String dualities and superpotential

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakulatät
der
Rheinischen Friedrich-Wilhelms-Universität
zu Bonn

vorgelegt von
Tae-Won Ha
aus
Cheong-Ju, Chung-Buk, Südkorea

Bonn 2010

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Universität Bonn.

1. Gutachter:	Prof. Dr. Albrecht Klemm
2. Gutachter:	PD. Dr. Stefan Förste
Tag der Promotion:	16. Juli 2010
Erscheinungsjahr:	2010

Abstract

The main objective of this thesis is the computation of the superpotential induced by D5-branes in the type IIB string theory and by five-branes in the heterotic string theory. Both superpotentials have the same functional form which is the chain integral of the holomorphic three-form. Using relative (co)homology we can unify the flux and brane superpotential. The chain integral can be seen as an example of the Abel-Jacobi map. We discuss many structures such as mixed Hodge structure which allows for the computation of Picard-Fuchs differential equations crucial for explicit computations. We blow up the Calabi-Yau threefold along the submanifold wrapped by the brane to obtain geometrically more appropriate configuration. The resulting geometry is non-Calabi-Yau and we have a canonically given divisor. This blown-up geometry makes it possible to restrict our attention to complex structure deformations. However, the direct computation is yet very difficult, thus the main tool for computation will be the lift of the brane configuration to a F-theory compactification. In F-theory, since complex structure, brane and, if present, bundle moduli are all contained in the complex structure moduli space of the elliptic Calabi-Yau fourfold, the computation can be dramatically simplified. The heterotic/F-theory duality is extended to include the blow-up geometry and thereby used to give the blow-up geometry a more physical meaning.

어머니에게…

Whenever a theory appears to you as the only possible one, take this as a sign that you have neither understood the theory nor the problem which it was intended to solve.

K. Popper,
Objective Knowledge: An Evolutionary Approach

Notations and abbreviations

CY	Calabi-Yau
CS	Chern-Simons
GW	Gromov-Witten
GV	Gopakumar-Vafa
PF	Picard-Fuchs
GD	Griffiths-Dwork
SYZ	Strominger-Yau-Zaslow
GKZ	Gelfand-Kapranov-Zelevinski
KN	Kodaira-Nakano theorem, A.1.4
HRR	Hirzebruch-Riemann-Roch theorem, A.1.5
GRR	Grothendieck-Riemann-Roch theorem, A.1.6
VEV	vacuum expectation value
X	CY threefold for the type IIB theory or the B-model
Y	elliptically fibered CY fourfold for F-theory
Z	elliptically fibered CY threefold for heterotic string
B_M	$(n-1)$-dimensional base of n-dimensional elliptically fibered CY manifold M
Δ_M	$(n+1)$-dimensional reflexive polyhedron in which the CY n-fold M can be given as hypersurface
Ω_M	holomorphic n-form of CY n-fold M
$\widehat{M}$	mirror CY manifold of M for the type IIA theory or the A-model
$\widetilde{M}$	blow-up of the manifold M along a submanifold which will be clear from the context
$\mathrm{Bl}_N M$	blow-up of the manifold M along the submanifold N
Σ	holomorphic curve in CY threefold wrapped by a D5-brane
$\mathcal{C}$	holomorphic curve in the base B_2 of Z wrapped by horizontal five-branes in the heterotic string
T_M	the holomorphic tangent bundle of M
$N_{N/M}$	the holomorphic normal bundle of the submanifold N in M
K_M	the canonical bundle of M
$H^k(M,V)$	$H^k(M,\mathcal{O}_M(V))$
Ω^k_M	$\mathcal{O}_M(\bigwedge^k T^*_M)$
$H^{p,q}(M,\mathbb{Z})$	$H^{p,q}(M)\cap H^{p+q}(M,\mathbb{Z})$
l_i	$\log z_i$

Contents

Preface

It is not enough to have a good mind. The main thing is to use it well.

R. Descartes,
Discourse on the Method

During my high school years my favorite subjects were obviously mathematics and physics. This could be due to the fact that my German was not mature enough to appreciate the language. Whatever the reason was, after graduating the high school, for university studies, I had to decide between mathematics and physics. I decided to combine both and opted for mathematical physics. By doing so, I hoped for being able to combine the best of both worlds.

At the university, I eventually took two courses in differential topology where I first met forms, cohomology theories, exact sequences and all that. I wanted to do research in a field where all these beautiful mathematics could be used. I found in string theory the answer. So, the main reason I started to work on string theory is that it is an excellent playground to apply mathematics I like and, especially, to discover *new* mathematics. For instance, the main topic of this thesis, mirror symmetry, was first discovered in string theory. There are many other instances where new mathematical structures were first found in string theory and then formalized by the mathematics community. Also, I must admit that I was partly lured by popular science books *(propaganda?)* written by famous string theorists.

Despite my initial motivation, during my Ph.D. period, unfortunately, I could not discover any *new* mathematics. However, I take comfort in G. Galilei saying[1]

> *"I value more finding a verity even though a minor question, than endlessly discussing great problems without ever getting to any truth."*

[1] *"Io stimo più il trovar un vero, benché di cosa leggiera, che 'l disputar lungamente delle massime questioni senza conseguir verità nissuna."* as quoted in Le Opere di Galileo. Edizione Nazionale, vol. IV.

I think that the results obtained in the three works published during the Ph.D. period constitute coherent and nice results. Obviously, I could not have completed them without the help of others.

Thus, here comes the acknowledgement. There are many people I want to thank. Firstly, I want to thank Albrecht. Without him, my Ph.D. studies would not have been possible because, obviously, he was my advisor. His experiences and insights for many problems were necessary for the completion of this work. Next, I want to thank my collaborators: Thomas W. Grimm and Denis Klevers. It was a pleasure and very fun to discuss, work, or just to have conversations with them. My thanks goes to the other members in the group: Murad Alim, Babak Haghighat, Daniel V. Lopes, Marco Rauch, Piotr Sułkowski and Thomas Wotschke. They made my time at the institute pleasurable. I want to thank Stefan for gladly accepting the role of the second examiner. Also, I want to thank Prof. M. Banagl, Prof. D. Huybrechts and Prof. R. Weissauer for patiently answering mathematics questions. There were three people who read this thesis with care and made comments. I am grateful for their patience and efforts. They were Babak, Ju Min Kim and Thomas W..

Additionally, I want to thank the German Excellence Initiative via Bonn Cologne Graduate School by partly being the reason why my bank account numbers did not slide to the realm of $\mathbb{Q}_{\leq 0}$.

My greatest thank goes to my family and Ju Min. Ju Min supported me, suffered with me and made me smile and laugh in difficult (and also in not so difficult) times of the Ph.D. period. Without her, the Ph.D. time would only have been a transition period to the next phase in my life. However, she made it to a very precious part of my life; each moment valuable and worth to remember. My family constantly encouraged me and cheered me up. I especially want to thank my mother. That I came this far is all due to her support. I could not have finished or even have started my studies without her love. *This is dedicated to her.*

Bonn 2010, T.-W. H.

1 Introduction

We are all agreed that your theory is crazy. The question which divides us is whether it is crazy enough to have a chance of being correct. My own feeling is that it is not crazy enough.

N. Bohr,
to W. Pauli after his presentation of Heisenberg's and Pauli's nonlinear field theory of elementary particles

During and since the so-called *second string revolution* in the late nineties many dualities have been uncovered in string theory: S-duality, T-duality, U-duality, the AdS/CFT correspondence etc.. These dualities connect many distinct perturbative string theories with each other if one takes all quantum corrections into account. The novelty of these dualities is the fact that one theory in the *weak coupling regime* is dual to another theory in the *strong coupling regime* which is not accessible perturbatively or due to technical difficulties. The most of these dualities are not (yet) proven, but remain conjectures. However, there are many decisive evidences indicating that they are valid. The existence of the web of dualities led Witten to conjecture a unifying theory of string theories, the so-called *M-theory*. Unfortunately, the details like the action of this theory still remain in mystery.

Mirror symmetry

One of the most prominent dualities is *mirror symmetry* (MS). This thesis contains works done in the framework of this duality. Since its discovery MS served as an excellent communication platform for mathematicians and string theorists. In MS the space of theories parameterized by deformations of the Lagrangian corresponds to geometric moduli space of the compactification manifold. However, for this correspondence to work, the *classical* moduli spaces had to be extended. This led to a new branch of mathematics, the so-called *quantum* geometry. MS is the benchmark example in which the collaboration of mathematicians and string theorists has been tremendously fruitful.

MS states that the complex structure moduli space of a CY manifold and the complexified Kähler moduli space of the mirror CY manifold are equivalent. Physically speaking, it states the equivalence of the topological A- and B-model on mirror dual CY manifolds. In ref. [1] the possibility of this duality was conjectured in the context of $\mathcal{N} = (2,2)$ superconformal field theory. The conventional MS involves only closed strings. Therefore, it is also called the *closed mirror symmetry*. The first comparison of the geometrical moduli space [2] and the first computations [3, 4] ignited many activities. First of all, the relation to topological field theories respective to topological string theories was a very important discovery [5, 6, 7]. This connection made it possible to consider the reduced topological field theories in order to study MS. Due to the vast amount of the literature, it is impossible to list all hitherto references in this field. We will only mention some of the works. As mentioned above, MS also had great impact on mathematics community. Let us mention only few examples: Using MS, the GW invariants could be computed, for which no general computational method was thitherto known, and new structures, e.g. the stability conditions, could be discovered with the help of MS. Due to combined efforts of mathematicians and string theorists [8, 9, 10, 11, 12, 13, 14, 15, 16] (and many others), MS could be mathematically proved for genus 0 [17, 18] and from the physical point of view [19]. There are excellent books on this matter [20, 21, 22, 23]. Let us mention that the higher genus contributions can be computed using the holomorphic anomaly equation of ref. [24], which is related to the quantum background independence [25], and the occurrence of gaps discovered in ref. [26].

Mirror symmetry with D-branes

Also during the time of the second revolution, new non-perturbative objects essential for many dualities have been found: the so-called D-branes [27]. Without knowing about D-branes, Kontsevich conjectured the categorical formulation of the MS, the *homological mirror symmetry*, in ref. [28]. Interestingly, the homological MS physically corresponds to MS including D-branes. This duality, also known as *open mirror symmetry*, is by far less explored than the closed MS. The homological MS is proved for elliptic curves [29] and for K3 surfaces [30]. On the string theory side the most desired quantity to compute in the open MS is the superpotential W which would result in disk instanton numbers. It corresponds to the disk amplitudes in the topological theories. The main topic of this thesis is to compute the superpotential induced by D-branes using dualities like MS and the heterotic/F-theory duality [31].

Initial advances were made using non-compact CY geometries in refs. [32, 33] where the superpotential was computed directly. Then, refs. [34, 35] found PF type operators for these superpotentials by using relative (co)homology to describe different superpotentials in an unified way. Also for non-compact geometries, open-closed duality has been found in refs. [36, 37] and also in ref. [38]. Recently, first computations in compact geometries have been performed in ref. [39] for involution A-branes. There, the superpotential is treated as normal function and inhomogeneous PF operator is determined and solved to compute the domain wall tension, i.e. superpotential at critical points. Works in this direction include refs. [40, 41, 42, 43, 44]. Extending the previous works [34, 35], the authors of refs. [45, 46, 47, 48, 49] extended the non-compact computation to compact geometries. There are also works using conformal field theoretical methods [50, 51].

Publications

Our own contributions consist of the following three publications:

- *The D5-brane effective action and superpotential in* $\mathcal{N} = 1$ *compactifications*
  with T. W. Grimm, A. Klemm and D. Klevers,
  Nucl. Phys. **B816** (2009) 139–184, `arXiv:0811.2996 [hep-th]`
- *Computing Brane and Flux Superpotentials in F-theory Compactifications*
  with T. W. Grimm, A. Klemm and D. Klevers,
  JHEP **04** (2010) 015, `arXiv:0909.2025 [hep-th]`
- *Five-brane superpotentials and heterotic/F-theory duality*
  with T. W. Grimm, A. Klemm and D. Klevers,
  accepted for publication in Nucl. Phys., `arXiv:0912.3250 [hep-th]`

In the first work we have computed the effective action of a D5-brane and thereby re-derived the superpotential. We analyzed the superpotential using the relative (co)homology, mixed Hodge structure and blow-up. Then, in the second work we lifted the non-compact D5-brane settings to F-theory compactifications and computed the superpotential for explicit examples. In the third work, combining the previous two works, we used the heterotic/F-theory duality to give the blow-up construction a sound physical ground and explicit examples were given. This thesis is based on the above three publications.

Outline of the thesis

We proceed as follows: In § 2 we will introduce superpotentials of different configurations we will be discussing: D5-branes in the type IIB theory, F-theory and the heterotic string theory. Since the superpotential is tightly connected to disk instantons and the BPS numbers and since these numbers will be used to check our computations, we will review the enumerative geometry involved, in both closed and open settings. For explicit computations we will start from non-compact geometries and use computations in them as benchmark results. Thus, we will review these non-compact geometries in § 3.

After having done so, in § 4, we will turn our attention to the D5-brane superpotential. We will first concentrate on the conceptual points. Using relative (co)homology, we unify the two parts of the superpotential, the flux and the brane contributions. Then, we will discuss the mixed Hodge structure in detail which underlies the open-closed moduli space. It is easier to work with divisors than with curves. Thus, we will describe how we can obtain a canonical divisor associated to the curve wrapped by the D5-brane. This can be achieved by blow-up of the CY threefold along the curve. After describing the blow-up and the associated mixed Hodge structure, we will describe an example of explicit blow-up in the non-compact setting and also how we can describe compact blow-ups in general. In the last section of this chapter we will discuss PF equations and the way to obtain them using the mixed Hodge structure and the GD pole reduction method.

To explicitly compute the superpotential of D5-branes, we will lift the setting to F-theory in § 5. We will describe the geometrical prerequisites, e.g. elliptic fibration, fibration structures in CY hypersurfaces in toric varieties etc. Then, we will describe in great detail the MS for higher dimensional CY manifolds. We will discuss A- and B-model operators, their Frobenius algebra structures and how to identify them in the large complex structure point in the moduli space. Also, as a byproduct, we observe new behavior of the periods near the conifold point in CY fourfolds. After having discussed all techniques required for the computation, we discuss an example for which we compute the superpotential explicitly.

We will discuss the heterotic/F-theory duality in § 6 combining the results of § 4 and § 5. Firstly, we will review the required technique, the spectral cover, for the application of the duality. We will apply the results of § 4 to the heterotic superpotential of five-branes. Heterotic five-branes are related to F-theory by blow-up of the base of the elliptic CY fourfold which we will describe. Next, we will argue, by using complete intersection description of the blown-up threefold and fourfold, that the blow-up threefold geometry is not only a tool for computation, but obtains physical meaning under the heterotic/F-theory duality. A map connecting superpotentials of both theories will be given. Then, we discuss explicit examples embodying the results we have discussed till then.

The appendix contains mathematical theorems and definitions, a note on the orientifold limit of F-theory and results of further CY fourfold examples.

2 Superpotentials

Does anyone believe that the difference between the Lebesgue and Riemann integrals can have physical significance, and that whether say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not care to fly in that plane.

R. W. Hamming,
paraphrased from *American Mathematics Monthly* (1998) **105**, 640-50

Theories with $\mathcal{N} = 1$ supersymmetry can be completely characterized by the following three functions:

- The superpotential $W(\Phi)$,
- The gauge kinetic function $f(\Phi)$,
- The Kähler potential $K(\Phi, \overline{\Phi})$

where Φ denotes chiral superfields of the theory. The Kähler potential K determines the kinetic terms of the chiral superfields by $G_{i\overline{J}} = \partial_i \partial_{\overline{J}} K$. As already indicated by $K(\Phi, \overline{\Phi})$, the Kähler potential is a *real* function which depends on both Φ and $\overline{\Phi}$. It is not protected by any non-renormalization theorem against corrections. Therefore, it is the least well-explored quantity in the theory. The function $f(\Phi)$ is the gauge kinetic function and determines the kinetic terms of the Super Yang-Mills fields. It is holomorphic in Φ and can be shown to receive only 1-loop corrections in the perturbation theory [55, 56]. In this work we will focus on the first quantity in the above list, the superpotential $W(\Phi)$. The superpotential is the most protected object in the theory. It does not receive any perturbative corrections beyond tree level. This important non-renormalization theorem was originally proved by supergraph method [57]. Seiberg significantly simplified the proof in ref. [58][1] by promoting the coupling constants to the VEVs of

[1] Cf. also textbooks on supersymmetry, for example [59, 60, 61].

superfields. The following three points led to great simplification: The holomorphicity of W as a function of the coupling constants, selection rules resulting from global symmetries obeyed by the superpotential for vanishing couplings and smoothness of various weak coupling limits. There is also a proof using string theory by treating the $\mathcal{N} = 1$ theories as low energy effective theories coming from string compactifications [62]. Clearly, there can be non-perturbative corrections to W and f. However, these are again constrained by the holomorphicity and selection rules.

In this chapter we introduce the relevant superpotentials appearing in different string theories. Firstly, we will discuss the main object of study in this work, the superpotential W_B induced by D5-branes. Tightly related to this, we will discuss the moduli of D5-branes and show that the superpotential is an example of the Abel-Jacobi map. Then, we will review the flux superpotential W_F in F-theory. This superpotential contains the flux and the D7-brane superpotential. Therefore, we will first go through these superpotentials. We will later exploit this and the fact that D7-branes can induce D5-brane charge via its worldvolume flux to compute the D5-brane superpotential. The last superpotential to discuss is the superpotential in the heterotic theory. We will argue that five-branes generate a superpotential of chain integral type, making it possible to apply tools we will be developing for the D5-brane superpotential.

All these superpotentials allow for interpretation in the enumerative geometry. We will use this fact to check the computation by comparing the extracted invariants with those calculated in the literature. Therefore, we will briefly review the basic facts of the enumerative geometry at the end of the chapter.

2.1 D5-brane superpotential

The D5-brane we will consider will fill the four-dimensional Minkowski spacetime. Thus, it wraps a two-dimensional submanifold Σ of the internal CY threefold X. For supersymmetric configurations Σ has to be holomorphic, i.e. it is a holomorphic curve. A D5-brane wrapping a holomorphic curve $\Sigma \in H_2(X, \mathbb{Z})$ induces a superpotential of the form

$$W_B = \int_\Gamma \Omega \quad \text{with} \quad \partial\Gamma = \Sigma - \Sigma_0 \tag{2.1}$$

where Σ_0 is a holomorphic reference curve. As already mentioned, W_B is the main object we will be studying. The above superpotential was first written down by Witten in ref. [63] using M-theory and a spacetime filling M5-brane. It can be also derived from the D9-brane superpotential given by the holomorphic CS action of ref. [64]: We perform a dimensional reduction of the holomorphic CS action down to a holomorphic curve Σ and obtain W_B as it was done in ref. [32]. Yet another way to derive the superpotential (up to linear order in the deformation of the D5-brane) is to perform dimensional reduction of the DBI action of a D5-brane [52].

To study the superpotential W_B means to study the moduli constrained by this superpotential. Thus, we will be studying the D5-brane moduli. To do this, naturally, we look at the deformation of Σ in X. Infinitesimal deformations of Σ in X are described by elements of the cohomology group $H^0(\Sigma, N_{\Sigma/X})$, i.e. by sections in the holomorphic normal bundle. The obstructions to those deformations are parameterized by the group $H^1(\Sigma, N_{\Sigma/X})$. We first want to compute the dimensions of $H^0(\Sigma, N_X\Sigma)$ and $H^1(\Sigma, N_{\Sigma/X})$. We employ (HRR, A.1.5) to do so. Therefore, we need the first Chern class of the normal bundle. It can be determined by using the analog of the adjunction formula for higher co-dimensional submanifolds [65, Prop. II.8.20]

$$K_\Sigma = K_X|_\Sigma \otimes \bigwedge^2 N_{\Sigma/X} \tag{2.2}$$

and the fact $c_1(\bigwedge^k E) = c_1(E)$ for a rank k vector bundle E. Applying (HRR, A.1.5) to Σ and $N_{\Sigma/X}$, we obtain vanishing Euler-Poincaré characteristic $\chi(\Sigma, N_{\Sigma/X})$. This means for a curve that

$$\dim H^0(\Sigma, N_{\Sigma/X}) = \dim H^1(\Sigma, N_{\Sigma/X}). \tag{2.3}$$

Thus, the expected dimension of deformations of Σ is zero since all infinitesimal deformations are obstructed. It should be emphasized that this does *not* necessarily mean that there are no brane moduli. There still can be infinitesimal deformations lifting to finite deformations, cf. also ref. [66]. This is similar to the complex structure deformations of CY manifolds. The infinitesimal deformations are given by elements of $H^1(X, T_X)$. The obstruction space $H^2(X, T_X)$ is in general non-trivial: It is dual to $H^{1,1}(X)$ for CY threefolds and therefore for compact CY manifolds non-zero and its dimension can be also rather large. Naively, this would mean that some or even all moduli are obstructed. The Bogomolov-Tian-Todorov theorem[2] [67] however guarantees that all infinitesimal deformations lift to finite deformations.

Abel-Jacobi map

From mathematical point of view, the superpotential W_B is an example of the Abel-Jacobi map or normal function. In the remainder of this section we describe this aspect of the superpotential. For more detailed discussion of the Abel-Jacobi map and normal functions see ref. [68] and also refs. [69, 70, 71]. Let M be a Fano fourfold. As usual, a section in K_M^{-1} describes a CY threefold X. Let $\varpi : \mathcal{X} \to B$ be a complex analytic family of hypersurfaces in M and let X be a typical member of $\mathcal{X}$. Infinitesimal complex structure deformations of X are described by $H^1(X, T_X) \cong H^{2,1}(X)$. Let us recall the Kodaira-Spencer infinitesimal displacement map $\rho_{d,t}$ [72, Def. 5.5]

$$\rho_{d,t}: \; T_B \longrightarrow H^0(X_t, N_{X_t/M}), \qquad \frac{\partial}{\partial t} \longmapsto \zeta_t \quad \text{with} \quad t \in B. \tag{2.4}$$

The Kodaira-Spencer infinitesimal deformation ρ_t is given

$$\rho_t = \delta^0 \circ \rho_{d,t}: \; T_B \xrightarrow{\qquad\qquad} H^1(X_t, T_{X_t}) \cong H^{2,1}(X_t), \qquad T_B \xrightarrow{\rho_{d,t}} H^0(X_t, N_{X_t/M}) \xrightarrow{\delta^0} H^1(X_t, T_{X_t}) \tag{2.5}$$

where δ^0 is the first connecting homomorphism present in the long cohomology sequence coming from the normal bundle sequence. Thus, the infinitesimal complex structure deformations of X_t are essentially given by $H^0(X_t, N_{X_t/M})$.

[2]For more details, see also the discussion in ref. [23, § 14].

Let $C \in H_2(X,\mathbb{Z})$ be a curve which is homologous to zero. This means that there is a three-chain Γ with $\partial\Gamma = C$. The map

$$AJ : C \longmapsto \int_\Gamma \quad \text{with} \quad AJ(C) \in J^3(X) = \frac{H^3(X,\mathbb{C})}{F^2H^3(X) + H^3(X,\mathbb{Z})}. \tag{2.6}$$

is called the *Abel-Jacobi map* where we denote the *middle intermediate Jacobian* of X by $J^3(X)$. Here, $F^2H^3(X) = H^{2,1}(X) \oplus H^{3,0}(X)$ is one of the Hodge filtration modules which will be described in § 4.1. We can insert Ω, the holomorphic three-form of X, into the functional $\int_\Gamma$ since $F^2H^3(X)$ is the dual space of $H^3(X,\mathbb{C})/F^2H^3(X)$. Now, let $X = X_{\tilde{t}}$ for some $\tilde{t} \in B$. As discussed above, the superpotential arising from a D5-brane wrapping holomorphically embedded curve $\Sigma \in H_2(X,\mathbb{Z})$ has the form of the chain integral (2.1). The superpotential is thus nothing but the Abel-Jacobi map

$$W_B = \int_\Gamma \Omega = AJ(\Sigma - \Sigma_0)(\Omega) \quad \text{with} \quad \partial\Gamma = \Sigma - \Sigma_0 \tag{2.7}$$

where Σ_0 is a holomorphic reference curve. We now take into account the complex structure deformations of X_t. Let therefore $\Xi \in H_4^{\text{prim}}(M,\mathbb{Q})$ be a primitive four-cycle of M. This is to say that $\Xi \cap X_t = 0$ in $H_2(X_t,\mathbb{Q})$, meaning that $\Xi(t) = \Xi \cap X_t = \partial\Gamma$. The superpotential now becomes a normal function

$$W_B = \nu_{\Xi(t)}(\Omega) = AJ(\Xi(t))(\Omega). \tag{2.8}$$

A normal function is a section in the family $\mathcal{J}^3(X)$ of intermediate Jacobian $J^3(X_t)$. A section in $\mathcal{J}^3(X)$ has to satisfy certain conditions to be a normal function. We refrain from enumerating them here and refer to ref. [68, p. 116].

Let now $\{\Sigma_u\}_{u\in U}$ be an analytic family of two-cycles homologous to a reference curve Σ_0. Here, U is a local neighborhood of the parameter space for Σ_u. Above discussion about primitive four-cycle Ξ shows that we only need to consider the family of primitive four-cycles $\{\Xi(u)\}_{u\in U}$ instead of $\{\Sigma_u\}_{u\in U}$. It can be shown that a primitive four-cycle gives rise to a normal function iff its dual lies in $H^4_{\text{prim}}(M,\mathbb{Q}) \cap H^{2,2}(M)$. Thus, we assume that $\Xi(0) \in H^4_{\text{prim}}(M,\mathbb{Q}) \cap H^{2,2}(M)$. We write $\Xi'(u) = \Xi(u) \cap X = \partial\Gamma_u = \Sigma_u - \Sigma_0$. Let us consider the following map

$$\phi :\ U \longrightarrow J^3(X)\ , \quad u \longmapsto \int_{\Gamma_u}. \tag{2.9}$$

It can be shown that the differential of ϕ factors as follows [73, (2.25)]

$$\begin{array}{ccc} \phi_* = \psi \circ \rho_{d,u} :\ T_{U,u} & \xrightarrow{\qquad\qquad\qquad} & H^{1,2}(X) \subset T_{J^3(X)} \\ & {\scriptstyle\rho_{d,u}}\searrow \quad \nearrow{\scriptstyle\psi} & \\ & H^0(\Xi'(u), N_{\Xi'(u)/X}) & \end{array} \tag{2.10}$$

where $\rho_{d,u}$ is the infinitesimal displacement map (2.4) for $\Xi'(u)$ in X and ψ (A.10) is given by the dual of the Poincaré residue operator, cf. appendix A.1.3. The variations of the superpotential due to brane moduli are described by elements of $H^{1,2}(X)$ which comes from the elements of $H^0(\Xi'(u), N_{\Xi'(u)/X})$. The main reason to consider $\Xi'(u)$ instead of Σ_u is that using $\Xi'(u)$, it is easy to take into account the complex structure deformations of the CY threefold: We only have to consider $\Xi'(t,u) = \Xi'(u) \cap X_t$ and for the superpotential we then obtain the following simple expression

$$W_B = \nu_{\Xi'(t,u)}(\Omega) = AJ(\Xi'(t,u))(\Omega). \tag{2.11}$$

This means that we can consider instead of Σ_u a four-cycle Ξ in the ambient space M or the intersection of Ξ with X. This can be advantageous since, usually, the ambient spaces of the CY manifolds are simpler and easier to describe.

2.2 Flux superpotential in F-theory

F-theory [31] allows for a non-perturbative description of the type IIB theory with D7-branes or more generally with (p,q) seven-branes. It is defined on elliptic CY manifolds and the holomorphically varying modulus of the fiber torus is identified with the axio-dilaton $\tau = C_0 + ie^{-\phi}$. Using the duality to M-theory, by turning on a four-form flux G_4, a superpotential W_F will be induced in F-theory [74]. This superpotential includes the flux and D7-brane superpotential of the type IIB theory in the weak coupling limit. We therefore first introduce the superpotentials in the type IIB theory and then the flux superpotential in F-theory.

Flux superpotential in the type IIB theory

Let us compactify the type IIB theory on a CY threefold X. The RR and NS fluxes of $F_3 = \langle dC_2 \rangle$ and $H_3 = \langle dB \rangle$ generate the following superpotential [74]

$$W_{\text{flux}} = \int_X G_3 \wedge \Omega \quad \text{with} \quad G_3 = F_3 - \tau H_3. \tag{2.12}$$

Using the periods (X^A, F_A) of X, we can rewrite the superpotential as a linear combination

$$W_{\text{flux}} = \widehat{N}_A X^A + \widehat{M}^A F_A \quad \text{with} \quad X^A = \int_{A^A} \Omega \quad \text{and} \quad F_A = \int_{B_A} \Omega. \tag{2.13}$$

Here, $(\widehat{N}_A, \widehat{M}^A) = (N_A - \tau \widetilde{N}_A, M^A - \tau \widetilde{M}^A)$ denote the complex flux quantum numbers: (M_A, N^A) of F_3 and $(\widetilde{M}_A, \widetilde{N}^A)$ of H_3. As usual, we have introduced a topological symplectic basis (A^A, B_A) of $H_3(X,\mathbb{Z})$ which is possible for CY threefolds.[3] The $(2h^{2,1}(X)+2)$ periods depend on the (VEVs of) $h^{2,1}(X)$ closed string moduli z_i, the complex structure moduli of X. These are scalars in $\mathcal{N} = 1$ chiral multiplet. Let us collectively denote the moduli by $\underline{z}$. This dependence of the periods on the complex structure moduli can be evaluated by solving a system of partial differential equations, the PF equations

$$\mathcal{L}_a X^A = \mathcal{L}_a F_A = 0. \tag{2.14}$$

Here, $\mathcal{L}_a$ are linear differential operators in the complex structure moduli $\underline{z}$. We will describe an algorithmic method how to determine them in § 4.6. PF systems are by far the most important computational tool for mirror symmetry. In principal they allow for straightforward

[3] For higher dimensional CY manifolds, especially for CY fourfold, this is not necessarily true. We will discuss this in detail later.

computation of periods and using mirror symmetry the enumerative invariants. It is important to point out that due to the special geometry [75, 76, 77] the non-trivial information about the F_A periods can be encoded by a single holomorphic function, the prepotential $F^0(X^A)$. The prepotential is homogeneous of degree two in the periods X^A such that the F_A can be written as $F_A = \partial F^0/\partial X^A$. We will review the most important properties of the special geometry in § 4.1.

Only a few general observations can be made about the flux superpotential W_F since the form of W_F will highly depend on the point on the complex structure moduli space at which it is evaluated. One special point is the large complex structure point corresponding to a large volume point of the type IIA theory by mirror symmetry. One should bear in mind that, strictly speaking, mirror symmetry is only valid in the limit of large complex structure: Mirror symmetry identifies the large complex structure point of Y with the large radius point of $\widehat{Y}$. There can be more than one of these points and their mirrors can be in principal different. Thus, mirror symmetry is about *limiting families* of CY manifolds.

By the known monodromy of the B-field in the type IIA theory we know that this point must be of maximal unipotent monodromy. This implies on the type IIB side a maximal logarithmic degeneration of periods near this point $\{\underline{z} = 0\}$ [4, 9]. In general we have the following logarithmic structure of the periods[4]

$$X^0 \propto \mathcal{O}(\underline{z}), \quad X^i \propto \log \underline{z} + \mathcal{O}(\underline{z}), \quad F_i \propto (\log \underline{z})^2 + \mathcal{O}(\underline{z}), \quad F_0 \propto (\log \underline{z})^3 + \mathcal{O}(\underline{z}). \tag{2.15}$$

The zeroth period X^0 is the fundamental period having no logarithmic dependence on $\underline{z}$. By the local Torelli theorem, the X^A periods can be seen as local homogeneous coordinates of the complex structure moduli space. The mirror map is then given by

$$t^i = \frac{X^i}{X^0} \tag{2.16}$$

where t^i is the world-sheet volume complexified with the NS B-field on the type IIA side, i.e. the complexified Kähler moduli. These t^i can be also used as affine coordinates of the complex structure moduli space.

Let us now come to the enumerative interpretation of W_{flux}. Mirror symmetry maps the log-terms to classical large radius contributions while the regular terms in the F_i encode the closed string world-sheet instanton corrections. The prepotential F^0 encodes the classical couplings as well as the genus zero world-sheet instantons and takes the following general form

$$F^0 = -\frac{1}{3!}\mathcal{K}_{ijk}\, t^i t^j t^k - \frac{1}{2!}\mathcal{K}_{ij}\, t^i t^j + \mathcal{K}_i t^i + \frac{1}{2}\mathcal{K}_0 + \sum_\beta n^0_\beta \mathrm{Li}_3(q^\beta). \tag{2.17}$$

Many remarks are in order. We write $q^\beta = e^{2\pi i \beta_j t^j}$ for a vector β with entries $\mathbb{Z}_{\geq 0}$. The lattice vector β corresponds to an element of the lattice spanned by integral cycles in $H_2(X,\mathbb{Z})$ to which the genus zero worldsheet instanton is mapped to. The function $\mathrm{Li}_3(q)$ or more generally $\mathrm{Li}_k(q)$ is defined as

$$\mathrm{Li}_k(q) = \sum_{n=1}^{\infty} \frac{q^n}{n^k}. \tag{2.18}$$

[4] This structure gives the weight filtration of the limiting mixed Hodge structure [20, 21].

Furthermore, the classical terms are given by

$$\begin{aligned}\mathcal{K}_{ijk} &= \int_{\widehat{X}} J_i \wedge J_j \wedge J_k, & \mathcal{K}_{ij} &= \frac{1}{2}\int_{\widehat{X}} J_i \wedge i_* c_1(J_j),\\ \mathcal{K}_j &= \frac{1}{2^2 3!}\int_{\widehat{X}} c_2(\widehat{X}) \wedge J_j, & \mathcal{K}_0 &= \frac{\zeta(3)}{(2\pi i)^3}\int_{\widehat{X}} c_3(\widehat{X})\end{aligned} \tag{2.19}$$

and are determined by the classical intersections of the mirror CY threefold $\widehat{X}$. Note that by $c_1(J_j)$ we mean the first Chern class of the divisor associated to J_j and i_* is the Gysin homomorphism described in appendix A.1.2. Spelling out the definition, we see that $i_* c_1(J_j)$ is a four-form. The constants n^0_β are integral GV invariants, BPS numbers, which can be computed explicitly for a given example by solving the PF differential equations. Inserting the form of the pre-potential (2.17) into the flux superpotential (2.12) with $\widehat{M}^0 = 0$, we find

$$W_{\text{flux}} = \widehat{N}_0 + \widehat{N}_i t^i - \widehat{M}^i \left[\frac{1}{2}\mathcal{K}_{ijk} t^j t^k + \mathcal{K}_{ij} t^j + \mathcal{K}_i + \sum_\beta \beta_i n^0_\beta \operatorname{Li}_2(q^\beta)\right]. \tag{2.20}$$

This means that in addition to a cubic classical polynomial, also instanton correction terms proportional to $\operatorname{Li}_2(q)$ are induced by non-vanishing flux M^i.

D7-brane superpotential

Let us now turn to the superpotential induced by D7-branes. Ideally, we would like to compute the functional dependence of W_{D7} on the D7-brane and complex structure deformation moduli by solving the open-closed PF systems, i.e. PF operators involving both open and closed moduli. As we will see in § 5, this can be indeed achieved by lifting the setup to an F-theory compactification on a CY fourfold Y. A D7-brane with worldvolume flux $F_2 = \langle dA\rangle$ induces the following contribution to the superpotential [78, 79][5]

$$W_{\text{D7}} = \int_{\mathcal{C}_5} F_2 \wedge \Omega_X. \tag{2.21}$$

In the above superpotential, $\mathcal{C}_5$ is a five-chain with $D \subset \partial\mathcal{C}_5$ and therefore carries the information about the embedding of the D7-brane into X.

We will summarize the enumerative geometry involved in the counting problem for discs in more detail in § 2.4. Here, let us only note the form of the superpotential induced by the open string world-sheets. Recall that under mirror symmetry, a type IIB compactification with D7-branes is mapped to a type IIA compactification with D6-branes. In a supersymmetric configuration these D6-branes wrap special Lagrangian cycles L in the mirror CY threefold $\widehat{X}$. The (real) dimension of $H^1(L,\mathbb{Z})$ is the number of classical deformations[6] $\tilde{t}$ of L. The superpotential on the type IIA side is induced by string world-sheet discs ending on L

$$W_{\text{D7}} = C_i t^i \tilde{t} + C_{ij} t^i t^j + C\tilde{t}^2 + \sum_{\beta,n} n^0_{\beta,n} \operatorname{Li}_2(q^\beta Q^n) \quad \text{with} \quad Q = e^{2\pi i \tilde{t}}. \tag{2.22}$$

[5] In ref. [78], the superpotential is derived in the framework of generalized geometry. However, ref. [79] computes W_{D7} from the F-theory flux superpotential in weak coupling limit, formally matching the functional form of the superpotential of ref. [78].

[6] Note that $\tilde{t}$ is a complexified variable by the Wilson lines on L.

The constants C, C_i, C_{ij} and $n^0_{\beta,n}$ are determined by the geometry of D7-brane and X, as well as by the flux F_2. We find that W_{D7} contains both classical terms as well as instanton corrections which again has the Li_2 structure as the flux superpotential in eq. (2.20).

F-theory flux superpotential

We finally turn to the flux superpotential in F-theory. It is well-known that F-theory admits a superpotential upon switching on the four-form flux G_4. Let now Y be an elliptically fibered CY fourfold, i.e.

$$T^2 \longrightarrow Y \xrightarrow{\pi_Y} B_Y \tag{2.23}$$

where B_Y is the base manifold of the fibration. To determine the F-theory superpotential, we use the duality between M- and F-theory [79, 80]. In an M-theory compactification on Y we encounter the *Gukov-Vafa-Witten superpotential* [74]

$$W_F = \int_Y G_4 \wedge \Omega_Y \tag{2.24}$$

where Ω_Y is the holomorphic four-form of Y and $G_4 = \langle dA_3 \rangle$. The superpotential W_F depends on the complex structure deformations of Y. As we will discuss presently, upon imposing restrictions on the allowed fluxes G_4, the superpotential W_F also provides the correct expression for an F-theory compactification. The consistency condition on the flux are the following: The first constraint comes from the quantization condition for G_4 which depends on the second Chern class of Y in the following way [81]

$$G_4 + \frac{c_2(Y)}{2} \in H^4(Y, \mathbb{Z}). \tag{2.25}$$

The more restrictive condition comes from the fact that G_4 has to be primitive, i.e. orthogonal to the Kähler form of Y. In the F-theory limit of vanishing elliptic fiber this yields the constraints

$$\int_Y G_4 \wedge J_i \wedge J_j = 0 \quad \forall J_i \in H^{1,1}(Y) \tag{2.26}$$

where J_i are generators of the Kähler cone. As we will elaborate in § 5.2, the (co)homology groups of CY fourfolds split into horizontal and vertical subspaces

$$H^4_H(Y, \mathbb{Z}) = \bigoplus_{k=0}^{4} H_H^{4-k,k}(Y, \mathbb{Z}), \quad H^4_V(Y, \mathbb{Z}) = \bigoplus_{k=0}^{4} H_V^{k,k}(Y, \mathbb{Z}) \tag{2.27}$$

where we write $H^{p,q}_{H/V}(Y, \mathbb{Z})$ for $H^{p,q}_{H/V}(Y) \cap H^{p+q}(Y, \mathbb{Z})$. Since the dimension is even, the group $H^{2,2}(Y, \mathbb{Z})$ contains both, horizontal and vertical, parts and splits accordingly [82]

$$H^{2,2}(Y) = H^{2,2}_H(Y) \oplus H^{2,2}_V(Y). \tag{2.28}$$

The other groups $H^{p,q\neq p}(Y)$ do not split into two parts. As we will describe mirror symmetry later in § 5.2 in detail, we will be brief here . Analogous to the two-dimensional case of K3 and in contrast to the CY threefold case, the derivatives of Ω_Y w.r.t. the complex structure generate only the horizontal subspace. The remaining part is the vertical subspace which is the natural ring of polynomials in the Kähler cone generators J_i. Mirror symmetry *exchanges* the vertical and the horizontal subspaces. A corollary of these statements is that the allowed fluxes in the flux superpotential (2.29) are in the horizontal subspace. On the other hand, Chern classes are in the vertical subspaces. Consequently, half integral flux quantum numbers are not allowed if condition (2.26) is to be met. As for the type IIB flux superpotential W_{flux}, we can expand W_F in fourfold periods as follows

$$W_F = \int_Y G_4 \wedge \Omega_Y = N^{(i)\,a} \Pi^{(4-i)\,b} \eta^{(i)}_{ab} = N^{(i)\,a} \Pi^{(4-i)}_a \tag{2.29}$$

with periods and flux numbers, respectively,

$$\Pi^{(i)\,a} = \int_{\gamma^{(i)}_a} \Omega_Y, \quad N^{(i)\,a} = \int_{\gamma^{(i)}_a} G_4. \tag{2.30}$$

The four-cycles $\gamma^{(i)}_a$ represent a basis of the integral homology group $H_4(Y,\mathbb{Z})$. In contrast to CY threefolds, $H^4(Y,\mathbb{Z})$ does *not* carry a symplectic structure. Thus, the introduction of the intersection matrix $\eta^{(i)}$ is necessary. We refrain from giving the full-fledged definition, but postpone the detailed discussion to § 5.2. Clearly, the most important task is to find the periods which correspond to the integral over an integral basis of $H_4(Y,\mathbb{Z})$.

We will explicitly compute W_F for specific elliptically fibered CY fourfolds in § 5. The result is then matched with the superpotentials W_{flux} and W_{D7} at weak string coupling such that [79]

$$\int_Y G_4 \wedge \Omega_Y \longrightarrow \int_X G_3 \wedge \Omega_X + \sum_m \int_{\mathcal{C}^m_5} F^m_2 \wedge \Omega_X \tag{2.31}$$

where m labels all D7-branes on divisors D_m carrying two-form fluxes F^m_2 and X is the CY threefold of the type IIB in the weak coupling limit. Already by a pure counting of the flux quanta encoded by $G_4 \in H^4(Y,\mathbb{Z})$, as well as $F_3, H_3 \in H^3(X,\mathbb{Z})$ and $F^m_2 \in H^2(D^m,\mathbb{Z})$, we will generically encounter a mismatch. This can be traced back to the restrictions on G_4 as discussed above.

2.3 Superpotential in the heterotic string theory

In this section we discuss the superpotentials of heterotic string compactifications. We will only consider the $E_8 \times E_8$ heterotic string and compactifications which allow for dual F-theory. The heterotic/F-theory duality will be extensively investigated in § 6.1. Therefore, let the compactification manifold Z be an elliptically fibered CY threefold, i.e.

$$T^2 \longrightarrow Z \xrightarrow{\pi_Z} B_Z \tag{2.32}$$

where B_Z is the base manifold of the fibration. A consistent vacuum of the heterotic string requires, in addition to Z, a choice of a stable background gauge bundle $E = E_1 \oplus E_2$. This background bundle determines subgroups preserved from the perturbative $E_8 \times E_8$. In general, we can have five-branes wrapping holomorphic curves C of Z. The following anomaly cancellation condition further constrains the possible background bundle and five-branes

$$\lambda(E_1) + \lambda(E_2) + [C] = c_2(Z). \tag{2.33}$$

Here, $\lambda(E_i)$ denotes the fundamental characteristic class of the vector bundle E_i which is $c_2(E_i)$ for $SU(N)$ bundles and $c_2(E_i)/60$ for E_8 bundles. This condition dictates consistent choices of the cohomology class $[C]$ of the curve C in the presence of non-trivial bundles to match the non-triviality of the tangent bundle of Z measured by $c_2(Z)$. In particular, it implies that C corresponds to an effective class in $H_2(Z,\mathbb{Z})$ [83].

The analysis of the moduli space of the heterotic string consisting of the triple (Z,E_i,C) requires the study of three a priori very different pieces. Firstly, we have the geometric moduli spaces of the threefold Z consisting of the complex structure as well as the Kähler moduli space. Secondly, there are the moduli of the bundles E_i parameterizing different gauge backgrounds on Z. Finally, the deformations of C within Z have to be taken into account analogously to the situation of D5-branes. The entire moduli space is in general very complicated and difficult to analyze. It is even very difficult to find a suitable triple since the construction of stable bundles is a very hard problem. However, if one focuses on elliptic CY threefolds, there are well established constructions. The authors of ref. [84] have given elegant constructions of the stable bundles on elliptically fibered manifolds. We will review these constructions in § 6. Moreover, the moduli space of five-branes on elliptically fibered CY threefolds has been discussed in great detail in ref. [85].

Small instanton transition

In general, the moduli space of the triple admits several different branches corresponding to the number and type of five-branes present. However, there are distinguished points in the moduli space corresponding to enhanced gauge symmetry [86, 87] of the heterotic string allowing for a clear physical interpretation. At these points a transition is possible where during which a five-brane completely dissolves into a finite size instanton of the bundle E_i and vice versa. To understand this phenomenon, let us start without five-branes. The anomaly condition (2.33) forces us to turn on a background bundle E_i with non-trivial second Chern class $c_2(E_i)$ in order to cancel $c_2(Z_3)$. Then, the bundle is necessarily topologically non-trivial and carries bundle instantons characterized by the topological second Chern number [88]

$$[c_2] = -\int_Z J\wedge\mathcal{F}\wedge\mathcal{F} \tag{2.34}$$

where J denotes the Kähler form on Z and $\mathcal{F}$ the field strength of the background bundle. For simplicity let us concentrate on the first factor of the gauge group. The gauge group G_1 in four dimensions is generically broken and given by the commutant of the holonomy group of the bundle E_1 in E_8. Varying the moduli of E_1, we can restore parts or all of the broken gauge symmetry by flattening out the bundle as much as possible [89]. This is achieved by first decomposing $c_2(E_1)$ into its components each of them dual to an irreducible curve C_i in Z. Since the invariant $[c_2]$ is kept fixed, the best we can do is to consecutively split off the components of $c_2(E_1)$ and to localize the curvature of E_1 on the corresponding curves C_i. This should be contrasted with the generic situation where the curvature is smeared out all over Z. In this localization limit the holonomy of the bundle around each individual curve C_i becomes trivial and the gauge group G enhances accordingly. Having reached this so-called small instanton configuration at the boundary of the moduli space of the bundle, the dynamics of (this part of) the gauge bundle can be completely described by a five-brane on C_i [86].

Small instanton configurations thus allow for transitions between branches of the moduli space with different numbers of five-branes mapping the bundle moduli to the five-brane moduli and vice versa [90]. This is precisely what we need for our later F-theory analysis. Note that this transition is consistent with the anomaly cancellation since we have only shifted irreducible components between the two summands $c_2(E_1)$ and $[C]$. Thus, in the following, we will not distinguish between small instantons and five-branes. In particular, pushing this transition to the extreme, i.e. for all components of $c_2(E)$, full perturbative $E_8 \times E_8$ can be restored. This means also that a setting with full $E_8 \times E_8$ gauge symmetry on Z has to contain five-branes to cancel the anomaly according to eq. (2.33). In our concrete example of § 6.3 we will encounter this situation guiding us to the interpretation of the F-theory flux superpotential in terms of a superpotential for a particular class of five-branes.

Vertical and horizontal five-branes

For elliptically fibered CY manifolds Z there are two kinds of five-branes: The vertical five-branes wrapping the elliptic fiber denoted by $F = T^2$ and the horizontal five-branes wrapping holomorphic curves $\mathcal{C}$ in the base B_Z

$$C = n_F F + \mathcal{C} \quad \text{with} \quad n_F \in \mathbb{Z}_{\geq 0}. \tag{2.35}$$

This decomposition is according to the projection π_Z. In the dual F-theory compactifications these five-branes play different roles: Vertical five-branes correspond to spacetime filling three-branes at a point in the base B_Y of the F-theory fourfold Y [84, 91] and horizontal five-branes completely map to the geometry of the F-theory compactification [92, 93]. The map of the horizontal five-brane will be crucial and it will be discussed more thoroughly in § 6.2.2. From now on, we will mostly restrict our attention to the horizontal five-branes. The small instanton transition implies a transition between bundle and five-brane moduli [90]. Since both types of moduli are generally obstructed by a superpotential, also the superpotentials for bundle and five-brane have to be connected by the transition.

Superpotentials

The perturbative superpotential for the bundle moduli is given by the holomorphic CS functional [88]

$$W_{\text{CS}} = \int_Z \Omega_Z \wedge \left(A\bar{\partial}A + \frac{2}{3} A \wedge A \wedge A \right) \tag{2.36}$$

where A denotes the gauge connection that depends on the bundle moduli. Now, we want to argue that the above CS functional reduces to the chain integral expression

$$W_{\text{M5}} = \int_\Gamma \Omega_Z \tag{2.37}$$

in this transition. Let us assume a single instanton solution $\mathcal{F}$ with $\mathcal{F} \wedge \mathcal{F}$ dual to an irreducible curve C. In the small instanton limit $\mathcal{F} \wedge \mathcal{F}$ reduces to the delta function δ_C [94] describing the position moduli of the instanton within $[C]$. Inserting the gauge configuration $\mathcal{F}$ into $W_{\rm CS}$, the holomorphic CS functional is effectively dimensionally reduced to the curve C as done in similar setting in ref. [32]. In the vicinity of C we may write $\Omega_Z = d\omega$. Inserting this into eq. (2.36) in the background $\mathcal{F}\wedge\mathcal{F}$, we obtain

$$W_{\rm CS} = \int_C \omega \tag{2.38}$$

after a partial integration. Adding a constant given by the integral of ω over the reference curve C_0 this precisely matches the chain integral (2.37). Applying the above discussion, we can think about the five-brane moduli in $W_{\rm M5}$ as the bundle moduli describing the position of the instanton configuration $\mathcal{F}$ that in the small instanton limit map to sections of $H^0(C, N_{C/Z})$. We will verify this matching of moduli explicitly from the perspective of the dual F-theory setup later on. In this way, employing the heterotic/F-theory duality, we show in the case of an example the equivalence of the small instanton/five-brane picture. A different chain integral expression for $W_{\rm M5}$ was given in ref. [95] where the chain contains the spectral cover as its boundary.

To be complete in the discussion of perturbative heterotic superpotentials, let us also mention the flux superpotential due to bulk fluxes. In general, the heterotic B-field can have a non-trivial[7] background field strength H_3 in $H^3(Z,\mathbb{Z})$ due to the flux quantization. The induced superpotential will be intimately linked to eqs. (2.37) and (2.36) due to the Bianchi identity

$$dH_3 = \mathrm{Tr}\,\mathcal{R}\wedge\mathcal{R} - \frac{1}{30}\mathrm{Tr}\,\mathcal{F}\wedge\mathcal{F} - \sum_i \delta_{\mathcal{C}_i} \tag{2.39}$$

which yields, with an appropriate definition of the traces, the anomaly cancellation condition (2.33) upon restricting to cohomology classes. The superpotential in terms of H_3 reads [97, 98]

$$W_{\rm het} = \int_Z H_3\wedge\Omega_Z = W^{\rm het}_{\rm flux} + W_{\rm CS} + W_{\rm M5} \tag{2.40}$$

where the different terms can be associated to the various contributions in H_3 in eq. (2.39). Obviously, the above expression can be expanded in periods analogously to $W_{\rm flux}$ in eq. (2.13).

2.4 Enumerative geometry

It is well-known that topological string amplitudes $F^{g,h}$, g and h being the genus and the number of holes, compute special F-terms in the effective space-time theory of string theory. This means that these F-terms receive only contributions from fixed topology of the world-sheet in contrast to usual terms which get contributions from infinite tower of world-sheet topologies and quantum corrections on them. For closed world-sheets, i.e. $h = 0$, the amplitudes $F^g = F^{g,0}$ describe a sequence of higher derivative F-terms in the effective four-dimensional theory of the type IIA theory [24, 99]

$$\int d^4x \int d^4\theta\, \mathcal{W}^{2g} F^g(\underline{t}) = g\mathcal{R}_+^2\mathcal{F}_+^{2g-2}F^g(\underline{t}) + \cdots \tag{2.41}$$

[7]Strictly speaking there is a back-reaction of H_3 which renders Z to be non-Kähler [96]. Since our main focus will be on the five-brane superpotential, we will not be concerned with this back-reaction in this work.

where $\mathcal{W}$ denote the chiral superfield of $\mathcal{N}=2$ gravity multiplet and $\mathcal{R}_+^2$ and $\mathcal{F}_+^{2g-2}$ denote the self-dual part of the Riemann tensor and of the graviphoton field strength, respectively. Especially, the genus zero amplitude F^0 is the prepotential of the $\mathcal{N}=2$ theory.

For non-zero h, i.e. Riemann surfaces with boundaries, the topological string theory involves D-branes, thus reducing the amount of supersymmetry to $\mathcal{N}=1$. The amplitudes $F^{g,h\neq 0}$ computes the following F-terms [100][8]

$$\int d^4x \int d^2\theta \mathcal{W}^{2g} (\mathrm{Tr}\mathscr{W}^2)^{h-1} F^{g,h}(\underline{t},\underline{\tilde{t}}) N \tag{2.42}$$

where $\mathscr{W}$ is the chiral superfield of $\mathcal{N}=1$ gauge multiplet and N the number of coincident D-branes. In particular, $F^{0,1}$ and $F^{0,2}$ compute the superpotential and the gauge kinetic function

$$\int d^4x \int d^2\theta F^{0,1}(\underline{t},\underline{\tilde{t}}), \quad \int d^4x \int d^2\theta \,\mathrm{Tr}\mathscr{W}^2 F^{0,2}(\underline{t},\underline{\tilde{t}}). \tag{2.43}$$

Thus, the main object of this thesis, the superpotential, can be computed by using topological string theory.[9] The topological versions of the type IIA and IIB theories are called the A- and B-model [6, 7]. Henceforth, we will use the terms type (IIA, IIB) and (A,B)-model synonymously. We will also use the terms A- and B-branes.

Now, we want to describe from the A-model perspective the relevant enumerative quantities calculated in this work in the B-model using mirror symmetry. Important circumstantial evidence for the open-string/fourfold duality approach [36, 46, 38] is the identical integral structure of the generating functions. However, as reviewed in the following, the absence of higher genus invariants on smooth CY fourfolds as opposed to the topological open string amplitudes might be a hint that this duality merely relies on an embedding of the open/closed moduli space into the closed moduli space as discussed, e.g. in ref. [52], rather than on a full duality of physical theories. A possibility to avoid this conclusion would be that we have in general to consider singular CY fourfolds, typical for an F-theory compactification with degenerate elliptic fiber along the zero locus of the discriminant.

2.4.1 Closed Gromov-Witten invariants

First, we review the theory of closed GW invariants, i.e. the theory of holomorphic maps

$$\phi : \Sigma_g \longrightarrow \widehat{M} \tag{2.44}$$

from an oriented closed genus g curve Σ_g into a CY manifold $\widehat{M}$. We do not consider marked points.[10] The theory can be defined mathematically rigorously in general [11]. The invariants can be explicitly calculated using localization techniques [105] if $\widehat{M}$ is represented e.g. by a hypersurface in a toric variety. We denote by $\beta \in H_2(\widehat{M},\mathbb{Z})$ the homology class of the image curve. We measure the multi-degrees of β by

$$\deg(\beta) = \int_\beta c_1(\mathcal{L}) = \sum_{i=1}^{h^{1,1}(\widehat{M})} d_i t^i \quad \text{with} \quad \beta = \sum_{i=1}^{h^{1,1}(\widehat{M})} d_i \beta_i \quad \text{and} \quad d_i \in \mathbb{Z}_{\geq 0} \tag{2.45}$$

[8] For the case $g=0$ and $h\neq 0$ see also ref. [24].

