Essays on Information Disclosure in Auctions and Contests

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Thomas Rieck

für Mama und Papa

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Introduction

In strategic decision making, the agents' decisions depend crucially on the information they have. Less information means more uncertainty about the other agents' possible behavior, making strategic decisions more complex. As information has a strategic value, it seems natural to expect that in competitive environments an agent has an advantage if his rivaling agents know as little as possible about him. In this thesis, we study competitive strategic situations where this is not necessarily the case: we look at first-price auctions and innovation contests in which the participating agents voluntarily reveal information about themselves.

The value of information depends very much on the specific context. Consider a second-price auction with independent private values for a single object: every bidder has the weakly dominant strategy to bid his own valuation – irrespective of his information on the number of other bidders, the a priori distributions of the other bidders' values or any signals about their realized valuations, no matter how precise. This changes drastically in a first-price auction, keeping everything else fixed: the number of other bidders and the belief about their valuation has a huge impact on bidding behavior. This makes information valuable, and controlling information gives a strategic advantage. An auctioneer for example might raise his revenue by publishing information on the common value of an object for sale. Or the agents might optionally acquire the right to investigate the object and receive a signal about its valuation at some cost. Then, it becomes part of the agents' strategic decision whether to acquire information – and how precise the information should be if costs increase with precision.

In the settings studied in this thesis, every agent controls some relevant information about his valuation (in an auction) or his performance (in a contest). It is the decision of the agent whether he keeps the information to himself, or, depending on the context, reveals this information completely or partially. We incorporate this decision in the strategic considerations of an agent and show that even in highly competitive environments as first-price auctions and innovation contests agents may benefit from the coordination that arises when information is revealed voluntarily.

In Chapter 1, information disclosure is discussed in the context of innovation contests. In an innovation contest, the contest designer seeks to buy a new product which does not exist so far. The firms participating in the contest have multiple opportunities to develop an idea, and the best of all the products resulting from these ideas wins the contest. The contest designer awards a fixed prize to the winning firm. The prize is known to the participants in advance.

Particularly, we use the basic model of an innovation contest according to Taylor (1995), where the research process is modeled as being random: in each round, firms can pay a fixed cost to get an innovation and the quality of the innovation is realized according to a draw from a probability distribution. The possible maximum number of research draws (the number of rounds) is fixed in advance, e.g. due to time constraints. The firms can make their decision whether to conduct research or not contingent on the results of the previous rounds. In our basic model, we focus on the case with two firms and two research rounds and add a new element: the revelation of intermediate research results. The main decision influenced by the presence of additional information is whether to continue research in the second period, after having developed a first innovation in period one. If the revelation of the first-period research results is mandatory, the ranking of the realized first-period research qualities is of major importance. If a firm has a medium quality, she will only possibly continue if she is behind – a leading firm with a medium quality will stop innovating and hopes to win without further effort. Furthermore, contrary to the case without information revelation, firms with very good research results are able to deter the opponent from continuing his research. Overall, due to this coordination effect firms need less research effort in expectation if the additional information on first-period results is available. If everything else is kept fixed, firms thus prefer to participate in a contest with mandatory information revelation compared to a contest without information revelation, as research costs are lower while the prize stays the same.

Nevertheless, from the contest designer's perspective the coordination effect on research is not large enough to make up for the loss in research effort by the firms. He thus prefers if firms do not reveal their intermediate research results, as the expected quality of the best innovation is higher without information disclosure. As a consequence, contest designer and research firms have opposing interests with regard to the revelation policy. Is it possible for the firms to establish voluntary revelation in an equilibrium, such that no firm has an incentive to keep the own information secret and profit from the information revealed by the other firm? We analyze two different ways of doing so: in the first way, firms can play a game ex ante, where they simultaneously decide whether they want to commit on revealing the quality of their first-period innovation after learning it. In the second way, the decision is made after the first period, when firms already learned the value of their first innovation. In both cases, voluntary revelation emerges in equilibrium. This result is robust to a different innovation technology, where the innovation in the second period is an improvement of the first-period innovation. Furthermore, we extend the result to n firms and to m periods.

In Chapter 1, firms either reveal the value of their first-period innovation completely or not at all. Alternatively, values could be revealed partially by sending informative signals. This approach is of particular interest when it is not beneficial for an agent to fully reveal his private information to the opponent. In Chapter 3, *Signaling in First-Price Auctions*, we study the incentives for partial information revelation of two bidders in a standard first-price auction with independent private values. It is quite intuitive that full revelation of the own valuation is not beneficial: if the valuation is high, this will increase the bid of the opponent. If the valuation is low, the opponent can safely reduce his own bid to increase his winnings, incuring only a minor increase in the risk of losing compared to the potential gain. Things are not so clear if a bidder sends out a signal about his own valuation which contains a medium amount of information. Such a signal can realize to a high value although the true valuation is low, making the opponent believe that the sending bidder is strong although he is weak – and vice versa. It is still a bad thing for the sending bidder if he is believed to be strong, may it be true or not, as this will increase the bid of his opponent. However, if the signal is of medium precision, chances are that the sending bidder is believed to be weak although he is strong. Then, the opponent will choose a lower bid, increasing the potential gain of the sending bidder. From a quantitative perspective, it is not clear which of these two effects is dominating without further analysis. The aim of Chapter 3 is to quantify the gain or loss from sending informative signals on the own valuation, depending on the nature of the signals.

It can be profitable to release an informative signal – this is the first major insight of the chapter. It is derived from a simple discrete auction model, where bidders' valuations are drawn from a finite set. For a particular class of signals, we calculate the expected revenue of using the signals depending on their precision. Starting from an uninformative signal, increasing the signaling precision also increases the expected revenue up to the optimal revenue, which is given for a signal of medium precision. If the precision is increased further, expected revenue declines – and the profitability of signals gets lost when they are too precise. This is the second major insight: signaling is not beneficial if it is very precise. In a general context, we show this result for an auction model with a continuous typespace and a wide range of possible signal distributions. In this model, the support of the signals is an interval around the true valuation. For different realizations of the valuation, the densities of the signals have the same shape, but they are shifted such that the true valuation is in the middle of the support. We use the length of the support interval as a measure for the precision of the signal. Depending on the signal distribution we derive a bound on the signal precision: if the signal is more precise than this bound, it is not profitable to use it. Such a bound exists for all signals with a positive density, leading to the general result that signaling is not profitable if it is too precise.

The major difficulty of analyzing the case of continuous typespaces is given by the asymmetric auction that is played after the release of a signal – the updating of beliefs according to the additional information breaks the symmetry of the bidders. To exactly pin down the quantitative effects of signaling for all precisions, explicit knowledge on the equilibrium strategies in the corresponding asymmetric auctions is needed. So far, only little is known about these explicit strategies. We are thus restricted to focus our analysis on uniform signaling distributions, where the equilibrium strategies are known: in this case, we show that signaling is never profitable, no matter how precise the signal is. However, it is likely that this negative result is driven by the assumption of a uniform distribution. Signals that extend the peak-shaped form of the signals in the discrete case are good candidates for being profitable in the continuous case as well.

Chapter 2, *Ascending Combinatorial Scoring Auctions*, takes a different perspective – the auctioneer is in the focus of the analysis. He is a manufacturer and wishes to acquire multiple components necessary to construct his product. Thus, he makes use of a reverse multi-unit auction with package bidding. Additionally, for him, not only the price of the components is important, but also their quality. Consequently, the bids of the potential suppliers specify prices and qualities. These price-quality combinations are evaluated by means of a function, the scoring rule. It is fixed in advance and aggregates the preferences of the auctioneer over a set of bids to a single number, the score. The winning bids maximize this score. We depart from the standard auction framework by allowing the qualities of the components to be interdependent: for example, a fast delivery time of one component is only valuable if the other components are delivered quickly as well. These interdependencies are reflected in the scoring rule. Particularly, the values of bids of different suppliers now dependent on each other as well: the score generated by a bid offering fast delivery may change drastically, depending whether the other bidders offer fast delivery on complementing components as well or not.

It is the goal of this chapter to provide an auction format that is able to deal with interdependent scoring rules. The first obvious candidate, the Vickrey-Clarke-Groves mechanism, is not able to guarantee the auctioneer a nonnegative payoff. We thus look at a different mechanism, the Ausubel-Milgrom proxy auction. In this auction, a proxy bidder bids in an ascending process on behalf of the real bidders according to a predefined strategy: it bids myopically on all packages currently promising the highest potential payoff. We adapt this auction to interdependent scoring rules basically by additionally allowing the proxy bidder to simultaneously submit several bids on the same set of items, but with different qualities. For example, a bidder could ask for a low price if he is allowed to deliver an object slowly, but simultaneously offer to deliver the object faster for a higher price.

We show that this modified version maintains some of the desirable properties of the Ausubel-Milgrom proxy auction: equilibria in profit-target strategies exist, the final allocation maximizes the surplus and the payoff vector is in the core. Furthermore, the scoring rule used to evaluate the bids may contain valuable information about the auctioneer for his competitors, providing an incentive not to reveal it. We discuss the properties of secret scoring rules: it is possible to keep the scoring rule secret without changing the outcome of the auction and a universal equilibrium strategy, truthful reporting, exists for a particular class of scoring rules. Additionally, for additive scoring rules there is a close connection to the original proxy auction.

The possibility of keeping the scoring rule secret connects Chapter 2 with Chapter 3: assume the manufacturer of Chapter 2 is himself competing in a first-price auction to sell his product. Before bidding in the first-price auction, he acquires the components for his product using our modified Ausubel-Milgrom proxy scoring auction. The quality of his product, and thus his type in the first-price auction, depends on the result of the Ausubel-Milgrom scoring auction. Particularly, a public scoring rule provides his competitors in the first-price auction with some noisy information about his type – this is corresponding to the informative signal of Chapter 3. As long as the result of the Ausubel-Milgrom scoring auction is the same for both a public and a secret scoring rule, the decision whether to make the scoring rule public or not depends on the profitability of the signals generated by a public scoring rule, as analyzed in Chapter 3.

Chapter 1

Information Disclosure in Innovation Contests

1.1 Introduction

Contests have been used to stimulate research in a variety of contexts: from refrigerators over computer programs to aerospace research. To win the contest, only the best final innovation of all competitors matters. Nevertheless, if the progress of the participating firms is publicly known, intermediate stages of the research process already reveal interim leaders. This knowledge influences future research efforts. It is thus important to identify the impact of intermediate information revelation both from the participants' and from the contest designer's viewpoint. Intuitively, information disclosure has two major opposing effects on research effort. On the one hand, the publication can serve as a kind of positive coordination device for the participants, prohibiting excessive research: a firm will decrease research effort due to the observation of a very valuable or a worthless innovation made by her opponent. On the other hand, the additional information can also expand research effort: if the competitor of a firm turns out to unexpectedly have a slightly better innovation, a firm might discover the need for an improvement.

From the firms' perspective, the disclosure policy leading to lower research costs is preferable. In contrast, the contest designer cares about the value of the best innovation. In this chapter, we use a contest model with multiple stochastic research opportunities to compare two settings: obligatory intermediate information revelation by the firms opposed to keeping their progress secret. Both the firms' and the contest designer's view are analyzed. Furthermore, we study the possibility of endogenous information revelation.

Examples for information disclosure in contests occur in different areas. When a new drug needs to be developed, different pharmaceutical firms conduct research. To test the effectiveness of a new drug – and thus its chances of beating the rivals' developments – firms conduct clinical trials. These trials can be publicly registered in a trial registry like clinicaltrials.gov, giving also the opportunity to post a short result summary. Specifically, for drugs, biologics and medical devices regulated by the US Food and Drug Administration, U.S. Public Law forces sponsors of clinical trials to post results on their effectiveness in such a trial registry[1]. Additionally, some voluntary disclosure of research results takes place in the trial registries and peer-reviewed journals. Similarly, the performance of participants in the Netflix Prize (www.netflixprize.com) can be seen on a public leaderboard. Netflix, a popular video renting company, pays a prize of $1,000,000 for a new algorithm to predict the movie preferences of a user based on the past ratings he submitted. The accuracy of an algorithm is measured by a single number, which can only be ascertained by submitting the algorithm to the website. Interestingly, the website publishes the best result of each contestant automatically.

To capture the influence of intermediate information revelation on the participants' incentives to innovate, we compare two settings in the framework of an innovation contest, which only differ in the treatment of intermediate information. We model an innovation contest in the spirit of Taylor (1995): two firms have the possibility to make stochastic innovations at a fixed cost. Firms can develop up to two independent innovations. They decide sequentially whether they innovate or not. As it is common in contests, only the best of all innovations wins a fixed prize. The main decision problem of a firm appears after the first innovation is made: how good are the chances to beat the other firm with the current innovation? Should a second one be developed? Of course, information on the quality of the opponent's innovation has significant impact on the firm's decision. Hence, we compare two different versions of the model: in the benchmark setting, following Taylor (1995), no information about the first innovations is revealed. In our basic setting, intermediate information disclosure is mandatory. We extend it to include the possibility of voluntary information revelation, the main focus of this chapter.

In most of the chapter, a key assumption is the independence of innovations. It is motivated by interpreting different innovations as substantially different ideas that have to be explored independently. Specifically, we treat one innovation as fully developed and neglect small improvements due to extended research on an already completed innovation[2]. Consequently, in case of information revelation, the model does not leave room for spillovers between the firms. In a sense, spillovers are assumed to be smaller than the difference between firms' innovation values and would thus have no effect on the contest winner anyway. This is also in line with the revelation policy in both examples. There, only simple summary statistics of the contestants' performances are publicly available. Hence, competitors know how good their opponents are – but they do not know how they did it, so no direct spillovers are possible. Furthermore, in an extension of the basic

[1] see e.g. Groves (2008)

[2] Another way to think about independent innovations is the proof of a theorem: one approach might fail and its a completely different one that will lead to a success.

model, we use a different interpretation of a multi-round innovation contest and model the innovation process as an improvement of a single idea over several stages.

Surprisingly, only very mild assumptions on the distribution of innovation values are needed for the analysis of the basic model, which has two firms and two periods. In fact, the results essentially turn out to hold true independent of the specific functional form of the distribution of innovation values. Instead, the relative size of the final prize to the cost of developing an innovation is the most important parameter for the firms' incentives. The analysis of the basic model with mandatory information disclosure shows that both firms innovate in the first period in case the prize is not too low compared to the costs of developing an innovation. Then, second-period equilibrium behavior depends on the value of the first-period innovation according to two cutoffs: if one firm has an innovation value in the high range, the leading firm is confident to win, while the probability for the following firm to develop something better is too low compared to the costs. Hence, both stop innovating. Similarly, if the highest innovation is in the intermediate range, only the follower continues to innovate – and if both innovations are below the lower cutoff, both firms continue. We show that the total number of innovations – and thus the research costs – is lower in this equilibrium compared to the equilibrium with secret innovation values. Thus, there is a coordination effect which is favorable for the firms: a contest with information disclosure leads to lower expected research costs and thus a higher expected payoff for the firms. Yet, this does not necessarily mean that the prize sponsor prefers the setting without information disclosure: he cares about the expected value of the highest innovation, which is different from the total number of innovations. As firms stop innovating when they observe a high innovation value, the coordination effect could be strong enough to compensate for the lower total number of innovations. We show that this is not the case if the prize/cost-ratio is sufficiently high. Consequently, the prize sponsor gets a higher expected innovation in the setting without information disclosure. If a prize sponsor is able to enforce this secrecy, he should thus do so. However, if he does not do so, firms might be willing to voluntarily reveal their first-period value. We pursue this question by modeling voluntary disclosure in two different ways: in the first version, the firms decide in an ex ante-game whether they are going to reveal after the first period or not. In the second version, the decision to disclose is delayed until firms learn their first-period innovation value. In both cases, it turns out that there is essentially a unique equilibrium in which both firms disclose. Continuing this train of thought, the voluntary revelation has consequences for the contest designer: if he chooses the size of the prize optimally, he should choose it with respect to the setting where information is revealed in case he does not prevent voluntary disclosure. We also prove the existence of an equilibrium with voluntary revelation in case there are n firms or m periods.

In the extension with improving innovations, given mandatory information revelation the decision whether to continue research in the second period does not only depend on the leader's value, but also on the value of the runner-up. The cutoffs identified for the basic model still exist qualitatively but change in their quantitative value. Particularly, the incentives to continue research increase for the firms, as it becomes more likely that the runner-up can produce a better second-period innovation. Nevertheless, the main result of this chapter carries over to this setting with improving innovations: there is an equilibrium where firms voluntary reveal their own value.

This chapter extends the analysis of research tournaments by Taylor (1995). In his model with a secret innovation process, there is a unique symmetric equilibrium in which firms continue to innovate if their best innovation value does not exceed a certain threshold. Due to the information disclosure, which we introduce in our version of the model, a second cutoff value arises – the contestants are able to coordinate. Of course, the approach followed by Taylor (1995) is not the only one to model research contests. For example, Che and Gale (2003) find the optimal contest to be an auction given a deterministic research technology, Schoettner (2008) builds on the famous model by Lazear and Rosen (1981) to show that given a stochastic innovation technology, fixed-prize contests may in fact be superior to a first-price auction. Also building on Taylor (1995), Fullerton et al. (2002) study auction-style research tournaments. Finally, Baye and Hoppe (2003) show that there is a strategic equivalence between different models of rent-seeking, patent races and innovation contests.

The idea of intermediate revelation of research results is also studied by Gill (2008) in the context of patent contests with exogenously given leader and follower. In his model, research is a two-stage process where both steps are necessary to develop a single innovation. We use value distributions similar to his distributions in our extension with improving innovations. In Gill's model, the leader decides whether or not to disclose his performance after the first stage. Then, the follower may choose to drop out after the first stage. Whether or not the leader discloses depends on the research costs. By contrast, in our model leader and follower are endogenously determined, as multiple innovations can be developed. Furthermore, in Gill (2008) the patent winner is determined randomly, while in our model the best innovation wins for sure. In Aoyagi (2010) all information on intermediate performance is controlled by the contest designer. Related to our model, performance is stochastic. Furthermore, it is additive over the two rounds, while we mostly consider multiple independent innovations. The optimal feedback policy to the participants regarding this information is derived – it depends on the shape of the cost function whether a no-feedback or a full-feedback policy is optimal. In a related paper, Gershkov and Perry (2009) study the design of midterm reviews. Given a fixed prize, it is always optimal to have such a review, if the results of intermediate and final review are optimally aggregated.

This chapter also connects to the literature on multiple-round contests. In Konrad and Kovenock (2009), contestants have to win several component contests, modeled as all-pay auctions, to win the overall prize. Contrary to our model, the follower is not fully discouraged from continuing the contest even if he is far behind. Moldovanu and Sela (2006) investigate how to split contestants over sub-contests where only the winners continue to compete. In Yildirim (2005), building on work by Dixit (1987), heterogeneous participants can split their effort over two rounds with observable first-round effort. Similar to one result in this chapter, information disclosure can be endogenized by an ex ante game: agents can choose between non-observable effort (which equals one-shot play there) or two-round effort with intermediate revelation. In equilibrium they decide to reveal effort. In our model, we also get voluntary revelation – however, it is revelation of (stochastic) innovation values and not of effort. Furthermore, our model does not boil down to one-shot play in case of secret intermediate results.

Finally, an experimental study of information disclosure is provided by Ludwig and Luenser (2008). They compare two settings with and without intermediate information release, where equilibrium play is not affected by the information structure. Nevertheless, subjects in the experiments behave differently if they observe their opponent's effort.

The chapter is organized as follows: the basic model and equilibrium behavior with information disclosure is presented in Section 1.2. We compare it to the benchmark case without disclosure in Section 1.3. In Section 1.4 we endogenize information revelation. Extensions with a second innovation that improves the first one and with n firms and m periods are considered in Section 1.5. Finally, we conclude in Section 1.6. Proofs can be found in Appendix 1.A.

1.2 The Model and Equilibrium Derivation

We consider two risk-neutral research firms, $i = 1, 2$. They compete in an innovation contest to win a fixed prize $p > 0$. Firms are assumed to know the prize sponsor's utility function over research outcomes. Both firms have an innovation technology similar to Taylor (1995): research is modeled as drawing an innovation x out of a probability distribution F with strictly positive density f. F is defined on $[0, b]$ with $b \leq \infty$. Each innovation draw is associated with a cost of $c > 0$ for each firm. Firms are not capital constrained. There are two periods $t = 1, 2$ in which firms may innovate. Innovation values x_i^t are independent across periods and firms. For each firm, only the best draw $(\max\{x_i^1, x_i^2\})$ is relevant for the contest. The firm with the highest draw wins the contest and the prize of p. Ties are randomly broken. We assume that innovations that do not win have a value of zero outside the contest, so that loosing innovations cannot be sold afterwards. In contrast to Taylor (1995), in the basic version of our model we assume in the spirit of Yildirim (2005) that first-period innovations become common knowledge after both firms have made their decision whether to conduct research or not, and have taken their draw.

We first analyze equilibrium behavior of the two firms. We look for subgame perfect Nash equilibria by backward induction and thus start with the second period. First note that for $p < c$ both firms would make a loss from conducting research. Thus, both do not conduct any research (neither in the first nor in the second period). Consequently, we focus on the case $c \leq p$. Additionally, we will narrow the reasonable prize/cost combinations further down later.

1.2.1 Second Period

Suppose at least one firm has taken a draw in the first period, such that one of the two firms has taken the lead, $x_H^1 > x_L^1 \geq 0$. H stands for the firm with the *higher* first round innovation (the leader) and L for the firm with the *lower* innovation (the follower). We calculate best responses:

If the follower does not continue to innovate, it is a best response for the leading firm to stop innovating as well – she will win in any case.

So suppose now the firm with the higher value does not draw again. Then, the firm with the lower value wants to continue if the following condition holds:

$$P\left(x_L^2 > x_H^1\right) p - c \geq 0 \iff \left(1 - F\left(x_H^1\right)\right) p - c \geq 0 \iff F\left(x_H^1\right) \leq 1 - \frac{c}{p}.$$

This inequality defines a threshold x^* indicating an innovation high enough to make all firms stop research. x^* solves the following equation:

$$F\left(x^*\right) = 1 - \frac{c}{p}. \tag{1.1}$$

Then, if some $x > x^*$ is drawn by any of the two firms, the contest stops immediately and no new research will be conducted in the second round: the follower has no incentive to draw again if the leader has already drawn such a high innovation. Then, the leader will obviously not draw again as well.

Now consider the case $x_H^1 \leq x^*$, such that the firm with the lower value wants to draw again if the leader does not. What is the best response of the leader against the drawing follower? The firm with the higher value wants to draw again as well if the following condition holds:

$$\begin{aligned} & \left[P\left(x_H^2 > x_L^2 > x_H^1\right) + P\left(x_H^1 > x_L^2\right)\right] p - c \geq P\left(x_H^1 > x_L^2\right) p \\ \iff & \frac{1}{2}\left(1 - F\left(x_H^1\right)\right)^2 p - c \geq 0 \\ \iff & F\left(x_H^1\right) \leq 1 - \sqrt{2\frac{c}{p}}. \end{aligned}$$

This inequality defines a threshold $\bar{x}$ making both firms innovate again if there is no innovation above it. $\bar{x}$ solves

$$F\left(\bar{x}\right) = 1 - \sqrt{2\frac{c}{p}} \tag{1.2}$$

and note that $\bar{x} < x^*$. What is the best response of the follower against a leader drawing again for $x_H^1 \leq \bar{x}$? Drawing again is a best response according to the following condition:

$$\begin{aligned} & P\left(x_L^2 > x_H^1, x_H^2\right) p - c \geq 0 \\ \iff \quad & \left[\frac{1}{2}\left(1 - F\left(x_H^1\right)\right)^2 + \left(1 - F\left(x_H^1\right)\right)\left(F\left(x_H^1\right)\right)\right] p - c \geq 0. \end{aligned} \tag{1.3}$$

We know that

$$\frac{1}{2}\left(1 - F\left(x_H^1\right)\right)^2 p - c \geq 0$$

because $x_H^1 \leq \bar{x}$. Hence, (1.3) is fulfilled. Consequently, the follower wants to draw again in the second round as well. This is intuitive: the leader already has an advantage after the first round, so incentives for the follower to draw again are even higher.
We summarize our findings in the following proposition:

Proposition 1.1 *Given first-period innovations $x_H^1 > x_L^1$, there are the following second-period equilibrium strategies:*

- *If $x_H^1 > x^*$ both firms stop their research effort and the contest ends after the first period.*
- *If $x^* \geq x_H^1 > \bar{x}$ only the follower conducts research in the second period.*
- *If $\bar{x} \geq x_H^1$ both firms conduct research in the second period.*

Note that for small prize values $p < 2c$ we get $\bar{x} < 0$, thus, the leader will never draw again in the second period. Furthermore, the proposition implies that there are no mixed equilibria:

Corollary 1.2 *Given first-period innovations $x_H^1 > x_L^1$ there is no second-period equilibrium in which players mix at values other than $\bar{x}$ and x^*. Thus, the equilibrium in Proposition 1.1 is almost everywhere unique.*

It follows immediately from Proposition 1.1 that a leading firm with $x_H^1 > \bar{x}$ does not do any research irrespective of the following firm's behavior and is thus playing a pure strategy. Similarly, a follower with $x_L^1 < \bar{x}$ will always do research. Thus, neglecting the cutoff values, there is always at least one firm playing a pure strategy, with a pure best reply by the other firm according to Proposition 1.1.
Let us now consider the case that both firms did not innovate in the first period, which is important for the calculation of first-period equilibrium behavior.

Proposition 1.3 *Suppose both firms did not innovate in the first period. Then, there are the following second-period equilibrium strategies:*

- *If $p \leq 2c$, there is an equilibrium where both firms do not conduct any research in the second period.*
- *If $p \geq 2c$ there is an equilibrium where both firms conduct research in the second period.*

The proof is given in Appendix 1.A. Note that if at least one firm takes a draw in the first period, a tie appears with zero probability, and thus second-period equilibrium play is almost everywhere unique in the sense of Corollary 1.2 for almost all possible first-period realizations. For this reason, we can safely skip the calculation of equilibria in case $x_1^1 = x_2^1$: this case will appear with zero probability given any first-period play and we will thus not need it in future calculations.

1.2.2 First Period

The first-period pure-strategy equilibria can be now derived, taking into account second-period equilibrium play. As the main focus of this chapter is on information revelation after the first period, we are especially interested in the conditions under which both firms start innovating in the first period. If they do not innovate in the first period, information revelation is only of minor interest. It turns out that the size of the prize compared to the innovation costs is the crucial parameter for first-period innovation to take place. We make use of the following short notations: $r := \frac{c}{p}$ and $s := \sqrt{2r}$.

Proposition 1.4 *Let v^* be the solution of the following equation:*

$$\frac{1}{6} - v^* + \frac{2}{3}v^*\sqrt{2v^*} - \frac{1}{2}(v^*)^2 - \frac{1}{2}(v^*)^3 = 0$$

Then, $v^* < \frac{1}{2}$ and in the first period, we get the following pure-strategy equilibrium behavior with firms continuing in the second period as described in Proposition 1.1:

- *For $r > \frac{1}{2}$ both firms do not conduct any research in the first period.*
- *For $0 < r < v^*$ both firms conduct research in the first period.*
- *For $\frac{1}{2} \geq r > v^*$ equilibrium behavior is asymmetric – one firm conducts research, the other does not.*

Proof See Appendix 1.A. □

Numerically, v^* is given by $v^* \approx 0.2428$ and by Proposition 1.4 both firms conduct research if $\frac{c}{p} = r < 0.2428$. This means that a prize value of $p \approx 4c$ is high enough to ensure the maximum amount of research in the first period.

The proposition shows that if the prize is too low compared to the costs, both firms will invest neither in the first nor in the second period. Additionally, there are two pure-strategy equilibria if r takes intermediate values. Furthermore, there is a more prominent symmetric mixed strategy equilibrium in this case as well, which we do not calculate here because we focus on $r < v^*$ in the following: we are interested in information revelation with firms in fact doing research in the first period. This problem has no meaning if the setting is such that firms do not have full incentives to invest in the first period – and these incentives are already given at a very reasonable prize level. There is thus no need to consider the mixed equilibrium here.

1.3 Comparison with No Information Release

In this section, we compare the setting with information release after the first period, which we just analyzed, with the setting known from the literature (Taylor 1995) where information is kept secret after the first period. We want to find the preferred setting for both the firms and the contest designer. First, we compare the settings from the perspective of the firms, then we turn to the contest designer.

1.3.1 Firms' Perspective

To analyze the firms' perspective, we compare the expected number of innovation draws in the setting with information revelation to no information revelation after the first period – firms prefer the setting with lower research costs, which means less innovation draws in this context. The first step is to calculate the expected number of draws $d_R(r)$ in the equilibrium with information release, given that both firms do research in the first period.

Proposition 1.5 *Given $r < v^*$ the expected number of draws in equilibrium fulfills $d_R(r) = 4 - 2s + r^2$.*

Proof See Appendix 1.A. □

We now come back to the setting of Taylor (1995), where no information is released. He shows that there is a unique equilibrium in which firms play a stopping strategy with stop value z: they take draws as long as they do not have an innovation that exceeds z and stop as soon as an innovation exceeds z. However, Taylor does not calculate the z explicitly but characterizes it implicitly. We rewrite his implicit characterization to make it suitable for our purposes. According to Proposition 2 in Taylor (1995), z is the solution of the following equation:

$$0 = p \int_z^b \left[F^2(z) + \left(1 - F^2(z)\right) \frac{F(x) - F(z)}{1 - F(z)} - F^2(z) \right] dF(x) - c.$$

Calculating the integral, this can be rewritten as follows:

$$\begin{aligned} 0 &= p\,(1+F(z))\left[\int_z^b F(x)f(x)dx - F(z)\int_z^b f(x)dx\right] - c \\ &= p\,(1+F(z))\left[\frac{1}{2}\left(1-F^2(z)\right) - F(z)\,(1-F(z))\right] - c \\ &= p\frac{1}{2}\,(1+F(z))\,(1-F(z))^2 - c \\ \iff \qquad 0 &= (1+F(z))\,(1-F(z))^2 - 2r \qquad (1.4) \end{aligned}$$

The first line follows by factoring out $1+F(z)$ and changing the notation of the integration. The second line uses integration by parts. Unfortunately, the explicit solution of this equation is quite messy. The following lemma gives a feeling of the size of z.

Lemma 1.6 *For $p > 2c$ the stop value in the setting without information release is between the two thresholds of the setting with information release, $\bar{x} < z < x^*$.*

Proof See Appendix 1.A. □

We make a comparison between the setting of Taylor (1995) and our setting. As the expected number of innovations a firm makes represents her cost, we compare the number of draws the firms take in expectation in each setting. For our case with information revelation we already calculated the expected number of draws ($d_R(r)$, Proposition 1.5). For the setting without information revelation, the expected number of draws can be written as $d_{NR}(r) = 2(1 + F(z))$ (a firm is drawing again if and only if the first period value did not exceed z, this happens with probability $F(z)$). z is implicitly defined by (1.4) for a given r.

Proposition 1.7 *Considering $0 < r < v^*$, the expected number of draws $d_{NR}(r)$ in case no information is revealed after the first period is larger then the expected number of draws $d_R(r)$ in case information is revealed, $d_{NR}(r) > d_R(r)$.*

Proof See Appendix 1.A. □

We immediately get the following corollary, as both players win in expectation $\frac{1}{2}p$ in equilibrium in both settings, but have lower costs in the setting with information disclosure because they take less draws:

Corollary 1.8 *For $0 < r < v^*$, both research firms prefer the setting with information disclosure over the setting without information disclosure.*

Note that $r < v^*$ is exactly the range of r-values guaranteeing research draws by both firms in the first period. This is the range we focus on as revelation decisions after the first period are only interesting if firms do innovate in the first period.

1.3.2 Designer's Perspective

From the prize sponsor's perspective, a higher number of innovation draws is in principle favorable, as more draws suggest a higher expected final prize. However, it is not obvious that this relationship really holds in this context: draws are taken conditional on already realized innovations. Thus, if a draw is not taken, a good innovation has already been made. But the equilibrium decision rules whether another draw is taken differ between the two settings. Thus, a higher number of draws is an indicator for a higher expected final innovation, but does not allow a sure conclusion.

The key to the comparison from the designer's perspective is the highest expected innovation generated by the two different settings. The designer prefers the setting yielding the higher one.

