Curvature-Dimension Bounds and Functional Inequalities: Localization, Tensorization and Stability

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Abstract

This work is devoted to the analysis of abstract metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying the curvature-dimension condition $\mathsf{CD}(K, N)$ presented by Sturm [Stu06a, Stu06b] and in a similar form by Lott and Villani [LV07, LV09].

In the first part, we introduce the notion of a *Borell-Brascamp-Lieb inequality* in the setting of metric measure spaces denoted by $\mathsf{BBL}(K, N)$. This inequality holds true on metric measure spaces fulfilling the curvature-dimension condition $\mathsf{CD}(K, N)$ and is *stable* under convergence of metric measure spaces with respect to the L_2-transportation distance.

In the second part, we prove that the local version of $\mathsf{CD}(K, N)$ is equivalent to a global condition $\mathsf{CD}^*(K, N)$, slightly weaker than the usual global one. This so-called *reduced curvature-dimension condition* $\mathsf{CD}^*(K, N)$ has the *localization property.* Furthermore, we show its *stability* and the *tensorization property.*

As an application we conclude that the fundamental group $\pi_1(\mathsf{M}, x_0)$ of a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is finite whenever it satisfies locally the curvature-dimension condition $\mathsf{CD}(K, N)$ with positive K and finite N.

In the third part, we study *cones* over metric measure spaces. We deduce that the *n-Euclidean cone* over an n-dimensional Riemannian manifold whose Ricci curvature is bounded from below by $n-1$ satisfies the curvature-dimension condition $\mathsf{CD}(0, n+1)$ and that the *n-spherical cone* over the same manifold fulfills $\mathsf{CD}(n, n+1)$.

Contents

Introduction

The analysis on singular spaces is one big challenge in mathematics. An important class of singular spaces are abstract metric measure spaces with generalized lower bounds on the Ricci curvature formulated in terms of optimal transportation. This is the class of spaces being under consideration in this work.

Many geometric and functional analytic results on Riemannian manifolds depend on lower bounds on the Ricci curvature and on upper bounds on the dimension. Hence, for a long time an ambitious aim in geometric analysis was to extend the notion of curvature and dimension to the class of abstract metric measure spaces. This problem was solved in a mathematically fruitful way at the beginning of the 21st century.

Already in 1951, Alexandrov [Ale51] introduced the concept of generalized lower bounds on the sectional curvature for abstract metric spaces. The definition of this concept is based on (triangle) comparisons with the Euclidean world. An important property of these bounds is their stability with respect to the Gromov-Hausdorff convergence of the underlying spaces. Moreover, families of these so-called Alexandrov spaces with given lower bounds on the generalized sectional curvature and given upper bounds on the Hausdorff dimension and diameter are compact [BGP92].

However, for many fundamental results in geometric analysis, the relevant ingredients are not bounds for the sectional curvature but bounds on the Ricci curvature: For instance, the Bishop-Gromov volume growth estimate, the Bonnet-Myers theorem on diameter bounds and the Lichnerowicz bound for the spectral gap depend on lower bounds on the Ricci curvature and on upper bounds on the dimension of the underlying manifolds.

The family of Riemannian manifolds with given lower bound on their Ricci curvature is neither closed under Gromov-Hausdorff convergence nor it is closed under any other notion of convergence. Thus, in order to bridge a gap in the field of geometric analysis, a generalized notion of lower Ricci curvature bounds for metric measure spaces, closed under a reasonable notion of convergence, had to be found.

In 2006 Sturm [Stu06a] presented a dimension-independent concept of lower 'Ricci' curvature bounds in the setting of abstract metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$. The definition introduced in [Stu06a] is based on optimal transportation, or more precisely, on convexity properties of the relative 'Shannon' entropy $\mathsf{Ent}(\cdot|\mathsf{m})$ considered as a function on the L_2-Wasserstein space $\mathcal{P}_2(\mathsf{M}, \mathsf{d})$ of probability measures on the metric space (M, d). An important benefit of this notion of curvature bounds is its stability under convergence with respect to the L_2-transportation distance $\mathbb{D}$, a complete length metric on the family of isomorphism classes of metric measure spaces.

Still in the same year Sturm [Stu06b] established an even further reaching concept: In addition to a generalized lower bound on the Ricci curvature he imposed a generalized upper bound on the dimension. This is the content of the so-called curvature-dimension condition $\mathsf{CD}(K, N)$. This condition depends on two parameters K and N, playing the role of a curvature and dimension bound, respectively. The curvature-dimension condition $\mathsf{CD}(K, N)$ as well is stable under convergence with respect to the L_2-transportation distance $\mathbb{D}$. Related concepts were studied by Lott and Villani [LV07, LV09].

The justification of the definition of $\mathsf{CD}(K, N)$ – as well as of the interpretation of the two parameters K and N leading to the name curvature-dimension condition – is its consistency with the Riemannian world: A complete Riemannian manifold satisfies $\mathsf{CD}(K, N)$ if and only if its Ricci curvature is bounded from below by K and its dimension from above by N.

Moreover, a broad variety of geometric and functional analytic statements can be deduced from the curvature-dimension condition $\mathsf{CD}(K, N)$. Among them are the Brunn-Minkowski inequality and the already mentioned theorems by Bishop-Gromov, Bonnet-Myers and Lichnerowicz. However, four relevant questions remained open:

- ▷ Is there a reasonable generalized formulation of the Borell-Brascamp-Lieb inequality known from the Euclidean setting in the framework of abstract metric measure spaces? And if the answer is 'yes' – what is its relation to the curvature-dimension condition? And moreover, is it stable with respect to the L_2-transportation distance $\mathbb{D}$?

- ▷ Is the curvature-dimension condition $\mathsf{CD}(K, N)$ for general K, N a local property? That means, does it hold true globally on the whole space $(\mathsf{M},\mathsf{d},\mathsf{m})$ whenever it holds true locally on a family of sets M_i covering M?

- ▷ Does the curvature-dimension condition fulfill a tensorization property? In other words, does a product space $\bigotimes_{i\in I}\mathsf{M}_i$ inherit the curvature-dimension condition $\mathsf{CD}(K,\sum_{i\in I}N_i)$ whenever $\mathsf{CD}(K,N_i)$ holds true on each factor M_i?
- ▷ Does a metric measure space pass the curvature-dimension condition on to its Euclidean cone in an appropriate way? Precisely, does the N-Euclidean cone $\mathsf{Con}(\mathsf{M})$ over a metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfy $\mathsf{CD}(0,N+1)$ whenever $(\mathsf{M},\mathsf{d},\mathsf{m})$ fulfills $\mathsf{CD}(N-1,N)$? What is true on spherical cones? Does the N-spherical cone $\Sigma(\mathsf{M})$ over a metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfy $\mathsf{CD}^*(N,N+1)$ whenever $(\mathsf{M},\mathsf{d},\mathsf{m})$ fulfills $\mathsf{CD}^*(N-1,N)$?

The goal of this work is to study these questions and to answer them – or at least to approach a solution.

There already exists a partly positive answer to the first question: In 2001 Cordero-Erausquin, McCann and Schmuckenschläger [CMS01] generalized the formulation of the Borell-Brascamp-Lieb inequality as well as the definition of the related Prékopa-Leindler inequality from the Euclidean to the Riemannian setting. According to the curvature of the underlying Riemannian manifold, the Riemannian versions of these inequalities involve a volume distortion coefficient which can be controlled via lower bounds on the Ricci curvature. The methods they used are based on optimal mass transportation on Riemannian manifolds.

About six years later in 2007, Bonnefont [Bon07] defined an approximated Brunn-Minkowski inequality generalizing the classical one for length spaces. Based only on distance properties, this definition provided a possibility to deal with discrete spaces. Bonnefont proved the stability of this approximated inequality as well as the stability of the classical one under $\mathbb{D}$-convergence of metric measure spaces. Additionally, he showed in the second part of his work that every metric measure space satisfying the Brunn-Minkowski inequality can be approximated by discrete spaces fulfilling an approximated Brunn-Minkowski inequality.

The answer to the second question is ‘yes’ in the particular cases $K=0$ and $N=\infty$. Locality of the curvature-dimension $\mathsf{CD}(K,\infty)$ was proved in [Stu06a] and, using analogous methods, locality of $\mathsf{CD}(0,N)$ by Villani [Vil09].

Similarly, the tensorization property – content of the third question – is known to be true, but as well only in the special case $\mathsf{CD}(K,\infty)$. This was proved in [Stu06a].

There are three motivating results in the context of the fourth question, namely

* a result by Cheeger and Taylor [CT82, Che83] saying that the punctured n-Euclidean cone based on a compact and complete n-dimensional Riemannian manifold with $\mathsf{Ric} \geq n-1$ is an incomplete $(n+1)$-dimensional Riemannian manifold whose Ricci curvature is bounded from below by 0

* a result by Ohta [Oht07b] stating that the so-called measure contraction property $\mathsf{MCP}(K,N)$ descends to Euclidean cones in an appropriate manner. Precisely, the N-Euclidean cone over a metric measure space satisfying $\mathsf{MCP}(N-1,N)$ fulfills $\mathsf{MCP}(0,N+1)$

* a result by Petean [Pet] saying that the n-spherical cone without north and south pole over a compact and complete n-dimensional Riemannian manifold with $\mathsf{Ric} \geq n-1$ is an incomplete $(n+1)$-dimensional Riemannian manifold whose Ricci curvature is bounded from below by n.

Furthermore, we would like to mention

* a result that can be found in the book by Burago, Burago and Ivanov [BBI01] saying that the Euclidean cone over an Alexandrov space whose 'sectional' curvature is bounded from below by 1 is again an Alexandrov space with 'sectional' curvature bounded from below by 0.

According to the posed questions, this work is divided into three main chapters dealing with the generalized Borell-Brascamp-Lieb inequality (Chapter 2), with the localization and tensorization property of the curvature-dimension condition (Chapter 3) and with cones over metric measure spaces (Chapter 4), respectively.

In Section 2.1 of Chapter 2 the definition of the Borell-Brascamp-Lieb inequality $\mathsf{BBL}(K,N)$ depending on two parameters K and N is introduced in the setting of metric measure spaces $(\mathsf{M},\mathsf{d},\mathsf{m})$ via weighted p-means. We say that the inequality $\mathsf{BBL}(K,N)$ is satisfied for $(\mathsf{M},\mathsf{d},\mathsf{m})$ whenever a pointwise inequality

$$h(z) \geq \mathcal{M}_t^p \left(\frac{f(x)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(x,y))}, \frac{g(y)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(x,y))} \right)$$

for non-negative integrable functions f,g,h, parameters $p \geq -\frac{1}{N}$ and $t \in [0,1]$ and all t-midpoints z of $x,y \in \mathsf{M}$, can be improved to an inequality of the corresponding integrals

$$\int_{\mathsf{M}} h \, d\mathsf{m} \geq \mathcal{M}_t^{p/(1+Np)} \left(\int_{\mathsf{M}} f \, d\mathsf{m}, \int_{\mathsf{M}} g \, d\mathsf{m} \right)$$

by paying the price of a smaller exponent $p/(1+Np)$. The non-Euclidean nature of this inequality is expressed via the coefficients $\tilde{\tau}_{K,N}^{(t)}(\cdot)$.

In Section 2.2 we show that the Borell-Brascamp-Lieb inequality is implied by the curvature-dimension condition – at least for compact and non-branching metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$. The key point of this proof is a pointwise characterization of the curvature-dimension condition $\mathsf{CD}(K, N)$ presented in [Stu06b]: Provided that for $\mathsf{m} \otimes \mathsf{m}$-almost every $(x, y) \in \mathsf{M} \times \mathsf{M}$ there exists a unique geodesic $t \mapsto \gamma_t(x, y)$ depending in a measurable way on the starting point x and the end point y, then for each pair $\nu_0 = \rho_0 \mathsf{m}$ and $\nu_1 = \rho_1 \mathsf{m}$ of absolutely continuous probability measures and each optimal transference plan q from ν_0 to ν_1 it holds that

$$\rho_t(\gamma_t(x_0, x_1)) \leq \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{\rho_0(x_0)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(x_0, x_1))}, \frac{\rho_1(x_1)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(x_0, x_1))} \right)$$

for all $t \in [0, 1]$ and q-almost every $(x_0, x_1) \in \mathsf{M} \times \mathsf{M}$, where ρ_t denotes the density of the push-forward measure of q under the map $(x_0, x_1) \mapsto \gamma_t(x_0, x_1)$.

Section 2.3 is devoted to the stability of the Borell-Brascamp-Lieb inequality under convergence of a sequence $((\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n))_{n \in \mathbb{N}}$ of compact metric measure spaces with respect to the L_2-transportation distance $\mathbb{D}$. According to embedding properties of this distance, the proof of stability boils down to show that $\mathsf{BBL}(K, N)$ is stable with respect to the weak convergence $\mathsf{m}_n \xrightarrow{w} \mathsf{m}$ of probability measures on a given metric space (M, d). Due to the definition of weak convergence, the requirement in the formulation of $\mathsf{BBL}(K, N)$ holds obviously true on the limit $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ if we restrict our attention to continuous bounded functions f, g and h. The general case requires technical approximations.

The Borell-Brascamp-Lieb inequality is strong enough to imply the geometric consequences mentioned above. In Section 2.4, the Bishop-Gromov theorem on the volume growth of concentric balls and spheres as well as the Bonnet-Myers theorem on compactness and diameter bounds for metric measure spaces are formulated as consequences of the Borell-Brascamp Lieb inequality.

In a certain sense, – we are particularly thinking of its stability property – $\mathsf{BBL}(K, N)$ might be used as an alternative characterization of metric measure spaces with curvature $\geq K$ and dimension $\leq N$.

The aim of Chapter 3 is to study metric measure spaces satisfying a local version of the curvature-dimension condition $\mathsf{CD}(K, N)$. In Section 3.4 we prove that the local version of $\mathsf{CD}(K,N)$ is equivalent to a global condition $\mathsf{CD}^*(K,N)$, slightly weaker than the usual global one $\mathsf{CD}(K,N)$. More precisely,

$$\mathsf{CD}_{\mathsf{loc}}(K-,N) \Leftrightarrow \mathsf{CD}^*_{\mathsf{loc}}(K-,N) \Leftrightarrow \mathsf{CD}^*(K,N).$$

This so-called reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ is introduced in Section 3.1.

Again the reduced curvature-dimension condition turns out to be stable under $\mathbb{D}$-convergence. This is the content of Section 3.2. Moreover, in Section 3.3, we prove the tensorization property for $\mathsf{CD}^*(K,N)$. Finally, also the reduced curvature-dimension condition allows to deduce all the familiar geometric and functional analytic inequalities (Bishop-Gromov, Bonnet-Myers, Lichnerowicz, etc), – however, with slightly worse constants as we will see in Section 3.5. Actually, this can easily be made plausible due to the fact that for $K > 0$

$$\mathsf{CD}(K,N) \Rightarrow \mathsf{CD}^*(K,N) \Rightarrow \mathsf{CD}(K^*,N)$$

with $K^* = \frac{N-1}{N}K$.

As an interesting application of these results we prove in Section 3.6 that the fundamental group $\pi_1(\mathsf{M},x_0)$ of a metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ is finite whenever it satisfies the local curvature-dimension condition $\mathsf{CD}_{\mathsf{loc}}(K,N)$ with positive K and finite N. Indeed, the local curvature-dimension condition for a given metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ carries over to its universal cover $(\hat{\mathsf{M}},\mathsf{d}_{\mathsf{c}},\hat{\mathsf{m}})$. The global version of the reduced curvature-dimension condition then implies Bonnet-Myers' diameter estimate (with non-sharp constants) and thus compactness of $\hat{\mathsf{M}}$.

Chapter 4 deals with cones. In Section 4.1 the Euclidean cone $\mathsf{Con}(\mathsf{M})$ over a space M is defined as the quotient of the product $\mathsf{M}\times[0,\infty)$ obtained by identifying all points in the fiber $\mathsf{M}\times\{0\}$. This point is called the origin O of the cone. Given a metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ with $\mathsf{diam}(\mathsf{M})\leq\pi$, we define a metric $\mathsf{d}_{\mathsf{Con}}$ on the cone via the cosine formula

$$\mathsf{d}_{\mathsf{Con}}((x,s),(y,t)) = \sqrt{s^2+t^2-2st\cos(\mathsf{d}(x,y))},$$

and a measure ν on it as the product $d\nu(x,s) := d\mathsf{m}(x)\otimes s^N ds$. The resulting metric measure space $(\mathsf{Con}(\mathsf{M}),\mathsf{d}_{\mathsf{Con}},\nu)$ is called the N-Euclidean cone over $(\mathsf{M},\mathsf{d},\mathsf{m})$. The definition of the metric $\mathsf{d}_{\mathsf{Con}}$ and the measure ν ensures that the n-Euclidean cone over the n-dimensional sphere $\mathbb{S}^n\subseteq\mathbb{R}^{n+1}$ equipped with the angular distance is the Euclidean space $\mathbb{R}^{n+1}$ endowed with the Euclidean metric and the Lebesgue measure expressed in spherical coordinates.

The equality '$\mathsf{Con}(\mathbb{S}^n) = \mathbb{R}^{n+1}$' can be expressed in a slightly different formulation emphasizing the Riemannian nature of $\mathbb{S}^n$ and $\mathbb{R}^{n+1}$: The cone $\mathsf{Con}(\mathbb{S}^n)$ over $\mathbb{S}^n$, an n-dimensional Riemannian manifold whose Ricci curvature is bounded from below by $n-1$, coincides with $\mathbb{R}^{n+1}$, an $(n+1)$-dimensional Riemannian manifold with Ricci curvature bounded from below by 0.

More generally, Cheeger and Taylor [CT82, Che83] – as already noted – were able to prove that the punctured n-Euclidean cone $\mathsf{Con}(\mathsf{M}) \setminus \{\mathsf{O}\}$ constructed over a compact and complete n-dimensional Riemannian manifold M with $\mathsf{Ric} \geq n-1$ is an $(n+1)$-dimensional Riemannian manifold with $\mathsf{Ric} \geq 0$. Of course, $\mathsf{Con}(\mathsf{M}) \setminus \{\mathsf{O}\}$ is not a complete manifold and in general, $\mathsf{Con}(\mathsf{M})$ on its own is not a smooth one. In particular, the Ricci curvature in the classical sense is not defined in its singularity O. In this situation, we prove in Section 4.3 that $\mathsf{Con}(\mathsf{M})$ satisfies the curvature-dimension condition $\mathsf{CD}(0, n+1)$. This result can be regarded as a further justification of the definition of the curvature-dimension condition.

Our further reaching conjecture is – but unfortunately, we failed in proving it up to now – that the above observations can be generalized in the following sense: Whenever $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is a metric measure space satisfying the curvature-dimension condition $\mathsf{CD}(N-1, N)$ then the corresponding N-Euclidean cone $(\mathsf{Con}(\mathsf{M}), \mathsf{d}_{\mathsf{Con}}, \nu)$ satisfies $\mathsf{CD}(0, N+1)$. We will abbreviate this conjecture by '$\mathsf{CD}(N-1, N) \rightsquigarrow \mathsf{CD}(0, N+1)$' in the sequel. The results [Oht07b] and [CT82, Che83] mentioned above lead in this direction. And there is another hint that '$\mathsf{CD}(N-1, N) \rightsquigarrow \mathsf{CD}(0, N+1)$' could be true, namely the tensorization property of the local version of the curvature-dimension condition. Ignoring the origin, the Euclidean cone is a product space. Moreover, optimal transport on the cone does not involve the origin, a statement that is proved in Section 4.2. However, the relevant differences from the product setting are the distance and the measure: In the framework of Euclidean cones we do not consider the 'product metric' and the 'product measure' but the 'Euclidean cone metric' and the weighted 'Euclidean cone measure', respectively. Therefore, the techniques which are available in the product setting and used in the proof of the tensorization property, cannot be applied here.

The key step in the proof of '$\mathsf{CD}(N-1, N) \rightsquigarrow \mathsf{CD}(0, N+1)$' will probably depend on an appropriately generalized result in the sense of Cheeger and Taylor. But a generalization of Cheeger' s and Taylor' s result in the framework of metric measure spaces is neither known to be true nor we were able to make it. Therefore, to prove the conjecture '$\mathsf{CD}(N-1, N) \rightsquigarrow \mathsf{CD}(0, N+1)$' remains a further challenge.

In Section 4.4 we consider a second object with a familiar Euclidean analogon:

The spherical cone $\Sigma(\mathsf{M})$ over a space M is defined as the quotient of the product space $\mathsf{M} \times [0, \pi]$ obtained by contracting all points in the fiber $\mathsf{M} \times \{0\}$ to the south pole $\mathcal{S}$ and all points in the fiber $\mathsf{M} \times \{\pi\}$ to the north pole $\mathcal{N}$. Given a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ with $\mathsf{diam}(\mathsf{M}) \leq \pi$, we define a metric d_Σ on the cone via

$$\cos\left(\mathsf{d}_\Sigma(p, q)\right) = \cos s \cos t + \sin s \sin t \cos\left(\mathsf{d}(x, y)\right)$$

for $p = (x, s), q = (y, t) \in \Sigma(\mathsf{M})$ and a measure $d\nu(x, s) := d\mathsf{m}(x) \otimes (\sin^N s ds)$ on it. In this way we construct a metric measure space $(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma, \nu)$ which we call N-spherical cone over $(\mathsf{M}, \mathsf{d}, \mathsf{m})$. The definition of d_Σ and ν guarantees that the n-spherical cone over the n-dimensional sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ coincides with $\mathbb{S}^{n+1}$.

From this point on we follow appropriately the line of argumentation presented in the example of Euclidean cones: Considering $\mathbb{S}^n$ and $\mathbb{S}^{n+1}$ as Riemannian manifolds, we can express the fact that '$\Sigma(\mathbb{S}^n) = \mathbb{S}^{n+1}$' in an involved manner as follows: The cone $\Sigma(\mathbb{S}^n)$ over $\mathbb{S}^n$, an n-dimensional Riemannian manifold whose Ricci curvature is bounded from below by $n - 1$, is $\mathbb{S}^{n+1}$, an $(n + 1)$-dimensional Riemannian manifold with Ricci curvature bounded from below by n.

Leaving the example of Euclidean spheres, Petean [Pet] proved that the n-spherical cone without poles $\Sigma(\mathsf{M}) \setminus \{\mathcal{S}, \mathcal{N}\}$ over a compact, complete n-dimensional Riemannian manifold M with $\mathsf{Ric} \geq n - 1$ is an $(n + 1)$-dimensional Riemannian manifold with $\mathsf{Ric} \geq n$. Again the punctured cone $\Sigma(\mathsf{M}) \setminus \{\mathcal{S}, \mathcal{N}\}$ is not a complete manifold and again, $\Sigma(\mathsf{M})$ on its own is not a smooth one, such that the classical Ricci curvature is not defined in the singularities $\mathcal{S}$ and $\mathcal{N}$. However, we prove in Section 4.6 combined with Section 4.5 that $\Sigma(\mathsf{M})$ satisfies the curvature-dimension condition $\mathsf{CD}(n, n + 1)$.

And again we dare to formulate a conjecture which is consistently denoted by '$\mathsf{CD}^*(N - 1, N) \rightsquigarrow \mathsf{CD}^*(N, N + 1)$': Whenever $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is a metric measure space satisfying the reduced curvature-dimension condition $\mathsf{CD}^*(N - 1, N)$ then the corresponding N-spherical cone $(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma, \nu)$ satisfies $\mathsf{CD}^*(N, N + 1)$. Having the tensorization property in mind we express this conjecture in terms of the reduced curvature-dimension condition instead of the original one. But again we emphasize the fact that there are two essential differences from the product setting, namely the 'spherical cone metric' and the weighted 'spherical cone measure'. Therefore, the arguments used in the product framework cannot be transferred in a simple way to the spherical cone setting.

Unfortunately, the conjecture '$\mathsf{CD}^*(N - 1, N) \rightsquigarrow \mathsf{CD}^*(N, N + 1)$' as well stays unproven in this work.

Chapter 1

Preliminaries

1.1 Terminology

We consider a metric space (M, d). In this framework, we denote the *open ball* centered at $x \in \mathsf{M}$ with radius $r > 0$ by $B_r(x) = \{y \in \mathsf{M} : \mathsf{d}(x, y) < r\}$ and its *closure* by $\overline{B_r(x)} \subseteq \{y \in \mathsf{M} : \mathsf{d}(x, y) \leq r\}$.

A *curve* connecting two points $x, y \in \mathsf{M}$ is a continuous map $\gamma : [0, 1] \to \mathsf{M}$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$. The *Length* $\mathsf{L}(\gamma)$ of γ is defined as

$$\mathsf{L}(\gamma) := \sup \sum_{k=1}^{n} \mathsf{d}(\gamma(t_{k-1}), \gamma(t_k)),$$

where the supremum runs over all $n \in \mathbb{N}$ and over all partitions $0 = t_0 < t_1 < \cdots < t_n = 1$. Automatically, we have $\mathsf{L}(\gamma) \geq \mathsf{d}(x, y)$. The curve γ is called *geodesic* if and only if $\mathsf{L}(\gamma) = \mathsf{d}(x, y)$. In this case, we always assume that γ has 'constant speed' meaning that $\mathsf{L}\left(\gamma|_{[s,t]}\right) = (t - s)\mathsf{L}(\gamma) = (t - s)\mathsf{d}(x, y)$ for $0 \leq s < t \leq 1$.

We denote by $\mathcal{G}(\mathsf{M})$ the space of geodesics $\gamma : [0, 1] \to \mathsf{M}$ in M. We regard $\mathcal{G}(\mathsf{M})$ as a subset of the set $\mathsf{Lip}([0, 1], \mathsf{M})$ of Lipschitz functions equipped with the topology of uniform convergence.

(M, d) is called a *length space* if and only if for all $x, y \in \mathsf{M}$,

$$\mathsf{d}(x, y) = \inf_{\gamma} \mathsf{L}(\gamma)$$

where the infimum ranges over all curves γ connecting x and y. It is said to be a *geodesic space* if and only if every $x, y \in \mathsf{M}$ are connected by a geodesic γ.

The following result on complete length spaces is taken from [BBI01]:

Lemma 1.1.1. *Let* (M, d) *be a complete length space and* $x \in \mathsf{M}$*. Then:*

(i) The closure of a ball $B_r(x)$ is the set $\{y \in \mathsf{M} : \mathsf{d}(x,y) \leq r\}$ for $r > 0$.

(ii) M is locally compact if and only if each closed ball in M is compact.

(iii) If M is locally compact then it is a geodesic space.

Throughout this work, the triple $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ denotes a *metric measure space* consisting of a complete separable metric space (M, d) and a locally finite measure m on $(\mathsf{M}, \mathcal{B}(\mathsf{M}))$ – locally finite in the sense that the volume $\mathsf{m}(B_r(x))$ of balls centered at x is finite for all $x \in \mathsf{M}$ and all sufficiently small $r > 0$. A metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is called *normalized* if and only if $\mathsf{m}(\mathsf{M}) = 1$. It is called *compact* or *locally compact* or *geodesic* if and only if the metric space (M, d) is compact or locally compact or geodesic, respectively.

A *non-branching* metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ consists of a geodesic metric space (M, d) such that for every tuple (z, x_0, x_1, x_2) of points in M for which z is the midpoint of x_0 and x_1 as well as of x_0 and x_2, it follows that $x_1 = x_2$.

The *diameter* $\mathsf{diam}(\mathsf{M})$ of a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is defined as the diameter of its support $(\mathsf{supp}(\mathsf{m}), \mathsf{d})$, that means

$$\mathsf{diam}(\mathsf{M}) := \mathsf{diam}(\mathsf{M}, \mathsf{d}, \mathsf{m}) := \sup\{\mathsf{d}(x,y) : x, y \in \mathsf{supp}(\mathsf{m})\}$$

with the support $\mathsf{supp}(\mathsf{m})$ of m being defined as the smallest closed set $\mathsf{M}_0 \subseteq \mathsf{M}$ such that $\mathsf{m}(\mathsf{M} \setminus \mathsf{M}_0) = 0$.

Two metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ and $(\mathsf{M}', \mathsf{d}', \mathsf{m}')$ are called *isomorphic* if and only if there exists an isometry $\psi : \mathsf{M}_0 \to \mathsf{M}'_0$ mapping from $\mathsf{M}_0 := \mathsf{supp}(\mathsf{m}) \subseteq \mathsf{M}$ onto $\mathsf{M}'_0 := \mathsf{supp}(\mathsf{m}') \subseteq \mathsf{M}'$ such that $\psi_* \mathsf{m} = \mathsf{m}'$. The family of isomorphism classes of metric measure spaces is denoted by $\mathbb{X}$ and the subfamily of isomorphism classes of normalized metric measure spaces with finite variances by $\mathbb{X}_1$.

The *variance* $\mathsf{Var}(\mathsf{M}, \mathsf{d}, \mathsf{m})$ of $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is defined by

$$\mathsf{Var}(\mathsf{M}, \mathsf{d}, \mathsf{m}) = \inf \int_{\mathsf{M}'} \mathsf{d}'^2(z', x') \, d\mathsf{m}'(x')$$

where the infimum runs over all metric measure spaces $(\mathsf{M}', \mathsf{d}', \mathsf{m}')$ which are isomorphic to $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ and over all $z' \in \mathsf{M}'$.

$(\mathcal{P}_2(\mathsf{M}, \mathsf{d}), \mathsf{d}_\mathsf{W})$ denotes the L_2-*Wasserstein space* of probability measures ν on $(\mathsf{M}, \mathcal{B}(\mathsf{M}))$ with finite second moments which means that $\int_\mathsf{M} \mathsf{d}^2(x_0, x) \, d\nu(x) < \infty$ for some – hence all – $x_0 \in \mathsf{M}$. The L_2-*Wasserstein distance* $\mathsf{d}_\mathsf{W}(\mu, \nu)$ between two probability measures $\mu, \nu \in \mathcal{P}_2(\mathsf{M}, \mathsf{d})$ is defined as

$$\mathsf{d}_\mathsf{W}(\mu, \nu) = \inf \left\{ \left(\int_{\mathsf{M} \times \mathsf{M}} \mathsf{d}^2(x,y) \, d\mathsf{q}(x,y) \right)^{1/2} : \mathsf{q} \text{ coupling of } \mu \text{ and } \nu \right\}.$$

Here the infimum ranges over all *couplings* of μ and ν which are probability measures on $\mathsf{M} \times \mathsf{M}$ with marginals μ and ν.

$(\mathcal{P}_2(\mathsf{M}, \mathsf{d}), \mathsf{d}_\mathsf{W})$ is a complete separable metric space. The subspace of m-absolutely continuous measures is denoted by $\mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$. It includes the space $\mathcal{P}_{2,\mathsf{b}}(\mathsf{M}, \mathsf{d}, \mathsf{m})$ of m-absolutely continuous measures with bounded support.

The L_2-*transportation distance* $\mathbb{D}$ is defined for two metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m}), (\mathsf{M}', \mathsf{d}', \mathsf{m}') \in \mathbb{X}_1$ by

$$\mathbb{D}((\mathsf{M}, \mathsf{d}, \mathsf{m}), (\mathsf{M}', \mathsf{d}', \mathsf{m}')) = \inf \left(\int_{\mathsf{M} \times \mathsf{M}'} \tilde{\mathsf{d}}^2(x, y') \, d\mathsf{q}(x, y') \right)^{1/2}.$$

The infimum is taken over all couplings q of m and m' and over all couplings $\tilde{\mathsf{d}}$ of d and d'. The pair $(\mathbb{X}_1, \mathbb{D})$ is a complete separable length space.

Given two metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ and $(\mathsf{M}', \mathsf{d}', \mathsf{m}')$, we say that a measure q on the product space $\mathsf{M} \times \mathsf{M}'$ is a *coupling* of m and m' if and only if

$$\mathsf{q}(A \times \mathsf{M}') = \mathsf{m}(A) \quad \text{and} \quad \mathsf{q}(\mathsf{M} \times A') = \mathsf{m}'(A')$$

for all $A \in \mathcal{B}(\mathsf{M})$ and all $A' \in \mathcal{B}(\mathsf{M}')$.

We say that a pseudo-metric $\tilde{\mathsf{d}}$ – meaning that $\tilde{\mathsf{d}}$ may vanish outside the diagonal – on the disjoint union $\mathsf{M} \sqcup \mathsf{M}'$ is a *coupling* of d and d' if and only if

$$\tilde{\mathsf{d}}(x, y) = \mathsf{d}(x, y) \quad \text{and} \quad \tilde{\mathsf{d}}(x', y') = \mathsf{d}'(x', y')$$

for all $x, y \in \mathsf{supp}(\mathsf{m}) \subseteq \mathsf{M}$ and all $x', y' \in \mathsf{supp}(\mathsf{m}') \subseteq \mathsf{M}'$.

1.2 Optimal Transportation

The problem of optimal mass transportation was formulated by Monge in 1781 for the first time. He was interested in the question whether there exists a map $T : \mathbb{R}^n \to \mathbb{R}^n$ which transports a given probability distribution μ to another one ν and minimizes the functional $\int c(x, T(x)) \, d\mu(x)$ – interpreted as the *total transportation cost* for the *transference map* T. Thanks to many distributions to this topic, the problem of optimal mass transportation was solved in a mathematically satisfactory way at the end of the 20th century. The concepts and techniques developed thereby have many applications, for instance in the theory of partial differential equations and of dynamical systems. And last but not least, the concepts of generalized 'Ricci' curvature bounds presented by Sturm in [Stu06a, Stu06b] as well as by Lott and

Villani in [LV07, LV09] are based on optimal transportation. That is the reason why we would like to give a brief introduction to this topic.

