Alternative Finestructural and Computational Approaches to Constructibility

Dissertation
zur Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn
vorgelegt von

Merlin Carl
aus
Düsseldorf

Bonn 2010

Angefertigt mit Genehmigung
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter: Prof. Dr. Peter Koepke
2. Gutachter: Prof. Dr. Stefan Geschke

Tag der Promotion: 10.06.2011

Erscheinungsjahr: 2011

Für Elisabeth Thölken,
Lux In Tenebris.

Danksagungen

Mein Dank an dieser Stelle hat jenen zu gelten, ohne deren Unterstützung und Vorbild diese Arbeit nicht entstanden wäre.

Ich danke meinem Betreuer Peter Koepke für seinen engagierten, aber niemals aufdringlichen Rat, seine Bereitschaft, seinen reichen Erfahrungsschatz zu teilen, sein tätiges Interesse an meinem akademischen Fortkommen vor, während und nach dieser Arbeit - und nicht zuletzt dafür, mich für die Mengenlehre gewonnen zu haben; meiner Mutter Heike Carl, die mich viel Wertvolles lehrte, zu viel für eine Aufzählung im einzelnen; meinem viel zu früh verstorbenen Vater Wolfgang König-Carl dafür, dass er mir sein Interesse am Grundsätzlichen, das später ein mathematisches werden sollte, hinterließ; meiner Lehrerin Inge Hachtel, die es auf die Mathematik, und meinem Lehrer Horst Frings, der es auf die Philosophie hin lenkte - erst durch sie erhielt es sein heutiges Gepräge; meinen Mitstudenten und Freunden Dominik Klein und Ulrike Endesfelder für Zuspruch und Aufmunterung in schwierigen Zeiten; Immanuel Kant, Friedrich Nietzsche und Martin Heidegger, die mich die großen und tiefen Ziele klarer zu sehen und nie aus dem Blick zu verlieren lehren; und meiner Gefährtin Elisabeth Thölken, mit der und durch die sie sich nicht bloß denken, sondern leben lassen.

1 Introduction

Axiomatic set theory serves as a foundation for pure mathematics. About 80 years after the formulation of the formal system for set theory by Zermelo and Fraenkel, ZFC, it seems that any theorem of common mathematics can be formulated as an ϵ-formula and that every argument acceptable as a proof in any area of pure mathematics with the exception of set theory itself has a formal counterpart within first-order predicate calculus, starting from the axioms of ZFC.
Apart from its intrinsic value and philosophical relevance, this has two consequences that are particularly remarkable: First, through the use of derivations within ZFC, mathematicians have a tool at hand that can in principle be used to reformulate proofs as syntactical objects that are mechanically checkable. Second, by giving such a precise definition of the notion of a proof, proofs itself could become a subject of mathematical study. Derivations within ZFC are thus in a similar relationship to proofs in the intuitive sense as Turing programs are to the informal idea of a 'method' or an 'algorithm'. Arguably, the most important advantage of this is the possibility of independence proofs: Having transformed proofs from the metatheory into objects of mathematical study, the statement that a certain mathematical statement ϕ has no proof becomes itself a mathematical statement, and so does the statement that ϕ is undecidable, i.e. that there are neither proofs for ϕ nor for $\neg\phi$.
The usual way to prove something like this is the construction of models: Assuming that some model M of ZFC is given, one describes how to turn it into a model of $ZFC+\phi$ and into another model of $ZFC+\neg\phi$. This implies the independence of ϕ from ZFC, provided ZFC is consistent.
There are essentially two methods for such constructions. The one, forcing, consists in adding extra objects to M in a controlled way, extending it to an outer model of ZFC and was invented by Paul Cohen. The other, historically first, method, is due to Gödel [39]: Here, one passes from a class-sized M to a potentially $\subset$-smaller, definable, class-sized submodel of M, a so-called inner model. The simplest inner model, the constructible hierarchy L, is generated by the following ordinal recursion:

1. (1) $L_0 = \emptyset$

2. (2) $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$

3. (3) $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$

Here, $\mathrm{Def}(x)$ is the set of sets definable by an $\in$-formula over x, i.e. of sets of the form $\{y \in x | x \models \phi(y, \vec{p})\}$ for some $\in$-formula ϕ and some finite vector $\vec{p} \subset x$. Then L is $\bigcup_{\alpha\in On} L_\alpha$. It is rather easy to check that L is a model of ZFC containing all ordinals, and in fact the $\subset$-smallest one. Because of its very concrete and uniform definition, L can be analyzed much easier and deeper than a general model of ZFC. An important and probably the most famous example is Cantor's continuum hypothesis (CH). The trouble with CH in ZFC seems to be that ZFC does not give a precise meaning to the notion of a subset of ω; hence, there are models where the size of the continuum becomes arbitrarily large. On the other hand, the set of real numbers in L is very clear. This allows one to demonstrate that all reals in L are already elements of $L_{\omega_1^L}$, which implies that $L \models CH$. In this way, Gödel demonstrated the relative consisteny of CH.
The research on L made a great step forwards through the invention of Jensen's fine structure theory. This studies in detail at which stages of the recursion new subsets of ordinals arise and thereby helps to reveal the combinatorial content of L. Roughly, the Σ_n-projectum ρ_α^n of L_α is the smallest ordinal ρ such that there is a Σ_n-formula ϕ and $\vec{p} \subset_{fin} L_\alpha$ with the property that $\{x \in L_\alpha | L_\alpha \models \phi(x, \vec{p})\} \cap \rho \notin L_\alpha$. The lexically minimal such parameter $\vec{p}$ is then the Σ_n-standard parameter. If $\rho_\alpha^n = \alpha$, the structure L_α is closed under Σ_n-comprehension, so the projectum can be seen as a measure for the closedness of a structure. Through the use of bounded truth predicates, the so called master codes, and relativized structures, reducts, these fine structural parameters become compatible with extension techniques for embeddings and condensation arguments. The theory becomes smoother by using the J-hierarchy instead of the L_α's. With these tools, strong combinatorial principles like the square principle or the existence of morasses could be demonstrated to hold in L.
However, these techniques lead to very complicated proofs. Typically for a finestructural construction, a lexically minimal triple $\langle \alpha, n, \vec{p} \rangle$ is considered

such that some interesting object (like the collapse of an ordinal or a cofinal sequence) is definable by a Σ_n-formula over J_α in parameter $\vec{p} \subset_{fin} J_\alpha$. It is then shown with considerable effort that these triples are preserved by sufficiently well-behaved maps like Mostowski collapses. Usually, the combinatorial intuition is encoded in the case where α is a limit ordinal and $n = 1$. The finestructural apparatus is then used to reduce every other case to this one. When one asks for simplifications of finestructure, it is therefore natural to try to set up a hierarchy in such a way that anything is Σ_1-definable over a level, so that the messy reduction procedure is avoided.

Several different fine structures based on this main idea have been proposed since the publication of [6]. Most notably are Jensens own Σ^*-theory, where the mentioning of codes is almost completely avoided by a clever modification of the Levy hierarchy. In [28] this is used for a comparably short proof of several combinatorial principles. Silver [19] used very slowly growing hierarchies of hull operators, known as Silver machines, to eliminate large parts of finestructural considerations from combinatorial proofs in L. A canonical choice of a Silver machine based on Gödels L_α-hierarchy lead to the hyperfine structure of Friedman and Koepke [16].

The F-hierarchy is another approach in this spirit. Its levels are defined by iterating a very limited comprehension operator, restricted to quantifier-free formulas in an extended set-theoretical language with build-in symbols for a well-ordering, a comprehension operator and Skolem functions. It was first described van Eijmeren in [15], and later on used by Koepke for a simplified proof of the covering lemma for L in [14].

In the first section, we introduce the F-hierarchy. The purpose of this part is two-fold: We give a complete account on results in the F-hierarchy and extend the range of these results, thereby exploring the possibilities and borders of this approach. First, we introduce the F-hierarchy and the basic related notions: We define hull operators, prove a condensation theorem for the F-hierarchy and introduce structure-preserving maps between F-levels, so called 'fine maps'. Simple combinatorics like GCH and versions of diamond are carried out. These proofs turn out to be rather stable between different hierarchies. Then, we consider direct limits of F-structures and introduce a technique for extending fine maps to larger domains. This is the basis for a proof of the covering lemma for L and a proof of the approximation lemma, claiming that any set of ordinals closed under the basic operations of the F-hierarchy is a union of countably many constructible sets, provided that $0^\sharp$ does not exist. Through the use of the new approach, there are considerable

simplifications in both proofs. In particular, due to the homogenity of the comprehension operator, no case distinction according to formula complexity necessary.
For stronger combinatorics, some extra effort is necessary. We take methods from Silver machines and hyperfine structure to incorporate the ideas behind the hyperfine proofs of square in the F-hierarchy. It turns out that hyperfine structure theory can be generalized when applied to the F-hierarchy, which gives a family of fine structural hierarchies, the so-called hyperings. Each hypering allows a proof of the square principle, two of which we work out. Also, in this setting, the construction of a gap-1-morass can be carried out directly from the general properties of hyperings without reference to a particular one. These proofs are adaptions of those given in [16] and [8] to the F-hierarchy. They are quite different in spirit from those given by Jensen. In particular, the construction of a gap-1-morass is based on a considerably different idea.

In the second part, we use methods of constructibility theory in the context of generalized recursion theory to determine the strength of Infinite Time Register Machines, a version of register machines computing along an ordinal time axis. Here, we make use of Jensen's J- and Gödel's L_α-hierarchy mainly for two reasons: First, the levels considered in this area are usually sufficiently closed anyway so that they appear in any of these hierarchies as a limit stage. Second, and more importantly, this allows us to use classical results of reverse mathematics and admissible recursion theory without the need to translate these into the F-hierarchy.
Computability theory on the ordinals can be seen as a kind of non-hierarchical approach to the constructible universe. The central operations that we build into our language to achieve simplifications when working with the fine or the hyperfine hierarchy can be viewed as recursive operators: For example, a Skolem function on a structure X corresponds to an exhaustive search for a witness through X. Recasting fine structure theory and its generalizations in the conceptual framework of generalized computability theory is a promising alternative to classical fine-structure and the hierarchical finestructures in general.

The third part deals with applications of alternative fine structures to relativizations. This is relevant for the construction and application of core models, L-like structures that can contain large cardinals that cannot exist in L. This is useful for estimating the consistency strength of combinatorial statements, which can be done according to the following strategy: Given a certain core model K, one first proves a covering lemma for K, stating that, in the absence of a large cardinal type transcending K, K is 'close to the universe' V. This can mean, e.g., that sets of ordinals in V from a certain size on are subsets of sets of ordinals in K of the same cardinality. As the combinatorics of K is simplified by the possibility to use fine structure, it is often possible to prove a certain principle in question to hold in K. Covering can then be used to show that it must also hold in V. Hence if the principle fails, the large cardinal must exist, which is an estimate of the consistency strength of the principle from below.

We present fragments of a hyperfine theory for the Dodd-Jensen core model K^{DJ}. Several central concepts of core model theory, like an iteration theory or the Dodd-Jensen lemma, are carried over. However, the approach encountered unexpected, massive difficulties, as a preservation of the extra operators used in hyperfine structure seems incompatible with the goal of controlling the target structure inside the source structure when forming an ultrapower. The theorem of Los, a triviality in the classical context, becomes an open challenge. This problem remains unsolved; however, we hope to have raised attention and interest for a central point easily and often overlooked in considerations about alternative approaches to core model theory.

2 Contents

3 Preliminaries and Notation

Most of our notation is standard and can be found in e.g. Jech's Book [12]. We make some common abbreviations.

If $f : X \to Y$ is a function, $a \subset X$, then $f[a]$ denotes $\{y \in Y | \exists x \in a | f(x) = y\}$.
Small greek letters like α, β, γ usually denote ordinals unless stated otherwise. κ and sometimes λ are used for an infinite (local) cardinal.
When talking about structures, we use boldface letters for structures and lightface letters for the underlying sets/classes. Thus, we would write $\mathbf{M}$ for $\langle L_\alpha, \in, <\rangle$ and M for L_α. However, this convention is sometimes violated when it is clear from the context what we mean.
We often denote the Mostowski collapse of a set X by $\pi : M \to_{coll^{-1}} X$; π is called the uncollapsing map or the inverse collapse. π^{-1} is called the collapsing map.
We write $p \subset_{fin} X$ to indicate that p is a finite subset of X, similar $p \subset_{cntbl} X$ for a countable subset. Also, if $p = \langle p_1, ..., p_n\rangle$ is a finite sequence, we make the nonstandard convention that $p \subset_{fin} X$ and $p \in X$ mean $\{p_1, .., p_n\} \subset_{fin} X$. If p and q are finite sequences, $p^\wedge q$ denotes their concatenation.
If $f : A \to B$ is a partial function, then $f(x) \downarrow$ means $x \in dom(f)$; $f(x) \downarrow= y$ means $f(x) \downarrow$ and $f(x) = y$. $f(x) \uparrow$ means $x \notin dom(f)$. If P is a programme, expressions like $P(x) \downarrow= y$ are to be read by replacing P by the function it computes. Usually, this will be functions from ω to ω.
If an equation $x = y$ or a map $\sigma : x \to y$ is given, LHS and RHS denote the left-hand side and the right-hand side of the term, respectively.
If B is a well-ordered set (usually a set of ordinals), then $A \downarrow_B$ means that $A \cap B$ is bounded in B. $A \uparrow_B$ means that $A \cap B$ is cofinal in B.
$lub(X)$ is the least upper bound of X, $sup(X)$ the supremum, $max(X)$ the maximum etc.
If $<$ is a well-ordering of a class X, then $<_{lex}$ is the well-ordering of $X^{<\omega}$ given by $p <_{lex} q \leftrightarrow max\{(p \cup q) - (p \cap q)\} \in q$ for $p, q \in X^{<\omega}$.
If $\mathbf{M}$ is an $\in$-structure, κ a cardinal in $\mathbf{M}$, then H_κ^M denotes the set of elements of $\mathbf{M}$ that are heriditarily of cardinality $< \kappa$, $\mathbf{H}_\kappa := H_\kappa^V$.
On is the class of ordinals. For $S \subset On$, $otp(S)$ denotes the order type of S.
$lim(\alpha)$ abbreviates the statement that α is a limit ordinal.

4 The F-Hierarchy

In this section, we introduce the fine hierarchy for the constructible universe. The approach, the definitions and results are - up to some technical modifications - due to Koepke and can be found in [14] and [15]. The general idea is to reduce the complexity of formulas under consideration and the overall development by directly integrating frequently used basic operations of fine structure into the underlying language. This will lead to a modified definition operator, and iterating this operator will give the F_α-hierarchy.
Our basic operations will be S_X and I_X, the Skolem function and the interpretation function for a well-founded ϵ-structure X. Furthermore, we will use the naming function N as a kind as inverse to I and the relations ϵ and a well-ordering $<_X$ of X. Thus the structures we work with will have the form $\langle X, I_X, S_X, N,_X, \epsilon, <_X \rangle$. For a subclass $\mathbf{S}_0$ of formulas ϕ of a language $\mathbf{S}$ yet to be specified, these are going to have the following definitions (let $\phi \in S_0$, $\in X$, $\vec{q} \subset_{fin} x$):

- $I(x, \phi, \vec{q}) = \{z \in x | \phi(z, \vec{q})\}$ for $\vec{q} \subset_{fin} x$, otherwise $\emptyset$ by default
- If ϕ has $m + n$ free variables and $\vec{q}$ is of length n, then $S(x, \phi, \vec{q}) =<_X -min\{z \in x | \exists x_2, ..., x_m \phi(z, x_2, ..., x_m, \vec{q})\}$ for $\vec{q} \subset_{fin} x$ if such z exists, otherwise $\emptyset$ by default.
- $\langle N(1, \_, z), N(2, \_, z), \langle N(3, i, z) : i < n \rangle \rangle =<_{lex} -min\{\langle y, \psi, \vec{p} \rangle | I(y, \psi, p) = z\}$ for $z \in X$, where n is the number of free variables in ϕ and $<_{lex}$ is the lexical ordering on triples $\langle a, b, c \rangle$, where $a \in X$, b is a suitable formula, $c \subset_{fin} X$ using $<_X$ on the first and third and some fixed, natural ordering of the formulas for the second component. Hence, N is a 3-adic function inverting the I-operator, where $N(1, \_, z)$ gives a $<_X$-minimal underlying set y, then $N(2, \_, z)$ the minimal index of a formula ψ (given y) and then $N(3, i, z)$ the i-th component of a $<_{lex}$-minimal $\vec{p} \subset_{fin} X$ (given y and ψ) such that $I(y, \psi, \vec{p}) = z$ for $i \in \omega$. If j is larger than the number of free variables of ψ, we set $N(3, j, z) = \emptyset$.

Remark: (1) These definitions make the implicite assumption that $\vec{q} \subset_{fin} x$ is of the right length so that an expression like $\phi(z, \vec{q})$ makes sense. For technical reasons, we extend it to the case where ϕ has only n free variables,

yet $\vec{q} = \langle q_1, .., q_m \rangle$ with $m > n$ by simply replacing $\vec{q}$ by its first n componenents. If $m < n$, we set $q_i := \emptyset$ for $m < i \leq n$.
(2) We will write $N(z)$ instead of $\langle N(1, \_, z), N(2, \_, z), \langle N(3, i, z) | i < n \rangle \rangle$. The reason for splitting the naming function into components is that we are going to work with structures that are not necessarily closed under the pairing function and will therefore not contain $N(z)$ even if they contain every single component. We will therefore also write $N(z) \in X$ to indicate that all components of $N(z)$ are elements of X. Hence, N will be treated as a unary function symbol, though it is formally 3-adic.
(3) By iteration of the S-operator, $x_2, ..., x_m$ can be found as well. $\langle z, x_2, .., x_m \rangle$ will in general not be an element of our structure. Nevertheless, where this simplifies the arguments, we usually treat the S-function as giving back the whole finite sequence in this case, understanding $\langle z, x_2, ..., x_m \rangle \in X$ as $\{z, x_2, .., x_m\} \subset X$ by our convention.
In practice, these general Skolem and comprehension operators will be restricted to triples where the first argument is a level of our finestructural hierarchy. Accordingly, $N(1, \_, x)$ will have to be a level of the hierarchy.

Definition 1: *The language for the fine hierarchy is a first-order language with countably many variable symbols v_i, the logical symbols $=$, $\exists$, $\forall$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, (,), the ternary function symbols I, S and N, the binary relation $\in$ and the ternary relation $<$.*
By $\boldsymbol{S}_0$, we denote the set of quantifier-free $\boldsymbol{S}$-formulas.
An $\boldsymbol{S}$-term ($\boldsymbol{S}_0$-term) over a set X is then of the form $I(x, \phi, y)$, $N(x)$ or $S(x, \phi, y)$, where ϕ is an $\boldsymbol{S}$-formula ($\boldsymbol{S}_0$-formula), $x \in X$ and $y \subset_{fin} X$.
We fix a canonical linear ordering $\langle \phi_i | i \in \omega \rangle$ of the $\boldsymbol{S}_0$-formulas in order type ω.

Remark: When we actually mention expressions of **S**, we commonly use variable symbols from the meta-language like x, y, z instead of $v_1, v_2, ...$. In our opinion, this makes the formulas much more readable.

We are now ready to define our hierarchy. The possibility of passing to the next level depends on some assumptions on the current level. We therefore formulate our definition with a stopping condition first and demonstrate that it leads to a proper class afterwards.

Definition 2: *(F-hierarchy)*

- *$F_i = V_i$ for $i \leq \omega$. I, N and S, restricted to F_ω, are empty functions. $S|F_\omega$, $I|F_\omega$, $N|F_\omega$ are simply $\emptyset$, $<_{F_\omega}$ is any linear well-ordering of F_ω in order type ω.*

- *If λ is a limit ordinal and F_ι is defined for $\iota < \lambda$, then $F_\lambda := \bigcup_{\iota<\lambda} F_\iota$. Furthermore, $S|F_\lambda$, $I|F_\lambda$, $N|F_\lambda$ and $<_{F_\lambda}$ are simply the unions of their respective predecessors.*

- *Now for the successor step: Suppose F_ι is defined for $\iota \leq \alpha$, and that all these F_ι are transitive, that $\iota_1 < \iota_2 \leq \alpha$ implies $F_{\iota_1} \subset F_{\iota_2}$ and $F_{\iota_1} \in F_{\iota_2}$. If any of these conditions fails, the construction stops. Then $F_{\alpha+1}$ is defined as $\{I(F_\alpha, \phi, \vec{p}) | \phi \in S_0 \wedge \vec{p} \subset_{fin} F_\alpha\}$. $I|F_{\alpha+1}$ and $S|F_{\alpha+1}$ only have to be defined for triples containing the new level, i.e. triples of the form $\langle F_\alpha, \phi, p \rangle$, ϕ an $\boldsymbol{S}_0$-formula, $p \subset_{fin} F_\alpha$. They are then given by the definition above. $N|F_{\alpha+1}$ has to be defined for $x \in F_{\alpha+1} - F_\alpha$; for such an x, $N(1, y, x) = F_\alpha$ (for every y), while $N(2, y, x)$ is (for every y) the minimal $\boldsymbol{S}_0$-formula ϕ such that, for some $\vec{q} \subset_{fin} F_\alpha$, we have $I(F_\alpha, \phi, \vec{q}) = x$, and finally $N(3, i, x)$ is the i-th component of the $<_{lex}$-minimal such vector $\vec{q}$. If $n \notin \{1, 2, 3\}$ or $n = 3$ and $i \notin \omega$ or $\vec{q}$ does not have an i-th component, then $N(n, i, x) := \emptyset$. $<_{F_{\alpha+1}}$ is defined as an end-extension of $<_{F_\alpha}$, where, for $a, b \in F_{\alpha+1}$, we set $a <_{F_{\alpha+1}} b$ iff $N(a) <_{lex} N(b)$, using ϵ on the first component, the ordering of formulas on the second and the lexical ordering according to $<_{F_\alpha}$ on the third.*

Convention: As the restricted well-orderings $<_{F_\alpha}$ are easily seen to be compatible in the sense that $\alpha < \beta$ implies $<_{F_\alpha} \subset <_{F_\beta}$, we can and will leave out the index F_α when the structure in question is clear from the context.

Theorem: *Lemma 3: Let $\alpha < \beta$ be ordinals. Then:*
(a) F_α is transitive
(b) $F_\alpha \subset F_\beta$
(c) $F_\alpha \in F_\beta$

Proof: By simultaneous induction on α. For $\alpha = 0$, all clauses are trivial. Assume that (a)-(c) hold for all $\gamma < \alpha$. If $x \in F_\alpha$, $\alpha \leq \delta < \beta$, then $x = \{x \in F_\delta | x \in a\} \in F_\beta$. Hence $F_\alpha \subset F_\beta$. Also, $F_\alpha = \{x \in F_\alpha | x = x\} = \{x \in F_\delta | x \in F_\alpha\}$, so $F_\alpha \in F_\beta$ as well. (a) is trivial if α is a limit ordinal. If

$\alpha = \eta + 1$, then F_α consists of sets of the form $I(F_\eta, \phi, q)$, all of which are subsets of F_η, which is a subset of F_α by induction. □

Corollary 4: *F_α exists for every $\alpha \in On$.*

Proof: □

Each set first-order definable over F_α appears in the F-hierarchy finitely many steps after α:

Lemma 5: *(1) Let $\phi(x_0, ..., x_n)$ be an ϵ-formula with all free variables shown. Then there is a natural number k and an $\boldsymbol{S}_0$-formula $\phi^*(x_0, ..., x_n, x_{n+1}, ..., x_{n+k})$ uniformly definable from ϕ such that for all $a_0, ..., a_n \in F_\alpha$:*

$$(F_\alpha, \epsilon) \models \phi(a_0, ..., a_n) \leftrightarrow \boldsymbol{F}_{\alpha+k} \models \phi^*(a_0, ..., a_n, F_\alpha, .., F_{\alpha+k-1}).$$

In particular, $\{\langle a_0, ..., a_n\rangle | (F_\alpha, \epsilon) \models \phi(a_0, ..., a_n)\} = \{\langle a_0, ..., a_n\rangle | \boldsymbol{F}_\alpha \models \phi^*(a_0, ..., a_n, F_\alpha, ..., F_{\alpha+k-1}\}$.

Proof: We prove this by induction on the complexity of ϕ: If ϕ is $\neg\psi$, $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, then let ϕ^* be $\neg\psi^*$, $\psi_1^* \wedge \psi_2^*$, $\psi_1^* \vee \psi_2^*$, respectively.
Now suppose ϕ is $\exists x \psi(x, a_0, ..., a_n)$. Then we first apply induction to ψ, then add extra F-levels as a parameters and finally use the S-operator to eliminate the existential quantifier. Hence:
$\mathbf{F}_\alpha \models \exists x \psi(x, a_0, ..., a_n)$ iff
$\exists x \in F_\alpha \psi(x, a_0, ..., a_n)$ iff
$\exists x \in F_\alpha \psi^*(x, a_0, ..., a_n, F_\alpha, ..., F_{\alpha+k-1}$ iff
$\exists x \in F_{\alpha+k} x \in F_\alpha \wedge \psi^*(x, a_0, ..., a_n, F_\alpha, ..., F_{\alpha+k-1})$.
The last formula can be expressed in $F_{\alpha+k+1}$ as an $\mathbf{S}_0$-formula using the S-operator.
□

This is basically what we need to see that the union of the F-hierarchy is actually L, the universe of constructible sets.

Theorem 6: $\bigcup_{\alpha \in On} F_\alpha = L$

Proof: $\subset$: The definition of the F-hierarchy can be carried out within L as the basic constructible operations are definable by ϵ-formulas. This allows us to show inductively that $F_\beta \in L$ for any $\beta \in On$. The definition of a new element over F_β by the I-operator can also be emulated in L. Hence, any element of $\bigcup_{\alpha \in On} F_\alpha$ will also be an element of L.

$\supset$: As L is the $\subset$-smallest transitive class model of ZFC and we have already seen $F := \bigcup_{\alpha \in On} F_\alpha \subset L$, we only have to check that the latter is a transitive class model of ZFC. Transitivity is clear, as any element of F is an element of some F_β, which is transitive. Also, F contains all ordinals and is hence a proper class.
To see that $F \models ZFC$, recall that a transitive class X is a model of ZFC if X is closed under first-order definability and almost universal, i.e. any subset of X is already a subset of an element of X. F is easily seen to be almost universal, since any subset of F will already be a subset of some $F_\gamma \in F$. Closure under first-order definability was demonstrated above. $\Box$

Remark: This result relativizes. See the next section.

It will be convenient in many places that we can restrict ourselves to parameters with elements of the form F_γ in definitions rather than general elements of F-levels. This is possible by the next lemma.

Lemma 7: *For each $x \in F_{\alpha+1}$, there are $\beta_0, ..., \beta_n$ and an $\boldsymbol{S}_0$-formula ψ such that $x = I(F_\alpha, \psi, \langle F_{\beta_0}, ..., F_{\beta_n} \rangle)$.*

Proof: By induction on α. Suppose the lemma is true for elements of F_α. Let $x = I(F_\alpha, \phi(x_1, ..., , x_k), p)$, $p = \{p_1, ..., p_k\} \subset F_\alpha$. By induction, there are $m \in \omega$, $\beta_i^j < \alpha$ and ϕ_j such that $p_i = I(F_{\beta_m^i}, \phi_j, \langle F_{\beta_1^i}, F_{\beta_2^i}, ..., F_{\beta_{m-1}^i} \rangle)$ for $0 < i \leq m$, $0 < j \leq k$. Then also
$x = \{z \in F_\alpha | \phi(z, x_1, ...x_n) \wedge x_1 = p_1 \wedge ... \wedge x_m = p_m\}$, which is of the desired form. $\Box$

Convention: $\underline{F}_\alpha$ is $\{F_\beta | \beta < \alpha\}$.

4.1 Relativization

Many important features of constructibility theory generalize to L-like structures which can make use of an extra predicate A in definitions. This can be carried over to the F-hierarchy. As we will not make use of relativized F-structures, we restrict ourselves to a short account. An approach to the relativization of the fine hierarchy was also made in [42].

Definition 8: *An $\boldsymbol{S}_0^A$-formula is a quantifier-free formula in the language $\boldsymbol{S}$ amended with an extra predicate symbol A for an arbitrary class.*

Definition 9: *Define the basic constructible operations and relations S^A, I^A, N^A, $<_A$ for F^A as in the first section with $\boldsymbol{S}_0^A$-formulas in place of $\boldsymbol{S}_0$-formulas.*

- $F_n^A := V_n$ *for* $n \in \omega$
- $F_{\alpha+1}^A = \{I^A(F_\alpha^A, \phi, \vec{x}) | \phi$ *is an* $\boldsymbol{S}_0^A$*-formula and* $\vec{x} \subset_{fin} F_\alpha^A\}$.
- *If* λ *is a limit ordinal, then* $F_\lambda^A = \bigcup_{\iota<\lambda} F_\iota^A$

All of the results and proofs given so far carry over to relativized levels without any extra effort. Also, we will not make use of relativized F-structures in this work. Nevertheless, we state and prove some of the following in relativized form where this might be relevant for future considerations and the relativization isn't trivial. In cases where relativization merely leads to a complication of notation, we leave it out.
We give a short account on how the F^A-hierarchy relates to Jensen's J^A-hierarchy. At the same time, we introduce the J-hierarchy, which will be used in the second part.

4.1.1 F^A and J^A

The J-hierarchy is advantegous to Gödels L_α-hierarchy in several respects when developing finestructure. We will need it in the second part for the study of infinite time register machines. A good account on the properties of this hierarchy can be found in [1]. Here, we restrict ourselves to the definition:

Definition 10: *Let X be any set, A a class. A function $f : X \to V$ is rudimentary in A (rudimentary if $A = \emptyset$) if it is generated by the following rules:*

- *For each $c \in X$, the constant function $g(x) = c$ for $x \in X$ is rudimentary*
- *For each $i \leq n \in \omega$, the projection function $g(\langle c_1, ..., , c_n \rangle) = c_i$ is rudimentary*
- *The function $f(x) = x \cap A$ is rudimentary*
- *The function $f(x, y) = \{x, y\}$ is rudimentary*
- *If $f(x_0, ..., x_{n-1})$ and $g_0(\vec{y}), ..., g_{n-1}(\vec{y})$ are rudimentary, then so is their composition $f(g_0(\vec{y}), ..., g_{n-1}(\vec{y}))$.*
- *If g is rudimentary, then so is $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$.*

The unique $\subset$-minimal superset of X closed under rudimentary functions is denoted by $rud(X)$.

Definition 11:

- $J_0^A = \emptyset$
- $J_{\alpha+1}^A = rud(J_\alpha^A \cup \{J_\alpha^A\})$
- *If λ is a limit ordinal, then $J_\lambda^A = \bigcup_{\iota < \lambda} J_\lambda^A$*

Theorem 12: *For every $\alpha \in On$, $A \subset V$, we have $F_{\omega\alpha}^A = J_\alpha^A$.*

Proof: It is easily checked that the proof by Koepke and van Eijmeren in [15] for the case $A = \emptyset$ relativizes. $\Box$

The F^A levels can thus be viewed as a refinement of the J^A-hierarchy. One should therefore expect that they form an appropriate setting for finestructural arguments.

4.2 Fine Maps and Hulls

A crucial technique of any fine structure theory is the formation of hulls and the consideration of maps between them that preserve basic functions and relations. The F-hierarchy is constructed in such a way that such hulls and maps become very canonical. A hull in the constructible hierarchy is basically a closure under the constructible operations I, N and S. However, by the way these operations were defined, this would be inconvenient as, for example, the hull of an ordinal α would be α itself, not F_α, as one should expect. Therefore we also close under the operations $\alpha \to F_\alpha$ and $F_\alpha \to \alpha$.

Definition 13: *A class $A \subset L$ is constructibly closed (cc) iff $F_\omega \subset A$ and A is closed under the basic constructible operations I, S and N.*
For $X \subset L$, $F\{X\}$ denotes the intersection of all cc-sets Z that are also closed under the operations $\alpha \to F_\alpha$ and $F_\alpha \to \alpha$ for $\alpha \in On$ such that $X \subset Z$. $F\{X\}$ is called the constructible hull or the constructible closure of X.

Equivalently, $F\{X\}$ is the $\subset$-smallest *cc*-superset of X.

In these hulls, the correspondence between levels and ordinals is now as desired.

Proposition 14: *Let $X \subset L$ be a set. Then, for an ordinal α, we have $\alpha \in X \leftrightarrow F_\alpha \in X$.*

Proof: By definition. $\square$

Proposition 15: *Let X be a set of ordinals. Then $max\{F\{X\} \cap On\} = max\{X\}$. In particular, if p is finite, $max\{p\} > \mu$, then $F\{\mu \cup p\} = F_\alpha$ implies that α is a successor ordinal.*

Proof: Let $\eta = max\{X\}$. For the first statement, observe that the hull operations can only add subsets of F_η, hence no ordinal $\geq \eta$ can be added. For the second, if α was a limit ordinal, there would be $max\{p\} < \rho < \alpha$ which could not be an element of $F\{\mu \cup p\}$ by the first observation. $\square$

Proposition 16: *For each $\alpha \geq \omega$, F_α is cc.*

Proof: $F_\omega \subset F_\alpha$ follows from the fact that the F-levels form a $\subset$-increasing sequence of sets. Closure under S follows from the transitivity of F_α, closure under I is clear since, if $F_\gamma \in F_\alpha$, then $F_{\gamma+1} \subset F_\alpha$, and all elements $\mathbf{S}_0$-definable over F_γ are elements of $F_{\gamma+1}$ by definition. Also, F_α is closed under N, as each element was formed as the interpretation of some name. $\square$

Definition 17: *Let $\mathtt{A}$ and $\mathtt{B}$ be $\boldsymbol{S}$-structures and suppose $\pi : \mathtt{A} \to \mathtt{B}$ is a map. The π is called fine if it preserves $\boldsymbol{S}_0$-formulas and $\pi|F_\omega = id|F_\omega$.*
If π is also bijective, we call it a fine isomorphism. In this case, we write $\pi : \mathtt{A} \simeq \mathtt{B}$. If there is a fine isomorphism between $\mathtt{A}$ and $\mathtt{B}$, we call them isomorphic and simply write $\mathtt{A} \simeq \mathtt{B}$.
For $\boldsymbol{S}^A$-structures, π is accordingly called A-fine if it additionally preserves $\boldsymbol{S}_0^A$-formulas. Usually, the A will not be mentioned when there is no ambiguity.

Fine maps roughly correspond to Σ_0-preserving maps in the classical fine structure theory. As there, it will be important to consider maps with stronger preservation properties. These correspond to hulls with stronger closure properties, which we now define.

Definition 18: *Let $Z \subset \boldsymbol{F}_\alpha$, $p \subset_{fin} F_\alpha$. Z is constructibly closed up to p (p-cc) iff Z is cc and for all $\boldsymbol{S}_0$-formulas ϕ, $q \subset_{fin} Z$ with $q <_{lex} p$:*

- *(a) $I(F_\alpha, \phi, q) \in F_\alpha$ implies $I(F_\alpha, \phi, q) \in Z$*
- *(b) $S(F_\alpha, \phi, q) \in F_\alpha$ implies $S(F_\alpha, \phi, q) \in Z$*

If Z is p-cc for all $p \subset_{fin} F_\alpha$, then Z is constructibly closed up to α, denoted α-cc.

The easiest way to get α-cc sets is by truncation:

Proposition 19: *If Z is a cc set and $F_\alpha \in Z$, then $Z \cap F_\alpha$ is α-cc.*

Proof: That $Z \cap F_\alpha$ is *cc* is clear. Since $F_\alpha \in Z$ and Z is *cc*, it follows that $I(F_\alpha, \phi, p) \in Z$ for any $p \subset_{fin} Z$ and any $\mathbf{S}_0$-formula ϕ. Therefore, if $I(F_\alpha, \phi, p)$ is an element of F_α, it is also an element of $Z \cap F_\alpha$. The same reasoning applies to S and N. So $F_\alpha \cap Z$ is α-cc. $\square$

The main reason why α-cc sets are interesting is that they reflect Σ_1.

Lemma 20: *Let $Z \subset \boldsymbol{F}_\alpha$ be α-cc, $a_0, ..., a_n \in Z$. Then any statement of the form $\exists y_0, ..., y_m \phi(a_0, ..., a_n, y_0, ..., y_m)$ with $\phi(x_0, ..., x_{m+n})$ an $\boldsymbol{S}_0$-formula with all free variables shown that has witnesses in F_α has witnesses in Z and vice versa.*

Proof: Suppose $b_0, ..., b_m \in F_\alpha$ such that $\phi(a_0, ..., a_n, b_0, ..., b_m)$. Then let $c_i = S(F_\alpha, \phi, \langle a_0, ..., a_n, c_0, ..., c_{i-1}\rangle)$ for $0 \leq i \leq m$. By the existence of $b_0, ..., b_m$, every c_i is defined and we have $\phi(a_0, .., a_n, c_0, ..., c_m)$. Since Z is α-cc and is hence closed under $S(F_\alpha, \_, \_)$, all c_i are elements of Z.
The other direction is immediate since $Z \subset \mathbf{F}_\alpha$ and $\mathbf{S}_0$-formulas are preserved upwards. $\square$

Now for the corresponding degree of preservation for maps.

Definition 21: *$\pi : F_\alpha \to F_\beta$ is fine up to $p \subset_{fin} F_\beta$ (p-fine) iff it is fine and $rng(\pi)$ is p-cc. If π is p-fine for every $p \subset_{fin} F_\beta$, i.e. if $rng(\pi)$ is β-cc, then π is fine up to F_β.*

The Σ_1-reflection of α-cc sets is then turned into a Σ_1-preservation property of the map fine up to F^A_α.

Lemma 22: *Let $\pi : F^A_\alpha \to F^B_\beta$ be fine up to F^B_β, $\phi(x_0, ..., x_m, ..., x_n)$ be an $\boldsymbol{S}_0$-formula with all free variables shown. Then, for all $a_0, ..., a_m \in F^A_\alpha$, there are $a_{m+1}, ..., a_n \in F^A_\alpha$ such that $\phi(a_0, ..., a_n)$ holds iff there are $b_{m+1}, ..., b_n \in F^B_\beta$ such that $\phi(\pi(a_0), ..., \pi(a_m), b_{m+1}, ..., b_n)$ holds.*

Proof: This follows from the last lemma: As $rng(\pi)$ is fine up to F^B_β, it reflects Σ_1, hence the witnesses appear in the range of π and therefore in the pre-image. So downwards preservation holds. Upwards preservation already follows from the fact that π is fine. $\square$

Maps with this higher degree of preservation have the nice property that they can be lifted to the next level. This fact is important for the condensation lemma, but also in more advanced applications like the covering lemma or the approximation theorem.