To calculate the highest expected innovation for the two settings, we need the respective distribution functions of the highest innovation. In the setting without information release, the two firms are innovating independently. Let Φ be the distribution of the highest innovation for a single firm. Then, the joint distribution is given by Φ^2. Using the result by Taylor (1995) regarding Φ, we get

$$\Phi^2(x) = \begin{cases} F^4(x) & \text{if } x \leq z \\ \left(F(x) - F(z) + F(z)F(x)\right)^2 & \text{if } x > z \end{cases}$$

For the setting with information revelation, the two firms do not innovate independently. The distribution Ψ of the joint highest innovation has the following structure, given the equilibrium play of the two firms – they both draw in the first period as we assume $r < v^*$:

$$\Psi(x) = \begin{cases} F^4(x) & \text{if } x \leq \bar{x} \\ A & \text{if } \bar{x} < x \leq x^* \\ B & \text{if } x^* < x \end{cases}$$

Denote the highest innovation in period j by $x^j_{(1)}$. Then, A and B are given according to

$$\begin{aligned} A &= P\left(x^1_{(1)} < \bar{x}\right) P\left(x^2_{(1)} < x\right) + P\left(\bar{x} < x^1_{(1)} \leq x\right) P\left(x^2_{(1)} < x\right) \\ &= F^2(x)F^2(\bar{x}) + F(x)\left(F(x)^2 - F(\bar{x})^2\right) \\ B &= P\left(x^1_{(1)} < \bar{x}\right) P\left(x^2_{(1)} < x\right) + P\left(\bar{x} < x^1_{(1)} < x^*\right) P\left(x^2_{(1)} < x\right) + P\left(x^* < x^1_{(1)}\right) \\ &= F(x)^2F(\bar{x})^2 - F(x)F(\bar{x})^2 + F(x)F(x^*)^2 + F(x^2) - F(x^*)^2 \end{aligned}$$

Given these distribution functions, we can calculate which setting provides the higher expected innovation – no information revelation is preferred if the following condition holds:

$$\int_0^b 1 - \Phi^2(x)dx \geq \int_0^b 1 - \Psi(x)dx \iff \int_0^b \Phi^2(x) - \Psi(x)dx \leq 0. \tag{1.5}$$

Note that, different to the results from the firms' perspective, it depends on F whether condition (1.5) is fulfilled or not. This is because the designer cares about the absolute value of the innovations, while the firms care about their relative ranking. Additionally, the size of r is crucial for the profitability of the settings. We provide a bound on r such that (1.5) is fulfilled independent of F. This bound is called v':

Theorem 1.9 *The expected value of the highest innovation is larger in the setting without information revelation if $r < v'$ holds. Then, this setting is preferred by the prize sponsor.*

The derivation of v' can be found in Appendix 1.A. It basically uses a stochastic dominance argument: the integrand of the integral on the left-hand side of (1.5) is shown to be negative on the whole interval $[0, b]$ when $r < v' = 0.1647$. However, this bound is in general not binding, as the solution to (1.5) (with equality) differs for each F. For example, for F being the uniform distribution on $[0, 1]$, a calculation of (1.5) shows that the designer prefers the setting without information revelation for all relecant r-values ($r < v^*$).

1.4 Endogenous Information Release

We have seen in the previous section that firms prefer the setting with information disclosure after the first draw to the setting without information disclosure. However, the contest designer has opposite preferences, and he is the one to choose the setup. This raises the question whether firms could play the information revelation setting by voluntary revelation of their first-period innovation value.[3] We take two approaches to model this: first, we extend our model by adding a stage zero where firms can ex ante decide whether to disclose the level of their innovation after the first draw or not. This is an extension in the spirit of the analysis in Yildirim (2005). Second, we consider an intermediate decision, where the firms only decide whether they disclose the information after having observed the value of the first-period innovation.

1.4.1 Ex Ante Decision

We add an initial stage zero in which the firms simultaneously decide whether to reveal their information (action R) or whether they do not reveal (action N). The decision is observable. It is our goal to identify equilibria of this simultaneous-move game to find out whether the analysis in the previous sections can be supported by endogenous information revelation. This would be the case if (R, R) is an equilibrium of this game. In case both firms play R, the contest following afterwards is the same as the one described in the previous sections. Hence, we already know the corresponding equilibrium strategies. The

[3] We implicitly assume that the contest designer either does not set rules to prevent voluntary revelation or is not able to enforce such rules.

same holds true in case both firms play N. Then, we are back in the setting of Taylor (1995). To derive the best responses in this initial stage, we need to deduce the equilibrium strategies in the case of asymmetric information revelation. In the resulting contest, one firm reveals her first draw, the other one does not. We will analyze equilibria by backward induction. To provide incentives for research, we focus on the main case $p > 2c$ in the following, and assume thus $r < 0.5$.
For the second-period equilibrium, we take the first draw as given. One firm has played R in the initial stage, we denote her draw by x_R^1 and call her firm R. The draw of the firm playing N (short: firm N) is denoted by x_N^1.

Proposition 1.10 *In the setting with asymmetric information release, given first-period innovations x_R^1 and x_N^1, there are the following second-period equilibrium strategies:*

- *Firm R takes a second draw iff $x_R^1 < z$.*
- *Firm N takes a second draw iff $x_N^1 < x_R^1 < x^*$ or $\bar{x} > x_N^1 > x_R^1$.*

In case firm N does not take a draw in the first period, it is the best reply for firm R to take a second draw iff $\bar{x} > x_R^1$. Firm N takes a draw in the second period if $x_R^1 < x^*$.
In case firm R does not take a draw in the first period, it is the best reply for firm N to take a second draw iff $\bar{x} > x_N^1$. Firm R always takes a draw in the second period.

Proof See Appendix 1.A. □

Roughly speaking, firm R thus behaves as in the setting with no information release, while firm N plays the same strategy as with full information release. If both firms do not innovate in the first period, they both take a draw in the second period, as we assumed $p > 2c$. Note that Proposition 1.10 w.l.o.g. ignores the case $x_R^1 = x_N^1$ for values larger than zero, as it appears with zero probability – it is thus not payoff relevant and we can safely omit it here.
The first-period equilibrium behavior can be summarized as follows (again, we do not calculate possible mixed equilibria, as we later on focus on r-values inducing an equilibrium with research in the first period):

Proposition 1.11 *Let $\hat{v}$ be the solution to the following equation:*

$$-\frac{1}{24} - \frac{1}{3}\hat{v}^3 + \frac{1}{2}F(z) - \frac{1}{4}F(z)^2 - \frac{1}{6}F(z)^3 + \frac{1}{8}F(z)^4 - F(z)\hat{v} = 0$$

where $F(z)$ is determined by (1.4) with $r = \hat{v}$.
Furthermore, let $\tilde{v}$ be the solution to

$$\frac{1}{6} - 2\tilde{v} - \frac{1}{2}\tilde{v}^2 - \frac{1}{6}\tilde{v}^3 + 2\sqrt{2\tilde{v}}\tilde{v} - \frac{1}{2}\sqrt{2\tilde{v}}\tilde{v}^2 = 0.$$

Then, in the first period of the contest with asymmetric information disclosure we get the following pure-strategy equilibrium behavior with firms continuing in the second period as described in Proposition 1.10:

- *For $r < \hat{v}$ there is an equilibrium where both firms draw in the first period.*
- *For $0.5 > r > \tilde{v}$ there is an equilibrium where firm R draws in the first period and firm N does not.*
- *For $0.5 > r > \hat{v}$ there is an equilibrium where firm N draws in the first period and firm R does not.*

The proof is given in Appendix 1.A. Numerically, we can approximate $\hat{v} \approx 0.2623$ and $\tilde{v} \approx 0.1722$. Note that firm R plays different strategies in the two equilibria involving a draw by firm R: as Proposition 1.10 shows, firm R will continue to innovate in less cases if firm N does not innovate in the first period. Consequently, the best reply of firm N is affected by the change in strategy, yielding two different equilibria involving a draw by firm R in the range $\tilde{v} < r < \hat{v}$. We focus in our analysis on the symmetric equilibrium involving draws by both firms. It is also unique for small r-values.
With this characterization of pure-strategy equilibria we are ready to address the main question of this section: are the two firms willing to ex ante commit to revealing their information after the first draw or not? The answer is given by the following theorem:

Theorem 1.12 *Let $\bar{v}$ solve*

$$\frac{5}{24} - 2\bar{v} + \frac{2\sqrt{2}}{3}\bar{v}^{\frac{3}{2}} - \frac{1}{6}\bar{v}^3 - \frac{1}{2}F(z) + \frac{1}{4}F(z)^2 + \frac{1}{6}F(z)^3 - \frac{1}{8}F(z)^4 = 0, \qquad (1.6)$$

where $F(z)$ is determined by (1.4) with $r = \bar{v}$.
For $r < \bar{v}$ there is a subgame perfect Nash equilibrium in which both firms ex ante commit to revealing their information after the first period. For $r < \tilde{v}$ it is unique.

The proof is given in Appendix 1.A. A numerical approximation gives $\bar{v} \approx 0.2325$. Hence, we have shown that the disclosure of information can be endogenized – the firms are voluntarily agreeing to it ex ante.

1.4.2 Intermediate Decision

So far, we modeled the revelation decision as taking place before any research is done by the firms. In that setup, firms need to be able to commit to their decision. In the following, we drop the assumption that ex ante commitment is possible – the revelation decision is postponed after the first period, when firms are able to observe their first innovation. As the revelation decision works as a kind of signaling device, a firm holds a belief on the value of the other firm's innovation. We thus refine our equilibrium concept to Perfect Bayesian equilibrium. Nevertheless, firms reveal the information voluntarily, as the following theorem shows:

Theorem 1.13 *If firms $i = 1, 2$ can make their revelation decision simultaneously after learning their first-period innovation value x_i, in a Perfect Bayesian equilibrium both firms reveal their value if $x_i \neq x^*$. If $x_i = x^*$ firm i is indifferent between revealing or not. The revelation decision in a Perfect Bayesian equilibrium is thus unique up to firms' behavior for value x^*. Off the equilibrium path, in case one firm does not reveal, the other firm believes the deviating firm has value x^* with probability 1 and reacts accordingly.*

The intuition for the proof is as follows: no firm has an incentive to hide her value – then, she would be treated as a firm with value x^*, which is no improvement no matter what the true value of the firm is. Revelation in combination with this punishment thus forms an equilibrium. To show the uniqueness, one has to consider the fact that a firm wants to show that she has a high type (and discourage lower types from continuing to innovate) or a low type (and make intermediate types stop innovating). For intermediate types, one can show that if a firm keeps the information secret, she does so for an interval of values. However, for the lowest of these values a firm has an incentive to reveal – she does not want to pool with higher values against which the other firm would more often like to continue innovating. The details of the proof are given in Appendix 1.A.

1.5 Extensions

1.5.1 Second Innovation as Improvement

So far, we modeled the two innovations in the two periods as substantially different ideas: the resulting innovation values do not depend on each other and represent fully developed innovations. A different way of thinking about a multi-period contest is to interpret the second-period innovation not as a new idea, but as an extension of the first-period innovation that improves the innovation value. As a consequence, the distributions of the innovation values in the two periods are not the same (as they have been in our model so far), but the second-period distribution depends on the value of the first-period innovation. In this section, we adapt our model to this interpretation and show that voluntary revelation also appears when the second-period innovation builds on the first-period innovation.

Notation and assumptions stay the same except for the distribution functions: for tractability reasons we assume in this section that F is a uniform distribution on $[0, 1]$ with $F(x) = x$. Furthermore, in the second period, firm i can improve her innovation by taking a draw from the distribution $F_i(x|x_i^1)$ at costs c. We assume that F_i is derived from F and fitted to the interval $[x_i^1, 1]$ according to

$$F_i(x|x_i^1) := F\left(\frac{x - x_i^1}{1 - x_i^1}\right) = \frac{x - x_i^1}{1 - x_i^1}.$$

We start our analysis by identifying the equilibrium behavior of the two firms in the second period in case information is revealed after the first period. Again, we assume that the prize is high enough compared to the costs such that both firms innovate in the first period, which surely guarantees $r < \frac{1}{2}$. The leading firm, with the higher first-period innovation value, is once more denoted by H, the following firm with the lower innovation value by L. Thus, we have $x_H^1 > x_L^1$, again omitting the equality case as it appears with zero probability and is thus not payoff relevant.

Proposition 1.14 *Given first-period innovations $x_H^1 > x_L^1$, there are the following second-period equilibrium strategies:*

- *If $x_H^1 > 1 - (1 - x_L^1)r$ both firms stop innovating and the contest ends after the first period.*
- *If $1 - (1 - x_L^1)2r < x_H^1 \leq 1 - (1 - x_L^1)r$ only firm L innovates in the second period.*
- *If $x_H^1 \leq 1 - (1 - x_L^1)2r$ both firms innovate in the second period.*

Proof See Appendix 1.A. □

Compared to the equilibrium with independent innovations, the continuation decision does not depend only on the leader's value, but also on the value of the runner-up. This leads to an increased amount of research. Particularly, as in the independent case, the runner-up will always continue to innovate if the leader's value is below $1 - r = x^*$ – but additionally, he will also continue to innovate for higher values of firm H if his own first-period value, x_L^1, is not too far behind. Similarly, the leading firm will always innovate if her own value is smaller than $1 - 2r > 1 - \sqrt{2r} = \bar{x}$, which is already a larger set than in the case with independent values (where firm H only continues for $x_H^1 \leq \bar{x}$). Furthermore, the leading firm will also continue if the runner-up is only close behind. This is a major strategic difference to the case with independent values: two innovations of approximately the same size are worth almost the same. It is much less important which firm has the lead.

What is the effect of this strategic difference on voluntary revelation? If the revelation decision is made after the first period, the equilibrium in Theorem 1.13 uses a maximum punishment idea: if firm i does not reveal its value firm j believes firm i has value x^*, making firm j continue to innovate for the largest possible set of values – which is a bad thing for the hiding firm i. On the contrary, if the second innovation builds on the first one, firm i with a first-period innovation value above x^* can in fact profit from keeping the value secret for some values of firm j in the top range: if firm j believes to face a firm i with value x^*, hiding goes along with an underestimation of i's value by firm j, making j stop innovating for these values. However, at the same time a firm j with a value at the lower end will continue to innovate although she would stop if she knew the true value of i. We thus have two opposing effects. In the following theorem we show that the latter effect is the dominating one and voluntary revelation extends to this model of improving innovations.

Theorem 1.15 *If firms $i = 1, 2$ can make their revelation decision simultaneously after learning their first-period innovation value x_i^1 and the second innovation always improves the first innovation, there is a Perfect Bayesian equilibrium in which both firms reveal their value. Off the equilibrium path, in case one firm does not reveal, the other firm believes the deviating firm has value x^* with probability 1 and reacts accordingly.*

Proof See Appendix 1.A. □

Note that there will be no equilibrium in which both firms always hide their value: there is always an interval at the lower end of possible values for which it is beneficial to reveal, showing the opponent that the own value is much lower than he expected. Compared to no revelation, this makes the opponent stop innovating for some medium values and is thus profitable for a firm with a low value realization.

1.5.2 n Firms and m Periods

Voluntary revelation of intermediate research results is not limited to the case of two firms and two periods we have studied in detail until now. In this section, we extend the main result with independent research draws and an intermediate revelation decision to n firms (and two periods) and m periods (and two firms).

We start with the case of n firms and two periods, otherwise the setting is the same as with two firms. Again, we assume that the prize is large enough compared to the cost to make all firms innovate in the first period. Particularly, all participating firms should make nonnegative profit as they would not innovate at all otherwise. We thus assume that

$$p > nc \iff r < \frac{1}{n}.$$

In second-period equilibrium play with information disclosure, compared to the case with only two firms, incentives to innovate are lower if more competing firms are present. Particularly, if firm i has a first-period innovation better than x^*, no other firm will try to beat firm i in the second period. Furthermore, as long as no other firm is continuing to innovate, the incentives for firm $j \neq i$ to draw in the second period are the same as in the case with only the two firms i and j. Hence, some research is going on in period two if the highest value of the first period, x_H^1, is smaller than x^*. In a pure strategy equilibrium, only one firm will continue to innovate for values slightly below x^*, and there will be additional thresholds at lower values of the leading firm for which more firms continue to innovate. As this type of equilibrium is asymmetric, it comes along with a coordination problem. We will thus focus on a symmetric equilibrium which is in mixed strategies: for values slightly lower than x^*, all firms will continue to innovate with a positive probability depending on x_H^1, $q\left(x_H^1\right)$. This probability is obviously fixed by making all firms that are not in the lead indifferent between drawing or not. The largest value of x_H^1 for which all other firms draw with probability one is denoted by $\hat{x}$:

$$\hat{x} := \max\{x_H^1 | q\left(x_H^1\right) = 1\}$$

By definition, all firms who are not in the lead make zero profit if $x_H^1 = \hat{x}$. Thus, as all these firms draw with probability one in this case, we can conclude that each of these firms wins the contest with probability $r = \frac{c}{p}$, as $p \cdot \frac{c}{p} - c = 0$. Consequently, the remaining winning probability is with the leading firm, who wins with probability $1 - (n-1)r$ and does not draw herself, as due to the current leadership the incentives to draw are strictly lower for this firm. Thus, the leading firm wins exactly if all drawing firms have a second-period value lower than $\hat{x}$, and we can conclude that

$$F\left(\hat{x}\right)^{n-1} = 1 - (n-1)r \iff F\left(\hat{x}\right) = \sqrt[n-1]{1-(n-1)r}.$$

We summarize these results in the following proposition[4].

Proposition 1.16 *Given the largest first-period innovation x_H^1, in the symmetric second-period equilibrium strategies*

- *no firm draws if $x_H^1 > x^*$,*
- *non-leading firms draw with probability $q\left(x_H^1\right)$ if $\hat{x} \leq x_H^1 \leq x^*$, with $q\left(x_H^1\right) \in (0,1)$ for $\hat{x} < x_H^1 < x^*$,*
- *non-leading firms draw if $\hat{x} > x_H^1$,*
- *the leading firm does not draw if $\hat{x} \leq x_H^1$.*

As in the previous sections, we endogenize the information disclosure by letting firms decide whether they reveal or not after learning their first-period value. Again, the equilibrium we derive builds on maximum punishment: if a firm hides her first-period innovation

[4]Note that Proposition 1.16 is not a full equilibrium characterization but contains only the parts necessary for our purposes.

value, the other firms believe that a hiding firm has value $\hat{x}$, as stated in the following theorem:

Theorem 1.17 *If firms $i = 1, 2, \ldots, n$ can make their revelation decision simultaneously after learning their first-period innovation value x_i^1, there is a Perfect Bayesian equilibrium in which all firms reveal their value. Off the equilibrium path, in case one firm does not reveal, the other firms believe the deviating firm has value $\hat{x}$ with probability 1 and reacts accordingly.*

Proof See Appendix 1.A. □

If all other firms have a value smaller than $\hat{x}$, they will all continue to innovate and the punishment is maximal. However, contrary to the case with two firms, there is potentially some room for benefiting from these beliefs about a hiding firm. Suppose the second highest value is x_L^1, and the values are ordered as follows: $x^* > x_H^1 > x_L^1 > \hat{x}$. If firm H hides her value, the remaining firms will believe that firm L is in fact the leading firm. Particularly, this will make firm L stop innovating in the second period – this is in the interest of firm H. Hence, as this constellation of values only happens with some probability, the main part of proving the effectiveness of the punishment is thus to show that the expected loss from the other value constellations outweighs this potential gain.

Next, we consider m periods and two firms. The prize is assumed to be large enough compared to the costs such that both firms innovate in the first period. Suppose first that revelation is mandatory. Then, if one firm has an innovation with a value above x^*, incentives to continue innovating are similar to the second period of the two period case and it is never beneficial to continue innovating. The following corollary is a direct consequence of the corresponding argument in Proposition 1.1.

Corollary 1.18 *Consider an innovation contest with two firms, m periods and mandatory information revelation. Suppose firm i made an innovation in period t with $x_i^t > x^*$. Then, in any following period both firms do not innovate. In case the firm i made the highest innovation in period t and $x_i^t = x^*$, firm $j \neq i$ is indifferent between innovating or not in any following period where $x_i^t = x^*$ is still the highest innovation.*

Now suppose the revelation decision of the firms is voluntary and they can decide after each period whether to reveal or not. As a consequence of Corollary 1.18, it is immediate to see that the threat of Theorem 1.13 has bite with m periods as well:

Corollary 1.19 *Suppose firms $i, j = 1, 2$ can make a revelation decision simultaneously after learning their innovation value of each period $t = 1, \ldots, m-1$. Then, there is a Perfect Bayesian equilibrium in which both firms always reveal their value. Off the equilibrium path, in case firm i does not reveal, firm j believes the deviating firm i has* *value x^* with probability 1. Then, firm j continues to innovate until she has an innovation better than x^* or firm i reveals such an innovation.*

For firm i, hiding the own value will lead to the maximum punishment, firm j innovates in the next period for all values smaller than x^*. This is always worse for firm i then revealing, as there is no potential future advantage of an additional innovation of firm j for firm i.

1.6 Conclusion

We show that in a basic innovation contest with multiple rounds, firms and contest designer have opposing interests regarding the revelation policy of intermediate research results. Although the contest designer prefers firms to keep intermediate information secret, they are able to establish voluntary revelation of their research progress. For most of our analysis of the basic model – which has two firms, two periods and independent innovations – only mild assumptions on the research technology are needed. Furthermore, our main result of voluntary revelation turns out to be very robust: we consider extensions to n firms, m periods and improving innovations. The possibility of voluntary revelation has an impact on the prize setting by the contest designer. Suppose he wants to set his prize optimally, uses a setting without information disclosure (which he prefers) and does not prevent voluntary revelation. Then, if the firms decide to disclose on their own, using the optimal prize with respect to secret information can lead to a lower payoff for the designer than the optimal prize with respect to mandatory information disclosure. Consequently, the contest designer should then choose his prize as if information disclosure was mandatory.

Considering further extensions of the model, the most prominent one would be a joint examination of n firms, m periods and improving innovations. The existing results suggest that voluntary revelation would extend to this setting as well. Furthermore, we did not fully characterize the equilibrium research behavior for multiple firms and periods in the setting with mandatory information disclosure. Particularly, we simply assumed that the prize is large enough compared to the costs such that all firms start innovating in the first period. From a quantitative perspective, it would be possible to explicitly calculate the respective critical prize/cost ratios, although it has no impact on the qualitative nature of the results. Furthermore, our extension with improving innovations only considers a uniform distribution – it would be interesting to see the impact of a change in distribution. A completely different extension could be made by considering heterogeneous firms with different research costs or different research technologies. As long as the heterogeneity is only mild, we do not expect qualitative effects on the results, although quantitatively heterogeneity will lead to different cutoffs for the firms.

1.A Appendix: Proofs

Proof of Proposition 1.3

Suppose firm i decides not to draw again. Then, not drawing again is a best response for firm $j \neq i$ in case

$$P\left(x_j^2 \geq x^1\right) p - c \leq \frac{1}{2} p \iff \left(1 - F\left(x^1\right)\right) p - c \leq \frac{1}{2} p \iff \frac{1}{2} - \frac{c}{p} \leq F\left(x^1\right) \qquad (1.7)$$

Thus, we get that both firms not drawing again is an equilibrium if (1.7) holds.
Here, we can directly see that both firms do not want to draw in the second period in case $p < 2c$. Even if both firms did not invest in the first period, and a firm could win for sure by conducting research, expected profit is higher if no research is done.
Let us get to the best response in case firm i decides to draw in the second period. Then, drawing is a best response for firm $j \neq i$ according to the following condition:

$$\frac{1}{2} p - c \geq P\left(x_i^2 \leq x^1\right) \frac{1}{2} p \iff \frac{1}{2} p - c \geq F\left(x^1\right) \frac{1}{2} p \iff 1 - 2\frac{c}{p} \geq F\left(x^1\right) \qquad (1.8)$$

Hence, both firms drawing again is an equilibrium if (1.8) is fulfilled, which is obviously the case for $p \geq 2c$. Again, we can see that a firm does not want to draw again in case $p < 2c$. □

Proof of Proposition 1.4

To derive first-period equilibrium play, first consider the case $p < 2c$. As we have seen, both firms will not invest in the second period in case no research is done in the first period. If research is conducted by at least one firm, only the lower firm might invest again, because $\bar{x} < 0$ if $p < 2c$. By backward induction, we can conclude that both firms will not draw in the first period: we have seen in the analysis of the second period that a single draw is too expensive for a firm even when it wins for sure. In the first period, incentives for conducting research are even lower. An investing firm will not win for sure, as the other firm might decide to invest in the second period. Hence, both firms will not invest in the first period if the prize is too low. This is no problem for the firms, as they make a positive expected profit of $\frac{1}{2}p$. It is a problem of the prize sponsor, who will get no research done but has to pay the prize anyway.
So let us consider the case $p \geq 2c$. What is the best response against an opponent not taking a draw in the first period? Note that we know the following:

$$P\left(x_i^2 > x^*\right) = 1 - F\left(x^*\right) = \frac{c}{p} = r$$

$$P\left(\bar{x} < x_i^2 \leq x^*\right) = F\left(x^*\right) - F\left(\bar{x}\right) = \sqrt{2\frac{c}{p}} - \frac{c}{p} = s - r$$

$$P\left(x_i^2 \leq \bar{x}\right) = F\left(\bar{x}\right) = 1 - \sqrt{2\frac{c}{p}} = 1 - s$$

We can thus write down the condition for player i taking a draw in the first round against a player $j \neq i$ not taking a draw in the first round, bearing in mind second-period equilibrium behavior:

$$\begin{aligned}
&\Bigg[P\left(x_i^1 > x^*\right) + P\left(\bar{x} < x_i^1 \leq x^*\right)\left(P\left(x_j^2 \leq \bar{x}\right) + \frac{1}{2}P\left(\bar{x} < x_j^2 \leq x^*\right)\right) \\
&\quad + P\left(x_i^1 \leq \bar{x}\right)\left(P\left(x_i^2 > \bar{x}\right)\left(P\left(x_j^2 \leq \bar{x}\right) + \frac{1}{2}P\left(x_j^2 > \bar{x}\right)\right)\right. \\
&\quad \left.+ \frac{2}{3}P\left(x_i^2 \leq \bar{x}\right)P\left(x_j^2 \leq \bar{x}\right)\right)\Bigg]p - c - P\left(x_i^1 \leq \bar{x}\right)c \geq \frac{1}{2}p - c \\
\iff \quad &\left[r + (s-r)\left((1-s) + \frac{1}{2}(s-r)\right) + (1-s)\left(s\left((1-s) + \frac{1}{2}s\right)\right.\right. \\
&\quad \left.\left. + \frac{2}{3}(1-s)(1-s)\right)\right]p - (1-s)c \geq \frac{1}{2}p \\
\iff \quad &\left[r + (s-r)\left(1 - \frac{1}{2}(s+r)\right) + (1-s)\left((s-r) + \frac{2}{3}(1-2s+2r)\right)\right]p \\
&\quad - (1-s)c \geq \frac{1}{2}p \\
\iff \quad &\left[s - r + \frac{1}{2}r^2 + \frac{2}{3} - \frac{1}{3}s + \frac{1}{3}r - \frac{2}{3}s + \frac{2}{3}r - \frac{1}{3}rs\right]p - (1-s)c \geq \frac{1}{2}p \\
\iff \quad &\left[\frac{1}{6} - \frac{1}{3}rs + \frac{1}{2}r^2\right] - (1-s)r \geq 0 \\
\iff \quad &\frac{1}{6} - r + \frac{2}{3}rs + \frac{1}{2}r^2 \geq 0 \\
\iff \quad &\frac{1}{6} - r + \frac{2}{3}\sqrt{2}r^{\frac{3}{2}} + \frac{1}{2}r^2 \geq 0
\end{aligned} \tag{1.9}$$

We thus have to show now that (1.9) holds. To check this, we calculate the minimum of the left side in (1.9) with the help of the substitution $t := \sqrt{r}$. The FOC with respect to r is

$$\begin{aligned}
&-1 + \sqrt{2}r^{\frac{1}{2}} + r = 0 \\
\iff \quad &t^2 + \sqrt{2}t - 1 = 0 \\
\implies \quad &t = -\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{2}} = \frac{\sqrt{3}-1}{\sqrt{2}}
\end{aligned}$$

Only the positive solution matters here, as $t = \sqrt{r}$ is restricted to be positive. Hence, we get $r = \frac{\left(\sqrt{3}-1\right)^2}{2} \approx 0.2679$, leading to an expected gain from drawing compared to not drawing of approximately $0.0654 > 0$, which is clearly a minimum on $[0; 0.5]$ ($r \leq 0.5$ holds as $p \geq 2c$). Hence, it is always a best response to draw in the first period if the opponent does not take a draw.

