Essays about Option Valuation under Stochastic Interest Rates

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Chapter 1

Introduction

Option pricing under stochastic interest rates is since the seminal work of Vasicek [1977], where he introduced an interest rate model to value plain-vanilla interest rate options, a field of still ongoing research. There are open questions concerning not only the valuation of more complex products but even the choice of the appropriate interest rate model to replicate market prices. This thesis consists of three essays on the valuation of options under stochastic interest rates, where we propose some answers to the mentioned problems.

In the first essay we examine multivariate models where the stochastic process of a log-normally distributed underlying depends on the evolution of correlated interest rate processes. This type of model has been introduced by Amin and Jarrow [1991] for the valuation of exchange rate options but can also be used to model inflation rates or stocks under stochastic interest rates and continuous dividends rates. There the correlation structure can change by constant factor volatilities, which influences not only the values of financial instruments but also their hedge strategies. We derive analytical formulae as well as numerical results on this topic.
In this model class we propose a unified framework for the pricing and hedging of chooser options under stochastic interest rates. Chooser options are exotic derivatives who give the owner the right to enter at their exercise date a call or a put option with the same underlying. The chosen multivariate framework allows to derive closed form solutions of the arbitrage price for different specifications of chooser options such as different strike prices or time to maturities. Furthermore, different hedge strategies like a dynamic hedge and a static superhedge are derived and compared according to their properties.

The second essay deals with the so called convexity correction of swap rates. A convexity correction needs to be computed in the case of constant maturity swaps (CMS) for example. The expectation is taken under a different measure than the assets martingale measure. Then, the expectation is not the forward value but the forward value plus a convexity correction. Several swap rate derivatives for instance CMS spread options can only be valued correctly after correcting for convexity.
One approach suggested in the literature, see Brigo and Mercurio [2007], is to use a Taylor series expansion to gain an analytical approximation but the result is neither a tradeable asset nor can the information of the swaption volatility cube be included to replicate market prices.
Another approach, Hagan [2003], suggests that the convexity correction as the price of a static portfolio of plain-vanilla swaptions. This portfolio approach has the advantage that the volatility cube can be incorporated by using a stochastic or local volatility models, but it is the solution of an integral over an infinite number of strike prices. We propose an algorithm to approximate the replication portfolio with a finite number and therefore a discrete set of strike prices. The accuracy of the method is examined using numerical examples and different valuation models as well as different sets of strike prices.

The modeling of multi-asset options within an interest rate model is the topic of the third essay. There, we consider the joint dynamic of a basket of n-assets with the application to CMS spread options in mind. Therefore we chose models where each asset is modeled as a forward swap rate. In particular we use a Swap Market Model (SMM) introduced by Jamshidian [1999] with deterministic volatility and the SABR model of Hagan et al. [2003] with stochastic volatility.
Using the Markovian Projection methodology introduced by Piterbarg [2006] to quantitative finance we approximate multivariate SMM/SABR dynamic with a univariate SMM/SABR dynamic to price caps and floors in closed form. This enables us to consider not only the asset correlation but, in the case of the SABR model, as well the skew, the cross-skew and the decorrelation in our approximation. If for example, spread options are considered the latter is not possible in alternative approximations.
We illustrate the method by considering the example where the underlyings are two constant maturity swap rates. There we examine the influence of the swaption volatility cube on CMS spread options and compare our approximation formulae to results obtained by Monte Carlo simulation and a copula approach. In the end we find, that despite some harsh approxima-

tions the Markovian Projection is very close to option values and hedge parameters and can be used for volatility and correlation calibration in a high dimensional setting.

Chapter 2

Chooser Options under Stochastic Interest Rates

2.1 Introduction

In the last years there has been a huge uncertainty about the future direction of the global financial markets due to the financial crisis. This is especially true for the US dollar/euro exchange rate since the crisis started in the US financial system. Also the stock markets changed their direction very often during the crisis due to an uncertainty if the crisis is over or not. And due to the huge amount of liquidity the central banks offered to the banking system in order to fight a deflation scenario for 2009, a future rise in inflation is anticipated, as can be seen in the yield curves. One financial instrument designed to insure against uncertainty of future market directions is the chooser option. A chooser option is a contract where the counterparty has the right to enter a call or a put option at a future date.

In the related literature using deterministic volatility to get analytical solutions under a unique martingale measure, exchange rates, inflation indices and stocks are modeled as log-normal assets, see e.g. Black and Scholes [1973] and Garman and Kohlhagen [1983]. In more sophisticated models these processes are correlated to stochastic interest rate processes under a forward risk adjusted measures. Therefore we derive the pricing formulas for chooser options in a framework of an international economy designed to value exchange rate options under stochastic interest rates. This class of models is introduced by Amin and Jarrow [1991] and belongs to the arbitrage free interest rate models defined by Heath et al. [1992]. In this framework the exchange rate, the domestic and the foreign bond prices are modeled as cor-

related lognormal stochastic processes. In this framework the pricing and hedge strategies for a variety of exotic exchange rate options are discussed in detail by Frey and Sommer [1996] where the change of numeraire technique proposed by Geman et al. [1995] is used. The application of this model class for the modeling of inflation indices was proposed by Jarrow and Yildirim [2003] where the exchange rate is interpreted as an inflation index and the foreign bond price as a bond price in the real economy. The application for stocks is straightforward by interpreting the exchange rate as stock and the foreign interest rate as a continuous dividend yield.
In this model class the options volatility function which influences the arbitrage price and the hedge strategy, depends implicitly on the multi-dimensional volatility vectors of the exchange, the domestic and the foreign interest rate with an embedded correlation structure. A change in the correlation structure can rise or even lower the options volatility function by leaving the factor volatilities constant. Numerical results show that this influence is larger on the hedge strategy than on the arbitrage price of a chooser option.

We extend the solutions of chooser options under deterministic interest rates on stocks into a unified framework capable to price different lognormal assets under deterministic or stochastic interest rates. If a threshold at the options maturity date where the put and call prices coincide is known and it can be derived numerically, the chooser option can be decomposed into zero strike compound options. Therefore a semi-analytic solution for chooser options under determinstic interest rates has for the case of stocks been derived by Rubinstein [1991] by using the results of Geske [1979] for compound options. Using this approach, we derive the pricing formulas in an unified framework capable of handling stochastic interest rates and a correlation structure for different specifications of chooser options. The special case of equal strike prices and maturity dates of the underlying options is solved analytically, while the case of different strike prices uses the semi-analytic approach as does the case of different maturity dates where an approximation is needed in the numerical part if stochastic interest rates are considered.

Concerning the risk management of chooser options the hedge strategy is different to that of a plain vanilla call or put option. Since the chooser option is written on both options the delta varies between -1 and +1 in dependence of the moneyness of the underlying options. We compute how a delta neutral position can be achieved in dependence of the strike price, but as a side effect this position must be readjusted relative often due to a high gamma factor. The chooser option can also be hedged by a static or dynamic portfolio of a call and a put option where the static hedge is also a superhedge to the chooser option and known in the market as a straddle. We show that the maximal difference in price is, when the deltas of the chooser option and the straddle coincide but the chooser option is then exposed to a higher gamma risk than the straddle.

The setup of the chapter is as follows. In Section (2.2) we will introduce a model of an international economy that is capable of modeling different lognormal assets under stochastic interest rates. Then a subsection is devoted to the correlation structure that is embedded in the volatility functions and its influence on pricing and hedging. Section (2.3) will derive the arbitrage prices for different specifications of chooser options that are valid for different types of lognormal underlyings. And Section (2.4) discusses the risk management of chooser options and a static superhedge.

2.2 A Model of an International Economy

To model different lognormal assets with stochastic drift components we will use a model class that is known in the literature as a model of an international economy. In this class an exchange rate and a domestic and foreign interest rate market are modeled as stochastic processes with an embedded correlation structure. We apply this class for the case of two countries as introduced by Amin and Jarrow [1991] and restrict the analysis to the so called Gaussian case, i.e. we concentrate on deterministic volatility functions. Therefore we apply the special situation of the Amin and Jarrow framework and model bond prices instead of forward rates as proposed by El Karoui et al. [1992a], which for e.g. has been studied by Frey and Sommer [1996].

Assumption 2.2.1. *Let* $(\Omega, \mathcal{F}, P_d^*)$ *be a probability space and* $\{W_d^*(t)\}$ *be an* $n-$*dimensional standard brownian motion under* P_d^**. Furthermore we consider the natural filtration* $\{\mathcal{F}_t\}_t$ *defined by the brownian motion* $\{W_d^*(t)\}$*, i.e.* $\mathcal{F}_t$ *is equal to the* σ*-algebra generated by the brownian motion up to time t:*

$$\mathcal{F}_{t_0} = \{\emptyset, \Omega\} \quad \textit{and} \quad \mathcal{F}_t = \mathcal{A}(W_d^*(s), s \leq t).$$

The stochastic domestic $B_d(t,\tau)$ *and the foreign* $B_f(t,\tau)$ *bond price processes and the ex-*

change rate $X(t)$ (domestic/foreign currency) are modeled as follows $\forall \tau \in [0,T]$:

$$\begin{aligned} dB_d(t,\tau) &= B_d(t,\tau)\cdot r_d(t)dt + B_d(t,\tau)\cdot \nu_d(t,\tau)dW_d^*(t), \\ dB_f(t,\tau) &= B_f(t,\tau)\cdot [r_f(t) - \sigma_X(t)\cdot \nu_f(t,\tau)]dt + B_f(t,\tau)\cdot \nu_f(t,\tau)dW_d^* \\ dX(t) &= X(t)\cdot [r_d(t) - r_f(t)]dt + X(t)\cdot \sigma_X(t)dW_d^*(t), \end{aligned}$$

where $\nu_i(t,\tau)\ ; i = \{d,f\}$ are the domestic and foreign bond price volatility functions, $\sigma_X(t)$ the exchange rates volatility function and $r_i(t)\ ; i = \{d,f\}$ the domestic and foreign short rate. All volatility functions are assumed to be deterministic $\mathbb{R}^n$-valued functions who fulfill the usual requirements.

The interpretation of P_d^* is that of the domestic martingale measure and the model is arbitrage free and of HJM type as shown by Amin and Jarrow [1991]. Given this extended Black-Scholes framework the implied stochastic process of the domestic and foreign short rates are determined by

$$\begin{aligned} r_d(t) &= -\frac{\partial \ln B_d(t_0,t)}{\partial t} + \int_{t_0}^t \nu_d(u,t)\cdot \frac{\partial \nu_d(u,t)}{\partial t}du - \int_{t_0}^t \frac{\partial \nu_d(u,t)}{\partial t}dW_d^*(u), \\ r_f(t) &= -\frac{\partial \ln B_f(t_0,t)}{\partial t} + \int_{t_0}^t (\nu_f(u,t) + \sigma_X(u))\cdot \frac{\partial \nu_f(u,t)}{\partial t}du - \int_{t_0}^t \frac{\partial \nu_f(u,t)}{\partial t}dW_d^*(u). \end{aligned}$$

Since the volatility functions are assumed to be deterministic, the exchange rate as well as the domestic and foreign zero coupon bond prices are lognormal distributed under the domestic martingale measure. In turn short rates are normal distributed. Furthermore the discounted foreign zero coupon bond is not a martingale under the domestic martingale measure. Instead the domestic value of a foreign zero-coupon bond shares this property, i.e.

$$d(X(t)\cdot B_f(t,\tau)) = X(t)\cdot B_f(t,\tau)\cdot r_d(t)dt + X(t)\cdot B_f(t,\tau)\cdot [\sigma_X(t) + \nu_f(t,\tau)]dW_d^*(t).$$

We have chosen this model since it can not only be used to model exchange rates under stochastic interest rates, but also other lognormal assets under stochastic interest rates. We can e.g. model inflation indices by interpreting the exchange rate as the inflation index, the domestic interest rate as the nominal interest rate and the foreign interest rate as the real interest rate. This interpretation is based on the work of Jarrow and Yildirim [2003] where an international economy model based on HJM is used to value derivatives on inflation indices. This methodology is in the literature also known as the foreign-currency analogy. Another application are equities where we can interpret the foreign interest rate as a stochastic continuous dividend yield. We therefore end up with an extended Black-Scholes model under stochastic interest rates and dividend yields. As a special case even the standard Black-Scholes model is included by setting $n = 1$, $v_d = 0$ and $v_f = 0$.

2.2.1 The Influence of Volatility and Correlation

As defined in the previous section the modeling framework of an international economy consists of correlated processes and this correlation structure has an influence on the arbitrage price and the hedge strategy. If we recall for the sake of completeness the price a plain-vanilla call option on an exchange rate

$$\begin{aligned} Call(X(t_0), K, t_0, T) &= X(t_0) \cdot B_f(t_0, T) \cdot \mathcal{N}(d_1) - K \cdot B_d(t_0, T) \cdot \mathcal{N}(d_2) \\ \text{with } d_{1/2} &:= \frac{\ln\left(\frac{X(t_0)B_f(t_0,T)}{KB_d(t_0,T)}\right) \pm \frac{1}{2}g^2(t_0, T, T)}{g(t_0, T, T)}, \\ g^2(t, T_1, T_2) &:= \int_t^{T_1} ||\sigma_X(u) + \nu_f(u, T_2) - \nu_d(u, T_2)||^2 du \end{aligned}$$

it can be seen that the difference to a model under deterministic interest rates as proposed by Garman and Kohlhagen [1983] reduces to the option volatility function $g^2(t, \tau, T)$. Since an option value is a monotonic increasing function in volatility this influence will have an impact on the arbitrage price. And therefore the robustness of the arbitrage price and the hedging strategy regarding a change in the correlation structure is of concern. This change can occur while the drift and volatility of the modeled processes remain unchanged and has an influence on the arbitrage price and even more on the hedge strategy.

When we reconsider the equation for the volatility function of a call or put option $g^2(t, T_1, T_2) = \int_t^{T_1} ||\sigma_X(u) + \nu_f(u, T_2) - \nu_d(u, T_2)||^2 du$ we see that it depends on the length of a vector. The influence of the vector components on the volatility function can be seen in Figure (2.1) where an example of the vectors is plotted. We can see, that in our example the volatility-vector under deterministic interest rates $||\sigma_X||$ is of greater length than the volatility-vector under stochastic interest rates $||\sigma_X + \nu_f - \nu_d||$ which will lead to a lower arbitrage price under stochastic interest rates. This result depends on the volatility-vectors of the bond prices and therefore on the bond price volatility functions and their correlation.

Since the function $g^2(t, \tau, T)$ depends on three volatilities it captures a variance-covariance structure. Concerning the correlation between the processes of the exchange rate and the

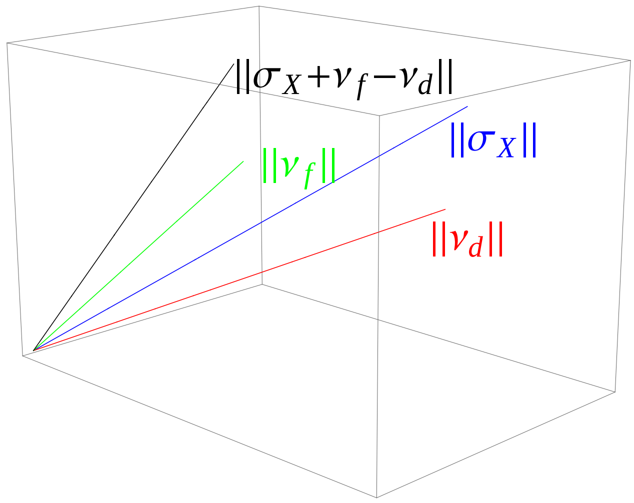

Figure 2.1: The length of different vectors of volatilities.

bond prices we can show how it is incorporated by reconsidering the volatility function as

$$\begin{aligned} g^2(t,\tau,T) &= \int_t^\tau ||\sigma_X(u)+\nu_f(u,T)-\nu_d(u,T)||^2 du \\ &= \int_t^\tau (||\sigma_X(u)||^2 + 2\rho_{x,f}(u,T)||\sigma_X(u)||\ ||\nu_f(u,T)|| \\ &\quad +||\nu_f(u,T)||^2 - 2\rho_{x,d}(u,T)||\sigma_X(u)||\ ||\nu_d(u,T)|| \\ &\quad -2\rho_{f,d}(u,T)||\nu_f(u,T)||\ ||\nu_d(u,T)|| + ||\nu_d(u,T)||^2)du \end{aligned} \tag{2.1}$$

where the correlation functions are defined as

$$\begin{aligned} \rho_{x,f}(t,T) &:= \frac{\text{cov}_{Q_d^T}[dX(t), dB_f(t,T)|F_t]}{(\text{var}_{Q_d^T}[dX(t)|F_t]\text{var}_{Q_d^T}[dB_f(t,T)|F_t])^{\frac{1}{2}}} = \frac{\sigma_X(t)\cdot\nu_f(t,T)}{||\sigma_X(t)||\ ||\nu_f(t,T)||}, \\ \rho_{x,d}(t,T) &:= \frac{\sigma_X(t)\cdot\nu_d(t,T)}{||\sigma_X(t)||\ ||\nu_d(t,T)||}, \\ \rho_{f,d}(t,T) &:= \frac{\nu_f(t,T)\cdot\nu_d(t,T)}{||\nu_f(t,T)||\ ||\nu_d(t,T)||} \end{aligned}$$

and can also be interpreted as the cosine of the angle between the vectors of the volatility functions, e.g. $\rho_{x,f}(t,T) = \cos(\measuredangle(||\sigma_X(t)||, ||\nu_f(t,T)||))$. To examine the influence of a change in the correlation between the price processes, i.e. an increasing angle between two vectors, we first concentrate on the general influence of a changing correlation on the arbitrage price. The value of a call or put option is monotonically increasing in $g(t, \tau, T)$ and combining this with Equation (2.1) one can show the following.

Corollary 2.2.1. *Given a model of an international economy, the correlation between the processes influence the price of a call/put option on exchange rates as follows:*

- $\rho_{x,f}(t,T) \uparrow \;\; \Rightarrow Call(X(t_0), K, t_0, T) \uparrow,$
- $\rho_{x,d}(t,T) \uparrow \;\; \Rightarrow Call(X(t_0), K, t_0, T) \downarrow$ *and*
- $\rho_{d,f}(t,T) \uparrow \;\; \Rightarrow Call(X(t_0), K, t_0, T) \downarrow.$

where the correlation functions are defined as above.

This is also valid for the case of chooser options, since a chooser option is a portfolio of a call and a synthetic put option.

Note, that the correlation structure can be changed without changing the length of the vectors of the volatility functions by changing the angles between them and therefore changing the length of $g^2(t, \tau, T)$. As an example let us assume the international economy can be described by a three factor model with the volatility functions $\sigma_x(t) = (\sigma_x, 0, 0)'$, $\nu_f(t,T) = (\nu_{f,1}(t,T), \nu_{f,2}(t,T), 0)'$ and $\nu_d(t,T) = (\nu_{d,1}(t,T), \nu_{d,2}(t,T), \nu_{d,3}(t,T))'$. Their correlations can be computed as

$$\begin{aligned}
\rho_{x,f}(t,T) &= \frac{\nu_{f,1}(t,T)}{||\nu_f(t,T)||},\\
\rho_{x,d}(t,T) &= \frac{\nu_{d,1}(t,T)}{||\nu_d(t,T)||},\\
\rho_{f,d}(t,T) &= \rho_{x,f}(t,T) \cdot \rho_{x,d}(t,T) \cdot \frac{\sqrt{1-\rho_{x,f}(t,T)^2}}{||\nu_d(t,T)||} \cdot \nu_{d,2}(t,T).
\end{aligned}$$

The correlation structure can be changed by leaving $||\dots||$ constant.

To quantify the effects we will use a three factor extended vasicek model for the domestic and foreign interest market with $\nu_{i,j}(t,T) = \frac{\sigma_{i,j}}{\alpha_{i,j}}\left(1 - \exp\{-\alpha_{i,j}(T-t)\}\right)$ and $i = d, f; j = 1, 2, 3$. Here we will change the correlation by leaving $||\nu_f||$, $||\nu_d||$ and $||\sigma_X||$ constant to see its influence on $||\sigma_X + \nu_f - \nu_d||$ and on the arbitrage price of a chooser option. In Table (2.1) we

Panel A	$\rho_{x,f}$	$\rho_{x,f}+5\%$	$\rho_{x,f}+10\%$	$\rho_{x,f}+20\%$	$\rho_{x,f}+30\%$
$\rho_{x,f}$	0,381	0,400	0,419	0,457	0,495
$\lvert\lvert\sigma_X+\nu_f-\nu_d\rvert\rvert$	0,195	0,196	0,198	0,202	0,206
%-change	0,00%	0,94%	1,89%	3,78%	5,68%
option value	0,179	0,181	0,182	0,184	0,185
%-change	0,00%	0,71%	1,31%	2,39%	2,99%
delta	-0,065	-0,063	-0,061	-0,058	-0,057
%-change	0,00%	-3,04%	-5,51%	-9,84%	-12,44%

Panel B	$\rho_{x,d}$	$\rho_{x,d}+5\%$	$\rho_{x,d}+10\%$	$\rho_{x,d}+20\%$	$\rho_{x,d}+30\%$
$\rho_{x,d}$	0,523	0,549	0,575	0,627	0,679
$\lvert\lvert\sigma_X+\nu_f-\nu_d\rvert\rvert$	0,195	0,192	0,190	0,186	0,182
%-change	0,00%	-1,11%	-2,20%	-4,32%	-6,32%
option value	0,179	0,178	0,177	0,175	0,173
%-change	0,00%	-0,63%	-1,32%	-2,61%	**-3,84**%
delta	-0,065	-0,067	-0,069	-0,072	-0,076
%-change	0,00%	2,71%	5,67%	11,29%	**16,78**%

Table 2.1: Panel A reports the changes due to a variation of $\rho_{x,f}$ and Panel B due to a variation of $\rho_{x,d}$. The model parameters are $t = 0$, $T = 2$, $||\nu_f|| = 0,103$, $||\nu_d|| = 0,092$ and $||\sigma_X|| = 0,2$. The chooser options parameters are given by $\tau = 1$, $K = 1$ and $X(t) = 1$ and the interest rate curves were exogenously given by analytical formulas.

report the numerical results of the changes due to a variation in the correlation structure. The singular effect of an increasing correlation between the foreign interest rate and the exchange rate under stochastic interest rates leads to a higher length of the volatility vector and therefore to a higher arbitrage price. For the correlation between the domestic interest rate and the exchange rate the opposite is true. These impacts on the arbitrage price are relatively small compared to the change in the correlation structure. As an example an increase in the correlation between the exchange rate and the domestic interest rate by 30% decreases the arbitrage price $3,84\%$. But a change in the correlation structure also influences the hedging strategy of the chooser option, since the hedging parameters also depend on $g^2(\tau, t, T)$. The before discussed increase in the correlation structure increases the hedgeparameter delta by $16,78\%$. Therefore the correlation structure has a considerable influence on the hedging strategy and changes in the structure are an additional risk that has to be taken into account.

2.3 Pricing of Currency Chooser Options

Following the definition of a chooser option payoff the arbitrage prices for different specifications of the chooser option will be derived. This specifications will include not only different strike prices of the underlying options but also different maturity dates.

We define the payoff for any possible asset underlying the call and put option as follows.

Definition 2.3.1. *Consider a currency chooser option with maturity τ and an underlying asset price process $X(t)$. Then the payoff is defined by*

$$\max\{Call[X(\tau), K_C, \tau, T]; Put[X(\tau), K_P, \tau, T]\} \tag{2.2}$$

where the $Call[X(\tau), K_C, \tau, T]$ denotes the arbitrage price of a currency call option with maturity $T \geq \tau$ and strike K_C at time τ and $Put[X(\tau), K_P, \tau, T]$ the arbitrage price of the put option with strike K_P.

The arbitrage price of the standard case with equal strike prices is given by the following theorem.

Theorem 2.3.1. *The arbitrage price of a chooser option under stochastic interest rates if the time to maturity and the exercise prices are equal is given by*

$$Chooser[X(t_0), K_C = K, K_P = K, t_0, \tau, T] =$$
$$X(t_0) \cdot B_f(t_0, T) \cdot \{\mathcal{N}(d_1) - \mathcal{N}(-\tilde{d}_1)\} - K \cdot B_d(t_0, T) \cdot \{\mathcal{N}(d_2) - \mathcal{N}(-\tilde{d}_2)\}$$

with

$$\begin{aligned} d_{1/2} &:= \frac{\ln\left(\frac{X(t_0)B_f(t_0,T)}{KB_d(t_0,T)}\right) \pm \frac{1}{2}g^2(t_0,T,T)}{g(t_0,T,T)}, \\ \tilde{d}_{1/2} &:= \frac{\ln\left(\frac{X(t_0)\cdot B_f(t_0,T)}{K\cdot B_d(t_0,T)}\right) \pm \frac{1}{2}g^2(t_0,\tau,T)}{\tilde{g}(t_0,\tau,T)} \text{ and} \\ g^2(t,\tau,T) &:= \int_t^\tau ||\sigma_X(u) + \nu_f(u,T) - \nu_d(u,T)||^2 du. \end{aligned}$$

Proof. To compute the arbitrage price we will use the former results and need therefore to value a standard call option and a synthetic put option with a stochastic strike. The price of standard call options in the framework is well known, see for instance Frey and Sommer [1996] and in a first step we can compute the arbitrage price of the chooser options payoff at time τ. The second step consists in computing the expected discounted payoff from time t_0 to time τ under the domestic martingale measure. In other words the idea is to apply Bayes Rule. To compute the arbitrage value of the chooser option at time $t_0 < \tau$ we have to consider a change of measure to the domestic τ-forward risk adjusted measure. With this in mind we now can turn to the computation of the arbitrage value for the chooser option at time t_0. Using put-call parity the arbitrage price can be computed as

$$\begin{aligned}
& \text{Chooser}[X(t_0), K_C = K, K_P = K, t_0, \tau, T] \\
= \; & E_{P_d^*}\left[\exp\left\{-\int_{t_0}^{\tau} r_d(u)du\right\} \max\left\{\text{Call}[X(\tau), K, \tau, T], \text{Put}[X(\tau), K, \tau, T]\right\}\right] \\
= \; & E_{P_d^*}\left[\exp\left\{-\int_{t_0}^{\tau} r_d(u)du\right\} \left(\text{Call}[X(\tau), K, \tau, T] + [KB_d(\tau, T) - X(\tau)B_f(\tau, T)]^+\right)\right] \\
= \; & E_{P_d^*}\left[\exp\left\{-\int_{t_0}^{\tau} r_d(u)du\right\} E_{P_d^*}\left[\exp\left\{-\int_{\tau}^{T} r_d(u)du\right\} [X(T) - K]^+ | \mathcal{F}_\tau\right]\right] \\
& + E_{P_d^*}\left[\exp\left\{-\int_{t_0}^{\tau} r_d(u)du\right\} [KB_d(\tau, T) - X(\tau)B_f(\tau, T)]^+\right] \\
= \; & B_d(t_0, T) E_{Q_d^T}\left[[X(T) - K]^+\right] + B_d(t_0, \tau) E_{Q_d^\tau}\left[[KB_d(\tau, T) - X(\tau)B_f(\tau, T)]^+\right] \\
= \; & X(t_0)B_f(t_0, T) \cdot \mathcal{N}(d_1(K, t_0, T)) - K \cdot B_d(t_0, T) \cdot \mathcal{N}(d_2(K, t_0, T)) \\
& + B_d(t_0, \tau) \cdot E_{Q_d^\tau}\left[[K \cdot B_d(\tau, T) - X(\tau) \cdot B_f(\tau, T)]^+\right].
\end{aligned}$$

The second expected value corresponds to the arbitrage price of an option with a stochastic strike. This type of option has been studied first by Margrabe [1978], with respect to stochastic interest rates by Jamshidian [1999] and in a more general situation by Frey and Sommer [1996]. One way to compute the value of this option is to apply the change of measure form the domestic $\tau-$forward risk adjusted measure to the $T-$forward risk adjusted measure. This implies

$$\begin{aligned}
& B_d(t_0, \tau) \cdot E_{Q_d^\tau}\left[[K \cdot B_d(\tau, T) - X(\tau) \cdot B_f(\tau, T)]^+\right] \\
= \; & B_d(t_0, \tau) \cdot E_{Q_d^\tau}\left[\frac{B_d(\tau, T)}{B_d(\tau, \tau)}\left[K - \frac{X(\tau) \cdot B_f(\tau, T)}{B_d(\tau, T)}\right]^+\right] \\
= \; & B_d(t_0, \tau) \cdot \frac{B_d(t_0, T)}{B_d(t_0, \tau)} \cdot E_{Q_d^T}\left[\left[K - \frac{X(\tau) \cdot B_f(\tau, T)}{B_d(\tau, T)}\right]^+\right] \\
= \; & B_d(t_0, T) \cdot \left(K \cdot \mathcal{N}(-\tilde{d}_2) - \frac{X(t_0)B_f(t_0, T)}{B_d(t_0, T)} \cdot \mathcal{N}(-\tilde{d}_1)\right).
\end{aligned}$$

□

In the standard case model independent results can be computed and are given by the following corollary.

Corollary 2.3.1. *Given a chooser option* $Chooser[X(t_0), K, t_0, ., T]$ *at time* t_0 *written on a call and a put option with equal strike prices* $K = K_C = K_P$ *and maturity dates* T*, then for* $t_0 < \tau < T$ *the chooser option can be duplicated by*

$$\begin{aligned} Chooser[X(t_0), K, t_0, t_0, T] &= \max\{Call[X(t_0), K, t_0, T]; Put[X(t_0), K, t_0, T]\} \\ Chooser[X(t_0), K, t_0, \tau, T] &= Call[X(t_0), K, t_0, T] + Put[X(t_0) \cdot B_d(t_0, T), K \cdot B_f(t_0, T), t_0, \tau] \\ Chooser[X(t_0), K, t_0, T, T] &= Call[X(t_0), K, t_0, T] + Put[X(t_0), K, t_0, T]. \end{aligned}$$

This can be derived by computing the payoff for the three values of the chooser options maturity and evaluating the arbitrage price by using the put-call parity for time τ.
The result implies that the payoff of a chooser option can be rewritten as a long position of a call option and a synthetic put option. But these model independent results are only valid for the case of equal strike prices and equal time to maturities of the underlying options.

The complex case where the options underlying the chooser option are allowed to have different strike prices, the chooser option can be valued semi-analytically.

