see [HSS00, Example 3.1.10]).

The spirit of the following definition is that we want to have the Ω-spectra as the fibrant objects.

Definition 1.3.7. A map $f\colon X \to Y$ is a *stable equivalence* if for every symmetric Ω-spectrum E the induced map

$$f^*\colon [Y,E]^{\text{level}} \to [X,E]^{\text{level}}$$

on morphism sets in the homotopy category of the level model structure is a bijection.

We remark that this is not the original definition given in [HSS00] but is taken from [MMSS01]. Of course both notions are equivalent.

Theorem 1.3.8. *The category Sp^Σ of symmetric spectra admits a stable simplicial cofibrantly generated model category structure in which the weak equivalences are the stable equivalences and the cofibrations the cofibrations from the level model structure. Fibrations are defined to be the maps which have the right lifting property with respect to the acyclic fibrations.*

See [HSS00, Theorem 3.4.4] for a proof.

Definition 1.3.9. A *symmetric ring spectrum* is a monoid in category Sp^Σ of symmetric spectra.

Given a symmetric ring spectrum R, one can consider modules over it.

Definition 1.3.10. Let R be a symmetric ring spectrum. A (left) *symmetric R-module spectrum* is a symmetric spectrum M together with an (left) action of R, *i.e.* a map $R \wedge M \to M$ of symmetric spectra, such that the usual diagrams commute (see [ML98, Chapter VII.4]). We denote the category of (left) modules over a symmetric spectrum by $R\text{-mod}_\Sigma$.

We now equip, for a given symmetric ring spectrum R, the category R-mod with a model structure. By lifting the stable model structure an symmetric along the adjunction

$$R \wedge (-)\colon\ Sp^\Sigma \rightleftarrows R\text{-mod}_\Sigma\ :U,$$

where U the forgetful functor.

Theorem 1.3.11. *Let R be a symmetric ring spectrum. The category of (left) module spectra over R admits the structure of a cofibrantly generated stable simplicial model category in which a map f is a weak equivalence or fibration if and only if the underlying map of symmetric spectra is one, and a cofibration if and only if it has the left lifting property with respect to acyclic fibrations.*

See [HSS00, Corollary 5.4.2] for a proof.

2. S-bimodules

In the last section we have seen that the definition of spectra as modules over a commutative monoid allows one to obtain a symmetric monoidal product for spectra. The price we had to pay was that we had to introduce actions of the symmetric groups in order to make the sphere spectrum a commutative monoid in an appropriate category. For most purposes this is no problem. A lot of spectra occurring in nature have these actions right away like for example the complex cobordism spectrum MU. Other spectra like Eilenberg-MacLane spectra have canonical models as symmetric spectra. But we are interested in the construction of an endomorphism ring spectrum for a given object in a stable model category. In this case it is a priori absolutely unclear how to get the symmetric groups acting on the spectrum one tries to construct. One solution to this problem is based on the observation that the induced smash product of spectra does not really depend on the fact that one has modules over a commutative monoid. The crucial point is that a left module over a commutative monoid admits a right action. So instead of taking modules over a commutative monoid we take the approach to use bimodules over an only associative monoid and will end up with a category which is monoidal though not symmetric monoidal. But this still suffices to define monoids.

2.1. Graded spaces and S-bimodules.

We now define the category of graded spaces and introduce a symmetric monoidal structure on it.

Definition 2.1.1. A *graded space* is a sequence X_n, $n \geq 0$, of pointed simplicial sets. A *map of graded spaces* between graded spaces X and Y is a is a simply a sequence of maps $f\colon X_n \to Y_n, n \geq 0$ of simplicial sets. We denote the category of graded spaces by $\mathrm{gr}\mathcal{S}_*$.

Note that the category $\mathrm{gr}\mathcal{S}_*$ is a bicomplete category where (co)limits are computed point-wise. Further, there is a the following monoidal structure on $\mathrm{gr}\mathcal{S}_*$.

Definition 2.1.2. For graded spaces X and Y we define their tensor product $X \otimes Y$ in degree n by

$$(X \otimes Y)_n = \bigvee_{p+q=n} X_p \wedge Y_q.$$

With this product the category of graded spaces becomes a symmetric monoidal category with unit object $(S^0, *, *, \dots)$. The associativity and twist isomorphisms are induced by the associativity and twist isomorphism of the monoidal structure of simplicial sets.

Example 2.1.3. The sequence of spheres $S^0, S^1, \dots, S^n, \dots$, which we denote by S, is given in degree n by the n-sphere. The natural isomorphisms $S^p \wedge S^q \cong S^{p+q}$ for $p, q \geq 0$ induce a map $S \otimes S \to S$ of graded spaces and endow S with the structure of an unital and associative monoid in graded spaces. We call S the *associative sphere spectrum*. Note that S is definitively not commutative for this would imply that the twist isomorphism $S^1 \wedge S^1 \to S^1 \wedge S^1$ is the identity which it is of course not.

Definition 2.1.4. Let K be a pointed simplicial set. We denote by $K[n]$ the graded space with K in degree n and $*$ else. This construction is functorial and left adjoint to the nth evaluation functor which sends a graded space X to the simplicial set X_n.

Definition 2.1.5. A map $f\colon X \to Y$ of graded spaces is called a *fibration* (respectively *weak equivalence*) if each f_n is a fibration (resp. weak equivalence) of simplicial sets. Further, f is a cofibration, if it has the the left lifting property with respect to all maps which are fibrations and weak equivalences.

With the definition above graded spaces becomes a model category. We refer to it as the *level model structure* and call a cofibration in the level structure a *projective cofibration.*

Now that we have defined a symmetric monoidal structure on graded spaces, we can consider monoids with respect to this structure. We already considered the associative sphere spectrum S in Example 2.1.3. To see that have we in fact an associative monoid note that S is actually the free associative monoid on the graded space $S^1[1] = (*, S^1, *, *, \dots)$, *i.e.* the tensor algebra on $S^1[1]$ with respect to $\otimes$.

Definition 2.1.6. Let R be an associative monoid in graded spaces. A *left module* M over R is a graded space M together with a map $R \otimes M \to M$ satisfying the usual associativity and unitality diagrams for a module. A *right module* over R is defined similarly.

Example 2.1.7. Consider the sphere spectrum S and a left module X over it. Having an action map $S \otimes X \to X$ means to have in degree n associative maps $S^p \wedge X_q \to X_{p+q}$ for all $p, q \geq 0$ with $p + q = n$. The associativity of these maps implies that it suffices to have the maps $S^1 \wedge X_n \to X_{n+1}$ which in turn means to have a sequential spectrum. We conclude that sequential spectra and left S-modules are isomorphic categories.

If we have a commutative monoid R we can, using the twist isomorphisms of the monoidal structure, view a left module M also as a right module and define a tensor product $M \otimes_R N$ of a two left modules M and N over R by the coequalizer

$$M \otimes R \otimes N \underset{\rho\otimes N}{\overset{M\otimes\lambda}{\rightrightarrows}} M \otimes N \longrightarrow M \otimes_R N$$

where we used the right modules structure ρ induced from the left module structure of M. The resulting object is a left R-module in a canonical way. If R is not commutative we still can apply the above construction given that M is a right and N a left module over R. But the resulting object will be neither a left R-module nor a right R-module. A way to circumvent this problem is to consider bimodules.

Definition 2.1.8. Let R be an associative monoid in $\mathrm{gr}\mathcal{S}_*$. A *bimodule M over R* is a graded space M together with a left and a right action map $\lambda\colon R \otimes M \to M$ and $\rho\colon M \otimes R \to M$ which, in addition to being associative and unital, commute with each other, *i.e.* the diagram

$$\begin{array}{ccc} R \otimes M \otimes R & \xrightarrow{\mathrm{id}\otimes\rho} & R \otimes M \\ \downarrow{\scriptstyle \lambda\otimes\mathrm{id}} & & \downarrow{\scriptstyle \lambda} \\ M \otimes R & \xrightarrow{\rho} & M \end{array}$$

is commutative.

We also have a level model structure for modules over a monoid in graded spaces. We omit the proof and refer to [BF78] where it is proved for the case of modules over S but the proof easily generalizes to modules over a different monoid.

Proposition 2.1.9. *Let R be a monoid in graded spaces. The category of (left) R-modules admits a cofibrantly generated simplicial model structure in which a map is a weak equivalence or fibration if it a weak equivalence or fibration respectively in each degree. Cofibrations are defined to be those maps which have the left lifting property with respect to maps which are acyclic fibrations in each degree. Generating acyclic cofibrations are the maps*

$$R \otimes (\Lambda^n_{k+})[m] \to R \otimes (\Delta^n_+)[m]$$

and generating cofibrations are

$$R \otimes (\partial\Delta^n_+)[m] \to R \otimes (\Delta^n_+)[m]$$

with $n, m \geq 0$ and $0 \leq k \leq n$.

Note that the model structure above also applies to bimodules as an R-bimodule is the same as a left module over $R \otimes R^{\mathrm{op}}$.

Having two bimodules over the same monoid, we can form their tensor product. Since we are mainly interested in bimodules over the sphere spectrum we make the

Definition 2.1.10. Let X and Y be S-bimodules. Define their tensor product $X \otimes_S Y$ by the coequalizer

$$X \otimes S \otimes Y \overset{X\otimes\lambda}{\underset{\rho\otimes Y}{\rightrightarrows}} X \otimes Y \longrightarrow X \otimes_S Y,$$

where $\rho\colon X \otimes S \to X$ and $\lambda\colon S \otimes Y \to Y$ denote the right and left action of S an X and Y respectively.

Note that a left S-module is exactly the same thing as a sequential spectrum in the sense of Definition 1.2.1. Of course a right S-module is also a sequential spectrum though a right sequential spectrum. Having a S-bimodule X means that we have a left and a right spectrum structure on X and we can form left and right homotopy groups associated to X.

Definition 2.1.11. Let X be a S-bimodule. Define for any integer q the qth left and right homotopy groups $\pi^{\mathrm{L}}_q X$ and $\pi^{\mathrm{R}}_q X$ by taking the colimits over the sequences

$$\cdots \to \pi_{q+n}(X_n) \xrightarrow{S^1\wedge(-)} \pi_{q+n+1}(S^1 \wedge X_n) \xrightarrow{(\lambda_n)_*} \pi_{q+n+1}(X_{n+1}) \to \cdots$$

and

$$\cdots \to \pi_{q+n}(X_n) \xrightarrow{(-)\wedge S^1} \pi_{q+n+1}(X_n \wedge S^1) \xrightarrow{(\rho_n)_*} \pi_{q+n+1}(X_{n+1}) \to \cdots$$

respectively.

To have a bimodule X over S explicitly just means that we have commutative diagrams

$$\begin{array}{ccc} S^1 \wedge X_n \wedge S^1 & \xrightarrow{\mathrm{id}\wedge\rho_n} & S^1 \wedge X_{n+1} \\ \downarrow{\scriptstyle \lambda_n\wedge\mathrm{id}} & & \downarrow{\scriptstyle \lambda_{n+1}} \\ X_{n+1} \wedge S^1 & \xrightarrow{\rho_{n+1}} & X_{n+2} \end{array}$$

for every $n \geq 1$. Denote the induced maps from the left and right action by

$$L\colon\ \pi_k X_i \to \pi_{k+1} X_{i+1} \quad \text{and} \quad R\colon\ \pi_k X_i \to \pi_{k+1} X_{i+1}$$

respectively. These maps are compatible in the sense that $RL = LR$ because of the commutative diagram above. Therefore we obtain induced endomorphisms $L\colon \pi^{\mathrm{R}}_* X \to \pi^{\mathrm{R}}_* X$ and $R\colon \pi^{\mathrm{L}}_* X \to \pi^{\mathrm{L}}_* X$ of right and left homotopy groups respectively.

More generally there is a functor

$$\Pi_q X\colon \mathbb{N}\times\mathbb{N}\to \mathcal{A}b$$

for every q, where we use the partial order to endow $\mathbb{N}$ with the structures of a category. It sends (k,l) to $\pi_{q+k+l}X_{k+l}$ and a morphism $(k_1\le k_2, l_1\le l_2)$ to the map

$$L^{k_2-k_1}R^{l_2-l_1}\colon\ \pi_{q+k_1+l_1}X_{k_1+l_1}\to \pi_{q+k_2+l_2}X_{k_2+l_2}$$

induced by the maps left and right structure maps of X. We recover the left and right homotopy groups by taking the colimit over the subcategories $\mathbb{N}\times 0$ and $0\times\mathbb{N}$ respectively. But of course we can also take the colimit over the whole category.

Definition 2.1.12. The *total homotopy groups* $\pi_*^{\mathrm{T}}X$ of a S-bimodule X are defined by taking the colimit of the functor $\Pi_* X\colon \mathbb{N}\times\mathbb{N}\to\mathcal{A}b$:

$$\pi_*^{\mathrm{T}}X = \mathrm{colim}_{\mathbb{N}\times\mathbb{N}}\,\Pi_* X.$$

The left and right action of a S-bimodule also induce an action on the total homotopy groups. If $\bar{x}$ is a class at the spot k,l so an element in $\pi_{q+k+l}X_{k+l}$ represented by x, we can take the class represented by x at the spot $l+1, k-1$ or at the spot $l-1, k+1$ and obtain endomorphisms $L\colon \pi_q^{\mathrm{T}}X\to\pi_q^{\mathrm{T}}X$ and $R\colon \pi_q^{\mathrm{T}}X\to\pi_q^{\mathrm{T}}X$ respectively of total homotopy groups. We see immediately from the definitions that $LR=\mathrm{id}=RL$, so L and R are automorphisms of total homotopy groups.

The inclusions of $\mathbb{N}\times 0$ and $0\times\mathbb{N}$ into $\mathbb{N}\times\mathbb{N}$ induce maps $\iota\colon \pi_*^{\mathrm{L}}X\to\pi_*^{\mathrm{T}}X$ and $\iota\colon \pi_*^{\mathrm{R}}X\to\pi_*^{\mathrm{T}}X$.

Lemma 2.1.13. *Let X be a S-bimodule.*

(i) *There exist endomorphisms R of π_*^{L} and L of π_*^{R} induced by the right and left action of X respectively.*
(ii) *There are automorphisms R and L of π_*^{T} satisfying $LR=\mathrm{id}=LR$.*
(iii) *The maps $\iota\colon \pi_*^{\mathrm{L}}X\to\pi_*^{\mathrm{T}}X$ and $\iota\colon \pi_*^{\mathrm{R}}X\to\pi_*^{\mathrm{T}}X$ induce isomorphisms*

$$(\pi_*^{\mathrm{L}}X)[R^{-1}]\cong\pi_*^{\mathrm{T}}X\cong(\pi_*^{\mathrm{R}}X)[L^{-1}].$$

In particular a morphism which induces an isomorphism on left or right homotopy groups also induces an isomorphism on total homotopy groups.

Proof. For an element $\bar{x}\in\pi_i^{\mathrm{L}}X$ represented by $x\in\pi_n X_{n+i}$ the element $\iota(\bar{x})$ is represented by $x\in(\Pi_n X)(i,0)=\pi_{n+i}X_i$ but also by $R(x)\in(\Pi_n X)(i,1)=\pi_{n+i+1}X_{i+1}$ considered as an elements in $\pi_n^{\mathrm{T}}X$. Hence $R(\iota(\bar{x}))$ in $\pi_*^{\mathrm{T}}X$ is represented by $R(x)\in(\Pi_n X)(i+1,0)$ and we conclude that $R\circ\iota=\iota\circ R$. But R acts as an isomorphism, so the map $\pi_*^{\mathrm{L}}X\to\pi_*^{\mathrm{T}}X$ extends to a map

$$\bar{\iota}\colon\ \pi_*^{\mathrm{L}}X\otimes_{\mathbb{Z}[R]}\mathbb{Z}[R^{\pm1}]\to\pi_*^{\mathrm{T}}X.$$

Since a functor indexed over $\mathbb{N}\times\mathbb{N}$ can be computed by first taking the colimit over the first and then over the second coordinate, the map $\bar{\iota}$ is an isomorphism. Similarly the other map $\iota\colon \pi_*^{\mathrm{R}}X\to\pi_*^{\mathrm{T}}X$ commutes with the operator L and extends to an isomorphism $\bar{\iota}\colon (\pi_*^{\mathrm{R}}X)[L^{-1}]\to\pi_*^{\mathrm{T}}X$. □

Example 2.1.14. Let X be a left S-module (*i.e.* a sequential spectrum). Define a right action of S by taking the maps $X_n\wedge S^1\to X_{n+1}$ to be trivial. Then the right homotopy groups of X are trivial and the operator R on left homotopy groups acts trivially. Therefore the total homotopy groups are trivial too.

Example 2.1.15. Let X be again a left S-module. Now we consider the free S-bimodule generated by X which is just $X \otimes S$. As a left S-module it decomposes as a wedge

$$X \otimes S = \bigvee_{n \geq 0} X \otimes S^n[n],$$

where $S^n[n]$ denotes the graded space with S^n concentrated in degree n. Hence there is an isomorphism of homotopy groups

$$\pi_*^{\mathrm{L}}(X \otimes S) \cong \bigoplus_{n \geq 0} \pi_*^{L}(X \otimes S^n[n]).$$

The left homotopy groups of the left S-module X are isomorphic to the left homotopy groups of $X \otimes S^n[n]$ via the isomorphism $\pi_n^{\mathrm{L}} X \to \pi_n^{\mathrm{L}}(X \otimes S^n[n])$ sending a homotopy class $[f]$ represented by a map $f\colon S^{i+n} \to X_i$ to the class represented by $f \wedge S^j\colon S^{i+n+j} \to X \wedge S^j$. The reader should be aware that this isomorphism is not induced by a map of left S-modules. Using this isomorphism we can identify the left homotopy groups of $X \otimes S$ as

$$\pi_n^{\mathrm{L}}(X \otimes S) \cong \bigoplus_{j \geq 0} \pi_n^{\mathrm{L}} X$$

and the right operator R on $\pi_n^{\mathrm{L}}(X \otimes S)$ with the shift operator

$$R\colon \bigoplus_{j \geq 0} \pi_n^{\mathrm{L}} X \to \bigoplus_{j \geq 0} \pi_n^{\mathrm{L}} X, \quad (x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots).$$

Therefore we have an isomorphism of $\mathbb{Z}[R]$-modules

$$\pi_n^{\mathrm{L}}(X \otimes S) \cong (\pi_n^{\mathrm{L}} X) \otimes \mathbb{Z}[R]$$

and an isomorphism of $\mathbb{Z}[R^{\pm 1}]$-modules

$$\pi_n^{\mathrm{T}}(X \otimes S) \cong (\pi_n^{\mathrm{L}} X) \otimes \mathbb{Z}[R^{\pm 1}].$$

Remark 2.1.16. Because of Lemma 2.1.13 we can consider left, right and total homotopy groups together as the data constituting a quasi coherent module on the projective line $\mathbb{P}^1_{\mathbb{Z}}$ over the integers and so taking homotopy groups as functor

$$S\text{-bimod} \to \text{q.c.}\, \mathcal{O}_{\mathbb{P}^1}\text{-mod}$$

lands in the category of quasi coherent sheaves over the projective line $\mathbb{P}^1_{\mathbb{Z}}$. This is the point of view taken in [HS06]. Since we are primarily not interested in this structure we do not further elaborate on this aspect.

The following example will be important for our theory later though it is interesting in its own right.

Example 2.1.17. Consider the category $\mathbb{S}$-modules, *i.e.* symmetric spectra. There is a forgetful functor $U\colon \mathbb{S}\text{-mod} \to S\text{-mod}$ which maps $\mathbb{S}$ to S and a symmetric spectrum to its underlying sequential spectrum. We can consider a left $\mathbb{S}$-module X as a right module by defining the the right action to be the composite

$$X \otimes^{\Sigma} \mathbb{S} \xrightarrow{\tau} \mathbb{S} \otimes X \xrightarrow{\sigma} X,$$

where τ is the twist isomorphism and σ the structure map of X. We want to know the right action on $U(X)$ explicitly. Recall that the twist isomorphism $A \otimes^{\Sigma} B \xrightarrow{\tau} B \otimes^{\Sigma} A$ has the property that for all $p, q \geq 0$ the diagram

$$\begin{array}{ccccc} A_p \wedge B_q & & \xrightarrow{\quad\quad\tau\quad\quad} & & B_q \wedge A_p \\ \downarrow & & & & \downarrow \\ (A \otimes^{\Sigma} B)_{p+q} & \xrightarrow[\tau]{} & (B \otimes^{\Sigma} A)_{p+q} & \xrightarrow[\chi_{p,q}]{} & (B \otimes^{\Sigma} A)_{q+p} \end{array}$$

commutes, where $\chi_{p,q}$ denotes the (p,q)-block permutation in Σ_{p+q}. Specializing to the case $A = X$, $B = \mathbb{S}$ and $q = 1$ we obtain the commutative diagram

$$\begin{array}{ccccccc} X_p \wedge S^1 & \longrightarrow & (X \otimes^{\Sigma} \mathbb{S})_{p+1} & \xrightarrow{\tau_{p+1}} & (\mathbb{S} \otimes^{\Sigma} X)_{p+1} & \xrightarrow{\sigma_{p+1}} & X_{p+1} \\ {\scriptstyle\tau}\downarrow & & & & \downarrow{\scriptstyle\chi_{p,1}} & & \downarrow{\scriptstyle\chi_{p,1}} \\ S^1 \wedge X_p & \longrightarrow & & \longrightarrow & (\mathbb{S} \otimes^{\Sigma} X)_{1+p} & \xrightarrow[\sigma_{1+p}]{} & X_{1+p} \end{array}$$

in which the upper horizontal composite is the right action $R\colon U(X)_p \wedge S^1 \to U(X)_{1+p}$ and the lower horizontal composite the left action $S^1 \wedge U(X)_p \to U(X)_{p+1}$ of the underlying S-bimodule X. We conclude that the right action can also be written as the composite

$$X_p \wedge S^1 \xrightarrow{\tau} S^1 \wedge X_p \xrightarrow{\lambda_{1,p}} X_{1+p} \xrightarrow{\chi_{1,p}} X_{p+1}.$$

The following property will be important later when we try to establish stable model structures on modules over a naive ring spectrum. To be more precise, a naive ring spectrum has an underlying S-bimodule and the category of modules over a naive ring spectrum admits a stable model structure if and only if the underlying S-bimodule is right stable (see 3.2.1 for details).