Assume that we have two sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$ of 'locations' – the locations of producers of a good and the locations of consumers – and probability measures μ on X as well as ν on Y describing the distribution of the producers and consumers, respectively. Furthermore, assume that we intend to transport the goods from the producers to the consumers according to a *transference plan* which is modelled by a coupling π of μ and ν. In our picture, $d\pi(x, y)$ measures the amount of goods transported from location $x \in X$ to location $y \in Y$. We denote the set of couplings of μ and ν by $\Pi(\mu, \nu)$. The transport of goods from μ to ν causes some effort depending on the transference plan π. This effort is expressed by the *total transportation cost*

$$\mathcal{I}_c(\pi) := \int_{X \times Y} c(x, y) \; d\pi(x, y)$$

associated with π due to a *cost function* $c : X \times Y \to \mathbb{R}_+$. Intuitively, $c(x, y)$ describes how much it costs to transport a good from a producer at location x to a consumer at location y. We have a natural interest in finding the *optimal transference plan* – optimal in the sense that the associated total transportation cost is minimal. At this point we reach the formulation of *Kantorovich's optimal transportation problem*: Minimize

$$\mathcal{I}_c(\pi) = \int_{X \times Y} c(x, y) \; d\pi(x, y)$$

for $\pi \in \Pi(\mu, \nu)$. The *optimal transportation cost* is denoted by

$$\mathcal{T}_c(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \mathcal{I}_c(\pi).$$

A transference plan π is called *optimal* if and only if $\mathcal{I}_c(\pi) = \mathcal{T}_c(\mu, \nu)$.

The problem Monge studied differs from the one of Kantorovich in one additional requirement: Monge assumes that goods produced at a location x are transported to a unique destination y. Transference plans π_T satisfying this requirement have a special form

$$\pi_T = (\mathsf{Id} \times T)_* \mu$$

where $T : X \to Y$ is a measurable map which pushes forward μ to ν meaning that $\nu = T_* \mu$. It is called the *transference map*. We formulate *Monge's optimal transportation problem*: Minimize

$$\mathcal{I}_c(T) = \int_X c(x, T(x)) \; d\mu(x)$$

over all measurable maps $T : X \to Y$ such that $\nu = T_*\mu$.

The following theorem by Gangbo and McCann [GM96] states that the transportation problems of Monge and Kantorovich coincide and admit a uniquely determined minimizer.

Theorem 1.2.1 (Optimal transportation theorem)**.** *Let $c : \mathbb{R}^n \to \mathbb{R}_+$ be a strictly convex, superlinear cost function on $\mathbb{R}^n$, and let μ, ν be probability measures on $\mathbb{R}^n$ such that $\mathcal{I}_c \not\equiv \infty$ on $\Pi(\mu, \nu)$. Moreover, let μ be absolutely continuous with respect to the Lebesgue measure. Then there exists a unique optimal transference plan π for the Kantorovich transportation problem from μ to ν with cost function c. It has the form*

$$\pi = (\mathsf{Id} \times T)_* \, \mu,$$

where the transference map T is uniquely determined μ-almost everywhere by the requirements

$$T_*\mu = \nu$$

and

$$T = \mathsf{Id} - \nabla c^* \left(\nabla \varphi \right)$$

for some c-concave function φ.

In the context of this theorem we identify the cost function c with the map $\tilde{c} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ defined by $\tilde{c}(x, y) := c(x - y)$.

The claim of uniqueness with respect to T in the formulation of Theorem 1.2.1 is to be understood in the sense that two maps with the above properties coincide μ-almost everywhere. The notions of *strictly convexity, superlinearity, c-concavity* and *conjugate functions c^** are defined at the end of this section.

Theorem 1.2.1 justifies to refer to the Monge or to the Kantorovich problem by *Monge-Kantorovich transportation problem.*

As a special case, we consider the transportation problem between absolutely continuous probability measures. In this setting, a prominent relation between the densities holds true – the *Jacobian equation.* A more general version of this theorem can be found in [Vil09].

Theorem 1.2.2 (Jacobian equation)**.** *Let $\mu_0 = \rho_0 \lambda_n$, $\mu_1 = \rho_1 \lambda_n$ be two absolutely continuous probability measures with respect to the Lebesgue measure λ_n on $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an injective and locally Lipschitz continuous map such that $\mu_1 = T_*\mu_0$. Then the Jacobian equation holds true μ_0-almost surely,*

$$\rho_0(x) = \rho_1(T(x)) \mathcal{J}_T(x),$$

where $\mathcal{J}_T(x)$ is the Jacobian determinant of T at x, defined by

$$\mathcal{J}_T(x) := \lim_{\varepsilon \downarrow 0} \frac{\lambda_n[T(B_\varepsilon(x))]}{\lambda_n[B_\varepsilon(x)]}.$$

The same holds true if T is only defined on the complement of a μ_0-negligible set and satisfies the above properties on its domain of definition.

Now we generalize the framework of optimal transportation: We replace the Euclidean pair $(\mathbb{R}^n, \lambda_n)$ by a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ where d^2 plays the role of the cost function c. According to [Stu06a, Lemma 2.11], we have

Theorem 1.2.3. *(i) For each pair $\mu, \nu \in \mathcal{P}_2(\mathsf{M}, \mathsf{d})$ there exists a coupling q - called* optimal coupling *- such that*

$$\mathsf{d}_\mathsf{W}^2(\mu, \nu) = \int_{\mathsf{M} \times \mathsf{M}} \mathsf{d}^2(x, y)\ d\mathsf{q}(x, y).$$

(ii) For each geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(\mathsf{M}, \mathsf{d})$, each $k \in \mathbb{N}$ and each partition

$$0 = t_0 < t_1 < \cdots < t_k = 1$$

there exists a probability measure $\hat{\mathsf{q}}$ on M^{k+1} with the following properties:

- *The projection on the i-th factor is $\Gamma(t_i)$ for all $i = 0, 1, \ldots, k$.*
- *For $\hat{\mathsf{q}}$-almost every $x = (x_0, \ldots, x_k) \in \mathsf{M}^{k+1}$ and every $i, j = 0, 1, \ldots, k$,*

$$\mathsf{d}(x_i, x_j) = |t_i - t_j| \mathsf{d}(x_0, x_k).$$

In particular, for every pair $i, j \in \{0, 1, \ldots, k\}$ the projection on the i-th and j-th factor is an optimal coupling of $\Gamma(t_i)$ and $\Gamma(t_j)$.

(iii) If (M, d) is a non-branching space, then we have in the case $k = 2$ for $\hat{\mathsf{q}}$-almost every (x_0, x_1, x_2) and (y_0, y_1, y_2) in M^3,

$$x_1 = y_1 \Rightarrow (x_0, x_2) = (y_0, y_2).$$

In this general framework, the notion of *cyclical monotonicity* plays an important role in the sense of Theorem 1.2.5 taken from [Vil09, Theorem 5.10]:

Definition 1.2.4 (Cyclical monotonicity)**.** *Let (M, d) be a metric space. A subset $\Xi \subset \mathsf{M} \times \mathsf{M}$ is called d-cyclically monotone if and only if for any $k \in \mathbb{N}$ and for any family $(x_1, y_1), \ldots, (x_k, y_k)$ of points in Ξ the inequality*

$$\sum_{i=1}^{k} \mathsf{d}(x_i, y_i) \le \sum_{i=1}^{k} \mathsf{d}(x_i, y_{i+1})$$

holds with the convention $y_{k+1} = y_1$.

Theorem 1.2.5 (Optimal transference plan)**.** *The optimal coupling* q *of two probability measures* $\nu_0, \nu_1 \in \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *is concentrated on a* d*-cyclically monotone set.*

We finish this section by recalling some definitions and basic facts which appear in the context of optimal mass transportation. For details we refer to [Vil03].

Definition 1.2.6 (Convex functions)**.** *(i) A function* φ *on* $\mathbb{R}^n$ *is called proper if and only if it maps onto* $\mathbb{R} \cup \{+\infty\}$*.*

(ii) A proper convex function φ *on* $\mathbb{R}^n$ *is a map* $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *with* $\varphi \not\equiv +\infty$ *such that for all* $x, y \in \mathbb{R}^n$ *and all* $t \in [0,1]$*,*

$$\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y). \tag{1.2.1}$$

(iii) A proper convex function φ *on* $\mathbb{R}^n$ *is called strictly convex if and only if equality in (1.2.1) implies* $x = y$ *or* $t = 0$ *or* $t = 1$*.*

(iv) The domain $\mathsf{Dom}(\varphi)$ *of a proper convex function* φ *is defined as the convex set of points where* φ *is finite,*

$$\mathsf{Dom}(\varphi) := \{x \in \mathbb{R}^n : \varphi(x) < +\infty\}.$$

A proper convex function φ on $\mathbb{R}^n$ is automatically continuous and locally Lipschitz continuous on $\mathsf{Int}(\mathsf{Dom}(\varphi))$, the interior of the domain of φ. Due to the theorem of Rademacher, its gradient $\nabla\varphi$ is λ_n-almost everywhere well-defined and locally bounded.

In fact, a convex function φ on $\mathbb{R}^n$ is automatically twice differentiable λ_n-almost everywhere on $\mathsf{Int}(\mathsf{Dom}(\varphi))$ according to the theorem of Aleksandrov.

Definition 1.2.7 (Subdifferentiability)**.** *The subdifferential* $\partial\varphi$ *of a convex function* φ *on* $\mathbb{R}^n$ *is a set-valued map defined by*

$$y \in \partial\varphi(x) \Leftrightarrow \left[\forall z \in \mathbb{R}^n : \ \varphi(z) \geq \varphi(x) + y \cdot (z - x)\right].$$

A convex function φ is differentiable at a point $x \in \mathbb{R}^n$ if and only if $\partial\varphi(x)$ consists of a single element, namely $\nabla\varphi(x)$.

Definition 1.2.8 (Conjugate functions)**.** *The convex conjugate function or Legendre transform* $\varphi*$ *of a proper function* φ *with* $\varphi \not\equiv +\infty$ *on* $\mathbb{R}^n$ *is defined for all* $y \in \mathbb{R}^n$ *by*

$$\varphi^*(y) := \sup_{x \in \mathbb{R}^n} \left[x \cdot y - \varphi(x)\right].$$

The Legendre transform φ^* of a proper function φ is proper convex and lower semi-continuous.

Proposition 1.2.9 (Characterization of the subdifferential)**.** *For a proper convex and lower semi-continuous function φ on $\mathbb{R}^n$ it holds that*

$$x \cdot y = \varphi(x) + \varphi^*(y) \Leftrightarrow y \in \partial\varphi(x) \Leftrightarrow x \in \partial\varphi^*(y)$$

for all $x, y \in \mathbb{R}^n$.

Definition 1.2.10 (Superlinear functions)**.** *A function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called superlinear if and only if it satisfies the following limit property,*

$$\lim_{|x|\to\infty} \frac{\varphi(x)}{|x|} = +\infty.$$

Definition 1.2.11 (c-concave functions)**.** *Let $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function on $\mathbb{R}^n \times \mathbb{R}^n$. A function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is called c-concave if and only if there exists $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ with $\psi \not\equiv -\infty$ such that for all $x \in \mathbb{R}^n$,*

$$\varphi(x) = \inf_{y\in\mathbb{R}^n} \left[c(x, y) - \psi(y)\right].$$

1.3 Generalized Bounds on the 'Ricci' Curvature and the Dimension

On the way to a notion of 'Ricci' curvature and dimension for a wider class of spaces than the one of Riemannian manifolds the essential steps were made by Sturm [Stu06a, Stu06b] as well as by Lott and Villani [LV07, LV09]. The concepts of [Stu06a, Stu06b] and [LV07, LV09] are similar – for non-branching spaces, they coincide. We will follow the notation of [Stu06a, Stu06b].

A dimension-independent concept of lower 'Ricci' curvature bounds is presented in the framework of metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ in [Stu06a]. The definition is based on convexity properties of the *relative Shannon entropy* $\mathsf{Ent}(\cdot|\mathsf{m})$ with respect to the reference measure m. The mapping $\nu \mapsto \mathsf{Ent}(\nu|\mathsf{m})$ is considered as a function on $\mathcal{P}_2(\mathsf{M}, \mathsf{d})$,

$$\mathsf{Ent}(\nu|\mathsf{m}) := \begin{cases} \lim\limits_{\varepsilon\to 0} \int_{\{\rho>\varepsilon\}} \rho \log \rho \, d\mathsf{m}, & \text{if } \nu = \rho\mathsf{m} \in \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m}) \\ +\infty, & \text{if } \nu \in \mathcal{P}_2(\mathsf{M}, \mathsf{d}) \setminus \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m}). \end{cases}$$

Precisely,

Definition 1.3.1. *A metric measure space* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies* $\mathsf{Curv}(\mathsf{M},\mathsf{d},\mathsf{m}) \geq K$ *for* $K \in \mathbb{R}$ *if and only if the relative entropy* $\mathsf{Ent}(\cdot|\mathsf{m})$ *is weakly* K*-convex on* $\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ *in the following sense: For any pair* $\nu_0,\nu_1 \in \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ *with* $\mathsf{Ent}(\nu_0|\mathsf{m}) < \infty$ *and* $\mathsf{Ent}(\nu_1|\mathsf{m}) < \infty$ *there exists a geodesic* $\Gamma : [0,1] \to \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ *connecting* $\nu_0 = \Gamma(0)$ *and* $\nu_1 = \Gamma(1)$ *and satisfying*

$$\mathsf{Ent}(\Gamma(t)|\mathsf{m}) \leq (1-t)\mathsf{Ent}(\Gamma(0)|\mathsf{m}) + t\mathsf{Ent}(\Gamma(1)|\mathsf{m}) - \frac{K}{2}t(1-t)\,\mathsf{d}_{\mathsf{W}}^2(\Gamma(0),\Gamma(1))$$

for all $t \in [0,1]$.

The starting point of this definition is the observation that for complete Riemannian manifolds M with Riemannian distance d and Riemannian volume $\mathsf{m} = \mathsf{vol}$, it holds that $\mathsf{Ent}(\cdot|\mathsf{m})$ is weakly K-convex on $\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ if and only if $\mathsf{Ric} \geq K$, an abbreviation indicating that $\mathsf{Ric}_{\mathsf{M}}(\xi,\xi) \geq K \cdot |\xi|^2$ for all $\xi \in \mathsf{TM}$.

The curvature-dimension condition $\mathsf{CD}(K,N)$ introduced in [Stu06b] is defined in terms of convexity properties of the lower semi-continuous *Rényi entropy functional*

$$\mathsf{S}_N(\nu|\mathsf{m}) := -\int_{\mathsf{M}} \rho^{-1/N} d\nu$$

on $\mathcal{P}_2(\mathsf{M},\mathsf{d})$ where ρ denotes the density of the absolutely continuous part ν^c in the Lebesgue decomposition $\nu = \nu^c + \nu^s = \rho\mathsf{m} + \nu^s$ of $\nu \in \mathcal{P}_2(\mathsf{M},\mathsf{d})$. For general (K,N) the condition $\mathsf{CD}(K,N)$ is quite involved. There are two cases which lead to significant simplifications: $N = \infty$ and $K = 0$.

→ The definition of the limiting case $\mathsf{CD}(K,\infty)$ for $K \in \mathbb{R}$ coincides with the definition of $\mathsf{Curv}(\mathsf{M},\mathsf{d},\mathsf{m}) \geq K$.

→ The condition $\mathsf{CD}(0,N)$ for $N \in [1,\infty)$ states that for each pair $\nu_0,\nu_1 \in \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ there exists a geodesic $\Gamma(t) = \rho_t\mathsf{m}$ in $\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting them such that the Rényi entropy functional

$$\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) := -\int_{\mathsf{M}} \rho_t^{1-1/N'} d\mathsf{m}$$

is convex in $t \in [0,1]$ for each $N' \geq N$.

For general $K \in \mathbb{R}$ and $N \in [1,\infty)$ the condition $\mathsf{CD}(K,N)$ is defined as follows:

Definition 1.3.2. *A metric measure space* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *fulfills the curvature-dimension condition* $\mathsf{CD}(K, N)$ *for two numbers* $K, N \in \mathbb{R}$ *with* $N \geq 1$ *if and only if for all* $\nu_0, \nu_1 \in \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *there exist an optimal coupling* q *of* $\nu_0 = \rho_0 \mathsf{m}$ *and* $\nu_1 = \rho_1 \mathsf{m}$ *and a geodesic* $\Gamma : [0, 1] \to \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *connecting* ν_0 *and* ν_1 *such that*

$$\begin{aligned}&\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq \\ &\quad \leq -\int_{\mathsf{M}\times\mathsf{M}} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0, x_1))\rho_0^{-1/N'}(x_0) + \tau_{K,N'}^{(t)}(\mathsf{d}(x_0, x_1))\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0, x_1)\end{aligned} \tag{1.3.1}$$

for all $t \in [0, 1]$ *and all* $N' \geq N$.

In order to define the *volume distortion coefficients* $\tau_{K,N}^{(t)}(\cdot)$ we introduce for $\theta \in \mathbb{R}_+$,

$$\mathfrak{S}_k(\theta) := \begin{cases} \frac{\sin(\sqrt{k}\theta)}{\sqrt{k}\theta} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \frac{\sinh(\sqrt{-k}\theta)}{\sqrt{-k}\theta} & \text{if } k < 0 \end{cases}$$

and set for $t \in [0, 1]$,

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} t\frac{\mathfrak{S}_{K/N}(t\theta)}{\mathfrak{S}_{K/N}(\theta)} & \text{if } K\theta^2 < N\pi^2 \\ \infty & \text{else} \end{cases}$$

as well as $\tau_{K,N}^{(t)}(\theta) := t^{1/N}\sigma_{K,N-1}^{(t)}(\theta)^{1-1/N}$.

For fixed $t \in (0, 1)$ and $\theta \in (0, \infty)$ the function $(K, N) \mapsto \tau_{K,N}^{(t)}(\theta)$ is continuous, non-decreasing in K and non-increasing in N.

The coefficient $\tau_{K,N}^{(t)}(\mathsf{d}(x_0, x_1))$ is a measure for the volume distortion due to the curvature K along the geodesic connecting x_0 and x_1.

The uniqueness of geodesics in non-branching metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying the curvature-dimension condition $\mathsf{CD}(K, N)$ for $K, N \in \mathbb{R}$ with $N \geq 1$ is proved in [Stu06b] as well as an equivalent characterization of $\mathsf{CD}(K, N)$. This is the content of the following two propositions.

Proposition 1.3.3. *Let* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *be a non-branching metric measure space satisfying the condition* $\mathsf{CD}(K, N)$ *for some numbers* $K, N \in \mathbb{R}$ *with* $N \geq 1$. *Then for every* $x \in \mathsf{supp}(\mathsf{m}) \subseteq \mathsf{M}$ *and* m*-almost every* $y \in \mathsf{M}$ *(with exceptional set depending on* x*) there exists a unique geodesic between* x *and* y.

Moreover, there exists a measurable map $\gamma : \mathsf{M} \times \mathsf{M} \to \mathcal{G}(\mathsf{M})$ *such that for* $\mathsf{m} \otimes \mathsf{m}$*-almost every* $(x, y) \in \mathsf{M} \times \mathsf{M}$ *the curve* $t \mapsto \gamma_t(x, y)$ *is the unique geodesic connecting* x *and* y.

Proposition 1.3.4. *Let* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *be a compact and non-branching metric measure space. Fix two numbers* $K, N \in \mathbb{R}$ *with* $N \geq 1$*. Then the following statements are equivalent:*

(i) $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfies the curvature-dimension condition* $\mathsf{CD}(K, N)$*.*

(ii) For each pair $\nu_0, \nu_1 \in \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *there exist an optimal coupling* q *of* ν_0 *and* ν_1 *and a geodesic* $\Gamma : [0, 1] \to \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *from* ν_0 *to* ν_1 *such that for all* $t \in [0, 1]$ *and all* $N' \geq N$*,*

$$\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq \tau_{K,N'}^{(1-t)}(\theta)\mathsf{S}_{N'}(\nu_0|\mathsf{m}) + \tau_{K,N'}^{(t)}(\theta)\mathsf{S}_{N'}(\nu_1|\mathsf{m}),$$

where

$$\theta := \begin{cases} \mathsf{q} - \operatorname{essinf}_{x_0,x_1}\mathsf{d}(x_0, x_1), & \text{if } K \geq 0, \\ \mathsf{q} - \operatorname{esssup}_{x_0,x_1}\mathsf{d}(x_0, x_1), & \text{if } K < 0, \end{cases}$$

denotes the minimal (in the case $K \geq 0$*) or maximal (if* $K < 0$*) transportation distance.*

(iii) For all $\nu_0, \nu_1 \in \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *and every optimal coupling* q *of* $\nu_0 = \rho_0\mathsf{m}$ *and* $\nu_1 = \rho_1\mathsf{m}$*,*

$$\rho_t(\gamma_t(x_0, x_1)) \leq \left[\tau_{K,N}^{(1-t)}(\mathsf{d}(x_0, x_1))\rho_0^{-1/N}(x_0) + \tau_{K,N}^{(t)}(\mathsf{d}(x_0, x_1))\rho_1^{-1/N}(x_1)\right]^{-N}$$

for all $t \in [0, 1]$ *and* q*-almost every* $(x_0, x_1) \in \mathsf{M} \times \mathsf{M}$*. For all* $t \in [0, 1]$*,* ρ_t *denotes the density with respect to* m *of the push-forward measure of* q *under the map* $(x_0, x_1) \mapsto \gamma_t(x_0, x_1)$*.*

1.4 Examples

Examples of metric measure spaces satisfying the curvature-dimension condition $\mathsf{CD}(K, N)$ include:

* Riemannian manifolds and weighted Riemannian spaces [OV00], [CMS01], [RS05], [Stu05, Stu06a, Stu06b]:

  Theorem 1.4.1 (Riemannian spaces)**.** *Let* M *be a complete Riemannian manifold with Riemannian distance* d *and Riemannian volume* m*. Let numbers* $K, N \in \mathbb{R}$ *with* $N \geq 1$ *be given.*

(i) Let $\mathsf{m}' = e^{-V}\mathsf{m}$ *with a* C^2*-function* $V : \mathsf{M} \to \mathbb{R}$*. Then*

$$\mathsf{Curv}(\mathsf{M}, \mathsf{d}, \mathsf{m}') \geq \inf\{\mathsf{Ric}_{\mathsf{M}}(\xi, \xi) + \mathsf{Hess}\ V(\xi, \xi) : \xi \in \mathsf{TM}, |\xi| = 1\}.$$

In particular, $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfies* $\mathsf{Curv}(\mathsf{M}, \mathsf{d}, \mathsf{m}) \geq K$ *if and only if its Ricci curvature is bounded from below by* K*.*

(ii) The metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfies* $\mathsf{CD}(K, N)$ *if and only if the Ricci curvature of the Riemannian manifold* M *is bounded from below by* K *and the dimension from above by* N*.*

(iii) Moreover, in this case for every measurable function $V : \mathsf{M} \to \mathbb{R}$ *the weighted space* $(\mathsf{M}, \mathsf{d}, V\mathsf{m})$ *satisfies* $\mathsf{CD}(K + K', N + N')$ *provided*

$$\mathsf{Hess}\ V^{1/N'} \leq -\frac{K'}{N'} V^{1/N'}$$

for numbers $K' \in \mathbb{R}$ *and* $N' > 0$*, in the sense that*

$$V(\gamma_t)^{1/N'} \geq \sigma^{(1-t)}_{K',N'}(\mathsf{d}(\gamma_0, \gamma_1)) V(\gamma_0)^{1/N'} + \sigma^{(t)}_{K',N'}(\mathsf{d}(\gamma_0, \gamma_1)) V(\gamma_1)^{1/N'}$$

for each geodesic $\gamma : [0, 1] \to \mathsf{M}$ *and each* $t \in [0, 1]$*.*

$*$ Finsler spaces [Oht]:

Theorem 1.4.2 (Finsler spaces)**.** *Let* $n \geq 2$ *and* $K \in \mathbb{R}$*. Let* (M, F) *be a connected, forward geodesically complete, n-dimensional* C^∞*-Finsler manifold, and let* m *be an arbitrary positive* C^∞*-measure on* M*.*

(i) For $N \in (n, \infty)$*,* $(\mathsf{M}, \mathsf{F}, \mathsf{m})$ *satisfies* $\mathsf{CD}(K, N)$ *if and only if*

$$\mathsf{Ric}_{\mathsf{M}}(v, v) + \partial_v^2 \mathcal{V} - \frac{1}{N - n}(\partial_v \mathcal{V})^2 \geq K$$

for every unit vector $v \in \mathsf{TM}$*.*

(ii) $(\mathsf{M}, \mathsf{F}, \mathsf{m})$ *satisfies* $\mathsf{CD}(K, n)$ *if and only if* $\mathsf{Ric} \geq K$ *and* $\partial_v \mathcal{V} = 0$ *hold for every unit vector* $v \in \mathsf{TM}$*.*

(iii) $(\mathsf{M}, \mathsf{F}, \mathsf{m})$ *satisfies* $\mathsf{CD}(K, \infty)$ *if and only if* $\mathsf{Ric}_{\mathsf{M}}(v, v) + \partial_v^2 \mathcal{V} \geq K$ *holds for every unit vector* $v \in \mathsf{TM}$*.*

Here it is set, given a unit vector $v \in \mathsf{T}_x\mathsf{M}$,

$$\mathcal{V}(v) := \log\left(\frac{\mathsf{vol}_{g_v}\left(B^+_{\mathsf{T}_x\mathsf{M}}(0, 1)\right)}{\mathsf{m}_x\left(B^+_{\mathsf{T}_x\mathsf{M}}(0, 1)\right)}\right),$$

where vol_{g_v} and m_x stand for Lebesgue measures on $\mathsf{T}_x\mathsf{M}$ induced from g_v and m, respectively, and $B^+_{\mathsf{T}_x\mathsf{M}}(0,1)$ is the forward open ball with center 0 and radius 1. For the precise definitions we refer to [Oht].

* Alexandrov spaces of generalized non-negative sectional curvature [Pet09]:

  Theorem 1.4.3 (Alexandrov spaces)**.** *An Alexandrov space of Hausdorff dimension n whose generalized sectional curvature is bounded from below by 0 equipped with the Hausdorff measure is a metric measure space satisfying $\mathsf{CD}(0,n)$.*

  A further reaching conjecture is only formulated in [Pet09], namely:

  Claim 1.4.4. *An n-dimensional Alexandrov space with generalized sectional curvature bounded from below by k endowed with the Hausdorff measure is a metric measure space satisfying $\mathsf{CD}((n-1)k,n)$.*

  So far, no proof exists for the last claim.

Chapter 2

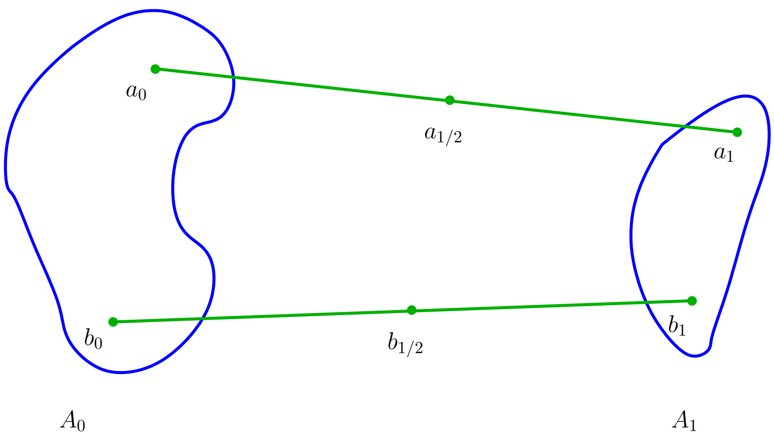
The Borell-Brascamp-Lieb Inequality

2.1 The Story of the Borell-Brascamp-Lieb Inequality

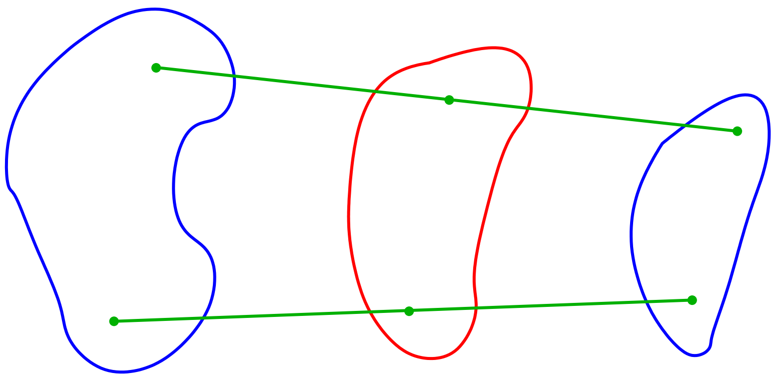
2.1.1 The Euclidean Setting

The story of the Borell-Brascamp-Lieb inequality begins in the Euclidean setting $(\mathbb{R}^n, |\cdot|, \lambda_n)$ with Euclidean distance $|\cdot|$ and n-dimensional Lebesgue measure λ_n.

We start our introduction to this topic with an inequality which is strongly related to the Borell-Brascamp-Lieb inequality, the so-called Brunn-Minkowski inequality: We consider two sets A_0 and A_1 in $\mathbb{R}^n$ and connect points in A_0 and A_1 via straight lines.



The midpoints lying on those geodesics form the set $A_{1/2} := \frac{1}{2}A_0 + \frac{1}{2}A_1$.



The Brunn-Minkowski inequality says that the volume of the set of midpoints is bounded from below by the $(1/n)$-mean of the volumes of A_0 and A_1:

$$\lambda_n(A_{1/2}) \geq \left(\tfrac{1}{2}\lambda_n(A_0)^{1/n} + \tfrac{1}{2}\lambda_n(A_1)^{1/n}\right)^n .$$

The Brunn-Minkowski inequality provides a way to solve the Euclidean isoperimetric theorem saying: Among all compact sets in $\mathbb{R}^n$ with given volume, the sphere has minimal surface. The proof goes like this: By definition, the surface $\mathcal{S}(A)$ of a set A in $\mathbb{R}^n$ is given by

$$\mathcal{S}(A) = \liminf_{\varepsilon \downarrow 0} \frac{\lambda_n(A + B_\varepsilon) - \lambda_n(A)}{\varepsilon}$$

using $B_r := B_r(0)$ for $r > 0$. Applying the Brunn-Minkowski inequality and the positive homogeneity of degree n of the Lebesgue measure yields

$$\begin{aligned}\mathcal{S}(A) &\geq \liminf_{\varepsilon \downarrow 0} \frac{(\lambda_n(A)^{1/n} + \varepsilon\lambda_n(B_1)^{1/n})^n - \lambda_n(A)}{\varepsilon} \\ &= n\lambda_n(A)^{(n-1)/n}\lambda_n(B_1)^{1/n}.\end{aligned}$$

Recalling that the surface of the unit sphere $\mathcal{S}(B_1)$ is equal to $n\lambda_n(B_1)$, we have

$$\begin{aligned}\left(\frac{\mathcal{S}(A)}{\mathcal{S}(B_1)}\right)^{1/(n-1)} &\geq \left(\frac{n\lambda_n(A)^{(n-1)/n}\lambda_n(B_1)^{1/n}}{n\lambda_n(B_1)}\right)^{1/(n-1)} \\ &= \left(\frac{\lambda_n(A)}{\lambda_n(B_1)}\right)^{1/n} .\end{aligned}$$

We end up with the statement of the isoperimetric theorem: Fixing a volume, say the volume of the unit sphere, we can see from the last inequality that the unit sphere itself has minimal surface.

In the Euclidean setting, we denote the Borell-Brascamp-Lieb inequality by $\mathsf{BBL}(0,n)$ where the parameter 0 stands for a lower bound on the Ricci curvature and n for the dimension of $(\mathbb{R}^n, |\cdot|, \lambda_n)$. $\mathsf{BBL}(0,n)$ consists of a family of inequalities labeled by a parameter $p \geq -\frac{1}{n}$. More precisely, the parameter p labels the so-called p-mean $\mathcal{M}_t^p(a,b)$ which appears in the formulation of $\mathsf{BBL}(0,n)$: For $p \in \mathbb{R} \setminus \{0\}$, $t \in [0,1]$ and $a, b \geq 0$ the p-mean $\mathcal{M}_t^p(a,b)$ is defined by

$$\mathcal{M}_t^p(a,b) := \begin{cases} ((1-t)a^p + tb^p)^{\frac{1}{p}} & \text{if } ab \neq 0 \\ 0 & \text{if } ab = 0. \end{cases}$$

In the cases $p \in \{0, \pm\infty\}$, the corresponding p-means are defined as limits:

$$\mathcal{M}_t^p(a,b) := \begin{cases} \min\{a,b\} & \text{if } p = -\infty \\ \max\{a,b\} & \text{if } p = +\infty \\ a^{1-t}b^t & \text{if } p = 0. \end{cases}$$

Using this notation, $\mathsf{BBL}(0,n)$ states that for every $p \geq -\frac{1}{n}$,

$$\int_{\mathbb{R}^n} h \, d\lambda_n \geq \mathcal{M}_t^{p/(1+np)} \left(\int_{\mathbb{R}^n} f \, d\lambda_n, \int_{\mathbb{R}^n} g \, d\lambda_n \right)$$

whenever $t \in [0,1]$ and $f, g, h \geq 0$ are non-negative integrable functions on $\mathbb{R}^n$ satisfying

$$h((1-t)x + ty) \geq \mathcal{M}_t^p \left(f(x), g(y) \right)$$

for all $x, y \in \mathbb{R}^n$. The number $p/(1+np)$ is interpreted in the 'obvious' way: It equals $-\infty$ for $p = -\frac{1}{n}$ and $\frac{1}{n}$ for $p = +\infty$.