Theorem 2.3.2. *Given a parameter value* Z^* *for which the underlying options share the same value at time* τ *the price of a chooser option with* $K_C \neq K_P$ *can be computed as*

$$\begin{aligned} &Chooser[X(t_0), K_C, K_P, t_0, \tau, T, Z^*] := \\ &\qquad X(t_0)B_f(t_0, T)\{\mathcal{N}_2(c_1, f_1(K_C); \rho) - \mathcal{N}_2(-c_1, -f_1(K_P); \rho)\} \\ &\quad - B_d(t_0, T)\{K_C \cdot \mathcal{N}_2(c_2, f_2(K_C); \rho) - K_P \cdot \mathcal{N}_2(-c_2, -f_2(K_P); \rho)\} \end{aligned}$$

with

$$\begin{aligned} c_{1/2} &:= \frac{\ln\left(\frac{X(t_0)B_f(t_0,T)}{B_d(t_0,T)Z^*}\right) \pm \frac{1}{2}g^2(t_0, \tau, T)}{g(t_0, \tau, T)}, \\ f_{1/2}(K) &:= \frac{\ln\left(\frac{X(t_0)B_f(t_0,T)}{B_d(t_0,T)K}\right) \pm \frac{1}{2}g^2(t_0, T, T)}{g(t_0, T, T)}, \\ \rho &:= \frac{g(t_0, \tau, T)}{g(t_0, T, T)}. \end{aligned}$$

Proof. (sketch of proof) In this case a decomposition due to the put call parity is no longer valid. The arbitrage price needs to be computed via its discounted expected payoff given by Equation (2.2) under the domestic martingale measure:

$$E^*_{P_d}[e^{-\int_{t_0}^{\tau} r_d(s)ds} \max\{\text{Call}[X(\tau), K_C, \tau, T]; \text{Put}[X(\tau), K_P, \tau, T]\}|\mathcal{F}_{t_0}]. \tag{2.3}$$

A semi-analytic solution of this equation can be derived in a three step procedure.
The idea is to substitute $\max\{\text{Call}[X(\tau), K_C, \tau, T]; \text{Put}[X(\tau), K_P, \tau, T]\}$ in Equation (2.3) with an indicator function. It is well known that the price functions of call and put options are convex functions of the underlying and that a call (put) option is monotonic increasing (decreasing) in the underlying. We can therefore conclude that there must exist one value Z^* for the underlying of the call and put options at which the options must have an identical value. Furthermore for every value above (below) Z^* the call (put) option has a higher value than the put (call) option as plotted in Figure (2.2).
In a first step we have to value the underlying options at time τ as already shown for the simple chooser option. The second step consists of finding a value for Z^* with a numerical algorithm. With a constant parameter Z^* and by defining $Z(\tau, T) = \frac{X(\tau)B_f(\tau,T)}{B_d(\tau,T)}$ we can

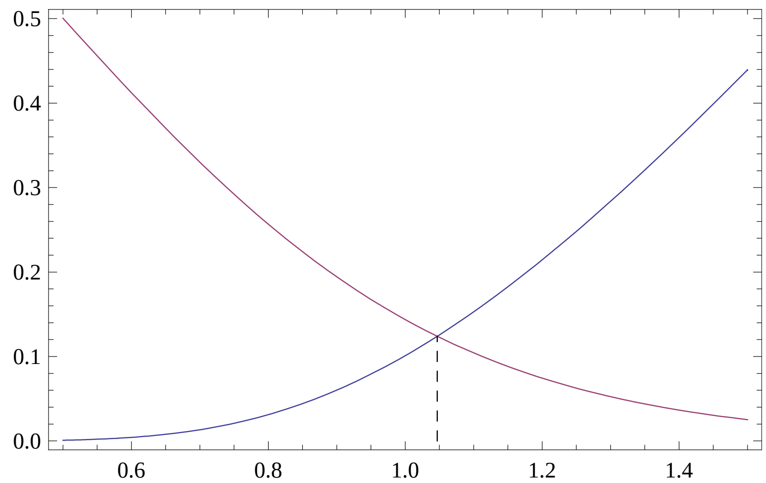

Figure 2.2: Values of a call and a put option and their relation to Z^*

decompose the pricing Equation (2.3) into

$$\begin{aligned} & E_{P_d^*}[e^{-\int_{t_0}^{\tau} r_d(s)ds}\text{Call}[X(\tau), K_C, \tau, T]1_{\{Z(\tau,T)>Z^*\}}|\mathcal{F}_{t_0}] \\ + \; & E_{P_d^*}[e^{-\int_{t_0}^{\tau} r_d(s)ds}\text{Put}[X(\tau), K_P, \tau, T]1_{\{Z(\tau,T)<Z^*\}}|\mathcal{F}_{t_0}]. \end{aligned} \tag{2.4}$$

The problem of solving these two terms in the third step is similar to the problem of pricing compound options which is solved in the case of options on stocks under deterministic interest

rates by Geske [1979]. A compound option is an option on an option which can be solved semi-analytic by the usage of a two-dimensional normal distribution. It is semi-analytical because a value for the underlying asset of the option for the maturity date of the compound option at which the underlying option has zero value needs to be evaluated numerically. In terms of a compound option our decomposed pricing equation can be described as zero-strike call compound option on a call and a put option with the exchange rate as their underlying and the conditions of Z^* as the semi-analytic part.
Since we are considering a model of an international economy under stochastic interest rates, we also have to deal with multi-factor processes of foreign and domestic bond prices correlated with the exchange rate. [1] To compute the value for Z^* numerically we will use the solutions for currency options at time τ to compute the following equation:

$$\begin{aligned} Z^*\mathcal{N}(d_1^*(K_C)) - K_C\mathcal{N}(d_2^*(K_C)) &= -Z^*\mathcal{N}(-d_1^*(K_P)) + K_P\mathcal{N}(-d_2^*(K_P)) \qquad (2.5) \\ \text{with} \quad d_{1/2}^*(K) &:= \frac{\ln\left(\frac{Z^*}{K}\right) \pm \frac{1}{2}g^2(\tau,T,T)}{g(\tau,T,T)}. \end{aligned}$$

A value for Z^* that solves Equation (2.5) can easily be found by a root search algorithm as implemented in most mathematical software packages. With the knowledge of Z^* we can even further decompose the pricing equation. Taking the first term of Equation (2.4) as an example and changing to the T domestic forward measure Q_d^T it results

[1] The case of a compound option on a stock under stochastic interest rates with a non-zero strike cannot be solved in a semi-analytic way as discussed by Frey and Sommer [1998].

$$
\begin{aligned}
&E_{P_d^*}\left[e^{-\int_{t_0}^{\tau} r_d(s)ds}\left(E_{P_d^*}[e^{-\int_{\tau}^{T} r_d(s)ds}(X(T)-K_c)1_{\{X(T)>K_c\}}|\mathcal{F}_\tau]\right)1_{\{Z(\tau,T)>Z^*\}}|\mathcal{F}_{t_0}\right] \\
&= B_d(t_0,T)\cdot E_{P_d^T}\left[Z(\tau,T)\int_{-d_1^*(K_C)}^{\infty} n(y)dy1_{\{Z(\tau,T)>Z^*\}}|\mathcal{F}_{t_0}\right] \\
&\quad -B_d(t_0,T)\cdot K_c\cdot E_{P_d^T}\left[\int_{-d_2^*(K_C)}^{\infty} n(y)dy1_{\{Z(\tau,T)>Z^*\}}|\mathcal{F}_{t_0}\right] \\
&= B_d(t_0,T)\int_{-\infty}^{\infty} Z(\tau,T)n(x)1_{Z(\tau,T)>Z^*}\int_{-d_1^*(K_C)}^{\infty} n(y)dydx \\
&\quad -B_d(t_0,T)\cdot K_c\int_{-\infty}^{\infty} n(x)1_{Z(\tau,T)>Z^*}\int_{-d_2^*(K_C)}^{\infty} n(y)dydx \\
&= B_d(t_0,T)\int_{-c_1}^{\infty} Z(\tau,T)n(x)\int_{-d_1^*(K_C)}^{\infty} n(y)dydx \qquad (2.6)\\
&\quad -B_d(t_0,T)\cdot K_c\int_{-c_2}^{\infty} n(x)\int_{-d_2^*(K_C)}^{\infty} n(y)dydx
\end{aligned}
$$

$$
\text{with } c_{1/2} := \frac{\ln\left(\frac{Z(t_0,T)}{Z^*}\right)\pm\frac{1}{2}g^2(t_0,\tau,T)}{g(t_0,\tau,T)}.
$$

These equations can now be solved using a two-dimensional normal distribution. But we have to remember that $d_{1/2}^*(K_C)$ is a function of $Z(\tau,T)$ and therefore the lower limit of the inner integral depends on the integration variable of the exterior integral. To deal with this we first solve the dynamic of $Z(\tau,T)$ under Q_d^T using Itô´s lemma

$$
\frac{X(\tau)B_f(\tau,T)}{B_d(\tau,T)} = \frac{X(t)B_f(t,T)}{B_d(t,T)}\exp\left\{-\frac{1}{2}g^2(t,\tau,T)+g(t,\tau,T)\cdot x\right\}
$$

with $x \sim N(0,1)$ and rewrite the lower limit as

$$
\begin{aligned}
d_2^*(K_C) &= \frac{\ln\left(\frac{z(t_0)}{K_C}\right)-\frac{1}{2}g^2(t_0,\tau,T)-\frac{1}{2}g^2(\tau,T,T)}{g(\tau,T,T)}+\frac{g(t_0,\tau,T)}{g(\tau,T,T)}\cdot x \\
&= \frac{\ln\left(\frac{z(t_0)}{K_C}\right)-\frac{1}{2}g^2(t_0,T,T)}{g(t_0,T,T)}\cdot\frac{g(t_0,T,T)}{g(\tau,T,T)}+\frac{g(t_0,\tau,T)}{g(\tau,T,T)}\cdot x \\
&= \frac{\ln\left(\frac{z(t_0)}{K_C}\right)-\frac{1}{2}g^2(t_0,T,T)}{g(t_0,T,T)}\cdot\sqrt{\frac{1}{1-\rho^2}}+\sqrt{\frac{\rho^2}{1-\rho^2}}\cdot x
\end{aligned}
$$

using the fact that $g^2(t_0,T,T) = g^2(t_0,\tau,T)+g^2(\tau,T,T)$ and defining the correlation coefficient ρ by

$$
\rho := \frac{g(t_0,\tau,T)}{g(t_0,T,T)}.
$$

Using integral substitution the double integral can be simplified to a two-dimensional normal distribution. In the following we take the second term in Equation (2.6) as an example. With the following substitution

$$\begin{aligned} f(y) &= \sqrt{1-\rho^2}\cdot y + \rho x \\ df &= \sqrt{1-\rho^2}dy \\ f(-d_2^*) &= -\frac{\ln\left(\frac{z(t_0)}{K_C}\right) - \frac{1}{2}g^2(t_0,T,T)}{g(t_0,T,T)} = -f_2 \end{aligned}$$

the double integral can be solved with $g(y) = \exp\left\{-\frac{1}{2}\left(\frac{y-\rho\cdot x}{\sqrt{1-\rho^2}}\right)\right\}$ as

$$\begin{aligned} & B_d(t_0,T)\cdot K_c\cdot\left[\int_{-c_2}^{\infty} n(x)\int_{-d_2^*(K_C)}^{\infty} n(y)dydx\right] \\ = \;& B_d(t_0,T)\cdot K_c\cdot\left[\int_{-c_2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}\int_{-d_2^*(K_C)}^{\infty}\frac{1}{\sqrt{2\pi}}g(f(y))f'(y)\frac{1}{\sqrt{1-\rho^2}}dydx\right] \\ = \;& B_d(t_0,T)\cdot K_c\cdot\left[\int_{-c_2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}\int_{-f_2}^{\infty}\frac{1}{\sqrt{2\pi}}g(z)\frac{1}{\sqrt{1-\rho^2}}dzdx\right] \\ = \;& B_d(t_0,T)\cdot K_c\cdot\left[\int_{-c_2}^{\infty}\int_{-f_2}^{\infty}\frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{x^2}{2}-\frac{(z-\rho\cdot x)^2}{2(1-\rho^2)}\right\}dzdx\right] \\ = \;& B_d(t_0,T)\cdot K_c\cdot\left[\int_{-c_2}^{\infty}\int_{-f_2}^{\infty}\frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2}(x^2-2\rho xz+z^2)\right\}dzdx\right] \\ = \;& B_d(t_0,T)\cdot K_c\cdot\mathcal{N}_2(c_2,f_2;\rho) \end{aligned}$$

In this way the whole pricing equation can now be solved. □

In the most general case not only the strike prices of the call and put options are different but also the times to maturities differ. We will denote the underlying call options maturity date with T_C and the put options with T_P. Due to this we can no longer evaluate a value for Z which fulfills the relation between call and put prices at time τ, since we have different forward values. We now have to find a value X^* for the following condition

$$\frac{B_d(\tau,T_C)}{B_d(\tau,T_P)}\cdot\left(\frac{X^*B_f(\tau,T_C)}{B_d(\tau,T_C)}\mathcal{N}(\hat{d}_1(X^*,K_C,T_C)) - K_C\cdot\mathcal{N}(\hat{d}_2(X^*,K_C,T_C))\right) =$$
$$\left(-\frac{X^*B_f(\tau,T_P)}{B_d(\tau,T_P)}\mathcal{N}(-\hat{d}_1(X^*,K_P,T_P)) + K_P\cdot\mathcal{N}(-\hat{d}_2(X^*,K_P,T_P))\right)$$
$$\text{with}\quad \hat{d}_{1/2}(X,K,T) := \frac{\ln\left(\frac{X\cdot B_f(\tau,T)}{K\cdot B_d(\tau,T)}\right) \pm \frac{1}{2}g^2(\tau,T,T)}{g(\tau,T,T)}.$$

In the case of deterministic interest rates this problem is again solvable by a search root algorithm. But in the case of stochastic interest rates this problem is analytical not solvable and we have to deal with four random variables at time τ namely $B_f(\tau, T_C), B_d(\tau, T_C), B_f(\tau, T_P)$ and $B_d(\tau, T_P)$. One possible approximation that is e.g. used in the context of the LIBOR market model is the approximation by low variance martingales. If we assume that the quotients $\frac{B_d(\tau,T_C)}{B_d(\tau,T_P)}, \frac{B_f(\tau,T_C)}{B_d(\tau,T_C)}$ and $\frac{B_f(\tau,T_P)}{B_d(\tau,T_P)}$ are martingales of low variance we can approximate their time τ values by their time t_0 values. This approximation works best if the domestic and foreign price processes are highly correlated and if the maturity dates are very close.

With a value of X^* we can compute the arbitrage price of the chooser option in a similar way as we have done for the complex case.

Corollary 2.3.2. *Given a value X^* for which the underlying options share the same value at time τ the price of a chooser option with $K_C \neq K_P$ and $T_C \neq T_P$ can be computed as*

$$\begin{aligned}
&\mathit{Chooser}[X(t_0), K_C, K_P, t_0, \tau, T_C, T_P, X^*] := \\
&B_d(t_0, T_C)\left\{Z(t_0, T_C)\mathcal{N}_2(c_1(T_C), f_1(K_C, T_C); \rho(T_C)) - K_C\mathcal{N}_2(c_2(T_C), f_2(K_C, T_C); \rho(T_C))\right\} - \\
&B_d(t_0, T_P)\left\{Z(t_0, T_P)\mathcal{N}_2(-c_1(T_P), -f_1(K_P, T_P); \rho(T_P)) - K_P\mathcal{N}_2(-c_2(T_P), -f_2(K_P, T_P); \rho(T_P))\right\}
\end{aligned}$$

with

$$\begin{aligned}
Z(t_0, T) &:= \frac{X(t_0) \cdot B_f(t_0, T)}{B_d(t_0, T)}, \quad \rho(T) := \frac{g(t, \tau, T)}{g(t, T, T)}, \\
c_{1/2}(T) &:= \frac{\ln\left(\frac{X(t) \cdot B_f(t,\tau)}{X^* \cdot B_d(t,\tau)}\right) \pm \frac{1}{2} g^2(t, \tau, T)}{g(t, \tau, T)} \quad \text{and} \\
f_{1/2}(K, T) &:= \frac{\ln\left(\frac{X(t) \cdot B_f(t,T)}{K \cdot B_d(t,T)}\right) \pm \frac{1}{2} g^2(t, T, T)}{g(t, T, T)}.
\end{aligned}$$

2.4 Hedging of Currency Chooser Options

A chooser option is an option with two degrees of underlyings. At the chooser options maturity date τ the payoff consists of the maximum between a call and a put option. Therefore as a first level, the chooser is written on a call and a put option as its underlyings. Since the call and the put option are written on the same underlying lognormal asset this asset is a second level underlying. A chooser option can therefore be hedged by a self-financing hedgeportfolio in either underlyings.

Since the chooser option consists of a call and a synthetic put option there is, as we show,

a relation between the strike and the forward price where the option is delta neutral. A desirable effect for a hedger, but the hedgeportfolio must be rehedged more often if the chooser option is close at this relation. If we remember that transaction costs exist and that rehedging can only be executed at discrete time steps, this can raise the hedgecosts above the arbitrage price in a practical implementation of this hedgestrategy. Therefore we will also consider the so called straddle. This is a long position in a call and a put option with the same specifications as the underlying options that can serve as a static superhedge.
In a first step the hedgeparameters of a self-financing hedgeportfolio consisting of the underlying asset are derived and discussed. In a second step the difference to the straddle in the arbitrage price and hedge strategy are examined in detail.

Corollary 2.4.1. *At time t_0 the delta of a currency chooser option with maturity date T is i) in the case of $K = K_C = K_P$ given by*

$$\Delta_{Chooser} = N(d_1) - N(-\tilde{d}_1).$$

ii) in the case of $K_C \neq K_P$ given by

$$\Delta_{Chooser} = N_2(c_1, f_1(K_C); \rho) - N_2(-c_1, -f_1(K_P); \rho).$$

Proof. As Equation (2.3) shows the arbitrage price of a chooser option in the case of $K = K_C = K_P$ can be written as a linear combination of a currency call option with volatility function $g^2(t_0, T, T)$ and a put option with volatility function $g^2(t_0, \tau, T)$. So we can apply the well known results for hedging currency options in a framework of an international economy where the delta specifies the nominal value of foreign bonds to be held.
For the second case the delta can be computed following El Karoui et al. [1992b] as will be shown for Equation (2.4)

$$\begin{aligned}
&\frac{\partial}{\partial[X(t_0)B_f(t_0,T)]} B_d(t_0,T) E_{Q_d^T}[\text{Call}[X(\tau), K_C, \tau, T] 1_{\{Z(\tau,T)>Z^*\}} | \mathcal{F}_{t_0}] \\
= \; & B_d(t_0,T) E_{Q_d^T}\left[\frac{\partial X(T)B_f(T,T)}{\partial[X(t_0)B_f(t_0,T)]} 1_{\{X(T)>K_C\}} 1_{\{Z(\tau,T)>Z^*\}} | \mathcal{F}_{t_0}\right] \\
= \; & N_2(c_1, f_1(K_C); \rho).
\end{aligned}$$

□

Since a chooser option consists of long positions in a call and a synthetic put option the delta varies between -1 and 1 depending on the moneyness of the options as can be seen in Figure (2.3).

In dependence of an equal strike K a delta neutral position of the chooser option or in other words the root of the deltafunction can be computed.

Corollary 2.4.2. *At time t_0 the delta of the chooser option with maturity date τ, $K = K_C = K_P$ is zero if:*

$$K = \underbrace{\frac{X(t_0) \cdot B_f(t_0, T)}{B_d(t_0, T)}}_{\textit{forward price}} \underbrace{\exp\left\{\frac{1}{2} \cdot g(t_0, \tau, T) \cdot g(t_0, T, T)\right\}}_{\geq 1}. \tag{2.7}$$

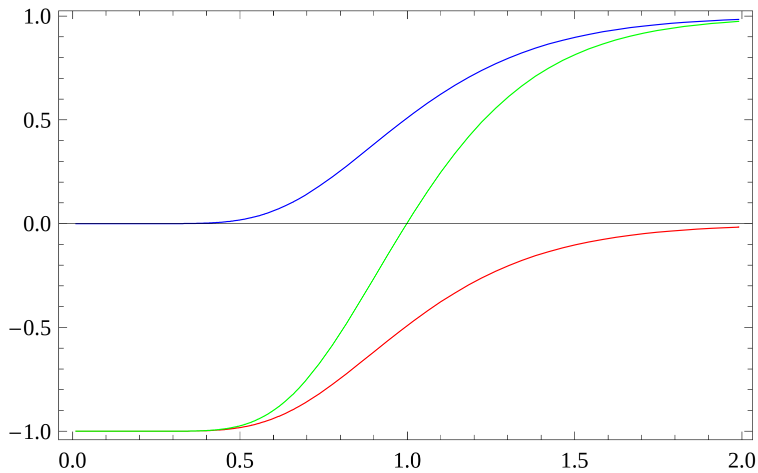

Figure 2.3: Delta of a chooser option (green line) and of a call (blue line) and a put (red line) option with a strike price of $K = 1$ in dependence of the underlying.

Proof.

$$
\begin{aligned}
\mathcal{N}(d_1) &= \mathcal{N}(-\tilde{d}_1) \\
d_1 &= -\tilde{d}_1 \\
\frac{\ln\left(\frac{X\cdot B_f(t_0,T)}{K\cdot B_d(t_0,T)}\right)+\frac{1}{2}g^2(t_0,T,T)}{g(t_0,T,T)} &= -\frac{\ln\left(\frac{X\cdot B_f(t_0,T)}{K\cdot B_d(t_0,T)}\right)+\frac{1}{2}g^2(t_0,\tau,T)}{g(t_0,\tau,T)} \\
\ln(X\cdot B_f(t_0,T)) &= \left(\left(\ln(K\cdot B_d(t_0,T))-\frac{1}{2}\cdot g^2(t_0,\tau,T)\right)g(t_0,T,T)\right. \\
&\quad \left.+\left(\ln(K\cdot B_d(t_0,T))-\frac{1}{2}\cdot g^2(t_0,T,T)\right)g(t_0,\tau,T)\right) \\
&\quad /(g(t_0,\tau,T)+g(t_0,T,T)) \\
\ln(X\cdot B_f(t_0,T)) &= \ln(K\cdot B_d(t_0,T)) \\
&\quad -\frac{1}{2}\cdot\frac{g(t_0,\tau,T)g(t_0,T,T)\cdot(g(t_0,\tau,T)+g(t_0,T,T))}{g(t_0,\tau,T)+g(t_0,T,T)} \\
\ln\left(\frac{X\cdot B_f(t_0,T)}{K\cdot B_d(t_0,T)}\right) &= -\frac{1}{2}\cdot g(t_0,\tau,T)g(t_0,T,T) \\
X(t_0) &= \frac{K\cdot B_d(t_0,T)}{B_f(t_0,T)}\exp\left\{-\frac{1}{2}\cdot g(t_0,\tau,T)\cdot g(t_0,T,T)\right\}.
\end{aligned}
$$

□

A necessary condition for the chooser option to be delta-neutral is that its strike price is larger or equal to the underlyings domestic forward value in dependence of the volatility.

As shown before the chooser option has a delta varying from -1 to 1 and the hedge can consist of long and short positions in the underlying. This can have an influence on the delta parameter and a measure for the amount of readjusting the hedge is the derivative of the delta, the so called gamma.

Corollary 2.4.3. *At time t_0 the gamma of a currency chooser option with maturity date τ is*
i) in the case of $K = K_C = K_P$ given by

$$
\Gamma_{Chooser} = \frac{n(d_1)}{X(t_0)\cdot B_f(t_0,T)\cdot g(t_0,T,T)}+\frac{n(\tilde{d}_1)}{X(t_0)\cdot B_f(t_0,T)\cdot g(t_0,\tau,T)}.
$$

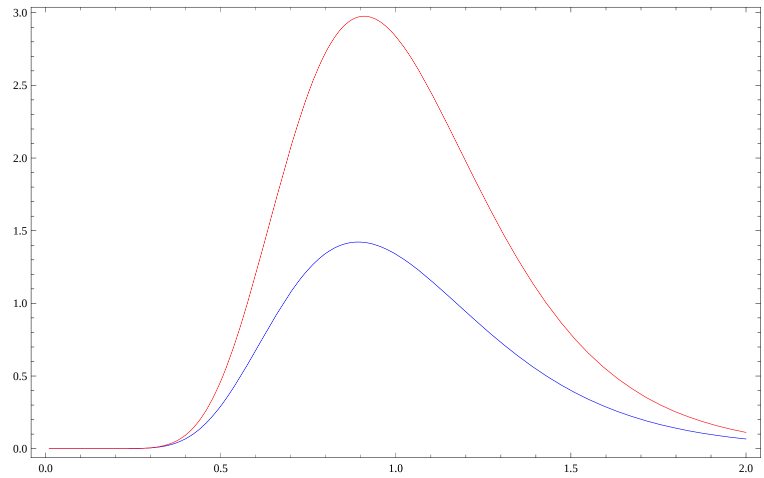

Figure 2.4: Gamma of a chooser option (red line) and of a call/put option (blue line) with a strike price of $K = 1$ in dependence of the underlying.

ii) in the case of $K_C \neq K_P$ given by

$$\Gamma_{Chooser} = \frac{n(c_1) \cdot N\left(\frac{f_1(K_C)-\rho\cdot c_1}{\sqrt{1-\rho^2}}\right)}{X(t_0)B_f(t_0,T)g(t_0,\tau,T)} + \frac{n(f_1(K_C)) \cdot N\left(\frac{c_1-\rho\cdot f_1(K_C)}{\sqrt{1-\rho^2}}\right)}{X(t_0)B_f(t_0,T)g(t_0,T,T)}$$
$$- \left(\frac{n(-c_1) \cdot N\left(\frac{-f_1(K_C)+\rho\cdot c_1}{\sqrt{1-\rho^2}}\right)}{X(t_0)B_f(t_0,T)g(t_0,\tau,T)} + \frac{n(-f_1(K_C)) \cdot N\left(\frac{-c_1+\rho\cdot f_1(K_C)}{\sqrt{1-\rho^2}}\right)}{X(t_0)B_f(t_0,T)g(t_0,T,T)}\right).$$

As wee can see in Figure (2.4) the chooser options gamma is not symmetric around the root of the chooser options delta as one could intuitively suggest. But since the chooser options gamma is a linear combination of the gamma of a call and a put option with different volatility functions it is not very different to a straddle´s gamma that can be computed by a call or put options gamma multiplied by a factor of two. The differences between them will be examined later.

We can see in Figure (2.4) that the highest gamma values are close to the root of the delta. We can therefore conclude that a delta neutral position in a chooser option is exposed to a relative high gamma risk.

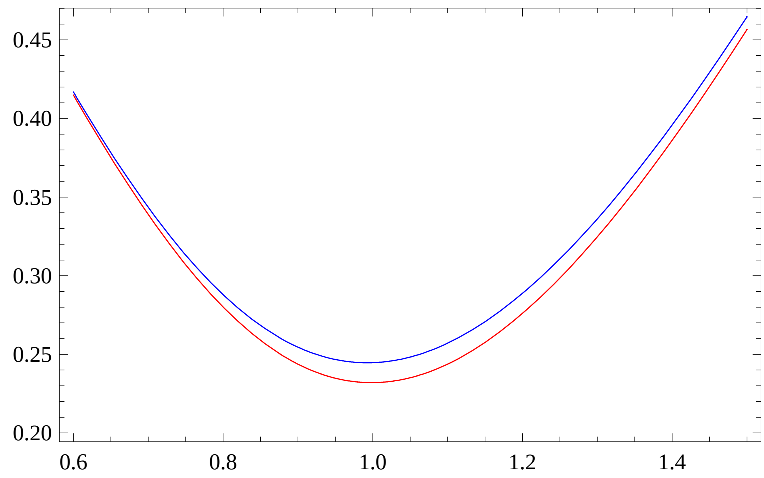

Figure 2.5: The value of a straddle (blue) and of a chooser option (red) in dependence of the exchange rate with a strike price of $K = 1$.

2.4.1 Differences to the Straddle

As can be seen in Figure (2.5) the straddle is a superhedge to the chooser option and it is very important to understand under which conditions the chooser option is lower in its arbitrage price and how the hedge strategies between these two instruments differ. The straddle consists of a call and a put option and its payoff at time τ is given by

$$[X(\tau) - K_C]^+ + [K_P - X(\tau)]^+.$$

In dependence of an equal strike the maximal difference in the values of a chooser option and a straddle can be computed analytically.

Corollary 2.4.4. *Given a chooser option and a straddle with the same strike prices $K_C = K_P = K$ and maturity date of the options T. If τ is fixed and not at its boundaries t_0 or T then the difference in both arbitrage values is maximal if*

$$K = \underbrace{\frac{X(t_0) \cdot B_f(t_0, T)}{B_d(t_0, T)}}_{\textit{forward price}} \underbrace{\exp\left\{-\frac{1}{2} \cdot g(t_0, \tau, T) \cdot g(t_0, T, T)\right\}}_{\leq 1}. \tag{2.8}$$

This value is very similar to the value of the strike for which the chooser option is delta neutral. The difference between both values is the sign in the exponent. As a conclusion the price difference can be maximal or is at least very high when the chooser option is delta neutral.

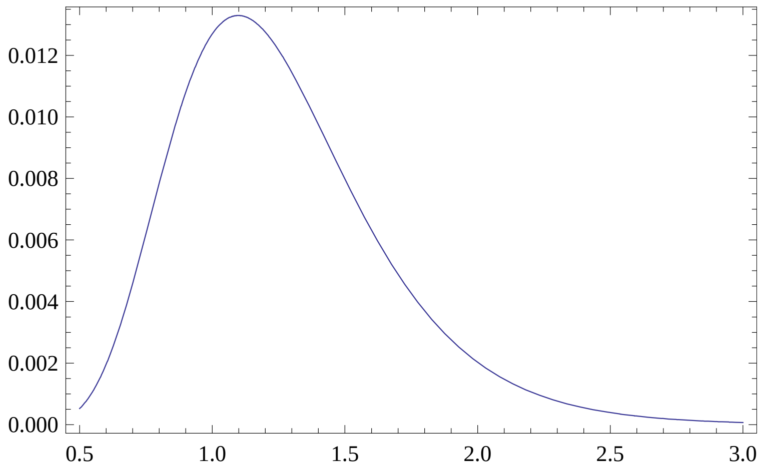

Figure 2.6: The difference of the values of a straddle and a chooser option in dependence of the exchange rate with a strike price of $K = 1$.

Proof. The difference between the straddle and the corresponding chooser option is given by

$$\begin{aligned} & \text{Call}[X(t_0), K, T, t_0] + \text{Put}[X(t_0), K, T, t_0] - \text{Chooser}[X(t_0), K, T, t_0, \tau, Z^*] \\ = \; & -X(t_0)B_f(t_0,T)(\mathcal{N}(-d_1) - \mathcal{N}(-\tilde{d}_1)) + K \cdot B_d(t_0,T)(\mathcal{N}(-d_2) - \mathcal{N}(-\tilde{d}_2)). \end{aligned}$$

□

If we use the proof of Equation (2.7) as given in the appendix we can easily calculate Equation (2.8). The differences between a Chooser Option and a Straddle in dependence of the spot exchange rate are plotted in Figure (2.6).

Since the straddle also needs to be hedged by its issuer we compare the hedge strategies of the straddle and of the chooser option. Assuming again equal strike prices for computational simplicity the difference in the delta parameters can be computed.

Corollary 2.4.5. *Given a chooser option and a straddle with equal strike prices and maturity dates of the underlying options at time t_0. In the case of $K = K_C = K_P$ the relation of both delta hedging strategies is given by*

$$\begin{aligned} \Delta_{Straddle} > \Delta_{Chooser} &\quad \leftrightarrow \quad X(t_0) < L \\ \Delta_{Straddle} = \Delta_{Chooser} &\quad \leftrightarrow \quad X(t_0) = L \\ \Delta_{Straddle} < \Delta_{Chooser} &\quad \leftrightarrow \quad X(t_0) > L \end{aligned}$$

with $L = \frac{K \cdot B_d(t_0,T)}{B_f(t_0,T)} \exp\left\{\frac{1}{2} \cdot g(t_0,\tau,T) \cdot g(t_0,T,T)\right\}$.

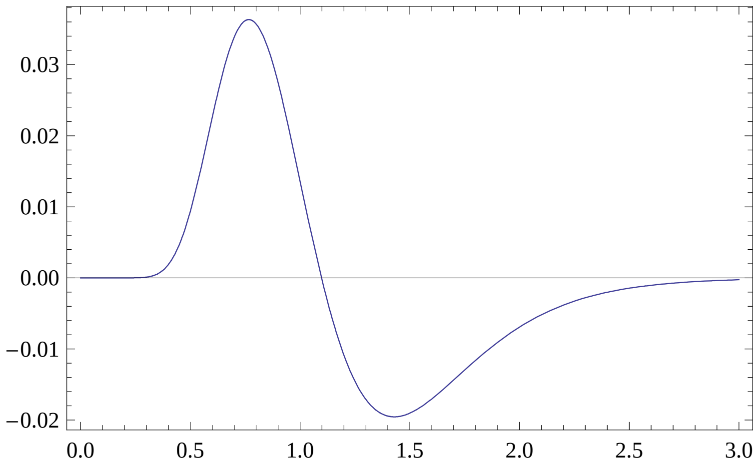

Figure 2.7: Difference between the delta of a straddle and a chooser option in dependence of the exchange rate.

Note, that the deltas coincide when the price difference is the highest, what can also be seen in Figure (2.7).

Proof. We first compute the delta of the straddle and of its difference to the chooser option

$$\begin{aligned}\Delta_{Straddle} &= 2 \cdot N(d_1) - 1 \\ \Delta_{Straddle} - \Delta_{Chooser} &= 2 \cdot N(d_1) - 1 - (N(d_1) - N(-\tilde{d}_1)) \\ &= N(d_1) - N(\tilde{d}_1) \lesseqgtr 0.\end{aligned}$$

We can now use the previous results, in particular Equation (2.8) to clarify how the difference behaves in dependence of the exchange rate. □

Of more concern is the difference in the gamma risk to see which option needs to be rehedged more often.

Corollary 2.4.6. *Given a chooser option and a straddle with the same strike prices and maturity dates of the underlying options at time t_0. In the case of $K = K_C = K_P$ the relation of both gamma hedging strategies is given by*

$$\begin{aligned}\Gamma_{Straddle} > \Gamma_{Chooser} &\leftrightarrow X(t_0) < M_1 \vee X(t_0) > M_2 \\ \Gamma_{Straddle} = \Gamma_{Chooser} &\leftrightarrow X(t_0) = M_1 \vee M_2 \\ \Gamma_{Straddle} < \Gamma_{Chooser} &\leftrightarrow X(t_0) > M_1 \wedge X(t_0) < M_2\end{aligned}$$

with

$$M_{1/2} := \exp\left\{ \pm\sqrt{g^2(t_0,T,T)g^2(t_0,\tau,T)\left(\frac{1}{4} - \frac{2\cdot\ln\left(\frac{g(t_0,T,T)}{g(t_0,\tau,T)}\right)}{g^2(t_0,\tau,T)-g^2(t_0,T,T)}\right)}\right\}.$$

Proof. The gamma of the straddle and of the difference are given by

$$\begin{aligned}
\Gamma_{Straddle} &= \frac{2n(d_1)}{X(t_0)B_f(t_0,T)g(t_0,T,T)} \\
\Gamma_{Straddle} - \Gamma_{Chooser} &= \frac{n(d_1)}{X(t_0)B_f(t_0,T)g(t_0,T,T)} - \frac{n(\tilde{d}_1)}{X(t_0)B_f(t_0,T)g(t_0,\tau,T)} \lesseqgtr 0.
\end{aligned}$$

In order to explain the sign of the difference we show that there is no difference in gamma if

$$\begin{aligned}
\Gamma_{Straddle} &= \Gamma_{Chooser} \\
\Leftrightarrow \exp\left\{-\frac{1}{2}(d_1)^2\right\} g(t_0,\tau,T) &= \exp\left\{-\frac{1}{2}(\tilde{d}_1)^2\right\} g(t_0,T,T) \\
\Leftrightarrow X(t_0) &= \frac{K\cdot B_d(t_0,T)}{B_f(t_0,T)}\cdot M_{1/2}
\end{aligned}$$

as also plotted in Figure (2.8). □

It can be seen that the gamma of the chooser option is larger for delta neutral chooser options than for the equivalent straddle. For chooser options with the maximal price difference the same is true. But if the price difference is low the gamma of the chooser option is lower than the gamma of the equivalent straddle.