Definition 2.1.18. An S-bimodule X is *right stable*, if the right operator $\rho\colon \pi_*^{\mathrm{L}} X \to \pi_*^{\mathrm{L}} X$ is an automorphism of left homotopy groups. Similarly one defines *left stable.* If X is both left and right stable, we say X is bistable.

Example 2.1.19. Let X be a S-bimodule which is a right Ω-spectrum, *i.e.* the adjoints of the right structure maps are weak equivalences. Then X is right stable. To see this recall that the operator R on $\pi_*^{\mathrm{L}} X$ is induced by the right structure maps from. But we can also use the adjoints of the right structure maps which are weak equivalences by assumption. Hence R operates as an isomorphism. Similarly L operates as an isomorphism on the right homotopy groups provided that X is a left Ω-spectrum. Obviously X is bistable if the underlying left and right modules are both Ω-spectra.

Lemma 2.1.20. *Let X be a semi-stable symmetric spectrum. Then the underlying S-bimodule $U(X)$ is bistable. In particular the maps*

$$\pi_*^{\mathrm{L}} U(X) \xrightarrow{\cong} \pi_*^{\mathrm{T}} U(X) \xleftarrow{\cong} \pi_*^{\mathrm{R}} U(X)$$

are isomorphisms.

Proof. We consider X as a left $\mathbb{S}$-module and show that $U(X)$ is left and right stable. Since X is semi-stable a fibrant replacement $X \to \bar{X}$ is a (left) π_*-isomorphism. Therefore X is left or right stable if and only if $\bar{X}$ is. So without loss of generality we may assume that X is in fact fibrant, *i.e.* an Ω-spectrum and by Example 2.1.19 left stable. From Example 2.1.17 we know that the right action, which is a map $\Sigma X \to \mathrm{sh}_1 X$ of left S modules, is given in degree n by

$$X_n \wedge S^1 \xrightarrow{\tau} S^1 \wedge X_n \xrightarrow{\sigma_n} X_{1+n} \xrightarrow{\chi_{1,n}} X_{n+1},$$

where σ_n the structure map of the symmetric spectrum X. By abuse of notation we will write X instead of $U(X)$. Instead of showing that $\Sigma X \to \mathrm{sh}_1 X$ induces an isomorphism on left homotopy groups, it suffices to show that the adjoint map $X \to \Omega\, \mathrm{sh}_1 X$ does so. But this adjoint is given by

$$X_n \xrightarrow{\sigma_n^{\natural}} \Omega X_{1+n} \xrightarrow{\Omega\chi_{1,n}} \Omega X_{n+1}.$$

The map $\sigma_n^{\natural}$ is a weak equivalence as adjoint structure map of the Ω-spectrum X and $\Omega\chi_{1,n}$ is a weak equivalence since it is in fact an isomorphism. Hence

$X \to \Omega \operatorname{sh}_1 X$ is a level equivalence and in particular a π_*^{L}-isomorphisms which finishes the proof. □

Example 2.1.21. Using the multiplication on the sphere spectrum S we can consider S as a S-bimodule. We can consider S as the underlying S-bimodule $U(\mathbb{S})$ of the symmetric sphere spectrum $\mathbb{S}$ which is well known to be semi-stable. Hence S is bistable by Lemma 2.1.20 above.

One can also see more directly that S is right stable using the characterization of right stable we give now. The compatibility of the left and the right action of a S-module says that the right action maps

$$X_n \wedge S^1 \to X_{n+1}$$

assemble to a map

$$\Sigma X \to \operatorname{sh}_1 X$$

of left S-modules. The induced map on homotopy groups fits into the a commutative diagram

$$\begin{array}{ccc} \pi_{n+1}^{\mathrm{L}} \Sigma X & \longrightarrow & \pi_{n+1}^{\mathrm{L}} \operatorname{sh}_1 X \\ \cong \downarrow & & \downarrow \cong \\ \pi_n^{\mathrm{L}} X & \xrightarrow{R} & \pi_n^{\mathrm{L}} X \end{array}$$

where the bottom map is the right operator on left homotopy groups. Therefore we have

Lemma 2.1.22. *A S-bimodule X is right stable if and only if the map*

$$\Sigma X \to \operatorname{sh}_1 X$$

is a stable equivalence of left S-modules.

In case of S the map $\Sigma S \to \operatorname{sh}_1 S$ is in fact an isomorphism, so in particular a stable equivalence and thus S is right stable.

2.2. **Stable model structure for S-bimodules.** We now give the category of S-bimodules a stable model structure. If one views the category of S-bimodules as the category of left S-modules with additional structure there is a canonical forgetful functor to left S-modules along which one can try to lift the standard model structure of Bousfield Friedlander spectra to S-bimodules.

Definition 2.2.1. Recall from Definition 1.2.8 that the map $\lambda_n\colon S \otimes S^1[n+1] \to S \otimes S^0[n]$ is defined to be the adjoint of the identity map $S^1 \to (S \otimes S^0[n])_{n+1}$. This is a map of left S-modules and given in degree m by the isomorphism $S^{m-1} \wedge S^1 \cong S^m$ for $m > n$ given by the right action of S. Define $\sigma_n^{\mathrm{L}}\colon S \otimes S^1[n+1] \otimes S \to S \otimes S^0[n] \otimes S$ to be $\lambda_n \otimes S$. Symmetrically we define a right version of λ_n and σ_n^R.

In fact λ_n is up to a shift the right action map $\Sigma S \to \operatorname{sh}_1 S$. This shows two things. First that the operator right operator R induces an isomorphism on left homotopy groups and second, that the map σ_n^{L} is a π_*^{L}-isomorphism. From Example 2.1.15 we conclude that σ_n^{L} also induces isomorphisms on left homotopy groups.

Definition 2.2.2. Denote by J_{L} the set of mapping cylinder inclusions of the maps σ_n^{L} for $n \geq 0$.

Lemma 2.2.3. *The maps in J_{L} are projective cofibrations which induce isomorphisms on left and total homotopy groups.*

Proof. We have already seen that σ_n^{L} induces isomorphisms on left homotopy groups. By 2.1.13 they also induce isomorphisms on total homotopy groups. Factoring a map using the mapping cylinder construction gives an inclusion followed by a simplicial homotopy equivalence. By 2-out-of-3 the inclusion also induces isomorphisms on left and total homotopy groups. Both, source and target of the maps σ_n^{L} are cofibrant in the level model structure as images of cofibrant simplicial sets under the functor $S \otimes (-) \otimes S$. It suffices to show that the inclusions in the mapping cylinders are cofibrations which then implies that the mapping cylinders themselves are cofibrant. But this is a general fact in a simplicial model category proved in the Lemma 1.1.7. □

Theorem 2.2.4. *The category of S-bimodules admits a proper cofibrantly generated simplicial stable model structure with the projective cofibrations as cofibrations and weak equivalences the maps, that induce isomorphisms on left homotopy groups.*

Proof. We lift the stable model structure along the functor $(-) \otimes S$ from left S-modules, *i.e.* Bousfield-Friedlander spectra, using [Hir03, Theorem 11.3.2]. In order to do so we have to check that every regular J_{L}-cell complex is a π_*^{L}-isomorphism. But left homotopy groups are preserved by cobase changes along cofibrations (five lemma) and commute with sequential colimits of cofibrations, and thus every J_{L}-cell complex is a π_*^{L}-isomorphism. Finally note that the (co)domains of the maps in J_L are small in the category of S-bimodules. Tensor and cotensor for simplicial sets are defined level-wise on underlying graded spaces. Since cofibrations are projective cofibrations so cofibrations in the level model structure, the pushout product axiom (Proposition 1.1.6) holds by Proposition 2.1.9. Hence the model structure is simplicial. For left properness and stability note that stable equivalences, suspension and loops are defined on underlying spectra and the forgetful functor commutes with pushouts . □

We refer to the above model structure as the *left stable model structure* or simply to the *stable model structure*. Note that the stably fibrant S-bimodules are the left Ω-spectra. Recall that a S-bimodule is right stable if and only if the map

$$\Sigma X \to \operatorname{sh}_1 X$$

is a stable equivalence. But this is the case if and only if the adjoint

$$X \to \Omega \operatorname{sh}_1 X$$

is a stable equivalence. If X is a left Ω-spectra, so is $\Omega \operatorname{sh}_1 X$. Hence the map is stable equivalence if and only if it is a level equivalence or equivalently X is a right Ω-spectrum. Hence we see that the stably fibrant right stable S-bimodules are exactly the bi-Ω-spectra.

Remark 2.2.5. There are other stable model structures on S-bimodules than the one we have established above. Of course on can use right homotopy groups instead of left homotopy groups. But one can also take weak equivalences to be maps that induce isomorphism on left and right homotopy groups as we will do later (see Theorem 3.3.2). A third possibility is to take as weak equivalences maps inducing isomorphisms on total homotopy groups. In this last case the fibrant objects are the bi-Ω-spectra, *i.e.* spectra such that the adjoints of the left and right structure maps are weak equivalences.

2.3. **Spectra with an endomorphism.** In the last section we endowed the category of S-bimodules with a stable model structure in a way that the forgetful functor form S-bimodules to left S-modules is a Quillen functor. This means that from the homotopy theoretic view we consider S-bimodules as Bousfield-Friedlander spectra with extra structure. We now make this point of view more precise and obtain a relationship to symmetric spectra which is very natural in this point of view. Recall that having a S-bimodule explicitly means to have a commutative diagram

$$\begin{array}{ccc} S^1 \wedge X_n \wedge S^1 & \xrightarrow{\mathrm{id} \otimes \rho_n} & S^1 \wedge X_{n+1} \\ \downarrow{\scriptstyle \lambda_n \otimes \mathrm{id}} & & \downarrow{\scriptstyle \lambda_{n+1}} \\ X_{n+1} \wedge S^1 & \xrightarrow{\rho_{n+1}} & X_{n+2} \end{array}$$

As already noted this data is exactly the same as to have a left S-module together with a morphism

$$\Sigma X \to \mathrm{sh}_1 X$$

of left S-modules. Up to homotopy this means, that one has a spectrum (*i.e.* left S-module) together with an endomorphism. This motivates our next result which says that S-bimodules with the stable model structure are Quillen equivalent to symmetric spectra together with an endomorphism. We will now make this statement precise and prove that S-bimodules with the stable model structure from Section 2.2 are Quillen equivalent to symmetric spectra together with an endomorphism. The category of symmetric spectra with an endomorphism has as objects pairs (X,t) where X is a symmetric spectrum and $t\colon X \to X$ is a map of symmetric spectra. Morphisms are maps of underlying symmetric spectra that commute with the given endomorphism. A somewhat nicer description of this category is as the category of $\mathbb{S}[t]$-modules where

$$\mathbb{S}[t] = F_0(\mathbb{N}_+) \cong \mathbb{S} \wedge \mathbb{N}_+ \cong \bigvee_{n\geq 0} \mathbb{S} \times n$$

is the monoid ring spectrum of the natural numbers, or in other words the free polynomial algebra on one generator with coefficients in $\mathbb{S}$ which justifies the notation $\mathbb{S}[t]$.

We can consider the commutative symmetric ring spectrum $\mathbb{S}[t]$ as an S-bimodule in the following way. The left action induced by the monoid map $\mathbb{S} \to \mathbb{S}[t]$ given by the inclusion of $\mathbb{S}$ in $\mathbb{S}[t]$ indexed by 0. The right action is given by a map of monoids $\mathbb{S} \to \mathbb{S}[t]$ which in degree k is given by the diagram

$$\mathbb{S}_k \xrightarrow{\cong} S^k \times k \xrightarrow{\iota_k} \bigvee_{n\geq 0} \mathbb{S}_k \times n.$$

Given an $\mathbb{S}[t]$-module X, we use the bimodule structure on $\mathbb{S}[t]$ and restrict to a $\mathbb{S}$-bimodule. Forgetting the symmetric group actions yields a S-bimodule $U(X)$ and so a functor $U\colon \mathbb{S}[t]\text{-mod} \to S\text{-bimod}$.

If we take $t = \mathrm{id}$ to be the identity of X we recover Example 2.1.17. For general t we can calculate as in the case where $t = \mathrm{id}$ and obtain as right structure maps of $U(X)$

$$X_q \wedge S^p \xrightarrow{\tau} S^p \wedge X_q \xrightarrow{\lambda_{p,q}} X_{p+q} \xrightarrow{\chi_{p,q}} X_{q+p} \xrightarrow{t^p_{q+p}} X_{q+p}.$$

For further reference we note the following lemma.

Lemma 2.3.1. *Let X be a $\mathbb{S}[t]$-module whose underlying symmetric spectrum is semi-stable. Then there is an isomorphism*

$$\pi^{\mathrm{L}}_* U(X) \cong \pi_* X$$

of the left homotopy groups of the underlying S-bimodule of X and the homotopy groups of X itself.

Proof. The underlying left $\mathbb{S}$-module structure of X is just given by the underlying symmetric spectrum. By assumption the underlying symmetric spectrum is semistable, so its homotopy groups are given by the underlying left S-modules. But the left homotopy groups of the underlying bimodule are by definition given by the homotopy groups of the underlying left S-module. □

The functor $U\colon \mathbb{S}[t]\text{-mod} \to S\text{-bimod}$ described above sending a symmetric spectrum to the S-bimodule with left and right action coming from a S-bimodule structure of $\mathbb{S}[t]$ has a left adjoint W we describe now. Recall that the left adjoint $G\colon \mathrm{gr}\mathcal{S}_* \to \mathcal{S}^\Sigma_*$ to the forgetful functor $\mathcal{S}^\Sigma_* \to \mathrm{gr}\mathcal{S}_*$ is given by $G(X)_n = (\Sigma_n)_+ \wedge X_n$. Now observe that $G(S)$ is the free associative monoid on $(*, S^1, *, *, \dots)$ in $\mathcal{S}^\Sigma_*$ with respect to $\otimes^\Sigma$. A free associative monoid always comes with a natural map to the free commutative monoid. So there is a natural quotient map $G(S) \to \mathbb{S}$ which is a map of monoids in $\mathcal{S}^\Sigma_*$. A short computation shows that G is actually monoidal so it extents to functor S-mod $\to G(S)$-mod. Now, since there is a monoid map $G(S) \to \mathbb{S}$, we have an extension of scalars functor

$$\mathbb{S} \otimes^\Sigma_{G(S)} (-)\colon\ G(S)\text{-mod} \to \mathbb{S}\text{-mod}.$$

Altogether, a left S-module X is mapped to $\mathbb{S} \otimes^\Sigma_{G(S)} G(X)$. This gives a left adjoint $V\colon\ S\text{-mod} \to \mathbb{S}\text{-mod}$ to the forgetful functor $U\colon\ \mathbb{S}\text{-mod} \to S\text{-mod}$ from symmetric to sequential spectra.

Composing the functor $U\colon\ \mathbb{S}[t]\text{-mod} \to S\text{-bimod}$ with the forgetful functor from S-bimodules to left S modules is obviously equal to the composite which first sends a $\mathbb{S}[t]$-module to its underlying (left) $\mathbb{S}$-module and then to the underlying sequential left spectrum. Therefore, if a left adjoint W exists, it must satisfy $W(A \otimes S) = \mathbb{S}[t] \wedge V(A)$ for any left S-module A. But an S-bimodule M is the same as the underlying left S-module ${}_SM$ together with a map ${}_SM \otimes S \to {}_SM$ of left S-modules given by the right action. So if we define the underlying symmetric spectrum of $W(M)$ to be $V({}_SM)$ then the right action would give a map

$$\mathbb{S}[t] \wedge V({}_SM) = W(M \otimes S) \to W(M) = V({}_SM)$$

of symmetric spectra and equip $V({}_SM)$ with the $\mathbb{S}[t]$-module action. This construction is only well defined if $V({}_S(M \otimes S)) = \mathbb{S}[t] \wedge V({}_SM)$ which is unfortunately not the case. Indeed one can check that $V({}_S(M \otimes S)) \cong FS \wedge V({}_SM)$ where FS is the symmetric spectrum given by the wedge

$$FS = \bigvee_{n\geq 0} F_n S^n.$$

The functors F_n are graded monoidal in the sense that $F_nK \wedge F_mL \cong F_{n+m}(K \wedge L)$ and therefore endow FS with a multiplication. The discussion above implies that the functor V on left S-modules N together with a map $N \otimes S \to N$ of left S-modules takes values in the category of left FS-modules.

Since taking the wedge over the canonical maps $F_nS^n \to \mathbb{S}$ give a map $FS \to S[t]$ of symmetric ring spectra, there is an extension of scalars functor

$$\mathbb{S}[t] \wedge_{FS} (-)\colon\ FS\text{-mod} \to \mathbb{S}[t]\text{-mod}.$$

Definition 2.3.2. For a S-bimodule M we define $W(M)$ to be the $\mathbb{S}[t]$-module

$$W(M) = \mathbb{S}[t] \wedge_{FS} V({}_SM)$$

where the symmetric spectrum $V({}_SM)$ carries the left FS action induced from the right action on M.

It is immediate that the right adjoint to W is exactly U. We now investigate the homotopical properties of this adjoint pair.

Theorem 2.3.3. *The adjoint pair*

$$W\colon\ S\text{-bimod} \rightleftarrows \mathbb{S}[t]\text{-mod} :U$$

is a Quillen equivalence.

Proof. First we show that U and W form a Quillen adjoint functor pair. We use the criterion of Dugger [Dug01, A.2] for which it suffices to show that U preserves acyclic fibrations and fibrations between fibrant objects. Since in the stable model structure of S-bimodules and $\mathbb{S}[t]$-modules the acyclic fibrations are the maps which are level fibrations and level equivalences, it is clear that U preserves these. The fibrant objects in $\mathbb{S}[t]$-modules are exactly the modules having underlying Ω-spectra. The functor U takes these to left Ω-spectra which are the fibrant objects in S-bimodules. Since every stable fibration of $\mathbb{S}[t]$-modules is in particular a level fibration by [HSS00, Lemma 3.4.12 and Corollary 3.4.16], U takes level fibrations between fibrant $\mathbb{S}[t]$-modules to level fibrations between left Ω-spectra. Given a level fibration $X \to Y$ between Ω-spectra, we have for each $n \geq 0$ a commutative diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\sim} & \Omega X_{n+1} \\ \downarrow & & \downarrow \\ Y_n & \xrightarrow{\sim} & \Omega Y_{n+1} \end{array}$$

in which every object is fibrant. Taking the pullback P of the lower right part gives in particular maps $X_n \to P$ and $P \to \Omega X_{n+1}$ from which the latter is a weak equivalence by right properness of simplicial sets. But then the first map is weak equivalence too by 2-out-of-3 and we conclude that the diagram above is homotopy pullback diagram. Therefore the map $X \to Y$ is a stable fibration by Theorem 1.2.4. This finishes the argument showing that U is a right Quillen functor.

It remains to show that U and W form a Quillen equivalence. A map between fibrant $\mathbb{S}[t]$-modules is a stable equivalence if and only if it induces isomorphisms on homotopy groups. But by Lemma 2.3.1 the left homotopy groups of $U(X)$ are isomorphic to the homotopy groups of the stably fibrant $\mathbb{S}[t]$-module X. Consequently, the right derived functor $\mathrm{R}U\colon\ \mathrm{Ho}(\mathbb{S}[t]\text{-mod}) \to \mathrm{Ho}(S\text{-bimod})$ detects isomorphisms since it detects stable equivalences between stably fibrant $\mathbb{S}[t]$-modules. In order to show that $\mathrm{R}U$ and $\mathrm{L}W$ form an equivalence of categories, it then suffices to show that for any object A in $\mathrm{Ho}(S\text{-bimod})$ the adjunction unit $A \to \mathrm{R}U(\mathrm{L}W(A))$ is an isomorphism. From the description of W we gave above, it is immediately clear that it takes the free rank one S-bimodule $S \otimes S$ to the free $\mathbb{S}[t]$-module of rank one which is of course $\mathbb{S}[t]$. Both of theses two are cofibrant in the respective model structure, so there in an isomorphism $W(S \otimes S) \to \mathbb{S}[t]$ of $\mathbb{S}[t]$-modules. We now show that its adjoint $S \otimes S \to \mathrm{R}U(\mathbb{S}[t])$ is a π_*-isomorphism. Since $\mathbb{S}[t]$ is semi-stable (see *e.g.* [Sch08, Example 4.6/4.7]), by Lemma 2.3.1 the left homotopy groups of $U(\mathbb{S}[t])$ are isomorphic to $\pi_*\mathbb{S}[t] \cong \pi_*(S) \otimes \mathbb{Z}[t]$. From Example 2.1.15 we already know that the left homotopy groups of $S \otimes S$ also have this structure. The semi-stability of $\mathbb{S}[t]$ implies that in suffices to show that the map $S \otimes S \to U(\mathbb{S}[t])$ is a left stable equivalence. But this map splits into a countable wedge and we check the claim on each summand. In wedge degree n the map is given by $S \otimes S^n[n] \to S$ which is ultimately the isomorphism $S^{m-n} \wedge S^n \cong S^m$ for growing degree m and hence induces an isomorphism on left homotopy groups.

Both homotopy categories are triangulated categories which are generated by the compact objects $S \otimes S$ and $\mathbb{S}[t]$ respectively. Therefore it suffices to show, that the adjunction unit $\eta_A\colon\ A \to \mathrm{R}U(\mathrm{L}W(A))$ is a natural transformation between exact functor which commutes with direct sums. By Theorem 1.1.3 $\mathrm{L}W$ and $\mathrm{R}U$ form an exact functor pair and $\mathrm{L}W$ commutes with sums as a left adjoint. But

RU also commutes with sums, since U preserves homotopy groups between stably fibrant objects. We already showed that η is an isomorphism on $S \otimes S$, hence it is one on every object of Ho(S-bimod). This shows that LW and RU are inverse equivalences of categories and thus W and U form a Quillen equivalence. □

We remark that by 2-out-of-3 for Quillen equivalences the category of S-bimodules is also Quillen equivalent to FS-modules since the map $FS \to \mathbb{S}[t]$ is as wedge of stable equivalences one and thus induces a Quillen equivalence between FS-mod and $\mathbb{S}[t]$-mod.

As corollary of the theorem above we give a characterization of right stable S-bimodules in terms of $\mathbb{S}[t]$-modules.

