Factorable Groups and their Homology

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Introduction

Group homology plays an important role in many areas of pure mathematics. The homology of a group G with coefficients in a G-module M is defined to be the homology of the complex $\mathcal{C}_* \otimes_{\mathbb{Z}G} M$, where $\mathcal{C}_*$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. It is well known, that the homology we obtain is independent of the choice of the resolution. The (normalized) bar resolution $E_*(G)$ provides a free resolution, but the chain groups of $E_*(G)$ are usually too big for computations.

In this work we introduce a family of normed groups, called factorable groups. Such a group G admits for each of its element g a factorization $g = \bar{g} \cdot g'$ into two factors $\bar{g}$, called remainder and g', called prefix, such that the norms of $\bar{g}$ and g' add up to the norm of g, and the norm of g' is minimal. Furthermore, there are axioms for the remainder and prefix of a product.
The factorization leads to a normal form, to geodesics on the Cayley graph, to a combing of G, to an automatic structure on G, and many more things. From the normal form one can obtain a collapsing scheme on the normalized bar resolution of G and therefore there are connections to discrete Morse theory and to re-writing systems (see [7], [8]). These topics are not subject of this thesis, but will be developed in the works of A. Hess, M. Rodenhausen, V. Ozornova, L. Stein and others.

We use the structure of a factorable group G to find a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ which has fewer generators than the bar resolution, making computations easier. Endowing the group G with a norm $N : G \to \mathbb{N}_0$ gives a filtration of the bar resolution $E_*(G)$ and of the bar complex $B_*(G) = E_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z}G$, which we call the *norm filtration.* More precisely, if $X = (g_q|\dots|g_1)$ is a q-dimensional generator of $B_*(G)$ in the inhomogeneous bar notation, then we set its norm or filtration degree equal to $N(g_q) + ... + N(g_1)$.
We study the spectral sequence associated to the norm filtration. The filtration quotients of degree h of the norm filtration of the bar complex are denoted by $\mathcal{N}_*(G)[h]$. More general, if the coefficient module M has an appropriate filtration itself, we obtain a spectral sequence converging to the homology of G with coefficients in M.

The complex $\mathcal{N}_*(G)[h]$ is bounded and the main result of the thesis, Theorem 4.1.1., is that for a factorable group G the homology of $\mathcal{N}_*(G)[h]$ is concentrated in the top degree $* = h$. Consequently, the spectral sequence associated to the norm filtration collapses after the E^2-term and the homology of the group G can be computed as the homology of the E^1-term

$$\dots \longrightarrow H_h(\mathcal{N}_*(G)[h]) \longrightarrow H_{h-1}(\mathcal{N}_*(G)[h-1]) \longrightarrow \dots$$

Moreover, we can identify generators of the homology groups $H_h(\mathcal{N}_*(G)[h])$ by introducing a new chain complex $\mathcal{R}_*(G)$ and maps $\kappa_h : \mathcal{R}_h(G) \to H_h(\mathcal{N}_*(G)[h])$ for each $h \geq 0$. The $\mathbb{Z}$-module $\mathcal{R}_q(G)$ is freely generated by q-tuples of elements of G with minimal non-zero norm and satisfying a monotonicity property. In contrast, the bar complex $B_q(G)$ is generated by all q-tuples of elements of G. We show that the maps κ_h are injective homomorphisms of $\mathbb{Z}$-modules and that κ_h is an isomorphism if the group G is the symmetric group Σ_p for any $p \geq 1$. In fact, using similar arguments, Rui Wang proves in [14] that κ_h is an isomorphism for any group G, provided that G has finitely many elements with minimal non-zero norm.

There are many examples of normed groups which are factorable. Indeed, factorability is a property with respect to the given norm N on the group G. It turns out that any group G is factorable with respect to the constant norm, i.e. the norm with $N(g) = 1$ for $g \neq 1$; namely we simply set $\bar{g} = 1$ and $g' = g$. But for this factorization the norm filtration coincides with the skeletal filtration and thus nothing new is gained. We show that direct-, semi-direct- and free-products of factorable groups are factorable, implying for example that free groups are factorable with respect to the word length norm and that dihedral groups are factorable.

Our most important examples of factorable groups are the symmetric groups Σ_p with the cycle norm, i.e. the word length norm with respect to all transpositions. The factorability of the symmetric groups has important applications in computation of the homology of moduli spaces of Riemann surfaces. In fact, the complexes $\mathcal{N}_*(\Sigma_p)[h]$ for a fixed value of h and for varying values of p appeared as the columns of a certain double-complex and the theory of factorable groups arose from the investigations of this double-complex, which we describe now in more detail.

Let $\mathfrak{M}^m_{g,n}$ denote the moduli space of compact, connected, oriented Riemann surfaces $F^m_{g,n}$ of genus $g \geq 0$ with $n \geq 1$ boundary curves and with $m \geq 0$ punctures. This moduli space is a non-compact manifold obtained as the classifying space of the mapping class group $\Gamma^m_{g,n}$, the group of isotopy classes of orientation-preserving diffeomorphisms fixing the boundary pointwise and permuting the punctures.

In [3] C.-F. Bödigheimer introduces a multi-simplicial space $\mathfrak{Par}(g, m, n)$, the space of parallel slit domains. He shows that $\mathfrak{Par}(g, m, n)$ is a vector bundle $\mathfrak{Par}(g, m, n) \to \mathfrak{M}^m_{g,n}$ over the moduli space, in particular this is a homotopy equivalence. More precisely, he considers a certain configuration space of $2h = 4g - 4 + 4n + 2m$ horizontal parallel slits in n disjoint complex planes. Cutting the planes along the slits and re-gluing these pieces according to certain gluing-rules one obtains a surface. These spaces appeared later for example in the works [2], [4], [5], [9], [10] and [13].

There is an equivalent, more combinatorial description of the moduli space (see [10], [13]). To describe a Riemann surface, one considers a $(q + 1)$-tuple $(\sigma_q, \dots, \sigma_0)$ of permutations in the symmetric group Σ_p satisfying certain conditions; here q varies from 1 to h, and p from 1 to $2h$. Some important conditions are:
(a) σ_0 has a special form, in particular it has n cycles,
(b) the sum of the norms of $\sigma_k \, \sigma_{k-1}^{-1}$ is h,
(c) σ_q has $m + 1$ cycles
This data determines a finite bi-graded complex $P(h, m, n)$. The vertical face operator ∂'_k is the bar-face for the group Σ_p, the horizontal face operators ∂''_k are the simultaneous application of

certain functions $D_k : \Sigma_p \to \Sigma_{p-1}$, i.e. $\partial_k''((\sigma_q, \ldots, \sigma_0)) = (D_k(\sigma_q), \ldots, D_k(\sigma_0))$. The functions D_k are not homomorphisms, but they satisfy the simplicial identities. If a face does not satisfy the conditions above it is in a subcomplex $P'(h, m, n)$ of "degenerate" surfaces. The quotient complex P/P' is homotopy equivalent to the Thom space of some vector bundle over moduli space $\mathfrak{M}^m_{g,n}$ if $h = 2g - 2 + 2n + m$. Thus P/P' is the Spanier-Whitehead dual to $\mathfrak{M}^m_{g,n}$, and the homology of one is Poincare dual to the cohomology of the other.

The simplicial chain complex of this pair is denoted by $\mathcal{S}_{*,*}(h, m, n)$. It is a complex of finite type, with top degree equal to h. Note that we are working with two sets of parameters: the genus g, the number n of boundary curves, and the number m of punctures determine the topological type of the surface, and therefore the moduli space; p determines the symmetric group Σ_p, and q is the homological degree in the bar resolution for the group (whereas $p + q$ is the homological degree for the moduli space), and h (the negative Euler characteristic of the surface) is the filtration degree. What we do is to amalgamate the bar resolutions of all symmetric groups to a bi-complex, and then select a specific filtration quotient.

R. Ehrenfried in [10] and J. Abhau in [1], and both authors together with Bödigheimer in [2], used the complex $\mathcal{S}_{*,*}(h, m, n)$ to compute the homology of $\mathfrak{M}^m_{g,n}$ in the case $n = 1$, and $h = 2g + m \leq 5$. However, the number of cells in these complexes grows very fast with h.

In those computations they have encountered an interesting phenomenon. Namely, the columns of the spectral sequence associated to this double complex have all homology concentrated in their respective top degree. Thus the spectral sequence collapses after the d_1-differential. Starting from this double complex $\mathcal{S}_{*,*}(h, m, n)$ we investigate this phenomenon.

We give now a more detailed description of the contents of the various chapters.

In **Chapter 1: Normed Groups** we introduce the notion of normed groups. The norm assigns to each group element $g \in G$ a natural number $N(g)$ such that
(1) $N(g \cdot h) \leq N(g) + N(h)$ for any $g, h \in G$,
(2) $N(g) = 0$ if and only if $g = 1$ and
(3) $N(g) = N(g^{-1})$ for all $g \in G$.
We give some examples of normed groups and constructions of norms, in particular norms on products and semi-direct products of normed groups.

In **Chapter 2: Norm Filtration** we investigate filtrations of various objects and spaces associated to a group. The norm induces a filtration of the group itself, but more important of the bar resolution $E_*(G)$, of the bar complex $B_*(G)$ and of the classyfing space BG of G. Note, that the corresponding filtration on the classifying space BG is similiar, but different from the Rips filtration.
We focus on the columns $\mathcal{N}_*(G)[h]$ of the spectral sequence associated to the norm filtration.
Similarly, if a G-module M has a *norm-admissible* filtration, then the complex $E_*(G, M)$ is filtered and we obtain a spectral sequence converging to the homology of G with coefficients in the module M. The columns of this spectral sequence are denoted by $\mathcal{N}_*(G, M)[h]$.

In **Chapter 3: Factorable Groups** we develop the theory of factorable groups. The strategy here is to construct maps $\mathcal{N}_q(G)[h] \to \mathcal{N}_{q+1}(G)[h]$, which behave like chain homotopies. They will

be defined by means of a norm-preserving factorization map $\eta : G \to G \times G$, that is a function $\eta(g) = (\bar{g}, g')$, such that
(1) $\bar{g} \cdot g' = g$,
(2) $N(\eta(g)) = N(g)$, that is $N(\bar{g}) + N(g') = N(g)$,
(3) $N(g')$ is the minimal non-zero value of the norm $N : G \to \mathbb{N}_0$,
(4) Consider the following diagram:

$$\begin{tikzcd} G \times G \arrow[r, "\eta \times id"] \arrow[d, "\mu"] & G \times G \times G \arrow[r, "id \times \mu"] & G \times G \arrow[r, "id \times \eta"] & G \times G \times G \arrow[d, "\mu \times id"] \\ G \arrow[rrr, "\eta"] & & & G \times G \end{tikzcd}$$

where $\mu : G \times G \to G$ is the multiplication of G. Denote the upper composition $(\mu \times id) \circ (id \times \eta) \circ (id \times \mu) \circ (\eta \times id)$ in the diagram by α_u and the lower composition $\eta \circ \mu$ by α_l. We require that for all pairs $(g, h) \in G \times G$ we have $N(\alpha_u((g, h))) = N((g, h))$ if and only if $N(\alpha_l((g, h))) = N((g, h))$ and for pairs where both compositions are norm-preserving $\alpha_u((g, h)) = \alpha_l((g, h))$.
We call a normed group *factorable*, or *norm-factorable*, if it admits such a factorization map. The basic examples of factorable groups are discussed: groups with constant norm, symmetric groups, free groups, and direct-, semi-direct- or free-products of factorable groups.

The main result of **Chapter 4: The Homology of Factorable Groups** is the following

Theorem 4.1.1.: *If G is a factorable group with respect to the norm N, then the homology of the complex $\mathcal{N}_*(G)[h]$ is concentrated in the top degree $m \cdot h$.*

Here m denotes the minimal non-zero value of the norm N.
The next question which naturally arises is the following: can we find generators for

$$H_h(\mathcal{N}_*(G)[h]) = \ker\{d : \mathcal{N}_h(G)[h] \to \mathcal{N}_{h-1}(G)[h]\}?$$

We give a partial answer by identifying a set $R_h(G)[h]$ of certain generators of the top dimensional module $\mathcal{N}_h(G)[h]$ and introducing a map $\kappa_h : \mathcal{R}_h(G)[h] \to H_h(\mathcal{N}_*(G)[h])$, where $\mathcal{R}_h(G)[h]$ is the free module generated by the set $R_h(G)[h]$. Our result is the following

Proposition 4.3.5.: *The map $\kappa_h : \mathcal{R}_h(G)[h] \to H_h(\mathcal{N}_*(G)[h])$ is a split injective homomorphism of modules.*

The case of non-trivial coefficients is also investigated here and depending on the filtration of the coefficient module M we obtain generalizations of Theorem 4.1.1..

Chapter 5: Symmetric Groups. We specialize here to the symmetric groups Σ_p with the cycle norm mentioned earlier and we prove

Theorem 5.2.1: *The symmetric group Σ_p is factorable with respect to the word length norm.*

We strengthen the general result about generators of $H_h(\mathcal{N}_*(\Sigma_p)[h])$ for symmetric groups. Note that the generators of $\mathcal{N}_h(\Sigma_p)[h]$ and therefore elements of $R_h(\Sigma_p)[h]$ are h-tuples of transpositions. We prove the following

Theorem 5.4.1.: *The map* $\kappa : \mathcal{R}_h(\Sigma_p)[h] \to H_h(\mathcal{N}_*(\Sigma_p)[h])$ *is an isomorphism.*

As mentioned above, meanwhile R.Wang found a proof for any group G with a finete number of norm one elements.
As an application we describe the complex $\mathcal{R}_*(\Sigma_p)[h]$ computing the homology of the symmetric group Σ_p in more detail.

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Chapter 1

Normed Groups

1.1 Basic Notions and Examples

In this section we introduce the notion of normed groups.

Definition 1.1.1. A *normed group* is a pair (G, N), where G is a group and N is a *norm* on G, that is a function $N : G \to \mathbb{N}$ satisfying the following properties:

$$N(g) = 0 \iff g = 1, \tag{$N1$}$$

$$N(g \cdot h) \leq N(g) + N(h) \text{ for any } g, h \in G, \tag{$N2$}$$

$$N(g^{-1}) = N(g) \text{ for any } g \in G. \tag{$N3$}$$

The notion of normed groups is closely related to that of metric groups: a norm on G induces a metric on G by setting $d(g, h) = N(g \cdot h^{-1})$. Vice versa, a right translation-invariant, integer valued metric (i.e. a metric for which $d(g \cdot h, g \cdot f) = d(h, f)$ holds for any g, h and f in G) determines a norm by setting $N(g^{-1} \cdot h) = d(g, h)$. Similarly, a left translation-invariant integer valued metric defines a norm by $N(g \cdot h^{-1}) = d(g, h)$. A metric, which is both left and right translation-invariant determines a conjugation-invariant norm, that is a norm N, which satisfies

$$N(g \cdot h \cdot g^{-1}) = N(h) \text{ for any } g, h \in G. \tag{$N4$}$$

Note that relaxing the symmetry condition $d(g, h) = d(h, g)$ in the definition of the metric, one obtains the notion of *quasi-metric* groups. For the corresponding *quasi-norm* condition $(N3)$ is not necessarily satisfied.

In the following we consider some examples of normed groups.

Example 1.1.2. The *constant norm* N_c on a group G assigns to each element $1 \neq g \in G$ a fixed non-zero value $c \in \mathbb{N}$. The norm N_c clearly satisfies conditions $(N1-N3)$ and it is also conjugation-invariant. The metric which determines N_c is the discrete metric on G.
If $c = 1$ we call the constant norm N_1 the *trivial norm*.

Other basic examples of normed and quasi-normed groups are the cyclic groups.

Example 1.1.3. The infinite cyclic group $\mathbb{Z}$, generated by t in a multiplicative notation, is a normed group with the norm $N(t^n) = |n|$ for each $n \in \mathbb{Z}$.

Example 1.1.4. Using a similar definition $N(t^n) = n$ for each $0 \leq n \leq p-1$ in the case of the finite cyclic group $\mathbb{Z}_p$ with generator t, one obtains a quasi-norm. Condition $(N2)$ is clearly satisfied, since the $(mod\ p)$ value of $n+m$ is less or equal than $n+m$, but this is only a quasi-norm, since it is not symmetric: $N(t^{-n}) = N(t^{p-n}) \neq N(t^n)$ if $2n \neq p$.

However, we can also define a norm for the group $\mathbb{Z}_p$.

Example 1.1.5. The finite cyclic group $\mathbb{Z}_p$ is a normed group with the norm $N(t^n) = min\{n, p-n\}$ for each $0 \leq n \leq p-1$. It follows from the definition, that $N(t^n) = N(t^{-n}) = N(t^{p-n})$, hence condition $(N3)$ is satisfied.

The sum $N_1 + N_2$ of two norms on G is again a norm. More general, the linear combination $\lambda_1 N_1 + \lambda_2 N_2$ of two norms is a norm for any $\lambda_1, \lambda_2 \in \mathbb{N}$.
Moreover, given a normed group (G, N) one obtains a norm on any subgroup $H \subseteq G$ by restricting N to H.

1.2 Norms from Word Length

A large class of normed groups arises from the following construction, where we prescribe the values of the norm only for a given generating set of the group. The prescribed values are called *weights* and they are extended to the entire group using minimally weighted presentations of group elements. More precisely, the method is described in the following

Construction 1.2.1. Assume that $S \subseteq G$ is a generating set of the group G and $w : S \to \mathbb{N}$ is weighting function, such that

$$w(s) > 0 \text{ for each } 1 \neq s \in S, \tag{1.2.1}$$

$$w(s_1 s_2 ... s_n) \leq \sum_{i=1}^{n} w(s_i) \text{ for any } s_1, s_2, ..., s_n \in S, \text{ for which } s_1 s_2 ... s_n \in S, \tag{1.2.2}$$

$$w(s) = w(s^{-1}) \text{ for each } s \in S \cap S^{-1}. \tag{1.2.3}$$

Then we can extend the function w to a norm, denoted by $N_{S,w}$ on the group G by defining $N_{S,w}(g)$ to be the minimum among the natural numbers

$$\sum_{j=1}^{k} |\alpha_j| w(s_{i_j})$$

for each presentation $g = s_{i_1}^{\alpha_1} \cdot ... \cdot s_{i_k}^{\alpha_k}$, where $s_{i_1}, \ldots, s_{i_k} \in S$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}$. The resulting function $N_{S,w} : G \to \mathbb{N}$ is a norm on G: property $(N2)$ clearly follows from the construction of $N_{S,w}$ and property $(N3)$ follows from the fact that $g = s_{i_1}^{\alpha_1} \cdot ... \cdot s_{i_k}^{\alpha_k}$ is a presentation of g with minimal weight if and only if $g^{-1} = s_{i_k}^{-\alpha_k} \cdot ... \cdot s_{i_1}^{-\alpha_1}$ is a presentation of g^{-1} with minimal weight. Note that if S has minimal cardinality among the generating sets of G, then any non-negative weighting function will induce a norm on G.

In particular, an important special case of the construction above is when the weighting function $w : S \to \mathbb{N}$ is the constant function $w(s) = 1$ for each $1 \neq s \in S$. The constant function satisfies the conditions (1.2.1-3) and the induced norm, which we denote by N_S^{wl} is called the *word length norm*, with respect to the generating set S.

Definition 1.2.2. The *word length* norm N_S^{wl} on a group G with respect to the generating set S of G is defined for each $g \in G$ to be the minimal number (with multiplicity) of generators from the set S needed to present g.

Remark 1.2.3. To obtain a norm N_S^{wl} which is conjugation-invariant it is enough to assume that the generating set S is closed under conjugation. Indeed, if the minimal presentation of $h \in G$ is $h = s_{i_1}^{\alpha_1} \cdot ... \cdot s_{i_k}^{\alpha_s}$, then $g \cdot h \cdot g^{-1}$ can be presented as $(g \cdot s_{i_1} \cdot g^{-1})^{\alpha_1} \cdot ... \cdot (g \cdot s_{i_k} \cdot g^{-1})^{\alpha_k}$ and hence condition $(N4)$ is fulfilled: $N_S^{wl}(g \cdot h \cdot g^{-1}) = N_S^{wl}(h)$.

Examples of the word length norm include also some examples of Section 1.1. The constant norm N_c on a group G is obtained by taking the generating set $S = G$ and the constant weighting function $w(g) = c$ for each $1 \neq g \in G$. Norms for cyclic groups defined in Example 1.1.3. and in Example 1.1.5. are also word length norms with respect to the generating sets $\{t\}$ and $\{t, t^{p-1}\}$, respectively. We also obtain new and important examples:

Example 1.2.4. The word length norm on the free group F_n of rank n, with respect to the set of free generators $S = \{t_1, ..., t_n\}$. The norm of a reduced word in F_n is then the length of the word. Note that this norm is not conjugation-invariant. More generally, we obtain a norm on the free product $G*H$ of two normed group (G, N_G) and (H, N_H). Here we use the generating set $S = G \cup H$ of the group $G * H$ with the weighting function $w(s) = N_G(s)$ if $s \in G$ and $w(s) = N_H(s)$ if $s \in H$. Since N_G and N_H are norms on the groups G and H respectively, it follows that w satisfies the conditions (1.2.1-3) and hence it induces a norm $N_{G*H} := N_{G\cup H,w}$ on the group $G * H$. We call this norm the *free product norm.*

Another important example is provided by the symmetric group Σ_p. Among various generating sets of Σ_p we mention here two important cases. For our notions regarding the symmetric groups and for a more thorough discussion of Example 1.2.6. we refer to Section 5.1.

Example 1.2.5. The symmetric group Σ_p is generated by the set of elementary transpositions $S_{el} = \{\sigma_i = (i, i+1) \mid 1 \leq i \leq p-1\}$. The relations between these generators are:

$$\sigma_i^2 = 1 \text{ for } 1 \leq i \leq p-1$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq p-2$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for all } 1 \leq i, j \leq p-1 \text{ with } |i-j| > 1$$

In the induced norm N a transposition $(i, j) \in \Sigma_p$, where $i < j$ has norm $N((i, j)) = 2(j-i) - 1$, since $(i, j) = \sigma_i \ldots \sigma_{j-2}\sigma_{j-1}\sigma_{j-2} \ldots \sigma_i$ is a presentation with minimal length. Note that the set S_{el} is not closed under conjugation, hence the resulting norm N does not satisfy condition $(N4)$.