Theorem 23: *Assume $\sigma : \boldsymbol{F}^D_\delta \to \boldsymbol{F}^E_\eta$ is fine up to F^E_η. Then there is a unique fine map $\sigma^+ : \boldsymbol{F}^D_{\delta+1} \to \boldsymbol{F}^E_{\eta+1}$ with the properties $\sigma^+|F^D_\delta = \sigma$ and $\sigma^+(F^D_\delta) = F^E_\eta$.*

Proof: The idea is of course to let $\sigma^+ = \sigma$ on F^D_δ and then to define $\sigma^+(I^D(F^D_\delta, \phi, p)) = I^E(F^E_\eta, \phi, \sigma[p])$ for ϕ a $\mathbf{S}_0$-formula and $p \subset_{fin} F^D_\delta$. (Recall that $\sigma[p]$ is the componentwise image of the finite sequence p under σ.) By the demands on σ^+ and the fact that, as a fine map, it must preserve the I-operation, it is clear that this is the only possible candidate for σ^+. So we know that there is at most one map of this kind. We have to see that it does indeed work. As $a \in F^D_\delta$ can be represented as $\{z \in F^D_\delta | z \in a\}$, we can take $\sigma^+(I^D(F^D_\delta, \phi, p)) = I^E(F^E_\eta, \phi, \sigma[p])$ to define σ^+ everywhere.

First, we need to see that it is well-defined, i.e. that there is only one 'image' of $x \in F^D_{\delta+1}$ under σ^+. So pick $\mathbf{S}_0$-formulas ϕ, ψ along with $p, q \subset_{fin} F^D_\delta$. Then, since σ preserves Σ_1-formulas:
$I^D(F^D_\delta, \phi, p) \neq I^D(F^D_\delta, \psi, q)$
$\leftrightarrow \mathbf{F}^D_\delta \models \exists x(\phi(x, p) \leftrightarrow \neg\psi(x, q))$
$\leftrightarrow \mathbf{F}^E_\eta \models \exists x(\phi(\sigma(x), \sigma(p) \leftrightarrow \neg\psi(\sigma(x), \sigma(q)))$
$\leftrightarrow I^E(F^E_\eta, \phi, \sigma(p)) \neq I^E(F^E_\eta, \psi, \sigma(q))$.
So $I^D(F^D_\delta, \phi, p) = I^D(F_\delta, \psi, q)$ iff $I^E(F^E_\eta, \phi, \sigma(p)) = I^E(F^E_\eta, \psi, \sigma(q))$, and σ^+ is really a well-defined map from F^D_δ to F^E_η.

Claim: $\sigma^+|F^D_\delta = \sigma$

Proof: Let $\in F^D_\delta$. Then $\sigma^+(a) = \sigma^+(I(F^D_\delta, x \in y, a)) = I(F^E_\eta, x \in y, \sigma(a)) = \sigma(a)$. □

Claim: $\sigma^+(F^D_\delta) = F^E_\eta$.

Proof: $\sigma^+(F^D_\delta) = \sigma^+(I(F^D_\delta, \emptyset = \emptyset, \emptyset)) = I(F^E_\eta, x = x, \emptyset) = F^E_\eta$. □

Claim: $z \in F^D_\delta \leftrightarrow \sigma^+(z) \in F^E_\eta$ for $z \in F^D_{\delta+1}$.

Proof: Let $N^D(z) = (F^D_\delta, \phi, \vec{x})$, $\sigma^+(z) = I^E(F^E_\eta, \phi, \sigma(\vec{x})) \in F^E_\eta$. By assumption on σ, $\sigma^+(z) \in rng(\sigma)$. So there is $y \in {F_\delta}^D$ with $\sigma^+(z) = \sigma(y)$. But then $\sigma^+(z) = \sigma^+(y)$, so $z \in {F_\delta}^D$. □

Claim: σ^+ is fine.

Proof: We need to show that σ^+ preserves D, $\in$, $<$, I, S and N. This is clear for elements of F^D_δ. If $x \in y \in F^D_\delta$, where $N(y) = \langle\gamma, \psi, \vec{q}\rangle$, then we have $F^D_\gamma \models \psi(x, \vec{q})$, $\gamma \leq \delta$. As σ is fine, it follows that $F^E_\eta = \sigma(F^D_\delta) \models \psi(\sigma(x)\vec{q})$. By definition of σ^+, $\sigma^+(y) = I(\sigma(N(y))$, hence $\sigma(x) \in \sigma(y)$. Preservation of I is immediate by construction of σ^+. From this, it also follows that $D(x) \leftrightarrow E(\sigma^+(x))$. The remaining operations will be handled separately.

Subclaim: σ^+ preserves N, i.e. for $x \in F^D_{\delta+1}$, $\sigma^+(N^{F^D_{\delta+1}}(x)) = N^{F^E_{\eta+1}}(\sigma^+(x))$.

Proof: As σ^+ preserves I, it is clear that, for any $x \in F^D_{\delta+1}$, $I^{F^E_{\eta+1}}(\sigma^+(N^{F^D_{\delta+1}}(x))) = \sigma^+(x)$. It remains to see that $\sigma^+(N^{F^D_{\delta+1}}(x))$ is lexically minimal with this property. Assume there is a lexically smaller name. We distinguish two cases according to whether the first component of this name is F^E_η or an earlier level of the F^E-hierarchy. In the former case, the existence of an $\mathbf{S}_0$-formula $\hat{\psi}$ and a parameter $\vec{q}$ which form a pair lexically smaller than the corresponding pair of $\sigma^+(N^{F^D_{\delta+1}}(x))$ can be expressed as a Σ_1-formula over F^E_η. In the latter, the same applies to the existence of all three components. In both cases, this Σ_1 statement is preserved downwards by σ as σ is fine up to F^E_η. This means that there is a name for x in F^D_δ which is lexically smaller than $N^{F^D_\delta}(x)$, a contradiction.
□

It is now clear that σ^+ preserves $<$, i.e. $\forall x, y \in F^D_{\delta+1}(x < y \leftrightarrow \sigma^+(x) < \sigma^+(y)$: $<$ is the lexical order on minimal names. Minimal names are preserved by the last subclaim, preservation of the order of ordinals and formulas is clear, and as the parameters must be elements of F^D_δ, the preservation of their order follows from the the fact that σ is fine.

Subclaim: σ^+ preserves $S^{F^D_{\delta+1}}$.

Proof: First, we need to see that, for β, $\vec{z} \in rng(\sigma^+)$, ψ an $\mathbf{S}_0$-formula, $\psi(x, \vec{z})$ has a witness in F^E_β iff there is one for $\psi(x, \sigma^{-1}(\vec{z}))$ in $F^D_{\sigma^{-1}(\beta)}$. But as β cannot be larger than δ, this follows again from the downwards preservation of Σ_1 by σ. Let w, $\hat{w}$ be the $<$-minimal witnesses on the F^D and the F^E-side, respectively.

We need to demonstrate that $\sigma^+(x) = \hat{x}$. As σ^+ preserves $\mathbf{S}_0$ and $<$, we must have $\sigma^+(x) \geq \hat{x}$. Again, $\hat{x} < \sigma^+(x)$, then this fact could be expressed as a Σ_1-statement over F^E_η and then be pulled back to F^D_δ, leading a contradiction.
□

All basic constructible operations are hence preserved by σ^+. Thus σ^+ is indeed fine.
□

This concludes the construction of σ^+ and hence the proof.
□

4.2.1 Condensation

We are now in the position to prove the following theorem, the condensation lemma for the F^A-hierarchy, an analogue to the condensation lemma for the J-hierarchy, which is one of the most powerful tools of finestructure theory.

Theorem 24: *Let A be a predicate, $Z \subset L^A$ be A-cc. Then there are $\alpha \in On$ and a class D such that $\boldsymbol{Z} := \langle Z, I^A|Z, S^A|Z, N^A|Z, \epsilon|Z, <^A |Z, A \cap Z\rangle$ is isomorphic to $\boldsymbol{F}^D_\alpha$ via a fine map π. Furthermore, can choose $D \subset F^D_\alpha$. Furthermore, π, α and D are unique with these properties.*
Also, if $A = \emptyset$, then $D = \emptyset$, so cc-subsets of L are isomorphic to levels of the F-hierarchy.

Proof: Uniqueness is clear, because if there is such a map at all, it can only be the Mostowski collapse, which determines D, α as well.
To see existence, we proceed by induction on $\beta := sup\{On \cap Z\}$. For $\beta \leq \omega$, the set Z will itself be transitive and there is nothing to show.
If β is a limit ordinal, assume the theorem is true for all $\eta < \beta$. Each $Z \cap F^A_\eta$ is A-cc and hence by induction isomorphic to some $F^{A(\eta)}_{\alpha(\eta)}$. If $x \in F^A_\gamma \cap F^A_\delta$ and $\pi_{\gamma,A}$, $\pi_{\delta,A}$ are the collapsing maps for these respective structures, then $\pi_{\gamma,A}(x) = \pi_{\delta,A}(x)$. Hence $F^{A(\gamma)}_{\alpha(\gamma)}$ is an initial segment of $F^{A(\delta)}_{\alpha(\delta)}$ for $\gamma < \delta$, thus we can form the union of these structures and let $\alpha^* = \bigcup_{\eta<\beta} \alpha(\eta)$, $A^* = \bigcup_{\eta<\beta} A(\eta)$, $\pi_{\beta,A} = \bigcup_{\eta<\beta} \pi_{\eta,A}$. $\pi_{\beta,A} : Z \to F^{A^*}_{\alpha^*}$ is then the desired structure.
On the other hand, if $\beta = \eta + 1$ is a successor, then $Z \cap F^A_\eta$ is isomorphic

to some $\mathbf{F}_{\eta^*}^{A^*}$ via a fine map π^* by induction. Furthermore, $Z \cap F_\eta^A$ is η-cc, hence π^* is fine up to F_η^A. Therefore we can lift π^* up to a fine map $\pi^+ : \mathbf{F}_{\eta^*+1}^{A^*} \to F_{\eta+1}^A$. This is already the structure we want, all that remains to show is that $rng(\pi^+) = Z$. But, as Z was cc, any element of Z had a name in Z which is an element of F_η^A and is hence unfolded in the lifting construction. On the other hand, if $z \in rng(\pi^+)$, it is constructed from elements of $Z \cap F_\eta^A$ by basic operations, and hence $\in Z$.
$\square$

The following general reflection properties will be relevant for the combinatorics in the section on elementary constructible combinatorics:

Lemma 25: *(1) Let ϕ be an arbitrary $\in$-formula. Then there is $n \in \omega$ such that, for every $\vec{x} \subset_{fin} F_\alpha$, $\vec{x} \subset X \subset F_\alpha$ we have $coll''F\{X \cup \{F_\alpha\} \cup \{F_{\alpha+1}\} \cup ... \cup \{F_{\alpha+n}\}\} \models \phi(\vec{\bar{x}}) \leftrightarrow F_\alpha \models \phi(\vec{x})$, where $\vec{\bar{x}}$ is the image of $\vec{x}$ under the collapsing map.*
(2) With ϕ, $\vec{x}$ as in (1), there is $n \in \omega$ such that each fine map $\pi : F_{\alpha+n} \to F_\beta$ preserves ϕ.

Proof: Using the S-operation for the extra F-levels, each $\in$-formula ϕ can be equivalently represented as an $\mathbf{S}_0$-formula ψ, which is preserved by fine maps, particularly collapsing maps. $\square$

4.3 Direct limits of F^A-structures

In the proof of the covering lemma and the approximation theorem, it will be crucial to extend a fine embedding to a larger domain. Suppose we have $\pi : \mathbf{F}_\alpha \to \mathbf{F}_\beta$ fine, and let $\delta > \alpha$. The method we use here is to present $\mathbf{F}_\delta$ as a direct limit of a system consisting of structures and maps in $\mathbf{F}_\alpha$, map the system over by π and unfold it again. To make this work, we need to know that direct limits of systems consisting of (relativized) F-levels are again (isomorphic to) F-levels.

Definition 26: *$\langle \mathtt{A}_i, \pi_{ij} | i \leq j \in \mathtt{I}\rangle$ is a fine system with index set $\langle \mathtt{I}, \leq\rangle$ iff, for $i \leq j$, $\pi_{ij} : \mathtt{A}_i \to \mathtt{A}_j$ is a fine map and $\leq$ is a directed partial order on $\mathtt{I}$, so that the structures and maps form a directed system.*

Proposition 27: *A fine system $\langle \mathtt{A}_i, \pi_{ij} | i \leq j \in \mathtt{I}\rangle$ has a direct limit $\mathtt{A}$ together with fine maps $\pi_i : \mathtt{A}_i \to \mathtt{A}$. The maps π_{ij} and π_k for $i, j, k \in \mathtt{I}$ commute in the sense that $\pi_j \circ \pi_{ij} = \pi_i$ for $i \leq j$. Furthermore, $\mathtt{A}$ and the π_i are unique up to isomorphism with this property.*

Proof: This is a general property of directed systems. See e.g. [12].□

Theorem 28: *Let $\langle \mathtt{A}_i, \pi_{ij} | i \leq j \in \mathtt{I}\rangle$ be a fine system with $\mathtt{A}_i = \boldsymbol{F}^{E_i}_{\alpha_i}$ and direct limit $\mathtt{A} = (A, I^*, S^*, N^*, \epsilon^*, E^*, <^*)$, $\pi_i | i \in \mathtt{I}$. If ϵ^* is well-founded, then there are α, E such that $\mathtt{A} \simeq \boldsymbol{F}^E_\alpha$.*

Proof: First, note that two elements of $\mathtt{A}$ are equal iff they have the same ϵ^*-predecessors: For if $a \neq b \in \mathtt{A}$, then there are $i, j \in \mathtt{I}$ and $\bar{a} \in \mathtt{A}_i$, $\bar{b} \in \mathtt{A}_j$ such that $\pi_i(\bar{a}) = a$, $\pi_j(\bar{b}) = b$. As $\leq$ is directed, there is $k \geq i, j$ in $\mathtt{I}$, let $a' = \pi_{ik}(\bar{a})$, $b' = \pi_{jk}(\bar{b})$. As $\mathtt{A}_k$ is an F-level, there must be $c \in \mathtt{A}_k$ such that $c \in a' \leftrightarrow c \notin b'$. Hence, $\pi_k(c) \in a \leftrightarrow \pi_k(c) \notin b$.
As $\mathtt{A}$ is also well-founded, we know now that there is a unique transitive set isomorphic to (A, ϵ^*), so we can, without loss of generality, assume that $\mathtt{A}$ is itself transitive and that ϵ^* is $\epsilon | \mathtt{A}$.
(1) $\epsilon | \{\pi_i(F^{E_i}_\gamma) | i \in \mathtt{I}, \gamma < \alpha_i\}$ is a well-order.

Proof: Obviously, ϵ is well-founded. To see that ϵ is transitive, we show that $\pi_i(F^{E_i}_\gamma)$ is transitive for $i \in \mathtt{I}$: Let $u \in v \in \pi_i(F^{E_i}_\gamma)$, pick $i \leq j \in \mathtt{I}$ such that u and v have pre-images $\bar{u}$ and $\bar{v}$ in $\mathtt{A}_i$ and $\mathtt{A}_j$, respectively, so that $\bar{u} \in \bar{v} \in \pi_{ij}(F^{E_i}_\gamma)$. The fine map π_{ij} preserves relativized F-levels and transitivity, so $\pi_{ij}(F^{E_i}_\gamma) = F^{E_j}_\zeta$ for some E_j, which implies $\bar{u} \in \pi_{ij}(F^{E_i}_\gamma)$ and so, by application of π_j, $u \in \pi_i(F^{E_i}_\gamma)$.
To obtain the linearity of ϵ, let $i, j \in \mathtt{I}$ and $\gamma < \alpha_i$, $\delta < \alpha_j$, then pick $k \geq i, j$ by directedness. As above, $\pi_{ik}(F^{E_i}_\gamma)$ and $\pi_{jk}(F^{E_j}_\delta)$ are $F^{E_k}_\rho$, $F^{E_k}_\tau$ for some $\rho, \tau \in On$ and E_k, respectively. By linearity of the F^{E_k}-hierarchy, we must have $\pi_{ik}(F_\gamma) \in \pi_{jk}(F^{E_j}_\delta)$ or vice versa, or equality, all of which would be preserved by the fine map π_k. □

We will now show that the basic constructible operations are preserved by the system maps π_i. This will be done by simultaneous induction on the well-founded relation ϵ.

We claim:

- (2) Each $\pi_i(F_\gamma^{E_i})$ is of the form F_ζ^E
- (3) I^*, S^*, $<^*$, ϵ^*, N^* are equal to $I, S, <, \epsilon, N$ on F_ζ^E
- (4) If ϕ is an $\mathbf{S}_0^E$-formula, $p \subset_{fin} F_\zeta^E$, then $I^*(F_\zeta^E, \phi, p) = I(F_\zeta^E, \phi, p)$
- (5) For $x, y \in F_\zeta^E$, $x <^* y$ iff $x <^E y$
- (6) If ϕ is an $\mathbf{S}_0^E$-formula, $p \subset_{fin} F_\zeta^E$, then $S^*(F_\zeta^E, \phi, p) = S(F_\zeta^E, \phi, p)$

Proof: By induction on $\epsilon|\{\pi_i(F_\gamma^{E_i})|i \in \mathtt{I}, \gamma < \alpha_i\}$. For $F_\gamma^{E_i} = \emptyset$, there is nothing to show. So let $i \in \mathtt{I}$, $\gamma < \alpha_i$ and assume that the statements hold for all ϵ-smaller elements of $\epsilon|\{\pi_i(F_\gamma^{E_i})|i \in \mathtt{I}, \gamma < \alpha_i\}$.

Claim: (2) and (3) hold at (i, γ).

Proof: To see this, we make a case distinction according to whether or not (i, γ) is a limit or a successor in the ordering $<$ of the directed system.

Case 1: First, suppose (i, γ) is a $<$-limit.
In this case, we can find for $(j, \delta) < (i, \gamma)$ a (k, η) which is between these elements. Consider $\bigcup_{(j,\delta)<(i,\gamma)} \pi_j(F_\delta^{E_j})$. By construction, this is a relativized fine level $F_\zeta^{\hat{E}}$. Using the fact that every element of $F_\zeta^{\hat{E}}$ appears in one of the $F_\delta^{E_j}$, we conclude that $\pi_i(F_\gamma^{E_i}) = F_\zeta^{\hat{E}}$. As the basic operations agree with the corresponding operations of the limit structure on each of these levels, the same applies to the union.

Case 2: If (i, γ) is a $<$-successor, proceed as follows. There must be (j, δ) whose immediate $<$-successor is (i, γ). Set $F_\rho^{\hat{E}} = \pi_j(F_\delta^{E_j})$, $\zeta := \rho + 1$. Without loss of generality, assume $j \geq i$. By using the induction hypothesis, we can deduce $\pi_i(F_\gamma^{E_i}) = F_\zeta^{\hat{E}}$. Preservation is now clear. □

Now we take care of the remaining statements (4) to (6). Here, no case distinction is necessary.
Claim: (4) holds at (i, γ).

Proof: Let $\phi(v_0, ..., v_n)$ be an $\mathbf{S}_0$-formula with all free variables shown, and suppose $\vec{y}$ is a vector of length n of elements of $F_\zeta^{\hat{E}}$. If $\bar{x} \in F_{\alpha_j}^{E_j}$, j from the

index set, we need to see that $\pi_j(\bar{x}) \in I^*(F^{\hat{E}}_\zeta, \phi, \vec{y}) \leftrightarrow \pi_j(\bar{x}) \in I(F^{\hat{E}}_\zeta, \phi, \vec{y})$. To see this, pick j large enough to guarantee that all elements of $\vec{y}$ have pre-images in $F^{E_j}_{\alpha_j}$, denote these pre-images by $\bar{y}_i$.
Consequently $\pi_j(\bar{x}) \in I^*(F^{\hat{E}}_\zeta, \phi, \vec{y}) \leftrightarrow \bar{x} \in I(F^{E_j}_\gamma, \phi, \vec{\bar{y}})$
But this means that $\phi(\bar{x}, \vec{\bar{y}})$ holds in $F^{E_j}_\gamma$, and using the preservation of the basic operations, this means that $\phi(\vec{y}, \pi_j(\bar{x}))$ holds in $F^{\hat{E}}_\zeta$, which is exactly the condition for elements of $I(F^{\hat{E}}_\zeta, \phi, \pi_j(\bar{x}))$. □

Claim: (5) holds at (i, γ).

Proof: This follows by induction hypothesis from the preservation of minimal names. □

Claim: (6) holds at (i, γ).

Proof: Let $\phi(x_0, ..., x_n)$ be an $\mathbf{S}_0$-formula with all free variables shown, and take $y_0, ..., y_{m-1}$ from F_ζ, where $m \leq n$. We handle the cases where S returns a default value or a minimal witness for the statement:
Case 1: $S^E(F^E_\zeta, \phi, p)$ has default value $\emptyset$.
Pick some index j large enough so that there are pre-image $\bar{y}_0$,...,$\bar{y}_{m-1}$ for y_0,...,y_{m-1} under the system map for j in $F^{E_j}_{\alpha_j}$. We must have $S^{E_j}(F^{E_j}_{\alpha_j}, \phi, \bar{y}_0, ..., \bar{y}_{m-1}) = \emptyset$ by default as well, and since π_j is fine, the same holds for S^*.
Case 2: $S^E(F^E_\zeta, \phi, p)$ is a witness for $\exists z \in F_\zeta \phi(z, p)$.
In this case, we proceed in the same manner: Pull back all of the (finitely many) relevant parameters to a component of the system with sufficiently large index, then use the fact that the limit maps are fine. □

All statements of the claim have been checked. □

Now the lemma is easy to prove. Considering $\bigcup \pi_j(F^{E_j}_\delta)$, where the union is over all (j, δ) from the directed system, we clearly get a structure of the form F^F_θ. (Here, F is $\bigcup \pi_j[E_j]$; note that, by the fact that the system maps are fine, we must have $\pi_i[E_i] \subset \pi_j[E_j]$ for $i \leq j$ in the ordering of indices.) If this is isomorphic to the direct limit of the system, we are done. It is also clear that this structure forms a subset of the direct limit structure $\mathtt{A}$. The only other possibility is hence that F^F_θ is a proper subset of $\mathtt{A}$. As the operations in the limit structure have been demonstrated to be equal to the constructible operations, we must then have $\mathtt{A} = F^F_{\theta+1}$, which is sufficient. □

5 Basic constructible combinatorics in the F-hierarchy

In this short section, we use the F-hierarchy of L to obtain some well-known combinatorial statements about L. We consider first the continuum hypothesis and the generalized continuum hypothesis originally proved to hold in L by Gödel before turning to various strengthenings invented by Jensen, the so-called $\Diamond$-principles. The combinatorial heart of the proof of all these statements is the fact that condensation holds in the F-hierarchy. The original proofs using Gödel's hierarchy can be found in e.g. [1] or [3].

5.1 GCH

Convention: If s is a set of ordinals, $\bar{s}$ denotes $\{F_\alpha | \alpha \in s\}$ for the rest of this section.

Definition 1: *By CH (Continuum Hypothesis), we denote the statement* $2^{\aleph_0} = \aleph_1$.
By GCH (Generalized Continuum Hypothesis), we denote the statement $card(P(X)) = card(X)^+$, *where* κ^+ *denotes the cardinal successor of* κ.

Theorem 2: *CH holds in L.*

Proof: : Let $s \subset \omega \in L$. Consider $\pi : F\{\bar{s}\} \rightarrow_{coll} F_\beta$; the hull on the LHS is countable, hence $\beta < \omega_1$. Obviously $\mathbf{F}_\omega := \bar{\omega} = \{F_i | i \in \omega\} \subset dom(\pi)$. Therefore, we have $\pi|_{\mathbf{F}_\omega} = id|_{\mathbf{F}_\omega}$, which implies $\pi(\bar{s}) = \bar{s}$, and therefore $\bar{s} \in F_\beta$. Thus we have $\bar{s} \in F_{\omega_1}$, hence $s \in F_{\omega_1}$ for each such s. But $card(F_{\omega_1}) = \omega_1$ in L, so $L \models card(\mathbf{P}(\omega)) \leq \omega_1$, i.e. $L \models card(\mathbf{P}(\omega)) = \omega_1$.$\Box$

Of course, this argument easily generalizes as usual to higher power sets:

Theorem 3: *GCH holds in L.*

Proof: Let $s \subset \kappa$, where κ is an infinite cardinal in L. Consider $\pi : F\{\bar{\kappa} \cup \bar{s}\} \rightarrow F_\beta$; since κ is a cardinal, the hull is again of cardinality $< \kappa^+$, so $\beta < \kappa^+$; since s is a subset of the transitive part of the hull, we have $\pi(s) = s$ as well, so $s \in F_\beta$, $card(\mathbf{P}(\kappa)) \leq \kappa^+$, thus $L \models card(\mathbf{P}(\kappa)) = \kappa^+$.$\Box$

5.2 The diamond principle

We recall some basic definitions. Keep in mind that, by our definition of hulls, we may and will confuse α with F_α wherever it is convenient to do so.

Convention: For $A \subset On$, $lim(A)$ denotes the set of limit points of A, i.e. $\{\beta \in On | sup\{A \cap \beta\} = \beta\}$.

Definition 4: *For α an ordinal, $C \subset \alpha$ is **club** in α if:*
(i) $B \subset C$, $sup(B) < \alpha$ implies $sup(B) \in C$.
(ii) If $\gamma < \alpha$, then there is $\beta \in C$ with $\beta > \gamma$.

Definition 5: *For κ a regular cardinal, $S \subset \kappa$ is **stationary** in κ if for every $C \subset \kappa$ club in κ, we have $S \cap C \neq \emptyset$.*

Definition 6: *For a regular uncountable cardinal κ, $\Diamond(\kappa)$ denotes the following statement:*
There is a sequence $\langle S_\alpha | \alpha < \kappa\rangle$ such that
(i) $S_\alpha \subset \alpha$ for $\alpha < \kappa$
(ii) For each $A \subset \kappa$, $\{\alpha < \kappa | A \cap \alpha = S_\alpha\}$ is stationary in κ

Theorem 7: *$\Diamond(\omega_1)$ holds in L.*

Proof: We simultanously define a sequence of clubs and a sequence of subsets in α, namely $\langle C_\alpha, S_\alpha\rangle$. Set $\langle C_0, S_0\rangle = \langle C_{\alpha+1}, S_{\alpha+1}\rangle = \langle\emptyset, \emptyset\rangle$. For α a limit ordinal, let $\langle C_\alpha, S_\alpha\rangle$ be the $<_L$-minimal pair such that C_α is club in α and $\{\mu < \alpha | S_\mu = S_\alpha \cap \mu\} \cap C_\alpha = \emptyset$ if it exists, and $\langle\emptyset, \emptyset\rangle$, otherwise. We will prove that $\langle S_\alpha | \alpha < \omega_1\rangle$ is a $\Diamond$-sequence.
Suppose for the sake of a contradiction that there is a $<_L$-minimal pair $\langle C, S\rangle$ in L such that C is club in ω_1, $S \subset \omega_1$ and for no $\gamma \in C$, we have $S \cap \gamma = S_\gamma$. Choose $n \in \omega$ large enough for the following to work. Consider $F\{\langle C, S\rangle \cup \{F_{\omega_1}, F_{\omega_1+1}, ..., F_{\omega_1+n}\}\} := H_1$. Since H_1 is countable, it is bounded below ω_1, say by β_1. If $H_1 \cap On \neq \beta_1$, construct $F\{H_1 \cup \bar{\beta}_1\} := H_2$. Define β_2 from H_2 as we defined β_1 from H_1. Continuing in this manner (eventually taking a union of all hulls if necessary), we get a countable hull H such that $\langle C, S\rangle \in H$, $H \cap \omega_1 := \beta \in On$, $H \models$'C is club in β', $H \models$'$S \subset \beta$' and H knows about the defining properties of $\langle C, S\rangle$. Obtain $\pi : H \rightarrow_{coll} F_\gamma$ so that $\pi(C) = C \cap \beta$, $\pi(S) = S \cap \beta$ and the $\langle\pi(C), \pi(S)\rangle$ has the same defining

properties as $\langle C, S\rangle$. So by absoluteness of the relevant notions, we have $S_\beta = S \cap \beta$ and $\beta \in C$ (since C is closed, $\pi(C) \subset C$ and $sup\{\pi(C)\} = \beta$). This contradicts our assumption on S and C. $\Box$

This proof strategy works again well in the general case.

Theorem 8: $\Diamond(\kappa)$ *holds in* L *for every regular uncountable cardinal* κ.

Proof: Define a sequence $\{\langle C_\alpha, S\alpha\rangle|\alpha < \kappa\}$ as above, letting $\langle C_\alpha, S_\alpha\rangle$ for limit α be the $<_L$-minimal pair such that C_α is club in α, $S_\alpha \subset \alpha$ and $\{\beta < \alpha|S_\beta = S_\alpha \cap \beta\} \cap C = \emptyset$ if it exists and $\langle \emptyset, \emptyset\rangle$ otherwise. $\langle S_\alpha|\alpha < \kappa\rangle$ will be our $\Diamond$-sequence.
Otherwise, take again $\langle C, S\rangle$ $<_L$-minimal witnessing the failure of $\Diamond(\kappa)$ as above. Choose $n \in \omega$ sufficiently large and form the hull
$H_1 := F\{\langle C, S\rangle \cup \{F_\kappa, F_{\kappa+1}, ..., F_{\kappa+n}\}\}$. Again iterate through ω if necessary to arrive at a hull H such that $H \cap \kappa \in On$. Take the collapse $\pi : H \to F_\gamma$ and continue as in the ω_1-case. $\Box$

An important strengthening of the $\Diamond$-principle is Jensens $\Diamond^+$. One application is his proof that the Kurepa hypothesis holds in L (indeed in all models of $ZFC + \Diamond^+$).
By $\mathcal{P}(X)$, we denote the powerset of X.

Definition 9: $\Diamond^+(\omega_1)$ *denotes the following statement:*
There is a sequence $\langle S_\alpha|\alpha < \omega_1\rangle$ *such that for* $\alpha < \omega_1$
(i) $S_\alpha \subseteq \mathcal{P}(\alpha)$
(ii) $card(S_\alpha) \leq \omega$
(iii) for each $X \subseteq (\omega_1)$, *there is* $\omega_1 \supseteq B \uparrow_{\omega_1}$ *with* $X \cap \alpha$, $B \cap \alpha \in S_\alpha$ *whenever* $\alpha = sup(B \cap \alpha)$.

We give a proof of $\Diamond^+$ closely following the usual strategy (see e.g. [1]) to demonstrate how more delicate combinatorics not relying on full fine structural techniques can be carried out in the F-hierarchy quite naturally. Extra stages are sometimes put into the hulls to ensure preservation of properties not $\mathbf{S}_0$-expressible at the stage one is currently working in. We make this idea precise in the following definition and convention before proceeding.

The following concept is useful for reflecting properties of high levels at lower stages of the hierarchy.

Definition 10: *Let z be the free variable of ϕ, $\alpha < \beta$ be ordinals. Then $S_\alpha(F_\beta, \phi, \bar{x}) := S(F_\beta, \phi \wedge z \in F_\alpha, \bar{x})$. For $X \subseteq F_\beta$, $F_\alpha\{X\}$ denotes the closure of X under I and S_α.*

Definition 11: *Let $X \subseteq F_\alpha$. Then we call $H^\omega_\alpha\{X\} := F_\alpha\{X \cup \{F_{\alpha+i} | i \in \omega\}\}$ the elementary hull of X in F_α. Members of $H^\omega_\alpha\{X\}$ are called F_α-definable from X, or simply F_α-definable if $X = \emptyset$.*

Proposition 12: *Let ϕ be any PL_1-formula, $X \subseteq F_\alpha$, $z \in H^\omega_\alpha$. Then $F_\alpha \models \phi(z) \leftrightarrow H^\omega_\alpha \models \phi(z)$.*

Proof: : By a previous lemma, any first order property over F_α can be expressed by $\mathbf{S}_0$-formulas with finetely many extra stages above F_α added as parameters. Therefore, the LHS is closed under Skolem functions for all PL_1-formulas and hence an elementary submodel of the RHS. □

The following basic lemma is especially important in application of fine structural arguments at low levels of the hierarchy.

Lemma 13: *Assume $V = L$. Then:*
(i) Let $Z \subseteq F_{\omega_1}$ be constructibly closed up to F_{ω_1}. Then there is $\alpha \leq \omega_1$ such that $Z = F_\alpha$.
(ii) Let $Z \subseteq F_{\omega_2}$ be such that $H^\omega_{\omega_2}\{Z\}$ is constructibly closed up to F_{ω_2}. Then there is $\alpha \leq \omega_1$ such that $Z \cap F_{\omega_1} = F_\alpha$.

Proof: : (i) By constructible closure, we can find for any F-stage in Z another F-stages that contains it. So the ordinal height of Z is a limit ordinal. Furthermore, Z satisfies the assumptions of the condensation lemma, and is thus isomorphic to a fine level. So we only have to show that this fine level is in fact Z itself, i.e. that Z is transitive. For this, take any $x \in Z$. Also take $F_\beta \in Z$ minimal such that $x \in F_\beta$. Since $x \in Z \subseteq F_{\omega_1}$, x is countable. So there is $g \in L$ such that $g : \omega \rightarrow_{surj} x$. This g is a subset of $\omega \times x \in F_{\omega_1}$. By the above proof of GCH, actually $g \in F_{\omega_1}$. Define f to be the $<_L |F_{\omega_1}$-minimal such g. Since enough stages above F_β are available, we can express the defining properties of the g in F_{ω_1} by an $\mathbf{S}_0$-formula and

thus use $S(F_{\omega_1}, \_, \_)$ with parameter x to find f. So $f \in Z$. Of course, we have $\omega \subset Z$. So we can use the Skolem function again to get $f(i)$ for each particular $i \in \omega$. But then $x = \{f(i) | i \in \omega \subset Z\}$, qed.
(ii) By Proposition 12, $Z \models$"F_{ω_1} is the only fine level indexed by an uncountable cardinal" and so we have $F_{\omega_1} \in Z$. Now it is obvious that $\bar{Z} := F_{\omega_1} \cap X$ is closed under $S(F_{\omega_1}, \_, \_)$ and therefore $\bar{Z}$ is elementary in F_{ω_1}, so we can apply (i) to conclude the proof. □

Theorem 14: $\diamondsuit_+(\omega_1)$ *holds in L.*

Proof: : By lemma 13 (i), we have for each $\alpha < \omega_1$ a $\beta := \beta(\alpha) < \omega_1$ such that $H^\omega_{\omega_1}\{\{F_\alpha\}\} = F_\beta$. By the corresponding proof, there is $f : \omega \rightarrow_{surj} \alpha$ in $H^\omega_{\omega_1}$, and α is countable inside that structure. Our $\diamondsuit^+(\omega_1)$-sequence $\langle S_\alpha | \alpha < \omega_1 \rangle$ can be thought to consist of the subsets of α definable from $\{\alpha\}$. This is made precise by setting $S_\alpha := \mathcal{P}(\alpha) \cap H^\omega_{\omega_1}$. This obviously satisfies (i) from definition 9 and (ii) as well (since already $H^\omega_{\omega_1}\{\alpha\}$ is countable). Also $\langle S_\alpha | \alpha < \omega_1 \rangle \in H^\omega_{\omega_2}\{\emptyset\}$, being definable by a PL_1-formula inside F_{ω_2}. We are going to prove that this sequence satisfies $\diamondsuit^+(\omega_1)$.
Suppose otherwise. So there is $Y \subset \omega_1$ such that for each $B \uparrow_{\omega_1}$, there is $\gamma \in lim(B)$ such that not both $B \cap \gamma$ and $X \cap \gamma$ are in S_γ. $Y \in L$ since we are assuming $V = L$ and so we can let X be the $<_L$-minimal such set. This X is F_{ω_2}-definable.
Next, we define a chain $\langle N_\eta | \eta < \omega_1 \rangle$ of elementary hulls in F_{ω_2} and a continous sequence $\langle \alpha_\eta | \eta < \omega_1 \rangle \uparrow_{\omega_1}$ of ordinals.
Set $N_0 := H^\omega_{\omega_2}\{\emptyset\}$, $\alpha_0 = \omega_1 \cap N_\emptyset$. For η a limit ordinal, $N_\eta := \bigcup_{\iota<\eta} N_\iota$, $\alpha_\eta := \bigcup_{\iota<\eta} \alpha_\iota$. At successor stages, set $N_{\eta+1} := H^\omega_{\omega_2}\{N_\eta \cup \{N_\eta\}\}$, $\alpha_\eta := \omega_1 \cap N_{\eta+1}$. Note that, by lemma 13 (ii), $F_{\omega_1} \cap N_\eta$ is a countable F-level, and in particular transitive. So $\omega_1 \cap N_\eta$ is really an ordinal in each step and the definition makes therefore sense. By Proposition 8, all these hulls satisfy the same first-order formulas. Denote by $\pi_\eta : N_\eta \rightarrow F_{\delta(\eta)}$ the transitive collapse, so $\pi_\eta(\omega_1) = \alpha_\eta$, $\pi_\eta(X) = X \cap \alpha_\eta$ for $\eta < \omega_1$. Let $B := \{\delta(\eta) | \eta < \omega_1\}$ for short. B is obviously an unbounded set of countable ordinals. We will show that it is a counterexample to our assumption that no such set can satisfy $\diamondsuit^+(\omega_1)$ for X.
Choose some limit point γ of B. Clearly, $\alpha_\eta < \delta(\eta)$ since α_η is an initial segment of the ordinals of a model of height $\delta(\eta)$. Furthermore $N_\eta \in N_{\eta+1}$ and $N_\eta \subset N_{\eta+1}$, so $F_{\delta(\eta)} \in N_{\eta+1}$. Since N_η is countable for each $\eta < \omega_1$, $\delta(\eta)$ is countable as well. But then, $\delta(\eta)$ is a countable ordinal of $N_{\eta+1}$ and hence,

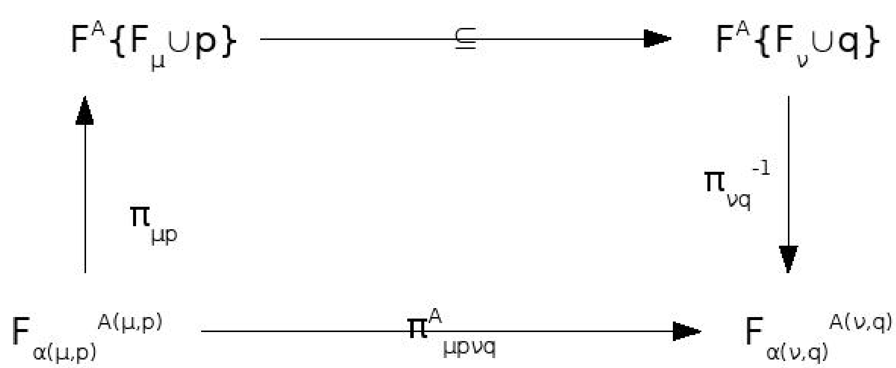
by definition of $\alpha_{\eta+1}$, a member of $\alpha_{\eta+1}$. Thus all limit points of B are also limit points of the sequence of the α's. In particular, we may suppose that $\gamma = \alpha_\theta$ for some limit ordinal $\theta < \omega_1$.
Since all concepts involved in this definition are absolute between ZF^--models, so we can carry it out inside N_θ with parameter $F_{\delta(\theta)}$ to get $B \cap \gamma$. Since $\gamma = \alpha_\theta = \pi_\theta(\omega_1)$, γ is uncountable in $F_{\delta(\theta)}$, while we have observed above that α is countable inside $F_{\beta(\alpha)}$ for each $\alpha < \omega_1$, so in particular γ is countable inside $F_{\beta(\gamma)}$. That means that a new bijection must have entered the defining process of the fine hierarchy between $\delta(\theta)$ and $\beta(\gamma)$ and thus $\delta(\theta) < \beta(\gamma)$. But $\beta(\gamma)$ is a limit ordinal and $F_{\beta(\gamma)}$ is a model of ZF^-, so carrying out the definition inside $F_{\beta(\gamma)}$ gives $B \cap \gamma \in F_{\beta(\gamma)}$. Furthermore $X \cap \gamma = \pi_\theta(X) \in F_{\delta(\gamma)} \subset F_{\beta(\alpha)}$. So we have $X \cap \gamma$, $B \cap \gamma \in S_\alpha$, which contradicts our assumption on X.
So there is no such X and $\Diamond^+(\omega_1)$ holds in L. $\Box$

6 Extensions of embeddings

Having proved the relevant properties of directed systems in section 4.3, we now turn to the extension construction. The main importance of this lies in the fact that it allows to code maps between class-sized structures by maps between initial segments of these classes that are sets. The obtained extenders are hence restrictions of the class-sized map to sets and hence sets themselves. These initial segments then satisfy a similar purpose as ultrafilters in the classical fine-structure theory of L and core models.