[9] For reviews on topological string theory, see for example refs. [101, 102, 103, 104].

[10] Marked points are needed to compactify the moduli space of maps, see ref. [11]. The dimension of the moduli space of eq. (2.47) increases by n if we have n marked points.

w.r.t. to an ample polarization $\mathcal{L}$ of $\widehat{M}$. In string theory and in the context of the mirror symmetry the volume of the curve β_i is complexified by the B-field. Thus, we define the complexified closed Kähler moduli

$$t^i = \int_{\beta_i} (B + i c_1(\mathcal{L})). \tag{2.46}$$

For smooth $\widehat{M}$ the expected or the virtual dimension of the moduli space of these maps are computed by (HRR, A.1.5) and reads

$$\mathrm{vir}\dim \mathcal{M}_g(\widehat{M},\beta) = \int_\beta c_1(\widehat{M}) + (\dim \widehat{M} - 3)(1-g) = (\dim \widehat{M} - 3)(1-g). \tag{2.47}$$

From this formula it is obvious that CY threefolds have a special property: For all genera the dimension of the moduli space is zero. Thus, we have a well-defined counting problem. For higher dimensional CY manifolds, there are no moduli space for $g \geq 2$. In particular for genus zero, this means that $\mathrm{vir}\dim \mathcal{M}_0(\widehat{M},\beta) = \dim \widehat{M} - 3$. Thus in order to define genus zero GW invariants one requires an incidence relation of the curve with $k = \dim \widehat{M} - 3$ surfaces to reduce the dimension of the moduli space to zero in order to arrive at a well-defined counting problem. For fourfolds we thus need *one* incidence surface and we denote the dual cycle of the surface by $\gamma \in H^{2,2}(\widehat{M}_4)$.

We define a generating function for each genus g GW invariants as follows:

$$F^g(\gamma_i) = \sum_{\beta \in H_2(\widehat{M},\mathbb{Z})} r^g_\beta(\gamma_1,\ldots,\gamma_k) q^\beta \quad \text{with} \quad q^\beta = \prod_{i=1}^{h^{1,1}(\widehat{M})} e^{2\pi i d_i t^i}. \tag{2.48}$$

They are labelled by g, β and for $\dim \widehat{M} \geq 4$ also by cycles γ_i dual to the incidence surfaces. We note that this is not just a formal power series,[11] but rather has finite region of convergence for large volumes of the curves β_i, i.e. for $\mathrm{Im}(t^i) \gg 0$. This puts a bound on the growth of the GW invariants $r^g_\beta(\gamma_i)$. The contributions of the maps is divided by their automorphism groups and the associated GW invariants $r^g_\beta(\gamma_1,\ldots,\gamma_k)$ are in general rational numbers.

Although the discussion of the dimension (2.47) indicates that the GW theory on higher dimensional CY manifolds is less rich than in the CY threefold case, there is an integrality structure associated to the invariants. In particular at genus zero, integer invariants $n^g_\beta(\gamma_1,\ldots\gamma_k) \in \mathbb{Z}$ for arbitrary $(k+3)$-dimensional CY manifolds can be defined as

$$F^0(\gamma_1,\ldots,\gamma_k) = \frac{1}{2} C^{0\,(1,1,(k+3)-2)}_{ab\gamma_1\cdots\gamma_k} t^a t^b + b^0_{a\gamma_1\cdots\gamma_k} t^a + a^0_{\gamma_1\cdots\gamma_k} + \sum_{\beta>0} n^g_\beta(\gamma_1,\ldots,\gamma_k)\,\mathrm{Li}_{3-k}(q^\beta) \tag{2.49}$$

where $C^{0\,(1,1,(k+3)-2)}_{ab\gamma_1\cdots\gamma_k}$ are the classical triple intersections and Li_k is defined in eq. (2.18). For CY threefolds an analogous formula was found in [4] and the multicovering was explained in ref. [106]. Note that $b^0_{a\gamma_1\cdots\gamma_k}$ and $a^0_{\gamma_1\cdots\gamma_k}$ are irrelevant for the quantum cohomology as it is defined by the second derivative of $F^0(\gamma_1,\cdots,\gamma_k)$.

[11] This is important for the interpretation of such terms in the effective action. In fact, analyticity allows us to define them beyond the large radius limit point in terms of period integrals on the mirror geometry.

Genus one GW invariants exist on CY manifolds of all higher dimensions with the need of incidence conditions as discussed above. For fourfolds the following integrality condition can be defined [107]

$$F^1 = \sum_{\beta>0} n^1_\beta \frac{\sigma(d)}{d} q^{d\beta} + \frac{1}{24}\sum_{\beta>0} n^0_\beta(c_2(\widehat{M}))\log(1-q^\beta) - \frac{1}{24}\sum_{\beta_1,\beta_2} m_{\beta_1,\beta_2}\log(1-q^{\beta_1+\beta_2}). \quad (2.50)$$

Here, the m_{β_1,β_2} are the so-called meeting invariants which are likewise integer as the $n^g_\beta(\cdot)$ and the function σ is defined by $\sigma(d) = \sum_{m|d} m$.

For CY threefolds there is the BPS state counting formula obtained by evaluating 1-loop Schwinger computation for M2-branes in the dual M-theory [108]

$$F(\lambda, q) = \sum_{g=0}^{\infty} \lambda^{2g-2} F^g = \sum_{n=1}^{\infty} \sum_{\substack{\beta>0,\\ r\geq 0}} n^r_\beta \frac{1}{n}\left(2\sin\frac{n\lambda}{2}\right)^{2r-2} q^{n\beta} \quad (2.51)$$

where r corresponds to the left spin of the M2-brane in five dimensions in a certain basis of the representation, where the little group is $SO(4) \cong SU(2)\times SU(2)$, and also to the genus of the curve wrapped by M2-branes.

2.4.2 Open Gromov-Witten invariants

Let us now come to the open GW invariants on CY threefolds. They arise in the open topological A-model. Let $\widehat{X}$ be a CY threefold and L be a special Lagrangian submanifold of $\widehat{X}$ wrapped by the middle-dimensional A-brane. We consider maps from oriented open Riemann surfaces of genus g and with h holes into $\widehat{X}$

$$\psi : \Sigma_{g,h} \longrightarrow (\widehat{X}, L). \quad (2.52)$$

Here, the Riemann surface is mapped with a given winding number into L such that the h boundary circles B_i of $\Sigma_{g,h}$ are mapped to non-trivial elements $\alpha = (\alpha_1,\ldots,\alpha_h) \in H^1(L,\mathbb{Z})^{\oplus h}$. As in the closed case we do not consider marked points. In ref. [109, Ex. 7.8] the virtual dimension of the moduli space is computed to

$$\text{vir dim}\,\mathcal{M}_{g,h}(\widehat{X}, L, \beta, \alpha, \mu) = 0 \quad \text{for} \quad \mu = 0 \quad (2.53)$$

where μ is the Maslov index. The Maslov index is required to be zero due to anomaly cancellation constraint. For details see for example ref. [22, § 39]. If $H^1(L,\mathbb{Z})$ is non-trivial, L has geometric deformation moduli. The open string moduli $\tilde{t}^j$ are complexifications of the geometric moduli by the Wilson-Loop integrals of the flat $U(1)$ gauge connection on the brane.

Open BPS state counting formula analogous to the formula for closed case (2.51) can be derived by counting degeneracies of open M2-branes ending on an M5-brane wrapping L or D4-branes wrapping L in the type IIA picture [100]. It is given by

$$\begin{aligned} F(t,U) &= \sum_{g=0}^{\infty}\sum_{h=1}^{\infty}\sum_{\alpha^i_k\in\mathbb{Z}} \lambda^{2g-2+h} F^{g,\alpha^i}(t) \prod_{k=1}^{h} \mathrm{Tr}_{\mathcal{R}} \prod_{i=1}^{b_1(L)} U_i^{\alpha^i_k} \\ &= i\sum_{n=1}^{\infty}\sum_{\mathcal{R}}\sum_{\substack{\beta>0,\\ r\in\mathbb{Z}/2}} \frac{n^r_{\beta,\mathcal{R}}}{2n\sin(n\lambda/2)} q_\lambda^{nr} q^{n\beta}\,\mathrm{Tr}_{\mathcal{R}}\prod_{i=1}^{b_1(L)} U_i^n \end{aligned} \quad (2.54)$$

where $\alpha^i = (\alpha^i_1, \dots, \alpha^i_h) \in H^1(L,\mathbb{Z})^{\oplus h}$ and $q_\lambda = e^{2\pi i \lambda}$. The numbers $n^r_{\beta,\mathcal{R}}$ are integers counting BPS particles coming from the M2-branes ending on the M5-branes in representation $\mathcal{R}$, spin r and β corresponds to the bulk charge as for the closed formula. The $\mathrm{Tr}\, U_i$ denote the holonomy of the gauge field along non-trivial one-cycles of L on the D4-brane. The numbers α^i_k are winding numbers of the k-th boundary along an element of $H^1(L,\mathbb{Z})$. The matrices U_i describe the brane and thus correspond to the brane moduli. Additionally, $\mathcal{R}$ denote the representation of the U_i. For more details see ref. [100].

The disk amplitude, as already discussed in eq. (2.43), gives rise to the superpotential and is given by

$$W = F^{0,1} = \sum_{\beta,m} n^r_\beta \,\mathrm{Li}_2(q^\beta Q^m) \quad \text{with} \quad Q = e^{2\pi i \tilde{t}} \tag{2.55}$$

where $\tilde{t}$ corresponds to the open modulus. For notational simplicity we have assumed only one open modulus in the above formula. Comparison with the $(g=0, h=0)$ amplitude in eq. (2.49) suggests that the counting problem of specific disk amplitudes can be mapped to the counting of rational curves in CY fourfolds since the integrality structure is the same and is given by the Li_2 structure.

3 Local Calabi-Yau geometries

Should I refuse a good dinner simply because I do not understand the process of digestion?

O. Heaviside,

replying to criticism over use of operators [before justified formally]

Non-compact toric CY threefolds are the most thoroughly studied geometries in the context of closed and also open mirror symmetry. Due to the simplicity of the non-compact geometries, there are many advances using these geometries. Let us mention a few of them: Firstly, the closed mirror symmetry can be studied and the topological amplitudes computed exactly in the framework of the topological vertex [110]. Recently, some works appeared extending the vertex to settings with involution A-branes [111, 112]. Also, GW invariants for non-compact orbifolds are defined in ref. [113].

There is an intricate duality between certain matrix models and topological string theory on these geometries [114]. Recently, new symplectic invariants could be defined on Riemann surfaces and using the Riemann surface appearing in the mirror construction, these invariants can be associated to analytic expressions for the topological string amplitudes [115, 116]. Most notably the Bergman kernel is identified with the annulus amplitude and gives a global definition of the gauge kinetic function.

Thus, it will prove as worthwhile to review and understand these geometries since apart from other progresses, also the open mirror symmetry is best understood in these geometries. The computation of the superpotential W_B for B-branes was ignited in ref. [32]. In ref. [33] the calculation was carried out to further geometries. We will concentrate on these two works and review the toric CY manifolds and Harvey-Lawson type A-branes. Also, an example based on $\mathbb{P}^2$ will be discussed which will be of importance for later chapters.

3.1 Toric Calabi-Yau manifolds and A-branes

In toric CY manifolds the so-called Harvey-Lawson type A-brane can be simply given. These branes are extensively studied in refs. [32, 33]. We will generically denote a toric variety by V. However, since the toric CY threefolds are used in the A-model due to the lack of complex structure deformations, we will use the notation $\widehat{V}$.

The toric variety $\widehat{V}$ is represented as a symplectic quotient $\mathbb{C}^m//G$ and can be specified by k charge vectors $\ell^{(i)}$. Firstly, we impose vanishing moment maps, i.e. D-term constraints

$$\sum_{j=1}^{m} \ell_j^{(i)} |X_j|^2 = r^i. \tag{3.1}$$

Secondly, we divide by the isometry or gauge group $G = U(1)^k$ as $X_j \mapsto e^{i\ell_j^{(i)}\epsilon_i} x_j$ [117]. Then, solving the D-term constraint and using coordinates $\{p_j = |X_j|^2,\ \theta_j\}$, the toric variety $\widehat{V}_{m-k}$ can be visualized as a T^{m-k} fibration over a real $(m-k)$-dimensional base M_{m-k} [118, 32]. The degeneration loci of the T^{m-k} fibration where one or more S^1 shrink are on the boundary of M_{m-k} which is determined by $p_j = 0$ or intersections thereof since $p_j \geq 0$. The condition for $\widehat{V}$ being a CY manifold is $\sum_j \ell_j^{(i)} = 0$.

In the A-model the Harvey-Lawson type branes wrap special Lagrangian cycles L which can be specified by r additional brane charge vectors $\widehat{\ell}^{(a)}$ restricting the p_i and the angles θ_j in the toric ambient variety $\widehat{V}$ such that [32]

$$\sum_{j=1}^{m} \widehat{\ell}_j^{(a)} |X_j|^2 = c^a, \quad \theta_i = \sum_{a=1}^{r} \widehat{\ell}_i^{(a)} \phi_a \tag{3.2}$$

for angular parameters ϕ_a. To fulfill the *special* condition of L equivalent to $\sum_i \theta_i = 0$, we demand $\sum_j \widehat{\ell}_j^{(a)} = 0$. These A-branes are graphically represented as real co-dimension r subspaces of the toric base M_3.

The case which was considered for the non-compact examples in ref. [33] is $r = 2$ where the non-compact three-cycle L is represented by a straight line ending on a point when projected onto the base M_3. The generic fiber is a T^2 so that the topology of L is just $\mathbb{R} \times S^1 \times S^1$. However, upon tuning the moduli c^a it is most convenient to move the L_a to the boundary of M_3 where two $\{p_j = 0\}$ planes intersect. Then, one of the two moduli is frozen, and one S^1 pinches such that the topology becomes $\mathbb{C} \times S^1$. These A-branes or D6-branes are mirror to non-compact D5-branes which intersect a Riemann surface at a point. Later on, we will use the D5-brane results of refs. [32, 33] in order to study the superpotential (2.21) of D7-branes with gauge flux F_2 on compact CY manifolds. The gauge flux induces an effective D5-brane charge on the D7-brane and we will be able to compare the D5-brane superpotential of refs. [32, 33] to the D7-brane superpotential with appropriate F_2 in the local limit.

Now, let us describe the mirror dual picture with D5-branes [119, 19, 32, 33]. The B-model description is given as follows

$$uv = W(y_i) \quad \text{with} \quad u, v \in \mathbb{C} \quad \text{and} \quad y_i \in \mathbb{C}^\times \tag{3.3}$$

where y_i are homogeneous coordinate w.r.t. an additional $C^\times$-action and subject to the following constraints

$$\prod_{j=1}^{m} y_j^{\ell_j^{(i)}} = z_i \quad \text{where} \quad i = 1, \ldots, n. \tag{3.4}$$

Note that we do *not* introduce the zeroth component ℓ_0^i to each charge vector since we are working with non-compact geometries. Solving the above constraints, we can rewrite $W(y_i)$ as follows

$$uv = W(x, y; z_i) \quad \text{with} \quad x, y \in \mathbb{C}^\times. \tag{3.5}$$

This geometry is a cylinder bundle which is pinched over a Riemann surface $\mathcal{Y}$ given by the zero locus of $W(x, y; z_i)$. The B-branes on holomorphic submanifolds $\mathcal{V}$ in V are specified by

$$\prod_{j=0}^{m} y_j^{\hat{\ell}_j^{(a)}} = \epsilon^a e^{-c^a} \quad \text{with} \quad a = 1, \ldots, r. \tag{3.6}$$

The phases ϵ^a are dual to the Wilson line background of the flat $U(1)$ connection on the special Lagrangian L and complexify the moduli c^a to the open moduli [120]. As it is clear from the above defining equation, the B-brane is supported over a holomorphic cycle $\mathcal{V}$ of complex codimension r. Thus, for the configuration $r = 2$ the mirror of the A-brane is a D5-Brane. Other cases can be considered as well, leading to mirror configurations given by D7-branes on divisors ($r = 1$) or D3-branes on points ($r = 3$).

Periods on the B-model side

The main simplification for the computation of the non-compact geometries is the dimensional reduction in the B-model geometry. The holomorphic three-form of V reduces to a meromorphic differential [119]

$$\lambda = \frac{dy}{y} \log x \tag{3.7}$$

on the Riemann surface $\mathcal{Y}$. The three-cycles in $H_3(V, \mathbb{Z})$ reduce either to one-cycles a_i, b^i with $i = 1, \ldots, g$ in $H_1(\mathcal{Y}, \mathbb{Z})$ or to one-cycles c_k enclosing the poles of λ at the points p_i, cf. ref. [121] for a nice exposition of this point in the language of relative homology. The flat closed string moduli, its mirror map and the closed string prepotential are encoded in periods of λ over paths in the homology of $\mathcal{Y} \setminus \{p_i\}$. It reduces to

$$\int_\alpha \lambda \quad \text{with} \quad \alpha = e'^j c_j + e^i a_i - m_k b^k. \tag{3.8}$$

The D5-brane reduces to a point x on $\mathcal{Y}$, such that the triple $(\mathcal{Y}, \lambda, x)$ contains the non-trivial information of the B-model geometry with one non-compact D5-brane. It provides the geometrical realization of the non-trivial superpotential. The latter is obtained by reduction of the chain integral of eq. (2.1) to the Riemann surface

$$W_B(x, z, m) = \int_{\alpha^x} \lambda(z, m) \tag{3.9}$$

where the integral is over a path α^x from an irrelevant reference point x_0 to x and is an element of the relative homology $H_1(\mathcal{Y},\{p_i\},\mathbb{Z})$ as we will explain in detail in § 4.2. Beside the open modulus x dependence whose domain is simply the Riemann surface $\mathcal{Y}$, the integral depends on the complex modulus z of $\mathcal{Y}$ and potentially on constants m_i, which are the non-vanishing residua of $\lambda(m,z)$. After applying the mirror map, $W_B(x,z,m)$ can be identified with the disk instanton generating function [33]. The evaluation of $W_B(x,z,m)$ or more generally the integral

$$\int_{\widehat{\alpha}}\lambda+\int_{\alpha^x}\lambda=\int_{\widehat{\Gamma}^x}\lambda \quad \text{with} \quad \widehat{\alpha}\in H_1(\mathcal{Y},\mathbb{Z}) \quad \text{and} \quad \alpha^x\in H_1(\mathcal{Y},\{p_i\},\mathbb{Z}) \tag{3.10}$$

is a simple example of a problem in relative homology. On the Riemann surface the above integral can be solved by evaluating the integrals directly [33]. As already mentioned in § 2.1, the superpotential W_B defines an Abel-Jacobi map, albeit with meromorphic one-form instead of the holomorphic one-form. The specific elements $H_1(\mathcal{Y},\{p_i\},\mathbb{Z})$ yielding the closed string flat coordinates, the closed string mirror flat coordinates and the superpotential have been identified and described in ref. [33].

3.2 Example

In the following we will discuss the local CY geometry in which the explicit computations of open and closed BPS numbers can be performed. It is the local $\mathbb{P}^2$, i.e. $\mathcal{O}_{\mathbb{P}^2}(-3)$. We will later consider the elliptically fibered CY threefold in the weighted projective space $\mathbb{P}^4_{1,1,1,6,9}$ containing the non-compact geometry in the limit of large elliptic fiber in § 5.3.1.

In ref. [33] the geometry given by $\mathcal{O}_{\mathbb{P}^2}(-3)$ with non-compact Harvey-Lawson branes was considered. The local CY threefold is defined as the toric variety $\widehat{V}$ characterized by the polyhedron

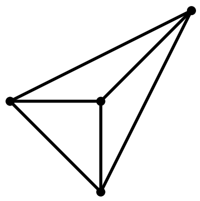
$$\left[\begin{array}{c|ccc|c|c} & & \Delta_{\widehat{V}} & & \ell^{(1)} & \\ \hline v_1 & 0 & 0 & 1 & -3 & X_0 \\ v_1^b & 1 & 1 & 1 & 1 & X_1 \\ v_2^b & -1 & 0 & 1 & 1 & X_2 \\ v_3^b & 0 & -1 & 1 & 1 & X_3 \end{array}\right] \tag{3.11}$$

where the superscript $(\cdot)^b$ denotes the two-dimensional basis $\mathbb{P}^2$ and the X_i denote homogeneous coordinates. The projected fan is shown in Figure 3.1(a). The D-term constraint for this geometry reads

$$-3|X_0|^2+|X_1|^2+|X_2|^2+|X_3|^2=0 \tag{3.12}$$

and $\widehat{V}$ can be viewed as a $(S^1)^3$ fibration over a three-dimensional base M_3. The degeneration loci of the fiber, $\{|X_i|=0\}$, are shown in Figure 3.1(b). The brane is defined torically by the following brane charge vectors

$$\widehat{\ell}^{(1)}=(1,0,-1,0), \quad \widehat{\ell}^{(2)}=(1,0,0,-1). \tag{3.13}$$



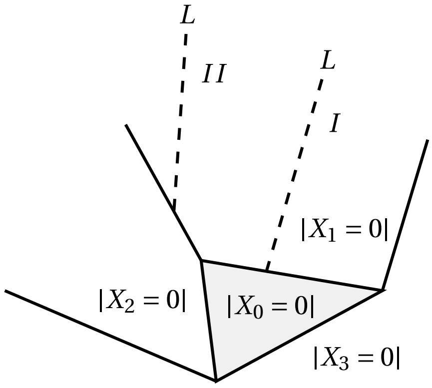


Figure 3.1: The left figure shows the projected fan of the local $\mathbb{P}^2$ and on the right the toric base and Harvey-Lawson Lagrangians are shown

These lead to the two constraints

$$|X_0|^2 - |X_2|^2 = c^1, \quad |X_0|^2 - |X_3|^2 = c^2 \tag{3.14}$$

where the c^a denote the open string moduli. The brane geometry is $\mathbb{C} \times S^1$ and can be described by a one dimensional half line in the three real dimensional base M_3 ending on a line where two of the three $\mathbb{C}^\times$ fibers degenerate. The A-brane has two inequivalent brane phases[1] I and II as indicated in Figure 3.1(b). Mirror Symmetry for this geometry was analyzed in ref. [33] where the disk instantons of the A-model were calculated. The mirror geometry is given by

$$uv = W(x, y; z) = x + 1 - z\frac{x^3}{y} + y. \tag{3.15}$$

This geometry effectively reduces to the Riemann surface $\mathcal{Y}$ defined by $W(x, y, z) = 0$. The A-brane is mapped under mirror symmetry to a D5-brane which intersects the Riemann surface $\mathcal{Y}$ in a point. In § 5 it will be this D5-brane picture which can be reformulated as a seven-brane with flux and embedded into an F-theory compactification.

[1]Note that our phase II is precisely phase III of ref. [33]. The phase II of ref. [33] has been omitted since it is equivalent to phase I by symmetry of $\mathbb{P}^2$.

4 D5-branes, mixed Hodge structure and blow-up

I cannot give their proof here because I do not understand it. Nevertheless, mathematicians I trust say that their argument is not only legitimate but brilliant, so let us assume they are right and continue.

S. Coleman,
Aspects of Symmetry

In $\mathcal{N} = 2$ compactifications the complex structure and Kähler moduli spaces locally decouple at generic point of the moduli space. Due to the special Kähler structure, it is possible to study the Kähler potential and the gauge kinetic function over the entire complex structure moduli space. They are related to the holomorphic prepotential. In this section we will discuss the less explored $\mathcal{N} = 1$ moduli space and concentrate on the superpotential induced by D5-branes. We will treat the open-closed moduli space in Hodge theoretical point of view. We will see that similar structure as for the special geometry appear, e.g. Hodge structure, PF equations etc.. Therefore, these structures are sometimes called $\mathcal{N} = 1$ special geometry in the literature. However, it should be noted that these structures are much less constrained as the usual $\mathcal{N} = 2$ special geometry.

Firstly, we will review the pure Hodge structure of the complex structure moduli space. Secondly, we will unify the flux and brane superpotential using relative (co)homology as in ref. [34, 35], thereby putting both superpotentials on the same geometrical footing. Then, we will discuss the mixed Hodge structure of the relative groups using the hypercohomology description. This will allow us to see analogous structure as in $\mathcal{N} = 2$ situations. To have better handle on the geometry, we will blow-up the curve wrapped by the D5-brane to a divisor. This will give us a canonical non-CY threefold with another mixed Hodge structure on the log cohomology groups. This blow-up geometry allows us to concentrate only on the complex structure deformations. The construction of the blow-up is done as complete intersection and we will discuss constructions of local as well as compact blow-ups. For practical computations, as usual in mirror symmetry, we need PF type differential equations. We will describe that these equations can be obtained exploiting the mixed Hodge structure and describe how this can be done in practice for general complete intersections. This chapter is based on ref. [52].

4.1 Pure Hodge structure

Since we will see many analogous structures later, we briefly review the complex structure moduli space, i.e. the vector multiplet moduli space of $\mathcal{N}=2$ compactification of the type IIB theory. Let us consider the type IIB theory compactified on a CY threefold X. This yields an $\mathcal{N}=2$ theory in four dimensions. The moduli space of the complex structures of X is a complex Kähler manifold and has the dimension $h^{2,1}(X)$. This means that X is a member of a family of CY manifolds $\varpi:\mathcal{X}\to\mathcal{M}_{\mathcal{N}=2}$ where $\mathcal{M}_{\mathcal{N}=2}$ denotes the moduli space. This moduli space is governed by the special geometry [75, 76, 2]. The main ingredient is the pure Hodge structure on $H^3(X,\mathbb{Z})$ and the Griffiths transversality both of which we will briefly review in the following. For more details on special geometry, cf. for example refs. [76, 77, 122] and for more Hodge theoretic treatment refs. [20, 21, 23].

The pure Hodge structure on $H^3(X,\mathbb{Z})$ consists of the decreasing Hodge filtration modules

$$F^m(X) = \bigoplus_{\substack{p+q=3,\\ p\geq m}} H^{p,q}(X). \tag{4.1}$$

Thus, each filtration module has the form

$$\begin{aligned}
F^3(X) &= H^{3,0}(X),\\
F^2(X) &= H^{3,0}(X)\oplus H^{2,1}(X),\\
F^1(X) &= H^{3,0}(X)\oplus H^{2,1}(X)\oplus H^{1,2}(X),\\
F^0(X) &= H^{3,0}(X)\oplus H^{2,1}(X)\oplus H^{1,2}(X)\oplus H^{0,3}(X) = H^3(X,\mathbb{C}).
\end{aligned} \tag{4.2}$$

We collectively denote the coordinates of $\mathcal{M}_{\mathcal{N}=2}$ by $\underline{z}$. The reason to consider the filtration modules is as follows: Each Dolbeault cohomology group $H^{p,q}(X_{\underline{z}})$ of the fiber does *not* fit together to a holomorphic bundle over $\mathcal{M}_{\mathcal{N}=2}$. However, each $F^k(X_{\underline{z}})$ forms a holomorphic bundle $\mathcal{F}^k(X)$. This holomorphic gauge is better suited if we do computations in the B-model since in this gauge the holomorphic dependence of the B-model correlation functions to be described in length in § 5.2 is easier to see [82]. The integral de Rham cohomology group $H^3(X_{\underline{z}},\mathbb{Z})$ only depends on the topology of the fiber and therefore does not depend on the complex structure moduli. Thus, $H^3(X_{\underline{z}},\mathbb{Z})$ forms a locally constant bundle $\mathcal{H}^3(X)$ over the moduli space. The flat connection ∇ of $\mathcal{H}^3(X)$ is called the Gauß-Manin connection. This connection satisfies the so-called *Griffiths transversality* property

$$\nabla:\mathcal{F}^k(X)\longrightarrow\mathcal{F}^{k-1}(X)\otimes\Omega^1_{\mathcal{M}_{\mathcal{N}=2}} \tag{4.3}$$

and one can show that $\{\mathcal{F}^3(X),\nabla\mathcal{F}^{k\leq 3}(X)\}$ span $\mathcal{H}^3(X)$. It is the combination of the Hodge filtration and the Griffiths transversality which allows for PF equations which are the main tools for computations.[1]

4.2 Relative (co)homology

It is possible to describe the flux and brane superpotential, given by eq. (2.12) and eq. (2.1) respectively, in an unified way. We need to use the language of relative (co)homology.[2] In contrary to the usual cohomology, the absolute cohomology, the relative cousin is a theory of a pair of manifolds. For our purpose the pair will consists of an ambient space X and a subspace $M \subset X$. To obtain the absolute cohomology groups $H^n(X)$, we consider the complex of i-forms

$$A^\bullet(X) = \{A^i(X), d\} \tag{4.4}$$

and its cohomology. Let M be a submanifold of X and $f : M \hookrightarrow X$ its embedding. For the relative cohomology groups $H^n(X,M)$ we construct a new complex

$$A^\bullet(X,M) = \{A^i(X) \oplus A^{i-1}(M), d\} \quad \text{with} \quad d(\omega,\theta) = (d\omega, f^*\omega - d\theta). \tag{4.5}$$

It can be easily checked that d squares to zero, thus, it is a differential. The relative cohomology groups $H^n(X,M)$ are constructed in usual way from the complex $A^\bullet(X,M)$. Note that the absolute cohomology can be obtained from $H^n(X,M)$ by setting $M = \emptyset$. It is also clear from the construction that $H^3(X) \subset H^3(X,M)$. Additionally, from the direct sum structure of $A^i(X,M)$ we have the following obvious short exact sequence

$$0 \longrightarrow A^{i-1}(M) \longrightarrow A^i(X,M) \longrightarrow A^i(X) \longrightarrow 0. \tag{4.6}$$

Consequently, we obtain the long exact sequence

$$\begin{array}{c} \cdots \longrightarrow H^{i-1}(M) \longrightarrow H^i(X,M) \longrightarrow H^i(X) \\ \xrightarrow{\;\delta\;} H^i(M) \longrightarrow H^{i+1}(X,M) \longrightarrow H^{i+1}(X) \longrightarrow \cdots \end{array} \tag{4.7}$$

where the connecting homomorphism δ is the pull-back of the embedding f [124, Claim 6.48]. From this long sequence we infer the following direct sum structure

$$H^i(X,M) = \mathrm{Ker}\Big(H^i(X) \to H^i(M)\Big) \oplus \mathrm{Coker}\Big(H^{i-1}(X) \to H^{i-1}(M)\Big). \tag{4.8}$$

We denote the first summand by $H^i_v(X)$ and the second by $H^{i-1}_v(M)$ for convenience.

Dual to the definition of relative cohomology, we can define the relative homology groups $H_n(X,M,\mathbb{Z})$. An element Γ of $H_n(X,M,\mathbb{Z})$ is an n-cycle of X whose boundary lies in M. Also for homology groups it is obvious that $H_n(X,\mathbb{Z}) \subset H_n(X,M,\mathbb{Z})$. As for absolute groups we can define a pairing between the relative cohomology and homology groups as follows

$$\langle \Gamma, \eta \rangle = \int_\Gamma \omega - \int_{\partial\Gamma} \theta \quad \text{with} \quad \Gamma \in H_i(X,M,\mathbb{Z}) \quad \text{and} \quad \eta = (\omega,\theta) \in H^i(X,M). \tag{4.9}$$

[1] It should be mentioned that there is also a limiting mixed Hodge structure at the point of maximal monodromy [21] which includes the log structure (2.15) of the solutions to the PF system.

[2] For more detailed discussion of relative groups and their other applications see for example [123, 124, 125].

It should be now clear how to use the relative (co)homology to unify the flux and the brane superpotential. We will only consider the RR part of the flux superpotential Since the three-chain Γ over which the holomorphic three-form is integrated is a non-trivial element of $H^3(X, M, \mathbb{Z})$, we can rewrite $W_F + W_B$ just as we have done it for W_F by using holomorphic volumes of relative three-cycles

$$W = W_F + W_B = N_A \int_{\gamma^A} \Omega + \widehat{N}_B \int_{\Gamma^B} \Omega = \widetilde{N}_A \widehat{\Pi}^A(\underline{z}, \underline{\hat{z}}) \tag{4.10}$$

where $\underline{\hat{z}}$ are the open moduli.

4.3 Mixed Hodge structure

The mixed Hodge structure is very central in the discussion of the brane superpotential and its computation as the pure Hodge structure is for the (closed) mirror symmetry. For more details on the subject we refer to [69, 126, 70, 71, 127]. Another very interesting application of this structure in physics can be found in ref. [128]. To discuss the mixed Hodge structure, the hypercohomology description of the relative group is more appropriate. We denote $\mathcal{O}_X(\bigwedge^k T_X^*)$ by Ω_X^k. Let us consider the following complex of sheaves

$$\Omega_f^\bullet = \{\Omega_X^i \oplus \Omega_M^{i-1}, \partial\} \quad \text{with} \quad \partial(\alpha, \beta) = (\partial\alpha, f^*\alpha - \partial\beta) \tag{4.11}$$

and the complex of cochains

$$C^\bullet(f, G) = \{C^i(X, G) \oplus C^{i-1}(M, G), \delta\} \quad \text{with} \quad \delta(\alpha, \beta) = (\delta\alpha, f^*\alpha - \delta\beta) \tag{4.12}$$

where G denotes the coefficients, e.g. $\mathbb{C}$, $\mathbb{Z}$, etc. Furthermore, we define the following double complex combining the above two complexes

$$C_f^{p,q} = C^p(\Omega_f^q) = \{C^p(X, \Omega_X^q) \oplus C^p(M, \Omega_M^{q-1}); \delta, \partial\} \tag{4.13}$$

From this double complex we construct the hypercohomology groups[3] $\mathbb{H}^k(\Omega_f^\bullet)$. We denote the cohomology groups $H^k(C^\bullet(i, G))$ built from the cochain complex $C^\bullet(f, G)$ by $H^k(i, G)$. Then, the following isomorphisms hold [69, p. 53]

$$H^k(i, \mathbb{C}) \cong H^k(X, M, \mathbb{C}) \cong \mathbb{H}^k(\Omega_f^\bullet). \tag{4.14}$$

It will be this definition of the relative groups which we will use to define the mixed Hodge structure on them. The hypercohomology spectral sequence[4] computing $\mathbb{H}^k(\Omega_f^\bullet)$ has the following first term

$$E_1^{p,q}(\Omega_f^\bullet) = H_\delta^q(\Omega_f^p) = \text{Ker}\left(\delta : C^q(\Omega_f^p) \to C^{q+1}(\Omega_f^p)\right) \Big/ \delta\left(C^{q-1}(\Omega_f^p)\right) \tag{4.15}$$

and degenerates at $E_2^{p,q}(\Omega_f^\bullet)$ term which has the following form

$$E_2^{p,q}(\Omega_f^\bullet) = H_\partial^p(H_\delta^q(\Omega_f^\bullet)) = \text{Ker}\left(\partial : H_\delta^q(\Omega_f^p) \to H_\delta^q(\Omega_f^{p+1})\right) \Big/ \partial\left(H_\delta^q(\Omega_f^{p-1})\right). \tag{4.16}$$

[3] For the discussion of hypercohomology see [129, § III.5].
[4] We refer to [129, 130] for details of spectral sequence.

This means that $\mathbb{H}^k(\Omega_f^\bullet)$ can be written as follows

$$\mathbb{H}^k(\Omega_f^\bullet) = \bigoplus_{p+q=k} E_2^{p,q}(\Omega_f^\bullet). \tag{4.17}$$

Analogously to the pure case, a decreasing filtration, the Hodge filtration, is needed to define a mixed Hodge structure. It is defined as follows [69, p. 53]

$$F^m\mathbb{H}^k(\Omega_f^\bullet) = \mathrm{Im}\left(\mathbb{H}^k(\Omega_f^{\geq m})\right) \tag{4.18}$$

where $\mathrm{Im}(\cdot)$ on the RHS is the image of the induced map on the cohomology from the embedding $\Omega_i^{\geq m} \hookrightarrow \Omega_i^\bullet$. Now, we want to write $F^m\mathbb{H}^k(\Omega_f^\bullet)$ in easier terms. We have

$$E_1^{p,q}(\Omega_f^{\geq m}) = \begin{cases} E_1^{p,q}(\Omega_f^\bullet) & \text{for } p \geq m, \\ 0 & \text{otherwise} \end{cases} \tag{4.19}$$

which follows trivially from eq. (4.15). Furthermore, we obtain for $E_2^{p,q}(\Omega_f^{\geq m})$

$$E_2^{p,q}(\Omega_f^{\geq m}) = \begin{cases} E_2^{p,q}(\Omega_f^\bullet) & \text{for } p > m, \\ \mathrm{Ker}\left(H_\delta^q(\Omega^p) \to H_\delta^q(\Omega^{p+1})\right) & \text{for } p = m, \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

which directly follows from the form of $E_1^{p,q}(\Omega_f^{\geq m})$ (4.19) and $E_1^{p,q}(\Omega_f^\bullet)$ (4.16). If we consider the image of $E_2^{m,q}(\Omega_f^{\geq m})$ in $\mathbb{H}^k(\Omega_f^\bullet)$, it is obvious that it equals $E_2^{p,q}(\Omega_f^\bullet)$. Thus,

$$F^m\mathbb{H}^k(\Omega_f^\bullet) = \mathrm{Im}\left(\mathbb{H}^k(\Omega_f^{\geq m})\right) = \bigoplus_{\substack{p+q=k,\\ p\geq m}} E_2^{p,q}(\Omega_f^\bullet). \tag{4.21}$$

For the case of interest, $k = 3$, we spell out the definition. By writing $F^m H^k = F^m\mathbb{H}^k(\Omega_f^\bullet)$ and $E_2^{p,q} = E_2^{p,q}(\Omega_f^\bullet)$ for convenience, we obtain

$$\begin{aligned} F^3 H^3 &= E_2^{3,0}, \\ F^2 H^3 &= E_2^{3,0} \oplus E_2^{2,1}, \\ F^1 H^3 &= E_2^{3,0} \oplus E_2^{2,1} \oplus E_2^{1,2}, \\ F^0 H^3 &= E_2^{3,0} \oplus E_2^{2,1} \oplus E_2^{1,2} \oplus E_2^{0,3} = H^3(X, M). \end{aligned} \tag{4.22}$$

The index structure is the same as the pure Hodge filtration (4.2) discussed earlier in § 4.1. To complete the definition of a mixed Hodge structure, in addition to the Hodge filtration, we need an increasing weight filtration.[5] The weight filtration for $H^k(X, M)$ is defined as follows

$$\begin{aligned} W_k\mathbb{H}^k(\Omega_f^\bullet) &= \mathbb{H}^k(\Omega_f^\bullet), \\ W_{k-1}\mathbb{H}^3(\Omega_f^\bullet) &= \mathrm{Im}\left(\mathbb{H}^k(\Omega_M^{\bullet-1}) \to H^3(\Omega_f^\bullet)\right), \\ W_{k-2}\mathbb{H}^3(\Omega_f^\bullet) &= 0. \end{aligned} \tag{4.23}$$

[5] As already mentioned in § 4.1, $\mathcal{M}_{\mathcal{N}=2}$ also has a mixed Hodge structure at the point of maximal unipotent monodromy. The increasing weight filtration corresponds to the increasing log structure of the solutions as shown in eq. (2.15).

For convenience we write $W_m H^k$ for $W_m\mathbb{H}^k(\Omega_f^\bullet)$. We want to show the following equality

$$W_2 H^k \cong \mathrm{Coker}\left(f^*: H^{k-1}(X,\mathbb{C}) \to H^{k-1}(Z,\mathbb{C})\right) = H_v^{k-1}(M,\mathbb{C}) \tag{4.24}$$

where f^* is the induced map appearing in the long exact cohomology sequence and the last definition is motivated from the decomposition (4.8) of $H^k(X,M)$. The spectral sequence $E_r^{p,q}(\Omega_M^{\bullet-1})$ computing $\mathbb{H}^k(\Omega_M^{\bullet-1})$ has the following first term $E_1^{p,q} = H_\delta^q(\Omega_M^{p-1})$ and degenerates at E_1-term. The second term of the spectral sequence for $\Omega_f^\bullet$ has the form

$$E_2^{p,q}(\Omega_f^\bullet) = \frac{\mathrm{Ker}\left(\partial: H^q(\Omega_X^p)\oplus H^q(\Omega_M^{p-1}) \to H^q(\Omega_X^{p+1})\oplus H^q(\Omega_M^p)\right)}{\partial\left(H^q(\Omega_X^{p-1})\oplus H^q(\Omega_M^{p-2})\right)} \tag{4.25}$$

where we have used the fact that $E_1^{p,q}(\Omega_f^\bullet) = H_\delta^q(\Omega_f^p) \cong H^q(\Omega_X^p)\oplus H^q(\Omega_M^{p-1})$. Furthermore, recall that $\mathrm{Coker}\, f^*$ is defined as $H^{k-1}(M)/\mathrm{Im}\, f^*$ meaning that $\mathrm{Coker}\, f^*$ consists of classes of $(k-1)$-forms on M which do not contain pull-back of $(k-1)$-forms on X. Obviously, the elements of $E_1^{p,q}(\Omega_M^{\bullet-1})$ are mapped to classes of $E_2^{p,q}(\Omega_f^\bullet)$ coming from the $H^q(\Omega_M^{p-1})$ part which are closed[6] under ∂ without involving classes of $H^q(\Omega_X^p)$. Additionally, we mod out classes of the form $\partial(\alpha,0) = (0, f^*\alpha)$ which are the images under f^*. Thus, it follows

$$\mathrm{Im}\left(\mathbb{H}^k(\Omega_M^{\bullet-1}) \to \mathbb{H}^k(\Omega_f^\bullet)\right) \cong \mathrm{Coker}\left(H^{k-1}(X,\mathbb{C}) \to H^{k-1}(M,\mathbb{C})\right). \tag{4.26}$$

We now define the graded weights as follows

$$\mathrm{Gr}_m^W \mathbb{H}^k(\Omega_f^\bullet) = W_m H^k / W_{m-1H^k}. \tag{4.27}$$

Using the decomposition (4.8), we can write

$$\begin{aligned} \mathrm{Gr}_k^W \mathbb{H}^k(\Omega_f^\bullet) &\cong \mathrm{Ker}\left(H^k(X,\mathbb{C}) \to H^k(M,\mathbb{C})\right), \\ \mathrm{Gr}_{k-1}^W \mathbb{H}^k(\Omega_f^\bullet) &\cong \mathrm{Coker}\left(H^{k-1}(X,\mathbb{C}) \to H^{k-1}(M,\mathbb{C})\right). \end{aligned} \tag{4.28}$$

Now, we have everything together to define the *mixed Hodge structure*. In general, it is defined on a free abelian group $H_\mathbb{Z}$ with a decreasing Hodge filtration $F^m H_\mathbb{C}$ and an increasing weight filtration $W_k H_\mathbb{C}$ where $H_\mathbb{C}$ is the complexification $H_\mathbb{Z}\otimes_\mathbb{Z}\mathbb{C}$. The case of interest for us is the relative group $H^3(X,M,\mathbb{Z})$. We have already defined the Hodge filtration $F^\bullet H^3 \equiv F^\bullet\mathbb{H}^3(\Omega_f^\bullet)$ and the weight filtration $W_\bullet H^3 \equiv W_\bullet\mathbb{H}^3(\Omega_f^\bullet)$. These two filtrations have to be defined such that the Hodge filtration $F^\bullet H^3$ induces a pure Hodge structure of weight k on the k-th graded weights $\mathrm{Gr}_k^W H^3 \equiv \mathrm{Gr}_k^W \mathbb{H}^3(\Omega_f^\bullet)$. The induced Hodge filtration on $\mathrm{Gr}_k^W H^3$ has the following form

$$F^p\,\mathrm{Gr}_k^W H^3 = (F^p H^3 \cap W_k H^3)/(F^p H^3 \cap W_{k-1} H^3) \cong \mathrm{Im}\left(F^p H^3 \cap W_k H^3 \to \mathrm{Gr}_k^W H^3\right). \tag{4.29}$$

Thus, for $H^3(X,M,\mathbb{C})$ where X is a CY threefold and M is a holomorphic curve in X, the following two induced filtrations

$$\begin{aligned} &F^3\cap H^3(X,\mathbb{C}) \subset F^2\cap H^3(X,\mathbb{C}) \subset F^1\cap H^3(X,\mathbb{C}) \subset F^0\cap H^3(X,\mathbb{C}) = H^3(X,\mathbb{C}), \\ &F^2\cap H_v^2(M,\mathbb{C}) \subset F^1\cap H_v^2(M,\mathbb{C}) \subset F^0\cap H_v^2(M,\mathbb{C}) = H_v^2(M,\mathbb{C}) \end{aligned} \tag{4.30}$$

[6]This means that if we would ignore the quotient by $\partial(E_1^{p-1,q}(\Omega_f^\bullet))$, then the image of $E_1^{p,q}(\Omega_M^{\bullet-1})$ would be just itself since $E_1^{p,q}(\Omega_M^\bullet) = E_\infty^{p,q}(\Omega_M^\bullet)$.

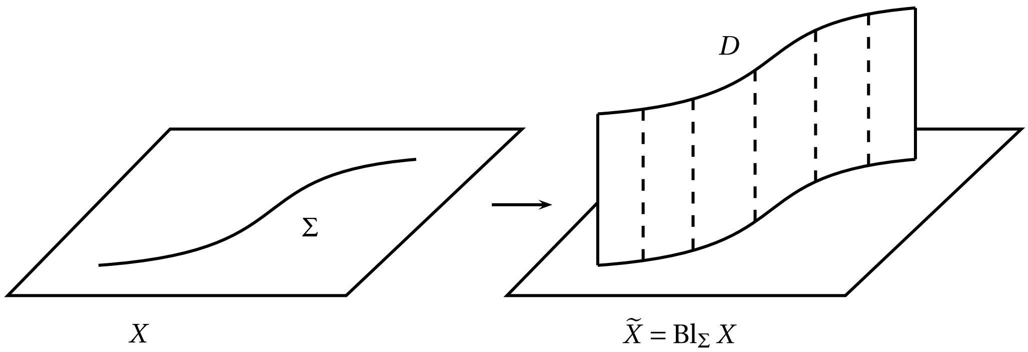


Figure 4.1: Blow-up of X along the curve Σ, resulting in $\widetilde{X}$ with exceptional divisor $D = \mathbb{P}(N_{\Sigma/X})$. The dashed vertical lines illustrate the fiber $\mathbb{P}^1$ of D.

form two pure Hodge structures of weight 3 and 2, respectively. Note that the first pure Hodge structure is the usual Hodge structure arising from the closed string sector giving rise to the special geometry.

4.4 Blow-up

Geometrically, it is difficult to work with curves in CY threefolds. Mainly, this is due to the fact that the objects at hand have co-dimension two. Thus, for conceptual and particularly for practical purposes, it is more adequate to consider co-dimension one objects, i.e. divisors, than curves. In this section we want to describe how we can obtain a canonically given divisor associated to the curve. In refs. [34, 35, 45, 46, 47, 48] a divisor which is not isolated is used to parameterize the deformations of the curve. The divisor we will consider will be isolated.

The main object of study is the relative group $H^3(X,\Sigma)$. Let us first use the Lefshetz and Poincaré duality to obtain

$$H^\bullet(X,\Sigma) \cong H_c^\bullet(X-\Sigma). \tag{4.31}$$

It can be shown [131, p. 37, C.][7] that the mixed Hodge structure of an open manifold U, i.e. in our case $U = X - \Sigma$, only depends on U and not on its compactification, i.e. in our case X. Hence, we can replace X and Σ by objects $\widetilde{X}$ and D satisfying

$$\widetilde{X} - D = U = X - \Sigma \tag{4.32}$$

meaning that we choose a different compactification of U. The deformations of the pair (X,Σ) which we denote by $\mathrm{Def}(X,\Sigma)$ are described more appropriately by an auxiliary pair $(\widetilde{X},D)$. The way to construct $\widetilde{X}$ and D is to blow-up X along Σ [129, § 4.6]. The blow-up procedure is depicted in Figure 4.1. The divisor D is the exceptional divisor. By construction, it is clear that $H^3(\widetilde{X} - D) = H^3(X - \Sigma)$. Furthermore, the deformation theory $\mathrm{Def}(X,\Sigma)$ is equivalent to $\mathrm{Def}(\widetilde{X},D)$ such that the variation of mixed Hodge structures of $H^3(X,\Sigma)$ and $H^3(\widetilde{X},D)$ over the moduli space are equivalent.

[7] See also [70, p. 214].

		1		
	g		g	
0		2		0
	g		g	
		1		

Table 4.1: Hodge diamond of an exceptional divisor with g being the genus of the curve

Exceptional divisor

Now, we want to discuss the geometry of $\widetilde{X}$ and D in more detail. Firstly, we turn to the exceptional divisor D. The exceptional divisor D is isomorphic to the ruled surface $\mathbb{P}(N_{\Sigma/X})$, i.e. a $\mathbb{P}^1$ bundle over Σ, and is a normal crossing divisor of $\widetilde{X}$. On any projectivization of a complex vector bundle there exists a natural line bundle which is called the tautological bundle T. It is the analog of $\mathcal{O}_{\mathbb{P}^n}(-1)$ of $\mathbb{P}^n$. Restricted to D, the bundle T is also the normal bundle of D. Since T does not have any holomorphic section, D is isolated and thus has no deformations in $\widetilde{X}$, i.e. $\dim H^0(D, N_{D/\widetilde{X}}) = 0$. Furthermore, the cohomology ring of D is generated by $\eta = c_1(T)$ as $H^\bullet(\Sigma)$ algebra with one relation

$$H^\bullet(D) = H^\bullet(\Sigma)\langle\eta\rangle \quad \text{with} \quad \eta^2 = c_1(N_{\Sigma/X}) \wedge \eta = -c_1(\Sigma) \wedge \eta. \tag{4.33}$$

Consequently, the Hodge diamond of D has a simple form and is shown in Table 4.4 where g is the genus of Σ. Here, the holomorphic one-forms are the Wilson lines a_I of Σ, the $(2,1)$-forms are of the form $a_I \wedge \eta$ and the two $(1,1)$-forms are given by η and $c_1(N_{\Sigma/D})$, the Poincaré dual of the curve Σ. Using twice the adjunction formula, one time for Σ as a divisor in D and another time for D as a divisor in $\widetilde{X}$, we obtain with the first Chern class of $\widetilde{X}$ (4.35), to be presented momentarily,

$$c_1(N_{\Sigma/D}) = -c_1(\Sigma) - 2\eta. \tag{4.34}$$

Naively, the moduli of Σ seem to have vanished after the blow-up since the exceptional divisor D is isolated as discussed above. However, this is not the case. The moduli of Σ have been transferred to the complex structure moduli of the ambient space $\widetilde{X}$. We will see this in explicit constructions of the blow-up in § 4.5.

Non-Calabi-Yau threefold

Next, we discuss the geometry of $\widetilde{X}$ in more detail. We first observe that the blow-up $\widetilde{X}$ is again a compact Kähler manifold [70, Prop. 3.24]. Secondly, $\widetilde{X}$ can still be embedded into $\mathbb{P}^N$ for some N, i.e. it is projective, since X, being CY, is projective. The first and the second Chern classes are affected by the blow-up as

$$c_1(\widetilde{X}) = \pi^* c_1(X) - (n-k-1)D, \quad c_2(\widetilde{X}) = \pi^*(c_2(X) + \eta_\Sigma) - \pi^* c_1(X) \cdot D \tag{4.35}$$

where n and k are the complex dimensions of X and Σ, respectively and $\eta_\Sigma \in H^4(X)$ the dual class of Σ. For X being CY, the formulas simplify to

$$c_1(\widetilde{X}) = -\eta, \quad c_2(\widetilde{X}) = \pi^*(c_2(X) + \eta_\Sigma). \tag{4.36}$$

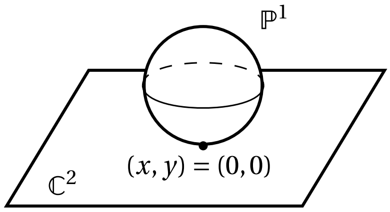


Figure 4.2: The blow-up of $\mathbb{C}^2$ at the origin

We can also determine the cohomology ring of $\widetilde{X}$ to be

$$H^\bullet(\widetilde{X}) = \pi^* H^\bullet(X) \oplus H^\bullet(D) \big/ \pi^* H^\bullet(Z) \tag{4.37}$$

where π is the natural projection from $\widetilde{X}$ to X. Since $H^{3,0}(\Sigma) = H^{3,0}(D) = 0$, it follows that $H^{3,0}(\widetilde{X}) \cong \pi^* H^{3,0}(X)$. One could worry that by pulling back one could lose the holomorphic three-form. However, this does not happen, since $h^{3,0}(\widetilde{X}) = h^{3,0}(X)$ [65, Thm. II.8.19]. The pulled-back holomorphic three-form $\widetilde{\Omega}$ has D as its zero locus, i.e.

$$\widetilde{\Omega}\big|_D = (\pi^*\Omega)\big|_D = 0. \tag{4.38}$$

This can be argued as follows: The first Chern class of a holomorphic vector bundle E describes the zero locus of a single section of the determinant line bundle $\det E$ [22, p. 47]. We can apply this for $E = T^*_{\widetilde{X}}$ by reading eq. (4.35) in terms of its Poincaré dual D and using $c_1(X) = 0$. This fact can be also easily seen in the local picture. Let us blow-up the origin of $\mathbb{C}^2$. The exceptional divisor is a $\mathbb{P}^1$. This blow-up is illustrated in Figure 4.2. If we denote the coordinates of $\mathbb{C}^2$ by (x, y) and the homogeneous coordinates of the $\mathbb{P}^1$ by (a, b), then the blown-up manifold $\widetilde{\mathbb{C}}^2$ is given by $ax = by$ in $\mathbb{C}^2 \times \mathbb{P}^1$. This means that we replace the point $\{x = y = 0\}$ by a $\mathbb{P}^1$. The blown-up manifold $\widetilde{\mathbb{C}}^2$ has two patches $U_a = \{a \neq 0\}$ and $U_b = \{b \neq 0\}$ since $\mathbb{P}^1$ has two. In U_a, we have $x = yb/a = \zeta x$ and thus we can take (ζ, y) as local coordinates. In these coordinates the original holomorphic two-form $\omega = dx \wedge dy$ changes to $\widetilde{\omega} = y d\zeta \wedge dy$. Thus, $\widetilde{\omega}$ is zero along the locus $\{y = 0\}$. For U_b we obtain the coordinates $(\kappa = xa/b, x)$ and $\widetilde{\omega} = x dx \wedge d\kappa$. The form $\widetilde{\omega}$ is compatible with the transition functions and thus it exists on the whole $\widetilde{\mathbb{C}}^2$. We explicitly see, that there is a holomorphic top-form which has the submanifold as its zero locus along which it was blown up.