Corollary 2.3.4. *Let M be a S-bimodule and X a $\mathbb{S}[t]$-module.*

(i) *If X is stably fibrant as a $\mathbb{S}[t]$-module, the underlying S-bimodule $U(X)$ is right stable if and only if the endomorphism $t\colon X \to X$ is a level equivalence.*
(ii) *If M is cofibrant as S-bimodule, then M is right stable if and only if the endomorphism $t\colon W(M) \to W(M)$ is a stable equivalence.*

Proof. (i) Let X be a stably fibrant $\mathbb{S}[t]$-module. This means that X has an underlying (left) Ω-spectrum or that the underlying S-bimodule $U(X)$ is a left Ω-spectrum. For $U(X)$ to be right stable means that the map $U(X) \to \Omega\,\mathrm{sh}_1\,U(X)$ adjoint to the right action of $U(X)$ is a π_*-isomorphisms of left S-modules. But both $U(X)$ and $\Omega\,\mathrm{sh}_1\,U(X)$ are left Ω-spectra, hence $U(X)$ is right stable if and only if $U(X) \to \Omega\,\mathrm{sh}_1\,U(X)$ is a level equivalence. But the right action of $U(X)$ is given in degree n by

$$X_n \wedge S^1 \xrightarrow{\tau} S^1 \wedge X_n \xrightarrow{\lambda_{1,n}} X_{1+n} \xrightarrow{\chi_{1,n}} X_{n+1} \xrightarrow{t_{n+1}} X_{n+1}$$

and so its adjoint by

$$X_n \xrightarrow{(\lambda_{1,n})^\natural} \Omega X_{1+n} \xrightarrow{\Omega\chi_{1,n}} \Omega X_{n+1} \xrightarrow{\Omega t_{n+1}} \Omega X_{n+1}.$$

Therefore the map in question is a level equivalence if and only if the map $\Omega t\colon \Omega X \to \Omega X$ is a level equivalence. By stability this is the case if and only if t itself is a stable equivalence between between Ω-spectra, so if and only if t is a level equivalence.

(ii) Let M be a cofibrant S-bimodule and $W(M) \to W(M)^f$ a stably fibrant replacement of $W(M)$ in the stable model structure of $\mathbb{S}[t]$-modules. The adjoint $M \to U(W(M)^f)$ is a stable equivalence of S-bimodules by Theorem 2.3.3 above. Hence M is right stable if and only if $U(W(M)^f)$ is right stable. Since $W(M)^f$ is stably fibrant as a $\mathbb{S}[t]$-module and we can apply part (i) and see that this is the case if and only if $t\colon W(M)^f \to W(M)^f$ is a level equivalence. But this is equivalent to the condition that $t\colon W(M) \to W(M)$ is a stable equivalence on underlying symmetric spectra. □

We finish this section with a result which says that the underlying symmetric spectrum of $W(M)$ is already determined by the left module structure on M for cofibrant S-bimodule M. The adjunction unit $M \to U(W(M))$ is a map of S-bimodules, hence it restricts to a map ${}_SM \to {}_SU(W(M))$. We obtain by adjoining the map above a map of symmetric spectra

$$\alpha_M\colon V({}_SM) \to W(M)$$

where $W(M)$ denotes, by abuse of language, the underlying symmetric spectrum of the $\mathbb{S}[t]$-module $W(M)$.

Lemma 2.3.5. *For a cofibrant S-bimodules M the map $\alpha_M\colon V({}_SM) \to W(M)$ is a stable equivalence of symmetric spectra.*

Proof. The target of the map is a left Quillen functor in M with respect to the left model structure of S-bimodules. Since the forgetful functor form S-bimodules to left S-modules preserves colimits, cofibrations and left homotopy groups, the source of the map is also a left Quillen functor with respect to the left stable model structure on S-bimodules. Therefore it suffices to verify the statement for $M = S \otimes S$, the free S-bimodule on the left module S. We know that

$$V(S \otimes S) \cong \bigvee_{n \geq 0} F_n S^n \quad \text{and} \quad W(S \otimes S) \cong \mathbb{S}[t]$$

For each $n \geq 0$ the canonical map $F_n S^n \to t^n \cdot \mathbb{S}$ is a stable equivalence, hence the map $\alpha_{S \otimes S}$ is a stable equivalence of symmetric spectra. □

2.4. **Tensor product of S-bimodules.** The tensor product of S-bimodules defined in Definition 2.1.10 is a monoidal though not symmetric monoidal product on the category of S-bimodules with the S-bimodule S as unit object. We are now interested in how far it is compatible with the model structures on S-bimodules considered so far. We begin with the level model structure. We say that a map of S-bimodule is a *left cofibration* if it is a cofibration of underlying left S-modules and similarly *right cofibrations.* Note that a cofibration of S-bimodules is both, a left and a right cofibration.

Lemma 2.4.1. *Let $i \colon K \to L$ be a map of right S-modules and $j \colon M \to N$ be a map of left S-modules.*

(a) *If i or j is a cofibration and the other map is injective then the pushout product map*

$$i \,\square\, j \colon \; L \otimes_S \cup_{K \otimes_S M} K \otimes_S N \to L \otimes_S N$$

is injective.

(b) *If i or j is a cofibration, the other map is injective and one of the maps is a level equivalence, then the pushout product map $i \,\square\, j$ is a level equivalence.*

(c) *If L is a cofibrant right S-module, then the functor $L \otimes_S (-)$ preserves level equivalences. If N is cofibrant as left S-module, then the functor $(-) \otimes_S N$ preserves level equivalences.*

Proof. (a) We treat the case where j is a cofibration of left S-modules. It suffices to show the case where j is a generating cofibration, *i.e.* of the form $S \otimes \partial\Delta^k_+[n] \to S \otimes \Delta^k_+[n]$. Then the pushout product map is isomorphic to

$$K \otimes \Delta^k_+[n] \cup_{K \otimes \partial\Delta^k_+[n]} L \otimes \partial\Delta^k_+[n] \to L \otimes \Delta^k_+[n],$$

which is injective by the pushout product axiom for pointed simplicial sets.

(b) Again we treat the case where j is a cofibration of left S-modules and i is injective map of right S-modules. If i is a level equivalence then it suffices to show the case where j is a generating cofibration. In this case the pushout product map $i\square j$ is a level equivalence for the same reason as in part (a). If j is a level equivalence then it suffices to show the case where i is a generating acyclic cofibration, *i.e.* a map of the form $S \otimes \Lambda^k_{i+}[n] \to S \otimes \Delta^k_+[n]$. Similar as before the pushout product axiom for pointed simplicial sets implies that the map in question is a level equivalence.

(c) We consider the case where N is a cofibrant left S-module. Suppose the claim holds for N and that the left S-module N' is obtained by cobase change along a generating cofibration of the form $S \otimes \partial\Delta^k_+[n] \to S \otimes \Delta^k_+[n]$. We have a pushout square

$$\begin{array}{ccc} K \otimes \partial\Delta^k_+[n] & \longrightarrow & K \otimes \Delta^k_+[n] \\ \downarrow & & \downarrow \\ K \otimes_S N & \longrightarrow & K \otimes_S N' \end{array}$$

and a similar one with K replaced by L. Now let $i\colon K \to L$ be a level equivalence of right S-modules. Then we have a commutative diagram

$$\begin{array}{ccccc} K \otimes_S N & \longleftarrow & K \otimes \partial\Delta^k_+[n] & \rightarrowtail & K \otimes \Delta^k_+[n] \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ L \otimes_S N & \longleftarrow & K \otimes \partial\Delta^k_+[n] & \rightarrowtail & K \otimes \Delta^k_+[n] \end{array}$$

in which the left vertical map is a weak equivalence by assumption and the middle and right vertical maps are weak equivalences by the pushout product axiom for pointed simplicial sets. Hence by [GJ99, Chapter II Lemma 8.2] the induced map $K \otimes_S N' \to L \otimes_S N'$ is a level equivalence. Therefore the class of right S-modules for which the claim holds is closed under cobase change along generating cofibrations. Since it is also closed under retracts and colimits over a sequence of cofibrations, it contains all cofibrant right S-modules. □

Lemma 2.4.2. *Let M be a right cofibrant S-bimodule. If M is right stable, then the functor $M \otimes_S (-)$ preserves π^{L}_*-isomorphisms. Similarly for left cofibrant and left stable S-bimodule M the functor $(-) \otimes_S M$ preserves π^{R}_*-isomorphisms.*

Proof. Let j be a map of left S-modules which induces isomorphisms on left homotopy groups. We can factor j as a cofibration followed by an equivalence in the level model structure of left S-modules. By Lemma 2.4.1 (c), $M \otimes_S (-)$ preserves level equivalences. So we may assume that j is a π^{L}_*-isomorphism and a cofibration of left S-modules. It suffices to check the case where j is a generating acyclic cofibration of the stable model structure. The level equivalences among these are taken care of by Lemma 2.4.1 (c) again, so it remains to check the case where j is σ^{L}_n. This follows if we show that the claim is true for $\lambda_n\colon S \otimes S^1[n+1] \to S \otimes S^0[n]$. But the map $M \otimes_S \lambda_n$ is isomorphic to the left S-module map $R\colon M \otimes S^1[n+1] \to M \otimes S^0[n]$ given by the right S-action of M. But this map induces an isomorphism of left homotopy groups since M was assumed to be right stable. □

Lemma 2.4.3. *The classes of cofibrant left stable, cofibrant right stable and cofibrant bistable S-bimodules is closed under $\otimes_S$*

Proof. If N is right stable and M right cofibrant and right stable, then the right action map $N \otimes S^1[1] \to N$ induces on isomorphism on left homotopy groups. By Lemma 2.4.2 above the map $M \otimes_S (N \otimes S^1[1]) \to M \otimes_S N$ induces an isomorphism of left homotopy groups. But this means that $M \otimes_S N$ is right stable. □

We finish this section with remarking that the left (or right) stable model structure on S-bimodules together with the monoidal product $\otimes_S$ is not a monoidal model structure.

Remark 2.4.4. Consider the S-bimodule ${}_{\mathrm{tr}}S$ which is equal to the sphere bimodule S as a right S-bimodule but has trivial left action. Let ${}_{\mathrm{tr}}S^c \to {}_{\mathrm{tr}}S$ be a cofibrant replacement in the level model structure of S-bimodules. Now the left and hence the total homotopy groups of ${}_{\mathrm{tr}}S^c$ are trivial, but the left and total homotopy groups of $(S \otimes S) \otimes_S {}_{\mathrm{tr}}S^c \cong S \otimes {}_{\mathrm{tr}}S^c$ are non-trivial. Therefore the pushout product axioms fails for the left (and of course also for the right) model structure. We also see that we do not have a chance to get a monoidal model structure if we take π_*^{T}-isomorphisms as weak equivalences for a model structure.

3. Naive ring spectra

Having set up a monoidal product on S-bimodules we now turn to monoids with respect to this monoidal structure. First we will give an explicit and surprisingly simple characterization of monoids in S-bimodules. Then we consider modules over these monoids and equip them with a stable model structure. Having described S-bimodules as symmetric spectra together with an endomorphism up to homotopy raises the question if one can characterize monoids in S-bimodules as monoids in $\mathbb{S}[t]$-modules, *i.e.* $\mathbb{S}[t]$-algebras. Unfortunately we are not able to lift the left stable model structure to monoids. But we will develop enough homotopy theory so that we can always replace an naive ring spectrum by a $\mathbb{S}[t]$-algebra (see Theorem 3.3.15).

3.1. **Monoids in S-bimodules.** For our treatment of Morita theory of stable model categories, monoids in S-bimodules are of essential importance and they get their own name.

Definition 3.1.1. A *naive ring spectrum* is a monoid R in S-bimodules with respect to tensor product $\otimes_S$ of S-bimodules.

It is tempting to call theses monoids "S-algebras", but this could be misleading since this would suggest a centrality condition, which is not the case. The name naive ring spectrum might be justified by the following characterization.

Lemma 3.1.2. *A naive ring spectrum is the same as a monoid in graded spaces with respect to the tensor product together with a monoid map $S \to R$. Explicitly this means R is a graded space equipped with associative and unital "multiplication maps"*

$$\mu_{p,q}\colon\ R_p \wedge R_q \to R_{p+q}$$

and a "unit map"

$$S^1 \to R_1$$

without any further compatibility conditions.

Proof. Given a monoid R in S-bimodules, we have a unit map $S \to R$ and a multiplication map $R \otimes_S R \to R$. Precomposing the multiplication map with the quotient map $R \otimes R \to R \otimes_S R$ gives a monoid in graded spaces. Using the unit isomorphism $S \otimes_S S \cong S$ one checks that the unit $S \to R$ is in fact a map of monoids. Such a map is determines and is uniquely determined by a map $S^1 \to R_1$ because S is by definition the free associative monoid on $S^1[1]$. Conversely, given a map $S \to R$ of monoids in graded spaces, we give R the structure of a S-bimodule using the composites

$$S \otimes R \to R \otimes R \to R \quad \text{and} \quad R \otimes S \to R \otimes R \to R$$

and observe that the associativity isomorphisms of the monoidal product $\otimes$ imply that the multiplication map $R \otimes R \to R$ factors over $R \otimes_S R \to R$. To check that $S \to R$ is in fact the unit of a monoid R in S-bimodules uses that the map in question is is a map of monoids in graded spaces. □

Definition 3.1.3. A naive ring spectrum is *right stable, left stable* or *bistable* if the underlying S-bimodule is.

In the next section we will consider modules over a naive ring spectrum R. We choose to work over left modules. These will have underlying left spectra and therefore we mostly consider left homotopy groups. Therefore we have the following convention.

Definition 3.1.4. Let R be an naive ring spectrum. Define the *homotopy groups* of R to be

$$\pi_n R = \pi_n^{\mathrm{L}} R$$

the left homotopy groups of the underlying S-bimodule.

3.2. **Modules over naive ring spectra.** Given a naive ring spectrum R one can consider the category of (naive) module spectra over R. We give necessary and sufficient conditions to lift the stable model structure from Bousfield-Friedlander spectra to module spectra over a naive ring spectrum.

Theorem 3.2.1. *Let R be a naive ring spectrum. Then the category of left R-modules has the structure of a model category with weak equivalences and fibrations defined on underlying spectra if and only R is right stable. Is this the case, the model structure is simplicial, proper, stable and cofibrantly generated. Further, for any R-module M, there is a natural isomorphism*

$$\pi_* M \cong [R, M]_*^{\mathrm{Ho}(R\text{-mod})}$$

of abelian groups and R is a compact generator of the triangulated homotopy category Ho(R-mod).

Proof. Suppose that (left) R-modules admit a model category structure with fibrations and weak equivalences defined on underlying (left) spectra. Then the forgetful functor R-mod $\to$ S-mod is a right Quillen functor. Therefore its left adjoint, the free R-module functor,

$$R \otimes_S (-)\colon\ S\text{-mod} \to R\text{-mod}$$

preserves weak equivalences between cofibrant objects. The map $\lambda_1\colon\ S \otimes S^1[1] \to S$ (the adjoint of the identity map $S^1 \to S_1$) induces an isomorphism on left homotopy groups. Under the free functor λ_1 is sent to $R \otimes S^1[1] \to R$ and hence a weak equivalence since it is a stable equivalence between cofibrant left S-modules. Now observe that the resulting map $R \otimes S^1[1] \to R$ is given by the right action $\mathrm{R}\colon\ \Sigma R \to \mathrm{sh}_1 R$ of R and so R is right stable as a S-bimodule.

To conversely establish the model structure on R-mod we use [SS00, Lemma 2.3]. We verify condition (2) of that lemma. Let J_R^{level} the set of maps of the form

$$R \otimes (\Lambda^i_{k+})[n] \to R \otimes (\Delta^i)_+[n]$$

for $n, i \geq 0$ and $i \geq k \geq 0$. These maps are point-wise injective and induce isomorphism on left homotopy groups. Now let J_R^{stable} be the set of simplicial mapping cylinder inclusions associated to the R-module maps of the form

$$R \otimes (\partial\Delta^i_+)[n] \cup_{R \otimes (S^1 \wedge \partial\Delta^i_+)[n+1]} R \otimes (S^1 \wedge \partial\Delta^i_+) \to R \otimes (\Delta^i_+)[n]$$

for $i, n \geq 0$. We assumed R to be right stable so the map $R \otimes S^1[n+1] \to R \otimes S^0[n]$ inducing the operator R on left homotopy groups is a point-wise injective π_*-isomorphism. Hence the maps in J_R^{stable} are point-wise injective π_*-isomorphisms. Therefore all maps in $J_R = J_R^{\mathrm{level}} \cup J_R^{\mathrm{stable}}$ are point-wise injective π_*-isomorphisms and Lemma 2.3 of [SS00] applies. Tensors and cotensors of R-modules and simplicial sets are defined on underlying graded spaces. The simplicial compatibility axiom (Proposition 1.1.6) since holds. To see this, let $K \to L$ be a cofibration of pointed simplicial sets and $M \to N$ a fibration of R-modules. We can view $M \to N$ as a fibration between left S-modules. Since cotensors are defined on underlying graded spaces, we can take the cotensor on underlying spectra and apply the pushout product axiom for spectra to see that the map

$$\mathrm{map}_R(L, M) \to \mathrm{map}_R(L, N) \times_{\mathrm{map}_R(K,N)} \mathrm{map}_R(K, M)$$

is a fibration of simplicial sets which is acyclic if $K \to L$ or $M \to N$ is.

Next we show properness. A cofibration of R-modules is a retract of a map obtained from maps of the form

$$R \otimes \partial\Delta^i_+[n] \to R \otimes \Delta^i_+[n]$$

by pushout and colimits. Thus every cofibration of R-modules is point-wise injective. Therefore a cofibration $M \to N$ gives rise to a long exact sequence of homotopy groups by Lemma 1.2.5. Hence left properness follows from the five lemma. Fibrations and weak equivalences are defined on underlying spectra, so right properness of R-modules follows from right properness of spectra (see Theorem 1.2.4).

To see that we have indeed a stable model structure, we consider the adjoint $M \to \Omega(\Sigma M)^f$ of a fibrant replacement $\Sigma M \to (\Sigma M)^f$ of ΣM. This map is a model for the derived adjunction unit. To see that this is indeed a stable equivalence of R-modules note that it suffices to show that it is a stable equivalence of underlying spectra. But tensor and cotensor are defined on underlying graded spaces, so we can equally consider $\Sigma M \to (\Sigma M)^f$ as a stable equivalence of spectra and then take the adjoint yielding a stable equivalence by stability of the stable model structure on spectra. Similarly we can show that the adjunction counit is an isomorphism in the homotopy category.

Now we show that there is a natural isomorphism of graded abelian groups

$$\pi^{\mathrm{L}}_* M \cong [R, M]^R_*.$$

Choose a fibrant replacement $M \to M^f$ of M in R-mod and note that we have $\pi^{\mathrm{L}}_* M \cong \pi^{\mathrm{L}}_* M^f$ and we may assume that M is fibrant. Let q be any integer and write $q = m - n$ with $n, m \geq 0$. Note that there is an isomorphism $S^m[n] \otimes S^n[0] \cong S^{m+n}[n]$. From this follows that $S \otimes S^m[n]$ is a model for $\Sigma^{m-n} S$. To see this we take the n-fold suspension of $S \otimes S^m[n]$ by applying $(-) \otimes S^n[0]$ and obtain $S \otimes S^{m+n}[n]$. The canonical map $S \otimes S^{m+n}[n] \to \Sigma^m S$ is an isomorphism on left homotopy groups since it is an isomorphisms from degree $\geq n$. Now we can, since M is an Ω-spectrum, calculate

$$\pi^{\mathrm{L}}_{m-n} M \cong \pi_m X_n = [S^m, M_n]^{\mathrm{Ho}(\mathcal{S}_*)}$$

and using derived adjunctions

$$[S^m, M_n]^{\mathrm{Ho}(\mathcal{S}_*)} \cong [S \otimes S^m[n], M]^{\mathrm{Ho}(\mathcal{S}p)} \cong [\Sigma^{m-n} R, M]^{\mathrm{Ho}(R\text{-mod})}.$$

which shows the formula.

Finally note that the formula above implies that if the graded abelian group $[R, M]^R_*$ vanishes, then M is stably contractible, *i.e.* trivial in Ho(R-mod). Thus the free R-module of rank one R is a weak generator in the homotopy category of R-modules. To show that R is in fact small, we must show that the canonical map

$$\bigoplus_I [R, M]^R \to [R, \coprod_I M_i]$$

for any family $M_i, i \in I$, of R-modules. Again by the formula for homotopy groups of R-modules the source of the map above is isomorphic to

$$\bigoplus_I \pi_0 M_i \cong \pi_0(\bigvee_I M_i).$$

Similarly the target is isomorphic to $\pi_0(\bigvee_I M_i)$ and hence the map in question is an isomorphism since coproducts are formed point-wise. □

Now we show that under the right stability condition the homotopy groups of a naive ring spectrum have additional structure: as expected they have the structure of a graded ring.

Lemma 3.2.2. *Let R be a right stable naive ring spectrum. Then the homotopy groups $\pi_* R$ form an graded ring.*

Proof. Since we assumed R to be right stable, the category of left R-modules admits a stable model structure such that we can identify

$$\pi_n R \cong [\Sigma^n R, R]^R$$

for all integers n. Now we the right hand side forms a graded ring using (graded) composition as the self maps of an object in a triangulated category always do. □

Remark 3.2.3. Let R be a right stable naive ring spectrum.

(1) Of course one can give a more direct and explicit description of the homotopy ring $\pi_* R$ of R using the multiplication maps $\mu_{p,q}\colon R_p \wedge R_q \to R_{p+q}$. For the sake of simplicity and without loss of generality we assume that R is fibrant as a (left) R-module, *i.e.* an Ω-spectrum. Using the identifications

$$\pi_n R \cong [S^n, R_0]^{\mathbb{S}_*} \cong [S^n, R]^{\mathcal{S}p} \cong [\Sigma^n R, R]^R$$

and similarly

$$\pi_{-n} R \cong [R, \Sigma^n R]^R$$

for $n \geq 0$ one can trace back that the composition on the right hand side translates to smashing maps and composing with the appropriate $\mu_{p,q}$ on the left hand side.

(2) For a R-bimodule M the formula $\pi_* M = [R, M]^R_*$ is actually not just an isomorphism of abelian groups but one of graded $\pi_* R$-modules.

We finish the section with showing the following homotopy invariance property for modules over naive ring spectra.

Theorem 3.2.4. *Suppose $\varphi\colon R \to R'$ is a stable equivalence of right stable naive ring spectra. Then restriction an extension of scalars*

$$R' \otimes_R (-)\colon R\text{-mod} \rightleftarrows R'\text{-mod} : f^*$$

forms a Quillen equivalence.