We begin by defining a directed system $\mathtt{D}$ with direct limit L^A and then a class of important subsystems that will be used for representing levels of the F^A-hierarchy.

Definition 1: *Set* $\mathtt{I}^A := \{(\mu, p) | \mu \in On, p \subset_{fin} \{F^A_\zeta | \zeta \in On\}\}$, $(\mu, p) \leq_{I^A} (\nu, q)$ *iff* $\mu \leq \nu$, $p \subseteq q$. *With each* $(\mu, p) \in \mathtt{I}^A$, *we associate the unique transitive structure* $\boldsymbol{F}^{A(\mu,p)}_{\alpha(\mu,p)} \simeq \boldsymbol{F}^A\{F_\mu \cup p\}$, *and let* $\pi_{\mu,p} : \boldsymbol{F}^{A(\mu,p)}_{\alpha(\mu,p)} \to \boldsymbol{F}^A\{F_\mu \cup p\}$ *be the inverse collapsing map. For* $(\mu, p) \leq_{I^A} (\nu, q) \in \mathtt{I}^A$, *the canonical embedding* $\pi^A_{\mu,p,\nu,q} : \boldsymbol{F}^{A(\mu,p)}_{\alpha(\mu,p)} \to \boldsymbol{F}^{A(\nu,q)}_{\alpha(\nu,q)}$ *is given by* $\pi^A_{\mu,p,\nu,q} := \pi^{-1}_{\nu,q} \circ \pi_{\mu,p}$ *as explained in the following diagram:*



$\mathtt{D}^A := \langle \boldsymbol{F}^{A(\mu,p)}_{\alpha(\mu,p)}, \pi^A_{\mu,p,\nu,q} | (\mu, p) \leq_{I^A} (\nu, q) \in \mathtt{I}^A \rangle$ *is then a fine system with direct limit* L^A.

Similarly, we can present initial segments of the F^A-hierarchy, granted appropriate conditions.

Definition 2: *F^A_α is a base for F^A_δ iff, for any $\mu < \alpha$, $p \subset_{fin} F^A_\delta$, the transitive collapses $A(\mu,p)$ and $F^{A(\mu,p)}_{\mu,p}$ of $A \cap \boldsymbol{F}^A\{F^A_\mu \cup p\}$ and $\boldsymbol{F}^A\{F^A_{\alpha(\mu\cup p)}\}$, respectively, are elements of F^A_α.*

Remark: Of course, if $A = \emptyset$, then the first condition is vacuous; hence F_α is a base for F_δ iff $coll''\mathbf{F}\{F_\mu \cup p\} \in F_\alpha$ for $\mu < \alpha$, $p \subset_{fin} F_\delta$.

Definition 3: *Let F^A_α be a base for F^A_δ. We set $\mathtt{I}^A_{\alpha\delta} := \{(\mu,p) | \mu < \alpha, p \subset_{fin} F^A_\delta\}$. Also, $(\mu,p) \leq_{I^A_{\alpha\delta}} (\nu,q)$ iff $\mu \leq \nu$ and $p \subset q$. With each $(\mu,p) \in \mathtt{I}^A_{\alpha,\delta}$, we associate the structure $\boldsymbol{F}^{A(\mu,p)}_{\alpha(\mu,p)}$ as before. For $(\mu,p) \leq (\nu,q) \in \mathtt{I}^A_{\alpha\delta}$, the map $\pi^A_{\mu,p,\nu,q} : \boldsymbol{F}^{A(\mu,p)}_{\mu,p} \to \boldsymbol{F}^{A(\nu,q)}_{\nu,q}$ is defined as above.*
Then $\mathtt{D}^A_{\alpha\delta} := \langle \boldsymbol{F}^{A(\mu,p)}_{\mu,p}, \pi^A_{\mu,p,\nu,q} | (\mu,p) \leq_{I^A_{\alpha\delta}} (\nu,q) \in \mathtt{I}^A_{\alpha\delta} \rangle$.

Convention: When it is clear which order relation is meant, the subscripts in $\leq_{I^A}$, $\leq_I$ etc. are usually dropped. The same applies to the sub- and superscripts for index sets, system maps and systems.

Now we describe the extension construction. Fix F^A_α, F^A_δ such that F^A_α is a base for F^A_δ and let $E : F^A_\alpha \to F^B_\beta$ be fine.
$\mathtt{D} := \mathtt{D}^A_{\alpha\delta}$ is then a directed system with limit F^A_δ. As F^A_α was a base for F^A_δ, we have that $F^{A(\mu,p)}_{\alpha(\mu,p)} \in \mathtt{D}$ implies $F^{A(\mu,p)}_{\alpha(\mu,p)} \in F^A_\alpha$. Also note that E-images of sets of the form $F^Z_\iota \in F^A_\alpha$ for $\iota < \alpha$ will be of the form $F^{Z'}_{\iota'}$ if $Z \in F^A_\alpha$, which is guaranteed for our $F^{A(\mu,p)}_{\alpha(\mu,p)}$ by the definition of a base.
The same applies to the system maps:

Lemma 4: *$(\mu,p) \leq (\nu,q)$ implies $\pi^A_{\mu,p,\nu,q} \in F^A_\alpha$.*

Proof: Let $\bar{p_1}$, $\bar{p_2}$ be the pre-images of p and q under $\pi^A_{\mu,p}$ and $\pi^A_{\eta,q}$, respectively. $\pi^A_{\mu,p,\nu,q}(x) = y$ holds for $x \in F^{A(\mu,p)}_{\alpha(\mu,p)}$, $y \in F^{A(\eta,q)}_{\alpha(\eta,q)}$ iff there are an $\mathbf{S}$-term t, $\vec{z} \subset_{fin} F^A_\mu$ with $x = t(\bar{p_1}, \vec{z})$, $y = t(\bar{p_2}, \vec{z})$. But the existence of such $\vec{z}$, t can be expressed as a Σ_1-formula over some F^A_τ for $\tau < \alpha$ sufficiently large (i.e. large enough so that it contains $\bar{p_1}$, $\bar{p_2}$, F^A_μ, $F^{A(\mu,p)}_{\alpha(\mu,p)}$ and $F^{A(\nu,q)}_{\alpha(\nu,q)}$), and hence as an $\mathbf{S}_0$-formula over $F^A_{\tau+1}$. This implies that $\pi^A_{\mu,p,\nu,q} \in F^A_{\tau+2} \subset F^A_\alpha$, since α is a limit ordinal. $\square$

This allows us to map each structure and map in $\mathtt{D}$ over by E. The image $\mathtt{D}^*$ will converge to the intended target structure. Set $F^{A^*(\mu,p)}_{\alpha^*(\mu,p)} = E(F^{A(\mu,p)}_{\mu,p})$, $\pi^{A^*(\mu,p)}_{\alpha^*(\mu,p)} = E(\pi^{A(\mu,p)}_{\mu,p}$.

Lemma 5: *$\mathtt{D}^*$ is a directed system.*

Proof: The property of the maps $\pi^A_{\mu,p,\nu,q}$ being fine can be expressed over $F^{A(\nu,q)}_{\alpha(\nu,q)}$ by a Π_1-formula. $\Box$

Denote the direct limit of $\mathtt{D}^*$ by $\mathtt{A} = \langle A, I^*, S^*, N^*, <^*, \in^* \rangle$, and let $(\pi^*_{\mu,p} | (\mu,p) \in \mathtt{I}^A_{\alpha\delta})$ be the system maps.
We define a fine embedding σ of F^B_β into $\mathtt{A}$: If $x \in F^B_\beta$, pick an index $(\mu, 0)$ such that $E(F^{A(\mu,0)}_{\alpha(\mu,0)}) = E(F^A_\mu) \ni x$, then let $\sigma(x) = \pi^*_{\mu,0}(x)$.

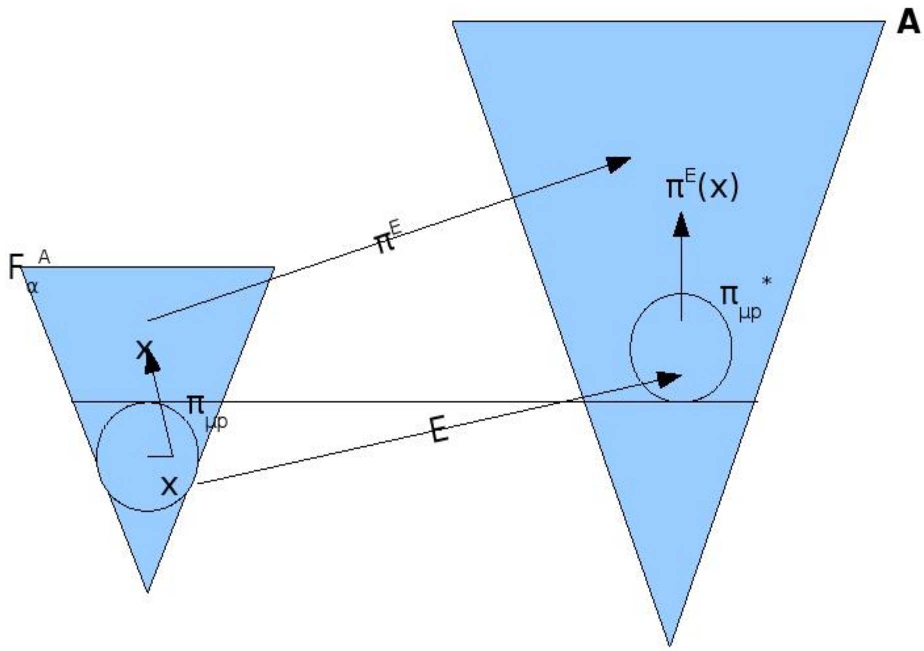
Claim: σ is well-defined, i.e. the definition of σ is independent of the choice of μ. Furthermore, σ is fine.

Proof: If we take $\mu_1 < \mu_2$ as required by the definition, $\pi^{A(\mu_2,0)}_{\mu_2,0}$ will be the identity on $F^{B(\mu_1,0)}_{\alpha(\mu_1,0)}$ and hence σ will lead the same image for elements of $F^{A(\mu,0)}_{\alpha(\mu,0)}$.
To see that σ is fine, it suffices to observe that it is the increasing union of the fine maps $\pi^*_{\mu,0}$. $\Box$

Claim: $\sigma[F^B_\beta]$ is an initial segment of $\mathtt{A}$, i.e. if $x \in F^B_\beta$ and $y \in \mathtt{A}$ with $y \in x$, then $y \in \sigma[F^B_\beta]$.

Proof: Let $(\mu, 0)$ be an index, $\pi^*_{\mu,0}(y)$ an element of F^B_β. Suppose that $z \in \pi^*_{\mu,0}(y)$. We need to see that $z \in F^B_\beta$. To see this, pick an index $(\nu, q) \geq (\mu, 0)$ such that z has a pre-image $\bar{z}$ under $\pi^*_{\nu,q}$. Now $\pi_{\mu,0,\nu,q} | F^{A(\mu,0)}_\mu = id | F^{A(\mu,0)}_\mu$, hence $\pi^*_{\mu,0,\nu,q} | F^{A^*(\mu,0)}_{\alpha^*(\mu,0)}$, so $\bar{z} \in y$ and hence, applying the fine map $\pi^*_{\nu,q}$, we get indeed $z = \pi^*_{\mu,0}(\bar{z}) \in \sigma[F^B_\beta]$. $\Box$

This means that, up to isomorphism, F^B_β is an initial segment of $\mathtt{A}$. So assume without loss of generality that F^B_β is an initial segment of $\mathtt{A}$.
Now we can define a map π^E from F^B_δ to $\mathtt{A}$: If $x \in F^A_\delta$, pick μ, p so that $x \in rng(\pi_{\mu,p})$ and let $\pi^E(x) = \pi^*_{\mu,p}(E(\pi^{-1}_{\mu,p}(x)))$.
The following diagram shows what is going on:



Claim: $\pi^E(x)$ does not depend on the choice of μ and p. Furthermore, π^E is fine.

Proof: First, let $x \in F_\delta^A$ and (μ_1, p_1) and (μ_2, p_2) be as required in the definition. Let x_1, x_2 be the pre-images of x under π_{μ_1,p_1} and π_{μ_2,p_2}, respectively. By the definition of the index set, $(max(\mu_1, \mu_2), p_1 \cup p_2) =: (\mu, p)$ is an index. Let $x_3 := \pi_{\mu,p}^{-1}(x)$. Then we must have $x_3 = \pi_{\mu_1,p_1,\mu,p}(x_1) = \pi_{\mu_2,p_2,\mu,p}(x_2)$, hence the definition of π^E gives the same value for both choices of an index. That π^E is fine now follows immediately from the above commutative diagram. $\square$

Claim: $\pi^E \supset E$.

Proof: Let $x \in F_\alpha^A$. Since α is a limit ordinal, there must be $\mu < \alpha$ such that $x \in F_\mu^A$. But $F_{\alpha(\mu,0)}^{A(\mu,0)} = F_\mu^A$ and $\pi_{\mu,0} = id|F_\mu^A$. Consequently $\pi_{\mu,0}^*$ is the identity on $E(F_\mu^A)$, hence $\pi^E(x) = E(x)$, as desired. $\square$

This concludes the construction.

Definition 6: *The structure* `M` *constructed above is the extension of* F_δ^A *by* E*, denoted* $Ext(F_\delta^A, E)$*. If this structure is well-founded, it is isomorphic to some* F_γ^C*, in which case we identify the extension with its transitive collapse.* $\pi^E : F_\delta^A \to$ `M` *is the extension map for* F_δ^A*,* E*.*

The next lemma holds as well for the relativized extension construction. But since the proof is virtually identical and we will only use it for the case where the extra predicate is $\emptyset$, we state and prove it only for this case.

Lemma 7: *Suppose* $Ext(F_\gamma, E)$ *is well-founded, say* $\simeq F_\delta$*. Then the extension map* $\pi^E : F_\gamma \to F_\delta$ *is fine up to* F_δ*.*

Proof: Clearly, $\pi^E[F_\gamma]$ is constructibly closed, as it is a fine image of a constructibly closed structure. So we need to check that interpretation and the Skolem function behave according to the definition.
Denote by D the directed system used to represent F_γ, and by D^* the mapped directed system with direct limit F_δ. $\pi_{\mu,p}$ and $\pi^*_{\mu,p}$ are the limit maps for components of D, D^*, respectively.
First, suppose we have an $\mathbf{S}_0$-formula ϕ and a finite vector $\vec{x} \subset_{fin} F_\gamma$ with the property that $v := I(F_\delta, \phi, \pi^E(\vec{z})) \in F_\delta$. Pick $(\mu, p) \in D^*$ with $\pi^E(\vec{z}) \cup \{v\} \subset rng(\pi^*_{\mu,p})$, let $\bar{z}$ be the pre-image of $\vec{z}$ under $\pi_{\mu,p}$, y the pre-image of v under $\pi^*_{\mu,p}$. This means that, in F_δ, the Π_1-formula $\forall x(x \in v \leftrightarrow \phi(E(\vec{z}), v))$ holds, which uniquely characterizes v. But $\pi^*_{\mu,p}$, being fine, preserves Π_1-statements downwards, so that $E(F_{\alpha(\mu,p)}) = F_{\alpha^*(\mu,p)} \models \forall x(x \in y \leftrightarrow \phi(E(\bar{z}), x))$. Now, there are ω many F-levels containing $F_{\alpha(\mu,p)}$ in the domain of the fine map E, so that E restricted to $F_{\alpha(\mu,p)}$ is elementary. Hence the statement $\psi := \exists z \forall x(x \in z \leftrightarrow \phi(E(\bar{z}), x))$ that holds in $F_{\alpha^*(\mu,p)}$ must also hold in $F_{\alpha(\mu,p)}$, and as y is the only witness for ψ, we must have $y \in E[F_{\alpha(\mu,p)}]$. Let $\bar{y} := E^{-1}(y)$, $\hat{y} := \pi_{\mu,p}(\bar{y})$. Then $\pi^E(\bar{y}) = y$ by definition of π^E, so $y \in rng(\pi^E)$, as desired.

Now we take care of the S-operation. Let $\phi(x_0, ..., x_n)$ be an $\mathbf{S}_0$-formula with all free variables shown, pick $a_0, ..., a_m \in rng(\pi^E) \subset F_\delta$, where $m < n$, and suppose that there is $b \in F_\delta$ such that there are $c_1, ..., c_{n-m-2}$ in F_δ with $\phi(a_0, ..., a_m, b, c_1, ..., c_{n-m-2})$. Equivalently, $S(F_\delta, \phi, (a_0, ..., a_m))$ is defined and not set to $\emptyset$ by default. Without loss of generality, let b be $<_{F_\delta}$-minimal with this property and hence equal to $S(F_\delta, \phi, (a_0, ..., a_m))$. Let $\bar{a}_0, ..., \bar{a}_m$ be the pre-images of $a_0, ..., a_m$ under π^E. Once again, take (μ, p) such that $\{\bar{a}_0, ..., \bar{a}_m\} \subset F_{\alpha(\mu,p)}$, $\{b, c_1, ..., c_{n-m-2}\} \subset rng(\pi^*_{\mu,p})$. Let $b^*, c_0^*, ..., c^*_{n-m-2}$ be

the pre-images of $b, c_0, ..., c_{n-m-2}$ under $\pi^*_{\mu,p}$. Then:
$F_{\alpha^*(\mu,p)} \models \exists z_1, ..., z_{n-m-2}\phi(a_0, ..., a_m, b^*, ..., z_{n-m-2})$. Also:
$F_\delta \models \forall x(x <_{F_\delta} b \to \neg\exists z_1, ..., z_{n-m-2}\phi(a_0, ..., a_m, x, ..., z_{n-m-2}))$.
This is preserved downwards by $\pi^*_{\mu,p}$, so
$F_{\alpha^*(\mu,p)} \models \forall x(x <_{F_{\alpha(\mu,p)}} b^* \to$
$\to \neg\exists z_1, ..., z_{n-m-2}\phi(\pi^*_{\mu,p}(a_0), ..., \pi^*_{\mu,p}(a_m), x, z_1, ..., z_{n-m-2}))$. By these two facts, b^* is uniquely characterized. As in the first part, E is elementary when restricted to $F_{\alpha(\mu,p)}$ and hence we must have $b^* \in E[F_{\alpha(\mu,p)}]$. If $\bar{b}$ is the pre-image of b^* under E, then $\hat{b} := \pi_{\mu,p}(\bar{b})$ is a pre-image of b under π^E. □

7 The Covering Lemma for L

7.1 The Statement

It is well-known that there can be no elementary embedding $\pi : V \to V$ from the universe to itself. It follows that the existence of an elementary embedding $\pi : L \to L$ implies that L and V must be quite different. This is indeed the case. Let us abbreviate the statement 'There is $\pi : L \to_{el} L$' by '$0^\sharp$'. One of many consequences of $0^\sharp$ is that all uncountable cardinals (of V) are inaccessible in L - so V and L have a very different structure.
It is natural to ask the other way round: How similar must V and L be in the absence of such an embedding, abbreviated $\neg 0^\sharp$? The answer to this question is given by Jensen's covering lemma (CL):

$\neg 0^\sharp$: Let $X \subset On$ be uncountable. Then there is $Y \in L$ such that $X \subset Y$ and $card^L(X) = card^L(Y)$.

This has numerous consequences on the cardinal and ordinal structure of L.
For example, suppose κ is a singular cardinal and let $X = \langle \gamma_i | i < \theta \rangle$ be a cofinal sequence in κ with $\theta < \kappa$. As κ is singular, we have $\kappa > \aleph_1$, so we can pass from X to another cofinal sequence $\hat{X}$ in κ with cardinality $\aleph_1 \leq card(\hat{X}) =: \hat{\theta} < \kappa$. Applying the covering lemma, we get $\hat{X} \subset \tilde{X} \in L$ such that $\tilde{\theta} := card(\tilde{X}) = card(\hat{X})$. Now, $\kappa \cap \tilde{X}$ will be a cofinal sequence in κ of length $\tilde{\theta} < \kappa$ in L, so κ is singular in L as well, and we have shown that, under $\neg 0^\sharp$, singular cardinals are also singular in L. (Compare this with the fact stated above that under $0^\sharp$, all cardinals are inaccessible in L.)

Now, let κ be a cardinal, $\lambda := (\kappa^+)^L$ be the L-successor of κ. It is easy to see that $cf(\lambda) \geq \kappa$: Otherwise, pick a cofinal sequence $X := \langle \gamma_i | i < \theta \rangle$ in λ with $\aleph_0 < \theta < \kappa$. Covering gives $Y \supset X$ with $card(Y) = card(X)$ and $Y \in L$. This means that λ is singular in L, but as $L \models ZFC$, no successor cardinal can be singular in L.

In particular, it follows that if κ is singular, then $(\kappa^+)^L = \kappa^+$: For $cf((\kappa^+)^L) \geq \kappa$ implies $cf((\kappa^+)^L) > \kappa$ for κ singular (as cofinalities are always regular), and hence we must have $cf((\kappa^+)^L) = (\kappa^+)^L$. Now $(\kappa^+)^L = \kappa^+$ is immediate.

The last fact, asserting that L-successors of singular cardinals are the same as in V, is also known as the weak covering property. While this seems to be a rather weak corollary in the realm of L, it becomes in fact the primary goal when one tries to generalize covering ideas to larger core models.

7.2 The Proof Idea

CL was originally proved in [23]. Other versions appeared using Silver machines [19], Σ_n-Skolem functions [11] and the fine hierarchy [14]. A good account on the classical proof is [31]. The general strategy is as follows: Assume $\neg 0^\sharp$ and that CL fails. Let $X \subset On$ be a counterexample with minimal supremum β, so that $X \subset F_\beta$. Form the constructible hull $F\{X\}$ of X and collapse it to some constructible level by $E : F_\alpha \to F\{X\} \subset F_\beta$. We attempt to extend E to a larger domain. Let us ignore for a moment the question of well-foundedness of the extension; as we are working under $\neg 0^\sharp$, it must be impossible to form $Ext(L, E)$. Hence there must be some stage F_δ such that F_α is not a base for F_δ. This is witnessed by a lexically minimal pair $(\mu, p) \in \alpha \times F_\delta^{<\omega}$. We will show that under these conditions, F_δ is already generated by these parameters. These parameters are elements of some F_γ, $\gamma < \delta$ and can hence be mapped by the extension map for $Ext(F_{\delta-1}, E)$, which exists by choice of δ. The hull of the images contains X as a subset and can be mapped bijectively into some ordinal $\zeta < \beta$. This allows us to construct a counterexample to covering with smaller supremum than β, a contradiction.
The problem is of course how to guarantee the well-foundedness of the extended structures. The natural approach is to add to X witnesses for potential ill-foundedness, so that each ill-founded sequence in an extension is actually in the range of the extension map and can hence be pulled back to L, giving a contradiction. This could be achieved e.g. by demanding that X is constructibly closed and that X is ω-closed, i.e. $X^\omega = X$. If it isn't, one can pass to an appropriate superset. If $card(X) = \kappa$, this superset can be chosen to be of cardinality $\leq \kappa^\omega$. However, if $\kappa^\omega > \kappa$, the covering set given by this strategy will be larger than the original set X. If we want a covering set of the same size, we need to avoid the use of ω-closure in the argument. This was used to ensure the well-foundedness of the extension by allowing to pull back every witness for non-well-foundedness of the extended structure

to the original structure, leading to a contradiction. The crucial observation for a proof of the full covering lemma is that it is quite unnecessary for this purpose to add all ω-sequences: A small number of canonical witnesses is sufficient. That is the idea behind the proof in the next section.

Some preliminary steps allow us to focus on a seemingly special case which is convenient in several respects. The following does not use any assumptions on $0^\sharp$.

Assume that CL fails. Then there is an uncountable set $Z \subset On$ such that for all $Y \supset Z$ with $Y \in L$, $card(Y) > card(Z)$. Let β be minimal such that there is a counterexample $X \subset On$ with $\beta := \sup\{X\}$. Now let additionally $X \subset \beta$ be a counterexample of minimal cardinality μ. As X is uncountable, we can pass from X to $X' := X \cup \aleph_1$ without increasing the cardinality of X, and any covering of X' will also cover X. Hence we may assume without loss of generality that $\aleph_1 \subset X$.

Theorem:

1. *There is no $X \subset Y \in L$ with $card^L(Y) < \beta$.*
2. *$\beta = card^L(\beta)$*
3. *$\mu^+ < \beta$*
4. *μ is regular.*

Proof:

1. Assume otherwise, let Y be a counterexample, $card^L(Y) = \kappa$, let $f \in L$ such that $f : \kappa \rightarrow_{surj} Y$. Let $\bar{Y}$ be the pre-image of Y under f. $card^L(\bar{Y}) = card^L(Y) = \kappa < \mu$, and $sup\{\bar{Y}\} \leq \kappa < \beta$, so by induction, there must be $\bar{Y} \subset \hat{Y} \in L$ with cardinality $\leq \kappa + \aleph_1$. This implies $X \subset f[\hat{Y}] \in L$, hence X is not a counterexample.
2. This follows by the same argument: If β could be collapsed to $\kappa < \beta$ by a map f in L, then X could be pulled back to κ, contradicting the minimality of β.

3. Since X is a counterexample to CL, and $X \subset \beta$, $card(X) = card(\beta)$ is impossible. So $\mu < \beta$. Now, as β is the supremum of X, β must be singular and cannot be a successor cardinal, in particular not μ^+.

4. We argue by contradiction. Suppose μ is singular, $cf(\mu) := \lambda < \mu$, then $X = \bigcup_{\zeta<\lambda} X_\zeta$ with $card(X_\zeta) < \mu$ and $X_\zeta \subset \beta$. Then no X_ζ can be cofinal in β, so already $sup\{X_\zeta\} < \beta$ for $\zeta < \lambda$. By minimality off β, each X_ζ can be covered: For $\zeta < \lambda$, pick $Y_\zeta \in L$ with $X_\zeta \subset Y_\zeta$ and $card(X_\zeta) = card(Y_\zeta)$. By passing to $Y_\zeta \cap \beta$ when necessary, we can make our choices in such a way that $Y_\zeta \subset On$ and $sup\{Y_\zeta\} < \beta$ for each $\zeta < \lambda$. Since $H^L_\beta = F_\beta$, this implies $\forall \zeta < \lambda Y_\zeta \in F_\beta$. Now $F_\beta = F\{\beta\}$ and β is an L-cardinal, so that $\beta \cdot \omega = \beta$ and $\beta^{<\omega} = \beta$. Each element of F_β is generated by a finite sequence of constructible operations applied to a finite sequence of elements of β, hence there is a canonical map $f : \beta \cdot \omega \rightarrow_{surj} F_\beta$, which is definable over F_β and hence an element of L. Also, there is $g \in L$ such that $g : \beta \rightarrow_{surj} \beta \cdot \omega$. Hence $h := f \circ g : \beta \rightarrow_{surj} F_\beta$ is in L as well and we can define $Z = \{h^{-1}(Y_\zeta) | \zeta < \lambda\}$, the set of pre-images of the Y_ζ under h. Then $sup\{Z\} = \beta$, but as $card(Z) = \lambda < \mu$, the minimality of μ implies that Z can be covered. So let $L \ni P \subset \beta$ be a superset of Z of cardinality $\leq \mu$. Then the following set is in L:
   $W := \bigcup\{h(\eta) | \eta \in P \wedge h(\eta) \subset On \wedge otp(h(\eta)) < \mu^+\}$.
   So W is the union of all h-images with pre-images in P that are subsets of β of order type $< \mu^+$. In particular, this includes all of the Y_ζ, so their union will be a subset of W. Hence $X \subset W \subset \beta$. But W is the union of at most μ many sets of cardinality $\leq \mu$, and μ is regular. So $card(W) = \mu = card(X)$, and W covers X. This contradicts the assumption that X was a counterexample to CL.

□

In this situation, we want to make use of the extension construction.

7.3 The Full Covering Lemma

Convention: For the next section, $\leq_S$ is the ordering of the directed system under consideration.

As we sketched above, we need a replacement for lemma 2 that leads to a better control of the size of the covering set. In order to apply the proof idea of theorem 2, we have to present some superset of X as the range of a fine embedding E of some F_α with the property that the extender construction applied to E leads well-founded structures whenever applicable. This motivates the following definition.

Definition 1: *A fine map $E : \boldsymbol{F}_\alpha \to \boldsymbol{F}_\beta$ is strong for F_δ provided:*

1. *α is a limit ordinal*
2. *$rng(E) \uparrow_{F_\beta}$*
3. *$Ext(\boldsymbol{F}_\delta, E)$ is strong whenever $\boldsymbol{F}_\alpha$ is a base for $\boldsymbol{F}_\delta$*

If E is strong for every $\boldsymbol{F}_\delta$, E is called strong for the fine hierarchy, or $\boldsymbol{F}$-strong.

In order to prevent non-well-foundedness, we need a way to reflect the well-foundedness of the extension to the base structure.

Definition 2: *Let α be a limit ordinal, $E : F_\alpha \to_{cof} F_\beta$ be fine. If there is a $\leq_S$-increasing sequence $\langle(\mu_i, p_i)|i \in \omega\rangle$ with $\mu_i < \alpha$, $p_i \subset F_\delta$ together with a sequence $\langle y_i|i \in \omega\rangle$ with $y_i \in F_\beta$ such that, for all i, we have $y_{i+1} \in E(\pi_{\mu_i p_i,\mu_{i+1},p_{i+1}})(y_i)$, then the sequence $\langle y_i|i \in \omega\rangle$ is called a vicious sequence for F_δ, E.*

Lemma 3: *Let α be a limit ordinal, $E : F_\alpha \to_{cof} F_\beta$ be fine such that E is not strong for F_δ. Then there is a vicious sequence for F_δ, E.*

Proof: If $Ext(F_\delta, E) =: \mathtt{A}$ is ill-founded, there are $\langle y_i|i < \omega\rangle \subset \mathtt{A}$ with $y_{i+1} \in^* y_i$, where $\in^*$ is the $\in$-relation of $\mathtt{A}$. Let $\pi^*_{\mu,p}$ be the system maps on the right-hand side of the extension. This sequence must be generated by the directed system $D_{\alpha\delta}$: So pick a sequence $\langle \mu_i, p_i, \bar{y}_i|i < \omega\rangle$ with $\mu_i < \alpha$, $p_i \subset_{fin} F_\delta$ so that $y_i = \pi^*_{\mu_i,p_i}(\bar{y}_i)$. Since $D_{\alpha,\delta}$ is a directed system, we may assume without loss of generality that $(\mu_i, p_i) <_{D_{\alpha,\delta}} (\mu_j, p_j)$ for $i < j$. This is already a vicious sequence for F_δ, E: For now,
$\bar{y}_{i+1} \in (\pi^*_{\mu_{i+1},p_{i+1}})^{-1} \circ \pi^*_{\mu_i,p_i}(\bar{y}_i) = E(\pi_{\mu_i,p_i,\mu_{i+1},p_{i+1}})(\bar{y}_i)$ for all $i \in \omega$, which is just the definition of a vicious sequence. $\Box$

We are ready to prove Jensens's covering lemma.

Definition 4: *An uncountable set $X \subset On$ is suitable iff X is cc and, with $\sigma : \boldsymbol{F}_\alpha \to_{coll} X$, σ is strong.*

Lemma 5: *Assume $\neg 0^\sharp$ and let $X \subset On$ be suitable with $sup\{X\} = \beta$. Then there is $X \subset Y \subset On$ such that $Y \in L$ and $card(Y) < card(\beta)$.*

Proof: Form the collapse $E : F_\alpha \to_{coll^{-1}} F_\beta$ with inverse collapsing map E. If $E = id$, then $X = F_\alpha \in L$, so X covers itself. Hence, without loss of generality, $E \neq id$.
Now it is clear that F_α cannot be a base for L: Otherwise, since X is suitable, E is strong and therefore could be lifted to $Ext(L, E)$, a non-trivial automorphism of L, so $0^\sharp$ would exist. Hence there must be some minimal ordinal γ such that F_α is not a base for F_γ. Pick $\mu < \alpha$ minimal then $p \subset_{fin} F_\gamma <_{lex}$-minimal such that $coll''F\{\mu \cup p\} \notin F_\alpha$. By condensation, there is $\bar{\gamma}$ such that $F_{\bar{\gamma}} \simeq F\{\mu \cup p\}$, let σ be the collapsing map. It follows that $F_{\bar{\gamma}} = F\{\sigma(\mu) \cup \sigma(p)\} \notin F_\alpha$, so F_α is not a base for $F_{\bar{\gamma}}$. Hence $\gamma \leq \bar{\gamma}$ by minimal choice of γ, so in fact $\bar{\gamma} = \gamma$, and $F\{\mu \cup p\} \simeq F_\gamma$. Furthermore, $\sigma(\mu) \leq \mu$, $\sigma(p) \leq_{lex} p$, but $coll''F\{\sigma(\mu) \cup \sigma(p)\} \notin F_\alpha$, so the minimal choices of μ and p imply that $\sigma(\mu) = \mu$, $\sigma(p) = p$, so that already $F\{\mu \cup p\} = F_\gamma$.
By Proposition 1.4.2, there must then be an ordinal δ with $\gamma = \delta + 1$. By minimality of γ, F_α is a base for F_δ, so E can be lifted to some embedding $\pi_E : F_\delta \to F_\eta$ fine up to F_η and further to a fine map $\pi_E \subset \pi_E^+ : F_\gamma \to F_{\eta+1}$.
The rest is elementary manipulation: As $E \subset \pi_E \subset \pi_E^+$ and $X \subset rng(E)$, we have $X \subset rng(\pi_E^+)$. But F_γ could be presented as $F\{\mu \cup p\}$, so $rng(\pi_E^+) = \pi_E^+[F_\gamma] = \pi_E^+[F\{\mu \cup p\}]$. π_E^+ is fine, so it commutes with the constructible operations and also with the hull operator; accordingly $\pi_E^+[F\{\mu \cup p\}] = \pi_E^+[F\{\pi_E^+[\mu] \cup \pi_E^+[p]\}]$. As $\mu < \alpha$, $\pi_E^+(\mu) < \beta$, so we may exchange $\pi_E^+[\mu]$ for $\pi_E^+(\mu) \supset \pi_E^+[\mu]$ without increasing the cardinality above β in the latter hull. Putting this together, we get $X \subset F\{\sigma(\mu) \cup \pi_E^+[p]\}$, where $\pi_E^+(\mu)$ is simply an ordinal, while $\pi_E^+[p]$ is finite, so both are elements of L. Consequently, $F\{\pi_E^+ \cup \pi_E^+[p]\} \in L$, but $card(F\{\pi_E^+(\mu) \cup \pi_E^+[p]\}) \leq card(\pi_E^+(\mu) + \aleph_0) < \beta$, so we have obtained a set of the desired kind. $\square$

The following lemma finishes the proof:

Lemma 6: *Let $X \subset On$ such that $X \uparrow_\beta$, β a limit ordinal such that* *card*$(X) =: \mu > \aleph_0$ *is regular. Then there is a suitable set* $On \supset Y \supset X$ *with* $card(Y) = card(X)$.

Proof: We construct Y by recursion. First, let $\langle x_\iota | \iota < \mu \rangle$ enumerate X in any order. Then then $Y_0 = E_0 = \emptyset$ and define the sets Y_ζ along with ordinals α_ζ and fine maps E_ζ as follows:

- If λ is a limit, $Y_\lambda = \bigcup_{\zeta<\lambda} Y_\zeta$.

- If $\zeta = \gamma + 1$, then Y_ζ is the constructible closure of $Y_\zeta \cup \{x_\zeta\} \cup s$, where s is a vicious sequence for E_ζ, F_δ, where δ is minimal such that E_ζ is not strong for F_δ. If E_ζ is strong for L, set $s := \emptyset$.
- $E_\zeta : F_{\alpha_\zeta} \to Y_\zeta$ is simply the inverse collapsing map.

Then let $Y = \bigcup_{\zeta<\mu} Y_\zeta$, $E : F_\alpha \to_{coll^{-1}} Y$. We claim that Y and E are as desired.
First, $X \subset Y$ follows from the construction of Y, as each element of X was included in some Y_ζ. Y is constructibly closed, so E is fine. By induction, $card(Y_\zeta) \leq \mu$ for $\zeta < \mu$, hence $card(Y) \leq \mu = card(X)$ as well. (For unions this is immediate; at successor steps, we add a countable set and form a constructible hull, hence $card(Y_{\zeta+1}) \leq card(Y_\zeta) \cdot \aleph_0$.)

We need to see that E is strong. Assume otherwise, and pick η minimal such that E is not strong for F_η: Hence F_α is a base for F_η, yet $Ext(F_\eta, E)$ is not well-founded. The idea is to reflect this fact in one of the Y_ζ where the vicious sequences have been explicitely included in the range E_ζ, thus contradicting the well-foundedness of F_{α_ζ}.
So pick a vicious sequence for F_η, E, say $s = \langle (z_i, \mu_i, p_i) | i \in \omega \rangle$, i.e. $\mu_i < \alpha$, $p_i \subset F_\eta$ and $z_{i+1} \in E(\pi_{\mu_i, p_i, \mu_{i+1}, p_{i+1}})(z_i)$ for $i \in \omega$.
Recall that F_α is the transitive collapse of $Y = \bigcup_{\zeta<\mu} Y_\zeta$ with collapsing map E. Hence, setting $\bar{Y}_\zeta := E^{-1}[Y_\zeta]$, we get $F_\alpha = \bigcup_{\zeta<\mu} \bar{Y}_\zeta$. Pick a regular cardinal κ large enough so that the transitivizations of E, X, Y, F_α, F_η are of cardinality $< \kappa$ and hence elements of $\mathbf{H}_\kappa$. Strongness of maps is absolute between V and H_κ, and so are the definitions of Y_ζ, E_ζ, Y, E etc.
Now, we form a sequence of elementary substructures of $\mathbf{H}_\kappa$:

- Z_0 is the smallest elementary substructure of $\mathbf{H}_\kappa$ containing F_α, F_η as elements and s and Y_0 as subsets.