Finally, we get to the best response of firm i in case the other agent $j \neq i$ takes a draw in the first period. We compare the expected profit of drawing as well (and thus playing the same strategy and sharing the prize) with the expected profit of not drawing in the first period. Note that we just calculated above the expected share of the prize a firm gets when taking a draw in the first period against a firm not taking a draw in the first period. We can thus subtract this share from the whole prize to get the share of the firm not drawing against a drawing firm.

$$\frac{1}{2}p - c - \left[P\left(x_i^1 \leq \bar{x}\right) P\left(x_j^1 \leq x^*\right) + \frac{1}{2}P\left(\bar{x} < x_i^1 \leq x^*\right) P\left(\bar{x} < x_j^1 \leq x^*\right)\right] c$$
$$\geq \left[1 - \left(\frac{2}{3} - \frac{1}{3}rs + \frac{1}{2}r^2\right)\right] p - P\left(x_j^1 \leq x^*\right) c \quad (1.10)$$

Computing the probabilities yields

$$\begin{aligned}
& \frac{1}{2}p - c - \left[(1-s)(1-r) + \frac{1}{2}(s-r)(s-r)\right] c \geq \left[\frac{1}{3} + \frac{1}{3}rs - \frac{1}{2}r^2\right] p - (1-r)c \\
\iff \quad & \frac{1}{2}p - c - \left[1 - s + \frac{1}{2}r^2\right] c \geq \left[\frac{1}{3} + \frac{1}{3}rs - \frac{1}{2}r^2\right] p - (1-r)c \\
\iff \quad & \left[\frac{1}{6} - \frac{1}{3}rs + \frac{1}{2}r^2\right] - \left[1 - s + r + \frac{1}{2}r^2\right] r \geq 0 \\
\iff \quad & \frac{1}{6} - r + \frac{2}{3}rs - \frac{1}{2}r^2 - \frac{1}{2}r^3 \geq 0 \qquad (1.11)
\end{aligned}$$

We can see that the left side of (1.11) is decreasing by checking the first derivative, bearing in mind that $r \in [0, 0.5]$:

$$-1 + \sqrt{2r} - r - \frac{3}{2}r^2 \leq -r - \frac{3}{2}r^2 \leq 0$$

Numerically, we get that the left side of (1.11) equals zero for $r \approx 0.2428$ – we call this boundary value v^*. Hence, drawing as well is a best response for all $r < v^* = 0.2428$. For larger r values, firm i does not want to draw in the first period if firm j takes a draw. □

Proof of Proposition 1.5

Both firms take a draw in the first period. At least one additional draw is taken in case no innovation has a value above x^*:

$$P(x_i^1 \leq x^*)P(x_j^1 \leq x^*) = (1-r)^2.$$

A second additional draw is taken in case both values are below $\bar{x}$:

$$P(x_i^1 \leq \bar{x})P(x_j^1 \leq \bar{x}) = (1-s)^2.$$

This gives us a total number of

$$d_R(r) = 2 + (1-r)^2 + (1-s)^2 = 4 + r^2 - 2s$$

concluding the proof. □

Proof of Lemma 1.6 First, we show that the right-hand side of (1.4) is decreasing in $F(z)$. Using $F(z) < 1$ and the substitution $y = F(z)$ we can write the first derivative as follows:

$$\frac{\mathrm{d}}{\mathrm{d}y}\big((1+y)(1-y)^2 - 2r\big) = 3y^2 - 2y - 1 < 0. \tag{1.12}$$

To get $x^* > z$, we plug (1.1) into the right-hand side of (1.4), yielding

$$(1+F(x^*))(1-F(x^*))^2 - 2r = (2-r)\,r^2 - 2r = 2r\,(r-1) - r^3 < 0,$$

which holds as $r < \frac{1}{2}$. We thus can conclude that

$$(1+F(x^*))(1-F(x^*))^2 - 2r < 0 = (1+F(z))(1-F(z))^2 - 2r$$

and consequently $F(x^*) > F(z)$ by (1.12). As F is increasing, this shows $x^* > z$. Similarly, for $\bar{x} < z$ we use (1.2):

$$(1+F(\bar{x}))(1-F(\bar{x}))^2 - 2r = (2-s)\,2r - 2r = 2r\,(1-s) > 0,$$

which holds as $s = \sqrt{2r} < 1$ by $r < \frac{1}{2}$. Consequently, $\bar{x} < z$ follows as above. □

Proof of Proposition 1.7
First note that

$$d_{NR}(r) > d_R(r) \iff 2(1+F(z)) > 4 - 2\sqrt{2r} + r^2 \iff F(z) > 1 - \sqrt{2r} + \frac{1}{2}r^2,$$

where $F(z)$ depends on r. To show the proposition, it is thus sufficient to prove $F(z) > 1 - \sqrt{2r} + \frac{1}{2}r^2$. In the proof of Lemma 1.6 we showed that the right-hand side of (1.4) is decreasing in $F(z)$. Hence, it is sufficient to plug $1 - \sqrt{2r} + \frac{1}{2}r^2$ into the right-hand side of (1.4) and show that the resulting expression is greater than 0. As a consequence, $F(z) > 1 - \sqrt{2r} + \frac{1}{2}r^2$ directly follows as $F(z)$ solves (1.4) (and thus yields a lower right-hand side than $1 - \sqrt{2r} + \frac{1}{2}r^2$).
Plugging $1 - \sqrt{2r} + \frac{1}{2}r^2$ into the right-hand side of (1.4) we get

$$\begin{aligned}
&\left(1+\left(1-\sqrt{2r}+\frac{1}{2}r^2\right)\right)\left(1-\left(1-\sqrt{2r}+\frac{1}{2}r^2\right)\right)^2 - 2r\\
=&2r - 2\sqrt{2}r^{\frac{3}{2}} - 2\sqrt{2}r^{\frac{5}{2}} + 3r^3 + \frac{1}{2}r^4 - \frac{3}{4}\sqrt{2}r^{\frac{9}{2}} + \frac{1}{8}r^6\\
>&2r - 2\cdot\frac{3}{4}r - 2\cdot\frac{3}{4}r^2 + 3r^3 + \frac{1}{2}r^4 - \left(\frac{3}{4}\right)^2 r^4 + \frac{1}{8}r^6\\
=&\frac{1}{2}r - \frac{3}{2}r^2 + 3r^3 - \frac{1}{16}r^4 + \frac{1}{8}r^6\\
>&\frac{1}{2}r - \frac{3}{8}r + 3r^3 - \frac{1}{64}r^3 + \frac{1}{8}r^6\\
>&0.
\end{aligned}$$

The third line follows by $r < v^* < 0.25$ and thus $-\sqrt{2r} > -\frac{3}{4}$. Similarly, the fifth line follows by $-r > -\frac{1}{4}$ and the last line by $r > 0$. □

Proof of Theorem 1.9

We derive a condition on r making $\int_0^b \Phi^2(x) - \Psi(x)dx < 0$ in a rather coarse way by looking for a non-positive integrand on the whole interval $[0, b]$. We proceed in several steps, cutting the interval into different parts:

i) $[0, \bar{x}]$

In this case, it is easy to see that $\int_0^{\bar{x}} \Phi^2(x) - \Psi(x)dx = \int_0^{\bar{x}} 0dx = 0$ holds.

ii) $(\bar{x}, z]$

Here, we get

$$\int_{\bar{x}}^{z} \Phi^2(x) - \Psi(x)dx = \int_{\bar{x}}^{z} \underbrace{\left(F(x)^2 - F(x)\right)}_{<0} \underbrace{\left(F(x)^2 - F(\bar{x})^2\right)}_{>0} dx < 0.$$

iii) $(z, x^*]$

First, we rewrite

$$\int_{z}^{x^*} \Phi^2(x) - \Psi(x)dx = \int_{z}^{x^*} \underbrace{F(z)^2 + F(x)\left(F(\bar{x})^2 - 2F(z) - 2F(z)^2\right) + F(x)^2\left(1 + 2F(z) + F(z)^2 - F(\bar{x})^2\right) - F(x)^3}_{=:h(x)} dx$$

We now show that the integrand $h(x)$ is negative by analyzing its first derivative, which is given as follows:

$$h'(x) = F(\bar{x})^2 - 2F(z) - 2F(z)^2 + 2F(x)\left(1 + 2F(z) + F(z)^2 - F(\bar{x})^2\right) - 3F(x)^2$$

At z, h' is positive:

$$h'(z) = \underbrace{\left(F(z)^2 - F(\bar{x})^2\right)}_{>0} \underbrace{(2F(z) - 1)}_{>0 \text{ for } F(z)>\frac{1}{2}}$$

As $F(z)$ is implicitly given by (1.4) we get

$$F(z) > \frac{1}{2} \iff r < \frac{1}{2} \cdot \frac{3}{2} \cdot \left(\frac{1}{2}\right)^2 = 0.1875$$

and consequently $h'(z)$ is positive in this case. Additionally, a numerical check shows that $h'(x^*)$ is positive as well (for all $z \in [0, b]$). Furthermore, h' is a quadratic function which has a maximum (this follows from $h'''(x) = -6$). Taking these facts together, we get that h' is positive on $[z, x^*]$ given $r < 0.1875$. Hence, h is increasing on $[z, x^*]$. A numerical check shows that $h(x^*) < 0$ for $r < 0.1647$ – thus, for these r-values h is negative on the whole interval (as it is largest at x^*).

iv) $(x^*, b]$

In this case, we get the following:

$$\int_{x^*}^{b} \Phi^2(x) - \Psi(x)dx$$
$$= \underbrace{\int_{x^*}^{b} F(z)^2 + F(x^*)^2 + F(x)\left(F(\bar{x})^2 - 2F(z) - 2F(z)^2 - F(x^*)^2\right) + F(x)^2\left(2F(z) + F(z)^2 - F(\bar{x})^2\right)}_{=:l(x)} dx$$

As $l(x^*) = h(x^*)$, we know that $l(x^*)$ is negative for $r < 0.1647$. Furthermore, l is a quadratic function having a minimum (as $l''(x) = 2\left(2F(z) + F(z)^2 - F(\bar{x})^2\right) > 0$). Hence, as $l(b) = 0$, l is negative on $(x^*, b]$.

Thus, we can conclude that the integrand (and thus the whole integral) is negative if $r < 0.1647 = v'$ holds. □

Proof of Proposition 1.10

First, we know from Proposition 1.1 that no firm will draw again in case she knows that an innovation larger than x^* has been drawn. The conclusion of this proposition applies to asymmetric information release as well: in the situation of Proposition 1.1 a firm does not want to draw again even if she knows that she is behind. If a firm with such a high draw does not know the opponent's draw, incentives for drawing again are even lower.
Additionally, Proposition 1.1 implies that firm N will not draw again if $x_N^1 > \bar{x}$ and $x_N^1 > x_R^1$. We first consider the following case: both firms have taken a draw in the first round. Firm R has a draw $\bar{x} < x_R^1 < x^*$ and faces the decision whether to draw again or not. For the moment we assume that firm N behaves according to Proposition 1.1 and thus draws again if she is behind (the case of equality of draws can be ignored from firm R's perspective as it is a zero probability event). It is beneficial for firm R to draw again if the following condition holds:

$$\left[P\left(x_N^1 < x_R^1\right)\left(P\left(x_N^2 < x_R^1\right) + \frac{1}{2}P\left(x_N^2 > x_R^1\right)P\left(x_R^2 > x_R^1\right)\right) + \frac{1}{2}P\left(x_N^1 > x_R^1\right)P\left(x_R^2 > x_R^1\right)\right]p - c \geq P\left(x_N^1 < x_R^1\right)P\left(x_N^2 < x_R^1\right)p$$

This yields the following probabilities:

$$\left[F\left(x_R^1\right)\left(F\left(x_R^1\right) + \frac{1}{2}\left(1 - F\left(x_R^1\right)\right)^2\right) + \frac{1}{2}\left(1 - F\left(x_R^1\right)\right)^2\right]p - c \geq F\left(x_R^1\right)^2 p$$
$$\iff \left(1 + F\left(x_R^1\right)\right)\left(1 - F\left(x_R^1\right)\right)^2 - 2\frac{c}{p} \geq 0 \qquad (1.13)$$

Note that (1.13) has the same structure as (1.4). Hence, firm R will draw again exactly in case her first draw is smaller than z, which solves both (1.4) and (1.13). We denote $F(z) =: w$ in the following.

For the calculation above, we assumed that firm N follows the strategy described in Proposition 1.1, but it is not clear that this strategy is a best reply. Obviously, it is a best reply in case firm N is leading, as $x_R^1 > \bar{x}$. Not drawing is then profitable even against an opponent who draws. However, it could be profitable for firm N to stop drawing in case she is behind and firm R has a draw $\bar{x} < x_R^1 < z$ with $x_R^1 > x_N^1$. In this case, firm R will draw again as well – she would not do so if she knew that she is in front, as it is the case in the situation of Proposition 1.1. We check whether it is anyway profitable to draw again for firm N:

$$\begin{aligned}
& P\left(x_N^2 > x_R^1\right)\left(P\left(x_R^2 < x_R^1\right) + \frac{1}{2}P\left(x_R^2 > x_R^1\right)\right)p - c \\
=& \left(1 - F\left(x_R^1\right)\right)\left(F\left(x_R^1\right) + \frac{1}{2}\left(1 - F\left(x_R^1\right)\right)\right)p - c \\
>& \frac{1}{2}\left(1 + F\left(x_R^1\right)\right)\left(1 - F\left(x_R^1\right)\right)^2 p - c \\
\geq& 0
\end{aligned}$$

The strict inequality holds by direct comparison (and $0 < F\left(x_R^1\right) < 1$). The last inequality holds as $x_R^1 < z$ in this case and (1.13) applies. Hence, it is in fact a best reply for firm N to follow the strategy derived in Proposition 1.1.

If the draw of firm R fulfills $x_R^1 < \bar{x}$, the incentives to draw again are the same for firm N as in Proposition 1.1. Hence, firm N behaves similarly here. For firm R, we consider an estimate of her profit from drawing again, looking only at the largest terms:

$$\begin{aligned}
&\left[P\left(x_N^1 < x_R^1\right)\left(P\left(x_N^2 < x_R^1\right) + \frac{1}{2}P\left(x_N^2 > x_R^1\right)P\left(x_R^2 > x_R^1\right)\right)\right. \\
&\qquad\left. + \frac{1}{2}P\left(x_N^1 > \bar{x} > x_R^1\right)P\left(x_R^2 > \bar{x} > x_R^1\right)\right]p - c \\
&= \left[F\left(x_R^1\right)\left(F\left(x_R^1\right) + \frac{1}{2}\left(1 - F\left(x_R^1\right)\right)^2\right) + \frac{1}{2}\left(1 - F\left(\bar{x}\right)\right)^2\right]p - c \\
&> \left[F\left(x_R^1\right)F\left(x_R^1\right) + \frac{1}{2}\left(1 - \left(1 - \sqrt{2\frac{c}{p}}\right)\right)^2\right]p - c \\
&= F\left(x_R^1\right)F\left(x_R^1\right)p
\end{aligned}$$

The latter is the expected profit of firm R without a second draw. Hence, drawing again is beneficial for firm R.

What happens if one of the firms plays a strategy where she does not take a draw in the first period? If firm N faces a firm R taking no draw, the second period behavior is similar to playing against a firm with a draw of zero. For firm R, things change: if she faces a firm not drawing in the first period, her best reply is similar as in the situation of full information release. Thus, if she believes with probability one that she faces a not-drawing firm, she plays the same strategy as firm N in that case: she will only draw again if $x_R^1 < \bar{x}$. □

Proof of Proposition 1.11

In the first period, both firms have to compare the expected profits of taking a draw with the expected profits of waiting one period. Consider first the case of firm R not drawing in the first round. What is the best reply of firm N? This is basically the same exercise as deriving inequality (1.9), with one slight difference: firm N is not able to discourage firm R from taking a draw in case $x_N^1 > x^*$. This slightly reduces the probability of winning the prize for firm N compared to the setting of full revelation: it is now possible that firm R beats firm N with a draw $x_R^2 > x_N^1 > x^*$. This is the case with probability $\frac{1}{2}\left(1-F\left(x^*\right)\right)^2 = \frac{1}{2}r^2$. We can include this probability change into (1.9) by subtracting $\frac{1}{2}r^2$, which gives us the following condition for a profitable draw in the first round:

$$\frac{1}{6} - r + \frac{2}{3}\sqrt{2}r^{\frac{3}{2}} \geq 0 \tag{1.14}$$

The analysis of the first order condition shows that the left side of (1.14) has a minimum at $r = \frac{1}{2}$. For $r = \frac{1}{2}$, equality holds in (1.14). Hence, taking a draw is profitable for firm N in the first period in this case.

What is the best reply for firm R against this strategy of firm N? We first calculate the probability for firm R to win the prize if she is taking a draw in the first period (and following the equilibrium strategy of the second period afterwards).

$$\begin{aligned}
&P\left(x_R^1 > x^*\right)\left[P\left(x_N^1 < x^*\right) + \frac{1}{2}P\left(x_N^1 > x^*\right)\right] \\
&+P\left(z < x_R^1 < x^*\right)\left[\frac{1}{2}P\left(z < x_N^1 < x^*\right)\left(\frac{2}{3}P\left(z < x_N^2 < x^*\right) + P\left(x_N^2 < z\right)\right)\right. \\
&\left.+ P\left(x_N^1 < z\right)\left(P\left(x_N^2 < z\right) + \frac{1}{2}P\left(z < x_N^2 < x^*\right)\right)\right] \\
&+P\left(\bar{x} < x_R^1 < z\right)\left[\frac{1}{2}P\left(x_N^1 > x^*\right)P\left(x_R^2 > x^*\right)\right. \\
&+ P\left(z < x_N^1 < x^*\right)\left(P\left(x_R^2 > x^*\right) + \frac{1}{2}P\left(z < x_R^2 < x^*\right)\right) \\
&+ P\left(\bar{x} < x_N^1 < z\right)\left(\frac{1}{2}\left(P\left(x_R^2 > z\right) + \frac{1}{3}P\left(\bar{x} < x_R^2 < z\right)\right)\right. \\
&+ \frac{1}{2}\left[P\left(x_R^2 > z\right)\left(\frac{1}{2}P\left(x_N^2 > z\right) + P\left(x_N^2 < z\right)\right)\right. \\
&+ P\left(\bar{x} < x_R^2 < z\right)\left(P\left(x_N^2 < \bar{x}\right) + \frac{3}{4}P\left(\bar{x} < x_N^2 < z\right)\right) \\
&\left.\left.+ P\left(x_R^2 < \bar{x}\right)\left(P\left(x_N^2 < \bar{x}\right) + \frac{2}{3}P\left(\bar{x} < x_N^2 < z\right)\right)\right]\right) \\
&+ P\left(x_N^1 < \bar{x}\right)\left(P\left(x_R^2 > z\right)\left(P\left(x_N^2 < z\right) + \frac{1}{2}P\left(x_N^2 > z\right)\right)\right. \\
&+ P\left(\bar{x} < x_R^2 < z\right)\left(P\left(x_N^2 < \bar{x}\right) + \frac{2}{3}P\left(\bar{x} < x_N^2 < z\right)\right)
\end{aligned}$$

$$+ P\left(x_R^2 < \bar{x}\right)\left(P\left(x_N^2 < \bar{x}\right) + \frac{1}{2}P\left(\bar{x} < x_N^2 < z\right)\right)\Bigg)\Bigg]$$
$$+P\left(x_R^1 < \bar{x}\right)\Bigg[\frac{1}{2}P\left(x_N^1 > \bar{x}\right)P\left(x_R^2 > \bar{x}\right)$$
$$+ P\left(x_N^1 < \bar{x}\right)\Bigg(P\left(x_R^2 > \bar{x}\right)\left(P\left(x_N^2 < \bar{x}\right) + \frac{1}{2}P\left(x_N^2 > \bar{x}\right)\right)$$
$$+ \frac{1}{2}P\left(x_R^2 < \bar{x}\right)P\left(x_N^2 < \bar{x}\right)\Bigg)\Bigg]$$

Using the short notations $P(x > \bar{x}) = s$, $P(x < z) = w$ and $P(x > x^*) = r$, simplifying and subtracting the costs, this reduces to the following expected profit:

$$\left(\frac{7}{24} - \frac{1}{3}r^3 + \frac{1}{6}s^3 + \frac{1}{2}w - \frac{1}{4}w^2 - \frac{1}{6}w^3 + \frac{1}{8}w^4\right)p - (1+w)\,c \tag{1.15}$$

Furthermore, we have to calculate the expected profit of firm R when she is waiting for the second period without taking a draw (and faces a drawing firm N):

$$\Bigg[\frac{1}{2}P\left(x_N^1 > \bar{x}\right)P\left(x_R^2 > \bar{x}\right) + P\left(x_N^1 < \bar{x}\right)\Bigg(\frac{1}{3}P\left(x_R^2 < \bar{x}\right)P\left(x_N^2 < \bar{x}\right)$$
$$+P\left(x_R^2 > \bar{x}\right)\left(P\left(x_N^2 < \bar{x}\right) + \frac{1}{2}P\left(x_N^2 > \bar{x}\right)\right)\Bigg)\Bigg]p - c$$
$$= \left[\frac{1}{3} + \frac{1}{6}s^3\right]p - c. \tag{1.16}$$

Drawing in the first period is thus profitable if the value of (1.15) is larger than the value of (1.16). Comparing these two terms, we get

$$\left(\frac{7}{24} - \frac{1}{3}r^3 + \frac{1}{6}s^3 + \frac{1}{2}w - \frac{1}{4}w^2 - \frac{1}{6}w^3 + \frac{1}{8}w^4\right)p - (1+w)\,c \geq \left[\frac{1}{3} + \frac{1}{6}s^3\right]p - c$$
$$\Longleftrightarrow \quad \left(-\frac{1}{24} - \frac{1}{3}r^3 + \frac{1}{2}w - \frac{1}{4}w^2 - \frac{1}{6}w^3 + \frac{1}{8}w^4\right)p - wc \geq 0$$
$$\Longleftrightarrow \quad -\frac{1}{24} - \frac{1}{3}r^3 + \frac{1}{2}w - \frac{1}{4}w^2 - \frac{1}{6}w^3 + \frac{1}{8}w^4 - wr \geq 0.$$

A numerical analysis shows that the left-hand side equals zero for $r \approx 0.2623$ – we call this critical value $\hat{v}$. For larger r values firm R prefers to wait for the second period to take her draw. In this case, we showed that there is an asymmetric equilibrium with firm N drawing in the first and firm R drawing in the second period. Firm N then follows her second-period equilibrium strategy.

For smaller r values, firm R takes a draw in the first period as well. To confirm that this constellation is consistent with an equilibrium behavior, we have to check the incentives of firm N to take a draw in this case. If she does not take a draw, her expected profit is

$$\begin{aligned}
&\Bigg[P\left(z < x_R^1 < x^*\right)\left(P\left(x_N^2 > x^*\right) + \frac{1}{2}P\left(z < x_N^2 < x^*\right)\right) \\
&+P\left(x_R^1 < z\right)\left(\frac{1}{3}P\left(x_N^2 < z\right)P\left(x_R^2 < z\right)\right. \\
&\left.\left. + \; P\left(x_N^2 > z\right)\left(P\left(x_R^2 < z\right) + \frac{1}{2}P\left(x_R^2 > z\right)\right)\right)\right] p - P\left(x_R^1 < x^*\right)c \\
= &\left[\frac{1}{2} - \frac{1}{2}w + \frac{1}{2}w^2 - \frac{1}{6}w^3 - \frac{1}{2}r^2\right] p - (1-r)c. \qquad (1.17)
\end{aligned}$$

We compare this with the expected profit of taking a draw. As part of (1.15), we already calculated the probability that firm R wins the contest in case both firms take a draw in the first round. Consequently, this number and the probability that firm N wins this contest add up to one. Hence, firm N makes an expected profit according to the following expression:

$$\left[\frac{17}{24} + \frac{1}{3}r^3 - \frac{1}{6}s^3 - \frac{1}{2}w + \frac{1}{4}w^2 + \frac{1}{6}w^3 - \frac{1}{8}w^4\right] p - \left(2 - s + \frac{1}{2}r^2\right)c. \qquad (1.18)$$

Comparing (1.18) with (1.17), we get the following condition for a profitable draw in the first period:

$$\begin{aligned}
&\left[\frac{17}{24} + \frac{1}{3}r^3 - \frac{1}{6}s^3 - \frac{1}{2}w + \frac{1}{4}w^2 + \frac{1}{6}w^3 - \frac{1}{8}w^4\right] p - (2 - s + \frac{1}{2}r^2)c \\
&\geq \left[\frac{1}{2} - \frac{1}{2}w + \frac{1}{2}w^2 - \frac{1}{6}w^3 - \frac{1}{2}r^2\right] p - (1-r)c \\
\Longleftrightarrow \; &\frac{5}{24} - \frac{1}{6}r^3 + \frac{1}{3}w^3 - \frac{1}{4}w^2 - \frac{1}{8}w^4 - \frac{1}{6}s^3 - \frac{1}{2}r^2 - r + rs \geq 0.
\end{aligned}$$

Again, a numerical analysis shows that the left hand side equals zero for $r \approx 0.2939$. For smaller r values, the inequality is fulfilled and drawing in the first period is profitable for firm N – we found an equilibrium in that case. For larger r values, firm N's best reply is not to draw in the first round. We thus have to check how firm R's best reply against a waiting firm N looks like (with respect to correct beliefs). Note that firm R will only draw again in the second-period equilibrium if $x_R^1 < \bar{x}$. Hence, incentives to draw are similar to the case of full information release and result in condition (1.9). The analysis of that condition showed that it is thus profitable for firm R to draw against a waiting firm N.

Finally, we have to analyze the incentives of the waiting firm N – is it profitable to draw against a drawing firm R who believes to face a firm N that does not draw? The expected profit of drawing can be calculated as follows:

$$\begin{aligned}
&\Big[\frac{1}{2}P\left(x_R^1>x^*\right)P\left(x_N^1>x^*\right)+P\left(\bar{x}<x_R^1<x^*\right)\Big(P\left(x_N^1>x^*\right)\\
&+P\left(\bar{x}<x_N^1<x^*\right)\left(\frac{1}{2}+\frac{1}{2}\left(P\left(x_N^2>x^*\right)+\frac{1}{3}P\left(\bar{x}<x_N^2<x^*\right)\right)\right)\Big)\\
&P\left(x_R^1<\bar{x}\right)\left(\frac{1}{2}P\left(x_N^1<\bar{x}\right)+P\left(x_N^1>\bar{x}\right)\left(\frac{1}{2}P\left(x_R^2>\bar{x}\right)+P\left(x_R^2<\bar{x}\right)\right)\right)\Big]p\\
&-\left(2-s+\frac{1}{2}r^2\right)c\\
=&\left[\frac{1}{2}-\frac{1}{2}s^2+\frac{2}{3}s^3+\frac{1}{3}r^3-\frac{1}{2}r^2s\right]p-\left(2-s+\frac{1}{2}r^2\right)c.
\end{aligned}$$

If firm N does not draw in the first period, she is in the same situation as in the right hand side of (1.10). We compare the expected profits of drawing and not drawing:

$$\left[\frac{1}{2}-\frac{1}{2}s^2+\frac{2}{3}s^3+\frac{1}{3}r^3-\frac{1}{2}r^2s\right]p-\left(2-s+\frac{1}{2}r^2\right)c\geq\left[\frac{1}{3}+\frac{1}{3}rs-\frac{1}{2}r^2\right]p-(1-r)\,c.$$

Simplifying and using $s=\sqrt{2r}$, we get that drawing is profitable in case

$$\frac{1}{6}-2r-\frac{1}{2}r^2-\frac{1}{6}r^3+2sr-\frac{1}{2}sr^2\geq 0.$$

This condition holds for $r<0.1722$, as a numerical analysis shows. We call this critical value $\tilde{v}$. Given this condition, we are back in the situation where both want to draw (and our previous analysis showed that this is an equilibrium for this range of r-values). For $r>0.1722$, firm N does not want to draw and we are hence in an equilibrium as well – the best reply for firm R against a firm N that does not draw is to draw. □

Proof of Theorem 1.12 We focus our analysis on the first equilibrium identified in Proposition 1.11. In this equilibrium, both firms take a draw in the first period and it is unique for $r<\tilde{v}$. In the initial stage zero, where firms choose whether to reveal or not, we now have to identify the best responses of the two firms. What is the best response of a firm, if the other firm chooses to play R? If she plays R as well, they share the prize in expectation and $2-s+\frac{1}{2}r^2$ research draws are taken by each of the firms. If a firm deviates to play N, the expected costs of drawing do not change (as she still gets the same information and plays the same strategy). However, it may happen that she receives in expectation less than half of the prize after the deviation, as given by the following condition ($w=F(z)$):

$$\left[\frac{17}{24}+\frac{1}{3}r^3-\frac{1}{6}s^3-\frac{1}{2}w+\frac{1}{4}w^2+\frac{1}{6}w^3-\frac{1}{8}w^4\right]p-\left(2-s+\frac{1}{2}r^2\right)c=\frac{1}{2}p,\qquad(1.19)$$

The left-hand side of (1.19) states the profit for the firm deviating to N, as derived in (1.18). A r-value of $\bar{v}\approx 0.2325$ solves (1.19) (which is equivalent to (1.6)), and for $r<\bar{v}$

a firm playing N against R receives in expectation less than half the share of the total prize. Combined with the fact that research costs do not change, it is the best response against a firm playing R to play R as well for these values.

What is the best response against a firm playing N? Playing N as well gives in expectation half of the prize while taking $1 + w$ draws. As we have just seen, a firm playing R receives in expectation more than half the prize against a firm playing N for $r < \bar{v}$. Additionally, she has to take the same number of draws in expectation. Hence, it is profitable to play R against a firm playing N.

A firm will thus always play reveal in the initial stage, no matter whether the other firm plays reveal as well or not. □

Proof of Theorem 1.13 We first show that it is in fact an equilibrium. Note that the point of revealing (or not revealing) is to make the other firm stop researching in as many cases as possible. Suppose firm i deviates and does not reveal her value. This deviation cannot be beneficial: if firm j has a value $x_j > x^*$, the reaction of this firm does not change – she always stops researching in this case. Additionally, if $x_j < x^*$, firm j will continue to do research, and thus goes on in the maximum number of cases. Revealing a value $x_i < x^*$ would have made a firm with value $x_j \in (x_i, x^*)$ stop researching, increasing the chances of firm i to win.

To show the uniqueness, suppose there is another equilibrium in which at least one firm hides a value different from x^*. Consider the strategy of firm i, and first assume that this firm always keeps the information secret in case $x_i \in X_1 \subset (x^*, b]$ (and reveals her value for $x_i \notin X_1$). Thus, in equilibrium, if firm j observes that firm i does not reveal any information, she correctly believes that $x_i > x^*$. Consequently, firm j stops innovating, no matter what value her first-period innovation has. This provides firm i with an incentive to always keep her information secret, as this will make firm j stop. Hence, in any equilibrium where information is kept secret for values in X_1, this has to be done also for some values $x_i \in X_2 \subset [0, x^*]$. Furthermore, X_2 has to be large enough such that firm j continues to innovate for some values x_j when receiving no information by firm i (and believing correctly that $x_i \in X_1 \cup X_2$). However, if $x_i \in X_1$, firm i has a profitable deviation by simply revealing her value and making firm j stop innovating in any case. Thus, there cannot be an equilibrium in which firm i with value $x_i \in (x^*, b]$ keeps this value secret.

A similar reasoning applies in case we assume that information is kept secret only for values $x_i \in X_3 \subset [0, \bar{x}]$ – firm j with any $x_j \in (\bar{x}, x^*)$ would stop innovating, and firm i with $x_i \in (\bar{x}, x^*)$ had an incentive to keep her information secret and make firm j stop for these x_j. Additionally, consider the case of a set $X_4 \subset (\bar{x}, x^*)$ for which values are kept secret on top of X_3 (making some $x_j \in (\bar{x}, x^*)$ continue to innovate): then, it is profitable for $x_i \in X_3$ to reveal and make firm j stop innovating for all $x_j \in (\bar{x}, x^*)$. Thus, there cannot be an equilibrium in which firm i with value $x_i \in [0, \bar{x}]$ keeps this value secret.

Finally, consider the case where information is kept secret by firm i for values $x_i \in X_5 \subset (\bar{x}, x^*)$. Then, firm j will continue to innovate for all $x_j < \inf X_5$ if she does not observe any information by firm i. As a firm i with a value $x_i \in X_5$ decides to keep her information secret, firm i in equilibrium cannot be better off by revealing (and making firm j continue for all $x_j < x_i$). Thus, there can be no set X_6 with $x_j \in X_6$ continuing to innovate in equilibrium and $X_6 \cap (\inf X_5, x^*)$ having a positive mass. Otherwise, there would be some $x_j' \in X_6 \cap (\inf X_5, x^*)$ dividing this set in two parts with a positive mass. Consequently, some $x_i \in X_5 \cap (\inf X_5, x_j')$ would exist for which firm i had a profitable deviation by revealing her type (and making firm j stop innovating in the part above x_j'). This shows that in equilibrium firm j does not continue to innovate for all $x_j \in (\inf X_5, x^*)$, if she does not observe information by firm i. Keeping this in mind, we can conclude that X_5 is in fact an interval of the form $(\inf X_5, x^*)$ (possibly including the end points, which we ignore for notational purpose). Suppose this were not the case. Then, there is some $x_i \in (\inf X_5, x^*)$ for which firm i would reveal her value. However, she could do strictly better for that value by keeping the information secret and making firm j stop innovating for all $x_j \in (\inf X_5, x_i)$.

So suppose firm i keeps the information secret for such an interval, $(x_i', x^*) \neq \emptyset$. Then, as we just showed, firm j does not continue to innovate for values $x_j \in (x_i', x^*)$ if she does not observe any information. Consider some $x_i'' \in (x_i', x^*)$. From the equilibrium derivation in case of full information revelation we know that any $x_j \in (x_i', x_i'')$ makes a positive profit against x_i'' by continuing to innovate. We denote the average expected profit of drawing for firm j against values in (x_i'', x^*) by δ (it is independent of the size of x_j, as long as $x_j < x_i''$). Against all values in (x_j, x_i''), the expected profit is even larger than δ. Now consider some fixed $x_j < x_i''$ for which the probability that firm j is in the lead if she does not receive any information is less or equal to ε. If firm j would deviate for x_j and continue to innovate, this would have two effects: on the one hand, she would make an expected profit of at least δ against firm i having a higher valuation (up to x^*). On the other hand, she could maximally waste the cost of drawing c if she faces a firm i with a value in (x_i', x_j), as the one-sided deviation of an additional draw cannot make firm j loose more often. This only happens with probability ε. Thus, innovating is profitable for firm j with value $x_j \in (x_i', x_i'')$, if the following condition holds:

$$(1 - \varepsilon)\delta - \varepsilon c > 0$$

As this condition is fulfilled for ε small enough, firm j has the profitable deviation to continue innovating. Thus our initial assumption is not true and we cannot have an equilibrium where any firm keeps the information secret for values other than x^*. In case their first-period value is x^*, firms are indifferent between revealing or not – but this event has zero probability. □

Proof of Proposition 1.14

Suppose first the leading firm H does not innovate in the second period. Then, it is (weakly) beneficial for firm L to innovate again iff

$$P\left(x_L^2 > x_H^1\right) p - c \geq 0 \iff \left(1 - F_L\left(x_H^1 | x_L^1\right)\right) p - c \geq 0 \iff F_L\left(x_H^1 | x_L^1\right) \leq 1 - \frac{c}{p}$$
$$\iff x_H^1 \leq 1 - (1 - x_L^1) r$$

For firm H, there is only a possible need of continuing to innovate if the other firm is also innovating (otherwise, firm H would win for sure anyway). Thus, firm H will do so iff

$$\begin{aligned} & \left[P\left(x_H^2 > x_L^2 > x_H^1\right) + P\left(x_H^1 > x_L^2\right)\right] p - c \geq P\left(x_H^1 > x_L^2\right) p \\ \iff & \frac{1}{2}\left(1 - F_L\left(x_H^1 | x_L^1\right)\right) p - c \geq 0 \\ \iff & F_L\left(x_H^1 | x_L^1\right) \leq 1 - 2\frac{c}{p} \\ \iff & x_H^1 \leq 1 - (1 - x_L^1) 2r \end{aligned}$$

The second line follows as firm H will always improve her first period innovation and beats a firm L that also improves upon x_H^1 in exactly half of the cases because we assumed a uniform distribution.