Proof. Let $\varphi\colon R \to R'$ be stable equivalence between right stable naive ring spectra. Since fibrations and stable equivalences are defined on underlying spectra both are preserved by the right adjoint which preserves the underlying spectra. Hence we have a Quillen pair. The left Quillen functor $(-) \wedge_R R'$ takes the free R-module R to the free R'-module R' and so does its left derived functor $(-) \wedge^{\mathrm{L}}_R R'\colon \mathrm{Ho}(R\text{-mod}) \to \mathrm{Ho}(R'\text{-mod})$. The map

$$(-) \wedge^{\mathrm{L}}_R R'\colon [R, R]^R_* \to [R \wedge^{\mathrm{L}}_R R', R \wedge^{\mathrm{L}}_R R']^{R'}_*$$

is isomorphic to the map induced by $\varphi\colon R \to R'$ on left homotopy groups by the formula from Theorem 3.2.1 above. Since this is a bijection by assumption, the total left derived functor $(-) \wedge^{\mathrm{L}}_R R'$ is faithfully full and the discussion above implies that it is also essentially surjective and thus an equivalence of categories. Hence $(-) \wedge_R R'$ is a left Quillen equivalence. □

3.3. **Symmetric $\mathbb{S}[t]$-algebras.** In this section we develop the necessary properties needed to replace a right stable naive ring spectrum by symmetric ring spectrum whose underlying naive ring has the same stable homotopy type to the give one. We actually construct a $\mathbb{S}[t]$-algebra rather than a symmetric ring spectrum.

Consider symmetric $\mathbb{S}[t]$-modules X and Y having t_X and t_Y as their endomorphisms respectively. Define their smash product $X \wedge Y$ to the smash product $X \wedge_{\mathbb{S}} Y$ of underlying symmetric spectra together with the endomorphism $t_X \wedge_{\mathbb{S}} t_Y$. This gives a symmetric monoidal product on the category of $\mathbb{S}[t]$-modules having the symmetric sphere spectrum endowed with the identity endomorphism as unit object.

The monoids in the symmetric monoidal category of $\mathbb{S}[t]$-modules are of course $\mathbb{S}[t]$-algebras which can be described as symmetric ring spectra together with a multiplicative and unital self map, *i.e.* an endomorphism in the category of symmetric ring spectra.

Lemma 3.3.1. *The smash product of $\mathbb{S}[t]$-modules satisfies the pushout product axiom with respect to the stable model structure. Further the category $\mathbb{S}[t]$-alg of symmetric ring spectra endowed with an endomorphism admits a stable model structure with weak equivalences and fibrations defined in the stable model structure of underlying symmetric spectra.*

Proof. To prove the first claim we must check that for cofibrations $i\colon A \to B$ and $j\colon C \to D$ of $\mathbb{S}[t]$-modules the pushout product map

$$A \wedge D \cup_{A \wedge C} B \wedge C \to B \wedge D$$

is a cofibration too. It suffices to check this for generating cofibrations, so we show this a little more general that for $i = \mathbb{S}[t] \wedge i'$ and $\mathbb{S}[t] \wedge j'$ with $i'\colon A' \to B'$ and $j'\colon C' \to D'$ cofibrations of symmetric spectra. In this case the pushout product has the form

$$\mathbb{S}[t] \wedge \mathbb{S}[t] \wedge (A' \wedge D' \cup_{A' \wedge C'} B' \wedge C') \xrightarrow{\mathbb{S}[t] \wedge \mathbb{S}[t] \wedge (i' \wedge j')} \mathbb{S}[t] \wedge \mathbb{S}[t] \wedge (B' \wedge D').$$

The pushout product axiom for holds for symmetric spectra, so the map in question will be a cofibration of $\mathbb{S}[t]$-modules, if $\mathbb{S}[t] \wedge \mathbb{S}[t]$ endowed with the endomorphism $t \wedge t$ is cofibrant as a $\mathbb{S}[t]$-module. But this smash product decomposes as a wedge of countably many copies of $\mathbb{S}[t]$ indexed by the set $\{\dots, t^2 \wedge 1, t \wedge 1, 1 \wedge 1, 1 \wedge t, 1 \wedge t^2, \dots\}$ and therefore is cofibrant as the wedge of cofibrant $\mathbb{S}[t]$-modules.

Every cofibration of $\mathbb{S}[t]$-modules is also a cofibration on underlying symmetric spectra. So if in addition i or j is a stable equivalences on underlying symmetric spectra, then so is the pushout product map $i \wedge j$ since the pushout product axiom holds for the smash product of symmetric spectra.

The monoid axiom ([SS00, Definition 3.3]) holds since the forgetful functor from $\mathbb{S}[t]$-modules to symmetric spectra preserves colimits, cofibrations, the smash product and stable equivalences. Now [SS00, Theorem 4.1] applies and endows the category of monoids of $\mathbb{S}[t]$-modules with respect to the smash product with a model structure created by the forgetful functor. □

In order to build a $\mathbb{S}[t]$-algebra from a right stable naive ring spectrum, we need to understand how right stable S-bimodules are built from elementary pieces. Here it is useful to consider the stable $\mathbb{P}^1$-model structure. The discussion of the $\mathbb{P}^1$-model structure is essentially taken from [HS06] and we include the proofs only to make the text as self contained as possible.

Theorem 3.3.2. *The category of S-bimodules admits a cofibrantly generated simplicial stable model structure with weak equivalences the maps inducing isomorphisms on left and right homotopy groups and acyclic fibrations the level acyclic fibrations. We refer to weak equivalences in this model structure as $\mathbb{P}^1$-equivalences.*

Definition 3.3.3. The maps σ_n^{L} and σ_n^{R} from Definition 2.2.1 are the adjoints of the two inclusions $S^1 \to S^1 \vee S^1 = (S \otimes S)_1$. Similarly there are maps $F_2(S^2) \to F_1(S^1)$ adjoint to the two inclusions $S^2 \to S^2 \vee S^2 = (F_1 S^1)_2$. These fit into a commutative diagram

$$\begin{array}{ccc} F_2(S^2) & \longrightarrow & F_1(S^1) \\ \downarrow & & \downarrow \\ F_1(S^1) & \longrightarrow & S \otimes S \end{array}$$

Denote by g the induced map

$$F_1(S^1) \vee^h_{F_2(S^2)} F_1(S^1) \to S \otimes S$$

from the simplicial homotopy pushout to $S \otimes S$. The simplicial mapping cylinder gives a factorization

$$\begin{array}{ccc} \mathrm{sh}_n(F_1(S^1) \vee^h_{F_2(S^2)} F_1(S^1)) & \xrightarrow{c_n} & Z_n \\ & \searrow^{g[-n]} & \downarrow{\scriptstyle r_n} \\ & & \mathrm{sh}_n(S \otimes S) \end{array}$$

for every $n \in \mathbb{N}$. Let K_n set set of maps consisting of the maps $c_n \square i_n$ for $n \in \mathbb{N}$ where i_n denotes the inclusion $\partial\Delta^n_+ \to \Delta^n_+$. Further let K be the union of all K_n and $J_{\mathbb{P}^1} = FJ \cup K$.

Lemma 3.3.4. *All maps in $J_{\mathbb{P}^1}$ are $\mathbb{P}^1$-equivalences.*

Proof. The maps in FJ are already known to be levelwise weak equivalences. A map in K is a $\mathbb{P}^1$-equivalence if and only if it is a π_*-isomorphism on underlying left and right modules. We show the left statement; the other is symmetric. It suffices to show that the maps c_n are π_*-isomorphisms of left S-modules since left S-modules have a simplicial model structure which implies that $c_n \square i_m$ is a π_*-isomorphism. We can assume $n = 0$ since otherwise the homotopy groups are only shifted. By definition r_0 is a simplicial homotopy equivalence, so we can reduce to the case to show that g is a π_* isomorphism. The two maps $F_2(S^2) \to F_1(S^1)$ occurring in the homotopy pushout defining g are level-wise injective. Hence the induced map

$$F_1(S^1) \overset{h}{\underset{F_2(S^2)}{\vee}} F_1(S^1) \to F_1(S^1) \vee_{F_2(S^2)} F_1(S^1)$$

is a level equivalence. But g factors through the map above and a map

$$\beta\colon\ F_1(S^1) \vee_{F_2(S^2)} F_1(S^1) \to S \otimes S$$

so we may assume that g is defined on the honest pushout. As a left module there are splittings

$$S \otimes S \cong S \otimes \vee \bigvee_{n \geq 1} (S \otimes S^n[n])$$

and

$$F_1(S^1) \vee_{F_2(S^2)} F_1(S^1) \cong (S \otimes S^1[1]) \vee \bigvee_{n \geq 1} (S \otimes S^n[n])$$

as left S-modules. Under these splittings β is the inclusion on the first summand and the identity else. Since the map $S \otimes S^1[1] \to S$ induces isomorphisms on left homotopy groups, so does β. Hence g is a left π_*-isomorphism as required. □

Lemma 3.3.5. *All maps in $J_{\mathbb{P}^1}$ are cofibrations of underlying left and right S-modules.*

Proof. Because of symmetry it suffices to consider the left case. The S-bimodule $F_1(S^1) = S \otimes S^1[1] \otimes S$ splits as a left module into an infinite wedge of free S-modules and hence is cofibrant as a left module. Similarly the other objects occurring in the definition of the maps in $J_{\mathbb{P}}$ are cofibrant as left modules. Since left S-modules form a simplicial model category, the mapping cylinder construction yields cofibrations when applied to maps between cofibrant objects by Lemma 1.1.7. □

Proof of Theorem 3.3.2. We use the recognition theorem [Hov99, Theorem 2.1.19]. First we show that any map in $J_{\mathbb{P}^1}$-cell is a cofibration of S-bimodules. Using the fact that the level model structure of S-bimodules is simplicial we see that all constructions used for defining the maps in $J_{\mathbb{P}^1}$ preserve cofibrations we conclude that $J_{\mathbb{P}^1}$ consists of cofibrations.

Next we show that a map is an acyclic $\mathbb{P}^1$-fibration if and only if it is a level acyclic fibration. The maps in $J_{\mathbb{P}^1}$ are $\mathbb{P}^1$-equivalences by Lemma 3.3.4 and hence are stable equivalences of underlying left and right S-modules as well. From Lemma 3.3.5 we know that these maps are also cofibrations of underlying left and right S-modules. Since cobase changes and sequential colimits preserve acyclic cofibrations of left and right modules the maps in $J_{\mathbb{P}^1}$-cell induce isomorphisms on left and right homotopy groups and are thus $\mathbb{P}^1$-equivalences.

Now we show that a map is an acyclic $\mathbb{P}^1$-fibration if and only if it is a level acyclic fibration on underlying graded spaces. Suppose p is a level acyclic fibration. By definition every $\mathbb{P}$-cofibration has the left lifting property with respect to p and in particular every acyclic $\mathbb{P}^1$-cofibration has the left lifting property with respect to p. Hence p is a $\mathbb{P}^1$-fibration which is an equivalence in each level and thus a $\mathbb{P}^1$-equivalence. Altogether p is an acyclic $\mathbb{P}^1$-fibration.

Conversely, suppose p is an acyclic $\mathbb{P}^1$-fibration. By the small object argument [Hov99, Theorem 2.1.14] p can be factored as $p = qi$ where i is an FI-cofibration and q a FI-injective map. Since q is a level equivalence it is a $\mathbb{P}^1$-equivalence and 2-out-of-3 for $\mathbb{P}^1$-equivalences implies that i is also a $\mathbb{P}^1$-equivalence. Therefore i is an acyclic $\mathbb{P}^1$-cofibration and has the left lifting property with respect to p. Such a lift implies that p is a retract of q and hence a level acyclic fibration.

It remains to show that the $\mathbb{P}^1$-model structure is stable. Let X be a cofibrant S-bimodule and $\Sigma X \to Y$ be a $\mathbb{P}^1$-fibrant replacement. The adjunction unit of suspension and loop functor on the homotopy category is modeled by the composition

$$X \to \Omega\Sigma X \to \Omega Y$$

which we have to show to be a $\mathbb{P}^1$-equivalence. This means we have to show that it is a stable equivalence of underlying left and right modules. We show the left case the other being completely symmetric. Let $Y \to Z$ be a stable replacement in the stable model structure of left S-modules. Then the composite

$$X \to \Omega Y \to \Omega Z$$

models the adjunction unit of the loop and suspension functor on the homotopy category of S-modules. Since the loop functor only shifts left homotopy groups the map $\Omega Y \to \Omega Z$ is an isomorphism on left homotopy groups so a stable equivalence of left S-modules. Stability of the stable model structure of left S-modules ensures that the above composite is a stable equivalence of left S-modules and by 2-out-of-3 so is the map $X \to \Omega Y$. This finishes the proof. □

We want to know how the $\mathbb{P}^1$-model structure relates to the left an right model structure. Since a $\mathbb{P}^1$-equivalence is in particular a left and a right stable equivalence, the identity functor descends to derived functors

$$\mathrm{Ho}^{\mathbb{P}^1}(S\text{-bimod}) \to \mathrm{Ho}^{\mathrm{L}}(S\text{-bimod}) \quad \text{and} \quad \mathrm{Ho}^{\mathbb{P}^1}(S\text{-bimod}) \to \mathrm{Ho}^{\mathrm{R}}(S\text{-bimod})$$

respectively. In case of the right model structure we have the following result.

Lemma 3.3.6. *The functor*

$$\mathrm{Ho}^{\mathbb{P}^1}(S\text{-bimod}) \to \mathrm{Ho}^{\mathrm{R}}(S\text{-bimod})$$

restricts to an equivalence between the full subcategory of right stable S-bimodules in $\mathrm{Ho}^{\mathbb{P}^1}(S\text{-bimod})$ *and the right homotopy category* $\mathrm{Ho}^{\mathrm{R}}(S\text{-bimod})$.

Proof. A stable equivalence in the right stable structure between fibrant objects is a level equivalence and hence a $\mathbb{P}^1$-equivalence. So the identity on S-bimodules derives to a functor from $\mathrm{Ho}^{\mathrm{R}}(S\text{-bimod})$ to $\mathrm{Ho}^{\mathbb{P}^1}(S\text{-bimod})$ giving a derived right adjoint. A fibrant object in the right stable model structure is a right Ω-spectrum and so by Example 2.1.19 right stable. So the right derived functor takes values in right stable S-bimodules. Given a right stable S-bimodule and take a fibrant replacement $f\colon M \to M^{rf}$ in the right stable structure. Then f induces isomorphisms on π_*^{R} by definition. By Lemma 2.1.13 it also induces an isomorphism on total homotopy groups. But both M and M^{rf} are right stable, so by the same lemma the left homotopy groups coincide with the total homotopy groups. Hence f also induces isomorphisms on left homotopy groups and therefore it is a $\mathbb{P}^1$-equivalence. This shows that the derived adjunction unit is an isomorphism on right stable S-bimodules. The derived adjunction counit is an isomorphism on all S-bimodules since a $\mathbb{P}^1$-equivalence is in particular a right stable equivalence. □

There is a fundamental right stable S-bimodule defined as follows.

Definition 3.3.7. Let $(S \otimes S)^{\mathrm{rst}}$ be the S-bimodule which is given as right S-module by the wedge

$$(S \otimes S)^{\mathrm{rst}} = \bigvee_{\mathbb{N}} S$$

with left action

$$S^1[1] \otimes \left(\bigvee_{\mathbb{N}} S \right) \to \bigvee_{\mathbb{N}} S$$

the map that takes the j-th copy of $S^1[1] \otimes S$ to the $(j+1)$-st first copy of S by the left action on the sphere. Further let $(S \otimes S)^{\mathrm{rst}}_c \to (S \otimes S)^{\mathrm{rst}}$ a cofibrant replacement in the level model structure of S-bimodules and denote by $\mathcal{H}$ the union of the set $J_{\mathbb{P}^1}$ of generating acyclic cofibrations of the $\mathbb{P}^1$-model structure and the maps

$$\partial\Delta^i_+[n] \otimes (S \otimes S)^{\mathrm{rst}} \to \Delta^i_+[n] \otimes (S \otimes S)^{\mathrm{rst}}$$

for $i, n \geq 0$.

Lemma 3.3.8. *The S-bimodule $(S \otimes S)^{\mathrm{rst}}$ is right stable.*

Proof. As a left S-module, $(S \otimes S)^{\mathrm{rst}}$ is isomorphic to

$$(S \otimes S)^{\mathrm{rst}} \cong \left(\bigvee_{j \in \mathbb{N}} S^j[j] \otimes S \right) \vee \left(\bigvee_{j>0} S \right),$$

hence the left homotopy groups are isomorphic to direct sum

$$\pi^{\mathrm{L}}_*(S \otimes S)^{\mathrm{rst}} = \bigoplus_{\mathbb{Z}} \pi^{\mathrm{L}}_* S$$

under the same identifications as we used in Example 2.1.15. We also see that the right action R is again given by shifting elements one degree up, so as a $\mathbb{Z}[R]$-module we have an isomorphism

$$\pi_*^{\mathrm{L}}(S \otimes S)^{\mathrm{rst}} \cong \pi_*^{\mathrm{L}} S \otimes \mathbb{Z}[R^{\pm 1}]$$

and $(S \otimes S)^{\mathrm{rst}}$ is in particular right stable. □

Note that:

- all maps in $\mathcal{H}$ are cofibrations between cofibrant S-modules;
- the cofiber of every map in $J_{\mathbb{P}}$ has trivial left and right homotopy groups, hence is in particular right stable;
- the cofiber of a maps $\partial\Delta_+^i[n] \otimes (S \otimes S)_c^{\mathrm{rst}} \to \Delta_+^i[n] \otimes (S \otimes S)_c^{\mathrm{rst}}$ is right stable since $(S \otimes S)_c^{\mathrm{rst}}$ is.

It follows that the cofiber of every $\mathcal{H}$-cell complex is right stable. Now we show that conversely every right stable S-bimodule is equivalent to a retract of a $\mathcal{H}$-cell complex.

Definition 3.3.9. For a S-bimodule M we denote by $M_{\mathcal{H}}$ the $\mathcal{H}$-cell complex obtained by applying the small object argument ([Hov99, Theorem 2.1.14]) to the map $* \to M$ with respect to $\mathcal{H}$. Then $M_{\mathcal{H}}$ is functorial in M and comes with a natural map $M_{\mathcal{H}} \to M$. We call $M_{\mathcal{H}}$ the *$\mathcal{H}$-approximation* of M.

Definition 3.3.10. We call a naive ring spectrum R *cofibrant*, if the map $* \to R$ has the left lifting property with respect to all maps of naive ring spectra which are acyclic fibrations on underlying graded spaces.

Denote by T the tensor algebra functor from S-bimodules to naive ring spectra which is left adjoint to the forgetful functor. Denote by $T(\mathcal{H})$ the set of maps obtained from $\mathcal{H}$ by applying T to each map. Further denote by $R_{\mathcal{H}} \to R$ the $T(\mathcal{H})$-approximation obtained by applying the small object argument with with respect to $T(\mathcal{H})$ to the unit map $S \to R$.

Lemma 3.3.11. (a) *Let f be a map between right stable S-bimodules which has the left lifting property for the set $\mathcal{H}$. Then f is a level acyclic fibration.*
(b) *Every cofibrant and right stable S-bimodule is a retract of an $\mathcal{H}$-cell complex.*
(c) *Every $T(\mathcal{H})$-cell complex is cofibrant right stable as an S-bimodule.*
(d) *Every cofibrant and right stable naive ring spectrum is a retract of a $T(\mathcal{H})$-cell complex.*

Proof. (a) Since $\mathcal{H}$ contains the acyclic generating cofibrations of the $\mathbb{P}^1$-model structure, the map f is in particular a fibration in the $\mathbb{P}$-model structure. Then F, denoting the fiber of the $\mathbb{P}$-fibration f, is fibrant in the $\mathbb{P}$-model structure. The right lifting property for the maps $\partial\Delta_+^i[n] \otimes (S \otimes S)_c^{\mathrm{rst}} \to \Delta_+^i[n] \otimes (S \otimes S)_c^{\mathrm{rst}}$ give by adjunction the right lifting property for the map $\mathrm{map}((S \otimes S)_c^{\mathrm{rst}}, F) \to *$ with respect to the maps $\partial\Delta_+^i \to \Delta_+^i$ implying that the mapping space is weakly contractible. Using stability we conclude that the groups

$$[(S \otimes S)^{\mathrm{rst}}, F]_n^{\mathrm{Ho}^{\mathbb{P}^1}(S\text{-bimod})}$$

vanish for all integers n. The source and the target of f are right stable and so is F. By Lemma 3.3.6 these groups are isomorphic to the groups

$$[(S \otimes S)^{\mathrm{rst}}, F]_n^{\mathrm{Ho}^R(S\text{-mod})}.$$

Since the map $S \otimes S \to (S \otimes S)^{\mathrm{rst}}$ is a weak equivalence in the right model structure, it induces an isomorphism

$$[(S \otimes S)^{\mathrm{rst}}, F]^{\mathrm{Ho}^R(S\text{-bimod})} \cong [S \otimes S, F]^{\mathrm{Ho}^R(S\text{-bimod})}$$

and by adjunction the left group is isomorphic to $\pi_*^{\mathrm{R}}F$. Since F is right stable the left homotopy groups of F also vanish. Therefore the fiber F is weakly contractible in the $\mathbb{P}^1$-model structure and so f an acyclic fibration, hence a level acyclic fibration.

(b) The $\mathcal{H}$-approximation map has the right lifting property for $\mathcal{H}$. Since both $M_{\mathcal{H}}$ and M are right stable, the approximation map p is a level acyclic fibration by part (a). Since M is cofibrant, it is a retract of the $\mathcal{H}$-cell complex $M_{\mathcal{H}}$.

(c) It suffices to show that right stability and the property of having cofibrant underlying bimodules is inherited when attaching $T(\mathcal{H})$-cells. Suppose R is a naive ring spectrum with these two properties and P is obtained from R by a pushout in the category of naive ring spectra

$$\begin{array}{ccc} T(A) & \longrightarrow & T(B) \\ \downarrow & & \downarrow \\ R & \longrightarrow & P \end{array},$$

where $A \to B$ is a wedge of maps in $\mathcal{H}$. By [SS00, Proof of Lemma 6.2] the underlying bimodule of P can be written as the colimit of a sequence of cofibrations of bimodules

$$R = P_0 \to P_1 \to \cdots \to P_n \to \cdots$$

with sub-quotients of the form

$$P_n/P_{n-1} \cong (R \otimes_S B/A)^{\otimes_S n} \otimes_S R.$$

Since R and B/A are cofibrant and right stable as bimodules, so is the sub-quotient by Lemma 2.4.3. Therefore by induction all the S-bimodules P_n are right stable and hence so is the colimit P.