- For $i \in \omega$, let ζ_i be minimal such that $Z_i \cap F_\alpha \subset \bar{Y}_{\zeta_i}$. Then Z_{i+1} is the smallest elementary substructure of $\mathbf{H}_\kappa$ such that $Z_i \cup \bar{Y}_{\zeta_i} \subset Z_{i+1}$

Then let $W := \bigcup_{i\in\omega} Z_i$. $W \prec_{el} \mathbf{H}_\kappa$, and $W \cap F_\alpha = \bar{Y}_\gamma$ for some $\gamma < \mu$. Form the collapse $\sigma : \bar{W} \to_{coll^{-1}} W$, where σ is the inverse of the collapsing map. σ collapses F_α to $\bar{Y}_\gamma$, but $\bar{Y}_\gamma \simeq Y_\gamma$ via E by definition, and the transitivization of Y_γ was F_{α_γ}. This implies that $\sigma^{-1}(F_\alpha) = F_{\alpha_\gamma}$.
The corresponding inverse collapsing map was $E_\gamma : F_{\alpha_\gamma} \to Y_\gamma$. This can be presented as the result of first uncollapsing F_{α_γ} by σ to $\bar{Y}_\gamma$, then applying E. Hence $E_\gamma = E \circ \sigma | F_{\alpha_\gamma}$.
Being a base is absolute between V and $\mathbf{H}_\kappa$, and further reflected by the elementary substructures. Hence, letting $\bar{\eta} = \sigma^{-1}(\eta)$, F_{α_γ} will be a base for $F_{\bar{\eta}}$.
Now, we can use σ to pull back the vicious sequence s to one for E_γ: Setting $(\bar{\mu}_n, \bar{p}_n, \bar{\pi}_{n,n+1}) = \sigma^{-1}((\mu_n, p_n, \pi_{\mu_n,p_n,\mu_{n+1},p_{n+1}}))$, we get for all $n \in \omega$:
$z_{n+1} \in E(\pi_{\mu_n,p_n,\mu_{n+1},p_{n+1}})(z_n) = E \circ \sigma(\pi_{\bar{\mu}_n,\bar{p}_n,\bar{\mu}_{n+1},\bar{p}_{n+1}})(z_n) = E_\zeta(\pi_{\bar{\mu}_n,\bar{p}_n,\bar{\mu}_{n+1},\bar{p}_{n+1}})(z_n)$, since the elementary map σ sends system maps to system maps. We have obtained a vicious sequence for E_γ. Hence E_γ is not strong for L.
So pick δ minimal such that E_γ is not strong for F_δ. By the construction of Y_γ, $Y_{\gamma+1}$ contains a vicious sequence $s_\gamma = \langle (\nu_i, q_i, y_i) | i \in \omega \rangle$ for E_γ, F_δ. This means that, for all $n \in \omega$:
$y_{n+1} \in E_\gamma(\pi_{\nu_n,q_n,\nu_{n+1},q_{n+1}})(y_n)$. This can now be pulled back to the left side of the extension, leading:
$E^{-1}(y_{n+1}) \in E^{-1} \circ E_\gamma(\pi_{\nu_n,q_n,\nu_{n+1},q_{n+1}})(E^{-1}(y_n)) = \sigma(\pi_{\nu_n,q_n,\nu_{n+1},q_{n+1}})(E^{-1}(y_n)) = \pi_{\sigma(\nu_n),\sigma(q_n),\sigma(\nu_{n+1}),\sigma(q_{n+1})}(E^{-1}(y_n))$.
This represents a non-wellfounded $\in$-sequence in the directed system representing F_η. Unfolding this via $\pi_{\sigma(\nu_{n+1}),\sigma(p_{n+1})}$ then gives us:
$\pi_{\sigma(\nu_{n+1}),\sigma(p_{n+1})}(E^{-1}(y_{n+1})) \in \pi_{\sigma(\nu_n),\sigma(p_n)}(E^{-1}(y_n))$, which is an ill-founded $\in$-sequence in F_η, a contradiction.
So E_γ, and consequently E, must have been strong after all. $\square$

8 An Approximation Theorem for L

8.1 Preliminary Remarks

The covering theorem shows how, in the absence of $0^\sharp$, uncountable sets X of ordinals can be approximated by elements of L from 'above'. It is a natural question whether something similar can be done from 'below', i.e. whether sets of ordinals can be 'exhausted' by elements of L. This turns out to be the case: By a theorem of Magidor [11], if $\neg 0^\sharp$ and X is a set closed under primitive recursive set functions, there are $X_i \in L$ such that $\bigcup_{i\in\omega} X_i = X$. We prove a similar theorem in the context of the fine hierarchy, replacing *pr*-closure by the (weaker) condition of constructible closure. Since we don't have to distinguish between definition complexities, the proof becomes considerably shorter and simpler.

8.2 Notation

Definition 1: *F_α is an ω-base for F_δ if, for $A \subset F_\delta$ countable, $\bar{\alpha} < \alpha$, and $\pi : F\{F_{\bar{\alpha}} \cup A\} \to_{coll} F_\beta$, then $\beta < \alpha$.*

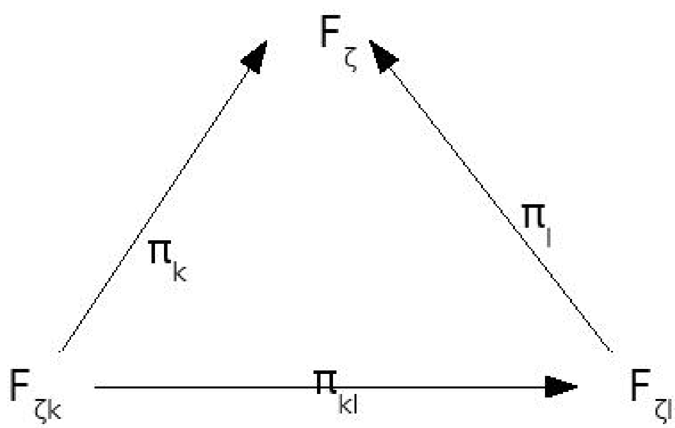
Convention: $\underline{F}_\alpha$ is $\{F_\beta | \beta < \alpha\}$; furthermore, we write $p \subset_{fin} X$ to indicate that p is a finite subset of X, similar $p \subset_{cntbl} X$ for a countable subset.

Obviously, being an ω-base is a stronger property than being a base. This gives some information on the wellfoundedness of extensions.

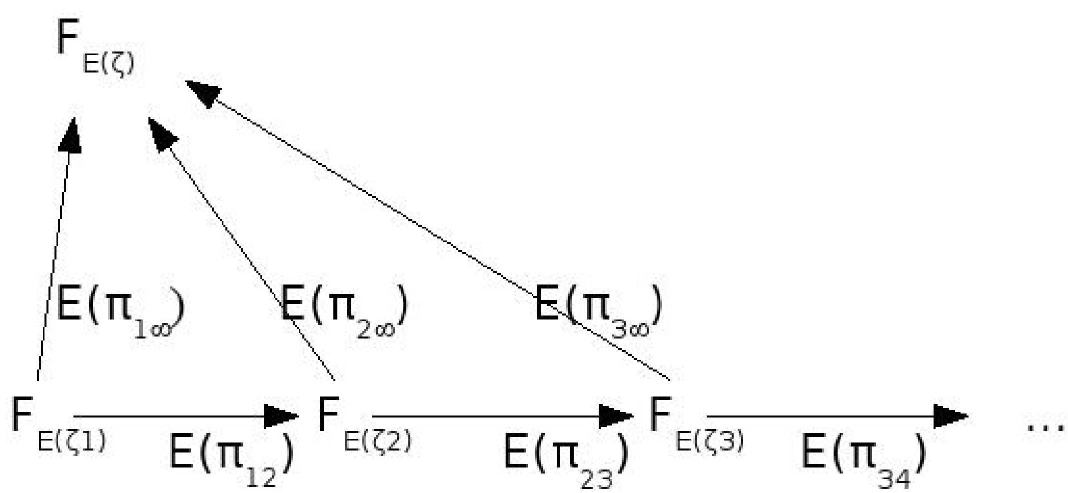
Lemma 2: *Let α be a limit ordinal such that $cf(\alpha) > \omega$. Suppose $E : F_\alpha \to F_\beta$ is fine and cofinal in β and that F_α is an ω-base for F_δ. Define the extension map π_E as in section* 6 *on extension of embeddings. Then the direct limit of the mapped directed system is well-founded.*

Proof: Define a directed system S: The index set
$I = \{(\bar{\alpha}, p) | \bar{\alpha} < \alpha, p \subset_{fin} \underline{F}_\delta\}$, associated structures $\pi_{\bar{\alpha},p} : F\{F_{\bar{\alpha}} \cup p\} \to_{coll} F_{\zeta(\bar{\alpha},p)}$ (where $\zeta(\bar{\alpha}, p) < \alpha$ because of the definition of ω-base), maps $\pi_{(\alpha_1,p_1),(\alpha_2,p_2)} : F_{\zeta(\alpha_1,p_1)} \to F_{\zeta(\alpha_2,p_2)}$ defined as usual. The direct limit of S is F_δ. Since the relevant structures and maps are easily seen to be elements of F_α, the system S lifts up by E to a directed system S_E. Now, suppose the limit of S_E is ill-founded, so there are $i_1 < i_2 < ... \in I$ such that (set $\zeta(i_k) = \zeta(k)$ etc. for con-

venience) the limit of the directed sequence $F_{E(\zeta(1))} \to_{E(\pi_{1,2})} F_{E(\zeta(2))} \to_{E(\pi_{2,3})}$ ... is ill-founded. Set $i_k = (\alpha_k, p_k)$, $\bar{\alpha} = \bigcup_{i\in\omega} \alpha_i$, $A = \bigcup_{i\in\omega} p_k$. So $A \subset_{cntbl} \underline{F}_\delta$ and $\bar{\alpha} < \alpha$ since $cf(\alpha) > \omega$.
Consider $h : F\{F_{\bar{\alpha}} \cup A\} \to_{coll} F_\zeta$; $\zeta < \alpha$ by definition of ω-base. Define $\pi_k := \pi_{(k,\infty)} : F_{\zeta(k)} \to F_\zeta$ by $h \circ \pi_{i_k}^{-1}$. $\pi_{k,\infty} \in F_\alpha$ by the usual argument: It is the identity on α_k, maps the condensed finite parameter correctly into $h[A]$ and respects $\mathbf{S}_0$-terms. Hence it is definable in, and, as α is a limit ordinal, an element of F_α. Hence all structures and maps mentioned are in F_α, and so the commutative diagram:



lifts up:



So the limit of the directed system can be embedded in $F_{E(\zeta)}$ and is thus not ill-founded.$\Box$

8.3 The approximation theorem

We state and prove the fine approximation theorem theorem; we assume $\neg 0^\sharp$ for this section.

Definition 3: *$X \subset \underline{F}_\infty$ is constructibly closed in the ordinals ($\Omega - cc$) if $F\{X\} \cap \underline{F}_\infty = X$.*

Remark/Convention: In the following, we freely confuse α, F_α and $\underline{F_\alpha}$ where this is possible and leads to a better readability. Note that, by our observations in the section on fine hulls, these all lead to the same hull.

Theorem 3: *Let X be constructibly closed in the ordinals. Then there are $\langle X_i | i \in \omega \rangle$ such that $X_i \in L$ and $X = \bigcup_{i\in\omega} X_i$.*

The rest of the section is devoted to the proof of this theorem.

Proof:
Suppose $X \subset \beta$ is as in the theorem statement, $\beta = sup(X)$. Let $E : F_\alpha \to F\{X\}$ be the inverse of the collapsing map.

Proposition 4: *$rng(E) \cap \beta = X$.*

Proof: Immediate by the fact that X is $\Omega - cc$.$\Box$

The proof will be an induction on β. So suppose the theorem to be true for $\gamma < \beta$.

If β is a successor, say $\beta = \gamma + 1$, then $X \cap \gamma$ is obviously $\Omega - cc$ and thus a countable union of constructible sets by induction hypothesis, say $X \cap \gamma = \bigcup_{i\in\omega} Z_i$. Then $X = (X \cap \gamma) \cup \{\gamma\} = \{\gamma\} \cup \bigcup_{i\in\omega} Z_i$ witnesses our claim.
For the same reason, we may without loss of generality assume that α is a limit ordinal.

If $cf(otp(X)) = cf(\alpha) = \omega$, then let $\langle \alpha_i | i \in \omega \rangle$ be cofinal in α. Induction applies to each $X \cap E(\alpha_i)$, giving us $X_{ij} \in L$ for $i, j \in \omega$ such that $X = \bigcup_{i\in\omega} X \cap E(\alpha_i) = \bigcup_{i\in\omega} \bigcup_{j\in\omega} X_{ij}$.
So assume $cf(\alpha) > \omega$ from now on.

Proposition 5: *We can assume without loss of generality that* $E|_{F_\alpha} \neq id$.

Proof: Otherwise, we have $X = \alpha \in L$ and the theorem statement is trivial.$\square$

Proposition 6: F_α *is not an* ω*-base for* F_∞.

Proof: Otherwise, by Lemma 1, we can lift E to a fine map $\pi_E : F_\infty \to F_\infty$ other than the identity, which contradicts the assumption $\neg 0^\sharp$.$\square$

So let ρ be minimal such that F_α is not an ω-base for F_ρ. Also, let $\bar{\alpha} < \alpha$ and $A \subset_{cntbl} F_\rho$ be such that $F\{F_{\bar{\alpha}} \cup A\} \to_{coll} F_\zeta$, $\zeta \geq \alpha$. Setting $\bar{A} = coll''A$, we get $F_\zeta = F\{F_{\bar{\alpha}} \cup \bar{A}\}$. Since ρ is minimal and $\zeta \leq \rho$, we actually have $\zeta = \rho$ and therefore may assume without loss of generality that $F_\rho = F\{F_{\bar{\alpha}} \cup A\}$. If ρ is a limit ordinal, then $F_\rho = \bigcup_{i\in\omega} F\{F_{\bar{\alpha}} \cup A_i\}$, where the A_i are such that A_n is an initial segment of A_{n+1} for $n \in \omega$ and $\bigcup_{i\in\omega} A_i = A$ (this is possible since A is countable). So we actually have $cf(\rho) = \omega$.

Now we distinguish two cases:

Case 1: ρ is a successor, let $\rho = \gamma + 1$.
By minimal choice of ρ, F_α is an ω-basis for F_γ, so there is a fine embedding $\pi_E : F_\gamma \to F_\delta$ extending E. In particular $\pi_E''\alpha = X$, so $(\pi_E''F_\gamma) \cap \beta = X$. By theorem 4.23, π_E is fine up to F_γ, so there is a lifting $\pi_E^+ : F_\rho \to F_{\delta+1}$ extending π_E; in particular, $\pi_E^{+''}\alpha = X$, $(\pi_E^{+''}F_\rho) \cap \beta = X$. This implies:
$X = \beta \cap rng(\pi_E^+) = \beta \cap \pi_E^{+''}F_\rho = \beta \cap \pi_E^{+''}F\{F_{\bar{\alpha}} \cup A\}$
$= \beta \cap F\{\pi_E^{+''}F_{\bar{\alpha}} \cup \pi_E^{+''}A\} = \beta \cap F\{\pi_E^{+''}\underline{F}_{\bar{\alpha}} \cup \pi_E^{+''}A\}$
$= \beta \cap F\{E''\underline{F}_{\bar{\alpha}} \cup \pi_E^{+''}A\}$.
Now $E''\underline{F}_{\bar{\alpha}}$ is $\Omega - cc$ (since it is a fine image of an $\Omega - cc$ set) and $\bar{\alpha} < \alpha$, so $E(\bar{\alpha}) < \beta$ and we get from the induction hypothesis that $E''\underline{F}_{\bar{\alpha}} = \bigcup_{i\in\omega} Y_i$, where $Y_i \in L$ for $i \in \omega$. We can assume without loss of generality that $i < j$ implies $Y_i \subset Y_j$. Thus:
$X = \beta \cap F\{\bigcup_{i\in\omega} Y_i \cup \pi_E^{+''}A\} = \bigcup_{i\in\omega} \beta \cap F\{Y_i \cup \pi_E^{+''}A\}$
$= \bigcup_{i\in\omega, q\in(\pi_E^{+''}A)^{<\omega}} F\{Y_i \cup q\} \cap \beta$,

and $\beta \cap F\{Y_i \cup q\} \in L$, so we are done in this case.

Case 2: ρ is a limit ordinal.
We have already seen that $cf(\rho) = \omega$ in this case. Let $\langle \rho_i | i \in \omega \rangle$ be a sequence of ordinals cofinal in ρ, so $\rho_i < \rho$ for $i \in \omega$ and $\rho = \bigcup_{i \in \omega} \rho_i$. We consider separately the countably many directed systems $S_k, k \in \omega$: I_k, the index set of S_k is given by $I_k := \{(\tau, q) | \tau < \alpha, q \subset_{fin} \underline{F}_{\rho_k}\}$, the structures and maps between them are defined as before.
Each system S_k has a direct limit isomorphic to the fine level F_{ρ_k}, remember $\bigcup_{i \in \omega} F_{\rho_k} = F_\rho$. Since ρ is minimal, F_α is an ω-base for each F_{ρ_k}, and by Lemma 1, the corresponding lifted direct limites are well-founded and hence isomorphic to fine levels. Name these levels $F_{\bar{\gamma}_k}$ and let $\pi_n^E : F_{\rho_k} \to F_{\bar{\gamma}_k}$ be the corresponding embeddings extending E, so $E \subseteq \pi_i^E$. We have $E''\underline{F_\alpha} = X$, and hence for each $i \in \omega$, we have $\pi_i^{E''}\underline{F_\alpha} = X$. Since these maps are order-preserving, this implies $rng(\pi_i^E) \cap \overline{F_\beta} = X$. Observe that $\underline{F_\alpha} \subseteq F_\rho = F\{\underline{F_{\bar{\alpha}}} \cup A\} = \bigcup_{i \in \omega} F\{\underline{F_{\bar{\alpha}}} \cup (A \cap \overline{F_{\rho_i}})\}$. Let $\beta_n := sup\{\pi_n^{E''} F\{F_{\bar{\alpha}} \cup (A \cap F_{\rho_n})\} \cap \beta\}$, so that $X = \bigcup_{n \in \omega} (\beta_n \cap \pi_n^{E''} F\{F_{\bar{\alpha}} \cup (A \cap F_{\rho_n})\})$. Let $X_n := (\beta_n \cap \pi_n^{E''} F\{F_{\bar{\alpha}} \cup (A \cap F_{\rho_n})\})$, so $X = \bigcup_{n \in \omega} X_n$ and $X_n = \beta_n \cap F\{\pi_n^{E''} F_{\bar{\alpha}} \cup \pi_n^{E''}(A \cap F_{\rho_n})\} = \beta_n \cap F\{E''\underline{F_{\bar{\alpha}}} \cup \pi_n^{E''}(A \cap F_{\rho_n})\}$. Since $\bar{\alpha} < \alpha$, $E''\underline{F_{\bar{\alpha}}} = \bigcup_{i \in \omega} Y_i$, where $Y_i \in L$ by induction. Assume again without loss of generality that $i < j$ implies $Y_i \subset Y_j$.
But then we can conclude:
$X_n = \beta_n \cap F\{\bigcup_{i \in \omega} Y_i \cup \pi_n^{E''}(A \cap (F_{\rho_n}))\} =$
$= \beta_n \cap \bigcup_{i \in \omega, q \in \pi_n^{E''}(A \cap F_{\beta_n})^{<\omega}} F\{Y_i \cup q\} =$
$= \bigcup_{i \in \omega, q \in \pi_n^{E''}(A \cap F_{\beta_n})^{<\omega}} \beta_n \cap F\{Y_i \cup q\}$,
which is a countable union of sets in L. So we have represented each X_n as a countable union of constructible sets, and hence $X = \bigcup_{n \in \omega} X_n$.

This concludes the proof of theorem 3.□

9 How to hyper the Fine Hierarchy

9.1 Introductory Remarks

The use of the F-hierarchy allowed proofs of basic theorems of constructible combinatorics and particularly short and simple proofs for the covering lemma and the approximation theorem. To make the F-hierarchy work for more advanced principles like global square or morass, we have to amend it with an even finer interpolation. A short preceeding remark is in order to explain why this is so.

In the proof of square using Jensen's fine structure theory, the singularization of an ordinal θ in L is associated with the minimal α such that a cofinal sequence of smaller order type is definable over J_α and the minimal n such that there is a Σ_n-formula ϕ defining such a sequence. The square sequence is then constructed by using master codes to reduce the consideration to the case of a Σ_1-formula over a relativized structure. The Σ_1-information of a structure can then be approximated from below: One chooses a canonical sequence converging to α, simultanously constructing the square sequence. This works due to the fact that, if the defining formula for a singularizing sequence is Σ_1, θ either singularizes over a limit stage or has a cofinal sequence of order type ω, which leads to several trivial special cases. If θ Σ_1-singularizes over $J_{\beta+1}$, then $cf^L(\theta) = \omega$.

This is not true for the F-hierarchy. Ordinals that are singularized over a successor stage by an $\mathbf{S}_0$-formula are not necessarily of countable cofinality. To see this, recall that $F_{\omega\alpha} = J_\alpha$. Now, if γ is any ordinal singular in L with uncountable cofinality singularized over J_α by a Σ_n-formula with $n \geq 3$, the singularization will be definable for the first time in the F-hierarchy over some successor stage between F_α and $F_{\alpha+\omega}$. And ordinals with this property are easily constructed.

Our solution to this problem is to refine the hull operators associated with the F-hierarchy; more precisely, we transfer the idea of hyperfine structure to the F-hierarchy. Since only $\mathbf{S}_0$-formulas are considered in constructing the next F-level, elementarity of important maps is achieved easier than in the classical development, leading to a more general and flexible approach.

There are many other potentially interesting modifications to classical hyperfine structure theory; for example, it is not necessary that the underlying enumeration of formulas is of order type ω. Limits in this enumeration would rather lead to an extra, but harmless limit case in applications. This way, hyperfine structure can be made compatible with the Levy-hierarchy, which could become relevant for the definition of fine-structural ultrapowers for levels of a hyperfine hierarchy. In this section, we will define a family of hyperfine structures rather than a single one. We exhibit in some detail three natural examples, thus proving versions of Jensens square principle.

9.2 Basic notions and facts

The standard F-hierarchy consisted of levels F_α, α an ordinal together with three-place Skolem functions $S(F_\beta, \phi, p)$, where ϕ is a $\mathbf{S}_0$-formula and $p \subset F_\beta$, $card(p) < \omega$ is a parameter. Our goal here is to insert intermediate structures by approximating the Skolem- and interpretation functions available at the next level. Let $\langle \phi_i \rangle_{i \in \omega}$ be any effective enumeration of the $\mathbf{S}_0$ formulas fixed for the rest of this section. (Note, however, that we do not require the closure of inital segments under subformulas or any other restrictions.) For a $\mathbf{S}_0$ formula ϕ, we denote by $[\phi]_{free}$ the number of free variables in ϕ.

Definition 1: *A* ***location*** *is a countable sequence of the form $\langle F_\alpha, x_1, x_2, ... \rangle$, where each x_i is either a finite sequence of elements of $\underline{F_\alpha}$ or Ω. If the ordinal in the first position is β, it is a β-****location****. If $s = \langle F_\alpha, x_1, x_2, ... \rangle$ is a location, the set $supp(s) := \{i \in \omega | x_i \neq \vec{0}\}$ is called the* ***support*** *of s.*

For convenience, we abbreviate the location $s = \langle \alpha, x_1, ..., x_n, ... \rangle$, where $x_i = \vec{0}$ for $i \neq n$ by $l^n_\alpha(x_n)$; also, $l_\alpha(x)$ means $\langle \alpha, x, x, ... \rangle$. The condensation lemma is a basic tool for any fine structure. We have to ensure that hulls in our hierarchy condense to levels of the hierarchy. By the condensation lemma for the F-levels, this can be achieved for structures closed under I, S and naming. This of makes sure that we will always have minimal names. In the context of hyperings, the unrestricted use of the naming function leads to technical problems, and we had to exclude it from the basic operations. Hence, some extra care is necessary to ensure that we still have names that are not necessarily minimal as the term was defined, but still minimal in some sense.

Definition 2: *A name $n = \langle F_\alpha, \phi, p\rangle$ is* ***appropriate*** *if there is no name $m = \langle F_\beta, \psi, p_1\rangle$ such that $I(n) = I(m)$ and $\beta < \alpha$. A name is an F-name if the parameter p consists only of F-stages; minimal and appropriate F-names have the obvious meaning. A set $X \subseteq L$ is said to be closed under appropriate F-names, abbreviated c.a.n., if it contains the componenents of an appropriate F-name for each of its elements.*

$N(x)$ is appropriate, of course. We can now define the structures we will be using; restriction of a function in an α-location to Ω means that the function is applied to the whole of F_α. When it is not clear from the context where the functions act, we will add this as an upper index; so the Skolem function $S(F_\alpha, \phi, y)$ in F_α for $\mathbf{S}_0$-formulas ϕ restricted to elements $y <_{lex} x$ will be $S^\alpha_\phi|x$. $I^\alpha_\phi(x)$ is $I(F_\alpha, \phi, x)$, if this is an element of F_α, and otherwise $\emptyset$, of course.

Definition 3: *Let $s = \langle F_\alpha, x_1, ...\rangle$ be a location. The* ***structure corresponding to s*** *is defined thus:*
$\langle F_\alpha, I, S, I|x_1, N \circ I|x_1, S|x_1, N \circ S|x_1, I|x_2, N \circ I|x_2, S|x_2, N \circ S|x_2, ...\rangle$
For $X \subset F_\alpha$, the s-hull of X, written $F_s\{X\}$, is the closure of X under all functions belonging to s.

Obviously, any set of the form $F_\alpha \cup \{F_{\beta_1}, ..., F_{\beta_n}\}$ is c.a.n.. Note that $I(x, y = y, \emptyset) = x$ characterizes levels of the F-hierarchy. Being c.a.n. is inherited from sets to their closures:

Proposition 4: *Let $X \subseteq L$ be c.a.n., s a location. Then $H := F_s\{X\}$ is c.a.n. as well.*

Proof: Each element of H is generated from elements of X by a finite series of applications of S, I and their restrictions. For the restricted functions, minimal, hence appropriate names are added by definition. Let $x = I(F_\beta, \phi, p)$, where F_β and (the components of) p have been generated before; if $\langle F_\beta, \phi, p\rangle$ is appropriate, we are done. Suppose otherwise, thus $x \in F_\beta$. Then $S(F_\beta, I(a, b, c) = d \wedge [a \text{ is an } F\text{-stage}] \wedge [c \text{ consists of } F\text{-stages}], \{x\})$ will find a minimal, hence an appropriate name. The fact that c consists of F-stages can be expressed as a $\mathbf{S}_0$-statement since c will be of fixed finite

cardinality, depending on ϕ.
Now suppose $x = S(F_\beta, \phi, p)$, the components being generated before. Then $x \in F_\beta$ by definition of S, so we can proceed as we did for I. In any case, we will have an appropriate name for x in our hull.□

From this, a weak condensation lemma already follows.

Proposition 5: *Let $X \subseteq F_\alpha$ be c.a.n., s an α-location and $\pi : F_s\{X\} \to M$ the Mostowski collapse, thus M is transitive. Then there is an ordinal β such that $M = F_\beta$ and π preserves I, S, $\in$ and $<_L$.*

Proof: By Proposition 4, $H := F_s\{X\}$ is closed under I, S and c.a.n.; so we are in the situation of the standard condensation lemma for the fine hierarchy, and the result follows.□

For arguments in HFS, it will be essential to canonically order (a subclass of) the locations. That this can be done in multiple ways gives our approach a nice flexibility. However, any reasonable ordering will respect the following.

Definition 6: *Let $s_1 = \langle \alpha, x_1, x_2, ... \rangle$ and $s_2 = \langle \beta, y_1, y_2, ... \rangle$ be locations. Then $s_1 <_{loc} s_2$ if $\alpha < \beta$ or $\alpha = \beta$ and $x_i \leq_{lex} y_i$ for $i \in \omega$.*

When passing to $<_{loc}$-larger locations, the corresponding hyperfine hulls are $\subset$-increasing:

Lemma 7 (Monotonicity):

- *(a) Suppose $s_1 <_{loc} s_2$ are α-locations, $X \subseteq F_\alpha$. Then $F_{s_1}\{X\} \subseteq F_{s_2}\{X\}$.*
- *(b) Suppose $X \subseteq F_\alpha$, s_1 and s_2 are α- and β-location, respectively, $\alpha < \beta$. Then $F_{s_1}\{X\} \subseteq F_{s_2}\{X \cup \{F_\beta\}\}$.*

Proof:

- (a) This is obvious, since we simply form a hull under extended functions.
- (b) Skolem functions and naming in F_β can be carried out on the RHS by using $S(F_\beta, \_, \_)$.

□

Definition 8: *Let $s_1 = \langle \alpha, x_1, ... \rangle, s_2 = \langle \alpha, y_1, ... \rangle$ be locations. Then s_2 is a successor of s_1 if $y_i = x_i$ or y_i is the immediate $<_{lex}$-successor of x_i for all $i \in \omega$. A chain from s_1 to s_2 is a sequence $\langle s_1 = l_0, l_1, ..., l_\gamma = s_2$ of α-locations such that for each $\beta^+ \leq \gamma$, l_{β^+} is a successor of l_β and for $lim(\beta)$, $\beta \leq \gamma$, $l_\beta =<_{loc}$-$sup\{l_\iota | \iota < \beta\}$.*

Note that a location can well be a successor and a successor of a successor of another location at the same time.

Proposition 9: *Let $X \subseteq F_\alpha$, s_1, s_2 be α-locations, where s_2 is a successor of s_1. Then there is a countable set p such that $F_{s_2}\{X\} \subseteq F_{s_1}\{X \cup p\}$. If $x_i = y_i$ for all but finitely many $i \in \omega$, there is a finite p with this property.*

Proof: Just let p be the set of the new values of the restricted Skolem functions added at the places where the scopes grow.□

So, we have monotonicity, a weak condensation property and an analogue of finiteness already in this general setting. However, to get continuity and a condensation lemma strong enough for applications like e.g. the square-principle, we have to specify further the kind of structure we are going to work in.
$\bar{S}$ denotes the language obtained from the usual language for the F-hierarchy by adding the function symbols $\alpha(s)$ and $p_n^i(s)$ giving the ordinal stage and the i-th element of the nth element of the location s, respectively. Also, denote by $p_n(s)$ the nth element of the location s. By $\bar{\mathbf{S}}_0$, we denote the quantifier-free formulas of $\bar{S}$. A set A of α-locations is $\bar{\mathbf{S}}_0$-definable over F_α if there is an $\bar{\mathbf{S}}_0$-formula ψ such that $s \in A \leftrightarrow \psi(s)$. A class of locations is uniformly $\bar{\mathbf{S}}_0$-definable if the sets of its α-locations are $\bar{\mathbf{S}}_0$-definable using the same ψ for each α. (In other words, if $s \in A \leftrightarrow F_{\alpha(s)} \models \psi(s)$ for some $\bar{\mathbf{S}}_0$-formula ψ.)

We will often have to apply embeddings of structures to locations. Since the locations are in general not elements of the structures, we make the following convention:

Convention: If $s = \langle \alpha, x_1, x_2, ... \rangle$ is an α-location, $\pi : F_\alpha \to F_\beta$ a fine map, then $\pi(s)$ denotes the β-location $\langle \beta, \pi(x_1), \pi(x_2), ... \rangle$.

Definition 10: *Suppose A is a class of locations, $<_H \subset A \times A$. Then $H := \langle A, <_H \rangle$ is a linear hypering of L if the following axioms are satisfied:*

1. (1) $<_H = <_{loc} |_H$ and the latter is a well-order
2. (2) For each location $l^n_\alpha(x)$, there is an α-location $s = \langle \alpha, y_1, y_2, ... \rangle$ in H such that $y_n = x$
3. (3) For each ordinal α, both $l_\alpha(\vec{0})$ and $l_\alpha(\Omega)$ are in H
4. (4) For $S \subset H$ a set of α-locations, $<_{loc}$-$sup\{S\} \in H$
5. (5) If $s_1, s_2 \in H$ are α-locations, H contains a (naturally unique) chain from s_1 to s_2
6. (6) If $\alpha < \beta$, $\pi : F_\alpha \to F_\beta$ a fine map, then $s \in H_\alpha$ iff $\pi(s) \in H_\beta$.

We say that a linear hypering $H_1 = \langle A_1, <_1 \rangle$ extends another linear hypering $H_2 = \langle A_2, <_2 \rangle$, written $H_1 \preceq H_2$, if $A_1 \subseteq A_2$.

Remark 1: There are also quite natural ways to define non-linear hyperings of L, i.e. classes H of locations for which (1) fails so that $<_{loc}$ is not total on H. However, it is not clear how to give a general treatment of these; some things, such as condensation, appear to require some new work in each case. We comment on these later; meanwhile, linear hyperings are completely sufficient for our needs.
Remark 2: (2) follows from (3) and (5); we add it to make the concept more transparent.
Remark 3: While (6) is apparently a convenient property, it is not used in our development of hyperfine structure; in particular, the condensation lemma can be proved without that assumption. So far, we have not encountered an example of a natural hypering that is not uniformly $\bar{\mathbf{S}}_0$-definable; but if necessary, it seems that this requirement might be dropped without difficulty. This changes when we work with general hyperings in the section on morasses, where it is needed to obtain the soundness property of the hypering. However, (6) seems to be a rather 'external' criterion. An alternative would be to demand that the set of α-locations in H is uniformly $\bar{\mathbf{S}}_0$-definable over F_α. Note, however, that $\bar{\mathbf{S}}_0$-definability is a considerably stronger property. (6) would e.g. still hold if H_α was given by a countable

scheme of such statements rather than merely a single one.

H_β will denote the set of β-locations in H. Now, the basic properties of HFS can be shown to hold for any hypering of L.

Definition 11: *Let H be a hypering of L, $s \in H$ an α-location, $X \subseteq F_\alpha$, and suppose that $\pi : F_s\{X\} \to F_\beta$ is the transitive collapse. Then $\bar{s}$, the condensed structure of X in s relative to H is defined as* $<_H$*-sup*$\{t \in H_\beta | \pi^{-1}(t) <_H s\}$.

Lemma 12 (Condensation): *Let $H = \langle A, <_H \rangle$ be a hypering of L, $s \in A$ an α-location, $X \subseteq F_\alpha$, $\bar{s}$ the condensed structure of X in s relative to H. Then $\pi : F_s\{X\} \to F_{\bar{s}}$ is an isomorphism, where π is the Mostowski collapse.*

Proof: By proposition 5, we already have that the image of π is an F-stage, say $F_{\bar{\alpha}}$, as well as preservation of $I, S, \in, <_L$. Preservation of minimal and appropriate names follows easily. It remains to show preservation for restricted Skolem- and intepretation functions.
So suppose $l^n_\alpha(x) \leq \bar{s}$ By (5) of Def.10, there is $r \in H$ such that $r <_{loc} \bar{s}$ and the n-th parameter of r is x. So $\pi^{-1}(r) = \langle \alpha, \pi^{-1}(x_1), \pi^{-1}(x_2), ... \rangle <_{loc} s$, since π preserves $<_L$ and hence $<_{lex}$. By (6) of Def.10, $\pi^{-1}(r) \in H_\alpha$. Then $a := I^\alpha_\phi(\pi^{-1}(x)) \in X$ and for some $b \in F_\alpha$, we have $F_\alpha \models \phi(b, \pi^{-1}(x)) \leftrightarrow F_{\bar{\alpha}} \models \phi(\pi(b), x)$. So $\pi(I^\alpha_\phi(\pi^{-1}(x)) = \pi(\{b \in F_\alpha | F_\alpha \models \phi(b, \pi^{-1}(x))\}) = \{\pi(b) \in F_{\bar{\alpha}} | F_{\bar{\alpha}} \models \phi(\pi(b), x)\} = I^{\bar{\alpha}}_\phi(x)$.
Furthermore, $a := S^\alpha_\phi(\pi^{-1}(x)) \in X$ and $F_\alpha \models \phi(a, \pi^{-1}(x) \leftrightarrow F_{\bar{\alpha}} \models \phi(\pi(a), x)$. That $\pi(a)$ is $<_L$-minimal follows from the preservation of $<_L$ under π, as well as the minimality of the condensed F-names. □

Remark: Note that this proof would still work if $\pi^{-1}(r)$ was any location $<_{loc} s$, not necessarily contained in H.

Definition 13: *A linear hypering H is called slow if for the successor s^+ of $s \in H$ in $<_H$, we have $s = s^+$ almost everywhere (i.e. only finitely many scopes move).*

Corollary 14: *For a slow linear hypering, Proposition 9 is true with "finite" instead of "countable".*

Lemma 15 (Continuity):

1. *Suppose* $lim(\alpha)$, $X \subseteq F_\alpha$; *then* $F_{l_\alpha(0)}\{X\} = F\{X\} = \bigcup_{\beta<\alpha} F_{l_\beta(0)}\{X \cap F_\beta\}$.

2. *Let* $X \subset F_\alpha$, $s := l_{\alpha+1}(0)$; *then* $F_s\{X \cup \{F_\alpha\}\} \cap F_\alpha =$
$F\{X \cup \{F_\alpha\}\} \cap F_\alpha = \bigcup\{F_r\{X\} | r <_{loc} s, r \in H_\alpha\}$

3. *Let* $s \in H$ *be a* $<_{loc}$*-limes not of the form* $l_\alpha(0)$, $X \subseteq F_\alpha$; *then* $F_s\{X\} = \bigcup\{F_r\{X\} | r \in H_\alpha, r <_{loc} s\}$.

Proof:

1. The hull is in this case just the closure under I, N and S; since any element of the LHS is generated by finitely many elements, it is already in such a hull of an initial segment, all of which are considered on the RHS.

2. The first equality follows as in the first part, since only I and S are considered. For the second, set $V := F\{X \cup \{F_\alpha\}\} \cap F_\alpha$ and $U := \bigcup\{F_r\{X\} | r <_{loc} s, r \in H_\alpha\}$.
(1) $U \subseteq V$
Each $z \in U$ is generated from elements of X by finite application $I, S, N \circ I_{\phi_i}|\Omega, N \circ S_{\phi_i}|\Omega$. Closure under I and S is clear, the restricted functions are given on the LHS by $I(F_\alpha, \phi_i, -)$ and $S(F_\alpha, \phi_i, -)$. By the results in the section on the fine hierarchy, the LHS is also closed under appropriate names; hence, the components of minimal names can be found using S.
(2) $V \subseteq U$
$z \in V$ is computed from elements of $X \cup F_\alpha$ by a finite sequence of applications of I and S. Suppose we have $\langle y_1, ..., y_n = z\rangle$, where each y_i is either an element of $X \cup F_\alpha$, or $z \in \{S(y_j, \phi, x)\} \cup \{I(y_j, \phi, x)\}$, where $j < i$ and the components of x appear earlier than y_i in the generating sequence. We prove by induction on i that if $y_i \in F_\alpha$, then $y_i \in U$. We distinguish three cases, depending on how y_i was generated.