Example 1.2.6. We can take for the symmetric group also the larger generating set S_{tr}, the set of all transpositions in the symmetric group Σ_p. The induced conjugation invariant word length norm will play central role in Chapter 5. and we investigate the properties of this norm there.

1.3 Direct and Semi-direct Products

We consider in this section the direct product, or more generally, the semi-direct product of normed groups.

Given two normed groups (G, N_G) and (H, N_H) we define a norm $N := N_{G\times H}$ on the direct product $G \times H$ by setting

$$N_{G\times H}((g, h)) = N_G(g) + N_H(h) \text{ for any } (g, h) \in G \times H. \tag{1.3.1}$$

Lemma 1.3.1. *The function $N_{G\times H}$ is a norm on the group $G \times H$.*

Proof: To verify condition $(N1)$ note that $N((g, h)) = N_G(g) + N_H(h) = 0$ if and only if $(g, h) = (1_G, 1_H)$, the identity element of the group $G \times H$. For the triangle inequality $(N2)$ we compute

$$N((g_1, h_1) \cdot (g_2, h_2)) = N((g_1 \cdot g_2, h_1 \cdot h_2)) = N_G(g_1 \cdot g_2) + N_H(h_1 \cdot h_2) \leq$$

$$\leq N_G(g_1) + N_G(g_2) + N_H(h_1) + N_H(h_2) = N((g_1, h_1)) + N((g_2, h_2)).$$

The symmetry condition $(N3)$ follows from the fact that $(g, h)^{-1} = (g^{-1}, h^{-1})$ and from the symmetry of N_G and N_H. □
We call the norm $N_{G\times H}$ the *direct product norm.*

Note that the norm $N_{G\times H}$ is again a special case of Construction 1.2.1. We can use the generating set $S = \{(g, 1_H) \mid g \in G\} \cup \{(1_G, h) \mid h \in H\}$ of the group $G \times H$ with the weighting function $w((g, 1)) = N_G(g)$ and $w((1, h)) = N_H(h)$. The induced norm $N_{S,w}$ is precisely the norm $N_{G\times H}$. In other words, restricting the direct product norm $N_{G\times H}$ to the subgroups G and H of $G \times H$ we obtain the original norms N_G and N_H, respectively.

Example 1.3.2. The group $\mathbb{Z}^n$ is a normed group with norm N induced by the norm of $\mathbb{Z}$ from Example 1.1.3. on the iterated direct product: the norm of an n-tuple $(t^{k_1}, \ldots, t^{k_n})$ is the sum $|k_1| + \cdots + |k_n|$. More general, if (G, N) is a normed group, we obtain a norm on the group G^n, where the norm of an n-tuple $(g_1, \ldots, g_n)$ is the sum $N(g_1) + \cdots + N(g_n)$.

In a similar way, if (G, N_G) and (H, N_H) are normed groups and $\varphi : H \to Aut(G)$ is a group homomorphism we can construct a norm $N := N_{G\rtimes H}$ on the semi-direct product $G \rtimes_\varphi H$ of G and H. However, the homomorphism φ must satisfy

$$N_G(g) = N_G(\varphi(h)(g)) \text{ for all } g \in G \text{ and } h \in H. \tag{1.3.2}$$

The norm N of a pair $(g, h) \in G \rtimes_\varphi H$ is then defined as

$$N((g, h)) = N_G(g) + N_H(h) \tag{1.3.3}$$

Lemma 1.3.3. *The function $N_{G\rtimes H}$ is a norm on the group $G \rtimes_\varphi H$.*

Proof: Condition $(N1)$ is obvious. For the triangle inequality we use (1.3.2) and (1.3.3):

$$N((g_1, h_1) * (g_2, h_2)) = N((g_1 \cdot \varphi(h_1)(g_2), h_1 \cdot h_2)) = N_G(g_1 \cdot \varphi(h_1)(g_2)) + N_H(h_1 \cdot h_2) \leq$$

$$\leq N_G(g_1) + N_G(\varphi(h_1)(g_2)) + N_H(h_1) + N_H(h_2) = N((g_1, h_1)) + N((g_2, h_2)),$$

where $*$ denotes the multiplication in the group $G \rtimes_\varphi H$.
The inverse of an element $(g, h) \in G \rtimes_\varphi H$ is the pair $(\varphi(h^{-1})(g^{-1}), h^{-1})$ and by equality (1.3.2) we have $N((g, h)^{-1}) = N_G(g^{-1}) + N_H(h^{-1}) = N_G(g) + N_H(h) = N((g, h))$, hence N is a norm. $\square$
We call the norm $N_{G \rtimes H}$ the *semi-direct product norm*.

The direct product $G \times H$ is of course a special case of the semi-direct product with $\varphi(h) = id_G$ for all $h \in H$, which is clearly a norm preserving automorphism.

Example 1.3.4. The dihedral group D_{2m} is the group of symmetries of a regular m-gon. It is generated by two elements r (rotation) and t (reflection) which satisfy the relations $r^m = t^2 = 1$ and $rtr = t$. Equivalently, the group D_{2m} can be regarded as the semi-direct product $\mathbb{Z}_m \rtimes_\varphi \mathbb{Z}_2$, where $\mathbb{Z}_m$ is generated by r and $\mathbb{Z}_2$ is generated by t. The homomorphism $\varphi : \mathbb{Z}_2 \to Aut(\mathbb{Z}_m)$ is given by $\varphi(t) : g \mapsto g^{-1}$ for all $g \in \mathbb{Z}_m$. It follows that $\varphi(t)$ is a norm-preserving homomorphism, hence we obtain a norm N on the dihedral group, where $N((r^k, 1)) = min\{k, m-k\}$ and $N((r^k, t)) = 1 + min\{k, m-k\}$. Note that this norm is the word length norm on D_{2m} with respect to the generating set $\{r, r^{-1}, t\}$.

Chapter 2

Norm Filtration

2.1 Filtration of the Group

The norm on a group can be used to define filtrations on various objects and spaces associated to the group, including the bar resolution and the classifying space of the group.
First of all, the norm N of a normed group G induces an increasing filtration

$$\{1\} = \mathcal{F}_0 G \subseteq \mathcal{F}_1 G \subseteq \cdots \subseteq \mathcal{F}_h G \subseteq \ldots$$

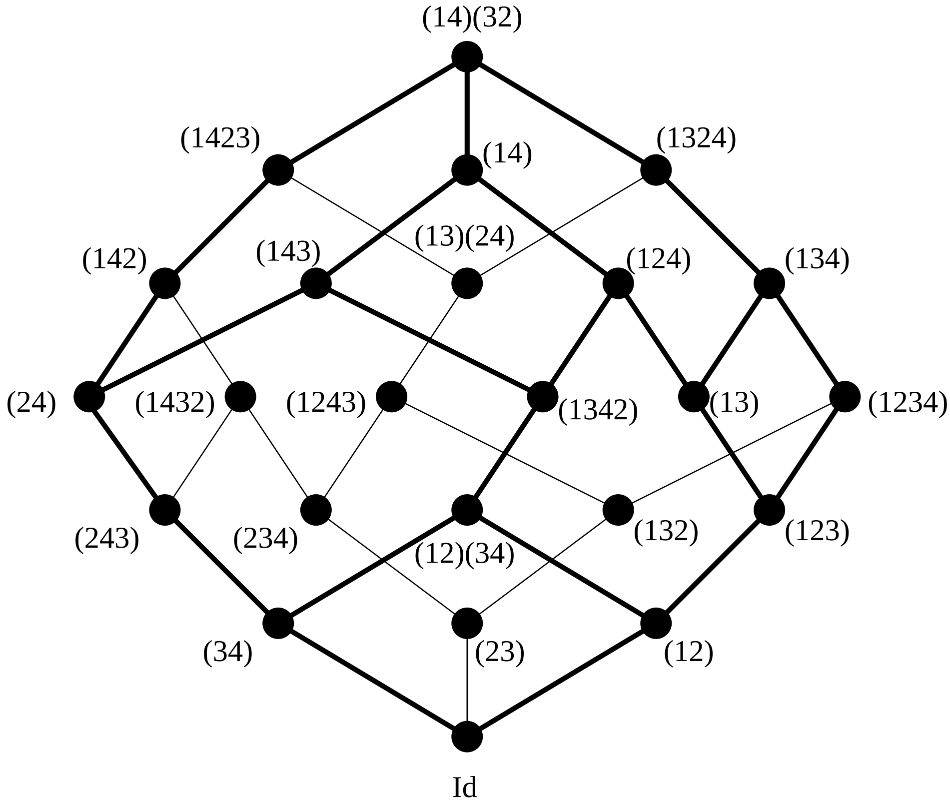
of G itself, where the sets $\mathcal{F}_h G$ are defined by $\mathcal{F}_h G = \{g \in G \mid N(g) \leq h\}$ for each $h \geq 0$.

The filtration stratum $T_h(G) = \mathcal{F}_h G \setminus \mathcal{F}_{h-1} G$ is the set of elements of G with norm h. In particular, the set $T(G) = T_m(G)$ will play an important role throughout the next chapters, where $m = min\{N(g) \mid g \in G, g \neq 1\}$ is the smallest non-zero value of the norm. Note that if the norm N is the word length norm N_S^{wl} with respect to the generating set S, then $T(G) = S \cup S^{-1}$.

In a similar way, the norm N filters the group ring $\mathbb{Z}G$, or more generally, the free $\mathbb{K}$-module $\mathbb{K}G$ generated by the elements of G, where $\mathbb{K}$ is any commutative ring. The submodule $\mathcal{F}_h \mathbb{K}G$ of $\mathbb{K}G$ is defined to be the free $\mathbb{K}$-module generated by the set $\mathcal{F}_h G$.

The sets $T_h(G)$ can be visualized on the Cayley graph. Assume that G is a group, with generating set $S \subset G$ which is closed with respect to inversion. The *Cayley graph* $\Gamma_S(G)$ of the group G with respect to the generating set S is defined as follows. The vertex set of $\Gamma_S(G)$ is the group G and there is an edge from the vertex g to the vertex $g \cdot s$ for each $s \in S$ and $g \in G$. We can turn the Cayley graph $\Gamma_S(G)$ of G into a metric space by requiring that each edge be isometric to the unit interval, and then taking the induced path metric. That is, the distance of two points x, y in $\Gamma_S(G)$ is the length of the shortest path joining them.
If we restrict this metric to the set of vertices, we obtain a metric on the group and even a norm as explained in Section 1.1. This norm is precisely the word length norm N_S^{wl} with respect to the generating set S. The vertices represented by the set $\mathcal{F}_h G$ are hence those vertices of $\Gamma_S(G)$ which are in the closed ball $B(1, h)$ of radius h centered at the vertex 1.

For example, consider the Cayley graph $\Gamma_S(\Sigma_4)$ of the symmetric group Σ_4 with respect to the generating set $S = \{(12), (23), (34)\}$, the set of elementary transpositions:



The graph is drawn in such a way that all vertices of $\Gamma_S(\Sigma_4)$ having norm h are in the same horizontal level for each $h = 0, 1, ..., 6$. Thus, for example $T_2(\Sigma_4)$, the set of vertices with norm two is the set $\{(243), (234), (12)(34), (132), (123)\}$. Note that using S_{tr}, the set of all transpositions as generating set, the Cayley graph we obtain contains the graph $\Gamma_S(\Sigma_4)$ as a subgraph but it has different induced level structure.

2.2 Filtration of the Bar Resolution

We have seen in Example 1.3.2. that if (G, N) is a normed group then the groups $G^q = G \times \cdots \times G$ are also normed groups for each $q \geq 1$. We will keep the notation N for the induced norm on the groups G^q. The norm of a q-tuple is thus the sum of the norms of the factors: $N((g_q, \ldots, g_1)) = N(g_q) + \cdots + N(g_1)$. The norm N induces a filtration on each of the modules $\mathbb{Z}G^q$, $q \geq 1$.

First let us consider the *standard resolution* $E_*(G)$ of $\mathbb{Z}$ over the group ring $\mathbb{Z}G$:

$$\cdots \xrightarrow{d} E_{q+1}(G) \xrightarrow{d} E_q(G) \xrightarrow{d} E_{q-1}(G) \xrightarrow{d} \ldots$$

In the homogeneous notation, $E_q(G)$ is the free $\mathbb{Z}$-module generated by all $(q+1)$-tuples $(g_q : ... : g_0)$ of elements of G. The boundary operator d is the alternating sum $d = \sum_{i=0}^{q}(-1)^i d_i$, where the face operators d_i are defined by $d_i((g_q : ... : g_0)) = (g_q : ... : \hat{g}_i : ... : g_0)$ for a generator $(g_q : ... : g_0) \in G^{q+1}$. The group G acts on $E_q(G)$ by multiplication from the right on each entry:

$(g_q : ... : g_0) \cdot \gamma = (g_q \gamma : ... : g_0 \gamma)$ for any $\gamma \in G$.

A $(q+1)$-tuple $(g_q : ... : g_0)$ is uniquely determined by the element g_0 and the successive quotients $h_i = g_i g_{i-1}^{-1}$. Writing $(g_q : ... : g_0)$ in the form $(g_q g_{q-1}^{-1} | \dots | g_1 g_0^{-1}) g_0$ we obtain the non-homogeneous, or bar- notation of $E_*(G)$. Thus in the non-homogeneous notation $E_q(G)$ is the free $\mathbb{Z}G$-module generated by all elements $(g_q | \dots | g_1) \in G^q$. The face operators in this notation are given by

$$d_i((g_q | \dots | g_1)) = \begin{cases} (g_q | \dots | g_2) g_1 & i = 0, \\ (g_q | \dots | g_{i+1} \cdot g_i | \dots | g_1) & 0 < i < q, \\ (g_{q-1} | \dots | g_1) & i = q. \end{cases} \tag{2.2.1}$$

We define a filtration of each $E_q(G)$ in the spirit of Section 2.1: the norm of a $\mathbb{Z}G$-generator $(g_q | \dots | g_1)$ is defined as the sum $N(g_q) + \dots + N(g_1)$ and we obtain the induced filtration on the free $\mathbb{Z}G$-module $E_q(G)$. Note that the norm is a G-equivariant function on $E_q(G)$, since $N((g_q | \dots | g_1) \cdot \gamma) = N((g_q | \dots | g_1))$ for any $\gamma \in G$.

Definition 2.2.1. The *norm filtration* on the standard resolution $E_*(G)$ is the filtration

$$\mathcal{F}_0 E_*(G) \subseteq \mathcal{F}_1 E_*(G) \subseteq \dots \subseteq \mathcal{F}_h E_*(G) \subseteq \dots$$

where $\mathcal{F}_h E_q(G)$ is the free $\mathbb{Z}G$-submodule of $E_q(G)$ generated by all q-tuples with norm at most h.

By property $(N2)$ of the norm the filtration of the complex $E_*(G)$ is stable with respect to the boundary d.

Lemma 2.2.2. *The boundary map $d : E_*(G) \to E_{*-1}(G)$ is filtration preserving.*

Proof: By the definition of $\mathcal{F}_h E_q(G)$ it is enough to show that for each face $d_i(c)$ of a generator $c = (g_q, \dots, g_1) \in E_q(G)$ the inequality $N(d_i(c)) \leq N(c)$ holds. From formula (2.2.1) this is obvious for the faces d_0 and d_q, and for $1 \leq i \leq q-1$ it follows from property $(N2)$ of the norm applied to the elements $g_{i+1}, g_i \in G$:

$$N(d_i(g_q | \dots | g_1)) = N((g_q | \dots | g_{i+1} \cdot g_i | \dots | g_1)) \leq N((g_q | \dots | g_1)).$$

Therefore, if $N(c) = h$, we have $d(c) \in \mathcal{F}_h E_{q-1}(G)$ and the lemma follows. □

The complex $E_*(G)$ is well known to be acyclic. The map $\zeta : E_q(G) \to E_{q+1}(G)$ defined as

$$\zeta((g_q | \dots | g_1) g_0) = (g_q | \dots | g_1 | g_0) \tag{2.2.2}$$

for generators of $E_q(G)$ is a contracting homotopy: it satisfies the equation $d \circ \zeta + \zeta \circ d = id$. Observe, that the contracting homotopy ζ is not a norm preserving map, hence it does not induce a contracting homotopy of the complexes $\mathcal{F}_h E_*(G)$.

The *normalized* bar-resolution is the quotient $\bar{E}_*(G) = E_*(G)/D_*(G)$, where $D_q(G)$ is the "degenerate" subcomplex of $E_q(G)$ generated by those q-tuples $(g_q | \dots | g_1)$ for which $g_i = 1$ for some i. The contracting homotopy ζ defined in (2.2.2) restricts to a contracting homotopy of $D_*(G)$, showing that the complex $\bar{E}_q(G)$ is acyclic.

The norm defined on G^q induces a filtration of the normalized complex $\bar{E}_*(G)$ as well. Namely, we can regard $\bar{E}_q(G)$ as the free $\mathbb{Z}G$-submodule of $E_q(G)$ generated by the set $(\bar{G})^q$, where $\bar{G} = G \setminus \{1_G\}$ and we restrict the filtration of $E_q(G)$ to the submodule $\bar{E}_q(G)$. The induced boundary $\bar{d} : \bar{E}_q(G) \to \bar{E}_{q-1}(G)$ is given as the alternating sum of the face operators $\bar{d}_i$, where $\bar{d}_i = d_i$ for $i = 0$ and $i = q$, and

$$\bar{d}_i((g_q|\dots|g_1)) = \begin{cases} (g_q|\dots|g_{i+1}\cdot g_i|\dots|g_1) & \text{if } g_{i+1}g_i \neq 1, \\ 0 & \text{if } g_{i+1}g_i = 1. \end{cases}$$

for $0 < i < q$. It follows, that the filtration obtained for $\bar{E}_*(G)$ is stable with respect to the induced boundary $\bar{d} : \bar{E}_q(G) \to \bar{E}_{q-1}(G)$.

2.3 Filtration of the Bar Complex

The norm defined on the bar resolution $E_*(G)$ is invariant under the action of G, therefore it induces a filtration on the bar complex $B_*(G) = E_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z}$ of the group G. Using the bar-notation, as a $\mathbb{Z}$-module $B_q(G)$ is generated by the elements of G^q and the face operators d_i are defined by

$$d_i((g_q|\dots|g_1)) = \begin{cases} (g_q|\dots|g_2) & i = 0, \\ (g_q|\dots|g_{i+1}\cdot g_i|\dots|g_1) & 0 < i < q, \\ (g_{q-1}|\dots|g_1) & i = q. \end{cases} \tag{2.3.1}$$

In other words, we regard $B_*(G)$ as the quotient $E_*(G)/G$. The arguments of Lemma 2.2.1. remain valid, thus the filtration of the complex $B_*(G)$ is stable with respect to the boundary d.

Lemma 2.3.1. *The boundary map $d : B_*(G) \to B_{*-1}(G)$ is filtration preserving.* □

Furthermore, we also obtain a filtration on the reduced bar complex $\bar{B}_*(G)$, which is the quotient of $B_*(G)$ by the "degenerate" subcomplex $D_*(G)$, or equivalently: $\bar{B}_*(G) = \bar{E}_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z}$. As in the case of $\bar{E}_q(G)$, we regard $\bar{B}_q(G)$ as the free $\mathbb{Z}$-submodule of $B_q(G)$ generated by the set $\bar{G}^q$. The filtration is again stable with respect to the induced boundary $\bar{d} : \bar{B}_q(G) \to \bar{B}_{q-1}(G)$.

The map ζ defined in (2.2.2) is not a homomorphism of $\mathbb{Z}G$-modules, hence it does not provide a contracting homotopy for $B_*(G)$. In fact, the bar complex $B_*(G)$ is not acyclic, the k-th homology $H_k(G;\mathbb{Z})$ of the group G with integer coefficients is defined as the k-th homology of the complex $(B_*(G), d)$. Since the functor $- \otimes_{\mathbb{Z}G} \mathbb{Z}$ from the category of G-modules to abelian groups sends any two free resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ to quasi-isomorphic chain complexes, we can equivalently use the smaller complex $(\bar{B}_*(G), \bar{d})$ to compute the homology groups of G.

We can study the homology of the group G via the spectral sequence associated to the norm-filtration of the normalized bar complex $\bar{B}_*(G)$. Let us denote by $\mathcal{N}_q(G)[h]$ the filtration quotient

$E^0_{h,q} = \mathcal{F}_h \bar{B}_q(G)/\mathcal{F}_{h-1}\bar{B}_q(G)$. Then the E^0-term of the spectral sequence has the following form:

$$\begin{array}{ccccc}
\cdots & \mathcal{N}_{q+1}(G)[h-1] & \mathcal{N}_{q+1}(G)[h] & \mathcal{N}_{q+1}(G)[h+1] & \cdots \\
 & \downarrow \bar{d}^0 & \downarrow \bar{d}^0 & \downarrow \bar{d}^0 & \\
\cdots & \mathcal{N}_{q}(G)[h-1] & \mathcal{N}_{q}(G)[h] & \mathcal{N}_{q}(G)[h+1] & \cdots \\
 & \downarrow \bar{d}^0 & \downarrow \bar{d}^0 & \downarrow \bar{d}^0 & \\
\cdots & \mathcal{N}_{q-1}(G)[h-1] & \mathcal{N}_{q-1}(G)[h] & \mathcal{N}_{q-1}(G)[h+1] & \cdots
\end{array}$$

where $\bar{d}^0$ is the map induced by the boundary operator $\bar{d} : \bar{B}_*(G) \to \bar{B}_{*-1}(G)$ on the filtration quotients.