4.4.1 Mixed Hodge structure on the log cohomology

After having blown-up X to $\widetilde{X}$, we have another structure available, namely the log cohomology and the mixed Hodge structure on it. Let D be a normal crossing divisor of $\widetilde{X}$, i.e. it can be locally written as $\{z_1 z_2 \cdots z_r = 0\}$ where $\{z_i\}$ are the complex coordinates of $\widetilde{X}$. For D we have the following isomorphism[8]

$$\phi : H^\bullet(\widetilde{X} - D, \mathbb{C}) \xrightarrow{\sim} \mathbb{H}^\bullet(\Omega^\bullet_{\widetilde{X}}(\log D)) \tag{4.39}$$

[8] For more details on $\Omega^\bullet_{\widetilde{X}}(\log D)$, see [129, p. 449].

where the first isomorphism is the Lefshetz and Poincaré duality. By $\Omega^k_{\widetilde{X}}(\log D)$ we mean holomorphic k-forms on $\widetilde{X}$ that are locally generated by e.g. dz^1, dz^2 and $d\log z_3 = dz^3/z_3$ with holomorphic functions as coefficients for a divisor locally given by $\{z_3 = 0\}$. Because of $d\log z_3$ these forms are denoted by $\Omega^1_{\widetilde{X}}(\log D)$. In general they have logarithmic singularities along D. As usual $\Omega^k_{\widetilde{X}}(\log D)$ is then given by the k-th exterior power of $\Omega^1_{\widetilde{X}}(\log D)$. For the hypercohomology of the log-complex there exists Hodge and weight filtration which give rise to a mixed Hodge structure. The filtrations are as follows [70, p. 208]

$$F^p H^k = \mathrm{Im}\left(\mathbb{H}^k(\Omega^{\geq p}_{\widetilde{X}}(\log D))\right), \quad W_q H^k = \mathrm{Im}\left(\mathbb{H}^k(W_q\Omega^\bullet_{\widetilde{X}}(\log D))\right) \tag{4.40}$$

where we write[9]

$$\begin{aligned} W_q\Omega^p_{\widetilde{X}}(\log D) &= \bigwedge^q \Omega^1_{\widetilde{X}}(\log D) \wedge \Omega^{p-q}_{\widetilde{X}} \\ &= \left\{\omega \in \Omega^p_{\widetilde{X}}(\log D) \mid \omega \text{ has a pole of order } \leq q \text{ along } D\right\}. \end{aligned} \tag{4.41}$$

On $H^k(\widetilde{X} - D)$, $F^\bullet H^k$ and $W_{\bullet+k}$ give a mixed Hodge structure. As for the Hodge filtration of a (general) pair § 4.3, we can write down $F^p H^k$ in easier terms using the same arguments as before

$$F^m H^k = \bigoplus_{\substack{p+q=k,\\ p\geq m}} E_1^{p,q}(\Omega^\bullet_{\widetilde{X}}(\log D)) = \bigoplus_{\substack{p+q=k,\\ p\geq m}} H^q(\Omega^p_{\widetilde{X}}(\log D)) \tag{4.42}$$

since the spectral sequence computing $\mathbb{H}^\bullet(\Omega^\bullet_{\widetilde{X}}(\log D))$ degenerates at the first term $E_1^{p,q}$.

From now on, we assume that D is smooth, i.e. it can locally be written as $\{z_n = 0\}$ where n is the (complex) dimension of $\widetilde{X}$. The weight filtration can then be described as follows

$$\begin{aligned} &W_{0+k}H^k = H^k(\widetilde{X}, \mathbb{C}), \\ &W_{1+k}H^k = H^k(\widetilde{X} - D, \mathbb{C}) \cong H^k(\widetilde{X}, D, \mathbb{C}), \\ &W_{2+k}H^k = 0 \end{aligned} \tag{4.43}$$

since $W_k\Omega^\bullet_{\widetilde{X}}(\log D)/W_{k-1}\Omega^\bullet_{\widetilde{X}}(\log D) \cong f_*\Omega^{\bullet-1}_D$ [70, Prop. 8.32] and $W_{1+k}H^k$ is the whole log complex. Thus, we obtain the graded weights

$$\mathrm{Gr}^W_{0+k} H^k \cong H^k(\widetilde{X}, \mathbb{C}), \quad \mathrm{Gr}^W_{1+k} H^k \cong H^{k-1}(D, \mathbb{C}). \tag{4.44}$$

4.5 Explicit blow-ups

4.5.1 Local geometries

In this section we construct an local example for which the blow-up procedure can be carried out explicitly. The non-compact CY manifold X is K_{F_n}, the canonical bundle of the n-th del

[9] In ref. [69] a slightly different notation is used: Meromorphic forms which are holomorphic on $\widetilde{X} - D$ are denoted by $\Omega^\bullet_{\widetilde{X}}(\log D)$. For example, refs. [129, 70] use $\Omega^\bullet_{\widetilde{X}}(*D)$ and denote the meromorphic forms which has a pole of order at most 1 along D by $\Omega^\bullet_{\widetilde{X}}(\log D)$. The hypercohomologies $\mathbb{H}^\bullet(\Omega^\bullet_{\widetilde{X}}(*D))$ and $\mathbb{H}^\bullet(\Omega^\bullet_{\widetilde{X}}(\log D))$ are however isomorphic [129, p. 453].

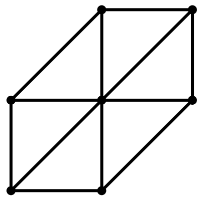


Figure 4.3: Polyhedron for F_3

Pezzo surface. The surface F_n is a $\mathbb{P}^2$ blown-up at n generic points to $\mathbb{P}^1$. We also wrap a space-time filling D5-brane on X such that it sits at $x \in F_n$ and also extends along the non-compact $\mathbb{C}$ fiber in X. The D5-brane can move on the del Pezzo surface which corresponds to moving the point x.

Let us first examine what is the minimal number of blow-ups in F_n for which the point x can be moved with respect to a fixed reference point $x_0 \in F_n$ such that the movement cannot be compensated by a coordinate redefinition. We count eight coordinate redefinition symmetries of $\mathbb{P}^2$ which is the dimension of $PGL(3,\mathbb{C})$ acting on the projective coordinates $\{x_1, x_2, x_3\}$. Hence, we have to mark at least four points in $\mathbb{P}^2$, each specified by two coordinates, to fix the coordinate freedom on $\mathbb{P}^2$. The movement of the fifth point then cannot be compensated by a coordinate redefinition. Thus, the fifth point gives rise to two complex open moduli describing its position in $\mathbb{P}^2$. Thus, we are lead to minimally consider F_3 with one fixed reference point x_0 in order to have open moduli.[10]

The canonical class of F_3 is given by $K_{F_3} = -3H + D_1 + D_2 + D_3$ where H is the hyperplane divisor and D_i are the three exceptional divisors. The CY threefold is then given by K_{F_3} and can be described torically by the following four charge vectors

$$\left[\begin{array}{c|ccccccc} \ell^{(1)} & -1 & -1 & 1 & 0 & 0 & 0 & 1 \\ \ell^{(2)} & -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ \ell^{(3)} & -1 & 0 & 1 & -1 & 1 & 0 & 0 \\ \ell^{(4)} & -1 & 1 & -1 & 1 & 0 & 0 & 0 \end{array}\right]. \tag{4.45}$$

The latter can be viewed as coefficients of linear relations among the vectors

$$(1,0,0), \quad (1,1,0), \quad (1,1,1), \quad (1,0,1), \quad (1,-1,0), \quad (1,-1,-1), \quad (1,0,-1) \tag{4.46}$$

which span the non-compact fan for X from the origin in $\mathbb{R}^3$. In the plane $(1, x, y)$, $(x, y) \in \mathbb{R}^2$ the fan contains the hexagonal polyhedron for F_3, see Figure 4.3. Each point in Figure 4.3 is associated to a coordinate $x_i \in \mathbb{C}$ and the Stanley-Reisner ideal SR is generated by all sets $\{x_{i_1} = \cdots = x_{i_r} = 0\}$ where $\{i_1, \ldots, i_r\}$ are not indices of a common triangle in the figure. Since X is a toric CY manifold, it has no complex structure moduli. However, once we include the D5-brane on the fiber at x (and fix the reference line at x_0), we find two complex open moduli $\{\zeta_1, \zeta_2\}$ which correspond to the two complex dimensions in which x can move on F_3 as we discussed above.

[10] This should be compared to the non-compact examples of § 3 where the D5-brane is a point on a Riemann surface $\mathcal{Y}$. If $\mathcal{Y}$ has genus $g = 1$, we need to fix the reference point x_0 to fix the freedom of coordinate choice.

Next, we want to use the insights of § 4.4 and blow up the curve Σ wrapped by the D5-brane and a reference curve Σ_0 into a divisor. We note that Σ intersects F_3 in the point x while a reference line Σ_0 intersects F_3 in the isolated point x_0. We recall that the blow-up divisor is the projectivization of the normal bundle $\mathbb{P}(N_{\Sigma/X})$ and $\mathbb{P}(N_{\Sigma_0/X})$. However, for x and x_0 not on the exceptional divisors in F_3, we can simply identify the blow-up divisors as the blow-ups of x and x_0 into two new $\mathbb{P}^1$. Therefore, the new base of $\widetilde{X}$ is the del Pezzo surface F_5. We can construct $\widetilde{X}$ as the total space of the line bundle $\nu : K_{F_3} \to F_5$ where $K_{F_3} = -3H + D_1 + D_2 + D_3$ only includes D_1, D_2, D_3 as in X. Now, however, the first Chern class does not vanish and equals $c_1(\widetilde{X}) = -\nu^* D_4 - \nu^* D_5$. This is in accord with the general formula (4.35) and matches our expectation that $\widetilde{X}$ is not CY.

We can also investigate what happened to the open moduli of the D5-brane in this set-up. Clearly, after blowing-up the exceptional divisors cannot be moved within F_5. This corresponds to the general fact the blow-up divisors are isolated. Thus, the two deformations $\{\zeta_1, \zeta_2\}$ of Σ have disappeared, but the del Pezzo surface F_5 has now two complex structure deformations $\{z_1, z_2\}$. These complex structure deformations can be canonically identified with $\{\zeta_1, \zeta_2\}$, and by studying the periods depending on $\{z_1, z_2\}$, we implicitly solve the original deformation problem for the curve Σ. Hence the complex structure moduli space of $\widetilde{X}$ captures the deformation space of the brane moduli on X.

Even for this non-compact CY threefolds, we have to ensure tadpole cancellation. In the case at hand we can explicitly give the orientifold involution. Since all directions normal to the D5-brane are compact, O5-planes with negative D5-brane charge have to be included in order to obtain a vanishing net RR charge. Therefore we consider the following involution on the del Pezzo base whose action on the basis (H, D_1, D_2, D_3) of the cohomology lattice is given by [132]

$$\sigma = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -0 \end{pmatrix}. \tag{4.47}$$

This involution has four fixpoints on the del Pezzo surface. We extend this involution to X by demanding it to act trivially on the fiber such that the O5-planes extend along the fiber and intersect F_3 in four points. Therefore, a consistent configuration requires eight D5-branes in the covering space. We conclude the example by noting that this non-compact situation can be generalized to compact examples. We replace the fibration of X with an elliptic fibration giving rise to a well-known elliptically fibered CY. The methods discussed in § 4.6.1 should be directly applicable to these examples and the open mirror symmetry can be studied in detail.

4.5.2 Global geometries

For concreteness, let us consider a CY threefold X described as the hypersurface $\{P = 0\}$ in a projective or toric ambient space V. Consider then a curve Σ specified by two additional constraints $\{h_1 = h_2 = 0\}$ in the ambient space intersecting transversally X. In general, the constraints h_1, h_2 describe divisors in the ambient space that descend to divisors[11] in X as well upon intersecting with $\{P=0\}$, called D_1 and D_2. Locally, (h_1, h_2) can be considered as normal coordinates to the curve Σ in X. Thus, the normal bundle $N_{\Sigma/X}$ of the curve takes the form $N_{\Sigma/X} = \mathcal{O}_X(D_1) \oplus \mathcal{O}_X(D_2)$. As the divisors D_i, also their line bundles $\mathcal{O}_X(D_i)$ are induced from the bundles $\mathcal{O}_V(D_i)$ on the ambient space V.

To describe the blown-up threefold $\widetilde{X}$, we introduce the total space of the projective bundle $\mathbb{P}(\mathcal{O}_V(D_1) \oplus \mathcal{O}_V(D_2))$. This total space describes a $\mathbb{P}^1$ fibration over the ambient space V on which we introduce the $\mathbb{P}^1$ coordinates $(l_1, l_2) \sim \lambda(l_1, l_2)$. Then, the blow-up $\widetilde{X}$ is given by the complete intersection[12]

$$P = 0, \quad Q = l_1 h_2 - l_2 h_1 = 0 \tag{4.48}$$

in this projective bundle. This is easily checked to describe $\widetilde{X}$. The first constraint depending only on the coordinates of the base V of the projective bundle restricts to the threefold X. The second constraint then fibers the $\mathbb{P}^1$ non-trivially over X to describe the blow-up along Σ. Away from $h_1 \neq 0$ or $h_2 \neq 0$ we can solve eq. (4.48) for l_1 or l_2 respectively. Thus, the two equations P and Q describes a point in the $\mathbb{P}^1$ fiber for every point in X away from the curve. However, if $h_1 = h_2 = 0$ the coordinates (l_1, l_2) are unconstrained and parameterize the full $\mathbb{P}^1$ which is fibered over Σ as its normal bundle $N_{\Sigma/X}$. Thus, we have replaced the curve by the exceptional divisor D that is given by the projectivization of its normal bundle in X, i.e. the ruled surface $D = \mathbb{P}(N_{\Sigma/X})$ over Σ. We denote the blow-down map by

$$\pi : \widetilde{X} \longrightarrow X. \tag{4.49}$$

As for a hypersurface CY manifold the holomorphic three-form $\widetilde{\Omega}$ of $\widetilde{X}$ can be represented by a residue integral [133, 73]

$$\widetilde{\Omega} = \int_{T(P,Q)} \frac{\Delta}{PQ} \tag{4.50}$$

where $T(P,Q)$ is the union of two S^1 bundles over the zero locus of P and Q in their normal bundles. The form Δ denotes a top-form on the ambient manifold $\mathbb{P}(\mathcal{O}_X(D_1) \oplus \mathcal{O}_X(D_2))$. For the type of ambient space we consider, the measure Δ takes the schematic form [134]

$$\Delta = \Delta_V \wedge (l_1 dl_2 - l_2 dl_1) \tag{4.51}$$

where Δ_V denotes the invariant form of V and (l_1, l_2) the coordinates of the $\mathbb{P}^1$ fiber. This makes it possible to study some of the properties of $\widetilde{\Omega}$ explicitly as we will see in the next section.

[11] The Lefshetz-Hyperplane theorem tells us that indeed any divisor and line bundle in X is induced from the ambient space [129].

[12] Abstractly, we can easily construct $\widetilde{X}$ without the help of the ambient space V. We only have to consider the equation $l_1 h_2 - l_2 h_1 = 0$ in $\mathbb{P}(\mathcal{O}_X(D_1) \oplus \mathcal{O}_X(D_2))$. However, this is not useful for practical purposes.

4.6 Picard-Fuchs equations

4.6.1 Two ways towards Picard-Fuchs equations

Let us summarize what we have learnt till now: The flux and the brane superpotential, W_{flux} and W_B, can be treated on the same footing using the relative (co)homology. The relative cohomology group has many different equivalent descriptions

$$H^3(X,\Sigma,\mathbb{C}) \cong \mathbb{H}^3(\Omega^\bullet_f)) \cong H^3(X-\Sigma,\mathbb{C}) \cong H^3(\widetilde{X}-D) \cong \mathbb{H}^3(\Omega^\bullet_{\widetilde{X}}(\log D)). \tag{4.52}$$

The closed and open moduli dependence can be characterized by PF equations. In the following, we will discuss two possible ways to derive these equations. The cases of most interest are those where X and $\widetilde{X}$ are described as complete intersections in (weighted) projective spaces, or more generally as toric variety, where powerful methods like residue representation of cohomology, GD reduction method etc. are available.

The first way is to use variation of mixed Hodge structure [135, 136]. We have already seen in eq. (4.30) that the graded weights $\mathrm{Gr}^W_k H^3$ induce pure Hodge structure of respective weight. The mixed Hodge structure on the log cohomology also induce the same structure. Using the graded weights (4.44), we obtain

$$\begin{aligned} &H^3(\widetilde{X})\cap F^3H^3 \subset H^3(\widetilde{X})\cap F^2H^3 \subset H^3(\widetilde{X})\cap F^1H^3 \subset H^3(\widetilde{X})\cap F^0H^3 = H^3(\widetilde{X}),\\ &H^2(D)\cap F^2H^3 \subset H^2(D)\cap F^1H^3 \subset H^2(D)\cap F^0H^3 = H^2_v(D). \end{aligned} \tag{4.53}$$

Here, for example, $H^2(D)\cap F^2H^3$ should be understood as follows: The second summand $H^2_v(D)$ of eq. (4.8) represents the part of $H^2(D)$ contained in the relative group $H^3(\widetilde{X},D)$. Thus, we use the isomorphism ϕ of eq. (4.39) to obtain their logarithmic counterparts. Then, we intersect $\phi(H^2(D))$ with F^2H^3. Analogously to the case of closed string moduli, $H^3(\widetilde{X}-D)$ forms a bundle $\mathcal{H}^3$ over the open-closed moduli space $\mathcal{M}$ with the Gauß-Manin connection ∇. Each F^mH^3 forms a subbundle $\mathcal{F}^m$ of $\mathcal{H}^3$. As already discussed, the Gauß-Manin connection has the following important transversality property

$$\nabla\mathcal{F}^p \subset \mathcal{F}^{p-1}\otimes\Omega^1_{\mathcal{M}}. \tag{4.54}$$

Combining this with eq. (4.30) and assuming that $\{\nabla_{\underline{z},\underline{u}}\mathcal{F}^k\}$ span $\mathcal{F}^{k-1}$, we see that

$$\begin{array}{ccccccc} \mathcal{H}^3(\widetilde{X})\cap\mathcal{F}^3 & \xrightarrow{\nabla_z} & \mathcal{H}^3(\widetilde{X})\cap\mathcal{F}^2 & \xrightarrow{\nabla_z} & \mathcal{H}^3(\widetilde{X})\cap\mathcal{F}^1 & \xrightarrow{\nabla_z} & \mathcal{H}^3(\widetilde{X})\cap\mathcal{F}^0 \\ & \searrow^{\nabla_u} & & \searrow^{\nabla_u} & & \searrow^{\nabla_u} & \downarrow{\scriptstyle \nabla_z,\nabla_u} \\ & & \mathcal{H}^2(D)\cap\mathcal{F}^2 & \xrightarrow[\nabla_z,\nabla_u]{} & \mathcal{H}^2(D)\cap\mathcal{F}^1 & \xrightarrow[\nabla_z,\nabla_u]{} & \mathcal{H}^2(D)\cap\mathcal{F}^0 \end{array} \tag{4.55}$$

where z denotes the closed string moduli and u the open string moduli. Here, again, we should understand the groups under the isomorphism ϕ, i.e. all forms occurring in eq. (4.55) are logarithmic three-forms. If we want to obtain a two-form representative of e.g. $\eta\in\mathcal{H}^2(D)\cap\mathcal{F}^2$, we consider $\phi^{-1}(\eta)$ which is an element of $H^2_v(D)$ and thus also an element of $H^2(D)$. As we can see the variations of the mixed Hodge structure has two levels: The closed string sector and a sector which mixes the open and closed moduli. As it has been pointed out in ref. [68], there exist differential equations obeyed by the relative periods of $H^3(X_t, D_t)$ where D_t denotes a family of divisors in the family of manifolds X_t. In particular, this covers our setting for the blow-up $\widetilde{X}$ by D. The resulting equations for the relative periods of $H^3(\widetilde{X}, D)$ are the advertised PF equations. One possible way to obtain these PF equations explicitly may be given by residue representations for the relative forms of $H^3(\widetilde{X}, D) \cong \bigoplus H^q(\widetilde{X}, \Omega^p_{\widetilde{X}} \log D)$ making explicit use of algebraic equations defining $\widetilde{X}$ and D. The main difficulty of this approach is to find explicit residue representation of $H^3(\widetilde{X}, D)$. Recently, there has been a progress in determining relative PF equations for a pair of manifold and a divisor [45, 47], not using toric methods.

The second ansatz relies on the study of the complex structure moduli of the blow-up $\widetilde{X}$. Since $\mathrm{Def}(\widetilde{X}, D)$ form a subset of deformations of $\widetilde{X}$, we can use the available techniques for ordinary complex structure deformations to describe the relevant PF equations. Using the algebraic equations for $\widetilde{X}$ as a complete intersection as in § 4.5.2, it is possible to apply the GD reduction method for residue representation of the unique holomorphic three-form $\widetilde{\Omega}$. We will describe in the next section how the GD algorithm works for complete intersections in great detail. We use the fact that $\widetilde{\Omega}$ vanishes on the exceptional divisor D as argued in § 4.4. This implies that $\widetilde{\Omega}$ is an element of $H^3_v(\widetilde{X})$ in the splitting (4.8) meaning that it can be represented as $(\widetilde{\Omega}, 0)$ in the relative cohomology on $\widetilde{X}$. Thus, eq. (4.37) allows us to represent $\widetilde{\Omega}$ as a pull-back form of $H^3(X)$. In this way we obtain PF operators $\mathcal{L}_i$ for $\widetilde{\Omega}$ with

$$\mathcal{L}_i \widetilde{\Omega} = d\alpha_i, \tag{4.56}$$

where α_i denote two-forms constructed by the GD method. Furthermore, we expect that the full effective superpotential W is a linear combination of the solutions to the corresponding PF system with the inhomogeneous piece given by functions obtained by integrating $d\alpha_i$ over chains. Indeed, we can replace all quantities occurring in the expansion of the superpotential into relative periods by corresponding relative periods on $\widetilde{X}$. First, we use the Lefshetz duality, cf. eq. (A.2) to replace $H_3(X, \Sigma)$ by $H_3(\widetilde{X}, D)$ as well as the corresponding integral basis Γ^{Σ}_i and Γ^D_j. Then, we replace the holomorphic three-form Ω by its pull-back $\widetilde{\Omega}$. This leads to the following expression for the superpotential,

$$W = \sum_j \widetilde{N}_j \, \langle \widetilde{\Omega}, \Gamma^D_j \rangle \,, \tag{4.57}$$

where $\widetilde{N}_j$ denote appropriately chosen integers. Thus, we used here the following for the flux superpotential

$$W_{\text{flux}} = \int_{\widetilde{X}} \widetilde{G}_3 \wedge \Omega = \int_{\widetilde{X}-D} G_3 \wedge \widetilde{\Omega} = \int_{\Gamma^D} \widetilde{\Omega}. \tag{4.58}$$

Next, we observe that the superpotential is annihilated by the PF operators $\mathcal{L}_i$ for $\widetilde{\Omega}$ as it just consists of the integral of $\widetilde{\Omega}$ over the relative cycles of $H_3(\widetilde{X}, D)$. Due to the isolation of the exceptional divisor D in $\widetilde{X}$ all deformations are now complex structure deformations of $\widetilde{X}$. Thus, we can choose a topological integral basis of $H_3(\widetilde{X}, D)$ not affected by the complex structure deformations on $\widetilde{X}$. This is in contrast to the original chains which depend on deformations of the boundary curves Σ in Y. It is a main advantage of the prescribed blow-up procedure that all moduli dependence of the relative periods of $\widetilde{\Omega}$ is captured by the dependence of $\widetilde{\Omega}$ itself.

The superpotential W is a linear combination of the solutions to the PF system on $\widetilde{X}$. In general there might be more complex structure deformations of $\widetilde{X}$ than $\mathrm{Def}(\widetilde{X},D)$, so that we have to identify the deformations, that correspond to the original deformation problem $\mathrm{Def}(\widetilde{X},D)$ and to restrict the dependence of the solutions to the PF system accordingly.

Comparison of the two methods reveals their advantages and drawbacks. On the one hand, it is necessary for the starting point of the first approach to find the residue representation of the logarithmic forms. Then, the remaining calculations should follow straight forwardly. On the other hand, it is clear for the second approach how to start, i.e. the residue representation of the holomorphic three-form of $\widetilde{X}$. However, the identification of the right moduli for the pair $(\widetilde{X},D)$ from the complex structure moduli $H^1(\widetilde{X},T_{\widetilde{X}})$ is crucial to obtain the relevant moduli dependence.

4.6.2 Picard-Fuchs equations of complete intersections

As we have seen in the previous section, the blow-up geometry can be represented as a complete intersection. There is a well-defined algorithm, GD method, to determine the PF system if the geometry is given as a hypersurface or as a complete intersection. In this section we describe the algorithm for complete intersections.

Let M be a threefold with $h^{3,0}(M)=1$. So, M can be a CY threefold or its blow-up. Furthermore, let us assume for simplicity that M is given by two equations f_1 and f_2 in $\mathbb{P}^5$. Let us denote $\{f_i=0\}$ by D_i. The generalization to general ambient toric variety is straight forward. The only difference is that there are more homogeneous coordinates with possibly different weights. Using the Griffiths residuum formula, we can represent the holomorphic three-form Ω_M as follows

$$\Omega_M=\int_{T(f_1,f_2)}\frac{1}{f_1f_2}\Delta=\mathrm{Res}_M\left(\frac{\Delta}{f_1f_2}\right)\quad\text{with}\quad\Delta=\sum_j(-)^jx_jdx^1\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^5 \tag{4.59}$$

where $\widehat{(\cdot)}$ denote the omission of the argument and $T(f_1,f_2)$ is the union of two tubular neighborhoods around D_i, i.e. S^1-bundles over D_i in $N_{D_i/\mathbb{P}^5}$. Generally, an element η of $H^3(M)$ can be represented by [137]

$$\eta=\mathrm{Res}_M\frac{p}{f_1^if_2^{n-i}}\Delta \tag{4.60}$$

and p is a homogeneous polynomial of appropriate degree. For notational brevity, we will omit Res_M from now on. To apply the GD method of reduction of pole order, we use the following two-form

$$\omega=\frac{1}{f_1^af_2^b}\sum_{i<j}(-)^{i+j}(x_jh_i-x_ih_j)\Lambda_{ij} \tag{4.61}$$

where $\Lambda_{ij}=dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^5$. Since $d\omega$ is exact, we obtain

$$\frac{a\sum_ih_iJ_1^i}{f_1^{a+1}f_2^b}+\frac{b\sum_ih_iJ_2^i}{f_1^af_2^{b+1}}=\frac{\sum_i\partial h_i/\partial x_i}{f_1^af_2^b}\quad\text{with}\quad J_i^j=\frac{\partial f_i}{\partial x_j} \tag{4.62}$$

up to exact forms. Let us now assume that η depends on a parameter ψ, then

$$\partial_\psi \eta = -\frac{i p \partial_\psi f_1}{f_1^{i+1} f_2^{n-i}} - \frac{(n-i) p \partial_\psi f_2}{f_1^i f_2^{n-i+1}} + \frac{\partial_\psi p}{f_1^i f_2^{n-i}}. \tag{4.63}$$

Thus, if the numerators of the first two terms of $\partial_\psi \eta$ are elements of the Jacobian ideal of f_1 and f_2, i.e. ideal of $\mathbb{C}[x_1, \cdots, x_5]$ spanned by partial derivatives of f_i, then we can reduce the pole order by using eq. (4.62). The strategy to determine PF equations is as follows: We differentiate Ω_M w.r.t. the parameter ψ. After taking k derivatives, we have an expression of the form

$$\eta_k = \frac{p}{f_1^a f_2^b} \Delta \quad \text{with} \quad a + b = k. \tag{4.64}$$

The pole order is k. Using the algorithm described above, we reduce the pole order of this expression by 1. From the resulting expression α_{k-1} we separate the part proportional to $\partial_\psi^{k-1} \Omega_M$ and denote the rest by η_{k-1}. Here, proportionality means

$$\beta = g \tilde{\beta} \quad \text{with} \quad g \in \mathbb{C}[\psi, \psi^{-1}]. \tag{4.65}$$

Then, we recursively apply the algorithm to η_{k-1} till we are left with an expression proportional to Ω_M. The result of this procedure is a PF equation

$$\partial_\psi^k \Omega_M = \sum_{i=0}^{k-1} q_i \partial_\psi^i \Omega_M. \tag{4.66}$$

If we apply the algorithm in practice, we proceed as follows. The k-th derivative of Ω_M with respect to the complex structure parameter ψ contains all possible pole order for f_i of the total pole order $k+2$. We then construct a vector with entries being the numerator of each pole order, i.e.

$$P_k^T = Q_k^T = \begin{pmatrix} p_{k+1,1} & p_{k,2} & \cdots & p_{2,k} & p_{1,k+1} \end{pmatrix} \tag{4.67}$$

where $p_{a,b}$ denotes the numerator (polynomial) of the term of pole order (a, b) in (f_1, f_2). For the reduction of the pole order, we use

$$K_k = \left(\begin{array}{ccccc|c|c} kJ_1 & & & & & & \\ J_2 & (k-1)J_1 & & & & & \\ & 2J_2 & \ddots & & & f_1 \cdot \mathbb{1}_{(k+1)\times(k+1)} & f_2 \cdot \mathbb{1}_{(k+1)\times(k+1)} \\ & & \ddots & J_1 & & & \\ & & & kJ_2 & & & \end{array} \right) \tag{4.68}$$

where J_i denote $(J_i^1, \cdots, J_i^m)$ with m being the number of homogeneous coordinates. This means that the matrix K_k is a $(k+1) \times (km)$ matrix. We then solve the matrix equation

$$P_k^T = K_k \cdot A \quad \text{with} \quad A = \left(\begin{array}{ccc|c|c} A_1 & \cdots & A_k & A_{k+1} & A_{k+2} \end{array} \right) \tag{4.69}$$

where $A_{i\leq k} = (A_i^1, \cdots, A_i^m)$, $A_{k+r} = (A_{k+r}^1, \cdots A_{k+r}^{k+1})$ and $A_i^j \in \mathbb{C}[x_1, \cdots, x_m]$. The entries of A correspond to h_i in eq. (4.62). Thus, from A, we build the following vector

$$\widetilde{Q}_{k-1} = \begin{pmatrix} \nabla A_1 + A_{k+1}^1 \\ \nabla A_2 + A_{k+1}^2 + A_{k+2}^2 \\ \vdots \\ \nabla A_{k-1} + A_{k+1}^k + A_{k+2}^k \\ \nabla A_k + A_{k+2}^{k+1} \end{pmatrix}. \tag{4.70}$$

As discussed above, we separate the part which is proportional (in the sense of eq. (4.65)) to P_{k-1} and denote the rest by Q_{k-1}. We then apply the same reduction algorithm to Q_{k-1} until we reach $\widetilde{Q}_0$ which necessarily is proportional to Ω_M. The algorithm can terminate earlier if $\widetilde{Q}_k - Q_k$ vanishes for some k. We described the case for one-parameter model.

For multi-parameter models, several different derivatives can produce terms of the same order of poles. We have to include all possible terms to determine the $\widetilde{Q}_k$ part. For brane geometry we also have to take also the exact pieces into account since we are integrating over chains instead of cycles. The technical difficulty arises since we have to perform multi-variable polynomial division. This is not possible in `Mathematica 7`. The algebra system `Macaulay 2` [138] can accomplish this, but not with rational functions in the parameter as coefficient. However, this can be easily circumvented. It would be interesting to determine the PF systems for examples and solve the system. This would enable us to check the blow-up proposal would allow for computations for large class of brane geometries, not restricted to branes given torically by charge vectors.

5 Lift to F-theory

Everything popular is wrong.

O. Wilde,
The Importance of Being Earnest

F-theory is a non-perturbative description of the type IIB theory with D7-branes [31]. For a nice “derivation” of F-theory from M-theory, see ref. [79]. It allows for holomorphically varying axio-dilaton $\tau = C_0 + ie^{-\phi}$ and the $SL(2,\mathbb{Z})$ symmetry of the type IIB theory is built right into the geometry. The main advantage of F-theory over the type IIB theory is that it geometrizes the axio-dilaton and D7-branes or their generalizations, (p,q) seven-branes, into a twelve-dimensional manifold, i.e. four complex dimensional internal manifold Y. The internal manifold has to have an extra structure, namely the elliptic fibration. If we want to embed D7-branes in a type IIB compactification, we have to include O7 orientifolds to cancel the tadpole. This means also that we have to find a consistent orientifold involution of the CY threefold which is a very difficult task in general. F-theory does this automatically once we have a suitable CY fourfold with desired seven-brane content. Sen provided an elegant way of obtaining consistent orientifold configurations from F-theory [139, 140, 141].

Recently, there have been lots of activities for F-theory GUT model building starting with refs. [142, 143, 144, 145]. These refs. construct non-compact, i.e. local, models. For compact examples, cf. for example refs. [146, 147, 148, 149, 150, 151].

Since D7-branes can have non-trivial worldvolume flux inducing D5-brane charge, we will argue and show in this chapter that we can lift the setting with a D5-brane to a F-theory setting. This means that we will construct an F-theory compactification with a CY fourfold which has the appropriate singularity and flux on the seven-brane worldvolume. Using this embedding and employing mirror symmetry for CY fourfolds, we will be able to compute the superpotential W_B. Explicit checks will be made using the BPS numbers computed for non-compact geometries.

This chapter is therefore organized as follows: Firstly, we will review the elliptic fibration and seven-branes in F-theory. Since we will be extensively using toric varieties and CY hypersurfaces in them, we quickly review the construction of elliptic CY threefolds and fourfolds. Our examples will have many CY fibration structures. Therefore, we will study how we can determine and construct such CY fibration structures in great detail. Then, we will describe mirror symmetry for CY fourfolds which differ from mirror symmetry for CY threefolds considerably. We will discuss the states and operators of A- and B-model and their underlying algebra structure, the Frobenius algebra. After having set all the required techniques, we will compute the superpotential for one main example. Results for further examples are relegated to appendix. This chapter is based on ref. [53].

5.1 F-theory and elliptic Calabi-Yau fourfolds

The computation of the F-theory flux superpotential (2.24) will be done for a class of CY fourfolds Y which we will introduce in this section. Our basic strategy in constructing a fourfold Y with a low number of complex structure moduli is first to construct its mirror $\widehat{Y}$ as a CY threefold fibration $\widehat{X}$ over a $\mathbb{P}^1$ base. The threefolds $\widehat{X}$ we are interested in are themselves elliptically fibered and admit a local limit yielding the non-compact geometries $K_{B_{\widehat{X}}}$ studied in ref. [33]. This fact will be exploited when we analyze the seven-brane content of the F-theory compactification Y and later on determine the F-theory flux superpotential which we split into flux and brane superpotential as in § 2.2.

Due to the importance of the involved geometries we will introduce the geometrical prerequisites here. We will first review the elliptic fibration and seven-branes in F-theory. Then, we will discuss the hypersurface description of CY manifolds in toric varieties since our examples will be of this type. Our example geometries admit rich CY fibration structure and thus, we will study in great detail how these structures arise and how we can determine them. We will see how we can construct CY fourfolds of desired type.

5.1.1 Elliptic fibration and seven-branes

Let us study the four-dimensional compactification of F-theory on an elliptically fibered CY fourfold Y over a Kähler three-dimensional base manifold B_Y. This corresponds to the type IIB string theory compactified on B_Y with an axio-dilaton $\tau = C_0 + ie^{-\phi}$ varying holomorphically over B_Y, i.e. F-theory on Y describes the type IIB theory on B_Y. Let us first discuss the elliptic fibration.

Elliptic fibration

We will quickly review the Weierstraß form of the elliptic fibration, cf. refs. [152, 89, 84]. Let us assume that we have a $\mathbb{P}^2_{1,2,3}$ fibration over the base manifold B

$$\mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3) \longrightarrow B \tag{5.1}$$

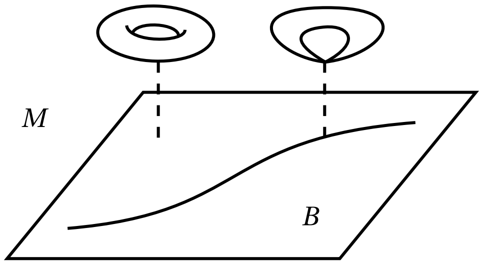


Figure 5.1: Elliptically fibered manifold M. Over the zero locus of the discriminant Δ depicted here as a curve the fiber tori degenerate.

where $\mathcal{L}$ is a line bundle over the base B. Let us denote the total space of the fibration by W. We have the following homogeneous coordinates of the fiber $\mathbb{P}^2_{1,2,3}$

$$x \in \Gamma(W, \mathcal{O}_W(2) \otimes \mathcal{L}^2), \quad y \in \Gamma(W, \mathcal{O}_W(3) \otimes \mathcal{L}^3), \quad z \in \Gamma(W, \mathcal{O}_W(1)). \tag{5.2}$$

The line bundle $\mathcal{O}_W(n)$ denote the n-th power of the inverse of the tautological bundle over W. We can see the corresponding weight of each homogeneous coordinate of $\mathbb{P}^2_{1,2,3}$ in eq. (5.2). The total Chern class of W is as following,[1] as explained in appendix A.1.7,

$$c(W) = c(B)(1+r)(1+2r+2c_1(\mathcal{L}))(1+3r+3c_1(\mathcal{L})), \tag{5.3}$$

i.e. $c_1(W) = c_1(B) + 6r + 6c_1(\mathcal{L})$. Here, we denote the hyperplane class of the fiber $\mathbb{P}^2_{1,2,3}$ by r. Since x, y, and z do not have common zeroes, we obtain

$$r(2r+2c_1(\mathcal{L}))(3r+3c_1(\mathcal{L})) = 0 \tag{5.4}$$

in the cohomology ring of W [84, § 7.2]. The Weierstraß equation is as follows

$$\mu = y^2 - x^3 - fxz^3 - gz^6 \quad \text{with} \quad f \in H^0(B, \mathcal{L}^4) \quad \text{and} \quad g \in H^0(B, \mathcal{L}^6). \tag{5.5}$$

Note that μ is a section of $\mathcal{O}_W(3) \otimes \mathcal{L}^6$.

The zero locus of μ is an elliptically fibered manifold. The fiber is a degree 6 hypersurface in $\mathbb{P}^2_{1,2,3}$, thus an elliptic curve or a torus. This will be the case for all our examples considered in this work even if M becomes singular. Let us denote the zero locus of μ by M. For M to be CY, we set $\mathcal{L} = K_B^{-1}$ since it can be shown that $N_{B/M} = \mathcal{L}^{-1}$ [89, p. 80]. From now on, we will assume that M is CY. The zero locus of the discriminant

$$\Delta = 4f^3 + 27g^2 \in H^0(B, \mathcal{L}^{12}) \tag{5.6}$$

gives the location where the fibration becomes singular. Figure 5.1 illustrates the setting. For later use we determine the second Chern class of M. In the cohomological relation (5.4), multiplication with $(6r + 6c_1(\mathcal{L}))$ means restriction to the CY hypersurface M. Consequently, $r(r + c_1(\mathcal{L})) = 0$ in the cohomology ring of M. For the second Chern class of M, we thus obtain

$$c_2(M) = 11c_1(B)^2 + c_2(B) + 12c_1(B)r. \tag{5.7}$$

Let us now come to the seven-branes in F-theory described by Δ.

[1] The formulas (5.3) and (5.7) are slightly different than those in ref. [84]. This is due to the use of $\mathbb{P}^2_{1,2,3}$ instead of $\mathbb{P}^2$.

Seven-branes in F-theory

As the axio-dilaton τ of the type IIB theory corresponds to the complex structure of the elliptic fiber, it can be specified by the value of the classical $SL(2,\mathbb{Z})$ modular invariant j-function expressed via f, g and Δ in eq. (5.5) as

$$j(\tau) = \frac{4(24f)^3}{\Delta}. \tag{5.8}$$

The function $j(\tau)$ admits a large $\mathrm{Im}\,\tau$ expansion $j(\tau) = e^{-2\pi i\tau} + 744 + \mathcal{O}(e^{2\pi i\tau})$ from which we can read off the monodromy of τ around a seven-brane.

The elliptic fibration is singular over the discriminant Δ. It can factorize into several components which individually correspond to divisors D_i in B_Y which are wrapped by seven-branes including the well-known D7-branes and O7-planes. The singularities of the elliptic fibration over the D_i determine the gauge group on the seven-branes. These can be determined explicitly using generalizations of the Tate formalism [153]. The weak coupling limit of F-theory is given by $\mathrm{Im}\,\tau \to \infty$ and yields a consistent orientifold setup with D7-branes on a CY manifold [139, 140, 141], cf. also refs. [154, 155, 156, 157] for recent treatments of this limit. It is important to note that the degeneration of the elliptic fibration can be so severe that the CY fourfold Y as given in eq. (5.5) becomes singular. In this case it is not possible to work with the singular space directly since the topological quantities such as the Euler characteristic and intersection numbers are not well-defined. To remedy this problem the singularities can be systematically blown up, the so-called crepant resolution, to obtain a smooth geometry [153]. In the cases considered in this chapter this is done using the methods of toric geometry [158, 153, 159, 160]. The resulting smooth geometry still contains the information about the gauge groups on the seven-branes and allows to analyze the compactification in detail.

In this chapter we will entirely focus on the complex structure sector of the CY fourfold Y. We will consider smooth spaces Y admitting only a small number of complex structure deformations, but are obtained from singular elliptically fibered CY fourfolds by multiple blow-ups. This affects only the number of Kähler moduli, which we will not discuss in the following. In order to compare to the type IIB weak coupling picture, the complex structure moduli can be split into three classes [79]:

- One complex modulus corresponding to the complex axio-dilaton τ parametrizing the complex structure of the elliptic fiber
- The moduli corresponding to the deformations of the seven-branes wrapped on divisors on B_Y
- The complex structure moduli corresponding to the deformations of the basis and its double covering CY threefold obtained in the orientifold limit

5.1.2 Calabi-Yau hypersurfaces

The most useful and powerful way to construct CY manifolds M and their mirrors $\widehat{M}$ for an arbitrary complex dimension n is by realizing them as hypersurfaces in toric varieties. These hypersurfaces are specified by reflexive polyhedra [158].

Let us start with the n-dimensional mirror CY manifold $\widehat{M}$ for the A-model. We realize $\widehat{M}$ as a hypersurface in a toric variety $\widehat{V}$. The hypersurface is constructed from a pair of reflexive polyhedra $\Delta_{\widehat{M}}$ and Δ_M in lattices M and N which are dual to each other, i.e. $\Delta_{\widehat{M}} = (\Delta_M)^*$. In general, the dual polyhedron Δ^* of a given polyhedron Δ in a lattice M is defined as the set of points p in the real span $N_{\mathbb{R}} = N \otimes \mathbb{R}$ of the dual lattice N such that

$$\Delta^* = \{p \in N_{\mathbb{R}} \mid \langle q, p\rangle \geq -1 \quad \forall q \in \Delta\}. \tag{5.9}$$

Let us assume that the combinatorics of $\widehat{V}$ associated to the polyhedron $\Delta_{\widehat{M}}$ is encoded in k charge vectors $\ell^{(j)}$ describing the relations among the $m = k + n$ vertices $\hat{v}_i$. The CY manifold $\widehat{M}$ is then given as the hypersurface $\{\hat{f} = 0\}$ in $\widehat{V}$ where $\hat{f}$ is given as the following polynomial [158]

$$\hat{f} = \sum_{q \in \Delta_M \cap N} \hat{a}_q \prod_i \hat{x}_i^{\langle \hat{v}_i, q\rangle + 1} \tag{5.10}$$

in the m homogeneous coordinates $\hat{x}_j$ of $\widehat{V}$ associated to each vertex $\hat{v}_j$. This formula provides a direct way to count the number of complex structure parameters $\hat{a}_q$ (up to automorphisms of $\widehat{V}$) by counting the integral points $q \in \Delta_M$. Furthermore, $\widehat{M}$ is CY using the general form of $H^0(\widehat{V}, \mathcal{O}_{\widehat{V}}(D))$ [161, p. 66] and the isomorphism between monomials built from $\{\hat{x}_i\}$ and elements of $H^0(\widehat{V}, \mathcal{O}_{\widehat{V}}(D))$ [162, Prop. 1.1], see [163, App. 3] for more details. Thus,

$$\hat{f} \in H^0(\widehat{V}, K_{\widehat{V}}^{-1}) \quad \text{with} \quad K_{\widehat{V}}^{-1} = \mathcal{O}_{\widehat{V}}\left(\textstyle\sum_{i=1}^m D_i\right) \quad \text{and} \quad D_i = \{\hat{x}_i = 0\}. \tag{5.11}$$

For the case of hypersurfaces in toric varieties the construction of the mirror CY manifold is realized in a very elegant way [158]. The mirror CY manifold M for the B-model is obtained by simply exchanging the roles of $\Delta_{\widehat{M}}$ and Δ_M such that eq. (5.10) describes M as the hypersurface in the toric variety V associated to the polyhedron Δ_M,

$$f = \sum_{p \in \Delta_{\widehat{M}} \cap M} a_p \prod_i x_i^{\langle v_i, p\rangle + 1}. \tag{5.12}$$

Here, we again associated the projective coordinates x_i to each vertex v_i of Δ_M. Indeed, the necessary requirements for mirror symmetry, $h^{1,1}(M) = h^{n-1,1}(\widehat{M})$ and $h^{n-1,1}(M) = h^{1,1}(\widehat{M})$, are fulfilled for this construction. This is obvious from the formulas for the Hodge numbers [158] for M and $\widehat{M}$

$$\begin{aligned} h^{n-1,1}(M) = h^{1,1}(\widehat{M}) &= l(\Delta_{\widehat{M}}) - (n+2) - \sum_{\dim \tilde{\theta} = n} l'(\tilde{\theta}) + \sum_{\mathrm{codim} \tilde{\theta}_i = 2} l'(\tilde{\theta}_i) l'(\theta_i), \\ h^{1,1}(M) = h^{n-1,1}(\widehat{M}) &= l(\Delta_M) - (n+2) - \sum_{\dim \theta = n} l'(\theta) + \sum_{\mathrm{codim} \theta_i = 2} l'(\theta_i) l'(\tilde{\theta}_i). \end{aligned} \tag{5.13}$$

In this expression θ ($\tilde{\theta}$) denote faces of Δ_M ($\Delta_{\widehat{M}}$) while the sum is over pairs $(\theta_i, \tilde{\theta}_i)$ of dual faces. The $l(\theta)$ and $l'(\theta)$ count the total number of integral points of a face θ and the number inside the face θ, respectively. Finally, $l(\Delta)$ is the total number of integral points of the polyhedron Δ. Using these formulas, we see that polyhedra with a small number of points will correspond to CY manifolds with few Kähler moduli, i.e. small $h^{1,1}$ and many complex structure moduli $h^{n-1,1}$. Since $h^{1,1}$ and $h^{n-1,1}$ are exchanged by mirror symmetry, CY fourfolds with small $h^{3,1}$

are obtained as mirror CY manifolds of hypersurfaces specified by a small number of lattice points in the polyhedron.

Let us now come to CY fourfolds. For the pair of CY fourfolds $(\widehat{Y}, Y)$ the complete list of model dependent Hodge numbers are $h^{1,1}(Y)$, $h^{3,1}(Y)$, $h^{2,1}(Y)$ and $h^{2,2}(Y)$. However, only three of these are independent due to (HRR, A.1.5) implying [164]

$$h^{2,2}(Y) = 2(22 + 2h^{1,1}(Y) + 2h^{3,1}(Y) - h^{2,1}(Y)). \tag{5.14}$$

Therefore, only $h^{2,1}(Y)$ has to be calculated to fix the basic topological data of $(\widehat{Y}, Y)$. Analogously to eq. (5.13), it is given by the symmetric expression

$$h^{2,1}(Y) = h^{2,1}(\widehat{Y}) = \sum_{\text{codim}\tilde{\theta}_i=3} l'(\tilde{\theta}_i)\, l'(\theta_i). \tag{5.15}$$

This finally enables us to calculate the Euler number of the CY fourfolds by

$$\chi(Y) = \chi(\widehat{Y}) = 6(8 + h^{3,1}(Y) + h^{1,1}(Y) - h^{2,1}(Y)). \tag{5.16}$$

5.1.3 Fibration structure of elliptic Calabi-Yau fourfolds

In the following we will discuss the construction and the fibration structure of elliptic CY fourfolds for which we want to compute the F-theory flux superpotential. Our strategy is to find fourfold examples admitting a small number of complex structure moduli such that we can evaluate the PF equations determining the holomorphic four-form Ω_Y. Candidate examples have already been considered in refs. [165, 164]. Moreover, we construct the CY fourfolds in such a way that they contain a local CY patch in which the effective D5-brane superpotential has been computed explicitly [32, 33].

The CY fourfolds studied in this work will be obtained as mirror to a CY threefold fibration over $\mathbb{P}^1$. We denote the CY threefold fiber by $\widehat{X}$. Summarizing, we can write

$$\widehat{X} \longrightarrow \widehat{Y} \longrightarrow \mathbb{P}^1. \tag{5.17}$$

We will later pick CY threefolds $\widehat{X}$ which are obtained by compactifications of local CY geometries which can support Harvey-Lawson type D6-branes as introduced in § 3. The compact CY threefolds $\widehat{X}$ have small number of Kähler moduli which is a feature inherited by $\widehat{Y}$. Moreover, since we want to study F-theory on the mirror CY fourfold Y, the CY threefold $\widehat{X}$ as well as the fibration structure of $\widehat{Y}$ will be chosen carefully such that Y has an elliptic fibration. This is achieved, for example, by choosing X elliptically fibered [93]. We have the following fibration structures

$$T^2 \longrightarrow X \longrightarrow B_X, \qquad T^2 \longrightarrow Y \longrightarrow B_Y \tag{5.18}$$

and

$$X \longrightarrow Y \longrightarrow \mathbb{P}^1, \qquad \mathbb{P}^1 \longrightarrow B_Y \longrightarrow B_X, \qquad \text{K3} \longrightarrow Y \longrightarrow B_X. \tag{5.19}$$

As discussed above, the generic elliptic fiber is given by a degree 6 hypersurface in $\mathbb{P}^2_{1,2,3}$ and is shared by X and Y. Figure 5.2 illustrates the fibration structures. Let us comment that the

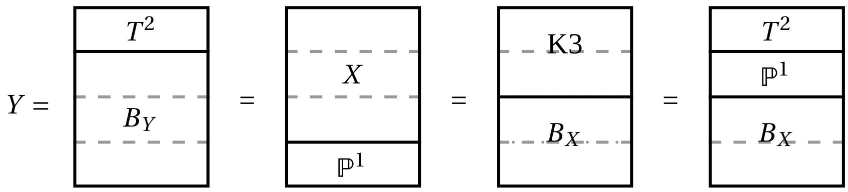


Figure 5.2: The fibration structure of the CY fourfold Y

two-dimensional base B_X cannot be the orientifold locus of Sen's limit, cf. appendix A.2.

To detect these fibration structures of a given mirror pair of CY fourfolds $(\widehat{Y}, Y)$ and in order to understand our construction more thoroughly, it turns out to be sufficient to study the toric data in the corresponding reflexive polyhedra $(\Delta_{\widehat{Y}}, \Delta_Y)$ without computing the intersection numbers [166]. In fact, for our example Y used in this chapter and further examples in appendix A.3.2 the intersection numbers will be hard to compute because of their huge number of Kähler moduli. In the remainder of this section we will recall the general theorem of ref. [166] and in § 5.3 we will apply it to our main example.

The theorem of ref. [166]

Suppose $(\widehat{Y}, Y)$ are given as hypersurfaces in toric varieties constructed from the reflexive pair $(\Delta_{\widehat{Y}}, \Delta_Y)$ in the pair of dual lattices (M, N). The statement of ref. [166] gives two equivalent conditions for the existence of a CY fibration structure of the given fourfold Y: Once in terms of Δ_Y and another time in terms of its dual polyhedron $\Delta_{\widehat{Y}}$. Assume there exists a $(n-k)$-dimensional lattice hyperplane in N through the origin such that $\Delta_F^k = H \cap \Delta_Y$ is a k-dimensional reflexive polyhedron. Then, this is equivalent to the existence of a projection P to a k-dimensional sublattice of M such that $P\Delta_{\widehat{Y}}$ is a k-dimensional reflexive polyhedron $\Delta_{\widehat{F}}^k$ which is the dual of Δ_F^k. If these conditions are satisfied, then the CY manifold Y obtained as a hypersurface of Δ_Y has a CY fibration whose $(k-1)$-dimensional fiber F_{k-1} is given by Δ_F^k. The crucial point of these two equivalent criteria is that we can turn things around: We can analyze the projection P in $\Delta_{\widehat{Y}}$ which is simpler by construction instead of analyzing Y and the hyperplane H in the complicated polyhedron Δ_Y. In both cases the base of the fibration can be found by considering the quotient polyhedron Δ_Y/Δ_F^k [160]. Here, this quotient polyhedron is obtained by first determining the quotient lattice in $M \supset \Delta_Y$ by dividing out the lattice generated by the integral points of Δ_F^k. The integral points of Δ_Y/Δ_F^k are the equivalence classes of integral points of Δ_Y in this quotient lattice. Schematically, the analysis of the fibration structure is summarized in Table 5.1. In the table the arrow $\leftrightarrow$ indicates the action of mirror symmetry interchanging projection and injection. Clearly, this analysis can be also used to determine CY fibers $\widehat{F}_{m-1}$ of the mirror $\widehat{Y}$. In general, it is not the case that mirror symmetry preserves fibration structures. However, in the constructions which we will analyze in § 5.3 we will find that both Y and $\widehat{Y}$ admit an intriguingly rich fibration structure.

fibration structure	$(\Delta_{\widehat{Y}}, \widehat{Y})$	$\leftrightarrow$	(Δ_Y, Y)
$\widehat{Y}$ admits CY fiber $\widehat{F}_{m-1}$	injection $\Delta^m_{\widehat{F}} = \widehat{H} \cap \Delta_{\widehat{Y}}$	$\leftrightarrow$	projection $\Delta^m_F = P\Delta_Y$
Y admits CY fiber F_{k-1}	projection $\Delta^k_{\widehat{F}} = P\Delta_{\widehat{Y}}$	$\leftrightarrow$	injection $\Delta^k_F = H \cap \Delta_Y$

Table 5.1: CY fibration structure of reflexive polyhedrons where we denote a m-dimensional CY manifold by F_m or $\widehat{F}_m$.

5.2 Mirror symmetry for Calabi-Yau fourfolds

In this section we will describe mirror symmetry for CY fourfolds. In its weak formulation it states the equivalence of the complex structure moduli space of Y and the (instanton corrected) Kähler moduli space of its mirror CY fourfold $\widehat{Y}$. As it was pointed out in ref. [7], this equivalence can be formulated in physical terms by considering topological field theories called the A- and B-model on the spaces $\widehat{Y}$ and Y. These theories are consistent cohomological truncations of some particular $\mathcal{N} = (2,2)$ superconformal field theories.[2] Their physical observables are subspaces of different cohomology groups. In particular, their marginal deformations coincide with the cohomology groups $H^{1,1}(\widehat{Y})$ for the A-model and $H^1(Y, T_Y)$ for the B-model. These are, in geometrical terms, precisely the infinitesimal deformations on the Kähler and complex structure moduli space of $\widehat{Y}$ and Y. Therefore, the physical statement of mirror symmetry is the equivalence of the A-model on $\widehat{Y}$ and the B-model on Y.

We will first describe the operator rings spanned by two- and three-point correlators of A- and B-model. Thereby, we will discuss in detail the geometrical quantities corresponding to those operators. These rings carry a natural Frobenius algebra structure. Mirror symmetry identifies the quantum cohomology ring of the A-model with the B-model ring. The precise matching is necessary for the enumerative predictions of the A-model and for the identification of the flux. Therefore, after having introduced the Frobenius algebra, we will describe how these rings of the A- and B-model are identified. It is crucial for this analysis to identify the integral basis of cohomology. One important step in this context is to determine the classical terms in the leading logarithmic period by means of analytic continuation to other points on the complex structure moduli space and a discussion of the monodromy. So in the last section we will study the behavior of the fourfolds periods near the conifold point of the moduli space. Since our discussion at several points can be generalized to arbitrary CY n-folds [82], we leave the dimension of the CY manifolds arbitrary in most cases.

5.2.1 States and correlation functions of the B-model

Let us consider a family of n-folds $\varpi : \mathcal{Y} \to \mathcal{M}$ with $\mathcal{M}$ being the complex structure moduli space. Let Y be a typical member of the family. The states[3] of the B-model are elements $B^{(j)}_k$ of

[2] For more details on the construction of cohomological field theories cf. refs. [6, 7] and also books and reviews e.g. refs. [19, 101, 103].