Concerning the price-difference between the straddle and the chooser option the highest difference is when both investments share the same delta but the chooser options gamma is at this point higher and it will therefore be more risky for a hedger. As for a delta neutral position in the chooser option this position has a high position of gamma risk which is even higher than the corresponding gamma risk of the straddle.

2.5 Conclusion

Assuming lognormal asset price processes and using a model of an international economy chooser options under stochastic interest rates or dividend yields can be valued in closed form for several underlying assets like exchange rates, inflation linked caps/floors or stocks. The framework allows thereby the valuation of several different specifications of chooser options.

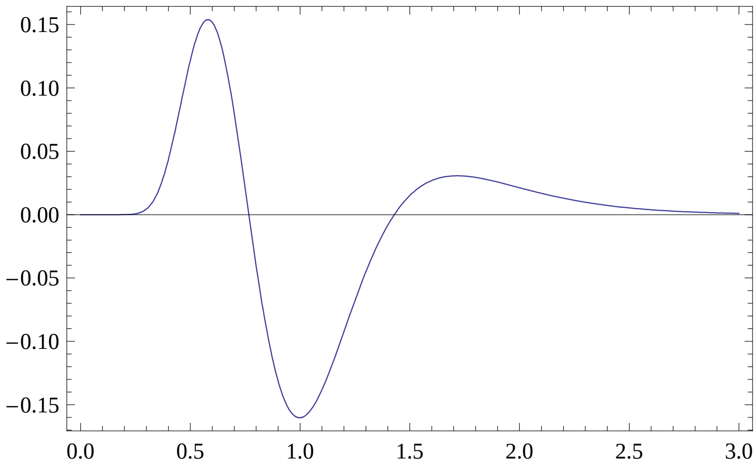

Figure 2.8: Difference between the gamma of a straddle and a chooser option in dependence of the exchange rate.

In this framework the correlation between the stochastic variables can change by leaving the factor volatilities constant. The effects of these changes on the arbitrage price and hedge parameters can be linked to the different factor correlations and influences the hedge parameter in a larger amount.
The hedgestrategy of chooser options is different to plain-vanilla call or put options. In particular the chooser options delta lies between -1 and 1 and the option can in dependence of the strike price be delta neutral without a hedge. But this gives rise to a high gamma risk and therefore a static hedge could be a useful alternative. A long position in a call and put option is a static hedge but the price difference to a chooser option is very large when the gamma is high.

Chapter 3

Convexity Correction of CMS rates

3.1 Introduction

Convexity correction in the fixed income markets is the quantity which describes a correction of computed values due to the convexity of the underlying process. The convexity correction can be distinguished in e.g. three different applications. There is the convexity correction in the bond market where the sensitivity of the bondprice in the yield the so called duration is computed as a tangent to a convex function. If the sensitivity for more than an instantaneous small interval needs to be computed an additional convexity correction has to be included. Another application is a convexity correction due to a so called timing lag, where the payment of a rate occurs at the start of its accrual period and not at its maturity. A typical example for this is a LIBOR in arrears caplet which pays at its maturity the positive difference of for instance a 3-month LIBOR rate that fixes at that date and a strike price. The third category is a convexity correction that arises when the expectation is taken under a different martingale measure than the assets measure. Here, the expectation is not the forward value but the forward value and a convexity correction. One typical example is the constant maturity swap (CMS) or CMS options. In CMS contracts the payoff consists of a single payment of the swap rate and not of the natural payments of the underlying swap. Therefore the expectation is under a forward measure but the swap is only a martingale under a so called annuity measure the numeraire which does not consist of only one but a portfolio of discount factors. In this work we will concentrate on the convexity correction due to a mismatch of measures and a timing lag for the case of CMS rates.

The standard approach in the literature, see Brigo and Mercurio [2007] or Pelsser [2003] is to compute the convexity correction as an analytical approximation. But several assumptions such like a parallel shift of rates and a freezing of the drift have to be imposed to approximate the value using a Taylor series expansion. Following this method, several approximation formulae have been derived to compute the convexity correction. A comparison of the existing approximations is done in Selic [2003] and Lu and Neftci [2003] provide a comparison by simulation in a LIBOR Market Model. One drawback of all these approaches is that the resulting convexity correction is not a tradeable asset and changes over time with varying forward rates. This is especially a problem for a hedger who cannot hedge his exposure to convexity by market instruments. A possible solution is the article of Hagan [2003] where he introduced a static replication portfolio of plain-vanilla swaptions to replicate the payoff of CMS caplets and floorlets. Using cap-floor parity a convexity correction for CMS rates can be computed by this portfolio of swaptions. Another advantage of this approach is that it is model independent and for the valuation of swaptions a model can be used that captions the volatility cube. The volatility cube is a matrix of implied volatilities for the deterministic volatility swaption model of Black [1976] to obtain market prices. This volatility cube contains therefore market information that should be included into the computation of the convexity correction. One model that is heavily used in the market to compute swaption prices taking into account the volatility cube is the SABR model of Hagan et al. [2003] for forward price processes. In this model the volatility is no longer deterministic but a stochastic process correlated with the asset price process. The popularity stems not only from its capability to replicate the volatility cube but also from its simple implementation since no numerical methods are needed to compute plain-vanilla option prices. The computation is done by the forward price, strike price and the time-to-maturity dependent volatility function which is then plugged into a Black76 formula to compute for instance the value of caps/floors or swaptions. In the end, the convexity correction for CMS rates is replicated by traded instruments thus incorporating additional market information.

Theoretically, the replication portfolio proposed by Hagan [2003] is a precise approximation to compute the convexity correction but it is calculated by integrating the swaption formula with respect to all strikes up to infinity. To compute the correction the integral has to be discretized by dividing it into buckets. Even though swaption prices are not quoted for strikes up to infinity the weight function is computed as if these swaptions were included in the portfolio. We propose a different approach to compute a discrete static replication portfolio.

For the computation we apply an iterative algorithm. Given a discrete interval of strike prices, it computes the weights for swaptions with given strikes in order to replicate the payoff of CMS caplets and floorlets and also accounting for the convexity correction for CMS rates. The formulae assume that the underlying swaptions are cash-settled which is standard in the European market and allow for both fixing in advance and in arrears. Since in the case of CMS caplets we use a portfolio of payer swaptions, we replicate a linear payoff with a concave payoff and the replication portfolio is therefore a superhedge to the CMS caplet. In the case of CMS floorlets the opposite is the case since receiver swaptions obey a convex payoff.

Numerical simulations show that a convergence of up to 8 digits for a forward swap rate of 0.03303 can be achieved for an upper bound of the underlying strike prices of about 0.1. The step size between the strike prices should be small but decreasing the step size does not necessarily lead to a better accuracy of the replication portfolio for low upper bounds . In the end it turns out that a small step size is more important than a high value for the upper bound. Concerning the valuation model the simulations show that an inclusion of the volatility cube does have a significant influence on the value of the convexity corrections, especially for high forward values.

The chapter is structured as follows. In Section (3.2) we describe interest rate swaps, swap rates and constant maturity swaps. The Black76 model and the SABR interest rate model are introduced in Section (3.3). Section (3.4) introduces the replication model with an infinite portfolio of swaptions. The replication portfolio for a finite number of swaption is introduced in detail in Section (3.5). The accuracy of the replication portfolio of the previous section is examined by several numerical simulations in Section (3.6). Section (3.7) concludes the chapter.

3.2 Interest Rate Swaps

To value constant maturity swaps and to discuss their properties standard interest rate swaps will be defined and valued in a first step. To ease the computations we will assume a default free economy and will therefore ignore basis risk and assume that one yield curve is sufficient to compute the present value of different cash flows.

An interest rate swap is a financial contract where, in the case of a payer swap, one party agrees to pay another party a fixed interest K at every payment date and receives a floating

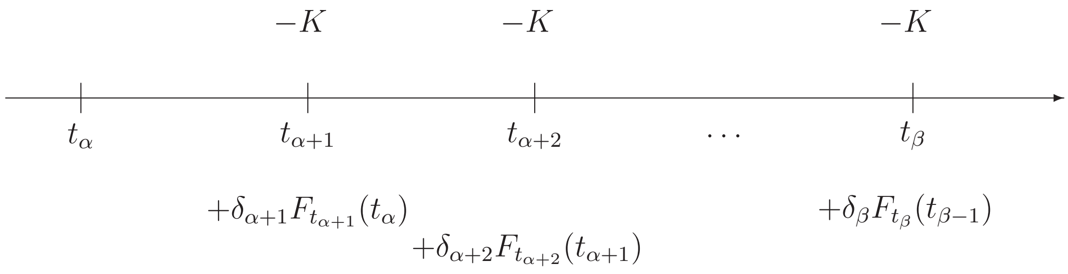

Figure 3.1: Illustration of the cash flows of a payer interest rate swap.

interest rate that is linked e.g. to the 3-month LIBOR rate with payment dates $i \in \{\alpha + 1, \ldots, \beta\}$ and a receiver swap is the reverse. A LIBOR rate $F(t, t_{i-1}, t_i)$ is defined as the time $t \leq t_{i-1}$ value of an interest rate fixed at time t_{i-1} and paying at time t_i.The trading date of a contract is time t, the first reset is at t_α, the first pay date at $t_{\alpha+1}$ and the contract matures at t_β with $0 \leq t \leq t_\alpha \leq \ldots \leq t_\beta$. The year fraction between the fixing and the payment dates is denoted by $\delta_i := t_i - t_{i-1}$. The cash flows of a payer interest rate swap are illustrated in Figure (3.1). Note, that here and in the following we assumed for simplicity that the payment dates of the fixed and floating payments coincide and we can use the same year fraction δ_i for both legs.

To compute the arbitrage price of an interest rate swap we decompose the contract into a fixed and a floating leg where the fixed leg corresponds to the fixed payments at interest K and the floating leg to the stochastic rates $F(t, t_{i-1}, t_i)$ with a simplified notation $F_{t_i}(t) := F(t, t_{i-1}, t_i)$ we use in the following. We further define the zerobond $B(t, T),\ t \leq T$ as the time t value of notional one paid at a future time T. Given an interest rate curve of today´s zerobond prices $B(0, \tau),\ \forall \tau \geq 0$ the forward LIBOR rates can be computed as:

$$F_{t_i}(0) \quad = \quad \frac{1}{\delta_i}\left(\frac{B(0, t_{i-1})}{B(0, t_i)} - 1\right). \tag{3.1}$$

As was shown by Jamshidian [1991] and Geman et al. [1995] a zerobond with maturity at time T can be used as a numeraire with its associated unique martingale measure called the time T forward measure P^T. Therefore the expected value at time 0 of a zerobond $B(t, t_{i-1})$ under the time t_i forward measure P^{t_i} and it´s numeraire is a martingale and given by:

$$\mathbb{E}^{t_i}\left[\frac{B(t, t_{i-1})}{B(t, t_i)}\middle|\mathcal{F}_0\right] \quad = \quad \frac{B(0, t_{i-1})}{B(0, t_i)}.$$

Using the forward measure Miltersen et al. [1997] showed that LIBOR rates $F_{t_i}(t)$ are a martingale under the time t_i adjusted forward measure

$$\begin{aligned}
F_{t_i}(0) &= \frac{1}{\delta_i}\left(\frac{B(0,t_{i-1})}{B(0,t_i)} - 1\right) \\
&= \frac{1}{\delta_i}\left(\mathbb{E}^{t_i}\left[\frac{B(t,t_{i-1})}{B(t,t_i)}\middle|\mathcal{F}_0\right] - 1\right) \\
&= \mathbb{E}^{t_i}\left[\frac{1}{\delta_i}\left(\frac{B(t,t_{i-1})}{B(t,t_i)} - 1\right)\middle|\mathcal{F}_0\right] \\
&= \mathbb{E}^{t_i}\left[F_{t_i}(t)|\mathcal{F}_0\right].
\end{aligned}$$

Given the above definitions, the fixed leg can in any interest rate model under the equivalent martingale measure P^* be valued as:

$$\begin{aligned}
\text{swap}_{fixed}(0,t_\alpha,\{t_{\alpha+1},\dots t_\beta\},K) &= \sum_{i=\alpha+1}^{\beta}\mathbb{E}^*\left[\exp\left\{-\int_0^{t_i} r(u)du\right\}\delta_i K\middle|\mathcal{F}_0\right] \\
&= \sum_{i=\alpha+1}^{\beta} B(0,t_i)\delta_i K.
\end{aligned}$$

The floating leg can be computed in the same way as:

$$\begin{aligned}
\text{swap}_{floating}(0,t_\alpha,\{t_{\alpha+1},\dots t_\beta\},F) &= \sum_{i=\alpha+1}^{\beta}\mathbb{E}^*\left[\exp\left\{-\int_0^{t_i} r(u)du\right\}\delta_i F_{t_i}(t_{i-1})\middle|\mathcal{F}_0\right] \\
&= \sum_{i=\alpha+1}^{\beta} B(0,t_i)\delta_i\mathbb{E}^{t_i}[F_{t_i}(t_{i-1})|\mathcal{F}_0] \\
&= \sum_{i=\alpha+1}^{\beta} B(0,t_i)\delta_i F_{t_i}(0) \\
&= \sum_{i=\alpha+1}^{\beta} B(0,t_i)\left(\frac{B(0,t_{i-1})}{B(0,t_i)} - 1\right) \\
&= B(0,t_\alpha) - B(0,t_\beta).
\end{aligned}$$

There we change from P^* to the time t_i forward measure P^{t_i} with the Radon-Nikodým derivative

$$\frac{dP^{t_i}}{dP^*} = \frac{\exp\left\{-\int_0^{t_i} r(u)du\right\}B(t_i,t_i)}{B(0,t_i)}$$

and use the fact that $F_{t_i}(t_{i-1})$ is a martingale under the time t_i forward risk adjusted measure. Note, that the value is independent of the chosen floating rate since we assume a default free economy without any restrictions on liquidity and therefore one interest rate curve is sufficient. For a general treatment on the valuation under basis risk, see Mercurio [2010]. Combining the results for the fixed and the floating leg the value of an interest rate swap is given by:

$$\begin{aligned} \text{swap}(0, t_\alpha, \{t_{\alpha+1}, \dots t_\beta\}, K) &= \omega\left(B(0,t_\alpha) - \left[\sum_{i=\alpha+1}^{\beta} B(0,t_i)\delta_i K + B(0,t_\beta)\right]\right) \\ &= \omega\left(B(0,t_\alpha) - CB(0, \{t_{\alpha+1}, \dots t_\beta\}, K)\right) \qquad (3.2) \end{aligned}$$

with $\omega = 1$ for a payer, $\omega = -1$ for a receiver swap and $CB(0, \{t_{\alpha+1}, \dots t_\beta\}, K)$ is the value of a coupon bond with fixed coupon payments of $K_i = K$ for every payment date $i \in \{t_{\alpha+1}, \dots t_\beta\}$ and redemption at time t_β.
The swap rate $S_{\alpha,\beta}(0)$ is the fixed interest rate which renders the swap a zero value and can also be interpreted as the coupon of a (forward starting) coupon bond which is issued at par. $S_{\alpha,\beta}(0)$ given by:

$$\begin{aligned} 0 &= \omega\left(B(0,t_\alpha) - CB(0, \{t_{\alpha+1}, \dots t_\beta\}, K)\right) \\ \Leftrightarrow 1 &= CB(0, \{t_{\alpha+1}, \dots t_\beta\}, K)/B(0,t_\alpha) \\ \Leftrightarrow S_{\alpha,\beta}(0) &= \frac{B(0,t_\alpha) - B(0,t_\beta)}{\sum_{i=\alpha+1}^{\beta} \delta_i B(0,t_i)}. \qquad (3.3) \end{aligned}$$

As shown by Jamshidian [1999], a numeraire under which the swap rate is a martingale is given by its so called annuity:

$$C_{\alpha,\beta}(0) = \sum_{i=\alpha+1}^{\beta} \delta_i B(0,t_i).$$

The annuity is a valid choice, since

$$C_{\alpha,\beta}(0) \cdot S_{\alpha,\beta}(0) = B(0,t_\alpha) - B(0,t_\beta)$$

is a tradeable asset. The corresponding measure $Q^{\alpha,\beta}$ is called the annuity measure and under this measure a lognormally modeled swap rate evolves as

$$dS_{\alpha,\beta}(t) = \sigma^{\alpha,\beta}(t) S_{\alpha,\beta}(t) dW^{\alpha,\beta}(t) \qquad (3.4)$$

where $\sigma^{\alpha,\beta}(t)$ is a time-dependend volatility and $W^{\alpha,\beta}$ a Brownian motion under $Q^{\alpha,\beta}$. Since the swap rate is a martingale under its annuity measure the expectation can be computed as:

$$E^{\alpha,\beta}[S_{\alpha,\beta}(t)|\mathcal{F}_0] = S_{\alpha,\beta}(0).$$

If the expectation is under a forward risk adjusted measure then the swap rate is no longer a martingale. To solve the expectation we must change to the annuity measure and can then solve the expectation as the forward value of the swaption and an additional term the so called convexity correction. The Radon-Nikodým derivative of this change of measure is given by:

$$\frac{dQ^{\alpha,\beta}}{dP^{t_i}} = \frac{C_{\alpha,\beta}(t)/C_{\alpha,\beta}(0)}{B(t,t_i)/B(0,t_i)}.$$

Definition 3.2.1. *Given a swap rate $S_{\alpha,\beta}(t)$ its expectation under a forward risk adjusted measure $Q^{\alpha+\Delta}$ can be computed as its time 0 value and an additional expectation, we call convexity correction, given in the following equation:*

$$\begin{aligned} \mathbb{E}^{\alpha+\Delta}[S_{\alpha,\beta}(t_\alpha)|\mathcal{F}_0] &= \mathbb{E}^{\alpha,\beta}\left[S_{\alpha,\beta}(t_\alpha)\frac{B(t_\alpha,t_{\alpha+\Delta})}{C_{\alpha,\beta}(t_\alpha)}\frac{C_{\alpha,\beta}(0)}{B(0,t_{\alpha+\Delta})}\middle|\mathcal{F}_0\right] \\ &= \mathbb{E}^{\alpha,\beta}\left[S_{\alpha,\beta}(t_\alpha)\middle|\mathcal{F}_0\right] + \mathbb{E}^{\alpha,\beta}\left[S_{\alpha,\beta}(t_\alpha)\left(\frac{B(t_\alpha,t_{\alpha+\Delta})}{C_{\alpha,\beta}(t_\alpha)}\frac{C_{\alpha,\beta}(0)}{B(0,t_{\alpha+\Delta})}-1\right)\middle|\mathcal{F}_0\right] \\ &= S_{\alpha,\beta}(0) + \textit{Convexity Correction}. \end{aligned} \tag{3.5}$$

Δ *is the time period between the fixing and the payment of the swap rate.*

This additional term due to the mismatch of measures is called convexity correction.
To illustrate the convexity, assume that forward bond prices can be computed by an annually forward swap rate $y(t)$ and are given as:

$$B^F(0,1,2) \approx \frac{(1+y(1))^{-1}}{B(0,1)}.$$

For three given equidistant forward bond prices the forward and expected swap rates are plotted in Figure (3.2). It can be seen that due to the convexity of the swap rate the forward swap rate is smaller than the expected swap rate, $y \leq E[y]$. The distance between both rates is the so called convexity adjustment. The computation of this convexity correction is discussed in Section (3.5).

A constant maturity swap (CMS) is also an interest rate swap but pays instead of a fixed or floating rate the m-year swap rate at every payment date. The cash flow of a CMS with a receiving fixed leg and a paying CMS leg is illustrated in Figure (3.3).

Assuming that the CMS leg is set-in-advance with pay dates $\tilde{t} \in \{t_{\alpha+1}, \dots t_\beta\}$, it can be

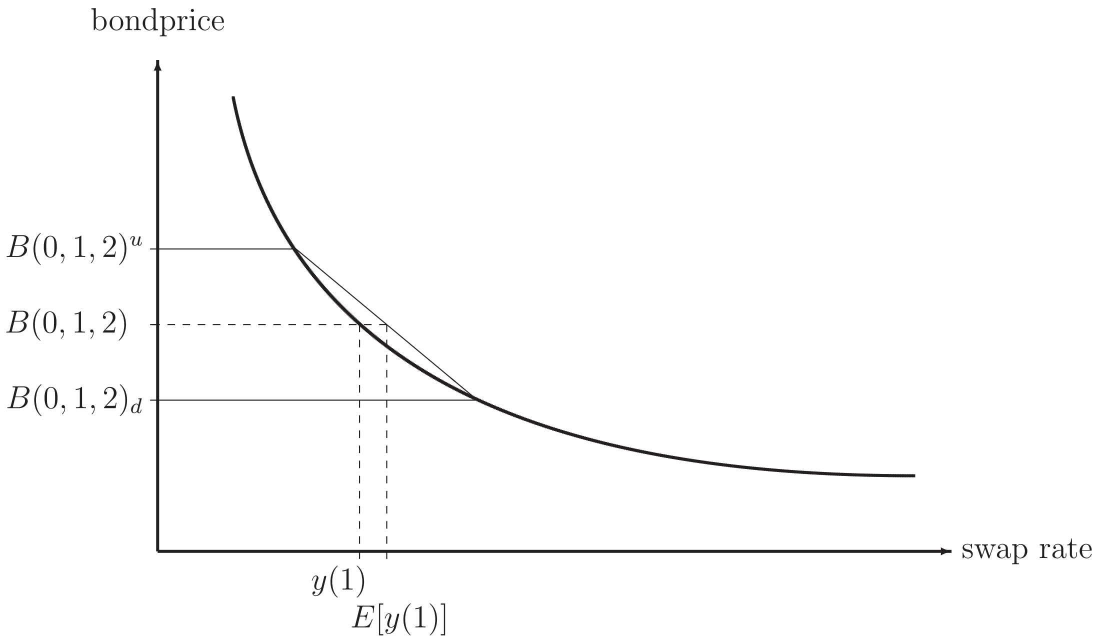

Figure 3.2: This figure shows, how forward swap rates and expected swap rates are related by the swap rate convexity.

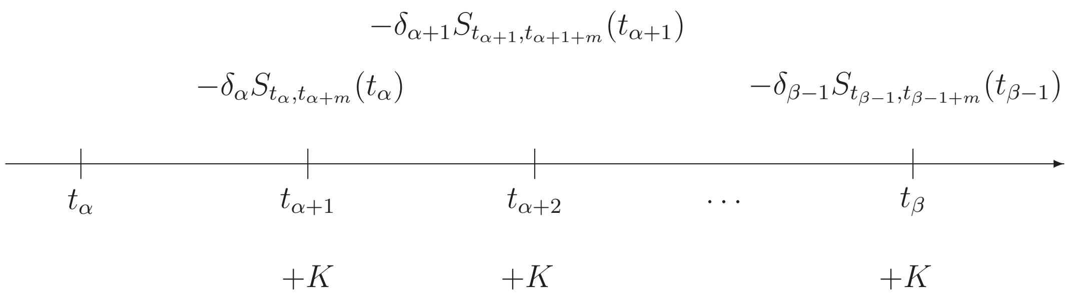

Figure 3.3: Illustration of a CMS cash flow.

computed as follows:

$$\begin{aligned}
\text{swap}_{CMS}(0, t_\alpha, \tilde{t}, m) &= \sum_{i=\alpha}^{\beta-1} \mathbb{E}^* \left[\exp\left\{ -\int_0^{t_{i+1}} r(u)du \right\} \delta_i S_{t_i, t_{i+m}}(t_i) \middle| \mathcal{F}_0 \right] \\
&= \sum_{i=\alpha}^{\beta-1} \delta_i B(0, t_{i+1}) \mathbb{E}^{t_{i+1}} \left[S_{t_i, t_{i+m}}(t_i) \middle| \mathcal{F}_0 \right] \\
&= \sum_{i=\alpha}^{\beta-1} \delta_i B(0, t_{i+1}) \left(S_{t_i, t_{i+m}}(0) + \text{Convexity Correction}(0) \right).
\end{aligned}$$

Since only the swap rate S is paid at every payment date and not a swap with this swap rate, S is not a martingale under the forward measure and it results in the forward starting value plus a convexity correction, which is shown in Eq. (3.5). This convexity correction can be approximated analytically or by a CMS caplet and a CMS floorlet replicated by a portfolio of swaptions using the cap-floor parity. In either case assumptions about the underlying stochastic process of the swap rate process have to be made and therefore an arbitrage free model has to be chosen.

3.3 Interest Rate Models

The stochastic process of a swap rate can be modeled by the assumption of a geometric Brownian motion with a deterministic volatility. The resulting closed form solution for an option on a swap, a so-called swaptions, is the Black76 formula. The Black76 model developed in Black [1976] is the interest rate equivalent to the classic model by Black and Scholes [1973].

A swaption is a European option where the owner of the contract has the right to enter a pre-defined interest rate swap at time t_α for a given fixed coupon K. In analogy to the interest rate swap these options are called payer and receiver swaptions. Their payoff at time t_α is defined as:

$$\begin{aligned} & \max\{\omega[B(t_\alpha,t_\alpha) - CB(t_\alpha,\{t_{\alpha+1},\dots t_\beta\},K)],0\} \\ = & \sum_{i=\alpha+1}^{\beta} \delta_i B(t_\alpha,t_i) \frac{\max\left\{\omega[1 - B(t_\alpha,t_\beta) - K\sum_{j=\alpha+1}^{\beta}\delta_j B(t_\alpha,t_j)],0\right\}}{\sum_{k=\alpha+1}^{\beta}\delta_k B(t_\alpha,t_k)} \\ = & \left(\sum_{i=\alpha+1}^{\beta}\delta_i B(t_\alpha,t_i)\right)\max\{\omega[S_{\alpha,\beta}(t_\alpha) - K],0\} \end{aligned}$$

where we used Eq. (3.2) and Eq. (3.3).

The Black76 formula can be derived by using the risk neutral pricing rule in combination with the change of numeraire technique. The swap rate is defined as in Eq. (3.4) and the measure change is to the corresponding annuity measure $Q^{\alpha,\beta}$ with $C_{\alpha,\beta}(0) = \sum_{i=\alpha+1}^{\beta}\delta B(0,t_i)$ as the new numeraire. The Radon-Nikodým derivative for a change of measure from P^* to $Q^{\alpha,\beta}$ is

$$\frac{dQ^{\alpha,\beta}}{dP^*} = \frac{\exp\left\{-\int_0^{t_\alpha} r(u)du\right\} C_{\alpha,\beta}(t_\alpha)}{C_{\alpha,\beta}(0)}.$$

Taking the discounted expected value of the swaptions payoff under P^*, the Black76 formula for a payer swaption can be derived as:

$$
\begin{aligned}
& \text{Swaption}_{payer}(S_{\alpha,\beta}(0), K, \sigma, 0, t_\alpha, \omega = 1) \\
= \; & E^* \left[\exp\left\{ -\int_0^{t_\alpha} r(u)du \right\} C_{\alpha,\beta}(t_\alpha)[S_{\alpha,\beta}(t_\alpha) - K]^+ \middle| \mathcal{F}_0 \right] \\
= \; & E^{\alpha,\beta} \left[\exp\left\{ -\int_0^{t_\alpha} r(u)du \right\} C_{\alpha,\beta}(t_\alpha)[S_{\alpha,\beta}(t_\alpha) - K]^+ \frac{C_{\alpha,\beta}(0)/C_{\alpha,\beta}(t_\alpha)}{\exp\left\{ -\int_0^{t_\alpha} r(u)du \right\}} \middle| \mathcal{F}_0 \right] \\
= \; & C_{\alpha,\beta}(0) E^{\alpha,\beta} \left[[S_{\alpha,\beta}(t_\alpha) - K]^+ | \mathcal{F}_0 \right] \\
= \; & C_{\alpha,\beta}(0) \left(E^{\alpha,\beta} \left[S_{\alpha,\beta}(t_\alpha) 1_{\{S_{\alpha,\beta}(t_\alpha) > K\}} | \mathcal{F}_0 \right] - K E^{\alpha,\beta} \left[1_{\{S_{\alpha,\beta}(t_\alpha) > K\}} | \mathcal{F}_0 \right] \right) \\
= \; & C_{\alpha,\beta}(0)(S_{\alpha,\beta}(0)N(d_1) - KN(d_2)) \\
& \text{with } d_1 = \frac{\ln\left(\frac{S_{\alpha,\beta}(0)}{K}\right) + \frac{1}{2}\sigma^2 t_\alpha}{\sigma\sqrt{t_\alpha}} \\
& \text{and } d_2 = \frac{\ln\left(\frac{S_{\alpha,\beta}(0)}{K}\right) - \frac{1}{2}\sigma^2 t_\alpha}{\sigma\sqrt{t_\alpha}}.
\end{aligned}
$$

And the value of a receiver swaption is given as

$$
\text{Swaption}_{receiver}(S_{\alpha,\beta}(0), K, \sigma, 0, t_\alpha, \omega = -1) \;=\; C_{\alpha,\beta}(0)(KN(-d_2) - S_{\alpha,\beta}(0)N(-d_1)).
$$

In the Swap Market Model for forward starting swap rates the same result is obtained for $\sigma\sqrt{t_\alpha} = \left(\int_0^{t_\alpha} \sigma^{\alpha,\beta}(u)^2 du \right)^{1/2}$ with $\sigma^{\alpha,\beta}(.)$ as the time dependent volatility of the forward swap rate in the Swap Market Model.

One typical feature of swaptions in European fixed income markets is the so called cash settlement. Here, the counterparty does not deliver an interest rate swap for the strike price but a cash payment. The computation of the value of the swap that will be paid is done under the assumption that the discounting of the swap payments is be done by the swap rate. Therefore, the participants have to agree on only one forward swap rate y and assume implicitly a flat yield curve. The corresponding annuity of a cash settled swaption is denoted by $A_{\alpha,\beta}(t, y)$ and can be computed for $\delta_i = \delta \; \forall i$ by using the approximation

$B(0, t_{\alpha+\delta}) \approx B(0, t_\alpha)(1 + \delta y)^{-1}$ as:

$$\begin{aligned} A_{\alpha,\beta}(0, y) &= B(0, t_\alpha)\frac{C_{\alpha,\beta}(0)}{B(0, t_\alpha)} \\ &= B(0, t_\alpha)\delta \sum_{i=\alpha+1}^{\beta} \frac{B(0, t_i)}{B(0, t_\alpha)} \\ &\approx B(0, t_\alpha)\delta \sum_{i=1}^{n} \frac{1}{(1 + \delta y)^i} \\ &= B(0, t_\alpha)\delta(1 + \delta y)^{-1} \left(\frac{1 - (1 + \delta y)^{-n}}{1 - (1 + \delta y)^{-1}}\right) \\ &= B(0, t_\alpha)\delta \left(\frac{1 - (1 + \delta y)^{-n}}{(1 + \delta y) - 1}\right) \\ &= \frac{B(0, t_\alpha)}{y} \left(1 - \frac{1}{(1 + \delta y)^n}\right) \qquad (3.6) \\ \text{with } n &= \frac{t_\beta - t_\alpha}{\delta} \end{aligned}$$

However it is not possible to impose a martingale dynamic for a swap under $A_{\alpha,\beta}$ since this is not a tradeable asset. To derive the value of a swaption within the Black76, the physical annuity measure $Q^{\alpha,\beta}$ needs to be taken and the value can only be approximated as:

$$\begin{aligned} \text{Swaption}(S_{\alpha,\beta}(0), K, \sigma, 0, t_\alpha, \omega) &\approx A_{\alpha,\beta}(0, S_{\alpha,\beta}(0))E^{\alpha,\beta}[\max\{\omega(S_{\alpha,\beta}(t_\alpha) - K), 0\}|\mathcal{F}_0] \\ &= A_{\alpha,\beta}(0, S_{\alpha,\beta}(0))[\omega(S_{\alpha,\beta}(0)N(\omega \cdot d1) - KN(\omega \cdot d2))] \qquad (3.7) \end{aligned}$$

where $\omega = 1$ denotes a payer and $\omega = -1$ a receiver swaption. Note, that this is no longer the unique arbitrage free price, since $A_{\alpha,\beta}(0, S_{\alpha,\beta}(0))$ is not the numeraire of the swap measure.