For later reference, let us consider the vertical complex $(\mathcal{N}_*(G)[h], \bar{d}^0)$ in detail. We can regard $\mathcal{N}_q(G)[h]$ as the submodule of $\bar{B}_q(G)$ freely generated by q-tuples of non-trivial group elements $(g_q, \ldots, g_1) \in \bar{G}^q$, such that the $N(g_q) + \cdots + N(g_1) = h$. This complex will play a central role in the following chapters, so to simplify notation, let us denote the boundary operator $\bar{d}^0$ simply by d. To avoid confusion, we recollect the terms of $d := \bar{d}^0$ in the following lemma:

Lemma 2.3.2. *The boundary $d : \mathcal{N}_q(G)[h] \to \mathcal{N}_{q-1}(G)[h]$ is given by $d = \sum_{i=1}^{q-1} d_i$, where*

$$d_i((g_q|\ldots|g_1)) = \begin{cases} (g_q|\ldots|g_{i+1}g_i|\ldots|g_1) & \text{if } N(g_{i+1}g_i) = N(g_{i+1}) + N(g_i), \\ 0 & \text{if } N(g_{i+1}g_i) < N(g_{i+1}) + N(g_i). \end{cases}$$

In fact, the face operators $\bar{d}_0 : (g_q|\ldots|g_1) \mapsto (g_q|\ldots|g_2)$ and $\bar{d}_q : (g_q|\ldots|g_1) \mapsto (g_{q-1}|\ldots|g_1)$ of the boundary map $\bar{d} : \bar{B}_q(G) \to \bar{B}_{q-1}(G)$ are strictly norm-decreasing, since we are working with normalized complexes: g_q and g_1 are non-trivial group elements, therefore $N(g_q) > 0$ and $N(g_1) > 0$. It follows that the faces $\bar{d}_0$ and $\bar{d}_q$ are zero in the quotient. Note that the implication $g_{i+1}g_i = 1 \implies \bar{d}_i = 0$ is included in the case $N(g_{i+1}g_i) < N(g_{i+1}) + N(g_i) \implies d_i = 0$, since if $g_{i+1}g_i = 1$ then $N(g_{i+1}g_i) = 0$.

Remark 2.3.3. Observe that each column in the E^0-term is a bounded chain complex: for fixed h we have $\mathcal{N}_q(G)[h] = 0$ for all $q > h$, since a normalized q-tuple has norm at least q. Hence, each term above the diagonal in the E^0-term is zero: $E^0_{p,q} = 0$ for all $p < q$.

2.4 Non-trivial Coefficients

Next we investigate the situation where we take homology with non-trivial coefficients, that is, we wish to define a filtration of the complex

$$\cdots \xrightarrow{d_M} E_{q+1}(G, M) \xrightarrow{d_M} E_q(G, M) \xrightarrow{d_M} E_{q-1}(G, M) \xrightarrow{d_M} \ldots$$

where (G, N_G) is a normed group, M is a (left) G-module and $E_q(G, M) = E_q(G) \otimes_{\mathbb{Z}G} M$. The boundary d_M is defined as $d \otimes_{\mathbb{Z}G} id$, where d is the boundary of the complex $E_*(G)$.

As a $\mathbb{Z}$-module we will regard $E_q(G,M)$ as $B_q(G) \otimes M$. In this notation the faces of a generator $(g_q|\dots|g_1) \otimes m$ are given by

$$(d_M)_i((g_q|\dots|g_1) \otimes m) = \begin{cases} (g_q|\dots|g_2) \otimes g_1 m & i = 0, \\ (g_q|\dots|g_{i+1} \cdot g_i|\dots|g_1) \otimes m & 0 < i < q, \\ (g_{q-1}|\dots|g_1) \otimes m & i = q. \end{cases} \tag{2.4.1}$$

To obtain a filtration of $E_q(G,M)$ we assume that the G-module M is filtered:

$$\{0\} = M_{-1} \subseteq M_0 \subseteq M_1 \subseteq \dots \subseteq M_h \subseteq \dots \subseteq M$$

such that each M_i is a subgroup of the abelian group M. Note that M_i is not necessarily G-invariant. We will need the following

Definition 2.4.1. Assume that (G,N) is a normed group. A G-module M has *norm-admissible filtration* if for each $g \in G$ the action of g on M, i.e. the map $L_g : M \to M$ defined as $L_g(m) = gm$, satisfies the condition $L_g(m) \in M_{i+N(g)}$.

Note that the function $N_M(m) := k$ for $m \in M_k \backslash M_{k-1}$ is a pseudo-norm on the abelian group M: it is a non-negative, symmetric function satisfying the triangle inequality, but $N_M(m) = 0$ is possible for $m \neq 0 \in M$. We introduce three important examples of G-modules with norm-admissible filtration:

Example 2.4.2. The *constant norm* on M is obtained for a given non-negative integer c from the filtration $M_i = M$ for $i \geq c$ and $M_i = \{0\}$ for $i < c$. Note that if $c \neq 0$ then N_M is indeed the constant norm on the group M.

Example 2.4.3. The G-action on M is trivial. Then any filtration of M is norm-admissible.

Example 2.4.4. $M = \mathbb{Z}G$ with G acting on M by multiplication from the left. M has a filtration induced by the norm N of G: M_k is defined as the free abelian group generated by the set $\mathcal{F}_k G = \{g \in G \mid N(g) \leq k\}$. By property (N2) of the norm N this filtration of $\mathbb{Z}G$ is norm-admissible.

Recall from Section 2.3 that the modules $B_q(G)$ are also filtered. On the tensor product $B_q(G) \otimes M$ of filtered modules we obtain an obvious filtration by defining $\mathcal{F}_h(B_q(G) \otimes M)$ to be the submodule of $B_q(G) \otimes M$ generated by all $(q+1)$-tuples $(g_q|\dots|g_1) \otimes m$ such that $(g_q|\dots|g_1) \in \mathcal{F}_i B_q(G)$, $m \in M_j$ and $i + j = h$. That is,

$$\mathcal{F}_h E_q(G,M) = \sum_{i+j=h} \mathcal{F}_i B_q(G) \otimes M_j$$

Lemma 2.4.5. *Let (G,N) be a normed group and M be a G-module with norm-admissible filtration. Then the boundary map $d_M : E_*(G,M) \to E_{*-1}(G,M)$ is filtration preserving.*

Proof: It follows from (2.4.1) that $(d_M)_i : \mathcal{F}_h E_q(G,M) \to \mathcal{F}_h E_{q-1}(G,M)$ for $1 \leq i \leq q$. Now assume that $(g_q|\dots|g_1) \otimes m \in \mathcal{F}_i B_q(G) \otimes M_j$ for some $i + j = h$. Then clearly $(g_q|\dots|g_2) \in \mathcal{F}_{i-N(g_1)} B_{q-1}(G)$ and since the filtration is norm-admissible, we have $g_1 m \in M_{j+N(g_1)}$. It follows that $(d_M)_0((g_q|\dots|g_1) \otimes m)) \in \mathcal{F}_h E_{q-1}(G,M)$. □

The filtration of the complexes $E_*(G)$ and of $B_*(G)$ are special cases of the filtration of $E_*(G,M)$ as we show in the following examples.

Example 2.4.6. $M = \mathbb{Z}$ with trivial G-action. Then $E_*(G, \mathbb{Z}) = B_*(G)$ and the filtration of $B_*(G)$ defined in Section 2.3. is obtained by using the constant zero norm on the G-module $\mathbb{Z}$.

Example 2.4.7. Using the constant zero norm on $M = \mathbb{Z}G$ we obtain the same filtration of $E_*(G) = E_*(G, \mathbb{Z}G)$ as discussed in Section 2.2.: $\mathcal{F}_h E_q(G)$ is the free abelian group generated by all $(q+1)$-tuples $(g_q|\dots|g_1)g_0$ with $N(g_q) + \dots + N(g_1) \leq h$.
However, using the filtration from Example 2.4.4. on $\mathbb{Z}G$ we obtain a different filtration $\mathcal{F}'$ of $E_*(G)$ where $\mathcal{F}'_h E_q(G)$ is the free abelian group generated by all $(q+1)$-tuples $(g_q|\dots|g_1)g_0$ with $N(g_q)+\dots+N(g_1)+N(g_0) \leq h$. Note that this second filtration is not invariant under the G-action on $E_*(G)$.

As before, to compute the homology of the complex $(E_*(G, M), d_M)$ we can consider the normalized complex $\bar{E}_*(G, M) = \bar{E}_* \otimes_{\mathbb{Z}G} M$ with the induced boundary operator $\bar{d}_M$. The filtration of $E_*(G, M)$ restricts to a filtration of $\bar{E}_*(G, M)$. The columns of the E^0-term of the spectral sequence associated to the filtration of $\bar{E}_*(G, M)$ form the chain complexes $(\mathcal{N}_*(G, M)[h], \bar{d}^0_M)$, where $\mathcal{N}_q(G, M)[h]$ denotes the filtration quotient $E^0_{h,q} = \mathcal{F}_h \bar{E}_q(G, M)/\mathcal{F}_{h-1}\bar{E}_q(G, M)$ and $\bar{d}^0_M$ is induced by the boundary operator $\bar{d}_M$ on the filtration quotients.

2.5 Filtration of the Classifying Space

The norm N of a normed group G induces a filtration of the weakly contractible G-space EG and of the classifying space BG as well. These spaces can be regarded as the geometric realisations of the simplicial sets underlying the chain complexes $E_*(G)$ and $B_*(G)$ respectively.
Explicitly, the q-th term $\mathcal{E}_q(G)$ of the simplicial set $\mathcal{E}_*(G)$ is the group G^{q+1}. The simplicial faces $d_i : \mathcal{E}_q(G) \to \mathcal{E}_{q-1}(G)$ are defined as:

$$d_i((g_q : \dots : g_0)) = (g_q : \dots : \hat{g}_i : \dots : g_0) \text{ for } 0 \leq i \leq q$$

and the simplicial degeneracies $s_i : \mathcal{E}_q(G) \to \mathcal{E}_{q+1}(G)$ are defined as

$$s_i((g_q : \dots : g_0)) = (g_q : \dots : g_i : g_i : \dots : g_0) \text{ for } 0 \leq i \leq q.$$

These maps satisfy the well-known simplicial identities

$$d_i \circ d_j = d_{j-1} \circ d_i \text{ if } i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ id & \text{if } i = j \text{ or } i = j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \text{ if } i \leq j$$

The G-action on $\mathcal{E}_q(G)$, given by $(g_q : \dots : g_0) \cdot g = (g_q g : \dots : g_0 g)$ commutes with the face and degeneracy maps, turning $\mathcal{E}_*(G)$ into a simplical G-space. The space EG is then the geometric realization of $\mathcal{E}_*(G)$:

$$EG = \bigsqcup_{q \geq 0} (\mathcal{E}_q(G) \times \triangle_q)/\sim$$

where $\triangle_q$ is the q-simplex $\{(t_q, \ldots, t_0) \mid 0 \le t_i \le 1, \sum t_i = 1\}$ and the equivalence relation $\sim$ is generated by

$$(g_q : ... : g_0, t_q, ..., t_0) \sim \begin{cases} (g_q : ... : \hat{g}_i : ... : g_0, t_q, ..., \hat{t}_i, ..., t_0) & \text{if } t_i = 0 \\ (g_q : ... : \hat{g}_i : ... : g_0, t_q, ..., t_{i+1} + t_i, ..., t_0) & \text{if } g_{i+1} = g_i. \end{cases}$$

The norm of a simplex $(g_q, \ldots, g_0)$ of the space EG is defined as the sum

$$N(g_q g_{q-1}^{-1}) + \cdots + N(g_1 g_0^{-1}),$$

geometrically a kind of "circumference" of the simplex.
We define $\mathcal{F}_h EG$ to be the union of all cells of EG with norm less or equal than h. Now by Lemma 2.2.2., $\mathcal{F}_h EG$ is a subcomplex of EG: each face of a cell $c \in \mathcal{F}_h EG$ is also in $\mathcal{F}_h EG$.

Observe that the free simplicial G action on EG is norm-preserving, hence we obtain a filtration of the classifying space $BG = EG/G$. Note that the space BG can be also defined as the geometric realization of a simplicial set $\mathcal{B}_*(G)$.

Example 2.5.1. The filtration of BG induced by the constant norm N_1 on G is precisely the skeletal filtration of the CW-complex BG: each q-simplex has norm q. For example the filtration on $B\mathbb{Z}_2 = \mathbb{R}P^\infty$ is necessarily the skeletal filtration of $\mathbb{R}P^\infty$.

2.6 The Rips Filtration

For a δ-hyperbolic group G a different filtration of the space EG can be used to obtain a finite Eilenberg-MacLane space $K(G, 1)$. This is the Rips-filtration, which is briefly introduced in this section using the material from [6] and from [12].

A path connected metric space (X, d) is called a *geodesic metric space* if for all points $x, y \in X$ there is an isometric map from the interval $[0, d(x, y)]$ to a path in X joining x to y. For example if G is a finitely generated group, then the Cayley graph of G with the usual path metric is a geodesic metric space.
Given three points x, y, z in a geodesic metric space X, choose geodesics joining them to form a triangle $\Delta(x, y, z)$. Denote the geodesic sides of this triangle by $[x, y]$, $[x, z]$ and $[y, z]$. We say that the geodesic triangle $\Delta(x, y, z)$ is δ-thin if the distance of any point on one side to the union of the other two sides is bounded above by δ. That is

$$d(p, [y, z] \cup [x, z]) \le \delta \text{ for all } p \in [x, y]$$

and similarly for any permutation of the sides.

A path-connected geodesic metric space X is called *δ-hyperbolic* if there exists $\delta \ge 0$ such that every geodesic triangle in X is δ-thin.

Definition 2.6.1. A finitely generated group G is called *hyperbolic* if the Cayley graph $\Gamma_S(G)$ is a δ-hyperbolic metric space.

It can be shown, that the property of being a hyperbolic group is independent of the generating set S.

Given a metric space (X, d) and a positive number $k > 0$, one can define an abstract simplicial complex $R_k(X)$, called the *Rips complex of* X, whose vertices are the points of X and whose q-simplices are all $(q+1)$-subsets of points in X with diameter at most k.
The Rips complex $R_k(G)$ of a group G is defined as the Rips complex of the subspace $G \subset \Gamma_S(G)$. Hence $R_k(G)$ has a q-simplex for each $(q+1)$-tuple of group elements $(g_q, \ldots, g_0)$ for which

$$diam((g_q, \ldots, g_0)) = max\{d(g_i, g_j) \mid 0 \leq i < j \leq q\} = max\{N_S^{wl}(g_i \cdot g_j^{-1}) \mid 0 \leq i < j \leq q\} \leq k$$

The following theorem is due to Rips, see [6]

Theorem 2.6.2. *Let G be a hyperbolic group. If k is large enough, then the Rips complex $R_k(G)$ is a locally finite, contractible, finite dimensional simplicial complex on which G acts faithfully, properly, simplicially and cocompactly. In particular, if G is torsion free, then the action is free and the quotient of this complex by G is a finite Eilenberg-MacLane space $K(G, 1)$.*

Since there is a bound on the number of points in any ball of a given radius in G, the complex $R_k(G)$ is indeed finite dimensional: it's dimension is one less than the cardinality of the largest set in G of diameter k. We also note, that the 1-skeleton of the Rips-complex $R_k(G)$ is the Cayley graph of G with respect to the generating set $T_k(G)$.

Chapter 3

Factorable Groups

3.1 Factoring Group Elements

In this and in the following chapter we investigate the homology groups of the complex $(\mathcal{N}_*(G)[h], d)$ defined in Section 2.3. More precisely, we want to find conditions for the normed group (G, N) which will ensure that these homology groups are trivial, except in the top degree. To obtain such a result, the standard trick would be to find a contracting homotopy $\mathcal{N}_*(G)[h] \to \mathcal{N}_{*+1}(G)[h]$, but the difficulty here is, that such a map must be norm-preserving. We present here a solution to this problem, by introducing a map $\eta : G \to G \times G$ which will "split" group elements into two factors in a nice and uniform way. In Chapter 4 we will show that the conditions we describe here are sufficient.

Recall from Section 2.1. that the sets $T_h(G) \subset G$ were defined as the filtration strata $\mathcal{F}_h G \setminus \mathcal{F}_{h-1} G$ and $T = T_m(G)$ was the set of group elements with minimal (non-zero) value m of the norm.

It will be convenient to introduce the notion of the graded object associated to the norm filtration of the group G. In general, if S is a filtered set

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_h \subseteq \ldots S$$

the graded object $\mathcal{G}r_*(S)$ associated to the filtered set S is defined as the wedge sum of the filtration quotients S_h/S_{h-1}. These quotients are the differences $S_h \setminus S_{h-1}$ plus an extra base point $+$ added. Thus $Gr_*(S)$ is - as a set - nothing but S plus an extra basepoint $+$.
A morphism $\varphi : S_1 \to S_2$ of filtered sets is a filtration preserving function of the underlying sets and it induces a graded map $\mathcal{G}r_*(\varphi) := \varphi_* : \mathcal{G}r_*(S_1) \to \mathcal{G}r_*(S_2)$ between the associated graded objects. That is, $\varphi_*(s) = +$ if the filtration degree of $\varphi(s)$ is less then that of s.
In particular, for a normed group G with the norm filtration we have

$$\mathcal{G}r_*(G) = \bigvee_{h \geq 0} \mathcal{F}_h G / \mathcal{F}_{h-1} G = \{+\} \sqcup \bigsqcup_{h \geq 0} T_h(G).$$

Since the multiplication map $\mu : G \times G \to G$ of the group G preserves the filtration by property $(N2)$ of the norm, it induces a graded map $\mu_* : \mathcal{G}r_*(G \times G) \to \mathcal{G}r_*(G)$ between graded objects. That is, $\mu_*((g, h)) = \mu((g, h)) = g \cdot h$ if $N(g \cdot h) = N(g) + N(h)$ and $\mu_*((g, h)) = +$ otherwise.

Definition 3.1.1. A function $\eta = (\bar{\eta}, \eta') : G \to G \times G$ mapping $g \in G$ to $(\bar{\eta}(g), \eta'(g))$ is called a *factorization map* if it satisfies the following properties:

$$\bar{\eta}(g) \cdot \eta'(g) = g, \tag{3.1.1}$$

$$N(\bar{\eta}(g)) + N(\eta'(g)) = N(g), \tag{3.1.2}$$

$$\eta'(g) \in T \text{ for any } g \neq 1. \tag{3.1.3}$$

The group element $\eta'(g)$ is called the *prefix* of g and $\bar{\eta}(g)$ is called the *remainder* of g. To shorten notation the prefix of g will be often denoted by g' and the remainder of g by $\bar{g}$.
In particular, from property (3.1.2) it follows that $\eta(1) = (1, 1)$ and it is also clear that $\eta(t) = (1, t)$ for any $t \in T$.

Remark 3.1.2. We have seen, that if G is equipped with the word length norm with respect to a generating set S, which is closed under taking inverses, then $T = S$. On the other hand, assume that a normed group G admits a factorization map η. We can decompose any $1 \neq g \in G$ as $g = \bar{g} \cdot g'$, where $g' \in T$ and $N(\bar{g}) < N(g)$. Using induction on the norm of g it follows that any $g \in G$ can be written as a product of elements of T, hence $T = T_m(G)$ is a generating set.

Definition 3.1.3. A normed group G with norm N is called *factorable* if it admits a factorization map η such that the following diagram of graded objects and graded maps commutes:

$$\begin{tikzcd}
\mathcal{G}r_*(G \times G) \arrow[r, "(\eta\times id)_*"] \arrow[d, "\mu_*"] & \mathcal{G}r_*(G \times G \times G) \arrow[r, "(id\times\mu)_*"] & \mathcal{G}r_*(G \times G) \arrow[r, "(id\times\eta)_*"] & \mathcal{G}r_*(G \times G \times G) \arrow[d, "(\mu\times id)_*"] \\
\mathcal{G}r_*(G) \arrow[rrr, "\eta_*"] & & & \mathcal{G}r_*(G \times G)
\end{tikzcd} \tag{3.1.4}$$

Note that factorability of G is with respect to a given norm N.

We describe the commutativity of diagram (3.1.4) in terms of the ungraded maps. Consider the two filtration-preserving compositions, first the upper composition

$$\alpha_u = (\mu \times id) \circ (id \times \eta) \circ (id \times \mu) \circ (\eta \times id) : G \times G \to G \times G$$

$$\alpha_u : (g, h) \mapsto (\bar{g}, g', h) \mapsto (\bar{g}, g' \cdot h) \mapsto (\bar{g}, \overline{g' \cdot h}, (g' \cdot h)') \mapsto (\bar{g} \cdot \overline{g' \cdot h}, (g' \cdot h)')$$

and then the lower composition

$$\alpha_l = \eta \circ \mu : G \times G \to G \times G.$$

$$\alpha_l : (g, h) \mapsto g \cdot h \mapsto (\overline{g \cdot h}, (g \cdot h)').$$

These two compositions induce the graded maps α_{u*} and α_{l*} respectively, which are precisely the two compositions in diagram (3.1.4). Investigating the equation $\alpha_{u*}((g, h)) = \alpha_{l*}((g, h))$ we have to distinguish the following two cases:
The first case is, when $\alpha_{u*}((g, h)) = \alpha_{l*}((g, h)) \in \mathcal{G}r_*(G \times G)$ is the base point, which means that the underlying maps α_u and α_l are both strictly norm-decreasing. Note that we do not require

that $\alpha_u((g,h)) = \alpha_l((g,h))$ in this case.
In the second case $\alpha_{u*}((g,h)) = \alpha_{l*}((g,h))$ is not the base point, and hence the underlying maps α_u and α_l are both norm-preserving and $\alpha_u((g,h)) = \alpha_l((g,h))$.

Summarizing the conditions for the ungraded maps α_u and α_l we obtain the following requirements: The first requirement is that the map α_u is norm-preserving if and only if the map α_l is norm-preserving, that is for all pairs $(g,h) \in G \times G$

$$N(\alpha_u((g,h))) = N(g) + N(h) \iff N(\alpha_l((g,h))) = N(g) + N(h). \tag{3.1.4.A}$$

The second requirement is that if $N(\alpha_l((g,h))) = N(g) + N(h)$ (and hence both α_u and α_l are norm-preserving) then $\alpha_u((g,h)) = \alpha_l((g,h))$ holds. Examining the values of $\alpha_u((g,h))$ and of $\alpha_l((g,h))$ we formulate the second requirement as

$$N(\alpha_l((g,h))) = N(g) + N(h) \implies (g \cdot h)' = (g' \cdot h)'. \tag{3.1.4.B}$$

Remark 3.1.4. The equality $\alpha_u((g,h)) = \alpha_l((g,h))$ is expressed here using the prefix maps as $(g \cdot h)' = (g' \cdot h)'$. We can equivalently use the remainder maps and obtain the equation $\overline{g} \cdot \overline{g' \cdot h} = \overline{g \cdot h}$.