Finally, we have to check that firm L has no incentives to refrain from innovating in the second period in the range of values where firm H innovates as well:

$$P\left(x_L^2 > x_H^2\right) p - c \geq 0 \iff \frac{1}{2}\left(1 - F_L\left(x_H^1 | x_L^1\right)\right) p - c \geq 0$$

which is the same condition as for firm H – both firms continuing to innovate is thus an equilibrium if this condition is fulfilled. □

Proof of Theorem 1.15

We check whether firm i has an incentive to deviate for a value x_i^1. Suppose first that $x_i^1 < x^* = 1 - r$. Then, hiding the value makes firm j continue to innovate for a strictly larger set of first-period values: Proposition 1.14 shows that firm j makes a second innovation for $x_j^1 \in [0, 1 - (1 - x^*)2r]$, which is a superset of the set of values for which firm j would draw if she knew x_i^1, $[0, 1 - (1 - x_i^1)2r]$. Firm i has thus no incentive to hide the value.

The more interesting case is given by $x_i^1 > x^*$. We first pin down the expected profit of firm i with given value x_i^1 when both firms reveal their true value. Applying Proposition 1.14 to determine the ranges for which the two firms continue to innovate and the resulting winning probabilities, the expected profit amounts to

$$\begin{aligned}
&\Bigg[P\left(x_j^1 < 1-\frac{1-x_i^1}{r}\right) + P\left(1-\frac{1-x_i^1}{r} < x_j^1 < 1-\frac{1-x_i^1}{2r} \wedge x_j^2 < x_i^1\right) \\
&\quad + P\left(1-\frac{1-x_i^1}{2r} < x_j^1 < x_i^1 \wedge \left(x_j^2 < x_i^1 \vee x_i^1 < x_j^2 < x_i^2\right)\right) \\
&\quad + P\left(x_i^1 < x_j^1 < 1-(1-x_i^1)2r \wedge x_i^2 > x_j^2\right) \\
&\quad + P\left(1-(1-x_i^1)2r < x_j^1 < 1-(1-x_i^1)r \wedge x_i^2 > x_j^1\right)\Bigg] p \\
&\quad - P\left(1-\frac{1-x_i^1}{2r} < x_j^1 < 1-(1-x_i^1)r\right) c \\
=&\Bigg[F\left(1-\frac{1-x_i^1}{r}\right) + \int_{1-\frac{1-x_i^1}{r}}^{1-\frac{1-x_i^1}{2r}} F_j\left(x_i^1|x_j^1\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1 \\
&\quad + \int_{1-\frac{1-x_i^1}{2r}}^{x_i^1} \left(F_j\left(x_i^1|x_j^1\right) + \frac{1}{2}\left(1-F_j\left(x_i^1|x_j^1\right)\right)\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1 \\
&\quad + \int_{x_i^1}^{1-(1-x_i^1)2r} \frac{1}{2}\left(1-F_i\left(x_j^1|x_i^1\right)\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1 \\
&\quad + \int_{1-(1-x_i^1)2r}^{1-(1-x_i^1)r} \left(1-F_i\left(x_j^1|x_i^1\right)\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1\Bigg] p \\
&\quad - \left[F\left(1-(1-x_i^1)r\right) - F\left(1-\frac{1-x_i^1}{2r}\right)\right] c
\end{aligned}$$

If firm i hides her own value, firm j believes that firm i has value x^* and acts accordingly. However, as we want to look at the one-sided deviation of firm i, firm j still reveals her value. Depending on the size of x_i^1, the decision whether to innovate in the second period or not changes. To write down the expected profit of firm i when hiding her value, we thus have to make a case distinction.

First case. We start with the case $x_i^1 < 1-(1-x^*)2r = 1-2r^2$, such that firm i will make the following expected profit:

$$\begin{aligned}
&\Bigg[P\left(0 < x_j^1 < 1-\frac{1-x_i^1}{2r} \wedge x_j^2 < x_i^1\right) \\
&\quad + P\left(1-\frac{1-x_i^1}{2r} < x_j^1 < x_i^1 \wedge \left(x_j^2 < x_i^1 \vee x_i^1 < x_j^2 < x_i^2\right)\right) \\
&\quad + P\left(x_i^1 < x_j^1 < 1-(1-x^*)2r \wedge x_i^2 > x_j^2\right) \\
&\quad + P\left(1-(1-x^*)2r < x_j^1 < 1-(1-x_i^1)r \wedge x_i^2 > x_j^1\right)\Bigg] p \\
&\quad - P\left(1-\frac{1-x_i^1}{2r} < x_j^1 < 1-(1-x_i^1)r\right) c \\
=&\Bigg[\int_0^{1-\frac{1-x_i^1}{2r}} F_j\left(x_i^1|x_j^1\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1 + \int_{1-\frac{1-x_i^1}{2r}}^{x_i^1} \left(F_j\left(x_i^1|x_j^1\right) + \frac{1}{2}\left(1-F_j\left(x_i^1|x_j^1\right)\right)\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1
\end{aligned}$$

$$+\int_{x_i^1}^{1-2r^2} \frac{1}{2}\left(1-F_i\left(x_j^1|x_i^1\right)\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1 + \int_{1-2r^2}^{1-(1-x_i^1)r} \left(1-F_i\left(x_j^1|x_i^1\right)\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1\Bigg] p$$
$$-\left[F\left(1-(1-x_i^1)r\right) - F\left(1-\frac{1-x_i^1}{2r}\right)\right] c$$

It is thus not profitable to hide the own value, iff

$$\Bigg[F\left(1-\frac{1-x_i^1}{r}\right) + \int_{1-2r^2}^{1-(1-x_i^1)2r} \frac{1}{2}\left(1-F_i\left(x_j^1|x_i^1\right)\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1$$
$$-\int_0^{1-\frac{1-x_i^1}{r}} F_j\left(x_i^1|x_j^1\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1 - \int_{1-2r^2}^{1-(1-x_i^1)2r} \left(1-F_i\left(x_j^1|x_i^1\right)\right) f\left(x_j^1\right)\,\mathrm{d}x_j^1\Bigg] p \geq 0$$
$$\iff 1-\frac{1-x_i^1}{r} - \int_{1-2r^2}^{1-(1-x_i^1)2r} \frac{1}{2}\left(1-\frac{x_j^1-x_i^1}{1-x_i^1}\right)\mathrm{d}x_j^1 - \int_0^{1-\frac{1-x_i^1}{r}} \frac{x_i^1-x_j^1}{1-x_j^1}\,\mathrm{d}x_j^1 \geq 0 \tag{1.20}$$

First note that $x_j^1 \geq x_i^1$ on $[1-2r^2, 1-(1-x_i^1)2r]$. Hence, we can estimate

$$\int_{1-2r^2}^{1-(1-x_i^1)2r} \frac{1}{2}\left(1-\frac{x_j^1-x_i^1}{1-x_i^1}\right)\mathrm{d}x_j^1 < \int_{1-2r^2}^{1-(1-x_i^1)2r} \frac{1}{2}\,\mathrm{d}x_j^1 = r(r-(1-x_i^1))$$

Furthermore, $\frac{x_i^1-x_j^1}{1-x_j^1}$ is decreasing in x_j^1, thus

$$\int_0^{1-\frac{1-x_i^1}{r}} \frac{x_i^1-x_j^1}{1-x_j^1}\,\mathrm{d}x_j^1 < \int_0^{1-\frac{1-x_i^1}{r}} x_i^1\,\mathrm{d}x_j^1 = \frac{x_i^1}{r}(r-(1-x_i^1))$$

As $x_i^1 > 1-r$, the condition (1.20) is thus fulfilled if

$$\left(\frac{1}{r} - r - \frac{x_i^1}{r}\right)(r-(1-x_i^1)) > 0 \iff x_i^1 < 1-r^2.$$

This is true, as by assumption $x_i^1 < 1-2r^2 < 1-r^2$. We can conclude that firm i does not want to deviate and hide her value for $x_i^1 < 1-2r^2$.

Second case. The next case is $1-4r^3 > x_i^1 > 1-2r^2$. The condition stems from requiring $1-\frac{1-x_i^1}{2r} < 1-2r^2$. Furthermore, the relationship $1-4r^3 > 1-2r^2$ is always fulfilled as $r < \frac{1}{2}$. If firm i hides her value, she makes the following expected profit:

$$\Bigg[P\left(0 < x_j^1 < 1-\frac{1-x_i^1}{2r} \wedge x_j^2 < x_i^1\right)$$
$$+P\left(1-\frac{1-x_i^1}{2r} < x_j^1 < 1-2r^2 \wedge \left(x_j^2 < x_i^1 \vee x_i^1 < x_j^2 < x_i^2\right)\right)$$
$$+P\left(1-2r^2 < x_j^1 < x_i^1\right) + P\left(x_i^1 < x_j^1 < 1-(1-x_i^1)r \wedge x_i^2 > x_j^1\right)\Big] p$$
$$-\left[P\left(1-\frac{1-x_i^1}{2r} < x_j^1 < 1-2r^2\right) + P\left(x_i^1 < x_j^1 < 1-(1-x_i^1)r\right)\right] c$$

$$= \left[\int_0^{1-\frac{1-x_i^1}{2r}} F_j\left(x_i^1|x_j^1\right) f\left(x_j^1\right) \,\mathrm{d}x_j^1 + \int_{1-\frac{1-x_i^1}{2r}}^{1-2r^2} \left(F_j\left(x_i^1|x_j^1\right) + \frac{1}{2}\left(1 - F_j\left(x_i^1|x_j^1\right)\right)\right) f\left(x_j^1\right) \,\mathrm{d}x_j^1 \right.$$

$$\left. + F(x_i^1) - F(1-2r^2) + \int_{x_i^1}^{1-(1-x_i^1)r} \left(1 - F_i\left(x_j^1|x_i^1\right)\right) f\left(x_j^1\right) \,\mathrm{d}x_j^1 \right] p$$

$$- \left[F\left(1-(1-x_i^1)r\right) - F(x_i^1) + F(1-2r^2) - F\left(1 - \frac{1-x_i^1}{2r}\right)\right] c$$

Thus, hiding is not profitable iff

$$\left[1 - \frac{1-x_i^1}{r} - \int_0^{1-\frac{1-x_i^1}{r}} \frac{x_i^1 - x_j^1}{1-x_j^1} \,\mathrm{d}x_j^1 + \int_{1-2r^2}^{x_i^1} \left(\frac{x_i^1 - x_j^1}{1-x_j^1} + \frac{1}{2} \cdot \frac{1-x_i^1}{1-x_j^1}\right) \,\mathrm{d}x_j^1 \right.$$

$$\left. - x_i^1 + 1 - 2r^2 - \int_{x_i^1}^{1-(1-x_i^1)2r} \frac{1}{2} \cdot \frac{1-x_j^1}{1-x_i^1} \,\mathrm{d}x_j^1 \right] p - \left[x_i^1 - 1 + 2r^2\right] c$$

$$\geq 0$$

By calculating the integrals and simplifying, the condition boils down to

$$2r - \frac{1}{2} - \frac{1}{2}\ln(2(1-x_i^1)) + \frac{1}{4}(1-2r)^2 - \frac{2r^3}{1-x_i^1} \geq 0$$

Thus, by using $1 - 2r^2 < x_i^1 < 1 - 4r^3$ we can get a lower bound of the left hand side and formulate the following sufficient condition for hiding to be non-profitable:

$$2r - \frac{1}{2} - \frac{1}{2}\ln(2 \cdot 2r^2) + \frac{1}{4}(1-2r)^2 - \frac{2r^3}{4r^3} \geq 0$$

$$\iff \qquad 2r - \ln(2r) + \frac{1}{4}(1-2r)^2 \geq 1$$

It remains to show that this condition is fulfilled. For $r = \frac{1}{2}$, it is obviously fulfilled with equality. We show that the left hand side is decreasing on $(0, \frac{1}{2})$ by looking at its first derivative with respect to r:

$$2 - \frac{1}{r} - (1-2r) \leq 0 \iff 1 - r - 2r^2 \geq 0,$$

which is true for $0 < r < \frac{1}{2}$.

Third case. The remaining case is $x_i^1 > 1 - 4r^3$. The expected profit of firm i from hiding the value amounts to

$$\left[P\left(0 < x_j^1 < 1 - 2r^2 \wedge x_j^2 < x_i^1\right) + P\left(1 - 2r^2 < x_j^1 < x_i^1\right)\right.$$

$$\left. + P\left(x_i^1 < x_j^1 < 1 - (1-x_i^1)r \wedge x_i^2 > x_j^1\right)\right] p - P\left(x_i^1 < x_j^1 < 1 - (1-x_i^1)r\right) c$$

$$= \left[\int_0^{1-2r^2} F_j\left(x_i^1|x_j^1\right) f\left(x_j^1\right) \,\mathrm{d}x_j^1 + F\left(x_i^1\right) - F(1-2r^2) \right.$$

$$\left. + \int_{x_i^1}^{1-(1-x_i^1)r} \left(1 - F_i\left(x_j^1|x_i^1\right)\right) f\left(x_j^1\right) \,\mathrm{d}x_j^1 \right] p - \left[F\left(1-(1-x_i^1)r\right) - F\left(x_i^1\right)\right] c$$

It is not profitable to hide the value iff

$$\left[1-\frac{1-x_i^1}{r}+\int_{1-\frac{1-x_i^1}{r}}^{x_i^1}\frac{x_i^1-x_j^1}{1-x_j^1}\,\mathrm{d}x_j^1+\int_{1-\frac{1-x_i^1}{2r}}^{x_i^1}\frac{1}{2}\cdot\frac{1-x_i^1}{1-x_j^1}\,\mathrm{d}x_j^1-\int_0^{1-2r^2}\frac{x_i^1-x_j^1}{1-x_j^1}\,\mathrm{d}x_j^1\right.$$
$$\left.-\int_{x_i^1}^{1-(1-x_i^1)2r}\frac{1}{2}\cdot\frac{1-x_j^1}{1-x_i^1}\,\mathrm{d}x_j^1-F\left(x_i^1\right)+F(1-2r^2)\right]p-\left[F\left(x_i^1\right)-F\left(1-\frac{1-x_i^1}{2r}\right)\right]c$$
$$\geq 0$$

Again, we calculate the integrals and simplify, to finally get the condition

$$\frac{3}{2}\ln(2r)+(1-2r)\left(\frac{3}{4}+\frac{1}{2}r\right)\leq 0.$$

It is fulfilled for $r=\frac{1}{2}$ and increasing on $(0,\frac{1}{2})$, as we can confirm by looking at the first derivative:

$$\frac{3}{2r}-1-2r\geq 0 \iff \frac{3}{2}-r-2r^2\geq 0,$$

which is true for $0<r<\frac{1}{2}$. The proof is thus complete – it is never profitable for one of the firms to deviate and hide the own value. □

Proof of Theorem 1.17

We show that firm i with value x_i^1 has no incentive to hide her value if all other firms reveal. This is easy to see in case $x_i^1>x^*$: revealing the value will make all opponents stop innovating. In case $x_i^1<\hat{x}$, there is no point in hiding – no other firm will be discouraged from drawing if her beliefs of firm i increase to $\hat{x}$ compared to x_i^1. Quite the contrary, for some values it could make a leading firm j with $\hat{x}>x_j^1>x_i^1$ continue to innovate although she would have stopped if she knew the true value of firm i.

Let us now suppose $\hat{x}\leq x_i^1\leq x^*$, the remaining case to show. If firm i is not leading, it does not make a difference whether she reveals or not as her value has no influence on the innovation behavior of the other firms. We thus have to check what happens if firm i is in the lead. First note that by Proposition 1.16 she will not continue to innovate then. If she does not hide her value, she thus wins the contest in case no other firm draws or all drawing firms have a lower value. Hence, her winning probability π can be written as

$$\pi=P\left(\max_{j\neq i}x_j^2<x_i^1\right)=\sum_{l=0}^{n-1}\binom{n-1}{l}\left(1-q\left(x_i^1\right)\right)^{n-1-l}\cdot q\left(x_i^1\right)^l\cdot F\left(x_i^1\right)^l.$$

If the leading firm i hides her value, the other firms believe she has value $\hat{x}$. If the second highest first-period innovation value is lower than $\hat{x}$, and firm i is thus still believed to be the leading firm, all other firms continue to innovate for sure. If it is higher, drawing behavior depends on its exact value. The winning probability of a hiding firm i, π_h, is given by

$$\begin{aligned}\pi_h &= P\left(\max_{j\neq i} x_j^2 < x_i^1\right)\\ &= F\left(\hat{x}\right)^{n-1} F\left(x_i^1\right)^{n-1}\\ &\quad + P\left(\hat{x} < \max_{j\neq i} x_j^1 < x_i^1\right)\sum_{l=0}^{n-2}\binom{n-2}{l}\left(1-q\left(\max_{j\neq i} x_j^1\right)\right)^{n-2-l} q\left(\max_{j\neq i} x_j^1\right)^l F\left(x_i^1\right)^l.\end{aligned}$$

We need to show that $\pi > \pi_h$. To do this, first note that

$$\begin{aligned}P\left(\hat{x} < \max_{j\neq i} x_j^1 < x_i^1\right) &< P\left(\hat{x} < \max_{j\neq i} x_j^1 < x^*\right)\\ &= F(x^*)^{n-1} - F(\hat{x})^{n-1}\\ &= (1-r)^{n-1} - (1-(n-1)r).\end{aligned}$$

Furthermore, we have $q\left(\max_{j\neq i} x_j^1\right) \geq q\left(x_i^1\right)$. If the same number of firms draws more often (with higher probability), this reduces the winning probability of the leading firm. Hence,

$$\begin{aligned}&\sum_{l=0}^{n-2}\binom{n-2}{l}\left(1-q\left(\max_{j\neq i} x_j^1\right)\right)^{n-2-l} q\left(\max_{j\neq i} x_j^1\right)^l F\left(x_i^1\right)^l\\ \leq &\sum_{l=0}^{n-2}\binom{n-2}{l}\left(1-q\left(x_i^1\right)\right)^{n-2-l} q\left(x_i^1\right)^l F\left(x_i^1\right)^l\end{aligned}$$

and we can conclude that

$$\begin{aligned}\pi_h &\leq (1-(n-1)r)F\left(x_i^1\right)^{n-1}\\ &\quad + \left((1-r)^{n-1} - (1-(n-1)r)\right)\sum_{l=0}^{n-2}\binom{n-2}{l}\left(1-q\left(x_i^1\right)\right)^{n-2-l} q\left(x_i^1\right)^l F\left(x_i^1\right)^l.\end{aligned}$$

Hence, to get $\pi > \pi_h$ it is sufficient to show

$$\begin{aligned}&\sum_{l=0}^{n-1}\binom{n-1}{l}\left(1-q\left(x_i^1\right)\right)^{n-1-l}\cdot q\left(x_i^1\right)^l\cdot F\left(x_i^1\right)^l - \Big[(1-(n-1)r)F\left(x_i^1\right)^{n-1}\\ &\quad + \left((1-r)^{n-1} - (1-(n-1)r)\right)\sum_{l=0}^{n-2}\binom{n-2}{l}\left(1-q\left(x_i^1\right)\right)^{n-2-l} q\left(x_i^1\right)^l F\left(x_i^1\right)^l\Big] > 0.\end{aligned} \tag{1.21}$$

We prove this statement by an induction argument, where we keep $F\left(x_i^1\right)$ fixed and take $q = q\left(x_i^1\right) \in [0,1]$ as variable. This approach does not use all available information, as it ignores the dependence of $q\left(x_i^1\right)$ and $F\left(x_i^1\right)$, but it is sufficient for our purposes.

We start with the basis, $n = 3$. For $q = 1$ the left-hand side of (1.21) boils down to

$$2rF\left(x_i^1\right)^2 - r^2F\left(x_i^1\right) > 0 \iff 2F\left(x_i^1\right) > r. \tag{1.22}$$

Note that the range of possible x_i^1 values in $[\hat{x}, x^*]$ depends on n and r. We thus need to make sure that the basis holds for all these combinations: (1.22) is true for $r < \frac{1}{n}$ and

$$F\left(x_i^1\right) > \sqrt[n-1]{1-(n-1)\frac{1}{n}} = \sqrt[n-1]{\frac{1}{n}} > \frac{1}{n} > r.$$

For $q \in [0, 1)$, we show that the left-hand side of (1.21) is monotone in q by looking at its first derivative with respect to q, which is given by

$$\begin{aligned}
&-2(1-q) + r^2 + F\left(x_i^1\right)\left(2 - 4q - r^2\right) + 2qF\left(x_i^1\right)^2 \\
&\quad = \left(2 - r^2 - 2q\left(1 - F\left(x_i^1\right)\right)\right)\left(F\left(x_i^1\right) - 1\right) \\
&\quad < 0.
\end{aligned}$$

The last step holds as $2 - r^2 - 2q\left(1 - F\left(x_i^1\right)\right) > 2F\left(x_i^1\right) - r^2 > 0$, which we already showed above. Hence, the left-hand side of (1.21) is decreasing in q, and as it is positive for $q = 1$, it is positive on the whole range.

We now get to the inductive step. Suppose we know that (1.21) is true for n firms. By multiplying with $1 - q + qF\left(x_i^1\right)$ (this is the change in π when adding another firm), (1.21) is equivalent to

$$\begin{aligned}
&\Bigg\{\sum_{l=0}^{n-1}\binom{n-1}{l}\left(1-q\left(x_i^1\right)\right)^{n-1-l}\cdot q\left(x_i^1\right)^l \cdot F\left(x_i^1\right)^l - \Big[(1-(n-1)r)F\left(x_i^1\right)^{n-1} \\
&\qquad + \left((1-r)^{n-1} - (1-(n-1)r)\right)\sum_{l=0}^{n-2}\binom{n-2}{l}\left(1-q\left(x_i^1\right)\right)^{n-2-l} q\left(x_i^1\right)^l F\left(x_i^1\right)^l\Big]\Bigg\} \\
&\qquad \cdot \left(1 - q + qF\left(x_i^1\right)\right) \\
&> 0 \\
\iff &\Bigg\{\sum_{l=0}^{n}\binom{n}{l}\left(1-q\left(x_i^1\right)\right)^{n-l}\cdot q\left(x_i^1\right)^l \cdot F\left(x_i^1\right)^l \\
&\qquad - \Big[(1-(n-1)r)F\left(x_i^1\right)^{n-1}\cdot\left(1-q+qF\left(x_i^1\right)\right) \\
&\qquad + \left((1-r)^{n-1} - (1-(n-1)r)\right)\sum_{l=0}^{n-1}\binom{n-1}{l}\left(1-q\left(x_i^1\right)\right)^{n-1-l} q\left(x_i^1\right)^l F\left(x_i^1\right)^l\Big]\Bigg\} \\
&> 0
\end{aligned}$$

Next, we subtract the left-hand side of this equation from the left-hand side of (1.21) with

$n+1$ firms, which amounts to

$$
\begin{aligned}
&(1-(n-1)r)F\left(x_i^1\right)^{n-1}\cdot\left(1-q+qF\left(x_i^1\right)\right)-(1-nr)F\left(x_i^1\right)^n\\
&\qquad-\left((1-r)^n-(1-nr)-(1-r)^{n-1}+(1-(n-1)r)\right)\\
&\qquad\cdot\sum_{l=0}^{n-1}\binom{n-1}{l}\left(1-q\left(x_i^1\right)\right)^{n-1-l}q\left(x_i^1\right)^l F\left(x_i^1\right)^l\\
&=(1-(n-1)r)F\left(x_i^1\right)^{n-1}\left(1-q+qF\left(x_i^1\right)-F\left(x_i^1\right)\right)+rF\left(x_i^1\right)^n\\
&\qquad-\left((1-r)^{n-1}(1-r-1)-r\right)\cdot\sum_{l=0}^{n-1}\binom{n-1}{l}\left(1-q\left(x_i^1\right)\right)^{n-1-l}q\left(x_i^1\right)^l F\left(x_i^1\right)^l\\
&=(1-(n-1)r)F\left(x_i^1\right)^{n-1}\left((1-q)\left(1-F\left(x_i^1\right)\right)\right)+rF\left(x_i^1\right)^n\\
&\qquad+\left((1-r)^{n-1}r+r\right)\cdot\sum_{l=0}^{n-1}\binom{n-1}{l}\left(1-q\left(x_i^1\right)\right)^{n-1-l}q\left(x_i^1\right)^l F\left(x_i^1\right)^l\\
&>0
\end{aligned}
$$

As the difference is positive, we showed that (1.21) is fulfilled for $n+1$ as well. This completes the inductive step and the proof. □

Chapter 2

Ascending Combinatorial Scoring Auctions

2.1 Introduction

In a procurement auction, the buyer is usually not only interested in getting an object as cheap as possible, but also cares about its quality. Scoring auctions provide the opportunity to submit bids that specify prices and quality attribute levels. These bids are evaluated with the help of a scoring rule (a function of quality attributes and price) and ranked according to the resulting scores. If the bidders know the scoring rule, this procedure resembles a classical auction with bids being scores. This simple relationship can get lost if the buyer wants to acquire multiple objects: his perception of an object's quality may heavily depend on the quality attributes of the other objects. In this chapter, the scoring rule can be an arbitrary increasing function of all quality attributes (but quasilinear in price). Especially, the overall quality may depend on the attribute levels of *all* items in a non-trivial way. Consequently, the score that a supplier is able to generate with his bid may depend on the bids of the other suppliers. Such an interdependency of bids does not appear in price-only auctions[1]. Suyama and Yokoo (2004) have shown that the presence of such quality interdependencies in the scoring rule is not innocuous: the Vickrey-Clarke-Groves mechanism may fail to guarantee a nonnegative payoff to the buyer. We analyze the properties of a different mechanism in the presence of an interdependent scoring rule: the Ausubel-Milgrom ascending proxy auction. In standard auctions, this mechanism does not suffer from certain weaknesses of the Vickrey-Clarke-Groves mechanism, e.g. regarding collusion. It is thus a suitable candidate to work well with an interdependent scoring

[1]Particularly, this model cannot be embedded in the standard package auction setting by treating each specific attribute configuration of an item as a distinct object and imposing additional restrictions on possible combinations of objects the buyer is allowed to acquire: it is necessary to have some kind of scoring rule to account for the differing quality of the objects.

rule.

To illustrate the role of the interdependency, think of a quality attribute like delivery time: the buyer may need several objects simultaneously. He thus only values a fast delivery time of one object if the other objects are delivered quickly as well – otherwise, the speed advantage of one supplier is worthless. If these preferences of the buyer are reflected in the scoring rule, it is difficult for the suppliers to estimate the impact of their bid on the overall quality in advance – it depends crucially on the bids of the other suppliers. Hence, a single bidder can be very influential, e.g. if he is the only one who can deliver a particular item very quickly.

Other problems arise if the buyer does not want to give out information on its scoring rule to the sellers, e.g. because he tries to avoid information spillovers to his competitors. This could be information about his preferences toward different suppliers which are reflected in the scoring rule, or information about the quality of an object he is able to produce out of the items he wants to buy in this scoring auction. Due to such reasons, the auctioneer may want to keep his scoring rule secret. In particular, we have an example like the following setting in mind: a manufacturing firm is facing two procurement situations. On the one hand, it wants to be the seller of a specific product, and has a competitor who is able to deliver a similar product. Revealing information about the firm's production abilities to the competitor would have a negative impact on the firm's revenue, because the other firm can profitably use this information in its pricing process. On the other hand, the firm wants to acquire the components to manufacture the product by means of an auction. Using a public scoring rule in this auction provides the competitor with an informative signal about the firm's production abilities. If the firm wants to avoid these signals, is it possible to adapt the Ausubel-Milgrom proxy auction to deal with secret scoring rules as well?

Our version of the Ausubel-Milgrom proxy auction works as follows: for each possible quality configuration for each package, a seller submits a minimum price at which he is willing to deliver. This can be interpreted as the seller's cost structure. With the help of this cost structure, the proxy bidder submits bids automatically on behalf of the seller. The proxy follows a simple bidding strategy: it bids on all possible quality configurations yielding the highest potential profits (with respect to the reported cost structure). Bidding is stopped in case this potential profit gets negative.

We show with direct proofs that main theorems for the Ausubel-Milgrom proxy auction extend to this mechanism. This includes, with respect to the reported preferences, surplus maximization and the core property for the final winning allocation, as well as existence of equilibria in profit-target strategies. The scoring rule can be kept secret without influencing the outcome. Particularly, for a specific class of scoring rules truthful bidding is an equilibrium strategy, making bidding behavior easy in case bidders know that the scoring rule is in this class.

Furthermore, we consider the special case of an additive scoring rule. A scoring rule is called additive if a score can be calculated for each item separately, and these scores are added up to generate the score for a package. Here, the auction procedure can stay essentially the same compared to the original price-only proxy auction, in case the scoring rule is public: each bidder calculates the maximum score he is able to generate for each single item and submits these scores (not necessarily truthfully) to the proxy. Similar to the price-only proxy auction, the proxy then bids myopically on packages of items. Consequently, results stay the same compared to the price-only proxy auction. We extend the bidding procedure to secret scoring rules by using price-quality bids. Although the scoring rule is not known, the outcome of the auction with public scoring rule is replicable with this bidding procedure. Particularly, this enables us to directly carry over some theory on the Ausubel-Milgrom proxy auction to secret scoring rules.

The literature on scoring auctions is surprisingly scarce[2], if one thinks of the variety of procurement settings where price *and* quality matter. There is a first strand of literature looking at optimal scoring auctions by adapting the scoring rule (Che 1993; Branco 1997; David et al. 2002a). Contrary to this approach, the scoring rule is fixed in our environment – we assume that the decision on the scoring rule has already been made.

Mueller et al. (2007) generalize Asker and Cantillon (2008) to combinatorial auctions: they show that the set of equilibria can be transferred from multi-dimensional price-only auctions to the corresponding scoring auctions. Mueller et al. (2007) use scoring rules for every possible package, which are not necessarily the sum of the scoring rules for the single items. The winning allocation is then determined by an allocation rule over scores. The additive scoring rule we use in part of this chapter is a special case of their setting. Our *general* scoring rule differs from their approach, as it allows for interdependencies of quality attributes for different items across bidders – there is just one single scoring rule for all objects. Such an interdependent scoring rule can also be found in Suyama and Yokoo (2004) and (2005) in the context of Vickrey-Clarke-Groves mechanisms. Which type of scoring rule one wants to use is a question of the context. Furthermore, mixed forms are possible as well.

It is well known that in some settings Vickrey-Clarke-Groves mechanisms do not achieve budget balancedness. In the context of general scoring rules, Suyama and Yokoo (2004) point out that a Vickrey-Clarke-Groves mechanism does not guarantee a nonnegative payoff for the buyer. They give a more detailed characterization in Suyama and Yokoo (2005). Even if the rule is publicly known, a score for bids on packages cannot be calculated – it depends on the bids submitted by the other bidders. Imagine a scoring rule that treats two quality attributes of different items as perfect complements. The score thus only increases if a bid raise is made on the attribute levels for *both* items. Hence, if this

[2] A good survey can be found in Strecker (2004).

raise is due to two different suppliers in the winning allocation, both suppliers have to be paid for it in the VCG mechanism – the buyer has to pay the raise *twice*. This may make the outcome too expensive for him, generating a negative final score – he would have preferred not to conduct the auction at all. We thus have to be careful when transferring results to scoring auctions with general scoring rules.

Secret scoring rules in the context of single-unit English auctions go back to David et al. (2002b). In their setting only monotonicity properties of the scoring rule with respect to the attribute values are announced to the sellers. Bids consist of price-quality combinations. As bidders do not know the scoring rule, submitted bids may be rejected if the score they generate is lower than the standing high bid. The auction ends when no sufficiently high bids are submitted in a prespecified period of time or all bidders stop their bidding activity. The set of possible attribute levels is assumed to be finite.

David et al. (2002b) show that in this setting, it is a dominant strategy for the suppliers to follow a *bid list strategy*: each bidder ranks the possible bids (there are only finitely many due to the finite attribute space) according to his own preferences (potential profit). Then, he submits his bids in order of decreasing profit. In case he is standing high bidder he suspends the submittance of bids. Otherwise, he submits bids until all bids on his list were submitted. We extend their approach to the multi-object case: a proxy bidder takes the role of submitting the multi-object counterpart of the bid list, a ranking of possible bids according to their potential profit. This is the extension of the Ausubel-Milgrom proxy auction to secret scoring rules.