One problem encountered when modeling derivatives like swaptions in a Swap Market Model and therefore using the Black [1976] formula is that the market prices for swaptions cannot be obtained with a constant volatility parameter as the model demands. Instead the volatility tends to rise if the option is out-of-the money. This results in the so called volatility smile describing the fact that implied Black volatility is strike-dependent. As can be seen in Table (3.1) and Figure (3.4) the quoted implied volatilities for plain-vanilla swaptions depend on the strike price and the time to maturity of the option. This kind of volatility surface was first observed after the 1989 crash where the market realized that normally distributed log-returns have a higher kurtosis than the normal distribution allows.

The problem with implied volatility is that it needs to be interpolated from market data

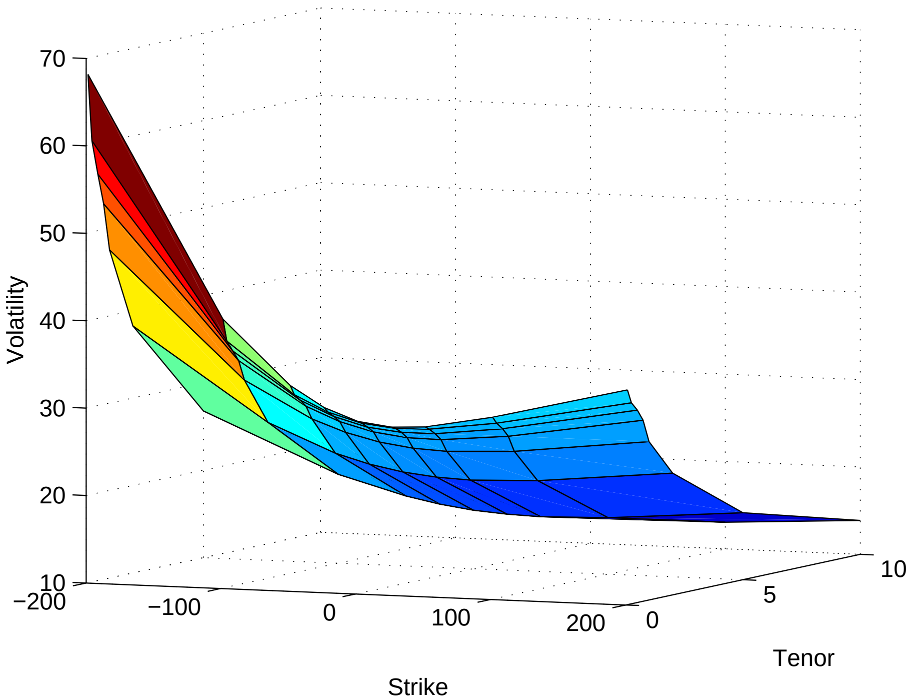

Figure 3.4: Visualization of the volcube of swaptions with an underlying swap rate tenor of 5 years as given in Table (3.1). The x-axis represents the deviation of the strike price from the at-the-money strike in basis points. The y-axis the time to maturity of the swaption.

and more important the assumption of a different model for each strike. With this in mind Dupire [1994] proposed the local volatility model. The advantage of this approach is that the model perfectly replicates the current market situation. But the approach behaves poorly in forecasting future dynamics and option pricing is not possible in closed form. Therefore Hagan et al. [2003] proposed the so called SABR model where a forward price process is modeled under its martingale measure with a correlated stochastic volatility process. One advantage of the SABR model is that there exist approximations to a volatility function which can be plugged into the Black76 formula to achieve option prices of a SABR distributed asset in the Black76 framework.

Assumption 3.3.1. *In the SABR model, the forward price process $F(t)$ and its volatility*

	-200	-100	-50	-25	0	25	50	100	200
1m	68.13	40.76	33.47	31.11	29.70	29.19	29.36	30.79	35.45
3m	60.45	38.17	32.34	30.50	29.40	28.96	29.02	30.02	32.97
6m	56.47	36.89	31.55	29.85	28.80	28.35	28.36	29.23	31.99
9m	52.96	35.64	30.73	29.13	28.10	27.61	27.54	28.23	30.70
1y	47.53	33.26	29.13	27.75	26.80	26.26	26.06	26.36	28.10
2y	38.25	27.83	24.65	23.56	22.80	22.36	22.19	22.43	23.92
5y	26.80	20.16	18.00	17.24	16.70	16.39	16.27	16.47	17.66
10y	20.32	15.77	14.25	13.70	13.30	13.05	12.95	13.05	13.89

Table 3.1: Table of annualized swaption volatilites with an underlying swap rate tenor of 5 years. The values depend on the deviation of the strike price from the at the money strike in basispoints and the time to maturity of the swaption. Source: ICAP, 11.03.2009.

$\alpha(t)$ *are modelled as:*

$$\begin{aligned} dF(t) &= \alpha(t)F(t)^{\beta}dW(t) \\ d\alpha(t) &= \nu\alpha(t)dZ(t) \\ F(0) &= f \\ \alpha(0) &= \alpha \\ d\langle W,Z\rangle_t &= \rho_{W,Z}dt. \end{aligned}$$

The implied Black76 volatility function using the SABR dynamics is given as:

$$\begin{aligned} \sigma_{SABR}(K,t,\ldots) &\approx \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}log^2 f/K+\frac{(1-\beta)^4}{1920}log^4 f/K\right\}}\left(\frac{z}{x(z)}\right) \\ &\quad \left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]t\right\} \\ \text{with} \qquad & z=\frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log f/K \\ \text{and} \qquad & x(z)=\log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}. \end{aligned}$$

$\alpha(t)$ *is the stochastic volatility,* ν *vol of the vol and* $Z(t)$ *the Brownian motion of the volatility process.* β *can be chosen to further specify the distribution of the forward price process.*

For example $\beta = 1$ constitutes a lognormal distribution and $\beta = 0$ a normal distribution under deterministic volatility and is also called the backbone of the diffusion process. The

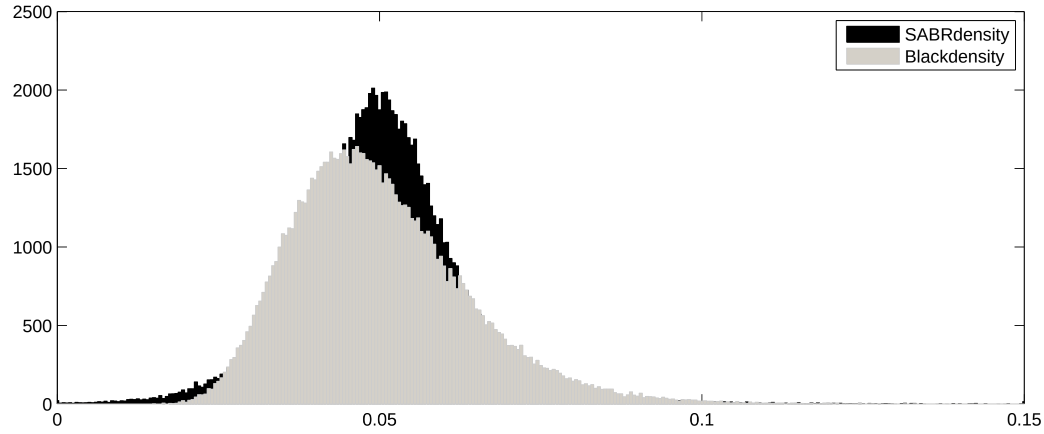

Figure 3.5: Simulated density of a SABR diffusion and a Black76 using the same ATM volatility and random variables.

	-200	-100	-50	-25	0	25	50	100	200
1m	0.418	-0.574	-0.352	-0.121	0.134	0.323	0.378	0.198	-0.548
3m	0.328	-0.474	-0.268	-0.085	0.111	0.253	0.298	0.173	-0.428
6m	0.290	-0.404	-0.243	-0.077	0.089	0.217	0.262	0.161	-0.370
9m	0.227	-0.301	-0.198	-0.072	0.059	0.168	0.211	0.149	-0.295
1y	0.184	-0.245	-0.165	-0.059	0.043	0.130	0.175	0.137	-0.237
2y	0.136	-0.174	-0.117	-0.047	0.025	0.090	0.125	0.104	-0.166
5y	0.115	-0.139	-0.093	-0.039	0.015	0.069	0.097	0.085	-0.126
10y	0.052	-0.062	-0.041	-0.019	0.003	0.024	0.049	0.047	-0.060

Table 3.2: Table of the differences between the swaption data of Table (3.1) and the results obtained by a functional SABR model calibrated to the same data set.

difference between a SABR density and a lognormal density is plotted in Figure (3.5). It can be seen that the SABR distribution has heavier tails in comparison to the lognormal distribution.

The volatility function $\sigma_{SABR}(K, t, \ldots)$ is time and strike dependent and can replicate the observed smile and skew effects very well as can be seen in Figure (3.6). The differences between implied swaption volatilities from market quotes and SABR volatilities calibrated to the same data set are given in Table (3.2). These differences are all below 1% which

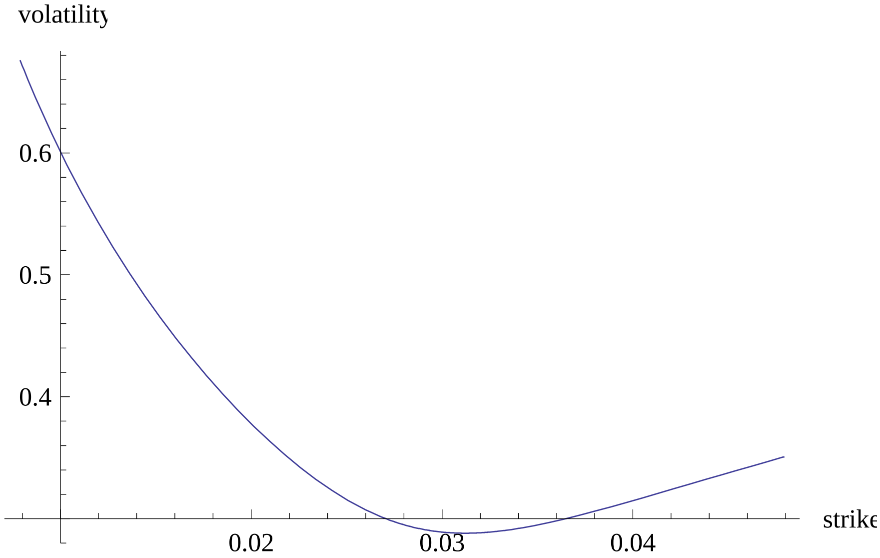

Figure 3.6: SABR volatility for the 1m5y swaption obtained by calibration to the market data of Table (3.2).

indicates a good fit of the SABR model to market prices. Therefore, option prices can be calculated in a well known pricing framework under consideration of the volatility cube using a strike- and time-to-maturity dependent volatility function as a calibration tool which is justified through the SABR model. Today, the SABR model has become a standard model in the financial industry because of these properties.

To compute the expected swap rate under a forward measure, we can use the cap-floor parity on CMS caplets and floorlets as will be discussed in the following sections. A CMS caplet/foorlet is the interest rate equivalent of a call or put option with a single swap rate payment for every caplet/floorlet as an underlying. The payoff at time t_α of a CMS caplet/floorlet is given by:

$$B(t_\alpha, t_{\alpha+\Delta}) \max\left\{\omega(S_{\alpha,\beta}(t_\alpha) - K), 0\right\}.$$

The bond price indicates that the payoff can be for a fixing in advance for a period of Δ and is equal to one for a fixing in arrears. To compute the Black 76 price, the convexity corrected expectation of the swaprate needs to be computed since

$$\mathbb{E}^{\alpha+\Delta}[(S_{\alpha,\beta}(t_\alpha)|\mathcal{F}_0] = S_{\alpha,\beta}(0) + \text{Convexity Correction}$$

as shown in Eq. (3.5).

One approach is an analytical approximation and another approach is the replication of the payoff by a portfolio of plain-vanilla swaptions. In the following sections we summarize the analytic approximation of the convexity correction and the replication portfolio. Then we extend the replication method to the case of a finite amount of swaptions to compute the replication portfolio.

3.4 Convexity Correction: Replication

This approach is introduced by Hagan [2003] and decomposes the value of a caplet/floorlet into a portfolio of plain vanilla swaptions. Using cap-floor parity a convexity corrected expected future swap rate can be computed by static positions in the forward starting swap rate and swaptions. Two major advantages of this method are that the convexity adjustment is tradeable and can be hedged in a consistent way to the swaption book of the trader and it is independent of the actual model chosen to value the swaptions.
Using the same approach as Hagan [2003] we first compute the static replication portfolios for the CMS caplets and floorlets.

Theorem 3.4.1. *A CMS caplet/floorlet can be replicated by a portfolio of plain-vanilla swaptions given by:*

$$\begin{aligned} \textit{CMS caplet} &= \frac{B(0,t_{\alpha+\Delta})}{C_{\alpha,\beta}(0)}\left((1+f'(K))PS(K)+\int_K^\infty f''(x)PS(x)dx\right) && (3.8)\\ \textit{CMS floorlet} &= \frac{B(0,t_{\alpha+\Delta})}{C_{\alpha,\beta}(0)}\left((1+f'(K))RS(K)-\int_{-\infty}^K f''(x)RS(x)dx\right) && (3.9)\\ \textit{with } f(x) &= [x-K]\left(\frac{G(x)}{G(S_{\alpha,\beta}(0))}-1\right). \end{aligned}$$

where PS(x) denotes a corresponding payer swaption with strike x and RS(x) a receiver swaption given by Eq. (3.7).

Before we prove the theorem we need a Lemma concerning the representation of functions.

Lemma 3.4.1. *We can define a smooth function $f(S)$ with $f(K)=0$ which can be represented as:*

$$f'(K)[S-K]^+ + \int_K^\infty [S-x]^+ f''(x)dx = \begin{cases} f(S), & K < S,\\ 0, & K \geq S. \end{cases}$$

Proof. Using partial integration and a function $g(x) = [S-x]^+$ we can show:

$$\begin{aligned}
\int_K^\infty g(x)f''(x)dx &= f'(x)g(x)|_K^\infty - \int_K^\infty g'(x)f'(x)dx \\
&= f'(x)g(x)|_K^S + f'(x)g(x)|_S^\infty - \int_K^\infty g'(x)f'(x)dx \\
&= \underbrace{f'(S)g(S)}_{=0} - f'(K)g(K) + \underbrace{f'(\infty)g(\infty)}_{=0} - \underbrace{f'(S)g(S)}_{=0} - \int_K^\infty g'(x)f'(x)dx \\
&= -f'(K)g(K) - \int_K^\infty g'(x)f'(x)dx \\
&= -f'(K)[S-K]^+ - \int_K^\infty g'(x)f'(x)dx.
\end{aligned}$$

To solve the last integral we have to remember that the function $g(x)$ has the payoff and derivative

$$\begin{aligned}
g(x) &= \begin{cases} S-x, & x < S, \\ 0, & x \geq S. \end{cases} \\
\Rightarrow g'(x) &= \begin{cases} -1, & x < S, \\ 0, & x \geq S. \end{cases}
\end{aligned}$$

Using this result we can solve the integral as

$$\begin{aligned}
-\int_K^S f'(x)dx &= -f(x)|_K^S \\
&= -(f(S) - f(K)) \\
&= -f(S).
\end{aligned}$$

Putting everything together we end up with the formula given in the Lemma. □

Now, we can turn to the valuation of CMS caplets and floorlets.

Proof. The time 0 value of a CMS caplet is given as:

$$\begin{aligned}
& B(0,t_{\alpha+\Delta})\mathbb{E}^{\alpha+\Delta}[[S_{\alpha,\beta}(t_\alpha) - K]^+|\mathcal{F}_0] \\
=\ & B(0,t_{\alpha+\Delta})\mathbb{E}^{\alpha,\beta}\left[[S_{\alpha,\beta}(t_\alpha) - K]^+ \left(\frac{B(t_\alpha,t_{\alpha+\Delta})}{B(0,t_{\alpha+\Delta})}\frac{C_{\alpha,\beta}(0)}{C_{\alpha,\beta}(t_\alpha)}\right)\middle|\mathcal{F}_0\right] \\
=\ & B(0,t_{\alpha+\Delta})\left(\mathbb{E}^{\alpha,\beta}[[S_{\alpha,\beta}(t_\alpha) - K]^+|\mathcal{F}_0]\right. \\
& \left.+\mathbb{E}^{\alpha,\beta}\left[[S_{\alpha,\beta}(t_\alpha) - K]^+ \left(\frac{B(t_\alpha,t_{\alpha+\Delta})}{B(0,t_{\alpha+\Delta})}\frac{C_{\alpha,\beta}(0)}{C_{\alpha,\beta}(t_\alpha)} - 1\right)\middle|\mathcal{F}_0\right]\right) \\
=\ & B(0,t_{\alpha+\Delta})\left(\frac{\mathrm{PS}(K)}{C_{\alpha,\beta}(0)} + \mathbb{E}^{\alpha,\beta}\left[[S_{\alpha,\beta}(t_\alpha) - K]^+ \left(\frac{B(t_\alpha,t_{\alpha+\Delta})}{B(0,t_{\alpha+\Delta})}\frac{C_{\alpha,\beta}(0)}{C_{\alpha,\beta}(t_\alpha)} - 1\right)\middle|\mathcal{F}_0\right]\right)
\end{aligned} \tag{3.10}$$

To compute the value of the CMS caplet the equation

$$\mathbb{E}^{\alpha,\beta}\left[[S_{\alpha,\beta}(t_\alpha)-K]^+\left(\frac{B(t_\alpha,t_{\alpha+\Delta})}{B(0,t_{\alpha+\Delta})}\frac{C_{\alpha,\beta}(0)}{C_{\alpha,\beta}(t_\alpha)}-1\right)\middle|\mathcal{F}_0\right]$$

will be replicated by a portfolio of swaptions.
To proceed, we assume that $\frac{B(t_\alpha,t_{\alpha+\Delta})}{C_{\alpha,\beta}(t_\alpha)}$ can be expressed as a function $G(S_{\alpha,\beta}(t_\alpha))$ of the swap rate. Using Eq. (3.6) the function is given by:

$$\begin{aligned} G(S_{\alpha,\beta}(t_\alpha)) &= \frac{B(t_\alpha,t_{\alpha+\Delta})}{C_{\alpha,\beta}(t_\alpha)} \\ &= S_{\alpha,\beta}(t_\alpha)\frac{(1+\tau S_{\alpha,\beta}(t_\alpha))^{n-\Delta}}{(1+\tau S_{\alpha,\beta}(t_\alpha))^n-1}. \end{aligned}$$

There we approximate $B(t_\alpha,t_{\alpha+\Delta})$ as in the case of cash-settlement as

$$B(t_\alpha,t_{\alpha+\Delta})\approx(1+\tau S_{\alpha,\beta}(t_\alpha))^{-\Delta}.$$

Using Lemma (3.4.1) and choosing $f(x)$ as

$$f(x)=[x-K]\left(\frac{G(x)}{G(S_{\alpha,\beta}(0))}-1\right)$$

we can compute:

$$\begin{aligned} &\mathbb{E}^{\alpha,\beta}\left[[S_{\alpha,\beta}(t_\alpha)-K]^+\left(\frac{G(S_{\alpha,\beta}(t_\alpha))}{G(S_{\alpha,\beta}(0))}-1\right)\middle|\mathcal{F}_0\right] \\ =\;& \mathbb{E}^{\alpha,\beta}\left[f(S_{\alpha,\beta}(t_\alpha))|\mathcal{F}_0\right] \\ =\;& \mathbb{E}^{\alpha,\beta}\left[f'(K)[S_{\alpha,\beta}(t_\alpha)-K]^+ + \int_K^\infty [S_{\alpha,\beta}(t_\alpha)-x]^+ f''(x)dx\middle|\mathcal{F}_0\right] \\ =\;& f'(K)\mathbb{E}^{\alpha,\beta}\left[[S_{\alpha,\beta}(t_\alpha)-K]^+|\mathcal{F}_0\right] + \int_K^\infty \mathbb{E}^{\alpha,\beta}\left[[S_{\alpha,\beta}(t_\alpha)-x]^+|\mathcal{F}_0\right] f''(x)dx \\ =\;& \frac{1}{C_{\alpha,\beta}(0)}\left(f'(K)PS(K)+\int_K^\infty PS(x)f''(x)dx\right) \end{aligned} \tag{3.11}$$

where $PS(x)$ is the value of a payer cash-settled swaption with strike x. Similarly,

$$\begin{aligned} &\mathbb{E}^{\alpha,\beta}\left[[K-S_{\alpha,\beta}(t_\alpha)]^+\left(\frac{G(S_{\alpha,\beta}(t_\alpha))}{G(S_{\alpha,\beta}(0))}-1\right)\middle|\mathcal{F}_0\right] \\ =\;& \mathbb{E}^{\alpha,\beta}\left[f'(K)[K-S_{\alpha,\beta}(t_\alpha)]^+ + \int_{-\infty}^K [x-S_{\alpha,\beta}(t_\alpha)]^+ f''(x)dx\middle|\mathcal{F}_0\right] \\ =\;& \frac{1}{C_{\alpha,\beta}(0)}\left(f'(K)RS(K)-\int_{-\infty}^K RS(x)f''(x)dx\right) \end{aligned}$$

where $RS(x)$, the value of a receiver cash-settled swaption with strike x.

Combining Eq. (3.10) and Eq. (3.11) the CMS caplet can be replicated by:

$$\frac{B(0,t_{\alpha+\Delta})}{C_{\alpha,\beta}(0)}\left((1+f'(K))\mathrm{PS}(K)+\int_K^\infty f''(x)\mathrm{PS}(x)dx\right).$$

Using the same arguments the value of a CMS floorlet can be computed as:

$$\frac{B(0,t_{\alpha+\Delta})}{C_{\alpha,\beta}(0)}\left((1+f'(K))\mathrm{RS}(K)-\int_{-\infty}^K f''(x)\mathrm{RS}(x)dx\right).$$

□

Corollary 3.4.1. *Using the replication portfolios for CMS caplets and floorlets the convexity correction of an expected swap rate $\mathbb{E}^{\alpha+\Delta}[S_{\alpha,\beta}(t_\alpha)|\mathcal{F}_0]$ can be computed as:*

$$\frac{1}{C_{\alpha,\beta}(0)}\left(\int_{S_{\alpha,\beta}(0)}^\infty f''(x)PS(x)dx+\int_{-\infty}^{S_{\alpha,\beta}(0)} f''(x)RS(x)dx\right) \quad (3.12)$$

$$\textit{with } f(x) = [x-S_{\alpha,\beta}(0)]\left(\frac{G(x)}{G(S_{\alpha,\beta}(0))}-1\right).$$

Proof. Using the cap-floor parity and choosing $S_{\alpha,\beta}(0)$ as strike price

$$\begin{aligned}[S-K]^+-[K-S]^+ &= S-K\\ \Rightarrow \text{CMS caplet} - \text{CMS floorlet} &= B(0,t_{\alpha+\Delta})\left(\mathbb{E}^{\alpha+\Delta}[S_{\alpha,\beta}(t_\alpha)|\mathcal{F}_0]-S_{\alpha,\beta}(0)\right)\end{aligned}$$

the convexity adjustment of an expected future swap rate can be computed by a portfolio of payer and receiver swaptions given by Eq. (3.8) and Eq. (3.9).

$$\mathbb{E}^{\alpha+\Delta}[S_{\alpha,\beta}(t_\alpha)|\mathcal{F}_0]=S_{\alpha,\beta}(0)+\underbrace{\frac{1}{C_{\alpha,\beta}(0)}\left(\int_{S_{\alpha,\beta}(0)}^\infty f''(x)\mathrm{PS}(x)dx+\int_{-\infty}^{S_{\alpha,\beta}(0)} f''(x)\mathrm{RS}(x)dx\right)}_{\textit{Convexity Correction}}.$$

□

With this method the volatility cube can be integrated into the convexity correction by using an interest rate model for the valuation of the swaptions that is capable of replicating the smile like the SABR model.
But to implement the replication portfolio the integral has to be discretized. One approach

is to discretize the integral into equally spaced buckets and to interpret the integral as a sum over plain-vanilla swaptions centered in each bucket. Since the value of a payer swaption is a decreasing function in its strike price we do not have to consider buckets till infinity and can use a algorithm that breaks off when the prices fall below a threshold. In the next section we derive an alternative algorithm to compute a discrete replication portfolio.

3.4.1 Analytical Approximation

If a Black76 model with deterministic volatility is chosen, Eq. (3.12) can be approximated in closed form.

Corollary 3.4.2. *Assuming a Black76 dynamic for the swaprate the convexity correction by a replication portfolio can be approximated in closed form and is given as:*

$$\begin{aligned} \textit{Convexity Correction} &\approx S_{\alpha,\beta}(0)\theta(\Delta)\left(\exp\{\sigma^2_{\alpha,\beta}t_\alpha\} - 1\right) \\ \textit{with } \theta(\Delta) &= 1 - \frac{\tau S_{\alpha,\beta}(0)}{1+\tau S_{\alpha,\beta}(0)}\left(\Delta + \frac{\beta-\alpha}{(1+\tau S_{\alpha,\beta}(0))^{\beta-\alpha}-1}\right). \end{aligned}$$

Proof. As in Hagan [2003] we expand $G(x)$ around the forward swaprate by a first order Taylor expansion, since the future realizations will be heavily centered around this value.

$$G(x) \approx G(S_{\alpha,\beta}(0)) + G'(S_{\alpha,\beta}(0))(x - S_{\alpha,\beta}(0))$$

Using the expanded $G(x)$ we can reformulate $f(x)$ as:

$$\begin{aligned} f(x) &= (x-K)\left(\frac{G(S_{\alpha,\beta}(0)) + G'(S_{\alpha,\beta}(0))(x - S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))} - 1\right) \\ &= (x-K)(x - S_{\alpha,\beta}(0))\frac{G'(S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))}. \end{aligned}$$

Using Eq. (3.8) the value of a CMS caplet can now be computed as:

$$\begin{aligned}
& \frac{B(0,t_{\alpha+\Delta})}{C_{\alpha,\beta}(0)}\left(\left(1+(K-S_{\alpha,\beta}(0))\frac{G'(S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))}\right)\mathrm{PS}(K)+\int_K^\infty 2\frac{G'(S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))}\mathrm{PS}(x)dx\right) \\
= & \frac{B(0,t_{\alpha+\Delta})}{C_{\alpha,\beta}(0)}\left(\mathrm{PS}(K)+\frac{G'(S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))}\left((K-S_{\alpha,\beta}(0))\mathrm{PS}(K)+\int_K^\infty 2\mathrm{PS}(x)dx\right)\right) \\
= & \frac{B(0,t_{\alpha+\Delta})}{C_{\alpha,\beta}(0)}\mathrm{PS}(K)+G'(S_{\alpha,\beta}(0))C_{\alpha,\beta}(0)\left((K-S_{\alpha,\beta}(0))\mathbb{E}[[S_{\alpha,\beta}(t_\alpha)-K]^+|\mathcal{F}_0]\right. \\
& \left.+\int_K^\infty 2\mathbb{E}[[S_{\alpha,\beta}(t_\alpha)-x]^+|\mathcal{F}_0]dx\right) \\
= & \frac{B(0,t_{\alpha+\Delta})}{C_{\alpha,\beta}(0)}\mathrm{PS}(K)+G'(S_{\alpha,\beta}(0))C_{\alpha,\beta}(0)\left((K-S_{\alpha,\beta}(0))\mathbb{E}[[S_{\alpha,\beta}(t_\alpha)-K]^+|\mathcal{F}_0]\right. \\
& \left.+\mathbb{E}[([S_{\alpha,\beta}(t_\alpha)-K]^+)^2|\mathcal{F}_0]\right) \\
= & \frac{B(0,t_{\alpha+\Delta})}{C_{\alpha,\beta}(0)}\mathrm{PS}(K)+G'(S_{\alpha,\beta}(0))C_{\alpha,\beta}(0)\left(\mathbb{E}[(S_{\alpha,\beta}(t_\alpha)-S_{\alpha,\beta}(0))[S_{\alpha,\beta}(t_\alpha)-K]^+|\mathcal{F}_0]\right).
\end{aligned}$$

In the same manner the value of the CMS floorlet can be computed as:

$$\frac{B(0,t_{\alpha+\Delta})}{C_{\alpha,\beta}(0)}\mathrm{RS}(K)-G'(S_{\alpha,\beta}(0))C_{\alpha,\beta}(0)\left(\mathbb{E}[(S_{\alpha,\beta}(0)-S_{\alpha,\beta}(t_\alpha))[K-S_{\alpha,\beta}(t_\alpha)]^+|\mathcal{F}_0]\right).$$

Assuming a Black76 model and using cap-floor parity again the convexity correction can be computed as:

$$\begin{aligned}
& \frac{G'(S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))}\mathbb{E}[(S_{\alpha,\beta}(t_\alpha)-S_{\alpha,\beta}(0))^2|\mathcal{F}_0] \\
= & \frac{G'(S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))}S^2_{\alpha,\beta}(0))(\exp\{\sigma^2_{\alpha,\beta}t_\alpha\}-1).
\end{aligned} \tag{3.13}$$

To proceed further, we need to compute $G'(S_{\alpha,\beta}(t_\alpha))$ as:

$$\begin{aligned}
G'(S_{\alpha,\beta}(t_\alpha)) = & \frac{(1+\tau S_{\alpha,\beta}(t_\alpha))^{n-\Delta}+\tau S_{\alpha,\beta}(t_\alpha)(n-\Delta)(1+\tau S_{\alpha,\beta}(t_\alpha))^{n-\Delta-1}}{(1+\tau S_{\alpha,\beta}(t_\alpha))^n-1} \\
& -\frac{n\tau S_{\alpha,\beta}(t_\alpha)(1+\tau S_{\alpha,\beta}(t_\alpha))^{2n-\Delta-1}}{((1+\tau S_{\alpha,\beta}(\alpha))^n-1)^2}.
\end{aligned}$$

and can therefore solve

$$\begin{aligned}
& \frac{G'(S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))} \\
= \quad & \frac{1}{S_{\alpha,\beta}(0)} \left\{1 + S_{\alpha,\beta}(0)(n-\Delta)\tau(1+\tau S_{\alpha,\beta}(0))^{-1}\right\} \\
& - \frac{1}{S_{\alpha,\beta}(0)} \left\{\frac{n\tau S_{\alpha,\beta}(0)(1+\tau S_{\alpha,\beta}(0))^{n-1}}{(1+\tau S_{\alpha,\beta}(0))^{n}-1}\right\} \\
= \quad & \frac{1}{S_{\alpha,\beta}(0)} \left\{1 - \Delta\tau S_{\alpha,\beta}(0)(1+\tau S_{\alpha,\beta}(0))^{-1}\right\} \\
& - \frac{1}{S_{\alpha,\beta}(0)} \left\{n\tau S_{\alpha,\beta}(0)\frac{(1+\tau S_{\alpha,\beta}(0))^{-1}}{(1+\tau S_{\alpha,\beta}(0))^{n}-1}\right\} \\
= \quad & \frac{1}{S_{\alpha,\beta}(0)} \left\{1 - \frac{\tau S_{\alpha,\beta}(0)}{1+\tau S_{\alpha,\beta}(0)}\left(\Delta + \frac{n}{(1+\tau S_{\alpha,\beta}(0))^{n}-1}\right)\right\}. \qquad (3.14)
\end{aligned}$$

Using Eq. (3.13) and Eq. (3.14) the convexity correction is

$$\begin{aligned}
\text{Convexity Correction} \quad & \approx \quad S_{\alpha,\beta}(0)\theta(\Delta)\left(\exp\{\sigma^2_{\alpha,\beta}t_\alpha\}-1\right) \\
\text{with } \theta(\Delta) \quad & = \quad 1 - \frac{\tau S_{\alpha,\beta}(0)}{1+\tau S_{\alpha,\beta}(0)}\left(\Delta + \frac{n}{(1+\tau S_{\alpha,\beta}(0))^{n}-1}\right).
\end{aligned}$$

□

This formula approximates the value of a convexity correction by a replication portfolio in closed form for fixing in arrears and in advance.

3.5 Convexity Correction: Discrete Replication

Another computational approach following the ideas of Ross [1976] is the static replication by a discrete portfolio of European swaptions [1]. The idea is to replicate the linear payoff of CMS caplets and CMS floorlets with the concave/convex payoff of European swaptions with different strike prices in such a way that the distance between both payoffs is minimized. Considering a CMS caplet, the problem at maturity t_α given N swaptions and their strike prices $\hat{K} = \{\hat{k}_1 = K, \ldots, \hat{k}_N\}$ is to compute the swaptions weights $\hat{\omega} = \{\hat{\omega}_1, \ldots, \hat{\omega}_N\}$, for $\hat{k}_N \geq S_{\alpha,\beta}(t_\alpha) \geq K$ in:

$$\hat{\omega} = \underset{\{\omega_1,\ldots,\omega_N\}}{\operatorname{argmin}} \left(\text{CMS}_{caplet}(S_{\alpha,\beta}(t_\alpha), K, t_\alpha, t_\alpha) - \sum_{i=1}^{N}\omega_i \text{Swaption}^{cs}_{payer}(S_{\alpha,\beta}(t_\alpha), \hat{k}_i, t_\alpha, t_\alpha)\right)^2.$$

[1]Note, that similar results to ours have independently been derived by Zheng and Kwok [2009].