   - $y \in X \cup F_\alpha$. Then either $y \in X$, in which case obviously $y \in U$, or $y = F_\alpha$ and so $y \notin F_\alpha$.
   - $y = I(F_\beta, \phi, x)$. By induction, $x \subset U$. If $\beta < \alpha$, we have $F_\beta \in U$ again by induction, so $I(F_\beta, \phi, x) \in U$ since U is closed under I.

If $\beta = \alpha$, but still $y \in F_\alpha$ (so we have taken a non-appropriate name), the restricted interpretation function $I^\alpha_\phi(x)$ generates the same element in U in some location on the RHS.

- $y = S(F_\beta, \phi, x)$; by induction, $x \subset U$. Now, if $\beta < \alpha$, $F_\beta \in U$ by induction, so $y \in U$ since U is closed under S. If $\beta = \alpha$, $S(F_\alpha, \phi, x) = S^\alpha_\phi(x) \in U$.

3. Since there is a chain in H from $l_\alpha(0)$ to s, all scopes on the LHS appear also on the RHS.

□

Remark: Trivially, any location also satisfies the following compactness property: Let $X \subset L_s$, $y \in L_s\{X\}$. Then there is a finite $x \subset_{fin} X$ such that $y \in L_s\{x\}$. (This follows from the fact that each element of the hull is generated by finite applications of s-operations to elements of X.)

When we make use of hyperings in our constructions, we will sometimes consider maps between locations that preserve more than just the basic operations of these locations. This will be particularly relevant for the construction of a morass. Therefore, we make the following definitions:

Definition: *If s is a location, then an s-formula is a first-order formula in the basic functions of s.*

Definition: *If s is a location, ψ an s-formula, $\pi : F_s \to F_t$ an embedding, then ψ_π, the π-mapped formula ψ, is obtained by replacing in ψ any function symbol of the form $S^{F_s}_\phi|\vec{q}$ by $S^{F_t}_\phi|\pi[\vec{q}]$, similarly for the other basic functions.*

Convention: When π is clear from the context, the superscript is usually dropped and we write ψ for the corresponding formula on both sides.

Definition: *Let $s = \langle \alpha, x_1, x_2, ... \rangle$ be a location. A Σ^s_0-formula is a quantifier-free formula in the operations associated with s.*
A formula of the form $\exists y \phi(y, p)$ with $p \subset_{fin} F_\alpha$ is a Σ^s_1-formula if ϕ is a Σ^s_0-formula.
An embedding $\pi : F_\alpha \to F_\beta$ is Σ^s_1-preserving iff, for any Σ^s_1-formula ψ, $p \subset_{fin} F_\alpha$, we have $F_\alpha \models \psi(p) \leftrightarrow F_\beta \models \psi(\pi(p))$.

9.2.1 Examples

To illustrate the above general notions, we give three linear and two non-linear hypering of L. In the next section, we let the usual argument for square in HFS run through the first three of them to illustrate how they work.

First example: The classical hypering
Let $H_1 := \{c^n_\alpha(x) := \langle \alpha, \Omega, \Omega, ..., \Omega, x, \vec{0}, \vec{0}, ...\}$ consist of all locations that allow unrestricted Skolem functions for the first n formulas, then a restriction to $\vec{x} \subset F_\alpha$ for the $(n+1)$th, and the empty function for the rest. The conditions for being a hypering are clearly fulfilled. H_1 is modelled after the original HFS of Friedman and Koepke for the L_α-hierarchy and is therefore referred to as the classical hypering.

Second example: The horizontal hypering
$H_2 := \{l_\alpha(x) | Ord(\alpha), x \subset F_\alpha\}$. This is in my view the most natural hypering; it is generated by simultanously increasing the restrictions step by step. The main problem with this hypering is that it is not slow.

Third example: The slowed-down horizontal hypering
As the name suggests, H_3 is defined similar to H_2, but between $l_\alpha(x)$ and $l_\alpha(x^+)$, we place the ω many locations $\langle \alpha, x^+, x, ...\rangle$, $\langle \alpha, x^+, x^+, x, ...\rangle$ etc. This is still rather natural, but superior to H_2 in being slow (thus satisfying the finiteness property).

Now for the non-linear examples:
Singleton fine structure
$H_4 := \{l^n_\alpha(x) | Ord(\alpha), x \subset F_\alpha\}$, with $r <_{H_4} s$ iff either $r <_{loc} s$ or $r = l^n_\alpha(x)$, $s = l^m_\alpha(y)$ and $n < m$. Singleton fine structure and its extensions are especially interesting since they measure very accuratly the complexity of constructible sets.

The finite support hypering
$H_5 := \{\langle \alpha, x, ..., x, \vec{0}, \vec{0}, ...\rangle | Ord(\alpha), x \subset F_\alpha\}$. To achieve unique minimal elements in any set of α-locations, we expand the order $<_{loc}$ by sorting finite sets of naturals by $<_{lex}$ and letting $s_1 <_{H_5} s_2$ iff either $supp(s_1) <_{lex} supp(s_2)$ or $supp(s_1) = supp(s_2)$ and $s_1 <_{loc} s_2$.

10 Two proofs of square

In this section, $sngl(\alpha)$ means that α is singular.

Theorem 16: *Suppose $V = L$. Then, there is $\langle C_\beta | sngl(\beta)\rangle$ such that*

- *(a) C_β is club in β*
- *(b) $otp(C_\beta) < \beta$*
- *(c) If $\bar{\beta}$ is a limit point of C_β, then $sngl(\bar{\beta})$ and $C_{\bar{\beta}} = \bar{\beta} \cap C_\beta$*

We present two proofs, using the classical and the slowed-down horizontal hypering. Both proofs have the basic strategy as well as some initial notions and lemmata in common; so we prove these first. Let H be a linear hypering for the rest of this paragraph.

Definition 17: *Let $s \in H$, $\alpha := \alpha(s)$, $\vec{p} \in F_\alpha$ finite, $\beta \leq \alpha$; then $fix(s, \beta, \vec{p}) := \{\bar{\beta} < \beta | \bar{\beta} = F_s\{\bar{\beta} \cup \vec{p}\} \cap \beta\}$ is the fixed point set of s below β relative to $\vec{p}$.*

Convention: We usually suppress the vector sign for $\vec{p}$ when the nature of the parameter is clear from the context. Recall that, for a set X, $X \downarrow_\beta$ means that X is bounded in β, i.e. there is $\bar{\beta} < \beta$ such that $X \cap \beta \subseteq \bar{\beta}$.

Lemma 18: *Suppose β is singular in L. Then there is a location $\langle \alpha, x_1, ..., \rangle \in H$ and a finite $\vec{p} \subset F_\alpha$ such that $fix(s, \beta, \vec{p}) \downarrow_\beta$*

Proof: By singularity, choose $f \in L$, $f : \alpha \to \beta$ cofinally, $\alpha < \beta$, and $\bar{\gamma}$ such that $f \in F_{\bar{\gamma}}$. Obviously $\bar{\gamma} \geq max\{\alpha, \beta\}$. Now $f(x) = y \leftrightarrow F_{\bar{\gamma}} \models \exists z(z \in f \wedge z = \langle x, y\rangle)$. This is equivalent to an $\mathbf{S}_0$-formula ϕ_i over $\bar{\gamma} + 2 =: \gamma$. Then images of each $\iota < \alpha$ under f can be found by $S(F_{\bar{\gamma}}, \phi, ...)$, and these are cofinal in β. Now $s = l_\gamma(0)$ (which is in H by (3) of Def. 10) and $\vec{p} = \{f, F_{\bar{\gamma}}\}$ witnesses our statement. □

We already know that we can restrict ourselves to parameters $\vec{p}$ consisting of F-stages only, which we do. Choose a $<_{lex}$-minimal $s \in H$ and then a $<_L$-minimal $\vec{p}$ satisfying Lemma 18. Each approach to the square principle is an attempt to approximate s in a canonical way to define the C_β.

Lemma 19: *If H is slow, then s is a limit location. This holds as well if $cf(\beta) > \omega$, regardless whether H is slow or not.*

Proof: If s is an immediate successor of $r \in H$ in a slow hypering, then by Proposition 9 (Finiteness), there is a finite set q such that $F_s\{\bar{\beta} \cup p\} \subseteq F_r\{\bar{\beta} \cup p \cup q\}$. But then, we have $\{\bar{\beta} < \beta | F_r\{\bar{\beta} \cup p \cup q\} \cap \beta = \bar{\beta}\} \subseteq \{\bar{\beta} < \beta | F_s\{\bar{\beta} \cup p\} \cap \beta = \bar{\beta}\} \downarrow_\beta$, which contradicts the minimality of s.
Now assume $cf(\beta) > \omega$ and let $s = t^+$ be a successor. We can then use the countability property to obtain a countable set c such that
$\{\bar{\beta} < \beta | F_t\{\bar{\beta} \cup c\} \cap \beta = \bar{\beta}\} \downarrow_\beta$. (Just take the set of values of the basic functions at the places where the s is larger than t.) As $cf(\beta) > \omega$, there is $\tilde{\beta} < \beta$ with $F_t\{\tilde{\beta} \cup c\} \uparrow_\beta$. (Otherwise, form for $\bar{\beta} < \beta$ the sequence given by $\beta_0 = \bar{\beta}$, $\beta_{i+1} = \sup F_t\{\beta_i \cup c\} \cap \beta$. Then $\bigcup_{i \in \omega} \beta_i =: \beta' < \beta$ as β has uncountable cofinality, but β' is a fixpoint. Contradiction.) Now, if $c|i$ denotes the first i elements of c in some enumeration of c in order-type ω, we have that $\langle \sup F_t\{\tilde{\beta} \cup c|i\} \cap \beta \rangle_{i \in \omega} \uparrow_\beta$ forms a cofinal ω-sequence in β, contradicting again our assumption. Hence s cannot be a successor.
□

10.1 The classical hypering

We start with a proof of full square in H_1, along the lines of [16]. Modifications are e.g. due to the fact that our restricted Skolem functions only allow search for witnesses for and interpretation of $\mathbf{S}_0$-formulas.

Claim 1: $s \neq c^0_\beta(\vec{0})$

Proof: Suppose otherwise, and choose (since β is limit and p is finite) $\beta_0 < \beta$ such that $p \subset F_{\beta_0}$. For $\beta_0 \leq \bar{\beta} < \beta$, we have:
$\bar{\beta} \subseteq F_s\{F_{\bar{\beta}} \cup p\} \cap \beta \subseteq F\{F_{\bar{\beta}} \cup p\} \cap \beta \subseteq F_{\bar{\beta}} \cap \beta = \bar{\beta}$, which contradicts the choice of s and p. □

Claim 2: *$s \neq c^0_\gamma(\vec{0})$ for γ a limit ordinal.*

Proof: Choose $\gamma_0 < \gamma$ such that $p \subset F_{\gamma_0}$ and let $s_0 = c^0_{\gamma_0}(\vec{0})$. Then it follows that:
$\{\bar{\beta} < \beta | \bar{\beta} = \bar{\beta} \cap F_{s_0}\{F_{\bar{\beta}} \cup p\}\} = \{\bar{\beta} < \beta | \bar{\beta} = \beta \cap F_s\{F_{\bar{\beta}} \cup p\}\} \downarrow \beta$, which contradicts the minimal choice of s. $\Box$

Special Case I: $s = c^0_{\alpha+1}(\vec{0})$ for some ordinal α.

Since p consists entirely of F-stages, we must have $F_\alpha \in p$; for otherwise, $\bar{s} := c^0_\alpha(\vec{0})$ would lead to the same hull, contradicting the minimality of $s >_{loc} \bar{s}$. Set $q := p - \{F_\alpha\}$.
Now define $\beta_0 := max\{\bar{\beta} < \beta | \bar{\beta} = \beta \cap F_s\{F_{\bar{\beta}} \cup p\}\} < \beta$ by assumption. For β_n given, let $\beta_{n+1} < \beta_n$ be minimal with the property that $\beta_{n+1} = \beta \cap F_{c^n_\alpha(\vec{0})}\{\beta_{n+1} \cup p\}$. By minimality of s, we have $\beta_n < \beta$ for $n \in \omega$. Set $\beta_\omega := \bigcup_{n\in\omega} \beta_n$. Thus:
$\beta \cap F_s\{\beta_\omega \cup p\} = \beta \cap F_s\{F_{\beta_\omega} \cup q \cup \{F_\alpha\}\}$
$= \beta \cap \cup\{F_r\{F_{\beta_\omega \cup q}\} | r \in H_{1,\alpha}\} = \bigcup_{n\in\omega} \beta \cap F_{c^n_\alpha(\vec{0})}\{F_{\beta_\omega} \cup q\}$
$= \bigcup_{n\in\omega} F_{c^n_\alpha(\vec{0})}\{F_{\beta_{n+1}} \cup q\} = \bigcup_{n\in\omega} \beta_{n+1} = \beta_\omega$.
Now $\beta_\omega > \beta_0$, so we must have $\beta_\omega = \beta$; thus, if we let $C_\beta := \{\beta_n | n \in \omega\}$, we are done in this case.

From now on, we assume $s = c^n_\gamma(\vec{x}) \neq c^0_\gamma(\vec{0})$.

Claim 3: *There is a finite set $\bar{p} \subset F_\gamma$ such that $F_s\{F_\beta \cup \bar{p}\} = F_\gamma$.*

Proof: $\pi : F_s\{F_\beta \cup p\} \to_{coll} F_{\bar{s}}$. Then $F_{\bar{s}} = F_{\bar{s}}\{F_\beta \cup \bar{p}\}$, where $\bar{p} := \pi p$.
Since $\pi|_{F_\beta} = id$ (F-stages are preserved), this implies $\beta \cap F_s\{F_{\bar{\beta}} \cup p\} = \beta \cap F_{\bar{s}}\{F_{\bar{\beta}} \cup \bar{p}\}$. Thus:
$\{\bar{\beta} < \beta | \bar{\beta} = \beta \cap F_{\bar{s}}\{F_{\bar{\beta}} \cup p\}\} = \{\bar{\beta} < \beta | \bar{\beta} = \beta \cap F_s\{F_{\bar{\beta}} \cup p\}\} \downarrow_\beta$, so $\bar{s} = s$ by minimality of s. Thus $F_s = F_s\{F_\beta \cup \bar{p}\} = F_\gamma$. $\Box$

We choose $p(\beta) \subset F_\gamma$ $<_{lex}$-minimal with this property. In particular, we have $p \subset F_s\{F_\beta \cup p(\beta)\}$, which immediatly gives us:

Claim 4: $\{\bar{\beta} < \beta | \bar{\beta} = \beta \cap F_s\{F_{\bar{\beta}} \cup p(\beta)\}\} \downarrow_\beta$.

Let $\beta_0 < \beta$ be the maximum of this set. Since $p(\beta)$ satisfies the properties we demanded of p, we set $p := p(\beta)$ from now on. A hull $F_s\{X\}$ contains parameters that can serve as components for locations; for $Y \subset F_\gamma$, we write $c^n_\gamma(\vec{y}) =: r \in Y$ for $\vec{y} \in Y$ (which, as we recall, in turn means that any component of $\vec{y}$ is in Y).

Proposition 20: *If $\bar{s}$ is the l.u.b. of $Y = F_s\{X\}$, then $F_s\{X\} = F_{\bar{s}}\{X\}$.*

Proof: Only the $\bar{s}$-operations are relevant for $F_s\{X\}$. Nothing else can be considered in the hulling process. □

Special Case II: $F_s\{F_\alpha \cup p\} \downarrow_s$ for $\alpha < \beta$.

In this case, define β_0 as in special case I and let $\beta_{n+1} = \bigcup\{\beta \cap F_s\{F_{\beta_n} \cup \{F_{\beta_n}\} \cup p\}\}$ By assumption, there is $r <_{loc} s$ such that $F_s\{F_{\beta_n} \cup \{F_{\beta_n}\} \cup p\} = F_r\{F_{\beta_n+1} \cup p\}$. But then, $\beta \cap F_r\{F_{\beta_n+1} \cup p\} \downarrow_\beta$, or s would not be minimal. Therefore, $\beta_{n+1} < \beta$; let $\beta_\omega = \bigcup_{n\in\omega} \beta_n$, then $\beta_\omega > \beta_0$ and:
$\beta_\omega \subseteq \beta \cap F_s\{F_{\beta_\omega} \cup p\} \subseteq \bigcup_{n\in\omega} \beta \cap F_s\{F_{\beta_n+1} \cup p\} \subseteq \bigcup_{n\in\omega} \beta_{n+1} = \beta_\omega$,
so $\beta_\omega = \beta$. So, once again, $C_\beta := \{\beta_n | n \in \omega\}$ will work.

So, from now on, we consider only situations where $F_s\{F_{\alpha_0} \cup p\} \uparrow_s$ for some $\alpha_0 < \beta$; choose such an $\alpha_0 = \alpha_0(\beta)$ minimal. If α_0 is not a limit or 0, we add its predecessor α'_0 to the parameters and find a (smaller) $\alpha_1(\beta)'$ with the required property for the new parameter. We continue with this process, constructing a falling sequence of ordinals, so we have to stop at some point; thus $\alpha_k(\beta)'$ is limit or 0 for some $k \in \omega$.

Special Case III: $\alpha_k = 0$.

Then $F_s\{F_{\alpha_k} \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\} = F_s\{p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$ is obviously countable, so there is a countable sequence $\langle s_n | n \in \omega\rangle \uparrow_s$. Now, define β_0 in the same way as above and let $\beta_{n+1} > \beta_n$ be minimal such that $\beta_{n+1} = \beta \cap F_{s_{n+1}}\{F_{\beta_{n+1}} \cup p\}$, which is $< \beta$ since s is minimal. Again, let $\beta_\omega := \bigcup_{n\in\omega} \beta_n$, then:
$\beta_\omega = \bigcup_{n\in\omega} \beta_{n+1} = \bigcup_{n\in\omega} \beta \cap F_{s_{n+1}}\{F_{\beta_{n+1}} \cup p\} = \beta \cap F_s\{F_{\beta_\omega} \cup p\}$, so, once again, $\beta_\omega = \beta$, and we can set $C_\beta := \{\beta_n | n \in \omega\}$.

The Generic Case: $lim(\alpha) \neq 0$, $s = c^n_\gamma(x) \neq c^0_\gamma(\vec{0})$

Then define $\{\beta_i(\beta)|i \leq \alpha\}$ and $\{s_i|0 < i \leq \alpha\}$ as follows:
First, let β_0 be defined as before. Now, for $0 < i \leq \alpha$, let
$s_i :=<_{loc} -sup\{F_s\{i \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}\} <_{loc} s$
and $\beta_0 < \beta_i = \beta_i(\beta)$ minimal with:
$\beta_i = \beta \cap F_{s_i}\{F_{\beta_i} \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$.
For $i < \alpha$, we have $\beta_i < \beta$ by minimality of s since $s_i < s$. $s_\alpha = s$ (by assumption) and $\beta_\alpha = \beta$, of course. Without loss of generality, we assume that the indices i start at the point where the s_i become γ-locations. Furthermore:

Claim 5: *$0 < i < j < \alpha$ implies $s_i <_{loc} s_j$ and $\beta_i \leq \beta_j$. (Remember that we are only dealing with γ-locations!)*

Claim 6: *$\{\beta_i|i < \alpha\}$ is club in β.*

Proof: Let $\bar{\alpha} \leq \alpha$ be a limit ordinal. We show $\beta_{\bar{\alpha}} = \bigcup_{i<\bar{\alpha}} \beta_i$. Since $\beta_{\bar{\alpha}} \geq \beta_i$ for $i < \bar{\alpha}$, it suffices to see that:
$\bigcup_{i<\bar{\alpha}} \beta_i = \bigcup_{i<\bar{\alpha}} \beta \cap F_{s_i}\{F_{\beta_i} \cup p \cup \{F_{\alpha_0}, ..., F_{\alpha_{k-1}}\}\}$
$= \beta \cap F_{s_{\bar{\alpha}}}\{\bigcup_{i<\bar{\alpha}} F_{\beta_i} \cup p \cup \{F_{\alpha_0}, ..., F_{\alpha_{k-1}}\}\}$, so $\bigcup \beta_i$ satisfies the definition of $\beta_{\bar{\alpha}}$, and is hence by minimality equal to $\beta_{\bar{\alpha}}$, so $\beta_{\bar{\alpha}} = \bigcup \beta_i$. $\Box$

Now we restrict the s_i to an appropriate endsegment to ensure absoluteness. Let $I(\beta)$ be the set of all i such that:

1. $\beta_i \geq max\{\alpha_0, ..., \alpha_{k-1}\}$ (this is possible since $\langle\beta_i|i < \alpha\rangle \uparrow_\beta$)
2. $\beta_j > j$ (Recall that we have $j < \alpha < \beta$)
3. The upper bounds of the intermediate construction steps $F_s\{F_{\alpha_l} \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{l-1}}\}\}$ are $<_{loc} s_i$
4. In case $\beta < \gamma$, we have $F_\beta \in F_{s_i}\{F_{\beta_i} \cup p\}$

$I(\beta)$ is easily seen to be an endsegment of α. Now set:
$C_\beta := \langle\beta_i|i \in I(\beta)\rangle$. Since C_β is just a 'shortening' of a club with the same properties, one immediatly gets:

Claim 7: *C_β is club in β, $otp(C_\beta) \leq \alpha < \beta$.*

It remains to show the coherence property, namely that for $\bar\beta$ a limit point of C_β, then $C_{\bar\beta} = \bar\beta \cap C_\beta$. So assume $\bar\beta$ to be a limit point of C_β, $\bar\alpha$ minimal such that $\beta_{\bar\alpha} = \bar\beta$. Then $\bar\alpha$ is a limit ordinal, $\beta_{\bar\alpha} > \bar\alpha$ (by (2) of the definition of $I(\beta)$) and β belongs to the generic case (for otherwise, C_β would be of order type ω and hence there could be no limit points).
Consider $\pi : F_{s_{\bar\alpha}}\{F_{\bar\beta} \cup p\} \rightarrow_{coll} F_{\bar s}$, $q := \pi'' p$, $\bar s$ a $\bar\gamma$-location.

Claim 8:

1. *$\pi|_{F_{\bar\beta}} = id$ (By definition of the transitive collapse)*
2. *$\gamma > \beta$ implies $\pi(F_\beta) = F_{\bar\beta}$. (By definition of $I(\beta)$, we have $F_\beta \in F_{s_{\bar\alpha}}\{F_{\bar\beta} \cup p\}$, $\bar\beta = \beta \cap F_{s_{\bar\alpha}}\{F_{\bar\beta} \cup p\}$.)*
3. *$\gamma = \beta$ implies $\bar\gamma = \bar\beta$ (Since equal ordinals are mapped to equal ordinals.)*

Claim 9: $\bar s = s(\bar\beta)$

Proof:

1. $'\geq'$: If $\beta_0 < \delta < \bar\beta$, then $\delta \neq \beta \cap F_{s_{\bar\alpha}}\{F_\delta \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$, (by minimal choice of $\bar\beta$), so $\delta \neq \bar\beta \cap F_{\bar s}\{F_\delta \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$. So $\bar s$ satisfies the requirement of $s(\bar\beta)$, and since $s(\bar\beta)$ is minimal with this property, we get $\bar s \geq_{loc} s(\bar\beta)$.
2. $'\leq'$: We show that $r <_{loc} \bar s$ implies $r <_{loc} s(\bar\beta)$. Let $r <_{loc} \bar s$, $\bar q \subset F_{\alpha(r)}$ finite, then $\pi^{-1}(r) <_{loc} s_i$ and $\pi^{-1}\bar q \subset F_{s_i}\{F_{\beta i} \cup p\}$ for $i < \bar\alpha$ large enough, since the s_i are unbounded in $s_{\bar\alpha}$, the β_i are unbounded in $\bar\beta$ and $F_{\bar s}\{F_{\bar\beta} \cup q\} = F_{\alpha(\bar s)}$ hence generates the finitely many elements of $\bar q$.
Now $\beta_i = \beta \cap F_{s_i}\{F_{\beta_i} \cup p\}$ and $r <_{loc} s_i$ (recall that s_i is a γ-location!), we arrive at $\beta_j = \bar\beta \cap F_r\{F_{\beta_j} \cup \bar q\}$ for $i < j < \bar\alpha$, so $r <_{loc} s(\bar\beta)$. This shows the implication we stated and hence proves $s(\bar\beta) \geq_{loc} \bar s$.

□

Claim 10: *$\bar\beta$ does not fall under Special Case I.*

Proof: $\bar{s}$ cannot be of the form $c^0_{\bar{\gamma}}(\vec{0})$ here, since condensation only leads to such a location if the condensed location is of this form itself. □

Claim 11: $q = p(\bar{\beta})$

Proof: (1) $F_{\bar{s}}\{F_{\bar{\beta}} \cup q\} = F_{\bar{\gamma}}$, thus $q \geq_{lex} p(\bar{\beta})$.
(2) Assume $q >_{lex} p(\bar{\beta})$; then $q \subset F_{\bar{s}}\{F_{\bar{\beta}} \cup p(\bar{\beta})\}$, so $p = \pi^{-1}[q] \subset F_s\{F_{\bar{\beta}} \cup \pi^{-1}p(\bar{\beta})\}$; but $\pi^{-1}p(\bar{\beta}) <_{lex} p = \pi^{-1}q$ and $\pi^{-1}p(\bar{\beta})$ satisfies the requirements for p, which contradicts the minimal choice of $p(\beta)$. □

By a previous remark, we have $F_{s_{\bar{\alpha}}}\{F_{\bar{\alpha}} \cup p\} = F_s\{F_{\bar{\alpha}} \cup p\}$, thus $F_{s_{\bar{\alpha}}}\{F_{\bar{\alpha}} \cup p\} \uparrow_{s_{\bar{\alpha}}}$, and $F_{\bar{s}}\{F_{\bar{\alpha}} \cup q\} \uparrow_{\bar{s}}$. Since also $\bar{\alpha} < \bar{\beta}$:

Claim 12: *$\bar{\beta}$ does not fall under Special Case II. (Since $\bar{\alpha} \neq 0$).*

Claim 13: *$\alpha_j(\beta) = \alpha_j(\bar{\beta})$ for $j < k$*

Proof: $\alpha_j(\beta)$ is the minimal μ, so that $F_s\{F_\mu \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{j-1}}\}\} \uparrow_s$; $F_s\{F_{\bar{\alpha}} \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\} \uparrow_{s_{\bar{\alpha}}}$. Furthermore $F_{\bar{\alpha}} \cup \{F_{\alpha'_j}, ..., F_{\alpha'_{k-1}}\} \subset F_{\alpha_j(\beta)}$ (since $\alpha_j(\beta)$ majorizes all of them), so $F_{\bar{s}}\{F_{\alpha_j(\beta)} \cup q \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{j-1}}\}\} \uparrow_{\bar{s}}$, thus $\alpha_j(\bar{\beta}) \geq \alpha_j(\beta)$.
On the other hand, $F_s\{F_{\alpha'_j} \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{j-1}}\}\} \downarrow_s$ by some $s' <_{loc} s_{\bar{\alpha}}$ by (3) of the definition of $I(\beta)$, hence by some location in $F_{s_{\bar{\alpha}}}\{F_{\bar{\beta}} \cup p\}$; therefore, $F_{\bar{s}}\{F_{\alpha'_j} \cup q \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{j-1}}\}\} \downarrow_{\bar{s}}$ by some $\bar{s}' <_{loc} \bar{s}$, so $\alpha_j(\bar{\beta}) \leq \alpha_j(\beta)$ as well. In summary, we have shown $\alpha_j(\bar{\beta}) = \alpha_j(\beta)$. □

Claim 14: $\alpha_k(\bar{\beta}) = \bar{\alpha}$

Proof: $F_{\bar{s}}\{F_{\bar{\alpha}} \cup q \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\} \uparrow_{\bar{s}}$, thus $\bar{\alpha} \geq \alpha_k(\bar{\beta})$. For $\alpha' < \alpha$, it follows that $F_{s_{\bar{\alpha}}}\{F_{\alpha'} \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\} \downarrow_{s_{\bar{\alpha}}}$ (since $\bar{\alpha}$ is minimal), hence $F_{\bar{s}}\{F_{\alpha'} \cup q \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\} \downarrow_{\bar{s}}$, and $\bar{\alpha} \leq \alpha_k(\bar{\beta})$. □

Claim 15: *$\beta_i(\beta) = \beta_i(\bar{\beta})$ for $i < \bar{\alpha}$.*

Proof: $\beta_0(\beta)$ was defined to be the largest $\delta < \beta$ such that $\delta = \beta \cap F_s\{F_\delta \cup p\}$. By definition of $\bar{\beta} = \beta_{\bar{\alpha}}$ and $s_{\bar{\alpha}}$, β_0 is the maximal $\delta < \bar{\beta}$ with $\delta = \bar{\beta} \cap F_{s_{\bar{\alpha}}}\{F_\delta \cup p\}$; this condenses to $\delta = \bar{\beta} \cap F_{\bar{s}}\{F_\delta \cup q\}$, which is the definition of $\beta_0(\bar{\beta})$.
Now, let $0 < i < \bar{\alpha}$. Then $s_i(\beta) := <_{loc} -l.u.b.\{F_s\{F_i \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}\} <$

s. By previous results, $s_i(\beta) :=<_{loc} -l.u.b.\{F_{s_{\bar{\alpha}}}\{F_i \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}\} <$ $s_{\bar{\alpha}}$; also, $s_i(\bar{\beta}) :=<_{loc} -l.u.b.\{F_{\bar{s}}\{F_i \cup q \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}\} < \bar{s}$.
Recall that $\beta_i(\beta)$ was minimal $> \beta_0$ with $\beta_i(\beta) = \beta \cap F_s\{F_{\beta_i(\beta)} \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$, i.e. with $\beta_i = \beta_i(\beta)$ also $\beta_i = \beta \cap F_{s'}\{F_{\beta i} \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$ for all $s' <_{loc} s_{\bar{\alpha}}(\beta)$ with $s' \in F_{s_{\bar{\alpha}}}\{F_i \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$; furthermore $\bar{\beta}_i = \beta_i(\bar{\beta})$ is minimal $> \beta_0$ so that $\bar{\beta}_i(\beta) = \bar{\beta} \cap F_{\bar{s}'}\{F_{\bar{\beta}_i(\beta)} \cup q \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$ for $s' <_{loc} \bar{s}$, $\bar{s}' \in F_{\bar{s}}\{F_i \cup q \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$.
Consider $\pi : F_{s_{\bar{\alpha}}}\{F_{\bar{\beta}} \cup p\} \to_{coll} F_{\bar{s}}\{F_{\bar{\beta}} \cup q\}$; $\pi|\bar{\beta} = id$, hence $\beta_i = \bar{\beta}_i$. $\square$

Observe that $I(\bar{\beta}) = \bar{\alpha} \cap I(\beta)$, so the coherence follows: The sequences coincide and start with the same index.

This concludes the first proof of square.

10.2 The horizontal hypering

Now, we demonstrate how the square principle can be proved using the horizontal hypering H_2 from the second example. Actually, as we already remarked there, H_2 has the defect of not being slow; this makes it unfeasable for some cases where $cf(\beta) = \omega$. The slowed-down hierarchy H_3 works fine, but is not as natural. What we will do is to mix the two hierarchies, using H_3 for some (trivial) special cases and using H_2 for the main part. Most of the proof above goes through without change. We repeat the crucial parts nevertheless, for the sake of completeness.
While at first sight the square sequence constructed here is as good as any other, including the one from the previous section, one might note that the sequence coming from H_3 does - apart from the special cases - not depend on the choice of the enumeration of the formula and is therefore somewhat more canonical. Our hope is that this effect might become useful in some way.

When checking the techniques underlying the proof of square in the classical hypering, the first thing to note is of course that lemma 19 does not seem to be applicable here, due to lack of slowness. We circumvent this difficulty by introducing a new special case.

Special Case I: $s = \bar{s}^+ = \langle \gamma, x^+, x^+, ... \rangle$ is a successor in $<_{H_2}$, where $\bar{s}$

is not of the form $\langle \zeta, \Omega, \Omega, ...\rangle$.

We make use of the slowed-down interpolation: set $h_i = \langle \gamma, x^+, ..., x^+, x, x, ...\rangle$, where the x are preceeded by i occurences of x^+. Then define
$\beta_0 := max\{\bar\beta < \beta | \bar\beta = \beta \cap F_s\{F_{\bar\beta} \cup p\}\}$, which is $< \beta$ by definition of s and p. Now choose $\beta_{n+1} > \beta_n$ minimal with the property that $\beta_{n+1} = \beta \cap F_{h_{n+1}}\{F_{\beta_{n+1}} \cup p\}$.
We claim that $\beta_n < \beta$ for every $n \in \omega$: Otherwise, let n be the smallest counterexample. Set $p_n := p \cup \{S^{\phi_1}_\gamma(x), ..., S^{\phi_n}_\gamma(x)\}$; then since the extra Skolem values that could enter the hulling process for h_i have now been put in anyway, we conclude $\{\bar\beta < \beta | F_{\bar s}\{F_{\bar\beta} \cup p_n\} \cup \beta = \bar\beta\} \downarrow_\beta$, contradicting the minimal choice of s.
Now put $\beta_\omega := \bigcup_{i\in\omega} \beta_i$, thus:
$\beta \cap F_s\{F_{\beta_\omega} \cup p\} = \beta \cap \bigcup_{i\in\omega} F_{h_i}\{F_{\beta_\omega} \cup p\}$
$= \bigcup_{i\in\omega} \beta \cap F_{h_i}\{F_{\beta_\omega} \cup p\} = \bigcup_{i\in\omega} \beta \cap F_{h_i}\{F_{\beta_i} \cup p\}$
$= \bigcup_{i\in\omega} \beta_{i+1} = \beta_\omega$,
and by definition of β_0, we get $\beta_\omega = \beta$.
So, we conclude in this case by setting $C_\beta = \langle \beta_n | n \in \omega\rangle$.

Therefore, we can assume from now on that s is indeed a limit location in $<_{H_2}$, having avoided lemma 19.

The rest of the proof is now very similar to the first one. We will mostly repeat the statements, but not the proofs, where these carry over verbatim (substituting l for c in the naming of locations).

Claim 1: $s \neq l^0_\beta(\vec{0})$

Proof: Classically. □

Claim 2:

1. *$s \neq l_\gamma(\vec{0})$ for γ a limit ordinal*
2. *$s \neq l_\gamma(0)$ for $\gamma = \delta + 1$ a successor ordinal*

Proof: (1) Classically.
(2) All functions involved in this hulling process can be emulated by those of $\bar s := l_\delta(\Omega)$ in case $F_\delta \in p$ and otherwise, so the fixed-point set for $\bar s$ with

parameter $q = p - \{F_\delta\}$ will in any case be a subset of the original one and hence be bounded below β, contradicting the choice of s. □

Remark: Note that this means in particular that special case I of the first proof does not occur; this seems plausible, since special cases typically lead to cofinality ω, and there was a cofinal ω-sequence for such locations in the classical hypering, but in the horizontal, there is usually not. The corresponding case is the case I we have just dealt with. So even though H_1 and H_2 share some locations, these may play fundamentally different rules in the constructions. This should not be too much of a surprise, because we choose minimal locations at which a certain effect happens, and of course, there can for example be one in H_2 below $l_\gamma(\Omega)$, but not in H_1.

From now on, we assume $s = l_\gamma(\vec{x}) \neq l_\gamma(0)$.

Claim 3: *There is a finite set $\bar{p} \subset F_\gamma$ such that $F_s\{F_\beta \cup \bar{p}\} = F_\gamma$.*

Proof: Classically. □

We choose $p(\beta) \subset F_\gamma$ $<_{lex}$-minimal with this property. In particular, we have $p \subset F_s\{F_\beta \cup p(\beta)\}$, which immediatly gives us:

Claim 4: $\{\bar{\beta} < \beta | \bar{\beta} = \beta \cap F_s\{F_{\bar{\beta}} \cup p(\beta)\}\} \downarrow_\beta$.

Let $\beta_0 < \beta$ be the maximum of this set. Since $p(\beta)$ satisfies the properties we demanded of p, we set $p := p(\beta)$ from now on. A hull $F_s\{X\}$ contains parameters that can serve as components for locations; for $Y \subset F_\gamma$, we write $l_\gamma(\vec{y}) =: r \in Y$ for $\vec{y} \in Y$ (which, as we recall, in turn means that any component of $\vec{y}$ is in Y).

Proposition 21: *If $\bar{s}$ is the l.u.b. of $Y = F_s\{X\}$, then $F_s\{X\} = F_{\bar{s}}\{X\}$.*

Proof: Classically. Note, however, that the statement has the same wording, but not the same meaning, as that of Proposition 20, since we are in a different hypering here. □

Special Case II: $F_s\{F_\alpha \cup p\} \downarrow_s$ for $\alpha < \beta$.
(Classically. No changes whatsoever here.)

Now we drop to a steering limit ordinal as in the classical case. We re-use the notation from there as well.

Special Case III: $\alpha_k = 0$.
(Classically.)

Until now, the proofs were very similar; the generic case works in the same fashion, the only difference being that we have to take care of excluding the new special case I from being obtained by condensation instead of the old one. Luckily, this is already a side-effect of the old construction.

The Generic Case: $lim(\alpha) \neq 0$, $s = c^n_\gamma(x) \neq c^0_\gamma(\vec{0})$

The construction is the same as before. As we did there, we can show:

Claim 5: $0 < i < j < \alpha$ *implies* $s_i <_{loc} s_j$ *and* $\beta_i \leq \beta_j$.

Claim 6: $\{\beta_i | i < \alpha\}$ *is club in* β.

Proof: Classically. □

Even though the definition of the endsegment does not change, we repeat it here to keep things together. Let $I(\beta)$ be the set of all i such that:

1. $\beta_i \geq max\{\alpha_0, ..., \alpha_{k-1}\}$ (this is possible since $\langle \beta_i | i < \alpha \rangle \uparrow_\beta$)
2. $\beta_j > j$ (Recall that we have $j < \alpha < \beta$)
3. The upper bounds of the intermediate construction steps $F_s\{F_{\alpha_l} \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{l-1}}\}\}$ are $<_{loc} s_i$
4. In case $\beta < \gamma$, we have $F_\beta \in F_{s_i}\{F_{\beta_i} \cup p\}$

$I(\beta)$ is easily seen to be an endsegment of α. Now set:
$C_\beta := \langle \beta_i | i \in I(\beta) \rangle$. Since C_β is just a 'shortening' of a club with the same properties, one immediatly gets:

Claim 7: C_β *is club in* β, *otp*$(C_\beta) \leq \alpha < \beta$.

It remains to show the coherence property, namely that for $\bar{\beta}$ a limit point of C_β, then $C_{\bar{\beta}} = \bar{\beta} \cap C_\beta$. So assume $\bar{\beta}$ to be a limit point of C_β, $\bar{\alpha}$ minimal such that $\beta_{\bar{\alpha}} = \bar{\beta}$. Then $\bar{\alpha}$ is a limit ordinal, $\beta_{\bar{\alpha}} > \bar{\alpha}$ (by (2) of the definition of $I(\beta)$) and β belongs to the generic case (for otherwise, C_β would be of order type ω and hence there could be no limit points).
Consider $\pi : F_{s_{\bar{\alpha}}}\{F_{\bar{\beta}} \cup p\} \rightarrow_{coll} F_{\bar{s}}$, $q := \pi'' p$, $\bar{s}$ a $\bar{\gamma}$-location.