(d) Since the approximation map $R_{\mathcal{H}} \to R$ has the right lifting property for $T(\mathcal{H})$, the underlying S-bimodule map has the right lifting property for $\mathcal{H}$. By part (c) the naive ring spectrum $R_{T(\mathcal{H})}$ is right stable and since R is also right stable, the approximation map is a level acyclic fibration of underlying S-bimodules by part (a). Since R is cofibrant it follows that it is a retract of the $T(\mathcal{H})$-cell complex $R_{\mathcal{H}}$. □

Let X and Y be $\mathbb{S}[t]$-modules. Then the maps

$$X_p \wedge Y_q \to (X \wedge Y)_{p+q}$$

assemble to a map

$$UX \otimes_S UY \to U(X \wedge Y)$$

of S-bimodules. For S-bimodules M and N the adjoint of the composition

$$M \otimes_S N \to U(WM) \otimes_S U(WN) \to U((WM) \wedge (WN))$$

yields a map

$$\lambda\colon W(M \otimes_S N) \to (WM) \wedge (WN)$$

of $\mathbb{S}[t]$-modules.

Proposition 3.3.12. *If M and N are cofibrant S-bimodules such that M is right stable, then the map*

$$\lambda\colon W(M \otimes_S N) \to (WM) \wedge (WN)$$

is a stable equivalence of $\mathbb{S}[t]$-modules.

Proof. If we fix a cofibrant bimodule M, then the target of the map λ is a left Quillen functor in N from the left model structure of S-bimodules to the category of $\mathbb{S}[t]$-modules. If M is also right stable, then $M \otimes_S (-)$ preserves π_*^{L}-isomorphisms by Lemma 2.4.2. Using the small object argument it suffices to show the case where $N = S \otimes S$. Then the map in question has the form

$$W(M \otimes_S (S \otimes S)) \xrightarrow{\cong} W(M \otimes S) \to \mathbb{S}[t] \wedge (WM)$$

and there is a natural isomorphism $W(M \otimes S) \cong V({}_S M) \wedge \mathbb{S}[t]$. The composite map

$$\mathbb{S}[t] \wedge V({}_S M) \xrightarrow{\cong} W(M \otimes S) \xrightarrow{\lambda} (WM) \wedge \mathbb{S}[t]$$

is the $\mathbb{S}[t]$-linear extension of the map $\alpha_M \wedge \eta\colon V({}_S M) \to (WM) \wedge \mathbb{S}[t]$ where η denotes the unit of the symmetric ring spectrum $\mathbb{S}[t]$. The underlying symmetric spectrum of he source of the map λ is a countable wedge of $V({}_S M)$ whereas the target is a countable wedge of $W(M)$. The nth component of this decomposition is given by

$$V({}_S M) \xrightarrow{\alpha_M} W(M) \xrightarrow{t^{\wedge n}} W(M).$$

By Lemma 2.3.5 the map α_M is a stable equivalence since M was assumed to be cofibrant. But the the composite above is a stable equivalence if $t\colon W(M) \to W(M)$ is one which is indeed the case by Corollary 2.3.4 □

The functor U from $\mathbb{S}[t]$-modules to S-bimodules is (lax) monoidal and therefore takes monoids to monoids. Since U preserves limits as a functor from $\mathbb{S}[t]$-algebras to naive ring spectra, there exists a left adjoint functor

$$\Lambda\colon \text{(naive ring spectra)} \to \mathbb{S}[t]\text{-mod}$$

from $\mathbb{S}[t]$-algebras to naive ring spectra.

Definition 3.3.13. For a naive ring spectrum R let

$$\beta\colon W(R) \to \Lambda(R)$$

be the $\mathbb{S}[t]$-module map adjoint to the underlying S-bimodule map of the adjunction unit $R \to U(\Lambda(R))$.

Lemma 3.3.14. *If R is a cofibrant ring spectrum whose underlying S-bimodule is right stable, then the map $\beta\colon W(R) \to \Lambda(R)$ is a stable equivalence of symmetric spectra.*

Proof. By Lemma 3.3.11 (d), every right stable cofibrant ring spectrum is a retract of a $T(\mathcal{H})$-cell complex. So it suffices to show the statement for $T(\mathcal{H})$-cell complexes. We will prove this by showing that the property of the map β being a stable equivalence is inherited by attaching $T(\mathcal{H})$-cells. Let R be a cofibrant and right stable naive ring spectrum for which the lemma has already been established. As in the proof of Lemma 3.3.11 (c), let P be the pushout of a diagram of naive ring spectra

$$\begin{array}{ccc} T(A) & \longrightarrow & T(B) \\ \downarrow & & \downarrow \\ R & \longrightarrow & P \end{array}$$

where $A \to B$ is a wedge of maps in $\mathcal{H}$ and the underlying S-bimodule of P can be written as the colimit of sequence of cofibrations of S-bimodules

$$R = P_0 \to P_1 \to \cdots \to P_n \to \cdots$$

with sub-quotients of the form

$$P_n/P_{n-1} \cong (R \otimes_S B/A)^{\otimes_S n} \otimes_S R.$$

Since W is a left adjoint it takes this filtration of S-bimodules to a filtration of $\mathbb{S}[t]$-modules. But Λ is a left adjoint as well, so it obtain a pushout diagram of $\mathbb{S}[t]$-algebras

$$\begin{array}{ccc} \Lambda(T(A)) & \longrightarrow & \Lambda(T(B)) \\ \downarrow & & \downarrow \\ \Lambda(R) & \longrightarrow & \Lambda(P). \end{array}$$

Moreover, $\Lambda(T(A))$ is isomorphic to $T(W(A))$, the tensor algebra with respect to the smash product $\wedge$ on the $\mathbb{S}[t]$-module $W(A)$. Again, as above, we can write $\Lambda(P)$ as the colimit of a sequence

$$\Lambda(R) = Q_0 \to Q_1 \to \cdots \to Q_n \to \cdots$$

of $\mathbb{S}[t]$-modules with sub-quotients of the form

$$\begin{aligned} Q_n/Q_{n-1} &\cong (\Lambda(R) \wedge W(B)/W(A))^{\wedge n} \wedge \Lambda(R) \\ &\cong (\Lambda(R) \wedge W(B/A))^{\wedge n} \wedge \Lambda(R). \end{aligned}$$

The natural map $W(P) \to \Lambda(P)$ restricts to maps $W(P_n) \to \Lambda(P_n)$ which we will prove to be weak equivalences by induction on n. For $n = 0$ the statement is true by assumption. Now consider the commutative diagram

$$\begin{array}{ccccc} W(P_{n-1}) & \longrightarrow & W(P_n) & \longrightarrow & W((R \otimes_S B/A)^{\otimes_S n} \otimes_S R) \\ \downarrow & & \downarrow & & \downarrow \\ Q_{n-1} & \longrightarrow & Q_n & \longrightarrow & (\Lambda(R) \wedge W(B/A))^{\wedge n} \wedge \Lambda(R) \end{array}$$

in which both rows are cofibrations sequences between cofibrant $\mathbb{S}[t]$-modules. The right vertical map factors as the composite

$$W((R \otimes_S B/A)^{\otimes_S n} R) \to (W(R) \wedge W(B/A))^{\wedge n} \wedge W(R) \to (\Lambda(R) \wedge W(B/A))^{\wedge n} \wedge \Lambda(R).$$

The first map is a stable equivalence by an iterated application of Proposition 3.3.12 since all objects involved are cofibrant right stable S-bimodules. The second map is a stable equivalence since the map $W(R) \to \Lambda(R)$ is a stable equivalence (by assumption) between cofibrant symmetric spectra and since $W(B/A)$ is cofibrant. Applying $[-, E]_*$ for any symmetric spectrum E to the diagram above together with the five-lemma shows that $W(Q_n) \to Q_n$ is a stable equivalence if $W(Q_{n-1}) \to Q_{n-1}$ is one, but this we assumed by induction. Now the colimit of a sequence of cofibrations between cofibrant object models the homotopy colimit of the diagram. Since homotopy colimits are designed to preserve weak equivalences, the map on colimits is a stable equivalence. □

Theorem 3.3.15. *Let R be a naive ring spectrum whose underlying S-bimodule is right stable. Then there exists a fibrant symmetric ring spectrum T and a chain of left π_*-isomorphisms of naive ring spectra between R and $U(T)$, the underlying naive ring spectrum of T.*

Proof. Let R be a naive ring spectrum whose underlying S-bimodule is right stable. Lemma 3.3.11 (a) shows that the $T(\mathcal{H})$-cell approximation map $R_{T(\mathcal{H})} \to R$ is a level acyclic fibration, and by part (*c*) of the same lemma the approximation is cofibrant and we may assume that R itself is cofibrant. Now let $\Lambda(R) \to \Lambda(R)^{\mathrm{f}}$ be a stably fibrant replacement of $\Lambda(R)$ in the stable model category structure of $\mathbb{S}[t]$-algebras. We claim that then the adjoint

$$R \to U(\Lambda(R)^{\mathrm{f}})$$

of the fibrant replacement map induces isomorphisms of left homotopy groups which then finishes the proof of the theorem. We have to show that the map of underlying S-bimodules is a left stable equivalence. By Theorem 2.3.3 the functor U is a right Quillen equivalence when viewed as a functor from $\mathbb{S}[t]$-modules to S-bimodules with respect to the left model structure. Therefore the map is a left π_*-isomorphism if and only if its adjoint $W(R) \to \Lambda(R)^f$ is a stable equivalence of $\mathbb{S}[t]$-modules. But this last map is the composite of $\beta\colon W(R) \to \Lambda(R)$ and the fibrant replacement map $\Lambda(R) \to \Lambda(R)^f$. The first map is a stable equivalence by Lemma 3.3.14 since R was assumed to be cofibrant, and the second map is a fibrant replacement map which is a stable equivalence by definition. □

4. Morita theory for stable model categories

In this section we apply the theory of S-bimodules and naive ring spectra to Morita theory of stable model categories. Our main result is a generalization of a theorem due to Schwede and Shipley. Their theorem states that a model category whose homotopy category admits a compact generator (in the sense of triangulated categories) is Quillen equivalent to a category of modules of a certain symmetric ring spectrum. This is proved by replacing the model category in question Quillen equivalently by a model category which is enriched over symmetric spectra in homotopically compatible way. The symmetric ring spectrum is then obtained by taking the enriched endomorphism object which is by definition a symmetric ring spectrum. The process of replacing the model category by one enriched over symmetric spectra requires technical prerequisites: Schwede and Shipley assume the model category to be simplicial and cofibrantly generated. Our aim is to remove the assumptions by avoiding the replacement process and directly construct the endomorphism ring spectrum. If one tries to construct a symmetric ring spectrum directly, it is not a priori clear where the symmetric group actions should come from. We therefore construct a naive endomorphism spectrum and replace it afterwards with a symmetric one. We like to remark that even though our result is not only a generalization of Schwede's and Shipley's result, it also uses a completely different approach.

4.1. **The abstract Morita theorem.** In this section we give an axiomatic approach to Morita theory. We will define the notion of a Morita context for a given object X in a stable model category and prove a Morita like theorem for the case where X is compact in the homotopy category.

Definition 4.1.1. A *Morita context* for a bifibrant object X in a stable model category consist of an Quillen adjunction

$$X \wedge_E (-)\colon\ E\text{-mod} \rightleftarrows \mathcal{C}\ :\mathrm{Hom}(X,-)$$

where E is a naive ring spectrum with underlying bi-Ω-spectrum such that

(i) $\mathrm{Hom}(X,X)$ is stably equivalent to E as a E-module spectrum and

(ii) the natural map

$$[X,Y]^{\mathcal{C}} \xrightarrow{\mathrm{RHom}(X,-)} [\mathrm{RHom}(X,X),\mathrm{RHom}(X,Y)]^E$$

is an isomorphism for all Y in $\mathrm{Ho}(\mathcal{C})$.

We call a Morita context a *Morita equivalence*, if the underlying Quillen pair is in fact a Quillen equivalence.

Note that a Morita equivalence for X in particular means that homotopy categories $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(E\text{-mod})$ are equivalent as triangulated categories.

Remark 4.1.2. Given a Morita context for X there is an isomorphism of abelian groups

$$[X,Y]^{\mathcal{C}} \cong [\mathrm{RHom}(X,X), \mathrm{RHom}(X,Y)]^E \cong [E, \mathrm{RHom}(X,Y)]^E \cong \pi_0 \,\mathrm{RHom}(X,Y)$$

resulting from condition (ii), a choice of a stable equivalence $E \simeq \mathrm{RHom}(X,X)$ from condition (i) and adjunction isomorphisms. Using appropriate suspensions of Y, we obtain a natural isomorphism

$$\pi_* \,\mathrm{RHom}(X,Y) \cong [X,Y]^{\mathcal{C}}_*$$

of graded abelian groups. This justifies the notation $\mathrm{Hom}(X,-)$. When setting $Y = X$ it is straightforward to check that the isomorphism above is compatible with composition in $\mathcal{C}$ so that we obtain an isomorphism

$$\pi_* E \cong [X,X]^{\mathcal{C}}_*$$

of graded rings which is well defined up to inner automorphisms. Therefore we often refer to E as the *endomorphism ring spectrum* of X. Under the identification of $\pi_* E$ with $[X,X]_*$, the isomorphism $\pi_* \,\mathrm{RHom}(X,Y)$ is one of graded $\pi_* E$ modules.

Before we can state the our version of the Morita theorem, we have to recall some notions from triangulated categories.

Definition 4.1.3. Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. A full triangulated subcategory of $\mathcal{T}$ is called *localizing*, if it is closed under arbitrary coproducts. An object X in $\mathcal{T}$ is *compact*, if for any family $A_i, i \in I$, of objects in $\mathcal{T}$ the natural map

$$\bigoplus_{i\in I} [X, A_i] \to [X, \coprod_{i \in I} A_i]$$

is an isomorphism. An object is called a *generator*, if the smallest localizing subcategory containing the object is the whole category.

Now we give the main theorem of this section.

Theorem 4.1.4. *Let $\mathcal{C}$ be a stable model category and X a bifibrant object in $\mathcal{C}$ and*

$$X \wedge_E (-) \colon\, E\text{-mod} \rightleftarrows \mathcal{C} \,: \mathrm{Hom}(X,-)$$

a Morita context for X.

(i) *If X is compact, then the localizing subcategory of* $\mathrm{Ho}(\mathcal{C})$ *generated by X is triangulated equivalent the* $\mathrm{Ho}(E\text{-mod})$*, the homotopy category of E-modules.*
(ii) *If X is a compact generator of* $\mathrm{Ho}(\mathcal{C})$*, the Quillen adjunction between $\mathcal{C}$ and E-mod is in fact a Quillen equivalence, so in particular* $\mathrm{Ho}(\mathcal{C})$ *and* $\mathrm{Ho}(E\text{-mod})$ *are triangulated equivalent.*

Proof. The Quillen adjoint pair gives rise to a derived adjunction whose functors we denote by $X \wedge^{\mathrm{L}}_E (-)$ and $\mathrm{RHom}(X,-)$. By assumption $\mathrm{RHom}(X,X) = \mathrm{Hom}(X,X)$ is fibrant and E is cofibrant in E-mod, so a fixed choice of an isomorphism $E \cong \mathrm{Hom}(X,X)$ can be represented by a map $\varphi\colon\, E \to \mathrm{Hom}(X,X)$. We show that the adjoint map $\varphi^\natural \colon\, X \wedge^{\mathrm{L}}_E E \to X$ is an isomorphism as well. Consider the commutative diagram

$$\begin{array}{ccc} [\mathrm{Hom}(X,X), \mathrm{Hom}(X,Y)] & \xrightarrow{\varphi^*} & [E, \mathrm{Hom}(X,Y)] \\ \cong \downarrow & & \downarrow \cong \\ [X,Y] & \xrightarrow[(\varphi^\natural)^*]{} & [X \wedge^{\mathrm{L}}_E E, Y] \end{array}$$

where Y is an arbitrary fibrant object in $\mathcal{C}$. The right vertical map is an adjunction isomorphism, the left vertical map is an isomorphism by condition (ii) of a Morita context, and the top horizontal map is an isomorphisms since φ is one by condition (i) of the Morita context. Hence the bottom map is an isomorphism for any fibrant Y and thus $\varphi^\natural$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. Our strategy is to show that the adjunction unit

$$\epsilon\colon\ M \to \mathrm{RHom}(X, X \wedge^{\mathrm{L}}_E M)$$

is an isomorphism for all E-modules M and the counit

$$\eta\colon\ X \wedge^{\mathrm{L}}_E \mathrm{RHom}(X, Y)$$

is an isomorphism for all Y in the localizing subcategory generated by X. To do so, it will suffice to show that ϵ and η are isomorphisms on E and X respectively and both functors $X \wedge^{\mathrm{L}}_E (-)$ and $\mathrm{RHom}(X, -)$ are exact functors which preserve coproducts. It is in general true that the derived functors of a Quillen pair of stable model categories descend to an exact pair between the triangulated homotopy categories. Of course $X \wedge^{\mathrm{L}}_E (-)$ preserves coproducts as a left adjoint. To see that $\mathrm{RHom}(X, -)$ also preserves coproduct, note that $X \wedge^{\mathrm{L}}_E E$ is compact in $\mathrm{Ho}(\mathcal{C})$ being isomorphic to X, so that we have isomorphisms

$$\begin{aligned}
[E, \mathrm{RHom}(X, \bigvee_i Y_i)]^E &\cong [X \wedge^{\mathrm{L}}_E E, \bigvee_i Y_i]^{\mathcal{C}} \\
&\cong \bigoplus_i [X \wedge^{\mathrm{L}}_E E, Y_i] \\
&\cong \bigoplus_i [E, \mathrm{RHom}(X, Y_i)]^E \\
&\cong [E, \bigvee_i \mathrm{RHom}(X, Y_i)]^E
\end{aligned}$$

natural in E. Since E is a generator, Yoneda tells us, that there is an isomorphism

$$\mathrm{RHom}(X, \bigvee_i Y_i) \cong \bigvee_i \mathrm{RHom}(X, Y_i)$$

in $\mathrm{Ho}(E\text{-mod})$. Now we show that the derived adjunction counit η is an isomorphism on X. Consider the commutative triangle

$$\begin{array}{ccc}
[X, Y]^{\mathcal{C}} & \xrightarrow{\mathrm{RHom}(X,-)} & [\mathrm{Hom}(X, X), \mathrm{RHom}(X, Y)]^E \\
& \searrow^{\eta_X^*} & \downarrow \cong \\
& & [X \wedge^{\mathrm{L}}_E \mathrm{RHom}(X, X), Y]^{\mathcal{C}}
\end{array}$$

in which the top horizontal map is an isomorphism by condition (ii) of a Morita context and the right vertical by adjunction. Hence η_X induces an isomorphism for all Y in $\mathrm{Ho}(\mathcal{C})$, so is an isomorphism itself. As already said this implies that η is an isomorphism for all objects in the localizing subcategory generated by X, because $X \wedge^{\mathrm{L}}_E (-)$ and $\mathrm{RHom}(X, -)$ are both exact functors that preserve sums. Finally consider the commutative diagram

$$\begin{array}{ccc}
[E, \mathrm{RHom}(X, X \wedge^{\mathrm{L}}_E E)]^E & & \\
\uparrow (\epsilon_E)_* & \searrow^{\mathrm{RHom}(X, \varphi^\natural)} & \\
[E, E]^E & \xrightarrow{\varphi} & [E, \mathrm{RHom}(X, X)]^E
\end{array}$$

in which the diagonal and bottom horizontal maps are isomorphisms since φ and $\varphi^\natural$ are. But E is a generator of $\mathrm{Ho}(E\text{-mod})$ and thus ϵ_E is an isomorphism. As above, since $X \wedge^{\mathrm{L}}_{E} (-)$ and $\mathrm{RHom}(X,-)$ are exact and preserve coproducts, ϵ is an isomorphism on all objects in the localizing subcategory generated by E which is the whole category in this case. □

Now we give an examples where the Morita theorem applies.

Corollary 4.1.5. *Let R be a semi-stable symmetric ring spectrum. Then there is a Quillen equivalence*

$$R\text{-mod}_{\Sigma} \simeq UR\text{-mod}$$

between the model category of symmetric R-module spectra and the category of naive modules over the underlying naive ring spectrum $U(R)$ of R.

Proof. Take a fibrant replacement $R \to \bar{R}$ which is a π_*-isomorphism since R is assumed to be semi-stable. This implies that $UR \to U\bar{R}$ is a stable equivalence of naive ring spectra. There are Quillen equivalences $R\text{-mod}_{\Sigma} \simeq \bar{R}\text{-mod}_{\Sigma}$ by [HSS00, Theorem 5.4.5] and $UR\text{-mod} \to U\bar{R}\text{-mod}$ by Theorem 3.2.4. Note that the model structures on UR-mod and $U\bar{R}$-mod exist since UR and $U\bar{R}$ are right stable by Lemma 2.1.20. Hence we can assume that R has an underlying Ω-spectrum. Now let $E = UR$, the underlying naive ring spectrum, $X = R$ and $\mathrm{Hom}(X,Y) = UY$. Since fibrations and stable equivalences are in both cases defined on underlying spectra, $\mathrm{Hom}(X,-)$ is a right Quillen functor. To check that we indeed have a Morita context it remains to show that that the map $[R,Y]^R_* \xrightarrow{U} [UR,UY]^{UR}_*$ is an isomorphism for all fibrant symmetric R-modules Y. In the commutative diagram

$$\begin{array}{ccccc}
[R,Y]^R & \xrightarrow{\cong} & [\mathbb{S},Y]^{\mathcal{S}p^{\Sigma}} & \xrightarrow{\cong} & [S^0,Y_0]^{\mathcal{S}_*} \\
\Big\downarrow{\scriptstyle U} & & \Big\downarrow{\scriptstyle U} & & \Big\downarrow{\scriptstyle U} \\
[UR,UY]^{UR} & \xrightarrow[\cong]{} & [S,UY]^{\mathcal{S}p} & \xrightarrow[\cong]{} & [S^0,(UY)_0]^{\mathcal{S}_*}
\end{array}$$

all vertical maps are derived adjunction isomorphisms. Obviously we have $U(Y_0) = (UY)_0$ so the rightmost vertical map is an isomorphisms being actually the identity. It is well known, that R is a compact generator in $R\text{-mod}_{\Sigma}$, so the Morita Theorem 4.1.4 applies which finishes the proof. □

As another example we could use the abstract Morita theorem to reprove Theorem 3.2.4.