[3] We use in the following the same symbol for states and operators.

the cohomology groups $H^j(Y,\bigwedge^j T_Y)$. Their cubic forms are defined as

$$C(B_a^{(i)},B_b^{(j)},B_c^{(k)})=\int_Y \Omega(B_a^{(i)}\wedge B_b^{(j)}\wedge B_c^{(k)})\wedge\Omega \tag{5.20}$$

and vanish unless $i+j+k=n$. Here, $\Omega(\cdot)$ denotes the contraction of the n upper indices of the argument with Ω producing an anti-holomorphic n-form on Y. Note that this is the usual isomorphism given for CY manifolds

$$H^i(Y,\bigwedge^j T_Y)\cong H^{n-j,i}(Y). \tag{5.21}$$

We denote the image of $B_k^{(i)}$ in $H^{n-i,i}(Y)$ by $b_k^{(i)}={}^{\Omega}(B_k^{(i)})$ and the inverse[4] by $B_k^{(i)}=(b_k^{(i)})^{\Omega}$. Now, we can define the hermitian metric

$$G(B_c^{(i)},\overline{B}_d^{(i)})=\int_Y b_c^{(i)}\wedge\overline{b}_d^{(i)}. \tag{5.22}$$

In contrast to CY threefold case, $H^n(Y)$ splits into two parts, the vertical and the horizontal parts

$$H_H^n(Y,\mathbb{Z})=\bigoplus_{k=0}^{n}H_H^{n-k,k}(Y,\mathbb{Z}),\quad H_V^n(Y,\mathbb{Z})=\bigoplus_{k=0}^{n}H_V^{k,k}(Y,\mathbb{Z}). \tag{5.23}$$

We consider only states $B_a^{(i)}$ for which the image $b_a^{(i)}$ is in the horizontal subspace and assume that the $b_a^{(i)}$ form a basis of this space. For $B_c^{(1)}\in H^1(Y,T_Y)$ the image spans all of $H^{n-1,1}(Y)$ and the hermitian metric (5.22) is the well-known Weil-Petersson metric on $\mathcal{M}$. In the CY threefold case we can simply define the periods of the holomorphic three-form as in eq. (2.13) by introducing an integral symplectic basis for the third integral homology group. In the CY fourfold case however this is not possible. Let us fix a base point $\underline{z}_0\in\mathcal{M}$ and assume that Y is the fiber. We introduce a graded topological basis $\gamma_a^{(i)}$ of $H_n^H(Y,\mathbb{Z})$ with the following properties in order to define the periods

$$\Pi^{(i)\,a}=\int_{\gamma_a^{(i)}}\Omega_Y\quad\text{with}\quad i=0,\dots,n\quad\text{and}\quad a=1,\dots,h_H^{n-i,i}(Y) \tag{5.24}$$

where $\gamma_a^{(i)}$ denote a graded basis of $H_n^H(Y)$ with grading given by the index (i) for each group $H_H^{n-i,i}(Y)$. Here, we also introduce the dual basis $\widehat{\gamma}_a^{(i)}$ of $H_H^n(Y,\mathbb{Z})$ with the following pairing

$$\int_{\gamma_a^{(i)}}\widehat{\gamma}_b^{(j)}=\delta^{ij}\delta_{ab}. \tag{5.25}$$

This cohomology basis satisfies

$$\int_Y\widehat{\gamma}_a^{(i)}\wedge\widehat{\gamma}_b^{(j)}=\begin{cases}\eta_{ab}^{(i)} & \text{for } i+j=n,\\ 0 & \text{for } i+j>n.\end{cases} \tag{5.26}$$

It should be noted that the intersection matrix $\eta_{ab}^{(i)}$ is moduli independent. Later in § 5.2.3 we will identify the grading given by (i) with the natural grading on the observables of the A-model given by the vertical subspaces $H_V^{i,i}(\tilde{Y})$ of the mirror cohomology. Note that as in the threefold case a direct definition of the integral basis $\gamma_a^{(i)}$ is impossible. However, the existence of such a basis can be inferred by using mirror symmetry at the large complex structure point as we will describe in § 5.2.3.

[4] The inversion is the contraction $(b_k^{(i)})^{\Omega}=\frac{1}{||\Omega||^2}\overline{\Omega}^{a_1\dots a_{n-i}b_1\dots b_i}(b_k^{(i)})_{a_1\dots a_{n-i}\overline{b}_1\dots\overline{b}_i}$ such that $(\Omega)^{\Omega}=1$. Formally, it is the multiplication with the inverse $\mathcal{L}^{-1}$ of the Kähler line bundle $\mathcal{L}=\langle\Omega\rangle$, see e.g. ref. [22].

The filtration structure

As usual, $\mathcal{M}$ carries a pure Hodge structure. The discussion for CY threefold in § 4.1 can be trivially generalized to CY n-folds by restricting to the horizontal parts of the cohomology groups. The Hodge filtration is thus given by

$$F^k H^n = \bigoplus_{\substack{p+q=k,\\ p\geq m}} H_H^{p,q}(Y) \tag{5.27}$$

and $H_H^n(Y,\mathbb{Z})$ forms a locally constant bundle $\mathcal{H}_H^n$ over $\mathcal{M}$. The basis $\hat{\gamma}_a^{(i)}$ serves as a local frame for $\mathcal{H}_H^n$. The $H_H^{p,q}(Y)$ do not fit to a holomorphic bundle over $\mathcal{M}$, but $F^k H^n$ do and we denote these bundles by $\mathcal{F}^k$. We introduce frames $\beta_a^{(k)}$ for these bundles as follows[5]

$$\beta_a^{(k)} = \hat{\gamma}_a^{(k)} + \sum_{p>k} \Pi_a^{(p,k)\ c}\hat{\gamma}_c^{(p)} \quad \text{for} \quad \mathcal{F}^{n-k}. \tag{5.28}$$

The normalization of $\hat{\gamma}_a^{(k)}$ of $\beta_a^{(k)}$ is needed to obtain the correct affine flat coordinates $\{t_a\}$ of $\mathcal{M}$ and therefore for enumerative predictions of the A-model. In these coordinates we can write for example

$$\beta^{(0)} = \Omega, \quad \beta_a^{(1)} = \frac{\partial}{\partial t^a}\Omega. \tag{5.29}$$

By construction, in the basis of $\beta_a^{(k)}$ the period matrix P takes an upper triangular form

$$P = \int_{\gamma_a^{(p)}} \beta_b^{(k)} = \begin{cases} \Pi_b^{(p,k)\,a} & \text{for } p>k, \\ \delta_{ab} & \text{for } p=k, \\ 0 & \text{for } p<k \end{cases} \tag{5.30}$$

where (p,k) is the bi-grade of the non-trivial periods. The moduli dependence of P is captured by $\beta_a^{(k)}$ since $\gamma_a^{(p)}$ are topological and thus locally constant. By using the basis consisting of $\beta_a^{(k)}$, we are choosing the holomorphic gauge since $F^k H^n$ vary holomorphically over $\mathcal{M}$. This gauge is more appropriate for the B-model computation than treating $H_H^{p,q}(Y)$ directly [82].

Flat coordinates

Now, we come to the affine flat coordinates $\{t^a\}$. We expand Ω as follows

$$\Omega = \Pi^{(p,0)\,a}\,\hat{\gamma}_a^{(p)} = \Pi^{(p)\,a}\,\hat{\gamma}_a^{(p)} \quad \text{with} \quad \Pi^{(p)\,a} = \Pi^{(p,0)\,a} = \int_{\gamma_a^{(p)}} \Omega \tag{5.31}$$

where for CY fourfolds we have $p = 0,\ldots,4$ and $a = 1,\ldots,h_H^{4-p,p}(Y)$. Note that we do not write the index b of $\Pi_b^{(p,0)\,a}$ since $F^n H^n$ is one-dimensional. For an arbitrarily normalized Ω the periods

$$(X^0, X^a) = (\Pi^{(0)\,1}, \Pi^{(1)\,a}) \quad \text{with} \quad a = 1,\ldots,h^{n,1}(Y) \tag{5.32}$$

[5] The index (k) of $\beta_a^{(k)}$ is such that $\beta^{(0)}$ corresponds to the holomorphic n-form.

form homogeneous coordinates of $\mathcal{M}$. The inhomogeneous coordinates are given by

$$t^a = \frac{X^a}{X^0} = \int_{\gamma_a^{(1)}} \Omega \Big/ \int_{\gamma_1^{(0)}} \Omega. \tag{5.33}$$

The projectivity comes from the fact that we can arbitrarily scale Ω. At the point of maximal unipotent monodromy X^0 and X^a are distinguished by their monodromies: X^0 is holomorphic and single-valued and $X^a \propto \log(z)$ has the monodromy $X^a \to X^a + 1$. In mirror symmetry the t^a are identified with the complexified Kähler coordinates of the mirror CY n-fold $\widehat{Y}$. Additionally, they are flat w.r.t. the Gauß-Manin connection, i.e. the covariant derivative ∇_a becomes ordinary partial derivative $\partial/\partial t^a$ in these coordinates. This can be seen from the gauge choice reflecting in the basis $\beta_a^{(k)}$ (5.28) combined with the Griffiths transversality, together implying $\nabla_a t^b = \delta_a^b$ [82]. Also, in these coordinates the particularly important three-point coupling $C_{abc}^{(1,k,n-k-1)}$ becomes

$$C_{abc}^{(1,k,n-k-1)} = C((\beta_a^{(1)})^\Omega, (\beta_b^{(k)})^\Omega, (\beta_c^{(n-k-1)})^\Omega) = \int_Y \beta_c^{(n-k-1)} \wedge \partial_a \beta_b^{(k)}. \tag{5.34}$$

The above formula can be seen by recalling the definition of $\beta_a^{(k)}$ and the fact that under the integral of eq. (5.20) [82]

$$(\beta_a^{(1)})^\Omega \wedge (\beta_b^{(k)})^\Omega = (\nabla_a \beta_b^{(k)})^\Omega. \tag{5.35}$$

Furthermore, it can be shown using the properties of the Frobenius algebra that all other triple couplings (5.20) can be expressed in terms of the above coupling. We will see this in the next section.

The topological two-point couplings $\eta_{ab}^{(k)}$ in the basis of $\beta_a^{(k)}$ read as follows

$$\eta_{ab}^{(k)} = \int_Y \beta_b^{(n-k)} \wedge \beta_a^{(k)} \tag{5.36}$$

since only the lowest $\widehat{\gamma}^{(p)}$ for $p = k$ in the upper-triangular basis transformation (5.28) contributes to the integral, cf. eq. (5.26). In particular it is important to note that $\eta_{ab}^{(k)}$ is moduli independent. From the above it is easy to see the basis expansion at the grade $(k+1)$

$$\partial_a \beta_b^{(k)} = C_{abc}^{(1,k,n-k-1)} \eta_{(n-k-1)}^{cd} \beta_d^{(k+1)} \tag{5.37}$$

where $\eta_{(p)}^{cd}$ is the inverse of $\eta_{cd}^{(p)}$.

Let us end this section with some comments on general properties of the periods. The period integrals P in eq. (5.30) obey differential and algebraic relations which are different from the special geometry relations of CY threefold periods. They have however exactly the same origin, namely the Griffiths transversality, conveniently written as follows

$$\int_Y \Omega \wedge \partial_{i_1} \cdots \partial_{i_k} \Omega = 0 \quad \text{for} \quad k < n. \tag{5.38}$$

However, since $a \wedge b = (-1)^n b \wedge a$ for a, b real n-forms, we have additional algebraic relations from the trivial equation

$$\int_Y \Omega \wedge \Omega = 0 \tag{5.39}$$

between the periods $\Pi^{(p)\,a}$ for n even, like $n = 4$. These are absent for n odd, in particular for CY threefold cases.

5.2.2 Frobenius algebra

As it was already mentioned in the previous section, the B-model operators form a Frobenius algebra. Since also the A-model classical cohomology operators as well as its quantum cohomology operators form such an algebra, it is worthwhile to describe the general structure [164] before discussing the precise matching in the next section.

A Frobenius algebra is a graded vector space $\mathcal{A} = \bigoplus_{i=1}^{n} \mathcal{A}^{(i)}$ with $\mathcal{A}^{(0)} = \mathbb{C}$ equipped with a non-degenerate symmetric bilinear form η, a cubic form

$$C^{(i,j,k)} : \mathcal{A}^{(i)} \otimes \mathcal{A}^{(j)} \otimes \mathcal{A}^{(k)} \longrightarrow \mathbb{C} \quad \text{with} \quad i,j,k \geq 0 \tag{5.40}$$

and the following properties:

(F1) *Degree:* $C^{(i,j,k)} = 0$ unless $i + j + k = n$

(F2) *Unit:* $C^{(0,i,j)}_{1bc} = \eta^{(i)}_{bc}$

(F3) *Non-degeneracy:* $C^{(1,i,j)}$ is non-degenerate in the second slot

(F4) *Symmetry:* $C^{(i,j,k)}_{abc} = C^{(\sigma(i,j,k))}_{\sigma(a,b,c)}$ under any permutation of the indices.

(F5) *Associativity:*

$$C^{(i,j,n-i-j)}_{abp} \eta^{pq}_{(n-i-j)} C^{((i+j),k,(n-i-j-k))}_{qef} = C^{(i,k,n-i-k)}_{aeq} \eta^{qp}_{(n-i-k)} C^{(i+k,j,n-i-j-k)}_{pbf} \tag{5.41}$$

where the sum over common indices is over a basis of the corresponding spaces

The product

$$\mathcal{O}^{(i)}_a \cdot \mathcal{O}^{(j)}_b = C^{(i,j,i+j)}_{abq} \eta^{qp}_{(i+j)} \mathcal{O}^{(i+j)}_p \quad \text{with} \quad \mathcal{O}^{(i)}_a \in \mathcal{A}^{(i)} \tag{5.42}$$

defines the Frobenius algebra for a basis $\mathcal{O}^{(i)}_k$. This product corresponds to the operator product expansion of the B-model operators. The product is easily seen to be commutative using the symmetry property. Note that the associativity implies that n-point correlators can be factorized in various ways in the three-point functions. Also not all three-point correlators are independent. By associativity, non-degeneracy and symmetry it can be shown [164] that all $C^{(i,j,k)}$ can be expressed in terms of the $C^{(1,r,n-r-1)}$, for the B-model given as three-point correlators in eq. (5.34). It is easy to see that the $B^{(i)}$ operators of the B-model with the correlators defined by eq. (5.20) or equivalently eq. (5.34) and eq. (5.36) fulfill the axioms of a Frobenius algebra.

The A-model ring

Let us now consider the A-model operators. An operator corresponds to an element in the vertical subspace $H^{p,p}_V(\widehat{Y})$. The vertical subspaces are generated by J_i with $i = 1, \ldots, h^{1,1}(\widehat{Y})$. They are naturally graded as follows

$$A^{(p)}_b = a^{i_1,\ldots,i_p}_b J_{i_1} \wedge \cdots \wedge J_{i_p} \in H^{p,p}_V(\widehat{Y}). \tag{5.43}$$

For the classical A-model the correlation functions are simply given by the geometrical intersections

$$C^{0\,(i,j,k)}_{abc} = C(A^{(i)}_a, A^{(j)}_b, A^{(k)}_c) = \int_{\widehat{Y}} A^{(i)}_a \wedge A^{(j)}_b \wedge A^{(k)}_c \tag{5.44}$$

which vanish unless $i+j+k=n$. The topological metrics are analogously defined by

$$\eta^{(k)}_{ab} = \int_{\widehat{Y}} A^{(k)}_a \wedge A^{(n-k)}_b . \tag{5.45}$$

Together with the above classical three-point couplings a Frobenius algebra is defined. Clearly, the $A^{(p)}_b$ are not freely generated by the J_i. The products $J_{i_1}\wedge\cdots\wedge J_{i_p}$ are set to zero if their pairings (5.45) with all other cohomology elements vanish. For CY n-folds given as a hypersurface in a toric variety this is easily calculated using toric techniques and reflects geometrical properties of $\widehat{Y}$ like for instance fibration structures.

In the A-model the classical intersections are extended to quantum cohomological intersections[6]

$$C(A^{(i)}_a, A^{(j)}_b, A^{(k)}_c) = C^{0\,(i,j,k)}_{abc} + \text{instanton corrections} \tag{5.46}$$

where the instanton corrections are from holomorphic curves with meeting conditions on the homology cycles dual to the $A^{(p)}_b$ as described in § 2.4. They are such that the correlator vanishes again unless $i+j+k=n$. Note that the $C(A^{(i)}_a, A^{(j)}_b, A^{(k)}_c)$ depend via the instantons on the complexified Kähler moduli of $\widehat{Y}$ while $\eta^{(k)}_{ab}$ is still purely topological. There are no instanton corrections present because in the moduli space for the two-pointed sphere not all zero modes are saturated due to the one remaining conformal Killing field on the sphere.

5.2.3 Matching the A- and B-model Frobenius algebras

In this section we describe the matching of the A- and B-model Frobenius algebras. Since all our explicit examples will be hypersurfaces in toric varieties, we will restrict our attention to these cases. At the large radius point of the Kähler structure, the correlation functions of the classical A-model can be calculated using toric intersection theory. We will match this information with the leading logarithmic behavior of the periods of the B-model at the point of large complex structure characterized by its maximal logarithmic degeneration. This degeneration leads to a maximal unipotent monodromy.

Let us now discuss the PF differential operators for the CY n-fold Y at the large complex structure point. The PF operators can be very easily determined for toric examples. To the Mori cone generators $\ell^{(a)}$ of the A-model we associate the canonical GKZ system of differential operators for the B-model as follows

$$\mathcal{D}_a = \prod_{\ell^{(a)}_i>0}\left(\frac{\partial}{\partial a_i}\right)^{\ell^{(a)}_i} - \prod_{\ell^{(a)}_i<0}\left(\frac{\partial}{\partial a_i}\right)^{-\ell^{(a)}_i} \tag{5.47}$$

where the derivative is taken w.r.t. the coefficients a_i of monomials in the CY constraint defining Y as in eq. (5.1.2). By the methods described in ref. [9] we obtain linear PF operators $\mathcal{L}_a(\theta,z)$ written in terms of the logarithmic derivatives $\theta_a = z_a\frac{\partial}{\partial z_a}$ w.r.t. the canonical complex variables z_a which vanish at the large complex structure point defined by eq. (5.81). We extract

[6] We denote the both the operators of the classical algebra and the operators of quantum cohomology algebra by $A^{(p)}_k$.

the leading θ piece of the differential operators as the formal limit $\mathcal{L}_i^{\lim}(\theta) = \lim_{\underline{z}\to 0}\mathcal{L}_i(\theta,\underline{z})$, $i = 1,\dots,r$ and consider the algebraic ring

$$\mathcal{R} = \mathbb{C}[\theta]/\mathcal{J} \quad \text{with} \quad \mathcal{J} = \{\mathcal{L}_1^{\lim},\dots,\mathcal{L}_r^{\lim}\}. \tag{5.48}$$

This ring is graded by the degree k in θ and we denote the ring at grade k by $\mathcal{R}^{(k)}$. The dimension of $\mathcal{R}^{(k)}$ is given by $h_H^{n-k,k}(Y) = h_V^{k,k}(\widehat{Y})$ for $k = 0,\dots,n$. We note that for $k = 0,1,n-1,n$ there is no splitting in vertical or horizontal parts. Let us explain in more detail how this ring connects the A- and the B-model structure at the large radius point:

The construction of the ring is up to normalization equivalent to the calculation of the intersection numbers of the classical A-model. In particular the n-fold intersections appear as coefficients of the up to a normalization unique top ring element

$$\mathcal{R}^{(n)} = \sum_{i_1\le\dots\le i_n} C^0_{i_1,\dots,i_n}\theta_{i_1}\cdots\theta_{i_n} \tag{5.49}$$

and similarly the $\mathcal{R}^{(k)}$ are generated by

$$\mathcal{R}^{(k)}_b = \sum_{i_1\le\dots\le i_k} a_b^{i_1,\dots,i_k}\theta_{i_1}\cdots\theta_{i_k} \quad \text{with} \quad a_b^{i_1,\dots,i_k}\mathcal{K}_{i_1,\dots,i_k,j_1,\dots,j_{n-k}} = C^0_{b,j_1,\dots,j_{n-k}} \tag{5.50}$$

where $\mathcal{K}_{i_1,\cdots,i_n}$ is the intersection number of the J_{i_k}. The ring $\mathcal{R}^{(k)}$ is in one-to-one correspondence of the solutions to the PF equations at large radius. As discussed before, the solutions are characterized by their monodromies, i.e. they are graded by their leading logarithmic structure. To the following given element of $\mathcal{R}^{(k)}$

$$\mathcal{R}^{(k)a} = \sum_{|\underline{\alpha}|=k}\frac{1}{(2\pi i)^k} m^a_{\underline{\alpha}}\theta_1^{\alpha_1}\cdots\theta_h^{\alpha_h} \tag{5.51}$$

we associate a solution of the form

$$\widetilde{\mathcal{R}}^{(k)a} = X^0(\underline{z})\left[\mathbb{L}^{(k)a} + \mathcal{O}(\log(\underline{z})^{|\alpha|-1})\right] \tag{5.52}$$

with leading logarithmic piece of order k

$$\mathbb{L}^{(k)a} = \sum_{|\underline{\alpha}|=k}\frac{1}{(2\pi i)^k}\widetilde{m}^a_{\underline{\alpha}}\log(z_1)^{\alpha_1}\cdots\log(z_h)^{\alpha_h} \quad \text{with} \quad \widetilde{m}^a_{\underline{\alpha}}\left(\prod_i \alpha_i!\right) = m^a_{\underline{\alpha}}. \tag{5.53}$$

In particular, we map the unique element 1 of $\mathcal{R}^{(0)}$ to the unique holomorphic solution $X_0(\underline{z}) = 1 + \mathcal{O}(z)$. The above map follows from the fact that all $\mathcal{L}_i^{\lim}$ in the ideal $\mathcal{J}$ must annihilate the leading logarithmic terms if $\Pi^{(k)a}$ are be solutions. This yields the same conditions as for $\mathcal{R}^{(k)}$ to be normal to $\mathcal{J}$ in eq. (5.48). This association of solutions implies mirror symmetry at the level of the classical couplings and can be proven for CY threefolds given as hypersurfaces in toric varieties [13].

By formally replacing the J_i with θ_i, we get a map

$$\mu : A_b^{(k)} \longmapsto \mathcal{R}_b^{(k)}\Omega\Big|_{\underline{z}=0} = a_b^{i_1,\dots,i_k}\theta_{i_1}\cdots\theta_{i_k}\Omega\Big|_{\underline{z}=0} = \beta_b^{(k)}\Big|_{\underline{z}=0} \tag{5.54}$$

which preserves the grading. This implies that we can think of the integral basis $\hat{\gamma}_a$ in terms of their corresponding differential operators $\mathcal{R}_b^{(k)}$ acting on Ω. Thus, the map μ provides a map between $\mathcal{R}_b^{(k)}$ and the classical A-model operators $A_b^{(k)}$ defined in eq. (5.43). This provides also the matching of the A- and B-model Frobenius structures at the large radius limit by identifying the periods of Ω and the solutions of the PF system in the following way: To a given element $\mathcal{R}_b^{(k)}$ we can associate an A-model operator $A_b^{(k)}$ by replacing θ_i with J_i and wedging of the J_i. However, to relate the two- and three-point correlators to the periods of the $\beta_a^{(k)}$ along the lines of § 5.2.1, we have to specify the topological homology basis $\gamma_a^{(k)}$ in terms of the A-model operators $A_a^{(k)}$ as well.

Let us see how this is accomplished. Firstly, we select a basis of solutions $\Pi^{(k)\,a}$ of the PF system with leading logarithm $\mathbb{L}^{(k)\,a}$ that is dual to the $A_a^{(k)}$ at large radius, i.e.

$$\mathcal{R}_a^{(k)}\mathbb{L}^{(k)\,b}=\delta_a^b \tag{5.55}$$

in the limit $\underline{z}\to 0$ [165]. Then, the $\gamma_a^{(k)}$ are fixed by setting the periods $\Pi^{(k)\,a}$ in the expansion of Ω (5.31). This provides a map between the $\mathbb{L}^{(k)\,a}$ and $\hat{\gamma}_a^{(k)}$. With these definitions the requirements (5.30) on the upper triangular basis $\beta_a^{(k)}$ are trivially fulfilled since

$$\beta_b^{(k)}=\mathcal{R}_b^{(k)}\Omega=\mathcal{R}_b^{(k)}\left(\Pi^{(k)\,a}\hat{\gamma}_a^{(k)}+\cdots\right)\longrightarrow\hat{\gamma}_b^k+\cdots \tag{5.56}$$

where the dots indicate forms $\hat{\gamma}_a^{(k)}$ at grade $k>q$ with higher logarithms. Let us exploit this matching by for example considering the B-model coupling $C_{abc}^{(1,1,n-2)}$. Using the particularly important three-point coupling $C_{abc}^{(1,k,n-k-1)}$ in eq. (5.34), we obtain for CY fourfolds that

$$C_{ab\gamma}^{(1,1,2)}=\partial_a\Pi_b^{(2,1)\,\delta}\eta_{\delta\gamma}^{(2)}=\partial_a\partial_b\Pi^{(2)\,\delta}\eta_{\delta\gamma}^{(2)}=\partial_a\partial_bF^0(\gamma)\quad\text{with}\quad\partial_a=\frac{\partial}{\partial t^a} \tag{5.57}$$

where $a,b=1,\ldots,h^{3,1}(Y)$ and γ labels the elements of $H_H^{2,2}(Y)$. Here, we used the upper triangular form (5.28) of $\beta_a^{(k)}$ and the intersection properties (5.26) of the $\hat{\gamma}_i$ for the first equality. Then, we replaced $\partial_a\beta^{(0)}=\beta_a^{(1)}$ for general dimension n which eq. ollows from eq. (5.37) and the property (F2) in flat coordinates. If we now let $\underline{z}\to 0$ and use the flat coordinates $\{t^a\}$ given by eq. (5.52), we obtain

$$t^i\propto\log z_i+\text{hol.}\longrightarrow\log z_i. \tag{5.58}$$

We see that in this limit the classical intersection $C_{ab\gamma}^{0\,(1,1,2)}$ of eq. (5.44) are reproduced. Once the matching is established in this large radius limit, we can define the full quantum cohomological Frobenius algebra structure by $C_{ab\gamma}^{(1,1,2)}$ in the flat coordinates.

For the case at hand the intersections are obtained from the second derivative of the holomorphic quantities $F^0(\gamma)$ in eq. (5.57) for each basis element $\beta_\gamma^{(2)}$ where γ labels elements of $H_H^{2,2}(Y)$. These are the analogs of the holomorphic prepotential F^0 familiar from the CY threefold case and they are obtained in the general discussion of § 2.4 from the generating functionals[7] of eq. (2.49) for $k=1$. The relation for $C_{ab\gamma}^{(1,1,2)}$ tells us that we obtain them simply from the

[7] We note that the terms $b_{a\gamma}^0$, a_γ^0 are irrelevant for the quantum cohomology, but important for the large radius limit of the superpotential in eq. (2.24).

PF equations as double-logarithmic solutions that we will identify below. However, as mentioned above the identification using the ring structure fixes the solutions of the PF system so far only up to normalizations. The normalization of the unique holomorphic solution is determined by the fact that the leading term of X^0 has to be 1. Also, the dual periods can be uniquely normalized by the classical n-fold intersections. The single-logarithmic solutions are normalized to reproduce the effect of a shift of the background NS B-field under which we obtain the shift $t^i \to t^i + 1$. This corresponds to the monodromy around $\underline{z} = 0$ and implies according to the definition of the flat coordinates (5.33) that $\tilde{m}^a_{\underline{\alpha}} = 1$ for $|\alpha| = 1$ in the leading logarithm (5.53).

Let us from now on concentrate on CY fourfolds. All further $\underline{t}$ dependent quantities are restricted further by the monodromy of the period vector of the holomorphic four-form around $\underline{z} = 0$.

$$\Pi^T = \begin{pmatrix} \Pi^{(0)} & \Pi^{(1)\,a} & \Pi^{(2)\,b} & \Pi^{(3)\,c} & \Pi^{(4)} \end{pmatrix}. \tag{5.59}$$

Let Σ be the matrix representing the following intersection

$$K = \int_Y \Omega \wedge \overline{\Omega} = \Pi \Sigma \Pi^\dagger. \tag{5.60}$$

Using the expansion of Ω (5.31) and the property of the basis $\hat{\gamma}^{(i)}_a$ (5.26), it is easy to see that the anti-diagonal of Σ is given by the blocks $(1, (\eta^{(1)})^T, \eta^{(2)}, \eta^{(1)}, 1)$. The monodromies act by

$$\Pi \longrightarrow M\Pi \quad \text{with} \quad M \in Sp(h^4_H(Y), \mathbb{Z}). \tag{5.61}$$

The invariance of K and the basis $\hat{\gamma}^{(p)}_a$ under the monodromy implies

$$M^T \Sigma M = \Sigma. \tag{5.62}$$

Using the monodromy at other points in the moduli space and analytic continuation, all a priori undetermined constants in the solutions to the PF system can be fixed. However, this can be very tedious. Useful information about some of the irrational constants appearing e.g. in the leading logarithmic solution follow from the Frobenius method [137, 13]. We will exemplify this for the conifold in the next section.

We conclude with some remarks about the enumerative geometry of the prepotentials $F^0(\gamma)$ in eq. (5.57). As in the CY threefold case there is an enumerative geometry or counting interpretation of mirror symmetry in higher dimensions for the A-model [107]. As we have seen in § 2.4.1, flux γ is necessary to obtain a well-defined counting problem of curves with the prepotential $F(\gamma)$ as a generating function, cf. eq. (2.49). The prepotential furthermore has a Li_2 structure and it is now possible to calculate the genus zero BPS invariants $n^0_\beta(\gamma)$ of § 2.4.1 for a suitable basis of $H^{2,2}_V(\hat{Y})$ and β in $H_2(\hat{Y}, \mathbb{Z})$.

5.2.4 New behavior near the conifold

Using the Frobenius method the leading logarithmic solution can be obtained. By applying the operator

$$D^{(4)} = \frac{1}{4!(2\pi i)^4} \mathcal{K}_{i_1 i_2 i_3 i_4} \partial_{\rho_{i_1}} \partial_{\rho_{i_2}} \partial_{\rho_{i_3}} \partial_{\rho_{i_4}}, \tag{5.63}$$

on the fundamental solution

$$\omega_0(\underline{z},\underline{\rho}) = \sum_{\underline{n}} c(\underline{n},\underline{\rho}) \underline{z}^{\underline{n}+\underline{\rho}} \quad \text{with} \quad c(\underline{n},\underline{\rho}) = \frac{\Gamma(-\sum_\alpha l_0^{(\alpha)}(n_\alpha+\rho_\alpha)+1)}{\prod_i \Gamma(\sum_\alpha l_i^{(\alpha)}(n_\alpha+\rho_\alpha)+1)} \tag{5.64}$$

and setting $\underline{\rho} = 0$. Here, $\underline{\rho}$ are auxiliary variables. The general leading logarithmic solution, i.e. with all possible admixtures of lower logarithmic solutions, for $X^0 = \omega_0(\underline{z})|_{\underline{\rho}=0}$ reads

$$\Pi^{(4)} = X^0 \left(\frac{1}{4!} \mathcal{K}_{ijkl} t^i t^j t^k t^l + \frac{1}{3!} a_{ijk} t^i t^j t^k + \frac{1}{2!} a_{ij} t^i t^j + a_i t^i + a_0 \right) \tag{5.65}$$

where as in the threefold case $\mathcal{K}_{ijkl}$ is the classical top intersection numbers. It was observed in ref. [13] for the threefold case that the Frobenius method reproduces some of the topological constants in eq. (2.17). In particular,

$$\begin{aligned} \int_X c_2(X) \wedge J_i &= \frac{3}{\pi^2} \mathcal{K}_{ijk}\, \partial_{\rho_j} \partial_{\rho_k} c(\underline{0},\underline{\rho}) \Big|_{\underline{\rho}=0}, \\ \int_X c_3(X) &= \frac{1}{3!\zeta(3)} \mathcal{K}_{ijk}\, \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k} c(\underline{0},\underline{\rho}) \Big|_{\underline{\rho}=0} \end{aligned} \tag{5.66}$$

where X is a CY threefold. If we generalize these to fourfolds, we obtain

$$\int_Y \frac{3}{4} c_2(Y)^2 + c_4(Y) = \frac{1}{4!\zeta(4)} \mathcal{K}_{ijkl}\, \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k} \partial_{\rho_l} c(\underline{0},\underline{\rho}) \Big|_{\underline{\rho}=0}. \tag{5.67}$$

These constants are expected to appear as coefficients of the lower order logarithmic solutions in $\Pi^{(4)}$. Similarly to the CY threefold case, as it was done in ref. [167], we can also use the K-theory charge Q [168, 169, 170]

$$Q \cdot \Pi = -\int_Y e^{-J} \mathrm{ch}(A) \sqrt{\mathrm{td}(Y)} = Z(A) \tag{5.68}$$

where A is the bundle on the brane wrapping Y and $Z(A)$ is the central charge of the brane. Combined with mirror symmetry we can obtain information about the sub-leading logarithmic terms in the periods.

Let us apply a more direct argument and use properties of the simplest CY fourfold, the sextic in $\mathbb{P}^5$. The PF operator of the mirror sextic having one complex structure modulus z is as follows, see e.g. ref. [107]

$$\theta^5 - 6z \prod_{k=1}^{5} (6\theta + k). \tag{5.69}$$

We can easily construct solutions at $z = 0$ using the Frobenius method, but let us first give a different basis of logarithmic solutions, namely

$$\hat{\Pi}^k = \frac{1}{(2\pi i)^k} \sum_{l=0}^{k} \binom{k}{l} \log(z)^l s_{k-l}(z) \tag{5.70}$$

with

$$\begin{aligned} X^0 = s_0 &= 1 + 720z + 748440z^2 + \cdots, \\ s_1 &= 6246z + 7199442z^2 + \cdots, \\ s_2 &= 20160z + 327001536z^2 + \cdots, \\ s_3 &= -60480z - 111585600z^2 + \cdots, \\ s_4 &= -2734663680z^2 - 57797926824960z^3 + \cdots. \end{aligned} \tag{5.71}$$

Under the mirror map we obtain $\widehat{\Pi}_k = t^k + \mathcal{O}(q)$ such that these solutions correspond to the leading volume term of branes of real dimension $2k$. The conifold locus of the sextic is at $\Delta = 1 - 6^6 z = 0$. Near that point the PF equation has the indicials $\left(0,1,2,3,\frac{3}{2}\right)$. It is easy to construct solutions and we choose a basis in which the solution to indicial $k \in \mathbb{Z}$ has the next power z^4. The only unique solution is the one with the branch cut

$$\nu = \Delta^{3/2} + \frac{17}{18}\Delta^{5/2} + \frac{551}{648}\Delta^{7/2} + \cdots . \tag{5.72}$$

The behavior of the periods at conifold is crucial for mirror symmetry and computations. Let us describe in the following the general situation and come back to the sextic example. Let us assume that we have a CY n-fold. At the conifold point there is a non-trivial monodromy between a cycle of topology T^n and a cycle of topology S^n. The former corresponds to the solution X^0 and therefore to the zero-dimensional D-brane in the A-model which is uniquely defined. The latter cycle corresponds to the solution $\Pi^{(n)}$, i.e. to the top-dimensional D-brane in the A-model. The topological intersection between these two cycles is 1. They form the fiber and the base of the SYZ fibration of the CY n-fold [120]. Let us concentrate on these two periods and write $\Pi_{\text{red}}^T = (\Pi^{(n)}, X^0)$. In odd dimensional CY manifolds the monodromy at the conifold acts on Π_{red} as follows

$$M_{\text{odd}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{5.73}$$

corresponding to the Lefshetz formula with vanishing cycle $\Pi^{(n)}$. This means that the volume of the CY manifold vanishes. For CY fourfolds Y we have a monodromy of order two. The only integral idempotent monodromy compatible with the intersection Σ introduced in eq. (5.62) is given by

$$M_Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.74}$$

Here, we see a new behavior at the conifold. The zero- and top-dimensional D-brane get *exchanged* by the conifold monodromy in four dimensions. This implies that we have the following[8]

$$X^0 = \eta - c\nu, \quad \Pi^{(4)} = \eta + c\nu. \tag{5.75}$$

Here, η is a combination of solutions at $\Delta = 0$ without any branch cut. We can determine η by analytic continuation of X^0 to the conifold and we obtain the combination corresponding to the correct integral choice of the geometric period $\Pi^{(4)}$ as

$$\Pi^{(4)} = 2c\nu + X^0 \quad \text{with} \quad c = \frac{1}{\sqrt{3}\pi^2} \tag{5.76}$$

from the uniquely defined periods X^0 and ν at $z = 0$ and $\Delta = 0$. The analytic continuation of ν to the large complex structure point fits nicely with our expectation from above and fixes most

[8] The sign is chosen such that the t^4 term in $\Pi^{(4)}$ comes out with positive sign.

of the numerical coefficients in eq. (5.65)

$$
\begin{aligned}
a_0 &= \frac{\zeta(4)}{2^4(2\pi i)^6}\int_Y 5c_2^2, & a_i &= -\frac{\zeta(3)}{(2\pi i)^3}\int_Y c_3\wedge J_i, \\
a_{ij} &= \frac{\zeta(2)}{2(2\pi i)^2}\int_Y c_2\wedge J_i\wedge J_j, & a_{ijk} &= \tilde{c}\int_Y J_j\wedge J_k\wedge i_* c_1(J_i)
\end{aligned}
\tag{5.77}
$$

where for convenience we write $c_i = c_i(T_Y)$. Note that $c_1(J_i)$ is the first Chern class of the divisor associated to J_i mapped to a four-form via the Gysin homomorphism, cf. A.1.2, of the embedding map of this divisor into the CY fourfold. This is the generalization of eq. (2.19) to the case of CY fourfolds. To be precise, the coefficients a_{ijk} are not fixed by the sextic example because it turns out to be zero in this case. This does not necessarily mean that it is absent in general. Rather, it implies that it is physically irrelevant for the sextic. The divisibility of the properly normalised triple-logarithmic solution allows an integral symplectic choice of the periods in which this term can be set to zero. This might not be the case in general and other hypersurface in weighted projective space indicate that $\tilde{c} = 1$. It is similarly possible to use the orbifold monodromy to fix the exact integral choice of the other periods. The principal form of the terms should again follow from the Frobenius method.

5.3 Example

In this section we will use all ingredients discussed in the previous sections to analyze one main example. Other examples are relegated to appendix not to overload the main text. We will start by describing in detail the compact CY threefold fiber geometry and show that it allows to embed the non-compact geometry discussed in § 3.2. Then, we will construct the fourfold in question. Special emphasis will be laid on the fibration structure and the complex structure moduli. After having described the geometry, we will compute the flux superpotential. We will apply mirror symmetry for CY fourfolds to check the computation against available BPS numbers in the literature.

5.3.1 The compact elliptic Calabi-Yau threefold

The non-compact CY threefold discussed in § 3.2 can easily be embedded into a compact CY threefold. The compactification can be understood as a replacement of the non-compact $\mathbb{C}$ fiber corresponding to v_1 of $\mathcal{O}_{\mathbb{P}^2}(-3)$ in eq. (3.11) by an elliptic fiber. Here, we choose the generic fiber to be the elliptic curve in $\mathbb{P}^2_{1,2,3}$ which we fiber over the $\mathbb{P}^2$ basis the same way as the non-compact $\mathbb{C}$ fiber before. Thus, the polyhedron of this compact threefold $\widehat{X}$, its charge vectors, the homogeneous coordinates $\hat{x}_i$ as well as the corresponding monomials for the mirror geometry X are given in Table 5.2. In the table, the points v_1, v_2, v_3 carry the information of the elliptic fiber where we added the inner point v_1 in order to recover the $\mathbb{P}^2_{1,2,3}$. In particular, its homogeneous coordinate $\hat{x}_1$ with weight one under the new $\mathbb{C}^\times$ action $\ell^{(2)}$. Furthermore, applying the insights displayed in Table 5.1, the elliptic fibration structure of $\widehat{X}$ is obvious from the fact that the polyhedron of $\mathbb{P}^2(1,2,3)$ occurs in the hyperplane $H = \{(0,0,a,b)\}$, but also as

	$\Delta_{\widehat{X}}$				$\ell^{(1)}$	$\ell^{(2)}$		
v_0	0	0	0	0	0	−6	$\widehat{x}_0$	$zxyu_1u_2u_3$
v_1	0	0	2	3	−3	1	$\widehat{x}_1$	$z^6u_1^6u_2^6u_3^6$
v_1^b	1	1	2	3	1	0	$\widehat{x}_2$	$z^6u_3^{18}$
v_2^b	−1	0	2	3	1	0	$\widehat{x}_3$	$z^6u_1^{18}$
v_3^b	0	−1	2	3	1	0	$\widehat{x}_4$	$z^6u_2^{18}$
v_2	0	0	−1	0	0	2	$\widehat{x}_5$	x^3
v_3	0	0	0	−1	0	3	$\widehat{x}_6$	y^2

Table 5.2: The toric data of $\widehat{X}$

a projection P on the $(3-4)$-plane indicating an elliptic fibration of the mirror X.[9]

The polyhedron $\Delta_{\widehat{X}}$ corresponds to the degree 18 hypersurface in the weighted projective space $\mathbb{P}^4_{1,1,1,6,9}$ blown up along the singular curve $\widehat{x}_2 = \widehat{x}_3 = \widehat{x}_4 = 0$ with exceptional divisor $D_1 = \{v_1 = 0\}$, writing $D_i = \{v_i = 0\}$. The basic topological data are as follows

$$\chi = -540, \quad h^{1,1} = 2, \quad h^{2,1} = 272. \tag{5.78}$$

The two Kähler classes $J_1 = D_2$ and $J_2 = 3D_2 + D_1$ correspond to the Mori vectors $\ell^{(1)}$ and $\ell^{(2)}$ in Table 5.2. They represent a curve in the hyperplane class of the $\mathbb{P}^2$ base and a curve in the elliptic fiber, respectively. The intersections of the dual divisors and the second Chern class are respectively computed to be[10]

$$\mathcal{C}_0 = 9J_2^3 + 3J_2^2J_1 + J_2J_1^2, \quad \mathcal{C}_2 = 102J_2 + 36J_1. \tag{5.79}$$

In this notation the coefficients of the top intersection ring $\mathcal{C}_0$ are the cubic intersection numbers $J_i \cap J_j \cap J_k$ while the coefficients of $\mathcal{C}_2$ are $[c_2(T_{\widehat{X}})] \cap J_i$.

Mirror symmetry for this example has been studied in refs. [9, 12]. In order to construct the mirror pair $(X, \widehat{X})$ as well as their defining equations, we need the dual polyhedron Δ_X which is given in Table 5.3(a). where again the basis was indicated by the superscript $(\cdot)^b$. We added the inner point v_1 to recover the polyhedron of $\mathbb{P}^2_{1,2,3}$ as the injection with $H = \{0, 0, a, b\}$, thus confirming the elliptic fibration of the mirror X. Here, we distinguish between the two-dimensional basis $B_X = \mathbb{P}^2$ and the elliptic fiber by denoting the homogeneous coordinates of $\mathbb{P}^2_{1,2,3}$ by (z, x, y) and of B_X by (u_1, u_2, u_3). The elliptic fibration structure reflects in particular in the constraint of X which takes the Weierstraß form,[11]

$$p_0 = a_6y^2 + a_5x^3 + a_0zxyu_1u_2u_3 + z^6\left(a_3u_1^{18} + a_4u_2^{18} + a_1u_1^6u_2^6u_3^6 + a_2u_3^{18}\right) = 0. \tag{5.80}$$

The generic fiber can be seen by setting the coordinates u_i of the basis B_X to some reference point, such that p_0 takes the form of a degree six hypersurface in $\mathbb{P}^2_{1,2,3}$. The basis itself is obtained as the section $\{z = 0\}$ of the elliptic fibration over B_X.

[9] Besides the above chosen $(2,3)$, which leads to an elliptic fibration with one section, the values $(1,2)$ and $(1,1)$ are also admissible in the sense that these choices lead to reflexive polyhedra. The corresponding elliptic fibration has two and three sections, respectively [164].

[10] In performing these toric computations we have used the Maple package `Schubert` [171].

[11] Eq. (5.80) is slightly different than eq. (5.5). However, we can redefine the coordinates (x, y, z) to transform from one to another.

	Δ_X				
v_1	0	0	1	1	z
v_1^b	−12	6	1	1	u_1
v_2^b	6	−12	1	1	u_2
v_3^b	6	6	1	1	u_3
v_2	0	0	−2	1	x
v_3	0	0	1	−1	y

(a)

y_0	v_0	$a_0\,zxyu_1u_2u_3$
y_1	v_1	$a_1\,z^6u_1^6u_2^6u_3^6$
y_2	v_1^b	$a_2\,z^6u_3^{18}$
y_3	v_2^b	$a_3\,z^6u_1^{18}$
y_4	v_3^b	$a_4\,z^6u_2^{18}$
y_5	v_2	$a_5\,x^3$
y_6	v_3	$a_6\,y^2$

(b)

Table 5.3: Polyhedron of X and the étale map

The complex structure dependence of X is evident from the dependence of p_0 on the parameters a_i which are coordinates of $\mathbb{P}^6$. However, they redundantly parameterize the complex structure of X due to the symmetries of $\mathbb{P}^4_{1,1,1,6,9}$. Indeed, there is a $(\mathbb{C}^\times)^6/(\mathbb{C}^\times)^2$ rescaling symmetry of the coordinates that enables us to eliminate four of the a_i recovering the two complex structure parameters that match $h^{1,1}(\widehat{X}) = h^{2,1}(X) = 2$. The appropriate coordinates z_i obeying $z_i = 0$ at the large complex structure/large volume point are completely determined by the phase of the A-model, i.e. the choice of charge vectors $\ell^{(i)}$ of $\Delta_{\widehat{X}}$. In general they are given by[12]

$$z_i = (-1)^{\ell_0^{(a)}} \prod_{j=0}^{m} a_j^{\ell_j^{(i)}} \quad \text{with} \quad \ell_0^{(a)} = \sum_{j=1}^{m} \ell_j^{(a)}. \tag{5.81}$$

For the situation at hand we obtain

$$z_1 = \frac{a_2 a_3 a_4}{a_1^3}, \quad z_2 = \frac{a_1 a_5^2 a_6^3}{a_0}. \tag{5.82}$$

Thus, we can use the $(\mathbb{C}^\times)^4$ action and the overall scaling to set $a_i = 1$ for $i = 2, \ldots, 6$ to obtain

$$p_0 = y^2 + x^3 + zxym_1 + z^6 m_6 \tag{5.83}$$

where we write

$$m_1 = z_2^{-1/6} z_1^{-1/18} u_1u_2u_3, \quad m_6 = u_1^{18} + u_2^{18} + u_3^{18} + z_1^{-1/3} u_1^6 u_2^6 u_3^6. \tag{5.84}$$

Alternatively and more directly, this result can be obtained by the mirror construction of ref. [19]. In this case we need the assignment of coordinates y_i to points of $\Delta_{\widehat{X}}$ given in Table 5.3(b). This defines the étale map that solves the following constraint automatically

$$W = \sum_{j=0}^{m} y_j \quad \text{with} \quad \prod_{j=0}^{m} y_j^{\ell_j^{(i)}} = z^i \quad \text{where} \quad i = 1, \ldots, n \tag{5.85}$$

when the assignments for z_i (5.82) hold. By setting $a_0 = z_2^{-1/6} z_1^{-1/18}$ and $a_1 = z_1^{-1/3}$ for $a_i = 1$, $i = 2, \ldots, 6$, we solve the assignments for z_i and W immediately reproduces p_0.

[12] The formula for z_i (5.81) contains also the zeroth component $\ell_0^{(a)}$ for each charge vector since we are working with a compact CY manifold. For non-compact manifold we do not need to add this component, as is apparent from eq. (3.4).

Local limit

Next, we show that the equation for X indeed gives back the local geometry which is a conic over a genus one Riemann surface discussed in § 3.2. The local limit in the A-model geometry is given by making the elliptic fiber infinitely large. This corresponds to $z_2 \to 0$ in the B-model geometry. We set $\varepsilon = z_2$ such that the local limit is given by $\varepsilon \to 0$. At the end we should obtain an affine equation. Using the two $\mathbb{C}^\times$ actions, we set the coordinates z and u_3 to one. By redefining the coordinates x and y as follows

$$y \to \varepsilon^{-1/2} y + k_1^{1/2}, \quad x \to \varepsilon^{-1/3} x + k_2^{2/3}, \tag{5.86}$$

the hypersurface equation for X becomes

$$p_0 = \frac{1}{\varepsilon}\widetilde{p}_0 + k_1^2 + k_2^2 + m_6 = 0 \tag{5.87}$$

where we set $z = 1$ and $u_3 = 1$. Now, we split the above equation

$$\widetilde{p}_0 = \varepsilon, \quad k_1^2 + k_2^2 + m_6 = -1. \tag{5.88}$$

If we now take the $\varepsilon \to 0$ limit, we obtain after appropriately redefining the k_i the equation for the local geometry of the form

$$uv = H(x,y) = x + 1 - \phi\frac{x^3}{y} + y. \tag{5.89}$$

The Riemann surface defined by the zero locus of $H(x,y)$ is isomorphic to the surface $\{m_6 = 0\}$ up to isogeny [172].

Let us comment on the singularities of X. If we use X for the compactification of the heterotic string, as we will do in § 6, there will be a large non-perturbative gauge group from the blown-up singularities of the elliptic fibration of X. Upon introducing the full set of coordinates, i.e. introducing the inner points in Δ_X, we see that the elliptic fibration not only degenerates over the curves $\{m_6 = 0\}$ and $\{432m_6 + m_1^6 = 0\}$ in B_X, but also over many curves described by the additional coordinates. These singularities will induce large non-perturbative gauge group in the heterotic string. The identification of the moduli of the gauge bundles with the complex structure moduli of Y can be performed by invoking the spectral cover construction. We will apply the heterotic/F-theory duality for X in § 6 and construct the F-theory dual. There, we will also analyze the identification of the moduli in more detail.

Before continuing with the construction of the CY fourfold, let us close with another comment on the use of the vectors $\widehat{\ell}^{(1)}$ and $\widehat{\ell}^{(2)}$ given in eq. (3.13). On $\widehat{X}$ they translate to

$$\widehat{\ell}^{(1)} = (0,1,0,-1,0,0,0), \quad \widehat{\ell}^{(2)} = (0,1,0,0,-1,0,0) \tag{5.90}$$

due to the new origin in the polyhedron given in Table 5.2. In fact, applying the defining equations for the B-brane (3.6) and using the étale map given in Table 5.3(b), they define the divisors

$$z_1^{-1/3} u_1^6 u_2^6 u_3^6 = \widehat{z}_1 u_1^{18}, \quad z_1^{-1/3} u_1^6 u_2^6 u_3^6 = \widehat{z}_2 u_2^{18} \tag{5.91}$$

in the compact X. Here, we introduced the moduli $\hat{z}_a$ corresponding to the charge vector $\hat{\ell}^{(a)}$. Note that in our F-theory compactification of the next section we will not consider seven-branes naively wrapped on these divisors as we would in a compactification of the type IIB theory on CY orientifolds. Rather, we will construct a CY fourfold with seven-branes on its discriminant possessing additional moduli. These additional moduli correspond to either $\hat{z}_1$ or $\hat{z}_2$ and allow deformations of the seven-brane constraint by the additional terms (5.91). Hence, $\hat{z}_i$ can be interpreted as deformations of the seven-brane divisors in Y, or as spectral cover moduli in the heterotic dual.

5.3.2 Construction of the elliptic Calabi-Yau fourfold

Having discussed the CY threefold geometry, we are now in position to construct and analyze the elliptic CY fourfold Y used in the F-theory compactification. Again, we start by first constructing the mirror geometry $\hat{Y}$. It is obtained by fibering the CY threefold $\hat{X}$ over a $\mathbb{P}^1$ such that one of the brane vectors $\hat{\ell}^{(a)}$ of the non-compact model (3.11) appears as a new charge vector. In the following we will exemplify our construction for a main example in detail and we list the toric and geometrical data necessary to reproduce our results. Further examples and results are relegated to appendix A.3.2.

The CY fourfolds pair $(Y, \hat{Y})$ is realized as hypersurfaces in toric varieties described by a pair of reflexive polyhedra $(\Delta_Y, \Delta_{\hat{Y}})$. The reflexive polyhedron $\Delta_{\hat{Y}}$ for a fibration of the toric variety constructed from $\Delta_{\tilde{X}}$ over $\mathbb{P}^1$ is specified as follows

$$\Delta_{\hat{Y}} = \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{\Delta_{\hat{X}}} & 0 \\ \hline -1 & 0 & 2 & 3 & -1 \\ 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 2 & 3 & 1 \end{array}\right]. \tag{5.92}$$

By construction we find $\Delta_{\tilde{X}}$ by intersecting the hyperplane $\hat{H} = (p_1, p_2, p_3, p_4, 0)$ with $\Delta_{\hat{Y}}$. Following Table 5.1, this indeed identifies $\hat{Y}$ as a $\hat{X}$ fibration. By performing the quotient $\Delta_{\hat{Y}}/\Delta_{\tilde{X}}$ the base is readily shown to be the toric variety $((-1),(1))$, i.e. a $\mathbb{P}^1$. It is crucial to note that the additional points which do not lie on $\hat{H}$ are constrained by two important conditions. Firstly, they are chosen to be of a form such that the mirror Y is elliptically fibered. This means, that using the projection to the third and fourth coordinates, we find the polyhedron of a torus in $\mathbb{P}^2_{1,2,3}$ just as in the threefold case in § 3.2. The fact that $\hat{Y}$ is also elliptically fibered is not crucial in the construction. In particular, a similar construction can also be performed for the quintic hypersurface fibered over a $\mathbb{P}^1$ since the mirror quintic admits an elliptic fibration with generic elliptic fiber being a torus in $\mathbb{P}^2$. Secondly, the remaining entries are inserted such that one charge vector for the CY fourfold is of the form $(\hat{\ell}^{(1)}, -, -, -)$. Adding this vector to form a higher-dimensional non-reflexive polyhedron was already proposed in refs. [36, 46][13] and the completion to reflexive polyhedron with similar computations was made in ref. [48].

In the following we will choose to realize the open string vector $\hat{\ell}^{(1)}$ as in eq. (5.92) to construct the $\mathbb{P}^1$ fibration.The CY fourfold $\hat{Y}$ is realized as a hypersurface in the toric variety de-

[13] Note that the interpretation of the construction in terms of the B-model in refs. [36, 46] seems different from the F-theory interpretation given here.

	$\Delta_{\widehat{Y}}$					$\ell_I^{(1)}$	$\ell_I^{(2)}$	$\ell_I^{(3)}$	$\ell_I^{(4)}$	$\ell_{II}^{(1)}$	$\ell_{II}^{(2)}$	$\ell_{II}^{(3)}$	$\ell_{II}^{(4)}$
v_0	0	0	0	0	0	0	−6	0	0	0	−6	0	0
v_1^b	0	0	2	3	0	−2	1	−1	−1	−3	0	1	−2
v_2^b	1	1	2	3	0	1	0	0	0	1	0	0	0
v_3^b	−1	0	2	3	0	0	0	1	−1	1	1	−1	0
v_4^b	0	−1	2	3	0	1	0	0	0	1	0	0	0
v_1	0	0	−1	0	0	0	2	0	0	0	2	0	0
v_2	0	0	0	−1	0	0	3	0	0	0	3	0	0
$\hat{v}_1$	−1	0	2	3	−1	1	0	−1	1	0	−1	1	0
$\hat{v}_2$	0	0	2	3	−1	−1	0	1	0	0	1	−1	1
$\hat{v}_3$	0	0	2	3	1	0	0	0	1	0	0	0	1

Table 5.4: Toric data for the main example

scribed by the polyhedron $\Delta_{\widehat{Y}}$ as discussed in § 5.1.2. Its topological numbers are computed to be

$$\chi = 16848, \quad h^{3,1} = 2796, \quad h^{1,1} = 4, \quad h^{2,1} = 0, \quad h^{2,2} = 11244. \tag{5.93}$$

Note that $\Delta_{\widehat{Y}}$ has three triangulations corresponding to non-singular CY phases which are connected by flop transitions. In the following we will consider two of these phases in detail. These phases will match the two brane phases in Figure 3.1(b) in the local CY threefold geometry.