Claim 8:

1. $\pi|_{F_{\bar{\beta}}} = id$ *(By definition of the transitive collapse)*
2. $\gamma > \beta$ *implies* $\pi(F_\beta) = F_{\bar{\beta}}$. *(By definition of* $I(\beta)$*, we have* $F_\beta \in F_{s_{\bar{\alpha}}}\{F_{\bar{\beta}} \cup p\}$, $\bar{\beta} = \beta \cap F_{s_{\bar{\alpha}}}\{F_{\bar{\beta}} \cup p\}$.*)*
3. $\gamma = \beta$ *implies* $\bar{\gamma} = \bar{\beta}$ *(Since equal ordinals are mapped to equal ordinals.)*

Claim 9: $\bar{s} = s(\bar{\beta})$

Proof:

- $'\geq'$: If $\beta_0 < \delta < \bar{\beta}$, then $\delta \neq \beta \cap F_{s_{\bar{\alpha}}}\{F_\delta \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$, (by minimal choice of $\bar{\beta}$), so $\delta \neq \bar{\beta} \cap F_{\bar{s}}\{F_\delta \cup p \cup \{F_{\alpha'_0}, ..., F_{\alpha'_{k-1}}\}\}$. So $\bar{s}$ satisfies the requirement of $s(\bar{\beta})$, and since $s(\bar{\beta})$ is minimal with this property, we get $\bar{s} \geq_{loc} s(\bar{\beta})$.
- $'\leq'$: We show that $r <_{loc} \bar{s}$ implies $r <_{loc} s(\bar{\beta})$. Let $r <_{loc} \bar{s}$, $\bar{q} \subset F_{\alpha(r)}$ finite, then $\pi^{-1}(r) <_{loc} s_i$ and $\pi^{-1}\bar{q} \subset F_{s_i}\{F_{\beta i} \cup p\}$ for $i < \bar{\alpha}$ large enough, since the s_i are unbounded in $s_{\bar{\alpha}}$, the β_i are unbounded in $\bar{\beta}$ and $F_{\bar{s}}\{F_{\bar{\beta}} \cup q\} = F_{\alpha(\bar{s})}$ hence generates the finitely many elements of $\bar{q}$.
Now $\beta_i = \beta \cap F_{s_i}\{F_{\beta_i} \cup p\}$ and $r <_{loc} s_i$ (recall that s_i is a γ-location!), we arrive at $\beta_j = \bar{\beta} \cap F_r\{F_{\beta_j} \cup \bar{q}\}$ for $i < j < \bar{\alpha}$, so $r <_{loc} s(\bar{\beta})$. This shows the implication we stated and hence proves $s(\bar{\beta}) \geq_{loc} \bar{s}$.

□

So claim 8 and claim 9 go through as well. The only thing we have to take care of is claim 10, for it now refers to a different special case; but this can easily be dealt with:

Claim 10: *$\bar{\beta}$ does not fall under Special Case I.*

Proof: $s_{\bar{\alpha}}$ is the limit of the sequence $\langle s_i | i < \bar{\alpha} \rangle$ and not of the form $l_\zeta(0)$, so $\bar{s}$ will be the limit of the images of these locations under the condensation map and thus not a successor. $\square$

From here, the proofs of claims 11 to 15, and hence the conclusion of the proof, are identical to the classcial case and can entirely be carried out in H_2.

This concludes the horizontal approach to square.

11 Morasses

Morasses were invented by Ronald Jensen as a strong tool for describing structures of a certain cardinality as limits of a small system of smaller structures. If the size of the approximated structure is the cardinal successor of the size of the system, we are in the simplest case of a gap-1-morass. A construction of such a structure using classical fine structure can be found in [1]. Higher gap morasses have been defined and constructed, yet these notions tend to become so complicated that few, if any, applications are known.
The first construction of a morass in L was given by Jensen using his fine-structure theory. Using Silver machines, the construction could be considerably simplified by Richardson in his PhD-thesis [20]. Later on, Koepke, Friedman and Piwinger constructed a morass in L using Friedman-Koepke hyperfine structure theory [8], [40], [17].
Here, we will exhibit morasses in the context of general hyperings. We start by showing that the horizontal hypering suffices for constructing what is known as a 'coarse morass'. Then, after giving some more general results on hyperings, we give an 'abstract' construction of a gap-1-morass only using general properties of hyperings, without referring to a particular instance. Doing so, it will become apparent which properties are important for the proof and where. This places hyperings between Silver machines and classical hyperfine structure in terms of generality.

We will now define the structures we are talking about.

Definition 1: *Let* $S_0, S_1 \subset \omega_2$, $\prec, \lhd \subset \omega_2 \times \omega_2$, $\pi_{\sigma\tau} \subset L \times L$, $S_\gamma := \{\sigma \in S_1 | \gamma_\sigma = \gamma\}$ *such that* $S_0 = \{\gamma_\sigma | \sigma \in S_1\}$. *The* ω_1*-morass axioms are:*

- *(M0)*
  - *(a) If* $\gamma \in S_0$, *then* S_γ *is closed in* $sup[S_\gamma]$.
  - *(b)* S_{ω_2} *is club in* ω_2.
  - *(c)* $S_0 \cap \omega_1 \uparrow_{\omega_1}$.
  - *(d)* $\lhd$ *is a tree-ordering on* S_1.

- *(M1) If* $\sigma \lhd \tau$*, then:*
  - *(a)* $\nu < \sigma$ *implies that* $\nu \in S_1 \leftrightarrow \pi_{\sigma\tau(\nu)} \in S_1$.
  - *(b)* $\pi_{\sigma\tau}$ *preserves* $\prec$*-minimality,* $\prec$*-limites and being a successor upwards and immediate predecessors downwards.*
- *(M2) If* $\bar{\sigma} \prec \sigma$ *and* $\sigma \lhd \tau$*, then, setting* $\bar{\tau} = \pi_{\sigma\tau}(\bar{\sigma})$*, we have* $\bar{\sigma} \lhd \bar{\tau}$ *and* $\pi_{\bar{\sigma}\bar{\tau}} = \pi_{\sigma\tau}|F_{s(\bar{\sigma})}$.
- *(M3) For* $\tau \in S_1$, $\{\gamma_\sigma|\sigma \lhd \tau\} \cup \{\gamma_\tau\}$ *is closed.*
- *(M4) If* $\tau \in S_1$ *has a* $\prec$*-successor, then* $\{\gamma_\sigma|\sigma \lhd \tau\} \uparrow_{\gamma_\tau}$.
- *(M5)* $\{\gamma_\sigma|\sigma \lhd \tau\} \uparrow_{\gamma_\tau}$ *implies* $\tau = \bigcup_{\sigma \lhd \tau} \pi_{\sigma\tau}[\sigma]$.
- *(M6) If* $\sigma \in S_1$ *is a* $\prec$*-limit,* $\sigma \lhd \tau$, $\lambda := sup \pi_{\sigma\tau}[\sigma]$ *and* $\lambda < \tau$*, then* $\sigma \lhd \lambda$ *and* $\pi_{\sigma\lambda}|\sigma = \pi_{\sigma\tau}|\sigma$.
- *(M7) If* $\sigma \in S_1$ *is a* $\prec$*-limit,* $\sigma \lhd \tau$, $\tau = sup \pi_{\sigma\tau}[\sigma]$ *and* $\alpha \in S_0$ *is such that, for each* $\bar{\sigma} \prec \sigma$*, we can find* $\bar{\nu} \in S_\alpha$ *with* $\bar{\sigma} \lhd \bar{\nu} \lhd \pi_{\sigma\tau}(\bar{\sigma})$*, then there is also* $\nu \in S_\alpha$ *with* $\sigma \lhd \nu \lhd \tau$.

Definition 2: *A structure* $M := (S_1, \sigma \lhd \tau, (\pi_{\sigma\tau})_{\sigma \lhd \tau})$ *satisfying (M0)-(M5) is a coarse morass. If* M *additionally satisfies (M6) and (M7), it is an* $(\omega_1, 1)$*-morass.*

11.1 Construction of a Coarse Morass

In this section, we show how to get a coarse morass using the horizontal hypering H_2. This will already use many of the central ideas of the general construction given in the next two sections.

Definition 3: *An ordinal $\sigma \leq \omega_2$ is a morass point if $F_\sigma \models ZF^- +$'There is exactly one uncountable cardinal'. We abbreviate the statement that σ is a morass point by $MP(\sigma)$. If σ is a morass point, let $\gamma_\sigma > \omega$ be the unique ordinal such that $F_\sigma \models' \gamma_\sigma$ is a cardinal'.*

Lemma 4: *Let σ be a morass point. Then there is a minimal $s =: s(\sigma) \in H_2$ such that, for some $p =: p_\sigma \subset_{fin} L_s$, we have $L_s\{\gamma_\sigma \cup p\} \uparrow_\sigma$.*

Proof: As $F_{\omega_1} \models$'There is no uncountable cardinal', $\sigma \neq \omega_1$, so σ must eventually be collapsed. That is, there are α, $f \in F_\alpha$, such that
$f : \gamma_\sigma \rightarrow_{surj} \sigma$. Take enough extra levels so that 'y is the f-image of x' becomes $\mathbf{S}_0$-expressable over $F_{\alpha+k}$ in the parameters $f, F_\alpha, ..., F_{\alpha+k-1}$. Then, setting $q := \langle F_\alpha, ..., F_{\alpha+1} \rangle$, the H_2-location $l_{\alpha+k}(q^+)$ will generate σ from γ_σ and the finite parameter q, and in particular the corresponding hull will be cofinal in σ. As H_2 is a linear hypering, $<_{loc}$ is a well-ordering on H_2, so there is also a minimal location with this property.
□

Remark: The location $s(\sigma)$ of lemma 4 will be called the H_2-collapsing location for σ. In the next sections, we will consider collapsing locations in arbitrary hyperings. Though it might be tempting to introduce a notation like $s^{H_2}(\sigma)$ to indicate what hypering is currently used, this is the sort of index that will immediately be dropped unless ambiguities arise, which will never be the case in this chapter.

Lemma 5: *Let σ be a morass point, $\gamma = \gamma_\sigma$, $s = s(\sigma)$ and $p = p_\sigma$ as given by the last lemma. Then:*

- *(a) $\sigma \subset F_s\{\gamma \cup p\}$*

- *(b)* $F_s = F_s\{\gamma \cup p\}$

Proof:
(a) Suppose $\zeta < \sigma$. If $\zeta < \gamma_\sigma$, the lemma is trivial. So suppose without loss of generality that $\gamma_\sigma \leq \zeta$. Since σ is a morass point, $F_\sigma \models ZF^-$ and γ_σ is the largest cardinal in F_σ. By definition of $s = s(\sigma)$, $F_s\{\gamma \cup p\} \uparrow_\sigma$. So there are $\zeta < \alpha < \sigma$ and a $<_L$-least function $f \in F_\sigma$ such that $f : \gamma_\sigma \rightarrow_{surj} \alpha$. Also, there are ω many F-stages above α in $F_s\{\gamma \cup p\}$, hence f is definable in, and in fact an element of $F_s\{\gamma \cup p\}$. Say $f \in F_\delta \in F_s\{\gamma \cup p\} =: Z$. Take enough F-stages above δ in Z and another one, say F_λ, on top of these so that the relation $f(x) = y$ becomes $\mathbf{S}_0$-expressable in Z over F_λ. Let $\phi(x, y, q)$ be the corresponding $\mathbf{S}_0$-formula, $q \subset_{fin} F_\lambda$. Then $S_\phi(F_\lambda, x^\wedge q)$ can be used to find the image of each $\iota < \gamma_\sigma$ under f. As Z is closed under this Skolem function and contains each element of γ_σ, we get $\alpha \subset Z$. As $\zeta < \alpha$, also $\zeta \subset Z$.
(b) By (a), $F_s\{\gamma \cup p\} = F_s\{\sigma \cup p\}$. Consider $\pi : F_s\{\sigma \cup p\} \rightarrow_{coll} F_{\bar{s}}$. Let $\bar{p} := \pi(p)$. As $\pi|\sigma = id$, it follows that $F_{\bar{s}}\{\gamma \cup \bar{p}\} \uparrow_\sigma$, so $\bar{s}$ collapses σ to γ_σ. Thus $s(\sigma) \leq_{loc} \bar{s}$. By minimality of s, also $\bar{s} \leq_{loc} s(\sigma)$, therefore $\bar{s} = s$. The same reasoning shows $\bar{p} = p$. But then, π can only be the identity. So $F_s\{\gamma \cup p\}$ is transitive, and hence equal to F_s.
□

Now for the construction:
Let $S_1 := \{\sigma < \omega_2 | MP(\sigma)\}$, $S_0 := \{\gamma | \exists \sigma \in S_1(\gamma = \gamma_\sigma)\}$. For $\alpha, \beta \in S_1$, write $\alpha \prec \beta$ iff $\alpha < \beta$ and $\gamma_\alpha = \gamma_\beta$.
Also, let $\alpha \lhd \beta$ iff $\alpha < \beta$ and there is a structure-preserving embedding $\pi_{\alpha\beta} : F_{s(\alpha)} \rightarrow F_{s(\beta)}$ satisfying:

- (i) $\pi_{\alpha\beta}|\gamma_\alpha = id$
- (ii) $\pi_{\alpha\beta}(\gamma_\alpha) = \gamma_\beta$
- (iii) $\pi_{\alpha\beta}(\alpha) = \beta$
- (iv) $\pi_{\alpha\beta}(p_\alpha) = p_\beta$
- (v) $\pi_{\alpha\beta}$ preserves $\Sigma_1^{s(\alpha)}$-formulas

It remains to show that this works.

Theorem 6: *The system $M := \langle S_0, S_1, \triangleleft, \prec, \pi_{\sigma\tau} | \sigma \triangleleft \tau\rangle$, constructed in H_2, is a coarse morass, i.e. satisfies the morass axioms (M0)-(M5).*

Proof: This will shown by a sequence of claims.

- (M0) holds for M.
  Proof:

  - (a) First, for any $\gamma \in S_0$, the set S_γ of morass points σ with $\gamma_\sigma = \gamma$ is closed in its supremum. This is easy to see: Suppose δ is a limit point of S_γ. As F_δ is a limit of limits of ZF^--models, it is itself a limit of ZF^--models. Also, γ is the only uncountable cardinal of F_δ: On the one hand, if there was another one, say κ, then there would be $\delta > \beta \in S_\gamma$ such that $\kappa \in F_\beta$. As $F_\beta \subset F_\delta$ and κ is supposed to be a cardinal in F_δ, κ cannot be collapsed in F_β. So κ is also an uncountable cardinal in F_β, but $\kappa \neq \gamma$ and β is a morass point, which is impossible. So there are no uncountable cardinals in F_δ besides γ. On the other hand, if γ was collapsed in F_δ, say by a function $f : \alpha \to_{surj} \gamma$ with $\alpha < \gamma$ and $f \in F_\delta$, then there would be $\delta > \beta \in S_\gamma$ such that $f \in F_\beta$. But this implies that γ is already collapsed in F_β, again a contradiction. So δ is indeed a morass point with $\gamma_\delta = \gamma$.

  - (b) That S_{ω_2} is closed in ω_2 follows from (a). By regularity of ω_2, there are unboundedly many $\alpha < \omega_2$ such that $F_\alpha \models ZF^-$, and hence there are unboundedly many limits of such α below ω_2 as well. For each $\omega_1 < \zeta < \omega_2$, there is $g \in F_{\omega_2}$ such that $g : \omega_1 \to_{surj} \zeta$. So for each $\alpha < \omega_2$ there is $\beta = \beta(\alpha) < \omega_2$ such that F_β is a limit of ZF^--models and no cardinal greater than ω_1 in F_α remains a cardinal in F_β. Setting $\alpha_0 = \alpha$, $\alpha_{n+1} = \beta(\alpha_n)$, $\hat{\alpha} := sup\{\alpha_i | i \in \omega\}$ will be an element of S_{ω_1} greater than α, so S_{ω_1} is indeed unbounded in ω_2.

  - (c) Let $\zeta < \omega_1$, we need to show that there are $\zeta < \alpha < \sigma < \omega_1$ such that α is the unique uncountable cardinal in F_σ. To see this, take a countable elementary submodel X of F_{ω_2} containing ζ as a subset and ω_1 as a singleton. Let $\pi : X \to_{coll} F_\mu$ by condensation. As X is countable, so is μ. By elementarity, $\pi(\omega_1)$ is the unique uncountable cardinal in F_μ and, since $\zeta \subset X$, we must have $\pi(\omega_1) > \zeta$.

- (d) Suppose $\sigma \lhd \tau$, $\nu \lhd \tau$ and assume without loss of generality that $\sigma < \nu$. Define $\pi : F_{s(\sigma)} \to F_{s(\nu)}$ by $\pi = \pi_{\nu\tau}^{-1} \circ \pi_{\sigma\tau}$. As both maps are Σ_1^s-preserving for their respective locations, π is $\Sigma_1^{s(\sigma)}$-preserving. As $\pi_{\sigma\tau}(\gamma_\sigma) = \gamma_\tau$ and $\pi_{\nu\tau}(\gamma_\nu) = \gamma_\tau$, we have $\pi_{\nu\tau}^{-1}(\gamma_\tau) = \gamma_\nu$ and hence $\pi(\gamma_\sigma) = \gamma_\tau$, as required. Also, $\pi(\sigma) = \nu$, for the same reason. Finally, as $\sigma < \tau$, we have $\pi_{\sigma\tau}|\gamma_\sigma = \pi_{\nu\tau}|\gamma_\sigma = id|\gamma_\sigma$, and so $\pi|\gamma_\sigma = id|\gamma_\sigma$. So π is a morass map, and we have $\sigma \lhd \nu$, which is what had to be shown.
  (The well-foundedness of $\lhd$ is immediate from the fact that, in a $\lhd$-falling sequence of morass points σ, the γ_σ form a $<$-falling sequence of ordinals.)

- (M1) holds for M.
  Proof:
  - (a) First, suppose $\nu < \sigma$ is a morass point. As σ is a morass point, there is $\nu < \alpha < \sigma$ such that $F_\alpha \models ZF^-$. As being a morass point is absolute between ZF^--models, $F_\alpha \models MP(\nu)$. As $\alpha < \sigma$ and σ is a limit ordinal, this statement will be preserved by $\pi_{\sigma\tau}$: So $\pi_{\sigma\tau}(F_\alpha) = F_{\pi_{\sigma\tau}(\alpha)} \models ZF^- + MP(\pi_{\sigma\tau}(\nu))$, and hence $\pi_{\sigma\tau}(\nu)$ really is a morass point.
    For the other direction, assume $MP(\pi_{\sigma\tau}(\nu))$. Since τ is a morass point, there is $\pi_{\sigma\tau}(\nu) < \beta < \tau$ such that $F_\beta \models ZF^- + MP(\pi_{\sigma\tau}(\nu))$. As enough stages are available, this statement can be expressed by a $\Sigma_0^{s(\tau)}$-formula. Hence the existence of such a β is $\Sigma_1^{s(\tau)}$-expressable and therefore preserved by $\pi_{\sigma\tau}$. Now, we can proceed as we did for the reverse direction to deduce that ν actually is a morass point.
  - (b) The properties of $\prec$-minimality, being a $\prec$-limit, a $\prec$-successor etc. are easily first-order expressable and hence preserved by $\pi_{\sigma\tau}$ by the same reasoning used for part (a). For the preservation of

immediate predecessors, observe that the existence of an intermediate morass point is as well PL_1-expressable, and hence preserved by suitable restrictions of $\pi_{\sigma\tau}$.

- (M2) holds for M.
Proof: Basically, we need to show that the collapsing locations $s(\bar{\sigma})$ and $s(\bar{\tau})$ are mapped to each other by $\pi_{\sigma\tau}$. As σ is a morass point, there is $\bar{\sigma} < \alpha < \sigma$ such that $F_\alpha \models ZF^-$. Inside F_α, construct $s(\bar{\sigma})$ and $p_{\bar{\sigma}}$. $\pi_{\sigma\tau}|F_\alpha$ is elementary, hence $\pi_{\sigma\tau}(F_\alpha) = F_{\pi_{\sigma\tau}(\alpha)} \models ZF^-$ and, inside $F_{\pi_{\sigma\tau}(\alpha)}$, $\pi_{\sigma\tau}(s_{\bar{\sigma}})$ is the collapsing location for $\pi_{\sigma\tau}(\bar{\sigma}) = \bar{\tau}$. Also, as $\gamma_{\bar{\sigma}} = \gamma_\sigma$ and $\gamma_{\bar{\tau}} = \gamma_\tau$, we get $\pi_{\sigma\tau}|\gamma_{\bar{\sigma}} = id$ and $\pi_{\sigma\tau}(\gamma_{\bar{\sigma}}) = \gamma_{\bar{\tau}}$; $\pi_{\sigma\tau}(\bar{\sigma}) = \bar{\tau}$ is true by definition of $\bar{\tau}$. That $\pi_{\sigma\tau}|F_{s(\bar{\sigma})}$ is $\Sigma_1^{s(\bar{\sigma})}$-preserving is immediate from the fact that it is a restriction of a Σ_1^s-map with $s >_{loc} s(\bar{\sigma})$.

- (M3) holds for M.
Proof: If $\{\gamma_\sigma|\sigma \lhd \tau\} \uparrow \gamma_\tau$, then the supremum, namely γ_τ, is in the considered set since we have added it. So we can without loss of generality assume that $\zeta < \gamma_\tau$ and ζ is a limit point of $\{\gamma_\sigma|\sigma \lhd \tau\}$. We need to show that ζ is the unique uncountable cardinal in some F_δ where δ is a morass point and $\delta \lhd \tau$. The natural attempt is to take $F_{s(\tau)}\{\zeta \cup \{\tau\} \cup \{p_\tau\}\}$ and collapse this to some $F_{\bar{s}}$ with collapsing map π, hoping that $\pi(\tau) =: \bar{\tau}$ will be a morass point with $\gamma_{\bar{\tau}} = \zeta$, $\bar{s}$ will be the collapsing location $s(\bar{\tau})$ for $\bar{\tau}$ with parameter $\bar{p} := \pi(p_\tau)$ and that π itself can be chosen as the morass map witnessing $\bar{\tau} \lhd \tau$. This is indeed the case, as we will now demonstrate step by step:

  - $\bar{\tau}$ is a morass point.

    Proof: We need to show that $\bar{\tau}$ is a limit of ordinals ν such that $F_\nu \models ZF^-$. Let $\eta < \bar{\tau}$, we will show that there is $\nu > \eta$ such that $F_\nu \models ZF^-$. As ζ is a limit of ordinals of the form γ_ι with $\iota \lhd \tau$ and $\zeta \in F_{s(\tau)}\{\gamma_\tau \cup p_\tau\} = F_{s(\tau)}$, there is $\sigma \lhd \tau$ such that $\zeta \in F_{s(\tau)}\{\gamma_\sigma \cup p_\tau\} = rng(\pi_{\sigma\tau})$. So ζ has a pre-image under $\pi_{\sigma\tau}$ in $F_{s(\tau)}$, say $\bar{\zeta}$. $\bar{\zeta} < \sigma$, of course, so by the fact that σ is a morass point, we get $\bar{\zeta} < \mu < \sigma$ with $F_\mu \models ZF^-$. Since there are enough F-stages above F_μ in $dom(\pi_{\sigma\tau})$ to ensure elementarity, $\pi_{\sigma\tau}(F_\mu) \models ZF^-$ is clear. Also, this structure is an element of $rng(\pi_{\sigma\tau}) \subset F_{s(\tau)}\{\zeta \cup p_\tau\}$ and hence in the range of the uncollapsing map π^{-1}. Pulling it back via π, which is elementary for F_μ to $F_{\bar{s}}$ gives $\pi(\mu)$ between ζ and $\bar{\tau}$ such that $F_{\pi(\mu)} \models ZF^-$, as desired.

- ζ is the unique uncountable cardinal of $F_{\bar{\tau}}$.
  Proof: First, we show that ζ is an uncountable cardinal in $F_{\bar{\tau}}$: Assume the contrary, so that there are $\beta < \zeta$ and a function $f \in F_{\bar{\tau}}$ such that $f : \beta \rightarrow_{surj} \zeta$. f must have a π-preimage $\hat{f}$ in $F_{s(\tau)}\{\zeta \cup \{\tau\} \cup p_\tau\}$, and so, by definition of ζ, there must be $\sigma < \tau$ such that $\hat{f} \in F_{s(\tau)}\{\gamma_\sigma \cup \{\tau\} \cup p_\tau\}$. Now, consider $\sigma' < \zeta$ large enough such that $\gamma_{\sigma'} > max\{\beta, \gamma_\sigma\}$. This must exist by definition of ζ. Then certainly $\hat{f} \in F_{s(\tau)}\{\gamma_{\sigma'} \cup \{\tau\} \cup p_\tau\}$, so $\gamma_{\sigma'}$ is collapsed to β in $F_{s(\sigma')}$, a contradiction.
  To see uniqueness, suppose ζ' is another uncountable cardinal in $F_{\bar{\tau}}$. Then there is again σ such that ζ, ζ' have π-preimages in $F_{s(\tau)}\{\gamma_\sigma \cup \{\tau\} \cup p_\tau\}$. But then, $F_{s(\sigma)}$ will also have two uncountable cardinals, again a contradiction. So ζ is indeed unique.□

  For the next two steps, we need the following:
  Claim: $Z := F_{s(\tau)}\{\zeta \cup \{\tau\} \cup p_\tau\} \cap \gamma_\tau = \zeta$. $(*)$

  Proof: That ζ is a subset of Z is clear. For the other direction, let $\lambda \in F_{s(\tau)}\{\zeta \cup \{\tau\} \cup p_\tau\}$. By definition of ζ, there must be $\delta < \tau$ such that $\lambda \in F_{s(\tau)}\{\gamma_\delta \cup \{\tau\} \cup p_\tau\}$. But, as γ_δ is the critical point of $\pi_{\delta\tau}$ and $\pi_{\delta\tau}(\gamma_\delta) = \gamma_\tau$, we must have (recall that $\tau = \pi_{\delta\tau}(\delta)$) $rng(\pi_{\delta\tau}) = F_{s(\tau)}\{\gamma_\delta \cup \{\tau\} \cup p_\tau\} \cap \gamma_\tau = \gamma_\delta$, hence indeed $\delta \in Z$. □

- $\bar{s}$ is the collapsing location for $\bar{\tau}$.
  Proof: We show this by two inequalities. As we have seen in the last part, $\bar{s}$ is a collapsing location for $\bar{\tau}$, so $s(\bar{\tau}) \leq_{loc} \bar{s}$ is clear.
  For the other direction, suppose $s(\bar{\tau}) <_{loc} \bar{s}$. Then there must be $\sigma \triangleleft \tau$ and $\hat{s}, \hat{p} \in F_{s(\sigma)}$ such that $s(\bar{\tau}) = \pi \circ \pi_{\sigma\tau}(\hat{s})$, $p_{\bar{\tau}} = \pi \circ \pi_{\sigma\tau}(\hat{p})$. Now, as $s(\bar{\tau}) <_{loc} \bar{s}$ by assumption and π and $\pi_{\sigma\tau}$ preserve $<_{lex}$ and (hence) $<_{loc}$, clearly $\hat{s} <_{loc} s(\sigma)$, so $F_{\hat{s}}\{\gamma_\sigma \cup \hat{p}\} \downarrow_\sigma$ by minimality of $s(\sigma)$. Let $\lambda := sup\{F_{\hat{s}}\{\gamma_\sigma \cup \hat{p}\} \cap \sigma\}$. Again by the preservation properties of $\pi_{\sigma\tau}$ and π, $F_{s(\bar{\tau})}\{\zeta \cup \bar{\tau} \cup p_{\bar{\tau}}\} \downarrow_{\bar{\tau}}$, because

we can simply map over the bound λ to $\pi \circ \pi_{\sigma\tau}(\lambda)$ and obtain an upper bound for the latter.
But now, the boundedness of $F_{s(\bar{\tau})}\{\zeta \cup \{\bar{\tau}\} \cup p_{\bar{\tau}}\}$ in $\bar{\tau}$ contradicts the definition of $s(\bar{\tau})$ and $p_{\bar{\tau}}$. So our assumption must have been wrong, and $s(\bar{\tau}) \geq_{loc} \bar{s}$.

- $\bar{p}$ is the minimal collapsing parameter $p_{\bar{\tau}}$ for $\bar{\tau}$.
  Proof: This follows by the same reasoning we used to obtain that $\bar{s}$ is the collapsing location for $\bar{\tau}$. Clearly, $F_{\bar{s}}\{\gamma_\tau \cup \{\tau\} \cup \bar{p}\} \uparrow_{\bar{\tau}}$ by condensation, so $p_{\bar{\tau}} \leq_{lex} \bar{p}$. For the other direction, if we had $\bar{p} <_{lex} p_{\bar{\tau}}$, we could again pull this back using $\pi \circ \pi_{\sigma\tau}$ for sufficiently $\lhd$-large σ to obtain a contradiction.

- $\pi : F_{s(\bar{\tau})} \to F_{s(\tau)}$ is a morass map.
  Proof: That $\pi|_\zeta$ is the identity map is clear from the fact that π is a collapsing map and ζ is contained in the pre-image of π as a subset. Also, it was just shown that π preserves the parameter, the unique uncountable cardinal and the morass point. The only property that requires an argument is $\Sigma_1^{\bar{s}}$-preservation. Upwards preservation of $\Sigma_1^{\bar{s}}$ follows from the fact that π is a collapsing map. If a $\Sigma_1^{s(\tau)}$-formula ψ with parameters from $rng(\pi)$ is true in $F_{s(\tau)}$, there must be $\sigma \lhd \tau$ such that ψ has a witness in $F_{s(\tau)}\{\gamma_\sigma \cup \{\tau\} \cup p_\tau\}$. This witness will then also be an element of $F_{\bar{s}}$.

- (M4) holds for M.
  Proof: To see this, assume that ν is a $\prec$-sucessor of τ, so $\nu \in S_{\gamma_\tau}$ and $\tau < \nu$, and let $\zeta < \gamma_\tau$. We need to show that there is $\sigma \lhd \tau$ with $\zeta < \gamma_\sigma < \gamma_\tau$. This is easily arranged: First, since ν is a morass point, we may pick $\tau < \eta < \nu$ such that $F_\eta \models ZF^-$ and $L_{s(\tau)} \in F_\eta$. Then, form a countable elementary submodel X of F_η that contains as subsets the hull $L_{s(\tau)}\{\zeta \cup p_\tau\}$ and the singleton $\{\tau\}$ and such that $X \cap \gamma_\tau$ is transitive and $< \gamma_\tau$. (By regularity of γ_τ in F_η, this can be achieved by repeatedly taking the transitive closure of the part below γ_τ and then forming the union.) Take the collapse $\pi : X \to_{coll} F_{\bar{\eta}}$, and let $\sigma = \pi(\tau)$, $\bar{p} = \pi(p_\tau)$. π will map ZF^--models to ZF^--models and hence σ is a morass point. The restriction of the uncollapsing map to $F_{s(\sigma)}$ will be

elementary, as there are infinitely many ZF^--models between F_τ and F_ν, so $\pi^{-1}|F_{s(\sigma)}$ is in particular $\Sigma_1^{s(\sigma)}$-preserving. It follows that $\sigma \triangleleft \tau$. τ will be collapsed to the unique uncountable cardinal γ_σ of F_σ, and, by transitivity of the hull below γ_τ, will be $> \zeta$. This is exactly what we wanted to obtain.

- (M5) holds for M.
  Proof: Suppose $\Gamma_\tau := \{\gamma_\sigma | \sigma \triangleleft \tau\} \uparrow_{\gamma_\tau}$, and let $\zeta \in \tau$. We need to find $\sigma < \tau$, $\bar{\zeta} < \sigma$ such that $\pi_{\sigma\tau}(\bar{\zeta}) = \zeta$. For this, recall that $L_{s(\tau)}\{\gamma_\tau \cup p_\tau\} = L_{s(\tau)} \supset \tau$, so by the $\triangleleft$-cofinality of Γ_τ in γ_τ, we can find $\sigma \triangleleft \tau$ such that $\zeta \in L_{s(\tau)}\{\gamma_\sigma \cup p_\tau\}$. But, as $\pi_{\sigma\tau}|\gamma_\sigma = id$ and $\pi_{\sigma\tau}(p_\sigma) = p_\tau$, $L_{s(\tau)}\{\gamma_\sigma \cup p_\tau\}$ is just $rng(\pi_{\sigma\tau})$, so we can simply let $\bar{\zeta}$ be $\pi_{\sigma\tau}^{-1}(\zeta)$.

□

11.2 More on hyperings

In this section, we generalize the ideas and lemmas used in the preceeding construction. Already in our account on the □-principle, we had to deal with singularizing locations. We generalize some of the observations made there. For this section, let H be a linear hypering.

Definition 7: *We say that $s \in H$ singularizes $\sigma \in On$ if there are $p \subset_{fin} L_s$, $\gamma < \sigma$ such that $\sigma \subset L_s\{\gamma \cup p\}$. In this case, we also say that σ is collapsed to γ at s in the parameter p.*
If s is $<_H$-minimal such that s singularizes σ, we call s the singularizing location for σ.

Lemma 8: *Let $\sigma \in On$, s be the singularizing location for σ, and suppose H is slow. Then s is a limit location.*

Proof: Suppose for a contradiction that $s = \bar{s}^+$ for some location $\bar{s} \in H$. Assume that σ is collapsed to γ in parameter p. As H is slow, there is

$q \subset_{fin} L_{\bar{s}}$ such that $L_s\{\gamma \cup p\} \subset L_{\bar{s}}\{\gamma \cup p \cup q\}$: For example, q can be the set of all the (finitely many) new Skolem values added when passing from $\bar{s}$ to s. This implies $\sigma \subset L_{\bar{s}}\{\gamma \cup (p \cup q)\}$, so σ is already collapsed at $\bar{s}$, contradicting the minimal choice of s.
□

Lemma 9: *Let $\sigma \in On$, s the singularizing location for σ. Suppose σ is collapsed to γ at s in parameter p. Then:*
$F_s = F_s\{\sigma \cup p\}$.

Proof: Consider $F_{\bar{s}} = coll'' F_s\{\sigma \cup p\}$, let π be the uncollapsing map, $\bar{s}$ the condensed location as given by definition 11, $\bar{p} := \pi^{-1}(p)$. We start by showing that for $q \in F_s\{\sigma \cup p\}$, if $S_{\phi_n}(q)$ can be formed in s, then, letting $\bar{q} := \pi^{-1}(q)$, $S_{\phi_n}(\bar{q})$ can be formed in $\bar{s}$.
Assume otherwise, so $p_n(\bar{s}) < \bar{q}$. Then $\pi(p_n(\bar{s})) <_{lex} q$, and $\pi(\bar{s}) <_{loc} s$ by linearity of H.
Suppose now that s is a limit location. Hence $\pi(\bar{s}^+) <_{loc} s$ as well. But, by clause (6) in the definition of a linear hypering, $\pi(\bar{s})$ is an H-location, and the property of being a successor location is preserved by π. So $\pi(\bar{s}^+) <_{loc} s$, and $\bar{s}^+ \in H$, which contradicts the definition of the condensed location.
Now we have that $F_{\bar{s}} = F_{\bar{s}}\{\sigma \cup \bar{p}\}$. Furthermore, we get that $\bar{s}$ collapses σ to γ.
Since s was minimal, it follows that $\bar{s} = s$. Since p was minimal, also $\bar{p} = p$. So $F_s = F_{\bar{s}} = F_{\bar{s}}\{\sigma \cup \bar{p}\} = F_s\{\sigma \cup p\}$. If s is a successor location, the proof is similar: The image of the condensed location will be strictly below s, so the image of the successor cannot be strictly greater, contradicting definition 11. In any case, the theorem is proved.
□

Corollary 10: *In the setting of the last lemma, we have in fact $F_s = F_s\{\gamma \cup p\}$.*

Proof: : By definition of s, $\sigma \subset F_s\{\gamma \cup p\}$, the rest follows from the last lemma.
□

Definition 11: *A hypering H with the property proved in the last corollary, i.e. that, if there is σ such that σ is collapsed to $\gamma < \sigma$ at $s \in H$ in parameter p implies $F_s = F_s\{\gamma \cup p\}$ is called a sound hypering, and this property is from now on referred to as the soundness of H.*

An important fact in constructible combinatorics is the isomorphy of hulls. The following shows that this is preserved by Σ_1^s-maps for a special kind of hulls that occurs frequently.

Lemma 12: *(Preservation of Isomorphy) Let s and t be H-locations such that $\pi : F_s \to F_t$ is Σ_1^s-preserving. Assume $s_0 \leq_{loc} s_1 \leq_{loc} s$ are limit locations of H and let $p_0 \subset_{fin} F_{s_0}$, $p_1 \subset_{fin} F_{s_1}$, $\alpha \leq \alpha(s_0)$. Then:*
$F_{s_0}\{\alpha \cup p_0\} \simeq F_{s_1}\{\alpha \cup p_1\} \leftrightarrow F_{\pi(s_0)}\{\pi(\alpha) \cup \pi(p_0)\} \simeq F_{\pi(s_1)}\{\pi(\alpha) \cup \pi(p_1)\}$.

Proof: Let $\sigma : F_{s_0}\{\alpha \cup p_0\} \simeq F_{s_1}\{\alpha \cup p_1\}$. We define a map $\hat{\sigma} : F_{\pi(s_0)}\{\pi(\alpha) \cup \pi(p_0)\} \to F_{\pi(s_1)}\{\pi(\alpha) \cup \pi(p_1)\}$ as the unique structure-preserving map such that $\hat{\sigma}|\pi(\alpha) = id|\alpha$ and $\hat{\sigma}(\pi(p_0)) = \pi(p_1)$, if it exists. We need to show that is does. So let τ_1 be an s_0-term, τ_2 be an s_1-term. In order for $\hat{\sigma}$ to be well-defined, we need to know that $R(\tau_1(\vec{x}, p_0), \tau_2(\vec{x}, p_0))$ for all $\vec{x} \subset_{fin} F_\alpha$ implies $R(\tau_1(\vec{x}, \pi(p_0)), \tau_2(\vec{x}, \pi(p_1)))$ for all $\vec{x} \subset_{fin} F_{\pi(\alpha)}$, where R is either $\in$ or $=$.
Assume otherwise: So there are $\tau_1, \tau_2, \vec{x}, R$ as above such that $\neg R(\tau_1(\vec{x}, \pi(p_0)), \tau_2(\vec{x}, \pi(p_1)))$, but $R(\tau_1(\vec{y}, p_0), \tau_2(\vec{y}, p_0))$ for all $\vec{y} \subset_{fin} F_\alpha$. But the former implies
$\exists x \neg R(\tau_1(\vec{x}, \pi(p_0)), \tau_2(\vec{x}, \pi(p_1)))$,
which is a Σ_1-formula and hence preserved by π. But then
$\exists \vec{y} \subset_{fin} F_\alpha \neg R(\tau_1(\vec{y}, p_0), \tau_2(\vec{y}, p_0))$, a contradiction. □

Remark: In the context of the last lemma, the isomorphism $\hat{\sigma}$ given there is called the canonical isomorphism.