4.2. **Endomorphism ring spectra.** Suppose that $\mathcal{C}$ is a stable simplicial model category. Given a bifibrant object X we can set $\omega^0 X = X$ and take simplicial loops $\Omega\omega^0 X$ which gives a fibrant though not cofibrant object. Therefore we take a cofibrant replacement $\omega^1 X \to \Omega\omega^0 X$. Inductively we get a sequence $\omega^n X$ together with maps $\omega^{n+1}X \to \Omega\omega^n X$, *i.e.* desuspensions of X. Note that the adjoint maps $\Sigma\omega^{n+1}X \to \omega^n X$ form a spectrum except that the degree is lowered rather than raised. Formally we can consider ωX as a spectrum in the opposite category $\mathcal{C}^{\mathrm{op}}$ and refer to such an object as a *cospectrum.* For a cospectrum X we refer to the maps $X^{n+1} \to \Omega X^n$ as the structure maps rather than their adjoints. The dual concept of an Ω-spectrum is therefore called an Σ-*spectrum*, *i.e.* a cospectrum such that the adjoints $\Sigma X^{n+1} \to X^n$ of the structure maps are weak equivalences and each $\omega^n X$ is cofibrant. Given a cospectrum X and any object Y in $\mathcal{C}$ gives rise to an spectrum $\mathrm{Hom}(X,Y)$ given in level n by $\mathrm{map}_{\mathcal{C}}(X^n, Y)$ with adjoint structure maps

$$\mathrm{map}(X^n,Y) \to \mathrm{map}(\Sigma X^{n+1},Y) \cong \Omega\,\mathrm{map}(X^{n+1})$$

induced from the structure maps $\Sigma X^{n+1} \to X^n$. Note that $\mathrm{Hom}(X,Y)$ is an Ω-spectrum provided Y is fibrant. This yields a functor $\mathrm{Hom}(X,-)\colon \mathcal{C} \to \mathcal{S}p$ as it is given in [SS02]. Our approach is to modify this construction to give a functor which takes values in End(X)-modules rather than S-modules where End(X) is a naive ring spectrum built out of ωX.

Since we do not restrict ourselves to the case of simplicial model categories we take an axiomatic approach and assume that we have a model category with some additional data and some properties.

Axioms 4.2.1. Let $\mathcal{C}$ be a closed simplicial category, *i.e.* (co)tensored and enriched over $\mathcal{S}_*$, which has a model structure and a full subcategory $\mathcal{L}$ in which every object is cofibrant.

(A1) The subcategory $\mathcal{L}$ is closed under tensors, *i.e.* $K \wedge L \in \mathcal{L}$ for all $L \in \mathcal{L}$ and $K \in \mathcal{S}_*$.

(A2) The adjoint pairs

$$L \wedge (-)\colon\ \mathcal{S}_* \rightleftarrows \mathcal{C} :\ \mathrm{map}(L, -),$$

$$(-) \wedge K\colon\ \mathcal{C} \rightleftarrows \mathcal{C} : (-)^K$$

are Quillen pairs for $K \in \mathcal{S}_*$ and $L \in \mathcal{L}$, and

$$\mathrm{map}(-, L)\colon\ \mathcal{C} \rightleftarrows\ \mathcal{S}_*^{\mathrm{op}} \colon L^{(-)}$$

is a Quillen pairs all fibrant $L \in \mathcal{L}$.

(A3) The suspension functor $\Sigma = (-) \wedge S^1$ detects weak equivalences in $\mathcal{L}$.

For the rest of this section we assume that $\mathcal{C}$ is a model category together with a full subcategory $\mathcal{L}$ which in addition is a simplicial category satisfying Axioms 4.2.1 above.

There are two cases we are interested in. The first case is that of a simplicial model category $\mathcal{C}$ where $\mathcal{L} = \mathcal{C}^c$ is the full sub category of cofibrant objects (this is the example one should have in mind, see section 4.4 for details). The other example is the case of cosimplicial objects in any model category where $\mathcal{L}$ is the class of so called *cosimplicial frames* which we need to construct desuspension cospectra for arbitrary model categories. See section 4.5 and 4.6 for details. By desuspension spectrum we shall mean the following:

Definition 4.2.2. A *desuspension cospectrum* for a bifibrant object X in $\mathcal{L}$ is a Σ-cospectrum ωX in $\mathcal{C}$ such that

(i) $\omega^0 X = X$,
(ii) $\omega^n X$ is an object in $\mathcal{L}$ and bifibrant for all $n \geq 0$
(iii) and the structure maps $\omega^{n+1} X \to \Omega\omega^n X$, $n \geq 0$ is a fibration in $\mathcal{C}$ which is a colocal equivalence with respect to $\mathcal{L}$, which means that the induced maps on mapping spaces

$$\mathrm{map}_{\mathcal{C}}(L, \omega^{n+1} X) \xrightarrow{\sim} \mathrm{map}_{\mathcal{C}}(L, \Omega\omega^n X)$$

are weak equivalence of simplicial sets for all $L \in \mathcal{L}$.

Condition (iii) should be thought of a colocal version of desuspensions of X.

Remark 4.2.3. Given a cospectrum Y such that each Y^n is fibrant and the structure maps $Y^{n+1} \to \Omega Y^n$ of induce a weak equivalence on $\mathrm{map}(L, -)$ for some $L \in \mathcal{L}$ and all $n \geq 0$. This already implies that the "iterated" structure maps $Y^{p+q} \to \Omega^q Y^p$ do so as well. We prove this by induction on q. For $q = 1$ we have assumed this for all p. If the claim is true for some $q > 1$. The map $Y^{p+q+1} \to \Omega^{q+1} Y^p$

factors as $Y^{p+q+1} \to \Omega Y^{p+q} \to \Omega^{q+1} Y^p$ so we have a commutative diagram

$$\begin{array}{ccccc} \operatorname{map}(L, Y^{p+q+1}) & \xrightarrow{\sim} & \operatorname{map}(L, \Omega Y^{p+q}) & \longrightarrow & \operatorname{map}(L, \Omega^{q+1} Y^p) \\ & & \cong\downarrow & & \downarrow\cong \\ & & \Omega \operatorname{map}(L, Y^{p+q}) & \xrightarrow{\sim} & \Omega \operatorname{map}(L, \Omega^q Y^p). \end{array}$$

The left top horizontal map is a weak equivalence by the case $q = 1$ and the bottom horizontal is one by induction hypotheses. So the top composite is a weak equivalence for all p. □

In the following construction we make use of the simplicial mapping space in cospectra. We denote the category of cospectra in $\mathcal{C}$ by $c\mathcal{S}p$ and define the simplicial tensor of a cospectrum X and a simplicial set K in $\mathcal{S}_*$ point-wise by $(X \wedge K)_n = X^n \wedge K$ using the simplicial tensor on $\mathcal{C}$. Then $\operatorname{map}_{c\mathcal{S}p}(X, Y)$ in degree n is given by

$$\operatorname{map}_{c\mathcal{S}p}(X, Y)_n = c\mathcal{S}p(X \wedge \Delta^n_+, Y).$$

Note that we do note require a model structure on $c\mathcal{S}p$ nor do we claim any homotopically properties of the mapping space at this point.

Construction 4.2.4. Suppose we have a bifibrant object X in $\mathcal{C}$ together with a choice of a desuspension cospectrum ωX for it. We now define a graded space $\operatorname{End}(X)$ by setting

$$\operatorname{End}(X)_n = \operatorname{map}_{c\mathcal{S}p}(\operatorname{sh}^n \omega X, \omega X)$$

where sh^n is the usual shift, *i.e.* $(\operatorname{sh}^n Z)^m = Z^{n+m}$ for any cospectrum Z. In order to endow $\operatorname{End}(X)$ with the structure of a naive ring spectrum, we have to define associative multiplication maps

$$\mu_{p,q} \colon \operatorname{End}(X)_p \wedge \operatorname{End}(X)_q \to \operatorname{End}(X)_{p+q}$$

and a unit map

$$S^1 \to \operatorname{End}(X)_1.$$

First we construct the multiplication maps. For non-negative integers p and q define $\mu_{p,q}$ as the composite

$$\operatorname{End}(X)_p \wedge \operatorname{End}(X)_q \to \operatorname{map}_{c\mathcal{S}p}(\operatorname{sh}^{p+q} \omega X, \operatorname{sh}^q \omega X) \wedge \operatorname{End}(X)_q \to \operatorname{End}(X)_{p+q},$$

where the first map is the q-th shift functor on the first and the identity on the second smash factor, and the second map the enriched composition. Now we come to the unit map. Let $\varphi \colon S^1 \to \operatorname{map}_{\mathcal{C}}(\omega^1 X, \omega^0 X)$ be the adjoint of the first structure map $S^1 \wedge \omega^1 X \to \omega^0 X$ of the cospectrum ωX. We have a canonical map

$$\operatorname{ev}_0 \colon \operatorname{map}_{c\mathcal{S}p}(\operatorname{sh}^1 \omega X, \omega X) \to \operatorname{map}_{\mathcal{C}}(\omega^1 X, \omega^0 X)$$

which is given by sending a map of cospectra to the map in degree zero. We would like to lift the map φ to a map $\phi \colon S^1 \to \operatorname{map}_{c\mathcal{S}p}(\operatorname{sh}^1 \omega X, \omega X) = \operatorname{End}(X)_1$ as indicated in the diagram

$$\begin{array}{ccc} & & \operatorname{map}_{c\mathcal{S}p}(\operatorname{sh}^1 \omega X, \omega X) \\ & \overset{\phi}{\nearrow} & \downarrow \operatorname{ev}_0 \\ S^1 & \xrightarrow[\varphi]{} & \operatorname{map}_{\mathcal{C}}(\omega^1 X, \omega^0 X). \end{array}$$

Such a lift exists provided that the map ev_0 is an acyclic fibration of simplicial sets. This is actually the case and is proved in Corollary 4.2.7 below.

In order to prove that the evaluation map in construction 4.2.4 above is an acyclic fibration we prove a slightly more general result.

Proposition 4.2.5. *Let X and Y be desuspension cospectra (for X^0 and Y^0) in $\mathcal{C}$. Then the evaluation map*

$$\mathrm{ev}_m\colon\ \mathrm{map}_{c\mathcal{S}p}(X,Y)\to \mathrm{map}_{\mathcal{C}}(X^m,Y^m)$$

is weak equivalence for all $m\geq 0$ and a fibration for $m=0$.

Proof. We first show that the statement is true for $m=0$. For a cospectrum Z we can consider the truncated cospectrum $Z_{\leq n}$ which is given in degree k by Z^k for $k\leq n$ and $*$ otherwise. There is a pullback diagram

$$\begin{array}{ccc}
\mathrm{map}_{c\mathcal{S}p}(X_{\leq n+1},Y_{\leq n+1}) & \xrightarrow{\mathrm{ev}_{n+1}} & \mathrm{map}_{\mathcal{C}}(X^{n+1},Y^{n+1}) \\
\downarrow & & \downarrow{\scriptstyle \sigma^n} \\
\mathrm{map}_{c\mathcal{S}p}(X_{\leq n},Y_{\leq n}) & \longrightarrow & \mathrm{map}_{\mathcal{C}}(X^{n+1},\Omega Y^n)
\end{array}$$

which explains how one gets from maps between cospectra truncated at n to cospectra truncated at $n+1$ by adding maps in degree $n+1$ in a compatible way. The right vertical map is induced by the nth structure map $\sigma^n\colon\ Y^{n+1}\to\Omega Y^n$ and the bottom map is given by the composite

$$\mathrm{map}_{c\mathcal{S}p}(X_{\leq n},Y_{\leq n})\xrightarrow{\mathrm{ev}_n}\mathrm{map}(X^n,Y^n)\xrightarrow{\Omega}\mathrm{map}(\Omega X^n,\Omega Y^n)\longrightarrow\mathrm{map}(X^{n+1},\Omega Y^n),$$

where the last map is induced by the nth structure map of X. The left vertical map is given on 0-simplices by forgetting the degree $n+1$ part of a map. From the diagram above we see that there is an isomorphism

$$\mathrm{map}_{c\mathcal{S}p}(X,Y)\cong\lim_{n\geq 0}\mathrm{map}_{c\mathcal{S}p}(X_{\leq n},Y_{\leq n}).$$

By assumption the map $Y_{n+1}\to\Omega Y^n$ is a fibration, hence the right vertical map in the diagram above is a fibration since X^{n+1} is assumed to be in $\mathcal{L}$. Also by assumption the induced map is a weak equivalence, hence it is an acyclic fibration. Since pullback preserves such maps, the left vertical map is an acyclic fibration as well. It follows that the projection to the degree zero part

$$\lim_{n\geq 0}\mathrm{map}_{c\mathcal{S}p}(X_{\leq n},Y_{\leq n})\to\mathrm{map}_{c\mathcal{S}p}(X_{\leq 0},Y_{\leq 0})$$

is an acyclic fibration of simplicial sets. This map can be identified up to isomorphism with the evaluation at degree zero map. Now let m be any non-negative integer. There is a commutative diagram

$$\begin{array}{ccc}
\mathrm{map}_{c\mathcal{S}p}(X,Y) & \xrightarrow{\mathrm{sh}^m} & \mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^m X,\mathrm{sh}^m Y) \\
{\scriptstyle \mathrm{ev}_0}\downarrow{\scriptstyle \sim} & \searrow^{\mathrm{ev}_m} & {\scriptstyle \mathrm{ev}_0}\downarrow{\scriptstyle \sim} \\
\mathrm{map}(X^0,Y^0) & & \mathrm{map}(X^m,Y^m) \\
\downarrow{\scriptstyle \sim} & & \downarrow \\
\mathrm{map}(\Sigma^m X^m,Y^0) & \xleftarrow{\cong} & \mathrm{map}(X^m,\Omega^m Y^0)
\end{array}$$

in which the left and right lower vertical maps are induced by the canonical maps $\Sigma^m X^m\to X^0$ and $Y^m\to\Omega^m Y^0$ respectively. The first map is a weak equivalence between cofibrant objects by the assumption that X is a Σ-cospectrum, hence induces a weak equivalence on $\mathrm{map}(-,Y^0)$ as indicated in the diagram. If we show that the second map induces a weak equivalence, then we know that top vertical map is one too, since the upper left and right vertical maps are weak equivalences by the first part of our proof. Then we can conclude, that evaluation at m is a weak equivalence as required. So far we have reduced our problem to the task to show, that the lower right vertical map induced by $Y^m \to \Omega^m Y^0$ induces a weak equivalence on $\mathrm{map}(X^m, -)$ which follows from Remark 4.2.3 for $p = 0$ and $q = m$ when we set $L = X^m$. □

Remark 4.2.6. The cautioned reader may have noticed that in the proof above we did not use that Y is a Σ-cospectrum as it is not needed. For the sake of simplicity of the statement of the above Proposition we assumed that both X and Y are desuspension cospectra, so in particular Σ-cospectra.

The following corollary completes Construction 4.2.4.

Corollary 4.2.7. *The evaluation map at zero map from construction 4.2.4 is an acyclic fibration.*

Proof. Just take $\omega X = \omega Y$ in the Proposition above. □

Now having finished the Construction 4.2.4, it is worth to inspect the multiplication on $\mathrm{End}(X)$ explicitly. This is done in the following

Remark 4.2.8. Let $\mathrm{End}(X)$ be a the ring spectrum obtained from Construction 4.2.4 for some X. Recall that the multiplication $\mu_{p,q}\colon \mathrm{End}(X)_p \wedge \mathrm{End}(X)_q \to \mathrm{End}(X)_{p+q}$ is given by

$$\mathrm{End}(X)_p \wedge \mathrm{End}(X)_q \to \mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^{p+q}\,\omega X, \mathrm{sh}^q\,\omega X) \wedge \mathrm{End}(X)_q \to \mathrm{End}(X)_{p+q}.$$

where the first map is essentially the shift functor sh^q and the second map (enriched) composition. The evaluation at zero map relates $\mathrm{End}(X)_n = \mathrm{map}(\mathrm{sh}^n\,\omega X, \omega X)$ with $\mathrm{map}(\omega^q X, \omega^0 X)$ by a weak equivalence and we want to know how multiplication on $\mathrm{End}(X)$ relates then "multiplication" on $\mathrm{map}(\omega^{(-)}X, \omega^0 X)$. For the composition part of the multiplication this is easy. There is a commutative diagram

$$\begin{array}{ccc}
\mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^{p+q}\,\omega X, \mathrm{sh}^q\,\omega X) \wedge \mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^q\,\omega X, \omega X) & \xrightarrow{\circ} & \mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^{p+q}\,\omega X, \omega X) \\
\downarrow{\scriptstyle \mathrm{ev}_0 \wedge \mathrm{ev}_0} & & \downarrow{\scriptstyle \mathrm{ev}_0} \\
\mathrm{map}(\omega^{p+q}X, \omega^q X) \wedge \mathrm{map}(\omega^q X, \omega^0 X) & \xrightarrow{\circ} & \mathrm{map}(\omega^{p+q}X, \omega^0 X).
\end{array}$$

The shift part of the multiplication is more interesting. Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{End}(X)_p \wedge \mathrm{End}(X)_q & \xrightarrow{\mathrm{sh}^q} & (\mathrm{sh}^{p+q}\,\omega X, \mathrm{sh}^q\,\omega X) \wedge \mathrm{End}(X)_q \\
\downarrow{\scriptstyle \mathrm{ev}_0 \wedge \mathrm{ev}_0} & & \downarrow{\scriptstyle \mathrm{ev}_0 \wedge \mathrm{ev}_0} \\
\mathrm{map}(\omega^p X, X) \wedge \mathrm{map}(\omega^q X, X) & & \mathrm{map}(\omega^{p+q}X, \omega^q X) \wedge \mathrm{map}(\omega^q X, X) \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
\mathrm{map}(\Sigma^q \omega^{p+q} X, X) \wedge \mathrm{map}(\omega^q X, X) & \xrightarrow{\cong} & \mathrm{map}(\omega^{p+q}X, \Omega^q X) \wedge \mathrm{map}(\omega^q X, X)
\end{array} \tag{4.2.1}$$

in which the lower left and right vertical maps are induced by the canonical maps $\Sigma^q\omega^{p+q} \to \omega^p X$ and $\omega^q X \to \Omega^q \omega^0 X = \Omega^q X$ and therefore are weak equivalences. Taking π_0 of the lower half of the square and using the isomorphism $\pi_0\,\mathrm{map}(\omega^m X, \omega^n X) \cong [\omega^m X, \omega^n X]$ we see that starting with two maps $f\colon \omega^p X \to \omega^0 X$ and $g\colon \omega^q X \to \omega^0 X$ in the homotopy category, we can compose f with the map $\Sigma^q \omega^{p+q} X \to \omega^p X$ and adjoin it to a map $\omega^{p+q} X \to \Omega^q \omega^p X \to \Omega^q \omega^0 X$. But the lower right vertical maps in the diagram above gets invertible under taking π_0. Therefore there exists a map $\omega^{p+q}X \to \omega^q X$ such that the resulting diagram

$$\begin{array}{ccccc} \omega^{p+q}X & \longrightarrow & \omega^q X & \xrightarrow{g} & \omega^0 X \\ \downarrow & & \downarrow & & \\ \Omega^q\omega^p X & \xrightarrow{\Omega^q f} & \Omega^q\omega^0 X & & \end{array} \tag{4.2.2}$$

commutes in the homotopy category. The top row of the diagram is then the product of f and g. As this discussion indicates, trying to define $\mathrm{End}(X)_n = \mathrm{map}(\omega^n X, X)$ does not work, since when it comes to multiplication on $\mathrm{End}(X)$, one has to invert a certain map. In fact the map in question is a fibration between Kan sets and hence surjective. This means, that multiplication involves the choice of a certain map. Passing to the cospectrum ωX means in this context that we fix compatible choices for the various degrees.

We want to prove that the underlying S-module of $\mathrm{End}(X)$ is bistable, *i.e.* is a bi-Ω-spectrum, We begin with a little lemma.

Lemma 4.2.9. *The adjoint* $\phi^\natural\colon \mathrm{sh}^1\,\omega X \wedge S^1 \to \omega X$ *of the map* ϕ *of Construction 4.2.4 is a weak equivalence in each degree.*

Proof. By definition of ϕ, we have a commutative diagram

$$\begin{array}{ccc} & & \mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^1\,\omega X, \omega X) \\ & \overset{\phi}{\nearrow} & \downarrow\sim \\ S^1 & \xrightarrow{\bar\varphi} & \mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^1\,\omega X, \Omega^\infty\omega^0 X) \\ & \underset{\varphi}{\searrow} & \downarrow\cong \\ & & \mathrm{map}_{\mathcal{C}}(\omega^1 X, \omega^0 X) \end{array}$$

so that the composite of the two right vertical map is evaluation in degree 0 map ev_0. By adjunction of the top triangle we obtain the commutative diagram

$$\begin{array}{ccc} & \overset{\phi^\natural}{\nearrow} & \omega X \\ & & \downarrow \\ \mathrm{sh}^1\,\omega X \wedge S^1 & \xrightarrow{\bar\varphi^\natural} & \Omega^\infty\omega^0 X \end{array}$$

having the adjoint of the identity $\omega^0 X \to \mathrm{Ev}_0\,\Omega^\infty\omega^0 X$ as right vertical map given in degree n by the canonical map $\omega^n X \to \Omega^n\omega^0 X$ coming from the structure maps of ωX. By definition of $\bar\varphi$ we obtain (using the fact that Ev_0 is left adjoint to Ω^∞) a commutative diagram

$$\begin{array}{ccccc} \omega^{n+1}X \wedge S^1 & \xrightarrow{\tau} & S^1 \wedge \omega^{n+1}X & \xrightarrow{\phi_n^\natural} & \omega^n X \\ \downarrow & & & & \downarrow \\ \Omega^n(\omega^1 X \wedge S^1) & \xrightarrow{\Omega^n\tau} & \Omega^n(S^1 \wedge \omega^n X) & \xrightarrow{\Omega^n(\sigma^1)^\natural} & \Omega^n(\omega^0 X). \end{array}$$

By adjoining again we get that $S^n \wedge \phi_n^\natural$ is weakly equivalent to $(\sigma^1)^\natural$ since both $\mathrm{sh}^1\,\omega X \wedge S^1$ and ωX are Σ-cospectra, so the adjoints of their structure maps, occurring as vertical maps in the diagram above, are weak equivalences. In particular $(\sigma^1)^\natural$ is a weak equivalence and so is $S^n \wedge \phi_n^\natural$ by 2-out-of-3. But the latter map is a map in $\mathcal{L}$ and we assumed in Axioms 4.2.1 (A3) that suspension detects weak equivalences in $\mathcal{L}$. Hence $\phi_n^\natural$ is itself a weak equivalence. □

We want to show that the naive ring spectrum $\mathrm{End}(X)$ has a bi-Ω-spectrum as underlying S-bimodule. When working out the adjoints of the structure maps we will make frequent use of the following simple result from enriched category theory.