To summarize the topological data of the CY fourfold for the two phases of interest, we specify the generators of the Mori cone $\ell_I^{(i)}$ and $\ell_{II}^{(i)}$ for $i = 1, \dots 4$. These data are shown in Table 5.4. The charge vectors are best identified in phase II. The vectors $\ell_{II}^{(1)}$ and $\ell_{II}^{(2)}$ are the extensions of the threefold charge vectors $\ell^{(1)}$ and $\ell^{(2)}$ in Table 5.2 to the fourfold. The brane vector $\widehat{\ell}^{(1)}$ is visible in phases II as a subvector of $\ell_{II}^{(3)}$. The remaining vector $\ell_{II}^{(4)}$ arises since we complete the polyhedron such that it becomes reflexive implying that $\widehat{Y}$ is a CY manifold. It represents the curve of the $\mathbb{P}^1$ basis of $\widehat{Y}$. Phase I is related to phase II by a flop transition of the curve associated to $\ell_I^{(3)}$. Hence, in phase I the brane vector is identified with $-\ell_I^{(3)}$. Furthermore, after the flop transition we have to set

$$\ell_{II}^{(3)} = -\ell_I^{(3)}, \quad \ell_{II}^{(1)} = \ell_I^{(1)} + \ell_I^{(3)}, \quad \ell_{II}^{(2)} = \ell_I^{(2)} + \ell_I^{(3)}, \quad \ell_{II}^{(4)} = \ell_I^{(4)} + \ell_I^{(3)}. \tag{5.94}$$

Note that $\ell_I^{(i)}$ and $\ell_{II}^{(i)}$ are chosen in such a way that they parameterize the Mori cone of $\widehat{Y}$. The dual Kähler cone generators for phase I are then given by

$$J_1 = D_2, \quad J_2 = D_1 + 2D_2 + D_3 + 2D_9, \quad J_3 = D_3 + D_9, \quad J_4 = D_9 \tag{5.95}$$

where as usual we write $D_i = \{x_i = 0\}$ for the toric divisors associated to the points $\Delta_{\widehat{Y}}$. In phase II we have the following

$$J_1 = D_2, \quad J_2 = D_1 + 2D_2 + D_3 + 2D_9, \quad J_3 = D_1 + 3D_2 + 2D_9, \quad J_4 = D_9. \tag{5.96}$$

The J_i provide a distinguished integral basis of $H^{1,1}(\widehat{Y})$ since in the expansion of the Kähler form J in terms of the J_i all coefficients will be positive parameterizing physical volumes of cycles in $\widehat{Y}$. The J_i are also canonically used as a basis in which we determine the topological data of $\widehat{Y}$. The complete set of topological data of $\widehat{Y}$ including the intersection ring as well as the non-trivial Chern classes are summarized in appendix A.3.1.

Fibration structure

The polyhedron $\Delta_{\widehat{Y}}$ has only few Kähler classes making it possible to identify part of the fibration structures from the intersection numbers. However, an analogous analysis is not possible for the mirror manifold Y since the dual polyhedron Δ_Y has more than two thousand Kähler classes. Therefore, we apply the methods reviewed in § 5.1.3 in analyzing both $\widehat{Y}$ and Y. As already mentioned above, $\Delta_{\widehat{Y}}$ intersected with the two hyperplanes

$$H_1 = (0,0,p_3,p_4,0), \quad H_2 = (p_1,p_2,p_3,p_4,0). \tag{5.97}$$

gives two reflexive polyhedra corresponding to the generic torus fiber and the generic CY threefold fiber $\widehat{X}$. The fibration structures of Y is studied by identifying appropriate projections to $\Delta^k_{\widehat{F}} \subset \Delta_{\widehat{Y}}$. Three relevant projections are

$$P_1(p) = (p_3,p_4), \quad P_2(p) = (p_1,p_2,p_3,p_4), \quad P_3(p) = (p_3,p_4,p_5) \tag{5.98}$$

where $p = (p_1,\ldots,p_5)$ are the columns in the polyhedron $\Delta_{\widehat{Y}}$. Invoking the theorem of § 5.1.3, we see from P_1 that Y is also elliptically fibered and since the polyhedron of $\mathbb{P}^2_{1,2,3}$ is self dual, the fibration is also of the same type. In addition, it is clear from P_2 that Y is CY threefold fibered. The fiber threefold is X, the mirror CY threefold to $\widehat{X}$. The fact, that the threefold fibers of Y and $\widehat{Y}$ are mirror to each other is special to this example since the subpolyhedra obtained by H_2 and P_2 are identical. Finally, note that Y is also K3 fibered as inferred from the projection P_3. This ensures the existence of a heterotic dual theory by fiberwise applying the duality of F-theory on K3 to the heterotic strings on T^2. Replacing the K3 fiber by an elliptic fiber, we find the CY threefold X. We will elaborate on this in § 6.

The hypersurface constraint for Y depends on the four complex structure moduli z_i. This dependence is already captured by only introducing twelve out of the many (blow-up) coordinates needed to specify a non-singular Y. This subset of points of Δ_Y is given in Table 5.5. In the table we have omitted the origin. Note that we have displayed in Table 5.5 the vertices of Δ_Y and added the inner points v_1 and v_2 to list all points necessary to identify the polyhedron Δ_X with vertices given in Table 5.3(a). The four-dimensional polyhedron Δ_X lies in the hyperplane orthogonal to $(0,0,0,0,1)$ and thus we have a CY threefold fibration with fiber X. The base of this fibration is given by the points labeled by the superscript $(\cdot)^b$. Note that $(0,0,1,1,0)$ is also needed to observe the elliptic fibration. The base of the elliptic fibration is obtained by performing the quotient $\Delta_{\text{base}} = \Delta_Y/(P_1\Delta_{\widehat{Y}})^*$ which amounts to simply dropping the third and fourth entry in the points of Δ_Y.

$\Delta_Y \supset$

v_1	0	0	1	1	0	z
v_2	−12	6	1	1	0	u_1
v_3	6	−12	1	1	0	u_2
v_4	6	6	1	1	0	u_3
v_5	0	0	−2	1	0	x
v_6	0	0	1	−1	0	y
v_1^b	−12	6	1	1	−6	x_1
v_2^b	−12	6	1	1	6	x_2
v_3^b	6	−12	1	1	−6	x_3
v_4^b	6	6	1	1	−6	x_4
v_5^b	0	−6	1	1	6	x_5
v_6^b	0	6	1	1	6	x_6

Table 5.5: The relevant subset of points of the polyhedron Δ_Y

Weierstraß form and moduli

Additionally, we can also see the elliptic fibration directly on the defining polynomial $\widetilde{\mu}$ of Y which can be written in a Weierstraß form. Indeed, if we apply eq. (5.12) for the points in Table 5.5 and all points of $\Delta_{\widehat{Y}}$ that are not on codimension one faces, we obtain a hypersurface of the form[14]

$$\widetilde{\mu} = a_6 y^2 + a_5 x^3 + \widetilde{m}_1(x_j, u_i) xyz + \widetilde{m}_6(x_j, u_i) z^6 = 0. \tag{5.99}$$

Here, x_j and u_i are the homogeneous coordinates on the base of the elliptic fibration while x, y, and z are the homogeneous coordinates of the $\mathbb{P}^2_{1,2,3}$ fiber. The polynomials $\widetilde{m}_1$ and $\widetilde{m}_6$ are given by

$$\begin{aligned} \widetilde{m}_1(x_j, u_i) &= a_0 u_1 u_2 u_3 x_1 x_2 x_3 x_4 x_5 x_6, \\ \widetilde{m}_6(x_j, u_i) &= u_1^{18} \left(a_7 x_1^{24} x_2^{12} x_3^6 x_4^6 + a_3 x_1^{18} x_2^{18} x_5^6 x_6^6 \right) + a_4 u_2^{18} x_3^{18} x_5^{12} + a_2 u_3^{18} x_4^{18} x_6^{12} \\ &\quad + u_1^6 u_2^6 u_3^6 \left(a_1 x_1^6 x_2^6 x_3^6 x_4^6 x_5^6 x_6^6 + a_9 x_2^{12} x_5^{12} x_6^{12} + a_8 x_1^{12} x_3^{12} x_4^{12} \right) \end{aligned} \tag{5.100}$$

where a_i denote coefficients encoding the complex structure deformations of Y. However, since $h^{3,1}(Y) = h^{1,1}(\widehat{Y}) = 4$, there are only four complex structure parameters rendering six of the a_i redundant. It is also straightforward to compare $\widetilde{m}_1$ and $\widetilde{m}_6$ for the fourfold Y with the corresponding threefold data given in eq. (5.83) and eq. (5.84).

For the different phases we can identify the complex structure moduli in the hypersurface constraint by using the charge vectors $\ell^{(i)}_{I/II}$ in Table 5.4. For phase I we find

$$z_1^{\mathrm{I}} = \frac{a_2 a_4 a_7}{a_1^2 a_8}, \quad z_2^{\mathrm{I}} = \frac{a_1 a_5^2 a_6^3}{a_0^6}, \quad z_3^{\mathrm{I}} = \frac{a_3 a_8}{a_1 a_7}, \quad z_4^{\mathrm{I}} = \frac{a_7 a_9}{a_1 a_3}. \tag{5.101}$$

[14] The polynomial $\widetilde{\mu}$ can be easily brought to the standard Weierstraß form by completing the square and the cube, i.e. $\widetilde{y} = y + \widetilde{m}_1 xz/2$ and $\widetilde{x} = x - \widetilde{m}_1^2 z^2/12$.

For phase II we find in accord with the rules for flop transition (5.94) that

$$z_1^{\mathrm{II}} = z_1^{\mathrm{I}} z_3^{\mathrm{I}}, \quad z_2^{\mathrm{II}} = z_2^{\mathrm{I}} z_3^{\mathrm{I}}, \quad z_3^{\mathrm{II}} = (z_3^{\mathrm{I}})^{-1}, \quad z_4^{\mathrm{II}} = z_4^{\mathrm{I}} z_3^{\mathrm{I}}. \tag{5.102}$$

In order to compare to the CY threefold X, we choose the gauge $a_i = 1$, $i = 2, \dots, 6$ and $a_8 = 1$ such that

$$a_0^6 = \frac{1}{(z_1^{\mathrm{II}})^{1/3} z_2^{\mathrm{II}} z_3^{\mathrm{II}}}, \quad a_1 = \frac{1}{(z_1^{\mathrm{II}})^{1/3}}, \quad a_7 = z_3^{\mathrm{II}} (z_1^{\mathrm{II}})^{1/3}, \quad a_9 = \frac{z_4^{\mathrm{II}}}{(z_1^{\mathrm{II}})^{2/3}}. \tag{5.103}$$

It is straightforward to find the expression for phase I by inserting the flop transition (5.102) into this expression for a_0, a_1 and a_7, a_9.

Having determined the defining equations for the CY fourfolds, we evaluate the discriminant $\Delta(Y)$ of the elliptic fibration. Using the formula for the discriminant (5.6), we find

$$\Delta(Y) = -\tilde{m}_6 (432 \tilde{m}_6 + \tilde{m}_1^6). \tag{5.104}$$

We conclude that there will be seven-branes on the divisors $\{\tilde{m}_6 = 0\}$ and $\{432\tilde{m}_6 + \tilde{m}_1^6 = 0\}$ in the base B_Y. The key observation is that in addition to a moduli independent part $\tilde{m}_6^0$ the full $\tilde{m}_6$ is shifted as

$$\tilde{m}_6 = \tilde{m}_6^0 + a_1 (u_1 u_2 u_3 x_1 x_2 x_3 x_4 x_5 x_6)^6 + a_7 u_1^{18} x_1^{24} x_2^{12} x_3^6 x_4^6 + a_9 u_1^6 u_2^6 u_3^6 x_2^{12} x_5^{12} x_6^{12}. \tag{5.105}$$

The moduli dependent part is best interpreted in phase II with a_1, a_7 and a_9 given in eq. (5.103). In fact, when setting the fourth modulus to $z_4^{\mathrm{II}} = 0$, we see that the deformation of the seven-brane locus $\{\tilde{m}_6 = 0\}$ is precisely parameterized by z_3^{II}. By setting $x_i = 1$, we fix a point in the base of Y viewed as fibration with fiber X. We are then in the position to compare the shift in eq. (5.105) with the first constraint in eq. (5.91) finding agreement if one identifies $\hat{z}_1 = z_3^{\mathrm{II}} (z_1^{\mathrm{II}})^{1/3}$. In the next section we will show that the open string BPS numbers of the local model with D5-branes of § 3.2 are recovered in the z_3^{II} direction. The shift of the naive open modulus $\hat{z}_1$ by the closed complex structure modulus z_1^{II} fits nicely with a similar redefinition made for the local models in ref. [33]. This leaves us with the interpretation that indeed z_3^{II} deforms the seven-brane locus and corresponds to an open string modulus in the local picture. As we will show in the next section, a z_3^{II} dependent superpotential is induced upon switching on fluxes on the seven-brane. It can be computed explicitly and matched with the local results for D5-branes for an appropriate choice of flux. A second interpretation of the shifts in eq. (5.105) by the monomials proportional to $z_3^{\mathrm{II}}, z_4^{\mathrm{II}}$ is via the heterotic dual theory on X and the spectral cover construction. This viewpoint will be treated in § 6.

As a side remark, let us again point out that the discriminant (5.104) with polynomials given in eq. (5.100) is not the full answer for the discriminant since we have set many of the blow-up coordinates to 1. However, we can use the toric methods of refs. [153, 159, 160] to determine the full gauge group in the absence of flux to be

$$G_Y = E_8^{25} \times F_4^{69} \times G_2^{184} \times SU(2)^{276}. \tag{5.106}$$

Groups of such large rank are typical for elliptically fibered CY fourfolds with many Kähler moduli corresponding to blow-ups of singular fibers [160].

5.3.3 Computation of the superpotential

For the matching of the flux and brane superpotentials (2.31) from the perspective of F-theory, we use the following strategy: We identify the periods of the threefold fiber X of Y among the fourfold periods. This implies a matching of all instanton numbers as well as the classical terms on the mirror CY threefold $\widehat{X}$. Furthermore, we explicitly identify fourfold periods that reproduce the physics of branes on the local geometry of $\widehat{X}$ discussed in § 3.2, namely all disk instanton numbers calculated in ref. [33]. This is equivalent to calculating the type IIB flux and the D7-brane superpotential for a specific brane flux from the F-theory flux superpotential where the closed BPS states of the CY fourfold are encoded in $F^0(\gamma)$. We explicitly show that there is an element $\hat{\gamma} \in H^{2,2}(Y)$ such that the enumerative geometry on the threefold mirror pair $(\widehat{X}, X)$ with and without Harvey-Lawson type branes is reproduced. The results presented below are of further importance for the discussion of the heterotic/F-theory duality in § 6 where the space X is promoted to the background geometry of the heterotic string.

Here, we will discuss the geometry Y introduced in § 5.3.2 and refer to appendix A.3.2 for further examples involving del Pezzo surfaces. The CY fourfold Y has four complex structure moduli z_a. The moduli dependence of the periods is determined by a complete set of six PF operators which are linear differential operators $\mathcal{L}_\alpha$ of order $(3,2,2,2,3,2)$. They can be obtained from the $\mathbb{C}^\times$ symmetries of period integrals associated to the charge vectors

$$\ell_I^{(1)}, \quad \ell_I^{(2)}, \quad \ell_I^{(3)}, \quad \ell_I^{(4)}, \quad \ell_I^{(1)} + \ell_I^{(3)}, \quad \ell_I^{(3)} + \ell_I^{(4)} \tag{5.107}$$

by the methods described in ref. [9]. We use logarithmic derivatives $\theta_a = z_a \frac{d}{dz_a}$. Here, we only write down the leading piece of the differential equations given by

$$\mathcal{L}_\alpha^{\lim} = \lim_{z_a \to 0} \mathcal{L}_\alpha(\theta_a, z_a) \quad \text{with} \quad \alpha = 1, \ldots, 6 \quad \text{and} \quad a = 1, \ldots, 4. \tag{5.108}$$

This means that for the case at hand we have

$$\begin{aligned} &\mathcal{L}_1^{\lim} = \theta_1^2(\theta_3 - \theta_1 - \theta_4), \quad \mathcal{L}_2^{\lim} = \theta_2(\theta_2 - 2\theta_1 - \theta_3 - \theta_4), \quad \mathcal{L}_3^{\lim} = (\theta_1 - \theta_3)(\theta_3 - \theta_4), \\ &\mathcal{L}_4^{\lim} = \theta_4(\theta_1 - \theta_3 + \theta_4), \quad \mathcal{L}_5^{\lim} = \theta_1^2(\theta_4 - \theta_3), \quad \mathcal{L}_6^{\lim} = \theta_4(\theta_1 - \theta_3). \end{aligned} \tag{5.109}$$

For the complete PF operators as well as the cohomology basis we extract from them, we refer to appendix A.3.1.

Applying the quotient ring construction $\mathcal{R}$ given in eq. (5.48), it is easy to see that there are $(1,4,6,4,1)$ generators for $\mathcal{R}$ of degree $\{0,\ldots,4\}$. These generators are shown in Table 5.6. These operators can be associated to solutions of the PF equations and to a choice of basis elements of the cohomology ring as explained in § 5.2.3. At grade $k = 2$, the leading solutions $\mathbb{L}^{(k)\,\alpha}$ of the PF system (A.27), normalized to obey $\mathcal{R}_\alpha^{(k)} \mathbb{L}^{(k)\,\beta} = \delta_\alpha^\beta$, are then given by

$$\begin{aligned} &\mathbb{L}^{(2)\,1} = l_1^2, && \mathbb{L}^{(2)\,2} = \frac{1}{2} l_4 (l_1 + l_3), \\ &\mathbb{L}^{(2)\,3} = \frac{1}{2} l_3 (l_1 + l_3), && \mathbb{L}^{(2)\,4} = \frac{1}{7} l_2 (3l_1 - 2(l_3 + l_4 - l_2)), \\ &\mathbb{L}^{(2)\,5} = \frac{1}{7} l_2 (-2l_1 + l_2 + 6l_4 - l_3), && \mathbb{L}^{(2)\,6} = \frac{1}{7} l_2 (-2l_1 + l_2 + 6l_3 - l_4) \end{aligned} \tag{5.110}$$

where we used the abbreviation $l_k = \log z_k$ and omitted the prefactor X^0. In comparison to the complete solutions $\Pi^{(2)\,\alpha}$ of the PF equations we omitted terms of order $\mathcal{O}(l_k)$ as in eq. (5.52)

$\mathcal{R}^{(0)}$	1
$\mathcal{R}^{(1)}_a$	$\theta_1,\ \theta_2,\ \theta_3,\ \theta_4$
$\mathcal{R}^{(2)}_\alpha$	$\theta_1^2,\ (\theta_1+\theta_3)\theta_4,\ (\theta_1+\theta_3)\theta_3,\ (\theta_1+2\theta_2)\theta_2,\ (\theta_2+\theta_4)\theta_2,\ (\theta_2+\theta_3)\theta_2$
$\mathcal{R}^{(3)}_a$	$(\theta_3+\theta_4)\left(\theta_1^2+\theta_1\theta_3+\theta_3^2\right),\ \theta_2\left(\theta_3^2+3\theta_2\theta_3+5\theta_2^2+\theta_1(\theta_2+\theta_3)\right),$ $\theta_2\left(\theta_1(\theta_2+\theta_4)+\theta_4(\theta_3+3\theta_2)+\theta_2(\theta_3+6\theta_2)\right),\ \theta_2\left(\theta_1^2+2\theta_1\theta_2+4\theta_2^2\right)$
$\mathcal{R}^{(4)}$	$\theta_4(\theta_1^2\theta_2+3\theta_1\theta_2^2+9\theta_2^3+\theta_1\theta_2\theta_3+3\theta_2^2\theta_3+\theta_2\theta_3^2)$ $+\theta_2\left(46\theta_2^3+15\theta_2^2\theta_3+4\theta_2\theta_3^2+\theta_3^3+\theta_1^2(2\theta_2+\theta_3)+\theta_1\left(11\theta_2^2+4\theta_2\theta_3+\theta_3^2\right)\right)$

Table 5.6: Ring of operators for the main example

and eq. (5.53). Since we are calculating the holomorphic potentials $F(\gamma)$ and the corresponding BPS-invariants, we have to change the basis of solutions such that to any operator $\mathcal{R}^{(2)}_\alpha$ in Table 5.6 we associate a solution with leading logarithm determined by the classical triple intersection $C^{0\ (1,1,2)}_{ab\alpha}$

$$\mathbb{L}^{(2)}_\alpha = \frac{1}{2}X^0 C^0_{\alpha ab} l_a l_b. \tag{5.111}$$

From the above classical intersection data in $\mathcal{R}^{(4)}$ we obtain the leading terms $\mathbb{L}^{(2)}_\alpha$ related to the leading periods $\mathbb{L}^{(2)\,\alpha}$ of the four-form Ω_Y by $\mathbb{L}^{(2)}_\alpha = \mathbb{L}^{(2)\,\beta}\eta^{(2)}_{\alpha\beta}$.

As discussed before, the choice of periods $\Pi^{(2)\,\alpha}$ with leading terms $\mathbb{L}^{(2)\,\alpha}$ corresponds to a particular choice of a basis $\hat{\gamma}^{(2)}_\alpha$ of $H^{2,2}_V(\hat{Y})$. In fact, by construction we find

$$\hat{\gamma}^{(2)}_\alpha = \mathcal{R}^{(k)}_\alpha \Omega_Y\Big|_{\underline{z}=0}. \tag{5.112}$$

However, this choice of basis for $H^{2,2}_V(\hat{Y})$ does not necessarily coincide with a basis of integral cohomology. An integral basis can be determined by an appropriate basis change. We first note that the Kähler generator J_4 can be identified as the class of the CY threefold fiber $\hat{X}$, cf. appendix A.3.1 for more details on this identification. Moreover, we find the identification of the CY fourfold Kähler generators J_i with the CY threefold generators $J_k(\hat{X})$ as

$$J_1+J_3 \leftrightarrow J_1(\hat{X}), \quad J_2 \leftrightarrow J_2(\hat{X}) \tag{5.113}$$

by comparing the coefficient of J_4 in the intersection form $\mathcal{C}_0(\hat{Y})$ given in eq. (A.24) with $\mathcal{C}_0(\hat{X})$ in eq. (5.79). A subset of the basis elements of the CY fourfold integral basis are now naturally induced from the CY threefold integral basis. Consequently, we identify the threefold periods $\partial_i F_{\hat{X}}$ with derivatives in the directions of $J_1(\hat{X})$ and $J_2(\hat{X})$, with an appropriate linear combination of the fourfold periods $\Pi^{(2)\,\alpha}$ [165]. In other words we determine a new basis $\hat{\gamma}^{(2)}_i$ such that

$$\partial_i F_{\hat{X}} = F^0(\gamma^{(2)}_i)\Big|_{z_4=0} = \Pi^{(2)}_i\Big|_{z_4=0}. \tag{5.114}$$

In this matching both the classical part of the periods as well as the threefold BPS invariants n_{d_1,d_2} and fourfold BPS invariants $n_{d_1,d_2,d_1,0}(\gamma)$ have to match in the large $\mathbb{P}^1$ base limit.

The above identification (5.114) is most easily performed by first comparing the classical parts of the periods. In fact, using the classical intersections of $\widehat{X}$ in eq. (5.79) we can deduce the leading parts of the threefold periods to be

$$\mathbb{L}_1(X) = \frac{1}{2}X_0\tilde{l}_2\big(2\tilde{l}_1+3\tilde{l}_2\big), \quad \mathbb{L}_2(X) = \frac{1}{2}X_0\big(\tilde{l}_1+3\tilde{l}_2\big)^2 \tag{5.115}$$

where $\tilde{l}_i = \log\tilde{z}_i$ correspond to the two threefold directions $J_k(\widehat{X})$ in the identification given in eq. (5.113). Using the identification and the matching of the periods, eq. (5.113) and eq. (5.114), we then find the appropriately normalized leading fourfold periods

$$\mathbb{L}_2^{(2)} = \frac{1}{2}X_0 l_2\,(2l_1+3l_2+2l_3)\,, \quad \mathbb{L}_5^{(2)} = \frac{1}{2}X_0\,(l_1+3l_2+l_3)^2\,. \tag{5.116}$$

A direct computation also shows that the threefold BPS invariants $d_i n_{d_1,d_2}$ and fourfold BPS invariants $n_{d_1,d_2,d_1,0}(\gamma_i)$ match in the large $\mathbb{P}^1$ base limit such that the matching (5.114) is established on the classical as well as on the quantum level. This match fixes corresponding integral basis elements of $H^{2,2}_V(\widehat{Y})$ as follows: Firstly, we determine those two ring elements $\widetilde{\mathcal{R}}_2^{(2)}$ and $\widetilde{\mathcal{R}}_5^{(2)}$ such that we obtain $\mathbb{L}_2^{(2)}$ and $\mathbb{L}_5^{(2)}$ using eq. (5.111). We complete them to a new basis of ring elements $\widetilde{\mathcal{R}}_\alpha^{(2)}$ by choosing

$$\begin{aligned}
&\widetilde{\mathcal{R}}_1^{(2)} = \theta_1^2, && \widetilde{\mathcal{R}}_2^{(2)} = \frac{1}{2}\theta_4\,(\theta_1+\theta_3)\,,\\
&\widetilde{\mathcal{R}}_3^{(2)} = \frac{1}{2}\theta_3\,(\theta_1+\theta_3)\,, && \widetilde{\mathcal{R}}_4^{(2)} = \frac{1}{7}\theta_2\,(3\theta_1-2\,(\theta_3+\theta_4-\theta_2))\,,\\
&\widetilde{\mathcal{R}}_5^{(2)} = \frac{1}{7}\theta_2\,(-2\theta_1+\theta_2+6\theta_4-\theta_3)\,, && \widetilde{\mathcal{R}}_6^{(2)} = \frac{1}{7}\theta_2\,(-2\theta_1+\theta_2+6\theta_3-\theta_4)\,.
\end{aligned} \tag{5.117}$$

These operators fix the two integral basis elements

$$\hat{\gamma}_2^{(2)} = \widetilde{\mathcal{R}}_2^{(2)}\Omega_Y\Big|_{\underline{z}=0}, \quad \hat{\gamma}_5^{(2)} = \widetilde{\mathcal{R}}_5^{(2)}\Omega_Y\Big|_{\underline{z}=0}. \tag{5.118}$$

which reproduce the corresponding part of the flux superpotential (2.20) on X for $\widehat{N}_i = 0$ when turning on four-form flux on Y in these directions

$$W_{\text{flux}} \equiv M^1F^0(\gamma_2^{(2)}) + M^2F^0(\gamma_5^{(2)}) = \int_Y \Omega\wedge G_4 = M^1\Pi_2^{(2)} + M^2\Pi_5^{(2)} \tag{5.119}$$

for the following choice of G_4 flux

$$G_4 = M^1\hat{\gamma}_2^{(2)} + M^2\hat{\gamma}_5^{(2)}. \tag{5.120}$$

For the choices $M^i = 1$ we extract the invariants $d_i n_{d_1,d_2}$ from this superpotential, i.e. from the prepotentials $F^0(\gamma_2^{(2)})$ and $F^0(\gamma_5^{(2)})$. We note that the above grade $k=2$ basis elements (5.117) become, under formal identification of l_i with θ_i, the leading solutions (5.110) of the PF system. Using the same identification, we find

$$\mathbb{L}^{(2)\,2} = X_0(l_1+l_3)l_4, \quad \mathbb{L}^{(2)\,5} = X_0(l_2+l_4)l_2 \tag{5.121}$$

as the leading behavior of corresponding periods $\Pi^{(2)\,\alpha}$. This agrees with the naive expectation from the large base limit that a partial factorization of the periods occurs as $t_4\cdot t_i^{\widehat{X}}$ for $t_{1/2}^{\widehat{X}}$, the two classes in $\widehat{X}$ [165].

d_1	$d_3=0$	$d_3=1$	$d_3=2$	$d_3=3$	$d_3=4$	$d_3=5$	$d_3=6$
0	0	1	0	0	0	0	0
1	1	n_1	−1	−1	−1	−1	−1
2	−1	−2	$2n_2$	5	7	9	12
3	1	4	12	$3n_3$	−40	−61	−93
4	−2	−10	−32	−104	$4n_4$	399	648
5	5	28	102	326	1085	$5n_5$	−4524
6	−13	−84	−344	−1160	−3708	−12660	$6n_6$

Table 5.7: BPS invariants $n_{d_1,0,d_3,0}(\hat{\gamma})$ for the disks. Identifying $m = d_3 - d_1$ and $d = d_1$ with winding and $\mathbb{P}^2$ degree, this agrees with Tab. 5 of ref. [33].

It is one crucial point of our whole analysis that we can extend this matching of threefold invariants even for disk invariants counting curves with boundaries on Lagrangian cycles L in $\hat{X}$. Having explained the F-theory origin of this fact before, we will here explicitly find the flux choice in $H^{2,2}_H(Y)$ for which the F-theory flux superpotential reproduces the brane superpotential. By construction the CY fourfold $\hat{Y}$ inherits the information of the fiber $\hat{X}$ and in particular the local geometry $\mathcal{O}_{\mathbb{P}^2}(-3)$. As noted earlier, the brane data is translated to the F-theory by the Mori cone generator $\ell^{(3)}$ and its dual divisor J_3 of $\hat{Y}$. Therefore, we expect to reproduce all classical terms as well as to extract the disk instantons of ref. [33] from the GW invariants $n_{d,0,d+k,0}$ of a period constructed via eq. (5.111) from operators of the form

$$\mathcal{R}^{(2)}_{\gamma} = \theta_3(\theta_1+\theta_3)+\cdots \tag{5.122}$$

of the basis of the solutions (5.110). However, the geometry at hand is more complicated and the ring element $\mathcal{R}^{(2)}_{\gamma}$ with this property is not unique. It takes the form

$$\mathcal{R}^{(2)}_{\gamma} = -\mathcal{R}^{(2)}_{1} + \frac{1}{3}\mathcal{R}^{(2)}_{2} + \mathcal{R}^{(2)}_{3} = -\theta_1^2 + \frac{1}{2}\theta_3(\theta_1+\theta_3) + \frac{1}{6}\theta_4(\theta_1+\theta_3) \tag{5.123}$$

which is the most convenient choice by setting the arbitrary coefficients of $\mathcal{R}^{(2)}_{\alpha}$ with $\alpha = 4,5,6$ to zero. We note that only the coefficient in front of $\mathcal{R}^{(2)}_{3}$ was fixed to 1 by the requirement of reproducing the disk instanton invariants. The other two coefficients were fixed by the requirement of reproducing the closed GW invariants n_d of $\mathcal{O}_{\mathbb{P}^2}(-3)$, computed in ref. [172], by the fourfold invariants $n_d = n_{d,0,d,0}$, i.e. for $m = 0$, as explained below. The relation between $\mathcal{R}^{(2)}_{\gamma}$ and the corresponding solution is $\hat{\gamma} = \mathcal{R}_{\gamma}\Omega_Y\big|_{\underline{z}=0}$, i.e. $\mathcal{R}^{(2)}_{\gamma}\Pi^{(2)\gamma} = 1$ such that

$$\mathbb{L}^{(2)\gamma} = -X_0 l_1^2, \quad \mathbb{L}^{(2)}_{\gamma} = \frac{1}{6}X_0 l_2(8l_1+9l_2+2l_3)\,. \tag{5.124}$$

This implies that we have explicitly calculated the D7-brane superpotential (2.22) from the fourfold superpotential (2.24) by turning on the following flux

$$W_{\mathrm{D7}} = F^0(\gamma) = \int_Y \Omega_Y \wedge \hat{\gamma} = \Pi^{(2)}_{\gamma} \Longrightarrow G_4 = \hat{\gamma}. \tag{5.125}$$

Table 5.7 shows the extracted numbers $n_{d_1,0,d_3,0}(\gamma)$ from $F^0(\gamma)$. The BPS invariants of the holomorphic disks depend only on the relative homology class. In the table $m = d_3 - d_1$ labels the winding number of the disks and $d = d_1$ the degree with respect to canonical class of $\mathbb{P}^2$. If the open string disk superpotential is in terms of the closed string parameter $q = e^{2\pi i t}$ and the open string string parameter $Q = e^{2\pi i \tilde{t}}$ for the outer brane defined as

$$W = a_{tt} t^2 + a_{t\tilde{t}} t\tilde{t} + a_{\tilde{t}\tilde{t}} \tilde{t}^2 + a_t t + a_{\tilde{t}} \tilde{t} + a_0 + \sum_{d=1}^{\infty} \sum_{m=-d}^{\infty} n_{d,m} \, \mathrm{Li}_2(q^d Q^m), \tag{5.126}$$

then $n_{d_1,0,d_3,0} = n_{d_1,d_3-d_1}$. Note that the numbers $n_{d_1,0}$ are not calculated in the framework of ref. [33]. However, it is natural and calculable in the topological vertex formalism [110] that they should be identified with $d n_d$ where n_d is the closed string genus zero BPS invariant defined via the prepotential as $F^0 = \sum_{d=1}^{\infty} n_d \, \mathrm{Li}_3(q^d)$. The factor of d comes by identifying $W = dF^0/dt$. This interpretation $n_{d,0,d,0} = d n_d$ can be consistently imposed and yields two further conditions as mentioned above.

To obtain the open BPS invariants of phase III of ref. [33], we use phase II of Table 5.4. In this phase the fiber class is not realized as a generator of the Kähler cone. However, we readily recover the classes of $\widehat{X}$ as

$$J_1 \leftrightarrow J_1(\widehat{X}), \quad J_2 + J_3 \leftrightarrow J_2(\widehat{X}) \tag{5.127}$$

by comparison of the Mori cone in Table 5.4 with the Mori cone in Table 5.2 of $\widehat{X}$. Then, we fix a basis $\mathcal{R}^{(2)}_\alpha$ of the ring at grade 2 as

$$\theta_1^2, \quad 2\theta_2\left(\theta_1 + 3\theta_3\right), \quad \theta_3\left(\theta_1 + 3\theta_3\right), \quad \theta_1\theta_4, \quad \theta_2^2, \quad \left(\theta_2 + \theta_3\right)\left(2\theta_3 + \theta_4\right) \tag{5.128}$$

from which we obtain a basis of dual solutions $\mathbb{L}^{(k)\,\alpha}$ to the PF system (A.34)

$$\begin{aligned}
\mathbb{L}^{(2)\,1} &= l_1^2, \\
\mathbb{L}^{(2)\,2} &= \frac{1}{140}\left(l_1\left(16 l_2 + 9 l_3\right) + 3\left(l_2\left(6 l_3 - 5 l_4\right) - l_3\left(l_3 + 5 l_4\right)\right)\right), \\
\mathbb{L}^{(2)\,3} &= \frac{1}{70}\left(l_1\left(9 l_2 + 16 l_3\right) - 3\left(l_3\left(-6 l_3 + 5 l_4\right) + l_2\left(l_3 + 5 l_4\right)\right)\right), \\
\mathbb{L}^{(2)\,4} &= l_1 l_4, \quad \mathbb{L}^{(2)\,5} = l_2^2, \quad \mathbb{L}^{(2)\,6} = \frac{1}{14}\left(l_2 + l_3\right)\left(-3 l_1 + l_3 + 5 l_4\right).
\end{aligned} \tag{5.129}$$

Next, we construct two solutions with leading logarithms matching the two threefold periods of eq. (5.115) for which we are able to match the threefold invariants $d_i n_{d_1,d_2}$ in the large base limit as well. The leading logarithms of these fourfold periods read

$$\begin{aligned}
\mathbb{L}^{(2)}_4 &= \frac{1}{2} X_0 \left(l_1 + 3\left(l_2 + l_3\right)\right)^2, \\
\mathbb{L}^{(2)}_6 &= \frac{1}{2} X_0 \left(l_2 + l_3\right)\left(2 l_1 + 3\left(l_2 + l_3\right)\right)
\end{aligned} \tag{5.130}$$

which is in perfect agreement with the threefold periods (5.115) under the identification of the classes given in eq. (5.127). We fix the corresponding operators $\widetilde{\mathcal{R}}^{(2)}_4$ and $\widetilde{\mathcal{R}}^{(2)}_6$ by matching the above two leading logarithms by the classical intersections $C^0_{\alpha ab}$ via eq. (5.111). We complete them to a basis of $\widetilde{\mathcal{R}}^{(2)}$ as follows

$$\begin{aligned}
\widetilde{\mathcal{R}}^{(2)}_1 &= \theta_1^2, \\
\widetilde{\mathcal{R}}^{(2)}_2 &= \frac{1}{140}\left(\theta_1\left(16\theta_2 + 9\theta_3\right) + 3\left(\theta_2\left(6\theta_3 - 5\theta_4\right) - \theta_3\left(\theta_3 + 5\theta_4\right)\right)\right), \\
\widetilde{\mathcal{R}}^{(2)}_3 &= \frac{1}{70}\left(\theta_1\left(9\theta_2 + 16\theta_3\right) - 3\left(\theta_3\left(-6\theta_3 + 5\theta_4\right) + \theta_2\left(\theta_3 + 5\theta_4\right)\right)\right), \\
\widetilde{\mathcal{R}}^{(2)}_4 &= \theta_1\theta_4, \quad \widetilde{\mathcal{R}}^{(2)}_5 = \theta_2^2, \quad \widetilde{\mathcal{R}}^{(2)}_6 = \frac{1}{14}\left(\theta_2 + \theta_3\right)\left(-3\theta_1 + \theta_3 + 5\theta_4\right)
\end{aligned} \tag{5.131}$$

d	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0	0	n_1	$2n_2$	$3n_3$	$4n_4$	$5n_5$	$6n_6$
1	−1	2	−5	32	−286	3038	−35870
2	0	1	−4	21	−180	1885	−21952
3	0	1	−3	18	−153	1560	−17910
4	0	1	−4	20	−160	1595	−17976
5	0	1	−5	26	−196	1875	−20644
6	0	1	−7	36	−260	2403	−25812

Table 5.8: BPS invariants $n_{k,0,i,0}(\gamma)$ for the disks of the second triangulation

where again this basis relates to the leading periods (5.129) by $\theta_i \leftrightarrow l_i$. The corresponding integral basis elements of $H_H^{2,2}(Y)$ read

$$\hat{\gamma}_4^{(2)} = \tilde{\mathcal{R}}_4^{(2)}\Omega_Y\Big|_{\underline{z}=0}, \quad \hat{\gamma}_6^{(2)} = \tilde{\mathcal{R}}_6^{(2)}\Omega_Y\Big|_{\underline{z}=0}. \tag{5.132}$$

Furthermore, we determine the ring element $\mathcal{R}_\gamma^{(2)}$ that matches the open superpotential by turning on four-form flux in the direction $\hat{\gamma} = \mathcal{R}_\gamma^{(2)}\Omega\Big|_{\underline{z}=0}$. Again we fix

$$\mathcal{R}_\gamma^{(2)} = a_1\mathcal{R}_2^{(2)} - \frac{1}{10}(1+6a_2)\mathcal{R}_3^{(2)} + \mathcal{R}_4^{(2)} + a_3\mathcal{R}_5^{(2)} + a_2\mathcal{R}_6^{(2)} \tag{5.133}$$

by extracting the disk invariants from the associated solution via eq. (5.111) which reads

$$\begin{aligned} \mathbb{L}^{(2)\gamma} &= c(a_1)X_0 l_2(l_1+3l_3)\,, \\ \mathbb{L}_\gamma^{(2)} &= \frac{1}{6}(l_2+l_3)(2l_1+3(l_2+l_3)) - \frac{1}{10}(l_1+3(l_2+l_3))(3l_1+29l_2+29l_3+10l_4)\,. \end{aligned} \tag{5.134}$$

Here, we explicitly displayed the dependence on the three free parameters a_i for $\mathbb{L}^{(2)\gamma}$ by

$$c(a_1) = \frac{7}{9+140a_1} \tag{5.135}$$

and evaluated $\mathbb{L}_\gamma^{(2)}$ for the convenient choice $a_i = 0$. In Table 5.8 we show some BPS invariants for phase II.

6 Heterotic/F-theory duality and five-brane superpotential

The profound study of nature is the most fertile source of mathematical discoveries.

J. B. J. Fourier,
The Analytical Theory of Heat

In this chapter the main theme will be the heterotic/F-theory duality [31]. Since the discovery of F-theory, the duality to the heterotic theory was investigated in detail, cf. for example refs. [173, 174, 84, 175, 176, 177, 178, 93, 179, 180]. This will give the blow-up construction described in § 4 a sound physical ground. To study the duality, we can start by blowing up the heterotic CY threefold along the curve wrapped by horizontal five-branes. Then, from the blow-up geometry we construct the CY fourfold for the F-theory compactification. In this way we will be able to map complex structure, gauge and brane moduli of the heterotic setup to the complex structure moduli of the CY fourfold. Thus, we slightly extend the heterotic/F-theory duality. This can be schematically illustrated as follows

heterotic string on CY threefold Z, vector bundle E, five-brane on $\mathcal{C}$	$\longleftrightarrow$	F-theory on CY fourfold Y blown up along $\mathcal{C}$, G_4-flux

$\searrow$ heterotic string on non-CY threefold $\widetilde{Z} = \mathrm{Bl}_{\mathcal{C}}\, Z$, vector bundle $\widetilde{E}$ $\nearrow$

where the horizontal arrows indicates the action of the heterotic/F-theory duality.

Following this strategy, we will first discuss the heterotic/F-theory duality. We will study the spectral cover construction from ref. [84] and how we have to identify the moduli in this duality. Then, we will investigate how the blow-up geometry enters the heterotic setting and how it is related to the horizontal five-branes and the blow-up of the base of the CY fourfold of F-theory. We will also argue that there is a map between heterotic and the F-theory flux superpotentials since both blown-up threefolds and the fourfolds can be given as complete intersections. In the last section, we will treat explicit examples of heterotic/F-theory pairs checking the moduli map. We will also construct the CY fourfold for F-theory explicitly from the blown-up CY threefold of the heterotic theory. As the last example, we will re-investigate the main example of § 5 in the light of this duality. This chapter is based on ref. [54].

6.1 Heterotic/F-theory duality

In this section we will describe the crucial ingredients for the heterotic/F-theory duality: The spectral cover construction and the identification of the moduli. The spectral cover construction is the only known method for general construction of stable vector bundles on elliptic CY manifolds. In addition, it plays a very important role in the heterotic/F-theory duality. Therefore, its importance cannot be overstated. Since the beginning of the duality many mappings of the moduli of both sides have been uncovered. We will describe some of the most important mappings which we will need later in our computations.

6.1.1 Spectral cover

A consistent compactification of the heterotic string on CY manifolds requires a gauge subbundle V of $E_8 \times E_8$ bundle to be *stable*. In complex coordinates the stability of a gauge bundle can be characterized as follows

$$\mathcal{F}_{ij} = \mathcal{F}_{\bar{\imath}\bar{\jmath}} = 0, \quad g^{i\bar{\jmath}}\mathcal{F}_{i\bar{\jmath}} = 0 \tag{6.1}$$

where $\mathcal{F}$ is the field strength of the gauge bundle. The first equation tells us that the bundle V and the connection has to be holomorphic. The second equation is the so-called Donaldson-Uhlenbeck-Yau equation. This has a unique solution if V is stable. The connection is constrained further by the Bianchi identity

$$dH = \mathrm{Tr}\,\mathcal{R} \wedge \mathcal{R} - \frac{1}{30}\mathrm{Tr}\,\mathcal{F} \wedge \mathcal{F} \tag{6.2}$$

which, however, we will not consider further in this work. Let us comment here that the stability of vector bundles required in the heterotic theory ignited a completely new branch of mathematics. Douglas formulated the Π stability based on the stability discussed above in ref. [181], cf. ref. [182, 183] for a nice review. Then, the topic was extended to the stability of BPS states in $\mathcal{N} = 2$ supersymmetric theories or the stability of D-branes in the string theory. Bridgeland mathematically formulated the Π stability in ref. [184]. Recently, there has been tremendous progress regarding the wall crossing behavior of the stability conditions of the BPS states, culminating in the Kontsevich-Soibelman wall crossing formula [185].

The condition of stability puts a very severe constraint on the possible choice of the gauge bundle V since it is a very difficult task to construct a stable bundle for arbitrary CY manifolds.

For a class of manifolds however, namely for elliptically fibered manifolds, there are constructive methods to obtain stable bundles. The seminal works [84, 186] use the method of spectral cover, del Pezzo surfaces, and the parabolics to construct stable bundles. We will review the spectral cover method of ref. [84] which works for $SU(n)$ and $Sp(n)$ bundles. We will concentrate on the $SU(n)$ case.

Stable bundles on elliptic curves

Let Z be an elliptically fibered manifold with a section σ, meaning

$$T^2 \longrightarrow Z \underset{\sigma}{\overset{\pi_Z}{\rightleftarrows}} B_Z. \tag{6.3}$$

The basic strategy of the spectral cover method is to use stable bundles over the elliptic curve E. We first construct stable $SU(n)$ bundles over E and pull it back to Z. Let V be a stable $SU(n)$ bundle on E. It is obvious from eq. (6.1) that V is flat. The fact that V is a flat $SU(n)$ bundle means the following

$$V = \bigoplus_{i=1}^{n} L_i, \quad \bigotimes_{i=1}^{n} L_i = \mathbb{1} \tag{6.4}$$

where L_i are line bundles and $\mathbb{1}$ denotes the trivial line bundle. It is a well-know fact that we can define a group law on points of E with the identity element being the distinguished point p on E. This group can be obtained by the degree zero Picard group whose elements are of the form $L_Q = \mathcal{O}_E(Q) \otimes \mathcal{O}_E(p)^{-1}$. It is clear that L_Q is of degree zero and thus flat. This means that there is an one-to-one correspondence between points of E and flat line bundles on E. The first equation in eq. (6.4) translates to

$$\sum_{i=1}^{n}{}' Q_i = p \tag{6.5}$$

where the primed sum denotes the sum under the group law just discussed. Now, it is easy to construct a stable $SU(n)$ bundle V on E: Choose a set of n points $\{Q_i\} \subset E$ and V is given by the direct sum of the L_{Q_i}. Consequently, the moduli space $\mathcal{M}$ for $SU(n)$ stable bundles on E is isomorphic to $\mathbb{P}(H^0(E, \mathcal{O}_E(np)) \cong \mathbb{P}^{n-1}$. This can be explained since an element of $H^0(E, \mathcal{O}_E(np))$ has a pole of order n at p and n zeroes corresponding to the Q_i. Usually, the sections of $H^0(E, \mathcal{O}_E(np))$ have zeroes of order n at p. However, we have the following short exact sequence for a divisor D in M, cf. for example ref. [187, p. 84]

$$0 \longrightarrow \mathcal{O}(-D) \stackrel{\alpha}{\longrightarrow} \mathcal{O}_M \longrightarrow \mathcal{O}_D \longrightarrow 0 \tag{6.6}$$

where the map α is given by multiplication with the (non-unique) non-trivial section s whose zero locus is D. The dual map

$$\mathcal{O}_M \longrightarrow \mathcal{O}(D) \tag{6.7}$$

is then given by division by s. Thus, we have to consider sections with pole of appropriate order since we are going to work with the coordinates of the ambient space. An element w in $\mathcal{M}$ has the following explicit form in the affine coordinates x and y

$$w = a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 x^2 y + \cdots + \begin{cases} a_n x^{n/2} & \text{for } n \text{ even,} \\ a_n x^{(x-3)/2} y & \text{for } n \text{ odd} \end{cases} \tag{6.8}$$

with a_i being the homogeneous coordinates of $\mathcal{M}$. The point p corresponds to the infinity in x and y.

For elliptically fibered manifolds

We are now in place to start to construct a stable bundle over an elliptically fibered manifold. In § 5.1 we have seen that we need a line bundle $\mathcal{L}$ to specify the elliptic fibration. We want to fiber $\mathcal{M}$ over the base manifold B to obtain the moduli space of stable bundles on Z. Now, the coefficient a_i becomes holomorphic sections of $\mathcal{K} \otimes \mathcal{L}^{-i}$, i.e. $a_i \in H^0(B, \mathcal{K} \otimes \mathcal{L}^{-i})$. We will momentarily explain the role of the line bundle $\mathcal{K}$. Each $\mathbb{P}^{n-1}_b$ with $b \in B$ fits to a $\mathbb{P}^{n-1}$ bundle over B denoted by $\mathcal{W}$, thus

$$\mathcal{W} = \mathbb{P}(\mathcal{K} \otimes (\mathcal{O}_B \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3} \oplus \cdots \oplus \mathcal{L}^{-n})). \tag{6.9}$$

Before we proceed further, let us summarize what we have learned: If we have a stable bundle V on Z, then it uniquely determines a section $s \in \mathcal{W}$. Reversely, if we choose a section $s \in \mathcal{W}$, then it determines a stable bundle, but not uniquely. The section s and the Weierstraß equation determine a hypersurface C in B which is a n-fold cover of B since s has n solutions corresponding to the n points. This hypersurface C is called the *spectral cover*. The line bundle $\mathcal{K}$ determines the class of the spectral cover in Z as $[C] = n\sigma + \mathcal{K}$ where we write again $\mathcal{K}$ for the associated divisor to the line bundle $\mathcal{K}$. We rephrase the correspondence between bundles and the sections of $\mathcal{W}$ as follows

$$\text{bundles} \underset{\text{not unique}}{\overset{\text{unique}}{\rightleftarrows}} \text{spectral cover.} \tag{6.10}$$

Let us assume that we are given a spectral cover C and want to construct the stable bundle from it. To do this, we need the fiber product, the Poincaré line bundle, and the push-forward of a vector bundle. Let us go through them in steps.

Fibert product

The *fiber product* is a central and general construction[1] in algebraic geometry. Here, we discuss the construction adapted to our need, namely only for elliptically fibered manifolds. For $\pi_Z : Z \to B$ we define the fiber product $Z \times_B Z$ as follows

$$Z \times_B Z = \{(z_1, z_2) \in Z \times Z \mid \pi_Z(z_1) = \pi_Z(z_2)\}. \tag{6.11}$$

There is a naturally defined divisor Δ in $Z \times_B Z$, the diagonal

$$\Delta = \{(z_1, z_2) \in Z \times_B Z \mid z_1 = z_2\}. \tag{6.12}$$

[1] Cf. for example [65, § III.3].

We can define three natural projections $\pi_{1/2}, \tilde{\pi}$

$$\begin{array}{ccc} Z \times_B Z & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\downarrow & \searrow^{\tilde{\pi}} & \downarrow{\scriptstyle \pi_Z} \\ Z & \xrightarrow[\pi_Z]{} & B \end{array} \tag{6.13}$$

where $\tilde{\pi} = \pi_Z \circ \pi_1 = \pi_Z \circ \pi_2$.

Poincaré line bundle

Having defined the fiber product, we now define the Poincaré line bundle $\mathcal{P}$ for the elliptic curve E and then construct it for Z using the fiber product. Here, we consider E as the elliptic fibration over a point. We have seen above that the degree zero line bundles of E are parameterized by points of E, i.e. by E itself. Therefore, we want to construct a line bundle $\mathcal{P}^E$, the *Poincaré line bundle*, on the direct product[2] $E \times E$ with the following property

$$\mathcal{P}^E\big|_{Q\times E} \cong \mathcal{O}_E(Q) \otimes \mathcal{O}_E(p)^{-1} \quad \forall Q \in E. \tag{6.14}$$

The line bundle $\mathcal{P}^E$ is the universal bundle for degree zero line bundles on E. If we set $\mathcal{P}^E = \mathcal{O}_{E\times E}(D_E)$ where $D_E = \Delta_E - E \times p - p \times E$ with Δ_E being the diagonal of $E \times E$, the above property is fulfilled. To obtain the Poincaré line bundle $\mathcal{P}$ over $Z \times_B Z$ which restricted to $E_b \times E_b$ is isomorphic to $\mathcal{P}^E$ and trivial restricted to $\sigma \times_B Z$, we set

$$\mathcal{P} = \mathcal{O}_{Z\times_B Z}(D) \otimes \tilde{\pi}^* \mathcal{L}^{-1} \tag{6.15}$$

where $D = \Delta - \sigma \times_B Z - Z \times_B \sigma$. The second factor in $\mathcal{P}$ is needed since $\mathcal{O}_{Z\times_B Z}(D)\big|_{\sigma\times_B X} \cong \tilde{\pi}^* \mathcal{L}$.

Push-forward

Now, we discuss the last point of our list: the *push-forward* of a vector bundle. Let $V \to M$ a vector bundle on M and $f : M \to N$ a map from M to N. The push-forward bundle $f_* V$ is a vector bundle on N and is defined as follows[3]

$$(f_* V)(U) = V(f^{-1}(U)) \,, \quad U \subset N \text{ open} \tag{6.16}$$

The relevant case V being a line bundle and M an n-fold cover of N is illustrated in Figure 6.1. For a line bundle on an n-fold cover M the resulting vector bundle on N is a rank n vector bundle.[4]

[2] This can be seen as the trivial fiber product over a point.

[3] The push-forward of a vector bundle might not be a vector bundle anymore if f is not surjective. For our case f will be a projection and thus surjective. For sheaves, containing the vector bundles as a subclass, this is not a problem and the push-forward operation is called the *direct image*, cf. ref. [65, § II.1].

[4] There is a Higgs bundle interpretation of the spectral cover, cf. for example refs. [188, 189].

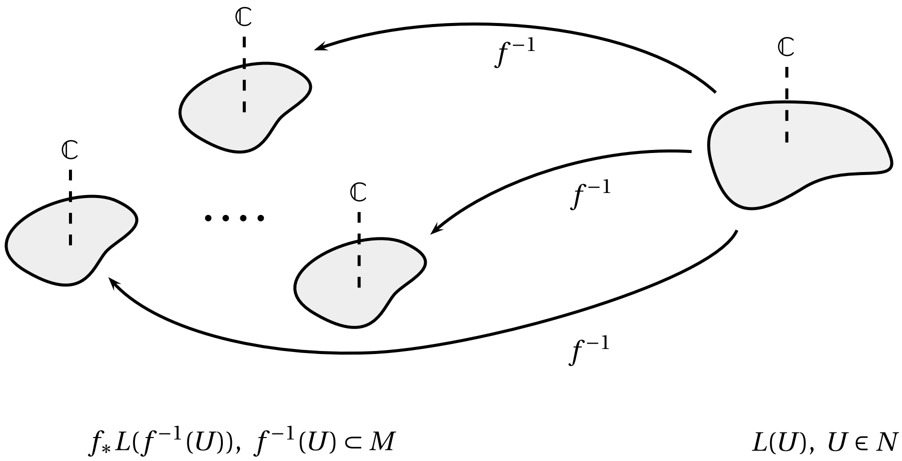


Figure 6.1: Push-forward of a line bundle

The bundle, finally

We are now in place to give the stable vector bundle corresponding to the spectral cover C. The data given are with the notation of eq. (6.13)

$$Z \longrightarrow B, \quad C \hookrightarrow B, \quad \mathcal{P} \longrightarrow Z \times_B Z, \quad \mathcal{N} \longrightarrow C \tag{6.17}$$

where $\mathcal{N}$ is an a-priori arbitrary line bundle on C called the *twisting line bundle*. From these data we obtain the following diagram

$$\begin{array}{ccccc} \mathcal{N} & & \mathcal{P} \otimes \pi_1^* \mathcal{N} & & \pi_{2*}(\mathcal{P} \otimes \pi_1^* \mathcal{N}) \\ \downarrow & & \downarrow & & \downarrow \\ C & \xleftarrow{\pi_1} & C \times_B Z & \xrightarrow{\pi_2} & Z \end{array} \tag{6.18}$$

and set $V = \pi_{2*}(\mathcal{P} \otimes \pi_1^* \mathcal{N})$. In recent literatures this construction is also called the *Fourier-Mukai transform*. The non-uniqueness in eq. (6.10) comes from the line bundle $\mathcal{N}$. Since a line bundle can always be locally trivialized, during the push-forward, $\mathcal{N}$ contributes only by tensoring a factor of $\mathbb{C}$.