11.3 Construction of Gap-1-Morasses

We have now gathered enough of the and facts about hyperings to construct a gap-1-morass in L directly from these. This 'meta'-finestructual proof makes the central steps of the morass construction very apparent. In particular, it should be observed that the crucial ingredient of slowness, which is a kind of a border between coarse and fine structure, is only relevant for (M6) and (M7). This explains why the 'coarser' hypering H_2 was sufficient for the construction of a coarse morass. We see here that the heart of the fine-structural power of a hypering is its slowness.

Luckily, most of the technical work has already been done. In particular, the construction given for a coarse morass makes sense for any linear hypering H, using H-singularizing locations $s^H(\sigma)$ instead of H_2-locations. A closer inspection of the proofs of (M0)-(M5) reveals that no particular property of H_2 besides being a linear hypering was used there. We will therefore not repeat the proofs but merely state which general properties of linear hyperings were used for the argument.
The remaining challenge is thus to show that (M6) and (M7) hold as well. As we already stated, this requires the slowness of the hypering.
Now for the argument. Until the end of this section, H denotes a slow, linear hypering.

Theorem 13: *The system $M := \langle S_0, S_1, \triangleleft, \prec, \pi_{\sigma\tau} | \sigma \triangleleft \tau\rangle$, constructed in H, is a gap-1-morass, i.e. satisfies the morass axioms (M0)-(M7).*

Proof:

- (M0) This does not make use of the power of hyperfine structure at all. It would work equally well if 'bare' F-stages were used instead of hyperfine levels.
- (M1) This only uses 'coarse' properties of the hierarchy as well and could also be carried out using only the F-levels without hyperfine interpolation.
- (M2) Here, clause (6) of the definition of linear hyperings, i.e. preservation of locations under fine maps, is invoced to guarantee that the

collapsing location can be mapped over so that its image is still an H-location.

- (M3) Requires soundness, continuity and clause (6).
- (M4) Requires continuity and (6).
- (M5) Requires the soundness of H.
- (M6) holds for M.

Proof: Consider $\hat{s}$, the least upper bound of $\{\pi_{\sigma\tau}(t) | t <_{loc} s(\sigma)$ is an H-location$\}$. This is an H-locations as hyperings are closed under least upper bounds.

Claim: $\lambda \cap F_{\hat{s}}\{\gamma_\tau \cup p_\tau\} = \lambda$

Proof: To see $\supset$, we use the usual argument: If $\delta < \lambda$, there must, by definition of λ, be $\delta' < \lambda$ such that $\delta' \geq \delta$ and $\delta' \in rng(\pi_{\sigma\tau})$. Pick such a δ' and let $\bar{\delta}'$ be the pre-image of δ' under $\pi_{\sigma\tau}$. Since σ is a morass point, there must be a function $f : \gamma_\sigma \rightarrow_{surj} \bar{\delta}'$ in $F_\sigma = F_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$. This is the first place where we need to use the slowness of H: Since $s(\sigma)$ is a collapsing location, it is a limit location. By continuity, there must be $t <_{loc} s(\sigma)$ such that already $f \in F_t\{\gamma_\sigma \cup p_\sigma\}$. Let $g := \pi_{\sigma\tau}(f)$, then g is a surjection from $\gamma_\tau = \pi_{\sigma\tau}(\gamma_\sigma)$ onto $\delta' = \pi_{\sigma\tau}(\bar{\delta}')$. But $\gamma_\tau \in F_{\hat{s}}\{\gamma_\tau \cup p_\tau\}$, so also $\delta, \subset F_{\hat{s}}\{\gamma_\tau \cup p_\tau\}$, hence, in particular, $\delta \in F_{\hat{s}}\{\gamma_\tau \cup p_\tau\}$.
For $\subset$, take $\delta \in F_{\hat{s}}\{\gamma_\tau \cup p_\tau\}$, find some $t <_H s(\sigma)$ such that this already holds at t, observe that $F_t\{\gamma_\sigma \cup p_\sigma\} \downarrow_\sigma$ by minimality of $s(\sigma)$ and that the bound maps over by $\pi_{\sigma\tau}$ to something $< \delta$. $\square$

Collapse $F_{\hat{s}}\{\gamma_\tau \cup p_\tau\}$ to $F_{\bar{s}}$ with collapsing map π and set $\bar{p} := \pi(p_\tau)$. As $\hat{s}$ is a limit location, so is $\bar{s}$.

Claim: $\bar{s} = s(\lambda)$.

Proof: $\geq_{loc}$ is clear, as $\bar{s}$ is a singularizing location for λ. If $\bar{s} >_{loc} s(\lambda)$, obtain a contradiction as follows:

As $p_\lambda \in F_{\bar{s}}\{\gamma_\tau \cup \bar{p}\}$ and $\bar{s}$ is a limit location, there is $t \in H$, strictly between $s(\lambda)$ and $\bar{s}$ in $<_{loc}$, with $p_\lambda \in F_t\{\gamma_\tau \cup \bar{p}\}$. $t >_{loc} s(\lambda)$ clearly singularizes λ, and the π-preimage of t must be $<_{loc} \bar{s}$. Then there is $r \in H$ with $r <_{loc} s(\sigma)$ and $\pi^{-1}(t) <_{loc} \hat{s}$. Furthermore $F_r\{\gamma_\sigma \cup p_\sigma\} \downarrow_\sigma$. The bound is preserved by $\pi_{\sigma\tau}$, so $F_{\pi_{\sigma\tau}(r)}\{\gamma_\tau \cup p_\tau\} \downarrow_\tau$, and further by π, so $F_t\{\gamma_\tau \cup \bar{p}\} \downarrow_\lambda$, which is impossible. □

Claim: $\bar{p} = p_\lambda$

Proof: Again, $\bar{p} \geq_{lex} p_\lambda$ is as in the last claim. If $\bar{p} >_{loc} p_\lambda$, $\bar{p} \in F_{s(\lambda)} = F_{s(\lambda)}\{\gamma_\lambda \cup p_\lambda\} = F_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\}$. Hence $\pi^{-1}(p_\lambda) <_{lex} p_\tau \in F_{\hat{s}}\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\}$, but p_τ was supposed to be minimal with this property. Contradiction. □

From the last two claims, we immediately get that $F_{\bar{s}}\{\gamma_\tau \cup \bar{p}\} = F_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\}$. To finish the proof, we need to find a morass map from $F_{s(\sigma)}$ to $F_{s(\lambda)}$. It is natural to try $\bar{\pi} := \pi \circ \pi_{\sigma\tau}$. $\bar{\pi}(\sigma) = \lambda$ and $\bar{\pi}(p_\sigma) = p_\lambda$ are clear. It only remains to check preservation of $\Sigma_1^{s(\sigma)}$-formulas. Upwards preservation is trivial. Downwards preservation by $\pi_{\sigma\tau}$ is also clear, but π is an isomorphism, so $\bar{p} = \pi \circ \pi_{\sigma\tau}$ is indeed $\Sigma_1^{s(\sigma)}$-preserving, and hence a morass map. Thus $\sigma \triangleleft \tau$. □

- (M7) holds for M.

Proof: Here, to obtain an intermediate morass point between σ and τ, the first candidate would be to consider e.g. $F_{s(\tau)}\{\alpha \cup \{\gamma_\tau\} \cup \{\tau\} \cup p_\tau\}$ and collapse this. τ will be send to some ν, the collapsing map will witness that $\nu \triangleleft \tau$ provided it is sufficiently preserving. That $\sigma \triangleleft \nu$ then follows from the fact that $\nu > \sigma$ and that $\triangleleft$ is a tree-ordering. The challenge is to see that $\nu \in S_\alpha$, i.e. that α is the unique uncountable cardinal in F_ν. For this, we need to know that γ_τ is collapsed to α. Hence, we start with the following important fact:

Claim: $F_{s(\tau)}\{\alpha \cup p_\tau\} \cap \gamma_\tau = \alpha$.

Proof: $\supset$ is obvious. To see $\subset$, suppose $\zeta \in F_{s(\tau)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$. As $\{\pi_{\sigma\tau}(t) | t <_{loc} s(\sigma)\}$ is $<_{loc}$-cofinal in $s(\tau)$, there is, by continuity, $\hat{s} <_H s(\sigma)$ such that already $\zeta \in F_{\pi_{\sigma\tau}(\hat{s})}\{\alpha \cup p_\tau\}$. $\hat{s} <_{loc} s(\sigma)$ and hence, $\hat{s}$ is not a collapsing location for σ. Collapse $F_{\hat{s}}\{\gamma_\sigma \cup p_\sigma\}$ to $F_{\bar{s}}$, let $\bar{\pi}$ be the collapsing map, $\bar{p} = \bar{\pi}^{-1}(p_\sigma)$. So $\bar{\pi}^{-1}(\sigma) < \sigma$, as σ is not collapsed. Also, $\bar{\pi}^{-1}(\sigma)$ is not collapsed in $F_{\bar{s}}$, for otherwise it would be collapsed to γ_σ and hence σ itself would be collapsed as well. (Note that sufficiently many F-levels are available in the structures in question by the fact that σ is a limit ordinal so that these properties are in fact preserved.) So $F_{\bar{s}}$ must come before F_σ in the ordering of the F-levels, and hence $F_{\bar{s}} \in F_\sigma$. σ is a $\prec$-limit, so there will be $\bar{\sigma} \prec \sigma$ with $\{F_{\bar{s}}, \bar{p}\} \subset F_{s(\bar{\sigma})} = F_{s(\bar{\sigma})}\{\gamma_\sigma \cup p_{\bar{\sigma}}\}$.

By preservation of isomorphy, we get that $F_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\} \simeq F_{\pi_{\sigma\tau}(\hat{s})}\{\gamma_\tau \cup p_\tau\}$. Call the corresponding canonical isomorphism $\hat{\pi}$. By the definition of the canonical isomorphism, no $\eta < \gamma_\tau$ is moved by $\hat{\pi}$, so we have $\zeta \in F_{\pi_{\sigma\tau}(\bar{s})}\{\alpha \cup \pi_{\sigma\tau}(\bar{p})\}$, and hence $\zeta \in F_{s(\pi_{\sigma\tau}(\bar{\sigma}))}\{\alpha \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\}$, as $\pi_{\sigma\tau}(\bar{p})$ is definable in, and hence an element, of the latter hull. Further, by the assumption of (M7), there is some $\bar{\eta} \in S_\alpha$ strictly $\triangleleft$-between $\bar{\sigma}$ and $\pi_{\sigma\tau}(\bar{\sigma})$. The map $\pi_{\eta\tau}$ will have critical point α, send α to γ_τ and will satisfy $rng(\pi_{\eta\tau}) = F_{s(\pi_{\sigma\tau}(\bar{\sigma}))}\{\alpha \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\}$. Hence $\zeta \in F_{s(\pi_{\sigma\tau}(\bar{\sigma}))}\{\alpha \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\} \cap \gamma_\tau = \alpha$. This was to be shown. $\square$

Form the collapse of the hull $F_{s(\tau)}\{\alpha \cup p_\tau\}$ to $F_{s'}$, let π be the collapsing map, $p' := \pi(p_\tau)$, $\bar{\tau} := \pi(\tau)$. The last claim implies that α is collapsed to γ_τ, so $\pi^{-1}(\alpha) = \gamma_\tau$. By the preservation properties of the maps considered, $\bar{\tau}$ is a morass point and in fact an element of S_α.
$\bar{\tau}$ is our designated morass point between σ and τ.

Claim: $s' = s(\bar{\tau})$.

Proof: Since s' is a collapsing location for $\bar{\tau}$, $s(\bar{\tau}) \leq_{loc} s'$. Suppose $s(\bar{\tau}) <_{loc} s'$. By continuity and compactness of H, there is $s_0 \in rng(\pi \circ \pi_{\sigma\tau})$ strictly $<_H$-between $s(\bar{\tau})$ and s' such that $p_{\bar{\tau}} \in F\{\alpha \cup p'\}$, let $\bar{s_0} = (\pi \circ \pi_{\sigma\tau})^{-1}(s_0)$. By minimality of $s(\sigma)$, $F_{\bar{s_0}}\{\gamma_\sigma \cup p_\sigma\} \downarrow_\sigma$; both π and $\pi_{\sigma\tau}$ preserve such bounds, so $F_{s_0}\{\alpha \cup p'\} \downarrow_{\bar{\tau}}$, but this contradicts the fact that s_0 was supposed to be $>_H s(\bar{\tau})$. $\square$

It remains to see that really $\sigma \lhd \bar{\tau} \lhd \tau$, i.e. that the corresponding morass maps exist. π^{-1} looks like a promising witness for the latter. If it is, the former follows from the fact that $\lhd$ is a tree-order and that we cannot have $\bar{\tau} \lhd \sigma$ as $\bar{\tau} \geq \sigma$.

Claim: π is $\Sigma_1^{s(\tau)}$-preserving.

Proof: π is a collapsing map. Hence, the non-trivial direction is downwards preservation. Take some $\Sigma_1^{s(\tau)}$-formula ψ, suppose ψ is $\exists \vec{x} \phi(\vec{x}, \vec{q})$, where ϕ is $\Sigma_0^{s(\tau)}$. Assume ψ has a witness $\vec{z}$ in $F_{s(\tau)}\{\gamma_{\bar{\tau}} \cup p_\tau\}$. By $<_H$-cofinality of $\pi_{\sigma\tau}$, pick $t <_H s(\sigma)$ with $\vec{q}, \vec{z} \in F_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\}$. Form the collapse $\pi_\sigma : F_t\{\gamma_\sigma \cup p_\sigma\} \rightarrow_{coll} F_{\bar{t}}$, $\bar{p_\sigma} := \pi_\sigma(p_\sigma)$. By preservation of isomorphy and $\pi_{\sigma\tau}(\gamma_\sigma) = \gamma_\tau$, we get $F_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\} \simeq F_{\pi_{\sigma\tau}(\bar{t})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p_\sigma})\}$. Let $\hat{\pi}$ be the canonical isomorphism, then the images of both $\vec{q}$ and $\vec{z}$ are elements of the latter hull. Again, using the minimality of collapsing locations, we can reflect this at an earlier stage, letting $\bar{\sigma} \lhd \eta \lhd \bar{\bar{\tau}}$ such that all relevant locations and parameters are included in $rng(\pi_{\eta\bar{\bar{\tau}}})$. But this is a morass map and hence it preserves Σ_1-formulas. Thus, in fact $\vec{z} \in F_{\pi_{\sigma\tau}(t)}\{\gamma_{\bar{\tau}} \cup p_\tau\} = rng(\pi_{\bar{\tau},\tau})$, so ψ is preserved downwards as well. □

Hence, (M7) holds. □

All properties of a morass have been checked, hence the proof is finished.

□

This concludes our account on morasses in particular and on constructible combinatorics in general.

12 Generalized Computability and Constructibility

In this chapter, we consider yet another way of analyzing the constructible hierarchy: Namely in terms of a notion of generalized computability, Infinite Time Register Machine computability ($ITRM$ computability). The basic operations of the fine structural frameworks considered so far, Skolem functions and interpretation, can be rephrased as recursive processes on appropriate structures: For example, a Skolem function giving the minimal witness for some statement according to a well-ordering of the underlying set is basically a μ-operator and can be interpreted as a generalized search procedure. Nested terms using these basic operators then correspond to more complex algorithms. Here, the question comes up whether this intuition can be made precise. A great advantage of analyzing L in terms of computability would be the great stability of results and constructions under changes of the underlying language; also, a lot of the strong intuitions behind programming Turing machines and its analogues can thus be made useful for the theory of constructibility.
Generalized recursion theory is an attempt to carry over the ideas behind notions like computability, decidability etc. into the realm of abstract set theory. These notions are in their classical sense connected to the natural numbers. Computability theory provides a tool for characterizing reals (subsets of ω) by the complexity of the means necessary to generate them. The power of this approach is due to a philosophical assumption known as the Church-Turing-Thesis (CTT):

(CTT): The notion of computability in the sense of recursion theory coincides with the intuitive notion of 'being capable to be generated according to a recipee'.

The CTT hence says that recursion theory adequatly catpures our (human) concept of finite constructions. The classical recursion theory limits the scope of mathematical objects by restrictring itself to operations on ω. As the real mathematics is much broader, the question arises whether a similar capture is feasible for objects given by infinite constructions. As we are able to communicate about such matters in a precise manner, we should expect a positive answer.

Intuitively, the set theory after Cantor is characterized by the step of transcending the concept of time underlying mathematical constructions in which everything is either the starting point or a successor. The obvious way to proceed would accordingly be to operate along an ordinal time axis. A similar consideration applies to the available space, which corresponds to memory in recursion theory and is e.g. modeled by the tape length in the case of Turing machines. This leads to the (α, β)-Turing machines of [2], which carry out β many steps on a tape of length α, where α and β are ordinals. As results of ordinal constructions, there is a natural connection between computations of ordinal Turing machines and L.

The different approaches to the classical theory of computation, like λ-calculus, recursive functions, Turing machines etc. have turned out to be equivalent, which is commonly seen as a consequence and a support for the Church-Turing thesis. Appropriately formulated, this definitional stability still holds in the ordinal case. However, it is unclear what the intuitive idea behind these concepts might be. Formulating a version of CTT for infinite time computability remains a philosophical challenge.

In this section, we present the concept of infinite time register machines, a generalization of unlimited registers machines (URM's) where computations are carried out along an ordinal time axis of a priori unlimited length, but with only finitely much memory. One might say that while the time is extended with respect to URM's, the hardware remains unaltered. We shall prove a theorem on the difference between recognizability and computability of reals with $ITRM$'s and show how $ITRM$'s are connected with the theory of constructibility. In particular, we give a precise analyis of the computational power of $ITRM$'s with certain numbers of registers. To this end, we introduce typed $ITRM$'s, where registers with simpler limit behaviour are used for the implementation of auxiliar functions.

12.1 Infinite Time Register Machines

Infinite Time Register Machines ($ITRM$'s) are meant to genalize computations with classical register machines to an ordinal time axis. We give here the definitions of [7], based on the unlimited register machines (URM's) of [38].

Definition 1: *An unlimited register machine (URM) has registers $R_0, R_1, \ldots$ which can hold natural numbers. An URM program is a finite list $P = I_0, I_1, \ldots, I_{s-1}$ of instructions, each of which may be of one of five kinds:*

- *the zero instruction $Z(n)$ changes the contents of R_n to 0, leaving all other registers unaltered;*
- *the successor instruction $S(n)$ increases the natural number contained in R_n by 1, leaving all other registers unaltered;*
- *the oracle instruction $O(n)$ replaces the content of the register R_n by the number 1 if the content is an element of the oracle, and by 0 otherwise;*
- *the transfer instruction $T(m, n)$ replaces the contents of R_n by the natural number contained in R_m, leaving all other registers unaltered;*
- *the jump instruction $J(m, n, q)$ is carried out as follows: the contents r_m and r_n of the registers R_m and R_n are compared, all registers are left unaltered; then, if $r_m = r_n$, the URM proceeds to the qth instruction of P; if $r_m \neq r_n$, the URM proceeds to the next instruction in P.*

Remarks/Conventions: It is clear that a URM-program P can only mention finitely many registers, which will be called the registers used by the program. The number of registers used can serve as a complexity measure of a program. Later, when we introduce typed programs, this will be replaced by triples of naturals. The position of the appearance of a certain instruction in a program is called its index; the indices of a program P are also called the program states of P.
At any ordinal time λ, the computation by P will be in a configuration which is given by the current program state $I^P(\lambda)$ and the sequence of register contents $\langle R_i^P(\lambda) | i < \omega \rangle$. The superscript P is dropped when the program is clear from the context; also, we will usually abbreviate the configuration by only mentioning registers used by P.

Definition 2: *Let $P = I_0, I_1, \ldots, I_{s-1}$ be an URM program. Let $Z \subseteq \omega$ be an oracle. A pair*

$$I : \theta \to \omega, R : \theta \to ({}^{\omega}\omega)$$

is an (infinite time register) computation by P if the following hold:

- *θ is an ordinal or $\theta = On$; θ is the length of the computation;*
- *$I(0) = 0$; the machine starts in state 0;*
- *If $\tau < \theta$ and $I(\tau) \notin s = \{0, 1, \ldots, s-1\}$ then $\theta = \tau + 1$; the machine halts if the machine state is not a program state of P;*
- *If $\tau < \theta$ and $I(\tau) \in s$ then $\tau + 1 < \theta$; the next configuration is determined by the instruction $I_{I(\tau)}$, with $I(\tau+1) = I(\tau)+1$ unless otherwise specified:*
  - *if $I_{I(\tau)}$ is the zero instruction $Z(n)$ then define $R(\tau+1) : \omega \to On$ by setting $R_k(\tau+1)$ to be 0 (if $k = n$) or $R_k(\tau)$ (otherwise).*
  - *if $I_{I(\tau)}$ is the successor instruction $S(n)$ then define $R_k(\tau+1)$ to be $R_k(\tau)+1$ (if $k = n$) or $R_k(\tau)$ (otherwise).*
  - *if $I_{I(\tau)}$ is the oracle instruction $O(n)$ then define $R_k(\tau+1)$ to be $R_k(\tau)$ (if $k \neq n$), or 1 (if $k = n$ and $R_k(\tau) \in Z$), or 0 (if $k = n$ and $R_k(\tau) \notin Z$).*
  - *if $I_{I(\tau)}$ is the transfer instruction $T(m, n)$ then define $R_k(\tau+1)$ to be $R_m(\tau)$ (if $k = n$) or $R_k(\tau)$ (otherwise).*
  - *if $I_{I(\tau)}$ is the jump instruction $J(m, n, q)$ then let $R(\tau+1) = R(\tau)$, and set $I(\tau+1) = q$ (if $R_m(\tau) = R_n(\tau)$) or $I(\tau+1) = I(\tau)+1$ (otherwise).*
- *If $\tau < \theta$ is a limit ordinal, then $I(\tau) = \liminf_{\sigma\to\tau} I(\sigma)$ and for all $k < \omega$*

$$R_k(\tau) = \begin{cases} \liminf_{\sigma\to\tau} R_k(\sigma), & \textbf{\textit{if}} \ \liminf_{\sigma\to\tau} R_k(\sigma) < \omega \\ 0, & \textbf{\textit{if}} \ \liminf_{\sigma\to\tau} R_k(\sigma) = \omega. \end{cases}$$

The computation is obviously determined recursively by the initial register contents $R(0)$, the oracle Z and the program P. We call it the (infinite time register) computation by P with input $R(0)$ and oracle Z. If the computation halts then $\theta = \beta + 1$ is a successor ordinal and $R(\beta)$ is the final register content. In this case we say that P computes $R(\beta)(0)$ from $R(0)$ and the oracle Z.

Convention: If a register R is reset to 0 at time λ due to the $\liminf$ being ω, we say that that R overflows at λ.

Definition 3: *An n-ary partial function $F : \omega^n \rightharpoonup \omega$ is computable if there is a register program P such that for every n-tuple $(a_0, \ldots, a_{n-1}) \in dom(F)$,*

$$P : (a_0, \ldots, a_{n-1}, 0, 0, \ldots), \emptyset \mapsto F(a_0, \ldots, a_{n-1}).$$

Here the oracle instruction is not needed.

Obviously any standard recursive function is computable.

Definition 4: *A subset $x \subseteq \omega$, i.e., a (single) real number, is computable if its characteristic function χ_x is computable.*

A subset $A \subseteq \mathcal{P}(\omega)$ is computable in the oracle Y if there is a register program P such that for all $Z \subseteq \omega$:

$$Z \in A \ \texttt{iff}\ P : (0, 0, \ldots), Y \times Z \mapsto 1\texttt{, and}\ Z \notin A\ \texttt{iff}\ P : (0, 0, \ldots), Y \times Z \mapsto 0$$

where $Y \times Z$ is the cartesian product of Y and Z with respect to the Cantor pairing function

$$(y, z) \mapsto \frac{(y+z)(y+z+1)}{2} + z.$$

Definition 5: *$x \subset \omega$ is computable in $y \in \omega$ if and only if there is an $ITRM$-program P such that $P^y(i) = 1$ if $i \in x$ and $P^y(i) = 0$, otherwise. In this case, we call x $ITRM$-reducible to y and write $x \leq_{ITRM} y$. If x and y are mutually reducible to each other, we write $x =_{ITRM} y$. If $x \leq_{ITRM} y$, but $x \neq_{ITRM} y$, then we write $x <_{ITRM} y$.*

The following basic fact of classical recursion theory will be used frequently.

Fact: A real $x \subset \omega$ is URM-computable if and only if it is computable by a URM with 3 registers. The same holds relative to any oracle.

The main sources for $ITRM$s are [7], [21]. The computational strength of $ITRM$'s is known to increase with the number of registers; in particular, there is $m \in \omega$ for each $n \in \omega$ such that an $ITRM$ with m registers can solve the halting problem for an $ITRM$ with n registers. This makes $ITRM$'s much stronger than classical register machines. On the other hand, the Infinite Time Turing Machines from [2] can simulate all $ITRM$'s and solve the halting problem for $ITRM$'s.

12.2 The Lost Melody Theorem for Infinite Time Register Machines

In this section, we state and prove an analogue to the Lost Melody Theorem for $ITTM$s from [2]. This proof was recently published in [21]. Intuitively, a lost melody theorem for a certain notion of computability says that there are reals that can be recognized, but not computed. Accordingly, let us call a real r *recognizable* if the set of reals $\{r\}$ is computable in the empty oracle $\emptyset$.

Definition 6: *A real $r \subset \omega$ is recognizable iff there is an $ITRM$-program P such that $P^x(0) \downarrow = 1$ if and only if $x = r$. If r is recognizable, but not computable by an $ITRM$, then r is called a lost melody.*

Theorem 7: *There is a real r which is recognizable, but not computable. Thus, the Lost Melody Theorem holds for ITRM's as well.*

The rest of this section is devoted to the proof of this theorem.

Roughly, the proof goes like this: We will choose a real $r \subset \omega$ that codes the minimal J_α such that $J_\alpha \models ZF^-$ and we will take the $<_L$-minimal such r. It will be easy to see that this r cannot be computable. However, to see that it is indeed recognizable, we will have to develop methods for, given r, evaluating truth predicates (i) within J_α, which is a comparably easy task and allows us to check that r codes a structure of the desired form, and (ii) for $J_{\alpha+2}$, which is necessary to identify r as the $<_L$-minimal such code. The latter makes it necessary to unfold the structure coded by r far enough so that properties of r itself become feasible. The idea is that, if there was a $<_L$-smaller real t with the same properties, it would have to be generated by Gödel functions over $J_{\alpha+1}$, and can hence be coded by parameters from J_α and a natural number coding a series of Gödel functions. The challenge remains to exhibit statements about the coded objects in terms of the codes with oracle r. This is the main concern of the following construction.

We need some notions from the Jensen fine structure theory of the constructible universe L. The canonical source for the following is [6].

Definition 8: *The Gödel basis functions are the following:*

1. $F_1(x, y) = \{x, y\}$
2. $F_2(x, y) = x \times y$
3. $F_3(x, y) = \{(u, v) : u \in x \wedge v \in y \wedge u \in v\}$
4. $F_4(x, y) = x - y$
5. $F_5(x, y) = x \cap y$
6. $F_6(x, y) = \bigcup x$
7. $F_7(x, y) = dom(x)$
8. $F_8(x, y) = \{(u, v) : (v, u) \in x\}$
9. $F_9(x, y) = \{(u, v, w) : (u, w, v) \in x\}$
10. $F_{10}(x, y) = \{(u, v, w) : (v, w, u) \in x\}$

Definition 9:

- $S_0 := \emptyset$
- $S_{\alpha+1} = \{F_i(x, y) | 1 \leq i \leq 10 \wedge \{x, y\} \subset S_\alpha\}$
- *If λ is a limit ordinal, then* $S_\lambda = \bigcup_{\iota<\lambda} S_\iota$

Fact: $J_\alpha = S_{\omega\alpha}$, where J_α is as defined in the first part. (See e.g. [6]).

From now on until the end of the proof, we define α to be the smallest ordinal such that $J_\alpha \models ZF^-$, where ZF^- is ZF without the power set axiom. Thus, α is a countable ordinal and J_α is itself countable. In fact, we have:

Lemma 10: *There is $s \in J_{\alpha+2}$ such that $s : \omega \to J_\alpha$ is surjective.*

Proof: Let M_α be the Σ_ω Skolem hull of $\{J_\alpha\}$ in $J_{\alpha+1}$. All elements of M_α are of the form $h(i, \{J_\alpha\})$, where h is the canonical Σ_1 Skolem function for $J_{\alpha+1}$ (which is Σ_1 over $J_{\alpha+1}$ and hence an element of $J_{\alpha+2}$). Also, we have $M_\alpha \in J_{\alpha+2}$. Let $\pi_\beta(x) : x \to y$ be the collapsing map for elements x of S_β, where y is the transitive collapse of x; this can be defined by induction on β and is easily seen to be an element of J_α for $\omega\alpha > \beta$. Furthermore, let π be the collapsing map for M_α, i.e. $\bigcup_\beta \pi_\beta$.

By the condensation lemma, the transitive collapse of M_α is of the form J_γ for some ordinal $\gamma \leq \alpha+1$. Since $J_{\alpha+1} \models$ 'there is a maximal J-stage' (namely J_α), the same holds in J_γ and so $\gamma = \delta + 1$ for some ordinal δ. Furthermore, for each axiom ϕ of ZF^-, $J_{\alpha+1} \models \phi^{J_\alpha}$, and hence $J_\gamma \models \phi^{J_\delta}$. Now, since α was minimal, it follows that $\alpha \leq \delta$, so we get $\alpha = \delta$. Now $f : i \to \pi(h(i, \{J_\alpha\}))$ is a partial surjection from ω onto $J_{\alpha+1}$. Define $s(x) = f(x)$ if $f(x) \in J_\alpha, \emptyset$ otherwise. Since J_α and $J_{\alpha+1}$ are in $J_{\alpha+2}$ and the latter is closed under rudimentary functions, f is easily seen to be an element of $J_{\alpha+2}$ as well and is the desired surjection.$\Box$

Convention: From now on, let $p : \omega \times \omega \to \omega$ be the Cantor pairing function.

Given a surjective map s as above, we can code J_α by a real r in a canonical way by simply putting $n = p(i, j)$ into r iff $s(i) \in s(j)$. Conversely, any real can be interpreted as a (possibly ill-founded) countable $\in$-structure in this way: Introduce countably many constants c_i and let $c_i E c_j \leftrightarrow p(i, j) \in r$. We say that r codes a model of ZF^- iff the $\in$-structure obtained in this way is such a model. (Obviously, any structure obtained in this way is transitive.) From now on, r denotes the $<_L$-minimal real that codes a $J_\alpha \models ZF^-$. Since a real coding J_α is easily generated from s as in Lemma 8 by applying some Gödel functions, we have $r \in J_{\alpha+2}$. From now on, we write $P^{\emptyset}(n)$ for the output that the program P generates from the input n in the empty oracle.

We start by proving:

Lemma 11: *r is not computable.*

Proof: Suppose for the sake of a contradiction that P computes r. Since computations are absolute between transitive models of ZF^-, there is an $\in$-formula $\phi(v)$ such that $P^{\emptyset}(n) = 1 \leftrightarrow J_\alpha \models \phi(n)$. Since comprehension holds in J_α, we have $r \in J_\alpha$. But then, since J_α satisfies replacement, the structure coded by r is itself an element of J_α, and we get $J_\alpha \in J_\alpha$, a contradiction. $\Box$

The algorithm for deciding whether or not the oracle number o is equal to r proceeds in three steps: First, it is checked whether the $\in$-structure R coded by o (in the sense mentioned above) is well-founded. This can be done by the algorithm testing for well-foundedness in [21]. If it doesn't succeed, we stop with negative result. If it does, we have to check whether all axioms of $ZF^- + V = L$ are valid in R and R is $\in$-minimal with this property. How to do this will follow easily from the effort taken for the last step: Assuming that the last step was successful (so o codes an $\in$-minimal model of $ZF^- + V = L$), we have to check whether o is $<_L$-minimal with this property. For this purpose, we fix the oracle number o for the rest of the proof.

Since it is checked by now that R is isomorphic to a transitive, well-founded, $\in$-minimal model of $ZF^- + V = L$, we may assume that R is of the form J_γ for some ordinal γ.

The J-hierarchy is obtained by iterating the process of closing $J_\beta \cup \{J_\beta\}$ under Gödel functions and taking unions at limits. Each element of $L = \bigcup_\beta J_\beta$ can therefore be represented as an iteration of Gödel functions applied to several of the S_γ. We view such a representation as a name for the element; if we restrict ourselves to an initial segment of L below a countable ordinal, this concept can be arithmetized, which will allow us to decide $\in$-formulae relativized to $J_{\alpha+2}$ when a J_α-oracle is given, where J_α is the minimal $ZF^- + V = L$-model as above.

We start by assigning natural numbers to the constituting elements of names; having a surjection s as in the introduction at our disposal, we let $3n$ code $s(n)$. $S_{\omega(\alpha+i)+j}$, $j \in \omega$, $i \in \{0, 1\}$ is represented by $3j + i + 1$. Names can now be coded by a suitable application of the pairing function p:

Definition 12: *A name is any number generated in the following way:*
(i) $p(2n, i)$ is a name for all $i, n \in \omega$
(ii) if a and b are names, $i \in \omega$, then so is $p(2i + 1, p(a, b))$.

Thus a name is an ordered pair $\langle a, b\rangle$ of naturals; the parity of the first element shows whether the name is flat, i.e. an S_β or an element of J_γ if a is even or whether and which Gödel function was applied. We explain the coding by giving the interpretation function I:

Definition 13: *The interpretation function I is defined as follows:*
(i) if $i = 3k + j$, $j \in \{1, 2\}$, then $I(p(2n, i)) = S_{\omega(\gamma+j-1)+k}$
(ii) otherwise, $I(p(2n, 3i)) = s(i)$
(iii) if $j = 10k + l$, $0 \leq l < 10$, then $I(p(2j + 1, p(a, b))) = F_{l+1}(I(a), I(b))$

Obviously, we assign multiple (and in fact infinitely many) names to each interpretation. However, this has a technically advantageous consequence:

Proposition 14: *Every natural number is a name.*

Proof: Trivial. □

This will allow us to search through $J_{\gamma+2}$ by searching through ω without any further checks.

The idea of a final constituent of a name is given by the following formal notion:

Definition 15: *The argument set $A(n)$ of a name n is given by the following recursive rules:*
(i) $A(p(2n,i)) = \{i\}$
(ii) $A(p(2k+1, p(a,b))) = A(a) \cup A(b)$

The following is our central tool for inductive arguments and definitions on names. For rational q, $\lceil q \rceil$ denotes the smallest integer n such that $n \geq q$.

Definition 16: *Let a be a name. Then $ps(a)$, the pseudostage of a, is defined as follows:*
(i) for $i = 3k + j$, $j \in \{1, 2\}$, $ps(p(2n, i)) = \omega(\gamma + j - 1) + 3k$
(ii) otherwise, $ps(p(2n, i)) = 0$
(iii) if $A(a), A(b) \subseteq \{3i | i \in \omega\}$, *then*
$ps(p(2k + 1, p(a, b))) = max\{ps(a), ps(b)\} + 1$
(iv) if $k = 10n + 1$ *for some* $n \in \omega$
$ps(p(2k + 1, p(a, b))) = max\{ps(a), ps(b)\} + 1$
(v) for $k = 10n + j$, $n \in \omega$, $j \in \{4, 5, 6\}$, *if* $max\{ps(a), ps(b)\} = \omega(\gamma + j) + t$, *let* $ps(p(2k + 1, p(a, b))) = \omega(\gamma + j) + 3[\frac{t}{3}] + 1$
(vi) for $k = 10n + j$, $n \in \omega$, $j \in \{2, 3, 7, 8, 9, 0\}$, *if* $max\{ps(a), ps(b)\} = \omega(\gamma + j) + t$, *let* $ps(p(2k + 1, p(a, b))) = \omega(\gamma + j) + 3\lceil \frac{t}{3} \rceil + 2$

We call a name m *minimal* if any name n with $I(n) = I(m)$ satisfies $ps(n) \geq ps(m)$.

In our arithmetization, the ordinal $\omega(\gamma + j) + i$ will be coded by $p(j, i)$. Accordingly, we slightly abuse our notation by viewing ps as a function taking naturals to naturals rather than to ordinals. If we talk about relations between pseudostages like $<$, we nevertheless mean the ordinals, and similarly for $ps(a)+2$ etc. Since the definition consists of easy recursive rules, which can be implemented even on a classical (finite) register machine, we note:

Proposition 17: *The pseudostage of a name can be computed by an ITRM-program in finite time.*

Proof: Trivial. $\Box$

From now on, if a and b are names, we write $a\tilde{\in}b$ and $a\tilde{=}b$ instead of $I(a) \in I(b)$ and $I(a) = I(b)$. Furthermore, we write $a <_{ps} b$ for $ps(a) < ps(b)$, similarly for $>, =$ etc. If β is an ordinal $a <_{ps} \beta$ means $ps(a) < \beta$. Sometimes we will write $a <_{ps+i} b$, $i \in \omega$ to indicate that $ps(a) + i < ps(b)$.

The following lemma is the main reason for the usefulness of the pseudostage. To enhance readability, we will e.g. write $\langle 8, x, y\rangle$ instead of $p(8, p(x, y))$.

Lemma 18: *Suppose a and b are names such that $I(a) \in I(b)$. Then:*
(i) If $ps(b) > 0$ then there is a name c such that $ps(c) < ps(b)$ and $I(c) = I(a)$. Thus, minimal names of elements of sets with names of $ps > 0$ have a strictly smaller pseudostage.
(ii) If $ps(b) = 0$ then there is a name c such that $ps(c) = 0$ and $I(c) = I(a)$.

Proof: (i) Easy induction on the pseudostage. To give a feeling for the kind of argument used here, we prove this for names of the form $\langle 8, x, y\rangle$. In the following, all names are chosen minimal. Consider $z \in \langle 8, x, y\rangle$, so that z is of the form $\langle v, u\rangle$, where $\langle u, v\rangle \in x$. By definition of ps, we have $x <_{ps+1} \langle 8, x, y\rangle$; now, by induction, $\langle u, v\rangle <_{ps} x$, $\{u, v\} <_{ps} \langle u, v\rangle$, $u <_{ps} \{u, v\}$, $v <_{ps} \{u, v\}$. Since pairing (i.e. application of F_1) increases the pseudostage by 1, we have $\langle v, u\rangle <_{ps} x <_{ps} \langle 8, x, y\rangle$.
(ii) By transitivity of J_γ.$\Box$

We will now define $a \in b$ and $a = b$ by induction on a partial order $\triangleleft$ of the tripels $\langle\sigma, a, b\rangle$, where $\sigma \in \{\in, =\}$, a,b names without using the interpretation function. This will allow a purely syntactical decision procedure for atomic formulae by inspection of the names.