Where we ignore constraints given by liquidity of the swaptions or transaction costs at this stage and examine these in Section (3.6).

Theorem 3.5.1. *Given a defined set of strike prices* $\{K = K_1, \ldots, K_n\}$ *with* $K_1 < K_2 < \ldots < K_n$ *a CMS caplet fixed in arrears can be replicated by a portfolio of corresponding swaptions weighted by:*

$$\omega_j = \frac{K_{j+1} - K_1}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_{j+1})(K_{j+1} - K_j)} - \sum_{i=1}^{j-1} \omega_i \frac{K_{j+1} - K_i}{K_{j+1} - K_j}.$$

The weights for a replication of a CMS floorlet can be computed with descending strike prices $K_1 > K_2 > \ldots > K_n$ *and are given as:*

$$\omega_j = \frac{K_1 - K_{j+1}}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_{j+1})(K_j) - K_{j+1}} - \sum_{i=1}^{j-1} \omega_i \frac{K_i - K_{j+1}}{K_j - K_{j+1}}$$

Proof. The computation of the weights can be done iteratively and we explain it in detail. At maturity t_α the payoff of a CMS caplet is given by:

$$\text{CMS}_{caplet}(S_{\alpha,\beta}(t_\alpha), K, t_\alpha, t_\alpha) = [S_{\alpha,\beta}(t_\alpha) - K]^+$$

and the payoff of a corresponding cash settled payer swaption is given by:

$$\text{Swaption}^{cs}_{payer}(S_{\alpha,\beta}(t_\alpha), K, t_\alpha, t_\alpha) = A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha))[S_{\alpha,\beta}(t_\alpha) - K]^+.$$

Since the payoff of the CMS caplet is linear and the payoff of the payer swaption is concave the CMS caplet cannot be replicated by only one payer swaption with the same strike price. Therefore additional payer swaptions have to be included with increasing strike prices. The n swaptions will be weighted by $\omega_1, \ldots, \omega_n$ and have the strike prices $K = K_1, \ldots, K_n$ with $K_1 < K_2 < \ldots < K_n$. The problem is illustrated in Figure (3.7) where we plotted the payoff of a CMS caplet and the payoff of replication portfolios consisting of 1,2 and 9 payer swaptions. It can clearly be seen how the distance of the concave payoff of the swaptions to the linear payoff is reduced by an increasing amount of swaptions and that the replication portfolio is an upper bound of the CMS caplet value. Theoretically, we should increase n to infinity to get a perfect fit, but an implementation of this strategy is not feasible as mentioned before.

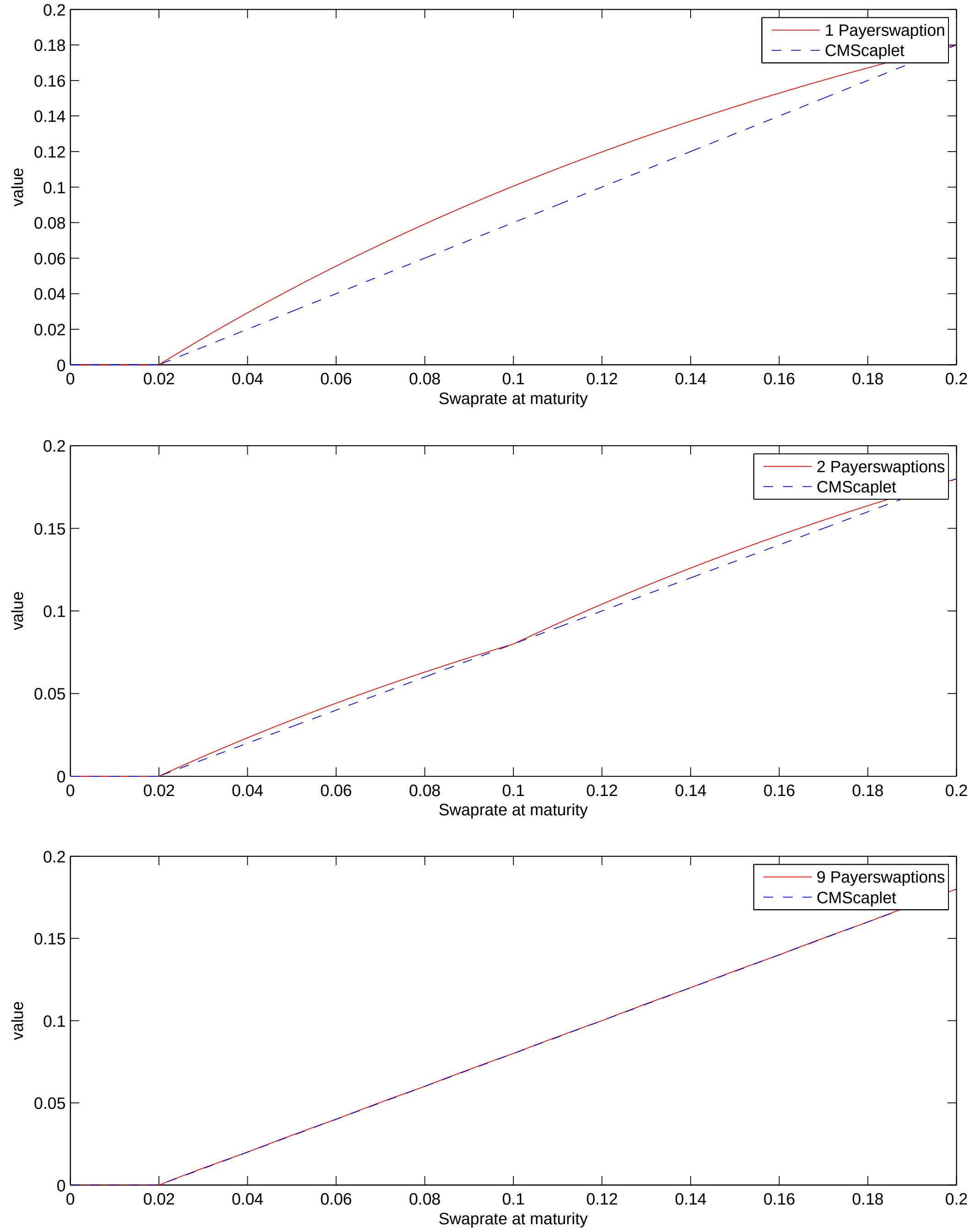

Figure 3.7: Three discrete replication portfolios with different numbers of payer swaptions to replicate the payoff of a CMS caplet with a strike price of 0.02. The underlying swap has a tenor of five years and the fixed leg pays semi-annually.

If we fix the number of swaptions to the number n, the portfolio weights can be computed iteratively by assuming that the realized future swap rate will be at the next swaptions strike price. If we e.g. assume that the swap rate will be equal to K_2 only the first swaption will be in the money. The values of the instruments and the weights can be calculated iterative as:

First step $(S_{\alpha,\beta}(t_\alpha) = K_2)$

$$\begin{aligned}
\mathrm{CMS}_{caplet}(S_{\alpha,\beta}(t_\alpha) = K_2, K_1) &= [K_2 - K_1]^+ = K_2 - K_1 \\
\mathrm{Portfolio}(S_{\alpha,\beta}(t_\alpha) = K_2, K_1) &= \omega_1 A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_2)[K_2 - K_1]^+ \\
&= \omega_1 A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_2)(K_2 - K_1)
\end{aligned}$$

and the first portfolio weight is given as:

$$\omega_1 = \frac{1}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_2)}.$$

Second step $(S_{\alpha,\beta}(t_\alpha) = K_3)$

$$\begin{aligned}
\mathrm{CMS}_{caplet}(S_{\alpha,\beta}(t_\alpha) = K_3, K_1) &= [K_3 - K_2]^+ = K_3 - K_2 \\
\mathrm{Portfolio}(S_{\alpha,\beta}(t_\alpha) = K_3, K_2) &= \omega_1 A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_3)(K_3 - K_1) \\
&\quad + \omega_2 A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_3)(K_3 - K_2) \\
\Rightarrow \omega_2 &= \frac{K_3 - K_1}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_3)(K_3 - K_2)} - \omega_1 \frac{K_3 - K_1}{K_3 - K_2} \\
&= \frac{2}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_3)} - 2 \cdot \omega_1
\end{aligned}$$

where the last line is only valid for equally spaced strike prices.

General case $(S_{\alpha,\beta}(t_\alpha) = K_{j+1})$
If we assume that the swap rate will be equal to K_{j+1} and have calculated the weights for

all j before, the weight ω_j must solve the following equations:

$$\begin{aligned}
\mathrm{CMS}_{caplet}(S_{\alpha,\beta}(t_\alpha) = K_{j+1}, K_1) &= K_{j+1} - K_1 \\
\mathrm{Portfolio}(S_{\alpha,\beta}(t_\alpha) = K_{j+1}, K_1) &= \sum_{i=1}^{j} \omega_i A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t) = K_{j+1})(K_{j+1} - K_i) \\
\Rightarrow \omega_j &= \frac{K_{j+1} - K_1}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_{j+1})(K_{j+1} - K_j)} \\
&\quad - \sum_{i=1}^{j-1} \omega_i \frac{K_{j+1} - K_i}{K_{j+1} - K_j} \\
&= \frac{j}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_{j+1})} - \sum_{i=1}^{j-1} \omega_i (j + 1 - i)
\end{aligned}$$

where the last line is again only valid for equally spaced strike prices.

The replication of a CMS floorlet can be computed in a similar way with descending strike prices $K_1 > K_2 > \ldots > K_n$. However, floorlets are not concave in the swap rate but convex as can be seen in Figure (3.8). For a finite number of strike prices the approximation of the replication portfolio will be a lower bound to the value of a CMS floorlet. The corresponding weights can be computed as

$$\begin{aligned}
\mathrm{CMS}_{floorlet}(S_{\alpha,\beta}(t_\alpha) = K_{j+1}, K_1) &= K_1 - K_{j+1} \\
\mathrm{Portfolio}(S_{\alpha,\beta}(t_\alpha) = K_{j+1}, K_1) &= \sum_{i=1}^{j} \omega_i A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_{j+1})(K_i - K_{j+1}) \\
\Rightarrow \omega_j &= \frac{K_1 - K_{j+1}}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_{j+1})(K_j) - K_{j+1}} \\
&\quad - \sum_{i=1}^{j-1} \omega_i \frac{K_i - K_{j+1}}{K_j - K_{j+1}} \\
&= \frac{j}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_{j+1})} - \sum_{i=1}^{j-1} \omega_i (j + 1 - i).
\end{aligned}$$

□

For both cases of caplets and floorlets the replication portfolio weights have the same structure as can be seen in Figure (3.9). The first weight that corresponds to the swaption with the same strike price as the CMS caplet or CMS floorlet that is to be replicated has the highest value. The further out of the money strikes have all a significant smaller value which is nearly identical. Concerning the replication of a convexity correction for a CMS rate, this

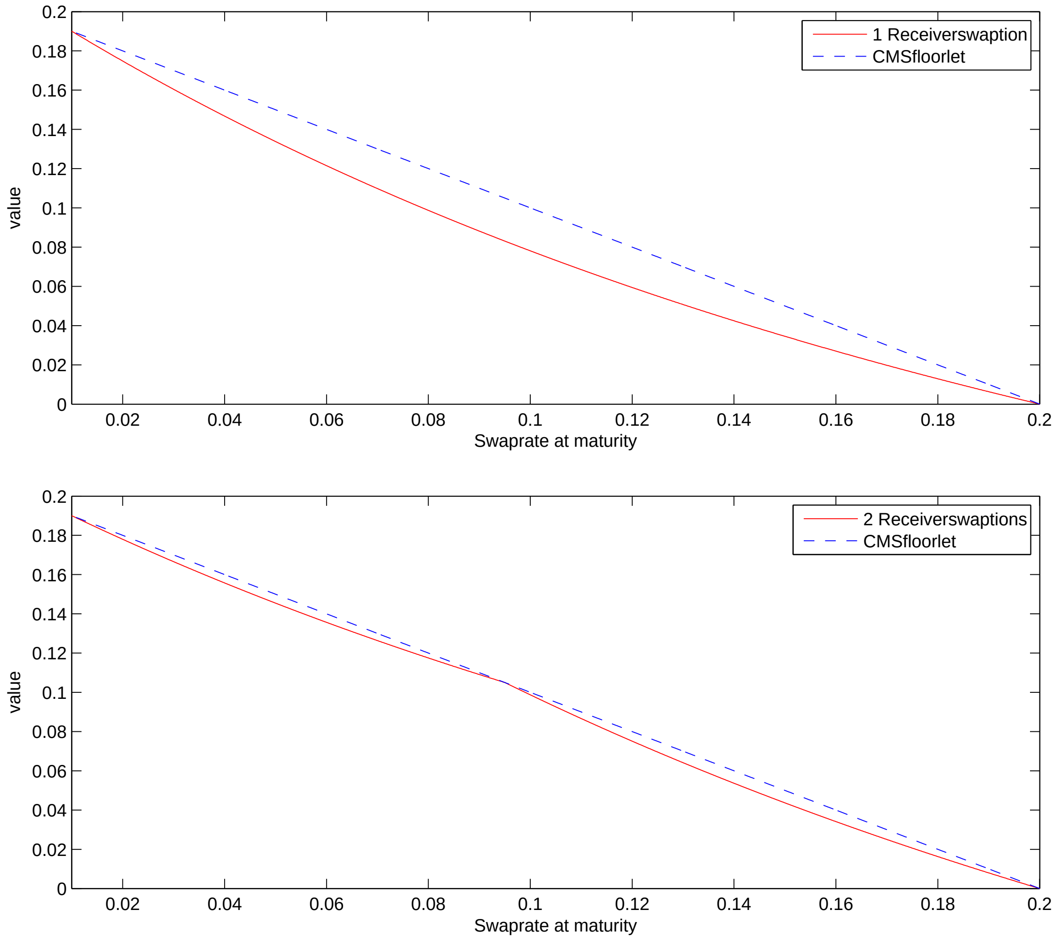

Figure 3.8: Two discrete replication portfolios with different numbers of receiver swaptions to replicate the payoff of a CMS floorlet with a strike price of 0.2. An unrealistic strike of 0.2 is chosen to illustrate the convexity of the swaptions. The underlying swap has a tenor of five years and the fixed leg pays semi-annually.

correction can, as in the continuous case of Hagan [2003], be computed by the caplet-floorlet parity:

$$\begin{aligned} S_{\alpha,\beta}(t_\alpha) &= K + [S_{\alpha,\beta}(t_\alpha) - K]^+ - [K - S_{\alpha,\beta}(t_\alpha)]^+ \\ \Rightarrow B(0,t_\alpha)\mathbb{E}^{t_\alpha}[S_{\alpha,\beta}(t_\alpha)|\mathcal{F}_0] &= B(0,t_\alpha)S_{\alpha,\beta}(0) \\ &\quad + \underbrace{\text{CMS}_{caplet}(S_{\alpha,\beta}(0), S_{\alpha,\beta}(0)) - \text{CMS}_{floorlet}(S_{\alpha,\beta}(0), S_{\alpha,\beta}(0))}_{\text{Convexity Correction}} \end{aligned}$$

Due to the payer-receiver parity for swaptions with same strike prices, the large negative

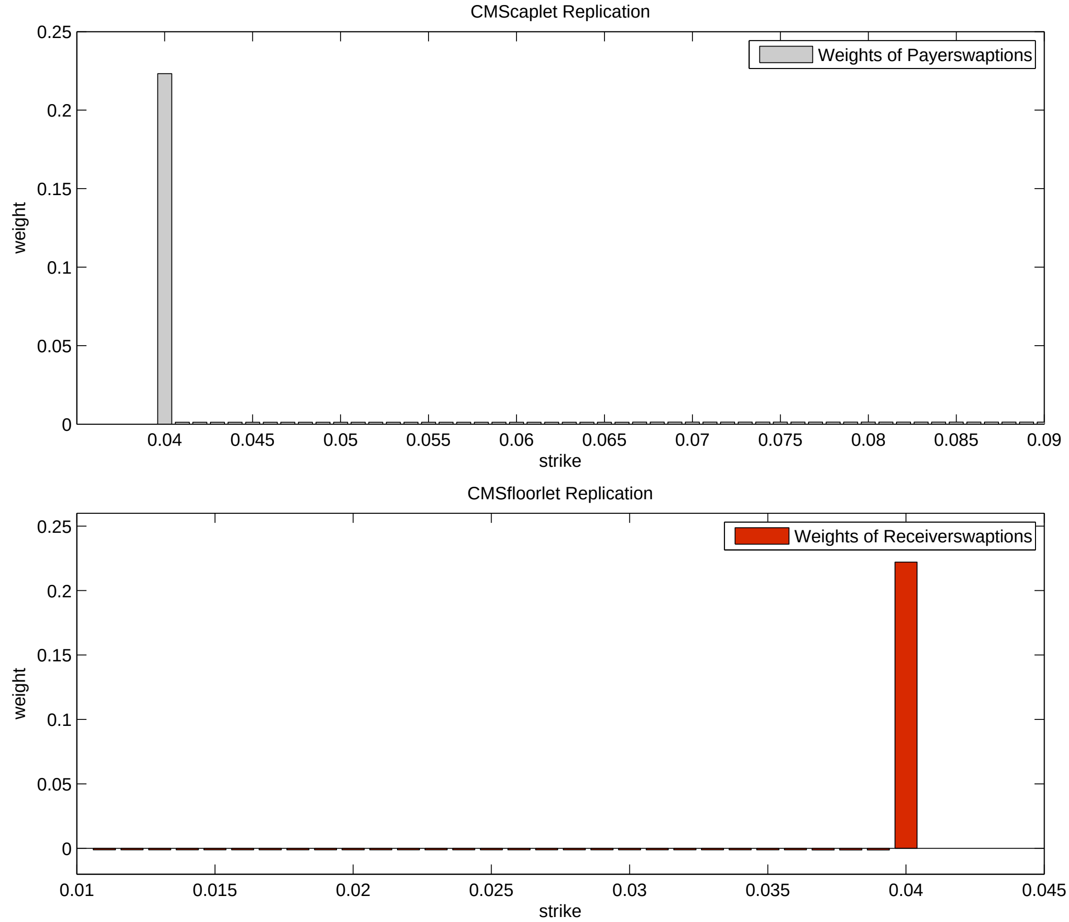

Figure 3.9: Weights of replication portfolios for a CMS caplet and a CMS floorlet. Both contacts have a strike K of 0.04 and mature in one year. The underlying CMS rate has a tenor of five years and the fixed leg pays semi-annually. The replication portfolios consist of 60 payer swaptions and 30 receiver swaptions with equidistant strike prices.

and positive first weights in the CMS caplet and CMS floorlet replication portfolio cancel themselves out to some degree when computing the convexity correction. In Figure (3.10) the portfolio weights for a CMS convexity correction are plotted by using the same scenario as in Figure (3.9) and consists of no peaks at the ATM value of 0.04. With a replication portfolio of swaptions the convexity correction due to a mismatch of measures when valuing CMS rates fixed in arrears can be calculated by a static hedge portfolio.

This methodology can be extended to the case where the CMS rate is fixed in advance. Here an additional convexity correction due to the mismatch of fixing and pay date has to be

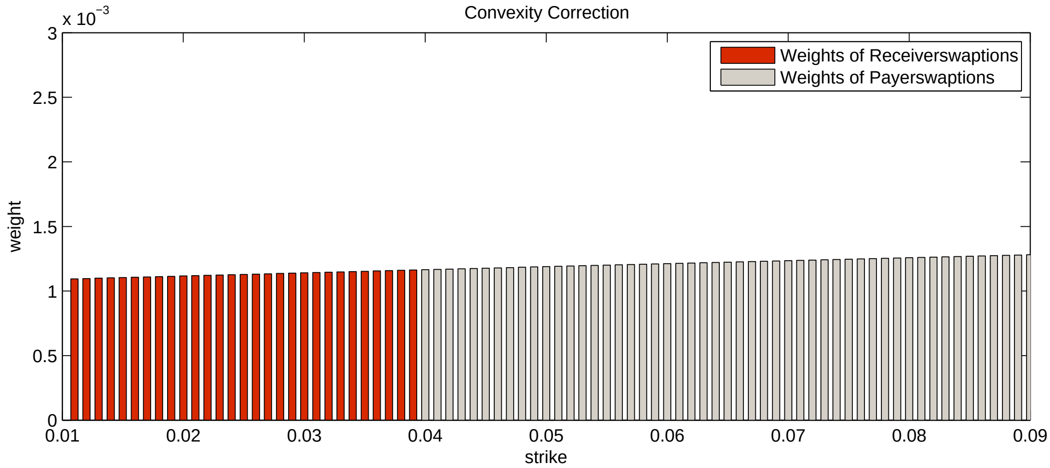

Figure 3.10: Weights of a replication portfolio for a CMS convexity correction. The portfolio consists of the difference of the CMS caplet and CMS floorlet replication portfolios of Figure (3.9).

incorporated into the replication weights.

Corollary 3.5.1. *Given a set of strike prices $\{K = K_1, \ldots, K_n\}$ with $K_1 < K_2 < \ldots < K_n$ a CMS caplet fixed in advance can be replicated by a portfolio of corresponding swaptions weighted by:*

$$\omega_j = (1 + \delta S_{\alpha,\beta}(t_\alpha))^{-n} \frac{K_{j+1} - K_1}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_{j+1})(K_{j+1} - K_j)} - \sum_{i=1}^{j-1} \omega_i \frac{K_{j+1} - K_i}{K_{j+1} - K_j}.$$

The weights for a replication of a CMS floorlet can be computed with descending strike prices $K_1 > K_2 > \ldots > K_n$ and are given as:

$$\omega_j = (1 + \delta S_{\alpha,\beta}(t_\alpha))^{-n} \frac{K_1 - K_{j+1}}{A_{\alpha,\beta}(t_\alpha, S_{\alpha,\beta}(t_\alpha) = K_{j+1})(K_j) - K_{j+1}} - \sum_{i=1}^{j-1} \omega_i \frac{K_i - K_{j+1}}{K_j - K_{j+1}}$$

Proof. At maturity time t the payoff of a CMS caplet fixed in advance is given by:

$$\mathrm{CMS}_{caplet}^{advance}(S_{\alpha,\beta}(t_\alpha), K, t_\alpha, t_{\alpha+\Delta}) = B(t, t_{\alpha+\Delta})[S_{\alpha,\beta}(t_\alpha) - K]^+.$$

As with the cash-settled swaptions we assume a flat yield curve at time t_α and discount the payoff with the swaprate. Therefore, the payoff of the CMS caplet can be approximated as:

$$\begin{aligned} \text{CMS}^{advance}_{caplet}(S_{\alpha,\beta}(t_\alpha), K, t_\alpha, t_{\alpha+\Delta}) &\approx (1+\delta S_{\alpha,\beta}(t_\alpha))^{-n}[S_{\alpha,\beta}(t_\alpha) - K]^+ \\ \text{with } n &= \frac{t_{\alpha+\Delta} - t_\alpha}{\delta} \end{aligned}$$

The computation of the weights for the replicating portfolios can be computed in the same iterative manner. Assuming strike prices $\{K = K_1, \ldots, K_n\}$ with $K_1 < K_2 < \ldots < K_n$ the weights are given for K_{j+1} as:

$$\begin{aligned} \omega_j &= (1+\delta S_{\alpha,\beta}(t_\alpha))^{-n} \frac{K_{j+1} - K_1}{A_{\alpha,\beta}(t, S_{\alpha,\beta}(t) = K_{j+1})(K_{j+1} - K_j)} - \sum_{i=1}^{j-1} \omega_i \frac{K_{j+1} - K_i}{K_{j+1} - K_j} \\ &= (1+\delta S_{\alpha,\beta}(t_\alpha))^{-n} \frac{j}{A_{\alpha,\beta}(t, S_{\alpha,\beta}(t) = K_{j+1})} - \sum_{i=1}^{j-1} \omega_i (j+1-i) \end{aligned}$$

where the last line is again only valid for equally spaced strike prices.

The replication of a CMS floorlet can be computed in a similar way with descending strike prices $K_1 > K_2 > \ldots > K_n$ and the corresponding weights are given as:

$$\begin{aligned} \omega_j &= (1+\delta S_{\alpha,\beta}(t_\alpha))^{-n} \frac{K_1 - K_{j+1}}{A_{\alpha,\beta}(t, S_{\alpha,\beta}(t) = K_{j+1})(K_j - K_{j+1})} - \sum_{i=1}^{j-1} \omega_i \frac{K_i - K_{j+1}}{K_j - K_{j+1}} \\ &= (1+\delta S_{\alpha,\beta}(t_\alpha))^{-n} \frac{j}{A_{\alpha,\beta}(t, S_{\alpha,\beta}(t) = K_{j+1})} - \sum_{i=1}^{j-1} \omega_i (j+1-i) \end{aligned}$$

The replication of a convexity correction for a CMS rate fixed in advance can be computed by the caplet-floorlet parity:

$$\begin{aligned} S_{\alpha,\beta}(t_\alpha) &= K + [S_{\alpha,\beta}(t_\alpha) - K]^+ - [K - S_{\alpha,\beta}(t_\alpha)]^+ \\ \Rightarrow B(0, t_{\alpha+\Delta})\mathbb{E}^{\alpha+\Delta}[S_{\alpha,\beta}(t_\alpha)|\mathcal{F}_0] &= B(0, t_{\alpha+\Delta}) S_{\alpha,\beta}(0) \\ &\quad + \underbrace{\text{CMS}_{caplet}(S_{\alpha,\beta}(0), S_{\alpha,\beta}(0)) - \text{CMS}_{floorlet}(S_{\alpha,\beta}(0), S_{\alpha,\beta}(0))}_{\text{Convexity Correction}}. \end{aligned}$$

□

Using the replication portfolio with discrete strikes, a convexity correction can be computed together with its static hedge for CMS rates fixed in advance and in arrears. But as already pointed out, the markets liquidity aspects and transaction costs must be taken into consideration and therefore only a finite number of strike prices will remain for the replication. But as payer/receiver swaption prices decrease with increasing/decreasing strike prices their influence on the replication also decreases and a cut off at a certain level has no considerable influence as discussed in Section (3.6).

3.6 Numerical Results

The accuracy of the replication method relies on the number of swaptions and on the step size of the underlying strike prices used in the replication portfolios. To quantify the influence of these factors we conduct several simulation studies. As a base scenario we set $S_{\alpha,\beta}(0) = 0.03303$, $t_\alpha = 1$, $t_\beta = 6$, $\delta_i = \delta = 1\ \forall i$, $\sigma = 0.2680$, $\Delta = 0$ and $B(0, t_\alpha) = 0.9883$.

In a first step we examine a variation of the highest strike price in the replication portfolio, we will further refer to as upper bound. Therefore we assume that the strike prices are equidistant and vary the upper bound for three different sets of step sizes, namely 100 basis points, 10 basis points and 1 basis point. With the results of these simulations we can examine up to which strike price swaptions have to be included into the replication portfolio. Since the value of payer swaptions tends to go to zero with increasing strike prices the convexity correction converges up to a specified amount of digits with an increasing upper bound. This is also dependent on the chosen step size as can be seen in Figure (3.11). Here the value of the replication portfolios clearly converge for an increasing upper bound. Due to the step size the replication portfolio with a step size of 1bp converges smoother than the portfolio with a step size of 100bp. The portfolios with the step size of 100bp converges to a much higher value than the other two portfolios which seem to converge to a similar value. We define for this simulation convergence after 8 digits and compute the values and their upper bound in Table (3.3).

In a second step we assume that the upper bound is fixed due to liquidity of the swaptions and examine the variation of step sizes. This is a realistic scenario for an implementation of the replication portfolio as an actual hedge. Given an upper bound a smaller step size should result in a better approximation of the convexity correction. To this end we set the upper bounds to the swap rate +150 basis points, +300 basis points and +500 basis points and vary the step sizes from 0.0001 to 0.01. The results are plotted in Figure (3.12). The

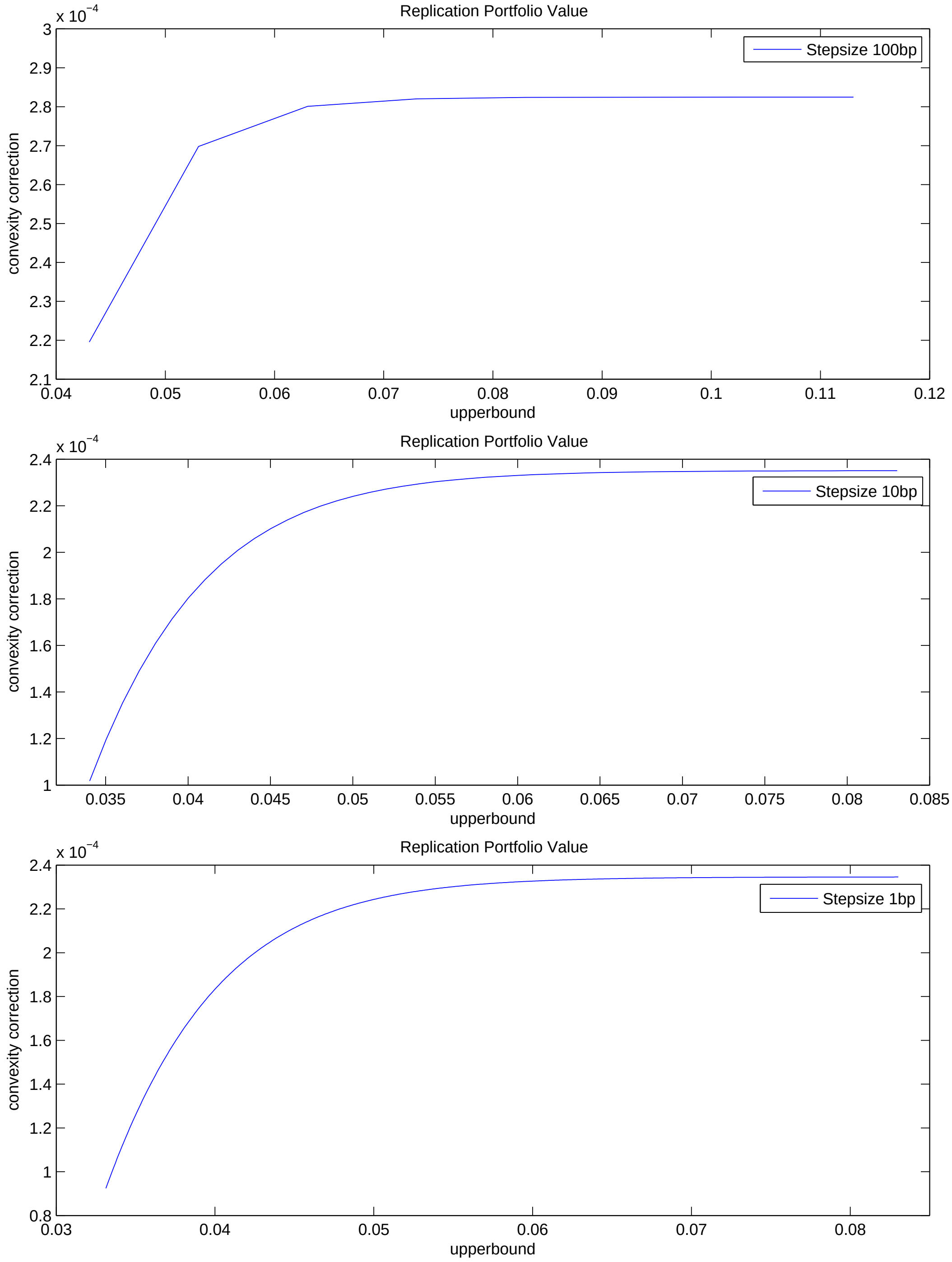

Figure 3.11: Simulations of CMS rate replication portfolios given three different step sizes of the equidistant strike prices. The values of the convexity correction are computed in dependence of the upper bound.