Characteristic classes

Later when we apply the heterotic/F-theory duality, the second Chern class of the bundles will play an important role. Also, on the way to the second Chern class, we can constrain the line bundle $\mathcal{N}$ by fixing its characteristic class. The computation of the characteristic classes involves the application of (GRR, A.1.6), for e.g. the projections π_2 and $\pi_Z|_C$, and is quite elaborate. We refrain from the lengthy derivation, but collect the most important results of the computation. The spectral cover C is given by a section $s \in \mathcal{W}$ and is an n-fold cover of B. This means that its class in Z is $n\sigma + \eta$ where $\eta = c_1(\mathcal{K})$. Thus,

$$\mathcal{O}_Z(C) = \mathcal{O}_Z(\sigma)^n \otimes \mathcal{K} \quad \text{with} \quad \eta = c_1(\mathcal{K}). \tag{6.19}$$

The characteristic class of $\mathcal{N}$ is given by

$$\pi_{Z*} c_1(\mathcal{N}) = -\frac{1}{2} \pi_{Z*}(c_1(C) - \pi_Z^* c_1(B)) \Rightarrow c_1(\mathcal{N}) = -\frac{1}{2}(c_1(C) - \pi_Z^* c_1(B)) + \gamma \tag{6.20}$$

where γ is a class in the kernel of π_{Z*}. Finally, the characteristic class of V is given by, writing also σ for $c_1(\mathcal{O}_Z(\sigma))$,

$$\lambda(V) = c_2(V) = \eta\sigma - \frac{c_1(\mathcal{L})^2(n^3 - n)}{24} - \frac{n\eta(\eta - nc_1(\mathcal{L}))}{8} - \frac{\pi_{Z*}(\gamma^2)}{2}. \tag{6.21}$$

For E_8 bundles, we have to use the del Pezzo surfaces or the method of parabolics. We refer to ref. [84] for those methods and only quote the corresponding formula for the characteristic class for an E_8 bundle V

$$\lambda(V) = \frac{c_2(V)}{60} = \eta\sigma - 15\eta^2 + 135\eta c_1(\mathcal{L}) - 310 c_1(\mathcal{L})^2. \tag{6.22}$$

We see that the class η is essential in the general construction of the spectral cover. In the next section we will see that, in the context of the heterotic/F-theory duality, η can be constructed from the dual F-theory manifold.

6.1.2 Identification of the moduli

Using the data gained by the spectral cover method, we can start to identify the moduli in the light of the heterotic/F-theory duality. Let us however first describe the geometries for which the duality is valid. The fundamental duality is the eight-dimensional equivalence of the heterotic string compactified on T^2 and F-theory on elliptic K3 [31]. The eight-dimensional gauge symmetry G is determined in the heterotic string as the commutant of an $E_8 \times E_8$ bundle on T^2 with structure group H. This precisely matches the singularity type G of the elliptic fibration of K3 in the F-theory. Using the adiabatic argument of ref. [190], it is possible to consider a family of dual eight-dimensional theories parameterized by a base manifold to obtain dualities between the heterotic string and F-theory in lower dimensions. In this way, a four-dimensional heterotic compactification on the elliptic threefold Z is equivalent to a F-theory compactification on an elliptic K3 fibered CY fourfold Y. Consequently, the base B_Y of Y has to be a $\mathbb{P}^1$ fibration over the base B_Z of Z. Thus, we have the following

$$\mathbb{P}^1 \longrightarrow B_Y = \mathbb{P}(\mathcal{O}_{B_Z} \oplus \mathcal{T}) \longrightarrow B_Z. \tag{6.23}$$

For later use, let us denote the first Chern class of $\mathcal{T}$ by t. It turns out that precisely this fibration data of B_Y is crucial for the construction of the dual heterotic theory, in particular the stable vector bundle E on Z that determines the four-dimensional gauge group G.

It should be noted that the identification of geometrical data, e.g. moduli, of both sides is, strictly speaking, only valid at the stable degeneration limit of the elliptic CY fourfold [191]. However, we will not take this subtlety into account since it will not be important in our considerations.

Three- and five-branes

There are further building blocks necessary to specify a consistent F-theory setup. This is due to the fact that a four-dimensional compactification generically has a three-brane tadpole of the form [192, 193, 194, 195]

$$\frac{\chi(Y)}{24} = n_3 + \frac{1}{2}\int_Y G_4 \wedge G_4. \tag{6.24}$$

In the most common case of non-zero Euler characteristic a given number n_3 of spacetime-filling three-branes on points in B_Y and a specific amount of quantized four-form flux G_4 have to be added in order to fulfill the above tadpole. For a generic setup with three-branes and flux, the four-dimensional gauge symmetry as determined by the seven-branes is not affected. However, if the three-brane happens to collide with a seven-brane, it can dissolve, by the same transition, as discussed in § 2.3, into a finite-size instanton on the seven-brane worldvolume that breaks the four-dimensional gauge group G. During this transition the number n_3 of three-branes jumps and a flux G_4 is generated describing the gauge instanton on the seven-brane worldvolume [79]. In particular, in case of a heterotic dual theory the three-branes on the F-theory side precisely correspond to vertical five-branes on the heterotic threefold [84]. Thus, under duality the three-brane/instanton transition is precisely the F-theory dual of the transition of a vertical five-brane into a finite size instanton breaking the gauge group on the heterotic side accordingly. However, we will not encounter this any further since we restrict our discussion to the case that the gauge bundle on those seven branes dual to the perturbative heterotic gauge group is trivial and no three-branes sit on top of their worldvolumes.

Fluxes and twisting data

The twisting data γ in eq. (6.20) of the line bundle $\mathcal{N}$ can be identified with the four-form flux G_4 via the *cylinder map* c [177]. There is a natural $\mathbb{P}^1$ bundle $p:\mathcal{Q}\to C$ over the spectral cover C. It corresponds to the points Q_k in the elliptic fiber of the spectral cover construction described in the previous section thickened to a $\mathbb{P}^1$ [143]. We have the following relation

$$G_4 = c(\gamma) = i_* p^*\gamma \quad \text{with} \quad i:\mathcal{Q}\hookrightarrow Y. \tag{6.25}$$

The inverse map is given by

$$\gamma = p_* i^* G_4 = \int_{\mathbb{P}^1 \text{ fiber}} i^* G_4. \tag{6.26}$$

Here, p_* is the Gysin homomorphism, cf. appendix A.1.2, and in this case it can be understood as integration over the fiber [196]. It should be noted that strictly speaking, as mentioned above, we should work in the stable degeneration limit, cf. refs. [177, 143, 197] for more detailed discussion about the identification of fluxes.

K3 fibration and bundle data

The heterotic bundle $E = E_1 \oplus E_2$ is specified by [84]

$$\eta(E_1) = 6c_1(B_Z) + t, \quad \eta(E_2) = 6c_1(B_Z) - t \tag{6.27}$$

meaning that the choice of $\mathbb{P}^1$ fibration uniquely determines the η-classes of the two bundles. This is a generalization of the formula for the six-dimensional duality, namely the heterotic string on an elliptic K3 and F-theory on elliptic fibration over Hirzebruch surfaces F_n [173], cf. also the nice exposition in ref. [89, §. 6]. There, the class t is replaced by n and it corresponds to the number of instantons on each E_8 factor. In particular, we note that the heterotic anomaly (2.33) is trivially fulfilled without the inclusion of any horizontal five-branes. So far, the above discussion is not the most general setup possible since it does not allow for the presence of horizontal five-branes. It turns out that the F-theory dual to the $E_8 \times E_8$ heterotic string has to a be analyzed more thoroughly in order to naturally include horizontal five-branes to the setup. This will be the topic of next sections.

Dimensions of the moduli spaces

We finish the discussion by a brief look at the moduli map in the heterotic/F-theory duality where we focus on the fate of the five-brane moduli in the just mentioned blow-up process. The first step in the moduli analysis is to relate the dimensions of the various moduli spaces in both theories and to point to possible mismatches where moduli of some ingredients are missing. In particular, this happens in the presence of heterotic five-branes. Indeed, it was argued in ref. [92] that the relation of $h^{3,1}(Y)$, $h^{2,1}(Y)$ and $h^{1,1}(Y)$ to $h^{2,1}(Z)$, $h^{1,1}(Z)$, the bundle moduli, and characteristic data has to be modified in the presence of five-branes. The extra contribution is due to deformation moduli of the curve $\mathcal{C}_i$ supporting the five-brane counted by $\dim H^0(\mathcal{C}_i, N_{\mathcal{C}_i/Z})$ as well as the blow-ups in B_Y increasing $h^{1,1}(B_Y)$ such that we obtain [92]

$$\begin{aligned} h^{3,1}(Y) &= h^{2,1}(Z) + I(E_1) + I(E_2) + h^{2,1}(Y) + 1 + \sum_i \dim H^0(\mathcal{C}_i, N_{\mathcal{C}_i/Z}), \\ h^{1,1}(Y) &= 1 + h^{1,1}(B_Y) + \mathrm{rk}(G). \end{aligned} \tag{6.28}$$

Here, the sum index i runs over all irreducible curves $\mathcal{C}_i$ wrapped by the five-branes and we denote the rank of the four-dimensional gauge group by $\mathrm{rk}(G)$. The index $I(E_{1/2})$ counts a topological invariant of the bundle moduli and is given by the following formula [84, 180]

$$I(E_i) = \mathrm{rk}(E_i) + \int_{B_Z} \big(4(\eta_i \sigma - \lambda(E_i)) + \eta_i c_1(B_Z)\big). \tag{6.29}$$

The formula for $h^{3,1}(Y)$ reflects the fact that the four-dimensional gauge symmetry G on the heterotic side is determined by the gauge bundle E whereas on the F-theory side G is due to the seven-brane content defined by the discriminant Δ sensitive to the change of complex structure. These formulas will be used for our examples in § 6.3.

6.2 Blow-ups and superpotentials

In this section we will discuss occurrences of blow-up in the context of the heterotic/F-theory duality. Firstly, as we have introduced and described the blow-up procedure in detail in § 4.4, we will quickly review this construction for the heterotic theory. Under the duality, the horizontal five-branes are completely mapped to the geometry of the F-theory. The three-dimensional base of the F-theory compactification is blown-up along the curve of the five-brane. We will discuss the consequences of this in the polynomial structure of the Weierstraß equation. After the discussion of blow-ups, we will argue that the superpotentials of both theories are mapped to each other under the duality using the complete intersection description of blown-up three-fold and the CY fourfold.

6.2.1 Blow-up in the heterotic string

In the following we will apply the blow-up procedure discussed in § 4 in the context of the type IIB theory to the heterotic setup. The idea is again the same: Find a purely geometric description that puts the dynamics of the five-brane and the geometry of Z on an equal footing. To achieve this, we blow up the curve $\mathcal{C}$ wrapped by the horizontal five-brane into a isolated divisor D in a non-CY threefold $\tilde{Z}$. This embeds the deformation moduli of $\mathcal{C}$ in Z as well as the complex structure deformations of Z into the deformation problem of only complex structures of $\tilde{Z}$. We will see explicitly later that this alternative view on the heterotic string with five-branes allows for a direct geometric interpretation of the fate of the five-brane dynamics in the heterotic/F-theory duality. Here, we will describe the blow-up of Z along a curve $\mathcal{C}$ in the heterotic setup which will be used later in our examples in § 6.3.

It was argued in § 4.4 that the complex structure moduli space of $\tilde{Z}$ contains the complex structure moduli of Z as well as the deformation moduli of $\mathcal{C}$ within Z. The basic reason for this is roughly that the complex structure deformations of the isolated divisor D contain the deformation moduli of the curve $\mathcal{C}$ and thus embed them into the complex structure of $\tilde{Z}$. This way the deformations of the pair $(Z,\mathcal{C})$ form a subsector of the geometrical deformations of $\tilde{Z}$. This allows for the study of the combined superpotential of five-branes and flux as well. First we use the formal unification of the two superpotentials in terms of the relative homology group $H_3(Z,\mathcal{C},\mathbb{Z})$ consisting of three-cycles $H_3(Z,\mathbb{Z})$ and three-chains $\Gamma^{\mathcal{C}}$ ending on the curve $\mathcal{C}$. Then the superpotential can be written as

$$W_{\text{flux}} + W_{\text{M5}} = \sum_i \tilde{N}^i \int_{\Gamma^i} \Omega \qquad (6.30)$$

with respect to an integral basis Γ^i of the relative group $H_3(Z,\mathcal{C},\mathbb{Z})$. Here the integers $\tilde{N}^i$ correspond to the three-form flux quanta and the five-brane windings. In particular Ω has to be interpreted as a relative form.

We have argued in § 4 that in the blow-up $\pi : \tilde{Z} \to Z$ the above superpotential (6.30) is lifted to $\tilde{Z}$. We quickly repeat it here for convenience. First, we have to replace Ω by its equivalent on $\tilde{Z}$, the pullback form

$$\tilde{\Omega} = \pi^*\Omega \quad \text{with} \quad \tilde{\Omega}\big|_D = 0 \qquad (6.31)$$

Consequently, we can write the heterotic superpotentials as

$$W_{\text{flux}} + W_{\text{M5}} = \int_{\tilde{Z}} H_3 \wedge \tilde{\Omega} = \int_{\tilde{Z}-D} H_3 \wedge \tilde{\Omega} = \int_{\Gamma(H_3)} \tilde{\Omega} \qquad (6.32)$$

such that it only depends on the topology of the open manifold $Z - \mathcal{C} = \tilde{Z} - D$. Here, we naturally obtain $\Gamma(H_3)$ as the Poincaré dual of the flux H_3 in the group $H_3(\tilde{Z} - D,\mathbb{Z})$. These replacements can also be understood in the language of relative (co)homology. On the one hand we can treat $\tilde{\Omega}$ as a relative form exploiting the fact that any element in the relative group $H^3(\tilde{Z},D,\mathbb{Z})$ can be represented by a form vanishing on D. The element $\Gamma(H_3)$ maps to the relative homology via the pairing introduced in eq. (4.9). This identification of (co-)homology groups gets completed by the equivalence $H_3(Z,\mathcal{C},\mathbb{Z}) = H_3(\tilde{Z},D,\mathbb{Z})$ telling us that we have consistently replaced all relevant topological quantities on Z by those on the blow-up $\tilde{Z}$. Finally, we expand the element $\Gamma(H_3)$ in a basis Γ^i_D of $H_3(\widetilde{Z}, D, \mathbb{Z})$ to obtain an expansion of the superpotential by relative periods of $\widetilde{\Omega}$ as

$$W_{\text{flux}} + W_{\text{M5}} = \sum_i \widetilde{N}^i \int_{\Gamma^i_D} \widetilde{\Omega} = \sum_i \widetilde{N}^i \int_{\widetilde{Z}} \widetilde{\Omega} \wedge \gamma^D_i . \tag{6.33}$$

Here, γ^D_i are the Poincaré duals in $H^3(\widetilde{Z}, D, \mathbb{Z})$.

Similar to the CY threefold case where every element in $H^3(Z,\mathbb{Z})$ can be obtained upon differentiating Ω w.r.t. the complex structure, it is possible to obtain a basis of $H^3(\widetilde{Z}, D, \mathbb{Z})$ the same way. More precisely, we can write the basis elements $\widehat{\gamma}^D_i$ as differentials of $\widetilde{\Omega}$ evaluated at the large complex structure point

$$\widehat{\gamma}^D_i = \mathcal{R}_i \widetilde{\Omega}\big|_{\underline{z}=0} . \tag{6.34}$$

The operators $\mathcal{R}_i$ are polynomials in the differentials $\theta_a = z_a \frac{d}{dz_a}$.

The crucial achievement of the blow-up to $\widetilde{Z}$ is the fact that all moduli dependence of the superpotential is now contained in the complex structure dependence of $\widetilde{\Omega}$. Thus, it is possible, analogous to the CY case, to derive PF type differential equations for $\widetilde{\Omega}$ by studying its complex structure dependence explicitly. Since we have the algebraic representation of $\widetilde{Z}$ as the complete intersection (4.48), it is now possible to find an explicit residue representation of $\widetilde{\Omega}$ such that GD pole reduction algorithm can be used to derive the desired differential equations for $\widetilde{\Omega}$ among whose solutions we find the superpotential W.

So far the discussion of the blow-up procedure and the determination of the brane and flux superpotential was entirely in the heterotic theory, i.e. in the CY threefold Z. However, we will shed more light on the connection between the brane geometry of $(Z, \mathcal{C})$ and the complex geometry of the blow-up $\widetilde{Z}$ in the context of the heterotic/F-theory duality. More precisely, we argue that the five-brane superpotential is mapped to a flux superpotential for F-theory compactified on a dual CY fourfold Y. Starting with $\widetilde{Z}$, the fourfold Y can be represented as a complete intersection generalizing eq. (4.48). However, in contrast to $\widetilde{Z}$ the fourfold $\widehat{Y}$ can also be represented as a hypersurface. This fact allows us to directly compute the flux superpotential. We already computed the F-theory flux superpotential in § 5 (and also in appendix A.3.2) and confirmed that the five-brane superpotential is naturally contained in the F-theory flux superpotential. In the next sections we will discuss how horizontal five-brane is mapped in this duality in detail and outline the construction of Y and the F-theory flux G_4.

6.2.2 Blow-up in F-Theory

In this section we will discuss the F-theory dual of horizontal five-branes [92, 93, 198] as will be essential for our understanding of the five-brane superpotential. As before, let B_Y be $\mathbb{P}(\mathcal{O}_{B_Z} \oplus \mathcal{T})$ where we denote the associated divisor to $\mathcal{T}$ by T and assume that $-T$ is an effective divisor of B_Z. This fibration $p: B_Y \to B_Z$ has two holomorphic sections denoted by C_0 and C_∞ with

$$C_\infty = C_0 - p^* T. \tag{6.35}$$

Then, the perturbative gauge group $G = G_1 \times G_2$ denoting the group factors from the first E_8 by G_1 and from the second E_8 by G_2 is realized by seven-branes over C_0 and C_∞ with singularity type G_1 and G_2, respectively [173, 175]. On the other hand, components of the discriminant on which Δ of vanishing order higher than 1 projecting onto curves $\mathcal{C}_i$ in B_Z correspond to heterotic five-branes on the same curves in Z [173, 153, 175]. Consequently, the corresponding seven-branes induce a gauge symmetry of non-perturbative nature due to five-branes on the heterotic side.

For the later application in § 6.3.3 we will consider the enhanced symmetry point with $G = E_8 \times E_8$ due to small instantons/five-branes such that the heterotic bundle is trivial. In general, an analysis of the local F-theory geometry near the five-brane curve $\mathcal{C}$ is possible [198] applying the method of stable degeneration [84, 191]. However, since the essential point in the analysis is the trivial heterotic gauge bundle, the results of ref. [198] carry over to our situation immediately.

As follows from the above form of C_∞ (6.35), the canonical bundle of the ruled base B_Y reads

$$K_{B_Y} = -2C_0 + p^*(K_{B_Z} + \mathcal{T}) = -C_0 - C_\infty + p^* K_{B_Z}. \tag{6.36}$$

From this we obtain the classes F, G and $\mathbf{\Delta}$ of the divisors defined by f, g and Δ. To match the heterotic gauge symmetry $G = E_8 \times E_8$, there have to be a II^* fibers over the divisors C_0 and C_∞ in B_Y. Since II^* fibers require f, g and Δ to vanish to order 4, 5 and 10 over C_0 and C_∞, their divisor classes split accordingly with remaining parts

$$\begin{aligned} F' &= F - 4(C_0 + C_\infty) = -4p^* K_{B_Z}, \\ G' &= G - 5(C_0 + C_\infty) = C_0 + C_\infty - 6p^* K_{B_Z}, \\ \mathbf{\Delta}' &= \mathbf{\Delta} - 10(C_0 + C_\infty) = 2C_0 + 2C_\infty - 12p^* K_{B_Z}. \end{aligned} \tag{6.37}$$

This generic splitting implies that the component Δ' can locally be described as a quadratic constraint in a local normal coordinate k to C_0 or C_∞, respectively. Thus, Δ' can be understood locally as a double cover over C_0 respectively C_∞ branching over each irreducible curve $\mathcal{C}_i$ of $\Delta' \cap C_0$ and $\Delta' \cap C_\infty$. In fact, near an irreducible curve $\mathcal{C}_i$ intersecting, say, C_0 the splitting (6.37) implies that the sections f and g take the form

$$f = k^4 f', \quad g = k^5(g_5 + k g_6) = k^5 g' \tag{6.38}$$

where f', g_5 and g_6 are sections of $K_{B_Y}^{-4}, K_{B_Y}^{-6} \otimes \mathcal{T}$ and $K_{B_Y}^{-6}$, respectively. The discriminant then takes the form $\Delta = k^{10}\Delta'$ where Δ' is calculated from f' and g'. Thus, the intersection curve is given by $\{g_5 = 0\}$ and the degree of the discriminant Δ rises by 2 over $\mathcal{C}_i$ with f' and g' vanishing to order 2 and 1. The singular curves $\mathcal{C}_i$ in Y that occur in g as above are the locations of the small instantons/horizontal five-branes in Z [92, 198]. In the CY fourfold Y the collision of a II^* and a I_1 singularity over $\mathcal{C}_i$ induces a singularity of Y exceeding Kodaira's classification of singularities. Thus, it requires a blow-up $\pi : \widetilde{B}_3 \to B_Y$ in the three-dimensional base of the curves $\mathcal{C}_i$ into divisors D_i. This blow-up is crepant, i.e. it can be performed without violating the CY condition since the shift in the canonical class of the base, $K_{\widetilde{B}_3} = \pi^* K_{B_Y} + D_i$, can be absorbed into a redefinition of the line bundle $\mathcal{L}' = \pi^* \mathcal{L} - D_i$ entering the Weierstraß equation such that

$$K_Y = p^*(K_{B_Y} + \mathcal{L}) = p^*(K_{\widetilde{B}_Y} + \mathcal{L}') = 0. \tag{6.39}$$

To describe this blow-up explicitly, let us restrict to the local neighborhood of one irreducible curve $\mathcal{C}_i$ of the intersection of $\mathbf{\Delta}$ and C_0. We note that the curve $\mathcal{C}_i$ in B_Z is given by the following two constraints

$$h'_1 = k = 0, \quad h'_2 = g_5 = 0 \tag{6.40}$$

for k and g_5 being sections of the normal bundle N_{C_0/B_Y} and of $K_{B_Y}^{-6} \otimes \mathcal{T}$, respectively. Then, if Y is given as a hypersurface $\{P' = 0\}$, we obtain the blow-up as the complete intersection

$$P' = 0, \qquad Q' = l_1 h'_2 - l_2 h'_1 = 0. \tag{6.41}$$

As in eq. (4.48), we have introduced coordinates $\{l_1, l_2\}$ parameterizing the $\mathbb{P}^1$ fiber. However, at least in a local description, we can introduce a local normal coordinate t to $\mathcal{C}_i$ in B_Z such that $g_5 = t g'_5$ for a section g'_5 which is non-vanishing at $t = 0$. Then, by choosing a local coordinate k_1 of the $\mathbb{P}^1$ fiber, we can solve the blow-up relation Q' to obtain $k = k_1 t$. This coordinate transformation can be inserted into the constraint $P' = 0$ of Y to obtain the blown-up fourfold $\widetilde{Y}$ as a hypersurface. The f', g' of this hypersurface are given by

$$f' = k_1^4 f, \quad g' = k_1^5 (g_5 + k_1 t\, g_6 + \cdots). \tag{6.42}$$

In particular, calculating the discriminant Δ' of $\widetilde{Y}$, it can be demonstrated that the I_1 singularity no longer hits the II^* singularity over C_0 [198]. This way we have one description of $\widetilde{Y}$ as a complete intersection and another as a hypersurface. Both will be of importance for the explicit examples discussed in § 6.3.

To summarize, the F-theory counterpart of a heterotic compactification with full perturbative gauge group is given by a CY fourfold with II^* fibers over the sections C_0 and C_∞ in B_Y. The component Δ' of the discriminant enhances the degree of Δ on each intersection curve $\mathcal{C}_i$ such that a blow-up in B_Y is necessary. On the other hand, as previously described in § 2.3, each blow-up corresponds to a small instanton in the heterotic bundle [173, 89], e.g. a horizontal five-brane on the curve $\mathcal{C}_i$ in the heterotic threefold Z. Indeed, this can be viewed as a consequence of the observation mentioned above that a vertical component of the discriminant with degree greater than 1 corresponds to a horizontal five-brane [175] as the degree of Δ' on C_0 and C_∞ is 2.

Let us now discuss how the moduli maps in eq. (6.28) change during the blow-up procedure. To actually perform the blow-up along the curve $\mathcal{C}_i$, it is necessary to first degenerate the constraint of Y such that Y develops a singularity over $\mathcal{C}_i$ described above. This requires a tuning of the coefficients entering the fourfold constraint, thus restricting the complex structure of Y accordingly lowering $h^{3,1}(Y)$. Then, we perform the actual blow-up by introducing a new Kähler class associated to the exceptional divisor D_i. Thus, we end up with a new CY fourfold $\widetilde{Y}$ with decreased $h^{3,1}(\widetilde{Y})$ and $h^{1,1}(\widetilde{B}_Y)$ increased by one. This is also clear from the general argument of ref. [175] that, by enforcing a given gauge group G in four dimensions, the complex structure moduli have to respect the form of Δ dictated by the singularity type G. Since the blow-up, dual to the heterotic small instanton/five-brane transition, enhances the gauge symmetry G, the form of the discriminant becomes more restrictive, thus fixing more complex structures. In this picture the blow-down can be understood as switching on moduli, decreasing the singularity type of the elliptic fibration.

Similarly, we can understand the moduli map (6.28) from the heterotic side. For each transition between small instanton and five-brane the bundle loses parts of its moduli since the small instanton is on the boundary of the bundle moduli space. Consequently, the index I reduces accordingly. In the same process, the five-brane in general gains moduli counted by $h^0(\mathcal{C}_i, N_{\mathcal{C}_i/Z})$ contributing to the moduli map.

We close this section by making a more refined and illustrative statement about the meaning of the Kähler modulus of the exceptional divisors D_i in the heterotic theory. To do so we have to consider the heterotic M-theory on $Z \times S^1/\mathbb{Z}_2$. In this picture the instanton/five-brane transition can be understood as follows: A spacetime-filling five-brane wrapping $\mathcal{C}_i$ moves on the $S^1/\mathbb{Z}_2$ and reaches the end-of-the-world brane where one perturbative E_8 gauge group is located [199]. There, it dissolves into a finite size instanton of the heterotic bundle E. With this in mind the distance of the five-brane on the interval $S^1/\mathbb{Z}_2$ away from the end-of-world brane precisely maps to the Kähler modulus of the divisor D_i resolving $\mathcal{C}_i$ in B_Y [198].

6.2.3 Duality of the heterotic and F-Theory superpotentials

Let us finally turn to the matching of the heterotic and F-theory superpotentials. Recall, that the heterotic superpotential (2.40), is formally given by

$$W_{\text{het}}(\underline{t}^c, \underline{t}^g, \underline{t}^b) = W_{\text{flux}}(\underline{t}^c) + W_{\text{CS}}(\underline{t}^c, \underline{t}^g) + W_{\text{M5}}(\underline{t}^c, \underline{t}^b) \tag{6.43}$$

where $\underline{t}^c, \underline{t}^g$ and $\underline{t}^b$ denote the complex structure, bundle and five-brane moduli respectively. The last two terms are not inequivalent, since tuning the $\underline{t}^g$ or $\underline{t}^b$ moduli, we can condense or evaporate five-branes and explore different branches of the heterotic moduli space. Clearly the moduli spaces parametrized by $\underline{t}^c$ and $\underline{t}^g$ do not factorize globally in complex structure and bundle moduli since the notion of a holomorphic gauge bundle on Z depends on the complex structure of Z. Similarly, $\underline{t}^c$ and $\underline{t}^b$ do not factorize as the notion of a holomorphic curve in Z does depend on the complex structure of Z. This is also reflected in the fact that flux and brane superpotential can be unified into one superpotential (6.30) for which the splitting into W_{M5} and W_{flux} is just a matter of basis choice of $H_3(Z, \mathcal{C}, \mathbb{Z})$.

The key point of our construction is the fact that we can map the set of heterotic moduli $\{\underline{t}^c, \underline{t}^g, \underline{t}^b\}$ to the complex structure moduli $\underline{t}$ of Y which are encoded in the fourfold periods. To make the equivalence

$$W_{\text{het}}(\underline{t}^c, \underline{t}^g, \underline{t}^b) = W_F(\underline{t}) \tag{6.44}$$

precise, we need to establish a dictionary between the topological data on the heterotic side consisting of the heterotic flux quanta, the topological classes of gauge bundles and the class of the curves $\mathcal{C}$, and the F-theory flux quanta. We will restrict our considerations to the map between five-brane moduli and complex structure deformations of Z to complex structure deformations of Y. This can be achieved by restricting the heterotic gauge bundle E to be of trivial $SU(1) \times SU(1)$ type. In this case one needs to include heterotic five-branes to satisfy the anomaly cancellation condition. In accord with the discussion of § 6.2.2 the dual fourfold Y

can be realized as a complete intersection blown up along the five-brane curves. As above, we will restrict the discussion to a single five-brane. We want to match this description with the heterotic theory on $\widetilde{Z}$. We can now identify the blow-up constraints

$$Q = l_1 g_5(\underline{u}) - l_2 \tilde{z} \longmapsto Q' = l_1 g_5(\underline{u}) - l_2 k \quad \text{with} \quad \tilde{z} \longmapsto k \tag{6.45}$$

where $\underline{u}$ denote coordinates on the base B_Z, $\{\tilde{z} = 0\}$ defines the base B_Z in Z, and $\{z = 0\} \cap \{k = 0\}$ defines the base B_Z in Y.[5] The above map is possible since both Z and Y share the twofold base B_Z with the curve $\mathcal{C}$. The identification of $\tilde{z}$ with k corresponds to the fact that in the heterotic/F-theory duality the elliptic fibration of Z is mapped to the $\mathbb{P}^1$ fibration of B_Y. Clearly, the map (6.45) identifies the deformations of $\mathcal{C}$ realized as coefficients in the constraint $\{Q = 0\}$ of $\widetilde{Z}$ with the complex structure deformations of Y realized as coefficient in $\{Q' = 0\}$. We also have to match the remaining constraints $\{P = 0\}$ and $\{P' = 0\}$ of $\widetilde{Z}$ and Y, respectively. Clearly, there will not be a general match. However, as was argued in ref. [93] for CY fourfold hypersurfaces, one can split $P' = 0$ as $P + V_E$ yielding a map

$$P + V_E \longmapsto P' \tag{6.46}$$

where V_E is describing the spectral cover of the dual heterotic bundles $E = E_1 \oplus E_2$. Again, this requires an identification of $\tilde{z}$ and k. For $SU(1)$ bundles this map was given in eq. (6.45), but can be generalized for non-trivial bundles. Note that the maps of eq. (6.45) and eq. (6.46) can also be formulated in terms of the GKZ systems of the complete intersections $\widetilde{Z}$ and Y. It implies that the $\ell_i^{(a)}$ of Y contain the GKZ system of Z and the five-brane charge vectors, similar to the situation encountered in § 5 and also in refs. [46, 48, 38, 49]. An explicit construction of the charge vectors of $\widetilde{Z}$ and its corresponding GKZ system will be given in ref. [200].

To match the superpotentials as in eq. (6.44), we first have to identify the integral basis of $H^3(\widetilde{Z}, D, \mathbb{Z})$ with elements of $H^4(Y, \mathbb{Z})$ and show that the relative periods of $\widetilde{\Omega}_Z$ can be identified with a subset of the periods of Ω_Y. In order to do that, we compare the residue integrals for $\widetilde{\Omega}_Z$ and Ω_Y represented as complete intersections. Using the above maps (6.45) and (6.46), we then show that each PF operator annihilating $\widetilde{\Omega}_Z$ is also annihilating Ω_Y. Hence, also a subset of the solutions to the PF equations can be matched accordingly. As a minimal check, we find that the periods of Ω_Z before the blow-up arise as a subset of the periods of Ω_Y in specific directions as we have already seen in § 5, cf. also ref. [165]. The map between the cohomologies $H^3(\widetilde{Z}, D, \mathbb{Z}) \to H^4(Y, \mathbb{Z})$ is also best formulated in terms of operators $\mathcal{R}_p^{(i)}$ applied to the forms $\widetilde{\Omega}_Z$ and Ω_Y.

$$\mathcal{R}_p^{(i)} \widetilde{\Omega}_Z(\underline{z}^c, \underline{z}^b)\Big|_{\underline{z}^c = \underline{z}^b = 0} \longmapsto \mathcal{R}_p^{(i)} \Omega_Y(\underline{z})\Big|_{\underline{z}=0}. \tag{6.47}$$

Note that the pre-image of this map will in general contain derivatives with respect to the variables $\underline{z}^b$ and hence is an element in relative cohomology. It was shown in refs. [45, 47] that one can find differential operators $\mathcal{R}_p^{(i)}$ which span the full space $H^3(\widetilde{Z}, D, \mathbb{Z})$. By identifying the heterotic and F-theory moduli at the large complex structure point $\underline{z} = 0$, we obtain an embedding map of the integral basis.

[5] Note that the $\mathbb{P}^1$ fibration $B_Y \to B_Z$ has actually two fibers. As in § 6.2.2, $k = 0$ is one of the two sections, say, the zero section.

One immediate application of this formalism is that if we know the classical quadratic terms in $W_{\rm het}$ we can fix the dual G_4 flux and use the periods of the fourfold to determine the instanton parts. In particular, for the five-brane superpotential $W_{M5}(\underline{t}^c, \underline{t}^b)$ we find that the dual flux $G_4^{\rm M5}$ can be expressed as

$$G_4^{\rm M5} = \sum_p N^{p\,(2)}\, \mathcal{R}_p^{(2)} \Omega_Y \Big|_{\underline{z}=0} \tag{6.48}$$

Note that for G_4 fluxes generated by operators $\mathcal{R}^{(2)}$ the superpotential yields an integral structure of the fourfold symplectic invariants at large volume of the mirror $\widehat{Y}$ of Y as [82, 165, 107]

$$W_{G_4}^{\rm inst} = \sum_{\beta \in H_2(\widehat{Y},\mathbb{Z})} n_\beta^0(\gamma_{G_4})\, \mathrm{Li}_2(q^\beta) \quad \text{with} \quad n_\beta^0(\gamma_{G_4}) \in \mathbb{Z} \tag{6.49}$$

where γ_{G_4} is co-dimension two cycle specified by the flux. For superpotential from five-branes wrapped on a curve $\mathcal{C}$ this matches naturally the disk multi-covering formula (2.54) since this part is mapped by mirror symmetry to disk instantons ending on special Lagrangians L mirror dual to $\mathcal{C}$.

Finally, there is geometric way to identify the flux which corresponds to a chain integral. The three-chain Γ can be mapped to a three-chain Γ in B_Y whose boundary two-cycles lie in the worldvolume of a seven-brane over which the cycles of the F-theory elliptic fiber degenerates. By fibering one-cycle of the elliptic fiber which vanishes at the seven-brane locus over Γ, we get a transcendental cycle in $H_4(Y,\mathbb{Z})$. Its dual form lies then in the horizontal part $H_H^4(Y,\mathbb{Z})$ and therefore yields the flux, cf. ref. [79] for a review on such constructions. For explicit constructions of these cycles in F-theory compactifications on elliptic K3 surfaces and elliptic CY threefolds see refs. [154, 156].

6.3 Examples

In this section we study concrete examples to demonstrate the concepts discussed in the earlier sections. We will examine two geometries in detail. The first F-theory CY fourfold, discussed in the first and the second parts, will have few Kähler moduli and many complex structure moduli. In this case we can use toric geometry to explicitly compute the intersection numbers, evaluate both sides of the dimensional matching (6.28) yielding the number of deformation moduli of the five-brane curve and check the anomaly formula (2.33). We also show that the CY fourfold can be explicitly constructed from the heterotic non-CY threefold obtained by blowing up the five-brane curve. The second CY fourfold example will have few complex structure moduli and many Kähler moduli. It is also the main example studied in detail in § 5. This allows us to identify the bundle moduli and five-brane moduli under duality by studying the Weierstraß equation. The F-theory flux superpotential for this configuration is computed in § 5 and we will discuss its heterotic dual.

6.3.1 Five-branes in the elliptic fibration over $\mathbb{P}^2$

We begin the discussion of our first example of the heterotic/F-theory duality by defining the geometric setup on the heterotic side. Following § 6.1, the heterotic theory is specified by an elliptic CY threefold Z with a stable holomorphic vector bundle $E = E_1 \oplus E_2$ obeying the anomaly constraint (2.33). We choose the threefold Z as the elliptic fibration over the base $B_Z = \mathbb{P}^2$ with generic torus fiber given by a degree 6 hypersurface in $\mathbb{P}^2_{1,2,3}$. This is the CY threefold of § 5.3.1. It is given as the hypersurface $\{P = 0\}$ in the toric variety whose reflexive polyhedron we repeat here for convenience

$$\Delta_Z = \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 3B+9H \\ 0 & -1 & 0 & 0 & 2B+6H \\ 3 & 2 & 0 & 0 & B \\ 3 & 2 & 1 & 1 & H \\ 3 & 2 & -1 & 0 & H \\ 3 & 2 & 0 & -1 & H \end{array}\right] \tag{6.50}$$

with the class of the hypersurface Z given by

$$[Z] = \sum D_i = 6B + 18H \tag{6.51}$$

as explained in § 5.1.2. Here, we denote the two linearly independent toric divisors D_i by H and B: The pull-back of the hyperplane class of the base $\mathbb{P}^2$ and the class of the base itself, respectively. From the toric data the basic topological numbers of Z are obtained as

$$\chi(Z) = 540, \quad h^{1,1}(Z) = 2, \quad h^{2,1}(Z) = 272. \tag{6.52}$$

The second Chern-class of Z is given in eq. (5.7) which we repeat here

$$c_2(Z) = 12c_1(B_Z)\cdot\sigma + 11c_1(B_Z)^2 + c_2(B_Z) \tag{6.53}$$

where $\sigma : B_Z \to Z$ is the section of the elliptic fibration. Here, we have $\sigma = B$ and thus obtain

$$c_2(Z) = 36H\cdot B + 102H^2. \tag{6.54}$$

To fulfill the anomaly formula (2.33), we have to construct the vector bundle $E_1 \oplus E_2$ and determine the characteristic classes $\lambda(E_i)$. Now, we first need to specify the classes $\eta_1, \eta_2 \in H^2(B_Z, \mathbb{Z})$ essential in the spectral cover construction. We furthermore restrict $E_1 \oplus E_2$ to be an $E_8 \times E_8$ bundle over Z and choose both classes as $\eta_1 = \eta_2 = 6c_1(B_Z)$. Then, we use the formula for the second Chern class of E_8 bundles (6.22) to obtain

$$\lambda(E_1) = \lambda(E_2) = 18H\cdot B - 360H^2. \tag{6.55}$$

The anomaly constraint then leads to conditions on the coefficients of the independent classes in $H^4(Z)$. The class of five-branes C will have the following general form

$$[C] = \sigma\cdot H^2(B_Z, \mathbb{Z}) + \cdots. \tag{6.56}$$

The first part represents horizontal five-branes, i.e. a curve in the base B_Z and the rest vertical five-branes wrapping the elliptic fiber. This implies that no horizontal five-branes are present

[180]. For the class $H \cdot B$ this is trivially satisfied by the choice of $\lambda(E_i)$. For the class of the fiber F the anomaly forces the inclusion of vertical five-branes in the class

$$[C] = c_2(B_Z) + 91 c_1(B_Z)^2 = 822 H^2 = n_F F. \tag{6.57}$$

Since F is dual to the base B_Z, the number of vertical five-branes is determined by integrating C over the base

$$n_F = \int_{\mathbb{P}^2} C = 822. \tag{6.58}$$

To conclude the heterotic side we compute the index $I(E_i)$ since it appears in the identification of the moduli (6.28) and thus is crucial for the analysis of the heterotic/F-theory duality. For Z we use the index formula (6.29) to obtain

$$I(E_1) = I(E_2) = 8 + 4 \cdot 360 + 18 \cdot 3 = 1502. \tag{6.59}$$

Next, we include horizontal five-branes to the setup by shifting the classes η_i appropriately. We achieve this by putting $\eta_2 = 6c_1(B) - H$. The class of the five-brane C can then be determined analogous to the above discussion by evaluating the characteristic class of the bundle and imposing the anomaly condition. It takes the following form

$$[C] = 91 c_1(B_Z)^2 + c_2(B_Z) - 45 c_1(B_Z) \cdot H + 15 H^2 + H \cdot B = 702 H^2 + H \cdot B. \tag{6.60}$$

As discussed above, this means that we have to include five-branes in the base on a curve $\mathcal{C}$ in the class H of the hyperplane of $\mathbb{P}^2$. Additionally, the number of five-branes on the fiber F is changed to $n_F = 702$. Accordingly, the shift of η_2 changes the second index to $I_2 = 1019$ whereas $I_1 = 1502$ remains unchanged.

Let us now turn to the dual F-theory description. We first construct the fourfold Y dual to the heterotic setup with no five-branes. In this case the base B_Y of the elliptic fourfold is $B_Y = \mathbb{P}^1 \times \mathbb{P}^2$. This can be seen from the form of η_i given in eq. (6.27) and the fibration structure of B_Y for E_8 bundles. Since both classes equal $6c_1(Z)$, we have $t = 0$ and thus the bundle $\mathcal{T} = \mathcal{O}_{\mathbb{P}^2}$ as well as the projective bundle $B_Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$. Then, the CY fourfold Y is constructed as the elliptic fibration over B_Y with generic fiber given by $\mathbb{P}^2_{1,2,3}[6]$. Again, Y is described as a hypersurface in the toric variety V_Y as described by the toric data in Table 6.1 if one drops the point $(3,2,-1,0,1)$ and sets the divisor D to zero. The class of Y is then given by

$$[Y] = \sum_i D_i = 6B + 18H + 12K \tag{6.61}$$

where the independent divisors are the base B_Y denoted by B, the pull-back of the hyperplane H in $\mathbb{P}^2$ and of the hyperplane K in $\mathbb{P}^1$. Then, the basic topological data reads

$$\chi(Y) = 19728, \quad h^{1,1}(Y) = 3, \quad h^{3,1}(Y) = 3277, \quad h^{2,1}(Y) = 0. \tag{6.62}$$

Now, we have everything at hand to discuss the heterotic/F-theory duality along the lines of § 6.2.2, in particular the map of the moduli formula (6.28). As discussed there, the complex

$$\Delta(\widetilde{Y}) = \left[\begin{array}{ccccc|c|c} -1 & 0 & 0 & 0 & 0 & 3D+3B+9H+6K & D_1 \\ 0 & -1 & 0 & 0 & 0 & 2D+2B+6H+4K & D_2 \\ 3 & 2 & 0 & 0 & 0 & B & D_3 \\ 3 & 2 & 1 & 1 & 0 & H & D_4 \\ 3 & 2 & -1 & 0 & 0 & H-D & D_5 \\ 3 & 2 & 0 & -1 & 0 & H & D_5 \\ 3 & 2 & 0 & 0 & 1 & K & D_7 \\ 3 & 2 & 0 & 0 & -1 & K+D & D_8 \\ 3 & 2 & -1 & 0 & 1 & D & D_9 \end{array}\right]$$

Table 6.1: Toric data of the CY fourfold $\widetilde{Y}$ blown up in the base

structure moduli of the F-theory fourfold are expected to contain the complex structure moduli of Z as well as the bundle and brane moduli of possible horizontal five-branes. Indeed, we obtain a complete matching by adding up all contributions

$$h^{3,1}(Y) = 3277 = 272 + 1502 + 1502 + 1 \tag{6.63}$$

where it is crucial that no horizontal five-branes with possible brane moduli are present.

To obtain the F-theory dual of the heterotic theory with horizontal five-branes, we have to apply the recipe discussed in § 6.2.2. We have to perform the described geometric transition. Firstly, by tuning the complex structure of Y, the fourfold becomes singular over the curve $\mathcal{C}$ which we then blow up into a divisor D. This way we obtain a new smooth CY fourfold denoted by $\widetilde{Y}$. The toric data of this fourfold are given in Table 6.1 where we included the last point $(3,2,-1,0,1)$ and a corresponding divisor $D_9 = D$ to perform the blow-up along the curve $\mathcal{C}$ as follows: Since the curve $\mathcal{C}$ on the heterotic theory is in the class H we have to blow-up over the hyperplane class of $\mathbb{P}^2$ in B_Y. Firstly, we project the polyhedron $\Delta(Y)$ to the base B_Y which is done just by omitting the first and second column in Table 6.1. Then, the last point maps to the point $(-1,0,1)$ that subdivides the two-dimensional cone spanned by $(-1,0,0)$ and $(0,0,1)$ in the polyhedron of B_Y. Thus, upon adding this point the curve $\mathcal{C} = H$ in B_Z corresponding to this cone is removed from B_Y and replaced by the divisor D corresponding to the new point. We see that the toric data in Table 6.1 contain this blown-up base $B_{\widetilde{Y}}$ in the last three columns. The CY fourfold is then realized as a generic constraint $\{P = 0\}$ in the class

$$[\widetilde{Y}] = 6B + 18H + 12K + 6D. \tag{6.64}$$

Note that this fourfold has now three different triangulations which correspond to the various five-brane phases on the dual heterotic side. The topological data for the new fourfold $\widetilde{Y}$ are given by

$$\chi(\widetilde{Y}) = 16848, \quad h^{1,1}(\widetilde{Y}) = 4, \quad h^{3,1}(\widetilde{Y}) = 2796, \quad h^{2,1}(\widetilde{Y}) = 0 \tag{6.65}$$

where the number of complex structure moduli has reduced in the transition as expected. If we now analyze the map of the moduli (6.28) in the heterotic/F-theory duality, we observe that we have to put $h^0(\mathcal{C}, N_{\mathcal{C}/Z}) = 2$ in order to obtain a matching. This implies, from the point of view of the heterotic/F-theory duality, that the horizontal five-brane wrapped on $\mathcal{C}$ has to have two deformation moduli. Indeed, this precisely matches the fact that the hyperplane class of $\mathbb{P}^2$ has two deformations since a general hyperplane is given by the linear constraint $a_1x_1 + a_2x_2 + a_3x_3 = 0$ in the three homogeneous coordinates x_i of $\mathbb{P}^2$. Discarding the overall scaling, it thus has two moduli parameterized by $\mathbb{P}^2$ with homogeneous coordinates a_i. In this way, we have found an explicit construction of an F-theory fourfold with complex structure moduli encoding the dynamics of heterotic five-branes.

In § 6.3.2 we provide further evidence for this identification by showing that we can also construct $\widetilde{Y}$ as a complete intersection starting with a heterotic non-CY threefold. Unfortunately, it will be very hard to compute the complete superpotential for the fourfold $\widetilde{Y}$ since it admits a large number of complex structure deformations. It would be interesting, however, to extract the superpotential for a subsector of the moduli including the two brane deformations.[6] Later on, we will take a different route and consider examples with only a few complex structure moduli which are constructed by using mirror symmetry.

6.3.2 Calabi-Yau fourfolds from heterotic non-Calabi-Yau threefolds

In this section we discuss the example of § 6.3.1 employing the blow-up procedure of § 4 which is reviewed in § 6.2.1 in the context of the heterotic theory. More precisely, we will explicitly construct a non-CY threefold $\widetilde{Z}$ obtained by blowing up the horizontal five-brane curve into a divisor. This transfers the deformations of $\mathcal{C}$ into new complex structure deformations of $\widetilde{Z}$. The F-theory CY fourfold $\widetilde{Y}'$ is then naturally obtained from the base of $\widetilde{Z}$ by an additional $\mathbb{P}^1$ fibration. This CY fourfold $\widetilde{Y}'$ is identical to the fourfold $\widetilde{Y}$ considered in § 6.3.1 despite the fact that it is now realized as a complete intersection.

Explicit blow-up of CY hypersurfaces in toric varieties

As in § 6.3.1 the starting point is the elliptic fibration Z over $B_Z = \mathbb{P}^2$ with a five-brane wrapping the hyperplane class of the base. Let us describe the explicit construction of $\widetilde{Z}$. The blow-up geometry $\widetilde{Z}$ is given by $\mathbb{P}(N_{\mathcal{C}/Z})$. In § 4.5.2 we have explained how to blow up in general. Here, we restrict our attention to the case where the CY manifold is given as a hypersurface in a Fano toric variety. So, let us assume that Z is given as a hypersurface $\{P = 0\}$ in a toric variety V_Z and the curve $\mathcal{C}$ as a complete intersection of two hypersurfaces in Z, i.e. $\mathcal{C} = \{h_1 = 0\} \cap \{h_2 = 0\} \subset Z$. The charge vectors of V_Z are given by $\{\ell^{(i)}\}$ with $i = 1, \ldots, k$. We are aiming to construct a five-dimensional toric variety which is given by $V_{\widetilde{Z}} = \mathbb{P}(N_{\mathcal{C}/V_Z})$ and use the blow-up equation described in § 4.5.2. Let us denote the divisor classes defined by h_i by H_i and the charges of h_i by $\mu_i = (\mu_i^{(1)}, \ldots, \mu_i^{(k)})$. Then, the coordinates l_i of N_{H_i/V_Z} transform with charge $\mu_i^{(m)}$ under the k scaling relations. The normal bundle $N_{\mathcal{C}/V_Z}$ is given by $N_{H_1/V_Z} \oplus N_{H_2/V_Z}$. Since we have to projectivize $N_{\mathcal{C}/V_Z}$, we have to include another $\mathbb{C}^\times$ action with charge vector $\ell^{(k+1)}_{V_{\widetilde{Z}}}$ acting non-trivially only on the new coordinates l_i. The new charge vectors of $V_{\widetilde{Z}}$ are thus given in Table 6.2. The blown-up geometry $\widetilde{Z}$ is now given as a complete intersection, cf. eq. (4.48),

$$P = 0, \quad l_1 h_2 - l_2 h_1 = 0 \tag{6.66}$$

analogously to eq. (4.48).

[6] If we consider exactly the mirror of $\widetilde{Y}$, as we will in fact do in § 6.3.3, it might be possible to embed this reduced deformation problem into the complicated deformation problem of $\widetilde{Y}$ constructed in this section.

	coordinates of V_Z	l_1	l_2
$\ell^{(1)}_{V_{\tilde{Z}}}$	$\ell^{(1)}$	$\mu_1^{(1)}$	$\mu_2^{(1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ell^{(k)}_{V_{\tilde{Z}}}$	$\ell^{(k)}$	$\mu_1^{(k)}$	$\mu_2^{(k)}$
$\ell^{(k+1)}_{V_{\tilde{Z}}}$	0	1	1

Table 6.2: The charges of the blow-up geometry

$$\Delta(\widetilde{Z}) = \left[\begin{array}{ccccc|c} -1 & 0 & 0 & 0 & 0 & 3B+3D+9H \\ 0 & -1 & 0 & 0 & 0 & 3B+3D+6H \\ 3 & 2 & 0 & 0 & 0 & B \\ 3 & 2 & 1 & 1 & 1 & H \\ 3 & 2 & -1 & 0 & 0 & H \\ 3 & 2 & 0 & -1 & 0 & H \\ 3 & 2 & 0 & 0 & -1 & D \\ 0 & 0 & 0 & 0 & -1 & H-D \end{array}\right]$$

Table 6.3: The toric data for the blow-up geometry $\widetilde{Z}$

To apply this to the elliptic fibration over $\mathbb{P}^2$ with the polyhedron in eq. (6.50), we pick the curve $\mathcal{C}$ given by $\{\tilde{z} = 0\}$ and $\{x_1 = 0\}$. The curve $\mathcal{C}$ has genus zero and we will find that the exceptional divisor D will be the first del Pezzo surface F_1 in accord with the discussion of § 4.4. We construct the five-dimensional ambient toric variety as explained above. The polyhedron $\Delta_{\widetilde{Z}}$ is shown in Table 6.3. Note that we have to include the inner point $(3,2,0,0,0)$ which corresponds to the base of the elliptic fibration $\widetilde{Z}$. Furthermore, the point $(0,0,0,0,1)$, required for the above scalings, can be omitted since the associated divisor does not intersect the complete intersection $\widetilde{Z}$. Explicitly, the complete intersection $\widetilde{Z}$ is given by a generic constraint in the class

$$[\widetilde{Z}] = (6B+6D+18H) \cap H. \tag{6.67}$$

The first divisor above is the sum of the first seven divisors in Table 6.3 and corresponds to the original CY constraint $\{P = 0\}$ in the defining equations of the blow-up (6.66). The second divisor is the sum of the last two divisors and is the class of the second equation of the blow-up equations. This complete intersection threefold has

$$\chi(\widetilde{Z}) = -538 = \chi(Z) - \chi(\mathbb{P}^1) + \chi(F_1). \tag{6.68}$$

We can check that the exceptional divisor D has the characteristic data of the first del Pezzo surface. This means that we have replaced the hyperplane isomorphic to $\mathbb{P}^1$ in the base with the exceptional divisor isomorphic to F_1. It can be readily checked that the first Chern class of $\widetilde{Z}$ is non-vanishing and equals $-D$.

Having described the heterotic blow-up geometry, we now turn to the construction of the fourfold $\widetilde{Y}'$ for F-theory. This CY fourfold will also be constructed as a complete intersection, but it will be the same manifold as the fourfold described in § 6.3.1, Table 6.1. We fiber an

$$\Delta_{\widetilde{Y}'} = \left[\begin{array}{cccccc|c|c} -1 & 0 & 0 & 0 & 0 & 0 & 3D+3B+9H+6K & D_1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2D+2B+6H+4K & D_2 \\ 3 & 2 & 0 & 0 & 0 & 0 & B & D_3 \\ 3 & 2 & 1 & 1 & 1 & 0 & H & D_4 \\ 3 & 2 & -1 & 0 & 0 & 0 & H & D_5 \\ 3 & 2 & 0 & -1 & 0 & 0 & H & D_6 \\ 3 & 2 & 0 & 0 & 0 & 1 & K & D_7 \\ 3 & 2 & 0 & 0 & 0 & -1 & K+D & D_8 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 & D & D_9 \\ 0 & 0 & 0 & 0 & -1 & 0 & H-D & D_{10} \end{array}\right]$$

Table 6.4: Toric data of $\widetilde{Y}'$ as a complete intersection

additional $\mathbb{P}^1$ over $\mathbb{P}(\Delta_{\widetilde{Z}})$ which is only non-trivially fibered along the exceptional divisor. This is analogous to the construction of the dual fourfold in the heterotic/F-theory duality where one also fibers $\mathbb{P}^1$ over the base twofold of the CY threefold to obtain the F-theory fourfold. Here, we proceed in a similar fashion but construct a $\mathbb{P}^1$ fibration over the base of the non-CY manifold $\widetilde{Z}$. This base is a complete intersection and thus leads to a realization of $\widetilde{Y}'$ as a complete intersection. Explicitly, Table 6.4 shows the polyhedron of $\widetilde{Y}'$. The fourfold $\widetilde{Y}'$ is given as the following complete intersection

$$[\widetilde{Y}] = (6B+6D+18H+12K) \cap H. \tag{6.69}$$

Note that this fourfold is indeed CY as it can be checked explicitly by analyzing the toric data. For complete intersections the CY constraint is realized via the two partitions, the so-called *nef partitions* as in refs. [201, 202]. The first nef partition yields the sum of the first eight divisors in Table 6.4 and gives the first constraint in eq. (6.69). The second nef partition yields the sum of the last two divisors, $D_9 + D_{10}$, in Table 6.4 and yield the second constraint in eq. (6.69). The divisors D_7 and D_8 correspond to the $\mathbb{P}^1$ fiber in the base of $\widetilde{Y}'$ obtained by dropping the first two columns in Table 6.4. This fibration is only non-trivial over the exceptional divisors $D = D_9$ in the second nef partition of Table 6.4. Note that if we simply drop K from the expression (6.69), we formally recover the constraint (6.67) of the non-CY threefold $\widetilde{Z}$. To check that the complete intersection $\widetilde{Y}'$ is precisely the CY fourfold $\widetilde{Y}$ constructed in § 6.3.1, we have to compute the intersection ring and Chern classes. In particular, it is not hard to show that also polyhedron shown in Table 6.4 has three triangulations matching the result of § 6.3.1.