Finally, Rezende (2009) makes use of *bias functions* instead of scoring rules to account for quality differences. In his setting, the release of information is always optimal. The ascending proxy auction that we use is introduced and discussed by Ausubel and Milgrom (2002). A final discount stage is added by Lamy (2007), ensuring that truthful bidding leads to a bidder-optimal point in the core. Ranger (2005) extends the proxy auction to a setting with externalities.

The chapter is organized as follows: first, we introduce the general framework in Section 2.2. The results on the proxy auction with a general scoring rule are developed in Section 2.3. Finally, we discuss the properties of secret scoring rules in the general context and in the presence of an additive scoring rule in Section 2.4, before we conclude in Section 2.5. Proofs can be found in Appendix 2.A.

2.2 The Model

We consider the set N of n suppliers in a procurement auction. They bid to provide all or some of the m indivisible goods in the set G. Each good $j \in G$ is specified by r_j quality attributes. The attributes may take different real-valued attribute levels. For a realization of these levels of good j, we denote the attribute vector by $q_j \in Q_j \subset \mathbb{R}^{r_j}$. Let $r = \sum_{j=1}^{m} r_j$ be the total number of attributes of all goods in G and $q = (q_1, q_2, \ldots, q_m) \in \bigtimes_{j=1}^{m} Q_j \subset \mathbb{R}^r$ the total attribute vector. Additionally, each bid on a good j specifies a price $p_j \in \mathbb{R}_+$. Let $p = \sum_{j=1}^{m} p_j$ be the total price of all goods. The set P of all possible prices and Q_j are assumed to be discrete and finite[3]. For a subset $G_j \subset G$, we denote the vector of attributes of the goods in G_j by q_{G_j}. r_{G_j} and p_{G_j} are defined analogously. The buyer has a valuation function v for the set G of goods with a quality vector q. We assume quasilinear utility for the buyer: if the set G is bought for a price p, the buyer has a total utility of $v(q) - p$.

In a *multi-object scoring auction*, the sellers submit bids (p_{G_j}, q_{G_j}) for sets G_j of goods. The evaluation of the bids is done on the basis of *scores* which are calculated by a scoring rule that does not necessarily need to match the buyers true valuation v.

We consider a quasilinear *scoring rule* $S : \mathbb{R}^{r+1} \to \mathbb{R}$, which takes the form

$$S\left(p, (q_1, \ldots, q_m)\right) = \Phi(q_1, \ldots, q_m) - p, \tag{2.1}$$

with an overall price of $p = \sum_{i=1}^{m} p_i$ and increasing in the vector of quality attributes $(q_1, \ldots, q_m)$. $\Phi(q)$ represents the quality level that is achieved by the attribute vector q. It is also the maximum payment the buyer is willing to make for q.

We denote the set of all possible allocations by $\mathcal{H}$. A (winning) allocation $q_H \in \mathcal{H}$ specifies a tuple $H = (H_1, \ldots, H_n)$ where every bidder i gets assigned a subset $H_i \subseteq G$ such that $\bigcup_{i=1}^{n} H_i \subseteq G$ and $H_i \cap H_j = \emptyset$ for all $i \neq j$. These are the items each supplier has to deliver. The set H_i may be empty. Additionally, q_H fixes the attribute levels for the items that each supplier got assigned. Our model covers the case $\bigcup_{i=1}^{n} H_i \subsetneq G$ where some items may remain unassigned. Then, when calculating the quality level, the attribute levels of the absent items take default attribute values, which can be specified by the buyer[4]. The winning allocation is the one that maximizes the overall score according to S with respect to the submitted bids. Note that we do not select a specific tie breaking rule; any rule will do for our purposes.

In our model, the suppliers differ with respect to their cost structure, which are private values. In general, cost functions are specified for each package. Thus, bidder i has a cost function $c_i\left(G_j, q_{G_j}\right)$ to produce quality q_{G_j} for a package G_j. We write in short $c_i\left(q_{G_j}\right)$

[3]This assumption is not very restrictive, as we will use an ascending auction procedure in the following, where it is common to use discrete bid increments. Furthermore, there are usually technical restrictions on possible attribute levels, and the auctioneer has a maximum price he is willing to pay. All in all, the attribute space is allowed to become very large, such that any realistic bid can be included.

[4]For example, if one item is optional from the buyer's perspective and bidders are offering this item only for very high prices, the buyer prefers to go without the item. This can e.g. be reflected by using attribute levels of zero (depending on the functional form of the scoring rule). Similarly, if one item is essential for the buyer, default attribute levels for the item in case of not getting it will lead to low or zero quality levels.

for this. The costs are assumed to be strictly increasing in each quality attribute, and each q_{G_j} may be delivered for some finite price. Furthermore, let $c_i(q_\emptyset) \equiv 0$.
The social surplus with respect to the scoring rule $W(q_H)$ that an allocation q_H achieves can thus be denoted as follows:

$$W(q_H) := \Phi(q_H) - \sum_{i \in N} c_i(q_{H_i}).$$

We illustrate this model by the following example:

Example 2.1 Consider three bidders (1, 2 and 3) and two objects (A and B). There is one quality attribute for each object, the negative[5] delivery time, represented by q_A and q_B, respectively. The buyer uses the scoring rule

$$S(p_A, p_B, q_A, q_B) := \underbrace{15 + \min\{q_A, q_B\}}_{=\Phi(q_A, q_B)} - p_A - p_B$$

We assume that bidders are able to produce the objects in either one or five days or not at all, according to the costs given in Table 2.1.

	Object A		Object B		Objects A and B			
Quality	-1	-5	-1	-5	$(-1,-1)$	$(-1,-5)$	$(-5,-1)$	$(-5,-5)$
Costs bidder 1	<u>5</u>	4	10	9	20	16	16	12
Costs bidder 2	10	9	<u>4</u>	3	19	15	15	13
Costs bidder 3	11	5	11	5	30	20	20	14

Table 2.1: Costs of the bidders.

The underlined costs mark the efficient allocation: bidder 1 delivers item A and bidder 2 delivers item B, both in time 1. We now look at a Vickrey-Clarke-Groves mechanism to determine the payments. Denote the winning allocation by q_H^* (maximizing $W(q_H)$) and the winning allocation if bidder i were not present by $q_{H_{-i}}^*$. Bidder i gets paid p_i according to the VCG payment rule:

$$p_i := W(q_H^*) - W\left(q_{H_{-i}}^*\right) + c_i\left(q_{H_i}^*\right)$$

Thus, each winning bidder gets paid for the surplus that he generates by his presence plus his costs. Bidder 1 has costs of 5 and generates a score of $((15-1)-(4+5)) - ((15-5)-(3+5)) = 3$ (if bidder 1 were not present, bidder 3 would deliver item A instead and delivery time would go up to 5). Hence, he gets paid $p_1 = 8$. Similarly, bidder 2 gets paid 8. With these payments we get a total score of $(15-1)-(8+8) = -2 < 0$. Consequently,

[5] We take the negative time to make the score increasing in quality.

the buyer would prefer not to buy the items at these payments – the VCG mechanism does not guarantee a nonnegative payoff for the buyer.[6]

2.3 Ascending Proxy Scoring Auctions

The auction format we use in this chapter is a generalization of the Ausubel-Milgrom ascending (proxy) package auction. We first describe how the ascending package auction extends to the scoring auction environment and then introduce the proxy bidder.

The auctioneer (buyer) publicly announces the items he wants to buy and the items' corresponding possible attributes and their levels. He decides on a scoring rule[7] S to evaluate the bids. In each round, a seller i bids according to the following general structure: first, he selects the packages of items he wants to bid on and decides for each package G_j which quality attribute levels q_{G_j} he wants to offer – he may offer different combinations of attribute levels for each package. For each of these offers, the bidder specifies a price bid $\beta_i\left(q_{G_j}\right)$ at which he is willing to sell. Then, he submits all of these bids $\left(\beta_i\left(q_{G_j}\right), q_{G_j}\right)$ simultaneously. Bids are treated as mutually exclusive – for each bidder, at most one bid will be selected by the auctioneer for the standing high bids, the winning allocation of each round. The auctioneer may also include bids of previous rounds in the standing high bids (e.g. of bidders that already stopped bidding).

For the first bidding round, the auctioneer specifies a maximum price $\bar{p}$ the sellers may ask for. For the following rounds, sellers have to lower their previous price offer according to the finite set of possible prices P on a particular package and attribute level configuration. We denote the corresponding maximum bid price by $m_i\left(q_{G_j}\right)$. A single bid is rejected if the maximum bid rule is not met, all bids are rejected if the resulting standing high bids would yield a negative score. Rejected bids are treated as a zero bid. Bidding ends if no new bids are submitted or all submitted bids violate the maximum bid rule.

In the proxy auction, the bidding process is automated with the help of a *proxy bidder*. It uses the following strategy, where asking for a price of ∞ corresponds to submitting no bid on this attribute level configuration:

Definition 2.2 The bidding strategy

$$\forall q_{G_j}: \quad \beta_i\left(q_{G_j}\right) := \begin{cases} m_i(q_{G_j}) & \text{if } q_{G_j} \in \arg\max_{q'_{G_j}} \left[m_i\left(q'_{G_j}\right) - c_i\left(q'_{G_j}\right)\right] \\ \infty & \text{if } q_{G_j} \notin \arg\max_{q'_{G_j}} \left[m_i\left(q'_{G_j}\right) - c_i\left(q'_{G_j}\right)\right] \end{cases}$$

is called *straightforward bidding strategy.* New bids are only submitted in case bidder i is not one of the standing high bidders.

[6] See Suyama and Yokoo (2005) for a more detailed discussion.

[7] The analysis of the decision process is not part of this chapter – any decision process is fine for our purposes.

Bidding stops in case $\arg\max_{q'_{G_j}} \left[m_i\left(q'_{G_j}\right) - c_i\left(q'_{G_j}\right)\right] < 0$.

According to this strategy, the bidder places the maximum bid on all attribute level configurations that yield the highest potential profit. Note that the straightforward bidding strategy does not depend on the scoring rule used by the auctioneer, but only on the cost structure of each bidder. Thus, using the straightforward bidding strategy is similar to sorting all bids according to their potential profit into a bid list and submitting one after the other (and all bids with the same profit at the same time).
In the proxy auction, each seller reports a cost structure c_i (not necessarily truthfully) to the proxy bidder. Then, the proxy submits bids on behalf of the seller following the straightforward bidding strategy with respect to the reported cost structure.

Example 2.3 We use the setting of Example 2.1 and apply the proxy scoring auction. Bidders costs are given by Table 2.1. The maximum starting price has to be chosen high enough – a price of 15 will do for our purposes. Bidders start by bidding myopically on the attribute level configuration with the lowest production cost, yielding the highest possible profit. Table 2.2 shows the first set of bids submitted by the proxy bidders. We assume that the bidders reported their costs truthfully to the proxy bidder.

	Object A		Object B		Objects A and B			
Quality	-1	-5	-1	-5	$(-1,-1)$	$(-1,-5)$	$(-5,-1)$	$(-5,-5)$
Bids bidder 1	–	15	–	–	–	–	–	–
Bids bidder 2	–	–	–	15	–	–	–	–
Bids bidder 3	–	15	–	15	–	–	–	–

Table 2.2: First round bids.

As no combination of these bids generates a positive payoff, all bids are rejected by the buyer. The proxies submit a new set of bids, uniformly lowering the potential profit – we assume a bid increment of 1 here. Table 2.3 shows the second set of bids.

	Object A		Object B		Objects A and B			
Quality	-1	-5	-1	-5	$(-1,-1)$	$(-1,-5)$	$(-5,-1)$	$(-5,-5)$
Bids bidder 1	15	14	–	–	–	–	–	–
Bids bidder 2	–	–	15	14	–	–	–	–
Bids bidder 3	–	14	–	14	–	–	–	–

Table 2.3: Second round bids.

Again, bids are rejected because no positive score is generated by the submitted bids. In the following rounds, the proxies will continue to lower the potential profit until a nonnegative score is generated by the submitted bids. This is the case for the bids in Table 2.4.

	Object A		Object B		Objects A and B			
Quality	-1	-5	-1	-5	$(-1,-1)$	$(-1,-5)$	$(-5,-1)$	$(-5,-5)$
Bids bidder 1	<u>7</u>	6	12	11	–	–	–	14
Bids bidder 2	13	12	<u>7</u>	6	–	–	–	–
Bids bidder 3	12	6	12	6	–	–	–	15

Table 2.4: Bids leading to a nonnegative score.

The winning bids are the underlined bids in Table 2.4 – bidder 3 will submit one more set of bids with a potential profit of zero, but these bids are not high enough to outbid the other two. The winning allocation is the same as in Example 2.1, but prices are lower – the buyer gets a nonnegative payoff.

For the following analysis, similar to Ausubel and Milgrom (2002), we assume that bid increments are negligibly small, such that we have a continuous price range. We think of bidding rounds as taking place at times $t \geq 0$.
To derive our main results, we need to make sure that all allocations that may theoretically win the auction are included in the bidding process. Particularly, bidding does not stop before e.g. bidder i starts submitting a bid on an attribute level configuration that is complementary to the others and would yield a higher score. This is established in the following lemma.

Lemma 2.4 *Consider any possible set of cost structures and any allocation q_H that possibly generates a positive score. Then, a sufficiently high starting price $\bar{p}$ exists, fulfilling the following: in each bidding round that yields a nonnegative score, every bidder who delivers one or more items in this allocation q_H submits a bid on his respective attribute level configuration.*

Proof See Appendix 2.A. □

In other words, the proxy starts bidding with a very high price, such that bids get rejected in the beginning of the auction. As soon as the score gets positive, and bids are not rejected any more, all allocations that are theoretically able to win the auction may be chosen by the auctioneer.
To analyze the properties of the proxy scoring auction, we take a look at the corresponding game in coalitional form. In particular, we want to show that the proxy auction leads to a core outcome of this game.

First, we denote the set of all participants in the auction by $N_s = N \cup \{0\}$. The buyer is player 0. A coalition is any subset $N_c \subset N_s$. The set of allocations for a coalition N_c is the set where all items get assigned to sellers in $N_c \setminus \{0\}$, denoted by $\mathcal{H}_c := \{q_H \in \mathcal{H} | \forall i \notin N_c \setminus \{0\} : H_i = \emptyset\}$. The coalitional value function w represents the profit a coalition N_c can achieve by producing and trading all items only within its members. Note that only coalitions including the buyer can achieve positive profits as he is paying the bill for the delivered items. For such a coalition, an allocation in $\mathcal{H}_c$ is chosen and and w takes the following form:

$$\forall N_c \subset N_s : w(N_c) := \begin{cases} \max_{q_H \in \mathcal{H}_c} \left[\Phi(q_H) - \sum_{i \in N_c \setminus \{0\}} c_i(q_{H_i}) \right] & \text{if } 0 \in N_c \\ 0 & \text{if } 0 \notin N_c \end{cases}$$

A payoff vector $\pi = (\pi_0, \ldots, \pi_n)$ is called *feasible* if its aggregate payoff does not exceed the value achievable by the coalition of everyone. It is called *unblocked* if no coalition is able to improve the payoff of its members on its own. The *core* is the set of feasible and unblocked payoff vectors:

$$\text{Core}(N_s, w) := \left\{ \pi \left| \sum_{i \in N_s} \pi_i = w(N_s) \wedge \forall N_c \subset N_s : \sum_{i \in N_c} \pi_i \geq w(N_c) \right. \right\}$$

Let $\tilde{\pi}^t$ denote the intermediate payoff vector in the auction at time t. We can now derive the following connection between core and proxy scoring auction:

Theorem 2.5 *The surplus with respect to the scoring rule and the reported cost structures $\Phi(q_H^*) - \sum c_i(q_H^*) = w(N_s)$ is maximized in the final winning allocation q_H^* of a proxy scoring auction. The final payoff vector at time $\bar{t}$ is in the core, $\tilde{\pi}^{\bar{t}} \in Core(N_s, w)$.*

Proof See Appendix 2.A. □

The strategies of the sellers in the proxy scoring auction describe what kind of cost structures they submit to the proxy. One particular type of strategy is the π_i-*profit-target* or *semi-sincere* strategy. Such a strategy guarantees bidder i a profit of π_i in case he is one winner of the auction. The strategy can be realized by submitting a cost structure $\tilde{c}_i = c_i + \pi_i$. Let $\Pi_i(\tilde{c}_i, \tilde{c}_{-i})$ denote the profit bidder i makes in the proxy scoring auction if he reports $\tilde{c}_i$ and the others report $\tilde{c}_{-i}$. Generalizing the results of Ausubel and Milgrom (2002), we first show that there is always a best reply which is a profit-target strategy.

Theorem 2.6 *For any bidder i and any reports $\tilde{c}_{-i}$ to the proxy by the other bidders, let $\bar{\pi}_i = \max_{\tilde{c}_i} \Pi_i(\tilde{c}_i, \tilde{c}_{-i})$. Then the $\bar{\pi}_i$-profit-target strategy is a best reply for bidder i in the proxy scoring auction.*

Proof See Appendix 2.A. □

To characterize a set of equilibria of the proxy scoring auction, we need the following definition.

Definition 2.7 A payoff vector π is called *bidder-optimal* if $\pi \in \text{Core}(N_s, w)$ and there exists no $\pi' \in \text{Core}(N_s, w)$ with $\pi'_{-0} \geq \pi_{-0}$ and $\pi'_{-0} \neq \pi_{-0}$.

Bidder-optimal points in the core are associated with Nash equilibria of the proxy scoring auction:

Theorem 2.8 *Suppose that π is bidder-optimal. Then the corresponding π_i-profit-target-strategies constitute a Nash equilibrium of the proxy scoring auction. Conversely, the payoff vector in any Nash equilibrium in profit-target strategies at which losing bidders bid sincerely is bidder-optimal.*

Given Theorem 2.6, the proof of Theorem 2.8 is now identical to the proof of the corresponding theorem in Ausubel and Milgrom (2002) (Theorem 4).
In general, there may be several bidder-optimal points in the core, yielding several equilibria. A condition guaranteeing a unique bidder-optimal point in the core is *bidder-submodularity* of the coalitional value function:

Definition 2.9 A coalitional value function w is called *bidder-submodular*, if for any bidder i and all coalitions N_1, N_2 that include the seller, $N_1 \subset N_2$,

$$w(N_1 \cup \{i\}) - w(N_1) \geq w(N_2 \cup \{i\}) - w(N_2)$$

holds.

Bidder-submodularity of w also relates the outcome of the proxy scoring auction to the outcome of the VCG mechanism, π^V:

Theorem 2.10 *Suppose w is bidder-submodular. Then, the strategy profile where every bidder i reports c_i truthfully to the proxy bidder is an equilibrium of the proxy scoring auction. Its payoff vector π is the unique bidder-optimal point in Core(N_s, w), and $\pi_i = \pi_i^V = w(N_s) - w(N_s \setminus \{i\}) = \max\{\pi_i | \pi \in Core(N_s, w)\}$.*

Given Theorem 2.5, the proofs of the corresponding theorems in Ausubel and Milgrom (2002) (Theorem 8) or Milgrom (2004) (Theorem 8.11) apply.
The theorem provides a sufficient condition for the proxy scoring to work well: with bidder-submodularity of the coalitional value function, truthtelling is an equilibrium and it is thus easy for the sellers to follow this strategy. Additionally, the unique bidder-optimal core point with respect to the true valuations is reached. Under these circumstances, the proxy scoring auction works as well as the VCG mechanism, as it reaches the same payoff vector. Note, however, that the proxy scoring auction additionally always guarantees a nonnegative payoff for the buyer, which the VCG mechanism does not.

2.4 Secret Scoring Rules

We now turn to the question of how far the auctioneer is able to keep the scoring rule secret. First, we discuss whether the analysis of the general model in Section 2.3 can be extended to secret scoring rules. Then, we consider a specific type of scoring rules: additive scoring rules, where the total score can be calculated as the sum of the scores of the individual items. For this type of scoring rules, a general result connecting the Ausubel-Milgrom proxy auction and the proxy scoring auction can be derived.

2.4.1 General Scoring Rules

How does the proxy scoring auction work with a secret scoring rule? Note that the auction procedure did not specifically rely on the scoring rule being public. Without knowledge of the scoring rule, sellers submit a cost structure to the proxy. Its bidding behavior stays the same: the proxy submits the bids myopically in order of the respective *seller's* preferences – it does not need the scoring information to do so, but only the submitted cost structure. Then, bids get evaluated according to the scoring rule – this can be done by a proxy as well to ensure that the auctioneer sticks to the scoring rule and does not change it during the auction process. The submitted bids do not need to be publicly announced. The minimum information that is necessary to make the procedure work is to let every bidder know when he is standing high bidder. Announcing this publicly has no impact on the outcome of the auction. However, any information the auctioneer reveals contains information about his scoring rule. Consequently, the more concerned he is with keeping it secret, the less information on bids and their evaluation should be given out.
We start the theoretical analysis with the observation that Theorem 2.5 holds even with a secret scoring rule, as the bidding behavior of the proxy does not change.

Corollary 2.11 *The proxy scoring auction with a secret scoring rule reaches the same outcome as the proxy scoring auction with a public scoring rule. Particularly, the final winning allocation maximizes the surplus with respect to the scoring rule and the reported cost structures. The final payoff vector is in the corresponding core.*

What is the impact of a secret scoring rule on the equilibrium analysis in Section 2.3? Let us assume the scoring rule to be secret, but fixed: the sellers take the rule as given without knowing anything about it. Then, the equilibrium analysis in Theorem 2.6 and Theorem 2.8 is in principle still valid: as the outcome stays the same, best replies are still best replies, no matter whether the scoring rule is secret or not. However, the assumption of a secret, but fixed scoring rule is problematic in the sense that, in contrast to the case of a public scoring rule, sellers are in general not able to calculate best replies themselves. Even if a seller knows the reported cost structure of his opponents, he cannot conclude

his best reply without some knowledge of the scoring rule – although the best replies and equilibria we characterized in Theorem 2.6 and Theorem 2.8 still exist.
Theorem 2.10 shows a way out of these problems for some particular scoring rules. If the auctioneer is able to credibly announce that his scoring rule is such that the coalitional value function is bidder-submodular, bidders have the simple equilibrium strategy of reporting their cost structure truthfully. Such an announcements corresponds to changing the scoring rule from being *secret, but fixed* to being *secret, but fixed from a menu where each possible scoring rule makes the coalitional value function bidder-submodular.* Particularly, in the latter case there is the universal best reply of reporting the true cost structure, making the above mentioned problems with a secret scoring rule vanish.

Corollary 2.12 *If the scoring rule is secret in a proxy scoring auction, but bidders know that the corresponding coalitional value function is bidder-submodular, all bidders reporting their true cost structure is an equilibrium of the auction.*

However, an announcement of bidder-submodularity by the buyer may be difficult to make: the coalitional value function depends on the bidders' cost structures. If the auctioneer does not know these cost structures, a general characterization for bidder-submodularity would be needed, enabling the auctioneer to deduce bidder-submodularity using only the scoring rule and, if necessary, some regularity conditions on the cost structures.

2.4.2 Additive Scoring Rules

We now analyze the special case of an *additive* scoring rule. Then, the score for each item j can be calculated individually by

$$S_j(p_j, q_j) := \phi_j(q_j) - c_i(q_j).$$

The total score for a set G_j is then given according to

$$S_{G_j}(p_{G_j}, q_{G_j}) = \sum_{l \in G_j} S_l.$$

Suppose in a first step that the scoring rule is publicly known. Note that, in contrast to our previous analysis, an additive scoring rule enables the bidders to calculate the value of their bids in terms of the score they generate. The bidding procedure can be substantially simplified in this setting: bidders need to submit only one score for each package they bid on. To show this, think of a bidder i who is one of the winners of the auction. He has to deliver the package G_j, and the auctioneer expects to get a score of t_j^i on this package. We suppose that bidder i has the freedom to provide a score of t_j^i in any way he likes. Analogously to statements in Asker and Cantillon (2008) and Mueller et al. (2007) we can formulate the following lemma:

Lemma 2.13 *The optimal level of quality $q^*_{G_j} \in Q_{G_j}$ that a supplier i with cost function c_{G_j} produces for a package G_j is independent of the score t^i_j he has to fulfill.*

Proof See Appendix 2.A. □

In our general setting, bidders needed to differentiate their bids by submitting different attribute levels and configurations for each package. Lemma 2.13 shows that, with an additive scoring rule, the suppliers are not interested in submitting different attribute level configurations for the same package – they produce the same configuration in any case. This leads us to the following corollary:

Corollary 2.14 *Consider a multi-object scoring auction with a publicly known additive scoring rule. Bidders have no restriction on how to deliver the requested score. Then, there is no difference between bidders submitting bids of price-quality combinations or bids of scores: both mechanisms lead to the same outcome – the same quality is delivered and the same price is paid.*

Hence, it is sufficient to let bidders submit a single score for each package instead of a price-quality combination as long as they know the scoring rule. For this setting with an additive and public scoring rule we thus assume for the sequel that bids consist of scores. Furthermore, Lemma 2.13 shows that each bidder has a maximum of social surplus he can generate with respect to the scoring rule. This leads to the following definition.

Definition 2.15 Suppose that bidder i has a cost function c_{G_j} for each $G_j \subset G$. Then

$$k_{G_j} := \max_{q_{G_j}} \left[\phi_{G_j}(q_{G_j}) - c_{G_j}(q_{G_j})\right] \tag{2.2}$$

is the *pseudotype* of bidder i for the package G_j.

Alternatively, the pseudotype can be interpreted as the maximum score a seller is able to deliver without losing any money. Regarding this interpretation, the pseudotype is similar to the valuation in price-only auctions – there, the valuation is the maximum amount of money a buyer can pay without obtaining an object at a loss.
In this context, the proxy scoring auction works as follows: the sellers submit a pseudotype vector (the pseudotype for each package) to the proxy. Then, the proxy submits mutually exclusive bids of scores according to the straightforward bidding strategy – he bids myopically on all packages promising the highest profit. The auctioneer selects the standing high bidder by selecting the allocation that maximizes the sum of submitted scores. We can compare this bidding procedure with the original Ausubel-Milgrom proxy auction:

Remark 2.16 The proxy scoring auction with a public additive scoring rule can be interpreted as the original Ausubel-Milgrom proxy auction with bidders submitting pseudotypes as valuation vectors and proxies submitting scores as bids. Particularly, if bidders types in the scoring auction are their pseudotypes and distributed as the types in the original scoring auction, all theorems that hold for the original proxy auction hold for the additive proxy scoring auction as well (in their corresponding reformulations).

Note that this is a general statement on the transferability of results to the scoring auction environment. Not only Theorems 2.5, 2.6, 2.8 and 2.10 hold, but all other statements that are true for the original proxy auction have their counterpart for the additive proxy scoring auction. For a general scoring rule we do not have this kind of general transferability – each theorem has to be proven in the new environment, as we did in Section 2.3.
However, transferring results for the additive scoring auction can be a bit more complicated in an independent private values model with incomplete information: if bidders have multidimensional types that do not represent the pseudotypes (but can be reduced to get them), it is not directly obvious that the strategic bidding behavior of each participant is the same as in the price-only auction. Nevertheless, Mueller et al. (2007) show that under mild regularity assumptions the set of equilibria of a price-only auction and the corresponding scoring auction is basically the same.
We now turn to the analysis of secret scoring rules in the context of the additive proxy scoring auction. In this auction, bids are submitted as scores generated out of the pseudotypes vectors of the bidders. As bidders need to know the scoring rule to calculate their pseudotype, reducing bids to scores is not possible. Consequently, the bidding procedure needs to be transformed in the presence of a secret scoring rule. A suitable bidding procedure is the one used in Section 2.3: each bidder submits a cost structure to the proxy, which generates bids on all possible attribute level configurations. The additive scoring rule imposes enough structure to connect the two different bidding procedures:

Theorem 2.17 *Consider a winning bidder and the associated winning package in any round of a proxy scoring auction with a secret additive scoring rule. Then, among the bids on the different attribute levels of that package by the bidder, the buyer chooses the bid using the optimal quality attribute level from the seller's perspective (if he knew the scoring rule).*

Proof See Appendix 2.A. □

What does this theorem tell us? Consider a seller i who decides to bid according to a cost structure c_i'. Suppose he submits this cost structure to the proxy bidder, it bids accordingly and the seller is a standing high bidder in one of the bidding rounds. Then, the auctioneer will select the *optimal* attribute level in this standing high bid – the one the seller would have chosen (according to his submitted cost structure) in case he would have only been forced to deliver a particular package and score, and not a particular attribute level configuration. Thus, the outcome of the auction is the same in case the scoring rule is public and each seller i uses similarly c_i' to calculate his pseudotype and decide on the attribute levels he will deliver.

Note one particular difference: if seller i decides to choose c_i' such that the relative costs of attribute levels for a specific package are changed, his true optimal quality might differ from the optimal quality implied by c_i'. In case of a public scoring rule, he would have an ex post incentive to supply his true optimal attribute levels after being told the package and score he has to deliver. With a secret scoring rule, he is forced to deliver the attribute levels chosen by the auctioneer. However, this distortion cannot appear when for each possible package G_j there is a π_{G_j} such that $c_i'(q_{G_j}) = c_i(q_{G_j}) + \pi_{G_j}$ – each bidder uses his true relative costs for a package. This kind of bidding behavior is a best reply: suppose a seller would distort his costs for a specific package and win the auction on that package with a profit π'. Then, he always makes at least the same profit by not distorting and uniformly asking for a profit-target of π' on that package. In fact, he could possibly even raise the profit-target and still win the auction, as he is able to generate a (weakly) higher score on the same package if he does not distort. Consequently, we can conclude:

Corollary 2.18 *The outcome of the proxy scoring auction with a public additive scoring rule can be reproduced using a secret scoring rule.*

This is particularly interesting as bidding with public additive scoring rule relied on the bidders' knowledge of the scoring rule. Thus, we showed a way to transfer theory regarding the original Ausubel-Milgrom proxy auction to scoring auctions with secret additive scoring rule, using the public additive scoring rule as an intermediate step. Of course, the restrictions on the use of secret scoring rules as mentioned in Section 2.4.1 still apply.

2.5 Conclusion

We showed that the Ausubel-Milgrom proxy auction can be extended to a combinatorial scoring auction setting. It is able to replicate the desirable outcome of the Vickrey-Clarke-Groves mechanism in case the coalitional value function is bidder-submodular, but does not suffer of the problems with negative payoff for the buyer that may appear in the context of a general scoring rule when the VCG mechanism is used. Thus, the Ausubel-Milgrom proxy scoring auction is suitable for auctioneers who care about interdependencies of quality attributes.

Furthermore, we discussed the possibility of keeping the scoring rule secret: the outcome stays the same, best replies are still best replies, and if it is publicly known that the coalitional value function is bidder-submodular, sellers are sure to have the universal truthful equilibrium bid without further knowledge of the scoring rule. For an additive scoring rule we derived a close connection to the original Ausubel-Milgrom proxy auction.