The inequalities associated with a label $p > 0$ were proved by Henstock and Macbeath [HM53] in the case n=1 and by Dinghas [Din57]. The case $p = 0$ was proved by Prékopa [Pré71, Pré73] and Leindler [Lei72]. In general version, $\mathsf{BBL}(0,n)$ was stated and proved independently by Borell [Bor75] and by Brascamp and Lieb [BL76].

One prominent member of this family of inequalities is the so-called Prékopa-Leindler inequality $\mathsf{PL}(0,n)$ having the label $p = 0$: Given $t \in [0,1]$ and three non-negative integrable functions f, g, h on $\mathbb{R}^n$, we assume that for all $x, y \in \mathbb{R}^n$

$$h((1-t)x + ty) \geq f(x)^{1-t} g(y)^t.$$

Then
$$\int_{\mathbb{R}^n} h \, d\lambda_n \geq \left(\int_{\mathbb{R}^n} f \, d\lambda_n\right)^{1-t} \left(\int_{\mathbb{R}^n} g \, d\lambda_n\right)^t .$$
We may interpret $\mathsf{PL}(0, n)$ as the functional version of the Brunn-Minkowski inequality $\mathsf{BM}(0, n)$ in the following sense: By applying $\mathsf{PL}(0, n)$ to indicator functions of non-empty measurable sets $A, B \subseteq \mathbb{R}^n$, we obtain for all $t \in [0, 1]$

$$\lambda_n((1-t)A + tB) \geq \lambda_n(A)^{1-t}\lambda_n(B)^t.$$

After suitable scaling and taking advantage of the positive homogeneity of the Lebesgue measure, we derive $\mathsf{BM}(0, n)$

$$\lambda_n((1-t)A + tB) \geq \left((1-t)\lambda_n(A)^{1/n} + t\lambda_n(B)^{1/n}\right)^n$$

provided that the Minkowski sum

$$(1-t)A + tB = \{(1-t)x + ty : x \in A, y \in B\}$$

is also measurable.

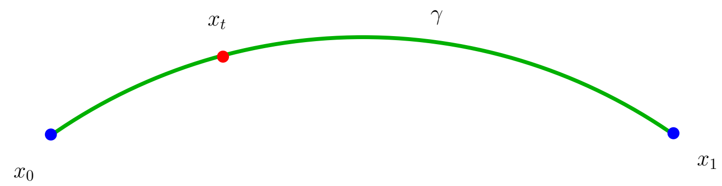
So far we essentially followed a work by Gardner [Gar02] who presented a guide explaining the relationship between the Prékopa-Leindler inequality and other inequalities in geometry and analysis and some of its recent applications in the Euclidean case.

2.1.2 The Riemannian Setting

An inspiring and important step beyond the Euclidean case was made by Cordero-Erausquin, McCann and Schmuckenschläger [CMS01]. In this work the authors generalize $\mathsf{BBL}(0, n)$ to $\mathsf{BBL}((n-1)k, n)$ in the setting of n-dimensional Riemannian manifolds whose Ricci curvature is bounded from below by $(n-1)k$ using methods of mass transportation theory.

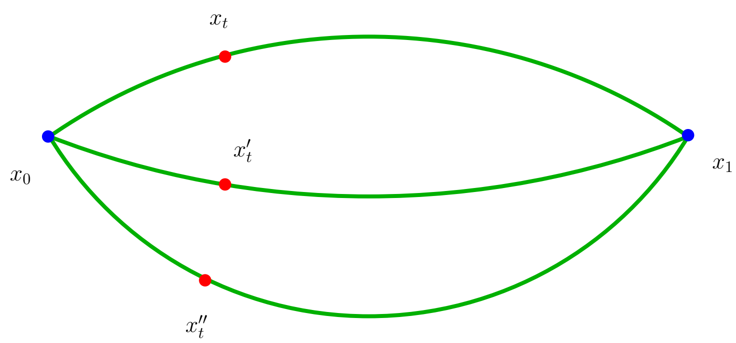
They consider a triple $(\mathsf{M}, \mathsf{d}, \mathsf{vol})$ consisting of a complete and connected, n-dimensional Riemannian manifold M with Riemannian distance d and Riemannian volume vol. The first challenge in this framework is to define a notion of 'midpoint' or 'barycenter' in order to be able to replace the weighted vector sum of two points appearing in the Euclidean case. Cordero-Erausquin, McCann and Schmuckenschläger (CMS) proceed as follows: On each geodesic γ connecting two points x_0 and x_1 in M lies a unique point x_t dividing this geodesic with ratio $t : (1-t)$,

$$\mathsf{d}(x_0, x_t) = t\mathsf{d}(x_0, x_1) \text{ and } \mathsf{d}(x_t, x_1) = (1-t)\mathsf{d}(x_0, x_1)$$



All the points fulfilling this property build the set $Z_t(x_0, x_1)$ of t-intermediate points,

$$Z_t(x_0, x_1) := \{z \in \mathsf{M} : \mathsf{d}(x_0, z) = t\mathsf{d}(x_0, x_1),\ \mathsf{d}(z, x_1) = (1-t)\mathsf{d}(x_0, x_1)\}.$$



Whenever the geodesic $\gamma : [0,1] \to \mathsf{M}$ joining x_0 and x_1 is unique, then

$$Z_t(x_0, x_1) = \{\gamma(t)\}.$$

Accordingly, we define

$$Z_t(A_0, A_1) := \bigcup_{x_0 \in A_0, x_1 \in A_1} Z_t(x_0, x_1)$$

playing the role of the Minkowski sum. The main result of CMS is the following:

Theorem 2.1.1. *Let* M *be a complete, connected* n*-dimensional Riemannian manifold with* $\mathsf{Ric} \geq (n-1)k$ *and* $t \in [0,1]$*. Let* $f, g, h \geq 0$ *be integrable functions on* M *satisfying* $\int_{\mathsf{M}} f \ \mathsf{dvol} = \int_{\mathsf{M}} g \ \mathsf{dvol} = 1$*. Assume that for every* $x, y \in \mathsf{M}$ *and every* $z \in Z_t(x, y)$*,*

$$h(z) \geq \mathsf{M}_t^{-1/n}\left(\frac{f(x)}{\zeta_k^{(1-t)}(x,y)}, \frac{g(y)}{\zeta_k^{(t)}(x,y)}\right).$$

Then

$$\int_{\mathsf{M}} h \ \mathsf{dvol} \geq 1.$$

This inequality reveals its non-Euclidean character by the use of the volume distortion coefficients $\zeta_k^{(t)}$: We define $\zeta_k^{(t)}(x,y)$ for $x,y\in\mathsf{M}$ and $t\in[0,1]$ by

$$\zeta_k^{(t)}(x,y) := \left(\frac{\mathfrak{S}_k(t\mathsf{d}(x,y))}{\mathfrak{S}_k(\mathsf{d}(x,y))}\right)^{n-1}.$$

In their main theorem, CMS identify the inequality with label $p=-\frac{1}{n}$, the so-called $(-\frac{1}{n})$-mean inequality, to be the strongest one in this whole class of inequalities. Certain properties of p-means, in particular a kind of Hölder-inequality, lead to the Riemannian version of the Borell-Brascamp-Lieb inequality:

Corollary 2.1.2 ($\mathsf{BBL}((n-1)k,n)$)**.** *Let* M *be a complete, connected n-dimensional Riemannian manifold with* $\mathsf{Ric}\geq(n-1)k$. *Fix* $p\geq-\frac{1}{n}$ *and* $t\in[0,1]$. *Let* $f,g,h\geq 0$ *be integrable functions on* M. *Assume that for every* $x,y\in\mathsf{M}$ *and every* $z\in Z_t(x,y)$,

$$h(z)\geq\mathcal{M}_t^p\left(\frac{f(x)}{\zeta_k^{(1-t)}(x,y)},\frac{g(y)}{\zeta_k^{(t)}(x,y)}\right).$$

Then

$$\int_{\mathsf{M}} h\ d\mathsf{vol}\geq\mathcal{M}_t^{p/(1+np)}\left(\int_{\mathsf{M}} f\ d\mathsf{vol}\int_{\mathsf{M}} g\ d\mathsf{vol}\right).$$

2.1.3 The Setting of Metric Measure Spaces

Leaving the Riemannian case we arrive at the setting of geodesic metric measure spaces $(\mathsf{M},\mathsf{d},\mathsf{m})$. According to the Riemannian setting, we define for $t\in[0,1]$ and for $a,b\in\mathsf{M}$ the set $Z_t(a,b)$ of t-intermediate points of a and b by

$$Z_t(a,b) := \{x\in\mathsf{M} : \mathsf{d}(a,x)=t\mathsf{d}(a,b),\mathsf{d}(x,b)=(1-t)\mathsf{d}(a,b)\}$$

and furthermore,

$$Z_t(A,B) := \bigcup_{a\in A,b\in B} Z_t(a,b)$$

for subsets $A,B\subseteq\mathsf{M}$. We define $\mathsf{BBL}(K,N)$ in this setting following the way of Cordero-Erausquin, McCann and Schmuckenschläger [CMS01]:

Definition 2.1.3. *A metric measure space* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies the Borell-Brascamp-Lieb inequality* $\mathsf{BBL}(K,N)$ *for two numbers* $K,N\in\mathbb{R}$ *with* $N\geq 1$ *if and only if for all* $p\geq-\frac{1}{N}$, $t\in[0,1]$ *and all non-negative integrable functions* f,g,h *on* M *satisfying*

$$h(z)\geq\mathcal{M}_t^p\left(\frac{f(x)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(x,y))},\frac{g(y)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(x,y))}\right)$$

for all $x, y \in \mathsf{M}$ and every $z \in Z_t(x,y)$, it holds that

$$\int_{\mathsf{M}} h \, d\mathsf{m} \geq \mathcal{M}_t^{p/(1+Np)} \left(\int_{\mathsf{M}} f \, d\mathsf{m}, \int_{\mathsf{M}} g \, d\mathsf{m} \right).$$

The volume distortion coefficients $\tilde{\tau}_{K,N}^{(t)}(\cdot)$ are defined for $t \in [0,1]$ and for $\theta \in \mathbb{R}_+$ by

$$\tilde{\tau}_{K,N}^{(t)}(\theta) := \begin{cases} \left(\frac{\mathfrak{S}_{K/(N-1)}(t\theta)}{\mathfrak{S}_{K/(N-1)}(\theta)} \right)^{N-1}, & \text{if } K\theta^2 < (N-1)\pi^2, \\ \infty, & \text{else.} \end{cases}$$

In the sequel we will take advantage of the fact that the $(-\frac{1}{N})$-mean inequality is the strongest one and that it suffices to consider normalized functions f and g meaning that $\int_{\mathsf{M}} f \, d\mathsf{m} = \int_{\mathsf{M}} g \, d\mathsf{m} = 1$ in the case $p = -\frac{1}{N}$ – the corresponding proof in [CMS01] can easily be generalized to the setting of metric measure spaces. We will focus on this case.

The aim of this chapter is to show the following statements:

- ▷ On a compact and non-branching metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying the curvature-dimension condition $\mathsf{CD}(K,N)$, the Borell-Brascamp-Lieb inequality $\mathsf{BBL}(K,N)$ holds true.

- ▷ The Borell-Brascamp-Lieb inequality $\mathsf{BBL}(K,N)$ is stable under convergence

$$(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n) \xrightarrow{\mathbb{D}} (\mathsf{M}, \mathsf{d}, \mathsf{m}), \quad n \to \infty,$$

  where $(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n)$ is assumed to be a compact metric measure space for every $n \in \mathbb{N}$. According to [Stu06a], $(\mathsf{M}_n, \mathsf{d}_n)$ can be embedded in (M, d) via an isometry ψ_n

$$\psi_n : (\mathsf{M}_n, \mathsf{d}_n) \hookrightarrow (\mathsf{M}, \mathsf{d}).$$

  Thus, it suffices to show that $\mathsf{BBL}(K,N)$ is stable with respect to the weak convergence $\mathsf{m}_n \xrightarrow{w} \mathsf{m}$ of probability measures on a given metric space (M, d). Due to the definition of weak convergence, the requirement formulated in Definition 2.1.3 would obviously hold true on the limit $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ if we considered only bounded continuous functions. The technical approximations which cope with the general case are prepared in Section 2.3.

2.2 The Relation to $\mathsf{CD}(K,N)$

The following theorem is an immediate consequence of Proposition 1.3.4.

Theorem 2.2.1 ($\mathsf{CD}(K,N) \Rightarrow \mathsf{BBL}(K,N)$)**.** *For each compact and non-branching metric measure space* $(\mathsf{M},\mathsf{d},\mathsf{m})$*, the curvature-dimension condition* $\mathsf{CD}(K,N)$ *with two real parameters* K *and* $N \geq 1$ *implies the Borell-Brascamp-Lieb inequality* $\mathsf{BBL}(K,N)$*.*

Proof. We fix $t \in (0,1)$ and three non-negative integrable functions f,g,h on M satisfying $\int_{\mathsf{M}} f\ d\mathsf{m} = \int_{\mathsf{M}} g\ d\mathsf{m} = 1$ and

$$h(z) \geq \mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(x)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(x,y))}, \frac{g(y)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(x,y))}\right)$$

for all $x,y \in \mathsf{M}$ and $z = \gamma_t(x,y)$ where $\gamma_t : \mathsf{M}\times\mathsf{M} \to \mathsf{M}$ denotes the map introduced in Proposition 1.3.3. We regard f and g as density functions of probability measures $\nu_0,\nu_1 \in \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ meaning

$$\nu_0 = f\mathsf{m},\ \ \nu_1 = g\mathsf{m}.$$

Let q be an optimal coupling of ν_0 and ν_1. According to Proposition 1.3.4 the following inequality holds true

$$\begin{aligned}\rho_t(\gamma_t(x,y)) &\leq \left[\tau_{K,N}^{(1-t)}(\mathsf{d}(x,y))f^{-1/N}(x) + \tau_{K,N}^{(t)}(\mathsf{d}(x,y))g^{-1/N}(y)\right]^{-N} \\ &= \mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(x)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(x,y))}, \frac{g(y)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(x,y))}\right)\end{aligned}$$

for q-almost every $(x,y) \in \mathsf{M}\times\mathsf{M}$ denoting by ρ_t the density of the push-forward measure of q under the map γ_t. Due to our assumption,

$$\rho_t(\gamma_t(x,y)) \leq h(\gamma_t(x,y))$$

for q-almost every $(x,y) \in \mathsf{M}\times\mathsf{M}$ which means that $\rho_t \leq h$ on a measurable set $Y_t \subseteq \mathsf{M}$ for which $\int_{Y_t} \rho_t d\mathsf{m} = 1$. This concludes the proof:

$$\int_{\mathsf{M}} h\ d\mathsf{m} \geq \int_{Y_t} h\ d\mathsf{m} \geq \int_{Y_t} \rho_t\ d\mathsf{m} = 1.$$

□

Proposition 2.2.2 ($\mathsf{BBL}(K,N) \Rightarrow \mathsf{BM}(K,N)$)**.** *Fix two parameters $K, N \in \mathbb{R}$ with $N \geq 1$. For each metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ the Borell-Brascamp-Lieb inequality $\mathsf{BBL}(K,N)$ implies the generalized Brunn-Minkowski inequality $\mathsf{BM}(K,N)$ in the following version: For all measurable sets $X, Y \subseteq \mathsf{M}$ and all $t \in [0,1]$,*

$$\mathsf{m}^*(Z_t(X,Y)) \geq \mathcal{M}_t^{\frac{1}{N}} \left(\tilde{\tau}_{K,N}^{(1-t)}(\theta) \cdot \mathsf{m}(X), \tilde{\tau}_{K,N}^{(t)}(\theta) \cdot \mathsf{m}(Y) \right)$$

with

$$\theta := \begin{cases} \inf\limits_{x \in X, y \in Y} \mathsf{d}(x,y), & K \geq 0 \\ \sup\limits_{x \in X, y \in Y} \mathsf{d}(x,y), & K < 0 \end{cases}$$

denoting by m^* the outer measure of m.

Remark 2.2.3. *(i) We use the concept of outer measures in the formulation of $\mathsf{BM}(K,N)$ in order to avoid difficulties arising from the fact that the t-intermediate set $Z_t(X,Y)$ of two measurable sets $X, Y \subseteq \mathsf{M}$ is not automatically measurable. Actually, in many applications X and Y are not only measurable but also compact. In this case the set $Z_t(X,Y)$ is closed, consequently measurable and the outer measure m^* appearing on the left-hand side of the above inequality can be replaced by m.*

(ii) In [Stu06b], Sturm derives the above version of the Brunn-Minkowski inequality for metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ from the curvature-dimension condition $\mathsf{CD}(K,N)$. The Brunn-Minkowski inequality implies further geometric consequences like the Bishop-Gromov volume growth estimate and the Bonnet-Myers theorem. These statements are also proved in [Stu06b].

Proof of Proposition 2.2.2. We fix $t \in (0,1)$ and measurable sets $X, Y \subseteq \mathsf{M}$. We assume that the t-intermediate set $Z_t(X,Y)$ is measurable and furthermore, that $\mathsf{m}(X) < \infty$ as well as $\mathsf{m}(Y) < \infty$. We associate X, Y and $Z_t(X,Y)$ with their indicator functions and define

$$f := I_X, \;\; g := I_Y, \;\; h := I_{Z_t(X,Y)}.$$

Then for $x, y \in \mathsf{M}$ and $z \in Z_t(x,y)$,

$$\begin{aligned} h(z) &\geq \mathcal{M}_t^k(f(x), g(y)) \\ &\geq \mathcal{M}_t^k \left(\frac{\tilde{\tau}_{K,N}^{(1-t)}(\theta) \cdot f(x)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(x,y))}, \frac{\tilde{\tau}_{K,N}^{(t)}(\theta) \cdot g(y)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(x,y))} \right) \end{aligned}$$

for all $k \in \mathbb{N}$ where θ is defined as above. The Borell-Brascamp-Lieb inequality implies that

$$\begin{aligned}\mathsf{m}(Z_t(X,Y)) &= \int_{\mathsf{M}} h \, d\mathsf{m} \\ &\geq \mathcal{M}_t^{k/(1+Nk)} \left(\tilde{\tau}_{K,N}^{(1-t)}(\theta) \cdot \int_{\mathsf{M}} f \, d\mathsf{m}, \tilde{\tau}_{K,N}^{(t)}(\theta) \cdot \int_{\mathsf{M}} g \, d\mathsf{m} \right) \\ &= \mathcal{M}_t^{k/(1+Nk)} \left(\tilde{\tau}_{K,N}^{(1-t)}(\theta) \cdot \mathsf{m}(X), \tilde{\tau}_{K,N}^{(t)}(\theta) \cdot \mathsf{m}(Y) \right) \\ &\underset{k\to\infty}{\longrightarrow} \mathcal{M}_t^{\frac{1}{N}} \left(\tilde{\tau}_{K,N}^{(1-t)}(\theta) \cdot \mathsf{m}(X), \tilde{\tau}_{K,N}^{(t)}(\theta) \cdot \mathsf{m}(Y) \right).\end{aligned}$$

In the general case, we consider measurable sets including $Z_t(X,Y)$ and approximate X and Y by sets of finite volume. □

2.3 Stability of the Borell-Brascamp-Lieb Inequality

In the following we fix two numbers $K, N \in \mathbb{R}$ with $N \geq 1$.

2.3.1 Some Technical Statements

Proposition 2.3.1. *Let* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *be a metric measure space and let* $C \subseteq \mathsf{M}$ *be a compact subset of* M*. Fix* $t \in (0,1)$*. Let* $f, g \geq 0$ *be non-negative continuous functions on* C*. Then the function* $\xi := \xi_t^C(f,g)$*, defined on* M *by*

$$\xi(x) := \sup_{\substack{a,b\in C \\ x\in Z_t(a,b)}} \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{f(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a,b))}, \frac{g(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a,b))} \right) \cdot I_{Z_t(C,C)}(x),$$

is upper semi-continuous on M*.*

Proof. It suffices to show that for a sequence $Z_t(C,C) \ni z_n \to z \in Z_t(C,C)$ for $n \to \infty$ the inequality $\eta := \limsup_{n\to\infty} \xi(z_n) \leq \xi(z)$ holds true. Given such a sequence there exists an appropriate subsequence – also denoted by $(z_n)_{n\in\mathbb{N}}$ – with the property that $\lim_{n\to\infty} \xi(z_n) = \eta$. We fix $\delta > 0$. For every $n \in \mathbb{N}$ there exist elements $a_n, b_n \in C$ for which $z_n \in Z_t(a_n, b_n)$ and

$$\xi(z_n) \leq \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{f(a_n)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a_n,b_n))}, \frac{g(b_n)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a_n,b_n))} \right) + \delta.$$

According to the compactness of $C \subseteq \mathsf{M}$ there exist $a, b \in C$ and appropriate subsequences $(a_{n_k})_{k\in\mathbb{N}}$ and $(b_{n_k})_{k\in\mathbb{N}}$ satisfying $a_{n_k} \to a$ as well as $b_{n_k} \to b$ for $k \to \infty$. Additionally,

$$\begin{array}{ccc} \mathsf{d}(a_{n_k}, z_{n_k}) & = & t\mathsf{d}(a_{n_k}, b_{n_k}) \\ \Big\downarrow & & \Big\downarrow k \to \infty \\ \mathsf{d}(a, z) & & t\mathsf{d}(a, b). \end{array}$$

In a similar way we can deduce that $\mathsf{d}(z, b) = (1 - t)\mathsf{d}(a, b)$ which means that $z \in Z_t(a, b)$. Moreover, the continuity of f and g on C implies:

$$\begin{aligned} \eta = \lim_{k\to\infty} \xi(z_{n_k}) &\leq \lim_{k\to\infty} \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{f(a_{n_k})}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a_{n_k}, b_{n_k}))}, \frac{g(b_{n_k})}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a_{n_k}, b_{n_k}))} \right) + \delta \\ &= \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{f(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a, b))}, \frac{g(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a, b))} \right) + \delta \\ &\leq \xi(z) + \delta. \end{aligned}$$

Since $\delta > 0$ can be chosen arbitrarily small this proves the claim. $\square$

Remark 2.3.2. *The statement of Proposition 2.3.1 holds true even in the case where $f, g \geq 0$ are not continuous but at least upper semi-continuous on C.*

Proposition 2.3.3. *Let $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ be a metric measure space and let $C \subseteq \mathsf{M}$ be a compact subset of M. Fix $t \in (0, 1)$. Let $f, g \geq 0$ be non-negative upper semi-continuous functions on C. Then there exist sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ of continuous functions on C with the property that $f_n \downarrow f$ and $g_n \downarrow g$, respectively, for $n \to \infty$. Moreover, the functions ξ_n and ξ, defined for $x \in \mathsf{M}$ by*

$$\xi_n(x) := \sup_{\substack{a,b\in C \\ x\in Z_t(a,b)}} \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{f_n(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a, b))}, \frac{g_n(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a, b))} \right) \cdot I_{Z_t(C,C)}(x), \quad n \in \mathbb{N},$$

$$\xi(x) := \sup_{\substack{a,b\in C \\ x\in Z_t(a,b)}} \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{f(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a, b))}, \frac{g(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a, b))} \right) \cdot I_{Z_t(C,C)}(x),$$

satisfy $\xi_n \downarrow \xi$ on $Z_t(C, C)$ for $n \to \infty$.

Proof. For the proof of existence of sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ satisfying $f_n \downarrow f$ and $g_n \downarrow g$, respectively, we refer to [AB76]. Here we only prove the convergence of $(\xi_n)_{n\in\mathbb{N}}$ to ξ. The inequality $\lim_{n\to\infty}\xi_n(z)\geq\xi(z)$ holds obviously true for every $z\in Z_t(C,C)$. In order to prove the converse inequality we consider a fixed element $z\in Z_t(C,C)$. For every $n\in\mathbb{N}$ there exist elements $a_n,b_n\in C$ for which $z\in Z_t(a_n,b_n)$ and

$$\xi_n(z)\leq\mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f_n(a_n)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a_n,b_n))},\frac{g_n(b_n)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a_n,b_n))}\right)+\tfrac{1}{n}.$$

The compactness of $C\subseteq\mathsf{M}$ implies the existence of elements $a,b\in C$ and of appropriate subsequences $(a_{n_k})_{k\in\mathbb{N}}$, $(b_{n_k})_{k\in\mathbb{N}}$ satisfying $a_{n_k}\to a$ and $b_{n_k}\to b$ for $k\to\infty$. Furthermore, it holds that $z\in Z_t(a,b)$. Finally, we deduce that

$$\begin{aligned}\lim_{k\to\infty}\xi_{n_k}(z)&\leq\limsup_{k\to\infty}\mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f_{n_k}(a_{n_k})}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a_{n_k},b_{n_k}))},\frac{g_{n_k}(b_{n_k})}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a_{n_k},b_{n_k}))}\right)\\&\leq\mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a,b))},\frac{g(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a,b))}\right)\leq\xi(z).\end{aligned}$$

□

Proposition 2.3.4. *Let* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *be a metric measure space and let* $C\subseteq\mathsf{M}$ *be a compact subset of* M. *Fix* $t\in(0,1)$. *Let* $f,g\geq 0$ *be non-negative, bounded and continuous functions on* M. *For* $i\in\mathbb{N}$ *and* $\varepsilon_i:=\frac{1}{i}$ *we denote by*

$$C_{\varepsilon_i}:=\{x\in\mathsf{M}:\mathsf{dist}(x,C)<\varepsilon_i\}$$

the open ε_i*-neighborhood of* C *in* M. *The closure of* C_{ε_i} *is denoted by* $\bar{C}_{\varepsilon_i}$. *For every* $x\in\mathsf{M}$ *we define*

$$\xi(x):=\sup_{\substack{a,b\in C\\ x\in Z_t(a,b)}}\mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a,b))},\frac{g(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a,b))}\right)\cdot I_{Z_t(C,C)}(x),$$

$$\xi_i(x):=\sup_{\substack{a,b\in\bar{C}_{\varepsilon_i}\\ x\in Z_t(a,b)}}\mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a,b))},\frac{g(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a,b))}\right)\cdot I_{Z_t(\bar{C}_{\varepsilon_i},\bar{C}_{\varepsilon_i})}(x)$$

and

$$\begin{aligned}\eta_i(x)&:=\inf\{q(x):q\text{ upper semi-continuous on }\mathsf{M},\ q\geq\xi_i\}\\&=\sup_{\substack{(x_k)_{k\in\mathbb{N}}\\ x_k\to x}}\limsup_{k\to\infty}\xi_i(x_k).\end{aligned}$$

Then, η_i *is upper semi-continuous for every* $i\in\mathbb{N}$ *and* $\eta_i\searrow\xi$ *on* $Z_t(C,C)$ *for* $i\to\infty$.

Proof. We refer to [AB76] for the proof of the upper semi-continuity of η_i. In order to show the convergence statement we consider an element $x \in Z_t(C,C)$. Obviously, $\lim_{i\to\infty} \eta_i(x) \geq \xi(x)$. In order to prove the reverse inequality we fix $\delta > 0$. For every $i \in \mathbb{N}$ there exists a sequence $(x_k^i)_{k\in\mathbb{N}}$ for which $x_k^i \to x$ for $k \to \infty$ and

$$\eta_i(x) \leq \limsup_{k\to\infty} \xi_i(x_k^i) + \delta.$$

Furthermore, for every $i, k \in \mathbb{N}$ there exist elements $a_k^i, b_k^i \in \bar{C}_{\varepsilon_i}$ satisfying $x_k^i \in Z_t(a_k^i, b_k^i)$ and

$$\xi_i(x_k^i) \leq \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{f(a_k^i)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a_k^i, b_k^i))}, \frac{g(b_k^i)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a_k^i, b_k^i))} \right) + \delta.$$

In addition, there exist elements $\tilde{a}_k^i, \tilde{b}_k^i \in C$ such that

$$\mathsf{d}(a_k^i, \tilde{a}_k^i) = \inf_{a\in C} \mathsf{d}(a_k^i, a) \leq \tfrac{1}{i}$$

and

$$\mathsf{d}(b_k^i, \tilde{b}_k^i) = \inf_{b\in C} \mathsf{d}(b_k^i, b) \leq \tfrac{1}{i}.$$

Since C is compact there exist $\tilde{a}^i, \tilde{a}, \tilde{b}^i, \tilde{b} \in C$ such that (considering appropriate subsequences)

$$\begin{array}{ccccc} \tilde{a}_k^i & \underset{k\to\infty}{\longrightarrow} & \tilde{a}^i, & \tilde{a}^i & \underset{i\to\infty}{\longrightarrow} & \tilde{a} \\ \tilde{b}_k^i & \underset{k\to\infty}{\longrightarrow} & \tilde{b}^i, & \tilde{b}^i & \underset{i\to\infty}{\longrightarrow} & \tilde{b} \end{array}$$

$(i \in \mathbb{N})$. For each $l \in \mathbb{N}$ we fix $i_l \in \mathbb{N}$ such that

$$\tfrac{1}{i_l} \leq \tfrac{1}{3l}, \quad \mathsf{d}(\tilde{a}^{i_l}, \tilde{a}) \leq \tfrac{1}{3l}, \quad \mathsf{d}(\tilde{b}^{i_l}, \tilde{b}) \leq \tfrac{1}{3l}$$

and afterwards we choose $k_l := k_l(i_l) \in \mathbb{N}$ for every $l \in \mathbb{N}$ such that

$$\mathsf{d}(\tilde{a}_{k_l}^{i_l}, \tilde{a}^{i_l}) \leq \tfrac{1}{3l}, \quad \mathsf{d}(\tilde{b}_{k_l}^{i_l}, \tilde{b}^{i_l}) \leq \tfrac{1}{3l}$$

as well as

$$\mathsf{d}(x_{k_l}^{i_l}, x) \leq \tfrac{1}{l}, \quad \eta_{i_l}(x) \leq \xi_{i_l}(x_{k_l}^{i_l}) + 2\delta.$$

For each $l \in \mathbb{N}$ we define $a_l := a_{k_l}^{i_l}$, $b_l := b_{k_l}^{i_l}$ and $x_l := x_{k_l}^{i_l}$. Then we have $a_l \to \tilde{a}$, $b_l \to \tilde{b}$ and $x_l \to x$ for $l \to \infty$ which can be verified as follows: For given $\epsilon > 0$ we choose $l_\epsilon \in \mathbb{N}$ such that $\frac{1}{l_\epsilon} \leq \epsilon$. For all $l \geq l_\epsilon$ we derive

$$\begin{aligned} \mathsf{d}(a_l, \tilde{a}) &= \mathsf{d}(a_{k_l}^{i_l}, \tilde{a}) \\ &\leq \mathsf{d}(a_{k_l}^{i_l}, \tilde{a}_{k_l}^{i_l}) + \mathsf{d}(\tilde{a}_{k_l}^{i_l}, \tilde{a}^{i_l}) + \mathsf{d}(\tilde{a}^{i_l}, \tilde{a}) \\ &\leq \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon, \end{aligned}$$

similarly, $\mathsf{d}(b_l,\tilde b)\le\epsilon$ and $\mathsf{d}(x_l,x)\le\epsilon$. The t-midpoint property of x_l implies that $x\in Z_t(\tilde a,\tilde b)$. Finally, we derive from the continuity of f and g:

$$\begin{aligned}\eta_{i_l}(x)&\le \xi_{i_l}(x_l)+2\delta\\&\le \mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(a_l)}{\tilde\tau_{K,N}^{(1-t)}(\mathsf{d}(a_l,b_l))},\frac{g(b_l)}{\tilde\tau_{K,N}^{(t)}(\mathsf{d}(a_l,b_l))}\right)+3\delta\\&\underset{l\to\infty}{\to}\mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(\tilde a)}{\tilde\tau_{K,N}^{(1-t)}(\mathsf{d}(\tilde a,\tilde b))},\frac{g(\tilde b)}{\tilde\tau_{K,N}^{(t)}(\mathsf{d}(\tilde a,\tilde b))}\right)+3\delta\\&\le \xi(x)+3\delta.\end{aligned}$$

Since $\delta>0$ can be chosen arbitrarily small,

$$\lim_{l\to\infty}\eta_{i_l}(x)\le\xi(x).$$

□

2.3.2 Characterization of $\mathsf{BBL}(K,N)$

For technical reasons we need an equivalent formulation of $\mathsf{BBL}(K,N)$ where we content ourselves with upper semi-continuous functions (instead of integrable ones). Precisely,

Proposition 2.3.5. *A normalized metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ with compact support M_0 satisfies $\mathsf{BBL}(K,N)$ as introduced in Definition 2.1.3 if and only if for all $t\in[0,1]$ and all non-negative upper semi-continuous functions f,g,h on M satisfying $\int_{\mathsf{M}} f\ d\mathsf{m}=\int_{\mathsf{M}} g\ d\mathsf{m}=1$ and*

$$h(z)\ge\mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(x)}{\tilde\tau_{K,N}^{(1-t)}(\mathsf{d}(x,y))},\frac{g(y)}{\tilde\tau_{K,N}^{(t)}(\mathsf{d}(x,y))}\right)$$

for all $x,y\in\mathsf{M}$ and every $z\in Z_t(x,y)$, it holds that

$$\int_{\mathsf{M}} h\ d\mathsf{m}\ge 1.$$

Proof. (i) The 'weak' formulation on M is equivalent to the 'weak' formulation on M_0: In order to prove this statement, we fix $t\in(0,1)$ and let $f,g,h\ge0$ be non-negative upper semi-continuous functions on M_0 for which $\int_{\mathsf{M}_0} f\ d\mathsf{m}=\int_{\mathsf{M}_0} g\ d\mathsf{m}=1$ and

$$h(z)\ge\mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(x)}{\tilde\tau_{K,N}^{(1-t)}(\mathsf{d}(x,y))},\frac{g(y)}{\tilde\tau_{K,N}^{(t)}(\mathsf{d}(x,y))}\right)$$

for all $x, y \in \mathsf{M}_0$ and every $z \in Z_t(x,y) \subseteq \mathsf{M}_0$. Regarding $f \cdot I_{\mathsf{M}_0}$ as well as $g \cdot I_{\mathsf{M}_0}$ as functions on M (extended by the value 0), we define for $x \in \mathsf{M}$

$$\begin{aligned}\xi(x) &:= \sup_{\substack{a,b\in\mathsf{M}\\ x\in Z_t(a,b)}} \mathcal{M}_t^{-\frac{1}{N}}\left(\frac{(f\cdot I_{\mathsf{M}_0})(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a,b))}, \frac{(g\cdot I_{\mathsf{M}_0})(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a,b))}\right)\\ &= \sup_{\substack{a,b\in\mathsf{M}_0\\ x\in Z_t(a,b)}} \mathsf{M}_t^{-\frac{1}{N}}\left(\frac{f(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a,b))}, \frac{g(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a,b))}\right)\cdot I_{Z_t(\mathsf{M}_0,\mathsf{M}_0)}(x).\end{aligned}$$

According to Proposition 2.3.1 and Remark 2.3.2, ξ is upper semi-continuous on M. We deduce from our assumption

$$\int_{\mathsf{M}_0} h\ d\mathsf{m} \geq \int_{\mathsf{M}_0} \xi\ d\mathsf{m} \geq \mathcal{M}_t^{-\infty}\left(\int_{\mathsf{M}_0} f\ d\mathsf{m}, \int_{\mathsf{M}_0} g\ d\mathsf{m}\right) = 1.$$

(ii) Thanks to step (i) we may assume without restriction that $\mathsf{M} = \mathsf{M}_0$. Furthermore, we assume that $(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfies the requirement concerning upper semi-continuous functions and consider $t \in (0,1)$ as well as three non-negative integrable functions f, g, h on M with $\int_{\mathsf{M}} f\ d\mathsf{m} = \int_{\mathsf{M}} g\ d\mathsf{m} = 1$ and

$$h(z) \geq \mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(x)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(x,y))}, \frac{g(y)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(x,y))}\right)$$

for all $x, y \in \mathsf{M}$ and every $z \in Z_t(x,y)$. Due to [AB76], both, f and g can be regarded as the pointwise supremum of upper semi-continuous (u.s.c.) functions

$$f = \sup_{\substack{\varphi \text{ u.s.c.}\\ \varphi\leq f}} \varphi, \quad g = \sup_{\substack{\psi \text{ u.s.c.}\\ \psi\leq g}} \psi.$$

We define for all upper semi-continuous functions $\varphi \leq f$ and $\psi \leq g$

$$\xi_{\varphi\psi}(x) := \sup_{\substack{a,b\in\mathsf{M}\\ x\in Z_t(a,b)}} \mathcal{M}_t^{-\frac{1}{N}}\left(\frac{\varphi(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a,b))}, \frac{\psi(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a,b))}\right), \quad x \in \mathsf{M}.$$

Due to our assumption and Remark 2.3.2, it holds that

$$\int_{\mathsf{M}} h\ d\mathsf{m} \geq \int_{\mathsf{M}} \xi_{\varphi,\psi}\ d\mathsf{m} \geq \mathcal{M}_t^{-\infty}\left(\int_{\mathsf{M}} \varphi\ d\mathsf{m}, \int_{\mathsf{M}} \psi\ d\mathsf{m}\right)$$

and therefore

$$\int_{\mathsf{M}} h \, d\mathsf{m} \geq \mathcal{M}_t^{-\infty}\left(\int_{\mathsf{M}} f \, d\mathsf{m}, \int_{\mathsf{M}} g \, d\mathsf{m}\right) = 1$$

considering $\int_{\mathsf{M}} f \, d\mathsf{m}$ – and similarly $\int_{\mathsf{M}} g \, d\mathsf{m}$ – as

$$\int_{\mathsf{M}} f \, d\mathsf{m} = \sup\left\{\int_{\mathsf{M}} \varphi \, d\mathsf{m} : \varphi \text{ u.s.c, } \varphi \leq f\right\}$$

following the way of [AB76].