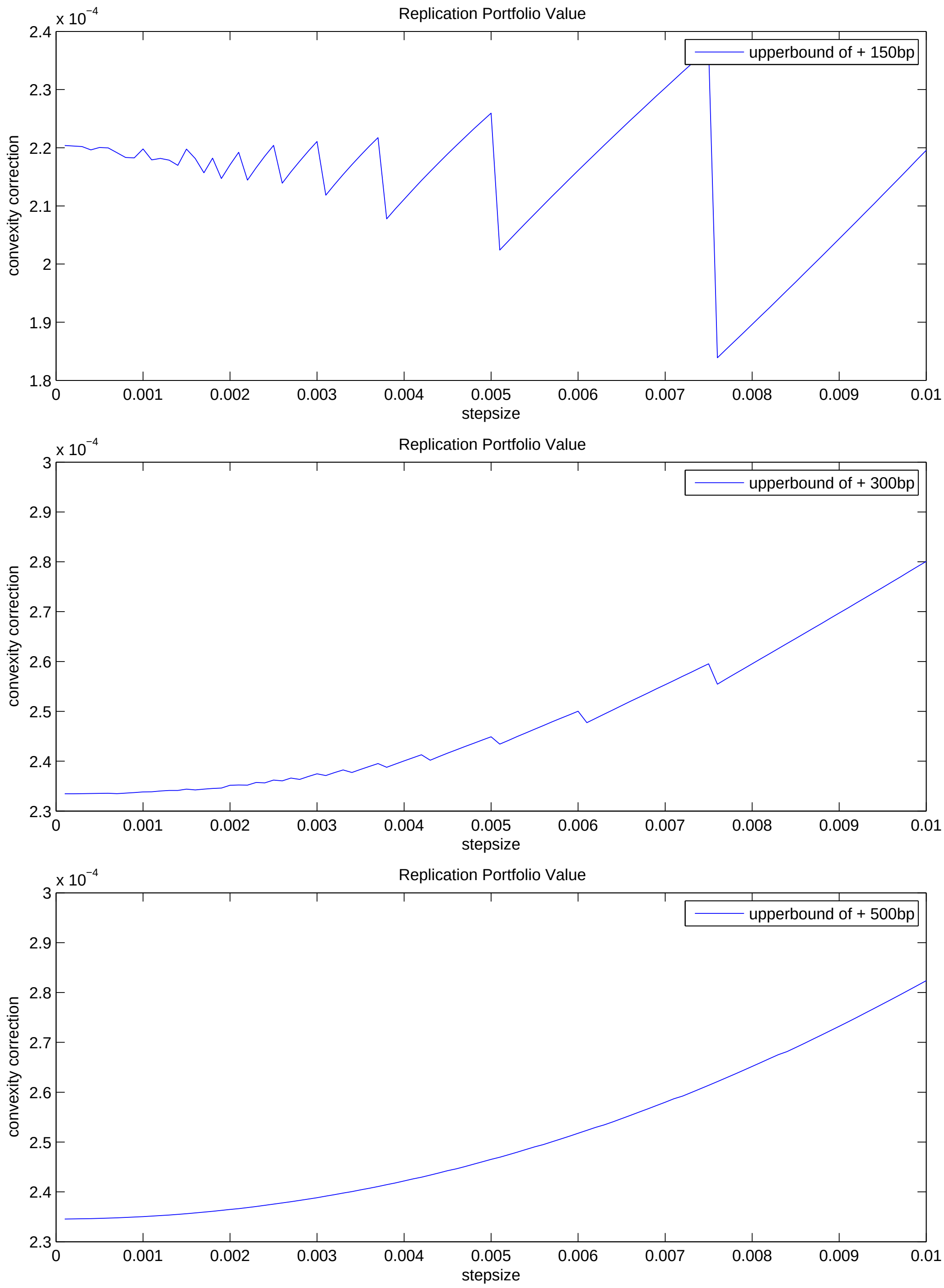

Figure 3.12: Simulations of CMS rate replication portfolios given three different upper bounds for the replicating swaptions. The values of the convexity correction are computed in dependence of the step size.

	correction	upperbound
100bp	$2.8246e^{-4}$	10.30%
10bp	$2.3508e^{-4}$	9.70%
1bp	$2.3461e^{-4}$	11.22%

Table 3.3: Values of a replication portfolio converged up to 8 digits at the given upper bound in dependence of the step size.

approximations for the lowest upper bound of swap rate + 150 basis points behaves in a strong zigzag pattern, which is dampened by a decreasing step size. This pattern can be explained by the payoff of the last swaption in the replication portfolio after the upper bound. The upper bound of swap rate + 300 basis points has the same pattern but in much more dampened form. For the largest upper bound this pattern is no longer recognizable and the approximation converges with a decreasing step size in a monotonic way. These results indicate that for a low upper bound a decrease of the step size does not automatically lead to a better approximation. The swaptions to include in the replication portfolio have therefore to be chosen with care if liquidity constrains are included.
To examine this further we simulate the difference between a replication portfolio with an upper bound of swap rate +2000 basis points and a step size of 0.0001 with an approximation including only the swaptions who are quoted in the volatility cube. In Figure (3.13) the simulation is plotted for varying swap rate and volatility. The results show that for small swap rates a replication can be done by using the swaptions of the volatility cube but the error due to the small number of swaptions is increasing with an increasing swap rate. Only if the volatility is low the error stays considerable low.

One of the advantages of the replication portfolio is that it is model independent in that sense that any model can be used to value the swaptions in the replication portfolio. The advantage herein is that all the market information of the swaption prices implicitly quoted in the volatility cube can be incorporated into the computation of the convexity correction. Since the volatility cube has no information about volatilities more than 200 basis points away from the forward swap rate and extrapolation is an unreliable task, a parametric volatility function for the whole surface is needed. This is given by the usage of the SABR model to value swaptions by the implied Black volatility function and is used in the following. To quantify the influence of the market information given by the swaption volatility cube we

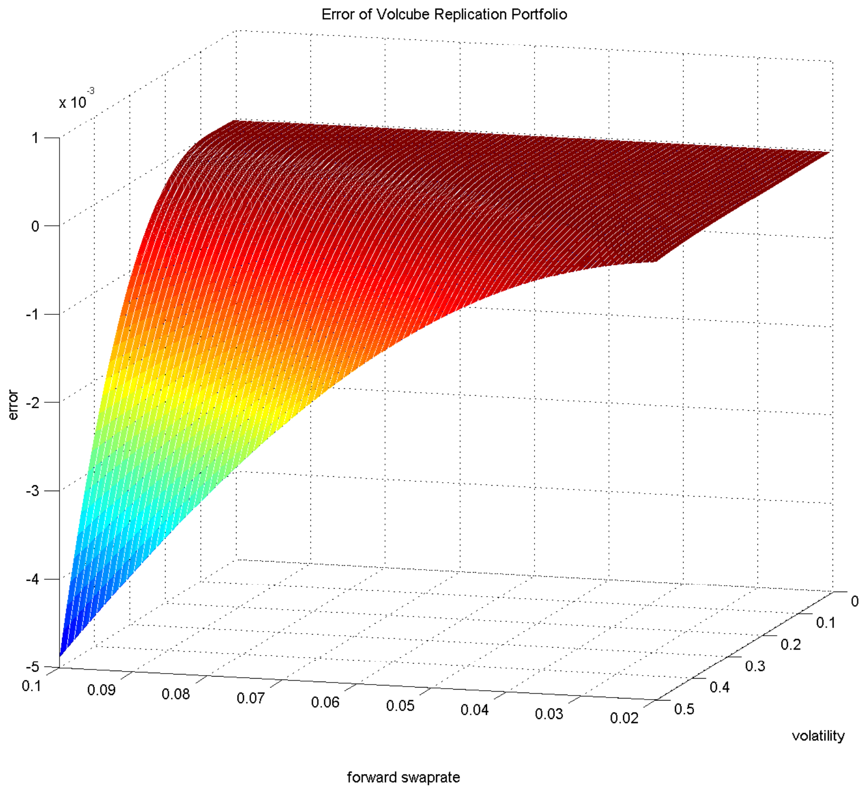

Figure 3.13: Difference between benchmark values and values obtained by only using the swaptions of the volatility cube in the replication portfolio.

calibrate the SABR model to the data set of the swaptions with an underlying swap tenor of five years. As a base scenario for the SABR model we set after calibration $\alpha = 0.9304$, $\beta = 0.7$, $\nu = 0.7108$ and $\rho = -0.1917$.

With this calibrated SABR model we compute the convexity correction for a swap rate starting in one year with a tenor of five years for a range of forward values of the swap rate. To get an nearly exact value we choose a step size of 1 basis point and an upper bound of forward swap rate + 300 basis points. The difference of the results and the same simulation using a deterministic Black76 model using only the at the money volatility are plotted in Figure (3.14). As already seen for the case of a replication with only the volatilities of the volatility cube, for high forward swap rates the computation of the convexity correction must be done with care, since this range is very sensible to misspecifications. As can be seen in Figure (3.14) and Table (3.4) for forward swap rates within 0.0001 to 0.00495 the error lies in $\approx |5e^{-5}|$ and for 0.0001 to 0.00555 within $\approx |1e^{-4}|$. As forward swap rates have been

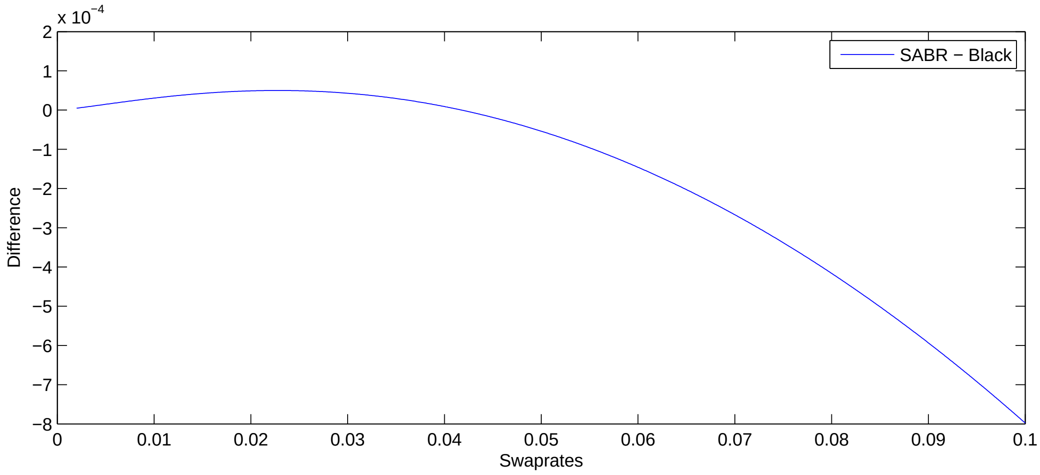

Figure 3.14: Figure of differences between between a SABR model and a Black76 model when computing the convexity correction by replication portfolio for a range of forward swap rates.

Forward Swap rate	Difference
0.002	$4.8716e^{-6}$
0.01	$3.0567e^{-5}$
0.02	$4.9050e^{-5}$
0.0227	$5.0017e^{-5}$
0.04	$9.0335e^{-6}$
0.0495	$-4.9975e^{-5}$
0.0555	$-1.0082e^{-4}$
0.06	$-1.4584e^{-4}$
0.08	$-4.1628e^{-4}$
0.1	$-7.9786e^{-4}$

Table 3.4: Table of differences between between a SABR model and a Black76 model when computing the convexity correction by replication portfolio for a set of forward swap rates.

historically beyond these ranges the inclusion of market information of the volatility cube is necessary to compute the convexity correction.

As already discussed for the case of a constant volatility, the step size is very important for a replication portfolio, since an increase in the step size does not necessarily increase the accuracy of the approximation. Since this is a realistic scenario for a hedger we reconsider the examination by including the volatility cube into the simulation by the SABR model. Since the change of the step size does not only change the strike prices but also the implied volatilities the effects observed for the Black76 model should be more pronounced in the SABR model.

And the effect of a zigzag pattern is clearly more pronounced as can be seen in Figure (3.15) in comparison to Figure (3.12). The pattern is not vanishing for the case of a SABR model with a higher upper bound and is still recognizable for small step sizes even for an upper bound of forward swap rate + 2000 basis points. These results underline that the accuracy of the convexity correction is more sensible to the construction of the replication portfolio in case of a SABR model.

3.7 Conclusion

The convexity correction of swap rates to compute constant maturity swaps and CMS options can be done by analytical approximations or by a replication strategy. The analytical approximations have the drawback that the resulting convexity correction is not a tradeable asset and that market information from the swaption volatility cube cannot be included in the computation.

An alternative computation is by a static replication portfolio of swaptions. In this case the convexity correction is a tradeable asset via the swaptions and the market information can be included by a pricing model like the SABR model. To implement this replication strategy an integral till infinity for the underlying swaptions has to be solved numerically. This is computationally costly and the swaptions market is not liquid enough to provide all prices. We have therefore shown how this replication portfolio can be implemented for the in Europe common type of swaptions, the cash-settled swaptions. An in depth analysis of the accuracy of the strategy showed that the information of the swaption volatlity cube is important to the computation and that a small step size between the strike prices is more important than a high upper bound of the strikes to be included.

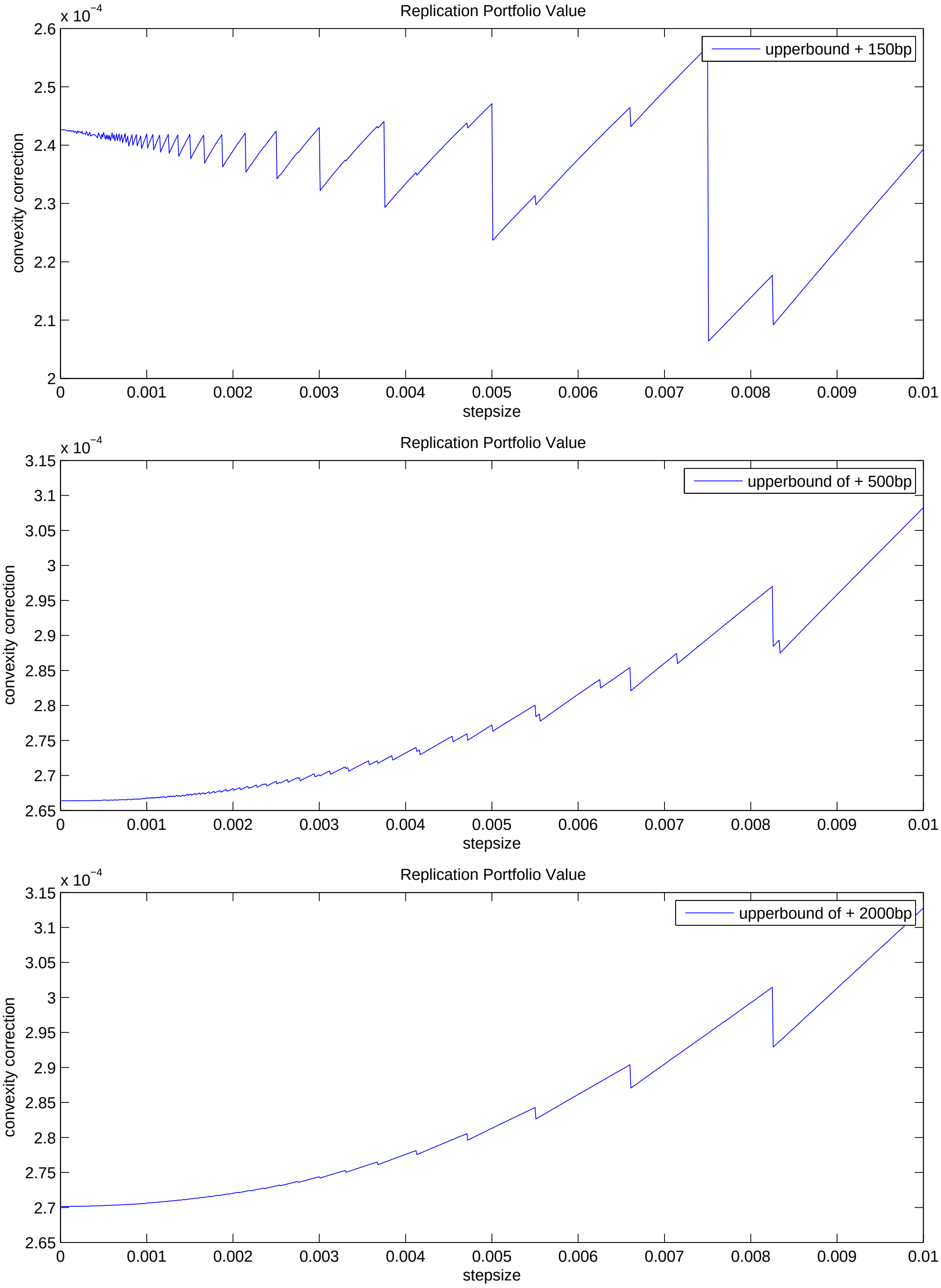

Figure 3.15: Simulations of CMS rate replication portfolios given three different upper bounds for the replicating swaptions using a SABR model. The values of the convexity correction are computed in dependence of the step size.

Chapter 4

Markovian Projection of Multivariate Diffusions

4.1 Introduction

One natural choice to model swap rates to value for instance CMS spread options is the Swap Market Model by Jamshidian [1999]. In this model several swap rates are modeled simultaneous under one pricing measure. The advantage is that the resulting formula to value swaptions coincides with the market standard Black [1976] formula. But the model is not capable to replicate the swaption volatility cube. The valuation of financial instruments taking into account the whole volatility cube can be done by for instance applying a model with stochastic volatility. This approach gained despite the fact that we are no longer in a complete markets setting popularity over the last years. One popular model to model forward price processes is the SABR model of Hagan et al. [2003]. This model assumes that the forward price process of an asset evolves under a stochastic volatility process correlated with the forward price process. One of the major advantages of the SABR model in comparison with other models with stochastic volatility is that an approximation of a strike- and time-to-maturity dependent volatility function exists. This approximation can be plugged into the well-known Black [1976] formula to calculate an arbitrage-free price.

In the setting of a basket consisting of forwards as underlying processes, an option on the basket can only be valued analytically in the case of two assets and a zero strike by the formula of Margrabe [1978]. For higher dimensions the arbitrage-free price needs to be computed numerically. One numerical method suited to these kind of problem is the Monte

Carlo simulation. But in the case of stochastic volatility, this procedure can be very time consuming. This is acceptable if only an arbitrage-free price is be computed, but it is a major problem if the concern is the calibration of a model to market prices. Therefore, approximation formulae for the contracts to be calibrated to should be available.

The Markovian Projection is a method introduced to quantitative finance by Piterbarg [2006] who applies the results of Gyoengy [1986]. It allows for an approximation of the terminal distribution of a multivariate diffusion process by a univariate diffusion process. This method is capable to incorporate stochastic volatility models with a correlation structure between all stochastic variables and has been applied by Antonov and Misirpashaev [2009] to project the spread of two Heston processes onto a displaced Heston process. Using the case of a multivariate SMM/SABR diffusion we show how the basket can be approximated by a displaced diffusion model along the lines of Rubinstein [1983] with e.g. a SABR style stochastic volatility. Given the approximated SDE, caps/floors on a basket of n-assets can be valued in closed form taking into account the volatility cube and a full correlation structure. However, this numerically fast valuation can only be achieved by some harsh approximations with very good results for short time to maturities as numerical simulations show.

A liquid financial instrument in the fixed income market that depends on two correlated forward price processes is the CMS spread option. The contract´s payoff depends on the spread of two CMS rates with different tenors. A CMS rate is a swap rate paid in one installment. Its name is derived from constant maturity swaps.
Regarding the valuation of spread options with non-zero strike several approximations and simulations are discussed in the related literature. Using deterministic volatility the valuation can be done by a semi-analytic conditioning technique, see Belomestny et al. [2008] or in a swap market model or a displaced diffusion swap market model by Monte Carlo simulation as shown by Leon [2006] and Joshi and Yang [2009]. Solutions for stochastic volatility models are given by Dempster and Hong [2000] who extended the FFT method to spread options. Antonov and Arneguy [2009] and Lutz and Kiesel [2010] consider a stochastic volatility LIBOR Market Model and approximations to the CMS rate as well as numerical integration methods.
One approach in a SABR framework is to use a Gaussian copula with the marginals being SABR processes as shown by Berrahoui [2004] and Benhamou and Croissant [2007]. The advantage of the Markovian Projection method we propose is that we can include a rich

correlation structure and derive a closed-form solution which can be extended to the n-asset case.

Concerning the valuation of products dependent on CMS rates, the expected value of the CMS rate under the forward measure is its forward starting value and a convexity correction independent of the chosen pricing model. This convexity correction can be computed by an analytical approximation as discussed in Lu and Neftci [2003] or by using a replication portfolio of European swaptions as proposed by Hagan [2003]. In the case of a Markovian projected spread diffusion the convexity correction can be approximated by the difference of the original CMS convexity corrections under a so-called spread measure.

Numerical results for CMS spread options show that the Markovian Projection of multivariate SMM/SABR diffusions is a good approximation which for example can be used for volatility and correlation calibration. For a liquid set of strike prices from 0 to 100 basis points the model prices and delta parameters lie close to the results obtained by Monte Carlo simulation and the prices even outperform a copula approach. But there are parameter sets for which the approximation is less accurate. This is for instance the case for a long time to maturity.
Concerning the properties of a CMS spread option the numerical studies show a significant influence of the swaption volatility cube and the correlation between the stochastic correlation parameters on the options price. This last issue cannot be modeled by the previously mentioned approximations.

The chapter is structured as follows: In Section (4.2), we first describe the multivariate SMM/SABR model. In a second step the approximated Markovian Projection is computed in Section (4.3) for the general case of a n-dimensional basket. Section (4.4) applies the results to the special case of a CMS spread option, where also the convexity correction of CMS rates and a copula approach are presented. The accuracy of the suggested approximations and the properties of CMS spread options are illustrated in Section (4.5) by numerical examples. Section (4.6) concludes the chapter.

4.2 Interest Rate Models

The underlying process of the options we model is a basket of assets which follow forward price processes and so we end up with so called basket options. Let N be the number of different correlated assets S_i and denoting their weights by ϵ_i, $i = 1, \ldots, N$. A basket of assets is defined as:

$$\sum_{i=1}^{N} \epsilon_i S_i.$$

For instance the case where $N = 2$, $\epsilon_1 = 1$ and $\epsilon_2 = -1$ constitutes a spread of assets underlying e.g. a spread option. To compute the value of a basket option the price processes have to be modeled simultaneously including a dependence structure. Since we restrict our attention to CMS options, we need to model co-initial forward swap rates. One natural model is the so-called Swap Market Model by Jamshidian [1999] where each forward swap rate follows a log-normal diffusion with a deterministic volatility. Another approach is the SABR model by Hagan et al. [2003] where each forward swap rate follows a CEV process with a correlated stochastic volatility process.

Together with the LIBOR Market Model (LMM), the SMM belongs to the class of the so called Market models, where multivariate forward price processes are modeled under one measure. This class of models justified the well-known pricing formulae for swaptions and caps/floors by the model of Black [1976] as arbitrage-free. In the following, we assume that the usual assumptions apply.

Definition 4.2.1. *A multi-dimensional Swap Market Model diffusion under one measure is given as follows. For each asset $S_i(t)$ with $i = \{1, \ldots, N\}$ let:*

$$\begin{aligned} dS_i(t) &= \mu_i(t) S_i(t) dt + \sigma_i S_i(t) dW_i(t) \\ S_i(0) &= s_i^0 \\ \sigma_i(0) &= \sigma_i^0 \\ \langle dW_i(t), dW_j(t) \rangle &= \rho_{ij} dt. \end{aligned} \tag{4.1}$$

with $\mu_i(t)$ the stochastic drift, σ_i the deterministic volatility and ρ_{ij} the correlation between the Brownian motions driving the asset price processes.

Note, that we do not further specify $\mu_i(t)$, the drift due to a measure change, since we do not need the precise form of $\mu_i(t)$ for further computations. The volatility σ_i is modeled by a constant parameter to use the implied Black76 volatilities from the swaption volatility cube to calibrate each model to market prices.

One problem encountered when modeling derivatives like swaptions in a Swap Market Model and therefore using the Black [1976] formula is that not all market prices for swaptions can be obtained with a constant volatility parameter in one SMM as the model demands. An alternative suggested by Hagan et al. [2003] is the so-called SABR model where a forward price process is modeled under its forward measure using a correlated stochastic volatility process. Since this model can replicate the volatility cube for swaptions we propose to use a multidimensional SABR model to model options on several swap rates like CMS spread options.

Definition 4.2.2. *A multi-dimensional SABR diffusion under one measure is given as follows. For each asset $S_i(t)$ with $i = \{1, \dots, N\}$ let:*

$$\begin{aligned} dS_i(t) &= \mu_i(t)S_i(t)dt + \alpha_i(t)S_i(t)^{\beta_i}dW_i(t) \\ d\alpha_i(t) &= \nu_i\alpha_i(t)dZ_i(t) \\ S_i(0) &= s_i^0 \\ \alpha_i(0) &= \alpha_i^0 \\ \langle dW_i(t), dW_j(t)\rangle &= \rho_{ij}dt \\ \langle dW_i(t), dZ_j(t)\rangle &= \gamma_{ij}dt \\ \langle dZ_i(t), dZ_j(t)\rangle &= \xi_{ij}dt. \end{aligned} \tag{4.2}$$

with $\mu_i(t)$ the stochastic drift, $\alpha_i(t)$ the stochastic volatility, ν_i the volatility of the volatility and $\gamma_{W,Z}$ is the correlation of the forward price and volatility process. β_i is the CEV parameter, ρ_{ij} the correlation between the Brownian motions driving the asset price processes, γ_{ij} the cross-skew and ξ_{ij} the so-called decorrelation between the stochastic volatilities.

The multidimensional SABR process models the dependency between all factors, which will be further examined in Section (4.5).

A major problem when valuing basket options is that only for $\beta_i = 0,\ i = 1, \dots, N$, the case of normally distributed assets if the drift does not depend on $S(t)$ and a SMM, the distribution of the basket is known and option prices can be computed in closed-form. For the special case of two assets with $\beta_{1,2} = 1$ for the SABR model and a zero strike a solution is given by the Margrabe [1978] formula. But for nonzero strikes and more than two assets only numerical methods and semi-analytic approximation formulae are known. In the following, we extend the framework by a projected multivariate SMM/SABR diffusion which can applied to the n-assets case.

4.3 Markovian Projection

The Markovian Projection is an approximation method introduced to quantitative finance by Piterbarg [2006]. It applies the results of Gyoengy [1986] to project multidimensional processes onto a reasonable simple process. Using this methodology we project a multidimensional SMM/SABR diffusion process onto a one-dimensional one.

The key result to approximate the multidimensional model of Eq. (4.1) or Eq. (4.2), is the following result derived by Gyoengy [1986].

Lemma 4.3.1. *Let $X(t)$ be given by*

$$dX(t) = \alpha(t)dt + \beta(t)dW(t), \tag{4.3}$$

where $\alpha(.), \beta(.)$ are adapted bounded stochastic processes such that Eq. (4.3) admits a unique solution. Define $a(t,x), b(t,x)$ by

$$\begin{aligned} a(t,x) &= E[\alpha(t)|X(t) = x] \\ b^2(t,x) &= E[\beta^2(t)|X(t) = x] \end{aligned}$$

Then, the SDE

$$\begin{aligned} dY(t) &= a(t, Y(t))dt + b(t, Y(t))dW(t), \\ Y(0) &= X(0), \end{aligned}$$

admits a weak solution $Y(t)$ that has the same one-dimensional distribution as $X(t)$.

In the following we use the result to project a multivariate SMM and SABR diffusion onto a one-dimensional diffusion.

4.3.1 Projection of the Swap Market Model

To project the multivariate SMM diffusion of Eq. (4.1) by Lemma (4.3.1), we choose a simple SDE given by:

$$\begin{aligned} dS(t) &= F(S(t))dW(t) \\ S(0) &= \sum_{i=1}^{N} s_i^0. \end{aligned} \tag{4.4}$$

where $F(S(t))$ is a function of the swap rate.
The computation involves approximations, which we explain in detail in the proof of the following Theorem.

Theorem 4.3.1. *The dynamics of a basket of assets following a multivariate SMM model, Eq. (4.1), is approximated by:*

$$\begin{aligned} dS(t) &= F(S(t))dW(t) \\ S(0) &= \sum_{i=1}^{N} s_i^0 \\ F(S(0)) &= p = \sum_{i=1}^{N} \sigma_i^2 S_i^2(0) + 2\sum_{i<j}^{N} \epsilon_{ij}\rho_{ij}\sigma_i\sigma_j S_i(0)S_j(0) \\ F'(S(0)) &= q = \frac{\sum_{i=1}^{N}\left(\rho_i S_i^2(0)\sigma_i^3 + \sum_{i\neq j}^{N} \rho_{ij}\rho_i\epsilon_{ij}S_i(0)S_j(0)\sigma_i^2\sigma_j\right)}{p^2} \\ \rho_i &= p^{-1}\sum_{j=1}^{N} S_j(0)\sigma_j\epsilon_j\rho_{ji}. \end{aligned}$$

Proof. The approximation is computed in several steps. First, we rewrite the original SMM diffusion of Eq. (4.1) as a single diffusion. Using the SDE for the individual assets, we find:

$$\begin{aligned} dS(t) &= \sum_{i=1}^{N} \epsilon_i dS_i(t) \\ &= \sum_{i=1}^{N} \epsilon_i(\mu_i S_i(t) + \sigma_i S_i(t)dW_i(t)). \end{aligned}$$

We also assume that one measure $\hat{Q}$ exists under which the basket has a zero drift. This assumption is further explained in Section (4.4) for the case of a spread option. Here we assume that this measure exists and the basket can be computed as:

$$dS(t) \approx \sum_{i=1}^{N} \epsilon_i\sigma_i S_i(t)d\hat{W}_i(t).$$

Choosing the Brownian motion such that

$$d\hat{W}(t) = \sigma^{-1}(t) \sum_{i=1}^{N} \epsilon_i \sigma_i S_i(t) d\hat{W}_i(t)$$

where $\hat{W}(t)$ is also a $\hat{Q}$-Brownian motion and we have the representation:

$$dS(t) = \sigma(t) d\hat{W}(t)$$

with $\epsilon_{ij} = \epsilon_i \cdot \epsilon_j$ and $\sigma(t)$ given by:

$$\sigma^2(t) = \sum_{i=1}^{N} \epsilon_i \sigma_i^2 S_i^2(t) + 2 \sum_{i<j}^{N} \epsilon_{ij} \rho_{ij} \sigma_i \sigma_j S_i(t) S_j(t).$$

Under this specification, the Lévy characterization gives that $\hat{W}(t)$ is a Brownian motion.

Now, we are in a position to apply the result of Gyoengy. Given the spread $dS(t) = \sigma(t) d\hat{W}(t)$ and the projection $dS(t) = F(S(t)) d\hat{W}(t)$, we set with the notation of Lemma (4.3.1)

$$b^2(t,x) = \mathbb{E}[\sigma^2(t) | S(t) = x].$$

Thus, we have

$$F^2(t,x) = \mathbb{E}\left[\sigma^2(t) | S(t) = x\right]. \tag{4.5}$$

To compute the conditional expectations of Eq. (4.5) and to ease notation we observe that $\sigma^2(t)$ is a linear combinations of the form:

$$f_{ij}(t) = S_i(t) S_j(t) \sigma_i \sigma_j(t)$$

and can be represented as follows:

$$\sigma^2(t) = \sum_{i=1}^{N} f_{ii}(t) + 2 \sum_{i<j}^{N} f_{ij}(t) \rho_{ij} \epsilon_{ij}.$$

To compute the conditional expectation, a first-order Taylor series expansion leads to

$$f_{ij} \approx S_i(0) S_j(0) \left(1 + \frac{1}{S_i(0)}(S_i(t) - S_i(0)) + \frac{1}{S_j(0)}(S_j(t) - S_j(0))\right). \tag{4.6}$$

Thus, to compute the conditional expectations of Eq. (4.5), we need simple expressions for

$$\mathbb{E}[S_i(t) - S_i(0) | S(t) = x].$$

To find a simple formula, we apply a Gaussian approximation to compute the expected values. The Gaussian approximation is a crude but reasonable approximation to get a solution in closed form and is given by:

$$
\begin{aligned}
dS_i(t) &\approx d\bar{S}_i(t) = S_i(0)\sigma_i d\hat{W}_i(t), \\
dS(t) &\approx d\bar{S}(t) = p d\bar{W}(t), \\
\bar{W} &= p^{-1}\sum_{i=1}^{N} \epsilon_i S_i(0)\sigma_i \hat{W}_i(t), \\
p &= \sum_{i=1}^{N} \epsilon_i \sigma_i^2 S_i^2(0) + 2\sum_{i<j}^{N} \epsilon_{ij}\rho_{ij}\sigma_i\sigma_j S_i(0)S_j(0).
\end{aligned} \tag{4.7}
$$

With the correlation structure:

$$
\langle d\bar{W}(t), d\hat{W}_i(t)\rangle = p^{-1}\sum_{j=1}^{N} S_j(0)\sigma_j\epsilon_j\rho_{ji}dt = \rho_i dt.
$$

The expected values can now be computed and we have:

$$
\begin{aligned}
\mathbb{E}[S_i(t) - S_i(0)|S(t) = x] &\approx \mathbb{E}[\bar{S}_i(t) - S_i(0)|\bar{S}(t) = x] \\
&= \mathbb{E}[S_i(t) - S_i(0)] + \frac{\text{covar}(\bar{S}_i(t), \bar{S}(t))}{\text{var}(\bar{S}(t))}(x - \mathbb{E}[\bar{S}(t)]) \\
&= \frac{\langle \bar{S}_i(t), \bar{S}(t)\rangle}{\langle \bar{S}(t), \bar{S}(t)\rangle}(x - S(0)) \\
&= \sigma_i S_i(0)\rho_i \frac{x - S(0)}{p}.
\end{aligned}
$$

Using these expressions we compute $F(t, x)$, we find:

$$
\begin{aligned}
F^2(x) \approx\ & p^2 + \frac{(x - S(0))}{p}\left\{\sum_{i=1}^{N} \sigma_i^3 S_i^2(0)\rho_i\right. \\
& \left. + \sum_{i<j}^{N} \epsilon_{ij}\rho_{ij}\sigma_i\sigma_j S_i(0)S_j(0)(\sigma_i\rho_i + \sigma_j\rho_j)\right\}.
\end{aligned}
$$

Given these solutions $F(S(0))$ and $F'(S(0))$ are given by:

$$
F(S(0)) = p \qquad\qquad F'(S(0)) = q
$$

with p given by Eq.(4.7) and q given by:

$$
q = \frac{\sum_{i=1}^{N}\left(\rho_i S_i^2(0)\sigma_i^3 + \sum_{i\neq j}^{N} \rho_{ij}\rho_i\epsilon_{ij}S_i(0)S_j(0)\sigma_i^2\sigma_j\right)}{p^2}.
$$

□

4.3.2 Projection of the SABR model

To project the multivariate SABR diffusion of Eq. (4.2) by Lemma 4.3.1, we choose a one-dimensional SABR style model given by:

$$\begin{aligned} dS(t) &= u(t)F(S(t))dW(t) \\ du(t) &= \eta u(t)dZ(t) \\ S(0) &= \sum_{i=1}^{N} s_i^0 \\ u(0) &= 1 \\ \langle dW(t), dZ(t)\rangle &= \gamma dt. \end{aligned} \tag{4.8}$$

γ denotes the correlation between the forward price and the volatility process and $F(S(t))$ is a function of the swap rate.