2.A Appendix: Proofs

Proof of Lemma 2.4

Let

$$\bar{p} := 2 \cdot \max_q S(0,q),$$

which is twice the maximum possible score (if all sellers would give away their items for free). Hence, bids get rejected at least until the first submitted prices reach $\frac{\bar{p}}{2}$. Now suppose we are in any bidding round after this point and consider any allocation q_H that generates a positive score with respect to the reported cost structures (these are the allocations that may theoretically win the auction). In particular, this means that the individual costs $c_i(q_{H_i})$ of each seller in this allocation are below $\frac{\bar{p}}{2}$. As this seller has already submitted at least one bid (on some attribute level configuration) with a price lower than $\frac{\bar{p}}{2}$, the maximum profit he may obtain in this round is lower than $\frac{\bar{p}}{2}$. He can make at least the same profit by asking for a price of $\bar{p}$ on q_{H_i}, because $\bar{p} - c_i(q_{H_i}) \geq \bar{p} - \frac{\bar{p}}{2}$. Hence, according to the straightforward bidding strategy, the proxy places a bid on q_{H_i} in this round. □

Proof of Theorem 2.5

We first show that at any time t the provisional payoff vector is unblocked by any coalition. Among all submitted bids at[8] time t, the auctioneer selects the allocation $q_H^t \in \mathcal{H}$ that maximizes the score[9]:

$$q_H^t \in \arg\max_{q_H \in \mathcal{H}} S\left((\beta_1(t, q_{H_1}), q_{H_1}), \ldots, (\beta_n(t, q_{H_n}), q_{H_n})\right)$$

Now, we can rearrange this score:

$$\begin{aligned}
\tilde{\pi}_0^t &= \max_{q_H \in \mathcal{H}} S\left((c_1(q_{H_1}) + \tilde{\pi}_1^t, q_{H_1}), \ldots, (c_n(q_{H_n}) + \tilde{\pi}_n^t, q_{H_n})\right) \\
&= \max_{N_c \subset N_s} \max_{q_H \in \mathcal{H}_c} \left(\Phi(q_H) - \sum_{i \in N_c \setminus \{0\}} \left(c_i(q_{H_i}) + \tilde{\pi}_i^t\right) \right) \\
&= \max_{N_c \subset N_s} \left[\max_{q_H \in \mathcal{H}_c} \left(\Phi(q_H) - \sum_{i \in N_c \setminus \{0\}} c_i(q_{H_i}) \right) - \sum_{i \in N_c \setminus \{0\}} \tilde{\pi}_i^t \right]
\end{aligned}$$

[8]Similar to the original Ausubel-Milgrom proxy auction, all bids up to time t can be included in the optimization problem of the auctioneer. However, as the proxy simultaneously lowers the price on all possible quality levels, the auctioneer will always prefer the latest bid submitted.

[9]Note that the auctioneer pays an amount of zero to every not winning bidder although the notation suggests something different. For the ease of a simple notation we stick to it.

$$= \max_{N_c \subset N_s} \left[w(N_c) - \sum_{i \in N_c \setminus \{0\}} \tilde{\pi}_i^t \right] \tag{2.3}$$

The second equality holds as bidders not in coalition N_c receive a payment of 0 and do not deliver an item. Using equation (2.3) we can directly see that the payoff vector is unblocked:

$$\begin{aligned} \forall N_c \subset N_s : \qquad \tilde{\pi}_0^t \geq w(N_c) - \sum_{i \in N_c \setminus \{0\}} \tilde{\pi}_i^t \\ \iff \forall N_c \subset N_s : \ \sum_{i \in N_c} \tilde{\pi}_i^t \geq w(N_c) \end{aligned} \tag{2.4}$$

We still need to show that the final payoff vector is indeed feasible. Denote the set of bidders in the final winning coalition at time $\bar{t}$ by W. Then, we get as final payoff vector $\tilde{\pi}^{\bar{t}}$:

$$\tilde{\pi}_i^{\bar{t}} = \begin{cases} \beta_i(\bar{t}, q_{H_i}^*) - c_i(q_{H_i}^*) & \text{if } i \in W \\ \Phi(q_H^*) - \sum_{j \in W} \beta_j(\bar{t}, q_{H_j}^*) & \text{if } i = 0 \\ 0 & \text{if } i \notin W \cup \{0\} \end{cases}$$

This payoff vector yields

$$w(N_s) \overset{(2.4)}{\leq} \sum_{i \in N_s} \tilde{\pi}_i^{\bar{t}} = \Phi(q_H^*) - \sum_{i \in W} c_i(q_H^*) \leq \max_{q_H \in \mathcal{H}_c} \left(\Phi\left(q_H\right) - \sum_{i \in N_s \setminus \{0\}} c_i(q_{H_i}) \right) = w(N_s).$$

Hence, feasibility and maximization of surplus with respect to the scoring rule and the reported cost structures are established. □

Proof of Theorem 2.6

Suppose c_i' is a cost structure that yields a profit of $\bar{\pi}_i$ for bidder i if the others report $\tilde{c}_{-i}$. Denote the associated quality attribute allocation by q'. Then, bidder i sells q_i' for a price of $c_i(q_i') + \bar{\pi}_i$, as $\bar{\pi}_i$ is the profit that he makes with respect to his true cost structure. We can then slightly change this strategy without altering the outcome of the auction: let $c_i''(q_i) = c_i'(q_i) - c_i'(q_i') + c_i(q_i') + \bar{\pi}_i$. The report c_i'' shifts the report c_i' such that the winning quality allocation q' makes a profit of zero with respect to the new cost structure c_i''. Especially, bidding behavior by the proxy is not changed with this alteration of the reported cost structure: the cost minimal quality allocation $\hat{q}$ stays the same, $\hat{p}$ stays the same and relative reported costs stay the same as well. The shift can only affect the potential profit, which is not visible for the auctioneer. Thus, decisions by the auctioneer stay the same in every round, and the final decision q' will stay the final decision with a report of c_i'' as well – only that the internal profit in the calculations of the proxy for this allocation is reduced to 0.

Theorem 2.5 showed that q' is surplus maximizing with respect to the scoring rule and cost structures $(c_i'', \tilde{c}_{-i})$. Hence, surplus cannot be increased by choosing an allocation excluding i. Thus, keeping the reports of the others fixed, with any cost structure $\tilde{c}_i$ that specifies $\tilde{c}_i(q'_i) = c''_i(q'_i)$ a quality allocation of q' is feasible and bidder i will be included in the winning allocation (either q' or some other allocation including i).

Now note that the $\bar{\pi}_i$-profit-target-strategy specifies a bid of $c_i(q'_i) + \bar{\pi}_i = c''_i(q'_i)$ for q'. So from our considerations above we know that bidder i will be included in the winning allocation using this strategy. Furthermore, as the $\bar{\pi}_i$-profit-target-strategy guarantees a profit of $\bar{\pi}_i$ in case i is in the winning allocation, the maximum possible profit of $\bar{\pi}_i$ is realized with this strategy. It is thus a best reply. □

Proof of Lemma 2.13

The bidder chooses to supply the price-quality combination $(p^*_{G_j}, q^*_{G_j})$ that maximizes his profit. Thus, his objective is

$$\max_{(p_{G_j}, q_{G_j})} \left[p_{G_j} - c(q_{G_j})\right] \quad \text{s.t.} \quad \phi_{G_j}(q_{G_j}) - p_{G_j} = t^i_j.$$

This can be rewritten as

$$\begin{aligned} &\max_{q_{G_j}} \left[\phi_{G_j}(q_{G_j}) - t^i_j - c(q_{G_j})\right] \\ =&\max_{q_{G_j}} \left[\phi_{G_j}(q_{G_j}) - c(q_{G_j})\right] - t^i_j. \end{aligned} \tag{2.5}$$

Note that the maximum exists and is unique. Furthermore, in (2.5) we can see that the optimal quality does not depend on the score to fulfill. □

Proof of Theorem 2.17

Consider the bids of a bidder i. For each possible attribute level configuration q_{G_j} for each possible package, a price exists such that a potential profit value π_i is realized. As the proxy continuously decreases the profit value during the auction process, for every quality level and package the proxy will simply set the bid price $\beta_i(q_{G_j})$ such that $\beta_i(q_{G_j}) := c_i(q_{G_j}) + \pi_i$. Hence, in each round the buyer receives a set of bids by each seller including every attribute configuration and the corresponding prices, all leading to the same profit for the seller. Every level q_{G_j} thus generates a score $\phi_{G_j}(q_{G_j}) - \beta_i(q_{G_j})$. As the buyer is rational, for every π_i he will prefer the bid $(\beta_i(q_{G_j}), q_{G_j})$ out of the bids of seller i that maximizes his score for a certain package. This is the bid that maximizes $\phi_{G_j}(q_{G_j}) - \beta_i(q_{G_j}) + \beta_i(q_{G_j}) - c_i(q_{G_j}) = \phi_{G_j}(q_{G_j}) - c_i(q_{G_j})$, as $\beta_i(q_{G_j}) - c_i(q_{G_j}) = \pi_i$ is a constant. But the maximum of $\phi_{G_j}(q_{G_j}) - c_i(q_{G_j})$ for a certain package is the bidder's pseudotype for this package. Hence, the optimal quality is chosen by the buyer. □

Chapter 3

Signaling in First-Price Auctions

3.1 Introduction

Can it be beneficial to reveal some information about one's own valuation to another bidder in a first-price auction with private values? On the first glance, the answer seems to be an obvious *no*: one bidder receives additional information while the revealing bidder's information level stays the same. In principle, the informed bidder should be able to use this information to his own advantage and take away part of the profit of the revealing bidder. On the second glance however, things are not so clear: a bidder wants to appear weak in the eyes of his opponent, such that the opponent tries to profit from this weakness by reducing his bid. This increases the chances of winning for the bidder who reveals to be weak. Of course, there is also an opposing effect if a bidder appears strong. It is the goal of this chapter to characterize circumstances under which it is profitable (or not profitable) to release an informative public signal while learning one's valuation.

A typical situation where an informative signal could emerge can be found in the context of procurement auctions. Consider a manufacturer who wants to compete in a first-price procurement auction[1] to sell a new product. Before he takes part in the auction he has to acquire information about his production costs and about the quality of the new product. Costs and quality depend on the production technology and the costs for buying the necessary components. If the competitors in the auction are able to observe which components the manufacturer buys for which price, they update their beliefs about quality and price of the manufacturer's product. Nevertheless, the manufacturer is the only one who knows his production technology, while the competitors observe only an informative signal. How a signal is perceived by the competitors and how their updating works depends very much on the context, the possible production technologies and the competitor's beliefs about

[1]We think of a multi-attribute auction where bids are price-quality combinations evaluated by a scoring rule. This auction is essentially strategically equivalent to a standard first-price auction (see Asker and Cantillon (2008)). It is thus safe to transfer the results of this chapter, which are obtained for standard first-price auctions, to procurement auctions.

these things. If the manufacturer buys the components secretly, no signal is released. Usually, the manufacturer has the power to decide whether he uses a secret buying process or whether he makes its results public. For example, if he uses a request for quotation to acquire the components, the manufacturer provides public information about the specifications of the components he intends to use. Alternatively, he would be free to secretly approach possible suppliers and get their offers without revealing any public information.

In our model, two bidders take part in a first-price auction with private values. One of the two bidders has the option to release a signal about his valuation while he learns it. Thus, he has to commit to releasing the signal before he knows his valuation. In case a signal is released, the receiving bidder updates his beliefs about the valuation of the sending bidder. As a consequence, the two bidders bid as in an asymmetric auction. Furthermore, for each signal realization the resulting beliefs differ and thus do the distributions of the players' valuations in the auction. This is the major difficulty of this chapter: to derive the expected profit of using these signals, an expectation over the bidders' payoffs of different asymmetric auctions has to be calculated. A closed-form solution for the bidders' equilibrium strategies is necessary to do this explicitly. Unfortunately, a general closed-form solution for asymmetric first-price auctions is not known.

A crucial element for the success of signaling is the structure of the signals. The results of this chapter show that a very precise signal is not favorable from the sender's point of view. Nevertheless, we provide a signaling structure for which signaling is favorable: such a structure contains some information about the valuation, but is not too precise. However, in general a signaling precision guaranteeing the success of signaling does not need to exist: for a different structure, we show that signaling is never favorable for the sender, no matter what the precision is. In particular, one setting where signaling may be favorable is a simple discrete first-price auction setting. Each bidder's valuation and the signal may be either high, medium, or low. The signal is informative in the sense that it will take the true value with a larger probability than the other two values, and the remaining two values are taken with equal probability. We show that releasing such a signal is beneficial for a bidder, as long as the signal is not too precise (the probability of revealing the true valuation is not close to one). Additionally, we derive the optimal probability of revealing the true valuation from the sending bidder's perspective.

Our other results are obtained in a continuous environment: the valuations of the two bidders are drawn from the same interval. Signals may realize in an interval around the true valuation. This interval is shifted for different realizations. The signal precision is given by the length of this interval – the shorter the interval, the more precise the signal. Using only mild assumptions on the signal distributions we give an explicit length of the interval such that signaling is not beneficial if the signals stem from an interval at most as long as this length. In our final setting, we assume all distributions to be uniform. This is the only continuous environment where a general explicit solution is known (Kaplan and

Zamir (2007)) – in particular, a solution is needed that allows for different supports of the distributions of the bidders' valuations. With this signaling structure it is not beneficial for a bidder to release a signal about his realized valuation, irrespective of the signal precision.

This problem has not been addressed in the literature so far. The most related paper is Hoerner and Sahuguet (2007). They explain bluffing and jump bidding in a model with two bidders and an initial stage. In this initial stage, one of the two bidders makes an opening bid and the other bidder has to match it to start the actual auction following this stage. A similar feature to our model is the fact that the beliefs of the bidders change depending on the opening bid and thus an asymmetric auction is played afterward. However, the opening bid has to be paid in any case. Thus, the signaling happening in the initial stage has a direct influence on the payoff. Hoerner and Sahuguet (2007) concentrate mostly on an all-pay auction for the second stage, but also briefly discuss a discrete first-price auction related to the one we look at in parts of this chapter. In a similar framework, Ye (2007) looks at the concept of indicative bidding. Potential bidders submit non-binding bids in a stage before the actual auction starts, which is related to the signals in our model. However, these bids are used to select the participants for the auction and thus have a direct influence on the payoffs. Furthermore, bidders only learn the highest rejected non-binding bid, such that the following auction is a symmetric one – and not asymmetric, as in our case.

Another related line of research is dealing with information acquisition in auctions. Bergemann and Valimaki (2002) study efficiency in a general mechanism design problem where agents do not know their type but may acquire a signal about it. More precise signals are more expensive. In contrast to our model, agents do not learn anything about the other agents, but only about themselves. Persico (2000) shows that agents acquire more information about their types in a first-price auction compared to a second-price auction. Compte and Jehiel (2007) compare sealed-bid and dynamic formats, where some bidders are informed and others are uninformed. In their model, more information is acquired in the dynamic format, which goes along with a higher revenue for the seller.

Furthermore, our chapter is connected to the literature on information disclosure by the seller. Milgrom and Weber (1982) show that a seller wants to disclose public information which is affiliated with the buyers' types. Eso and Szentes (2007) give a similar result when information is given to the bidders privately by the seller. Board (2009) studies the English auction where the seller may be worse off in some cases when releasing information. Looking for the optimal auction, in Bergemann and Pesendorfer (2007) the seller has full control how the buyers learn their types. Finally, Kaplan and Zamir (2000) explore the role of commitment.

The main difficulty of this chapter lies in solving an asymmetric auction. We use the explicit solution for two bidders with uniform distributions and a general support by

Kaplan and Zamir (2007). Plum (1992) provides the differential equations characterizing a general solution when the support of both bidders' distributions has the same lower bound. He also provides an explicit solution for power distributions. Numerical solutions are provided by Gayle and Richard (2008) and the general questions of uniqueness and existence are examined by Maskin and Riley (2000a, 2000b, 2003) and Lebrun (1999, 2006).

This chapter is organized as follows: in Section 3.2 we introduce signaling in a discrete first-price auction. The general model with continuous typespaces is studied in Section 3.3 and a special case of this model with uniform distributions is given in Section 3.4. We conclude in Section 3.5. We derive the equilibrium for a discrete asymmetric auction in Appendix 3.A and proofs are given in Appendix 3.B.

3.2 Signaling in a Discrete Environment

We consider a first-price auction with two bidders, $i = 1, 2$, and discrete valuations $v_i \in V := \{0, 1, 2\}$. The valuations are independently distributed and private information of the bidders. $f_i(v_i)$ is the probability that valuation v_i is realized for bidder i. Bidder 1 may send a signal $s \in S := \{0, 1, 2\} = V$ about his realized valuation. The signal is common knowledge to both agents. The decision whether to send a signal or not is made before he knows his valuation. For a given $v_1 \in V$, we denote the probability of sending a signal value of s by $h(s|v_1)$. As the signaling should reveal some information about the true realization, we assume that $h(v_1|v_1) > f_1(v_1)$ and for $s \neq v_1$ we assume $h(s|v_1) < f_1(v_1)$. Consequently, bidder 2 updates his beliefs about bidder 1's true valuation to the posteriors $g(v_1|s)$ according to

$$g(v_1|s) = \frac{h(s|v_1) \cdot f_1(v_1)}{\sum_{j=0}^{2} h(s|j) \cdot f_1(j)}. \tag{3.1}$$

As a result, an asymmetric auction is played. To be able to study the consequences of signaling in a first-price auction, we need to know some properties of the equilibrium in this asymmetric auction. By Proposition 2 in Maskin and Riley (2000b) we know that a monotonic equilibrium exists in this setting if a *Vickrey tie-breaking rule* is used. According to this rule, ties are broken by performing a Vickrey auction among the bidders with the same bid. The resulting payment of the Vickrey tie-breaking auction has to be paid on top of the winning bid of the actual first-price auction. Ties in the Vickrey auction are broken by randomizing with equal probability. This kind of tie-breaking rule ensures that in equilibrium a bidder with a higher valuation may submit the same bid as another bidder with a lower valuation and still win the auction with probability one (while two bidders with the same valuation and the same bid win with equal probability). We assume a Vickrey tie-breaking rule in the following and concentrate on monotonic equilibria. The detailed derivation of the equilibrium, which is in mixed strategies, is given in Appendix 3.A.

For concreteness, when studying signaling we assume that the a priori-distribution of both bidders' valuations is uniform, $f_i(v_i) = \frac{1}{3}$ for $i = 1, 2$ and $v_i \in V$. Furthermore, we assume that signaling is of the following form: both signal realizations not meeting the true valuation are equally likely, $h(s|v_1) = h(s'|v_1) < h(v_1|v_1)$ for $s \neq s' \neq v_1 \neq s$. Additionally, the probability of sending a signal containing the true valuation, the *signal precision* r, is assumed to be the same irrespective of the valuation. Hence, for all $v_1, v_1' \in V$ it holds that $r := h(v_1|v_1) = h(v_1'|v_1')$. Consequently, the posterior in (3.1) becomes $g(v_1|s) = h(s|v_1)$, as $\sum_{j=0}^{2} h(s|j) = 1$.

With the help of Proposition 3.16 in Appendix 3.A we are able to calculate the expected revenue of using signals with precision r, $\pi_1^s(r)$. For each possible signal realization, different posteriors arise, and hence essentially a different asymmetric auction is played. The detailed profit of the bidders is derived in Appendix 3.B, the overall profit is summarized in the following lemma.

Lemma 3.1 *Bidder 1's expected profit in this auction setting when he uses signals with precision r is given by*

$$\pi_1^s(r) = \frac{7}{36} + \frac{1}{6}r + \frac{1}{18}r\sqrt{13-12r} - \frac{1}{12}r^2 + \frac{1-r}{12} \cdot \frac{3 + 32r - 3r^2 + (1+r)\sqrt{9+78r+9r^2}}{3 - 3r + \sqrt{9+78r+9r^2}}.$$

Next, we derive the optimal signal precision r from bidder 1's perspective. This is done by maximizing bidder 1's expected profit as given in Lemma 3.1. We use the short notation $a := \sqrt{9+78r+9r^2}$ and $b := \sqrt{13-12r}$. Then, the first order condition amounts to

$$\frac{(54+18b)r^4 + (375 - 6ab + 105b - 18a)r^3 - (9ab + 47a + 713 + 519b)r^2}{ab(-3+3r-a)^2} + \frac{(\frac{76}{3}a + 107b - 12ab + 245)r + 13a + 11ab + 33b + 39}{ab(-3+3r-a)^2} = 0 \qquad (3.2)$$

and we can state the following theorem:

Theorem 3.2 *The optimal signaling precision r^* in the discrete auction model is given by the solution to* (3.2)*, with $r^* \approx 0.5462$. Signaling is beneficial for all r fulfilling $\frac{1}{3} < r < r'$ with r' being the larger solution of $\pi_1^s(r') - \frac{4}{9} = 0$. This yields $r' \approx 0.7572$.*

Proof $r^* \approx 0.5462$ is the unique solution to the first order condition (3.2). We furthermore need to show that it is in fact associated with a maximum: by continuity of the left hand side of (3.2) the uniqueness of the solution yields that a local maximum is a global maximum as well. Furthermore, a numerical calculation as in Figure 3.1 shows that there are r-values above and below r^* leading to a lower profit than r^*. Because of the continuity this is sufficient to show that r^* is a local maximum, and hence a global maximum.

To show the second part of the theorem, we note that the profit of using no signals (or signaling with a precision of $r = \frac{1}{3}$) yields an expected profit of $\frac{4}{9}$ for bidder 1. By our analysis of the first order condition we have essentially seen that $\pi_1^s(r)$ is monotonically increasing on $\left(\frac{1}{3}, r^*\right)$ and monotonically decreasing on $(r^*, 1)$. Hence, the zeros of $\pi_1^s(r) - \frac{4}{9}$ describe the boundaries of the interval for which signaling is beneficial. $\pi_1^s(r) - \frac{4}{9}$ has two zeros, the lower one being $\frac{1}{3}$ and the larger one being $r' \approx 0.7572$. □

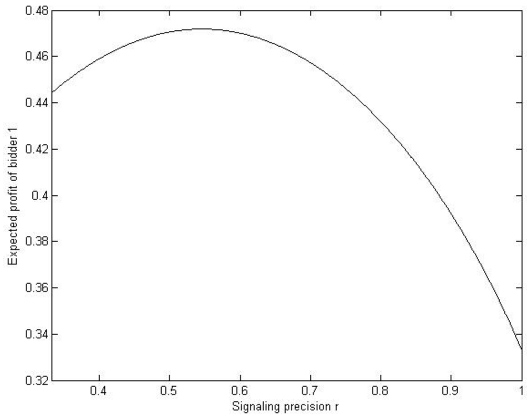

Figure 3.1: Expected profit of bidder 1 depending on the signaling precision.

As illustrated by Figure 3.1, the expected revenue of the signaling bidder is increasing as soon as informative signaling is introduced. There is a unique optimal signaling precision given the signaling structure we use. Furthermore, a general pattern of signaling is already visible here: if signaling gets too precise, it is not beneficial any more. Particularly, if the precision is very high, the revenue decrease is substantial. Nevertheless, as shown in Theorem 3.2, signaling is beneficial for quite a wide range of parameters.

If we increase the number of possible valuations in the set V, this basic insight does not change. In principle, the same analysis can be repeated for any number of valuations. In the natural extension of our example, the ex ante distribution of types is uniform, the average value stays the same and the signaling structure does not change: the signal takes the true value with a high probability and the remaining values with a smaller probability, which is equal across all remaining types. However, a general statement is difficult to make, as we do not have an explicit general characterization of the equilibrium with n discrete types. We thus limit the explicit analysis to this small example and omit the detailed characterization of signaling with other numbers of types. Qualitatively, a basic analysis shows that the revenue without signaling is decreasing in the number of types, and it suggests that the interval of precisions for which signaling is profitable gets shorter in absolute and relative terms. The same is true for the maximum gain of signaling, which is achieved by using the optimal signaling precision. However, it is not clear how the profitability of signaling will develop in the limit for a large number of discrete types. Note that the shape of the signaling distribution becomes flatter with an increasing number of types – it is likely that a more peak-shaped form of the signals, like in the original three-type example, keeps up the profitability of signaling. Nevertheless, for the reasons mentioned above, we cannot prove this type of general statements for larger numbers of types.

3.3 Signaling in a Continuous Environment

We now introduce signaling when the agents have continuous type spaces. Valuations v_i are independently drawn from an interval $V = [\underline{v}, \bar{v}]$ and are private information of the bidders. $F_i(v_i)$ is the cumulative distribution function of bidder i's valuation with associated strictly positive density $f_i(v_i)$. Bidder 1 may send a signal $s \in S = [v_1 - d, v_1 + d]$ about his realized valuation, with $d \in \mathbb{R}_+$. The signal is common knowledge to both agents. We call d the *precision* of the signal. As in the discrete case, the decision whether to send a signal or not is made before the bidder learns his valuation. Given that a valuation v_1 is realized, the conditional distribution of the signal s with precision d is denoted by $H_d(s|v_1)$ and the corresponding density by $h_d(s|v_1)$. Note that the signals may be up to d higher (respectively lower) than the actual maximal (minimal) possible valuation.
After receiving s, bidder 2 correctly updates his beliefs that bidder 1's valuation is distributed on $[\max\{\underline{v}, s - d\}, \min\{\bar{v}, s + d\}] =: [\underline{s}(s, d), \bar{s}(s, d)]$ according to a cumulative posterior distribution function $G_d(v_1|s)$ with strictly positive density $g_d(v_1|s)$. We write $\underline{s}$ and $\bar{s}$ in short for $\underline{s}(s, d)$ and $\bar{s}(s, d)$ where the reference to s and d is clear. The overall expected profit of using signals is denoted by π^s, if no signals are used the expected profit is π. The expected profit of bidder 1, when he has valuation v_1 and a signal s has realized, is denoted by $\pi_d(v_1|s)$. As lower signal realizations lead to lower beliefs of bidder 2 and thus lower equilibrium bids with a higher profit of bidder 1, we concentrate on signaling structures fulfilling the following assumption, which is true for example for the uniform signaling presented in Section 3.4 (see Proposition 3.11).

Assumption 3.1 *Lower signal realizations increase the profit: $\pi_d(v_1|s)$ is weakly decreasing in s given fixed values of v_1 and d.*

Note that in the current section we do not further restrict the signal to take a specific form. Its informativeness comes from the fact that the true valuation of bidder 1 is determined by the signal with a precision of d.
Maskin and Riley (2000b) showed that in such a setting a pure-strategy equilibrium of the first-price auction with monotonic bid functions exists. We denote the monotonic equilibrium bidding strategy of agent i in case no signal is revealed by $\beta_i(v_i)$. In case the signal realization is s and the signal precision is d, we denote the strategy of agent i

by $\beta_i(v_i|s,d)$. We focus on undominated equilibrium strategies and thus make use of the following assumption, similar to Maskin and Riley (2003):

Assumption 3.2 *Bidder i never bids more than his type v_i in equilibrium.*

Adapting a lemma of Maskin and Riley (2003) to our context, we can characterize the bid of the lowest possible type of bidder 1. Note that this lowest possible type depends on the signal realization.

Lemma 3.3 *If Assumption 3.2 holds, for any $d \in \mathbb{R}$ and any possible signal realization s, the lowest possible type $\underline{s}(s,d)$ of bidder 1 has an equilibrium bid of*

$$b_*(\underline{s}(s,d)) = \beta_1(\underline{s}(s,d)|s,d) = \max \arg\max_b F_2(b)(\underline{s}(s,d)-b).$$

Note that in case $\underline{s}(s,d) = \underline{v}$, $b_*(\underline{s}(s,d)) = \underline{v}$ holds. The following simple lemma shows that the highest possible type of bidder 1 always wins the auction:

Lemma 3.4 *The highest type $\bar{s}$ wins the auction with probability* 1 *in equilibrium.*

We now come to our main result of this section:

Theorem 3.5 *Assume*

$$d \leq \frac{1}{2} \cdot \int_{\underline{v}}^{\bar{v}} \left[F_2\left(\beta_2^{-1}\left(b_*(v_1)\right)\right) - F_2\left(b_*(v_1)\right)\right] \cdot \left(v_1 - b_*(v_1)\right) f_1(v_1)\, dv_1,$$

then it is more profitable for bidder 1 not to reveal additional information about his valuation than revealing a signal s with precision d.

Proof Consider the lowest possible valuation of bidder 1, $\underline{s}$, with an equilibrium profit of $\pi_d(\underline{s}|s)$. Furthermore, recall from Lemma 3.4 that the highest type wins the auction with probability 1. In equilibrium, it is not profitable for $\underline{s}$ to imitate the bidding behavior of the highest type. Hence, it holds that

$$\pi_d(\underline{s}|s) \geq \underline{s} - \beta_1(\bar{s}|s,d).$$

We now compare the profit of the lowest and the highest type:

$$\begin{aligned} \pi_d(\bar{s}|s) - \pi_d(\underline{s}|s) &\leq (s + d - \beta_1(\bar{s}|s,d)) - (s - d - \beta_1(\bar{s}|s,d)) \\ &= 2d. \end{aligned} \tag{3.3}$$

For a signal s, any type $v_1 \in [\underline{s}, \bar{s}]$ makes a profit

$$\pi_d(v_1|s) \leq \pi_d(v_1|v_1 - d) \tag{3.4}$$

$$\leq \pi_d(\underline{s}(v_1 - d, d)|v_1 - d) + 2d \tag{3.5}$$

$$\leq \pi_d(v_1|v_1 + d) + 2d. \tag{3.6}$$

Here, (3.4) holds by Assumption 3.1 and (3.5) holds by using (3.3) as $v_1 = \bar{s}(v_1 - d, d)$. Finally, (3.6) follows directly from Lemma 3.3: the profit of the lowest type given there is obviously increasing in the value of the lowest type. Clearly, this increase in profit applies here as $v_1 = \underline{s}(v_1 + d, d) \geq \underline{s}(v_1 - d, d)$.

As a consequence, we can derive a bound on the expected profit bidder 1 makes in case the signal is sent. We can write the expected profit in case bidder 1 uses signals in the following way:

$$\begin{aligned}
\pi^s &= \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}-d}^{\bar{v}+d} \pi_d(v_1|s) h_d(s|v_1) f_1(v_1) \, \mathrm{d}s \, \mathrm{d}v_1 \\
&= \int_{\underline{v}}^{\bar{v}} f_1(v_1) \int_{\underline{v}-d}^{\bar{v}+d} \pi_d(v_1|s) h_d(s|v_1) \, \mathrm{d}s \, \mathrm{d}v_1 \\
&\leq \int_{\underline{v}}^{\bar{v}} f_1(v_1) \int_{\underline{v}-d}^{\bar{v}+d} \left(\pi_d(v_1|v_1 + d) + 2d\right) h_d(s|v_1) \, \mathrm{d}s \, \mathrm{d}v_1 \\
&= \int_{\underline{v}}^{\bar{v}} f_1(v_1) \left(\pi_d(v_1|v_1 + d) + 2d\right) \int_{\underline{v}-d}^{\bar{v}+d} h_d(s|v_1) \, \mathrm{d}s \, \mathrm{d}v_1 \\
&= \int_{\underline{v}}^{\bar{v}} f_1(v_1) \left(\pi_d(v_1|v_1 + d) + 2d\right) \cdot 1 \, \mathrm{d}v_1 \\
&= \int_{\underline{v}}^{\bar{v}} f_1(v_1) \pi_d(v_1|v_1 + d) \, \mathrm{d}v_1 + 2d \\
&= \int_{\underline{v}}^{\bar{v}} f_1(v_1) F_2(b_*(v_1))(v_1 - b_*(v_1)) \, \mathrm{d}v_1 + 2d. \qquad (3.7)
\end{aligned}$$

The last line holds by Lemma 3.3, as v_1 is the lowest possible type given a signal $v_1 + d$ and wins exactly against all opponent's types that are lower than his bid.

Now suppose to the contrary that revealing a signal s with precision d is more profitable than not revealing such a signal. Given that no signal is revealed, consider the following strategy β_1^+ of bidder 1: if his type realization is $v_1 \in [\underline{v}, \bar{v}]$, he plays as if a signal $v_1 + d$ was realized such that v_1 is the lowest possible type given this signal. By Lemma 3.3 we therefore get $\beta_1^+(v_1) = b_*(v_1)$. Our proof now proceeds as follows: we show that β_1^+ would be a profitable deviation for bidder 1 in comparison to his equilibrium strategy without signal realization, β_1.