□

As a consequence of Proposition 2.3.5 and its proof:

Proposition 2.3.6 (Isomorphism). *Let $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ and $(\mathsf{M}', \mathsf{d}', \mathsf{m}')$ be two isomorphic normalized metric measure spaces with compact supports. Then $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfies $\mathsf{BBL}(K, N)$ if and only if $(\mathsf{M}', \mathsf{d}', \mathsf{m}')$ satisfies $\mathsf{BBL}(K, N)$.*

2.3.3 Stability under Convergence

Theorem 2.3.7 (Convergence). *Let $((\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n))_{n\in\mathbb{N}}$ be a sequence of compact and normalized metric measure spaces converging to a normalized metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ with compact support,*

$$(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n) \xrightarrow{\mathbb{D}} (\mathsf{M}, \mathsf{d}, \mathsf{m}), \quad n \to \infty.$$

For every $n \in \mathbb{N}$ let $(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n)$ satisfy $\mathsf{BBL}(K, N)$ for two numbers $K, N \in \mathbb{R}$ with $N \geq 1$. Then $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfies $\mathsf{BBL}(K, N)$ as well.

Proof. According to [Stu06a, Proof of Theorem 3.6] we may assume without restriction that $(\mathsf{M}_n, \mathsf{d}_n)$ is embedded in (M, d) via an isometry ψ_n

$$\psi_n : (\mathsf{M}_n, \mathsf{d}_n) \hookrightarrow (\mathsf{M}, \mathsf{d}), \quad n \in \mathbb{N},$$

and that $\mathsf{d_W}(\tilde{\mathsf{m}}_n, \mathsf{m}) \to 0$ for $n \to \infty$, denoting by $\tilde{\mathsf{m}}_n$ the push-forward measure of m_n under ψ_n. According to Proposition 2.3.5 and part (i) of its proof, it suffices to show that $(\mathsf{M}_0, \mathsf{d}, \mathsf{m})$ with $\mathsf{M}_0 := \mathsf{supp}(\mathsf{m})$ satisfies $\mathsf{BBL}(K, N)$. Therefore, we consider $t \in (0, 1)$ and non-negative upper semi-continuous functions f, g, h on M_0 with $\int_{\mathsf{M}_0} f \, d\mathsf{m} = \int_{\mathsf{M}_0} g \, d\mathsf{m} = 1$ and

$$h(z) \geq \mathcal{M}_t^{-\frac{1}{N}}\left(\frac{f(x)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(x,y))}, \frac{g(y)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(x,y))}\right)$$

for all $x, y \in \mathsf{M}_0$ and every $z \in Z_t(x,y) \subseteq \mathsf{M}_0$. There exist sequences $(f_l)_{l\in\mathbb{N}}$, $(g_l)_{l\in\mathbb{N}}$ of continuous functions on M_0 satisfying $f_l \downarrow f$ and $g_l \downarrow g$ on M_0 for $l \to \infty$. For every $l \in \mathbb{N}$ let $F_l : \mathsf{M} \to \mathbb{R}_+$ and $G_l : \mathsf{M} \to \mathbb{R}_+$ be non-negative bounded continuous extensions of f_l and g_l, respectively, on M. Such extensions exist due to [Mor05]. For $\varepsilon > 0$ we denote by

$$\mathsf{M}_0^\varepsilon := \{x \in \mathsf{M} : \mathsf{dist}(x, \mathsf{M}_0) < \varepsilon\}$$

the open ε-neighborhood of M_0 in M. The closure of M_0^ε is denoted by $\bar{\mathsf{M}}_0^\varepsilon$. For fixed $l \in \mathbb{N}$ and $\varepsilon > 0$ we define functions ξ_l^ε and η_l^ε for every $x \in \mathsf{M}$ by

$$\xi_l^\varepsilon(x) := \sup_{\substack{a,b\in\bar{\mathsf{M}}_0^\varepsilon \\ x\in Z_t(a,b)}} \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{F_l(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a,b))}, \frac{G_l(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a,b))} \right) \cdot I_{Z_t(\bar{\mathsf{M}}_0^\varepsilon, \bar{\mathsf{M}}_0^\varepsilon)}(x)$$

and

$$\begin{aligned} \eta_l^\varepsilon(x) &:= \inf\{q(x) : q \text{ upper semi-continuous on } \mathsf{M},\ q \geq \xi_l^\varepsilon\} \\ &= \sup_{\substack{(x_k)_{k\in\mathbb{N}} \\ x_k \to x}} \limsup_{k\to\infty} \xi_l^\varepsilon(x_k). \end{aligned}$$

According to Proposition 2.3.6 stating the stability of $\mathsf{BBL}(K,N)$ under isomorphisms, $(\mathsf{M}, \mathsf{d}, \tilde{\mathsf{m}}_n)$ satisfies $\mathsf{BBL}(K,N)$. Therefore, for all $n \in \mathbb{N}$

$$\begin{aligned} \int_{\mathsf{M}} \eta_l^\varepsilon \, d\tilde{\mathsf{m}}_n &\geq \mathcal{M}_t^{-\infty} \left(\int_{\bar{\mathsf{M}}_0^\varepsilon} F_l \, d\tilde{\mathsf{m}}_n, \int_{\bar{\mathsf{M}}_0^\varepsilon} G_l \, d\tilde{\mathsf{m}}_n \right) \\ &\geq \mathcal{M}_t^{-\infty} \left(\int_{\mathsf{M}_0^\varepsilon} F_l \, d\tilde{\mathsf{m}}_n, \int_{\mathsf{M}_0^\varepsilon} G_l \, d\tilde{\mathsf{m}}_n \right). \end{aligned}$$

The weak convergence of the sequence $(\tilde{\mathsf{m}}_n)_{n\in\mathbb{N}}$ to m implies that

$$\begin{aligned} \int_{\mathsf{M}} \eta_l^\varepsilon \, d\mathsf{m} &\geq \limsup_{n\to\infty} \int_{\mathsf{M}} \eta_l^\varepsilon \, d\tilde{\mathsf{m}}_n \\ &\geq \liminf_{n\to\infty} \mathcal{M}_t^{-\infty} \left(\int_{\mathsf{M}_0^\varepsilon} F_l \, d\tilde{\mathsf{m}}_n, \int_{\mathsf{M}_0^\varepsilon} G_l \, d\tilde{\mathsf{m}}_n \right) \\ &\geq \mathcal{M}_t^{-\infty} \left(\int_{\mathsf{M}_0} f_l \, d\mathsf{m}, \int_{\mathsf{M}_0} g_l \, d\mathsf{m} \right). \end{aligned}$$

We introduce the notation

$$\xi_l(x) := \sup_{\substack{a,b \in \mathsf{M}_0 \\ x \in Z_t(a,b)}} \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{f_l(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a,b))}, \frac{g_l(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a,b))} \right) \cdot I_{Z_t(\mathsf{M}_0,\mathsf{M}_0)}(x),$$

and additionally,

$$\xi(x) := \sup_{\substack{a,b \in \mathsf{M}_0 \\ x \in Z_t(a,b)}} \mathcal{M}_t^{-\frac{1}{N}} \left(\frac{f(a)}{\tilde{\tau}_{K,N}^{(1-t)}(\mathsf{d}(a,b))}, \frac{g(b)}{\tilde{\tau}_{K,N}^{(t)}(\mathsf{d}(a,b))} \right) \cdot I_{Z_t(\mathsf{M}_0,\mathsf{M}_0)}(x).$$

At first, letting ε tend to 0, we have due to Proposition 2.3.4,

$$\begin{aligned} \int_{\mathsf{M}_0} \xi_l \, d\mathsf{m} &= \lim_{\varepsilon \to 0} \int_{\mathsf{M}_0} \eta_l^\varepsilon \, d\mathsf{m} \\ &\geq \mathcal{M}_t^{-\infty} \left(\int_{\mathsf{M}_0} f_l \, d\mathsf{m}, \int_{\mathsf{M}_0} g_l \, d\mathsf{m} \right). \end{aligned}$$

Finally, letting l tend to ∞, we deduce from Proposition 2.3.3,

$$\begin{aligned} \int_{\mathsf{M}_0} h \, d\mathsf{m} &\geq \int_{\mathsf{M}_0} \xi d\mathsf{m} \\ &= \lim_{l \to \infty} \int_{\mathsf{M}_0} \xi_l \, d\mathsf{m} \\ &\geq \lim_{l \to \infty} \mathcal{M}_t^{-\infty} \left(\int_{\mathsf{M}_0} f_l \, d\mathsf{m}, \int_{\mathsf{M}_0} g_l \, d\mathsf{m} \right) \\ &= \mathcal{M}_t^{-\infty} \left(\int_{\mathsf{M}_0} f \, d\mathsf{m}, \int_{\mathsf{M}_0} g \, d\mathsf{m} \right) = 1. \end{aligned}$$

This concludes the proof. □

2.4 Geometric Consequences

The Borell-Brascamp-Lieb inequality $\mathsf{BBL}(K,N)$ is a relatively weak statement – compared with the curvature-dimension condition $\mathsf{CD}(K,N)$. But nevertheless, implying the Brunn-Minkowski inequality $\mathsf{BM}(K,N)$, it is strong enough to lead to the geometric results Sturm formulated and proved in [Stu06b, Theorem 2.3 – Corollary 2.6]. In the following we content ourselves with listing these results as consequences of $\mathsf{BBL}(K,N)$ without their proofs.

For a fixed point $x_0 \in \mathsf{supp}(\mathsf{m})$ we study the growth of the volume of closed balls centered at x_0 and the growth of the volume of the corresponding spheres

$$v(r) := \mathsf{m}\left(\overline{B_r(x_0)}\right) \quad \text{and} \quad s(r) := \limsup_{\delta \to 0} \tfrac{1}{\delta}\mathsf{m}\left(\overline{B_{r+\delta}(x_0)} \setminus B_r(x_0)\right),$$

respectively.

Theorem 2.4.1 (Generalized Bishop-Gromov volume growth estimate)**.** *If* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfies the Borell-Brascamp-Lieb inequality* $\mathsf{BBL}(K, N)$ *for some* $K, N \in \mathbb{R}$ *with* $N \geq 1$*, then each bounded set* $M_b \subseteq \mathsf{M}$ *has finite volume. Moreover, either* m *is supported by one point, or all points and all spheres have mass 0.*

To be more precise, if $K > 0$ *then for each fixed* $x_0 \in \mathsf{supp}(\mathsf{m})$ *and all* $0 < r < R \leq \pi\sqrt{(N-1)/K}$

$$\frac{s(r)}{s(R)} \geq \left(\frac{\sin(r\sqrt{K/(N-1)})}{\sin(R\sqrt{K/(N-1)})}\right)^{N-1} \tag{2.4.1}$$

and

$$\frac{v(r)}{v(R)} \geq \frac{\int_0^r \sin\left(t\sqrt{K/(N-1)}\right)^{N-1} dt}{\int_0^R \sin\left(t\sqrt{K/(N-1)}\right)^{N-1} dt}. \tag{2.4.2}$$

In the case $K < 0$*, analogous inequalities hold true (where the right-hand side of (2.4.1) and (2.4.2), respectively, is replaced by analogous expressions according to the definition of the coefficients* $\tau_{K,N}^{(t)}(\cdot)$ *for negative* K*). If* $K = 0$ *then*

$$\frac{s(r)}{s(R)} \geq \left(\frac{r}{R}\right)^{N-1} \quad \textit{and} \quad \frac{v(r)}{v(R)} \geq \left(\frac{r}{R}\right)^{N}.$$

Definition 2.4.2 (Doubling property)**.** *Consider* $C \in \mathbb{R}_+$*. We say that a metric measure space* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfies the doubling property with doubling constant* C *if and only if for all* $x \in \mathsf{supp}(\mathsf{m})$ *and all* $r \in \mathbb{R}_+$*,*

$$\mathsf{m}(B_{2r}(x)) \leq C\mathsf{m}(B_r(x)).$$

Corollary 2.4.3 (Doubling)**.** *For each metric measure space* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfying* $\mathsf{BBL}(K, N)$ *for* $K, N \in \mathbb{R}$ *with* $N \geq 1$*, the doubling property holds true on each bounded set* $M_b \subseteq \mathsf{supp}(\mathsf{m})$*. Particularly, each bounded closed subset* $M_{b,c} \subseteq \mathsf{supp}(\mathsf{m})$ *is compact. In the case* $K \geq 0$*, the doubling constant* C *satisfies* $C \leq 2^N$*. Otherwise, it can be estimated in terms of* K*,* N *and the diameter* L *of* M_b *as follows,*

$$C \leq 2^N \cosh\left(L\sqrt{\frac{-K}{N-1}}\right)^{N-1}.$$

Corollary 2.4.4 (Generalized Bonnet-Myers theorem)**.** *Fix two real parameters* $K > 0$ *and* $N \geq 1$. *Each metric measure space* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfying* $\mathsf{BBL}(K, N)$ *has compact support and its diameter* L *has an upper bound,*

$$L \leq \pi \sqrt{\frac{N-1}{K}}.$$

In the future, our understanding of the relations between the various inequalities presented above and the curvature-dimension condition could be improved. For instance, up to now, the implications '$\mathsf{BBL}(K, N) \Rightarrow \mathsf{CD}(K, N)$' or '$\mathsf{BM}(K, N) \Rightarrow \mathsf{BBL}(K, N)$' are neither proved nor appropriate counterexamples are known – as far as we are informed.

Chapter 3

The Localization and Tensorization Property of the Curvature-Dimension Condition

In two similar but independent approaches, Sturm [Stu06a, Stu06b] as well as Lott and Villani [LV07, LV09] present a concept of generalized lower 'Ricci' curvature bounds for metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$. The full strength of these concepts appears if the notion of curvature bounded from below, say by K, is combined with a kind of upper bound N for the dimension. This leads to the so-called curvature-dimension condition $\mathsf{CD}(K, N)$ which makes sense for each pair of numbers $K \in \mathbb{R}$ and $N \in [1, \infty)$. The precise definition can be found in Chapter 1.

This chapter focuses on two open questions in the context of the curvature-dimension condition, namely

- ▷ whether the curvature-dimension condition $\mathsf{CD}(K, N)$ for general (K, N) is a *local property*, in the sense that $\mathsf{CD}(K, N)$ for all subsets M_i with $i \in I$ of a covering of M implies $\mathsf{CD}(K, N)$ for a given space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ and
- ▷ whether the curvature-dimension condition $\mathsf{CD}(K, N)$ has the *tensorization property*, in other words whether $\mathsf{CD}(K, N_i)$ for each factor M_i with $i \in I$ implies $\mathsf{CD}(K, \sum_{i \in I} N_i)$ for the product space $\mathsf{M} = \bigotimes_{i \in I} \mathsf{M}_i$.

3.1 The Reduced Curvature-Dimension Condition

Before we give the precise definition of the reduced curvature-dimension condition $\mathsf{CD}^*(K, N)$ which is obtained from $\mathsf{CD}(K, N)$ by replacing the coefficients $\tau_{K,N}^{(t)}(\cdot)$

by the slightly smaller ones $\sigma_{K,N}^{(t)}(\cdot)$, we summarize two properties of the latter coefficients. These statements can be found in [Stu06b].

Lemma 3.1.1. *For all $K, K' \in \mathbb{R}$, all $N, N' \in [1, \infty)$ and all $(t, \theta) \in [0, 1] \times \mathbb{R}_+$,*

$$\sigma_{K,N}^{(t)}(\theta)^N \cdot \sigma_{K',N'}^{(t)}(\theta)^{N'} \geq \sigma_{K+K',N+N'}^{(t)}(\theta)^{N+N'}.$$

Remark 3.1.2. *For fixed $t \in (0, 1)$ and $\theta \in (0, \infty)$ the function $(K, N) \mapsto \sigma_{K,N}^{(t)}(\theta)$ is continuous, non-decreasing in K and non-increasing in N.*

Definition 3.1.3. *Let two numbers $K, N \in \mathbb{R}$ with $N \geq 1$ be given.*

(i) We say that a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfies the reduced curvature-dimension condition $\mathsf{CD}^*(K, N)$ (globally) if and only if for each pair $\nu_0, \nu_1 \in \mathcal{P}_{2,\mathsf{b}}(\mathsf{M}, \mathsf{d}, \mathsf{m})$ there exist an optimal coupling q of $\nu_0 = \rho_0 \mathsf{m}$ and $\nu_1 = \rho_1 \mathsf{m}$ and a geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ connecting ν_0 and ν_1 such that

$$\begin{aligned} &\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq \\ &- \int_{\mathsf{M}\times\mathsf{M}} \left[\sigma_{K,N'}^{(1-t)}(\mathsf{d}(x_0, x_1))\rho_0^{-1/N'}(x_0) + \sigma_{K,N'}^{(t)}(\mathsf{d}(x_0, x_1))\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0, x_1) \end{aligned} \tag{3.1.1}$$

for all $t \in [0, 1]$ and all $N' \geq N$.

(ii) $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfies the reduced curvature-dimension condition $\mathsf{CD}^*(K, N)$ locally - denoted by $\mathsf{CD}^*_{\mathsf{loc}}(K, N)$ - if and only if each point x of M has a neighborhood $M(x)$ such that for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ supported in $M(x)$ there exist an optimal coupling q of ν_0 and ν_1 and a geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ connecting ν_0 and ν_1 satisfying (3.1.1) for all $t \in [0, 1]$ and all $N' \geq N$.

Remark 3.1.4. *(i) Actually, the condition $\mathsf{CD}^*(K, N)$ implies property (3.1.1) for all measures $\nu_0, \nu_1 \in \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$. We refer to Lemma 3.1.10. Only for technical reasons we prefer the formal restriction to measures with bounded support in the formulation of $\mathsf{CD}^*(K, N)$.*

(ii) Note that we do not require that $\Gamma(t)$ is supported in $M(x)$ for $t \in (0, 1)$ in part (ii) of Definition 3.1.3.

(iii) One can deduce that a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying $\mathsf{CD}^*_{\mathsf{loc}}(K, N)$ has a locally compact support using analogous arguments as in [Stu06b, Corollary 2.4].

Proposition 3.1.5. *(i)* $\mathsf{CD}(K,N) \Rightarrow \mathsf{CD}^*(K,N)$*: For each metric measure space* $(\mathsf{M},\mathsf{d},\mathsf{m})$*, the curvature-dimension condition* $\mathsf{CD}(K,N)$ *for* $K,N \in \mathbb{R}$ *with* $N \geq 1$ *implies the reduced curvature-dimension condition* $\mathsf{CD}^*(K,N)$*.*

(ii) $\mathsf{CD}^*(K,N) \Rightarrow \mathsf{CD}(K^*,N)$*: If* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *fulfills the reduced curvature-dimension condition* $\mathsf{CD}^*(K,N)$ *for some* $K > 0$ *and* $N \geq 1$*, then* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies* $\mathsf{CD}(K^*,N)$ *for* $K^* = \frac{K(N-1)}{N}$*.*

Proof. (i) Due to Lemma 3.1.1 we have for all $K',N' \in \mathbb{R}$ with $N' \geq 1$ and all $(t,\theta) \in [0,1] \times \mathbb{R}_+$,

$$\tau_{K',N'}^{(t)}(\theta)^{N'} = t \cdot \sigma_{K',N'-1}^{(t)}(\theta)^{N'-1} = \sigma_{0,1}^{(t)}(\theta) \cdot \sigma_{K',N'-1}^{(t)}(\theta)^{N'-1} \geq \sigma_{K',N'}^{(t)}(\theta)^{N'}$$

which means

$$\tau_{K',N'}^{(t)}(\theta) \geq \sigma_{K',N'}^{(t)}(\theta).$$

Now we consider two probability measures $\nu_0, \nu_1 \in \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$. Due to $\mathsf{CD}(K,N)$ there exist an optimal coupling q of $\nu_0 = \rho_0\mathsf{m}$ and $\nu_1 = \rho_1\mathsf{m}$ and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 such that

$$\begin{aligned}
&\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \\
&\leq -\int_{\mathsf{M}\times\mathsf{M}} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) + \tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0,x_1) \\
&\leq -\int_{\mathsf{M}\times\mathsf{M}} \left[\sigma_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) + \sigma_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0,x_1)
\end{aligned}$$

for all $t \in [0,1]$ and all $N' \geq N$.

(ii) Put $K^* := \frac{K(N-1)}{N}$ and note that $K^* \leq \frac{K(N'-1)}{N'}$ for all $N' \geq N$. Comparing the relevant coefficients $\tau_{K^*,N'}^{(t)}(\theta)$ and $\sigma_{K,N'}^{(t)}(\theta)$, yields

$$\tau_{K^*,N'}^{(t)}(\theta) = \tau_{\frac{K(N'-1)}{N'},N'}^{(t)}(\theta) = t^{1/N'} \left(\frac{\sin(t\theta\sqrt{K/N'})}{\sin(\theta\sqrt{K/N'})}\right)^{1-1/N'} \leq \sigma_{K,N'}^{(t)}(\theta) \tag{3.1.2}$$

for all $\theta \in \mathbb{R}_+$, $t \in [0,1]$ and $N' \geq N$.

According to our curvature assumption, for every $\nu_0, \nu_1 \in \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ – at this point we have to refer to Lemma 3.1.10 again – there exist an optimal coupling

q of $\nu_0 = \rho_0 \mathsf{m}$ and $\nu_1 = \rho_1 \mathsf{m}$ and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ from ν_0 to ν_1 with property (3.1.1). From (3.1.2) we deduce

$$
\begin{aligned}
&\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \\
&\leq - \int_{\mathsf{M}\times\mathsf{M}} \left[\sigma^{(1-t)}_{K,N'}(\mathsf{d}(x_0,x_1)) \rho_0^{-1/N'}(x_0) + \sigma^{(t)}_{K,N'}(\mathsf{d}(x_0,x_1)) \rho_1^{-1/N'}(x_1) \right] d\mathsf{q}(x_0,x_1) \\
&\leq - \int_{\mathsf{M}\times\mathsf{M}} \left[\tau^{(1-t)}_{K^*,N'}(\mathsf{d}(x_0,x_1)) \rho_0^{-1/N'}(x_0) + \tau^{(t)}_{K^*,N'}(\mathsf{d}(x_0,x_1)) \rho_1^{-1/N'}(x_1) \right] d\mathsf{q}(x_0,x_1)
\end{aligned}
$$

for all $t \in [0,1]$ and all $N' \geq N$. This proves property $\mathsf{CD}(K^*, N)$.

□

From now on until the end of this chapter, we consider metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ where (M, d) is a length space. We summarize two interesting properties of the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$. The analogous results for metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying the 'original' curvature-dimension condition $\mathsf{CD}(K,N)$ are proved in [Stu06b] and cited in Chapter 1.

The first result states the uniqueness of geodesics:

Proposition 3.1.6 (Geodesics). *Let $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ be a non-branching metric measure space satisfying the condition $\mathsf{CD}^*(K,N)$ for some numbers $K, N \in \mathbb{R}$ with $N \geq 1$. Then for every $x \in \mathsf{supp}(\mathsf{m}) \subseteq \mathsf{M}$ and m-almost every $y \in \mathsf{M}$ - with exceptional set depending on x - there exists a unique geodesic between x and y.*

Moreover, there exists a measurable map $\gamma : \mathsf{M} \times \mathsf{M} \to \mathcal{G}(\mathsf{M})$ such that for $\mathsf{m} \otimes \mathsf{m}$-almost every $(x,y) \in \mathsf{M} \times \mathsf{M}$ the curve $t \mapsto \gamma_t(x,y)$ is the unique geodesic connecting x and y.

The second one provides equivalent characterizations of the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ in analogy to Proposition 1.3.4:

Proposition 3.1.7 (Equivalent characterizations). *Fix $K, N \in \mathbb{R}$ with $N \geq 1$. For each locally compact non-branching metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$, the following statements are equivalent:*

(i) $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfies* $\mathsf{CD}^*(K,N)$.

(ii) For all $\nu_0, \nu_1 \in \mathcal{P}_{2,\mathsf{b}}(\mathsf{M}, \mathsf{d}, \mathsf{m})$ there exist an optimal coupling q of ν_0 and ν_1 and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ from ν_0 to ν_1 such that for all $t \in [0,1]$ and all $N' \geq N$,

$$
\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq \sigma^{(1-t)}_{K,N'}(\Theta) \mathsf{S}_{N'}(\nu_0|\mathsf{m}) + \sigma^{(t)}_{K,N'}(\Theta) \mathsf{S}_{N'}(\nu_1|\mathsf{m}),
$$

where

$$\Theta := \begin{cases} \mathsf{q}-\operatorname{essinf}_{x_0,x_1}\mathsf{d}(x_0,x_1), & \text{if } K\geq 0,\\ \mathsf{q}-\operatorname{esssup}_{x_0,x_1}\mathsf{d}(x_0,x_1), & \text{if } K<0, \end{cases} \tag{3.1.3}$$

denotes the minimal (in the case $K \geq 0$) or maximal (if $K < 0$) transportation distance.

(iii) For all $\nu_0, \nu_1 \in \mathcal{P}_{2,\mathsf{b}}(\mathsf{M},\mathsf{d},\mathsf{m})$ there exists a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 such that for all $t \in [0,1]$ and all $N' \geq N$,

$$\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq \sigma_{K,N'}^{(1-t)}(\theta)\mathsf{S}_{N'}(\nu_0|\mathsf{m}) + \sigma_{K,N'}^{(t)}(\theta)\mathsf{S}_{N'}(\nu_1|\mathsf{m}), \tag{3.1.4}$$

where

$$\theta := \begin{cases} \inf_{x_0\in\mathcal{S}_0,x_1\in\mathcal{S}_1}\mathsf{d}(x_0,x_1), & \text{if } K\geq 0,\\ \sup_{x_0\in\mathcal{S}_0,x_1\in\mathcal{S}_1}\mathsf{d}(x_0,x_1), & \text{if } K<0, \end{cases} \tag{3.1.5}$$

denoting by $\mathcal{S}_0$ and $\mathcal{S}_1$ the supports of ν_0 and ν_1, respectively.

(iv) For all $\nu_0, \nu_1 \in \mathcal{P}_{2,\mathsf{b}}(\mathsf{M},\mathsf{d},\mathsf{m})$ and each optimal coupling q of them there exists a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 and satisfying (3.1.1) for all $t \in [0,1]$ and all $N' \geq N$.

(v) For all $\nu_0, \nu_1 \in \mathcal{P}_{2,\mathsf{b}}(\mathsf{M},\mathsf{d},\mathsf{m})$ and each optimal coupling q of them there exists a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 such that for all $t \in [0,1]$ and all $N' \geq N$,

$$\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq \sigma_{K,N'}^{(1-t)}(\Theta)\mathsf{S}_{N'}(\nu_0|\mathsf{m}) + \sigma_{K,N'}^{(t)}(\Theta)\mathsf{S}_{N'}(\nu_1|\mathsf{m}),$$

where Θ is defined as in (3.1.3).

(vi) For all $\nu_0, \nu_1 \in \mathcal{P}_{2,\mathsf{b}}(\mathsf{M},\mathsf{d},\mathsf{m})$ there exists an optimal coupling q of $\nu_0 = \rho_0\mathsf{m}$ and $\nu_1 = \rho_1\mathsf{m}$ such that

$$\rho_t^{-1/N}(\gamma_t(x_0,x_1)) \geq \sigma_{K,N}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N}(x_0) + \sigma_{K,N}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N}(x_1) \tag{3.1.6}$$

for all $t \in [0,1]$ and q-almost every $(x_0,x_1) \in \mathsf{M}\times\mathsf{M}$. Here for all $t \in [0,1]$, ρ_t denotes the density with respect to m of the push-forward measure of q under the map $(x_0,x_1) \mapsto \gamma_t(x_0,x_1)$.

Proof. (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v): These three implications follow from the fact that

$$\sigma_{K,N'}^{(t)}(\theta_\alpha) \geq \sigma_{K,N'}^{(t)}(\theta_\beta)$$

for all $t \in [0,1]$, all N' and all $\theta_\alpha, \theta_\beta \in \mathbb{R}_+$ with $K\theta_\alpha \geq K\theta_\beta$.