Suppose $\mathcal{D}$ be a closed $\mathcal{C}$-module category, where $\mathcal{C}$ is a closed symmetric monoidal category with product $\wedge$ and internal function object $\mathrm{Hom}(A,B)$ for A and B in $\mathcal{C}$, cotensor $\mathrm{Hom}(A,X)$ and tensor $A\wedge X$ for X in $\mathcal{D}$. Also denote by $\mathrm{Hom}(X,Y)$ the enriched function object in $\mathcal{C}$ for X and Y in $\mathcal{D}$. Given a map $\lambda\colon A\wedge X\to Y$, we can form the two adjoints $\lambda^\natural\colon A\to\mathrm{Hom}(X,Y)$ and $\lambda^\flat\colon X\to\mathrm{Hom}(A,Y)$. Then we have the

Lemma 4.2.10. *In the situation above*

(a) *the adjoint of the map*

$$A\wedge\mathrm{Hom}(Y,Z)\xrightarrow{\lambda^\natural\wedge\mathrm{id}}\mathrm{Hom}(X,Y)\wedge\mathrm{Hom}(Y,Z)\xrightarrow{\circ}\mathrm{Hom}(X,Z)$$

fits into the commutative diagram

$$\begin{array}{ccc}\mathrm{Hom}(Y,Z) & \longrightarrow & \mathrm{Hom}(A,\mathrm{Hom}(X,Z))\\ & \searrow^{\lambda^*} & \downarrow\cong\\ & & \mathrm{Hom}(A\wedge X,Z),\end{array}$$

(b) *and the adjoint of the composite*

$$\mathrm{Hom}(Z,X)\wedge A\xrightarrow{\mathrm{id}\wedge\lambda^\natural}\mathrm{Hom}(Z,X)\wedge\mathrm{Hom}(X,Y)\xrightarrow{\circ}\mathrm{Hom}(Z,Y)$$

fits into the commutative diagram

$$\begin{array}{ccc}\mathrm{Hom}(Z,X) & \longrightarrow & \mathrm{Hom}(A,\mathrm{Hom}(Z,Y))\\ \lambda^\flat_*\downarrow & & \downarrow\cong\\ \mathrm{Hom}(Z,\mathrm{Hom}(A,Y)) & \xrightarrow{\cong} & \mathrm{Hom}(Z\wedge A,X).\end{array}$$

Proposition 4.2.11. *Given a desuspension spectrum ωX for a bifibrant object X in $\mathcal{L}$, the endomorphism spectrum $\mathrm{End}(X)$ obtained from ωX by Construction 4.2.4 has an underlying bi-Ω-spectrum.*

Proof. We start with the right action on $\mathrm{End}(X)$. Consider the following diagram

$$\begin{array}{ccc}\mathrm{End}(X)_n\wedge S^1 & \xrightarrow[\sim]{\mathrm{ev}_1\wedge\mathrm{id}} & \mathrm{map}(\omega^{n+1}X,\omega^1X)\wedge S^1\\ \mathrm{id}\wedge\phi\downarrow & & \downarrow\mathrm{id}\wedge\varphi\\ \mathrm{End}(X)_n\wedge\mathrm{End}(X)_1 & \xrightarrow[\sim]{\mathrm{ev}_1\wedge\mathrm{ev}_0} & \mathrm{map}_{\mathcal{C}}(\omega^{n+1}X,\omega^1X)\wedge\mathrm{map}_{\mathcal{C}}(\omega^1X,\omega^0X)\\ \mu_{n,1}\downarrow & & \downarrow\circ\\ \mathrm{End}(X)_{n+1} & \xrightarrow[\sim]{\mathrm{ev}_0} & \mathrm{map}_{\mathcal{C}}(\omega^{n+1}X,\omega^0X)\end{array}$$

in which the upper square commutes by the construction of ϕ and φ and the lower square by the definition of the multiplication map $\mu_{n,1}$ of $\mathrm{End}(X)$. Note that the horizontal maps are all weak equivalences since the evaluation maps are acyclic fibrations by Corollary 4.2.7 and smashing with simplicial sets preserves weak equivalences. By adjunction we obtain from the outer square another commutative square

$$\begin{array}{ccc} \mathrm{End}(X)_n & \xrightarrow[\sim]{\mathrm{ev}_1} & \mathrm{map}_{\mathcal{C}}(\omega^{n+1}X, \omega^1 X) \\ \downarrow & & \downarrow \\ \Omega\,\mathrm{End}(X)_{n+1} & \xrightarrow[\sim]{\Omega\,\mathrm{ev}_0} & \Omega\,\mathrm{map}_{\mathcal{C}}(\omega^{n+1}X, \omega^0 X) \end{array}$$

in which the horizontal maps are still weak equivalences since Ω preserves acyclic fibrations. Our task is to prove that the left vertical map is a weak equivalence which is the case if and only if the right vertical map is one. The right vertical map is given by the composite

$$\begin{array}{ccc} \mathrm{map}(\omega^{n+1}X, \omega^1 X) & \xrightarrow{\mathrm{map}(-,\omega^0 X)} & \mathrm{map}\big(\mathrm{map}(\omega^1 X, \omega^0 X), \mathrm{map}(\omega^{n+1}X, \omega^0 X)\big) \\ & & \downarrow{\scriptstyle \varphi^*} \\ & & \Omega\,\mathrm{map}(\omega^{n+1}X, \omega^0 X). \end{array}$$

A short diagram chase shows that the map above fits into the following commutative square

$$\begin{array}{ccc} \mathrm{map}(\omega^{n+1}X, \omega^1 X) & \longrightarrow & \Omega\,\mathrm{map}(\omega^1 X, \omega^0 X) \\ {\scriptstyle \sigma^1_*}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathrm{map}(\omega^{n+1}X, \Omega\omega^0 X) & \xrightarrow{\cong} & \mathrm{map}(S^1 \wedge \omega^{n+1}X, \omega^0 X), \end{array}$$

where $\sigma^1 \colon \omega^1 X \to \Omega\omega^0 X$ is the first structure map of ωX (from which φ is obtained by two adjunctions), and two remaining maps in the diagrams are adjunction isomorphisms. Part (ii) of Definition 4.2.2 of a desuspension cospectrum ensures that σ^1 induces a weak equivalence on $\mathrm{map}(\omega^{n+1}X, -)$. We conclude that the map in question is a weak equivalence which completes the proof that $\mathrm{End}(X)$ is a right Ω-spectrum.

Now we turn to the left S-module structure of $\mathrm{End}(X)$. For this we consider the diagram

$$\begin{array}{ccc} S^1 \wedge \mathrm{End}(X)_n & \xrightarrow[\sim]{\mathrm{id} \wedge \mathrm{ev}_0} & S^1 \wedge \mathrm{map}(\omega^n X, \omega^0 X) \\ {\scriptstyle \phi\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \phi\wedge\mathrm{id}} \\ \mathrm{End}(X)_1 \wedge \mathrm{End}(X)_n & \xrightarrow[\sim]{\mathrm{id} \wedge \mathrm{ev}_0} & \mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^1\,\omega X, \omega X) \wedge \mathrm{map}_{\mathcal{C}}(\omega^n X, \omega^0 X) \\ {\scriptstyle \mu_{1,n}}\downarrow & & \downarrow \\ \mathrm{End}(X)_{n+1} & \xrightarrow[\sim]{\mathrm{ev}_0} & \mathrm{map}_{\mathcal{C}}(\omega^{n+1}X, \omega^0 X), \end{array}$$

in which the lower right vertical map is given by composing

$$\mathrm{End}(X)_n \wedge \mathrm{map}_{\mathcal{C}}(\omega^n X, \omega^0 X) \xrightarrow{\mathrm{ev}_n \wedge \mathrm{id}} \mathrm{map}_{\mathcal{C}}(\omega^{n+1}X, \omega^n X) \wedge \mathrm{map}_{\mathcal{C}}(\omega^n X, \omega^0 X)$$

with the enriched composition map. Note that the left vertical composite is actually $(\mathrm{ev}_n\,\phi \wedge \mathrm{id})$ composed with enriched composition and the adjoint of $\mathrm{ev}_n\,\phi$ is $(\phi^\natural)_n$, the degree n part of the adjoint of ϕ. Since we consider, as in case of the right structure, the adjoint of the outer square, the resulting map on the right hand side, by Lemma 4.2.10, turns out to be the composite

$$\operatorname{map}_{\mathcal{C}}(\omega^n X, \omega^0 X) \xrightarrow{(\phi_n^\natural)^*} \operatorname{map}_{\mathcal{C}}(S^1 \wedge \omega^{1+n} X, \omega^0 X) \cong \Omega \operatorname{map}_{\mathcal{C}}(\omega^{1+n} X, \omega^0 X).$$

By Lemma 4.2.9 the degree n part of the adjoint $\phi^\natural\colon S^1 \wedge \operatorname{sh}^1 \omega X \to \omega X$ of ϕ is a weak equivalence. Therefore the adjoints of the left structure maps of $\operatorname{End}(X)$ are weak equivalences as well, which completes the proof. □

4.3. **Morita contexts.** We now construct a Morita context for a (bifibrant) object X in $\mathcal{L}$. As in case of the construction of endomorphism ring spectra we take an axiomatic point of view. More precisely we assume that we are given a desuspension cospectrum ωX for X and then define $\operatorname{Hom}(X, -)$ in a quite similar fashion as we did with $\operatorname{End}(X)$.

Construction 4.3.1. Suppose we are given a desuspension cospectrum ωX for a bifibrant object X in $\mathcal{C}$. Using Construction 4.2.4 we obtain an naive ring spectrum $\operatorname{End}(X)$. First we construct the functor $\operatorname{Hom}(X, -)\colon \mathcal{C} \to \operatorname{End}(X)$-mod. For any object Y we define $\operatorname{Hom}(X, Y)$ in degree n by

$$\operatorname{Hom}(X, Y)_n = \operatorname{map}_{\mathcal{C}}(\omega^n X, Y)$$

obtaining a graded space. We have to show that $\operatorname{Hom}(X, Y)$ is in fact a (left) module over $\operatorname{End}(X)$ for which we have to provide a associative and unital maps

$$\alpha_{p,q}\colon \operatorname{End}(X)_p \wedge \operatorname{Hom}(X, Y)_q \to \operatorname{Hom}(X, Y)_{p+q}.$$

These are obtained by composing the map

$$\operatorname{End}(X)_q \wedge \operatorname{Hom}(X, Y)_q \xrightarrow{\operatorname{ev}_q \wedge \operatorname{id}} \operatorname{map}_{\mathcal{C}}(\omega^{p+q} X, \omega^q X) \wedge \operatorname{map}_{\mathcal{C}}(\omega^q X, Y)$$

with the enriched composition. It is clear from the definition of the multiplication of $\operatorname{End}(X)$ that these maps are associative and unital. For the behavior on maps we just note that $\operatorname{map}_{\mathcal{C}}(\omega^n X, -)$ is a functor by birthright. The functor $\operatorname{Hom}(X, -)$ has a left adjoint, denoted by $X \wedge_{\operatorname{End}(X)} (-)$, and which is given by coequalizer of the diagram

$$\bigvee_{p,q \geq 0} \omega^{p+q} X \wedge \operatorname{End}(X)_p \wedge Y_q \rightrightarrows \bigvee_{n \geq 0} \omega^n X \wedge Y_n$$

in which the upper map is induced by the left action of E on Y and the lower by composing

$$\omega^{p+q} X \wedge \operatorname{End}(X)_p \wedge Y_q \xrightarrow{\operatorname{sh}^q} \operatorname{map}_{c\mathcal{S}p}(\operatorname{sh}^{p+q} \omega X, \operatorname{sh}^q \omega X) \wedge Y_q$$

with

$$\omega^{p+q} X \wedge \operatorname{map}_{c\mathcal{S}p}(\operatorname{sh}^{p+q} \omega X, \operatorname{sh}^q \omega X) \wedge Y_q \xrightarrow{\operatorname{ev}_0} \omega^{p+q} X \wedge \operatorname{map}_{\mathcal{C}}(\omega^{p+q} X, \omega^q X) \wedge Y_q$$

and the formal evaluation map

$$\omega^{p+q} X \wedge \operatorname{map}_{\mathcal{C}}(\omega^{p+q} X, \omega^q X) \wedge Y_q \to \omega^q X \wedge Y_q.$$

As expected we obtain a Quillen pair. Note that the stable model structure on $\operatorname{End}(X)$-mod exists since $\operatorname{End}(X)$ is right stable by Proposition 4.2.11.

Lemma 4.3.2. *The adjoint pair* $X \wedge (-)\colon \operatorname{End}(X)\text{-mod} \rightleftarrows \mathcal{C} : \operatorname{Hom}(X, -)$ *obtained from Construction 4.3.1 applied to a desuspension spectrum is a Quillen adjoint pair.*

Proof. Each $\omega^n X$ is in $\mathcal{L}$ and hence $\mathrm{map}_{\mathcal{C}}(\omega^n X, -)$ preserves (acyclic) fibrations. Since the stable acyclic fibrations in the stable model structure of $\mathrm{End}(X)$-modules are level acyclic fibrations, the functor $\mathrm{Hom}(X, -)$ preserves acyclic fibrations. By a criterion of Dugger (see [Dug01, A.2]) it remains to show that it preserves fibrations between fibrant objects. In order to show this we first show that $\mathrm{Hom}(X, -)$ maps fibrant objects to Ω-spectra. Again since $\omega^n X$ is cofibrant, $\mathrm{map}_{\mathcal{C}}(\omega^n X, Y)$ is fibrant for each X. To see this note, that the adjoint $\mathrm{map}_{\mathcal{C}}(\omega^n X, Y) \to \Omega\,\mathrm{map}_{\mathcal{C}}(\omega^{n+1} X, Y)$ of the n-th structure map

$$S^1 \wedge \mathrm{Hom}(X,Y)_n \xrightarrow{\phi\wedge\mathrm{id}} \mathrm{End}(X)_1 \wedge \mathrm{Hom}(X,Y)_n \xrightarrow{\alpha_{1,n}} \mathrm{Hom}(X,Y)_{1+n}$$

is up to the adjunction isomorphism $\Omega\,\mathrm{map}_{\mathcal{C}}(\omega^{n+1}X, Y) \cong \mathrm{map}_{\mathcal{C}}(S^1 \wedge \omega^{n+1} X)$ induced by the degree n part of $\phi^\natural$ which is a weak equivalence by Lemma 4.2.9. Hence the induced map is a weak equivalence too and $\mathrm{Hom}(X,Y)$ is an Ω-spectrum and therefore fibrant in the stable model structure of $\mathrm{End}(X)$-modules. Now suppose $Y \to Z$ is a fibration between fibrant objects in $\mathcal{C}$. We know that the induced map on $\mathrm{Hom}(X,-) = \mathrm{map}_{\mathcal{C}}(X,-)$ is a level fibration. We factor it into a acyclic stable cofibration $i\colon \mathrm{Hom}(X,Y) \to W$ and a stable fibration $p\colon W \to \mathrm{Hom}(X,Z)$. As we showed above $\mathrm{Hom}(X,Z)$ is fibrant and hence W is fibrant too. Therefore i is an acyclic cofibration between Ω-spectra, thus a level equivalence. But this means that i is an acyclic cofibration in the level model structure and therefore has the right lifting property with respect to the level fibration $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$. From this we see that $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$ is a stable fibration as retract of the stable fibration p which finishes the proof. □

Theorem 4.3.3. *Given a desuspension spectrum ωX for a bifibrant object X in $\mathcal{L}$, there exists a Morita context.*

Proof. From Construction 4.2.4 we obtain a naive ring spectrum $E = \mathrm{End}(X)$ which has an underlying bi-Ω-spectrum by Proposition 4.2.11. Construction 4.3.1 gives us an adjunction with right adjoint $\mathrm{Hom}(X,-)\colon \mathcal{C} \to E$-mod with $\mathrm{Hom}(X,Y)_n = \mathrm{map}_{\mathcal{C}}(\omega^n X, Y)$ and $\mathrm{End}(X)_n = \mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^n\,\omega X, \omega X)$. We know from 4.2.5 that the evaluation map

$$\mathrm{ev}_0\colon\ \mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^n\,\omega X, \omega X) \xrightarrow{\sim} \mathrm{map}_{\mathcal{C}}(\omega^n X, X)$$

gives a level equivalence $\mathrm{End}(X) \to \mathrm{Hom}(X,X)$ of graded spaces. It is immediate from the definitions that this map is compatible with the $\mathrm{End}(X)$-module structure so that we have a stable equivalence of $\mathrm{End}(X)$-modules as required by part (i) of the definition of a Morita context. For part (ii) we consider the composite

$$[X,Y]^{\mathcal{C}} \to [\mathrm{Hom}(X,X), \mathrm{Hom}(X,Y)] \cong [E, \mathrm{Hom}(X,Y)]^E \cong [S, \mathrm{Hom}(X,Y)]^{\mathcal{S}p}$$

sending the class of a map $f\colon\ X \to Y$ to the class of the composite

$$S \xrightarrow{\iota} E \xrightarrow{\varphi} \mathrm{Hom}(X,X) \xrightarrow{f_*} \mathrm{Hom}(X,Y),$$

where we wrote E for $\mathrm{End}(X)$ and assumed, for the sake of simplicity and without loss of generality, that Y is fibrant. Here $\iota\colon S \to E$ is the unit of the naive ring spectrum E and φ the evaluation at 0 map. The composition of these two should be though of as "selecting the identity" in $\mathrm{Hom}(X,X)$. Indeed, if one considers this map in spectrum level zero, the resulting composite

$$S^0 \to \mathrm{map}_{c\mathcal{S}p}(\mathrm{sh}^0\,\omega X, \omega X) \to \mathrm{map}_{\mathcal{C}}(\omega^0 X, X)$$

is the map adjoint to the 0th structure map of the cospectrum ωX, which is, up to coherence unit isomorphism, the identity on $\omega^0 X = X$. The upshot of this discussion is that, using the adjunction isomorphism, evaluation at level zero of a spectrum gives isomorphisms

$$[S, \mathrm{Hom}(X,Y)]^{\mathcal{S}p} \xrightarrow{\cong} [S^0, \mathrm{map}_{\mathcal{C}}(\omega^0 X, Y)]^{\mathcal{S}_*} \xrightarrow{\cong} [X,Y]^{\mathcal{C}}$$

sending the composite $S \xrightarrow{\iota} E \xrightarrow{\varphi} \mathrm{Hom}(X,X) \xrightarrow{f_*} \mathrm{Hom}(X,Y)$ given above back to f. Now the 2-out-of-3 property for isomorphisms shows that

$$[X,Y]^{\mathcal{C}} \xrightarrow{\mathrm{RHom}(X,-)} [\mathrm{RHom}(X,X), \mathrm{RHom}(X,Y)]$$

is in fact an isomorphism as required. □

4.4. **The simplicial case.** As promised in Section 4.2 we ensure the existence of desuspension spectra in a stable simplicial model category $\mathcal{C}$. For the rest of this section we fix a stable simplicial model category $\mathcal{C}$ and set $\mathcal{L} = \mathcal{C}^c$ to be the full subcategory of cofibrant objects. Being a simplicial model category, it is immediate that $\mathcal{C}$ satisfies (A1) and (A2) from Axioms 4.2.1. Axiom (A3) follows from the stability of the model structure on $\mathcal{C}$.

Lemma 4.4.1. *Given a bifibrant object X in a simplicial stable model category $\mathcal{C}$. Then there exists a desuspension spectrum ωX for X.*

Proof. We define a cospectrum ωX by setting $\omega^0 X = X$ and inductively $\omega^n X$ by the factorization

$$* \rightarrowtail \omega^{n+1} X \xrightarrow{\sim} \Omega \omega^n X$$

of the unique map from the zero object to the loop object of $\omega^n X$ by a cofibration followed by an acyclic fibration. We claim that we have constructed a desuspension cospectrum in the sense of Definition 4.2.2. By definition we have $\omega^0 X = X$ so condition (i) holds. We assumed X to be bifibrant. By definition $\omega^n X$ is cofibrant. If $\omega^n X$ is fibrant, so is $\Omega\omega^n X$ and hence is $\omega^{n+1} X$ since the map $\omega^{n+1} X \to \Omega\omega^n X$ is a fibration which proves part (ii) and the first statement of part (iii). The second half of part (iii) holds since the above mentioned map is actually an acyclic fibration which induces an acyclic fibration on the mapping spaces in question of our simplicial model category since $\omega^n X$ is cofibrant. By stability of $\mathcal{C}$ the adjoints of the structure maps of ωX are weak equivalences and so ωX is in fact a Σ-cospectrum. □

Putting everything together we have proved the following Morita theorem for compactly generated stable simplicial model categories.

Theorem 4.4.2. *For any stable simplicial model category $\mathcal{C}$ with a compact generator X, there exist a Morita equivalence*

$$X \wedge_E (-)\colon\ E\text{-mod} \rightleftarrows \mathcal{C} :\ \mathrm{Hom}(X,-)$$

and thus $\mathcal{C}$ is Quillen equivalent to a category of modules over a symmetric ring spectrum.

Proof. Under our assumptions there exists by Proposition 4.6.1 a desuspension cospectrum for X. By Theorem 4.3.3 we have a Morita context for X. Since X is assumed to be compact we can apply Theorem 4.1.4 to see that we actually have a Morita equivalence. Finally there exist a fibrant symmetric ring spectrum R together with a weak equivalence $E \to U(R)$ by Theorem 3.3.15. Therefore the invariance Theorem 3.2.4 implies that E-mod is Quillen equivalent to $U(R)$-mod which is in turn Quillen equivalent to R-mod_Σ by Corollary 4.1.5. □

4.5. **Frames and stable frames.** Now we turn to the case of an arbitrary model category. In order to have mapping spaces available we have to discuss cosimplicial (stable) frames.