We first calculate the profit π^+ of deviating to β_1^+:

$$\begin{aligned}
\pi^+ &= \int_{\underline{v}}^{\bar{v}} F_2\left(\beta_2^{-1}\left(\beta_1^+(v_1)\right)\right)\left(v_1 - \beta_1^+(v_1)\right) f_1(v_1) \, \mathrm{d}v_1 \\
&= \int_{\underline{v}}^{\bar{v}} \left[F_2\left(\beta_2^{-1}\left(\beta_1^+(v_1)\right)\right) - F_2\left(\beta_1^+(v_1)\right)\right]\left(v_1 - \beta_1^+(v_1)\right) f_1(v_1) \, \mathrm{d}v_1 \\
&\qquad + \int_{\underline{v}}^{\bar{v}} f_1(v_1) F_2\left(\beta_1^+(v_1)\right)\left(v_1 - \beta_1^+(v_1)\right) \mathrm{d}v_1 \\
&\overset{(3.7)}{\geq} \int_{\underline{v}}^{\bar{v}} \left[F_2\left(\beta_2^{-1}\left(\beta_1^+(v_1)\right)\right) - F_2\left(\beta_1^+(v_1)\right)\right]\left(v_1 - \beta_1^+(v_1)\right) f_1(v_1) \, \mathrm{d}v_1 + \pi^s - 2d
\end{aligned}$$

$$\geq \int_{\underline{v}}^{\bar{v}} \left[F_2\left(\beta_2^{-1}\left(\beta_1^+(v_1)\right)\right) - F_2\left(\beta_1^+(v_1)\right)\right]\left(v_1 - \beta_1^+(v_1)\right) f_1(v_1)\,\mathrm{d}v_1 + \pi - 2d. \qquad (3.8)$$

The last line holds because by assumption, the expected profit given no signaling takes place, π, is smaller than the expected profit with signaling, $\pi^s \geq \pi$. If we rearrange (3.8) we get the following:

$$\pi^+ - \pi \geq \int_{\underline{v}}^{\bar{v}} \left[F_2\left(\beta_2^{-1}\left(\beta_1^+(v_1)\right)\right) - F_2\left(\beta_1^+(v_1)\right)\right]\left(v_1 - \beta_1^+(v_1)\right) f_1(v_1)\,\mathrm{d}v_1 - 2d.$$

Here, as $\beta_1^+(v_1) = b_*(v_1)$, we can see that the deviation to β_1^+ is profitable if

$$\int_{\underline{v}}^{\bar{v}} \left[F_2\left(\beta_2^{-1}\left(b_*(v_1)\right)\right) - F_2\left(b_*(v_1)\right)\right]\left(v_1 - b_*(v_1)\right) f_1(v_1)\,\mathrm{d}v_1 \geq 2d, \qquad (3.9)$$

leading to $\pi^+ - \pi \geq 0$. In these cases, we get a contradiction to the fact that β_1 is an equilibrium strategy. Thus, our initial assumption that revealing a signal with a precision d as in (3.9) must have been false and bidder 1 prefers not to reveal a signal. □

The theorem shows that a bidder never likes to use a signal that is too precise in the sense of d being very small. This bound on d we derived is independent of the precise distribution used for signaling (as long as Assumption 3.1 is fulfilled). However, it depends on the original distributions of the bidder's valuations. Note that the result does not say whether signaling is profitable or not for higher values of d. In the following example, we calculate the size of the bound for a uniform distribution.

Example 3.6 Suppose valuations are drawn from a uniform distribution on $[\underline{v}, \bar{v}] = [0, 1]$, hence $F_i(v_i) = v_i$ and $f_i(v_i) = 1$. It is commonly known that equilibrium bids in a first-price auction are then given by $\beta_i(v_i) = \frac{v_i}{2}$. Furthermore, by Lemma 3.3 we know $b_*(v_1) = \max \arg\max_b F_2(b)(v_1 - b) = b(v_1 - b) = \frac{v_1}{2}$. This fixes the bound on the precision d according to Theorem 3.5:

$$d \leq \frac{1}{2} \cdot \int_0^1 \left[2 \cdot \frac{v_1}{2} - \frac{v_1}{2}\right] \cdot \left(v_1 - \frac{v_1}{2}\right) \cdot 1\,\mathrm{d}v_1 = \frac{1}{2} \cdot \int_0^1 \frac{v_1^2}{4} = \frac{1}{24}.$$

Thus, for a signaling interval length smaller than $2d = \frac{1}{12}$ it is not profitable to make use of the signals.

3.4 Signaling via Uniform Distributions

We now consider the only class of distributions for which a complete characterization of equilibrium strategies in the asymmetric auction exists: the uniform distribution. This is a special case of the general continuous environment in Section 3.3. The aim of this section is to analyze the profitability of signaling for all possible signal precisions d. Ex ante, the valuations for both bidders are identically and independently distributed according to a uniform distribution on $[\underline{v}, \bar{v}]$. Accordingly, the cumulative distribution function F is given by $F(v) = \frac{v-\underline{v}}{\bar{v}-\underline{v}}$ and its density by $f(v) = \frac{1}{\bar{v}-\underline{v}}$. Bidder 1 has the option to ex ante commit to sending a signal s with a precision d after his valuation v_1 is realized. Specifically, the signal s is distributed uniformly on $[v_1 - d, v_1 + d]$. The corresponding cumulative distribution function is given by $H_d(s|v_1) = \frac{s-(v_1-d)}{2d}$ and its density by $h_d(s|v_1) = \frac{1}{2d}$. Hence, from an ex ante-perspective, we can derive the density $h_d(s)$ for a realization of signal s by the law of total probability:

$$h_d(s) = \int_{\underline{s}(s,d)}^{\bar{s}(s,d)} f(v_1) h_d(s|v_1)\,\mathrm{d}v_1 = \int_{\underline{s}(s,d)}^{\bar{s}(s,d)} \frac{1}{\bar{v}-\underline{v}} \cdot \frac{1}{2d}\,\mathrm{d}v_1 = \frac{\bar{s}(s,d) - \underline{s}(s,d)}{(\bar{v}-\underline{v})2d} \tag{3.10}$$

After observing a signal s, bidder 2 updates his belief to the posterior probability distribution $G_d(v_1|s)$ with density $g_d(v_1|s)$, which can be derived as follows, using Bayes' law:

$$g_d(v_1|s) = \frac{h_d(s|v_1) f(v_1)}{h_d(s)} = \frac{\frac{1}{2d} \cdot \frac{1}{\bar{v}-\underline{v}}}{\frac{\bar{s}(s,d)-\underline{s}(s,d)}{(\bar{v}-\underline{v})2d}} = \frac{1}{\bar{s}(s,d) - \underline{s}(s,d)}.$$

Thus, the posterior is distributed uniformly on $[\underline{s}(s,d), \bar{s}(s,d)]$. Given a signal realization s, the two bidders face the situation of an asymmetric auction with uniform distributions. The two bidders play as if bidder 1's value had been drawn uniformly from $[\underline{s}(s,d), \bar{s}(s,d)]$ and bidder 2's value from $[\underline{v}, \bar{v}]$. We denote the expected profit of bidder 1 in this auction by $\pi_1(s,d)$. The general inverse bidding strategies for this asymmetric auction have been derived by Kaplan and Zamir (2007) and can be found in Appendix 3.B. Again, we denote the bidding strategy of bidder i by $\beta_i(v_i|s,d)$ as the bid depends on the realized valuation v_i, the realized signal s and the precision of the signal d. For notational simplicity, we write $\beta_i(v_i)$ whenever s and d are fixed. The expected profit is given as follows, using the substitution $(\beta_1)^{-1}(b) = v_1$ with boundaries $\underline{b}(s,d) = \beta_1(\underline{s}(s,d))$ and $\bar{b}(s,d) = \beta_1(\bar{s}(s,d))$:

$$\begin{aligned}
\pi_1(s,d) &= \int_{\underline{s}(s,d)}^{\bar{s}(s,d)} (v_1 - \beta_1(v_1)) \cdot F\left(\beta_2^{-1}(\beta_1(v_1))\right) g_d(v_1|s)\,\mathrm{d}v_1 \\
&= \int_{\underline{b}(s,d)}^{\bar{b}(s,d)} (\beta_1^{-1}(b) - b) \cdot F\left(\beta_2^{-1}(b)\right) \left(\beta_1^{-1}\right)'(b) \frac{1}{\bar{s}(s,d) - \underline{s}(s,d)}\,\mathrm{d}b \\
&= \int_{\underline{b}(s,d)}^{\bar{b}(s,d)} (\beta_1^{-1}(b) - b) \cdot \frac{\beta_2^{-1}(b) - \underline{v}}{\bar{v}-\underline{v}} \cdot \frac{\left(\beta_1^{-1}\right)'(b)}{\bar{s}(s,d) - \underline{s}(s,d)}\,\mathrm{d}b.
\end{aligned} \tag{3.11}$$

The ex ante expected profit of bidder 1 from using signals with a precision d, π_1^s, can be expressed as

$$\pi_1^s(\underline{v}, \bar{v}, d) = \int_{\underline{v}-d}^{\bar{v}+d} h_d(s) \pi_1(s,d)\,\mathrm{d}s. \tag{3.12}$$

Our main goal is to analyze whether signaling is profitable. To simplify the analysis, we first formulate a series of lemmas enabling us to restrict attention on F being uniform on

$[0, 1]$. We formulate these lemmas in the general framework with bidders having valuations distributed on $[\underline{v}_i, \bar{v}_i]$. The proofs for all lemmas are given in Appendix 3.B.

Lemma 3.7 *Suppose the supports of the valuations $[\underline{v}_i, \bar{v}_i]$ are transformed to $[\underline{v}_i^+, \bar{v}_i^+] = [\alpha\underline{v}_i + k, \alpha\bar{v}_i + k]$ with $\alpha, k \in \mathbb{R}_+$. Then, the inverse bidding strategies are transformed accordingly: $\underline{b}^+ = \alpha\underline{b} + k$, $\bar{b}^+ = \alpha\bar{b} + k$ and for all $\alpha b + k =: b^+ \in [\underline{b}^+, \bar{b}^+]$ it holds that $\left(\beta_i^+\right)^{-1}(b^+) = \alpha\beta_i^{-1}(b) + k$.*

Making use of this result, we can make a statement about a bidder's payoffs depending on the distribution parameters. Denote bidder i's payoff by $\pi_i(\underline{v}_1, \bar{v}_1, \underline{v}_2, \bar{v}_2)$.

Lemma 3.8 *Given the situation of Lemma 3.7, the expected profit changes according to*

$$\pi_i(\underline{v}_1^+, \bar{v}_1^+, \underline{v}_2^+, \bar{v}_2^+) = \pi_i(\alpha\underline{v}_1 + k, \alpha\bar{v}_1 + k, \alpha\underline{v}_2 + k, \alpha\bar{v}_2 + k) = \alpha\pi_i(\underline{v}_1, \bar{v}_1, \underline{v}_2, \bar{v}_2).$$

Transforming $[\underline{v}, \bar{v}]$ to $[\underline{v}^+, \bar{v}^+] := [\alpha\underline{v} + k, \alpha\bar{v} + k]$ and the signal precision d to $d^+ := \alpha d$, it is immediate to see that the bounds for valuations possibly generating a signal $s^+ = \alpha s + k$ change according to $\underline{s}^+(s^+, d^+) = \alpha\underline{s}(s, d) + k$ and $\bar{s}^+(s^+, d^+) = \alpha\bar{s} + k$. We can apply this to get the last lemma:

Lemma 3.9 *The expected profit from using signals changes according to*

$$\pi_1^s(\underline{v}^+, \bar{v}^+, d^+) = \pi_1^s(\alpha\underline{v} + k, \alpha\bar{v} + k, \alpha d) = \alpha\pi_1^s(\underline{v}, \bar{v}, d).$$

We can summarize our findings to state the following proposition:

Proposition 3.10 *Signaling is not profitable for valuations drawn from $[\underline{v}, \bar{v}]$ if and only if it is not profitable for valuations drawn from $[0, 1]$,*

$$\pi_1^s(\underline{v}, \bar{v}, (\bar{v} - \underline{v})d) < \pi_1(\underline{v}, \bar{v}, \underline{v}, \bar{v}) \iff \pi_1^s(0, 1, d) < \pi_1(0, 1, 0, 1).$$

Proof We use Lemmas 3.8 and 3.9 with $\alpha = \bar{v} - \underline{v}$ and $k = \underline{v}$ to conclude

$$\begin{aligned} & \pi_1^s(0, 1, d) < \pi_1(0, 1, 0, 1) \\ \iff \quad & (\bar{v} - \underline{v})\pi_1^s(0, 1, d) < (\bar{v} - \underline{v})\pi_1(0, 1, 0, 1) \\ \iff \quad & \pi_1^s(\underline{v}, \bar{v}, (\bar{v} - \underline{v})d) < \pi_1(\underline{v}, \bar{v}, \underline{v}, \bar{v}). \end{aligned}$$

□

The following proposition shows that a better (lower) signal realization leads to higher profits and thus Assumption 3.1 made in Section 3.3 holds in this signaling structure.

Proposition 3.11 *Suppose bidder 1 has a valuation v_1 drawn from a uniform distribution on the support $[\underline{v}_1, \bar{v}_1]$ with $\underline{v}_1 \geq 0$ and bidder 2's valuation is drawn uniformly from $[0,1]$. Then, the profit of bidder 1 with valuation v_1 is weakly lower if v_1 is a realization from a uniform distribution on $[\underline{v}_1^+, \bar{v}_1^+]$ with $\underline{v}_1^+ \geq \underline{v}_1$ and $\bar{v}_1^+ \geq \bar{v}_1$ with one of the two inequalities being strict.*

The following theorem leads to the main result of this section.

Theorem 3.12 *For $d \geq \frac{\bar{v}-\underline{v}}{2}$, the expected profit $\pi_1^s(\underline{v}, \bar{v}, d)$ is monotonically increasing in d with*

$$\lim_{d\to\infty} \pi_1^s(\underline{v}, \bar{v}, d) = \pi_1(\underline{v}, \bar{v}, \underline{v}, \bar{v}).$$

Proof By Proposition 3.10 it is sufficient to show the results for $[\underline{v}, \bar{v}] = [0,1]$. The expected profit of signaling for $d \geq 0.5$ is given as follows:

$$\begin{aligned}
\pi_1^s(0,1,d) &\overset{(3.12)}{=} \int_{-d}^{1+d} h_d(s)\pi_1(s,d)\,\mathrm{d}s \\
&\overset{(3.10)}{=} \int_{-d}^{1+d} \frac{\bar{s}(s,d)-\underline{s}(s,d)}{2d}\pi_1(\underline{s}(s,d), \bar{s}(s,d), 0, 1)\,\mathrm{d}s \\
&= \int_{-d}^{1-d} \frac{s+d}{2d}\pi_1(0, s+d, 0, 1)\,\mathrm{d}s + \int_{1-d}^{d} \frac{1}{2d}\pi_1(0,1,0,1)\,\mathrm{d}s \\
&\qquad + \int_{d}^{1+d} \frac{1-(s-d)}{2d}\pi_1(s-d, 1, 0, 1)\,\mathrm{d}s \\
&= \frac{1}{2d}\left(\int_0^1 t\pi_1(0,t,0,1)\,\mathrm{d}t + (d-(1-d))\pi_1(0,1,0,1) + \int_0^1 (1-t)\pi_1(t,1,0,1)\,\mathrm{d}t\right) \\
&= \pi_1(0,1,0,1) + \frac{1}{2d}\underbrace{\left(\int_0^1 t\pi_1(0,t,0,1)\,\mathrm{d}t - \pi_1(0,1,0,1) + \int_0^1 (1-t)\pi_1(t,1,0,1)\,\mathrm{d}t\right)}_{=:\tilde{c}}
\end{aligned}$$

$\tilde{c}$ is constant, and thus $\lim_{d\to\infty} \pi_1^s(\underline{v}, \bar{v}, d) = \pi_1(\underline{v}, \bar{v}, \underline{v}, \bar{v})$. A calculation of $\tilde{c}$ shows $\tilde{c} \approx -0.03 < 0$. Hence, $\pi_1^s(0,1,d)$ is increasing. □

This theorem already proofs part of our main result:

Result 3.13 *For any precision of signals $d > 0$, signaling is less profitable:*

$$\pi_1^s(\underline{v}, \bar{v}, d) < \pi_1(\underline{v}, \bar{v}, \underline{v}, \bar{v}).$$

This result is a generalization of Theorem 3.12 (for the case $d \geq \frac{\bar{v}-\underline{v}}{2}$) and Example 3.6 as an application of Theorem 3.5 (for the case $d \leq \frac{\bar{v}-\underline{v}}{24}$). For the remaining parameter values, we give a proof in Appendix 3.B. The proof uses the assertion that an increase in the upper or lower end point of the support of bidder 1's uniform distribution also increases his expected profit. We do not provide a formal proof of this assertion. Nevertheless, a numerical calculation shows directly that the profit is increasing in d for all values in d and the result thus holds.

3.5 Conclusion

We showed that a bidder in a first-price auction might voluntarily commit to revealing an informative signal about his valuation. However, whether he does so or not depends on several parameters, particularly the distribution and precision of the signals. As a general pattern, bidders have no incentive to reveal an informative signal if it is very precise. In a setting with only three possible valuations – high, medium or low – we derived the optimal signal and the range of precision for which signaling is beneficial. The analysis relies on a closed-form solution of the equilibrium strategies. Such an analysis is in principle feasible for other discrete sets of valuations and other shapes of signaling distributions as well. However, general statements for higher numbers of valuations are difficult to make without an explicit general characterization of discrete asymmetric equilibria. Nevertheless, the key insight can already be gleaned from the small example with three valuations: the voluntary release of an informative signal about one's own valuation can be beneficial.
It is likely that a similar shaped distribution of signals as in the discrete case would also make signaling profitable in the continuous setting. The distributions in such a family should be single-peaked on the same interval, differing in the position of the peak. Unfortunately, the explicit equilibrium strategies for a family of signals having that peaked shape is not known so far – and without knowledge of the explicit strategies it is difficult to estimate the expected revenue, as the auctions played differ with each signal realization. Hence, we chose to introduce informativeness of the signals by altering the support of the possible signals depending on the realized valuation. This enables us to get both, a result for a general class of distributions on a restricted set of signal precisions and a result for all signal precisions using uniform distributions. In these settings, signaling is not profitable for a bidder.

3.A Appendix: Equilibrium of a Discrete Asymmetric Auction

We derive the necessary equilibrium properties of the asymmetric auction used in Section 3.2: a first-price auction with two bidders, $i = 1, 2$, with private values $v_1, v_2 \in V = \{0, 1, 2\}$, independently drawn according to the probabilities p^{v_1} and q^{v_2} respectively, using a Vickrey tie-breaking rule. Note that compared to Section 3.2, we change the notation of the probabilities. This is to avoid confusion: depending on the specific probabilities in the asymmetric auction, bidder 1 in Section 3.2 may take the role of either bidder 1 or bidder 2 in this appendix and a different notation minimizes the risk of mixing them up.
To start with the equilibrium analysis, first note that the equilibrium is in mixed strategies:

Lemma 3.14 *In this discrete first-price auction with Vickrey tie-breaking rule no pure-strategy equilibrium exists.*

Proof Consider two bidders with valuation 2 and suppose there is a (monotonic) pure-strategy equilibrium in which they bid differently. Then, the bidder submitting the strictly higher bid has an incentive to undercut his own bid such that he decreases his payment but still wins for sure. This cannot happen in equilibrium. In the same way, if both bidders submit the same highest bid, both of them have an incentive to slightly overcut the other bidder – the additional payment can be made arbitrary low, while the winning probability will make a fixed jump upwards (with the new bid, the bidder will always win the auction while he lost with positive probability before). □

A mixed equilibrium has the following structure:

Lemma 3.15 *In a mixed equilibrium of this discrete first-price auction with Vickrey tie-breaking rule*

1. *both bidders submit the same maximum bid* b^**;*
2. *there cannot be an interval* (b', b'') *with* $0 < b' < b'' < b^*$ *in which any of the two bidders does not submit a bid;*
3. *bidders do not use atoms in* $(0, b^*]$.

Proof To prove the first part, we use the fact that for a given valuation bidders have to be indifferent between all bids they possibly submit. Hence, the maximum bid b^* has to be the same for both bidders – otherwise, the bidder with the higher one could profitably deviate from his maximum bid by slightly undercutting. For the second part, suppose that such an interval (b', b'') in which bidder i does not submit a bid would exist. Then, bidder j would not submit bids on this interval either and hence no bids at all would be submitted on this interval. Suppose bidder j does not place an atom on b''. Then, bidder i had a profitable deviation from his bid b'' by deviating to a bid in the interval (b', b''), lowering the price to pay in case of winning without losing any winning probability. If bidder j has an atom on b'', then either bidder i has no atom, and the argument above applies for bidder j – or bidder i has an atom as well. In this case, both bidders necessarily have a positive winning probability with their bid b'' and make profit using it[2] (otherwise, they would have a profitable deviation in the interval (b', b'')). However, as a consequence they have a profitable deviation by slightly increasing their bid, making a jump upwards in the winning probability on the expense of an arbitrarily low increase in payment. This cannot be the case in equilibrium. Finally, we note that bidders do not use atoms in $(0, b^*]$: as already shown above, it is not possible that both bidders place an atom on the same

[2]Note that there is a positive mass of bidders with valuation 0 who always submit a bid of 0.

bid in equilibrium. Similarly, if only bidder i places an atom on some $b' \in (0, b^*]$, bidder j has an incentive to bid slightly above b' instead of bidding in an interval $(b' - \varepsilon, b')$ for ε small enough. This increases the winning probability by at least the mass of the atom, while the payment is only increased by at most ε. Consequently, this atom cannot be part of an equilibrium in case $v_j > b'$. However, $v_j = b'$ cannot be part of an equilibrium as well, as bidder j would earn a profit arbitrarily close to 0 with the bid $b' - \varepsilon$, while he could get a fixed positive amount by simply bidding 0. We can thus conclude that bidders possibly only use atoms when bidding 0. □

Hence, we look for equilibria with bids on the whole interval $[0, b^*]$. Additionally, we assume w.l.o.g. that bidder 1 has a higher probability of having valuation 2, $p^2 \geq q^2$. The lowest possible equilibrium bid of bidder i with valuation v_i is denoted by $b_i(v_i)$, and bidder 1's winning probability with his bid $b_1(2)$ is q'. Similarly, his winning probability with a bid $b_1(1) = 0$ is given by $q'' \geq q^0$. $p'' \geq p^0$ is the respective probability for bidder 2. We are now ready to derive some necessary equilibrium conditions. A bidder with valuation 2 has the opportunity to win against all others for sure by submitting a bid of b^*. Then, he makes a profit of $2 - b^*$. All other bids submitted with valuation 2 have to generate the same profit. Consequently, in equilibrium both bidders mix symmetrically on $[\max\{b_1(2), b_2(2)\}, b^*]$. Hence, as we assumed that $p^2 \geq q^2$, it holds that $b_1(2) \leq b_2(2)$ and the first equilibrium condition is given by

$$2 - b^* = (1 - q^2)(2 - b_2(2)), \tag{3.13}$$

as bidder 1 wins the auction with a bid of $b_2(2)$ exactly against all bidder 2 types with a valuation of 1 or 0. If p^2 is strictly larger than q^2, bidder 1's lowest bid fulfills $b_1(2) < b_2(2)$, and bidder 1 with valuation 2 sometimes loses against bidder 2 who has valuation 1. We get a second condition involving bidder 1's winning probability with his bid $b_1(2)$, q':

$$2 - b^* = q'(2 - b_1(2)). \tag{3.14}$$

Similarly, bidder 2 with valuation 1 has to be indifferent between submitting a bid of $b_2(2)$ and $b_1(2)$ according to

$$(1 - q^2)(1 - b_2(2)) = (1 - p^2)(1 - b_1(2)). \tag{3.15}$$

Additionally, he gets the same profit by submitting a bid of $b_2(1) = 0$, having a winning probability of $p'' \geq p^0$:

$$(1 - p^2)(1 - b_1(2)) = p''(1 - 0). \tag{3.16}$$

Bidder 1 with valuation 1 is indifferent between submitting a bid of $b_1(2)$ or $b_1(1) = 0$, winning with probability $q'' \geq q^0$ in the latter case:

$$q'(1 - b_1(2)) = q''(1 - 0). \tag{3.17}$$

Note that due to the Vickrey tie-breaking rule, a bidder with valuation 1 wins against all opponents with valuation 0 in case he submits a bid of 0. Furthermore, at least one of $p'' = p^0$ and $q'' = q^0$ is always true: it cannot be the case that both bidders bid 0 with a positive probability when having valuation 1 – facing a bidder with the same valuation, tie-breaking will let them win only in half of the cases. Increasing the bid slightly would hence be a profitable deviation. Given these additional conditions, we have a linear equation system with five equations and five unknowns, pinning down the bidding intervals for the different valuations as stated in the following proposition:

Proposition 3.16 *In this discrete asymmetric auction setting with $p^2 \geq q^2$, bidder i's equilibrium bids have the following properties:*

1. *With valuation 2, bidder i mixes his bids on $[b_i(2), b^*]$;*
2. *with valuation 1, bidder i mixes his bids on $[0, b_i(2)]$, with possibly a mass point on 0;*
3. *with valuation 0, bidder i bids 0.*

The boundaries of the bidding intervals and the probability of bidding 0 are given as follows:

1. *In case $p'' = p^0$*

$$\begin{aligned} b^* &= 1 - p^0 + q^2 \\ b_1(2) &= 1 - \frac{p^0}{1 - p^2} \\ b_2(2) &= 1 - \frac{p^0}{1 - q^2} \\ q'' &= p^0 \cdot \frac{1 + p^0 - q^2}{1 - p^2 + p^0}. \end{aligned}$$

2. *In case $q'' = q^0$*

$$\begin{aligned} b^* &= 2 - q^0 - q' \\ b_1(2) &= 1 - \frac{q^0}{q'} \\ b_2(2) &= 2 - \frac{q^0 + q'}{1 - q^2} \\ p'' &= (1 - p^2) \cdot \frac{q^0}{q'} \end{aligned}$$

with $q' = \frac{q^1}{2} + \sqrt{\left(\frac{q^1}{2}\right)^2 + (1 - p^2)q^0}$.

Proof We start with the case $p'' = p^0$. It follows directly from (3.16) that

$$b_1(2) = 1 - \frac{p^0}{1-p^2}.$$

Plugging this into (3.15), we get

$$b_2(2) = 1 - \frac{p^0}{1-q^2}.$$

Then, (3.13) yields

$$b^* = 2 - (1-q^2)\left(1 + \frac{p^0}{1-q^2}\right) = 1 - p^0 + q^2.$$

The winning probabilities follow from (3.14) (for q') and (3.17) (for q''):

$$q' = \frac{1+p^0-q^2}{1+\frac{p^0}{1-p^2}} = \frac{(1+p^0-q^2)(1-p^2)}{1-p^2+p^0}$$
$$q'' = p^0 \cdot \frac{1+p^0-q^2}{1-p^2+p^0}.$$

Next, we focus on the case $q'' = q^0$. Starting with (3.17), we get

$$b_1(2) = 1 - \frac{q^0}{q'}. \tag{3.18}$$

Combining (3.13) and (3.14) we can write

$$b_2(2) = 2 - \frac{q'}{1-q^2} \cdot \left(1 + \frac{q^0}{q'}\right) = 2 - \frac{q'+q^0}{1-q^2}. \tag{3.19}$$

The probability q' can be calculated by plugging (3.18) and (3.19) into (3.15):

$$(1-q^2)\left(\frac{q'+q^0}{1-q^2} - 1\right) = (1-p^2) \cdot \frac{q^0}{q'}$$
$$\iff \quad (q')^2 - q^1 \cdot q' - (1-p^2)q^0 = 0$$
$$\implies \quad q' = \frac{q^1}{2} + \sqrt{\left(\frac{q^1}{2}\right)^2 + (1-p^2)q^0}.$$

Plugging q' in (3.18) and (3.19) yields the expressions stated in the proposition. (3.13) fixes b^* according to

$$b^* = 2 - (1-q^2) \cdot \frac{q'+q^0}{1-q^2} = 2 - q^0 - q'.$$

Finally, according to (3.16) we get

$$p'' = (1-p^2) \cdot \frac{q^0}{q'}.$$

□

Particularly, the proposition allows us to pin down the equilibrium profit of the bidders, which is all we need for calculating the profit of signaling. Hence, there is no need for a full characterization of equilibrium strategies in this place.
Finally, we give a characterization which of the cases $p'' = p^0$ or $q'' = q^0$ in Proposition 3.16 is the relevant one for some specific probability distributions. This lemma will be useful in the next section.

Lemma 3.17 *In Proposition 3.16, the case $p'' = p^0$ is relevant if $p^0 > q^0$. Furthermore, the case $q'' = q^0$ is relevant if either $p^2 > 1/3$, $p^1 = p^0 = \frac{1-p^2}{2}$, $q^2 = q^1 = q^0 = \frac{1}{3}$ or if $p^2 = p^1 = p^0 = \frac{1}{3}$, $q^0 > \frac{1}{3}$, $q^1 = q^2 = \frac{1-q^0}{2}$.*

Proof First note that $1 - p^2 \le q'$: substituting the left-hand side of (3.14) with the right-hand side of (3.13) and dividing (3.15) by the resulting equation yields

$$\frac{1-p^2}{q'} \cdot \frac{1-b_1(2)}{2-b_1(2)} = \frac{1-b_2(2)}{2-b_2(2)} \quad \Longleftrightarrow \quad \frac{1-p^2}{q'} = \frac{2-b_1(2)-2b_2(2)+b_1(2)b_2(2)}{2-2b_1(2)-b_2(2)+b_1(2)b_2(2)}.$$

As

$$2-b_1(2)-2b_2(2)+b_1(2)b_2(2) \le 2-2b_1(2)-b_2(2)+b_1(2)b_2(2) \quad \Longleftrightarrow \quad b_1(2) \le b_2(2),$$

we know that $1 - p^2 \le q' \iff b_1(2) \le b_2(2)$, while the latter is true by our initial assumption $p^2 \ge q^2$. Consequently, by comparing (3.16) and (3.17) we get the general condition $p'' \le q''$. Hence, if $p^0 > q^0$ is fulfilled, it can never be the case that $q'' = q^0$ because it would yield the contradiction $q'' = q^0 < p^0 \le p''$.
In the case $p^2 > 1/3$, $p^1 = p^0 = \frac{1-p^2}{2}$, $q^2 = q^1 = q^0 = \frac{1}{3}$ the above argumentation cannot be applied as $q^0 > p^0$. We thus take a different approach and show that if $p'' = p^0$ were true, $q'' \ge q^0 = \frac{1}{3}$ would be violated. According to Proposition 3.16, q'' is given by

$$q'' = \frac{1-p^2}{2} \cdot \frac{1 + \frac{1-p^2}{2} - \frac{1}{3}}{1 - p^2 + \frac{1-p^2}{2}} = \frac{1}{3} \cdot \left(\frac{3}{2} - \frac{p^2}{2} - \frac{1}{3}\right).$$

We get the contradiction $q'' < q^0 = \frac{1}{3}$ in case

$$\frac{1}{3} \cdot \left(\frac{3}{2} - \frac{p^2}{2} - \frac{1}{3}\right) < \frac{1}{3} \quad \Longleftrightarrow \quad \frac{1}{3} < p^2,$$

which is true by our assumption.
Similarly, the case $p^2 = p^1 = p^0 = \frac{1}{3}$, $q^0 > \frac{1}{3}$, $q^1 = q^2 = \frac{1-q^0}{2}$ can be analyzed. Here, we have

$$q'' = \frac{1}{3} \cdot \frac{1 + \frac{1}{3} - \frac{1-q^0}{2}}{1 - \frac{1}{3} + \frac{1}{3}} = \frac{5}{18} + \frac{q^0}{6}.$$

Again, the contradiction $q'' < q^0$ is given iff

$$\frac{5}{18} + \frac{q^0}{6} < q^0 \quad \Longleftrightarrow \quad \frac{1}{3} < q^0,$$

which is true in the case we are analyzing. □

3.B Appendix: Proofs

Proof of Lemma 3.1

First note that each signal realizes with probability $\frac{1}{3}$. We will thus proceed by calculating the expected profit given a signal realization $s \in S$, denoted by $\pi_1(s,r)$, and then take the average of these profits. Suppose that a signal $s = 2$ is received. Then, $g(2|2) = r > \frac{1}{3} = f_2(2)$ and bidder 1 is associated with the p-probabilities in Proposition 3.16, while bidder 2 is associated with the q's. Hence, the two bidders are playing an asymmetric auction with posterior probabilities $p^2 = r$, $p^1 = p^0 = \frac{1-r}{2}$ and $q^0 = q^1 = q^2 = \frac{1}{3}$. As we assumed $r > \frac{1}{3}$, Lemma 3.17 tells us that $q'' = q^0$ has to hold in Proposition 3.16. The expected profit can be calculated according to

$$\begin{aligned}\pi_1(2,r) = p^2\,(2-b^*) + p^1 q^0\,(1-0) &= r\left(\frac{1}{3} + \frac{1}{6} + \sqrt{\left(\frac{1}{6}\right)^2 + \frac{1}{3}(1-r)}\right) + \frac{1-r}{2}\cdot\frac{1}{3} \\ &= \frac{1}{3}r + \frac{1}{6}r\sqrt{13-12r} + \frac{1}{6}.\end{aligned}$$