In summary, we have found that there is a natural construction of $\widetilde{Y}'$ as a complete intersection with the base obtained from the heterotic non-CY threefold $\widetilde{Z}$. Let us stress that this construction will straightforwardly generalize to dual heterotic/F-theory setups with other toric base spaces and different types of bundles. For example, to study the bundle configurations on Z of § 6.3.1 with $\eta_{1/2} = 6c_1(B_Z) \pm kH$ with $k = 0,1,2$, we only have to replace

$$D_4 \longrightarrow (3,2,1,1,k) \ , \quad D_4 \longrightarrow (3,2,1,1,1,k) \tag{6.70}$$

in the polyhedra given in Table 6.1 and Table 6.4, respectively. Moreover, also bundles which are not of the type $E_8 \times E_8$ can be included by generalizing the form of the $\mathbb{P}^1$ fibration just as in the standard construction of dual F-theory fourfolds.

6.3.3 Five-brane superpotential in the heterotic/F-Theory duality

Let us now discuss an example for which the F-theory flux superpotential can be computed explicitly since the F-theory fourfold admits only few complex structure moduli. We will proceed analogously to § 5.3. To start with, let us consider the heterotic string theory on the *mirror* of the CY threefold which is an elliptic fibration over $\mathbb{P}^2$. Thus, the heterotic compactification manifold Z is given in Table 5.3(a). Since Z is elliptically fibered, it is at least in principle possible to construct the bundles explicitly. The Weierstraß form of Z is given as follows

$$\mu_Z = x^3 + y^2 + xy\tilde{z}a_0 u_1 u_2 u_3 + \tilde{z}^6 \left(a_1 u_1^{18} + a_2 u_2^{18} + a_3 u_3^{18} + a_4 u_1^6 u_2^6 u_3^6\right). \qquad (6.71)$$

The coordinates $\{u_i\}$ are the homogeneous coordinates of B_Z. Note that we find that the elliptic fibration is highly degenerate over B_Z. The CY threefold is nevertheless non-singular since the singularities are blown up by many divisors in the ambient toric variety of Z. In writing the Weierstraß form, many of the coordinates parameterizing these additional divisors[7] have been set to 1. Turning to the perturbative gauge bundle $E_1 \oplus E_2$, we will restrict in the following to the simplest bundle $SU(1) \times SU(1)$ thus preserving the full perturbative $E_8 \times E_8$ gauge symmetry in four dimensions. To nevertheless satisfy the anomaly condition (2.33), we also have to include five-branes. In particular, we consider a five-brane in Z given by the equations

$$h_1 = b_1 u_1^{18} + b_2 u_1^6 u_2^6 u_3^6 = 0, \quad h_2 = \tilde{z} = 0. \qquad (6.72)$$

The curve $\mathcal{C}$ wrapped by the five-brane is thus in the base B_Z. Unfortunately, it is hard to check the anomaly constraint explicitly as in the example of § 6.3.1 since Z has too many Kähler classes. However, we can proceed to construct the associated CY fourfold Y encoding a consistent completion of the setup.

The associated fourfold Y cannot be constructed as it was done in § 6.3.1. However, we can employ mirror symmetry to first obtain the mirror fourfold $\widehat{Y}$ as CY fibration

$$\widehat{Z} \longrightarrow \widehat{Y} \longrightarrow \mathbb{P}^1 \qquad (6.73)$$

where $\widehat{Z}$ is the mirror of the heterotic threefold Z [93]. This naturally leads to identify Y as the mirror to the fourfold $\widehat{Y}$ which is given in Table 6.1. This fourfold is also the main example discussed in detail in § 5.3. In the following we will check that this is indeed the correct identification by using the formalism of refs. [93, 198]. The Weierstraß form of Y can be computed using the dual polyhedron of polyhedron given in Table 6.1 yielding

$$\mu_Y = y^2 + x^3 + m_1(u_i, w_j, k_m)xyz + m_6(u_i, w_j, k_m)z^6 = 0, \qquad (6.74)$$

where

$$\begin{aligned} m_1(w_j, u_i) &= a_0 u_1 u_2 u_3 w_1 w_2 w_3 w_4 w_5 w_6 k_1 k_2, \\ m_6(w_j, u_i) &= a_1 (k_1 k_2)^6 u_1^{18} w_1^{18} w_2^{18} w_5^6 w_6^6 + a_2 (k_1 k_2)^6 u_2^{18} w_3^{18} w_5^{12} \\ &\quad + a_3 (k_1 k_2)^6 u_3^{18} w_4^{18} w_6^{12} + a_4 (k_1 k_2)^6 (u_1 u_2 u_3 w_1 w_2 w_3 w_4 w_5 w_6)^6 \\ &\quad + b_1 k_2^{12} u_1^{18} w_1^{24} w_2^{12} w_3^6 w_4^6 + b_2 k_2^{12} (u_1 u_2 u_3)^6 (w_1 w_3 w_4)^{12} \\ &\quad + c_1 k_1^{12} (u_1 u_2 u_3)^6 (w_2 w_5 w_6)^{12}. \end{aligned} \qquad (6.75)$$

[7] Note that the blow-down of these divisors induces a large non-perturbative gauge group in the heterotic compactification.

The coordinates u_i are the coordinates of the base twofold B_Z as before and w_i, k_1, k_2 are additional coordinates of the base threefold B_Y. Again, note that we have set many coordinates to 1. The chosen coordinates correspond to divisors which include the vertices of Δ_Y, hence completely determine the polyhedron. In particular, we find that k_1, k_2 are the coordinates of the fiber $\mathbb{P}^1$ over B_Z. The coefficients a_i, b_1, b_2, c_1 denote coefficients encoding the complex structure deformations of Y. However, since $h^{3,1}(Y) = 4$, there are only four complex structure parameters rendering six of the a_i redundant.

As the first check that Y is indeed the correct geometry, we use the stable degeneration limit [84, 191, 203] and write μ_Y in a local patch with an appropriate coordinate redefinition as follows [93]

$$\mu_Y = p_0 + p_+ + p_- \tag{6.76}$$

where

$$\begin{aligned} p_0 &= x^3 + y^2 + xy\tilde{z}a_0 u_1 u_2 u_3 + \tilde{z}^6\left(a_1 u_1^{18} + a_2 u_2^{18} + a_3 u_3^{18} + a_4 u_1^6 u_2^6 u_3^6\right), \\ p_+ &= v\tilde{z}^6\left(b_1 u_1^{18} + b_2 u_1^6 u_2^6 u_3^6\right), \\ p_- &= v^{-1}\tilde{z}^6 c_1 u_1^6 u_2^6 u_3^6. \end{aligned} \tag{6.77}$$

The coordinate v is the affine coordinate of the fiber $\mathbb{P}^1$. In the stable degeneration limit $\{p_0 = 0\}$ describes the CY threefold of the heterotic string. In this case p_0 coincides with μ_Z meaning that the heterotic CY threefold of Y is precisely Z. This shows that the geometric moduli of Z are correctly embedded in Y. The polynomials $p_\pm$ encode the perturbative bundles. Their explicit form shows a trivial $SU(1) \times SU(1)$ bundle. This fact can also be directly checked by analyzing the polyhedron of Y using the methods of refs. [153, 159, 160]. Over each divisor $k_i = 0$ in B_Y a full E_8 gauge group is realized. Since the full $E_8 \times E_8$ gauge symmetry is preserved, we are precisely in the situation of § 6.2.2. Recall that a smooth CY fourfold Y contains a blow-up corresponding to a heterotic five-brane.

We will now check that this allows us to identify the brane moduli in the duality. Let us now make contact to the discussion in § 6.2.2. To make the perturbative $E_8 \times E_8$ gauge group visible in μ_Y, we have to include new coordinates $\{\widetilde{k}_1, \widetilde{k}_2\}$ replacing $\{k_1, k_2\}$. This can be again understood by analyzing the toric data using the methods of refs. [159, 153, 160]. We denote by $(3, 2, \vec{\mu})$ the toric coordinates of the divisor corresponding to $\widetilde{k}_1$ in the Weierstraß model. Then the resolved E_8 singularity corresponds to the points[8]

$$\begin{array}{llll} (0,0,\vec{\mu}), & & (1,0,n\vec{\mu}) \text{ with } & n=1,2, \\ (1,1,n\vec{\mu}) \text{ with } & n=1,2,3, & (2,1,n\vec{\mu}) \text{ with } & n=1,\dots,4, \\ (3,2,n\vec{\mu}) \text{ with } & n=1,\dots,6. & & \end{array} \tag{6.78}$$

While $(3, 2, 6\vec{\mu})$, corresponding to k_1, is a vertex of the polyhedron, $(3, 2, \vec{\mu})$ corresponding to $\widetilde{k}_1$ is an inner point. Using the inner point for $\widetilde{k}_1$, the Weierstraß form μ_Y slightly changes

[8] Note that we have chosen the vertices in the $\mathbb{P}^2_{1,2,3}$ to be $(-1,0), (0,-1), (3,2)$ to match the discussion in refs. [153, 159]. However, if one explicitly analyses the polyhedron of Y we find that we have to apply a $GL(2,\mathbb{Z})$ transformation to find a perfect match. This is due to the fact that Y, in comparison to its mirror $\widehat{Y}$, actually contains the dual torus as elliptic fiber.

while the polynomials p_0, p_+ and p_- can still be identified in the stable degeneration limit. To determine the polynomial g_5 in eq. (6.38), we compute g of the Weierstraß form in a local patch where $\tilde{k}_2 = 1$

$$g = \tilde{k}_1^5 \left(b_1 u_1^{18} + b_2 u_1^6 u_2^6 u_3^6 + \tilde{k}_1 \left(a_1 u_1^{18} + a_2 u_2^{18} + \dots\right)\right). \tag{6.79}$$

The dots contain only terms of order zero or higher in $\tilde{k}_1$. Comparing this with eq. (6.42), it is obvious that g_5 is given by

$$g_5 = b_1 u_1^{18} + b_2 u_1^6 u_2^6 u_3^6. \tag{6.80}$$

This identifies $\{g_5 = 0\}$ with the curve of the five-brane in the base B_Z and is in accord with the defining equations of $\mathcal{C}$ (6.72). We can conclude that Y is indeed a correct fourfold associated to Z with the given five-brane. As we can see from g_5, the five-brane has one modulus. If we compare g_5 with p_+, we see that $p_+ = v\tilde{z}^6 g_5$. This nicely fits with the bundle description. In our configuration, p_+ and p_- should describe $SU(1)$ bundles since we have the full unbroken perturbative $E_8 \times E_8$ bundle as described above. The $SU(1)$ bundles do not have any moduli such that the moduli space corresponds just to one point [84]. In the explicit discussion of the Weierstraß form in our setting, p_+ has one modulus which corresponds to the modulus of the five-brane. Note that the CY fourfold Y is already blown up along the curve $\tilde{k}_1 = g_5 = 0$ in the base B_Y. This blow-up can be equivalently described as a complete intersection as we discussed in the previous sections. A simple example of such a construction was presented in § 6.3.2.

Finally, we consider the computation of the flux superpotential. The flux superpotential is computed in § 5. The different triangulations of $\hat{Y}$ correspond to different five-brane configurations. The four-form flux, for one five-brane configurations, was shown to be given in the base elements

$$\hat{\gamma}_1^{(2)} = \frac{1}{2}\,\theta_4(\theta_1 + \theta_3)\Omega_Y|_{\underline{z}=0}, \quad \hat{\gamma}_1^{(2)} = \frac{1}{7}\,\theta_2(\theta_2 - 2\theta_1 + 6\theta_4 - \theta_3)\Omega_Y|_{\underline{z}=0} \tag{6.81}$$

where as usual $\theta_i = z_i \frac{d}{dz_i}$. The moduli z_1, z_2 can be identified as the deformations of the complex structure of the heterotic threefold Z while z_3 corresponds to the deformation of the heterotic five-brane.[9] A non-trivial check of this identification is already provided in § 5 where we show that the F-theory flux superpotential in the directions (6.81) matches the superpotential for a five-brane configuration in a local CY threefold obtained by decompactifying Z. This non-compact five-brane can be described by a point on a Riemann surface in the base B_Z. Using the heterotic/F-theory duality as in § 6.2 we can now argue that the above flux (6.81) actually describes a compact heterotic five-brane setup and the induced superpotential.

[9] The deformation z_4 describes the change in p_-.

7 Conclusions

I remember one occasion when I tried to add a little seasoning to a review, but I wasn't allowed to. The paper was by Dorothy Maharam, and it was a perfectly sound contribution to abstract measure theory. The domains of the underlying measures were not sets but elements of more general Boolean algebras, and their range consisted not of positive numbers but of certain abstract equivalence classes. My proposed first sentence was: "The author discusses valueless measures in pointless spaces."

P. R. Halmos,
I want to be a mathematician

In this thesis we have studied the superpotential induced by D5-branes. Main tools for the computation was string dualities, namely mirror symmetry for CY three- and fourfolds and the heterotic/F-theory duality. The superpotential in study is given by the chain integral of the holomorphic three-form. Its functional form is quite universal meaning that it occurs also in the heterotic string theory for five-branes.

We first discussed the different superpotentials occurring in string theories: The type IIB, F-theory and the heterotic string theory. By doing so, we described the chain integral as the Abel-Jacobi map. Also, we argued that for five-branes of the heterotic theory the same superpotential is induced using the small instanton transition. Since the main computations were done in F-theory, we reviewed the flux superpotential of F-theory and how it contains both the flux and seven-brane superpotential. The D7-brane superpotential was important because the worldvolume flux on seven-branes induces D5-brane charge which enables us to use F-theory configuration to compute the D5-brane superpotential. Since the non-compact CY geometries represent the benchmark computations and we used those results to check our calculations, we reviewed these geometries with D-branes.

We studied the superpotential in the framework of relative (co)homology and discussed its underlying mixed Hodge structure. This structure allows for PF type equations which is *the* main tool for calculations. Then, we discussed the blow-up geometry which is advantageous since co-dimension 1 objects, i.e. divisors, are easier to handle than higher co-dimensional objects. Using the blown-up geometry, it was possible to embed the deformations of the pair consisting of the CY threefold and the D5-brane into the complex structure deformations of the non-CY blown-up threefold. We also described the technical details how to obtain the PF operators in general for general complete intersection CY manifolds. This was important because the blow-up geometry can be represented as a complete intersection.

However, to deal with the relative (co)homology or with the blow-up geometry is technically yet challenging. Thus, our the main technical tool for explicit computation was the lift of the brane configurations to F-theory compactifications, i.e. to elliptically fibered CY fourfold. We used the fact that the complex structure moduli space encompasses the complex structure moduli of the CY threefold and also the D5-brane moduli. After having described the required techniques, e.g. mirror symmetry for higher dimensional CY manifolds, the underlying Frobenius algebra structure of the operator rings and matching of the correlators of the A- and B-model, we computed the flux superpotential of F-theory for examples containing the two-dimensional complex projective space and the first two del Pezzo surfaces. We then identified the D5-brane superpotential computed in the non-compact geometries in the F-theory superpotential by comparing the integer BPS invariants.

For D5-brane in the type IIB theory the blow-up geometry seems to be only an auxiliary construction. However, using the heterotic/F-theory duality, it can be given a physical ground. We described the occurrences of blow-ups in both F-theory and the heterotic theory involving horizontal five-branes. It could be shown that these constructions fit well with the existing mappings of the moduli under the duality. In addition, we showed that we can directly construct the CY fourfold from the complete intersection description of blow-up geometry. This means that we extended the heterotic/F-theory duality using the blow-up geometry.

Future directions

Let us now come to possible future research directions. First of all, it would be essential to construct the GKZ system for the blow-up geometry directly. This is work in progress [200]. This would allow for direct and greatly simplified determination of the PF operators for toric branes, i.e. branes given by charge vectors. Also, for other branes, not given torically, the determination of PF operators using the methods outlined in this thesis, namely the GD pole reduction method for complete intersections, would be a very important task. This would allow for computations of even larger class of D5-branes, not restricted to torically given branes. One should determine the solutions to the PF equations and compare the results with the literature to confirm the computation method and compute new examples.

We have observed a new qualitatively different behavior of periods at the conifold point for CY fourfolds. It would be very interesting and necessary to embark a research along the lines of refs. [204, 205] to obtain a physical interpretation.

Another direction would be to deepen the understanding of the heterotic/F-theory duality in the context of the blow-up geometry. The geometry we investigated was given torically. It would be interesting to extend the construction of the CY fourfold from the blown-up threefold for more general geometries. It would be very interesting to study what happens to the (stable) bundle data after the blow-up and to the integral structure of the generating function.

A Appendices

I believe there are 15 747 724 136 275 002 577 605 653 961 181 555 468 044 717 914 527 116 709 366 231 425 076 185 631 031 296 protons in the universe and the same number of electrons.

A. S. Eddington,
The Philosophy of Physical Science (1939), 170

A.1 Mathematics pool

In this appendix we collect definitions, theorems and formulas we employ and use in the main text.

A.1.1 Topological dualitity theorems

The following Poincaré duality map for X of complex dimension n

$$\mathrm{PD}_X: \; H_{2n-k}(X,\mathbb{Z}) \xrightarrow{\sim} H_c^k(X,\mathbb{Z}) \tag{A.1}$$

is an isomorphism [125, Thm. 3.35]. Here, X may be non-compact and therefore the duality involves, $H_c^k(\cdot)$, the cohomology with compact support. For compact X, obviously, $H^\bullet(\cdot) \equiv H_c^\bullet(\cdot)$. Another useful duality is the Lefshetz duality [69, Lec. 5]. Let Y be a closed subset of X. This duality connects the relative cohomology of the pair (X,Y) to the homology of the open manifold $X - Y$ as follows

$$H^k(X,Y,\mathbb{Z}) \cong H_{2n-k}(X - Y,\mathbb{Z}). \tag{A.2}$$

A.1.2 Gysin homomorphism

The Gysin homomorphism on the cohomology for $f : Y \to X$ is defined as

$$f_* = \mathrm{PD}_X \circ f_* \circ \mathrm{PD}_Y^{-1} \tag{A.3}$$

where f_* on the RHS is the push-forward of homology classes, cf. for example [206, § 23].

A.1.3 Poincaré residue operator

Let M be a complex manifold of complex dimension n. Consider on M an analytic family $\{Z_\lambda\}_{\lambda\in\Delta}$ where Δ is a disc and Z_λ are complex q-codimensional submanifolds of M which are homologous to zero. For convenience we set $p = n-q$. Let $Z = Z_0$. We have the normal bundle sequence

$$0 \longrightarrow T_Z \longrightarrow T_M|_Z \longrightarrow N_{Z/M} \longrightarrow 0. \tag{A.4}$$

Dualizing this sequence and applying $\mathcal{O}_Z(\cdot)$ we obtain

$$0 \longrightarrow \mathcal{O}_Z((N_{Z/M})^*) \longrightarrow \Omega^1_M|_Z \longrightarrow \Omega^1_Z \longrightarrow 0. \tag{A.5}$$

Generally, if we have a short exact sequence of vector spaces

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{A.6}$$

with $\dim B = n, \dim C = p$ and $\dim A = q = n-p$, then we have the following canonical exact sequence [73, (2.23)]

$$\bigwedge^2 A \otimes \bigwedge^{p-2} B \longrightarrow \bigwedge^{p+1} B \longrightarrow A \otimes \bigwedge^p C \longrightarrow 0. \tag{A.7}$$

Applying this to eq. (A.5), we obtain

$$\Omega^{p+1}_M|_Z \xrightarrow{\alpha} \Omega^p_Z \otimes \mathcal{O}((N_{Z/M})^*) \longrightarrow 0. \tag{A.8}$$

The map α is called the *Poincaré residue operator* [207, p. 106]. We also have a natural map from Ω^{p+1}_M to $\Omega^{p+1}_M|_Z$ which yields a map $H^p(M,\Omega^{p+1}_M) \to H^p(Z,\Omega^{p+1}_M|_Z)$. Together with eq. (A.8), we get a commutative diagram

$$\begin{array}{lll} H^p(M,\Omega^{p+1}_M) & \overset{\psi^*}{\searrow} & \\ \downarrow & & \\ H^p(Z,\Omega^{p+1}_M|_Z) & \longrightarrow & H^{p,p}(Z,(N_{Z/M})^*) = H^0(Z,N_{Z/M})^* \end{array} \tag{A.9}$$

where $H^{p,p}(Z,(N_{Z/M})^*) = H^p(Z,\Omega^p_Z \otimes \mathcal{O}((N_{Z/M})^*))$. The dual map of ψ^* gives us the map

$$H^0(Z,N_{Z/M}) \xrightarrow{\psi} H^p(M,\Omega^{p+1}_M)^*. \tag{A.10}$$

A.1.4 Kodaira-Nakano theorem

If $E \otimes K_X^{-1}$ is positive, then $H^k(X,E)$ vanish for all $i > 0$ [206, Thm. 18.2.2]. Thus, the Euler-Poincaré characteristic $\chi(X,E)$ equals $\dim H^0(X,E)$.

A.1.5 Hirzebruch-Riemann-Roch theorem

The Euler-Poincaré characteristic $\chi(X,E)$ can be computed as follows [206, Thm. 21.1.1]

$$\chi(X,E) = \sum_{i=0}^{n} (-1)^i \dim H^i(X,E) = \int_X \mathrm{ch}(E)\,\mathrm{td}(X) = T(X,E) \tag{A.11}$$

where X is a complex n-dimensional manifold, E a holomorphic vector bundle over X, and $T(X,E)$ the corresponding Todd genus.

A.1.6 Grothendieck-Riemann-Roch theorem

Let $f : X \to Y$ be a holomorphic map and E a coherent sheaf on X, then [206, Thm. 23.4.3]

$$\mathrm{ch}(f_! E)\,\mathrm{td}(Y) = f_*(\mathrm{ch}(E)\,\mathrm{td}(X)). \tag{A.12}$$

The map $f_!$ denotes the following

$$f_! E = \sum_i (-1)^i R^i f_* E \tag{A.13}$$

where $R^i f_*$ is the i-th right derived direct image functor.

A.1.7 Chern classes of projective bundles

Let $E \to B$ be a vector bundle and

$$\mathbb{P}^n \longrightarrow P(E) \xrightarrow{\ p\ } B \tag{A.14}$$

its projectivization. To compute the Chern classes of $T_{P(E)}$, we split the tangent vectors of $P(E)$ to horizontal and vertical tangent vectors. The horizontal vectors are tangent to B and the vertical vectors tangent to the fiber. Thus,

$$T_{P(E)} = p^* T_B \oplus T_F \tag{A.15}$$

where $\oplus$ denotes the Whitney sum. The task is now to compute T_F. Over $P(E)$ there is a canonical line bundle $k^1 \to P(E)$ whose fiber over $x \in P(E)$ is just the vectors of the line x. Furthermore, we have the complement bundle $k^\perp \to P(E)$. We have

$$k^1 \oplus k^\perp = p^* E \tag{A.16}$$

where $p^* E \to P(E)$ is the pulled back bundle of $E \to B$. Furthermore we have $T_F = \mathrm{Hom}(k^1, k^\perp)$ [208, Lem. 4.4]. The line bundle $\mathrm{Hom}(k^1, k^1)$ has a nowhere-vanishing section and thus is a trivial line bundle e^1 over $P(E)$ [208, in the proof of Thm. 4.5]. Thus,

$$\begin{aligned} T_F \oplus e^1 &= \mathrm{Hom}(k^1, k^\perp) \oplus e^1 = \mathrm{Hom}(k^1, k^\perp) \oplus \mathrm{Hom}(k^1, k^1) = \mathrm{Hom}(k^1, k^1 \oplus k^\perp) \\ &= \mathrm{Hom}(k^1, p^* E). \end{aligned} \tag{A.17}$$

Now, we can determine the Chern class of $P(E)$

$$c(T_{P(E)}) = p^* c(B) c(\mathrm{Hom}(k^1, p^* E)). \tag{A.18}$$

We compute the Chern class of $P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ as an example where $\mathcal{O}_{\mathbb{P}^1}$ is the trivial bundle over $\mathbb{P}^1$ and $\mathcal{O}_{\mathbb{P}^1}(n)$ the n-th power of the hyperplane bundle over $\mathbb{P}^1$. We obtain, using $\mathrm{Hom}(V,W) \cong V^* \otimes W$ and suppressing $p^*(\cdot)$,

$$\begin{aligned} c(P) &= c(\mathbb{P}^1)c(\mathrm{Hom}(k^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))) = (1+\omega)^2 c(k^{-1} \otimes (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))) \\ &= (1+\omega)^2 c((k^{-1} \otimes \mathcal{O}_{\mathbb{P}^1}) \oplus (k^{-1} \otimes \mathcal{O}_{\mathbb{P}^1}(n))) \\ &= (1+\omega)^2 (1+\eta)(1+\eta+n\omega) \end{aligned} \tag{A.19}$$

where ω and η are the hyperplane classes of the basis $\mathbb{P}^1$ and the fiber $\mathbb{P}^1$, respectively.

A.2 Note on the orientifold limit of F-theory

In this appendix we argue that the base twofold of a K3 fibered CY fourfold cannot be the orientifold limit of Sen. Let Y be a CY fourfold upon which we compactify F-theory. We furthermore assume that the base B_Y is a $\mathbb{P}^1$ fibration over a twofold B_2, i.e.

$$\begin{array}{ccccc} T^2 & & \mathbb{P}^1 & & \\ \downarrow & & \downarrow & & \\ Y & \longrightarrow & B_Y = \mathbb{P}(\mathcal{O}_{B_2} \oplus \mathcal{T}) & \longrightarrow & B_2 . \end{array} \tag{A.20}$$

We want to determine whether it is possible for B_2 to be the orientifold locus of Sen's limit [140, 141]. The parameterization of f and g of the Weierstraß equation is as follows

$$f = -3h^2 + C\eta, \quad g = -2h^3 + Ch\eta + C^2\eta \tag{A.21}$$

where $h \in \Gamma(B_Y, K_{B_Y}^{-2})$. The first Chern class of $K_{B_Y}^{-1}$ can be determined to be $c_1(B_Y) = c_1(B_2) + 2r + t$ where r is the class of the fiber $\mathbb{P}^1$ and t the class of the line bundle $\mathcal{T}$. Using the adjunction formula, we furthermore obtain the following

$$K_{B_2} = (K_{B_Y} \otimes \mathcal{O}_{B_Y}(B_2))|_{B_2} \Longrightarrow c_1(O_{B_Y})(B_2) = 2r + t. \tag{A.22}$$

This means that the section σ describing B_2 in B_Y has the class $2r + t$ in contrast to h which has the class $2c_1(B_2) + 4r + 2t$. Thus, B_2 *cannot* be the orientifold locus in Sen's limit.

A.3 Data and results of further Calabi-Yau fourfolds

In this appendix we collect topological data and results omitted in the main text in order to keep the main text clear.

A.3.1 Further topological data of the main example

Here, we supply the topological data of the fourfold $\widehat{Y}$ omitted in the main text. Besides the intersection rings we will also present the full PF system at the large complex structure point. These determine as explained in § 5.2 the primary vertical subspace $H_V^{p,p}(\widehat{Y})$ of the A-model.

As it was mentioned before, there are four triangulations whereas only three yield nonsingular varieties. Again we restrict our exposition to the two triangulations mentioned in

§ 5.3.2. For the following we label the points of the polyhedron $\Delta_{\hat{Y}}$ given in Table 5.4 consecutively by ν_i, $i = 0,\dots,9$ and the associate homogeneous coordinates x_i to each ν_i. Then the toric divisors are given by $D_i = \{x_i = 0\}$.

Phase I

In phase I of the toric variety defined by the polyhedron $\Delta_{\hat{Y}}$ in Table 5.4 we have the following Stanley-Reisner ideal

$$SR = \{D_3D_8, D_7D_9, D_8D_9, D_1D_5D_6, D_2D_3D_4, D_2D_4D_7\}. \tag{A.23}$$

From this we compute by standard methods of toric geometry the intersection numbers

$$\begin{aligned} \mathcal{C}_0 &= J_4(J_1^2J_2 + J_1J_3J_2 + J_3^2J_2 + 3J_1J_2^2 + 3J_3J_2^2 + 9J_2^3) + J_1^2J_3J_2 + J_1J_3^2J_2 + J_3^3J_2 \\ &\quad + 2J_1^2J_2^2 + 4J_1J_3J_2^2 + 4J_3^2J_2^2 + 11J_1J_2^3 + 15J_3J_2^3 + 46J_2^4, \\ \mathcal{C}_2 &= 24J_1^2 + 36J_1J_4 + 48J_1J_3 + 36J_4J_3 + 48J_3^2 + 128J_1J_2 + 102J_2J_4 + 172J_2J_3 + 530J_2^2, \\ \mathcal{C}_3 &= -660J_1 - 540J_4 - 900J_3 - 2776J_2. \end{aligned} \tag{A.24}$$

Here, we denoted generators of the Kähler cone of eq. (5.95) dual to the Mori cone by J_i as before. The notation for the $\mathcal{C}_k$ is as follows: Denoting the dual two-forms to J_i by ω_i the coefficients of the top intersection ring $\mathcal{C}_0$ are the quartic intersection numbers

$$J_i \cap J_j \cap J_k \cap J_l = \int_{\hat{Y}} \omega_i \wedge \omega_j \wedge \omega_k \wedge \omega_l \tag{A.25}$$

while the coefficients of $\mathcal{C}_2$ and $\mathcal{C}_3$ are

$$[c_2(\hat{Y})] \cap J_i \cap J_j = \int_{\hat{Y}} c_2 \wedge \omega_i \wedge \omega_j, \quad [c_3(\hat{Y})] \cap J_i = \int_{\hat{Y}} c_3 \wedge \omega_i, \tag{A.26}$$

respectively. As before, we write c_i for $c_i(\hat{Y})$.

As reviewed in § 5.2.3, the PF operators of the mirror fourfold Y at the large complex structure point are calculated by the methods described in ref. [9]. In the appropriate coordinates z_i defined by eq. (5.81) and evaluated in eq. (5.101), we obtain the full PF system for Y

$$\begin{aligned} \mathcal{L}_1^I &= -\theta_1^2(\theta_1 + \theta_4 - \theta_3) \\ &\quad - (-1 + \theta_1 - \theta_3)(-2 + 2\theta_1 + \theta_4 + \theta_3 - \theta_2)(-1 + 2\theta_1 + \theta_4 + \theta_3 - \theta_2)z_1, \\ \mathcal{L}_2^I &= \theta_2(-2\theta_1 - \theta_4 - \theta_3 + \theta_2) - 12(-5 + 6\theta_2)(-1 + 6\theta_2)z_2, \\ \mathcal{L}_3^I &= (\theta_1 - \theta_3)(-\theta_4 + \theta_3) - (1 + \theta_1 + \theta_4 - \theta_3)(-1 + 2\theta_1 + \theta_4 + \theta_3 - \theta_2)z_3, \\ \mathcal{L}_4^I &= \theta_4(\theta_1 + \theta_4 - \theta_3) - (-1 + \theta_4 - \theta_3)(-1 + 2\theta_1 + \theta_4 + \theta_3 - \theta_2)z_4. \end{aligned} \tag{A.27}$$

Now, we calculate the ring $\mathcal{R}$ given by the orthogonal complement of the ideal of PF operators defined as the quotient ring (5.48). Using the isomorphism $\theta_i \mapsto J_i$ discussed in § 5.2.3, we obtain the topological basis of $H_V^{p,p}(\hat{Y})$ by identification with the graded ring $\mathcal{R}^{(p)}$. Since the J_i form the trivial basis of $H^{1,1}(\hat{Y})$ and $H^{3,3}(\hat{Y})$ is fixed by duality to $H^{1,1}(\hat{Y})$, the non-trivial part is the cohomology group $H_V^{2,2}(Y)$. We calculate the ring $\mathcal{R}^{(2)}$ by choosing the basis

$$\begin{aligned} &\mathcal{R}_1^{(2)} = \theta_1^2, &&\mathcal{R}_2^{(2)} = \theta_4(\theta_1 + \theta_3), &&\mathcal{R}_3^{(2)} = \theta_3(\theta_1 + \theta_3), \\ &\mathcal{R}_4^{(2)} = \theta_2(\theta_1 + 2\theta_2), &&\mathcal{R}_5^{(2)} = \theta_2(\theta_4 + \theta_2), &&\mathcal{R}_6^{(2)} = \theta_2(\theta_3 + \theta_2). \end{aligned} \tag{A.28}$$

Then, we can use the intersection ring $\mathcal{C}_0$ to determine the topological metric

$$\eta_I^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 4 & 3 & 3 \\ 0 & 0 & 0 & 14 & 6 & 8 \\ 0 & 0 & 0 & 18 & 10 & 10 \\ 4 & 14 & 18 & 230 & 124 & 137 \\ 3 & 6 & 10 & 124 & 64 & 73 \\ 3 & 8 & 10 & 137 & 73 & 80 \end{pmatrix}. \tag{A.29}$$

The entries are the integrals

$$\mathcal{R}_\alpha^{(2)}\mathcal{R}_\beta^{(2)} = \int_{\widehat{Y}} (\mathcal{R}_\alpha^{(2)}\mathcal{R}_\beta^{(2)})\Big|_{\theta_i \mapsto J_i} \tag{A.30}$$

where we think of it in terms of the Poincaré duals and the quartic intersections are given in $\mathcal{C}_0$. The basis $\mathcal{R}_i^{(p)}$ at grade $p=3$ is determined by requiring $\eta_{ab}^{(3)} = \delta_{a,h^{1,1}(\widehat{Y})-b+1}$ where $h^{1,1}(\widehat{Y}) = 4$. The basis then reads

$$\begin{aligned} &\mathcal{R}_1^{(3)} = \theta_1(-\theta_1\theta_4 - \theta_2\theta_4 + \theta_2\theta_3), \quad &&\mathcal{R}_2^{(3)} = \theta_1(-\theta_1\theta_4 + \theta_1\theta_2 + \theta_2\theta_4 - \theta_2\theta_3), \\ &\mathcal{R}_3^{(3)} = \theta_1^2\theta_4, &&\mathcal{R}_4^{(3)} = \theta_1(-2\theta_1\theta_4 - \theta_1\theta_2 + \theta_2\theta_3). \end{aligned} \tag{A.31}$$

Finally, we choose a basis of $\mathcal{R}^{(4)}$ by requiring $\eta_{a_0,b_0}^{(4)} = 1$ for $\mathcal{R}^{(0)} = 1$

$$\mathcal{R}^{(4)} = \frac{1}{103}\,\mathcal{C}_0|_{J_i \mapsto \theta_i}\,. \tag{A.32}$$

Phase II

Turning to phase II of Table 5.4 the Stanley-Reisner ideal and the intersection numbers read

$$\begin{aligned} SR &= \{D_1D_7, D_7D_9, D_8D_9, D_1D_5D_6, D_2D_3D_4, D_2D_4D_7, D_3D_5D_6D_8\}, \\ \mathcal{C}_0 &= J_1^2J_4J_3 + 2J_1^2J_3^2 + 3J_1J_4J_3^2 + 12J_1J_3^3 + 9J_4J_3^3 + 54J_3^4 + J_1^2J_2J_4 + 2J_1^2J_3J_2 \\ &\quad + 3J_1J_2J_3J_4 + 12J_1J_3^2J_2 + 9J_2J_3^2J_4 + 54J_3^3J_2 + 2J_1^2J_2^2 + 3J_1J_4J_2^2 + 12J_1J_3J_2^2 \\ &\quad + 9J_4J_3J_2^2 + 54J_3^2J_2^2 + 11J_1J_2^3 + 9J_4J_2^3 + 51J_3J_2^3 + 46J_2^4, \\ \mathcal{C}_2 &= 24J_1^2 + 36J_1J_4 + 138J_1J_3 + 102J_4J_3 + 618J_3^2 + 128J_1J_2 + 102J_2J_4 \\ &\quad + 588J_3J_4 + 530J_4^2, \\ \mathcal{C}_3 &= 660J_1 - 540J_4 - 3078J_3 - 2776J_2 \end{aligned} \tag{A.33}$$

where the Kähler cone generators were given in eq. (5.96). The complete PF system consists of four operators given by

$$\begin{aligned} \mathcal{L}_1^{II} &= -\theta_1^2(\theta_1 + \theta_2 - \theta_3) \\ &\quad - (-3 + 3\theta_1 - \theta_3 + 2\theta_4)(-2 + 3\theta_1 - \theta_3 + 2\theta_4)(-1 + 3\theta_1 - \theta_3 + 2\theta_4)\,z_1, \\ \mathcal{L}_2^{II} &= -\theta_2(\theta_1 + \theta_2 - \theta_3)(\theta_2 - \theta_3 + \theta_4) - 12(-5 + 6\theta_2)(-1 + 6\theta_2)(-1 + \theta_2 - \theta_3)\,z_2, \\ \mathcal{L}_3^{II} &= -(\theta_2 - \theta_3)(-3\theta_1 + \theta_3 - 2\theta_4) - (1 + \theta_1 + \theta_2 - \theta_3)(1 + \theta_2 - \theta_3 + \theta_4)\,z_3, \\ \mathcal{L}_4^{II} &= \theta_4(\theta_2 - \theta_3 + \theta_4) - (-2 + 3\theta_1 - \theta_3 + 2\theta_4)(-1 + 3\theta_1 - \theta_3 + 2\theta_4)\,z_4. \end{aligned} \tag{A.34}$$

This enables us to calculate $H_V^{p,p}(\widehat{Y})$ as before. The basis at grade $p=2$ reads

$$\begin{aligned} &\mathcal{R}_1^{(2)} = \theta_1^2, && \mathcal{R}_2^{(2)} = \theta_2(2\theta_1+6\theta_3), && \mathcal{R}_3^{(2)} = \theta_3(\theta_1+3\theta_3), \\ &\mathcal{R}_4^{(2)} = \theta_1\theta_4, && \mathcal{R}_5^{(2)} = \theta_2^2, && \mathcal{R}_6^{(2)} = \theta_3(2\theta_2+2\theta_3+\theta_4)+\theta_2\theta_4 \end{aligned} \tag{A.35}$$

for which the topological metric $\eta^{(2)}$ is given by

$$\eta_{II}^{(2)} = \begin{pmatrix} 0 & 12 & 6 & 0 & 2 & 10 \\ 12 & 2240 & 1120 & 20 & 328 & 1512 \\ 6 & 1120 & 560 & 10 & 174 & 756 \\ 0 & 20 & 10 & 0 & 3 & 12 \\ 2 & 328 & 174 & 3 & 46 & 228 \\ 10 & 1512 & 756 & 12 & 228 & 1008 \end{pmatrix}. \tag{A.36}$$

Again the basis of $H^{3,3}(\widehat{Y})$ is fixed by $\eta_{ab}^{(3)} = \delta_{a,h^{1,1}(\widehat{Y})-b+1}$ to be

$$\begin{aligned} \mathcal{R}_1^{(3)} &= -\frac{1}{91}\left(182\theta_1^2+25\theta_2^2+\theta_1(-225\theta_2+85\theta_3)\right)(\theta_1+\theta_2+\theta_3+\theta_4), \\ \mathcal{R}_2^{(3)} &= \frac{1}{91}\left(91\theta_1^2+10\theta_2^2+\theta_1(\theta_2-57\theta_3)\right)(\theta_1+\theta_2+\theta_3+\theta_4), \\ \mathcal{R}_3^{(3)} &= -\theta_1(\theta_2-\theta_3)(\theta_1+\theta_2+\theta_3+\theta_4), \\ \mathcal{R}_4^{(3)} &= -\frac{1}{91}\left(273\theta_1^2+23\theta_2^2+\theta_1(-207\theta_2+60\theta_3)\right)(\theta_1+\theta_2+\theta_3+\theta_4). \end{aligned} \tag{A.37}$$

We conclude with the basis of $H^{4,4}(\widehat{Y})$ as follows

$$\mathcal{R}^{(4)} = \frac{1}{359}\,\mathcal{C}_0|_{J_i\mapsto\theta_i}\,. \tag{A.38}$$

A.3.2 Further examples of fourfolds

In this appendix, we consider a broader class of CY fourfolds $(\widehat{Y}, Y)$ constructed as described in § 5.1 by fibering CY threefolds $\widehat{X}$ over $\mathbb{P}^1$. The threefolds we consider here are itself elliptically fibered over Hirzebruch surfaces F_0 and F_1, i.e. in Figure 5.2 the base B_X is F_n. In the following we will present the toric data of the threefolds $\widehat{X}$ and fourfolds $\widehat{Y}$ including some of their topological quantities. Then, we will determine the complete system of PF differential operators at the large complex structure point of the mirror CY fourfold and calculate the holomorphic prepotential $F^0(\gamma)$. From this we extract the invariants n_β^g which are integer in all considered cases. Furthermore, we show that there exists a subsector for these invariants that reproduces the closed and open GW invariants of the local CY threefolds obtained by a suitably decompactifying the elliptic fiber of the original compact CY threefold. This matching allows us to determine the four-form flux G_4 for the F-theory compactification such that the F-theory flux superpotential (2.24) admits the split (2.31) into the type IIB flux and brane superpotentials.

Fourfold with F_0

We start with an elliptically fibered CY threefold $\widehat{X}$ with base given by the toric Fano basis of the zeroth Hirzebruch surface $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Its polyhedron and charge vectors read

$$\left[\begin{array}{c|cccc|ccc} & & \Delta_{\widehat{X}} & & & \ell^{(1)} & \ell^{(2)} & \ell^{(3)} \\ \hline v_0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 \\ v_1^b & 0 & 0 & 2 & 3 & 1 & -2 & -2 \\ v_2^b & 1 & 0 & 2 & 3 & 0 & 1 & 0 \\ v_3^b & -1 & 0 & 2 & 3 & 0 & 1 & 0 \\ v_4^b & 0 & 1 & 2 & 3 & 0 & 0 & 1 \\ v_5^b & 0 & -1 & 2 & 3 & 0 & 0 & 1 \\ v_1 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ v_2 & 0 & 0 & 0 & -1 & 3 & 0 & 0 \end{array}\right], \qquad (A.39)$$

where points in the base are again labelled by a superscript $(\cdot)^b$. There is one triangulation for which the Stanley-Reisner ideal in terms of the toric divisors $D_i = \{x_i = 0\}$ takes the form

$$SR = \{D_2D_3, D_4D_5, D_1D_6D_7\}. \qquad (A.40)$$

This threefold $\widehat{X}$ has

$$\chi = -480, \quad h_{1,1} = 3, \quad h_{2,1} = 243 \qquad (A.41)$$

where the three Kähler classes correspond to the elliptic fiber and the two $\mathbb{P}^1$ of F_0. The intersection ring for this triangulation in terms of the Kähler cone generators

$$J_1 = D_1 + 2D_2 + 2D_4, \quad J_2 = D_2, \quad J_3 = D_4 \qquad (A.42)$$

reads $\mathcal{C}_0 = 8J_1^3 + 2J_1^2J_3 + 2J_1^2J_2 + J_1J_2J_3$ and $\mathcal{C}_2 = 92J_1 + 24J_2 + 24J_3$.

In the local limit K_{F_0}, Harvey-Lawson type branes described by the brane charge vectors

$$\widehat{\ell}^{(1)} = (-1, 0, 1, 0, 0), \quad \widehat{\ell}^{(1)} = (-1, 0, 0, 1, 0) \qquad (A.43)$$

were studied in ref. [33]. To construct the CY fourfold $\widehat{Y}$ we use the construction described in § 5.1 with the brane vector $\widehat{\ell}^{(1)}$. We extend $\Delta_{\widehat{X}}$ to the polyhedron five-dimensional polyhedron $\Delta_{\widehat{Y}}$ and determine the five Mori cone generators $\ell^{(i)}$ for the four different triangulations of the corresponding CY phases. Table A.1 shows one of the four triangulations on which we focus our following analysis. In the triangulation shown in the table the Stanley-Reisner ideal takes the form

$$SR = \{D_2D_3,\ D_2D_8,\ D_3D_9,\ D_4D_5,\ D_8D_{10},\ D_9D_{10},\ D_1D_6D_7\}. \qquad (A.44)$$

The generators of the Kähler cone of the fourfold $\widehat{Y}$ in the given triangulation are

$$J_1 = D_1 + 2D_{10} + D_2 + D_3 + 2D_4, \quad J_2 = D_{10}, \quad J_3 = D_4, \quad J_4 = D_{10} + D_3, \quad J_5 = D_2\,, \qquad (A.45)$$

	$\Delta_{\widehat{Y}}$					$\ell^{(1)}$	$\ell^{(2)}$	$\ell^{(3)}$	$\ell^{(4)}$	$\ell^{(5)}$
v_0	0	0	0	0	0	−6	0	0	0	0
v_1	0	0	2	3	0	1	−1	−2	−1	−1
v_2	1	0	2	3	0	0	1	0	0	0
v_3	−1	0	2	3	0	0	0	0	1	−1
v_4	0	1	2	3	0	0	0	1	0	0
v_5	0	−1	2	3	0	0	0	1	0	0
v_6	0	0	−1	0	0	2	0	0	0	0
v_7	0	0	0	−1	0	3	0	0	0	0
v_8	−1	0	2	3	−1	0	1	0	−1	1
v_9	0	0	2	3	−1	0	−1	0	1	0
v_{10}	0	0	2	3	1	0	0	0	0	1

Table A.1: Toric data of the CY fourfold based on F_0

for which the intersections are determined to be

$$\begin{aligned}\mathcal{C}_0 &= 42J_1^4 + 8J_1^3J_2 + 7J_1^3J_3 + 2J_1^2J_2J_3 + 12J_1^3J_4 + 2J_1^2J_2J_4 + 3J_1^2J_3J_4 + J_1J_2J_3J_4 \\ &\quad + 2J_1^2J_4^2 + J_1J_3J_4^2 + 8J_1^3J_5 + 2J_1^2J_2J_5 + 2J_1^2J_3J_5 + J_1J_2J_3J_5 + 2J_1^2J_4J_5 + J_1J_3J_4J_5, \\ \mathcal{C}_2 &= 92J_1J_2 + 486J_1^2 + 24J_2J_3 + 82J_1J_3 + 24J_3J_5 + 92J_1J_5 + 24J_2J_5 \\ &\quad + 24J_2J_4 + 138J_1J_4 + 36J_3J_4 + 24J_4J_5 + 24J_4^2, \\ \mathcal{C}_3 &= -2534J_1 - 480J_2 - 420J_3 - 720J_4 - 480J_5.\end{aligned} \tag{A.46}$$

We calculate the core topological quantities to be

$$\chi = 15408, \quad h_{3,1} = 2555, \quad h_{2,1} = 0, \quad h_{1,1} = 5. \tag{A.47}$$

We note that the intersection numbers reveal the fibration structure of $\widehat{Y}$. We find the Euler number of the threefold $\widehat{X}$ as the coefficient of J_2 and J_5 in $\mathcal{C}_3$ and the fact that both J_2 and J_5 appear at most linear in $\mathcal{C}_0, \mathcal{C}_2$. This is consistent with the fact that the fiber F of a fibration has intersection number 0 with itself which implies $c_3(F) = c_3(\widehat{Y})$ using the adjunction formula as well as $c_1(F) + c_1(N_{F/\widehat{Y}}) = c_1(N_{F/\widehat{Y}}) = 0$ for $\widehat{Y}$ being CY. Thus, we observe a fibration of $\widehat{X}$ represented by the classes J_2 and J_5 over the base curves corresponding to $\ell^{(2)}$ and $\ell^{(5)}$, respectively. The PF operators are determined as before

$$\begin{aligned}\mathcal{L}_1 &= \theta_1(\theta_1 - \theta_2 - 2\theta_3 - \theta_4 - \theta_5) - 12(-5 + 6\theta_1)(-1 + 6\theta_1)\, z_1, \\ \mathcal{L}_2 &= \theta_2(\theta_2 - \theta_4 + \theta_5) - (-1 + \theta_2 - \theta_4)(-1 - \theta_1 + \theta_2 + 2\theta_3 + \theta_4 + \theta_5)\, z_2, \\ \mathcal{L}_3 &= \theta_3^2 - (1 + \theta_1 - \theta_2 - 2\theta_3 - \theta_4 - \theta_5)(2 + \theta_1 - \theta_2 - 2\theta_3 - \theta_4 - \theta_5)\, z_3, \\ \mathcal{L}_4 &= (\theta_2 - \theta_4)(\theta_4 - \theta_5) - (1 + \theta_2 - \theta_4 + \theta_5)(-1 - \theta_1 + \theta_2 + 2\theta_3 + \theta_4 + \theta_5)\, z_4, \\ \mathcal{L}_5 &= \theta_5(\theta_2 - \theta_4 + \theta_5) - (1 + \theta_1 - \theta_2 - 2\theta_3 - \theta_4 - \theta_5)(1 + \theta_4 - \theta_5)\, z_5.\end{aligned} \tag{A.48}$$

We can now proceed with determining the basis of $H_V^{(p,p)}(\widehat{Y})$ at each grade p by determining the ring $\mathcal{R}$ as given in eq. (5.48). We choose a basis at grade $p=2$ as

$$\begin{aligned} &\mathcal{R}_1^{(2)} = \theta_1(\theta_1+\theta_5)\,, \quad \mathcal{R}_2^{(2)} = \theta_1(\theta_1+\theta_2)\,, \quad \mathcal{R}_3^{(2)} = \theta_1(2\theta_1+\theta_3)\,, \\ &\mathcal{R}_4^{(2)} = \theta_1(\theta_1+\theta_4)\,, \quad \mathcal{R}_5^{(2)} = \theta_2\theta_3, \qquad\quad \mathcal{R}_6^{(2)} = (\theta_2+\theta_4)(\theta_4+\theta_5)\,, \\ &\mathcal{R}_7^{(2)} = \theta_3\theta_4, \qquad\quad \mathcal{R}_8^{(2)} = \theta_3\theta_5. \end{aligned} \tag{A.49}$$

The basis of solution dual to this basis choice is given by

$$\begin{aligned} &\mathbb{L}_1^{(2)} = \frac{1}{8}l_1(l_1-l_2-2l_3-l_4+7l_5)\,, && \mathbb{L}_2^{(2)} = \frac{1}{8}l_1(l_1+7l_2-2l_3-l_4-l_5)\,, \\ &\mathbb{L}_3^{(2)} = \frac{1}{4}l_1(l_1-l_2+2l_3-l_4-l_5)\,, && \mathbb{L}_4^{(2)} = \frac{1}{8}l_1(l_1-l_2-2l_3+7l_4-l_5)\,, \\ &\mathbb{L}_5^{(2)} = l_2l_3, && \mathbb{L}_6^{(2)} = \frac{1}{4}(l_2+l_4)(l_4+l_5)\,, \\ &\mathbb{L}_7^{(2)} = l_3l_4, && \mathbb{L}_8^{(2)} = l_3l_5 \end{aligned} \tag{A.50}$$

where we write $l_i = \log(z_i)$ as before. The topological two-point coupling between the $\mathcal{R}_\alpha^{(2)}$ in the chosen basis reads

$$\eta^{(2)} = \begin{pmatrix} 58 & 60 & 109 & 64 & 3 & 8 & 4 & 2 \\ 60 & 58 & 109 & 64 & 2 & 8 & 4 & 3 \\ 109 & 109 & 196 & 118 & 4 & 20 & 6 & 4 \\ 64 & 64 & 118 & 68 & 3 & 8 & 4 & 3 \\ 3 & 2 & 4 & 3 & 0 & 0 & 0 & 0 \\ 8 & 8 & 20 & 8 & 0 & 0 & 0 & 0 \\ 4 & 4 & 6 & 4 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 3 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.51}$$

The basis of $\mathcal{R}^{(3)}$ determining $H^{3,3}(\widehat{Y})$ that is fixed by Poincaré duality to the Kähler cone generators satisfying $\eta_{ab}^{(3)} = \delta_{a,h^{1,1}(\widehat{Y})-b+1}$ is given by

$$\begin{aligned} &\mathcal{R}_1^{(3)} = \frac{1}{4}(9\theta_1\theta_5 - 2\theta_1\theta_3 - \theta_3^2)\theta_3 + \theta_2\theta_3^2 - \theta_1\theta_2\theta_5, \\ &\mathcal{R}_2^{(3)} = \frac{1}{8}(\theta_1\theta_3 + 2\theta_3^2 - 10\theta_1\theta_5)\theta_3 - \theta_2\theta_3^2 - \theta_1\theta_2\theta_5, \\ &\mathcal{R}_3^{(3)} = \theta_1(\frac{1}{2}\theta_3^2 - \theta_3\theta_5 - 2\theta_2\theta_5), \\ &\mathcal{R}_4^{(3)} = \theta_1\theta_2\theta_5, \\ &\mathcal{R}_5^{(3)} = \frac{1}{8}\theta_3(2\theta_3^2 - 3\theta_1\theta_3 - 10\theta_1\theta_5 - 4\theta_2\theta_3) - \theta_1\theta_2\theta_5. \end{aligned} \tag{A.52}$$

We choose the basis of $H^{4,4}(\widehat{Y})$ such that the volume is normalized as $\eta_{a_0,b_0}^{(4)} = 1$ for $\mathcal{R}^{(0)} = 1$, i.e.

$$\mathcal{R}^{(4)} = \frac{1}{96}\,\mathcal{C}_0|_{J\mapsto\theta}\,. \tag{A.53}$$

In order to fix the integral basis of $H_V^{2,2}(\widehat{Y})$ we again match the threefold periods from the fourfold periods via eq. (5.114). The first step is to identify the Kähler classes of $\widehat{X}$. As discussed above, J_5 represents the class of the CY fiber $\widehat{X}$. The intersection forms of $\widehat{X}$ are obtained from eq. (A.46) upon the identification

$$J_1 \leftrightarrow J_1(\widehat{X}), \quad J_2 + J_4 \leftrightarrow J_2(\widehat{X}), \quad J_3 \leftrightarrow J_3(\widehat{X}). \tag{A.54}$$

With this in mind we calculate the leading logarithms $\mathbb{L}_\alpha(X)$ on the threefold given by

$$\mathbb{L}_1(X) = \frac{1}{2}X^0(2\tilde{l}_1 + \tilde{l}_2)(2\tilde{l}_1 + \tilde{l}_3), \quad \mathbb{L}_2(X) = \frac{1}{2}X^0\tilde{l}_1(\tilde{l}_1 + \tilde{l}_3), \quad \mathbb{L}_3(X) = \frac{1}{2}X^0\tilde{l}_1(\tilde{l}_1 + \tilde{l}_2). \tag{A.55}$$