In the remaining part of the section we investigate the question under which conditions are the maps α_l and α_u norm-preserving. We will need the following definition:

Definition 3.1.5. Assume that (G,N) is a normed group. A pair (g,h) of group elements is called a *geodesic pair* if $N(g \cdot h) = N(g) + N(h)$.

In the composition $\alpha_u = (\mu \times id) \circ (id \times \eta) \circ (id \times \mu) \circ (\eta \times id)$ the maps $id \times \eta$ and $\eta \times id$ are norm preserving by property (3.1.2) of the factorization map η. Therefore α_u is norm preserving if both pairs (g',h) and $(\overline{g}, \overline{g' \cdot h})$ are geodesic, that is

$$N(g' \cdot h) = N(g') + N(h) \text{ and}$$
$$N(\overline{g} \cdot \overline{g' \cdot h}) = N(\overline{g}) + N(\overline{g' \cdot h}).$$

Similarly, the composition $\alpha_l = \eta \circ \mu$ is norm preserving if the pair (g,h) is geodesic, thus

$$N(g \cdot h) = N(g) + N(h).$$

The statement of the following lemma will be a basic tool in computations with geodesic pairs in normed groups. The lemma will also serve for introducing norm level diagrams which will be used in the proofs.

Lemma 3.1.6. *For any three elements x, y and z of a normed group G the pairs (x, yz) and (y, z) are geodesic if and only if the pairs (xy, z) and (x, y) are geodesic.*

Proof: Consider the following diagram of tuples of group elements:

$$\begin{tikzcd}
(x, yz) \arrow[r, dotted, "(b)"] & (x, y, z) \\
(xyz) \arrow[u, dotted, "(a)"] \arrow[r, dotted, "(c)"] & (xy, z) \arrow[u, dotted, "(d)"']
\end{tikzcd}$$

Each arrow in the diagram denotes a decomposition of a product into two factors. By property (N2), the norm is non-decreasing along such an arrow. Thus the norm remains constant along (a) and (b) if and only if it remains constant along (c) and (d). $\square$

It is also possible to express the requirements of (3.1.4.A) in the following norm level diagram, where $(g, h) \in G \times G$

$$\begin{tikzcd}
(\bar{g}, \overline{g'h}, (g'h)') & (\bar{g}, g'h) \arrow[l, dotted, "="'] \arrow[r, dotted, "(a)"] & (\bar{g}, g', h) \\
(\bar{g}\overline{g'h}, (g'h)') \arrow[u, dotted, "(b)"] & (gh) \arrow[l, dotted, "(c)"'] \arrow[r, dotted, "(e)"] \arrow[u, dotted, "(d)"] & (g, h) \arrow[u, dotted, "="]
\end{tikzcd}$$

An equality sign over an arrow denotes a norm-preserving decomposition. Here we have the obvious geodesic pairs $(\overline{g'h}, (g'h)')$ and $(\bar{g}, g')$. Now condition (3.1.4.A) can be formulated as follows: (e) is an equality if and only if (a) and (b) are equalities. Notice that by using Lemma 3.1.6., if (e) is an equality, then (a) and (d) are both equalities and hence (b) and (c) are both equalities. Thus we obtain the following

Corollary 3.1.7. *Assume that $\eta : G \to G \times G$ is a factorization map and that $(g, h) \in G \times G$ is a geodesic pair. Then the pairs (g', h) and $(\overline{g}, \overline{g' \cdot h})$ are also geodesic. In partcular, if α_l is norm-preserving, then α_u is norm preserving.*

Considering examples it will be convient to check whether conditions (3.1.4.A) and (3.1.4.B) are satisfied for pairs $(g, t) \in G \times T$, where T is the set of group elements with minimal non-zero norm. The fact, that it is sufficient to consider only this smaller set of pairs is provided by the following

Proposition 3.1.8. *Assume that (G, N) is a normed group with a factorization map $\eta : G \to G \times G$. If diagram (3.1.4) commutes for all pairs $(g, t) \in G \times T$, then it commutes for all pairs $(g, h) \in G \times G$.*

Proof: We use induction on the norm of h. Clearly, if $h = 1 \in G$ then diagram (3.1.4) commutes: $(g, 1)$ is a geodesic pair for any $g \in G$, hence (3.1.4.A) is satisfied, and from $g' \in T$ it follows that $(g' \cdot 1)' = (g \cdot 1)'$, thus (3.1.4.B) is fulfilled. If $h \in T$ then diagram (3.1.4) commutes by our assumption. Now assume that $N(h) \geq 2$.
First we deal with condition (3.1.4.B): our aim is to show that $(g' \cdot h)' = (g \cdot h)'$. Consider the following sequence of equalities:

$$(g \cdot h)' = (g \cdot \overline{h} \cdot h')' = ((g \cdot \overline{h})' \cdot h')' = ((g' \cdot \overline{h})' \cdot h')' = (g' \cdot \overline{h} \cdot h')' = (g' \cdot h)'$$

In the first and in the last equation property (3.1.1) of the factorization map is used: $h = \overline{h} \cdot h'$. The second and the fourth equation follow from the assumption applied to the geodesic pairs $(g \cdot \overline{h}, h') \in G \times T$ and $(g' \cdot \overline{h}, h') \in G \times T$, respectively. Similarly the pair $(g', \overline{h})$ is a geodesic pair and using the induction step for $\overline{h}$ which has norm $N(\overline{h}) < N(h)$, we obtain the third equation. The fact that all pairs claimed to be geodesic are indeed geodesic pairs can be seen from the following norm level diagram:

$$
\begin{tikzcd}
 & (\bar{g}, g', \bar{h}, h') & \\
(\bar{g}, g', \bar{h}\cdot h') \arrow[ur, dotted, "="] & (\bar{g}, g'\cdot\bar{h}, h') \arrow[u, dotted] & (\bar{g}\cdot g', \bar{h}, h') \arrow[ul, dotted, "="'] \\
(\bar{g}, g'\cdot\bar{h}\cdot h') \arrow[u, dotted] \arrow[ur, dotted] & (\bar{g}\cdot g', \bar{h}\cdot h') \arrow[ul, dotted, "="] \arrow[ur, dotted, "="'] & (\bar{g}\cdot g'\cdot\bar{h}, h') \arrow[ul, dotted] \arrow[u, dotted] \\
 & (\bar{g}\cdot g'\cdot\bar{h}\cdot h') \arrow[ul, dotted] \arrow[u, dotted, "(a)"] \arrow[ur, dotted] &
\end{tikzcd}
\tag{3.1.5}
$$

Since the pair (g,h) is geodesic, arrow (a) represents a geodesic decomposition. Hence applying Lemma 3.1.6. in each square of the diagram it follows that all arrows in the diagram represent geodesic decompositions.

Now we consider condition (3.1.4.A): we wish to show that if the pairs (g',h) and $(\overline{g},\overline{g'\cdot h})$ are both geodesic, then (g,h) is geodesic. (Note, that the other direction of the equivalence in (3.1.4.A) is true for any factorization map by Corollary 3.1.7.). Applying Lemma 3.1.6. to the upper left square of diagram (3.1.5) it follows that if (g',h) is geodesic, then the pairs $(g',\overline{h})$ and $(g'\cdot\overline{h},h')$ are geodesic. Since $(g'\cdot\overline{h},h')\in G\times T$, by (3.1.4.A) the pair $(\overline{g'\overline{h}},\overline{(g'\overline{h})'h'})$ is geodesic, and by (3.1.4.B)

$$\overline{g'\overline{h}}\cdot\overline{(g'\overline{h})'h'} = \overline{g'h}$$

Hence we can construct the following norm level diagram:

$$
\begin{tikzcd}
(\overline{g},\overline{g'h}) \arrow[r, dotted, "="] & (\overline{g},\overline{g'\overline{h}},\overline{(g'\overline{h})'h'}) \\
(\overline{g}\cdot\overline{g'h}) \arrow[u, dotted, "="] \arrow[r, dotted] & (\overline{g}\cdot\overline{g'\overline{h}},\overline{(g'\overline{h})'h'}) \arrow[u, dotted]
\end{tikzcd}
\tag{3.1.6}
$$

where the left arrow is norm preserving since $(\overline{g},\overline{g'\cdot h})$ is geodesic by our assumption. It follows that $(\overline{g},\overline{g'\overline{h}})$ is a geodesic pair and we can use the induction step: the pairs $(g',\overline{h})$ and $(\overline{g},\overline{g'\overline{h}})$ are geodesic, hence the pair $(g,\overline{h})$ is geodesic. Moreover, using (3.1.4.A) we obtain that $(g'\cdot\overline{h})' = (g\cdot\overline{h})'$ and equivalently $\overline{g}\cdot\overline{g'\overline{h}} = \overline{g\overline{h}}$. Thus the lower right entry $(\overline{g}\cdot\overline{g'\overline{h}},\overline{(g'\overline{h})'h'})$ of diagram (3.1.6), which is a geodesic pair by Lemma 3.1.6. can be written as $(\overline{g\overline{h}},\overline{(g\overline{h})'h'})$. On the other hand, the pair $(g'\overline{h},h')$ is geodesic, hence the pair $((g'\overline{h})',h') = ((g\overline{h})',h')$ is geodesic. Using the induction step again we obtain that $(g\overline{h},h')$ is geodesic.
We summarize the results in the following norm level diagram:

$$
\begin{tikzcd}
(g,h) \arrow[r, dotted, "="] & (g,\overline{h},h') \\
(g\cdot h) \arrow[u, dotted] \arrow[r, dotted] & (g\cdot\overline{h},h') \arrow[u, dotted]
\end{tikzcd}
$$

We have seen that $(g\overline{h}, h')$ is geodesic and $(g, \overline{h})$ is geodesic. It follows that the pair (g, h) is geodesic. $\square$

3.2 Examples

Now we can investigate some of the examples of normed groups from Chapter 1. By Proposition 3.1.8. it is enough to consider pairs $(g, t) \in G \times T$. Note that usually we will use the ungraded version of diagram (3.1.4).

Example 3.2.1. Goups with constant norm are factorable. We show, that any group G with the constant norm $N(g) = m > 0$ for all $1 \neq g \in G$ is factorable. We simply define $\eta(g) = (1, g)$ for any $g \in G$. The ungraded version of diagram (3.1.4) has the following form:

$$\begin{tikzcd}
(g,h) \arrow[r, mapsto] \arrow[d, mapsto] & (1,g,h) \arrow[r, mapsto] & (1, g\cdot h) \arrow[r, mapsto] & (1,1,g\cdot h) \arrow[d, mapsto] \\
(g\cdot h) \arrow[rrr, mapsto] & & & (1, g\cdot h)
\end{tikzcd}$$

and it is obviously commutative. However, a pair (g, h) is geodesic if and only if $g = 1 \in G$ or $h = 1 \in G$, thus the commutativity of the ungraded diagram is necessary only in these cases.
Note that using the constant norm the filtration quotients $\mathcal{N}_q(G)[h]$ are non-zero if and only if $h = k \cdot m$ for some $k \in \mathbb{N}$ and $q = k$. It means that the homology of the complex $(\mathcal{N}_*(G)[h], d)$ is in fact concentrated in the top degree $* = k$ and $\mathcal{N}_k(G)[k \cdot m]$ is the entire module $\bar{B}_k(G)$. In other words, in the spectral sequence associated to the norm filtration $E^0_{p,q} = ... = E^{m-1}_{p,q}$ holds and the chain complex $(E^m_{*,*}, d^m)$ is isomorphic to the reduced bar resolution.

Example 3.2.2. Free groups are factorable. The free group F_n of rank n will serve as our first example of a factorable group with respect to a non-trivial norm. The norm we will consider here is the word length norm with respect to the set of free generators $X = \{x_1, \ldots, x_n\}$ and their inverses.

The set T of group elements with minimal non-zero norm is then the generating set $S = X \cup X^{-1}$ and there is an obvious candidate for a factorization map $\eta : F_n \to F_n \times F_n$. Each non-trivial reduced word $w \in F_n$ has the form $\tilde{w}x_i^\epsilon$ for some $1 \leq i \leq n$, with $\epsilon \in \{1, -1\}$. For $w = \tilde{w}x_i^\epsilon$ we set

$$\eta(w) = (\tilde{w}, x_i^\epsilon)$$

For the empty word $1 \in F_n$ we define $\eta(1) = (1, 1)$. The map η defined this way is obviously a factorization map, it remains to show that conditions (3.1.4.A) and (3.1.4.B) are satisfied.
Assume that $(w, x_j^{\epsilon_2}) = (\tilde{w}x_i^{\epsilon_1}, x_j^{\epsilon_2}) \in F_n \times T$. There are two case to distinguish:
The first case is, when $x_i^{\epsilon_1} \neq x_j^{-\epsilon_2}$. Here we have the following diagram:

$$\begin{tikzcd}
(\tilde{w}x_i^{\epsilon_1}, x_j^{\epsilon_2}) \arrow[r, mapsto] \arrow[d, mapsto] & (\tilde{w}, x_i^{\epsilon_1}, x_j^{\epsilon_2}) \arrow[r, mapsto] & (\tilde{w}, x_i^{\epsilon_1}x_j^{\epsilon_2}) \arrow[r, mapsto] & (\tilde{w}, x_i^{\epsilon_1}, x_j^{\epsilon_2}) \arrow[d, mapsto] \\
(\tilde{w}x_i^{\epsilon_1}x_j^{\epsilon_2}) \arrow[rrr, mapsto] & & & (\tilde{w}x_i^{\epsilon_1}, x_j^{\epsilon_2})
\end{tikzcd}$$

Hence $\alpha_u((w, x_j^{\epsilon_2})) = \alpha_l((w, x_j^{\epsilon_2})) = (w, x_j^{\epsilon_2})$ for these pairs, the maps in the diagram are norm-preserving and the diagram commutes.
The second case is, when $x_i^{\epsilon_1} = x_j^{-\epsilon_2}$. Then the compositions α_u and α_l have the following form:

$$\alpha_u : (\tilde{w}x_i^{\epsilon_1}, x_j^{\epsilon_2}) \mapsto (\tilde{w}, x_i^{\epsilon_1}, x_j^{\epsilon_2}) \mapsto (\tilde{w}, 1) \mapsto (\tilde{w}, 1, 1) \mapsto (\tilde{w}, 1)$$

$$\alpha_l : (\tilde{w}x_i^{\epsilon_1}, x_j^{\epsilon_2}) \mapsto (\tilde{w}) \mapsto (\overline{\eta}(\tilde{w}), \eta'(\tilde{w}))$$

Here α_u and α_l are not norm-preserving, because the pairs $(x_i^{\epsilon_1}, x_j^{\epsilon_2})$ and $(\tilde{w}x_i^{\epsilon_1}, x_j^{\epsilon_2})$ are not geodesic. Note that $(\tilde{w}, 1) = (\overline{\eta}(\tilde{w}), \eta'(\tilde{w}))$ if and only if $\tilde{w} = 1$.

In particular, the infinite cyclic group $\mathbb{Z} =< t >$ is factorable with respect to the word length norm $N(t^n) = |n|$. The factorization map $\eta : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is defined as

$$\eta(t^n) = \begin{cases} (t^{n-1}, t) & \text{if } n > 0, \\ (1, 1) & \text{if } n = 0, \\ (t^{n+1}, t^{-1}) & \text{if } n < 0. \end{cases}$$

Example 3.2.3. Finite cyclic groups are not factorable. We can not use a similar construction for finite cyclic groups of rank $p \geq 4$, as we show now. The norm we investigate here is the one from Example 1.1.5: the word length norm with respect to the generating set $S = \{t, t^{-1}\}$. The cases $p = 2$ and $p = 3$ are excluded, since the word length norm on the groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ is the constant one norm.
Since the set T of norm one elements is the set of generators S, it is clear that the factorization map η must be of the form

$$\eta(t^n) = \begin{cases} (1, 1) & \text{if } n = 0, \\ (t^{n-1}, t) & \text{if } 0 < n < \frac{p}{2}, \\ (t^{n+1}, t^{p-1}) & \text{if } \frac{p}{2} < n, \end{cases}$$

and in the case $p = 2k$ we have to make an additional choice whether $\eta(t^k) = (t^{k-1}, t)$, or $\eta(t^k) = (t^{k+1}, t^{2k-1})$. In both cases, it is easy to find pairs violating condition (3.1.4.A). Consider for example the case $p = 2k + 1$ with $k \geq 2$ and the pair (t^k, t). Then $\alpha_u((t^k, t)) = (t^k, t)$, while $\alpha_l((t^k, t)) = (t^{k+2}, t^{p-1})$, hence α_u is norm-preserving, but α_l is not.

An interesting example of a factorable group will be the symmetric group Σ_p with the word length norm with respect to the set of all tranpositions as a generating set. This important example will be discussed in Chapter 5 in detail.

3.3 Direct, Semi-direct and Free Products

Our next aim is to consider the question, under which conditions are semi-direct or free products of factorable groups again factorable. Throughout this section we will assume that in each normed group G we consider, the set $T_1(G)$ is not empty. First we investigate the case of semi-direct products. We have seen in Section 1.3, that there is a natural way to define a norm $N_F = N_{G \rtimes H}$ on the semi-direct product $F := G \rtimes_\varphi H$ of normed groups (G, N_G) and (H, N_H) where $\varphi : H \to Aut(G)$ is a group homomorphism satisfying condition (1.3.2):

$$N_G(g) = N_G(\varphi(h)(g)) \text{ for all } g \in G \text{ and } h \in H.$$

The norm N_F of a pair $(g, h) \in F$ was then defined as $N_F((g, h)) = N_G(g) + N_H(h)$. The direct product $G \times H$ we consider here as the special case $\varphi(h) = id_G$ for all $h \in H$.
Now assume that both groups G and H are factorable. We prove the following

Proposition 3.3.1. *If G and H are factorable groups and $\varphi : H \to Aut(G)$ is a homomorphism satisfying (1.3.2), then the semi-direct product $G \rtimes_\varphi H$ is factorable with respect to the norm $N_{G \rtimes H}$.*

Proof: The set of group elements in F with norm one can be written as

$$T(F) = \{(g, 1) \mid g \in T(G)\} \cup \{(1, h) \mid h \in T(H)\}.$$

Denote the factorization maps of the groups G and H by η_G and by η_H respectively. We define the factorization map $\eta_F : F \to F \times F$ for the semi-direct product $F = G \rtimes_\varphi H$ by

$$\eta_F((g, h)) = \begin{cases} ((g, \bar{\eta}_H(h)), (1, \eta'_H(h))) & \text{if } h \neq 1, \\ ((\bar{\eta}_G(g), 1), (\eta'_G(g), 1)) & \text{if } h = 1. \end{cases} \tag{3.3.1}$$

Using properties of η_G and of η_H, we have that

$$(g, \bar{\eta}_H(h))(1, \eta'_H(h)) = (g \cdot \varphi(\bar{\eta}_H(h))(1), h) = (g, h) \text{ and}$$

$$(\bar{\eta}_G(g), 1)(\eta'_G(g), 1) = (\bar{\eta}_G(g) \cdot \varphi(1)(\eta'_G(g)), 1) = (g, 1),$$

showing that condition (3.1.1) is satisfied. For condition (3.1.2) regarding the norms, we have

$$N_F((g, \bar{\eta}_H(h)), (1, \eta'_H(h))) = N_G(g) + N_G(1) + N_H(\bar{\eta}_H(h)\eta'_H(h)) = N_F((g, h)) \text{ and}$$

$$N_F((\bar{\eta}_G(g), 1), (\eta'_G(g), 1)) = N_G(g) + N_H(1) = N_F((g, 1)).$$

For condition (3.1.3) note that the prefix $\eta'_F((g, h))$ of (g, h) is clearly in $T(F)$ in both cases of (3.3.1), provided $(g, h) \neq (1, 1)$. It remains to show that diagram (3.1.4) commutes. We analyze the two compositions α_u and α_l for elements in $F \times T(F)$ in the following four cases. The map μ_F denotes the multiplication in F.

Case 1: $((g, h), (t, 1)) \in F \times T(F)$, where $h \neq 1$ and $t \in T(G)$.
Then α_u is the composition of the following maps:

$$\eta_F \times id : ((g, h), (t, 1)) \mapsto ((g, \overline{h}), (1, h'), (t, 1)),$$

$$id \times \mu_F : ((g, \overline{h}), (1, h'), (t, 1)) \mapsto ((g, \overline{h}), (\varphi(h')(t), h')),$$

$$id \times \eta_F : ((g, \overline{h}), (\varphi(h')(t), h')) \mapsto ((g, \overline{h}), (\varphi(h')(t), 1), (1, h')),$$

using that $N_H(h') = 1$ and therefore $\eta_H(h') = (1, h')$, and

$$\mu_F \times id : ((g, \overline{h}), (\varphi(h')(t), 1), (1, h')) \mapsto ((g \cdot \varphi(h)(t), \overline{h}), (1, h')),$$

using that $\varphi(\overline{h}) \circ \varphi(h') = \varphi(h)$. On the other hand

$$\alpha_l = \eta_F \circ \mu_F : ((g,h),(t,1)) \mapsto (g \cdot \varphi(h)(t), h) \mapsto ((g \cdot \varphi(h)(t), \overline{h}), (1, h'))$$

and thus $\alpha_u = \alpha_l$ in this case.