The computation of the projection and in particular of $F(S(t))$ involves as in the case of the SMM approximations which we explain in detail in the proof of the following Theorem.

Theorem 4.3.2. *The dynamics of a basket of assets following a multivariate SABR model, Eq. (4.2), is approximated by:*

$$\begin{aligned} dS(t) &= u(t)F(S(t))dW(t) \\ du(t) &= \eta u(t)dZ(t) \\ S(0) &= \sum_{i=1}^{N} s_i^0 \\ \langle dW(t), dZ(t)\rangle &= \gamma dt = \frac{1}{\eta p^3}\sum_{i=1}^{N}\sum_{k=1}^{N}\left(p_i^2\nu_i + \sum_{i\neq j}^{N}\epsilon_{ij}\rho_{ij}p_ip_j\nu_i\right)p_k\epsilon_k\gamma_{ik}dt \\ F(S(0)) &= p = \sqrt{\sum_{i=1}^{N}p_i^2 + 2\sum_{i<j}^{N}\rho_{ij}p_ip_j\epsilon_{ij}} \\ F'(S(0)) &= q = \frac{\sum_{i=1}^{N}\left(p_i^2 q_i\rho_i + \sum_{i\neq j}^{N}p_iq_i\rho_{ij}\rho_i\epsilon_{ij}p_j\right)}{p^2} \\ p_i &= \alpha_i(0)S_i(0)^{\beta_i} \\ q_i &= \alpha_i(0)\beta_iS_i(0)^{\beta_i-1}. \end{aligned}$$

Proof. The approximation is computed in several steps. First, we rewrite the original SABR diffusion of Eq. (4.2) as a single diffusion with stochastic volatility driven by a Brownian motion. To preserve the starting values of the process we rescale the volatility of Eq. (4.2) by:

$$\begin{aligned} u_i(t) &= \frac{\alpha_i(t)}{\alpha_i(0)} \\ \varphi(S_i(t)) &= \alpha_i(0)S_i(t)^{\beta_i} \\ \Rightarrow dS_i(t) &= u_i(t)\varphi(S_i(t))dW_i(t). \end{aligned}$$

Furthermore, we ease the notation by introducing:

$$\begin{aligned} \varphi(S_i(0)) = p_i &= \alpha_i(0)S_i(0)^{\beta_i} \\ \varphi'(S_i(0)) = q_i &= \alpha_i(0)\beta_i S_i(0)^{\beta_i - 1}. \end{aligned}$$

First, using the SDE for the individual assets, we find:

$$\begin{aligned} dS(t) &= \sum_{i=1}^{N} \epsilon_i dS_i(t) \\ &\approx \sum_{i=1}^{N} \epsilon_i u_i(t)\varphi(S_i(t))d\hat{W}_i(t) \end{aligned}$$

under a common measure as in the case of the SMM. Choosing the Brownian motion such that:

$$d\hat{W}(t) = \sigma^{-1}(t)\sum_{i=1}^{N} \epsilon_i u_i \varphi(S_i(t))d\hat{W}_i(t)$$

we have the representation

$$dS(t) = \sigma(t)d\hat{W}(t)$$

with $\epsilon_{ij} = \epsilon_i \cdot \epsilon_j$ and $\sigma(t)$ given by:

$$\sigma^2(t) = \sum_{i=1}^{N} \epsilon_i u_i^2 \varphi^2(S_i(t)) + 2\sum_{i<j}^{N} \epsilon_{ij}\rho_{ij}u_i u_j \varphi(S_i(t))\varphi(S_j(t)).$$

Under this specification, the Levy characterization gives that $\hat{W}(t)$ is a Brownian motion.

To apply the result of Gyoengy [1986] we need to compute the variance of Eq. (4.8) on which Eq. (4.2) is to be projected. We compute $u^2(t)$ as:

$$u^2(t) = \frac{1}{p^2}\left(2\sum_{i<j}^{N} p_i p_j u_i(t)u_j(t)\rho_{ij}\epsilon_{ij} + \sum_{i=1}^{N} \epsilon_i p_i^2 u_i(t)^2\right) \tag{4.9}$$

$$\text{with } p = \sqrt{\sum_{i=1}^{N} \epsilon_i p_i^2 + 2\sum_{i<j}^{N} \rho_{ij}p_i p_j \epsilon_{ij}}. \tag{4.10}$$

The factor $1/p$ is necessary to ensure $u(0) = 1$.

Now, we are again in a position to apply the result of Gyoengy. With the notation of Lemma (4.3.1) we set $b(t,x) = \mathbb{E}[\sigma^2(t)|S(t) = x]$ and on the other hand $b(t,x) = \mathbb{E}[u^2(t)|S(t) = x] \cdot F^2(x)$. Thus, we have

$$F^2(x) = \frac{\mathbb{E}[\sigma^2(t)|S(t) = x]}{\mathbb{E}[u^2(t)|S(t) = x]}. \tag{4.11}$$

To compute the conditional expectations of the nominator and the denominator we observe that $\sigma^2(t)$ and $u^2(t)$ are linear combinations of the form

$$\begin{aligned} f_{ij}(t) &= \varphi(S_i)\varphi(S_j)u_i(t)u_j(t) \\ g_{ij}(t) &= \frac{p_i p_j u_i(t)u_j(t)}{p^2} \end{aligned} \tag{4.12}$$

and can be represented as follows:

$$\begin{aligned} \sigma^2(t) &= \sum_{i=1}^{N} f_{ii}(t) + 2\sum_{i<j}^{N} f_{ij}(t)\rho_{ij}\epsilon_{ij} \\ u^2(t) &= \sum_{i=1}^{N} g_{ii}(t) + 2\sum_{i<j}^{N} g_{ij}(t)\rho_{ij}\epsilon_{ij}. \end{aligned}$$

To compute the conditional expectations, a first-order Taylor series expansion leads to

$$f_{ij} \approx p_i p_j \left(1 + \frac{q_i}{p_i}(S_i(t) - S_i(0)) + \frac{q_j}{p_j}(S_j(t) - S_j(0)) + (u_i(t) - 1) + (u_j(t) - 1)\right) \tag{4.13}$$

and

$$g_{ij} \approx \frac{p_i p_j}{p^2}\left(1 + (u_i(t) - 1) + (u_j(t) - 1)\right). \tag{4.14}$$

Thus, to compute the conditional expectations of Eq. (4.11) we need simple expressions for

$$\begin{aligned} &\mathbb{E}[S_i(t) - S_i(0)|S(t) = x] \\ &\quad \mathbb{E}[u_i(t) - 1|S(t) = x]. \end{aligned} \tag{4.15}$$

To find a simple formula we apply a Gaussian approximation to compute the expected values and include the stochastic volatility process:

$$\begin{aligned} dS_i(t) &\approx d\bar{S}_i(t) = p_i d\hat{W}_i(t), \\ du_i(t) &\approx d\bar{u}_i(t) = \nu_i d\hat{Z}(t), \\ dS(t) &\approx d\bar{S}(t) = p d\bar{W}(t), \\ d\bar{W} &= p^{-1}\sum_{i=1}^{N} p_i \epsilon_i d\hat{W}_i(t). \end{aligned}$$

We have the correlation structure:

$$\begin{aligned}\langle d\bar{W}(t), d\hat{W}_i(t)\rangle &= p^{-1}\sum_{j=1}^{N} p_j\epsilon_j\rho_{ji}dt = \rho_i dt \\ \langle d\bar{W}(t), d\hat{Z}_i(t)\rangle &= p^{-1}\sum_{j=1}^{N} p_j\epsilon_j\gamma_{ji}\rho_{j+N}dt = \rho_{i+N}dt.\end{aligned}$$

The expected values can now be computed and we have

$$\mathbb{E}[\bar{S}_i(t) - S_i(0)|\bar{S}(t) = x] = \frac{\langle \bar{S}_i(t), \bar{S}(t)\rangle}{\langle \bar{S}(t), \bar{S}(t)\rangle}(x - S(0)) = p_i\rho_i\frac{x - S(0)}{p}$$

and

$$\mathbb{E}[\bar{u}_i(t) - 1|\bar{S}(t) = x] = \nu_i\rho_{i+N}\frac{x - S(0)}{p}.$$

Using these expressions we compute $F(t, x)$. Denoting the coefficient appearing in the denominator by A_d and the numerator by A_u we find:

$$F^2(x) = \frac{\mathbb{E}[\sigma^2(t)|S(t) = x]}{\mathbb{E}[u^2(t)|S(t) = x]} \approx \frac{p^2 + A_u(x - S(0))}{1 + A_d(x - S(0))} \tag{4.16}$$

with:

$$\begin{aligned}A_u &= \frac{2}{p}\left\{\sum_{i=1}^{N} p_i^2(q_i\rho_i + \nu_i\rho_{i+N}) \right.\\ &\quad \left. + \sum_{i<j}^{N} \epsilon_{ij}\rho_{ij}p_ip_j(q_i\rho_i + q_j\rho_j + \nu_i\rho_{i+N} + \nu_j\rho_{j+N})\right\}, \\ A_d &= \frac{2}{p^3}\left\{\sum_{i=1}^{N} p_i^2\nu_i\rho_{i+N}\sum_{i<j}^{N}\epsilon_{ij}\rho_{ij}p_ip_j(\nu_i\rho_{i+N} + \nu_j\rho_{j+N})\right\}.\end{aligned}$$

Given these solutions $F(S(0))$ and $F'(S(0))$ are given by:

$$F(S(0)) = p \qquad\qquad F'(S(0)) = q$$

with p given by Eq.(4.10) and q given by:

$$q = \frac{\sum_{i=1}^{N}\left(p_i^2 q_i\rho_i + \sum_{i\neq j}^{N} p_iq_i\rho_{ij}\rho_i\epsilon_{ij}p_j\right)}{p^2}.$$

Finally, we need to derive a SABR-like diffusion for the stochastic volatility and apply the Itô formula to derive the SDE for $u(t)$. Only using first-order approximations and replacing the quotients $\frac{u_i(t)u_j(t)}{u^2(t)}$ with the expected value,

$$\mathbb{E}\left[\frac{u_i^2(t)}{u^2(t)}\right] = \mathbb{E}\left[\frac{u_i(t)u_j(t)}{u^2(t)}\right] = 1$$

we find:

$$\begin{aligned}\frac{du(t)}{u(t)} &= \frac{1}{p^2}\sum_{i=1}^{N}\left(p_i^2\frac{u_i^2}{u^2}+\sum_{i\neq j}^{N}\epsilon_{ij}\rho_{ij}p_ip_j\frac{u_iu_j}{u^2}\right)\nu_i d\hat{Z}_i(t)\\ &= \frac{1}{p^2}\sum_{i=1}^{N}\left(p_i^2+\sum_{i\neq j}^{N}\epsilon_{ij}p_ip_j\right)\nu_i d\hat{Z}_i(t). \end{aligned} \tag{4.17}$$

For more accurate approximations we may keep the higher order terms. This results in a more complex expression and drift terms. In this case we can apply the results for the λ-SABR model, see Labordere [2005].
Thus, by computing the (simple) approximation we obtain a SDE for $u(t)$ which we denote by:

$$du(t) = \eta u(t) d\hat{Z}(t).$$

For the Brownian motion $\hat{Z}(t)$ we have

$$\begin{aligned} d\hat{Z}(t) &= \frac{\sum_{i=1}^{N}\left(p_i^2\nu_i+\sum_{i\neq j}^{N}\epsilon_{ij}\rho_{ij}p_ip_j\nu_i\right)d\hat{Z}_i(t)}{\eta p^2},\\ \eta^2 &= Var\left(\frac{\sum_{i=1}^{N}\left(p_i^2\nu_i+\sum_{i\neq j}^{N}\epsilon_{ij}\rho_{ij}p_ip_j\nu_i\right)d\hat{Z}_i(t)}{p^2}\right)\end{aligned}$$

with η such that $\hat{Z}(t)$ scales to $\langle\hat{Z}(t)\rangle = t$. We determine the correlation between the dynamics of the forward price process and the stochastic volatility as:

$$\begin{aligned}\gamma &= \frac{\langle d\hat{W}(t), d\hat{Z}(t)\rangle}{dt}\\ &\approx \frac{\langle d\bar{W}(t), d\hat{Z}(t)\rangle}{dt}\\ &= \frac{1}{\eta p^3}\sum_{i=1}^{N}\sum_{k=1}^{N}\left(p_i^2\nu_i+\sum_{i\neq j}^{N}\epsilon_{ij}\rho_{ij}p_ip_j\nu_i\right)p_k\epsilon_k\gamma_{ik}. \end{aligned}\tag{4.18}$$

□

4.4 Application to CMS Spread Options

Constant maturity swaps (CMS) are interest rate swaps where the fixed leg pays a swap rate with a constant time to maturity at every payment date. These are liquid financial instruments that allow to take positions on future long-term rates due to the constant maturity of the fixed-leg payments. The underlying swap rates are also an important building block of structured products in today´s fixed income markets. Such products incorporate a CMS structure with payment dates similar to a swap but use the constant maturity swap rates as an underlying for embedded options. Common CMS payments in fixed income structured products are

- Capped / Floored CMS Coupons, (S_1),
- Capped / Floored CMS Spread Coupons, $(S_1 - S_2)$ and
- Capped / Floored CMS Swing Coupons, $(S_2 - S_3) - (S_1 - S_2)$.

The subscripts indicate that the underlying CMS yields are for different times-to-maturity with $1 > 2 > 3$. The structure of CMS spread and swing options allows to express views on future changes of the shape of the yield curve. Particularly, steepening or flattening is traded using spread options and the curvature of the yield curve using swing options. Therefore, such options can be used as hedges of swap rate correlation risk.

To give comparable numerical results for liquid CMS options priced by the Markovian Projetion approach in a multidimensional SMM/SABR model we restrict the implementation to the case of a CMS spread option.

Definition 4.4.1. *The payoff of a CMS spread option at maturity date t is as follows:*

$$\max\{S_1(t) - S_2(t) - K, 0\}.$$

For the special case of zero-strike options, $K = 0$, the option can be valued analytically using the formula for exchange options, see Margrabe [1978]. For $K \neq 0$ an analytical solution is only feasible if the spread is modeled as a normally distributed random variable

$$\begin{aligned} S_1(t) - S_2(t) &= \bar{S}(t) \\ \text{with } d\bar{S}(t) &= \bar{\sigma} d\bar{W}(t). \end{aligned}$$

This framework is too simple to consistently price CMS spread options since implicitly a perfect correlation is assumed. And it is also not taking into account the smile and the skew effects. But we included it for the sake of completeness since the market quotes spread options by their implied normal volatilities such as swaptions are quoted by their implied Black volatility.

One numerical method capable of utilizing the SABR model is the copula approach. In the following we present the copula approach of Berrahoui [2004] and Benhamou and Croissant [2007], show how to project the spread onto a displaced diffusion using a SMM/SABR model and how to value an option on the projected spread.

4.4.1 Copula Method

One way to approximate spreads in a SABR model is the copula approach. The idea is that the payoff of spread options with two correlated price processes can be decomposed into a portfolio of digital options and is given as:

$$\max\{S_1(t) - S_2(t) - K, 0\} = \int_0^\infty 1_{[S_1(t)>x+K]} 1_{[S_2(t)<x]} dx.$$

Here the indicator functions $1_{[.]}$ can be interpreted as the payoff of digital call options. Now taking the discounted expectation under the risk-adjusted measure P^t, the fair value can be computed by using numerical integration:

$$\begin{aligned} & B(0,t)E^t\left[\int_0^\infty 1_{[S_1(t)>x+K]} 1_{[S_2(t)<x]} dx\right] \\ = \; & B(0,t)\int_0^\infty P^t(S_1(t) > x + K, S_2(t) < x)dx. \end{aligned} \tag{4.19}$$

The joint probability function $P^t(.)$ can be computed using a Gaussian copula with SABR margins, but the swap rates have to be convexity corrected since their are not a martingale under the forward measure. The procedure consists of two steps. First, we have to compute the margins of the SABR distributions and then have to map the quantiles onto a log-normal distribution as shown in Figure (4.1). The second step uses the Gaussian copula to obtain the joint probability function. As stated above, we first need the margins $P^t_{SABR}(S_i(t) > x_i)$ which can be computed numerically or replicated using digital options. A digital option in the SABR model is given by using the formula of Black [1976] and the implied SABR volatility function σ_{SABR}:

$$\begin{aligned} P^t_{SABR}(S_i(t) > x_i) &= \frac{Digital_{SABR}(\hat{S}_i(0), x_i, t)}{B(0,t)} \\ &= \frac{Digital_{Black}(\hat{S}_i(0), x_i, t, \sigma_{SABR}) - Vega_{Black}(\hat{S}_i(0), x_i, t, \sigma_{SABR})\frac{\partial \sigma_{SABR}}{\partial x_i}}{B(0,t)} \end{aligned}$$

where $x_1 = x + K$ and $x_2 = K$. Note, that the probability is taken under the forward measure and therefore the swap rates $S(0)$ have to be convexity corrected to $\hat{S}(0)$, see e.g.

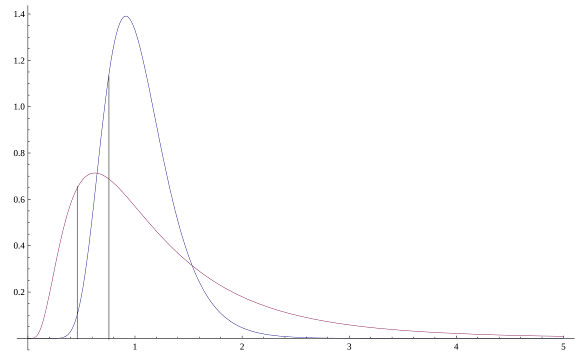

Figure 4.1: Quantile mapping of a SABR distribution onto a log-normal distribution

Hagan [2003]. To map the SABR distribution onto a log-normal for a given SABR quantile we compute the equivalent Black quantile using the implied SABR volatility and solve for $\bar{x}_i$:

$$\begin{aligned}
P^t_{SABR}(S_i(t) > x_i) &= P^t_{Black}(S_i(t) > \bar{x}_i | \sigma = \sigma_{SABR}) \\
\Rightarrow P^t_{SABR}(S_i(t) > x_i) &\approx P^t\left(\hat{S}_i(0)\exp\left\{-\frac{1}{2}\sigma^2_{SABR} + \sigma_{SABR}\sqrt{t}Z > \bar{x}_i\right\}\right) \\
\Leftrightarrow P^t_{SABR}(S_i(t) > x_i) &= N\left(\frac{\ln\frac{\hat{S}_i(0)}{\bar{x}_i} - \frac{1}{2}\sigma^2_{SABR}}{\sigma_{SABR}\sqrt{t}}\right) \\
\Leftrightarrow N^{-1}(P^t_{SABR}(S_i(t) > x_i)) &= \frac{\ln\frac{\hat{S}_i(0)}{\bar{x}_i} - \frac{1}{2}\sigma^2_{SABR}}{\sigma_{SABR}\sqrt{t}} \\
\Leftrightarrow \bar{x}_i &= \hat{S}_i(0)\exp\left\{N^{-1}(P^t_{SABR}(S_i(t) > x_i))\sigma_{SABR}\sqrt{t} - \frac{1}{2}\sigma_{SABR}t\right\}.
\end{aligned}$$

With Z as standard normally distributed variable and $N(.)$ the cumulative standard normal distribution. The joint probability distribution can with the knowledge of the $\bar{x}_i$ be computed by using a Gaussian copula C and correlation $\langle dW_1(t), dW_2(t)\rangle = \rho dt$. It is given by:

$$\begin{aligned}
& P^t(S_1(t) > x + K, S_2(t) < x) \\
=\; & P^t(S_2(t) < x) - C(P^t(S_1(t) < x + K), P^t(S_2(t) < x)) \\
\approx\; & N(d_2) - N_2(d_1, d_2, \rho) \\
& \text{with } d_i = \frac{1}{\sigma_{SABR}\sqrt{t}}\left(\ln\left(\frac{\bar{x}_i}{\tilde{S}_i(0)}\right) + \frac{1}{2}\sigma^2_{SABR}t\right). \qquad (4.20)
\end{aligned}$$

To compute the approximated arbitrage-free price of the CMS spread option, we need to apply Eq. (4.20) and substitute it into Eq. (4.19). Finally we use a numerical integration

method.
We can alter the correlation structure using a different copula, for instance the t-copula with heavier tail dependence. As will be shown in Section (4.5) the copula approach prices the CMS Spread Options fairly accurate, but there are still some drawbacks of this method:

- The copula method is static and we have no process of the spread dynamics.
- The numerical integration is time-consuming.
- The decorrelation and cross skews are assumed to be uncorrelated.
- The methodology cannot be extended to CMS options with more than two CMS rates.

4.4.2 Markovian Projection

In this subsection, we apply the general results obtained in Section (4.3) to the case of a CMS spread option. The guiding idea is to compute a SDE for the spread dynamics which approximates the joint SMM/SABR dynamics at maturity.

Swap Market Model

Theorem 4.4.1. *The dynamics of the spread can be approximated by*

$$\begin{aligned}
dS(t) &= F(S(t))dW(t) \\
S(0) &= S_1(0) - S_2(0) \\
F(S(0)) &= p = \sqrt{S_1^2(0)\sigma_1^2 + S_2^2(0)\sigma_2^2 - 2\rho_{12}S_1(0)S_2(0)\sigma_1\sigma_2} \\
F'(S(0)) &= q = \frac{S_1(0)\sigma_1^2\rho_1^2 - S_2(0)\sigma_2^2\rho_2^2}{p} \\
\rho_1 &= \frac{1}{p}[\sigma_1 S_1(0) - \sigma_2 S_2(0)\rho_{12}] \\
\rho_2 &= \frac{1}{p}[\sigma_1 S_1(0)\rho_{12} - \sigma_2 S_2(0)].
\end{aligned}$$

Proof. We consider the dynamics of Eq. (4.1) and compute the diffusion for the spread taking $N = 2, \epsilon_1 = 1$ and $\epsilon_2 = -1$. Since the payoff of a CMS spread option is a single payment at maturity time t the expectation has to be taken under the time T forward measure. Since we model swap rates with different tenor structures, we cannot model them as driftless processes under the same measure. If they are modeled under one measure, they obtain a drift term $\mu_i(t)$. In fact, they are only driftless under their own annuity measure

P^{A_i}. Since we need driftless processes to use a SABR-like displaced diffusion formula for option valuation, we follow the approach of Antonov and Arneguy [2009] and change to a so-called spread measure P^S. At this stage, we only have to assume that the measure exists to project the diffusions but do not need to compute it since we change back to the annuity measures when we compute the option value. Under this assumption the spread SDE is driftless and given by:

$$\begin{aligned} dS(t) &= dS_1(t) - dS_2(t) \\ &= (\mu_1(t)dt + \sigma_1 S_1(t)dW_1(t)) - (\mu_2(t)dt + \sigma_2 S_2(t)dW_2(t)) \\ &\approx \sigma(t)dW^S(t) \end{aligned}$$

with:

$$\begin{aligned} dW^S(t) &= \frac{1}{\sigma(t)}\left(\sigma_1 S_1(t)dW_1^S - \sigma_2 S_2(t)dW_2^S\right) \\ \sigma^2(t) &= \sigma_1^2 S_1(t)^2 + \sigma_2^2 S_2(t)^2 \\ &- 2\rho_{12}\sigma_1\sigma_2 S_1(t)S_2(t). \end{aligned} \tag{4.21}$$

At this point we have two representations for the spread SDE:

$$dS(t) = \sigma(t)dW^S(t) \quad \text{and} \quad dS(t) = F(S(t))dW^S(t).$$

With the first equation being the original spread SDE and the second one the approximating SDE both under the spread measure. We now have to compute the parameters of the approximating SDE that mimic the terminal one-dimensional distribution of the original SDE. Applying the Gyoengy [1986] result, we have to choose $F^2(t,x)$ such that:

$$F^2(x) = \mathbb{E}\left[\sigma^2(t)|S(t) = x\right]. \tag{4.22}$$

To proceed, we use Eq. (4.6) to further simplify the notation and compute the volatility as:

$$\sigma^2(t) = f_{11}(t) + f_{22}(t) - 2\rho_{12}f_{12}(t).$$

To be able to compute the conditional expectations, we use the first-order Taylor series approximation as in Eq. (4.6). This reduces the problem to the computation of the conditional expectations for $S_i(t)$. To make the calculations more explicit we apply a Gaussian approximation. Using the approximation, we can simplify the conditional expectations given as follows:

$$\begin{aligned}
\mathbb{E}[\bar{S}_i(t) - S_i(0)|\bar{S}(t) = x] &= \sigma_i S_i(0)\rho_i \frac{(x - S(0))}{p} \\
\rho_1 &= \frac{1}{p}[\sigma_1 S_1(0) - \sigma_2 S_2(0)\rho_{12}] \\
\rho_2 &= \frac{1}{p}[\sigma_1 S_1(0)\rho_{12} - \sigma_2 S_2(0)].
\end{aligned}$$

This leads to a simple expression for equation (4.22):

$$\begin{aligned}
\mathbb{E}\left[\sigma^2(t)|S(t) = x\right] &\approx p^2 + (x - S(0))A \\
\text{with } A &= \frac{2}{p}[\sigma_1^3 S_1^2(0)\rho_1 + \sigma_2^3 S_2^2(0)\rho_2 - \sigma_1\sigma_2\rho_{12}S_1(0)S_2(0)(\sigma_2\rho_2 + \sigma_1\rho_1)].
\end{aligned}$$

□

In Figure (4.2) simulated densities for the spread in a SMM are plotted where we compare the application of Markovian Projection and of Monte Carlo simulation. Since we model a spread option with only a small probability of negative realizations we can assume that this probability is approximated appropriately. The influence of the difference between the Markovian Projection and the Monte Carlo simulation on the price of options in the case of SMM diffusions is discussed in Section (4.5).

SABR Model

In the following, we derive the projected SABR diffusion of a CMS spread to value a CMS spread option. If the spread is modeled by a projected SABR model the resulting valuation formula for CMS spread options also captures the volatility smile as it can be seen in Figure (4.3).

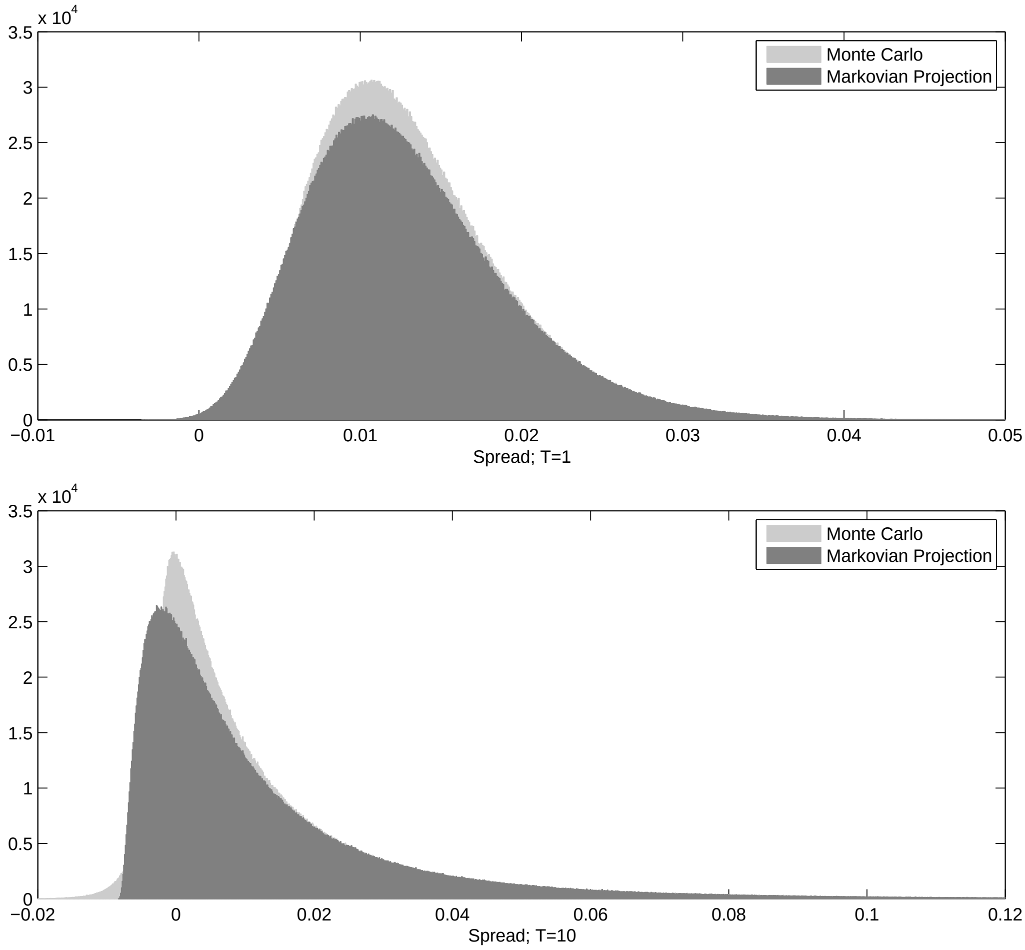

Figure 4.2: Densities of the spread of two SMM diffusions using a Monte Carlo simulation and a Markovian Projection onto a displaced diffusion for different first-fixing dates T.

Theorem 4.4.2. *The dynamics of the spread can be approximated by*

$$\begin{aligned}
dS(t) &= u(t)F(S(t))dW(t) \\
du(t) &= \eta u(t)dZ(t) \\
F(S(0)) &= p = \sqrt{p_1^2 + p_2^2 - 2\rho_{12}p_1p_2} \\
F'(S(0)) &= q = \frac{p_1q_1\rho_1^2 - p_2q_2\rho_2^2}{p} \\
\eta &= \sqrt{\frac{1}{p^2}\left[(p_1\nu_1\rho_1)^2 + (p_2\nu_2\rho_2)^2 - 2\xi_{12}p_1\nu_1\rho_1p_2\nu_2\rho_2\right]} \\
\gamma &= \frac{1}{\eta p^2}\left(p_1^2\nu_1\rho_1\gamma_{11} + p_2^2\nu_2\rho_2\gamma_{22} - p_1p_2\nu_2\rho_2\gamma_{21} - p_1p_2\nu_1\rho_1\gamma_{12}\right).
\end{aligned}$$

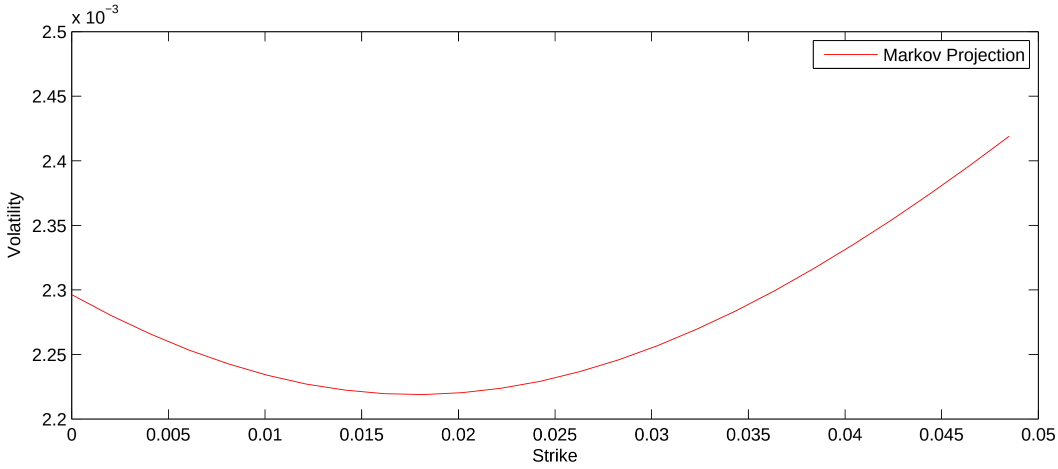

Figure 4.3: Strike dependent implied CMS spread call volatilities of prices obtained by a Markov Projection with $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\xi_{1,2} = 0.75$, $\beta = 0.7$, $\nu_1 = 0.4$, $\nu_2 = 0.4$ and $T = 10$.