If the signal realizes to $s = 1$, posteriors are given by $g(2|1) = g(0|1) = \frac{1-r}{2}$, $g(1|1) = r$ and $f_2(0) = f_2(1) = f_2(2) = \frac{1}{3}$. Hence, $g(2|1) < f_2(2)$ and in the language of Proposition 3.16 bidder 1 and bidder 2 switch roles. Consequently, to get $\pi_1(1,r)$ we have to calculate the profit of the bidder 2-role in Proposition 3.16 in an asymmetric auction with $p^0 = p^1 = p^2 = \frac{1}{3}$ and $q^2 = q^0 = \frac{1-r}{2}$, $q^1 = r$. As $q^0 = \frac{1-r}{2} < \frac{1}{3} = p^0$, by Lemma 3.17 $p'' = p^0$ holds in Proposition 3.16. Thus, we get

$$\begin{aligned}\pi_1(1,r) = q^2\,(2-b^*) + q^1 p^0 &= \frac{1-r}{2}\left(1 + \frac{1}{3} - \frac{1-r}{2}\right) + \frac{1}{3}r \\ &= \frac{1}{6}r + \frac{5}{12} - \frac{1}{4}r^2.\end{aligned}$$

The last possible signal realization is $s = 0$. Then, posteriors are $g(2|0) = g(1|0) = \frac{1-r}{2}$, $g(0|0) = r$ and $f_2(0) = f_2(1) = f_2(2) = \frac{1}{3}$. Again, $g(2|0) < f_2(2)$ and bidder 1 takes the role of bidder 2 when we apply Proposition 3.16. The according probabilities in the asymmetric auction are thus given by $p^0 = p^1 = p^2 = \frac{1}{3}$ and $q^2 = q^1 = \frac{1-r}{2}$, $q^0 = r$. Hence,

$q'' = q^0$ holds and the expected profit in this case amounts to

$$\begin{aligned}\pi_1(0,r) &= q^2\left(2-b^*\right)+q^1p'' \\ &= \frac{1-r}{2}\left(r+\frac{1-r}{4}+\sqrt{\left(\frac{1-r}{4}\right)^2+\frac{2}{3}r}\right)+\frac{1-r}{2}\cdot\frac{2}{3}\cdot\frac{r}{\frac{1-r}{4}+\sqrt{\left(\frac{1-r}{4}\right)^2+\frac{2}{3}r}} \\ &= \frac{1-r}{4}\cdot\frac{3+32r-3r^2+(1+r)\sqrt{9+78r+9r^2}}{3-3r+\sqrt{9+78r+9r^2}}.\end{aligned}$$

Calculating

$$\pi_1^s(r) = \frac{1}{3}\left(\pi_1(0,r)+\pi_1(1,r)+\pi_1(2,r)\right)$$

and simplifying yields the result. □

Proof of Lemma 3.4

Suppose $\bar{s}$ wins with a probability less than 1 in equilibrium. Then, a set of types of the opponent with a positive mass must submit the same bid as $\bar{s}$ – their bid cannot be higher, as they had a profitable deviation to a lower bid in this continuous setting otherwise. As $\bar{s}$ makes positive profits (this e.g. follows from Lemma 3.3), he than would have a profitable deviation by slightly increasing his bid and win with probability 1. This deviation will increase his profit if the bid increase is chosen small enough, such that the gain in winning probability makes up for the loss coming from a higher bid. As this profitable deviation cannot exist in equilibrium, $\bar{s}$ must win with probability 1. □

Inverse bidding strategies according to Kaplan and Zamir (2007), Proposition 1. We assume that bidder i's valuation is uniformly distributed on $[\underline{v}_i, \bar{v}_i]$ with $\underline{v}_2 < \underline{v}_1$ and $\underline{v}_1 < 2\bar{v}_2 - \underline{v}_2$.[3] Without the latter regularity assumption, bidder 2 always loses in equilibrium and the analysis is trivial. Hence, in equilibrium, both bidders have a positive chance of winning on the same interval, $[\underline{b}, \bar{b}]$. These boundaries are given according to

$$\underline{b} = \frac{\underline{v}_1+\underline{v}_2}{2} \qquad \text{and} \qquad \bar{b} = \frac{\bar{v}_1\cdot\bar{v}_2 - \left(\frac{\underline{v}_1+\underline{v}_2}{2}.\right)^2}{\bar{v}_1-\underline{v}_1+\bar{v}_2-\underline{v}_2} \tag{3.20}$$

If bidder 2 has a value $v_2 < \underline{b}$ we assume that he bids truthfully. For all $b \in [\underline{b}, \bar{b}]$, the inverse bid functions $\beta_i^{-1}(b)$ are given by

$$\beta_1^{-1}(b) = \underline{v}_1 + \frac{(\underline{v}_2-\underline{v}_1)^2}{(\underline{v}_1+\underline{v}_2-2b)c_1 e^{\frac{\underline{v}_2-\underline{v}_1}{\underline{v}_1+\underline{v}_2-2b}} + 4(\underline{v}_2-b)} \tag{3.21}$$

$$\beta_2^{-1}(b) = \underline{v}_2 + \frac{(\underline{v}_2-\underline{v}_1)^2}{(\underline{v}_1+\underline{v}_2-2b)c_2 e^{\frac{\underline{v}_1-\underline{v}_2}{\underline{v}_1+\underline{v}_2-2b}} + 4(\underline{v}_1-b)} \tag{3.22}$$

[3]Note that the roles of bidder 1 and 2 are exchanged compared to Kaplan and Zamir (2007) for consistency reasons with the rest of this chapter.

with constants

$$c_1 = \frac{\frac{(\underline{v}_2-\underline{v}_1)^2}{\bar{v}_1-\underline{v}_1} + 4(\bar{b}-\underline{v}_2)}{-2(\bar{b}-\underline{b})} e^{\frac{\underline{v}_2-\underline{v}_1}{2(\bar{b}-\underline{b})}} \qquad \text{and} \qquad c_2 = \frac{\frac{(\underline{v}_2-\underline{v}_1)^2}{\bar{v}_2-\underline{v}_2} + 4(\bar{b}-\underline{v}_1)}{-2(\bar{b}-\underline{b})} e^{\frac{\underline{v}_1-\underline{v}_2}{2(\bar{b}-\underline{b})}}. \tag{3.23}$$

This solution does not cover the case $\underline{v}_1 = \underline{v}_2 = \underline{v}$, which was already solved by Griesmer et al. (1967) in the context of reverse auctions. A generalization is given by Plum (1992) for the class of power distributions. The inverse bid functions can be written as follows, for $b \in [\underline{b}, \bar{b}]$ as in (3.20):

$$\beta_1^{-1}(b) = \underline{v} + \frac{2(b-\underline{v})}{1+b^2c-2bc\underline{v}+c\underline{v}^2} \tag{3.24}$$

$$\beta_2^{-1}(b) = \underline{v} + \frac{2(b-\underline{v})}{1-b^2c+2bc\underline{v}-c\underline{v}^2}. \tag{3.25}$$

The constant c is defined by

$$c = \frac{1}{(\bar{v}_1-\underline{v})^2} - \frac{1}{(\bar{v}_2-\underline{v})^2}. \tag{3.26}$$

Proof of Lemma 3.7

We first calculate $\underline{b}^+$ and $\bar{b}^+$ using (3.20):

$$\underline{b}^+ = \frac{\alpha\underline{v}_1 + k + \alpha\underline{v}_2 + k}{2} = \alpha\frac{\underline{v}_1+\underline{v}_2}{2} + k = \alpha\underline{b} + k$$

$$\bar{b}^+ = \frac{(\alpha\bar{v}_1+k)\cdot(\alpha\bar{v}_2+k) - \left(\frac{\alpha\underline{v}_1+k+\alpha\underline{v}_2+k}{2}\right)}{\alpha\bar{v}_1+k-\alpha\underline{v}_1-k+\alpha\bar{v}_2+k-\alpha\underline{v}_2-k} = \alpha\frac{\bar{v}_1\cdot\bar{v}_2 - \left(\frac{\underline{v}_1+\underline{v}_2}{2}\right)}{\bar{v}_1-\underline{v}_1+\bar{v}_2-\underline{v}_2} + k = \alpha\bar{b} + k.$$

Now consider the case $\underline{v}_1 < \underline{v}_2$. First note, using (3.23), that the constants c_1 and c_2 are invariant with respect to the transformation:

$$c_1^+ = \frac{\frac{(\alpha\underline{v}_2+k-\alpha\underline{v}_1-k)^2}{\alpha\bar{v}_1+k-\alpha\underline{v}_1-k} + 4(\alpha\bar{b}+k-\alpha\underline{v}_2-k)}{-2(\alpha\bar{b}+k-\alpha\underline{b}-k)} e^{\frac{\alpha\underline{v}_2+k-\alpha\underline{v}_1-k}{2(\alpha\bar{b}+k-\alpha\underline{b}-k)}} = \frac{\frac{(\underline{v}_2-\underline{v}_1)^2}{\bar{v}_1-\underline{v}_1} + 4(\bar{b}-\underline{v}_2)}{-2(\bar{b}-\underline{b})} e^{\frac{\underline{v}_2-\underline{v}_1}{2(\bar{b}-\underline{b})}} = c_1.$$

A similar calculation is true for c_2. Hence, we can calculate the inverse bidding function for bidder 1 according to (3.21):

$$\begin{aligned}
&\left(\beta_1^+\right)^{-1}(b^+) \\
&= \alpha\underline{v}_1 + k + \frac{(\alpha\underline{v}_2+k-\alpha\underline{v}_1-k)^2}{(\alpha\underline{v}_1+k+\alpha\underline{v}_2+k-2(\alpha b+k))c_1 e^{\frac{\alpha\underline{v}_2+k-\alpha\underline{v}_1-k}{\alpha\underline{v}_1+k+\alpha\underline{v}_2+k-2(\alpha b+k)}} + 4(\alpha\underline{v}_2+k-\alpha b-k)} \\
&= \alpha\left(\underline{v}_1 + \frac{(\underline{v}_2-\underline{v}_1)^2}{(\underline{v}_1+\underline{v}_2-2b)c_1 e^{\frac{\underline{v}_2-\underline{v}_1}{\underline{v}_1+\underline{v}_2-2b}} + 4(\underline{v}_2-b)}\right) + k \\
&= \alpha\beta_1^{-1}(b) + k.
\end{aligned}$$

Again, the calculation for bidder 2, using (3.22), is similar.

Finally, consider the case $\underline{v}_1 = \underline{v}_2 = \underline{v}$. We first calculate the constant c^+ according to (3.26):

$$c^+ = \frac{1}{(\alpha\bar{v}_1 + k - \alpha\underline{v} - k)^2} - \frac{1}{(\alpha\bar{v}_2 + k - \alpha\underline{v} - k)^2} = \frac{1}{\alpha^2}\left(\frac{1}{(\bar{v}_1 - \underline{v})^2} - \frac{1}{(\bar{v}_2 - \underline{v})^2}\right) = \frac{c}{\alpha^2}.$$

Hence, with (3.24) the inverse bidding strategy for bidder 1 can be written as

$$\begin{aligned}\left(\beta_1^+\right)^{-1}(b^+) &= \alpha\underline{v} + k + \frac{2(\alpha b + k - \alpha\underline{v} - k)}{1 + (\alpha b + k)^2\frac{c}{\alpha^2} - 2(\alpha b + k)\frac{c}{\alpha^2}(\alpha\underline{v} + k) + \frac{c}{\alpha^2}(\alpha\underline{v} + k)^2} \\ &= \alpha\left(\underline{v} + \frac{2(b - \underline{v})}{1 + b^2 c - 2bc\underline{v} + c\underline{v}^2}\right) + k \\ &= \alpha\beta_1^{-1}(b) + k.\end{aligned}$$

The inverse bidding strategy for bidder 2 can be derived in the same way using (3.25). □

Proof of Lemma 3.8

Using (3.11), the profit of bidder 1 for the transformed support can be written as

$$\pi_1(\underline{v}_1^+, \bar{v}_1^+, \underline{v}_2^+, \bar{v}_2^+) = \int_{\underline{b}^+}^{\bar{b}^+} \left(\left(\beta_1^+\right)^{-1}(b^+) - b^+\right) \cdot \frac{\left(\beta_2^+\right)^{-1}(b^+) - \underline{v}_2^+}{\bar{v}_2^+ - \underline{v}_2^+} \cdot \frac{\left(\left(\beta_1^+\right)^{-1}\right)'(b^+)}{\bar{v}_1^+ - \underline{v}_1^+}\, \mathrm{d}b^+$$

$$= \int_{\underline{b}}^{\bar{b}} \left(\left(\beta_1^+\right)^{-1}(\alpha b + k) - \alpha b - k\right) \cdot \frac{\left(\beta_2^+\right)^{-1}(\alpha b + k) - \alpha\underline{v}_2 - k}{\alpha\bar{v}_2 + k - \alpha\underline{v}_2 - k} \cdot \frac{\left(\left(\beta_1^+\right)^{-1}\right)'(\alpha b + k)}{\alpha\bar{v}_1 + k - \alpha\underline{v}_1 - k} \cdot \alpha\, \mathrm{d}b \tag{3.27}$$

$$= \int_{\underline{b}}^{\bar{b}} (\alpha\beta_1^{-1}(b) - \alpha b) \cdot \frac{\alpha\beta_2^{-1}(b) - \alpha\underline{v}_2}{\alpha\bar{v}_2 - \alpha\underline{v}_2} \cdot \frac{\left(\beta_1^{-1}\right)'\left(\frac{(\alpha b + k) - k}{\alpha}\right)}{\alpha\bar{v}_1 - \alpha\underline{v}_1} \cdot \alpha\, \mathrm{d}b \tag{3.28}$$

$$\begin{aligned}&= \alpha\int_{\underline{b}}^{\bar{b}} (\beta_1^{-1}(b) - b) \cdot \frac{\beta_2^{-1}(b) - \underline{v}_2}{\bar{v}_2 - \underline{v}_2} \cdot \frac{\left(\beta_1^{-1}\right)'(b)}{\bar{v}_1 - \underline{v}_1}\, \mathrm{d}b \\ &= \alpha\pi_1(\underline{v}_1, \bar{v}_1, \underline{v}_2, \bar{v}_2).\end{aligned}$$

(3.27) holds by using the substitution $b^+ = \alpha b + k$. (3.28) follows from Lemma 3.7 and

$$\left(\left(\beta_1^+\right)^{-1}\right)'(b^+) = \frac{\mathrm{d}\left(\alpha\beta_1^{-1}(b) + k\right)}{\mathrm{d}b^+} = \alpha\frac{\mathrm{d}\beta_1^{-1}(b)}{\mathrm{d}b}\frac{\mathrm{d}b}{\mathrm{d}b^+} = \alpha\cdot\left(\beta_1^{-1}\right)'(b)\cdot\frac{1}{\alpha} = \left(\beta_1^{-1}\right)'\left(\frac{b^+ - k}{\alpha}\right)$$

applied to $b^+ = \alpha b + k$. A similar calculation with changed indices gives the result for bidder 2. □

Proof of Lemma 3.9

Using (3.12) and (3.10), the expected profit after the transformation can be written as

$$\pi_1^s(\underline{v}^+,\bar{v}^+,d^+) = \int_{\underline{v}^+-d^+}^{\bar{v}^++d^+} \frac{\bar{s}^+(s^+,d^+)-\underline{s}^+(s^+,d^+)}{(\bar{v}^+-\underline{v}^+)2d^+}\pi_1(\underline{s}^+(s^+,d^+),\bar{s}^+(s^+,d^+),\underline{v}^+,\bar{v}^+)\,\mathrm{d}s^+$$

$$= \int_{\underline{v}-d}^{\bar{v}+d} \frac{\bar{s}^+(\alpha s+k,\alpha d)-\underline{s}^+(\alpha s+k,\alpha d)}{(\alpha\bar{v}+k-\alpha\underline{v}-k)2\alpha d}\pi_1(\underline{s}^+(\alpha s+k,\alpha d),\bar{s}^+(\alpha s+k,\alpha d),\alpha\underline{v}+k,\alpha\bar{v}+k)\cdot\alpha\,\mathrm{d}s \tag{3.29}$$

$$= \int_{\underline{v}-d}^{\bar{v}+d} \frac{\alpha\bar{s}(s,d)-\alpha\underline{s}(s,d)}{\alpha(\bar{v}-\underline{v})2\alpha d}\pi_1(\alpha\underline{s}(s,d)+k,\alpha\bar{s}(s,d)+k,\alpha\underline{v}+k,\alpha\bar{v}+k)\cdot\alpha\,\mathrm{d}s$$

$$= \alpha\int_{\underline{v}-d}^{\bar{v}+d} \frac{\bar{s}(s,d)-\underline{s}(s,d)}{(\bar{v}-\underline{v})2d}\pi_1(\underline{s}(s,d),\bar{s}(s,d),\underline{v},\bar{v})\,\mathrm{d}s \tag{3.30}$$

$$= \alpha\pi_1^s(\underline{v},\bar{v},d).$$

(3.29) follows from the substitution $s^+ = \alpha s + k$, (3.30) from Lemma 3.8. □

Proof of Proposition 3.11

The proof proceeds in several steps.

First step: the maximum bid increases, $\bar{b}^+ > \bar{b}$.

We use (3.20) to calculate the difference $\bar{b}^+ - \bar{b}$:

$$\bar{b}^+ - \bar{b} = \frac{\bar{v}_1^+\cdot 1 - \left(\frac{\underline{v}_1^++0}{2}\right)^2}{\bar{v}_1^+ - \underline{v}_1^+ + 1 - 0} - \frac{\bar{v}_1\cdot 1 - \left(\frac{\underline{v}_1+0}{2}\right)^2}{\bar{v}_1 - \underline{v}_1 + 1 - 0}$$

$$= \frac{(\bar{v}_1-\underline{v}_1+1)\left(\bar{v}_1^+ - \left(\frac{\underline{v}_1^+}{2}\right)^2\right) - (\bar{v}_1^+-\underline{v}_1^++1)\left(\bar{v}_1 - \left(\frac{\underline{v}_1}{2}\right)^2\right)}{\underbrace{(\bar{v}_1^+-\underline{v}_1^++1)}_{>0}\underbrace{(\bar{v}_1-\underline{v}_1+1)}_{>0}}.$$

As the denominator is positive, we only need to calculate the sign of the numerator to see whether $\bar{b}^+ > \bar{b}$ or not:

$$(\bar{v}_1-\underline{v}_1+1)\left(\bar{v}_1^+ - \left(\frac{\underline{v}_1^+}{2}\right)^2\right) - (\bar{v}_1^+-\underline{v}_1^++1)\left(\bar{v}_1 - \left(\frac{\underline{v}_1}{2}\right)^2\right)$$

$$= \bar{v}_1^+\left(1-\frac{1}{2}\underline{v}_1\right)^2 + (\underline{v}_1^+-\underline{v}_1)\frac{\underline{v}_1^+\underline{v}_1}{4} - \frac{1}{4}\left((\underline{v}_1^+)^2-\underline{v}_1^2\right) - \bar{v}_1\left(1-\frac{1}{2}\underline{v}_1^+\right)^2$$

$$\geq \bar{v}_1\left(1-\frac{1}{2}\underline{v}_1\right)^2 + (\underline{v}_1^+-\underline{v}_1)\frac{\underline{v}_1^+\underline{v}_1}{4} - \frac{1}{4}\left((\underline{v}_1^+)^2-\underline{v}_1^2\right) - \bar{v}_1\left(1-\frac{1}{2}\underline{v}_1^+\right)^2$$

$$= (\underline{v}_1^+-\underline{v}_1)\left(\bar{v}_1 + \frac{\underline{v}_1^+\underline{v}_1}{4} - \frac{1}{4}(\underline{v}_1^++\underline{v}_1)(1+\bar{v}_1)\right)$$

$$\geq (\underline{v}_1^+-\underline{v}_1)\left(\bar{v}_1 - \frac{1}{2}\bar{v}_1(1+\bar{v}_1)\right)$$

$$= (\underline{v}_1^+-\underline{v}_1)\left(\frac{1}{2}\bar{v}_1(1-\bar{v}_1)\right)$$

$$\geq 0.$$

The first inequality is strict if $\bar{v}_1^+ > \bar{v}_1$, the second inequality is strict if $\underline{v}_1^+ > \underline{v}_1$. As at least one of these two statements is true by assumption, we get $\bar{b}^+ - \bar{b} > 0$.

Second step: the bids of bidder 2 increase: for all $b \in [\underline{b}^+, \bar{b}]$ it holds that $\left(\beta_2^+\right)^{-1}(b) \leq \beta_2^{-1}(b)$.

First note that $\beta_2^{-1}(\bar{b}) = 1$ and $\left(\beta_2^+\right)^{-1}(b) < 1$ as $\bar{b} < \bar{b}^+$ by the first step. Hence, the assertion is true at the top. Now assume that the assertion fails for some lower b. Then, by continuity of the bid functions, there is a largest b^* in the interior of the interval where the two inverse bid functions cross,

$$b^* := \max_{b \in (\underline{b}^+, \bar{b})} \{b | \left(\beta_2^+\right)^{-1}(b) = \beta_2^{-1}(b)\}.$$

To come to a contradiction, we look at two different cases regarding the inverse bid function of bidder 1. The first case is $\left(\beta_1^+\right)^{-1}(b^*) < \beta_1^{-1}(b^*)$.

By the first-order conditions of the maximization problems of the two bidders, we get directly the following differential equations[4]:

$$\left(\beta_1^{-1}\right)'(b)\left(\beta_2^{-1}(b) - b\right) = \beta_1^{-1}(b) - \underline{v}_1$$
$$\left(\beta_2^{-1}\right)'(b)\left(\beta_1^{-1}(b) - b\right) = \beta_2^{-1}(b) - \underline{v}_2.$$

Applying this to our setting, as $\underline{v}_2 = 0$ the following equation holds at b^*:

$$\left(\left(\beta_2^+\right)^{-1}\right)'(b^*)\left(\left(\beta_1^+\right)^{-1}(b^*) - b^*\right) = \left(\beta_2^+\right)^{-1}(b^*) = \beta_2^{-1}(b^*) = \left(\beta_2^{-1}\right)'(b^*)\left(\beta_1^{-1}(b^*) - b^*\right). \tag{3.31}$$

By assumption, we have $\left(\beta_1^+\right)^{-1}(b^*) < \beta_1^{-1}(b^*)$. For (3.31) to hold, it is thus necessary that $\left(\left(\beta_2^+\right)^{-1}\right)'(b^*) > \left(\beta_2^{-1}\right)'(b^*)$. This leads to a contradiction: by construction of b^* we know that for all $\tilde{b} > b^*$ the inequality $\left(\beta_2^+\right)^{-1}\left(\tilde{b}\right) < \beta_2^{-1}\left(\tilde{b}\right)$ is true. Thus, at b^*, with $\left(\beta_2^+\right)^{-1}(b^*) = \beta_2^{-1}(b^*)$, we get that β_2^{-1} is at least as steep as $\left(\beta_2^+\right)^{-1}$. Consequently, $\left(\left(\beta_2^+\right)^{-1}\right)'(b^*) \leq \left(\beta_2^{-1}\right)'(b^*)$ holds, which contradicts the conclusion from above.

Thus, only the remaining case $\left(\beta_1^+\right)^{-1}(b^*) \geq \beta_1^{-1}(b^*)$ is possible. However, we will come to a contradiction in this case as well. We make use of an equilibrium condition derived by Kaplan and Zamir (2007) from the differential equations. This is equation (6) in their paper:

$$\beta_1^{-1}(b) = \frac{b\beta_2^{-1}(b) - (\underline{v}_1 + \underline{v}_2)b + \frac{(\underline{v}_1+\underline{v}_2)^2}{4}}{\beta_2^{-1}(b) - b}. \tag{3.32}$$

We apply this equation to our setting and conclude that at b^*

$$\frac{b^*\beta_2^{-1}(b^*) - \underline{v}_1 b^* + \frac{\underline{v}_1^2}{4}}{\beta_2^{-1}(b^*) - b^*} = \beta_1^{-1}(b^*) \leq \left(\beta_1^+\right)^{-1}(b^*) = \frac{b^*\left(\beta_2^+\right)^{-1}(b^*) - \underline{v}_1^+ b^* + \frac{\left(\underline{v}_1^+\right)^2}{4}}{\left(\beta_2^+\right)^{-1}(b^*) - b^*}.$$

[4] see e.g. Kaplan and Zamir (2007), equation (2)

As by assumption $\left(\beta_2^+\right)^{-1}(b^*) = \beta_2^{-1}(b^*)$, this reduces to

$$-\underline{v}_1 b^* + \frac{\underline{v}_1^2}{4} \leq -\underline{v}_1^+ b^* + \frac{\left(\underline{v}_1^+\right)^2}{4} \qquad \Longleftrightarrow \qquad b^*\left(\underline{v}_1^+ - \underline{v}_1\right) \leq \frac{1}{4}\left(\underline{v}_1^+ - \underline{v}_1\right)\left(\underline{v}_1^+ + \underline{v}_1\right).$$

In case the lower end of the interval strictly increases, $\underline{v}_1^+ > \underline{v}_1$, we conclude

$$b^* \leq \frac{1}{4}\left(\underline{v}_1^+ + \underline{v}_1\right) < \frac{\underline{v}_1^+}{2}.$$

This is a contradiction to the fact that $b^* > \underline{b}^+ = \frac{\underline{v}_1^+}{2}$. In case the lower end of the interval stays the same, $\underline{v}_1^+ = \underline{v}_1$, by (3.32) we can directly see that $\left(\beta_1^+\right)^{-1}(b^*) = \beta_1^{-1}(b^*)$ needs to hold. We look at the explicit solution of the equilibrium bid functions, (3.21) and (3.22) or, in case $\underline{v}_1^+ = \underline{v}_1 = 0$, (3.24) and (3.25). Using the fact that $b^* > \underline{b}^+$, it follows from $\left(\beta_1^+\right)^{-1}(b^*) = \beta_1^{-1}(b^*)$ that respectively $c_1 = c_1^+$ and $c_2 = c_2^+$ or $c = c^+$ need to hold. But this is not consistent with the true values of these constants – it would e.g. follow that the bid functions are the same for both intervals. We thus arrived at a contradiction and finished the proof of the second step.

Third step: the profit of bidder 1 with valuation v_1 is weakly decreasing.

Suppose to the contrary that the expected profit of bidder 1 with valuation v_1 is higher after the shift of the interval. Furthermore, assume b and b^+ are such that $\beta_1^{-1}(b) = v_1 = \left(\beta_1^+\right)^{-1}(b^+)$. By the second step[5], we know that $\left(\beta_2^+\right)^{-1}(b^+) \leq \beta_2^{-1}(b^+)$. Hence, as bidder 2's valuation is distributed uniformly on $[0, 1]$, we conclude that

$$(v_1 - b)\beta_2^{-1}(b) < (v_1 - b^+)\left(\beta_2^+\right)^{-1}(b^+) \leq (v_1 - b^+)\beta_2^{-1}(b^+).$$

This would be a profitable deviation for bidder 1 to b^+ in the case with the unshifted interval, a contradiction, as bidding b is equilibrium behavior by assumption. This concludes the proof. □

Proof of Result 3.13

For the second case, $d < 0.5$, we rewrite the expected profit with signaling as follows:

$$\begin{aligned}
\pi_1^s(0,1,d) &\overset{(3.12)}{=} \int_{-d}^{1+d} h_d(s)\pi_1(s,d)\,\mathrm{d}s \\
&\overset{(3.10)}{=} \int_{-d}^{1+d} \frac{\bar{s}(s,d) - \underline{s}(s,d)}{2d}\pi_1(\underline{s}(s,d), \bar{s}(s,d), 0, 1)\,\mathrm{d}s \\
&= \int_{-d}^{d} \frac{s+d}{2d}\pi_1(0, s+d, 0, 1)\,\mathrm{d}s + \int_{d}^{1-d} 1 \cdot \pi_1(s-d, s+d, 0, 1)\,\mathrm{d}s \\
&\qquad + \int_{1-d}^{1+d} \frac{1-(s-d)}{2d}\pi_1(s-d, 1, 0, 1)\,\mathrm{d}s
\end{aligned}$$

[5] Technically, we did not show $b^+ \leq \bar{b}$, and in case $b^+ > \bar{b}$ the inverse $\beta_2^{-1}(b^+)$ is not well defined – no type of bidder 2 will bid so high. However, a bid of b^+ will win with probability 1, and it is thus sufficient to identify $\beta_2^{-1}(b^+)$ with the highest possible valuation of bidder 2, which is 1. The inequality is thus trivially fulfilled in this case.

$$= \frac{1}{2d}\int_0^{2d} t\pi_1(0,t,0,1)\,\mathrm{d}t + \int_0^{1-2d} \pi_1(t,t+2d,0,1)\,\mathrm{d}t + \frac{1}{2d}\int_{1-2d}^{1} (1-t)\pi_1(t,1,0,1)\,\mathrm{d}t.$$

We now check for all three summands whether they are increasing or decreasing in d by using Leibniz' rule and the assertion that an increase in the upper or lower end point of the support of bidder 1's uniform distribution also increases his expected profit. We start with the first one:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}d}\frac{1}{2d}\int_0^{2d} t\pi_1(0,t,0,1)\,\mathrm{d}t &= \frac{-1}{2d^2}\int_0^{2d} t\pi_1(0,t,0,1)\,\mathrm{d}t + \frac{1}{2d}\cdot\frac{\mathrm{d}}{\mathrm{d}d}\int_0^{2d} t\pi_1(0,t,0,1)\,\mathrm{d}t\\
&\geq \frac{-1}{2d^2}\pi_1(0,2d,0,1)\int_0^{2d} t\,\mathrm{d}t + \frac{1}{2d}\cdot\left(\int_0^{2d}\frac{\mathrm{d}}{\mathrm{d}d}t\pi_1(0,t,0,1)\,\mathrm{d}t + 2d\pi_1(0,2d,0,1)\cdot 2\right)\\
&= \pi_1(0,2d,0,1)\\
&> 0.\end{aligned}$$

The first summand is thus increasing. The second summand is decreasing:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}d}\int_0^{1-2d}\pi_1(t,t+2d,0,1)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}d}\int_0^{1-2d}\pi_1(1-2d-t,1-t,0,1)\,\mathrm{d}t\\
&= \int_0^{1-2d}\underbrace{\frac{\mathrm{d}}{\mathrm{d}d}\pi_1(1-2d-t,1-t,0,1)}_{<0}\,\mathrm{d}t + \pi_1(0,2d,0,1)\cdot(-2)\\
&< 0.\end{aligned}$$

The third summand is increasing:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}d}\frac{1}{2d}\int_{1-2d}^{1}(1-t)\pi_1(t,1,0,1)\,\mathrm{d}t &= \frac{-1}{2d^2}\int_{1-2d}^{1}(1-t)\pi_1(t,1,0,1)\,\mathrm{d}t\\
&\quad + \frac{1}{2d}\left(\int_{1-2d}^{1}\frac{\mathrm{d}}{\mathrm{d}d}(1-t)\pi_1(t,1,0,1)\,\mathrm{d}t - 2d\pi_1(1-2d,1,0,1)\cdot(-2)\right)\\
&\geq \frac{-1}{2d^2}\pi_1(1,1,0,1)\int_{1-2d}^{1}(1-t)\,\mathrm{d}t + 2\pi_1(1-2d,1,0,1)\\
&\geq -\pi_1(1,1,0,1) + 2\pi_1(0,1,0,1) = -0.25 + \frac{1}{3}\\
&> 0.\end{aligned}$$

To show that $\pi_1^s(0,1,d) < \pi_1(0,1,0,1) = \frac{1}{6}$, we calculate the summands for different d values and use the results from above for the values in between. The following table gives simple (rounded) upper bounds for the values of the summands.

d	summand 1	summand 2	summand 3
0.00	0	0.09	0
0.26	0.01	0.06	0.06
0.36	0.02	0.04	0.075
0.44	0.035	0.02	0.09
0.5	0.05	0	0.095

Given the fact that summands one and three are increasing, and summand 2 is decreasing, we can thus estimate:

- For $d \leq 0.26$: $\pi_1^s(0, 1, d) < 0.01 + 0.09 + 0.06 < \frac{1}{6}$
- For $0.26 \leq d \leq 0.36$: $\pi_1^s(0, 1, d) < 0.02 + 0.06 + 0.075 < \frac{1}{6}$
- For $0.36 \leq d \leq 0.44$: $\pi_1^s(0, 1, d) < 0.035 + 0.04 + 0.09 < \frac{1}{6}$
- For $0.44 \leq d \leq 0.50$: $\pi_1^s(0, 1, d) < 0.05 + 0.02 + 0.095 < \frac{1}{6}$. □

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