Case 2: $((g,1),(t,1)) \in F \times T(F)$, where $t \in T(G)$.
Consider the norm-preserving homomorphism $\iota : G \to F$, defined by $\iota(g) = (g,1)$. From (3.3.1) it follows that $(\iota \times \iota) \circ \eta_G = \eta_F \circ \iota$ thus we obtain the following commutative diagrams:

$$\begin{tikzcd}
\mathcal{G}r_*(G \times T(G)) \arrow[r, "(\eta_G \times id)_*"] \arrow[d, "(\iota\times\iota)_*"] & \mathcal{G}r_*(G \times G \times T(G)) \arrow[r, "(id \times \mu_G)_*"] \arrow[d, "(\iota\times\iota\times\iota)_*"] & \mathcal{G}r_*(G \times G) \arrow[r, "(id \times \eta_G)_*"] \arrow[d, "(\iota\times\iota)_*"] & \mathcal{G}r_*(G \times G \times G) \arrow[r, "(\mu_G \times id)_*"] \arrow[d, "(\iota\times\iota\times\iota)_*"] & \mathcal{G}r_*(G \times G) \arrow[d, "(\iota\times\iota)_*"] \\
\mathcal{G}r_*(F \times T(F)) \arrow[r, "(\eta_F \times id)_*"] & \mathcal{G}r_*(F \times F \times T(F)) \arrow[r, "(id \times \mu_F)_*"] & \mathcal{G}r_*(F \times F) \arrow[r, "(id \times \eta_F)_*"] & \mathcal{G}r_*(F \times F \times F) \arrow[r, "(\mu_F \times id)_*"] & \mathcal{G}r_*(F \times F)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{G}r_*(G \times T(G)) \arrow[r, "(\mu_G)_*"] \arrow[d, "(\iota\times\iota)_*"] & \mathcal{G}r_*(G) \arrow[r, "(\eta_G)_*"] \arrow[d, "\iota_*"] & \mathcal{G}r_*(G \times G) \arrow[d, "(\iota\times\iota)_*"] \\
\mathcal{G}r_*(F \times T(F)) \arrow[r, "(\mu_F)_*"] & \mathcal{G}r_*(F) \arrow[r, "(\eta_F)_*"] & \mathcal{G}r_*(F \times F)
\end{tikzcd}$$

That is, if G is factorable, diagram (3.1.4) commutes for pairs $((g,1),(t,1)) \in F \times T(F)$.

Case 3: $((g,h),(1,t)) \in F \times T(F)$, where $h \neq 1$ and $t \in T(H)$.
In this case we use the maps $s_g : H \to F$ defined for any $g \in G$ as $s_g(h) = (g,h)$. Note that unless $g = 1$ the map s_g is not a homomorphism of groups and that the induced graded map $(s_g)_* : \mathcal{G}r_*(H) \to \mathcal{G}r_*(F)$ has degree $N_G(g)$. However, these maps induce again maps of diagrams, and the commutativity of diagram (3.1.4) for pairs $((g,h),(1,t)) \in F \times T(F)$ follows from the factorability of H. For example for $N_G(g) = k$ the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{G}r_*(H \times H \times H) \arrow[r, "(\mu_H \times id)_*"] \arrow[d, "(s_g \times s_1 \times s_1)_*"] & \mathcal{G}r_*(H \times H) \arrow[d, "(s_g \times s_1)_*"] \\
\mathcal{G}r_{*+k}(F \times F \times F) \arrow[r, "(\mu_F \times id)_*"] & \mathcal{G}r_{*+k}(F \times F)
\end{tikzcd}$$

follows from the equation $s_g \circ \mu_H = \mu_F \circ (s_g \times s_1)$.

Case 4: $((g,1),(1,t)) \in F \times T(F)$, where $t \in T(H)$.
Straightforward computation of α_u and α_l shows that $\alpha_u((g,1),(1,t)) = \alpha_l((g,1),(1,t))$. $\square$

As mentioned earlier, the case of the direct product of groups is covered by the special case $\varphi(h) = id_G$ for all $h \in H$. This homomorphism clearly norm-preserving, hence we have the following

Corollary 3.3.2. *The direct product $G \times H$ of two factorable normed groups G and H is factorable with respect to the norm $N_{G \times H}$.*

Now we consider the case of the free product of factorable groups. We have introduced in Example 1.2.4. a norm $N = N_{G*H}$ on the free product $F := G * H$ of normed groups (G, N_G) and (H, N_H) by defining the norm $N(w)$ of a reduced expression $w \in F$ to be the sum of the respective norms of its 'letters'. The empty word, the identity 1_F of F has norm zero by definition. For G and H being factorable groups we prove the following result:

Proposition 3.3.3. *The free product $G * H$ of factorable normed groups G and H is factorable with respect to the norm N_{G*H}.*

Proof: Here the set of group elements in F with norm one is the set $T(F) = T(G) \cup T(H)$. The factorization map $\eta_F : F \to F \times F$ has a simple definition: if $w = w_1 w_2 ... w_n \in F$ is a reduced word, then we define

$$\eta_F(w) = \begin{cases} (w_1 w_2 ... w_{n-1} \bar{\eta}(w_n), \eta'(w_n)) & \text{if } w \neq 1_F, \\ (1_F, 1_F) & \text{if } w = 1_F. \end{cases} \tag{3.3.2}$$

where $\eta = \eta_G$ if $w_n \in G$ and $\eta = \eta_H$ if $w_n \in H$, respectively. Note that unless $w = 1_F$ we have $w_n \neq 1$, thus $\eta'_F(w) = \eta'(w_n) \in T(F)$, as required in (3.1.3). Since the maps η_G and η_H are factorization maps, it follows that η_F is a factorization map. For diagram (3.1.4) we have to distinguish four cases depending whether in a tuple $(w_1 w_2 ... w_n, t) \in F \times T(F)$ the letters w_n and t are in G or in H. By symmetry, we can assume that $w_n \in G$ and we have the following two cases:

If $t \in T(H)$, then in diagram (3.1.4) the following (ungraded) maps occour:

$$\alpha_u : (ww_n, t) \mapsto (w\bar{\eta}_G(w_n), \eta'_G(w_n), t) \mapsto (w\bar{\eta}_G(w_n), \eta'_G(w_n)t) \mapsto (w\bar{\eta}_G(w_n), \eta'_G(w_n), t) \mapsto (ww_n, t)$$

$$\alpha_l : (ww_n, t) \mapsto ww_n t \mapsto (ww_n, t)$$

Hence $\alpha_u = \alpha_l = id$ and diagram (3.1.4) commutes.
On the other hand, the case $t \in T_G$ is clearly equivalent to the factorability of the group G □

Note that the norm we obtained in Example 3.2.2. for the free group F_n is the same norm one obtains using formula (3.3.2) for the iterated free product $F_n \cong \mathbb{Z} * \cdots * \mathbb{Z}$, where we consider $\mathbb{Z}$ with the word length norm as in Example 1.1.3.
Proposition 3.3.1. provides a large class of factorable groups. In fact, we can take any two groups G and H, both with the constant norm and we obtain that the group $G \rtimes_\varphi H$ is factorable with repsect to some non-constant norm.

Example 3.3.4. It is easy to see that the dihedral group D_{2m} with the word length norm from Example 1.3.5. is not factorable. On the other hand, regarding D_{2m} as the semi-direct product of the normed groups $(\mathbb{Z}_m, N_1)$ and $(\mathbb{Z}_2, N_1)$, where N_1 is the trivial norm, we obtain a non-constant norm $\tilde{N}$ different from the word length norm. Using the notation of Example 1.3.5. we have $\tilde{N}((r^k, 1)) = 1$ and $\tilde{N}((r^k, t)) = 2$ for all $1 \leq k \leq m - 1$. By Proposition 3.3.1. the normed group $(D_{2m}, \tilde{N})$ is factorable.

Chapter 4

The Homology of Factorable Groups

4.1 The Main Theorem

As mentioned in the previous chapter, our aim is to study the holomolgy of the complex $(\mathcal{N}_*(G)[h], d)$. The existence of a factorization map $\eta : G \to G \times G$ in a normed group (G, N), together with the commutativity of diagram (3.1.4) provides a sufficent condition to prove that the homology of the complex $\mathcal{N}_*(G)[h]$ vanishes in degrees lower than $m \cdot h$, where m is the minimal non-zero value of the norm N.

Theorem 4.1.1. *If G is a factorable group with respect to the norm N, then the homology of the complex $\mathcal{N}_*(G)[h]$ is concentrated in the top degree $m \cdot h$.*

The proof is based on a spectral-sequence, induced by a filtration on $\mathcal{N}_*(G)[h]$, which we introduce in the following section. The idea to use this filtration, the so called partition-type filtration, is due to R. Ehrenfried (see [11]).

4.2 The partition-type ordering

The *partition-type* of a generator $c = (g_q|\dots|g_1) \in \mathcal{N}_q(G)[h]$ is defined to be the tuple of natural numbers $Pt(c) := (N(g_q), \dots, N(g_1))$. Consider the set of ordered, positive partitions of h of arbitary length,

$$\mathcal{P}(h) = \{P = (l_q, \dots, l_1) \mid \sum_{i=1}^{q} l_i = h,\ l_i > 0 \text{ for all } i\}$$

On $\mathcal{P}(h)$ we introduce the following relation:

Definition 4.2.1. For any two elements $P = (l_r, \dots, l_1)$ and $P' = (l'_s, \dots, l'_1)$ of $\mathcal{P}(h)$ we define $(l_r, \dots, l_1) \prec (l'_s, \dots, l'_1)$ if there exists an index $j \geq 0$ such that $l_{r-i} = l'_{s-i}$ for $i = 0, 1, \dots, j-1$ and $l_{r-j} > l'_{s-j}$. The relation clearly induces a total order on the set $\mathcal{P}(h)$, and we call this order the *partition-type order* of $\mathcal{P}(h)$. Note, that there are exactly 2^h elements in $\mathcal{P}(h)$; they are sometimes abbreviated by $P_1 \prec \dots. \prec P_{2^h}$. The minimal element in the order is $P_1 = (h)$ and the maximal element is $P_{2^h} = (1, 1, \ldots, 1)$ (the number of ones is h).

Example 4.2.2. For $h = 4$ we obtain the following partition-type order of $\mathcal{P}(4)$:

$$(4) \prec (3,1) \prec (2,2) \prec (2,1,1) \prec (1,3) \prec (1,2,1) \prec (1,1,2) \prec (1,1,1,1)$$

Let us denote the free $\mathbb{Z}$-module generated by tuples of a fixed partition type $P = (l_q, ..., l_1)$ by $\mathcal{N}_q(G)[l_q, ..., l_1]$, or shortly by $\mathcal{N}_q(G)[P]$. Note that the homological degree q is already determined by the partition-type. Also note, that the singleton $P = (h) \in \mathcal{P}(h)$ will occour as $\mathcal{N}_1(G)[h]$ and it is in fact the same object as $\mathcal{N}_q(G)[h]$ for $q = 1$.

We obtain an increasing filtration of the complex $\mathcal{N}_*(G)[h]$

$$0 \subseteq \mathcal{F}_1\mathcal{N}_*(G)[h] \subseteq \cdots \subseteq \mathcal{F}_{2^{h-1}}\mathcal{N}_*(G)[h] = \mathcal{N}_*(G)[h]$$

of length 2^{h-1} (the cardinality of $\mathcal{P}(h)$) by defining

$$\mathcal{F}_i\mathcal{N}_*(G)[h] = \bigoplus_{j=1}^{i} \mathcal{N}_*(G)[P_j]$$

where P_j denotes the j-th partition in the partition-type order of $\mathcal{P}(h)$. This filtration is called the *partition-type filtration.*

The following lemma relating the faces of a generator of $\mathcal{N}_*(G)[h]$ is an easy consequence of the definitions.

Lemma 4.2.3. *Let c be a generator of $\mathcal{N}_q(G)[h]$ and $1 \le i < j \le q$. If $d_i(c)$ and $d_j(c)$ are non-zero, then $Pt(d_j(c)) \prec Pt(d_i(c)) \prec Pt(c)$ holds.* □

Corollary 4.2.4. *The boundary operator d strictly lowers the filtration degree, thus $d : \mathcal{F}_i\mathcal{N}_*(G)[h] \to \mathcal{F}_{i-1}\mathcal{N}_{*-1}(G)[h]$. In particular the partition-type filtration is stable with respect to the boundary d.*

¿From now on, we will assume that $m = \min\{N(g) \mid g \in G, g \neq 1\} = 1$, but the proofs would work equally well for arbitary values of m. We will need the following technical lemma:

Lemma 4.2.5. *Assume that c and c' are generators of $\mathcal{N}_q(G)[h]$ with partition-type $Pt(c') \prec Pt(c) = (l_q, \ldots, l_r, 1, \ldots, 1)$. Then $Pt(d_i(c')) \prec Pt(d_j(c))$, where $d_i(c)$ is any non-zero face of c and $d_j(c')$ is a non-zero face of c' for some $j \le r - 1$.*

Proof: Denote the partition-tpye of c' by $(l'_q, \ldots, l'_1)$. By assumption there is an index $k \ge 0$, such that $l_{q-i} = l'_{q-i}$ for $i = 0, \ldots, k-1$ and $l_{q-k} < l'_{q-k}$. Clearly $Pt(d_i(c')) \prec Pt(d_j(c))$ for any non-zero faces provided $i, j < q - k - 1$. On the other hand

$$l_{q-k-1} + l_{q-k-2} + \cdots + l_1 > l'_{q-k-1} + l'_{q-k-2} + \cdots + l'_1 \ge q - k - 1$$

and thus $q-k-1 > r-1$. Hence $Pt(d_i(c')) \prec Pt(d_j(c))$ if $i, j \leq r-1$ where by Lemma 4.2.3. the condition $i \leq r-1$ can be relaxed. $\square$

Now we can employ the factorization map η. By conditions (3.1.2) and (3.1.3) η induces a map

$$\eta_i^q = \eta_i : \mathcal{N}_q(G)[l_q, \dots, l_i, \dots, l_1] \to \mathcal{N}_{q+1}(G)[l_q, \dots, l_i - 1, 1, \dots, l_1]$$

for any $q < h$ and for all i with $l_i \geq 2$, defined for generators by applying η for the i-th factor of the tuple and then extended linearly to be a map of modules. More precisely, η_i is defined as

$$\eta_i : (g_q|\dots|g_i|\dots|g_1) \mapsto (g_q|\dots|\overline{\eta}(g_i)|\eta'(g_i)|\dots|g_1)$$

for any generator $(g_q|\dots|g_1)$ of $\mathcal{N}_q(G)[l_q, \dots, l_i, \dots, l_1]$ with $N(g_i) = l_i \geq 2$.
The composition $f_i = \eta_i d_i : \mathcal{N}_q(G)[l_q, \dots, l_{i+1}, l_i, \dots, l_1] \to \mathcal{N}_q(G)[l_q, \dots, l_{i+1} + l_i - 1, 1, \dots, l_1]$ is then defined for $1 \leq i \leq q-1$.

Lemma 4.2.6. *The map f_i satisfies the following equalities:*

$$d_i f_i = d_i \tag{4.2.1}$$

$$d_j f_i = f_i d_j \text{ for } i+2 \leq j \leq q \tag{4.2.2}$$

$$d_j f_i = f_{i-1} d_j \text{ for } 1 \leq j \leq i-2 \tag{4.2.3}$$

$$d_{i+1} f_i f_{i+1} = f_i d_i \tag{4.2.4}$$

Proof: First we show that the map η_i satisfies the following equalities:

(1) $d_i \eta_i = id$,
(2) $d_j \eta_i = \eta_i d_{j-1}$ for $i+2 \leq j \leq q$,
(3) $d_j \eta_i = \eta_{i-1} d_j$ for $1 \leq j \leq i-2$,
(4) $d_i \eta_{i-1} d_{i-1} \eta_i = \eta_{i-1} d_{i-1}$.

The first equality follows from property (3.1.1) of the factorization map η and equalities (2) and (3) are obvious. The last equality follows from the commutativity of diagram (3.1.4). More precisely, from (3.1.4.A) it follows that $d_i \eta_{i-1} d_{i-1} \eta_i(c) = 0$ if and only if $\eta_{i-1} d_{i-1}(c) = 0$ while from (3.1.4.B) it follows that the left and right hand sides of (4) are equal in that case too, if they are non-zero. Using the simplicial equalities $d_i d_j = d_j d_{i+1}$ for $j \leq i$ and $d_i d_j = d_{j-1} d_i$ for $j > i$ we obtain equalities (4.2.1-4). $\square$

Now let us consider the map $F_r : \mathcal{N}_q(G)[l_q, \dots, l_r, \dots, l_1] \to \mathcal{N}_{q+1}(G)$ defined for all partition-types with $l_r \geq 2$ as the composition

$$(-1)^r (id - f_{r-1} + ... + (-1)^{r-i} f_i f_{i+1} ... f_{r-1} + ... + (-1)^{r-1} f_1 f_2 ... f_{r-1}) \circ \eta_r$$

Note that if we restrict F_r to $\mathcal{N}_q(G)[l_q, \dots, l_r, 1, \dots, 1]$ we obtain a map

$$F_r : \mathcal{N}_q(G)[l_q, \dots, l_r, 1, \dots, 1] \to \mathcal{N}_{q+1}(G)[l_q, \dots, l_r - 1, 1, \dots, 1].$$

We use the notation $d_{(r)}$ to denote the signed sum of the first r face operators: $d_{(r)} = \sum_{i=1}^{r} (-1)^i d_i$.

Lemma 4.2.7. *The map F_r satisfies the following two equalities:*
(i) $d_{(r)}F_r = id - F_{r-1}d_{(r-1)}$,
(ii) $d_jF_r = F_rd_{j-1}$ for $r+2 \leq j \leq q$.

Proof: Assume that j and r are fixed and $j \leq r$. We compute the terms of d_jF_r:
If $i \geq j+2$, we can use (4.2.3) multiple times to obtain

$$d_jf_if_{i+1}...f_{r-1}\eta_r = ... = f_{i-1}...f_{r-2}d_j\eta_r = f_{i-1}...f_{r-2}\eta_{r-1}d_j$$

In the case $i = j$ and $i = j+1$ we find that by (4.2.1) the terms $d_jf_{j+1}...f_{r-1}\eta_r$ and $d_jf_jf_{j+1}...f_{r-1}\eta_r$ are equal, thus they cancel in the sum, since they have opposite signs.
In the case $i \leq j-1$, first using (4.2.2) we have

$$d_jf_if_{i+1}...f_{r-1}\eta_r = ... = f_if_{i+1}...f_{j-2}d_jf_{j-1}f_j...f_{r-1}\eta_r$$

Here we can use (4.2.4) to sneak d_j through f_{j-1} and we get

$$f_i...f_{j-2}f_{j-1}d_{j-1}f_{j+1}...f_{r-1}\eta_r = ... = f_i...f_{r-2}\eta_{r-1}d_{j-1}$$

Building the sum $d_{(r)}F_r$ one obtains the formula (i).
For (ii) we can use (4.2.2) for all the summands. $\square$

We consider now the (homological) spectral sequence associated to the partition-type filtration $\mathcal{F}_*$. The E^0-term has the following form:

$$\begin{tikzcd}
\mathcal{F}_i\mathcal{N}_{q+1}(G)[h]/\mathcal{F}_{i-1}\mathcal{N}_{q+1}(G)[h] \arrow[d, "d^0"] & \mathcal{F}_{i+1}\mathcal{N}_{q+1}(G)[h]/\mathcal{F}_i\mathcal{N}_{q+1}(G)[h] \arrow[d, "d^0"] \\
\mathcal{F}_i\mathcal{N}_q(G)[h]/\mathcal{F}_{i-1}\mathcal{N}_q(G)[h] \arrow[d, "d^0"] & \mathcal{F}_{i+1}\mathcal{N}_q(G)[h]/\mathcal{F}_i\mathcal{N}_q(G)[h] \arrow[d, "d^0"] \\
\mathcal{F}_i\mathcal{N}_{q-1}(G)[h]/\mathcal{F}_{i-1}\mathcal{N}_{q-1}(G)[h] & \mathcal{F}_{i+1}\mathcal{N}_{q-1}(G)[h]/\mathcal{F}_i\mathcal{N}_{q-1}(G)[h]
\end{tikzcd}$$

Remark. Since $d : \mathcal{F}_i\mathcal{N}_q(G)[h] \to \mathcal{F}_{i-1}\mathcal{N}_{q-1}(G)[h]$, we find that d^0, the boundary induced by d on the quotient is the zero map, thus the E^1-term has the following form:

$$\begin{tikzcd}
\mathcal{F}_i\mathcal{N}_{q+1}(G)[h]/\mathcal{F}_{i-1}\mathcal{N}_{q+1}(G)[h] & \mathcal{F}_{i+1}\mathcal{N}_{q+1}(G)[h]/\mathcal{F}_i\mathcal{G}_{q+1}(G)[h] \arrow[dl, "d^1"] \\
\mathcal{F}_i\mathcal{N}_q(G)[h]/\mathcal{F}_{i-1}\mathcal{N}_q(G)[h] & \mathcal{F}_{i+1}\mathcal{N}_q(G)[h]/\mathcal{F}_i\mathcal{N}_q(G)[h] \arrow[dl, "d^1"] \\
\mathcal{F}_i\mathcal{N}_{q-1}(G)[h]/\mathcal{F}_{i-1}\mathcal{N}_{q-1}(G)[h] & \mathcal{F}_{i+1}\mathcal{N}_{q-1}(G)[h]/\mathcal{F}_i\mathcal{N}_{q-1}(G)[h]
\end{tikzcd}$$

Also note, that since the partition-type uniquely determines the homological degree q, each column of the E^1-term has all but one non-zero term. For example for $h = 4$ we obtain the following E^1-term:

$$
\begin{tikzcd}
0 & 0 & 0 & 0 & 0 & 0 & 0 & {[1,1,1,1]} \arrow[dl, "d^1"] \\
0 & 0 & 0 & {[2,1,1]} \arrow[dl, "d^1"] & 0 & {[1,2,1]} \arrow[dl, "d^1"] & {[1,1,2]} & 0 \\
0 & {[3,1]} \arrow[dl, "d^1"] & {[2,2]} & 0 & {[1,3]} & 0 & 0 & 0 \\
{[4]} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{tikzcd}
$$

Here we supressed the terms $\mathcal{N}_*(G)$ from the notation and only the partition-types are indicated.