(iii) $\Rightarrow$ (i): We consider two measures $\nu_0 = \rho_0 \mathsf{m}$, $\nu_1 = \rho_1 \mathsf{m} \in \mathcal{P}_2(B_R(o), \mathsf{d}, \mathsf{m}) \subseteq \mathcal{P}_{2,\mathsf{b}}(\mathsf{M}, \mathsf{d}, \mathsf{m})$ for some $o \in \mathsf{M}$ and $R > 0$ and choose an arbitrary coupling $\tilde{\mathsf{q}}$ of them. For each $\epsilon > 0$, there exists a finite covering $(C_i)_{i=1,\ldots,n\in\mathbb{N}}$ of $M_c := \overline{B_{2R}(o)}$ by disjoint sets $C_1, \ldots, C_n$ with diameter $\leq \epsilon/2$ due to the compactness of M_c which is ensured by the local compactness of M. Now, we define probability measures ν_0^{ij} and ν_1^{ij} for $i,j = 1, \ldots, n$ on (M_c, d) by

$$\nu_0^{ij}(A) := \frac{1}{\alpha_{ij}} \tilde{\mathsf{q}}((A \cap C_i) \times C_j) \quad \text{and} \quad \nu_1^{ij}(A) := \frac{1}{\alpha_{ij}} \tilde{\mathsf{q}}(C_i \times (A \cap C_j)),$$

provided that $\alpha_{ij} := \tilde{\mathsf{q}}(C_i \times C_j) \neq 0$. Then

$$\mathsf{supp}(\nu_0^{ij}) \subseteq \overline{C_i} \quad \text{and} \quad \mathsf{supp}(\nu_1^{ij}) \subseteq \overline{C_j}.$$

By assumption there exists a geodesic $\Gamma^{ij} : [0,1] \to \mathcal{P}_2(M_c, \mathsf{d}, \mathsf{m})$ connecting $\nu_0^{ij} = \rho_0^{ij} \mathsf{m}$ and $\nu_1^{ij} = \rho_1^{ij} \mathsf{m}$ and satisfying

$$\begin{aligned} &\mathsf{S}_{N'}(\Gamma^{ij}(t)|\mathsf{m}) \\ &\quad \leq - \int_{\mathsf{M}\times\mathsf{M}} \Big[\sigma_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1) \mp \epsilon) \rho_0^{ij}(x_0)^{-1/N'} + \\ &\qquad\qquad + \sigma_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1) \mp \epsilon) \rho_1^{ij}(x_1)^{-1/N'} \Big] d\mathsf{q}^{ij}(x_0,x_1) \end{aligned}$$

for all $t \in [0,1]$ and all $N' \geq N$, with $\mp$ depending on the sign of K and with q^{ij} being an optimal coupling of ν_0^{ij} and ν_1^{ij}. We define for each $\epsilon > 0$ and all $t \in [0,1]$,

$$\mathsf{q}^{(\epsilon)} := \sum_{i,j=1}^{n} \alpha_{ij} \mathsf{q}^{ij} \quad \text{and} \quad \Gamma^{(\epsilon)}(t) := \sum_{i,j=1}^{n} \alpha_{ij} \Gamma^{ij}(t).$$

Then $\mathsf{q}^{(\epsilon)}$ is an optimal coupling of ν_0 and ν_1 and $\Gamma^{(\epsilon)}$ defines a geodesic connecting them. Furthermore, since $\Gamma^{ij}(t)$ is a t-midpoint of ν_0^{ij} and ν_1^{ij}, since the $\nu_0^{ij} \otimes \nu_1^{ij}$ are mutually singular for different choices of $(i,j) \in \{1, \ldots, n\}^2$ and since $(M_c, \mathsf{d}, \mathsf{m})$ is non-branching, the $\Gamma^{ij}(t)$ are as well mutually singular for different choices of $(i,j) \in \{1, \ldots, n\}^2$ and for each fixed $t \in [0,1]$. Hence, for all N',

$$\mathsf{S}_{N'}(\Gamma^{(\epsilon)}(t)|\mathsf{m}) = \sum_{ij} \alpha_{ij}^{1-1/N'} \mathsf{S}_{N'}(\Gamma^{ij}(t)|\mathsf{m}).$$

Compactness of $(M_c, \mathsf{d}, \mathsf{m})$ implies that there exists a sequence $(\epsilon(k))_{k\in\mathbb{N}}$ converging to 0 such that $(\mathsf{q}^{(\epsilon(k))})_{k\in\mathbb{N}}$ converges to some q and such that $(\Gamma^{(\epsilon(k))})_{k\in\mathbb{N}}$

converges to some geodesic Γ in $\mathcal{P}_2(M_c, \mathsf{d}, \mathsf{m})$. Therefore, for fixed $\varepsilon > 0$, all $t \in [0,1]$ and all $N' \geq N$,

$$\begin{aligned}
&\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \\
&\leq \liminf_{k\to\infty} \mathsf{S}_{N'}(\Gamma^{(\epsilon(k))}(t)|\mathsf{m}) \\
&\leq -\limsup_{k\to\infty} \int \Big[\sigma_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1) \mp \varepsilon)\rho_0^{-\frac{1}{N'}}(x_0) + \\
&\qquad\qquad + \sigma_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1) \mp \varepsilon)\rho_1^{-\frac{1}{N'}}(x_1)\Big] d\mathsf{q}^{(\epsilon(k))}(x_0,x_1) \\
&\leq -\int_{\mathsf{M}\times\mathsf{M}} \Big[\sigma_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1) \mp \varepsilon)\rho_0^{-\frac{1}{N'}}(x_0) + \\
&\qquad\qquad + \sigma_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1) \mp \varepsilon)\rho_1^{-\frac{1}{N'}}(x_1)\Big] d\mathsf{q}(x_0,x_1)
\end{aligned}$$

where the proof of the last inequality is similar to the proof of [Stu06b, Lemma 3.3]. In the limit $\varepsilon \to 0$ the claim follows due to the theorem of monotone convergence.

The implication (v) $\Rightarrow$ (ii) is trivial.

The equivalence (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) is obtained by following the arguments of the proof of [Stu06b, Proposition 4.2] replacing the coefficients $\tau_{K,N}^{(t)}(\cdot)$ by $\sigma_{K,N}^{(t)}(\cdot)$. □

Remark 3.1.8. *(i) In order to be honest, we suppressed an argument in the proof of Proposition 3.1.7, (iii) $\Rightarrow$ (i): In fact, the compactness of $(M_c, \mathsf{d}, \mathsf{m})$ implies the compactness of $\mathcal{P}_2(M_c, \mathsf{d})$ and therefore, we can deduce the existence of a limit Γ of $(\Gamma^{(\epsilon(k))})_{k\in\mathbb{N}}$ - using the same notation as in the above proof - in $\mathcal{P}_2(M_c, \mathsf{d})$! A further observation ensures that Γ is not only in $\mathcal{P}_2(M_c, \mathsf{d})$ but also in $\mathcal{P}_2(M_c, \mathsf{d}, \mathsf{m})$ - as claimed in the above proof: The characterizing inequality of $\mathsf{CD}^*(K,N)$ implies the characterizing inequality of the property $\mathsf{Curv}(\mathsf{M}, \mathsf{d}, \mathsf{m}) \geq K$ (at this point we refer to [Stu06a],[Stu06b]). Thus, the geodesic Γ satisfies*

$$\mathsf{Ent}(\Gamma(t)|\mathsf{m}) \leq (1-t)\mathsf{Ent}(\Gamma(0)|\mathsf{m}) + t\mathsf{Ent}(\Gamma(1)|\mathsf{m}) - \frac{K}{2}t(1-t)\mathsf{d}_{\mathsf{W}}^2(\Gamma(0),\Gamma(1))$$

for all $t \in [0,1]$. This implies that $\mathsf{Ent}(\Gamma(t)|\mathsf{m}) < +\infty$ and consequently, $\Gamma(t) \in \mathcal{P}_2(M_c, \mathsf{d}, \mathsf{m})$ for all $t \in [0,1]$. In the sequel, we will use similar arguments from time to time without emphasizing on them explicitly.

(ii) There are analogous characterizations of the local reduced curvature-dimension condition $\mathsf{CD}^*_{\mathsf{loc}}(K,N)$. In particular, the following statements are equivalent for locally compact and non-branching metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$:

(α) $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies* $\mathsf{CD}^*(K,N)$ *locally.*

(β) *Each point* $x \in \mathsf{M}$ *has a neighborhood* $M(x)$ *such that for each pair* $\nu_0,\nu_1 \in \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ *supported in* $M(x)$ *and each optimal coupling* q *of them there exists a geodesic* $\Gamma : [0,1] \to \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ *connecting* ν_0 *and* ν_1 *such that for all* $t \in [0,1]$ *and all* $N' \geq N$,

$$\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq \sigma_{K,N'}^{(1-t)}(\Theta)\mathsf{S}_{N'}(\nu_0|\mathsf{m}) + \sigma_{K,N'}^{(t)}(\Theta)\mathsf{S}_{N'}(\nu_1|\mathsf{m}),$$

where Θ *is defined as in (3.1.3).*

Proposition 3.1.9 (Midpoints). *A locally compact, non-branching metric measure space* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies* $\mathsf{CD}^*(K,N)$ *if and only if for all* $\nu_0,\nu_1 \in \mathcal{P}_{2,\mathsf{b}}(\mathsf{M},\mathsf{d},\mathsf{m})$ *there exists a midpoint* $\eta \in \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ *of* ν_1 *and* ν_1 *satisfying*

$$\mathsf{S}_{N'}(\eta|\mathsf{m}) \leq \sigma_{K,N'}^{(1/2)}(\theta)\mathsf{S}_{N'}(\nu_0|\mathsf{m}) + \sigma_{K,N'}^{(1/2)}(\theta)\mathsf{S}_{N'}(\nu_1|\mathsf{m}), \tag{3.1.7}$$

for all $N' \geq N$ *where* θ *is defined as in (3.1.5).*

Proof. We only consider the case $K > 0$. The general case requires analogous calculations. Due to Proposition 3.1.7, we have to prove that the existence of midpoints with property (3.1.7) for all $N' \geq N$ implies the existence of geodesics satisfying property (3.1.4) for all $N' \geq N$. Given $\Gamma(0) := \nu_0$ and $\Gamma(1) := \nu_1$, we define $\Gamma(\frac{1}{2})$ as a midpoint of $\Gamma(0)$ and $\Gamma(1)$ with property (3.1.7) for all $N' \geq N$. Then we define $\Gamma(\frac{1}{4})$ as a midpoint of $\Gamma(0)$ and $\Gamma(\frac{1}{2})$ satisfying (3.1.7) for all $N' \geq N$ and accordingly, $\Gamma(\frac{3}{4})$ as a midpoint of $\Gamma(\frac{1}{2})$ and $\Gamma(1)$ with (3.1.7) for all $N' \geq N$. By iterating this procedure, we obtain $\Gamma(t)$ for all dyadic $t = l2^{-k} \in [0,1]$ for $k \in \mathbb{N}$ and odd $l = 0,\ldots,2^k$ satisfying

$$\begin{aligned}&\mathsf{S}_{N'}\left(\Gamma\left(l2^{-k}\right)|\mathsf{m}\right) \leq \\ &\leq \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\mathsf{S}_{N'}\left(\Gamma\left((l-1)2^{-k}\right)|\mathsf{m}\right) + \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\mathsf{S}_{N'}\left(\Gamma\left((l+1)2^{-k}\right)|\mathsf{m}\right)\end{aligned}$$

for all $N' \geq N$ where θ is defined as above.

Now, we consider $k > 0$. By induction, we are able to pass from level $k-1$ to level k: Assuming that $\Gamma(t)$ satisfies property (3.1.4) for all $t = l2^{-k+1} \in [0,1]$ and all $N' \geq N$, we have for an odd number $l \in \{0,\ldots,2^{-k}\}$,

$$\begin{aligned}&\mathsf{S}_{N'}\left(\Gamma\left(l2^{-k}\right)|\mathsf{m}\right) \leq \\ &\leq \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\mathsf{S}_{N'}\left(\Gamma\left((l-1)2^{-k}\right)|\mathsf{m}\right) + \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\mathsf{S}_{N'}\left(\Gamma\left((l+1)2^{-k}\right)|\mathsf{m}\right) \\ &\leq \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\left[\sigma_{K,N'}^{\left(1-(l-1)2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(0)|\mathsf{m}) + \sigma_{K,N'}^{\left((l-1)2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(1)|\mathsf{m})\right] + \\ &\quad + \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\left[\sigma_{K,N'}^{\left(1-(l+1)2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(0)|\mathsf{m}) + \sigma_{K,N'}^{\left((l+1)2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(1)|\mathsf{m})\right]\end{aligned}$$

for all $N' \geq N$. Calculating the prefactor of $\mathsf{S}_{N'}(\Gamma(0)|\mathsf{m})$ yields

$$
\begin{aligned}
&\sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\sigma_{K,N'}^{\left(1-(l-1)2^{-k}\right)}(\theta)+\sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\sigma_{K,N'}^{\left(1-(l+1)2^{-k}\right)}(\theta)=\\
&=\sin\left(2^{-k}\theta\sqrt{K/N'}\right)\times\\
&\qquad\times\frac{\left[\sin\left(\left(1-(l-1)2^{-k}\right)\theta\sqrt{K/N'}\right)+\sin\left(\left(1-(l+1)2^{-k}\right)\theta\sqrt{K/N'}\right)\right]}{\sin\left(2^{-k+1}\theta\sqrt{K/N'}\right)\sin\left(\theta\sqrt{K/N'}\right)}\\
&=\frac{2\sin\left(\left(1-l2^{-k}\right)\theta\sqrt{K/N'}\right)\cos\left(2^{-k}\theta\sqrt{K/N'}\right)}{2\cos\left(2^{-k}\theta\sqrt{K/N'}\right)\sin\left(\theta\sqrt{K/N'}\right)}=\\
&=\frac{\sin\left(\left(1-l2^{-k}\right)\theta\sqrt{K/N'}\right)}{\sin\left(\theta\sqrt{K/N'}\right)}=\sigma_{K,N'}^{\left(1-l2^{-k}\right)}(\theta),
\end{aligned}
$$

and calculating the one of $\mathsf{S}_{N'}(\Gamma(1)|\mathsf{m})$ gives

$$
\begin{aligned}
&\sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\sigma_{K,N'}^{\left((l-1)2^{-k}\right)}(\theta)+\sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\sigma_{K,N'}^{\left((l+1)2^{-k}\right)}(\theta)=\\
&=\frac{\sin\left(2^{-k}\theta\sqrt{K/N'}\right)\cdot\left[\sin\left((l-1)2^{-k}\theta\sqrt{K/N'}\right)+\sin\left((l+1)2^{-k}\theta\sqrt{K/N'}\right)\right]}{\sin\left(2^{-k+1}\theta\sqrt{K/N'}\right)\sin\left(\theta\sqrt{K/N'}\right)}\\
&=\frac{2\sin\left(l2^{-k}\theta\sqrt{K/N'}\right)\cos\left(2^{-k}\theta\sqrt{K/N'}\right)}{2\cos\left(2^{-k}\theta\sqrt{K/N'}\right)\sin\left(\theta\sqrt{K/N'}\right)}=\\
&=\frac{\sin\left(l2^{-k}\theta\sqrt{K/N'}\right)}{\sin\left(\theta\sqrt{K/N'}\right)}=\sigma_{K,N'}^{\left(l2^{-k}\right)}(\theta).
\end{aligned}
$$

Combining the above results leads to property (3.1.4),

$$
\mathsf{S}_{N'}\left(\Gamma\left(l2^{-k}\right)|\mathsf{m}\right)\leq\sigma_{K,N'}^{\left(1-l2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(0)|\mathsf{m})+\sigma_{K,N'}^{\left(l2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(1)|\mathsf{m})
$$

for all $N' \geq N$. The continuous extension of $\Gamma(t)$ – t dyadic – yields the desired geodesic due to the lower semi-continuity of the Rényi entropy. $\square$

Lemma 3.1.10. *Fix two real parameters K and $N \geq 1$. If $(\mathsf{M},\mathsf{d},\mathsf{m})$ is non-branching then the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ implies that for all $\nu_0,\nu_1 \in \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ there exist an optimal coupling q of $\nu_0 = \rho_0\mathsf{m}$ and $\nu_1 = \rho_1\mathsf{m}$ and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 and satisfying (3.1.1) for all $N' \geq N$.*

Proof. We assume that $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfies $\mathsf{CD}^*(K, N)$. Let $\nu_0 = \rho_0 \mathsf{m}, \nu_1 = \rho_1 \mathsf{m} \in \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ and an optimal coupling q of ν_0 and ν_1 be given. We consider an exhausting sequence $(Q_n)_{n \in \mathbb{N}}$ of $\mathsf{M} \times \mathsf{M}$ by bounded sets, that means

$$Q_1 \subseteq Q_2 \subseteq \ldots Q_n \subseteq \ldots$$

and $\cup_{n \in \mathbb{N}} Q_n = \mathsf{M} \times \mathsf{M}$. For each $n \in \mathbb{N}$, we define q_n as the restriction of q to $Q_n \setminus Q_{n-1}$, that is for $A \subseteq \mathsf{M} \times \mathsf{M}$,

$$\mathsf{q}_n(A) = \frac{1}{\alpha_n} \mathsf{q}(A \cap (Q_n \setminus Q_{n-1}))$$

using the notations $\alpha_n := \mathsf{q}(Q_n \setminus Q_{n-1})$ and $Q_0 := \emptyset$. Moreover, we denote by μ_0^n and μ_1^n the marginals of q_n which are probability measures with bounded support. According to $\mathsf{CD}^*(K, N)$, for each $n \in \mathbb{N}$, there exist an optimal coupling $\tilde{\mathsf{q}}_n$ of $\mu_0^n = \rho_0^n \mathsf{m}$ and $\mu_1^n = \rho_1^n \mathsf{m}$ and a geodesic $\Gamma_n : [0, 1] \to \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ joining them such that

$$\begin{aligned} &\mathsf{S}_{N'}(\Gamma_n(t)|\mathsf{m}) \leq \\ &\leq - \int_{\mathsf{M} \times \mathsf{M}} \left[\sigma_{K,N'}^{(1-t)}(\mathsf{d}(x_0, x_1)) \rho_0^n(x_0)^{-1/N'} + \sigma_{K,N'}^{(t)}(\mathsf{d}(x_0, x_1)) \rho_1^n(x_1)^{-1/N'} \right] d\tilde{\mathsf{q}}_n(x_0, x_1) \end{aligned}$$

for all $t \in [0, 1]$ and all $N' \geq N$. We define for all $t \in [0, 1]$,

$$\tilde{\mathsf{q}} := \sum_{n=1}^{\infty} \alpha_n \tilde{\mathsf{q}}_n \quad \text{and} \quad \Gamma(t) := \sum_{n=1}^{\infty} \alpha_n \Gamma_n(t).$$

Then $\tilde{\mathsf{q}}$ is an optimal coupling of ν_0 and ν_1 and Γ defines a geodesic connecting them. Furthermore, since $\Gamma_n(t)$ is a t-midpoint of ν_0^n and ν_1^n, since the $\nu_0^n \otimes \nu_1^n$ are mutually singular for different choices of $n \in \mathbb{N}$ and since M is non-branching, the $\Gamma_n(t)$ are as well mutually singular for different choices of $n \in \mathbb{N}$ and for each fixed $t \in [0, 1]$. Due to the lower semi-continuity of the Rényi entropy functional, we have

$$\begin{aligned} &\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq \\ &\leq - \int_{\mathsf{M} \times \mathsf{M}} \left[\sigma_{K,N'}^{(1-t)}(\mathsf{d}(x_0, x_1)) \rho_0^{-1/N'}(x_0) + \sigma_{K,N'}^{(t)}(\mathsf{d}(x_0, x_1)) \rho_1^{-1/N'}(x_1) \right] d\tilde{\mathsf{q}}(x_0, x_1) \end{aligned}$$

for all $t \in [0, 1]$ and all $N' \geq N$. ☐

3.2 Stability under Convergence

Theorem 3.2.1. *Let* $((\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n))_{n\in\mathbb{N}}$ *be a sequence of normalized metric measure spaces with the property that for each* $n \in \mathbb{N}$ *the space* $(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n)$ *satisfies the reduced curvature-dimension condition* $\mathsf{CD}^*(K_n, N_n)$*. Assume that for* $n \to \infty$,

$$(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n) \xrightarrow{\mathbb{D}} (\mathsf{M}, \mathsf{d}, \mathsf{m})$$

as well as $(K_n, N_n) \to (K, N)$ *for some* $(K, N) \in \mathbb{R}^2$*. Then the space* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *fulfills* $\mathsf{CD}^*(K, N)$*.*

Under the additional assumption of uniformly bounded diameters by a constant L_0 with $KL_0^2 < N\pi^2$, the proof essentially follows the line of argumentation in [Stu06b, Theorem 3.1] where the coefficients $\tau_{K,N}^{(t)}(\cdot)$ have to be replaced by $\sigma_{K,N}^{(t)}(\cdot)$. In short, we consider $\nu_0, \nu_1 \in \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ and approximate them by probability measure $\nu_{0,n}$ and $\nu_{1,n}$ in $\mathcal{P}_2(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n)$ satisfying the relevant equation (3.1.1) with an optimal coupling q_n and a geodesic $\Gamma_{t,n}$ due to the curvature-dimension condition on $(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n)$. Via a map $\mathcal{Q} : \mathcal{P}_2(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n) \to \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ introduced in [Stu06a, Lemma 4.19] we define an 'ε-approximative' geodesic $\Gamma_t^\varepsilon := \mathcal{Q}(\Gamma_{t,n})$ from ν_0 to ν_1 satisfying (3.1.1) for an 'ε-approximative' coupling q^ε of ν_0 and ν_1. Compactness of M yields the existence of limits Γ_t and q, respectively, inheriting property (3.1.1).

To get rid of the additional assumption mentioned above, we may follow the argumentation in [LV07].

3.3 Tensorization

Theorem 3.3.1 (Tensorization)**.** *Let* $(\mathsf{M}_i, \mathsf{d}_i, \mathsf{m}_i)$ *be non-branching metric measure spaces satisfying the reduced curvature-dimension condition* $\mathsf{CD}^*(K, N_i)$ *with two real parameters* K *and* $N_i \geq 1$ *for* $i = 1, \ldots, k$ *with* $k \in \mathbb{N}$*. Then*

$$(\mathsf{M}, \mathsf{d}, \mathsf{m}) := \bigotimes_{i=1}^{k} (\mathsf{M}_i, \mathsf{d}_i, \mathsf{m}_i)$$

fulfills $\mathsf{CD}^*\left(K, \sum_{i=1}^k N_i\right)$.

Proof. Without restriction we assume that $k = 2$. We consider $\nu_0 = \rho_0 \mathsf{m}, \nu_1 = \rho_1 \mathsf{m} \in \mathcal{P}_{2,b}(\mathsf{M}, \mathsf{d}, \mathsf{m})$. In the first step, we treat the special case

$$\nu_0 = \nu_0^{(1)} \otimes \nu_0^{(2)} \quad \text{and} \quad \nu_1 = \nu_1^{(1)} \otimes \nu_1^{(2)}$$

with $\nu_0^{(i)} = \rho_0^{(i)}\mathsf{m}_i, \nu_1^{(i)} = \rho_1^{(i)}\mathsf{m}_i \in \mathcal{P}_{2,b}(\mathsf{M}_i, \mathsf{d}_i, \mathsf{m}_i)$ for $i = 1, 2$. According to our curvature assumption, there exists an optimal coupling q_i of $\nu_0^{(i)}$ and $\nu_1^{(i)}$ such that

$$\begin{aligned}
&\rho_t^{(i)}\left(\gamma_t^{(i)}\left(x_0^{(i)}, x_1^{(i)}\right)\right)^{-1/N_i} \geq \\
&\qquad \geq \sigma_{K,N_i}^{(1-t)}\left(\mathsf{d}_i\left(x_0^{(i)}, x_1^{(i)}\right)\right)\rho_0^{(i)}\left(x_0^{(i)}\right)^{-1/N_i} + \sigma_{K,N_i}^{(t)}\left(\mathsf{d}_i\left(x_0^{(i)}, x_1^{(i)}\right)\right)\rho_1^{(i)}\left(x_1^{(i)}\right)^{-1/N_i}
\end{aligned}$$

for all $t \in [0, 1]$ and q_i-almost every $\left(x_0^{(i)}, x_1^{(i)}\right) \in \mathsf{M}_i \times \mathsf{M}_i$ with $i = 1, 2$. As in Proposition 3.1.7, for all $t \in [0, 1]$, $\rho_t^{(i)}$ denotes the density with respect to m_i of the push-forward measure of q_i under the map $\left(x_0^{(i)}, x_1^{(i)}\right) \mapsto \gamma_t^{(i)}\left(x_0^{(i)}, x_1^{(i)}\right)$ for $i = 1, 2$. We introduce the map

$$\begin{aligned}
\mathsf{T} : \mathsf{M}_1 \times \mathsf{M}_1 \times \mathsf{M}_2 \times \mathsf{M}_2 &\to \mathsf{M}_1 \times \mathsf{M}_2 \times \mathsf{M}_1 \times \mathsf{M}_2 = \mathsf{M} \times \mathsf{M} \\
\left(x_0^{(1)}, x_1^{(1)}, x_0^{(2)}, x_1^{(2)}\right) &\mapsto \left(x_0^{(1)}, x_0^{(2)}, x_1^{(1)}, x_1^{(2)}\right),
\end{aligned}$$

we put $\tilde{\mathsf{q}} := \mathsf{q}_1 \otimes \mathsf{q}_2$ and define q as the push-forward measure of $\tilde{\mathsf{q}}$ under the map T, that means $\mathsf{q} := \mathsf{T}_*\tilde{\mathsf{q}}$. Then q is an optimal coupling of ν_0 and ν_1 and for all $t \in [0, 1]$, $\rho_t(x, y) := \rho_t^{(1)}(x) \cdot \rho_t^{(2)}(y)$ is the density with respect to m of the push-forward measure of q under the map

$$\begin{aligned}
\gamma_t : \mathsf{M} \times \mathsf{M} &\to \mathsf{M} = \mathsf{M}_1 \times \mathsf{M}_2 \\
\left(x_0^{(1)}, x_0^{(2)}, x_1^{(1)}, x_1^{(2)}\right) &\mapsto \left(\gamma_t^{(1)}\left(x_0^{(1)}, x_1^{(1)}\right), \gamma_t^{(2)}\left(x_0^{(2)}, x_1^{(2)}\right)\right).
\end{aligned}$$

Moreover, for q-almost every $x_0 = \left(x_0^{(1)}, x_0^{(2)}\right), x_1 = \left(x_1^{(1)}, x_1^{(2)}\right) \in \mathsf{M}$ and all $t \in [0, 1]$, it holds due to Lemma 3.1.1

$$\begin{aligned}
&\sigma_{K,N_1+N_2}^{(1-t)}(\mathsf{d}(x_0, x_1))\rho_0(x_0)^{-1/(N_1+N_2)} + \sigma_{K,N_1+N_2}^{(t)}(\mathsf{d}(x_0, x_1))\rho_1(x_1)^{-1/(N_1+N_2)} = \\
&= \sigma_{K,N_1+N_2}^{(1-t)}(\mathsf{d}(x_0, x_1))\rho_0^{(1)}\left(x_0^{(1)}\right)^{-1/(N_1+N_2)} \cdot \rho_0^{(2)}\left(x_0^{(2)}\right)^{-1/(N_1+N_2)} + \\
&\qquad + \sigma_{K,N_1+N_2}^{(t)}(\mathsf{d}(x_0, x_1))\rho_1^{(1)}\left(x_1^{(1)}\right)^{-1/(N_1+N_2)} \cdot \rho_1^{(2)}\left(x_1^{(2)}\right)^{-1/(N_1+N_2)} \\
&\leq \prod_{i=1}^{2} \sigma_{K,N_i}^{(1-t)}\left(\mathsf{d}_i\left(x_0^{(i)}, x_1^{(i)}\right)\right)^{N_i/(N_1+N_2)} \rho_0^{(i)}\left(x_0^{(i)}\right)^{-1/(N_1+N_2)} + \\
&\qquad + \prod_{i=1}^{2} \sigma_{K,N_i}^{(t)}\left(\mathsf{d}_i\left(x_0^{(i)}, x_1^{(i)}\right)\right)^{N_i/(N_1+N_2)} \rho_1^{(i)}\left(x_1^{(i)}\right)^{-1/(N_1+N_2)}
\end{aligned}$$

And using Hölder's inequality,

$$
\begin{aligned}
&\sigma_{K,N_1+N_2}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0(x_0)^{-1/(N_1+N_2)} + \sigma_{K,N_1+N_2}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1(x_1)^{-1/(N_1+N_2)} \\
&\leq \prod_{i=1}^{2}\Bigg[\sigma_{K,N_i}^{(1-t)}\left(\mathsf{d}_i\left(x_0^{(i)},x_1^{(i)}\right)\right)\rho_0^{(i)}\left(x_0^{(i)}\right)^{-1/N_i} + \\
&\qquad\qquad + \sigma_{K,N_i}^{(t)}\left(\mathsf{d}_i\left(x_0^{(i)},x_1^{(i)}\right)\right)\rho_1^{(i)}\left(x_1^{(i)}\right)^{-1/N_i}\Bigg]^{N_i/(N_1+N_2)} \\
&\leq \prod_{i=1}^{2}\rho_t^{(i)}\left(\gamma_t^{(i)}\left(x_0^{(i)},x_1^{(i)}\right)\right)^{-1/(N_1+N_2)} \\
&= \rho_t\left(\gamma_t^{(1)}\left(x_0^{(1)},x_1^{(1)}\right),\gamma_t^{(2)}\left(x_0^{(2)},x_1^{(2)}\right)\right)^{-1/(N_1+N_2)} = \rho_t\left(\gamma_t(x_0,x_1)\right)^{-1/(N_1+N_2)}.
\end{aligned}
$$

In the second step, we consider $o \in \mathsf{supp}(\mathsf{m})$ and $R > 0$ and set $M_b := B_R(o) \cap \mathsf{supp}(\mathsf{m})$ as well as $M_c := \overline{B_{2R}(o)} \cap \mathsf{supp}(\mathsf{m})$. We consider arbitrary probability measures $\nu_0, \nu_1 \in \mathcal{P}_2(M_b, \mathsf{d}, \mathsf{m})$ and $\varepsilon > 0$. There exist

$$\nu_0^\varepsilon = \rho_0^\varepsilon m = \frac{1}{n}\sum_{j=1}^{n}\nu_{0,j}^\varepsilon$$

with mutually singular product measures $\nu_{0,j}^\varepsilon$ and

$$\nu_1^\varepsilon = \rho_1^\varepsilon m = \frac{1}{n}\sum_{j=1}^{n}\nu_{1,j}^\varepsilon$$

with mutually singular product measures $\nu_{1,j}^\varepsilon$ for $j = 1, \ldots, n$ and $n \in \mathbb{N}$ such that

$$
\begin{aligned}
\mathsf{S}_{N_1+N_2}\left(\nu_0^\varepsilon|\mathsf{m}\right) &\leq \mathsf{S}_{N_1+N_2}(\nu_0|\mathsf{m}) + \varepsilon, \\
\mathsf{S}_{N_1+N_2}\left(\nu_1^\varepsilon|\mathsf{m}\right) &\leq \mathsf{S}_{N_1+N_2}(\nu_1|\mathsf{m}) + \varepsilon
\end{aligned}
$$

as well as

$$\mathsf{d_W}\left(\nu_0,\nu_0^\varepsilon\right) \leq \varepsilon, \quad \mathsf{d_W}\left(\nu_1,\nu_1^\varepsilon\right) \leq \varepsilon$$

and

$$\mathsf{d_W}\left(\nu_0^\varepsilon,\nu_1^\varepsilon\right) \geq \left[\frac{1}{n}\sum_{j=1}^{n}\mathsf{d}_\mathsf{W}^2\left(\nu_{0,j}^\varepsilon,\nu_{1,j}^\varepsilon\right)\right]^{1/2} - \varepsilon.$$

Moreover,

$$\theta := \begin{cases} \inf\limits_{\substack{x_0\in\operatorname{supp}(\nu_0),\\ x_1\in\operatorname{supp}(\nu_1)}} \mathsf{d}(x_0,x_1) \le \inf\limits_{\substack{x_0\in\operatorname{supp}\left(\nu_{0,j}^{\varepsilon}\right),\\ x_1\in\operatorname{supp}\left(\nu_{1,j}^{\varepsilon}\right)}} \mathsf{d}(x_0,x_1), & \text{if } K \ge 0,\\ \sup\limits_{\substack{x_0\in\operatorname{supp}(\nu_0),\\ x_1\in\operatorname{supp}(\nu_1)}} \mathsf{d}(x_0,x_1) \ge \sup\limits_{\substack{x_0\in\operatorname{supp}\left(\nu_{0,j}^{\varepsilon}\right),\\ x_1\in\operatorname{supp}\left(\nu_{1,j}^{\varepsilon}\right)}} \mathsf{d}(x_0,x_1), & \text{if } K < 0.\end{cases}$$

Since ν_0^ε is the sum of mutually singular measures $\nu_{0,j}^\varepsilon$ for $j = 1,\dots,n$,

$$\mathsf{S}_{N_1+N_2}\left(\nu_0^\varepsilon|\mathsf{m}\right) = \left(\frac{1}{n}\right)^{1-1/(N_1+N_2)} \sum_{j=1}^{n} \mathsf{S}_{N_1+N_2}\left(\nu_{0,j}^\varepsilon|\mathsf{m}\right)$$

and analogously,

$$\mathsf{S}_{N_1+N_2}\left(\nu_1^\varepsilon|\mathsf{m}\right) = \left(\frac{1}{n}\right)^{1-1/(N_1+N_2)} \sum_{j=1}^{n} \mathsf{S}_{N_1+N_2}\left(\nu_{1,j}^\varepsilon|\mathsf{m}\right).$$

Due to the first step, for each $j = 1,\dots,n$ there exists a midpoint $\eta_j^\varepsilon \in \mathcal{P}_2(M_c,\mathsf{d},\mathsf{m})$ of $\nu_{0,j}^\varepsilon$ and $\nu_{1,j}^\varepsilon$ satisfying

$$\mathsf{S}_{N_1+N_2}\left(\eta_j^\varepsilon|\mathsf{m}\right) \le \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_{0,j}^\varepsilon|\mathsf{m}\right) + \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_{1,j}^\varepsilon|\mathsf{m}\right).$$

Since M is non-branching and since the measures $\nu_{0,j}^\varepsilon$ for $j = 1,\dots,n$ are mutually singular, also the η_j^ε are mutually singular for $j = 1,\dots,n$. Therefore,

$$\eta^\varepsilon := \frac{1}{n}\sum_{j=1}^{n} \eta_j^\varepsilon$$

satisfies

$$\mathsf{S}_{N_1+N_2}\left(\eta^\varepsilon|\mathsf{m}\right) = \left(\frac{1}{n}\right)^{1-1/(N_1+N_2)} \sum_{j=1}^{n} \mathsf{S}_{N_1+N_2}\left(\eta_j^\varepsilon|\mathsf{m}\right)$$

and consequently,

$$\begin{aligned}\mathsf{S}_{N_1+N_2}\left(\eta^\varepsilon|\mathsf{m}\right) &\le \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_0^\varepsilon|\mathsf{m}\right) + \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_1^\varepsilon|\mathsf{m}\right)\\ &\le \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_0|\mathsf{m}\right) + \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_1|\mathsf{m}\right) + 2\varepsilon.\end{aligned}$$

Moreover, η^ε is an approximate midpoint of ν_0 and ν_1,

$$\begin{aligned}\mathsf{d_W}\left(\nu_0,\eta^\varepsilon\right) \le \mathsf{d_W}\left(\nu_0^\varepsilon,\eta^\varepsilon\right) + \varepsilon &\le \left[\frac{1}{n}\sum_{j=1}^{n} \mathsf{d}_{\mathsf{W}}^2\left(\nu_{0,j}^\varepsilon,\eta_j^\varepsilon\right)\right]^{1/2} + \varepsilon\\ &\le \frac{1}{2}\mathsf{d_W}\left(\nu_0^\varepsilon,\nu_1^\varepsilon\right) + 2\varepsilon \le \frac{1}{2}\mathsf{d_W}\left(\nu_0,\nu_1\right) + 3\varepsilon,\end{aligned}$$

a similar calculation holds true for $\mathsf{d_W}(\eta^\varepsilon, \nu_1)$. According to the compactness of (M_c, d), the family $\{\eta^\varepsilon : \varepsilon > 0\}$ of approximate midpoints is tight. Hence, there exists a suitable subsequence $(\eta^{\varepsilon_k})_{k\in\mathbb{N}}$ converging to some $\eta \in \mathcal{P}_2(M_c, \mathsf{d}, \mathsf{m})$. Continuity of the Wasserstein distance $\mathsf{d_W}$ and lower semi-continuity of the Rényi entropy functional $\mathsf{S}_{N_1+N_2}(\cdot|\mathsf{m})$ imply that η is a midpoint of ν_0 and ν_1 and that

$$\mathsf{S}_{N_1+N_2}(\eta|\mathsf{m}) \leq \sigma^{(1/2)}_{K,N_1+N_2}(\theta)\mathsf{S}_{N_1+N_2}(\nu_0|\mathsf{m}) + \sigma^{(1/2)}_{K,N_1+N_2}(\theta)\mathsf{S}_{N_1+N_2}(\nu_1|\mathsf{m}).$$

Applying Proposition 3.1.7 finally yields the claim. □

3.4 From Local to Global

Theorem 3.4.1 ($\mathsf{CD}^*_{\mathsf{loc}}(K,N) \Leftrightarrow \mathsf{CD}^*(K,N)$)**.** *Let $K, N \in \mathbb{R}$ with $N \geq 1$ and let $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ be a non-branching metric measure space. We assume additionally that $\mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is a geodesic space. Then $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfies $\mathsf{CD}^*(K,N)$ globally if and only if it satisfies $\mathsf{CD}^*(K,N)$ locally.*

Proof. We confine ourselves to treating the case $K > 0$. The general one follows by analogous calculations.