Suppose $\mathcal{C}$ is any pointed model category. Given a cosimplicial object X in $\mathcal{C}$ we can consider the simplicial set

$$n \mapsto \mathcal{C}(X^n, Y)$$

for any $Y \in \mathcal{C}$. This construction is natural in Y, so we have a functor

$$\mathcal{C}(X, -)\colon\ \mathcal{C} \to \mathcal{S}_*$$

and we can ask under which circumstances this is a right Quillen functor. First of all we need a left adjoint. Since X is itself a functor $\Delta \to \mathcal{C}$, we can define a functor

$$X(-)\colon\ \mathcal{S}_* \to \mathcal{C}$$

to be the left Kan extension (see [ML98, X.3]) of X along the standard cosimplicial space $\Delta\colon\ \Delta \to \mathcal{S}_*$ (sending n to the simplicial set Δ^n_+). A way to describe $X(-)$ evaluated on a simplicial set K is the coend

$$X(K) = \int^{n \in \Delta} X^n \wedge K_n.$$

This functor turns out to be left adjoint to the one we described above. We ask under which conditions this is a Quillen functor. Assuming that it is a Quillen functor it must preserve weak equivalences between cofibrant objects. In particular the weak equivalences $\Delta^n_+ \to \Delta^0_+$ must be mapped to weak equivalences $X^n \to X^0$ and each X_n must be cofibrant. From this we see that all structure maps of X must be weak equivalences. Further the map $X \wedge \partial\Delta^n_+ \to X \wedge \Delta^n_+$ must be a cofibration. It turns out that these conditions are not only necessary but in fact sufficient.

Definition 4.5.1. A cosimplicial object X in a model category $\mathcal{C}$ is *Reedy cofibrant* if each X^n is cofibrant and the map $X \wedge \partial\Delta^n_+ \to X \wedge \Delta^n_+$ is a cofibration. A map $f\colon\ X \to Y$ of cosimplicial objects is a *Reedy cofibration* if the map

$$X \wedge \Delta^n_+ \cup_{X \wedge \partial\Delta^n_+} Y \wedge \partial\Delta^n_+ \to Y \wedge \Delta^n_+$$

is a cofibration in $\mathcal{C}$ for all $n \geq 0$. A *cosimplicial frame* X is a Reedy cofibrant cosimplicial object X which is homotopically constant in the sense that every structure map $X^m \to X^n$ is a weak equivalence.

Example 4.5.2. Let K be a pointed simplicial set. Then $K \wedge \Delta^n_+$ is a cosimplicial object in simplicial sets for varying n. This is always a cosimplicial frame in $\mathcal{S}_*$ since $\mathcal{S}_*$ is a simplicial model category and therefore $K \wedge (-)$ is a left Quillen functor since every simplicial set is cofibrant.

Theorem 4.5.3. *Let $\mathcal{C}$ be a model category. The category $\mathcal{C}^\Delta$ of cosimplicial objects in $\mathcal{C}$ admits a model structure with weak equivalences the level equivalences, cofibrations the Reedy cofibrations and fibrations the maps with the right lifting property with respect to acyclic Reedy cofibrations.*

Proof. For a proof see [Hov99, Theorem 5.2.5]. □

The advantage of $\mathcal{C}^\Delta$ compared to $\mathcal{C}$ is, that we have a closed simplicial category.

Definition 4.5.4. If X is a cosimplicial object in $\mathcal{C}$ and K a pointed simplicial set, we define a cosimplicial object $X \wedge K$ by setting

$$(X \wedge K)^n = X \wedge (K \wedge \Delta^n_+)$$

with the structure maps induced by $\Delta^{(-)}$. In the special case where $K = S^1$ we denote $X \wedge S^1$ by ΣX and call it the suspension of X. For cosimplicial objects X and Y in $\mathcal{C}$ we define $\mathrm{map}_{\mathcal{C}^\Delta}(X, Y)$ to be the simplicial set given in degree n by

$$\mathrm{map}_{\mathcal{C}^\Delta}(X, Y) = \mathcal{C}^\Delta(X \wedge \Delta^n_+, Y)$$

and refer to it as the simplicial mapping space of X and Y.

Lemma 4.5.5. *Let $\mathcal{C}$ be a pointed model category.*

(i) *For any simplicial set K the functor $(-) \wedge K$ preserves (acyclic) Reedy cofibrations and level equivalences between Reedy cofibrant objects. In particular so does the suspension functor $\Sigma = (-) \wedge S^1$.*
(ii) *If X is a cosimplicial frame, then $\mathcal{C}(X, -)$ takes (acyclic) fibrations to (acyclic) fibrations in $\mathcal{S}_*$.*
(iii) *If Y is a fibrant object in $\mathcal{C}$, then the functor $\mathcal{C}(-, Y)$ takes (acyclic) Reedy cofibrations to (acyclic) fibrations of simplicial sets and level equivalences between Reedy cofibrant objects to weak equivalences of simplicial sets.*

Proof. (i) For a map $f\colon X \to Y$ of cosimplicial objects the map

$$(X \wedge K) \wedge \Delta^n_+ \cup_{(X\wedge K)\wedge \partial\Delta^n_+} (Y \wedge K) \wedge \partial\Delta^n_+ \to (Y \wedge K) \wedge \Delta^n_+$$

in $\mathcal{C}$ is isomorphic to the pushout product $f \,\square\, i$ where i is the inclusion $K \wedge \partial\Delta^n_+ \to K \wedge \Delta^n_+$. If f is a Reedy cofibration, $f \,\square\, i$ is one in $\mathcal{C}$. Hence $X \wedge K \to Y \wedge K$ is a cofibration by [Hov99, 5.7.1]. In cosimplicial degree n the map $f \wedge K$ is given by $f \wedge (K \wedge \Delta^n_+)$. So if f is a Reedy acyclic cofibration, then $f \wedge (K \wedge \Delta^n_+)$ is an acyclic cofibration in $\mathcal{C}$. Thus $f \wedge K$ is also a level equivalence. By Ken Browns Lemma [Hov99, 1.1.12] suspension preserves level equivalences between Reedy cofibrant objects. If X is a cosimplicial frame, $X \wedge (-)$ preserves acyclic cofibrations by [Hov99, 5.7.2] so in particular the acyclic cofibration $K \wedge \Delta^n_+ \to K \wedge \Delta^0_+$ and hence $(X \wedge K)^n \to (X \wedge K)^0$ is a weak equivalence and X homotopically constant. Since K is cofibrant we know that $X \wedge K$ is Reedy cofibrant from what we have already proved above.

(ii) and (iii) This follows from [Hov99, Proposition 5.7.1] and adjointness and Ken Browns Lemma as above. □

We would like to have that the constant object functor from $\mathcal{C}$ to $\mathcal{C}^\Delta$ is a Quillen equivalence, so we could apply Construction 4.2.4 to obtain our endomorphism ring spectrum. But one can only expect that the full subcategory of homotopically constant objects in the homotopy category of $\mathcal{C}^\Delta$ is equivalent to the homotopy category of $\mathcal{C}$ itself. One could of course try to localize the Reedy model structure of $\mathcal{C}^\Delta$ such that the fibrant objects are exactly the homotopically constant Reedy fibrant objects which is the approach in [RSS01]. But to do so one needs again technical assumptions, which is for localization at least cofibrantly generated, we are trying to get rid of. But even if this localized model structure does not exists as a honest model structure, part of the axioms are true. And since we are always starting with homotopically constant objects (in fact honestly constant objects) all properties we need are satisfied.

For the next result note that evaluating a cosimplicial object in degree zero is right adjoint to taking constant simplicial objects. To see this recall that taking colimit is left adjoint to the constant diagram functor and since Δ has $[0]$ as terminal object taking colimit equals evaluation at 0.

Proposition 4.5.6. *The adjoint pair*

$$\mathrm{ev}_0\colon \mathcal{D}^\Delta \rightleftarrows \mathcal{D} : c(-)$$

is a Quillen functor between the Reedy model structure on $\mathcal{D}$ and $\mathcal{D}$ itself. Moreover the derived adjunction restricts to an equivalence between $\mathrm{Ho}(\mathcal{D})$ *and the full subcategory of homotopically constant objects in* $\mathrm{Ho}(\mathcal{D}^{\Delta})$.

Proof. Since Reedy cofibrations are in particular level cofibrations and Reedy weak equivalences are level equivalences by definition, it is immediate that ev_0 preserves (acyclic) cofibrations. Hence we have a Quillen functor.

We consider the derived adjunction counit. For this let X be Reedy cofibrant and take a fibrant replacement $X^0 \to X^0_f$ of X^0 in $\mathcal{D}$. The adjoint $X \to c(X^0_f)$ is a model for the derived adjunction unit and a weak equivalence if and only if X is homotopically constant. So the derived adjunction unit is an isomorphism only on homotopically constant object. Now let Y be a fibrant object in $\mathcal{D}$ and take a Reedy cofibrant replacement $(cY)_c \to cY$. The adjoint modeling the derived adjunction counit is just a cofibrant replacement $(cY)^0_c \to Y$ of Y in $\mathcal{D}$ so always a weak equivalence. □

It turns out that the suspension functor is compatible with the equivalence between homotopically constant objects and objects in $\mathcal{D}$.

Lemma 4.5.7. *The derived suspension functor* $\mathrm{L}\Sigma$ *on cosimplicial objects in a stable model category is fully faithful on homotopically constant objects,* i.e. *for homotopically constant objects X and Y the map*

$$\mathrm{L}\Sigma\colon [X,Y] \to [\mathrm{L}\Sigma X, \mathrm{L}\Sigma Y]$$

is an isomorphism.

Proof. We may assume that X is Reedy cofibrant and Y Reedy fibrant. Denote by $(\Sigma Y)_{\mathrm{f}}$ a Reedy fibrant replacement of ΣY. There is a commutative diagram

$$\begin{array}{ccc} [X,Y] & \xrightarrow{\mathrm{L}\Sigma} & [\Sigma X, (\Sigma Y)_{\mathrm{f}}] \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ [X^0, Y^0] & \xrightarrow{\mathrm{L}\Sigma} & [\Sigma X^0, (\Sigma Y)^0_{\mathrm{f}}], \end{array}$$

where we used that $(\Sigma Z)^0 = \Sigma(Z^0)$ and so $(\Sigma Y)^0_{\mathrm{f}}$ is a fibrant replacement for $\Sigma(Y^0)$. By the stability assumption on $\mathcal{C}$ the bottom horizontal map is an isomorphism. By Proposition 4.5.6 above the vertical maps are isomorphisms. Hence the top horizontal map is an isomorphism as required. □

Recall from the discussion at the beginning of this section that given a cosimplicial object X in $\mathcal{C}$ there is an associated functor $X \wedge (-)\colon \mathcal{S}_* \to \mathcal{C}$. Evaluating such a functor on the standard simplices Δ^n on the other hand gives a cosimplicial object. Recall that the functor $X(-)$ is a left Quillen functor if and only if X is a cosimplicial frame.

Proposition 4.5.8. *The category $\mathcal{C}^{\Delta}$ of cosimplicial objects in a stable model category $\mathcal{C}$ equipped with the Reedy model structure satisfies Axioms 4.2.1 where $\mathcal{L}$ is defined to be the class of cosimplicial frames,* i.e. *the Reedy cofibrant and homotopically constant objects.*

Proof. In order to not get confused let us denote the category of cosimplicial objects $\mathcal{C}^{\Delta}$ in $\mathcal{C}$ by $\mathcal{D}$. For a simplicial set K we have defined a cosimplicial object $X \wedge K$ in Definition 4.5.4 above. In order to prove Axiom (A1) we have to show that $X \wedge K$ is a cosimplicial frame. But this is the case if and only if the associated functor $(X \wedge K)(-)\colon \mathcal{S}_* \to \mathcal{C}$ is a left Quillen functor. But we can write $(X \wedge K)(-)$ as the composite $X(K \wedge (-))$ of the left Quillen functors $X(-)$ and $K \wedge (-)$.

For (A2) assume that X is a cosimplicial frame in $\mathcal{C}$. Consider the cosimplicial object in $\mathcal{D}$ obtained by sending n to $X \wedge \Delta^n_+$ (a bicosimplicial object in $\mathcal{C}$). We claim that the associated functor $(X \wedge \Delta_+)(-)\colon \mathcal{S}_* \to \mathcal{D}$ is a Quillen functor and hence $X \wedge \Delta_+$ a cosimplicial frame in $\mathcal{D}$. But there is an isomorphisms $(X \wedge \Delta_+)(K)) \cong X(\Delta_+(K))$ which is natural in K and the functor $\Delta_+(-)$ is the identity functor on simplicial sets. Now the claim follows from the fact that $X(-)$ is a Quillen functor. Recall that for cosimplicial objects in X and Y in $\mathcal{C}$ the simplicial mapping space $\mathrm{map}(X, Y)$ is given by

$$\mathrm{map}_{\mathcal{D}}(X, Y) = \mathcal{D}(X \wedge \Delta_+, Y)$$

and so Lemma 4.5.5 (ii) applies to show that $\mathrm{map}_{\mathcal{D}}(X, -)$ is a right Quillen functor on the Reedy model structure on $\mathcal{D} = \mathcal{C}^{\Delta}$. Part (iii) of the same lemma ensures that for Reedy fibrant Y the functor $\mathrm{map}_{\mathcal{D}}(-, Y)$ is a right Quillen functor and for the functor $(-) \wedge K$ we use part (i).

Finally for (A3) note that a map between cosimplicial frames lives in the full subcategory of homotopically constant objects in the homotopy category of cosimplicial objects in $\mathcal{C}$. But this subcategory is equivalent to the homotopy category of the stable model category $\mathcal{C}$ by Proposition 4.5.6. In the first part of the proof we saw that $\Sigma X = X \wedge S^1$ is a cosimplicial frame if X is. So the suspension functor on $\mathcal{D}$ restricts to homotopically constant objects and the degree 0 part of ΣX is a model for the suspension of the object X^0. Hence by stability of $\mathcal{C}$ a map between cosimplicial frames is a weak equivalence if and only if the suspended map is one. □

Lemma 4.5.9. *Let X be a Reedy fibrant cosimplicial frame and $\varphi\colon \Sigma X \to Y$ be a Reedy fibrant replacement of the cosimplicial frame ΣX. Then the adjoint map $\varphi^\natural\colon X \to \Omega Y$ induces a weak equivalence of simplicial sets*

$$\mathrm{map}(Z, X) \to \mathrm{map}(Z, \Omega Y)$$

for any cosimplicial frame Z.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{map}(Z, X) & \xrightarrow{(\varphi^\natural)} & \mathrm{map}(Z, \Omega Y) \\ {\scriptstyle\Sigma}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathrm{map}(\Sigma Z, \Sigma X) & \xrightarrow{\varphi} & \mathrm{map}(\Sigma Z, Y) \end{array}$$

in which the right vertical map is an adjunction isomorphism. If we apply π_0 to the above diagram, we obtain that the resulting composite of the left vertical map with the bottom map models the derived suspension functor

$$\mathrm{L}\Sigma\colon [Z, X] \to [\Sigma Z, \Sigma X] = [\Sigma Z, Y]$$

which is an isomorphism by Lemma 4.5.7. The left vertical maps of course remains an isomorphism under application of π_0 and hence the map $(\varphi^\natural)_*\colon \mathrm{map}(Z, X) \to \mathrm{map}(Z, \Omega Y)$ induces an isomorphism on π_0 for any cosimplicial frame Z. Hence for any suspension of Z. Using the the isomorphism $\pi_0 \mathrm{map}(\Sigma^n Z, X) \cong \pi_n \mathrm{map}(Z, X)$ we conclude that $\mathrm{map}(Z, X) \to \mathrm{map}(Z, \Omega Y)$ induces isomorphisms on π_n for $n \geq 0$ and so is a weak equivalence of simplicial sets as required. □

The following result is essentially taken from [SS02, Lemma 6.4] where a slightly stronger result is given. We give it in the version sufficient for our purposes and give the proof for the sake of completeness of our treatment.

Lemma 4.5.10. *For any Reedy fibrant cosimplicial frame Y in a stable model category $\mathcal{C}$ there exists a Reedy fibrant cosimplicial frame X and a weak equivalence $\Sigma X \to Y$ such that its adjoint $X \to \Omega Y$ is a Reedy fibration.*

Proof. Since $\mathcal{C}$ is stable, there exists a cofibrant object Z^0 in $\mathcal{C}$ such that $\Sigma(Z^0) \cong Y^0$ in $\mathrm{Ho}(\mathcal{C})$. Let Z be Reedy cofibrant replacement of $\mathrm{c}Z^0$ the constant cosimplicial object on Z^0. By [Hov99, Chapter 5.2] this can be done by a map having the identity in degree zero which justifies our notation. Note that $(\Sigma Z)^0$ is a model for the suspension of Z^0, so we have an isomorphism $(\Sigma Z)^0 \cong \Sigma(Z^0) \cong Y^0$ in $\mathrm{Ho}(\mathcal{C})$. Because $(\Sigma Z)^0$ is cofibrant in $\mathcal{C}$ and Y^0 fibrant in $\mathcal{C}$ (for Y was assumed to be Reedy fibrant), this isomorphism can be realized by a weak equivalence $(\Sigma Z)^0 \xrightarrow{\sim} Y^0$. By adjunction we obtain a map $\Sigma Z \xrightarrow{\sim} \mathrm{c}Y^0$ which is a level equivalence since ΣZ is homotopically constant. Recall that for the homotopically constant and Reedy fibrant Y the canonical map $Y \to \mathrm{c}Y^0$ adjoint to the identity on Y^0 is an acyclic Reedy fibration. Now ΣZ is Reedy cofibrant and thus we can find a lift in the diagram

$$\begin{array}{ccc} & & Y \\ & \overset{\bar{\varphi}}{\nearrow} & \big\downarrow \sim \\ \Sigma Z & \xrightarrow{\sim} & \mathrm{c}Y^0 \end{array}$$

as indicated. By 2-out-of-3 we know that $\bar{\varphi}$ is also a Reedy weak equivalence. But we do not know whether the adjoint $Z \to \Omega Y$ is a Reedy fibration. We factor this map

$$Z \overset{\sim}{\rightarrowtail} X \twoheadrightarrow \Omega Y$$

by an acyclic cofibration followed by a fibration. Since Σ preserves acyclic cofibrations, adjoining back yields

$$\Sigma Z \overset{\sim}{\rightarrowtail} \Sigma X \overset{\varphi}{\twoheadrightarrow} Y.$$

The composite is of course $\bar{\varphi}$, so by 2-out-of-3 φ is a weak equivalence and we found a map as required. To finish the proof note that ΩY is Reedy fibrant, which implies that X is Reedy fibrant, too. □

4.6. **The general case.** To prove the Morita theorem in general, *i.e.* for an arbitrary stable model category (which is not simplicial) we consider cosimplicial frames and desuspension cospectra of those which might be called *stable frames.* In order to avoid confusion we fix an arbitrary model category $\mathcal{D}$. We set $\mathcal{C}$ to be $\mathcal{D}^{\Delta}$ the category of cosimplicial objects in $\mathcal{D}$. This was shown to be closed simplicial category admiting a model structure which with $\mathcal{L}$ as the class of cosimplicial frames satisfies Axioms 4.2.1. Now it remains to take a cosimplicial frame on the compact object X in $\mathcal{D}$ and build a desuspension cospectrum on a chosen cosimplicial frame and apply our axiomatic theory of Morita contexts. We begin ensuring the existence of desuspension cospectra for cosimplicial frames.

Proposition 4.6.1. *For any bifibrant cosimplicial frame X in $\mathcal{D}^{\Delta}$ there exists a desuspension spectrum ωX.*

Proof. Inductively, using Lemma 4.5.10, we obtain Reedy fibrant cosimplicial frames $\omega^n X$ together with weak equivalences $\Sigma\omega^{n+1}X \to \omega^n X$ such that the adjoints $\omega^{n+1}X \to \Omega\omega^n X$ are fibrations. In this way we obtain a fibrant Σ-cospectrum in cosimplicial objects such that each $\omega^n X$ is a fibrant cosimplicial frame. By Lemma 4.5.9 the structure map $\omega^n X \to \Omega\omega^{n+1}X$ induces a weak equivalence $\mathrm{map}(Z, \omega^n X) \xrightarrow{\sim} \mathrm{map}(Z, \Omega\omega^{n+1}X)$ for any cosimplicial frame Z. Altogether we have constructed a desuspension cospectrum for X. □

Theorem 4.6.2. *Let $\mathcal{D}$ be any stable model category and X a bifibrant compact object in it. Then there exists a Morita context for X.*

Proof. Let $\omega^0 X$ be a fibrant choice of a cosimplicial frame on X. Then there exists a desuspension cospectrum ωX for $\omega^0 X$ by Proposition 4.6.1. Now by Theorem 4.3.3 applied to ωX and the category $\mathcal{D}^\Delta$, there exist a Morita context for $\omega^0 X$ with a naive ring spectrum which we denote $\mathrm{End}(X)$ rather than by $\mathrm{End}(\omega^0 X)$ or even by E for the sake of simplicity of notation. There is a natural isomorphism

$$\mathrm{map}_{\mathcal{D}^\Delta}(\omega^n X, \mathrm{c}Y) \cong \mathrm{map}_{\mathcal{D}}(\omega^n X^0, Y)$$

of simplicial sets for any Y in $\mathcal{D}$. The adjoint pair

$$(-)^0 \colon \mathcal{D}^\Delta \rightleftarrows \mathcal{D} : \mathrm{c}(-)$$

composed with the Morita context

$$X \wedge_{\mathrm{End}(X)} (-) \colon \mathrm{End}(X)\text{-mod} \rightleftarrows \mathcal{D}^\Delta : \mathrm{Hom}(X, -) = \mathrm{map}_{\mathcal{D}^\Delta}(\omega^* X, -)$$

from Construction 4.3.1 gives a Morita context for X between $\mathcal{D}$ and $\mathrm{End}(X)$-mod, where we used Proposition 4.5.6 to verify conditions (i) and (ii) for a Morita context. This finishes the proof. □

Theorem 4.6.3. *For any stable model category $\mathcal{D}$ with a compact generator X, there exist a Morita equivalence*

$$X \wedge_E (-) \colon E\text{-mod} \rightleftarrows \mathcal{D} : \mathrm{Hom}(X, -)$$

and thus $\mathcal{D}$ is Quillen equivalent to a category of modules over a symmetric ring spectrum.

Proof. Of course we apply the abstract Morita theorem 4.1.4 to the Morita context just obtained in Theorem 4.6.2 above. Now we can replace the obtained naive endomorphism ring spectrum $\mathrm{End}(X)$ by a symmetric one as done in the proof of Theorem 4.4.2. □

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