Proof of Theorem 4.1.1.: Our aim is to show, that $E^\infty_{p,q} = 0$ if $q < h$. Here $E^\infty_{p,q}$ is defined as

$$E^\infty_{p,q} = Z^\infty_{p,q}/(Z^\infty_{p-1,q} + B^\infty_{p,q}),$$

where

$$Z^\infty_{p,q} = \{c \in \mathcal{F}_p\mathcal{N}_q(G)[h] \mid d(c) = 0\} \text{ and}$$
$$B^\infty_{p,q} = \{c \in \mathcal{F}_p\mathcal{N}_q(G)[h] \mid c \in \mathrm{Im}(d : \mathcal{N}_{q+1}(G)[h] \to \mathcal{N}_q(G)[h])\}$$

Assume that $q < h$ and $c \in Z^\infty_{p,q} \subset \mathcal{F}_p\mathcal{N}_q(G)[h] = \bigoplus_{i=1}^p \mathcal{N}_q(G)[P_i]$, where P_i is the i-th partition in the partition-tpye order of $\mathcal{P}(h)$. Thus we can uniquely decompose c as $c = c_1 + ... + c_p$, where $c_i \in \mathcal{N}_q(G)[P_i]$.
If $c_p = 0$, then $c \in Z^\infty_{p-1,q}$, thus we can assume, that $c_p \neq 0$, in particular that P_p is a q-partition of h. Here we use that $q < h$, hence there exists an index r such that P_p has the form $(l_q, \dots, l_r, 1, \dots, 1)$ with $l_r \geq 2$.
We claim that $d_i(c_p) = 0$ for all $i < r$. By Lemma 4.2.5. we have that $Pt(d_k(c_l)) \prec Pt(d_i(c_p))$ for any $l < p$, $1 \leq k \leq q-1$ and $i < r$, provided that the faces in question are non-zero. This means, that all partition-types of the terms in $d(c - c_p) = d(c_1 + ... + c_{p-1})$ are strictly smaller than the partition types $d_i(c_p)$, $i = 1, ..., r-1$. On the other hand $d(c - c_p) = -d(c_p)$, thus the claim follows. In particular $d_{(r-1)}(c_p) = 0$. Now using Lemma 4.2.7.(i) we obtain $d_{(r)}F_r(c_p) = c_p - F_{r-1}d_{(r-1)}(c_p) = c_p$, and hence $dF_r(c_p) - c_p \in \mathcal{F}_{p-1}\mathcal{N}_q(G)[h]$. Since $c - c_p$ is in $\mathcal{F}_{p-1}\mathcal{N}_q(G)[h]$, it follows that $c - dF_r(c_p) \in Z^\infty_{p-1,q}$, while obviously $dF_r(c_p) \in B^\infty_{p,q}$ and hence $c \in Z^\infty_{p-1,q} + B^\infty_{p,q}$. $\square$

4.3 Applications

Recall from Section 2.3 that the homological spectral sequence associated to the norm-filtration of the reduced bar complex $\bar{B}_*(G)$ of a group G has the following E^0-term:

$$
\begin{tikzcd}
\cdots & \mathcal{N}_{q+1}(G)[h-1] \arrow[d, "\bar{d}^0"] & \mathcal{N}_{q+1}(G)[h] \arrow[d, "\bar{d}^0"] & \mathcal{N}_{q+1}(G)[h+1] \arrow[d, "\bar{d}^0"] & \cdots \\
\cdots & \mathcal{N}_{q}(G)[h-1] \arrow[d, "\bar{d}^0"] & \mathcal{N}_{q}(G)[h] \arrow[d, "\bar{d}^0"] & \mathcal{N}_{q}(G)[h+1] \arrow[d, "\bar{d}^0"] & \cdots \\
\cdots & \mathcal{N}_{q-1}(G)[h-1] & \mathcal{N}_{q-1}(G)[h] & \mathcal{N}_{q-1}(G)[h+1] & \cdots
\end{tikzcd}
$$

where $\bar{d}^0$ is the map induced by the boundary operator $\bar{d} : \bar{B}_*(G) \to \bar{B}_{*-1}(G)$ on the filtration quotients. If in addition the group G is factorable, then by Theorem 4.1.1. the E^1-term is concentrated in the diagonal:

$$
\begin{tikzcd}
\cdots & 0 & 0 \arrow[dl, "\bar{d}^1"] & H_{h+1}(\mathcal{N}_*(G)[h+1]) \arrow[dl, "\bar{d}^1"] & \cdots \\
\cdots & 0 & H_h(\mathcal{N}_*(G)[h]) \arrow[dl, "\bar{d}^1"] & 0 \arrow[dl, "\bar{d}^1"] & \cdots \\
\cdots & H_{h-1}(\mathcal{N}_*(G)[h-1]) & 0 & 0 & \cdots
\end{tikzcd}
$$

It follows, that the spectral sequence collapses after the E^2-term, hence the holomogy of the group G can be computed as the homology of the complex

$$
\cdots \xrightarrow{\bar{d}^1} H_h(\mathcal{N}_*(G)[h]) \xrightarrow{\bar{d}^1} H_{h-1}(\mathcal{N}_*(G)[h-1]) \xrightarrow{\bar{d}^1} \ldots \tag{4.3.1}
$$

Our next aim is to find generators of the homology groups $H_h(\mathcal{N}_*(G)[h])$. We have proved that if $c \in \mathcal{N}_q(G)[h]$ is a cycle and $q < h$ then c is also a boundary, thus $c = d(b)$ for some $b \in \mathcal{N}_{q+1}(G)[h]$. We can go actually a little bit further by showing that b is a chain in a module generated by certain special generators of $\mathcal{N}_{q+1}(G)[h]$, namely those ones, which are in the image of some η_i. The idea here is based on the observation that in each summand of the maps F_r, the last map used in the compositions is some η_i. To make this statement precise, we divide the set of free generators of $\mathcal{N}_q(G)[h]$ into two disjoint parts: by $Q_q(G)[h]$ we denote those q-tuples which are in the image of some $\eta_i^{q-1} : \mathcal{N}_{q-1}(G)[l_{q-1}, \ldots, l_i, \ldots, l_1] \to \mathcal{N}_q(G)[h]$ and by $R_q(G)[h]$ the complementary set of $Q_q(G)[h]$. Equivalently, using the maps $f_i = \eta_i d_i$, we find that

$$
Q_q(G)[h] = \{c = (g_q|\ldots|g_1) \in \bar{G}^q \mid \sum_{i=1}^{q} N(g_i) = h, f_i(c) = c \text{ for some } 1 \leq i \leq q-1\},
$$

$$
R_q(G)[h] = \{c = (g_q|\ldots|g_1) \in \bar{G}^q \mid \sum_{i=1}^{q} N(g_i) = h, f_i(c) \neq c \text{ for all } 1 \leq i \leq q-1\}.
$$

Denote by $\mathcal{Q}_q(G)[h]$ and by $\mathcal{R}_q(G)[h]$ the free modules generated by the sets $Q_q(G)[h]$ and $R_q(G)[h]$, respectively. We have the injective maps $\iota_{\mathcal{Q},q} : \mathcal{Q}_q(G)[h] \to \mathcal{N}_q(G)[h]$ and $\iota_{\mathcal{R},q} : \mathcal{R}_q(G)[h] \to$

$\mathcal{N}_q(G)[h]$ with their images denoted by $\mathcal{Q}^*_q(G)[h]$ and by $\mathcal{R}^*_q(G)[h]$ respectively. Clearly $\mathcal{Q}^*_q(G)[h] \bigoplus \mathcal{R}^*_q(G)[h] = \mathcal{N}_q(G)[h]$. We prove the following

Lemma 4.3.1. *If c is a cycle in $\mathcal{N}_q(G)[h]$ with $q < h$, then $c = d(b)$, where $b \in \mathcal{Q}^*_{q+1}(G)[h]$.*

Proof: Assume that $c \in \mathcal{F}_p\mathcal{N}_q(G)[h]$, where $1 \leq p \leq 2^{h-1}$. We will use induction on the filtration degree p. As in the proof of Theorem 4.1.1. the cycle c decomposes as $c = c_1 + ... + c_p$. For the first step of the induction, assume that $P_p = (h - q + 1, 1, \ldots, 1)$ is the smallest q-partition of h in the partition-type order, hence $c = c_p$. By Lemma 4.2.7. we have $dF_q(c) = d_{(q)}F_q(c) = c - F_{q-1}d_{(q-1)}(c) = c$, since c has homological degree q and $F_q(c)$ has homological degree $q+1$. By definition, $F_q(c) \in \mathcal{Q}^*_{q+1}(G)[h]$, hence the statement of Lemma 4.3.1. is proved in this case.
In general, as shown at the end of the proof of Theorem 4.1.1. we have $c - dF_r(c_p) \in Z^{\infty}_{p-1,q}$, where r is determined by the partition-type of c_p. Thus, by induction $c - dF_r(c_p) = d(b)$, where $b \in \mathcal{Q}^*_{q+1}(G)[h]$, hence $c = d(b + F_r(c_p))$, finishing the proof of the lemma. □

Remark 4.3.2. Note that the chain $b_c = b$ from Lemma 4.3.1. can be constructed inductively: from the decomposition $c = c_1 + ... + c_p$ we obtain $c - dF_{r_p}(c_p) \in \mathcal{F}_{p-1}\mathcal{N}_q(G)[h]$, where r_p is determined by the partition-type of c_p. Hence in the next step we can decompose $c - dF_{r_p}(c_p)$ as a sum $c'_1 + ... + c'_{p-1}$ and obtain $c - dF_{r_p}(c_p) - dF_{r_{p-1}}(c'_{p-1}) \in \mathcal{F}_{p-2}\mathcal{N}_q(G)[h]$. The process terminates after at most p steps and we find the following formula:

$$b_c = \sum_{i=0}^{p-1} F_{r_{p-i}}(c^{(i)}_{p-i}),$$

where each $c^{(i)}_{p-i}$ can be determined step by step.

We can use these results to obtain generators of $H_h(\mathcal{N}_*(G)[h]) = \ker\{d : \mathcal{N}_h(G)[h] \to \mathcal{N}_{h-1}(G)[h]\}$ by constructing a map $\kappa_h : \mathcal{R}_h(G)[h] \to H_h(\mathcal{N}_*(G)[h])$ as follows. For each element $c \in R_h(G)[h]$ consider $d(c) \in \mathcal{N}_{h-1}(G)[h]$. By Lemma 4.3.1. we know that the cycle $d(c)$ is the boundary of some $b \in \mathcal{Q}^*_h(G)[h]$, where $b = b_{d(c)}$ can be constructed from $d(c)$ as described in Remark 4.3.2.. We define $\kappa_h(c)$ to be the cycle $c - b$ and then extend this map linearly to a map of modules. We compute in the following example the explicit formula of $\kappa_h(c)$ in the case $h = 3$.

Example 4.3.3. For $c \in R_3(G)[3]$ we have $d(c) = d_2(c) - d_1(c) \in \mathcal{F}_3\mathcal{N}_2(G)[3]$. Since the partition-type order of $\mathcal{P}(3)$ has the form $(3) < (2,1) < (1,2) < (1,1,1)$ we obtain $d(c) = c_1 + c_2 + c_3$, where $c_1 = 0$, $c_2 = d_2(c)$ and $c_3 = -d_1(c)$. Using the notation of Remark 4.3.2., $P_3 = (1,2)$, thus $r_3 = 1$ and $F_{r_3} = F_1 = -\eta_1$, hence $d(c) - d(F_{r_3}(c_3)) = d(c) - d(f_1(c)) \in \mathcal{F}_2\mathcal{N}_2(G)[3]$. In the next step, $d(c) - d(f_1(c)) = c'_1 + c'_2$, where $c'_1 = 0$ and $c'_2 = d_2(c) - d_2(f_1(c))$. Now $P_2 = (2,1)$ and therefore we apply $F_{r_2} = F_2$ to obtain $b = F_1(-d_1(c)) + F_2(d_2(c - f_1(c)))$. The chain $\kappa_3(c)$ has then the following form:

$$\kappa_3(c) = c - f_1(c) - f_2(c) + f_2f_1(c) + f_1f_2(c) - f_1f_2f_1(c).$$

It is easy to verify that this is indeed a cycle.

We can generalize the observations of Example 4.3.3. and of Remark 4.3.2.: if $c \in R_h(G)[h]$, then the partition type of a non-zero face $d_i(c)$ of c is $(1,\dots,1,2,1,\dots,1)$, where the number two is in the i-th position. We obtain $\kappa_h(c)$ in $h-1$ steps inductively: start with the chain $c(1) := c$. In the j-th step we lower the filtration degree of the boundary of the chain $c(j) \in \mathcal{N}_h(G)[h]$ by defining $c(j+1) := c(j) - F_j \circ (-1)^j d_j(c(j)) = (id - (-1)^j F_j \circ d_j)(c(j))$. After the last step $c(h) = \kappa_h(c)$, hence we obtain the follow ing

Lemma 4.3.4. *The cycle $\kappa_h(c) \in H_h(\mathcal{N}_*(G)[h])$ has the following explicit formula:*

$$\kappa_h(c) = (id - (-1)^{h-1} F_{h-1} \circ d_{h-1})(id - (-1)^{h-2} F_{h-2} \circ d_{h-2}) \dots (id - F_2 \circ d_2)(id + F_1 \circ d_1)(c)$$

As an example, Lemma 4.3.4. gives in the case $h = 3$ the formula

$$\kappa_3(c) = (id - f_2 + f_1 f_2)(id - f_1)(c),$$

which coincides with the formula in Example 4.3.3.

Remarks: Since each F_i has i summands, it follows that in general $\kappa_h(c)$ has $h!$ terms.
Notice that the formula for κ_h of Lemma 4.3.4. makes sense if we apply it to $c \in \mathcal{R}_{h'}(G)[h']$ for any $h' \geq h$. Denote the resulting map by $\kappa_{h',h} : \mathcal{R}_{h'}(G)[h'] \to \mathcal{N}_{h'}(G)[h']$. By this formulation

$$\kappa_h = \kappa_{h,h} = (id - (-1)^{h-1} F_{h-1} \circ d_{h-1}) \circ \kappa_{h,h-1}.$$

We give here two more formulas computing κ_h. Consider the following sequences of maps defined inductively as

$$G_0^{(r)} := id,\ G_1^{(r)} := id - f_1 \circ id \text{ and } G_i^{(r)} := id - f_i \circ G_{i-1}^{(r)} \text{ for } i \in \mathbb{N},$$

$$G_0^{(l)} := id,\ G_1^{(l)} := id - id \circ f_1 \text{ and } G_i^{(l)} := id - G_{i-1}^{(l)} \circ f_i \text{ for } i \in \mathbb{N}.$$

Then we have

$$\kappa_h = G_1^{(r)} \circ G_2^{(r)} \circ \dots \circ G_{h-1}^{(r)} = G_{h-1}^{(l)} \circ G_{h-2}^{(l)} \circ \dots \circ G_1^{(l)} \tag{4.3.2}$$

Observe, that any $c \in R_h(G)[h]$ will occour with multiplicity one in the cycle $\kappa_h(c)$, since $\kappa_h(c) - c \in \mathcal{Q}_h^*(G)[h]$, while $c \in \mathcal{R}_h^*(G)[h]$ by definition. In particular $\kappa_h(c) \neq 0$ follows. Moreover, by the same argument, for any non-zero chain $c \in \mathcal{R}_h(G)[h]$ we have $\kappa_h(c) \neq 0$, thus we have proved injectivity in the following

Proposition 4.3.5. *The map $\kappa_h : \mathcal{R}_h(G)[h] \to H_h(\mathcal{N}_*(G)[h])$ is a split injective homomorphism of modules.*

Proof: That the image of κ_h is a direct summand in $H_h(\mathcal{N}_*(G)[h])$ follows from the following diagram:

$$\mathcal{R}_h(G)[h] \xrightarrow{\kappa_h} H_h(\mathcal{N}_*(G)[h]) \xrightarrow{i_h} \mathcal{N}_h(G)[h] \xrightarrow{\pi_h} \mathcal{R}_h(G)[h]$$

where i_h is the inclusion $ker\{\mathcal{N}_h(G)[h] \to \mathcal{N}_{h-1}(G)[h]\} \hookrightarrow \mathcal{N}_h(G)[h]$ and π_h is the projection $\mathcal{N}_h(G)[h] = \mathcal{Q}_h^*(G)[h] \bigoplus \mathcal{R}_h^*(G)[h] \xrightarrow{pr_2} \mathcal{R}_h^*(G)[h] \simeq \mathcal{R}_h(G)[h]$.
Clearly $\pi_h \circ i_h \circ \kappa_h = id : \mathcal{R}_h(G)[h] \to \mathcal{R}_h(G)[h]$. $\square$

In Theorem 5.4.1. we will prove that in the case G is the symmetric group Σ_p, the map κ_h is an isomorphism. Using similar arguments to those ones used in the proof of Theorem 5.4.1. R. Wang shows in [14] that κ_h is an isomorphism if the set T of group elements with minimal non-zero norm is finite. In this case we obtain a new complex

$$\cdots \xrightarrow{\partial} \mathcal{R}_h(G)[h] \xrightarrow{\partial} \mathcal{R}_{h-1}(G)[h-1] \xrightarrow{\partial} \dots \tag{4.3.3}$$

which we will denote shortly by $(\mathcal{R}_*(G), \partial)$. The boundary map ∂ is defined as the composition $\kappa_{h-1}^{-1} \circ \bar{d}^1 \circ \kappa_h$ in the following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \mathcal{R}_h(G)[h] \arrow[r, "\partial"] \arrow[d, "\kappa_h"] & \mathcal{R}_{h-1}(G)[h-1] \arrow[r, "\partial"] \arrow[d, "\kappa_{h-1}"] & \cdots \\
\cdots \arrow[r, "\bar{d}^1"] & H_h(\mathcal{N}_*(G)[h]) \arrow[r, "\bar{d}^1"] & H_{h-1}(\mathcal{N}_*(G)[h-1]) \arrow[r, "\bar{d}^1"] & \cdots
\end{tikzcd}$$

It follows that $(\mathcal{R}_*(G), \partial)$ is chain-isomorphic to the complex in (4.3.1) and therefore its homology is the homology of the group G.

4.4 Non-trivial Coefficients

Next we investigate the situation where we take homology with non-trivial coefficients. Recall from Section 2.4 that M is assumed to be G-module with a norm-admissible filtration, that is

$$\{0\} = M_{-1} \subseteq M_0 \subseteq \dots \subseteq M$$

and for each $g \in G$ and $i \geq 0$ we have $g : M_i \to M_{i+N(g)}$.
The filtration of the module $\bar{E}_q(G, M) = \bar{E}_q(G) \otimes_{\mathbb{Z}G} M \cong \bar{B}_q(G) \otimes M$ is defined as

$$\mathcal{F}_h \bar{E}_q(G, M) = \sum_{i+j=h} \mathcal{F}_i \bar{B}_q(G) \otimes M_j$$

To apply the results of Section 4.2 first we have to analyze the filtration quotients $\mathcal{N}_q(G, M)[h] = \mathcal{F}_h \bar{E}_q(G, M)/\mathcal{F}_{h-1} \bar{E}_q(G, M)$.

Lemma 4.4.1.

$$\mathcal{N}_q(G, M)[h] \cong \bigoplus_{i+j=h} (\mathcal{F}_i \bar{B}_q(G)/\mathcal{F}_{i-1} \bar{B}_q(G)) \otimes (M_j/M_{j-1}).$$

Proof: Denote by S the set of free generators of the $\mathbb{Z}$-module $\bar{B}_q(G)$ and by S_i the set of free generators of the $\mathbb{Z}$-module $\mathcal{F}_i \bar{B}_q(G)$ for each $i \geq 0$. As an abelian group $\bar{B}_q(G) \otimes M$ is generated then by elements $s \otimes m$, where $s \in S$ and $m \in M$. The only relations among these generators are of the form $s \otimes m + s \otimes m' \sim s \otimes (m + m')$. It follows that

$$\sum_{s_i \neq s_j} s_i \otimes m_i = 0 \text{ if and only if } m_i = 0 \text{ for all } i$$

More general,

$$\sum_{s_i \neq s_j} s_i \otimes m_i \in \mathcal{F}_h(\bar{B}_q(G) \otimes M)$$

if and only if $s_i \otimes m_i \in \mathcal{F}_h(\bar{B}_q(G) \otimes M)$ for all i.
Denote the projection $C_h \to C_h/C_{h-1}$ by $[.]_h$ in a filtered module C and define a map

$$\pi : \mathcal{F}_h(\bar{B}_q(G) \otimes M)/\mathcal{F}_{h-1}(\bar{B}_q(G) \otimes M) \to \bigoplus_{i+j=h} (\mathcal{F}_i\bar{B}_q(G)/\mathcal{F}_{i-1}\bar{B}_q(G)) \otimes (M_j/M_{j-1})$$

on generators by $\pi([s \otimes m]_h) := [s]_i \otimes [m]_{h-i}$ for $s \in S_i \setminus S_{i-1}$. First we claim that

$$[s \otimes m]_h = 0 \iff \pi([s \otimes m]_h) = 0. \tag{4.4.1}$$

Assume that $s \in S_i \setminus S_{i-1}$. If $s \otimes m \in \mathcal{F}_{h-1}(\bar{B}_q(G) \otimes M)$, then $m \in M_{h-i-1}$, in particular $[m]_{h-i} = 0$. Vice versa, if $[s]_i \otimes [m]_{h-i} = 0$, then since $\mathcal{F}_i\bar{B}_q(G)/\mathcal{F}_{i-1}\bar{B}_q(G)$ is freely generated by the set $S_i \setminus S_{i-1}$ it follows that $[m]_{h-i} = 0$, thus $m \in M_{h-i-1}$ and $s \otimes m \in \mathcal{F}_{h-1}(\bar{B}_q(G) \otimes M)$.