This together with the requirement of matching the instanton numbers[1] n_{d_1,d_2,d_3} of $\widehat{X}$ via the invariants $n_{d_1,d_2,d_3,d_2,0}$ on $\widehat{Y}$ fixes unique solutions of the PF system

$$\mathbb{L}^{(2)}_1 = \frac{1}{2}X^0(2l_1 + l_3)(2l_1 + l_2 + l_4), \quad \mathbb{L}^{(2)}_6 = \frac{1}{2}X^0 l_1(l_1 + l_3), \quad \mathbb{L}^{(2)}_8 = \frac{1}{2}X^0 l_1(l_1 + l_2 + l_4) \tag{A.56}$$

that after the matching of threefold and fourfold classes given in eq. (A.54) coincide with the threefold solutions. This fixes three ring elements $\tilde{\mathcal{R}}^{(2)}_\alpha$ for $\alpha = 1,6,8$, by the map induced from eq. (5.111) that we complete to a new basis

$$\begin{aligned}
&\widetilde{\mathcal{R}}^{(2)}_1 = \frac{1}{8}\theta_1(\theta_1 - \theta_2 - 2\theta_3 - \theta_4 + 7\theta_5), \quad &&\widetilde{\mathcal{R}}^{(2)}_2 = \frac{1}{8}\theta_1(\theta_1 + 7\theta_2 - 2\theta_3 - \theta_4 - \theta_5),\\
&\widetilde{\mathcal{R}}^{(2)}_3 = \frac{1}{4}\theta_1(\theta_1 - \theta_2 + 2\theta_3 - \theta_4 - \theta_5), \quad &&\widetilde{\mathcal{R}}^{(2)}_4 = \frac{1}{8}\theta_1(\theta_1 - \theta_2 - 2\theta_3 + 7\theta_4 - \theta_5),\\
&\widetilde{\mathcal{R}}^{(2)}_5 = \theta_2\theta_3, &&\widetilde{\mathcal{R}}^{(2)}_6 = \frac{1}{4}(\theta_2 + \theta_4)(\theta_4 + \theta_5),\\
&\widetilde{\mathcal{R}}^{(2)}_7 = \theta_3\theta_4, &&\widetilde{\mathcal{R}}^{(2)}_8 = \theta_3\theta_5.
\end{aligned} \tag{A.57}$$

Then, the integral basis elements are given by

$$\hat{\gamma}^{(2)}_1 = \widetilde{\mathcal{R}}^{(2)}_1\Omega_Y\Big|_{\underline{z}=0}, \quad \hat{\gamma}^{(2)}_6 = \widetilde{\mathcal{R}}^{(2)}_6\Omega_Y\Big|_{\underline{z}=0}, \quad \hat{\gamma}^{(2)}_8 = \widetilde{\mathcal{R}}^{(2)}_8\Omega_Y\Big|_{\underline{z}=0} \tag{A.58}$$

where again the new grade $p = 2$ basis is obtained by replacing $l_i \leftrightarrow \theta_i$ in the dual solutions of eq. (A.50). We conclude by presenting the leading logarithms of the periods $\Pi^{(2)\,\alpha}$ when integrating Ω_Y over the dual cycles $\gamma^{(2)\,\alpha}$ for $\alpha = 1,6,8$. They are given by

$$\mathbb{L}^{(2)\,1} = X_0 l_1(l_1 + l_5), \quad \mathbb{L}^{(2)\,6} = X_0(l_2 + l_4)(l_4 + l_5), \quad \mathbb{L}^{(2)\,8} = X_0 l_3 l_5. \tag{A.59}$$

Finally, we determine a $\hat{\gamma}$ flux in $H^{2,2}_H(Y)$ matching the disk invariants of ref. [33] for both classes of the local geometry F_{F_0} with the brane. Furthermore, we reproduce the closed invariants of computed in ref. [172] for the two $\mathbb{P}^1$ classes for zero brane winding $m = 0$. Firstly, we identify in Table A.1 the vector $\ell^{(4)}$ to correspond to the brane vector. Then, we expect to recover the disk invariants from the fourfold invariants $n_{0,d_1,d_2,d_1+m,0}$. The flux $\hat{\gamma}$ deduced this way still contains a freedom of three parameters and takes the form

$$\hat{\gamma} = \left(-\mathcal{R}^{(2)}_5 + \frac{1}{4}\mathcal{R}^{(2)}_6 + \mathcal{R}^{(2)}_7 + \frac{1}{2}\mathcal{R}^{(2)}_8\right)\Omega_Y\Big|_{\underline{z}=0} \tag{A.60}$$

[1] We note here that by only matching the threefold instantons the solution of the fourfold could not be fixed. The two free parameters could only be determined by matching also the classical terms.

where we choose the free parameters a_i in front of $\mathcal{R}_1^{(2)}$, $\mathcal{R}_2^{(2)}$, $\mathcal{R}_3^{(2)}$ and $\mathcal{R}_4^{(2)}$ to be zero. Note that $a_7 = 1$ is fixed by the requirement of matching the disk invariants. For this parameter choice the leading logarithmic structures of the corresponding period and of the solution matching the invariants are given by

$$\mathbb{L}^{(2)\gamma} = X^0 (l_2 + l_4)(l_4 + l_5), \quad \mathbb{L}^{(2)}_\gamma = \frac{1}{2} X^0 l_1 (4l_1 + 3l_2 + 2l_3 + l_4). \tag{A.61}$$

Fourfold with F_1

We consider as our last example the elliptically fibered CY threefold $\widehat{X}$ with the base $F_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ which is the blow-up of $\mathbb{P}^2$ at one point. The polyhedron and charge vectors read

$$\left[\begin{array}{c|cccc|ccc} & & \Delta_{\widehat{X}} & & & \ell^{(1)} & \ell^{(2)} & \ell^{(3)} \\ \hline v_0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 \\ v_1^b & 0 & 0 & 2 & 3 & -1 & 1 & -2 \\ v_2^b & 1 & 1 & 2 & 3 & 1 & 0 & 0 \\ v_3^b & -1 & 0 & 2 & 3 & 1 & 0 & 0 \\ v_4^b & 0 & 1 & 2 & 3 & -1 & 0 & 1 \\ v_5^b & 0 & -1 & 2 & 3 & 0 & 0 & 1 \\ v_1 & 0 & 0 & -1 & 0 & 0 & 2 & 0 \\ v_2 & 0 & 0 & 0 & -1 & 0 & 3 & 0 \end{array}\right]. \tag{A.62}$$

where the labels by a superscript $(\cdot)^b$ again denote points in the base. There are two CY phases and for the triangulation given above the Stanley-Reisner ideal reads

$$SR = \{D_2 D_3, D_4 D_5, D_1 D_6 D_7\}. \tag{A.63}$$

This threefold has

$$\chi = 480, \quad h_{1,1} = 3, \quad h_{2,1} = 243 \tag{A.64}$$

where the three Kähler classes correspond to the elliptic fiber and the two $\mathbb{P}^1$ of the base F_1. The intersection ring for this phase in terms of the Kähler cone generators

$$J_1 = D_2, \quad J_2 = D_1 + 3D_2 + 2D_4, \quad J_3 = D_2 + D_4 \tag{A.65}$$

reads $\mathcal{C}_0 = 2J_1J_2^2 + 8J_2^3 + J_1J_2J_3 + 3J_2^2J_3 + J_2J_3^2$ and $\mathcal{C}_2 = 24J_1 + 92J_2 + 36J_3$. For the second CY phase we have the following data

$$\begin{gathered} \left[\begin{array}{c|cccccccc} \ell^{(1)} & -6 & 0 & 1 & 1 & -1 & 0 & 2 & 3 \\ \ell^{(2)} & 0 & -3 & 1 & 1 & 0 & 1 & 0 & 0 \\ \ell^{(3)} & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \end{array}\right], \\ SR = \{D_1 \cdot D_4, D_4 \cdot D_5, D_1 \cdot D_6 \cdot D_7, D_2 \cdot D_3 \cdot D_5, D_2 \cdot D_3 \cdot D_6 \cdot D_7\}, \\ J_1 = D_1 + 3D_2 + 2D + 4, \quad J_2 = D_2 + D_4, \quad J_3 = D_1 + 3D_2 + 3D_4, \\ \mathcal{C}_0 = 8J_1^3 + 3J_1^2J_2 + J_1J_2^2 + 9J_1^2J_3 + 3J_1J_2J_3 + J_2^2J_3 + 9J_1J_3^2 + 3J_2J_3^2 + 9J_3^3, \\ \mathcal{C}_2 = 92J_1 + 36J_2 + 102J_3. \end{gathered} \tag{A.66}$$

Harvey-Lawson type branes were considered in ref. [33] for the brane charge vectors

$$\widehat{\ell}^{(1)} = (-1,1,0,0,0), \quad \widehat{\ell}^{(1)} = (-1,0,0,1,0) \tag{A.67}$$

for the non-compact model K_{F_1}. The CY fourfold $\widehat{Y}$ is constructed from the brane vector $\hat{\ell}^{(1)}$ for which there are eleven triangulations. Again, we restrict our attention to one triangulation with the following data

$$\left[\begin{array}{c|ccccccccccc} \ell^{(1)} & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \ell^{(2)} & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ \ell^{(3)} & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \\ \ell^{(4)} & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \ell^{(5)} & -6 & 1 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 \end{array}\right],$$

$$\begin{aligned} SR &= \{D_2\cdot D_3,\ D_2\cdot D_8,\ D_3\cdot D_9,\ D_4\cdot D_5,\ D_8\cdot D_{10},\ D_9\cdot D_{10},\ D_1\cdot D_6\cdot D_7\},\\ J_1 &= D_2, \quad J_2 = D_1+2D_{10}+D_2+D_3+2D_4, \quad J_3 = D_4, \quad J_4 = D_{10}, \quad J_5 = D_{10}+D_3\\ \mathcal{C}_0 &= J_1J_2J_4J_5 + J_2^2J_4J_5 + J_1J_3J_4J_5 + J_2J_3J_4J_5 + J_1J_4^2J_5 + J_2J_4^2J_5\\ &\quad + 2J_1J_2J_5^2 + 2J_2^2J_5^2 + 2J_1J_3J_5^2 + 2J_2J_3J_5^2 + 3J_1J_4J_5^2 + 4J_2J_4J_5^2\\ &\quad + 2J_3J_4J_5^2 + 2J_4^2J_5^2 + 8J_1J_5^3 + 12J_2J_5^3 + 8J_3J_5^3 + 11J_4J_5^3 + 42J_5^4,\\ \mathcal{C}_2 &= 24J_1J_2 + 24J_2^2 + 24J_1J_3 + 24J_2J_3 + 36J_1J_4 + 48J_2J_4 + 24J_3J_4 + 24J_4^2\\ &\quad + 92J_1J_5 + 138J_2J_5 + 92J_3J_5 + 128J_4J_5 + 486J_5^2,\\ \mathcal{C}_3 &= -480J_1 - 270J_2 - 480J_3 - 660J_4 - 2534J_5,\\ \chi &= 15408, \quad h^{3,1} = 2555, \quad h^{2,1} = 0, \quad h^{1,1} = 5. \end{aligned} \tag{A.68}$$

Again, the Euler number of the threefold $\widehat{X}$ appears in $\mathcal{C}_3$ in front of J_1 and J_3 confirming the fibration structure. By comparing the coefficient polynomial of J_1 and J_3 with the threefold intersection rings presented in appendix A.3.2 and A.3.2, we infer that J_1 is precisely the elliptic fibration over F_1, i.e. $\widehat{X} = T^2 \to F_1$, whereas J_3 is $\widehat{X}' = T^2 \to F_0$. Since we discussed F_0 in detail before, we will just concentrate on the fibration structure involving F_1. The PF operators of Y read as

$$\begin{aligned} \mathcal{L}_1 &= \theta_1(\theta_1-\theta_2+\theta_3) - (-1+\theta_1-\theta_2)(-1+\theta_1+\theta_2+2\theta_4-\theta_5)z_1,\\ \mathcal{L}_2 &= (\theta_1-\theta_2)(\theta_2-\theta_3) - (1+\theta_1-\theta_2+\theta_3)(-1+\theta_1+\theta_2+2\theta_4-\theta_5)z_2,\\ \mathcal{L}_3 &= -\theta_3(\theta_1-\theta_2+\theta_3) - (1+\theta_2-\theta_3)(-1+\theta_3-\theta_4)z_3,\\ \mathcal{L}_4 &= \theta_4(-\theta_3+\theta_4) - (-2+\theta_1+\theta_2+2\theta_4-\theta_5)(-1+\theta_1+\theta_2+2\theta_4-\theta_5)z_4,\\ \mathcal{L}_5 &= \theta_5(-\theta_1-\theta_2-2\theta_4+\theta_5) - 12(-5+6\theta_5)(-1+6\theta_5)z_5. \end{aligned} \tag{A.69}$$

We determine the basis of $\mathcal{R}^{(2)}$ from the operators as

$$\begin{aligned} &\mathcal{R}_1^{(2)} = (\theta_1+\theta_2)(\theta_2+\theta_3), && \mathcal{R}_2^{(2)} = \theta_1\theta_4, && \mathcal{R}_3^{(2)} = \theta_5(\theta_1+\theta_5),\\ &\mathcal{R}_4^{(2)} = \theta_2\theta_4, && \mathcal{R}_5^{(2)} = \theta_5(\theta_2+\theta_5), && \mathcal{R}_6^{(2)} = \theta_4(\theta_3+\theta_4),\\ &\mathcal{R}_7^{(2)} = \theta_3\theta_5, && \mathcal{R}_8^{(2)} = \theta_5(\theta_4+2\theta_5) \end{aligned} \tag{A.70}$$

with the two-point coupling

$$\eta^{(2)} = \begin{pmatrix} 0 & 0 & 8 & 0 & 8 & 0 & 0 & 20 \\ 0 & 0 & 3 & 0 & 4 & 0 & 1 & 7 \\ 8 & 3 & 58 & 5 & 64 & 6 & 10 & 114 \\ 0 & 0 & 5 & 0 & 5 & 0 & 1 & 9 \\ 8 & 4 & 64 & 5 & 68 & 6 & 10 & 123 \\ 0 & 0 & 6 & 0 & 6 & 0 & 0 & 8 \\ 0 & 1 & 10 & 1 & 10 & 0 & 0 & 18 \\ 20 & 7 & 114 & 9 & 123 & 8 & 18 & 214 \end{pmatrix}. \tag{A.71}$$

The dual basis of solutions reads

$$\begin{aligned} &\mathbb{L}^{(2)1} = \frac{1}{4}(l_1+l_2)(l_2+l_3), && \mathbb{L}^{(2)2} = l_1 l_4, \\ &\mathbb{L}^{(2)3} = \frac{1}{7} l_5(6l_1 - l_2 - 2l_4 + l_5), && \mathbb{L}^{(4)1} = l_2 l_4, \\ &\mathbb{L}^{(2)5} = \frac{1}{7} l_5(-l_1 + 6l_2 - 2l_4 + l_5), && \mathbb{L}^{(2)6} = \frac{1}{2} l_4(l_3+l_4), \\ &\mathbb{L}^{(2)7} = l_3 l_5, && \mathbb{L}^{(2)8} = \frac{1}{7} l_5(-2l_1 - 2l_2 + 3l_4 + 2l_5). \end{aligned} \tag{A.72}$$

We determine $H^{(3,3)}(\widehat{Y})$ by duality to the canonical basis of $H^{(1,1)}(\widehat{Y})$ by the basis choice of $\mathcal{R}^{(3)}$ given as

$$\begin{aligned} &\mathcal{R}^{(3)}_1 = \theta_1\theta_2\theta_4, && \mathcal{R}^{(3)}_2 = -2\theta_1\theta_2\theta_4 + \theta_1\theta_2\theta_5, \\ &\mathcal{R}^{(3)}_3 = -\theta_1\theta_2\theta_5 + \theta_2\theta_4\theta_5 - \theta_3\theta_4\theta_5, && \mathcal{R}^{(3)}_4 = -\theta_1\theta_2\theta_4 + \theta_1\theta_4\theta_5 - \theta_2\theta_4\theta_5 + \theta_3\theta_4\theta_5, \\ &\mathcal{R}^{(3)}_5 = -\theta_1\theta_2\theta_4 - \theta_1\theta_4\theta_5 + \theta_2\theta_4\theta_5. \end{aligned} \tag{A.73}$$

Our choice for a basis of $H^{(4,4)}(\widehat{Y})$ is given by

$$\mathcal{R}^{(4)} = \frac{1}{106}\, \mathcal{C}_0|_{J_i \mapsto \theta_i}\,. \tag{A.74}$$

We fix the integral basis of $H^{2,2}(\widehat{Y})$ by the requirement of recovering the threefold periods from the fourfold ones. We readily identify the Kähler classes of the threefold $\widehat{X}$ among the fourfold classes as

$$J_2 + J_3 \leftrightarrow J_1(\widehat{X}), \quad J_5 \leftrightarrow J_2(\widehat{X}), \quad J_4 \leftrightarrow J_3(\widehat{X}) \tag{A.75}$$

which matches the threefold intersections by identifying $J_1 \sim \widehat{X}$ in the fourfold intersections (A.68). Then, we calculate the classical terms of the threefold periods to be

$$\mathbb{L}_1(X) = \tilde{l}_2(\tilde{l}_2 + \tilde{l}_3), \quad \mathbb{L}_2(X) = \frac{1}{2}(2\tilde{l}_2 + \tilde{l}_3)(2\tilde{l}_1 + 4\tilde{l}_2 + \tilde{l}_3), \quad \mathbb{L}_3(X) = \frac{1}{2} l_2(2l_1 + 3l_2 + 2l_3). \tag{A.76}$$

On the fourfold Y we determine the periods that match this leading logarithmic structure. They are given by

$$\begin{aligned} &\mathbb{L}^{(2)}_1 = X_0 l_5(l_4 + l_5), && \mathbb{L}^{(2)}_2 = \frac{1}{2} X_0(l_4 + 2l_5)(2(l_2 + l_3) + l_4 + 4l_5), \\ &\mathbb{L}^{(2)}_3 = \frac{1}{2} X_0 l_5(2(l_2 + l_3) + 2l_4 + 3l_5) \end{aligned} \tag{A.77}$$

and immediately coincide with the threefold result using eq. (A.75). It can be shown explicitly that the instanton series contained in the corresponding full solution matches the series on the threefold as well. The threefold invariants n_{d_1,d_2,d_3} are obtained as n_{0,d_1,d_1,d_3,d_2} from the fourfold invariants. To these solutions we associate using eq. (5.111) ring elements $\mathcal{R}^2_\alpha$ for $\alpha = 1,3,2$ that we complete to a new basis as

$$\begin{aligned}
\widetilde{\mathcal{R}}^{(2)}_1 &= \frac{1}{4}(\theta_1+\theta_2)(\theta_2+\theta_3), & \widetilde{\mathcal{R}}^{(2)}_2 &= \theta_1\theta_4,\\
\widetilde{\mathcal{R}}^{(2)}_3 &= \frac{1}{7}\theta_5(6\theta_1-\theta_2-2\theta_4+\theta_5), & \widetilde{\mathcal{R}}^{(2)}_4 &= \theta_2\theta_4,\\
\widetilde{\mathcal{R}}^{(2)}_5 &= \frac{1}{7}\theta_5(-\theta_1+6\theta_2-2\theta_4+\theta_5), & \widetilde{\mathcal{R}}^{(2)}_6 &= \frac{1}{2}\theta_4(\theta_3+\theta_4),\\
\widetilde{\mathcal{R}}^{(2)}_7 &= \theta_3\theta_5, & \widetilde{\mathcal{R}}^{(2)}_8 &= \frac{1}{7}\theta_5(-2\theta_1-2\theta_2+3\theta_4+2\theta_5)
\end{aligned} \tag{A.78}$$

where we note that the basis of dual solutions and the new ring basis coincide by the identification $l_i \leftrightarrow \theta_i$. The integral basis elements read

$$\hat{\gamma}^{(2)}_1 = \widetilde{\mathcal{R}}^{(2)}_1\Omega_Y\Big|_{\underline{z}=0}, \quad \hat{\gamma}^{(2)}_2 = \widetilde{\mathcal{R}}^{(2)}_2\Omega_Y\Big|_{\underline{z}=0}, \quad \hat{\gamma}^{(2)}_3 = \widetilde{\mathcal{R}}^{(2)}_3\Omega_Y\Big|_{\underline{z}=0} \tag{A.79}$$

such that we obtain the full solution with the above leading parts $\mathbb{L}^{(2)}_\alpha$. The leading behavior of the periods $\Pi^{(2)\,\alpha}$ is then given as

$$\mathbb{L}^{(2)\,1} = X_0(l_1+l_2)(l_2+l_3), \quad \mathbb{L}^{(2)\,2} = X_0 l_1 l_4, \quad \mathbb{L}^{(2)\,3} = X_0 l_5(l_1+l_5). \tag{A.80}$$

We conclude by determining the flux element $\hat{\gamma}$ in $H^{2,2}_H(Y)$ reproducing the disk invariants in phase II of ref. [33] where the local geometry K_{F_1} is considered. Firstly, we identify $\ell^{(2)}$ of the toric data in eq. (A.68) as the vector encoding the brane physics. Therefore, we expect the fourfold invariants $n_{0,m+d_1,d_1,d_2,0}$ to coincide with the disk invariants what can be checked in a direct calculation. The ring element yielding this result reads

$$\hat{\gamma} = \mathcal{R}^{(2)}_4 \tag{A.81}$$

where the free coefficients in front of the other ring elements were chosen to zero. The leading logarithmic parts of the period and of the solution $\Pi^{(2)}_\gamma = W_{\mathrm{D7}}$ respectively read

$$\mathbb{L}^{(2)}_\gamma = X_0 l_5(l_1+l_2+l_3+l_4+2l_5), \quad \mathbb{L}^{(2)\,\gamma} = X_0 l_2 l_4. \tag{A.82}$$

Bibliography

[1] W. Lerche, C. Vafa, and N. P. Warner, "Chiral Rings in N=2 Superconformal Theories," *Nucl. Phys.* **B324** (1989) 427.

[2] P. Candelas and X. C. de la Ossa, "Moduli space of Calabi-Yau manifolds,". Prepared for XIII International School of Theoretical Physics: The Standard Model and Beyond, Szczyrk, Poland, 19-26 Sep 1989.

[3] P. Candelas, X. C. De la Ossa, P. S. Green, and L. Parkes, "An Exactly soluble superconformal theory from a mirror pair of Calabi-Yau manifolds," *Phys. Lett.* **B258** (1991) 118–126.

[4] P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes, "A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory," *Nucl. Phys.* **B359** (1991) 21–74.

[5] E. Witten, "Topological Sigma Models," *Commun. Math. Phys.* **118** (1988) 411.

[6] E. Witten, "Topological Quantum Field Theory," *Commun. Math. Phys.* **117** (1988) 353.

[7] E. Witten, "Mirror manifolds and topological field theory," in *Essays on Mirror Manifolds*, pp. 121–160. 1991. `arXiv:hep-th/9112056`.

[8] P. Candelas, X. De La Ossa, A. Font, S. H. Katz, and D. R. Morrison, "Mirror symmetry for two parameter models. I," *Nucl. Phys.* **B416** (1994) 481–538, `arXiv:hep-th/9308083`.

[9] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau, "Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces," *Commun. Math. Phys.* **167** (1995) 301–350, `hep-th/9308122`.

[10] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, "The Monomial divisor mirror map," `arXiv:alg-geom/9309007`.

[11] M. Kontsevich and Y. Manin, "Gromov-Witten classes, quantum cohomology, and enumerative geometry," *Commun. Math. Phys.* **164** (1994) 525–562, `hep-th/9402147`.

[12] P. Candelas, A. Font, S. H. Katz, and D. R. Morrison, "Mirror symmetry for two parameter models. 2," *Nucl. Phys.* **B429** (1994) 626–674, `arXiv:hep-th/9403187`.

[13] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau, "Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces," *Nucl. Phys.* **B433** (1995) 501–554, `hep-th/9406055`.

[14] D. R. Morrison and M. Ronen Plesser, "Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties," *Nucl. Phys.* **B440** (1995) 279–354, `arXiv:hep-th/9412236`.

[15] P. Berglund, S. H. Katz, and A. Klemm, "Mirror symmetry and the moduli space for generic hypersurfaces in toric varieties," *Nucl. Phys.* **B456** (1995) 153–204, `arXiv:hep-th/9506091`.

[16] A. Givental, "Equivariant Gromov-Witten invariants," *International Mathematics Research Notices* **1996** (1996) 613–663.

[17] A. Givental, "A Mirror Theorem for Toric Complete Intersections," in *Topological field theory, primitive forms and related topics.* Birkhäuser, 1998. `alg-geom/9701016`.

[18] B. H. Lian, K. F. Liu, and S.-T. Yau, "Mirror principle. 1," *Asian J. Math.* **1** (1997) 729–763.

[19] K. Hori and C. Vafa, "Mirror symmetry," `arXiv:hep-th/0002222`.

[20] C. Voisin, *Mirror Symmetry.* American Mathematical Society, 1999.

[21] D. A. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry.* American Mathematical Society, 1999.

[22] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, and R. Vakil, *Mirror Symmetry.* American Mathematical Society, 2003.

[23] M. Gross, D. Huybrechts, and D. Joyce, *Calabi-Yau Manifolds and Related Geometries.* Springer-Verlag, 2003.

[24] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," *Commun. Math. Phys.* **165** (1994) 311–428, `arXiv:hep-th/9309140`.

[25] E. Witten, "Quantum background independence in string theory," `arXiv:hep-th/9306122`.

[26] M.-x. Huang, A. Klemm, and S. Quackenbush, "Topological String Theory on Compact Calabi-Yau: Modularity and Boundary Conditions," *Lect. Notes Phys.* **757** (2009) 45–102, `arXiv:hep-th/0612125`.

[27] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, `arXiv:hep-th/9510017`.

[28] M. Kontsevich, "Homological Algebra of Mirror Symmetry," in *Proceedings of the International Congress of Mathematicians*, pp. 335–368. Birkhäuser, 1995. `alg-geom/9411018`.

[29] A. Polishchuk and E. Zaslow, "Categorical mirror symmetry: The Elliptic curve," *Adv. Theor. Math. Phys.* **2** (1998) 443–470, `arXiv:math/9801119`.

[30] P. Seidel, "Homological mirror symmetry for the quartic surface," `math/0310414`.

[31] C. Vafa, "Evidence for F-Theory," *Nucl. Phys.* **B469** (1996) 403–418, `arXiv:hep-th/9602022`.

[32] M. Aganagic and C. Vafa, "Mirror symmetry, D-branes and counting holomorphic discs," `arXiv:hep-th/0012041`.

[33] M. Aganagic, A. Klemm, and C. Vafa, "Disk instantons, mirror symmetry and the duality web," *Z. Naturforsch.* **A57** (2002) 1–28, `arXiv:hep-th/0105045`.

[34] W. Lerche, P. Mayr, and N. Warner, "Holomorphic N = 1 special geometry of open-closed type II strings," `arXiv:hep-th/0207259`.

[35] W. Lerche, P. Mayr, and N. Warner, "N = 1 special geometry, mixed Hodge variations and toric geometry," `hep-th/0208039`.

[36] W. Lerche and P. Mayr, "On N = 1 mirror symmetry for open type II strings," `arXiv:hep-th/0111113`.

[37] P. Mayr, "N = 1 mirror symmetry and open/closed string duality," *Adv. Theor. Math. Phys.* **5** (2002) 213–242, `arXiv:hep-th/0108229`.

[38] M. Aganagic and C. Beem, "The Geometry of D-Brane Superpotentials," `arXiv:0909.2245 [hep-th]`.

[39] J. Walcher, "Opening mirror symmetry on the quintic," *Commun. Math. Phys.* **276** (2007) 671–689, `arXiv:hep-th/0605162`.

[40] D. R. Morrison and J. Walcher, "D-branes and Normal Functions," `arXiv:0709.4028 [hep-th]`.

[41] D. Krefl and J. Walcher, "Real Mirror Symmetry for One-parameter Hypersurfaces," *JHEP* **09** (2008) 031, `arXiv:0805.0792 [hep-th]`.

[42] J. Knapp and E. Scheidegger, "Towards Open String Mirror Symmetry for One-Parameter Calabi-Yau Hypersurfaces," `arXiv:0805.1013 [hep-th]`.

[43] J. Knapp and E. Scheidegger, "Matrix Factorizations, Massey Products and F-Terms for Two-Parameter Calabi-Yau Hypersurfaces," `arXiv:0812.2429 [hep-th]`.

[44] J. Walcher, "Calculations for Mirror Symmetry with D-branes," *JHEP* **09** (2009) 129, `arXiv:0904.4905 [hep-th]`.

[45] H. Jockers and M. Soroush, "Effective superpotentials for compact D5-brane Calabi-Yau geometries," *Commun. Math. Phys.* **290** (2009) 249–290, `arXiv:0808.0761 [hep-th]`.

[46] M. Alim, M. Hecht, P. Mayr, and A. Mertens, "Mirror Symmetry for Toric Branes on Compact Hypersurfaces," *JHEP* **09** (2009) 126, `arXiv:0901.2937 [hep-th]`.

[47] H. Jockers and M. Soroush, "Relative periods and open-string integer invariants for a compact Calabi-Yau hypersurface," *Nucl. Phys.* **B821** (2009) 535–552, arXiv:0904.4674 [hep-th].

[48] M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens, and M. Soroush, "Hints for Off-Shell Mirror Symmetry in type II/F-theory Compactifications," arXiv:0909.1842 [hep-th].

[49] S. Li, B. H. Lian, and S.-T. Yau, "Picard-Fuchs Equations for Relative Periods and Abel-Jacobi Map for Calabi-Yau Hypersurfaces," arXiv:0910.4215 [math.AG].

[50] M. Baumgartl, I. Brunner, and M. R. Gaberdiel, "D-brane superpotentials and RG flows on the quintic," *JHEP* **07** (2007) 061, arXiv:0704.2666 [hep-th].

[51] M. Baumgartl and S. Wood, "Moduli Webs and Superpotentials for Five-Branes," *JHEP* **06** (2009) 052, arXiv:0812.3397 [hep-th].

[52] T. W. Grimm, T.-W. Ha, A. Klemm, and D. Klevers, "The D5-brane effective action and superpotential in N=1 compactifications," *Nucl. Phys.* **B816** (2009) 139–184, arXiv:0811.2996 [hep-th].

[53] T. W. Grimm, T.-W. Ha, A. Klemm, and D. Klevers, "Computing Brane and Flux Superpotentials in F-theory Compactifications," *JHEP* **04** (2010) 015, arXiv:0909.2025 [hep-th].

[54] T. W. Grimm, T.-W. Ha, A. Klemm, and D. Klevers, "Five-Brane Superpotentials and Heterotic/F-theory Duality," arXiv:0912.3250 [hep-th]. Accepted for publication in *Nucl. Phys.*

[55] M. A. Shifman and A. I. Vainshtein, "Solution of the Anomaly Puzzle in SUSY Gauge Theories and the Wilson Operator Expansion," *Nucl. Phys.* **B277** (1986) 456.

[56] H. P. Nilles, "The Role Of Classical Symmetries In The Low-Energy Limit Of Superstring Theories," *Phys. Lett.* **B180** (1986) 240.

[57] M. T. Grisaru, W. Siegel, and M. Rocek, "Improved Methods for Supergraphs," *Nucl. Phys.* **B159** (1979) 429.

[58] N. Seiberg, "Naturalness Versus Supersymmetric Non-renormalization Theorems," *Phys. Lett.* **B318** (1993) 469–475, arXiv:hep-ph/9309335.

[59] S. Weinberg, *The Quantum Theory of Fields*, vol. III. Cambridge University Press, 2000.

[60] J. Terning, *Modern Supersymmetry*. Oxford University Press, 2005.

[61] M. Dine, *Supersymmetry and String Theory*. Cambridge University Press, 2007.

[62] M. Dine and N. Seiberg, "Nonrenormalization Theorems in Superstring Theory," *Phys. Rev. Lett.* **57** (1986) 2625.

[63] E. Witten, "Branes and the dynamics of QCD," *Nucl. Phys.* **B507** (1997) 658–690, arXiv:hep-th/9706109.

[64] E. Witten, "Chern-Simons gauge theory as a string theory," *Prog. Math.* **133** (1995) 637–678, `arXiv:hep-th/9207094`.

[65] R. Hartshorne, *Algebraic Geometry*. Springer-Verlag, 1977.

[66] S. Kachru, S. H. Katz, A. E. Lawrence, and J. McGreevy, "Open string instantons and superpotentials," *Phys. Rev.* **D62** (2000) 026001, `arXiv:hep-th/9912151`.

[67] G. Tian, "Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and its Peterson-Weil Metric," in *Mathematical Aspects of String Theory*, S. T. Yau, ed., pp. 629–646. World Scientific, 1986.

[68] P. A. Griffiths, "A Theorem Concerning the Differential Equations Satisfied by Normal Functions Associated to Algebraic Cycles," *American Jounal of Mathematics* **101** (1979) no. 1, 94–131.

[69] M. L. Green, *Algebraic Cycles and Hodge Theory*, ch. Infinitesimal Methods in Hodge Theory. Springer-Verlag, 1993.

[70] C. Voisin, *Hodge Theory and Complex Algebraic Geometry*, vol. I. Cambridge University Press, 2002.

[71] C. Voisin, *Hodge Theory and Complex Algebraic Geometry*, vol. II. Cambridge University Press, 2002.

[72] K. Kodaira, *Complex Manifolds and Deformations of Complex Structures*. Springer-Verlag, 1986.

[73] P. A. Griffiths, "Periods of Integrals on Algebraic Manifolds, II," *American Jounal of Mathematics* **90** (1968) no. 3, 805–865.

[74] S. Gukov, C. Vafa, and E. Witten, "CFT's from Calabi-Yau four-folds," *Nucl. Phys.* **B584** (2000) 69–108, `arXiv:hep-th/9906070`.

[75] L. J. Dixon, V. Kaplunovsky, and J. Louis, "On Effective Field Theories Describing (2,2) Vacua of the Heterotic String," *Nucl. Phys.* **B329** (1990) 27–82.

[76] A. Strominger, "Special Geometry," *Commun. Math. Phys.* **133** (1990) 163–180.

[77] D. S. Freed, "Special Kaehler manifolds," *Commun. Math. Phys.* **203** (1999) 31–52, `arXiv:hep-th/9712042`.

[78] L. Martucci, "D-branes on general N = 1 backgrounds: Superpotentials and D-terms," *JHEP* **06** (2006) 033, `arXiv:hep-th/0602129`.

[79] F. Denef, "Les Houches Lectures on Constructing String Vacua," `arXiv:0803.1194 [hep-th]`.

[80] M. Haack and J. Louis, "M-theory compactified on Calabi-Yau fourfolds with background flux," *Phys. Lett.* **B507** (2001) 296–304, `arXiv:hep-th/0103068`.

[81] E. Witten, "On flux quantization in M-theory and the effective action," *J. Geom. Phys.* **22** (1997) 1–13, `arXiv:hep-th/9609122`.

[82] B. R. Greene, D. R. Morrison, and M. R. Plesser, "Mirror manifolds in higher dimension," *Commun. Math. Phys.* **173** (1995) 559–598, `arXiv:hep-th/9402119`.

[83] R. Donagi, A. Lukas, B. A. Ovrut, and D. Waldram, "Holomorphic vector bundles and non-perturbative vacua in M-theory," *JHEP* **06** (1999) 034, `arXiv:hep-th/9901009`.

[84] R. Friedman, J. Morgan, and E. Witten, "Vector bundles and F theory," *Commun. Math. Phys.* **187** (1997) 679–743, `arXiv:hep-th/9701162`.

[85] R. Donagi, B. A. Ovrut, and D. Waldram, "Moduli spaces of fivebranes on elliptic Calabi-Yau threefolds," *JHEP* **11** (1999) 030, `arXiv:hep-th/9904054`.

[86] E. Witten, "Small Instantons in String Theory," *Nucl. Phys.* **B460** (1996) 541–559, `arXiv:hep-th/9511030`.

[87] O. J. Ganor and A. Hanany, "Small E_8 Instantons and Tensionless Non-critical Strings," *Nucl. Phys.* **B474** (1996) 122–140, `arXiv:hep-th/9602120`.

[88] E. Witten, "New Issues in Manifolds of SU(3) Holonomy," *Nucl. Phys.* **B268** (1986) 79.

[89] P. S. Aspinwall, "K3 surfaces and string duality," `arXiv:hep-th/9611137`.

[90] E. Buchbinder, R. Donagi, and B. A. Ovrut, "Vector bundle moduli and small instanton transitions," *JHEP* **06** (2002) 054, `arXiv:hep-th/0202084`.

[91] B. Andreas and G. Curio, "Three-branes and five-branes in N = 1 dual string pairs," *Phys. Lett.* **B417** (1998) 41–44, `arXiv:hep-th/9706093`.

[92] G. Rajesh, "Toric geometry and F-theory/heterotic duality in four dimensions," *JHEP* **12** (1998) 018, `arXiv:hep-th/9811240`.

[93] P. Berglund and P. Mayr, "Heterotic string/F-theory duality from mirror symmetry," *Adv. Theor. Math. Phys.* **2** (1999) 1307–1372, `arXiv:hep-th/9811217`.

[94] D. Tong, "TASI lectures on solitons," `arXiv:hep-th/0509216`.

[95] H. Jockers, P. Mayr, and J. Walcher, "On N=1 4d Effective Couplings for F-theory and Heterotic Vacua," `arXiv:0912.3265 [hep-th]`.

[96] A. Strominger, "Superstrings with Torsion," *Nucl. Phys.* **B274** (1986) 253.

[97] S. Gukov, "Solitons, superpotentials and calibrations," *Nucl. Phys.* **B574** (2000) 169–188, `arXiv:hep-th/9911011`.

[98] K. Behrndt and S. Gukov, "Domain walls and superpotentials from M theory on Calabi-Yau three-folds," *Nucl. Phys.* **B580** (2000) 225–242, `arXiv:hep-th/0001082`.

[99] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, "Topological amplitudes in string theory," *Nucl. Phys.* **B413** (1994) 162–184, `arXiv:hep-th/9307158`.

[100] H. Ooguri and C. Vafa, "Knot invariants and topological strings," *Nucl. Phys.* **B577** (2000) 419–438, `arXiv:hep-th/9912123`.

[101] M. Marino, “Les Houches lectures on matrix models and topological strings,” hep-th/0410165.

[102] A. Neitzke and C. Vafa, “Topological strings and their physical applications,” hep-th/0410178.

[103] A. Klemm, “Topological string theory on Calabi-Yau threefolds,” *PoS* **RTN2005** (2005) 002.

[104] M. Vonk, “A mini-course on topological strings,” hep-th/0504147.

[105] M. Kontsevich, “Enumeration of rational curves via Torus actions,” arXiv:hep-th/9405035.

[106] P. S. Aspinwall and D. R. Morrison, “Topological field theory and rational curves,” *Commun. Math. Phys.* **151** (1993) 245–262, arXiv:hep-th/9110048.

[107] A. Klemm and R. Pandharipande, “Enumerative geometry of Calabi-Yau 4-folds,” *Commun. Math. Phys.* **281** (2008) 621–653, arXiv:math/0702189.

[108] R. Gopakumar and C. Vafa, “M-theory and topological strings. II,” arXiv:hep-th/9812127.

[109] C.-C. M. Liu, “Moduli of J-Holomorphic Curves with Lagrangian Boundary Conditions and Open Gromov-Witten Invariants for an S^1-Equivariant Pair,” math/0210257.

[110] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, “The topological vertex,” *Commun. Math. Phys.* **254** (2005) 425–478, arXiv:hep-th/0305132.

[111] D. Krefl and J. Walcher, “The Real Topological String on a local Calabi-Yau,” arXiv:0902.0616 [hep-th].

[112] D. Krefl, S. Pasquetti, and J. Walcher, “The Real Topological Vertex at Work,” *Nucl. Phys.* **B833** (2010) 153–198, arXiv:0909.1324 [hep-th].

[113] V. Bouchard, A. Klemm, M. Marino, and S. Pasquetti, “Topological open strings on orbifolds,” arXiv:0807.0597 [hep-th].

[114] R. Dijkgraaf and C. Vafa, “Matrix models, topological strings, and supersymmetric gauge theories,” *Nucl. Phys.* **B644** (2002) 3–20, arXiv:hep-th/0206255.

[115] M. Marino, “Open string amplitudes and large order behavior in topological string theory,” *JHEP* **03** (2008) 060, arXiv:hep-th/0612127.

[116] V. Bouchard, A. Klemm, M. Marino, and S. Pasquetti, “Remodeling the B-model,” *Commun. Math. Phys.* **287** (2009) 117–178, arXiv:0709.1453 [hep-th].

[117] E. Witten, “Phases of N = 2 theories in two dimensions,” *Nucl. Phys.* **B403** (1993) 159–222, arXiv:hep-th/9301042.

[118] N. C. Leung and C. Vafa, “Branes and toric geometry,” *Adv. Theor. Math. Phys.* **2** (1998) 91–118, arXiv:hep-th/9711013.

[119] S. H. Katz, A. Klemm, and C. Vafa, "Geometric engineering of quantum field theories," *Nucl. Phys.* **B497** (1997) 173–195, arXiv:hep-th/9609239.

[120] A. Strominger, S.-T. Yau, and E. Zaslow, "Mirror symmetry is T-duality," *Nucl. Phys.* **B479** (1996) 243–259, arXiv:hep-th/9606040.

[121] A. Bilal and S. Metzger, "Special geometry of local Calabi-Yau manifolds and superpotentials from holomorphic matrix models," *JHEP* **08** (2005) 097, arXiv:hep-th/0503173.

[122] B. Craps, F. Roose, W. Troost, and A. Van Proeyen, "What is special Kaehler geometry?," *Nucl. Phys.* **B503** (1997) 565–613, arXiv:hep-th/9703082.

[123] E. H. Spanier, *Algebraic Topology.* Springer-Verlag, 1966.

[124] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology.* Springer-Verlag, 1982.

[125] A. Hatcher, *Algebraic Topology.* Cambridge University Press, 2002.

[126] V. S. Kulikov, *Mixed Hodge structures and singularities.* Cambridge University Press, 1998.

[127] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge Structures.* Springer-Verlag, 2008.

[128] S. Bloch and D. Kreimer, "Mixed Hodge Structures and Renormalization in Physics," *Commun. Num. Theor. Phys.* **2** (2008) 637–718, arXiv:0804.4399 [hep-th].

[129] P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* John Wiley and Sons, Inc., 1978.

[130] J. McCleary, *A User's Guide to Spectral Sequences.* Cambridge University Press, 1985.

[131] P. Deligne, "Théorie de Hodge : II," *Publications Mathématiques de l'IHÉS* **30** (1971) 5–57.

[132] R. Blumenhagen, M. Cvetic, P. Langacker, and G. Shiu, "Toward realistic intersecting D-brane models," *Ann. Rev. Nucl. Part. Sci.* **55** (2005) 71–139, arXiv:hep-th/0502005.

[133] P. A. Griffiths, "Periods of Integrals on Algebraic Manifolds, I," *American Jounal of Mathematics* **90** (1968) no. 3, 568–626.

[134] E. Witten, "Non-Perturbative Superpotentials In String Theory," *Nucl. Phys.* **B474** (1996) 343–360, arXiv:hep-th/9604030.

[135] J. Steenbrink and S. Zucker, "Variation of mixed Hodge structrure. I," *Invent. math.* **80** (1985) 489–542.

[136] S. Zucker, "Variation of mixed Hodge structrure. II," *Invent. math.* **80** (1985) 543–565.

[137] A. Libgober and J. Teitelbaum, "Lines on Calabi-Yau complete intersections, mirror symmetry, and Picard-Fuchs equations," arXiv:alg-geom/9301001.

[138] "Macaulay 2,". http://www.math.uiuc.edu/Macaulay2/.

[139] A. Sen, "F-theory and Orientifolds," *Nucl. Phys.* **B475** (1996) 562–578, arXiv:hep-th/9605150.

[140] A. Sen, "Orientifold limit of F-theory vacua," *Phys. Rev.* **D55** (1997) 7345–7349, arXiv:hep-th/9702165.

[141] A. Sen, "Orientifold limit of F-theory vacua," *Nucl. Phys. Proc. Suppl.* **68** (1998) 92–98, arXiv:hep-th/9709159.

[142] C. Beasley, J. J. Heckman, and C. Vafa, "GUTs and Exceptional Branes in F-theory - I," *JHEP* **01** (2009) 058, arXiv:0802.3391 [hep-th].

[143] R. Donagi and M. Wijnholt, "Model Building with F-Theory," arXiv:0802.2969 [hep-th].

[144] C. Beasley, J. J. Heckman, and C. Vafa, "GUTs and Exceptional Branes in F-theory - II: Experimental Predictions," *JHEP* **01** (2009) 059, arXiv:0806.0102 [hep-th].

[145] R. Donagi and M. Wijnholt, "Breaking GUT Groups in F-Theory," arXiv:0808.2223 [hep-th].

[146] B. Andreas and G. Curio, "From Local to Global in F-Theory Model Building," arXiv:0902.4143 [hep-th].

[147] J. Marsano, N. Saulina, and S. Schafer-Nameki, "F-theory Compactifications for Supersymmetric GUTs," *JHEP* **08** (2009) 030, arXiv:0904.3932 [hep-th].

[148] J. Marsano, N. Saulina, and S. Schafer-Nameki, "Monodromies, Fluxes, and Compact Three-Generation F-theory GUTs," *JHEP* **08** (2009) 046, arXiv:0906.4672 [hep-th].

[149] R. Blumenhagen, T. W. Grimm, B. Jurke, and T. Weigand, "Global F-theory GUTs," *Nucl. Phys.* **B829** (2010) 325–369, arXiv:0908.1784 [hep-th].

[150] J. Marsano, N. Saulina, and S. Schafer-Nameki, "Compact F-theory GUTs with $U(1)_{PQ}$," arXiv:0912.0272 [hep-th].

[151] T. W. Grimm, S. Krause, and T. Weigand, "F-Theory GUT Vacua on Compact Calabi-Yau Fourfolds," arXiv:0912.3524 [hep-th].

[152] A. Kas, "Weierstrass Normal Forms and Invariants of Elliptic Surfaces," *Transactions of the American Mathematical Society* **225** (1977) 259–266.

[153] M. Bershadsky *et al.*, "Geometric singularities and enhanced gauge symmetries," *Nucl. Phys.* **B481** (1996) 215–252, arXiv:hep-th/9605200.

[154] A. P. Braun, A. Hebecker, and H. Triendl, "D7-Brane Motion from M-Theory Cycles and Obstructions in the Weak Coupling Limit," *Nucl. Phys.* **B800** (2008) 298–329, arXiv:0801.2163 [hep-th].

[155] A. Collinucci, F. Denef, and M. Esole, "D-brane Deconstructions in IIB Orientifolds," *JHEP* **02** (2009) 005, arXiv:0805.1573 [hep-th].

[156] A. P. Braun, S. Gerigk, A. Hebecker, and H. Triendl, "D7-Brane Moduli vs. F-Theory Cycles in Elliptically Fibred Threefolds," arXiv:0912.1596 [hep-th].

[157] A. P. Braun, *F-Theory and the Landscape of Intersecting D7-Branes.* PhD thesis, University of Heidelberg, 2010. arXiv:1003.4867 [hep-th].

[158] V. V. Batyrev, "Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties," *J. Alg. Geom.* **3** (1994) 493–545, alg-geom/9310003.

[159] P. Candelas and A. Font, "Duality between the webs of heterotic and type II vacua," *Nucl. Phys.* **B511** (1998) 295–325, arXiv:hep-th/9603170.

[160] P. Candelas, E. Perevalov, and G. Rajesh, "Toric geometry and enhanced gauge symmetry of F- theory/heterotic vacua," *Nucl. Phys.* **B507** (1997) 445–474, arXiv:hep-th/9704097.

[161] W. Fulton, *Introduction to Toric Varieties.* Princeton University Press, 1993.

[162] D. A. Cox, "The Homogeneous Coordinate Ring of a Toric Variety, Revised Version," alg-geom/9210008.

[163] T.-W. Ha, "Toric Geometry and String Model Building," diploma thesis, University of Heidelberg, 2007. http://www.taewon.de/files/diploma.pdf.

[164] A. Klemm, B. Lian, S. S. Roan, and S.-T. Yau, "Calabi-Yau fourfolds for M- and F-theory compactifications," *Nucl. Phys.* **B518** (1998) 515–574, arXiv:hep-th/9701023.

[165] P. Mayr, "Mirror symmetry, N = 1 superpotentials and tensionless strings on Calabi-Yau four-folds," *Nucl. Phys.* **B494** (1997) 489–545, arXiv:hep-th/9610162.

[166] A. C. Avram, M. Kreuzer, M. Mandelberg, and H. Skarke, "Searching for K3 fibrations," *Nucl. Phys.* **B494** (1997) 567–589, arXiv:hep-th/9610154.

[167] P. Mayr, "Phases of supersymmetric D-branes on Kaehler manifolds and the McKay correspondence," *JHEP* **01** (2001) 018, arXiv:hep-th/0010223.

[168] M. B. Green, J. A. Harvey, and G. W. Moore, "I-brane inflow and anomalous couplings on D-branes," *Class. Quant. Grav.* **14** (1997) 47–52, arXiv:hep-th/9605033.

[169] R. Minasian and G. W. Moore, "K-theory and Ramond-Ramond charge," *JHEP* **11** (1997) 002, arXiv:hep-th/9710230.

[170] Y.-K. E. Cheung and Z. Yin, "Anomalies, branes, and currents," *Nucl. Phys.* **B517** (1998) 69–91, arXiv:hep-th/9710206.

[171] "Schubert,". http://folk.uib.no/nmasr/schubert/.

[172] B. Haghighat, A. Klemm, and M. Rauch, "Integrability of the holomorphic anomaly equations," *JHEP* **10** (2008) 097, arXiv:0809.1674 [hep-th].

[173] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – I," *Nucl. Phys.* **B473** (1996) 74–92, arXiv:hep-th/9602114.

[174] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – II," *Nucl. Phys.* **B476** (1996) 437–469, `arXiv:hep-th/9603161`.

[175] M. Bershadsky, A. Johansen, T. Pantev, and V. Sadov, "On four-dimensional compactifications of F-theory," *Nucl. Phys.* **B505** (1997) 165–201, `arXiv:hep-th/9701165`.

[176] B. Andreas, G. Curio, and D. Lust, "N = 1 dual string pairs and their massless spectra," *Nucl. Phys.* **B507** (1997) 175–196, `arXiv:hep-th/9705174`.

[177] G. Curio and R. Y. Donagi, "Moduli in N = 1 heterotic/F-theory duality," *Nucl. Phys.* **B518** (1998) 603–631, `arXiv:hep-th/9801057`.

[178] B. Andreas, "N = 1 heterotic/F-theory duality," *Fortsch. Phys.* **47** (1999) 587–642, `arXiv:hep-th/9808159`.

[179] B. Andreas, G. Curio, and A. Klemm, "Towards the standard model spectrum from elliptic Calabi- Yau," *Int. J. Mod. Phys.* **A19** (2004) 1987, `arXiv:hep-th/9903052`.

[180] B. Andreas and G. Curio, "Horizontal and vertical five-branes in heterotic / F- theory duality," *JHEP* **01** (2000) 013, `arXiv:hep-th/9912025`.

[181] M. R. Douglas, "Lectures on D-branes on Calabi-Yau manifolds,". Prepared for ICTP Spring School on Superstrings and Related Matters, Trieste, Italy, 2-10 Apr 2001.

[182] M. R. Douglas, "Dirichlet branes, homological mirror symmetry, and stability," in *Proceedings of the International Congress of Mathematicians: Beijing 2002*, vol. III, pp. 395–408. Higher Education Press, 2003. `arXiv:math/0207021`.

[183] P. S. Aspinwall, "D-branes on Calabi-Yau manifolds," `arXiv:hep-th/0403166`.

[184] T. Bridgeland, "Spaces of stability conditions," in *Proceedings of symposia in pure mathematics*, vol. 80. American Mathematical Society, 2009. `math/0611510`.

[185] M. Kontsevich and Y. Soibelman, "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations," `0811.2435 [math.AG]`.

[186] R. Friedman, J. W. Morgan, and E. Witten, "Vector bundles over elliptic fibrations," `arXiv:alg-geom/9709029`.

[187] D. Huybrechts, *Complex Geometry*. Springer-Verlag, 2004.

[188] R. Donagi, "Taniguchi lecture on principal bundles on elliptic fibrations," `arXiv:hep-th/9802094`.

[189] R. Donagi and M. Wijnholt, "Higgs Bundles and UV Completion in F-Theory," `arXiv:0904.1218 [hep-th]`.

[190] C. Vafa and E. Witten, "Dual string pairs with N = 1 and N = 2 supersymmetry in four dimensions," *Nucl. Phys. Proc. Suppl.* **46** (1996) 225–247, `arXiv:hep-th/9507050`.

[191] P. S. Aspinwall and D. R. Morrison, "Point-like instantons on K3 orbifolds," *Nucl. Phys.* **B503** (1997) 533–564, `arXiv:hep-th/9705104`.

[192] C. Vafa and E. Witten, "A One loop test of string duality," *Nucl. Phys.* **B447** (1995) 261–270, arXiv:hep-th/9505053.

[193] K. Becker and M. Becker, "M-Theory on Eight-Manifolds," *Nucl. Phys.* **B477** (1996) 155–167, arXiv:hep-th/9605053.

[194] S. Sethi, C. Vafa, and E. Witten, "Constraints on low-dimensional string compactifications," *Nucl. Phys.* **B480** (1996) 213–224, arXiv:hep-th/9606122.

[195] K. Dasgupta and S. Mukhi, "A note on low-dimensional string compactifications," *Phys. Lett.* **B398** (1997) 285–290, arXiv:hep-th/9612188.

[196] W. Fulton, *Intersection theory*. Springer-Verlag, 1997.

[197] H. Hayashi, R. Tatar, Y. Toda, T. Watari, and M. Yamazaki, "New Aspects of Heterotic–F Theory Duality," *Nucl. Phys.* **B806** (2009) 224–299, arXiv:0805.1057 [hep-th].

[198] D.-E. Diaconescu and G. Rajesh, "Geometrical aspects of five-branes in heterotic/F-theory duality in four dimensions," *JHEP* **06** (1999) 002, arXiv:hep-th/9903104.

[199] M. R. Douglas, S. H. Katz, and C. Vafa, "Small instantons, del Pezzo surfaces and type I' theory," *Nucl. Phys.* **B497** (1997) 155–172, arXiv:hep-th/9609071.

[200] T. W. Grimm, A. Klemm, and D. Klevers, "Work in progress," arXiv:10XX.XXXX [hep-th].

[201] V. V. Batyrev and L. A. Borisov, "On Calabi-Yau complete intersections in toric varieties," arXiv:alg-geom/9412017.

[202] V. V. Batyrev and E. N. Materov, "Mixed toric residues and Calabi-Yau complete intersections," arXiv:math/0206057.

[203] P. S. Aspinwall, "Aspects of the hypermultiplet moduli space in string duality," *JHEP* **04** (1998) 019, arXiv:hep-th/9802194.

[204] A. Strominger, "Massless black holes and conifolds in string theory," *Nucl. Phys.* **B451** (1995) 96–108, arXiv:hep-th/9504090.

[205] B. R. Greene, D. R. Morrison, and A. Strominger, "Black hole condensation and the unification of string vacua," *Nucl. Phys.* **B451** (1995) 109–120, arXiv:hep-th/9504145.

[206] F. Hirzebruch, *Topological Methods in Algebraic Geometry*. Springer-Verlag, 1978.

[207] P. A. Griffiths, "Some Results on Algebraic Cycles on Algebraic Manifolds," in *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*, pp. 93–191. Oxford University Press, 1969.

[208] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*. Princeton University Press, 1974.