For each number $k \in \mathbb{N} \cup \{0\}$ we define a set I_k of points in time,

$$I_k := \{l2^{-k} : l = 0, \ldots, 2^k\}.$$

For a given geodesic $\Gamma : [0,1] \to \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ we denote by $\mathcal{G}^\Gamma_k$ the set of all geodesics $[x] := (x_t)_{0\leq t\leq 1}$ in M satisfying $x_t \in \mathsf{supp}(\Gamma(t)) =: \mathcal{S}_t$ for all $t \in I_k$.

We consider $o \in \mathsf{supp}(\mathsf{m})$ and $R > 0$ and set $M_b := B_R(o) \cap \mathsf{supp}(\mathsf{m})$ as well as $M_c := \overline{B_{2R}(o)} \cap \mathsf{supp}(\mathsf{m})$. Now, we formulate a property $\mathsf{C(k)}$ for every $k \in \mathbb{N} \cup \{0\}$:

$\mathsf{C(k)}$: For each geodesic $\Gamma : [0,1] \to \mathcal{P}_{2,\mathsf{b}}(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying $\Gamma(0), \Gamma(1) \in \mathcal{P}_2(M_b, \mathsf{d}, \mathsf{m})$ and for each pair $s, t \in I_k$ with $t - s = 2^{-k}$ there exists a midpoint $\eta(s,t) \in \mathcal{P}_{2,\mathsf{b}}(\mathsf{M}, \mathsf{d}, \mathsf{m})$ of $\Gamma(s)$ and $\Gamma(t)$ such that

$$\mathsf{S}_{N'}(\eta(s,t)|\mathsf{m}) \leq \sigma^{(1/2)}_{K,N'}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) + \sigma^{(1/2)}_{K,N'}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}),$$

for all $N' \geq N$ where

$$\theta_{s,t} := \inf_{[x]\in\mathcal{G}^\Gamma_k} \mathsf{d}(x_s, x_t).$$

Our first claim is:

Claim 3.4.2. *For each $k \in \mathbb{N}$, $\mathsf{C(k)}$ implies $\mathsf{C(k-1)}$.*

In order to prove this claim, let $k \in \mathbb{N}$ with property $\mathsf{C(k)}$ be given. Moreover, let a geodesic Γ in $\mathcal{P}_{2,\mathsf{b}}(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfying $\Gamma(0), \Gamma(1) \in \mathcal{P}_2(M_b,\mathsf{d},\mathsf{m})$ and numbers $s,t \in I_{k-1}$ with $t-s = 2^{1-k}$ be given. We put $\theta := \inf_{[x]\in\mathcal{G}^{\Gamma}_{k-1}} \mathsf{d}(x_s,x_t)$, and we define iteratively a sequence $(\Gamma^{(i)})_{i\in\mathbb{N}\cup\{0\}}$ of geodesics in $\mathcal{P}_2(M_c,\mathsf{d},\mathsf{m})$ coinciding with Γ on $[0,s]\cup[t,1]$ as follows:

Start with $\Gamma^{(0)} := \Gamma$. Assuming that $\Gamma^{(2i)}$ is already given, let $\Gamma^{(2i+1)}$ be any geodesic in $\mathcal{P}_2(M_c,\mathsf{d},\mathsf{m})$ which coincides with Γ on $[0,s]\cup[t,1]$, for which $\Gamma^{(2i+1)}\left(s+2^{-(k+1)}\right)$ is a midpoint of $\Gamma(s) = \Gamma^{(2i)}(s)$ and $\Gamma^{(2i)}\left(s+2^{-k}\right)$ and for which $\Gamma^{(2i+1)}\left(s+3\cdot 2^{-(k+1)}\right)$ is a midpoint of $\Gamma^{(2i)}\left(s+2^{-k}\right)$ and $\Gamma(t) = \Gamma^{(2i)}(t)$ satisfying

$$\begin{aligned}&\mathsf{S}_{N'}\left(\Gamma^{(2i+1)}\left(s+2^{-(k+1)}\right)|\mathsf{m}\right) \leq \\ &\quad\leq \sigma^{(1/2)}_{K,N'}\left(\theta^{(2i+1)}\right)\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) + \sigma^{(1/2)}_{K,N'}\left(\theta^{(2i+1)}\right)\mathsf{S}_{N'}\left(\Gamma^{(2i)}\left(s+2^{-k}\right)|\mathsf{m}\right)\end{aligned}$$

for all $N' \geq N$ where

$$\theta^{(2i+1)} := \inf_{[x]\in\mathcal{G}^{\Gamma^{(2i)}}_{k}} \mathsf{d}\left(x_s, x_{s+2^{-k}}\right) \geq \tfrac{1}{2}\theta,$$

that is,

$$\begin{aligned}&\mathsf{S}_{N'}\left(\Gamma^{(2i+1)}\left(s+2^{-(k+1)}\right)|\mathsf{m}\right) \leq \\ &\qquad\leq \sigma^{(1/2)}_{K,N'}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) + \sigma^{(1/2)}_{K,N'}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}\left(\Gamma^{(2i)}\left(s+2^{-k}\right)|\mathsf{m}\right)\end{aligned}$$

for all $N' \geq N$ and accordingly,

$$\begin{aligned}&\mathsf{S}_{N'}\left(\Gamma^{(2i+1)}\left(s+3\cdot 2^{-(k+1)}\right)|\mathsf{m}\right) \leq \\ &\qquad\leq \sigma^{(1/2)}_{K,N'}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}\left(\Gamma^{(2i)}\left(s+2^{-k}\right)|\mathsf{m}\right) + \sigma^{(1/2)}_{K,N'}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m})\end{aligned}$$

for all $N' \geq N$. Such midpoints exist due to $\mathsf{C(k)}$.

Now let $\Gamma^{(2i+2)}$ be any geodesic in $\mathcal{P}_2(M_c,\mathsf{d},\mathsf{m})$ which coincides with Γ on $[0,s]\cup[t,1]$ and for which $\Gamma^{(2i+2)}\left(s+2^{-k}\right)$ is a midpoint of $\Gamma^{(2i+1)}\left(s+2^{-(k+1)}\right)$ and $\Gamma^{(2i+1)}\left(s+3\cdot 2^{-(k+1)}\right)$ satisfying

$$\begin{aligned}&\mathsf{S}_{N'}\left(\Gamma^{(2i+2)}\left(s+2^{-k}\right)|\mathsf{m}\right) \leq \\ &\quad\leq \sigma^{(1/2)}_{K,N'}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}\left(\Gamma^{(2i+1)}\left(s+2^{-(k+1)}\right)|\mathsf{m}\right) + \\ &\qquad\qquad + \sigma^{(1/2)}_{K,N'}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}\left(\Gamma^{(2i+1)}\left(s+3\cdot 2^{-(k+1)}\right)|\mathsf{m}\right)\end{aligned}$$

for all $N' \geq N$. Again such a midpoint exists according to $\mathsf{C}(\mathsf{k})$. This yields a sequence $(\Gamma^{(i)})_{i\in\mathbb{N}\cup\{0\}}$ of geodesics. Combining the above inequalities we obtain

$$\begin{aligned}
&\mathsf{S}_{N'}\left(\Gamma^{(2i+2)}\left(s+2^{-k}\right)|\mathsf{m}\right) \leq \\
&\quad \leq 2\sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)^2 \mathsf{S}_{N'}\left(\Gamma^{(2i)}\left(s+2^{-k}\right)|\mathsf{m}\right) + \\
&\qquad\qquad + \sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)^2 \mathsf{S}_{N'}\left(\Gamma(s)|\mathsf{m}\right) + \sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)^2 \mathsf{S}_{N'}\left(\Gamma(t)|\mathsf{m}\right)
\end{aligned}$$

and by iteration,

$$\begin{aligned}
&\mathsf{S}_{N'}\left(\Gamma^{(2i)}\left(s+2^{-k}\right)|\mathsf{m}\right) \leq \\
&\quad \leq 2^i\sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)^{2i} \mathsf{S}_{N'}\left(\Gamma\left(s+2^{-k}\right)|\mathsf{m}\right) + \\
&\qquad\qquad + \tfrac{1}{2}\sum_{k=1}^{i}\left(2\sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)^2\right)^k \left[\mathsf{S}_{N'}\left(\Gamma(s)|\mathsf{m}\right) + \mathsf{S}_{N'}\left(\Gamma(t)|\mathsf{m}\right)\right]
\end{aligned}$$

for all $N' \geq N$. By compactness of $\mathcal{P}_2(M_c, \mathsf{d})$, there exists a suitable subsequence of $\left(\Gamma^{(2i)}\left(s+2^{-k}\right)\right)_{i\in\mathbb{N}\cup\{0\}}$ converging to some $\eta \in \mathcal{P}_2(M_c, \mathsf{d})$. Continuity of the distance implies that η is a midpoint of $\Gamma(s)$ and $\Gamma(t)$ and the lower semi-continuity of the Rényi entropy functional implies

$$\mathsf{S}_{N'}(\eta|\mathsf{m}) \leq \sigma_{K,N'}^{(1/2)}(\theta)\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) + \sigma_{K,N'}^{(1/2)}(\theta)\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m})$$

for all $N' \geq N$. This proves property $\mathsf{C}(\mathsf{k}-1)$. At this point, we do not want to suppress the calculations leading to this last implication: For all $N' \geq N$, we have

$$\begin{aligned}
\sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right) &= \frac{\sin\left(\frac{1}{4}\theta\sqrt{K/N'}\right)}{\sin\left(\frac{1}{2}\theta\sqrt{K/N'}\right)} = \frac{\sin\left(\frac{1}{4}\theta\sqrt{K/N'}\right)}{2\sin\left(\frac{1}{4}\theta\sqrt{K/N'}\right)\cos\left(\frac{1}{4}\theta\sqrt{K/N'}\right)} \\
&= \frac{1}{2\cos\left(\frac{1}{4}\theta\sqrt{K/N'}\right)}.
\end{aligned}$$

In the case $2\sigma_{K,N'}^{(1/2)}\left(\frac{1}{2}\theta\right)^2 < 1$,

$$\begin{aligned}
\tfrac{1}{2}\lim_{i\to\infty}\sum_{k=1}^{i}\left(2\sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)^2\right)^k &= \tfrac{1}{2}\left[\left(1-2\sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)^2\right)^{-1} - 1\right] \\
&= \tfrac{1}{2}\left[\left(\frac{2\cos^2\left(\frac{1}{4}\theta\sqrt{K/N'}\right)-1}{2\cos^2\left(\frac{1}{4}\theta\sqrt{K/N'}\right)}\right)^{-1} - 1\right]
\end{aligned}$$

Calculating the fraction appearing on the right-hand side of the last equation yields

$$\begin{aligned}\tfrac{1}{2}\lim_{i\to\infty}\sum_{k=1}^{i}\left(2\sigma^{(1/2)}_{K,N'}\left(\tfrac{1}{2}\theta\right)^2\right)^k &= \tfrac{1}{2}\left[\frac{2\cos^2\left(\frac{1}{4}\theta\sqrt{K/N'}\right)}{\cos\left(\frac{1}{2}\theta\sqrt{K/N'}\right)}-1\right]\\ &= \tfrac{1}{2}\left[\frac{\cos\left(\frac{1}{2}\theta\sqrt{K/N'}\right)+1-\cos\left(\frac{1}{2}\theta\sqrt{K/N'}\right)}{\cos\left(\frac{1}{2}\theta\sqrt{K/N'}\right)}\right]\\ &= \frac{1}{2\cos\left(\frac{1}{2}\theta\sqrt{K/N'}\right)} = \sigma^{(1/2)}_{K,N'}(\theta)\,.\end{aligned}$$

According to our curvature assumption, each point $x \in \mathsf{M}$ has a neighborhood $M(x)$ such that probability measures in $\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ which are supported in $M(x)$ can be joined by a geodesic in $\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfying (3.1.1). By compactness of M_c, there exist $\lambda > 0$, $n \in \mathbb{N}$, finitely many disjoint sets $L_1, L_2, \ldots, L_n$ covering M_c, and closed sets $M_j \supseteq B_\lambda(L_j)$ for $j = 1,\ldots,n$, such that probability measures in $\mathcal{P}_2(M_j,\mathsf{d},\mathsf{m})$ can be joined by geodesics in $\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfying (3.1.1). Choose $\kappa \in \mathbb{N}$ such that

$$2^{-\kappa}\mathsf{diam}(M_c,\mathsf{d},\mathsf{m}) \le \lambda.$$

Our next claim is:

Claim 3.4.3. *Property* $\mathsf{C}(\kappa)$ *is satisfied.*

In order to prove this claim, we consider a geodesic Γ in $\mathcal{P}_{2,\mathsf{b}}(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfying $\Gamma(0),\Gamma(1) \in \mathcal{P}_2(M_b,\mathsf{d},\mathsf{m})$ and numbers $s,t \in I_\kappa$ with $t-s = 2^{-\kappa}$. Let $\hat{\mathsf{q}}$ be a coupling of $\Gamma(l2^{-\kappa})$ for $l = 0,\ldots,2^\kappa$ on $\mathsf{M}^{2^\kappa+1}$ such that for $\hat{\mathsf{q}}$-almost every $(x_l)_{l=0,\ldots,2^\kappa} \in \mathsf{M}^{2^\kappa+1}$ the points x_s, x_t lie on some geodesic connecting x_0 and x_1 with

$$\mathsf{d}(x_s,x_t) = |t-s|\mathsf{d}(x_0,x_1) \le 2^{-\kappa}\mathsf{diam}(M_c,\mathsf{d},\mathsf{m}) \le \lambda. \tag{3.4.1}$$

Define probability measures $\Gamma_j(s)$ and $\Gamma_j(t)$ for $j = 1,\ldots,n$ by

$$\Gamma_j(s)(A) := \frac{1}{\alpha_j}\Gamma(s)(A\cap L_j) = \frac{1}{\alpha_j}\hat{\mathsf{q}}\underbrace{(\mathsf{M}\times\cdots\times(\underset{\substack{\uparrow\\ s\text{-th factor}}}{A}\cap L_j)\times\mathsf{M}\times\cdots\times\mathsf{M})}_{(2^\kappa+1)\text{ factors}}$$

and

$$\Gamma_j(t)(A) := \frac{1}{\alpha_j}\hat{\mathsf{q}}(\mathsf{M}\times\cdots\times L_j\times \underset{\substack{\uparrow\\ t\text{-th factor}}}{A}\times\cdots\times\mathsf{M})$$

provided that $\alpha_j := \Gamma_s(L_j) \neq 0$. Otherwise, define $\Gamma_j(s)$ and $\Gamma_j(t)$ arbitrarily. Then $\mathsf{supp}(\Gamma_j(s)) \subseteq \overline{L_j}$ which combined with inequality (3.4.1) implies

$$\mathsf{supp}(\Gamma_j(s)) \cup \mathsf{supp}(\Gamma_j(t)) \subseteq \overline{B_\lambda(L_j)} \subseteq M_j.$$

Therefore, for each $j \in \{1, \ldots, n\}$, the assumption '$(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfies $\mathsf{CD}^*(K, N)$ locally' can be applied to the probability measures $\Gamma_j(s)$ and $\Gamma_j(t) \in \mathcal{P}_2(M_j, \mathsf{d}, \mathsf{m})$. It yields the existence of a midpoint $\eta_j(s,t)$ of $\Gamma_j(s)$ and $\Gamma_j(t)$ with the property that

$$\mathsf{S}_{N'}(\eta_j(s,t)|\mathsf{m}) \leq \sigma_{K,N'}^{(1/2)}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma_j(s)|\mathsf{m}) + \sigma_{K,N'}^{(1/2)}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma_j(t)|\mathsf{m}) \tag{3.4.2}$$

for all $N' \geq N$ where

$$\theta_{s,t} := \inf_{[x] \in \mathcal{G}_\kappa^\Gamma} \mathsf{d}(x_s, x_t).$$

Define

$$\eta(s,t) := \sum_{j=1}^n \alpha_j \eta_j(s,t).$$

Then, $\eta(s,t)$ is a midpoint of $\Gamma(s) = \sum_{j=1}^n \alpha_j \Gamma_j(s)$ and $\Gamma(t) = \sum_{j=1}^n \alpha_j \Gamma_j(t)$. Moreover, since the $\Gamma_j(s)$ are mutually singular for $j = 1, \ldots, n$ and since M is non-branching, also the $\eta_j(s,t)$ are mutually singular for $j = 1, \ldots, n$. Therefore, for all $N' \geq N$,

$$\mathsf{S}_{N'}(\eta(s,t)|\mathsf{m}) = \sum_{j=1}^n \alpha_j^{1-1/N'} \mathsf{S}_{N'}(\eta_j(s,t)|\mathsf{m}) \tag{3.4.3}$$

and

$$\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) = \sum_{j=1}^n \alpha_j^{1-1/N'} \mathsf{S}_{N'}(\Gamma_j(s)|\mathsf{m}), \tag{3.4.4}$$

whereas

$$\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \geq \sum_{j=1}^n \alpha_j^{1-1/N'} \mathsf{S}_{N'}(\Gamma_j(t)|\mathsf{m}), \tag{3.4.5}$$

since the $\Gamma_j(t)$ are not necessarily mutually singular for $j = 1, \ldots, n$. Summing up (3.4.2) for $j = 1, \ldots, n$ and using (3.4.3)–(3.4.5) yields

$$\mathsf{S}_{N'}(\eta(s,t)|\mathsf{m}) \leq \sigma_{K,N'}^{(1/2)}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) + \sigma_{K,N'}^{(1/2)}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m})$$

for all $N' \geq N$. This proves property $\mathsf{C}(\kappa)$.

In order to finish the proof let two probability measures $\nu_0, \nu_1 \in \mathcal{P}_2(M_b, \mathsf{d}, \mathsf{m})$ be given. By assumption there exists a geodesic Γ in $\mathcal{P}_{2,\mathsf{b}}(\mathsf{M}, \mathsf{d}, \mathsf{m})$ connecting them.

According to our second claim, property $\mathsf{C}(\kappa)$ is satisfied and according to our first claim, this implies $\mathsf{C}(\mathsf{k})$ for all $k = \kappa - 1, \kappa - 2, \ldots, 0$. Property $\mathsf{C}(0)$ finally states that there exists a midpoint $\eta \in \mathcal{P}_{2,\mathsf{b}}(\mathsf{M}, \mathsf{d}, \mathsf{m})$ of $\Gamma(0) = \nu_0$ and $\Gamma(1) = \nu_1$ with

$$\mathsf{S}_{N'}(\eta|\mathsf{m}) \leq \sigma_{K,N'}^{(1/2)}(\theta)\mathsf{S}_{N'}(\Gamma(0)|\mathsf{m}) + \sigma_{K,N'}^{(1/2)}(\theta)\mathsf{S}_{N'}(\Gamma(1)|\mathsf{m}),$$

for all $N' \geq N$ where

$$\theta := \inf_{x_0 \in \mathcal{S}_0, x_1 \in \mathcal{S}_1} \mathsf{d}(x_0, x_1).$$

This proves Theorem (3.4.1). $\square$

Corollary 3.4.4 ($\mathsf{CD}^*_{\mathsf{loc}}(K-, N) \Leftrightarrow \mathsf{CD}^*(K, N)$)**.** *Fix two numbers $K, N \in \mathbb{R}$. A metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ fulfills the reduced curvature-dimension condition $\mathsf{CD}^*(K', N)$ locally for all $K' < K$ if and only if it satisfies the condition $\mathsf{CD}^*(K, N)$ globally.*

Proof. Given any $K' < K$, the condition $\mathsf{CD}^*(K', N)$ is deduced from $\mathsf{CD}^*_{\mathsf{loc}}(K', N)$ according to the above localization theorem. Due to the stability of the reduced curvature-dimension condition stated in Theorem 3.2.1, $\mathsf{CD}^*(K', N)$ for all $K' < K$ implies $\mathsf{CD}^*(K, N)$. $\square$

Proposition 3.4.5 ($\mathsf{CD}^*_{\mathsf{loc}}(K-, N) \Leftrightarrow \mathsf{CD}_{\mathsf{loc}}(K-, N)$)**.** *Fix two numbers $K, N \in \mathbb{R}$ with $N \geq 1$. $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ fulfills the reduced curvature-dimension condition $\mathsf{CD}^*(K', N)$ locally for all $K' < K$ if and only if it satisfies the original condition $\mathsf{CD}(K', N)$ locally for all $K' < K$.*

Proof. As remarked before in the past, we content ourselves with the case $K > 0$. Again, the general one can be deduced from analogous calculations. The implication '$\mathsf{CD}^*_{\mathsf{loc}}(K-, N) \Leftarrow \mathsf{CD}_{\mathsf{loc}}(K-, N)$' follows from analogous arguments leading to part (i) of Proposition 3.1.5. The implication '$\mathsf{CD}^*_{\mathsf{loc}}(K-, N) \Rightarrow \mathsf{CD}_{\mathsf{loc}}(K-, N)$' is based on the fact that the coefficients $\tau_{K,N}^{(t)}(\theta)$ and $\sigma_{K,N}^{(t)}(\theta)$ are 'almost identical' for $\theta \ll 1$: In order to be precise, we consider $0 < K' < \tilde{K} < K$ and $\theta \ll 1$ and compare the relevant coefficients $\tau_{K',N}^{(t)}(\theta)$ and $\sigma_{\tilde{K},N}^{(t)}(\theta)$:

$$\begin{aligned}
\left[\tau_{K',N}^{(t)}(\theta)\right]^N &= t \left[\frac{\sin\left(t\theta\sqrt{\frac{K'}{N-1}}\right)}{\sin\left(\theta\sqrt{\frac{K'}{N-1}}\right)}\right]^{N-1} \\
&= t^N \left[\frac{1 - \frac{1}{6}t^2\theta^2\frac{K'}{N-1} + O(\theta^4)}{1 - \frac{1}{6}\theta^2\frac{K'}{N-1} + O(\theta^4)}\right]^{N-1} \\
&= t^N \left[1 + \tfrac{1}{6}(1-t^2)\theta^2\tfrac{K'}{N-1} + O(\theta^4)\right]^{N-1} \\
&= t^N \left[1 + \tfrac{1}{6}(1-t^2)\theta^2 K' + O(\theta^4)\right].
\end{aligned}$$

And accordingly,

$$\begin{aligned}\left[\sigma^{(t)}_{\tilde K,N}(\theta)\right]^N &= \left[\frac{\sin\left(t\theta\sqrt{\frac{\tilde K}{N}}\right)}{\sin\left(\theta\sqrt{\frac{\tilde K}{N}}\right)}\right]^N\\ &= t^N\left[\frac{1-\frac16 t^2\theta^2\frac{\tilde K}{N}+O(\theta^4)}{1-\frac16\theta^2\frac{\tilde K}{N}+O(\theta^4)}\right]^N\\ &= t^N\left[1+\tfrac16(1-t^2)\theta^2\tfrac{\tilde K}{N}+O(\theta^4)\right]^N\\ &= t^N\left[1+\tfrac16(1-t^2)\theta^2\tilde K+O(\theta^4)\right].\end{aligned}$$

Now we choose $\theta^*>0$ in such a way that

$$\tau^{(t)}_{K',N}(\theta)\le\sigma^{(t)}_{\tilde K,N}(\theta)$$

for all $0\le\theta\le\theta^*$ and all $t\in[0,1]$. According to our curvature assumption, each point $x\in\mathsf{M}$ has a neighborhood $M(x)\subseteq\mathsf{M}$ such that every two probability measures $\nu_0,\nu_1\in\mathcal{P}_2(M(x),\mathsf{d},\mathsf{m})$ can be joined by a geodesic in $\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfying (3.1.1). In order to prove that $(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfies $\mathsf{CD}(K',N)$ locally, we set for $x\in\mathsf{M}$,

$$M'(x):=M(x)\cap B_{\theta^*}(x)$$

and consider $\nu_0,\nu_1\in\mathcal{P}_2(M'(x),\mathsf{d},\mathsf{m})$. As indicated above, due to $\mathsf{CD}^*_{\mathsf{loc}}(\tilde K,N)$ there exist an optimal coupling q of $\nu_0=\rho_0\mathsf{m}$ and $\nu_1=\rho_1\mathsf{m}$ and a geodesic $\Gamma:[0,1]\to\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 such that

$$\begin{aligned}&\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m})\le\\ &\quad\le-\int_{\mathsf{M}\times\mathsf{M}}\left[\sigma^{(1-t)}_{\tilde K,N'}(\underbrace{\mathsf{d}(x_0,x_1)}_{\le\theta^*})\rho_0^{-1/N'}(x_0)+\sigma^{(t)}_{\tilde K,N'}(\underbrace{\mathsf{d}(x_0,x_1)}_{\le\theta^*})\rho_1^{-1/N'}(x_1)\right]d\mathsf{q}(x_0,x_1)\\ &\quad\le-\int_{\mathsf{M}\times\mathsf{M}}\left[\tau^{(1-t)}_{K',N'}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N'}(x_0)+\tau^{(t)}_{K',N'}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\right]d\mathsf{q}(x_0,x_1)\end{aligned}$$

for all $t\in[0,1]$ and all $N'\ge N$. $\square$

3.5 Geometric and Functional Analytic Results

3.5.1 Geometric Consequences

The weak versions of the geometric statements derived from $\mathsf{CD}(K,N)$ in [Stu06b] follow by using analogous arguments replacing the coefficients $\tau^{(t)}_{K,N}(\cdot)$ by $\sigma^{(t)}_{K,N}(\cdot)$.

Proposition 3.5.1 (Weak Brunn-Minkowski inequality). *Assume that* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfies the condition* $\mathsf{CD}^*(K, N)$ *for two real parameters* K *and* $N \geq 1$. *Then for all measurable sets* $A_0, A_1 \subseteq \mathsf{M}$ *with* $\mathsf{m}(A_0), \mathsf{m}(A_1) > 0$ *and all* $t \in [0,1]$,

$$\mathsf{m}(Z_t(A_0, A_1)) \geq \sigma_{K,N}^{(1-t)}(\Theta) \cdot \mathsf{m}(A_0)^{1/N} + \sigma_{K,N}^{(t)}(\Theta) \cdot \mathsf{m}(A_1)^{1/N} \tag{3.5.1}$$

where Θ *denotes the minimal/maximal length of geodesics starting in* A_0 *and ending in* A_1,

$$\Theta := \begin{cases} \inf_{x_0 \in A_0, x_1 \in A_1} \mathsf{d}(x_0, x_1), & K \geq 0 \\ \sup_{x_0 \in A_0, x_1 \in A_1} \mathsf{d}(x_0, x_1), & K < 0, \end{cases}$$

and m^* *denotes the outer measure of* m.

Remark 3.5.2. *In the case* $K \geq 0$, *we get back the Euclidean version of the Brunn-Minkowski inequality: For all measurable* $A_0, A_1 \subseteq \mathsf{M}$ *and all* $t \in [0,1]$,

$$\mathsf{m}(Z_t(A_0, A_1)) \geq \mathcal{M}_t^{\frac{1}{N}}\left(\mathsf{m}(A_0), \mathsf{m}(A_1)\right).$$

Proof. Without restriction we may assume that $N > 1$.

(i) Furthermore, we assume that $A_0, A_1 \subseteq \mathsf{M}$ are bounded sets fulfilling $0 < \mathsf{m}(A_0), \mathsf{m}(A_1) < +\infty$. We associate two probability measures ν_0 and ν_1 with A_0 and A_1, respectively, namely

$$\nu_i = \frac{1}{\mathsf{m}(A_i)} I_{A_i} \mathsf{m}$$

for $i = 0, 1$. Applying the condition $\mathsf{CD}^*(K, N)$ to ν_0 and ν_1 yields the existence of a geodesic $\Gamma_t = \rho_t \mathsf{m}$ with $t \in [0,1]$, connecting ν_0 and ν_1 and satisfying

$$\int_{Z_t(A_0, A_1)} \rho_t^{1-1/N} d\mathsf{m} \geq \sigma_{K,N}^{(1-t)}(\Theta) \mathsf{m}(A_0)^{1/N} + \sigma_{K,N}^{(t)}(\Theta) \mathsf{m}(A_1)^{1/N}. \tag{3.5.2}$$

Due to Jensen's inequality the left-hand side of (3.5.2) is bounded from above by $\mathsf{m}\left(Z_t(A_0, A_1)\right)^{1/N}$.

(ii) The general case follows by approximation of A_i for $i = 0, 1$ by bounded sets of finite volume.

$\square$

The Brunn-Minkowski inequality $\mathsf{BM}^*(K,N)$ implies further geometric consequences, for example the Bishop-Gromov volume growth estimate and the Bonnet-Myers theorem.

For a fixed point $x_0 \in \mathsf{supp}(\mathsf{m})$ we study the growth of the volume of closed balls centered at x_0 and the growth of the volume of the corresponding spheres

$$v(r) := \mathsf{m}\left(\overline{B_r(x_0)}\right) \quad \text{and} \quad s(r) := \limsup_{\delta \to 0} \tfrac{1}{\delta} \mathsf{m}\left(\overline{B_{r+\delta}(x_0)} \setminus B_r(x_0)\right),$$

respectively.

Theorem 3.5.3 (Weak Bishop-Gromov volume growth estimate)**.** *If* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies the condition* $\mathsf{CD}^*(K,N)$ *for real* K *and* $N \geq 1$*, then each bounded set* $M_b \subseteq \mathsf{M}$ *has finite volume. Moreover, either* m *is supported by one point, or all points and all spheres have mass 0.*

To be more precise, if $K > 0$ *then for each fixed* $x_0 \in \mathsf{supp}(\mathsf{m})$ *and all* $0 < r < R \leq \pi\sqrt{N/K}$

$$\frac{s(r)}{s(R)} \geq \left(\frac{\sin(r\sqrt{K/N})}{\sin(R\sqrt{K/N})}\right)^N \quad \text{and} \quad \frac{v(r)}{v(R)} \geq \frac{\int_0^r \sin\left(t\sqrt{K/N}\right)^N dt}{\int_0^R \sin\left(t\sqrt{K/N}\right)^N dt}. \tag{3.5.3}$$

In the case $K < 0$*, analogous inequalities hold true (where the right-hand sides of (3.5.3) are replaced by analogous expressions according to the definition of the coefficients* $\sigma_{K,N}^{(t)}(\cdot)$ *for negative* K*). If* $K = 0$ *then*

$$\frac{s(r)}{s(R)} \geq \left(\frac{r}{R}\right)^N \quad \text{and} \quad \frac{v(r)}{v(R)} \geq \left(\frac{r}{R}\right)^{N+1}.$$

Proof. We will content ourselves with deriving the first inequality of (3.5.3) in the case $K > 0$. The parts of the above statement left unproven in this work follow by using analogous arguments as Sturm did in [Stu06b, Theorem 2.3], replacing the coefficients $\tau_{K,N}^{(t)}(\cdot)$ by $\sigma_{K,N}^{(t)}(\cdot)$.