The map π is well defined: if $[s \otimes m]_h = [s' \otimes m']_h$ then $s \otimes m - s' \otimes m' \in \mathcal{F}_{h-1}(\bar{B}_q(G) \otimes M)$. We distinguish the following two cases:
If $s \neq s'$, then both $s \otimes m$ and $s' \otimes m'$ are in $\mathcal{F}_{h-1}(\bar{B}_q(G) \otimes M)$, thus $[s \otimes m]_h = [s' \otimes m']_h = 0$ and $\pi([s \otimes m]_h) = \pi([s' \otimes m']_h) = 0$.
If $s = s'$, then $m - m' \in M_{h-1-i}$, thus $[m]_{h-i} = [m']_{h-i}$ and $\pi([s \otimes m]_h) = \pi([s' \otimes m']_h)$.
Extending π linearly to $\mathcal{F}_h(\bar{B}_q(G) \otimes M)/\mathcal{F}_{h-1}(\bar{B}_q(G))$ is also well defined, since

$$\pi([s \otimes m]_h + [s \otimes m']_h) = [s]_i \otimes [m]_{h-i} + [s]_i \otimes [m']_{h-i} = [s]_i \otimes [m + m']_{h-i} = \pi([s \otimes (m + m')]_h)$$

By (4.4.1) we know that π is injective and it is clearly surjective. □

Denote the quotient M_h/M_{h-1} by $M[h]$. Applying Lemma 4.4.1., we obtain that $\mathcal{N}_q(G, M)[h]$ is isomorphic (as a module) to the direct sum

$$\bigoplus_{i+j=h} \mathcal{N}_q(G)[i] \otimes M[j]$$

hence each summand is generated by $(q+1)$-tuples $(g_q|...|g_1)[m]_j$, where $[m]_j \in M[j]$ is an equivalence class, and $N(g_q) + ... + N(g_1) + j = h$.

Since for each $g \in G$ and $m \in M_h$ we have $gm \in M_{h+N(g)}$ it follows that $g \in G$ defines a homomorphism $M[h] \to M[h + N(g)]$ by $g[m]_h := [gm]_{h+N(G)}$. To shorten notation, let us denote the boundary $\bar{d}^0_M : \mathcal{N}_q(G, M)[h] \to \mathcal{N}_{q-1}(G, M)[h]$ simply by d_M. It is the signed sum of the following face operators:

$$d_0((g_q| \dots |g_1)[m]_j) = (g_q| \dots |g_2)[g_1 m]_{N(g_1)+j}$$

and for $1 \leq i \leq q-1$

$$d_i((g_q|\dots|g_1)[m]_j) = \begin{cases} (g_q|\dots|g_{i+1}g_i|\dots|g_1)[m]_j & \text{if } N_G(g_{i+1}g_i) = N_G(g_{i+1}) + N_G(g_i), \\ 0 & \text{if } N_G(g_{i+1}g_i) < N_G(g_{i+1}) + N_G(g_i). \end{cases}$$

Note that for each $1 \leq i \leq q-1$ we have $d_i : \mathcal{N}_q(G)[i] \otimes M[j] \to \mathcal{N}_{q-1}(G)[i] \otimes M[j]$, while the image of d_0 is in a different summand.

Example 4.4.2. Using the constant norm with value c on M we obtain that $\mathcal{N}_q(G, M)[h]$ is isomorphic (as a module) to $\mathcal{N}_q(G)[h-c] \otimes M$. Since $gm \in M_c = M$ for each $g \in G$ and $m \in M$ it follows that $d_0 = 0$, hence as a complex $(\mathcal{N}_*(G, M)[h], d_M)$ is isomorphic to the complex $(\mathcal{N}_*(G)[h-c] \otimes M, d \otimes \mathrm{id})$.

¿From the universal coefficient theorem and from Theorem 4.1.1. we obtain:

Corollary 4.4.3. *Assume that G is a factorable group and M is a G-module with constant zero norm. Then the homology of the complex $(\mathcal{N}_*(G, M)[h], d_M)$ is concentrated in the top degree $* = h$ and $H_h((\mathcal{N}_*(G, M)[h], d_M)) = H_h((\mathcal{N}_*(G)[h], d)) \otimes M$.* □

We conclude, that the spectral sequence associated to the norm filtration of $\bar{E}_*(G, M)$ collapses after the E^2-term and the homology of the group G with coefficients in the G-module M can be computed as the homology of the complex

$$\cdots \xrightarrow{\bar{d}^1_M} H_h(\mathcal{N}_*(G)[h]) \otimes M \xrightarrow{\bar{d}^1_M} H_{h-1}(\mathcal{N}_*(G)[h-1]) \otimes M \xrightarrow{\bar{d}^1_M} \cdots$$

Moreover, if $T = T(G)$ has only finitely many elements and hence $\kappa_h : \mathcal{R}_h(G)[h] \to H_h(\mathcal{N}_*(G)[h])$ is an isomorphism we obtain a chain-isomorphic complex

$$\cdots \xrightarrow{\partial_M} \mathcal{R}_h(G, M)[h] \xrightarrow{\partial_M} \mathcal{R}_{h-1}(G, M)[h-1] \xrightarrow{\partial_M} \cdots$$

As a module $\mathcal{R}_h(G, M)[h] = \mathcal{R}_h(G)[h] \otimes M$, but $\partial_M = \kappa_{h-1}^{-1} \circ \bar{d}^1_M \circ \kappa_h$ and therefore it is different from $\partial \otimes id$. Applying these results to $M = \mathbb{Z}G$ with the constant zero norm we obtain that the complex $\mathcal{R}_*(G, \mathbb{Z}G)$ is a resolution of the group G and that $\mathcal{R}_*(G) = \mathcal{R}_*(G, \mathbb{Z}G) \otimes_{\mathbb{Z}G} \mathbb{Z}$ as complexes.

Example 4.4.4. More general, if the action of G on M is trivial, then $d_0 = 0$ and hence the complex $(\mathcal{N}_*(G, M)[h], d_M)$ splits as a direct sum of complexes:

$$(\mathcal{N}_*(G, M)[h], d_M) \cong \bigoplus_i (\mathcal{N}_*(G)[h-i] \otimes M[i], d \otimes \mathrm{id})$$

Again, from the universal coefficient theorem we obtain that in each degree $0 \leq k \leq h$

$$H_k((\mathcal{N}_*(G, M)[h], d_M)) = H_k((\mathcal{N}_*(G)[k], d)) \otimes M[h-k]$$

In the remaining of the section we apply the ideas of Section 4.2. to the complex $\mathcal{N}_*(G, M)[h]$. The partition-type of a generator $(g_q|\dots|g_1)[m]_j$ of $\mathcal{N}_q(G, M)[h]$ is defined to be the tuple $(N(g_q), N(g_{q-1}), ..., N(g_1), j)$, where the last entry may be zero.
Using Definition 4.2.1. we can extend the partition-type order to the set

$$\mathcal{P}^+(h) = \{(l_q, ..., l_1, l_0) \mid \sum_{i=0}^{q} l_i = h,\ l_i > 0 \text{ for } i \geq 1,\ l_0 \geq 0\}.$$

Let us denote by $\mathcal{N}_q(G, M)[P] = \mathcal{N}_q(G, M)[l_q, ..., l_1, l_0]$ the submodule of $\mathcal{N}_q(G, M)[h]$ generated by all $(q+1)$-tuples $(g_q|\dots|g_1)[m]_{l_0}$ with partition-type $P = (l_q, ..., l_1, l_0)$.
Since the set of generators of a given parition-type P is closed under the relations $(g_q|\dots|g_1)[m]_j + (g_q|\dots|g_1)[m']_j = (g_q|\dots|g_1)[m+m']_j$, it follows that as a module

$$\mathcal{N}_q(G, M)[h] = \bigoplus_{P \in \mathcal{P}^+(h)} \mathcal{N}_q(G, M)[P].$$

The definition of the partition-type filtration, the observations about the E^1-term of the associated homological spectral sequence are identical with the original case.
If G is factorable, then we have the homomorphisms

$$\eta_i^{(G)} : \mathcal{N}_q(G)[l_q, ..., l_i, ..., l_1] \to \mathcal{N}_{q+1}(G)[l_q, ..., l_i - 1, 1, ..., l_1],$$

where $l_i \geq 2$. These maps give rise to homomorphisms

$$\eta_i := \eta_i^{(G)} \otimes id : \mathcal{N}_q(G, M)[l_q, ..., l_i, ..., l_1, l_0] \to \mathcal{N}_{q+1}(G, M)[l_q, ..., l_i - 1, 1, ..., l_1, l_0]$$

for all $i \geq 1$ for which $l_i \geq 2$.
Assume that the minimal non-zero value of the norm N on G is one and that M_0 is non-trivial. ¿From the proof of Theorem 4.1.1. we obtain the following generalization:

Proposition 4.4.5. *Let (G, N) be a factorable group and M be a G-module, with norm-admissible filtration. Assume moreover that for each partition-type $(l_q, ..., l_i, ..., l_0)$ with $l_0 \geq 1$ we have a homomorphism*

$$\eta_0 : \mathcal{N}_q(G, M)[l_q, ..., l_0] \to \mathcal{N}_{q+1}(G, M)[l_q, ..., l_0, 0],$$

such that η_0 satisfies the following three equalities:
(1) $d_0\eta_0 = id$,
(2) $d_j\eta_0 = \eta_0 d_{j-1}$ for $2 \leq j \leq q$ and
(3) $d_1\eta_0 d_0 \eta_1 = \eta_0 d_0$.
Then the homology of the complex $(\mathcal{N}_*(G, M)[h], d_M)$ is concentrated in the top degree $* = h$. □

In particular, $\eta_0 : \mathcal{N}_q(G)[h-j] \otimes M[j] \to \mathcal{N}_{q+1}(G)[h] \otimes M[0]$ is a homomorphism for each summand $\mathcal{N}_q(G)[h-j] \otimes M[j]$ of $\mathcal{N}_q(G, M)[h]$, where $j \geq 1$ and $M[j] \neq 0$.

Example 4.4.6. $M = \mathbb{Z}G$ and the filtration of M comes from the norm on G. To distinguish from the case where we take the constant zero norm on $\mathbb{Z}G$ as in Example 4.4.2., we will denote the filtration quotients by $\mathcal{N}_q(G, \widetilde{\mathbb{Z}G})[h]$. For each $j \geq 0$ we have that $M[j]$ is the free $\mathbb{Z}$-module generated by elements of G with norm j. It follows that $\mathcal{N}_q(G, \widetilde{\mathbb{Z}G})[h]$ is free and we define η_0 for the free generators of $\mathcal{N}_q(G, \widetilde{\mathbb{Z}G})[h]$ by

$$\eta_0 : (g_q|...|g_1) \otimes g_0 \mapsto (g_q|...|g_1|g_0) \otimes 1,$$

where $1 \in G$. Extending η_0 linearly we obtain a module homomorphism which clearly satisfies requirement (1) and (2) in Proposition 4.4.5.. That requirement (3) is also satisfied follows from the (graded) commutativity of the following diagram:

$$\begin{tikzcd}
(g_q|\dots|g_1)\otimes g_0 \arrow[r, mapsto, "\eta_1"] \arrow[d, mapsto, "d_0"] & (g_q|\dots|\bar{g}_1|g_1')\otimes g_0 \arrow[r, mapsto, "d_0"] & (g_q|\dots|\bar{g}_1)\otimes g_1' g_0 \arrow[r, mapsto, "\eta_0"] & (g_q|\dots|\bar{g}_1|g_1' g_0)\otimes 1 \arrow[d, mapsto, "d_1"] \\
(g_q|\dots|g_2)\otimes g_1 g_0 \arrow[rrr, mapsto, "\eta_0"] & & & (g_q|\dots|g_1 g_0)\otimes 1
\end{tikzcd}$$

By Proposition 4.4.5. the homology of the complex $\mathcal{N}_*(G,\tilde{\mathbb{Z}G})[h]$ is concentrated in degree $* = h$. Assume that $T = T(G)$ is finite. Since the coefficients are now in $\mathbb{Z}G$ we can apply the results of Section 4.3. and the result of Wang to obtain that $H_h(\mathcal{N}_*(G,\tilde{\mathbb{Z}G})[h]) \cong \mathcal{R}_h(G,\tilde{\mathbb{Z}G})$, where $\mathcal{R}_h(G,\tilde{\mathbb{Z}G})$ is freely generated by the set

$$R_q(G,\tilde{\mathbb{Z}G})[h] = \{c = (g_q|...|g_1)\otimes 1 \mid \sum_{i=1}^{q} N(g_i) = h,\ \eta_i d_i(c) \neq c \text{ for all } 0 \leq i \leq q-1\}.$$

Since $\eta_0 d_0 : (g_q|...|g_1)\otimes 1 \mapsto (g_q|...|g_1)\otimes 1$ for any q-tuple $(g_q|...|g_1)$ it follows that $R_q(G,\tilde{\mathbb{Z}G})[h] = \emptyset$ and thus the complex $\mathcal{N}_*(G,\tilde{\mathbb{Z}G})[h]$ is acyclic.

Chapter 5

Symmetric Groups

5.1 The Word Length as a Norm

Our principal examples of factorable groups are the symmetric groups Σ_p, with the word-length norm induced by the set of all transpositions as a generating set. In this section we fix our notation dealing with the symmetric groups, and we investigate in detail the word-length norm N^{wl} on the group Σ_p.

We consider Σ_p as the group of permutations of the set $I_p = \{1, 2, \ldots, p\}$. We write permutations $\sigma \in \Sigma_p$ in the cycle-notation $(a, \sigma(a), ...)$. Cycles of length one (i.e. the fixed-points of σ) are often omitted. Multiplication of permutations we will write as composition of functions, thus $\sigma_1 \cdot \sigma_2$ is the permutation obtained by first applying σ_2 and then σ_1.
It will be convenient to have a second description of the world-length norm: we define a function $N = N_p : \Sigma_p \to \mathbb{N}$ for each $\sigma \in \Sigma_p$ as $N(\sigma) = p - \mathrm{cyc}(\sigma)$, where $\mathrm{cyc}(\sigma)$ denotes the number of cycles of the permutation σ, also counting cycles of length one.

Lemma 5.1.1. *Assume that $\sigma \in \Sigma_p$ and $\tau = (a, b) \in \Sigma_p$ is a transposition. Then*

$$N(\tau \cdot \sigma) = N(\sigma) + 1 = N(\sigma \cdot \tau)$$

if the letters a and b are in different cycles of σ and

$$N(\tau \cdot \sigma) = N(\sigma) - 1 = N(\sigma \cdot \tau)$$

if the letters a and b are both in the same cycle of σ.

Proof: Denote the set of cycles of σ by $\mathcal{C}(\sigma) = \{C_1, C_2, ..., C_m\}$. First consider the case where $C_i = (..., \sigma^{-1}(a), a, \sigma(a), ...)$ and $C_j = (..., \sigma^{-1}(b), b, \sigma(b), ...)$ are different cycles. Then $\mathcal{C}(\tau \cdot \sigma)$, the set of cycles of $\tau \cdot \sigma$ is $\mathcal{C}(\sigma) \setminus \{C_i, C_j\}$ and a new cycle $(..., \sigma^{-1}(b), a, \sigma(a), ..., \sigma^{-1}(a), b, \sigma(b), ...)$, hence the claim follows. In the second case assume that $C_i = (..., \sigma^{-1}(a), a, \sigma(a), ..., \sigma^{-1}(b), b, \sigma(b), ...)$. Then $\mathcal{C}(\tau \cdot \sigma)$ is $\mathcal{C}(\sigma) \setminus \{C_i\}$ and two new cycles $(..., \sigma^{-1}(a), b, \sigma(b), ...)$ and $(..., \sigma^{-1}(b), a, \sigma(a), ...)$. The product $\sigma \cdot \tau$ is treated similarly. □

Lemma 5.1.2. *The function N_p is a norm on the group Σ_p and it coincides with the world-length norm N^{wl} induced by the set of all transpositions as a generating set.*

Proof: As before, denote the set of cycles of $\sigma \in \Sigma_p$ by $\mathcal{C} = \{c_1, c_2, ..., c_m\}$ and denote the length of $c_i \in \mathcal{C}$ by l_i. Then $l_1 + l_2 + ... + l_m = p$ and $N(\sigma) = p - m = \sum_{j=1}^{m}(l_j - 1)$. Since a cycle $(a_1, a_2, ..., a_k)$ of length k can be written as a product of $k-1$ transpositions $(a_1, a_2, ..., a_k) = (a_1, a_k) \cdot (a_1, a_{k-1}) \cdot ... \cdot (a_1, a_2)$ it follows that $N(\sigma) \geq N^{wl}(\sigma)$.
On the other hand, if $\sigma \in \Sigma_p$ is a product of k transpositions then $N(\sigma) \leq k$. This follows inductively from Lemma 5.1.1., hence $N(\sigma) \leq N^{wl}(\sigma)$, finishing the proof. □

We will call the norm $N_p = N^{wl}$ the *cycle norm* on the group Σ_p.

5.2 Factorability

In this section we investigate the question whether the symmetric group is factorable with respect to the norm N_p. The positive answer is given by the following

Theorem 5.2.1. *The symmetric group Σ_p is factorable with respect to the cycle norm N_p.*

Proof: First we introduce a *height* function $ht : \Sigma_p \setminus \{1\} \to I_p$, which assigns to a permutation $\sigma \neq 1$ the largest element of the set I_p, which is not a fixed-point of σ:

$$ht(\sigma) = \max\{j \in I_p \mid \sigma(j) \neq j\}.$$

Now we can define the splitting map $\eta = \eta_p : \Sigma_p \to \Sigma_p \times \Sigma_p$ as follows. For $1 \neq \sigma \in \Sigma_p$ the two factors of η are given by

$$\eta'(\sigma) = \sigma' = (i, \sigma^{-1}(i)) \tag{5.2.1}$$

where $i = ht(\sigma)$ and by

$$\bar{\eta}(\sigma) = \bar{\sigma} = \sigma \cdot \eta'(\sigma). \tag{5.2.2}$$

As usual, for $1 \in \Sigma_p$ we set $\eta(1) = (1, 1)$. Now we show that η is a splitting function.
First note, that if $\sigma \neq 1$, then $i \neq \sigma^{-1}(i)$ for $i = ht(\sigma)$, thus $\eta'(\sigma) \in T$ and condition (3.1.3) is fulfilled. The equation $\bar{\eta}(\sigma) \cdot \eta'(\sigma) = \sigma$ is obvious.
The letters i and $\sigma^{-1}(i)$ are in one cycle of σ, thus by the second part of Lemma 5.1.1. we have $N(\bar{\eta}(\sigma)) = N(\sigma) - 1$ and hence condition (3.1.2) is satisfied: $N(e\bar{t}a(\sigma)) + N(\eta'(\sigma)) = N(\sigma)$. It is now also clear, that roughly speaking η has the following effect: in each permutation σ the largest non-fixed-point of σ is turned into a fixed-point by multiplying σ with an appropiate transposition.
It remains to check the conditions for diagram (3.1.4). Recall that the compositions α_u and α_l were defined as $(\mu \times id) \circ (id \times \eta) \circ (id \times \mu) \circ (\eta \times id)$ and $\eta \circ \mu$ respectively, where $\mu : \Sigma_p \times \Sigma_p \to \Sigma_p$ denotes the multiplication in Σ_p. By Proposition 3.1.8. it is enough to check two things:
Condition (3.1.4.A): for every pair $(\sigma, \tau) \in \Sigma_p \times T$ we have $N((\sigma, \tau)) = N(\alpha_u((\sigma, \tau))) \iff N((\sigma, \tau)) = N(\alpha_l((\sigma, \tau)))$; and
Condition (3.1.4.B): for all those pairs (σ, τ) where the two compositions do not change the norm $\alpha_u((\sigma, \tau)) = \alpha_l((\sigma, \tau))$ holds.
For the group $G = \Sigma_p$ we can show actually a bit more: (3.1.4.B) holds for any pair $(\sigma, \tau) \in \Sigma_p \times T$ for which $\tau \neq \eta'(\sigma)$, regardless whether α_u and α_l are norm-preserving, or not. In fact, for these pairs then (3.1.4.A) follows automatically.

We consider first the exceptional case $\tau = \eta'(\sigma)$ using the notation $\eta'(\sigma) = \sigma'$, $\bar{\eta}(\sigma) = \bar{\sigma}$. The two compositions in question are:

$$\alpha_u : (\sigma, \sigma') \mapsto (\bar{\sigma}, \sigma', \sigma') \mapsto (\bar{\sigma}, 1) \mapsto (\bar{\sigma}, 1, 1) \mapsto (\bar{\sigma}, 1) \text{ and}$$
$$\alpha_l : (\sigma, \sigma') \mapsto \bar{\sigma} \mapsto (\overline{\overline{\sigma}}, (\overline{\sigma})').$$

It is clear that none of the two compositions are norm-preserving: $N((\sigma, \sigma')) = N(\sigma) + 1$, while $N(\alpha_u((\sigma, \sigma'))) = N((\bar{\sigma}, 1)) = N(\sigma) - 1$ and $N(\alpha_d((\sigma, \sigma'))) \leq N(\bar{\sigma}) = N(\sigma) - 1$. Hence, condition (3.1.4.A) is satisfied in this case.

Now we turn to the case $\tau \neq \eta'(\sigma)$. We will need the following

Lemma 5.2.2. *If $(\sigma, \tau) \in \Sigma_p \times T_p$ and $\tau \neq \sigma'$, then*

$$(\sigma \cdot \tau)' = \begin{cases} \tau & \text{if } ht(\sigma) < ht(\tau), \\ \sigma' & \text{if } ht(\sigma) = ht(\tau), \\ \tau \cdot \sigma' \cdot \tau & \text{if } ht(\sigma) > ht(\tau). \end{cases}$$

Proof: Assume that $\tau = (i, j)$ and $\sigma' = (k, l)$, where $i < j = ht(\tau)$ and $k < l = ht(\sigma)$. Clearly those elements of I_p which are fixed points of both τ and σ are fixed points of $\sigma \cdot \tau$, hence $ht(\sigma \cdot \tau) \leq \max\{ht(\tau), ht(\sigma)\}$.
If $l < j$, then $(\sigma \cdot \tau)^{-1}(j) = \tau^{-1}(j) = i$, since $\sigma(j) = j$. It follows, that $ht(\sigma \cdot \tau) = j$, thus $(\sigma \cdot \tau)' = (i, j) = \tau$ in this case.
If $l = j$, then $(\sigma \cdot \tau)^{-1}(l) = \tau^{-1}(k) = k$, since $(i, j) \neq (k, l)$ by our assumption. Thus $ht(\sigma \cdot \tau) = l = j$ and $(\sigma \cdot \tau)' = (k, l) = \sigma'$.
Similarly, if $l > j$, then $(\sigma \cdot \tau)^{-1}(l) = \tau^{-1}(k) \neq l$, since $\tau \neq \sigma'$, hence $ht(\sigma \cdot \tau) = l$. To compute $(\sigma \cdot \tau)' = (\tau^{-1}(k), l)$ we have to distinguish the following three cases:

$$(\sigma \cdot \tau)' = \begin{cases} (k, l) & \text{if } k \neq i, j, \\ (j, l) & \text{if } k = i, \\ (i, l) & \text{if } k = j. \end{cases}$$

These three cases can be summarized by the equation $(\sigma \cdot \tau)' = \tau \cdot \sigma' \cdot \tau$. $\square$

The final step of the proof of Theorem 5.2.1. is now obvious: using that $ht(\sigma) = ht(\sigma')$ and that $(\sigma')' = \sigma'$ by our definition of a splitting map, it is clear that $(\sigma' \cdot \tau)' = (\sigma \cdot \tau)'$ proving condition (3.1.4.B). $\square$

5.3 The Alternating Group

The aim of this section is to show that the alternating group A_p is factorable. The group A_p is the group of even permutations of the set $I_p = \{1, 2, ..., p\}$. Note, that $(-1)^{N_p(\sigma)} = \text{sign}(\sigma)$, thus A_p is the subgroup of Σ_p consisting of all permutations of even cycle norm.