Proof. As in the case of the SMM, we compute the diffusion of the spread and change both measures to a so-called spread measure P^S under which their spread SDE is driftless and given by:

$$\begin{aligned} dS(t) &= dS_1(t) - dS_2(t) \\ &= (\mu_1 dt + u_1(t)\varphi(S_1(t))dW_1(t)) - (\mu_2 dt + u_2(t)\varphi(S_2(t))dW_2(t)) \\ &\approx \sigma(t)dW^S(t) \end{aligned}$$

with:

$$\begin{aligned} dW^S(t) &= \frac{1}{\sigma(t)}\left(u_1(t)\varphi(S_1(t))dW_1^S - u_1(t)\varphi(S_2(t))dW_2^S\right) \\ \sigma^2(t) &= u_1^2(t)\varphi(S_1(t))^2 + u_2^2(t)\varphi(S_2(t))^2 \\ &- 2\rho_{12}u_1(t)u_2(t)\varphi(S_1(t))\varphi(S_2(t)). \end{aligned} \tag{4.23}$$

In a second step we compute the variance of the approximating SDE as given by Eq. (4.23):

$$\begin{aligned} u^2(t) &= \frac{1}{p^2}\Big(p_1^2u_1^2(t) + p_2^2u_2^2(t) - 2\rho_{12}p_1p_2u_1(t)u_2(t)\Big) \\ \text{with } p = \sigma(0) &= \sqrt{p_1^2 + p_2^2 - 2\rho_{12}p_1p_2}. \end{aligned}$$

At this point we have two representations for the spread SDE:

$$dS(t) = \sigma(t)dW^S(t) \quad \text{and} \quad dS(t) = u(t)F(S(t))dW^S(t).$$

With the first equation being the original spread SDE and the second one the approximating SDE under the spread measure. Applying the Gyoengy [1986] result, we have in the case of the SABR model to choose $F^2(t,x)$ such that:

$$F^2(x) = \frac{\mathbb{E}\left[\sigma^2(t)|S(t)=x\right]}{\mathbb{E}\left[u^2(t)|S(t)=x\right]}. \tag{4.24}$$

To proceed, we use Eq. (4.12) to further simplify the notation for both volatility functions. Thus, we can compute the volatilities as:

$$\begin{aligned}\sigma^2(t) &= f_{11}(t) + f_{22}(t) - 2\rho_{12} f_{12}(t)\\ u^2(t) &= g_{11}(t) + g_{22}(t) - 2\rho_{12} g_{12}(t).\end{aligned}$$

To be able to compute the conditional expectations, we use the first-order Taylor series approximation as in Eq. (4.13) and Eq. (4.14). This reduces the problem to the computation of the conditional expectations for $S_i(t)$ and $u_i(t)$, Eq. (4.15). To make the calculations more explicit we apply a Gaussian approximation. Using the approximation we can simplify the conditional expectations as follows:

$$\mathbb{E}[\bar{S}_i(t) - S_i(0)|\bar{S}(t) = x] = p_i\rho_i\frac{(x - S(0))}{p}$$

and

$$\mathbb{E}[\bar{u}_i(t) - 1|\bar{S}(t) = x] = \nu_i\rho_{i+2}\frac{(x - S(0))}{p}.$$

This leads to a simple expression for the numerator and the denominator of equation (4.24):

$$\begin{aligned}\mathbb{E}\left[\sigma^2(t)|S(t)=x\right] &\approx p^2 + (x - S(0))A_u\\ \text{with } A_u &= \frac{2}{p}\Big(p_1^2(q_1\rho_1 + \nu_1\rho_3) + p_2^2(q_2\rho_2 + \nu_2\rho_4)\\ &\quad -p_1p_2\rho_{12}\big(q_1\rho_1 + q_2\rho_2 + \nu_1\rho_3 + \nu_2\rho_4\big)\Big),\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[u^2(t)|S(t)=x\right] &\approx 1 + (x - S(0))A_d\\ \text{with } A_d &= \frac{2}{p^3}\Big(\nu_1p_1(p_1 - p_2\rho_{12})\rho_3 + \nu_2(p_2 - p_1\rho_{12})\rho_3\Big).\end{aligned}$$

We can compute the approximating SDE for $S(t)$. To compute the dynamics of $u(t)$, we apply Eq. (4.17) to get:

$$\begin{aligned}du(t) &= \frac{1}{p^2}\Big(p_1^2\nu_1\frac{u_1^2}{u^2} - \rho_{12}p_1p_2\nu_1\frac{u_1u_2}{u^2}\Big)u(t)dZ_1(t)\\ &+ \frac{1}{p^2}\Big(p_2^2\nu_2\frac{u_2^2}{u^2} - \rho_{12}p_1p_2\nu_2\frac{u_1u_2}{u^2}\Big)u(t)dZ_2(t)\\ \Leftrightarrow \frac{du(t)}{u(t)} &= \Big(\frac{p_1\nu_1\rho_1}{p}dZ_1(t) + \frac{p_2\nu_2\rho_2}{p}dZ_2(t)\Big)\end{aligned}$$

and we have the SDE by setting:

$$
\begin{aligned}
Z(t) &= \frac{1}{\eta p}\Big(p_1\nu_1\rho_1 dZ_1 - \rho_2\nu_2\rho_2 dZ_2\Big) \\
\eta^2 &= \frac{1}{p^2}\left[(p_1\nu_1\rho_1)^2 + (p_2\nu_2\rho_2)^2 - 2\xi_{12}p_1\nu_1\rho_1p_2\nu_2\rho_2\right].
\end{aligned}
$$

The correlation between the projected forward price process $S(t)$ and its stochastic volatility process $u(t)$ can be computed using Eq. (4.18):

$$
\gamma = \frac{1}{\eta p^2}\left(p_1^2\nu_1\rho_1\gamma_{11} + p_2^2\nu_2\rho_2\gamma_{22} - p_1p_2\nu_2\rho_2\gamma_{21} - p_1p_2\nu_1\rho_1\gamma_{12}\right).
$$

□

Pricing

In the following we apply the Markovian Projection to the valuation of CMS spread caplets resp. floorlets. In the setting of a projected SMM we linearize $F(S(t))$ as:

$$
\begin{aligned}
F(S(t)) &= (S(t)+a)q \\
\text{with } a &= \frac{p}{q} - S(0) \\
F(S(0)) &= p \\
F'(S(0)) &= q.
\end{aligned}
$$

This is to rewrite the projected SDE as a displaced diffusion. A displaced diffusion is a reasonable choice, since in case of spread options negative realizations of the spread must have positive probabilities.
For a displaced diffusion, a caplet/floorlet on asset $S(t)$ with a terminal log-normal distribution and displacement parameter a can be valued under its measure by the Rubinstein [1983] formula.

To compute the price of a CMS spread caplet/floorlet using the displaced diffusion model, the expectation of the approximated spread at maturity needs to be computed. But since we discount the expectation of a single payment it is under the forward measure, while the approximated spread is under the spread measure. Therefore the expectation is not under the numeraire measure of the spread diffusion. This can be solved by changes of measure and using a convexity correction. Denoting by $A_i(t)$ the numeraire of the annuity measure

P^{A_i} and by $SN(t)$ the numeraire of the spread measure the expectation can be computed as:

$$
\begin{aligned}
& E_{P^T}\left[S(T)\right] \\
= \; & E_{P^S}\left[S(T)\frac{B(T,T)}{SN(T)}\frac{SN(0)}{B(0,T)}\right] \\
= \; & E_{P^S}\left[S(T)\right] + E_{P^S}\left[S(T)\left(\frac{B(T,T)}{SN(T)}\frac{SN(0)}{B(0,T)} - 1\right)\right] \\
= \; & S(0) + E_{P^{A_1}}\left[S_1(T)\left(\frac{B(T,T)}{SN(T)}\frac{SN(0)}{B(0,T)} - 1\right)\frac{SN(T)}{A_1(T)}\frac{A_1(0)}{SN(0)}\right] \\
& - E_{P^{A_2}}\left[S_2(T)\left(\frac{B(T,T)}{SN(T)}\frac{SN(0)}{B(0,T)} - 1\right)\frac{SN(T)}{A_2(T)}\frac{A_2(0)}{SN(0)}\right] \\
= \; & S(0) + E_{P^{A_1}}\left[S_1(T)\left(\frac{B(T,T)}{A_1(T)}\frac{A_1(0)}{B(0,T)} - 1\right)\right] - E_{P^{A_1}}\left[S_1(T)\left(\frac{SN(T)}{A_1(T)}\frac{A_1(0)}{SN(0)} - 1\right)\right] \\
& - E_{P^{A_2}}\left[S_2(T)\left(\frac{B(T,T)}{A_2(T)}\frac{A_2(0)}{B(0,T)} - 1\right)\right] + E_{P^{A_2}}\left[S_2(T)\left(\frac{SN(T)}{A_2(T)}\frac{A_2(0)}{SN(0)} - 1\right)\right] \\
= \; & \{S_1(0) - S_2(0)\} + \{\text{convexity corr}(S_1) - \text{convexity corr}(S_2)\} \\
& - \left\{E_{P^{A_1}}\left[S_1(T)\left(\frac{SN(T)}{A_1(T)}\frac{A_1(0)}{SN(0)} - 1\right)\right] - E_{P^{A_2}}\left[S_2(T)\left(\frac{SN(T)}{A_2(T)}\frac{A_2(0)}{SN(0)} - 1\right)\right]\right\} \\
\approx \; & \{S_1(0) - S_2(0)\} + \{\text{convexity correction}(S_1) - \text{convexity correction}(S_2)\}.
\end{aligned}
$$

The terms $E_{P^{A_i}}\left[S_i(T)\left(\frac{B(T,T)}{A_i(T)}\frac{A_i(0)}{B(0,T)} - 1\right)\right]$ denote the convexity correction, see for instance Hagan [2003], of the swap rate S_i which can be computed e.g. by a replication portfolio. The difference

$$
\left\{E_{P^{A_1}}\left[S_1(T)\left(\frac{SN(T)}{A_1(T)}\frac{A_1(0)}{SN(0)} - 1\right)\right] - E_{P^{A_2}}\left[S_2(T)\left(\frac{SN(T)}{A_2(T)}\frac{A_2(0)}{SN(0)} - 1\right)\right]\right\} \approx 0
$$

is approximated with a zero value, since the corrections due to the mismatch of the annuity measures and the spread measure can be assumed to be close to zero with nearly identical values for both expectations. Using convexity corrected swap rates the valuation of a CMS spread caplet or floorlet in the SMM is now possible.

This method can be extended to a projected SABR model and we define a displaced SABR diffusion as:

Definition 4.4.2. *A displaced SABR diffusion for $\beta = 1$ is given by:*

$$\begin{aligned} dS(t) &= \alpha(t)F(S(t))dW(t) \\ d\alpha(t) &= \nu\alpha(t)dZ(t) \\ \langle dW(t), dZ(t)\rangle &= \gamma dt \\ \textit{with } F(S(t)) &= p + q(S(t) - S(0)) \\ p &= F(S(0)) \\ q &= F'(S(0)) \end{aligned} \tag{4.25}$$

where γ denotes the correlation between the forward price and the volatility process.

Using the implied SABR volatility function σ_{SABR}, the solution of the projected SDE can be written as an asset in a Black [1976] framework and therefore the closed-form displaced diffusion formula can be used. The volatility function σ_{SABR}, is given by:

$$\begin{aligned} \sigma_{SABR} &= \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24} log^2 f/K + \frac{(1-\beta)^4}{1920} log^4 f/K\right\}} \left(\frac{z}{x(z)}\right) \\ & \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right] t\right\} \\ \text{with} \quad & z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log f/K, \\ & f = E_{P^T}[S(T)], \\ & \alpha = q \\ \text{and} \quad & x(z) = \log\left\{\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right\}. \end{aligned}$$

4.5 Numerical Results

To illustrate the approximation in the case of a basket option using the Copula and the Markovian Projection approach, we apply the results obtained in Section (4.4) for valuation of spread options in a SMM/SABR model. As a benchmark we apply a Monte Carlo simulation for the multivariate SMM/SABR model.
In the following, we consider as a base scenario: $F_1 = 0.045$, $F_2 = 0.032$, $\sigma_1 = \alpha_1 = 0.2$, $\sigma_2 = \alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\xi_{1,2} = 0.75$, $\beta = 0.7$, $\nu_1 = 0.4$, $\nu_2 = 0.4$ and $T = 10$.

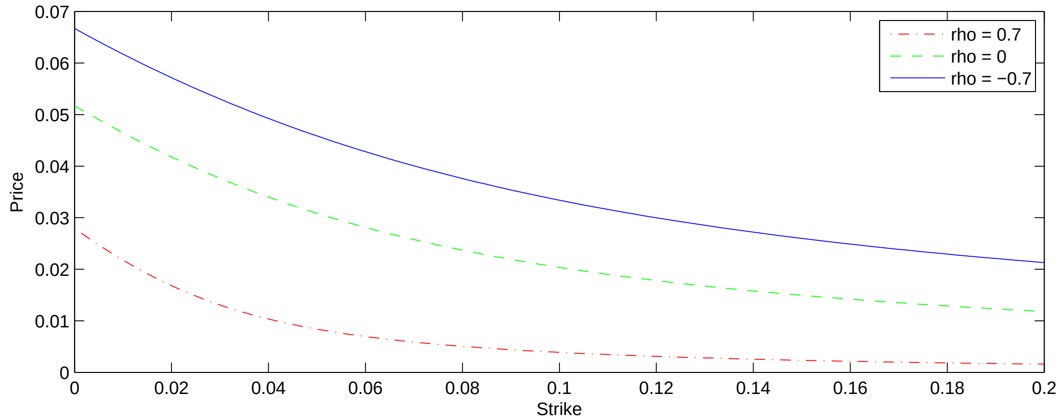

Figure 4.4: Strike-dependent CMS spread caplet prices in dependence of different correlations between the CMS rates.

We start the simulation studies by examining the influence of the correlation between the two CMS rates on the price of a spread option. Since a spread option goes in-the-money if the spread widens a negative correlation should lead to a higher option value and vice versa. As can be seen in Figure (4.4) where we used the Markovian Projection this is fulfilled by the projection model.

To examine the influence of the swaption volatility cube on the prices of CMS spread options we simulate a SMM and a SABR model by using the Markovian Projection. Therefore we use the data set from the base scenario and use the same ATM volatilities for both models. The shape of the volatility cube should lead to a higher spread volatility in the case of the SABR model. Since option values are monotonically increasing in spread volatility this should lead to higher option values in the case of stochastic Volatility by using a SABR model than for the case of deterministic volatility by using the SMM. In the numerical simulation we consider spread option values in dependence of the strike price. In Figure (4.5) can be seen that for a given strike range the influence of the volatility cube is significant. The values by using the SABR model are strictly above the values of the SMM and increase with the time to maturity T. Therefore, the volatility cube must be incorporated into the valuation of CMS spread options for long times to maturity.

For a projected SABR model, we first study the effect of changing the time to maturity and strike prices on the option prices. To this end, we price CMS spread calls and change the

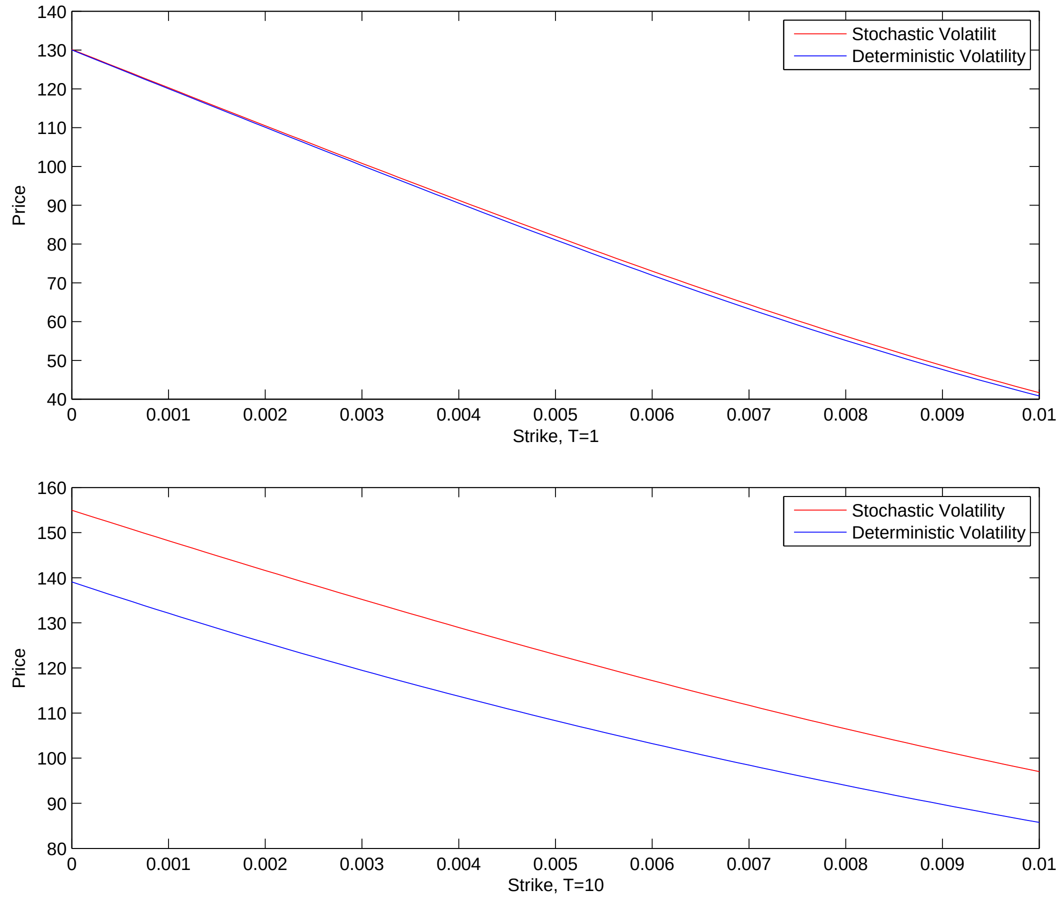

Figure 4.5: Strike-dependent CMS spread call prices using SMM (deterministic volatility) and a SABR model (stochastic volatility) with $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\xi_{1,2} = 0.75$, $\beta = 0.7$, $\nu_1 = 0.4$ and $\nu_2 = 0.4$.

time to maturity and the strike prices. In Figure (4.6) the numerical results of the Copula approach, the Markovian Projection approach and a Monte Carlo simulation are plotted.
It can be seen that the fit of the Copula approach and the Markovian Projection approach is reasonably good for five years to maturity. For ten years to maturity the goodness of the approximations is still good but the reference prices of the Monte Carlo simulation are clearly not in line with them. Both prices lie strictly below the Monte Carlo simulation but the Markovian Projection outperforms the Copula approach. As a result the approximations depend on the time to maturity and therefore should for longer times to maturity only be used with care for the calibration to market prices.

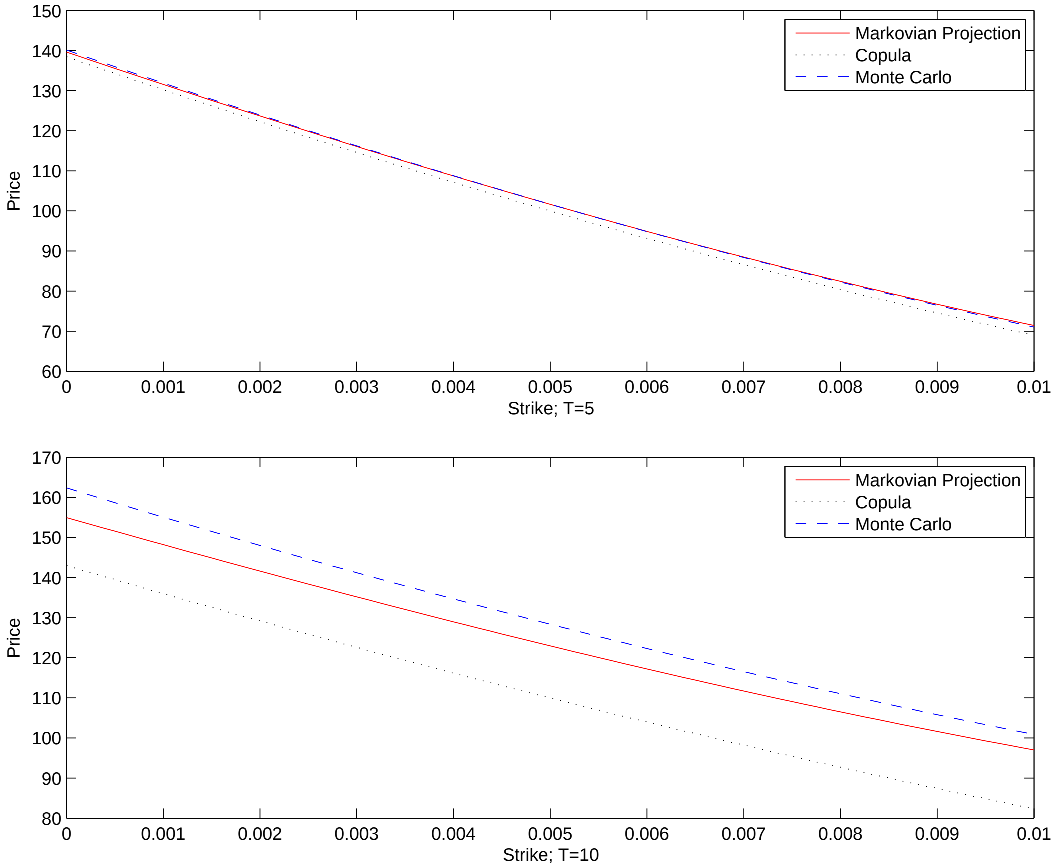

Figure 4.6: Strike-dependent CMS spread call prices of a Markov Projection, a Copula and a Monte Carlo simulation with $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\xi_{1,2} = 0.75$, $\beta = 0.7$, $\nu_1 = 0.4$, and $\nu_2 = 0.4$. The first Figure is plotted with $T = 5$ and the second with $T = 10$.

One advantage of the projected SABR model in comparison to the Copula approach is that the cross skew and the decorrelation are incorporated into the pricing. The influence of these parameters on the arbitrage-free price is significant as shown in Figure (4.8). There, arbitrage-free prices are plotted in dependence of the strike prices for different parameter values. The decorrelation parameter ξ shifts the prices parallel with a negative decorrelation leading to the lowest prices. This is due to the dependency of the spread distribution to the decorrelation. A lower decorrelation parameter shifts mass into the tails of the distribution. This becomes clear by considering Figure (4.7) where two histograms of a SABR spread

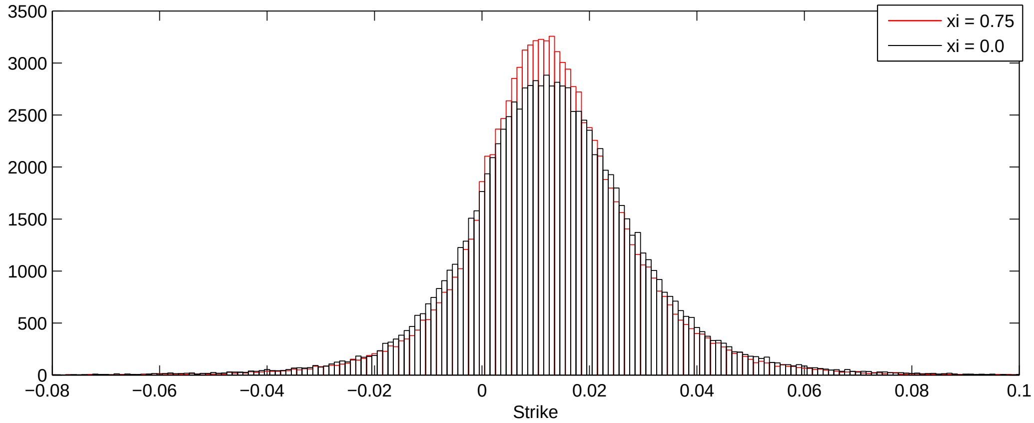

Figure 4.7: Histograms of CMS spread densities using a two-dimensional SABR model with $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\beta = 0.7$, $\nu_1 = 0.4$, $\nu_2 = 0.4$ and $T = 10$.

density are plotted for different values of ξ. If we change both cross skews $\gamma = \gamma_1 = \gamma_2$ simultaneously, the divergence in prices is smaller than by changing the decorrelation ξ with a slightly twist. As a result, if a multivariate SABR model is used to price baskets the decorrelation and cross skew parameters have a significant influence on the price.

In a last step we simulate the hedge parameter delta of a CMS spread caplet in the Markovian Projection model and the SMM. Since an option value is, in arbitrage free pricing the value of a self-financing hedge portfolio, the hedge parameter delta of the Markovian projection should also be close to the benchmark value given by a Monte Carlo simulation. The deltas are, as can be seen in Figure (4.9), very close to the benchmark values even for a long time to maturity.

Since the Markovian Projection is an approximation which is less accurate for long times to maturity, a proper valuation of a basket should in this case be done by a Monte Carlo simulation using the Markovian Projection for calibration. But the calibration is numerically very fast, since the Markovian Projection is an analytical approximation, while the Monte Carlo simulation and the Copula approach are plain numerical methods.

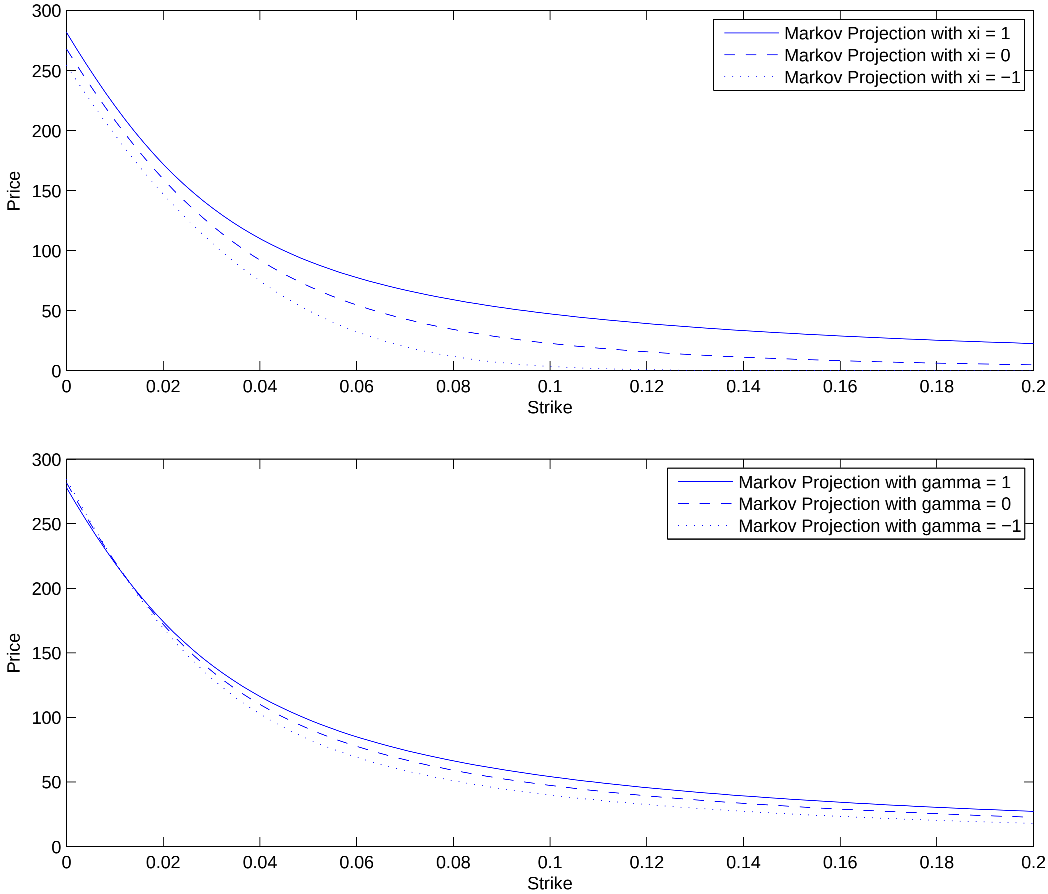

Figure 4.8: Strike-dependent CMS spread call prices of a Markov Projection with $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\xi_{1,2} = 0.75$, $\beta = 0.7$, $\nu_1 = 0.4$, $\nu_2 = 0.4$ and $T = 10$ for different cross skew and decorrelation parameters.

4.6 Conclusion

We have presented the application of the Markovian Projection technique to the SMM and the SABR stochastic volatility model in multiple dimensions. As an example, we have applied it to a popular interest rate derivative, the CMS spread option that significantly depends on the swaption volatility cube. The proposed technique uses harsh approximations but is capable to model a basket of n- assets taking into account all parameters modeling the dependence structure. Such as the correlation of the underlying forward CMS rates, the correlation between the rates and the volatility processes and the correlation between the

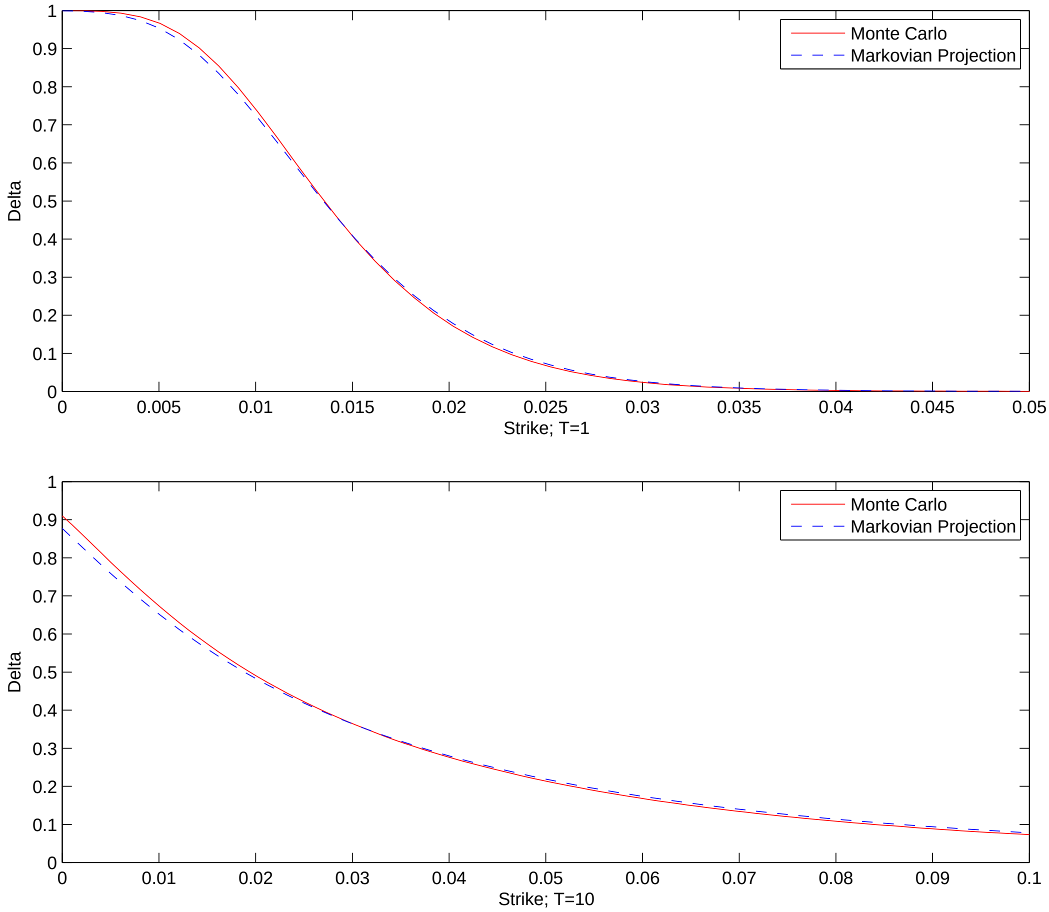

Figure 4.9: Strike-dependent CMS spread caplet deltas using Markovian Projection and Monte Carlo simulation with $F_1 = 0.045$, $F_2 = 0.032$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$ and $\rho_{1,2} = 0.8$.

volatility processes.

We find for option values and delta parameters a good match with results obtained using Monte Carlo simulation. However, there are parameter sets where the fit is not reasonable. In particular changing the time to maturity makes the fit worse. We found that for short time to maturities the approximation is good whereas for large values the approximation gets weak. Since the Markovian Projection is only an approximation it does not give the exact arbitrage free price of an option but can be used for calibration of the volatility and correlation parameters due to its numerical efficiency.

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