We consider $x_0 \in \mathsf{supp}(\mathsf{m})$. We fix numbers $0 < r < R \leq \pi\sqrt{N/K}$, set $t := r/R$ and choose $\varepsilon > 0$ as well as $\delta > 0$. Our aim is to apply the Brunn-Minkowski inequality $\mathsf{BM}^*(K,N)$ to $A_0 := B_\varepsilon(x_0)$ and $A_1 := \overline{B_{R+\delta R}(x_0)} \setminus B_R(x_0)$. The set $Z_t(A_0,A_1)$ of t-intermediate points of A_0 and A_1 satisfies

$$Z_t(A_0,A_1) \subseteq \overline{B_{t(R+\delta R+\varepsilon)}(x_0)} \setminus B_{t(R-\varepsilon)}(x_0)$$

and the minimal distance Θ between A_0 and A_1 satisfies $R-\varepsilon \le \Theta \le R+\delta R+\varepsilon$. Therefore, Proposition 3.5.1 implies that

$$\begin{aligned}
&\mathsf{m}\left(\overline{B_{r+\delta r+\varepsilon r/R}(x_0)} \setminus B_{r-\varepsilon r/R}(x_0)\right)^{1/N} \ge \\
&\quad\ge \sigma_{K,N}^{(1-r/R)}(R-\delta R-\varepsilon)\mathsf{m}\left(B_\varepsilon(x_0)\right)^{1/N} + \\
&\qquad\qquad + \sigma_{K,N}^{(r/R)}(R-\delta R-\varepsilon)\mathsf{m}\left(\overline{B_{R+\delta R}(x_0)} \setminus B_R(x_0)\right)^{1/N}.
\end{aligned}$$

In the limit $\varepsilon \to 0$, we have

$$\begin{aligned}
&\mathsf{m}\left(\overline{B_{(1+\delta)r}(x_0)} \setminus B_r(x_0)\right) \ge \\
&\quad\ge \Bigg(\sigma_{K,N}^{(1-r/R)}((1-\delta)R)\mathsf{m}\left(\{x_0\}\right)^{1/N} + \\
&\qquad\qquad + \sigma_{K,N}^{(r/R)}((1-\delta)R)\mathsf{m}\left(\overline{B_{(1+\delta)R}(x_0)} \setminus B_R(x_0)\right)^{1/N}\Bigg)^N \\
&\quad\ge \sigma_{K,N}^{(r/R)}((1-\delta)R)^N \mathsf{m}\left(\overline{B_{(1+\delta)R}(x_0)} \setminus B_R(x_0)\right).
\end{aligned} \tag{3.5.4}$$

Inequality (3.5.4) can be restated as

$$\begin{aligned}
&\frac{1}{\delta r}\mathsf{m}\left(\overline{B_{(1+\delta)r}(x_0)} \setminus B_r(x_0)\right) \ge \\
&\qquad\ge \frac{1}{\delta R}\mathsf{m}\left(\overline{B_{(1+\delta)R}(x_0)} \setminus B_R(x_0)\right)\left(\frac{\sin((1-\delta)r\sqrt{K/N})}{\sin((1-\delta)R\sqrt{K/N})}\right)^N.
\end{aligned}$$

Passing to the lim sup for $\delta \to 0$ yields (3.5.3). □

Corollary 3.5.4 (Doubling)**.** *For each metric measure space* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfying the condition* $\mathsf{CD}^*(K,N)$ *for* $K,N \in \mathbb{R}$ *with* $N \ge 1$ *the doubling property holds true on each bounded set* $M_b \subseteq \mathsf{supp}(\mathsf{m})$*. Particularly, each bounded closed subset* $M_{b,c} \subseteq \mathsf{supp}(\mathsf{m})$ *is compact. In the case* $K \ge 0$*, the doubling constant* C *satisfies* $C \le 2^{N+1}$*. Otherwise, it can be estimated in terms of* K*,* N *and the diameter* L *of* M_b *as follows*

$$C \le 2^{N+1}\cosh\left(L\sqrt{\frac{-K}{N}}\right)^N.$$

Proof. The doubling property can be deduced from the second inequality in (3.5.3) of Theorem 3.5.3: In the case $K<0$ it yields,

$$\frac{\mathsf{m}\left(B_{2r}(x)\right)}{\mathsf{m}\left(B_r(x)\right)} \le \frac{2\int_0^r \sinh\left(2t\sqrt{-K/N}\right)^N dt}{\int_0^r \sinh\left(t\sqrt{-K/N}\right)^N dt} \le 2^{N+1}\cosh\left(r\sqrt{-K/N}\right)^N.$$

The doubling property itself implies the compactness of bounded and closed sets $M_{b,c} \subseteq \mathsf{supp}(\mathsf{m})$. We refer to [Stu06a]. $\square$

Remark 3.5.5. *Using similar arguments, one can derive that a metric measure space* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfying* $\mathsf{CD}^*_{\mathsf{loc}}(K, N)$ *has a locally compact support.*

Corollary 3.5.6 (Weak Bonnet-Myers theorem)**.** *Fix two real parameters* $K > 0$ *and* $N \geq 1$*. Each metric measure space* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfying the condition* $\mathsf{CD}^*(K, N)$ *has compact support and its diameter* L *has an upper bound,*

$$L \leq \pi \sqrt{\frac{N}{K}}.$$

Proof. Assuming that $L > \pi\sqrt{N/K}$, we choose $\varepsilon > 0$ and $x_0, x_1 \in \mathsf{supp}(\mathsf{m})$ with $\mathsf{d}(x_0, x_1) \geq \pi\sqrt{N/K} + 4\varepsilon$ and $0 < \mathsf{m}\left(B_\varepsilon(x_i)\right) < \infty$ for $i = 0, 1$. We set $A_i := B_\varepsilon(x_i)$ for $i = 0, 1$. Then there exists a finite radius $R > 0$ with $Z_{1/2}(A_0, A_1) \subseteq B_R(x_0)$. The Brunn-Minkowski inequality $\mathsf{BM}^*(K, N)$ with parameter $\Theta > \pi\sqrt{N/K}$ (here Θ is to be understood as the minimal distance between elements in A_0 and A_1) implies that $\mathsf{m}\left(Z_{1/2}(A_0, A_1)\right) = \infty$, whereas Theorem 3.5.3 yields that $\mathsf{m}\left(B_R(x_0)\right) < \infty$. This contradiction shows that $\mathsf{d}(x_0, x_1) \leq \pi\sqrt{N/K}$ for all $x_0, x_1 \in \mathsf{supp}(\mathsf{m})$. The boundedness of the support implies its compactness due to Corollary 3.5.4. $\square$

Note that in the sharp version of this estimate the factor N is replaced by $N - 1$.

3.5.2 Lichnerowicz Estimate and Poincaré Inequality

In this subsection we follow the presentation of Lott and Villani in [LV07]. We denote by $\mathsf{Lip}(\mathsf{M})$ the set of Lipschitz functions $f : \mathsf{M} \to \mathbb{R}$.

Definition 3.5.7. *Given* $f \in \mathsf{Lip}(\mathsf{M})$*, we define* $|\nabla^- f|$ *by*

$$|\nabla^- f|(x) := \limsup_{y \to x} \frac{[f(y) - f(x)]_-}{\mathsf{d}(x, y)}$$

where for $a \in \mathbb{R}$*,* $a_- := \max(-a, 0)$*.*

Theorem 3.5.8. *Let* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfy* $\mathsf{CD}^*(K, N)$ *for two real parameters* $K > 0$ *and* $N \geq 1$*. Then for each positive Lipschitz function* $\rho_0 \in \mathsf{Lip}(\mathsf{M})$ *with* $\int_{\mathsf{M}} \rho_0 d\mathsf{m} = 1$ *it holds that*

$$N + N\mathsf{S}_N(\rho_0 \mathsf{m}|\mathsf{m}) \leq \int_{\mathsf{M}} \theta^{(K,N)}(\rho_0, |\nabla^- \rho_0|) d\mathsf{m},$$

where for $r, g \geq 0$,

$$\theta^{(K,N)}(r,g) := r \sup_{\alpha \in [0,\pi]} \Bigg[\frac{N-1}{N} \frac{g}{r^{1+1/N}} \sqrt{\frac{N}{K}} \alpha + N \left(1 - \left(\frac{\alpha}{\sin(\alpha)} \right)^{1-1/N} \right) + \\ + (N-1) \left(\frac{\alpha}{\tan(\alpha)} - 1 \right) r^{-1/N} \Bigg].$$

Lemma 3.5.9. *We have for* $x \in [0, \pi]$,

$$\frac{x}{\tan(x)} \leq 1 - \frac{x^2}{3} \quad \text{and} \quad 1 - \left(\frac{x}{\sin(x)} \right)^{1-1/N} \leq - \left(1 - \frac{1}{N} \right) \frac{x^2}{6}.$$

For the proofs of Theorem 3.5.8 and Lemma 3.5.9, respectively, we refer to [LV07, Theorem 5.3 and Lemma 5.13].

Theorem 3.5.10 (Sobolev-type inequality)**.** *Let* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *fulfill* $\mathsf{CD}^*(K, N)$ *for two real parameters* $K > 0$ *and* $N \geq 1$. *Then for each non-negative Lipschitz function* $\rho_0 \in \mathsf{Lip}(\mathsf{M})$ *with* $\int_\mathsf{M} \rho_0 d\mathsf{m} = 1$ *it holds that*

$$N + N \mathsf{S}_N(\rho_0 \mathsf{m} | \mathsf{m}) \leq \frac{1}{2K} \frac{N-1}{N} \int_\mathsf{M} \frac{\rho_0^{-1-2/N}}{\frac{1}{3} + \frac{2}{3} \rho_0^{-1/N}} |\nabla^- \rho_0|^2 d\mathsf{m}.$$

Proof. Due to Lemma 3.5.9, the function $\theta^{(K,N)}$ appearing in Theorem 3.5.8 is bounded from above by

$$\begin{aligned} \theta^{(K,N)}(r,g) &\leq r \sup_{\alpha \in [0,\pi]} \left[\frac{N-1}{N} \frac{g}{r^{1+1/N}} \sqrt{\frac{N}{K}} \alpha - \frac{N-1}{6} \alpha^2 \left(1 + 2r^{-1/N} \right) \right] \\ &\leq r \sup_{\alpha \in [0,\pi]} \left[\frac{\frac{1}{4} \left(\frac{N-1}{N} \right)^2 \frac{N}{K} r^{-2-2/N} g^2 \alpha^2}{\frac{N-1}{2} \left(\frac{1}{3} + \frac{2}{3} r^{-1/N} \right) \alpha^2} \right] \\ &= \left[\frac{1}{2K} \frac{N-1}{N} \frac{r^{-1-2/N}}{\frac{1}{3} + \frac{2}{3} r^{-1/N}} \right] \cdot g^2 \end{aligned}$$

for all $r, g \geq 0$. Applying Theorem 3.5.8 in combination with the above estimate finishes the proof. □

Theorem 3.5.11 (Lichnerowicz estimate, Poincaré inequality)**.** *Let* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfy* $\mathsf{CD}^*(K, N)$ *for two real parameters* $K > 0$ *and* $N \geq 1$. *Then for every* $f \in \mathsf{Lip}(\mathsf{M})$ *fulfilling* $\int_\mathsf{M} f \, d\mathsf{m} = 0$ *the following inequality holds true,*

$$\int_\mathsf{M} f^2 d\mathsf{m} \leq \frac{1}{K} \int_\mathsf{M} |\nabla^- f|^2 d\mathsf{m}. \tag{3.5.5}$$

Proof. Without restriction we may assume that $\max_{x\in\mathsf{M}}|f(x)| \le 1$. Given $\varepsilon \in (-1,1)$, we define $\rho_0 := 1+\varepsilon f$. Then $\rho_0 > 0$ and $\int_{\mathsf{M}} \rho_0 \, d\mathsf{m} = 1$. For small ε, it holds that

$$N + \mathsf{S}_N(\rho_0\mathsf{m}|\mathsf{m}) = \varepsilon^2 \frac{N-1}{N} \int_{\mathsf{M}} f^2 d\mathsf{m} + O(\varepsilon^3)$$

and

$$\frac{1}{2K}\frac{N-1}{N}\int_{\mathsf{M}} \frac{\rho_0^{-1-2/N}}{\frac{1}{3}+\frac{2}{3}\rho_0^{-1/N}} |\nabla^- \rho_0|^2 d\mathsf{m} = \frac{\varepsilon^2}{2K}\frac{N-1}{N}\int_{\mathsf{M}} |\nabla^- f|^2 d\mathsf{m} + O(\varepsilon^3).$$

The result can be deduced from Theorem 3.5.10. □

Remark 3.5.12. *In 'regular' cases, $\mathcal{E}(f,f) := \int_{\mathsf{M}} |\nabla^- f|^2 \, d\mathsf{m}$ is a quadratic form which – by polarization – then defines uniquely a bilinear form $\mathcal{E}(f,g)$ and a self-adjoint operator L ('generalized Laplacian') through the identity $\mathcal{E}(f,g) = -\int_{\mathsf{M}} f \cdot Lg \, d\mathsf{m}$.*

The Inequality (3.5.5) means that L admits a spectral gap $\lambda_1(-L)$ of size at least K,

$$\lambda_1(-L) \ge K.$$

In the sharp version, corresponding to the case where $(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfies $\mathsf{CD}(K,N)$, the spectral gap is bounded from below by $K\frac{N}{N-1}$.

3.6 Universal Coverings of Metric Measure Spaces

3.6.1 Coverings and Liftings

Let us recall some basic definitions and properties of coverings of metric (or more generally, topological) spaces. For further details we refer to [BBI01].

Definition 3.6.1. *Let X be a topological space.*

(i) We call X pathwise connected if and only if for every two points $x, y \in X$ there exists a curve connecting them.

(ii) X is called simply connected if and only if X is pathwise connected and for every point $x_0 \in X$ the fundamental group is trivial: $\pi_1(X, x_0) = \{1\}$.

(iii) X is locally pathwise connected if and only if for every point $x \in X$ and every neighborhood $U \subseteq X$ of x there is a smaller neighborhood $x \in V \subseteq U$ such that every two points $v, w \in V$ can be connected by a curve in U.

(iv) *X is called semi-locally simply connected if and only if every point $x_0 \in X$ has a neighborhood $U \subseteq X$ such that the image of the fundamental group $\pi_1(U, x_0)$ under the homomorphism $\pi_1(U, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $U \hookrightarrow X$ is trivial.*

Definition 3.6.2 (Covering). *(i) Let E and X be topological spaces and $p : E \to X$ a continuous map. An open set $V \subseteq X$ is said to be evenly covered by p if and only if its inverse image $p^{-1}(V)$ is a disjoint union of sets $U_i \subseteq E$ such that the restriction of p to U_i is a homeomorphism from U_i to V for each i in a suitable indexset I. The map p is a covering map (or simply covering) if and only if every point $x \in X$ has an evenly covered neighborhood. In this case, X is called the base of the covering and E the covering space.*

(ii) A covering map $p : E \to X$ is called a universal covering if and only if E is simply connected. In this case, E is called universal covering space for X.

The existence of a universal covering is guaranteed under some weak topological assumptions. More precisely:

Theorem 3.6.3. *If a topological space X is connected, locally pathwise connected and semi-locally simply connected, then there exists a universal covering $p : E \to X$.*

Example 3.6.4. *(i) The map $p : \mathbb{R} \to \mathsf{S}^1$ given by $p(x) = (\cos(x), \sin(x))$ is a covering map.*

(ii) The universal covering of the torus by the plane $P : \mathbb{R}^2 \to \mathsf{T}^2 := \mathsf{S}^1 \times \mathsf{S}^1$ is given by $P(x, y) := (p(x), p(y))$ where $p(x) = (\cos(x), \sin(x))$ is defined as in (i).

We consider a covering $p : E \to X$. For $x \in X$ the set $p^{-1}(x)$ is called the *fiber* over x. This is a discrete subspace of E and every $x \in X$ has a neighborhood V such that $p^{-1}(V)$ is homeomorphic to $p^{-1}(x) \times V$. The disjoint subsets of $p^{-1}(V)$ mapped homeomorphically onto V are called the *sheets* of $p^{-1}(V)$. If V is connected, the sheets of $p^{-1}(V)$ coincide with the connected components of $p^{-1}(V)$. If E and X are connected, the cardinality of $p^{-1}(x)$ does not depend on $x \in X$ and is called the *number of sheets*. This number may be infinity.

Every covering is a local homeomorphism which implies that E and X have the same local topological properties.

Remark 3.6.5. *Consider length spaces (E, d_E) and (X, d_X) and a covering map $p : E \to X$ which is additionally a local isometry. If X is complete, then so is E.*

We list two essential lifting statements in topology referring to [BS] for further details and the proofs.

Definition 3.6.6. *Let $\alpha, \beta : [0,1] \to X$ be two curves in X with the same end points meaning that $\alpha(0) = \beta(0) = x_0 \in X$ and $\alpha(1) = \beta(1) = x_1 \in X$. We say that α and β are homotopic relative to $\{0,1\}$ if and only if there exists a continuous map $H : [0,1] \times [0,1] \to X$ satisfying $H(t,0) = \alpha(t)$, $H(t,1) = \beta(t)$ as well as $H(0,t) = x_0$ and $H(1,t) = x_1$ for all $t \in [0,1]$. We call H a homotopy from α to β relative to $\{0,1\}$.*

Theorem 3.6.7 (Path lifting theorem)**.** *Let $p : E \to X$ be a covering and let $\gamma : [0,1] \to X$ be a curve in X. We assume that $e_0 \in E$ satisfies $p(e_0) = \gamma(0)$. Then there exists a unique curve $\alpha : [0,1] \to E$ such that $\alpha(0) = e_0$ and $p \circ \alpha = \gamma$.*

Theorem 3.6.8 (Homotopy lifting theorem)**.** *Let $p : E \to X$ be a covering, let $\gamma_0, \gamma_1 : [0,1] \to X$ be two curves in X with starting point $x_0 \in X$ and terminal point $x_1 \in X$, and let $\alpha_0, \alpha_1 : [0,1] \to E$ be the lifted curves such that $\alpha_0(0) = \alpha_1(0)$. Then every homotopy $H : [0,1] \times [0,1] \to X$ from γ_0 to γ_1 relative to $\{0,1\}$ can be lifted in a unique way to a homotopy $H' : [0,1] \times [0,1] \to E$ from α_0 to α_1 relative to $\{0,1\}$ satisfying $H'(0,0) = \alpha_0(0) = \alpha_1(0)$.*

We consider a universal covering $p : E \to X$ and distinguished points $x_0 \in X$ as well as $e_0 \in p^{-1}(x_0) \subseteq E$. The above lifting theorems enable us to define a function

$$\Phi : \pi_1(X, x_0) \to p^{-1}(x_0)$$

such that for $[\gamma] \in \pi_1(X, x_0)$, $\Phi([\gamma])$ is the (unique) terminal point of the lift of γ to E starting at e_0. Then Φ has the following property:

Theorem 3.6.9 (Cardinality of fibers)**.** *The function Φ is a one-to-one correspondence of the fundamental group $\pi_1(X, x_0)$ and the fiber $p^{-1}(x_0)$.*

3.6.2 Lifted Metric Measure Spaces

We consider now a non-branching metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ locally for two real parameters $K > 0$ and $N \geq 1$ and a distinguished point $x_0 \in \mathsf{M}$. Moreover, we assume that (M, d) is a semi-locally simply connected length space. Then, according to Theorem 3.6.3, there exists a universal covering $p : \hat{\mathsf{M}} \to \mathsf{M}$. The covering space $\hat{\mathsf{M}}$ inherits the length structure of the base M in the following way: We say that a curve $\hat{\gamma}$ in

$\hat{\mathsf{M}}$ is 'admissible' if and only if its composition with p is a continuous curve in M. The length $\mathsf{L}(\hat{\gamma})$ of an admissible curve in $\hat{\mathsf{M}}$ is set to the length of $p \circ \hat{\gamma}$ with respect to the length structure in M. For two points $x, y \in \hat{\mathsf{M}}$ we define the associated distance $\mathsf{d_c}(x, y)$ between them to be the infimum of lengths of admissible curves in $\hat{\mathsf{M}}$ connecting these points:

$$\mathsf{d_c}(x,y) := \inf\{\mathsf{L}(\hat{\gamma}) | \hat{\gamma} : [0,1] \to \hat{\mathsf{M}} \text{ admissible}, \hat{\gamma}(0) = x, \hat{\gamma}(1) = y\}. \tag{3.6.1}$$

Endowed with this metric, $p : (\hat{\mathsf{M}}, \mathsf{d_c}) \to (\mathsf{M}, \mathsf{d})$ is a local isometry.

Now, let ξ be the family of all sets $\hat{E} \subseteq \hat{\mathsf{M}}$ such that the restriction of p onto $\hat{E}$ is a local isometry from $\hat{E}$ to a measurable set $E := p(\hat{E})$ in M. This family ξ is stable under intersections, and the smallest σ-algebra $\sigma(\xi)$ containing ξ is equal to the Borel-σ-algebra $\mathcal{B}(\hat{\mathsf{M}})$ according to the local compactness of $(\hat{\mathsf{M}}, \mathsf{d_c})$. We define a function $\hat{\mathsf{m}} : \xi \to [0, \infty[$ by $\hat{\mathsf{m}}(\hat{E}) = \mathsf{m}(p(\hat{E})) = \mathsf{m}(E)$ and extend it in a unique way to a measure $\hat{\mathsf{m}}$ on $(\hat{\mathsf{M}}, \mathcal{B}(\hat{\mathsf{M}}))$.

Definition 3.6.10. *(i) We call the metric $\mathsf{d_c}$ on $\hat{\mathsf{M}}$ defined in (3.6.1) the lift of the metric d on M.*

(ii) The measure $\hat{\mathsf{m}}$ on $(\hat{\mathsf{M}}, \mathcal{B}(\hat{\mathsf{M}}))$ constructed as described above is called the lift of m.

(iii) We call the metric measure space $(\hat{\mathsf{M}}, \mathsf{d_c}, \hat{\mathsf{m}})$ the lift of $(\mathsf{M}, \mathsf{d}, \mathsf{m})$.

Theorem 3.6.11 (Lift)**.** *Assume that $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is a non-branching metric measure space satisfying $\mathsf{CD}^*_{\mathsf{loc}}(K, N)$ for two real parameters $K > 0$ and $N \geq 1$ and that (M, d) is a semi-locally simply connected length space. Let $\hat{\mathsf{M}}$ be a universal covering space for M and let $(\hat{\mathsf{M}}, \mathsf{d_c}, \hat{\mathsf{m}})$ be the lift of $(\mathsf{M}, \mathsf{d}, \mathsf{m})$. Then,*

(i) $(\hat{\mathsf{M}}, \mathsf{d_c}, \hat{\mathsf{m}})$ has compact support and its diameter has an upper bound

$$\mathsf{diam}(\hat{\mathsf{M}}) \leq \pi \sqrt{\frac{N}{K}}.$$

(ii) The fundamental group $\pi_1(\mathsf{M}, x_0)$ of $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is finite.

Proof. (i) Due to the construction of the lift, the local properties of $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ are transferred to $(\hat{\mathsf{M}}, \mathsf{d_c}, \hat{\mathsf{m}})$. That means, $(\hat{\mathsf{M}}, \mathsf{d_c}, \hat{\mathsf{m}})$ is a non-branching metric measure space $(\hat{\mathsf{M}}, \mathsf{d_c}, \hat{\mathsf{m}})$ satisfying $\mathsf{CD}^*(K, N)$ locally. Theorem 3.4.1 implies that $(\hat{\mathsf{M}}, \mathsf{d_c}, \hat{\mathsf{m}})$ satisfies $\mathsf{CD}^*(K, N)$ globally and therefore, the diameter estimate of Bonnet-Myers – Corollary 3.5.6 – can be applied.

(ii) If the fundamental group $\pi_1(\mathsf{M}, x_0)$ were infinite then $\hat{\mathsf{M}}$ could not be compact according to Theorem 3.6.9.

$\square$

At the end of this chapter we arrive at the following chain of equivalent relations:

$$\mathsf{CD}_{\mathsf{loc}}(K-, N) \Leftrightarrow \mathsf{CD}^*_{\mathsf{loc}}(K-, N) \Leftrightarrow \mathsf{CD}^*(K, N).$$

The question whether the equivalence '$\mathsf{CD}(K, N) \Leftrightarrow \mathsf{CD}_{\mathsf{loc}}(K, N)$' holds or not or equivalently, whether the implication '$\mathsf{CD}^*(K, N) \Rightarrow \mathsf{CD}(K, N)$' is true or not remains unanswered in this chapter. It is still an open problem.

Chapter 4

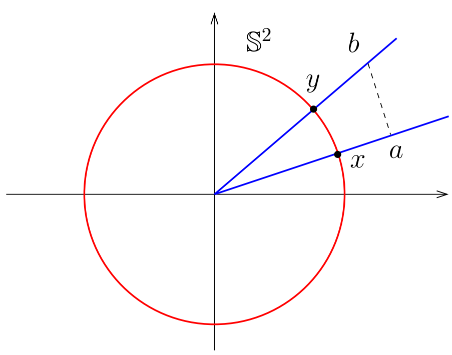
Cones over Metric Measure Spaces

4.1 Euclidean Cones

4.1.1 Euclidean Cones over Metric Spaces

The *Euclidean cone* $\mathsf{Con}(X)$ over a topological space X is the quotient of the product $X \times [0, \infty)$ obtained by identifying all points in the fiber $X \times \{0\}$. This point is called the *origin* of the cone.

A question which might arise is how a reasonable notion of distance on $\mathsf{Con}(X)$ can look like, if we start with a metric space (X, d) instead of a general topological one. In order to answer this question, we will have a look on the most prominent example in this setting: We consider the unit sphere $\mathbb{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3$ endowed with the spherical angular metric $\mathsf{d}_\measuredangle$ that is, the distance between two points in $\mathbb{S}^2$ is given by the Euclidean angle between them. To construct the Euclidean cone over $\mathbb{S}^2$, we draw a ray from the origin 0 in $\mathbb{R}^3$ through every point $x \in \mathbb{S}^2$. A point $a \in \mathsf{Con}(\mathbb{S}^2)$ can be described by a pair (x, t) where x is a point in $\mathbb{S}^2$ belonging to the ray $0a$ and $t = |a|$ is the Euclidean distance from the origin. By this construction we obtain the whole Euclidean space $\mathbb{R}^3$.



There is a way to describe the Euclidean distance $|a-b|$ between two points $a=(x,t)$ and $b=(y,s)$ in $\mathsf{Con}(\mathbb{S}^2)$ in terms of the angular metric $\mathsf{d}_\measuredangle$ and the lengths t and s. This relation is given by the cosine formula: We consider the triangle $\triangle 0ab$. We have $|a|=t$, $|b|=s$ and the angle $\measuredangle a0b$ coincides with the angular distance $\mathsf{d}_\measuredangle(x,y)$ in $\mathbb{S}^2$. Thus, by the cosine formula

$$|a-b| = \sqrt{t^2+s^2-2ts\cos(\mathsf{d}_\measuredangle(x,y))}.$$

This formula can be used to define Euclidean cone distances for general metric spaces (X,d). We just have to replace the angular metric $\mathsf{d}_\measuredangle$ by the abstract one d:

Definition 4.1.1. *Let (X,d) be a metric space with $\mathsf{diam}(X)\le\pi$. The Euclidean cone distance $\mathsf{d}_{\mathsf{Con}}$ on $\mathsf{Con}(X)$ is given by the formula*

$$\mathsf{d}_{\mathsf{Con}}(p,q) = \sqrt{t^2+s^2-2ts\cos(\mathsf{d}(x,y))},$$

where $p,q\in\mathsf{Con}(X)$, $p=(x,t)$, $q=(y,s)$.

Proposition 4.1.2. *If (X,d) is a metric space with $\mathsf{diam}(X)\le\pi$, then $\mathsf{d}_{\mathsf{Con}}$ is a metric on $\mathsf{Con}(X)$. Moreover, $(\mathsf{Con}(X),\mathsf{d}_{\mathsf{Con}})$ is geodesic if and only if (X,d) is geodesic.*

4.1.2 Euclidean Cones over Large Spaces

We would like to get rid of the assumption that $\mathsf{diam}(X)\le\pi$. The above definition of $\mathsf{d}_{\mathsf{Con}}$ is useless for a notion of distance on $\mathsf{Con}(X)$ in the framework of metric spaces (X,d) with $\mathsf{diam}(X)>\pi$ because the triangle inequality possibly fails on $(\mathsf{Con}(X),\mathsf{d}_{\mathsf{Con}})$. In this setting there are two reasonable requirements on a notion of distance on $\mathsf{Con}(X)$: Equipped with it, $\mathsf{Con}(X)$ shall form a metric space and the above formula shall hold for 'small' distances in X. These demands are satisfied by the following definition:

Definition 4.1.3. *Consider a metric space (X,d). The Euclidean cone distance $\mathsf{d}_{\mathsf{Con}}(p,q)$ between two points $p=(x,t)$, $q=(y,s)$ in $\mathsf{Con}(X)$ is defined as*

$$\mathsf{d}_{\mathsf{Con}}(p,q) := \sqrt{t^2+s^2-2ts\cos(\min\{\mathsf{d}(x,y),\pi\})}.$$

Theorem 4.1.4. *If (X,d) is a metric space, then $\mathsf{d}_{\mathsf{Con}}$ is a metric on $\mathsf{Con}(X)$.*

For the proofs of Proposition 4.1.2 and Theorem 4.1.4 as well as for more detailed information about Euclidean cones over metric spaces we refer to [BBI01].

4.1.3 N-Euclidean Cones over Metric Measure Spaces

We fix an arbitrary $N \geq 1$. For a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ the *N-Euclidean cone* $(\mathsf{Con}(\mathsf{M}), \mathsf{d}_{\mathsf{Con}}, \nu)$ is a metric measure space defined as follows:

- ◇ $\mathsf{Con}(\mathsf{M}) := \mathsf{M} \times [0, \infty)/\mathsf{M} \times \{0\}$
- ◇ For $(x, s), (x', t) \in \mathsf{M} \times [0, \infty)$

$$\mathsf{d}_{\mathsf{Con}}((x, s), (x', t)) := \sqrt{s^2 + t^2 - 2st \cos\left(\min\{\mathsf{d}(x, x'), \pi\}\right)}$$

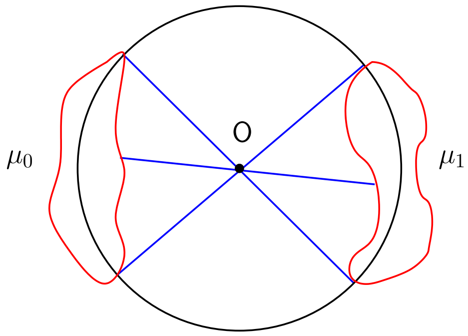
- ◇ $d\nu(x, s) := d\mathsf{m}(x) \otimes s^N ds$.

The first part of this chapter is devoted to the question whether a metric measure space hands the curvature-dimension condition on to its Euclidean cone in a certain manner known from the setting of spheres: We recall that the n-Euclidean cone $\mathsf{Con}(\mathbb{S}^n)$ over the sphere $\mathbb{S}^n$, an example of an n-dimensional Riemannian manifold with Ricci curvature bounded from below by $n - 1$, is the Euclidean space $\mathbb{R}^{n+1}$, an $(n + 1)$-dimensional Riemannian manifold with Ricci curvature bounded from below by 0. Due to this observation and a generalization by Cheeger and Taylor [CT82, Che83] in the framework of punctured Euclidean cones over Riemannian manifolds, we dare to say that '$\mathsf{CD}(N - 1, N) \rightsquigarrow \mathsf{CD}(0, N + 1)$' holds true generally, meaning that the N-Euclidean cone $(\mathsf{Con}(\mathsf{M}), \mathsf{d}_{\mathsf{Con}}, \nu)$ over a general metric measure spaces $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying $\mathsf{CD}(N - 1, N)$ fulfills $\mathsf{CD}(0, N + 1)$ – a conjecture which remains unproven in this chapter. Nevertheless, a satisfactory step in the setting of Riemannian manifolds is made in Section 4.3.