Clearly, the restriction of the cycle norm N_p to the subgroup A_p of Σ_p is itself a norm on the alternating group and we keep the notation N_p for this norm. Thus $T(A_p)$, the set of group elements in A_p with minimal non-zero norm is the set of all permutations with norm two.
Any factorization map $\eta : A_p \to A_p \times A_p$ has to satisfy condition (3.1.3): the prefix $\eta'(\sigma)$ of any $1 \neq \sigma \in A_p$ is an element of $T(A_p)$. The obvious candidate for such a map is the (first) iteration ζ of the factorization map η defined for the symmetric group in (5.2.1) and (5.2.2). That ζ makes the alternating group factorable follows from the next proposition, concerning iterations of factorization maps.

Consider a normed group (G, N) together with a factorization map $\eta : G \to G \times G$. The (first) iteration ζ of η is defined as the composition

$$G \xrightarrow{\eta} G \times G \xrightarrow{\eta \times id} G \times G \times G \xrightarrow{id \times \mu} G \times G$$

where $\mu : G \times G \to G$ denotes the multiplication in the group G. That is,

$$\zeta(g) := (\bar{\zeta}(g), \zeta'(g)) = (\bar{\eta}(\bar{\eta}(g)), \eta'(\bar{\eta}(g)) \cdot \eta'(g)) \text{ for any } g \in G.$$

Using our other notation for prefix and remainder we have $\zeta(g) = (\overline{\overline{g}}, (\overline{g})' \cdot g')$. Clearly the prefix $\zeta'(g)$ of g is not necessarily an element of $T(G)$, since the product of two norm one elements may have norm two; thus condition (3.1.3) of a factorization map may be violated. However, ζ satisfies condition (3.1.1) and (3.1.2): $\bar{\zeta}(g) \cdot \zeta'(g) = \overline{\overline{g}} \cdot (\overline{g})' \cdot g' = \overline{g} \cdot g' = g$ and the following norm level diagram

$$\begin{tikzcd}
(\overline{g}, g') \arrow[r, Rightarrow, dotted, "="] & (\overline{\overline{g}}, (\overline{g})', g') \\
(g) \arrow[u, Rightarrow, dotted, "="] \arrow[r, Rightarrow, dotted] & (\overline{\overline{g}}, (\overline{g})' \cdot g') \arrow[u, Rightarrow, dotted]
\end{tikzcd} \qquad (5.3.1)$$

shows that $N(\overline{\overline{g}}) + N((\overline{g})' \cdot g') = N(g)$.

Now assume that η makes the group G factorable. In this case we can show, that the first iteration ζ of η behaves similarly:

Proposition 5.3.1. *Assume that the factorization map $\eta : G \to G \times G$ turns the group G into a factorable group. Let H be a subgroup of G, such that the first iteration ζ of η restricts to a factorization map of H. Then H is factorable.*

Proof: We show that diagram (3.1.4) commutes using ζ. As discussed in Section 3.1 the commutativity of diagram (3.1.4) is equivalent with condition (3.1.4.A) and (3.1.4.B) to hold. Since the group G is factorable, these conditions hold for the factorization map η: by (3.1.4.A) we have that for any pair $(g, h) \in G \times G$

$$\text{If } (g', h) \text{ and } (\overline{g}, \overline{g'h}) \text{ are geodesic pairs} \Rightarrow (g, h) \text{ is a geodesic pair} \qquad (5.3.2)$$

and by (3.1.4.B) we have that

$$\text{If } (g, h) \text{ is a geodesic pair} \Rightarrow (g'h)' = (gh)'. \qquad (5.3.3)$$

In the present situation we have to show these conditions using ζ instead of η. Thus we are going to prove the following statements

$$\text{If } ((\overline{g})'g', h) \text{ and } (\overline{\overline{g}}, \overline{\overline{(\overline{g})'g'h}}) \text{ are geodesic pairs} \Rightarrow (g, h) \text{ is a geodesic pair} \tag{5.3.4}$$

$$\text{If } (g, h) \text{ is a geodesic pair} \Rightarrow (\overline{(\overline{g})'g'h})' \cdot ((\overline{g})'g'h)' = (\overline{gh})' \cdot (gh)' \tag{5.3.5}$$

First we prove (5.3.4) by considering the following norm level diagram:

$$\begin{tikzcd}
((\overline{g})'g', h) \arrow[r, dotted, "(a)"] & ((\overline{g})', g', h) \\
((\overline{g})'g'h) \arrow[u, dotted, "(b)"] \arrow[r, dotted, "(c)"] & ((\overline{g})', g'h) \arrow[u, dotted, "(d)"']
\end{tikzcd}$$

Decomposition (b) is norm-preserving by our assumption and decomposition (a) is norm-preserving by diagram (5.3.1). It follows that the pairs (g', h) and $((\overline{g})', g'h)$ are geodesic pairs.
Now applying (5.3.3) to the geodesic pairs $(\overline{g}, g')$ and $((\overline{g})'g', h)$ we obtain

$$((\overline{g})'g' \cdot h)' = (((\overline{g})'g')' \cdot h)' = ((\overline{g}g')' \cdot h)' = (g' \cdot h)'$$

and hence

$$\overline{(\overline{g})'g'h} = (\overline{g})' \cdot \overline{g'h}.$$

To shorten notation we set $x := \overline{g}$ and $y := \overline{g'h}$.
Since $((\overline{g})', g'h)$ is a geodesic pair, it follows that in particular $((\overline{g})', \overline{g'h}) = (x', y)$ is also a geodesic pair. On the other hand, by the assumption in (5.3.4) the pair $(\overline{\overline{g}}, \overline{\overline{(\overline{g})'g'h}}) = (\overline{\overline{g}}, \overline{(\overline{g})' \cdot \overline{g'h}}) = (\overline{x}, \overline{x'y})$ is geodesic. Applying (5.3.2) to (x', y) and to $(\overline{x}, \overline{x'y})$ we obtain that $(x, y) = (\overline{g}, \overline{g'h})$ is geodesic. Hence we can use (5.3.2) again: (g', h) and $(\overline{g}, \overline{g'h})$ being geodesic pairs implies that (g, h) is a geodesic pair, finishing the proof of (5.3.4).

It remains to show (5.3.5). Assume that the pair (g, h) is geodesic. Using the following norm level diagram

$$\begin{tikzcd}
(\overline{g}, g'h) \arrow[r, dotted] & (\overline{g}, g', h) \\
(gh) \arrow[u, dotted] \arrow[r, dotted, "="] & (g, h) \arrow[u, dotted, "="]
\end{tikzcd}$$

we obtain that $(\overline{g}, g'h)$ is a geodesic pair, for which we apply (5.3.3):

$$((\overline{g})' \cdot g'h)' = (\overline{g} \cdot g'h)' = (gh)' = (g'h)'.$$

Equivalently, for the remainder part of $(\overline{g})'g'h$ we obtain $\overline{(\overline{g})'g'h} = (\overline{g})' \cdot \overline{g'h}$. Thus

$$(\overline{(\overline{g})'g'h})' = ((\overline{g})' \cdot \overline{g'h})' = (\overline{g} \cdot \overline{g'h})' = (\overline{gh})'$$

where we used that if (g, h) is a geodesic pair, then $(\overline{g}, \overline{g'h})$ is a geodesic pair (Corollary 3.1.7.) and that by Remark 3.1.4. we have $\overline{g} \cdot \overline{g'h} = \overline{gh}$. Hence $(\overline{(\overline{g})'g'h})' \cdot ((\overline{g})'g'h)' = (\overline{gh})' \cdot (gh)'$ as required. $\square$

Corollary 5.3.2. *The alternating group A_p is factorable with respect to the cycle norm.*

Proof: Consider the first iteration ζ of the factorization map $\eta : \Sigma_p \to \Sigma_p \times \Sigma_p$. We claim that ζ restricts to a factorization map of the alternating group A_p. From diagram (5.3.1) it follows that $((\overline{\sigma})', \sigma')$ is a geodesic pair, thus $N_p(\zeta'(\sigma)) = N_p((\overline{\sigma})'\sigma') = N_p((\overline{\sigma})') + N_p(\sigma')$. Now $N_p(\sigma') = 1$ for all $1 \neq \sigma \in \Sigma_p$ and $N_p((\overline{\sigma})') = 1$ unless $\overline{\sigma} = 1$, that is $\sigma \in T(\Sigma_p)$. Hence for $1 \neq \sigma \in A_p$ we obtain that $N_p(\zeta'(\sigma)) = 2$, thus $\zeta'(\sigma) \in T(A_p)$ and by (3.1.2) that $\bar{\zeta}(\sigma) \in A_p$. By Proposition 5.3.1. the group A_p is then factorable. □

5.4 Homology of Symmetric Groups

In Proposition 4.3.5. we have identified a direct summand of the homology groups $H_h(\mathcal{N}_*(G)[h])$ as the image of the injective module-homomorphism $\kappa_h : \mathcal{R}_h(G)[h] \to H_h(\mathcal{N}_*(G)[h])$. The map κ_h is not surjective in general, but in the case $G = \Sigma_p$ we can show the following

Theorem 5.4.1. *The map $\kappa_h : \mathcal{R}_h(\Sigma_p)[h] \to H_h(\mathcal{N}_*(\Sigma_p)[h])$ is an isomorphism.*

Proof: One way to prove this statement would be to show directly, that κ_h is surjective, for example by showing that each cycle $c \in \mathcal{N}_h(\Sigma_p)[h]$ has at least one summand in $R_h(\Sigma_p)[h]$, the set of free generators of $\mathcal{R}_h(\Sigma_p)[h]$. Here we choose a different approach: we show that the two finitely generated free modules $\mathcal{R}_h(\Sigma_p)[h]$ and $H_h(\mathcal{N}_*(\Sigma_p)[h])$ have the same rank. The rank of $\mathcal{R}_h(\Sigma_p)[h]$ is just the cardinality of it's generating set $R_p(h) := R_h(\Sigma_p)[h]$ and the rank of $H_h(\mathcal{N}_*(\Sigma_p)[h])$ can be computed from the Euler-characteristic of the complex $(\mathcal{N}_*(\Sigma_p)[h], d)$. In fact, since by Theorem 4.1.1. $H_q(\mathcal{N}_*(\Sigma_p)[h]) = 0$ for $q \neq h$, we have

$$\chi_p(h) := \chi(\mathcal{N}_*(\Sigma_p)[h]) = (-1)^h \mathrm{rank}(H(\mathcal{N}_*(\Sigma_p)[h])) = \sum_{q=1}^{h} (-1)^q s_{p,q}(h),$$

where $s_{p,q}(h)$ denotes the number of free generators of $\mathcal{N}_q(\Sigma_p)[h]$. Our aim is thus to show that the equation $r_p(h) := |R_p(h)| = (-1)^h \chi_p(h)$ holds for all $p \geq 2$ and $h \geq 1$.

This we will do by showing that both double-sequences $\{r_p(h)\}_{p\geq 2, h\geq 1}$ and $\{|\chi_p(h)|\}_{p\geq 2, h\geq 1}$ satisfy the same recursive formula and the same starting conditions for the recursion. Let us consider a third double-sequence $\{St_p(h)\}_{p\geq 2, h\geq 0}$ defined by the formula

$$St_p(h) = \begin{cases} (p-1)St_p(h-1) + St_{p-1}(h) & p \geq 3, h \geq 1, \\ 1 & p = 2, \\ 1 & h = 0. \end{cases} \tag{5.4.1}$$

Double-sequences of this form are actually re-indexed Sirling numbers of the second kind. The Stirling number of the second kind $S(n, k)$, defined for $n \geq k \geq 1$ enumerates partitions of an n-set into k non-empty subsets and can be given recursively by $S(n, k) = k \cdot S(n-1, k) + S(n-1, k-1)$ for $n > k > 1$ and they satisfy $S(n, 1) = S(n, n) = 1$. By reindexing as $St_p(h) := S(h+p-1, p-1)$, the recursion is clearly translated to the form of (5.4.1).

First we consider the set $R_p(h)$ in detail. In Section 4.3 the set $R_h(G)[h]$ was defined in general as

$$R_h(G)[h] = \{c = (g_h|...|g_1) \in \bar{G}^q \mid \ N(g_i) = 1 \text{ for all } i, f_i(c) \neq c \text{ for all } 1 \leq i \leq h-1\}.$$

Since $f_i(c) = (g_h|\dots|\bar{\eta}(g_{i+1}\cdot g_i)|\eta'(g_{i+1}\cdot g_i)|\dots|g_1)$, we need that $\eta'(g_{i+1}\cdot g_i) \neq g_i$ for all $1 \leq i \leq h-1$. In the special case $G = \Sigma_p$ it follows from Lemma 5.2.2., that for any h-tuple of transpositions $\tau = (\tau_h|\dots|\tau_1)$ we have

$$\tau \in R_p(h) \iff ht(\tau_h) \geq ht(\tau_{h-1}) \geq \dots \geq ht(\tau_1).$$

Assume now that $(\tau_h|\dots|\tau_1) \in R_p(h)$. If $ht(\tau_h) < p$, then we can regard the h-tuple $(\tau_h|\dots|\tau_1)$ as an element of the set $R_{p-1}(h)$, by identifying the group Σ_{p-1} with the subgroup of Σ_p consisting all permutations leaving the number $p \in I_p$ invariant. On the other hand, if $ht(\tau_h) = p$, then the $(h-1)$-tuple $(\tau_{h-1}|\dots|\tau_1)$ is clearly an element of $R_p(h-1)$. In the symmetric group Σ_p there are $p-1$ transpositions τ with the property $ht(\tau) = p$, hence we obtain the following recursive formula:

$$r_p(h) = (p-1)r_p(h-1) + r_{p-1}(h) \tag{5.4.2}$$

for all $p \geq 3$ and $h \geq 2$. After extending the definition of $r_p(h)$ to the value $h = 0$ by setting $r_p(0) = 1$ for all $p \geq 2$, equation (5.4.2) will also hold for $h = 1$. In fact $R_p(1) = T_p$, the set of transpositions in the group Σ_p, hence the formula $r_p(1) = (p-1)r_p(0) + r_{p-1}(1)$ is valid. Note that $R_2(h)$ consits of only one element for any value of h, namely the h-tuple $((1,2)|(1,2)|\dots|(1,2))$. Hence $r_p(h)$ has the form of (5.4.1), in particular $r_p(h) = St_p(h)$.

Now we compute the values of the Euler-characteristic $\chi_p(h)$. The number $s_{p,q}(h)$ was defined as the number of free generators of $\mathcal{N}_q(\Sigma_p)[h]$, hence as the the number of q-tuples of permutations in the symmetric group Σ_p such that the norm of the tuple is h.
Note, that in the special case $q = 1$, the number $s_{p,1}(h)$, which will be denoted by $s_p(h)$ to shorten notation, is the signless Stirling number of the first kind $c(p, p-h)$ - the number of permutations in the group Σ_p with $p-h$ cycles.
Again, we extend the definition of $\chi_p(h)$ to the value $h = 0$ by setting $\chi_p(0) = 1$ for all $p \geq 2$. Then $\chi_p(h)$ satisfies the following recursive formula:

$$-\chi_p(h) = \sum_{j=1}^{h} s_p(j)\chi_p(h-j) \tag{5.4.3}$$

This can be shown by similar considerations as in the case of $R_p(h)$. Assume, that the q-tuple of permutations $(\sigma_q|\dots|\sigma_1)$ is a generator of $\mathcal{N}_q(\Sigma_p)[h]$ and that $N(\sigma_q) = i$. Then the $(q-1)$-tuple $(\sigma_{q-1}|\dots|\sigma_1)$ has norm $h-i$, hence it is a generator of $\mathcal{N}_{q-1}(\Sigma_p)[h-i]$. Since there are $s_p(i)$ permutations in Σ_p with norm i, we obtain the formula

$$s_{p,q}(h) = \sum_{i=1}^{h-1} s_p(i)s_{p,q-1}(h-i) \tag{5.4.4}$$

for $h \geq 2$, $q \geq 2$. The recursion starts with the values $s_{p,q}(1) = 0$ for all $q \geq 2$ and with the numbers $s_p(h)$, which can be also computed recursively by the following forumula:

$$s_p(h) = (p-1)s_{p-1}(h-1) + s_{p-1}(h) \text{ for all } h \geq 1. \tag{5.4.5}$$

Substituting equation (5.4.4) into the definition of $\chi_p(h)$ we obtain:

$$\chi_p(h) = -s_p(h) + \sum_{i=2}^{h}(-1)^i \sum_{j=1}^{h-1} s_p(j)s_{p,i-1}(h-j) =$$

$$= -s_p(h) - \sum_{j=1}^{h-1} s_p(j) \sum_{i=1}^{h-1}(-1)^i s_{p,i}(h-j) =$$

$$= -s_p(h)\chi_p(0) - \sum_{j=1}^{h-1} s_p(j)\chi_p(h-j),$$

which is equivalent to (5.4.3). Here, in the last equation we have used that $s_{p,i}(h-j) = 0$ if $i > h-j$.

Note, that $\chi_2(h) = (-1)^h$, since $s_{2,i}(h) = 0$ for $i \neq h$ and $s_{2,h}(h) = 1$. We also note that the formula (5.4.3) is one of the recursive formulas relating Stirling numbers of the first and second kind, but we prefer to show directly that $(-1)^h\chi_p(h)$ satisfies (5.4.1), or equivalently, that $\chi_p(h)$ satisfies

$$-\chi_p(h) + \chi_{p-1}(h) = (p-1)\chi_p(h-1)$$

Let us denote the expression $-\chi_p(h)+\chi_{p-1}(h)$ by $A_p(h)$. We wish to show by induction on h, that $A_p(h) = (p-1)\chi_p(h-1)$. We have

$$A_p(h) = \sum_{i=1}^{h}((p-1)s_{p-1}(i-1) + s_{p-1}(i))\chi_p(h-i) - s_{p-1}(i)\chi_{p-1}(h-i) =$$

$$= (p-1)\chi_p(h-1) + \sum_{i=1}^{h-1} s_{p-1}(i)((p-1)\chi_p(h-i-1) - A_p(h-i))$$

and using the induction steps, each term in the sum is zero, therefore $A_p(h) = (p-1)\chi_p(h-1)$ as claimed.

Therefore the proof of the equation $(-1)^h\chi_p(h) = St_p(h) = r_p(h)$ is complete. $\square$

Let us now return to the E^1-term

$$\cdots \xrightarrow{\bar{d}^1} H_h(\mathcal{N}_*(\Sigma_p)[h]) \xrightarrow{\bar{d}^1} H_{h-1}(\mathcal{N}_*(\Sigma_p)[h-1]) \xrightarrow{\bar{d}^1} \cdots$$

of the spectral sequence associated to the norm-filtration of the bar complex.

Recall that $H_h(\mathcal{N}_*(\Sigma_p)[h])$ is a submodule of the free module $\mathcal{N}_h(\Sigma_p)[h]$ generated by h-tuples of transpositions. The group Σ_p is atomic: for any two transpositions τ_2 and τ_1

$$N(\tau_2\tau_1) = \begin{cases} 0 & \text{if } \tau_2 = \tau_1, \\ 2 & \text{otherwise.} \end{cases}$$

It follows that $\bar{d}^1$ is the signed sum of the following two face operators: $d_0 : (\tau_h|\ldots|\tau_1) \mapsto (\tau_q|\ldots|\tau_2)$ and $d_h : (\tau_h|\ldots|\tau_1) \mapsto (\tau_{h-1}|\ldots|\tau_1)$.

Since by Theorem 5.4.1. the map $\kappa_h : \mathcal{R}_h(\Sigma_p)[h] \to H_h(\mathcal{N}_*(\Sigma_p)[h])$ is an isomorphism we obtain a chain-isomorphic complex

$$\cdots \xrightarrow{\partial} \mathcal{R}_h(\Sigma_p)[h] \xrightarrow{\partial} \mathcal{R}_{h-1}(\Sigma_p)[h-1] \xrightarrow{\partial} \ldots$$

where $\partial = \kappa_{h-1}^{-1} \circ \bar{d}^1 \circ \kappa_h = \kappa_{h-1}^{-1} \circ ((-1)^h d_h + d_0) \circ \kappa_h$. In (4.3.2) we obtained the following inductive formula computing κ_h:

$$\kappa_h = G_1^{(r)} \circ G_2^{(r)} \circ \cdots \circ G_{h-1}^{(r)},$$

where $G_1^{(r)} = id - f_1$ and inductively $G_i^{(r)} := id - f_i \circ G_{i-1}^{(r)}$ for $i \geq 2$. It is clear, that $d_h f_j = f_j d_h$ and therefore $d_h G_j^{(r)} = G_j^{(r)} d_h$ for all $1 \leq j \leq h-2$. Hence

$$\kappa_{h-1}^{-1} \circ d_h \circ \kappa_h = \kappa_{h-1}^{-1} \circ G_1^{(r)} \circ \cdots \circ G_{h-2}^{(r)} \circ d_h \circ G_{h-1}^{(r)} = d_h \circ G_{h-1}^{(r)},$$

reducing the complexity of the boundary formula ∂.

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