4.2 Optimal Transport on Euclidean Cones

In order to approach the question whether '$\mathsf{CD}(N - 1, N) \rightsquigarrow \mathsf{CD}(0, N + 1)$' holds true or not, we have to study optimal transport on Euclidean cones. Therefore, we consider a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ with full support $\mathsf{M} = \mathsf{supp}(\mathsf{m})$ satisfying the curvature-dimension condition $\mathsf{CD}(N - 1, N)$ for some $N \geq 1$. The diameter estimate by Bonnet-Myers implies that $\mathsf{diam}(\mathsf{M}) \leq \pi$. We denote by $(\mathsf{Con}(\mathsf{M}), \mathsf{d}_{\mathsf{Con}}, \nu)$ the N-Euclidean cone over $(\mathsf{M}, \mathsf{d}, \mathsf{m})$. For each pair of probability measures μ_0 and μ_1 in $\mathcal{P}_2(\mathsf{Con}(\mathsf{M}), \mathsf{d}_{\mathsf{Con}}, \nu)$ there exists a geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(\mathsf{Con}(\mathsf{M}), \mathsf{d}_{\mathsf{Con}})$ connecting them. The probability measures $\Gamma(t)$ for $0 < t < 1$ are not necessarily absolutely continuous because $\mathcal{P}_2(\mathsf{Con}(\mathsf{M}), \mathsf{d}_{\mathsf{Con}}, \nu)$ is not necessarily a geodesic space.

Thus, theoretically, it might happen that all mass is transported from μ_0 to μ_1 through the origin.



But due to Theorem 4.2.1, this phenomenon does not occur. We consider the partition $0 = t_0 < t_{1/2} = \frac{1}{2} < t_1 = 1$ of $[0,1]$. Due to Theorem 1.2.3, there exists a probability measure $\hat{\mathsf{q}}$ on $\mathsf{Con}(\mathsf{M})^3$ with the following properties:

* The projection on the i-th factor is $\Gamma(t_i)$ for all $i = 0, \frac{1}{2}, 1$.

* For $\hat{\mathsf{q}}$-almost every $x = (x_0, x_{1/2}, x_1) \in \mathsf{Con}(\mathsf{M})^3$, the point $x_{1/2}$ is a midpoint of x_0 and x_1. In particular, the projection on the i-th and j-th factor is an optimal coupling of $\Gamma(t_i)$ and $\Gamma(t_j)$ for $i, j = 0, \frac{1}{2}, 1$.

In the sequel we use the notation $\mathsf{O} := \mathsf{M} \times \{0\} \in \mathsf{Con}(\mathsf{M})$. The following theorem states that the optimal transport from μ_0 to μ_1 does not touch the origin.

Theorem 4.2.1. *It holds that*

$$\hat{\mathsf{q}}\left(\{(x_0, x_{1/2}, x_1) \in \mathsf{Con}(\mathsf{M})^3 : x_{1/2} = \mathsf{O}\}\right) = 0.$$

Proof. This proof is divided into three parts: Each part starts with the formulation of a lemma.

Lemma 4.2.2. *Let two points $x_0 = (\phi_0, r)$ and $x_1 = (\phi_1, s)$ in $\mathsf{Con}(\mathsf{M})$ be given and let $\gamma : [0,1] \to \mathsf{Con}(\mathsf{M})$ be a geodesic connecting them, meaning $\gamma(0) = x_0$ and $\gamma(1) = x_1$. If $\gamma_{1/2} := \gamma(\frac{1}{2}) = \mathsf{O}$, then ϕ_0 and ϕ_1 are antipodes in M in the sense that $\mathsf{d}(\phi_0, \phi_1) = \pi$.*

Proof of Lemma 4.2.2. Due to the definition of $\mathsf{d}_{\mathsf{Con}}$, we have first of all

$$r = \mathsf{d}_{\mathsf{Con}}(x_0, \gamma_{1/2}) = \tfrac{1}{2}\mathsf{d}_{\mathsf{Con}}(x_0, x_1) = \mathsf{d}_{\mathsf{Con}}(\gamma_{1/2}, x_1) = s,$$

and secondly,

$$r^2 = \mathsf{d}_{\mathsf{Con}}(x_0, \gamma_{1/2})^2 = \frac{1}{4}\mathsf{d}_{\mathsf{Con}}(x_0, x_1)^2$$
$$= \tfrac{1}{4}\left[2r^2 - 2r^2\cos(\mathsf{d}(\phi_0, \phi_1))\right] = \tfrac{1}{2}\left[1 - \cos(\mathsf{d}(\phi_0, \phi_1))\right] r^2$$

which implies that

$$\cos(\mathsf{d}(\phi_0, \phi_1)) = -1.$$

Due to [Stu06b, Corollary 2.6], the generalized Bonnet-Myers theorem on diameter bounds of metric measure spaces yields

$$\mathsf{d}(\phi_0, \phi_1) \leq \pi.$$

Therefore, we conclude that $\mathsf{d}(\phi_0, \phi_1) = \pi$. □

Lemma 4.2.3 (Ohta)**.** *The set* $\mathsf{S}(\phi, \pi) := \{\phi_a \in \mathsf{M} : \mathsf{d}(\phi, \phi_a) = \pi\}$ *of antipodes of* ϕ *consists of at most one point for every* $\phi \in \mathsf{M}$*.*

For a proof of Lemma 4.2.3, we refer to [Oht07a, Theorem 4.5].

Lemma 4.2.4. *Either* $\{(x_0, x_{1/2}, x_1) \in \mathsf{supp}(\hat{\mathsf{q}}) : x_{1/2} = \mathsf{O}\}$ *is the empty set or it coincides with* $\{(\mathsf{O}, \mathsf{O}, \mathsf{O})\}$ *or there exists at most one pair* (ϕ_0, ϕ_1) *of antipodes in* M *with the following property: If* $(\mathsf{O}, \mathsf{O}, \mathsf{O}) \neq a = (a_0, a_{1/2}, a_1) \in \mathsf{supp}(\hat{\mathsf{q}}) \subseteq \mathsf{Con}(\mathsf{M})^3$ *satisfies* $a_{1/2} = \mathsf{O}$ *then* $a_0 = (\phi_0, r)$ *and* $a_1 = (\phi_1, r)$ *for some* $r \in (0, \infty)$*.*

Proof of Lemma 4.2.4. We assume that there are two different pairs (ϕ_0, ϕ_1) and (φ_0, φ_1) of antipodes in M such that there exist $a = (a_0, a_{1/2}, a_1), b = (b_0, b_{1/2}, b_1) \in \mathsf{supp}(\hat{\mathsf{q}})$ fulfilling $a_{1/2} = \mathsf{O} = b_{1/2}$ as well as $a_i = (\phi_i, r)$ and $b_i = (\varphi_i, s)$ for $i = 0, 1$ and some $r, s \in (0, \infty)$. We denote by q the projection of $\hat{\mathsf{q}}$ on the first and third factor, formally

$$\mathsf{q} := (\mathsf{p}_{01})_* \hat{\mathsf{q}},$$

where

$$\mathsf{p}_{01} : \mathsf{Con}(\mathsf{M})^3 \to \mathsf{Con}(\mathsf{M})^2$$
$$(x_0, x_{1/2}, x_1) \mapsto (x_0, x_1).$$

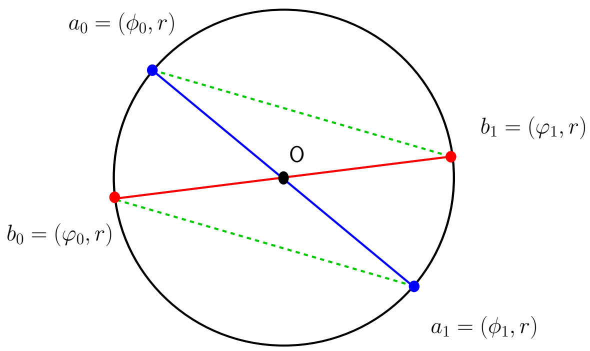
Then q is an optimal coupling of μ_0 and μ_1:

$$\hat{\mathsf{d}}_{\mathsf{W}}^2(\mu_0, \mu_1) = \int_{\mathsf{Con}(\mathsf{M}) \times \mathsf{Con}(\mathsf{M})} \mathsf{d}_{\mathsf{Con}}(x_0, x_1)^2 d\mathsf{q}(x_0, x_1).$$

Lemma 4.2.2 and Lemma 4.2.3 imply

$$
\begin{aligned}
&\mathsf{d}_{\mathsf{Con}}(a_0, b_1) + \mathsf{d}_{\mathsf{Con}}(b_0, a_1) \\
&\quad = \sqrt{r^2 + s^2 - 2rs\cos\big(\underbrace{\mathsf{d}(\phi_0, \varphi_1)}_{<\pi}\big)} + \sqrt{r^2 + s^2 - 2rs\cos\big(\underbrace{\mathsf{d}(\varphi_0, \phi_1)}_{<\pi}\big)} \\
&\quad < 2(r+s) = \mathsf{d}_{\mathsf{Con}}(a_0, a_1) + \mathsf{d}_{\mathsf{Con}}(b_0, b_1).
\end{aligned}
$$

This contradicts the fact that the support of q is $\mathsf{d}_{\mathsf{Con}}$-cyclically monotone due to Theorem 1.2.5. In the illustrating picture below we set $r = s$.



□

Lemma 4.2.4 finishes the proof of Theorem 4.2.1. □

4.3 Application to Riemannian Manifolds. I

We consider a compact and complete n-dimensional Riemannian manifold $(\mathsf{M}, \mathsf{d}, \mathsf{vol})$ with $\mathsf{Ric} \geq n-1$ denoting by d the Riemannian distance and by vol the Riemannian volume.

Theorem 4.3.1. *The n-Euclidean cone $(\mathsf{Con}(\mathsf{M}), \mathsf{d}_{\mathsf{Con}}, \nu)$ of a compact and complete n-dimensional Riemannian manifold $(\mathsf{M}, \mathsf{d}, \mathsf{vol})$ with $\mathsf{Ric} \geq n-1$ satisfies $\mathsf{CD}(0, n+1)$.*

Proof. We consider two measures $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{Con}(\mathsf{M}), \mathsf{d}_{\mathsf{Con}}, \nu)$. Then there exists a geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathcal{P}_2(\mathsf{Con}(\mathsf{M}), \mathsf{d}_{\mathsf{Con}})$ connecting μ_0 and μ_1. As above, we consider the partition

$$0 = t_0 < t_{1/2} = \tfrac{1}{2} < t_1 = 1$$

of $[0,1]$ and a probability measure $\hat{\mathsf{q}}$ on $\mathsf{Con}(\mathsf{M})^3$ satisfying the appropriate properties of Theorem 1.2.3. For $\varepsilon > 0$ we denote by $\hat{\mathsf{q}}_\varepsilon$ the restriction of $\hat{\mathsf{q}}$ to $\mathsf{Con}(\mathsf{M})^3_\varepsilon := (\mathsf{Con}(\mathsf{M}) \setminus B_\varepsilon(\mathsf{O}))^3$, meaning that

$$\hat{\mathsf{q}}_\varepsilon(A) = \frac{1}{\hat{\mathsf{q}}(\mathsf{Con}(\mathsf{M})^3_\varepsilon)}\hat{\mathsf{q}}(A \cap \mathsf{Con}(\mathsf{M})^3_\varepsilon)$$

for $A \subseteq \mathsf{Con}(\mathsf{M})^3$. Furthermore, we define μ_i^ε as the projection of $\hat{\mathsf{q}}_\varepsilon$ on the i-th factor

$$\mu_i^\varepsilon := (\mathsf{p_i})_* \hat{\mathsf{q}}_\varepsilon$$

where

$$\begin{aligned} \mathsf{p_i} : \mathsf{Con}(\mathsf{M})^3 &\to \mathsf{Con}(\mathsf{M}) \\ (x_0, x_{1/2}, x_1) &\mapsto x_i \end{aligned}$$

for $i = 0, \frac{1}{2}, 1$, and $\mathsf{q}_\varepsilon := (\mathsf{p_{01}})_* \, \hat{\mathsf{q}}_\varepsilon$ where

$$\begin{aligned} \mathsf{p_{01}} : \mathsf{Con}(\mathsf{M})^3 &\to \mathsf{Con}(\mathsf{M})^2 \\ (x_0, x_{1/2}, x_1) &\mapsto (x_0, x_1). \end{aligned}$$

Then for every $\varepsilon > 0$, q_ε is an optimal coupling of μ_0^ε and μ_1^ε and $\mu_{1/2}^\varepsilon$ is a midpoint of them. We derive from Theorem 4.2.1 that the following convergence statements hold true,

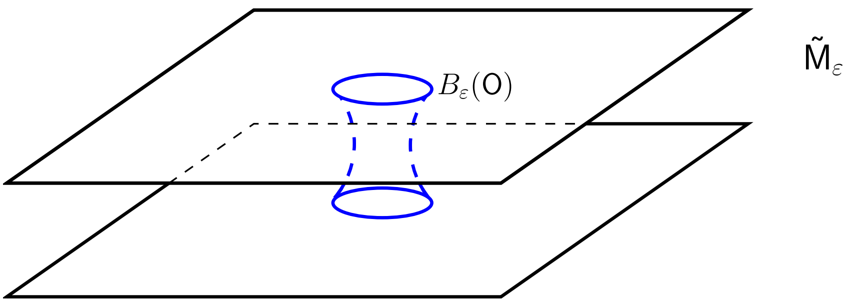
$$\mathsf{q}_\varepsilon(B) \underset{\varepsilon \to 0}{\longrightarrow} \mathsf{q}(B) \quad \text{and} \quad \mu_i^\varepsilon(C) \underset{\varepsilon \to 0}{\longrightarrow} \mu_i(C)$$

for $i = 0, \frac{1}{2}, 1$, $B \subseteq \mathsf{Con}(\mathsf{M})^2$ and $C \subseteq \mathsf{Con}(\mathsf{M})$, respectively, where $\mathsf{q} := (\mathsf{p_{01}})_* \, \hat{\mathsf{q}}$.

A more than 20-year old result by Cheeger and Taylor [CT82, Che83] states that the Euclidean cone $\mathsf{Con}(\mathsf{M})$ without origin is an incomplete, $(n+1)$-dimensional Riemannian manifold whose Ricci curvature is bounded from below by 0.

Lemma 4.3.2 (Cheeger/Taylor)**.** *$(\mathsf{Con}(\mathsf{M}) \setminus \{\mathsf{O}\}, \mathsf{d_{Con}}, \nu)$ is an $(n+1)$-dimensional Riemannian manifold with $\mathsf{Ric} \geq 0$.*

For fixed $\varepsilon > 0$ we embed $\mathsf{Con}(\mathsf{M}) \setminus B_\varepsilon(\mathsf{O})$ in a complete Riemannian manifold $\tilde{\mathsf{M}}_\varepsilon$ whose Ricci curvature is bounded from below:



The inclusion $\mathsf{Con}(\mathsf{M})\backslash B_\varepsilon(\mathsf{O}) \subseteq \tilde{\mathsf{M}}_\varepsilon$ in a complete Riemannian manifold implies that $\mu^\varepsilon_{1/2}$ is the unique midpoint of μ^ε_0 and μ^ε_1 and satisfies

$$\mathsf{S}_{n'}(\mu^\varepsilon_{1/2}|\nu) \leq \tfrac{1}{2}\mathsf{S}_{n'}(\mu^\varepsilon_0|\nu) + \tfrac{1}{2}\mathsf{S}_{n'}(\mu^\varepsilon_1|\nu)$$

for all $\varepsilon > 0$. Passing to the limit $\varepsilon \to 0$ yields according to the convergence statements,

$$\mathsf{S}_{n'}(\mu_{1/2}|\nu) \leq \tfrac{1}{2}\mathsf{S}_{n'}(\mu_0|\nu) + \tfrac{1}{2}\mathsf{S}_{n'}(\mu_1|\nu)$$

for all $n' \geq n+1$. □

4.4 Spherical Cones

4.4.1 Spherical Cones over Metric Spaces

There are further objects with famous Euclidean ancestors – among them is the *spherical cone* or *suspension* over a topological space X. We begin with a familiar example: In order to construct the Euclidean sphere $\mathbb{S}^{n+1}$ out of its equator $\mathbb{S}^n$ we add two poles $\mathcal{S}$ and $\mathcal{N}$ and connect them via semicircles, the *meridians*, through every point in $\mathbb{S}^n$.

In the general case of abstract spaces X, we consider the product $X \times I$ of X and a segment $I = [0, a]$ and contract each of the fibers $X \times \{0\}$ and $X \times \{a\}$ to a point, the *south* and the *north pole*, respectively. The resulting space is denoted by $\Sigma(X)$ and is called the spherical cone over X.

If (X, d) is a length space with $\mathsf{diam}(X) \leq \pi$, we choose $a = \pi$ and define the *spherical cone metric* d_Σ on $\Sigma(X)$ by the formula

$$\cos(\mathsf{d}_\Sigma(p, q)) = \cos s \cos t + \sin s \sin t \cos(\mathsf{d}(x, y))$$

for $p = (x, s), q = (y, t) \in \Sigma(X)$.

4.4.2 N-Spherical Cones over Metric Measure Spaces

The N-*spherical cone* $(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma, \nu)$ over a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying $\mathsf{diam}(\mathsf{M}) \leq \pi$ is a metric measure space defined as follows:

- ◇ $\Sigma(\mathsf{M}) := \mathsf{M} \times [0, \pi] \big/ \mathsf{M} \times \{0\}, \mathsf{M} \times \{\pi\}$
- ◇ For $(x, s), (x', t) \in \mathsf{M} \times [0, \pi]$

$$\cos(\mathsf{d}_\Sigma((x, s), (x', t))) := \cos s \cos t + \sin s \sin t \cos(\mathsf{d}(x, x'))$$

- ◇ $d\nu(x, s) := d\mathsf{m}(x) \otimes (\sin^N s ds)$.

The n-spherical cone $\Sigma(\mathbb{S}^n)$ over the sphere $\mathbb{S}^n$, an n-dimensional Riemannian manifold with Ricci curvature bounded from below by $n - 1$, is $\mathbb{S}^{n+1}$, an $(n+1)$-dimensional Riemannian manifold with Ricci curvature bounded from below by n. Thanks to a generalization by Petean [Pet] we are able to enter a more abstract level and prove in Section 4.6: The n-spherical cone based on a compact and complete n-dimensional Riemannian manifold with $\mathsf{Ric} \geq n - 1$ satisfies $\mathsf{CD}(n, n + 1)$.

4.5 Optimal Transport on Spherical Cones

This section is structured in the same manner as the corresponding section devoted to optimal transport on Euclidean cones. Again we consider a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ with full support $\mathsf{M} = \mathsf{supp}(\mathsf{m})$ satisfying the curvature-dimension condition $\mathsf{CD}(N - 1, N)$ for some $N \geq 1$. Then the diameter estimate by Bonnet-Myers implies that $\mathsf{diam}(\mathsf{M}) \leq \pi$.

We denote by $(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma, \nu)$ the N-spherical cone over $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ with poles $\mathcal{S} := \mathsf{M} \times \{0\}$ and $\mathcal{N} := \mathsf{M} \times \{\pi\}$. For each pair of probability measures μ_0 and μ_1 in $\mathcal{P}_2(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma, \nu)$ there exists a geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma)$ connecting them. The critical case in this situation would be if all mass was transported from μ_0 to μ_1 through the poles. But Theorem 4.5.1 excludes this scenario.

We fix $0 < s < 1$ and consider the partition $0 = t_0 < t_s = s < t_1 = 1$ of $[0,1]$. Due to Theorem 1.2.3, there exists a probability measure $\tilde{\mathsf{q}}$ on $\Sigma(\mathsf{M})^3$ with properties listed in Section 4.2 - with the only difference that in the current situation the time point $\frac{1}{2}$ is replaced by s.

The following theorem states that the optimal transport from μ_0 to μ_1 does not involve the poles.

Theorem 4.5.1. *It holds that*

$$\tilde{\mathsf{q}}\left(\{(x_0, x_s, x_1) \in \Sigma(\mathsf{M})^3 : x_s = \mathcal{S} \text{ or } x_s = \mathcal{N}\}\right) = 0.$$

Proof. We restrict our attention to the proof of the statement

$$\tilde{\mathsf{q}}\left(\{(x_0, x_s, x_1) \in \Sigma(\mathsf{M})^3 : x_s = \mathcal{S}\}\right) = 0. \tag{4.5.1}$$

Analogous calculations lead to the complete statement of Theorem 4.5.1. The proof of (4.5.1) consists of two steps:

Lemma 4.5.2. *Let two points $x_0 = (\phi_0, r)$ and $x_1 = (\phi_1, t)$ in $\Sigma(\mathsf{M})$ be given and let $\gamma : [0,1] \to \Sigma(\mathsf{M})$ be a geodesic connecting them. If $\gamma_s := \gamma(s) = \mathcal{S}$, then ϕ_0 and ϕ_1 are antipodes in M.*

Proof of Lemma 4.5.2. Due to the definition of d_Σ, it holds that

$$r = \mathsf{d}_\Sigma(x_0, \gamma_s) = s\mathsf{d}_\Sigma(x_0, x_1)$$

as well as

$$t = \mathsf{d}_\Sigma(\gamma_s, x_1) = (1-s)\mathsf{d}_\Sigma(x_0, x_1)$$

and consequently, $t = \frac{1-s}{s}r$. Inserting this equality in the expression for $\cos\left(\frac{r}{s}\right)$ we obtain

$$\begin{aligned}\cos\left(\tfrac{r}{s}\right) &= \cos\left(\mathsf{d}_\Sigma(x_0, x_1)\right)\\ &= \cos r \cos\left(\tfrac{1-s}{s}r\right) + \sin r \sin\left(\tfrac{1-s}{s}r\right)\cos\left(\mathsf{d}(\phi_0, \phi_1)\right).\end{aligned}$$

This leads to

$$\begin{aligned}\cos(\mathsf{d}(\phi_0, \phi_1)) &= \frac{\cos\left(\frac{r}{s}\right) - \cos r \cos\left(\frac{1-s}{s}r\right)}{\sin r \sin\left(\frac{1-s}{s}r\right)}\\ &= \frac{\cos\left(\frac{r}{s}\right) - \frac{1}{2}\left[\cos\left(\frac{2s-1}{s}r\right) + \cos\left(\frac{r}{s}\right)\right]}{\frac{1}{2}\left[\cos\left(\frac{2s-1}{s}r\right) - \cos\left(\frac{r}{s}\right)\right]}\\ &= \frac{\frac{1}{2}\left[\cos\left(\frac{r}{s}\right) - \cos\left(\frac{2s-1}{s}r\right)\right]}{\frac{1}{2}\left[\cos\left(\frac{2s-1}{s}r\right) - \cos\left(\frac{r}{s}\right)\right]} = -1.\end{aligned}$$

Finally, we deduce from $\mathsf{d}(\phi_0, \phi_1) \le \pi$ that $\mathsf{d}(\phi_0, \phi_1) = \pi$. $\square$

Lemma 4.5.3. *Either* $\{(x_0, x_s, x_1) \in \mathsf{supp}(\tilde{\mathsf{q}}) : x_s = \mathcal{S}\}$ *is the empty set or it coincides with* $\{(\mathcal{S}, \mathcal{S}, \mathcal{S})\}$ *or there exists at most one pair* (ϕ_0, ϕ_1) *of antipodes in* M *with the following property: If* $(\mathcal{S}, \mathcal{S}, \mathcal{S}) \neq a = (a_0, a_s, a_1) \in \mathsf{supp}(\tilde{\mathsf{q}}) \subseteq \Sigma(\mathsf{M})^3$ *satisfies* $a_s = \mathcal{S}$ *then* $a_0 = (\phi_0, r)$ *and* $a_1 = (\phi_1, \frac{1-s}{s} r)$ *for some* $r \in (0, \pi)$.

Proof of Lemma 4.5.3. We assume that there are two different pairs (ϕ_0, ϕ_1) and (φ_0, φ_1) of antipodes in M such that there exist $a = (a_0, a_s, a_1), b = (b_0, b_s, b_1) \in \mathsf{supp}(\tilde{\mathsf{q}})$ fulfilling $a_s = \mathcal{S} = b_s$ as well as $a_0 = (\phi_0, r)$, $a_1 = (\phi_1, \frac{1-s}{s} r)$ and $b_0 = (\varphi_0, t)$, $b_1 = (\varphi_1, \frac{1-s}{s} t)$ for $i = 0, 1$ and some $r, t \in (0, \pi)$. Lemma 4.5.2 and Lemma 4.2.3 imply

$$
\begin{aligned}
&\mathsf{d}_\Sigma(a_0, b_1) + \mathsf{d}_\Sigma(b_0, a_1) \\
&\quad = \arccos \left[\cos r \cos\left(\tfrac{1-s}{s} t\right) + \sin r \sin\left(\tfrac{1-s}{s} t\right) \cos \underbrace{(\mathsf{d}(\phi_0, \varphi_1))}_{<\pi} \right] + \\
&\qquad\qquad + \arccos \left[\cos\left(\tfrac{1-s}{s} r\right) \cos t + \sin\left(\tfrac{1-s}{s} r\right) \sin t \cos \underbrace{(\mathsf{d}(\varphi_0, \phi_1))}_{<\pi} \right] \\
&\quad < \arccos \left[\cos r \cos\left(\tfrac{1-s}{s} t\right) - \sin r \sin\left(\tfrac{1-s}{s} t\right) \right] \\
&\qquad\qquad + \arccos \left[\cos\left(\tfrac{1-s}{s} r\right) \cos t - \sin\left(\tfrac{1-s}{s} r\right) \sin t \right] \\
&\quad = \arccos \left[\cos\left(r + \tfrac{1-s}{s} t\right) \right] + \arccos \left[\cos\left(\tfrac{1-s}{s} r + t\right) \right] \\
&\quad = \tfrac{r}{s} + \tfrac{t}{s} = \mathsf{d}_\Sigma(a_0, a_1) + \mathsf{d}_\Sigma(b_0, b_1).
\end{aligned}
$$

This contradicts the fact that the support of $\mathsf{q} := (\mathsf{p}_{01})_* \tilde{\mathsf{q}}$ being an optimal coupling of μ_0 and μ_1 is d_Σ-cyclically monotone where $\mathsf{p}_{01} : \Sigma(\mathsf{M})^3 \to \Sigma(\mathsf{M})^2, (x_0, x_s, x_1) \mapsto (x_0, x_1)$. ☐

At the end of the second step, Theorem 4.5.1 is proved. ☐

4.6 Application to Riemannian Manifolds. II

We consider a compact and complete n-dimensional Riemannian manifold $(\mathsf{M}, \mathsf{d}, \mathsf{vol})$ with $\mathsf{Ric} \geq n-1$ denoting by d the Riemannian distance and by vol the Riemannian volume.

Theorem 4.6.1. *The n-spherical cone* $(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma, \nu)$ *of a compact and complete n-dimensional Riemannian manifold* $(\mathsf{M}, \mathsf{d}, \mathsf{vol})$ *with* $\mathsf{Ric} \geq n-1$ *satisfies* $\mathsf{CD}(n, n+1)$.

Proof. We consider two measures $\mu_0, \mu_1 \in \mathcal{P}_2(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma, \nu)$. Then there exists a geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathcal{P}_2(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma)$ connecting μ_0 and μ_1. As before, we consider for a fixed but arbitrary $0 < s < 1$ the partition

$$0 = t_0 < t_s = s < t_1 = 1$$

of $[0, 1]$ and a probability measure $\tilde{\mathsf{q}}$ on $\Sigma(\mathsf{M})^3$ satisfying the appropriate properties of Theorem 1.2.3. For $\varepsilon > 0$ we denote by $\tilde{\mathsf{q}}_\varepsilon$ the restriction of $\tilde{\mathsf{q}}$ to $\Sigma(\mathsf{M})^3_\varepsilon := [\Sigma(\mathsf{M}) \setminus (B_\varepsilon(\mathcal{S}) \cup B_\varepsilon(\mathcal{N}))]^3$, meaning that

$$\tilde{\mathsf{q}}_\varepsilon(A) = \frac{1}{\tilde{\mathsf{q}}(\Sigma(\mathsf{M})^3_\varepsilon)} \tilde{\mathsf{q}}(A \cap \Sigma(\mathsf{M})^3_\varepsilon)$$

for $A \subseteq \Sigma(\mathsf{M})^3$. Furthermore, we define μ_i^ε as the projection of $\tilde{\mathsf{q}}_\varepsilon$ on the i-th factor

$$\mu_i^\varepsilon := (\mathsf{p_i})_* \tilde{\mathsf{q}}_\varepsilon$$

for $i = 0, s, 1$ and q_ε as the projection of $\tilde{\mathsf{q}}_\varepsilon$ on the first and third factor

$$\mathsf{q}_\varepsilon := (\mathsf{p_{01}})_* \, \tilde{\mathsf{q}}_\varepsilon.$$

Then for every $\varepsilon > 0$, q_ε is an optimal coupling of μ_0^ε and μ_1^ε and μ_s^ε is an s-intermediate point of them. We derive from Theorem 4.5.1 that the following convergence statements hold true,

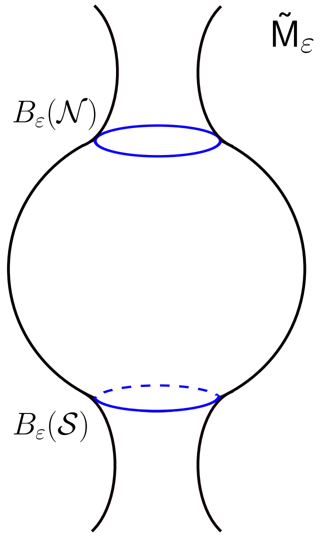
$$\mathsf{q}_\varepsilon(B) \underset{\varepsilon\to 0}{\to} \mathsf{q}(B) \quad \text{and} \quad \mu_i^\varepsilon(C) \underset{\varepsilon\to 0}{\to} \mu_i(C)$$

for $i = 0, s, 1$, $B \subseteq \Sigma(\mathsf{M})^2$ and $C \subseteq \Sigma(\mathsf{M})$, respectively, where $\mathsf{q} := (\mathsf{p_{01}})_* \, \tilde{\mathsf{q}}$.

The core of our proof is shown by Petean [Pet], namely that the spherical cone $\Sigma(\mathsf{M})$ without north and south pole is an incomplete $(n+1)$-dimensional Riemannian manifold whose Ricci curvature is bounded from below by n.

Lemma 4.6.2 (Petean). *$(\Sigma(\mathsf{M}) \setminus \{\mathcal{S}, \mathcal{N}\}, \mathsf{d}_\Sigma, \nu)$ is an $(n+1)$-dimensional Riemannian manifold with $\mathsf{Ric} \geq n$.*

For fixed $\varepsilon > 0$ we embed $\Sigma(\mathsf{M}) \setminus (B_\varepsilon(\mathcal{S}) \cup B_\varepsilon(\mathcal{N}))$ in a complete Riemannian manifold $\tilde{\mathsf{M}}_\varepsilon$ whose Ricci curvature is bounded from below:



This inclusion $\Sigma(\mathsf{M}) \setminus (B_\varepsilon(\mathcal{S}) \cup B_\varepsilon(\mathcal{N})) \subseteq \tilde{\mathsf{M}}_\varepsilon$ implies that μ_s^ε is the unique s-intermediate point of μ_0^ε and μ_1^ε and satisfies

$$\mathsf{S}_{n'}(\mu_s^\varepsilon|\nu) \leq \tau_{n-1,n'}^{(1-s)}(\theta)\mathsf{S}_{n'}(\mu_0^\varepsilon|\nu) + \tau_{n-1,n'}^{(s)}(\theta)\mathsf{S}_{n'}(\mu_1^\varepsilon|\nu),$$

where

$$\theta := \inf_{\substack{x_0 \in \mathsf{supp}(\mu_0), \\ x_1 \in \mathsf{supp}(\mu_1)}} \mathsf{d}_\Sigma(x_0, x_1)$$

for all $\varepsilon > 0$. Passing to the limit $\varepsilon \to 0$ yields according to the convergence statements,

$$\mathsf{S}_{n'}(\mu_s|\nu) \leq \tau_{n-1,n'}^{(1-s)}(\theta)\mathsf{S}_{n'}(\mu_0|\nu) + \tau_{n-1,n'}^{(s)}(\theta)\mathsf{S}_{n'}(\mu_1|\nu)$$

for all $n' \geq n+1$. $\square$

Because of Theorem 4.6.1 we can apply the Lichnerowicz theorem in order to obtain a lower bound on the spectral gap of the Laplacian on the spherical cone:

Corollary 4.6.3 (Lichnerowicz estimate, Poincaré inequality)**.** *Let* $(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma, \nu)$ *be the n-spherical cone of a compact and complete n-dimensional Riemannian manifold* $(\mathsf{M}, \mathsf{d}, \mathsf{vol})$ *with* $\mathsf{Ric} \geq n-1$. *Then for every* $f \in \mathsf{Lip}(\Sigma(\mathsf{M}))$ *fulfilling* $\int_{\Sigma(\mathsf{M})} f \, d\nu = 0$ *the following inequality holds true,*

$$\int_{\Sigma(\mathsf{M})} f^2 d\nu \leq \tfrac{1}{n+1} \int_{\Sigma(\mathsf{M})} |\nabla f|^2 d\nu.$$

Since the spherical cone $\Sigma(\mathsf{M})$ is a smooth manifold outside of the poles, every Lipschitz continuous function $f \in \mathsf{Lip}(\Sigma(\mathsf{M}))$ is differentiable almost everywhere and $\nabla^- f$ coincides with the gradient ∇f in the usual sense almost everywhere.

The Lichnerowicz estimate implies that the Laplacian Δ on the spherical cone $(\Sigma(\mathsf{M}), \mathsf{d}_\Sigma, \nu)$ defined by the identity

$$\int_{\Sigma(\mathsf{M})} f \cdot \Delta g \; d\nu = - \int_{\Sigma(\mathsf{M})} \nabla f \cdot \nabla g \; d\nu$$

admits a spectral gap $\lambda_1(-\Delta)$ of size at least $n+1$,

$$\lambda_1(-\Delta) \geq n+1.$$

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