Definition 19: *For triples as mentioned above, let* $m_a = max\{ps(a_1), ps(a_2)\}$, $m_b = max\{ps(b_1), ps(b_2)\}$. *Then define* $\langle\sigma_1, a_1, a_2\rangle \triangleleft \langle\sigma_2, b_1, b_2\rangle$ *iff one of the following holds:*
(i) $m_a < m_b$
(ii) $m_a = m_b$, *and left triple satisfies that* σ_1 *is* $\in$ *and* $a_1 <_{ps} a_2$, *while the analogous proposition for the right triple is not true*
(iii) $m_a = m_b$ *and the left, but not the right triple satisfies that* σ_1 *is* $=$.

Thus, given that we already know what $\in$ and $=$ mean for names with $ps < \beta$, we first explain $a \in b$ for $ps(a) < \beta$. Now, since elements of sets have names of smaller pseudostage then the sets themselves, we can define $a = b$ for names with $ps \leq \beta$ and then also $a \in b$ for $\beta = ps(a) > ps(b)$, since this is only possible if there is a name c with $ps(c) < \beta$ and $c = a$.

This approach leads to a meaningful definition: Since the maximum of the pseudostages cannot increase when going down in $\triangleleft$, and since we can go down at most two steps while preserving the maximum and these maxima are ordinals, we have the following:

Proposition 20: $\triangleleft$ *is well-founded.*

Proof: Trivial. $\square$

We will now give a formal version of the above sketch by induction on $\triangleleft$.

Definition 21: *For the sake of brevity, we abbreviate names and write e.g.* $a\tilde{\in}x, y$, *where we really mean* $a\tilde{\in}\langle 1, x, y\rangle$, *and similar for ordered pairs and triples. The replacement function* η *assigns (codes of) formulae to (codes of) formulae as follows:*
(i) $a\tilde{=}b \longmapsto \forall x <_{ps} max\{1, ps(a), ps(b)\}(x\tilde{\in}a \leftrightarrow x\tilde{\in}b)$
(ii) if $ps(a) = ps(b) = 0$, *then* $(a\tilde{\in}b) \longmapsto$ *true, if* $p(a, b) \in o$, *false, otherwise*
(iii) if $ps(a) > 0$, $ps(a), ps(b) \in \omega$, *then* $(a\tilde{\in}b) \longmapsto (\exists a_0, b_0 =_{ps} 0(a_0\tilde{=}a \wedge b_0\tilde{=}b \wedge a_0\tilde{\in}b_0))$

If $ps(b) = \beta \notin \omega$, $ps(a) < \beta$:
(i) $(a\tilde{\in}b = \langle 1, x, y\rangle) \longmapsto (a\tilde{=}x \vee a\tilde{=}y)$

(ii)$(a\tilde{\in}b=\langle 2,x,y\rangle)\longmapsto(\exists t_1<_{ps}x\exists t_2<_{ps}y(t_1\tilde{\in}x\wedge t_2\tilde{\in}y\wedge\langle t_1,t_2\rangle\tilde{=}a))$
(iii)$(a\tilde{\in}b=\langle 3,x,y\rangle)\longmapsto(\exists t_1<_{ps}x\exists t_2<_{ps}y(t_1\tilde{\in}x\wedge t_2\tilde{\in}y\wedge\langle t_1,t_2\rangle\tilde{=}a\wedge t_1\tilde{\in}t_2))$
(iv)$a\tilde{\in}b=\langle 4,x,y\rangle\longmapsto a\tilde{\in}x\wedge a\tilde{\notin}y$
(v)$a\tilde{\in}b=\langle 5,x,y\rangle\longmapsto a\tilde{\in}x\wedge a\tilde{\in}y$
(vi) $a\tilde{\in}b=\langle 6,x,y\rangle\longmapsto\exists z<_{ps}x(z\tilde{\in}x\wedge a\tilde{\in}z)$
(vii) $a\tilde{\in}b=\langle 7,x,y\rangle\longmapsto\exists u,v<_{ps+3}x\exists z<_{ps}x(z\tilde{=}\langle u,v\rangle\wedge a\tilde{=}u)$
(viii) $a\tilde{\in}b=\langle 8,x,y\rangle\longmapsto\exists z<_{ps}x\exists u,v<_{ps+1}z(z\tilde{=}\langle u,v\rangle\wedge a\tilde{=}\langle v,u\rangle\wedge z\tilde{\in}x)$
(ix) $a\tilde{\in}b=\langle 9,x,y\rangle\longmapsto\exists z<_{ps}x\exists u<_{ps+1}z\exists v,w<_{ps+3}z(z\tilde{=}\langle u,w,v\rangle\wedge z\tilde{\in}x\wedge a\tilde{=}\langle u,v,w\rangle)$
(x) $a\tilde{\in}b=\langle 10,x,y\rangle\longmapsto\exists z<_{ps}x\exists v<_{ps+1}z\exists w,u<_{ps+3}z(z\tilde{=}\langle v,w,u\rangle\wedge z\tilde{\in}x\wedge a\tilde{=}\langle u,v,w\rangle)$
(xi)b is an S-stage: $j\in\{1,2\},a\tilde{\in}b=p(2n,3k+j)\longmapsto\exists c\leq_{ps}a(c\tilde{=}a\wedge c<_{ps}b)$

If $ps(\alpha)\geq ps(\beta)$*:*
$a\tilde{\in}b\longmapsto\exists c<_{ps}b(a\tilde{=}c\wedge c\tilde{\in}b)$

If $0=ps(a)<ps(b)\in\omega$*, then the above almost works. Just replace each* $<_{ps}x$ *by* $<_{ps}max\{1,ps(x)\}$ *and terms like* $\langle u,v\rangle$ *by their definition (so if, for example,* $\{u,v\}$ *appears, replace it by a new variable c and add the condition* $\forall x<_{ps}max\{1,ps(c)\}(x\in c\leftrightarrow x=u\vee x=v)$*; similarly for ordered pairs and triples.)*

This function produces for every tripel $\langle s,x,y\rangle$, where s is $\tilde{\in}$ or $\tilde{=}$, an equivalent formula which is only based on $\triangleleft$-smaller atomic formulas. This procedure can be implemented on an ITRM.

For this, an arithmetization of the appearing formulas is needed: So set $a(x\tilde{\in}y)=5p(x,y)$, $a(x\tilde{=}y)=5p(x,y)+1$, $a(\phi\wedge\psi)=5p(a(\phi),a(psi))+2$, $a(\neg\phi)=5a(\phi)+3$, $a(\exists t_i\psi)=5p(p(i,a(\psi))+4$. A formula of the form $\exists t_i<_{ps+j}\phi$ is viewed as $\exists t_i(\text{ps}(t_i)+j<\text{ps}(x)\wedge\phi)$ in this respect; this will, in connection with the fact that the implementation considers conjunctions from left to right, lead to the termination of the algorithm. For the sake of uniformity, we introduce the symbol Ω and write the unbounded quantifiers as $\exists x\phi$ as $\exists x<_{ps}\Omega\phi$.

We now describe a stack algorithm for deciding $\in$-formulas in $J_{\gamma+2}$.

The implementation essentially uses only two registers, one of which contains a sequence of (codes of) $\in$-formulas coded by iterating the pairing function, while the others holds a status for the most recently processed element of this sequence (true, false, unknown, represented by 0,1,2, respectively). In addition, numerous auxiliary registers are used for calculating the auxiliary functions. We leave out those details.

For the description, we use sequences of pairs of the form $\langle f_1, s_1\rangle \longmapsto \langle f_2, s_2\rangle$, where the first element represents the sequence of formulas, the second the status; the reader will easily convince himself that the described development of the stack contents can be generated by a standard register machine without assigning other values to the two central registers in between. $\langle\rangle$ is the empty sequence, $\langle S|e\rangle$, $S = \langle s_1, ..., s_n\rangle$ denotes the sequence $\langle s_1, ..., s_n, e\rangle$; $\phi[x/i]$ for $i \in \omega$ is the formula derived from ϕ by replacing every free occurence of x in ϕ by i.

Base cases:
$\langle\langle\rangle, 1\rangle : output = true;$, $\langle\langle\rangle, 0\rangle : output = false$, $\langle\langle\rangle, ?\rangle : output = true$
$\langle\langle S|false\rangle, ?\rangle \longmapsto \langle S, 0\rangle$
$\langle\langle S|true\rangle, ?\rangle \longmapsto \langle S, 1\rangle$
Atomic formulas, $s \in \{\in, =\}$:
$\langle\langle S|s(x, y)\rangle, ?\rangle \longmapsto \langle\langle S|\eta(s(x, y))\rangle, ?\rangle$ (where η is the replacement function defined above; we assume that the formula on the right hand side is rewritten in such a way that in contains only $\exists$, $\neg$ and $\wedge$ as logical symbols.)
Conjunction:
$\langle\langle S|\phi \wedge \psi\rangle, ?\rangle \longmapsto \langle\langle\langle S|\phi \wedge \psi\rangle|\phi\rangle, ?\rangle$
$\langle\langle S|\phi \wedge \psi\rangle, 0\rangle \longmapsto \langle S, 0\rangle$
$\langle\langle S|\phi \wedge \psi\rangle, 1\rangle \longmapsto \langle\langle S|\psi\rangle, ?\rangle$
Negation:
$\langle\langle S|\neg\phi\rangle, ?\rangle \longmapsto \langle\langle\langle S|\neg\rangle|\psi\rangle, ?\rangle$
$\langle\langle S|\neg\rangle, 0\rangle \longmapsto \langle S, 1\rangle$
$\langle\langle S|\neg\rangle, 1\rangle \longmapsto \langle S, 0\rangle$
Existential quantifier:
$\langle\langle S|\exists x\phi\rangle, ?\rangle \longmapsto \langle\langle\langle S|\langle\exists x\phi, 0\rangle\rangle|\phi[x/0]\rangle, ?\rangle$
$\langle\langle S|\langle\exists x\phi, k\rangle\rangle, 1\rangle \longmapsto \langle S, 1\rangle$
$\langle\langle S|\langle\exists x\phi, k\rangle\rangle, 0\rangle \longmapsto \langle S, 0\rangle \longmapsto$
$\langle\langle\langle S|\langle\exists x\phi, k+1\rangle\rangle|\phi[x/k+1]\rangle, ?\rangle$

We will now show that this algorithm, given the input $\langle\langle\phi\rangle, ?\rangle$, ϕ an $\in$-

formula without free variables, always terminates and returns the truth value of $J_{\alpha+2} \models \phi$. We do this by induction on a well-order on these formulae.

In the following, $at(\phi)$ is the set of atomic subformulas of ϕ, written in the form $\langle \in, x, y \rangle$ etc. First, write ϕ in prenex normal form and bound all unbounded quantifiers with the help of the symbol Ω as introduced above.

For $\beta = \omega\gamma + j$, we set $\beta - i = \omega\gamma + (j - i)$ for $i \leq j$ and otherwise $\beta - i = \omega\gamma$.

Definition 22: *For such a formula ψ we define $pt(\psi)$, the potential of ψ, as follows:*
(i) $pt(\exists x <_{ps+i} y\phi) = \triangleleft{-}max\{\psi[x/ps(y) - i] | \psi \in at(\phi)\}$
(ii)$pt(\neg\phi)$ =$pt(\phi)$
(iii)$pt(\phi \wedge \psi) = \triangleleft{-}max\{pt(\phi), pt(\psi)\}$

Intuitively, pt(ψ) is an upper bound for the complexity of an atomic formula that has to be decided in order to evaluate ψ. For our purposes, a slightly finer order is necessary:

Definition 23: *If ϕ and ψ are formulae as described above, we let $\phi <_{\mathrm{F}} \psi$ iff one of the following cases occurs:*
(i) $pt(\phi) \triangleleft pt(\psi)$
(ii) $pt(\phi)$ and $pt(\psi)$ are incomparable in $\triangleleft$ and ϕ is a proper subformula of ψ.

Proposition 24: *$<_{\mathrm{F}}$ is a well-order on formulae of this kind.*

Proof: Trivial. □

By case distinction and the definition of the replacement function:

Lemma 25: *Whenever the algorithm puts a new formula ϕ on the stack on top of the formula ψ, we have $\phi <_{\mathrm{F}} \psi$.*

Proof: Trivial. (See the remark above.) □

Therefore, finally:

Lemma 26: *The algorithm terminates and gives the correct result.*

Proof: By induction on $<_{\mathrm{F}}$ with the help of the last lemma and the last proposition; observe that the bounding of a quantifier is always processed as the first conjunct by the way the algorithm treats conjunctions and that the complexity drop is therefore mirrored by the processing steps. The only interesting case is existential quantification: If $\exists x\phi$ is true, a witnessing x will be found, the formula will be taken off the stack, and the status register will be set to 1. If it is false, the seemingly pointless step in the last line in the description of the algorithm forces the occurence of a limit state, in which the formula is of the stack and the status register contains a 0.□

Thus, we are now able to decide arbitrary $\in$-formula in $J_{\alpha+2}$.

Finally, we have to check the $<_L$-minimality of o. Since in this case, we have $o = r$ and we know that $r \in J_{\alpha+2}$, we can do this by finding a name n for o and then checking for each name u whether $I(u) <_L r$ and u codes an $\in -minimal$ model of ZF^-. We just gave a procedure for the latter; the well-order $<_L$ of the constructible hierarchy (restricted to $J_{\alpha+2}$) can be expressed by an $\in$-formula in $J_{\alpha+2}$ and thus computed by the same method. Since there are only countably many names, we will have a way to test for $<_L$-minimality of a real given in the oracle as soon as we can tell how to find n such that $I(n) = o$. Again, since the number of names is countable, it suffices to be able to test for some given name m and some oracle number z whether or not $I(m) = z$.

For this, we first run through all the names until we find one, say y_0, such that $\neg\exists t(t \in I(y_0))$, that is, $I(y_0) = \emptyset$ and save it in a separate register.

Definition 27: *For $k \in \omega$ the canonical name $cn(k)$ of k is defined as follows:*
(1) $cn(0) = y_0$
(2) $cn(k+1) = \langle 6, \langle 1, cn(k), \langle 1, cn(k), cn(k)\rangle\rangle, 0\rangle$

Proposition 28: *$I(cn(k)) = k$ for $k \in \omega$*

Proof: If $k = 0$, this follows from the definition of y_0.
Otherwise, $I(\mathrm{cn}(k+1))$ is just $\bigcup\{I(cn(k)), \{I(cn(k)), I(cn(k))\}\}$, which, by induction, equals $\bigcup\{k, \{k, k\}\} = \bigcup\{k, \{k\}\} = k \cup \{k\} = k+1$.□

cn(k) is obviously easy to compute. So we can check for a name m whether $I(m) \in \omega$ simply by checking for any $i \in \omega$ wether $I(m) = I(\mathrm{cn}(i))$, at the same time finding the corresponding i in case of success. From this, one constructs an algorithm for checking $I(m) \subset \omega$ by running through the names and testing for being element of $I(m)$ and of ω.

To find out if $z \subseteq I(m)$, run through ω, checking by oracle call for every $k \in \omega$ whether $k \in z$, then, if so, whether $c(k) \in I(m)$.

Finally, check $I(m) \subseteq z$ by finding out if $I(m) \subseteq \omega$ and, if so, running through the canonical names of all $k \in \omega$ and calling the oracle for each $\mathrm{cn}(k) \in m$ to see if $k \in z$.

This concludes the description of the algorithm, and thus the proof of the lost melody theorem.

12.3 The Computational Strength of Infinite Time Register Machines

We now turn to the question which real numbers can be computed by ITRMs. This has been done in [5] for non-resetting infinite time register machines, a variant of $ITRM$s which stop in case of a register overflow. Also, the strength of an $ITRM$ with an arbitrary number of registers has been determined in [22].
Here, we consider the computational strength of $ITRM$s with up to n overflowing registers, where $n < \omega$. Let $\mathbf{R}_n$ be the set of reals computable by an $ITRM$ with at most n overflowing registers. The non-resetting $ITRM$s are thus the case $n = 0$, while the answer for arbitrarily many registers is $\bigcup_{i\in\omega} \mathbf{R}_i$.

When writing programs for $ITRM$s, it soon becomes apparent that a certain number of registers is never used in its limit behaviour, but rather deals with organizing operations like pushing/popping a stack etc. This is disadvantegous for a complexity analysis in terms of the number of registers, as naturally the real power of an $ITRM$ lies in the overflowing registers, not in the administrative ones. This could probably be accomodated by expanding the programming language in a suitable way, including certain tools as basic commands. However, a detailed choice of extra commands is always somewhat arbitrary, and a canonical one is not known so far. One might expect that such basic operations could be carried out without extra registers with a clever implementation. However, the problem is that, when taking limits, seemingly harmless intermediate calculations can vastly change the behaviour of the machine. We therefore introduce typed programs, where different registers behave differently at limit stages according to their type. A program can then be characterized by the number of registers of each types used in it, which gives more information than the mere overall number of registers. There is an imperfect analogy to second-order logic here, where the first-order quantifier complexity is rather irrelevant and only the second-order quantifiers are taken into account for the classification of a formula.

Definition 29: *(Register types) A register behaves just like a classical register at successor steps. Let R be a register, λ a limit ordinal.*

- *R is of type 1 if one of the following holds:*

- *(1) There is a finite, nonempty set T of registers of type 2 or 3 such that, between any two successive states of the computation where the content of an element of T is changed, the content of R is changed only finitely many times. Then, if λ is such that the times of changes of content for an element of T are cofinal in λ, we have $R(\lambda) = 0$. In all other cases where λ is a limit time, the content of R has changed only finitely many times and is hence eventually constant. R_λ is then defined to be this eventually constant value.*
- *(2) If there is no such set T and R has changed its value infinitely often at time λ, then the computation stops.*

- *R is of type 2 if $R(\lambda) = \liminf\{R(\iota)|\iota < \lambda\}$ in case this limit is $< \omega$. Otherwise $R(\lambda)$ is undefined and the computation stops*

- *R is of type 3 if $R(\lambda) = \liminf\{R(\iota)|\iota < \lambda\}$ in case this limit is $< \omega$ and $R(\lambda) = 0$, otherwise.*

The idea behind type 1 registers is to serve as auxiliar registers for sub-routines. A typical use would be to carry out stack operations for a register of higher type, but they may not carry information over limit steps of the registers involved in their intended sub-routine. Thus, type 1 registers correspond to finite register machines, type 2 registers to the non-overflowing registers of the former notion of *ITRM* from, e.g. [5], while type 3 registers behave as described in the first section. We can measure the complexity of an algorithm by the number of algorithms of each kind it uses. The main reason for the introduction of type 1 registers is to make complexity considerations stable with respect to minor changes in the definition of a machine. For example, adding commands for the organization of stacks should not change the computational complexity of an object. Recall that there is a universal classical register machine with 3 registers, so that sub-tasks for particular stack registers can also be carried out by at most 3 type 1-registers.

Convention: In plain text, we often call type 1 registers 'classical', type 2 registers 'non-overflowing' and type 3 registers 'overflowing'

Definition 30: *A machine type (sometimes simply 'type') is a triple of the form $t = \langle n_1, n_2, n_3\rangle \in \omega^3$. The τ-typed register machine program P^τ is* *the pair $\langle P, \tau\rangle$ consisting of a register machine program P together with a function $\tau : \omega \to \{1, 2, 3\}$. A type-$\tau$-specification of P is a pair $\rho = \langle \tau, f\rangle$, where f is a function from the set of type-1-registers to the powerset of the set of type-2 and type-3-registers. A computation according to P^ρ is performed in the obvious way, where the ith register used in P is evaluated according to its type $\tau(i)$ at limit stages, where the set relevant for a type-1-register with index i is $f(i)$. A typed program P^τ is of type $t = \langle n_1, n_2, n_3\rangle$ if, letting U be the set of register indices mentioned in P, $n_i = card\{j \in U|\tau(j) = i\}$ for $i \in \{1, 2, 3\}$.*

The specifications are not relevant for our complexity analysis, and whenever we use typed machines, they are clear. Hence, we only talk about types rather than their specifications. In some contexts, even the precise type is not as interesting as, e.g., the number of registers of the strongest type involved. Therefore we generalize the notion of a type to account for this concept.

Definition 31: *A generalized type is a triple of the form $t = \langle x_1, x_2, x_3\rangle$, where x_1, x_2, x_3 are either natural numbers or $'?'$. A type $\langle a_1, a_2, a_3\rangle$ is an instance of the generalized type t if $a_i \leq x_i$ whenever $x_i \neq ?$.*

Convention: If $<$ is an order relation, then $WO(<)$ abbreviates the statement that $<$ is a well-ordering.

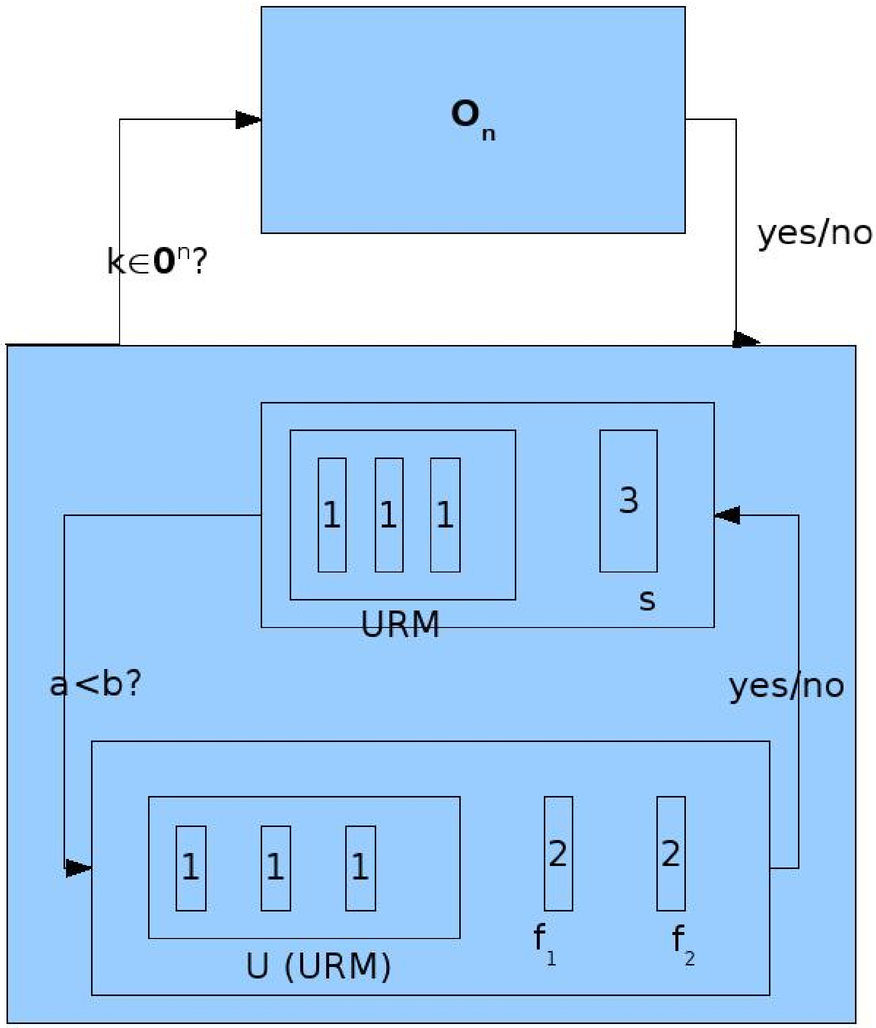
Definition 32: *Let $a \in 2^\omega$ be a real. For R_n a register machine program, let $<^a_n$ be the relation $\{\langle m, n\rangle|R^{\omega,B}_n(\langle m, n\rangle) = 1\}$, where $R^{\omega,a}_n$ is the (partial) function from ω to ω computed by R^a_n in finite time. (And is thus what a finite register machine with oracle a would compute.) Then a^+, the hyperjump of a, is defined to be the set $\{k \in \omega|WO(<^a_k)\}$.*
Also, $a^0 := a$, $a^{n+1} := (a^n)^+$. a^n is called the nth hyperjump of a.
$\emptyset^n$ is called the nth hyperjump, also written 0^n.

The following trick will be used many times in algorithms in this section: To determine whether the computation is at a limit stage, we use 2 type 2-registers R_1, R_2 with initial content 1 and 0, respectively. After that, we exchange the contents in each successor step. Then, the contents will be both equal to 0 if and only if the computation is in a limit state.

Theorem 33: *0^n can be computed by an ITRM with n overflowing registers.*

Proof: We prove this by induction. Obviously, a 0-ITRM can compute $0^0 = 0$. Now, assuming that we have an ITRM with n overflowing registers computing 0^n, we show how to construct an ITRM with $n+1$ overflowing registers computing 0^{n+1}. This will be done by a refined version of the stack algorithm for deciding well-orderings given in [21].
Suppose $\mathbf{O}_n$ is an ITRM such that, for $k \in \omega$, $\mathbf{O}_n(k) = 1$ if $k \in 0^n$, and otherwise 0. By induction, $\mathbf{O}_n$ can be assumed to have $\leq n$ overflowing registers. Also, let U be a universal finite register machine. We need to flag registers f_1 and f_2 to decide the halting problem for U. These will only contain the values 1 or 0 and in particular they will not overflow. The stack register, s, will be the new overflowing register used to store initial segments of a possible non-wellfounded branch.
Now, the algorithm proceeds like this: Suppose it is to be checked whether $m \in 0^{n+1}$ for some $m \in \omega$. Initially, let f_1 contain 0, f_2 contain 1. The content of s is set to 1. Now, use depth-first search to look for an infinite non-wellfounded branch in $<_m^{0^n}$, storing the intermediate results in s. If there's an infinite non-wellfounded branch, then there's one not containing 0 and 1, so we can without loss of generality assume that these are not used. Whenever a backtracking with stack content $\langle x_1, ..., x_i, x_{i+1}\rangle$ occurs, switch the stack content first to $\langle x_1, ..., x_i\rangle$ and then to $\langle x_1, ..., x_i, x_{i+1}+1\rangle$. If $i = 0$, i.e. the stack only contains one element, switch the content to 1 and then to $x_1 + 1$. So 0 can only become the stack content by overflow, which means that the search was successful. On the other hand, 1 can only occur in s at a limit stage when all possible first elements have been tried and there's no infinite non-wellfounded branch. Whenever a new number (expect 1) is put on the stack, we need to check whether the stack content $\langle x_1, ..., x_n\rangle$ forms a $<_m^{0^n}$-decreasing sequence. For this, it is sufficient to be able to decide whether $a <_m^{0^n} b$ for $a, b \in \omega$. This can be done using U, O_n and the flag registers f_1, f_2: We need to decide whether $R_m^{0^n}(\langle a, b\rangle) = 1$. For this, we simulate R_m by U with input $\langle a, b\rangle$. After each calculation step of R_m, we switch the register contents of f_1 and f_2. Oracle calls by R_m can be dealt with using $\mathbf{O}_n$. If $R_m^{0^n}$ terminates on this input with outcome 1, give a positive answer. Otherwise, there will either be a different result after finite time, in which case we output a negative answer, or $R_m^{0^n}$ does not halt. But then, f_1 and f_2 will be flashed infinitely often and a limit state will occur with f_1 and f_2

both containing 0. In this case, also a negative answer is returned.
This diagram explains the role of the registers and the information flow:



□

For the next step, we will need the notion of admissibility. As usually $On(x)$ abbreviates the statement that x is an ordinal. Good sources for these and related concepts are [13] and [27].

Definition 34: *The axiom system of Kripke-Platek set theory (KP) is a* *subtheory of ZFC consisting of the following axioms:*

1. *(Extensionality)* $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow (x = y))$
2. *(Induction Schema) Let* $\phi(x, \vec{a})$ *be an* ϵ*-formula with all free variables shown. Then* $\forall \vec{a} (\forall x ((\forall y \in x) \phi(y, \vec{a}) \rightarrow \phi(x, \vec{a})) \rightarrow \forall x \phi(x, \vec{a}))$
3. *(Pairing)* $\forall x \forall y \exists z \forall w ((w \in z) \leftrightarrow (w = x \vee w = y))$
4. *(Union)* $\forall x \exists y \forall z ((z \in y) \leftrightarrow (\exists u \in x)(z \in u))$
5. *(Infinity)* $\exists x (On(x) \wedge (x \neq \emptyset) \wedge (\forall y \in x)(\exists z \in x)(y \in z))$
6. *(Cartesian Product)* $\forall x \forall y \exists z \forall u ((u \in x) \leftrightarrow (\exists a \in x)(\exists b \in y)(u = (a, b))))$*, where* (a, b) *is the pair as defined in (3)*
7. *(Σ_0-Comprehension Schema) Let* $\phi(\vec{a}, x)$ *be a* Σ_0*-formula with all free variables shown. Then* $\forall \vec{a} \forall x \exists y \forall z ((z \in y) \leftrightarrow (z \in x \wedge \phi(\vec{a}, z)))$
8. *Let* $\phi(x, y, \vec{a})$ *be a* Σ_0*-formula with all free variables shown. Then* $\forall \vec{a} (\forall x \exists y \phi(y, x, \vec{a}) \rightarrow \forall u \exists v \forall x \in u \exists y \in v \phi(y, x, \vec{a}))$

Definition 35: *An ordinal α is admissible iff $L_\alpha \models KP$. For a class A, α is A-admissible iff $L_\alpha^A \models KP$.*

Remark 1: There are many equivalent formulations of this concept. For example, α is admissible iff $\rho_{L_\alpha} = \alpha$, where ρ_{L_α} denotes the Σ_1-projectum of L_α.

Remark 2: We will also use the relativized notion of admissibility and admissibly ordinals. For $A \subset \omega$, $\alpha \in On$ is admissible if $L_\alpha[A] \models ZF^-$.

Convention: For the rest of this chapter, ω_n^{CK} denotes the nth admissible ordinal greater than ω. Also, we let $\mu := sup\{\omega_n^{CK} | n \in \omega\}$. When relativized admissibles are considered, we drop the superscript CK and write e.g. ω_n^X instead of $\omega_n^{X\,CK}$.

We can now start by proving an upper bound for the halting time of an n-machine. The next two results are adaptions of results by Koepke [22] to typed machines.

Theorem 36: *A program of type $\langle ?, ?, 0\rangle$ either halts before ω_1^{CK} or does not halt at all.*

Proof: See [5] or below. □

This can be used as the base case of the following stronger version:

Theorem 37: *A program of type $\langle ?, ?, n\rangle$ either halts before ω_{n+1}^{CK} or does not halt at all.*

Proof: See [7] or below. □

Also, we can already determine the computational power of two rather broad types.

Theorem 38:

1. *$x \subset \omega$ is computable in type $\langle ?, ?, 0\rangle$ iff it is an element of $L_{\omega_1^{CK}}$.*
2. *$x \subset \omega$ is computable in type $\langle ?, ?, ?\rangle$ iff it is an element of L_μ.*

Proof: See [5] for (1) and [7] for (2). The adaption to typed machines is straightforward. □

Our goal is to achieve a finer analysis of the computational power of various types. For this, we need some notions and results from hyperarithmetical theory. Since everything we use can be found in [27] or [13], we assume some familiarity with the central ideas of admissible sets, recursive well-orders and ordinals. We write $x \leq_h y$ if x is hyperarithmetically reducible to y.

Definition 39: *For $Z \subset \omega$, $\mathcal{O}^Z$ denotes the set of indices e for Z-recursive functions such that e^Z computes a well-order.*

Definition 40: *If X is a set, then the $\subset$-smallest transitive admissible set M such that $X \in M$ is the admissible hull of X, which we denote by $Adh(X)$.*

Lemma 41: *A real is hyperarithmetic iff it is Δ_1^1-definable. The hyperarithmetical reals are exactly those in $L_{\omega_1^{CK}}$. Hence, a real is hyperarithmetic iff it is computable in type $\langle ?, ?, 0\rangle$. The relativization of this is also true.*

Proof: □

For the proof of lemma 44, I am indebted to Philip Welch for a private e-mail on this topic. We will use the following standard facts:

1. ω_1^Y is the supremum of the order types of all well-orders recursive in Y
2. $X \leq_h Y$ iff $X \in Adh(Y)$
3. If $Y \subset \omega$, then $Adh(Y) = L_{\omega_1^Y}[Y]$
4. If $X \subset \omega$, then $\mathtt{0}^X$ is $\Pi_1^{1,X}$-complete (that is, every $\Pi_1^{1,X}$-set of integers is many-one-reducible to $\mathtt{0}^X$)
5. (Spector-Gandy Theorem) If $Z \subset \omega$ and $A \subset \omega$ is $\Pi_1^{1,Z}$, then there is an arithmetical predicate ϕ such that $i \in A \leftrightarrow \exists Y \leq_h Z\phi(Y, i)$

Lemma 42: *There is a recursive bijection between the* Σ_1*-theory of* $\langle L_{\omega_1^Z}[Z], \epsilon, Z\rangle$ *and* $\mathtt{0}^Z$*. In particular,* $\mathtt{0}^Z$ *is* $\Sigma_1(\langle L_{\omega_1^Z}[Z], \epsilon, Z\rangle)$*-definable, and hence an element of* $L_{\omega_1^Z+1}[Z]$*.*

Proof: For the reduction of $\mathtt{0}^Z$ to the Σ_1-theory, observe that, as $\mathtt{0}^Z$ is $\Pi_1^{1,Z}$, Spector-Gandy gives us ϕ such that $e \in \mathtt{0}^Z \leftrightarrow \exists Y \leq_h Z\phi(Y, e)$. By (2) and (3), the right-hand side is absolute to $L_{\omega_1^Z}[Z]$, hence we can replace it by $L_{\omega_1^Z}[Z] \models \exists Y\phi(Y, e)$, which is a Σ_1-statement.
In the other direction, let ϕ be a Σ_1-sentence with $L_{\omega_1^Z}[Z] \models \phi$. There must be $\alpha < \omega_1^Z$ such that already $L_\alpha[Z] \models \phi$.
The set of codes $E \subset \omega \times \omega$ of ω-models with $\langle\omega, E\rangle \models KP + V = L$ is lightface Borel with some rank $\gamma < \omega_1^Z$. Therefore:

$$L_{\omega_1^Z}[Z] \models \phi \leftrightarrow \exists e \in \omega(e \in \mathtt{0}^Z \wedge \forall E(\langle\omega, E\rangle \models (e \in \mathtt{0}^Z \wedge KP + V = L) \rightarrow \langle\omega, E\rangle \models \phi)).$$

The direction from left to right is clear: If the Σ_1-statement ϕ is true in $L_{\omega_1^Z}[Z]$, then there is some Z-recursive ordinal α such that $\beta \geq \alpha \rightarrow L_\beta[Z] \models \phi$. Then, let e be an index for α.
For the converse, observe that the RHS states the existence of a recursive ordinal such that all L-stages containing that ordinal are models of ϕ, which is sufficient by upwards preservation of Σ_1.
□

The following lemma connects hyperjumps with admissible levels and is thus an important tool for a fine analysis of the computational strength of $ITRM$s.

Lemma 43: *For $k \in \omega$, $0^k \in L_{\omega_k^{CK}+1}$.*

Proof: This is proved by induction on k, using Fact (5) in each step to see that 0^k is Σ_1 over $L_{\omega_k^{CK}}$, and hence an element of $L_{\omega_k^{CK}+1}$, as desired. □

Lemma 44: *$X \subset \omega$ is hyperarithmetic in the nth hyperjump if and only if $X \in L_{\omega_{n+1}^{CK}}$.*

Proof: We show this by induction on n.
First, let $n = 0$, and $X \subset \omega$ by hyperarithmetic. Then $X \in Adh(\emptyset)$ by Fact (2). But $Adh(\emptyset) = L_{\omega_1^{CK}}$ by Fact (3). So $X \in L_{\omega_1^{CK}}$. The other direction follows by simply following the equivalences in the reverse direction.
Now, suppose $n = m+1$ for some $m \in \omega$, $X \leq_h 0^m$. Then $X \in L_{\omega_1^{0^m}}[0^m]$ by the last lemma. In particular, $0^m \in L_{\omega_1^{0^m}}$, which is admissible. Therefore, we must have $L_{\omega_1^{0^m}}[0^m] = L_{\omega_1^{0^m}}$. As all these steps are in fact equivalences, we deduce that $X \leq_h 0^m$ if and only if $X \in L_{\omega_1^{0^m}}$. But $L_{\omega_1^{0^m}}$ is admissible and has ordinal height larger than ω_m^{CK}, so we get $\omega_1^{0^{CK}} \geq \omega_{m+1}$.
Observe that $\omega_1^{0^m} > \omega_{m+1}^{CK}$ is impossible, as $0^m \in L_{\omega_{m+1}^{CK}}$ and $L_{\omega_1^{0^m}}$ was minimal with the property of being admissible and containing 0^m, so we would otherwise have $0^m \in L_{\omega_{m+1}^{CK}}$, a contradiction.
Therefore, we indeed get $L_{\omega_1^{0^m}} = L_{\omega_{m+1}}$. □

We can now distinguish very sharply between the computational powers of types according to the number of resetting registers.

Theorem 45: *The halting problem for type $\langle ?, ?, n\rangle$ can be solved in type $\langle ?, ?, n+1\rangle$.*

Proof: We saw that the reals computable in type $\langle ?, ?, n+1\rangle$ are exactly those in $L_{\omega_{n+1}^{CK}}$. Now, if P is a program of type $\langle ?, ?, n\rangle$, then, denoting by $P\downarrow^{\sigma}$ the statement that P applied to the empty input stops after σ many steps and assuming that P uses d many registers in total, we have:

$P\downarrow \leftrightarrow \exists \sigma \in \omega_{n+1}^{CK}(P\downarrow_{\sigma})$.

But $P\downarrow^{\sigma} \leftrightarrow \exists f : \sigma \to \omega^{d+1}$($f$ is a computation according to P and $I(\sigma) = STOP$). Denote the last part by $\phi(f, P, \sigma)$. If such f exists, it will be an element of $L_{\omega_{n+1}^{CK}}$ by a simple induction. Hence $P\downarrow \sigma \leftrightarrow \exists f \in L_{\omega_{n+1}^{CK}} \phi(f, P, \sigma)$. So $P\downarrow \leftrightarrow \exists \sigma, f \in L_{\omega_{n+1}^{CK}} \phi(f, P, \sigma)$, which is $\Sigma_1(L_{\omega_{n+1}^{CK}})$.
Therefore, $\{P \in \omega$ is a program of type $\langle ?, ?, n\rangle$ such that $P\downarrow\} \in L_{\omega_{n+1}^{CK}+1} \subset L_{\omega_{n+2}^{CK}}$. This number is computable in type $\langle ?, ?, n+1\rangle$ and codes the halting problem for type $\langle ?, ?, n\rangle$, as desired. $\square$

12.4 More on unresetting register machines

In contrast to the general $ITRM$s, there are universal types if we restrict ourselves to unresetting register machines, i.e. machines of generalized type $\langle ?, ?, 0\rangle$.

Definition 46: *Let $\tau = \langle a, b, c\rangle$ be a generalized type. An instance τ_0 of τ is universal for τ (τ-universal) if any real x computable in type τ is also computable in type τ_0.*

Theorem 47: *There is a universal $\langle ?, ?, 0\rangle$-type.*

Proof: With $k \in \omega$, write D_k for the kth computably enumerable binary relation on ω. Suppose x is $\langle ?, ?, 0\rangle$-computable. Then $x \in L_{\omega_1^{CK}}$. So $\{x\}$ is lightface Δ^1_1, and in particular Π^1_1. It is well-known that lightface Π^1_1-reals can be represented in the form

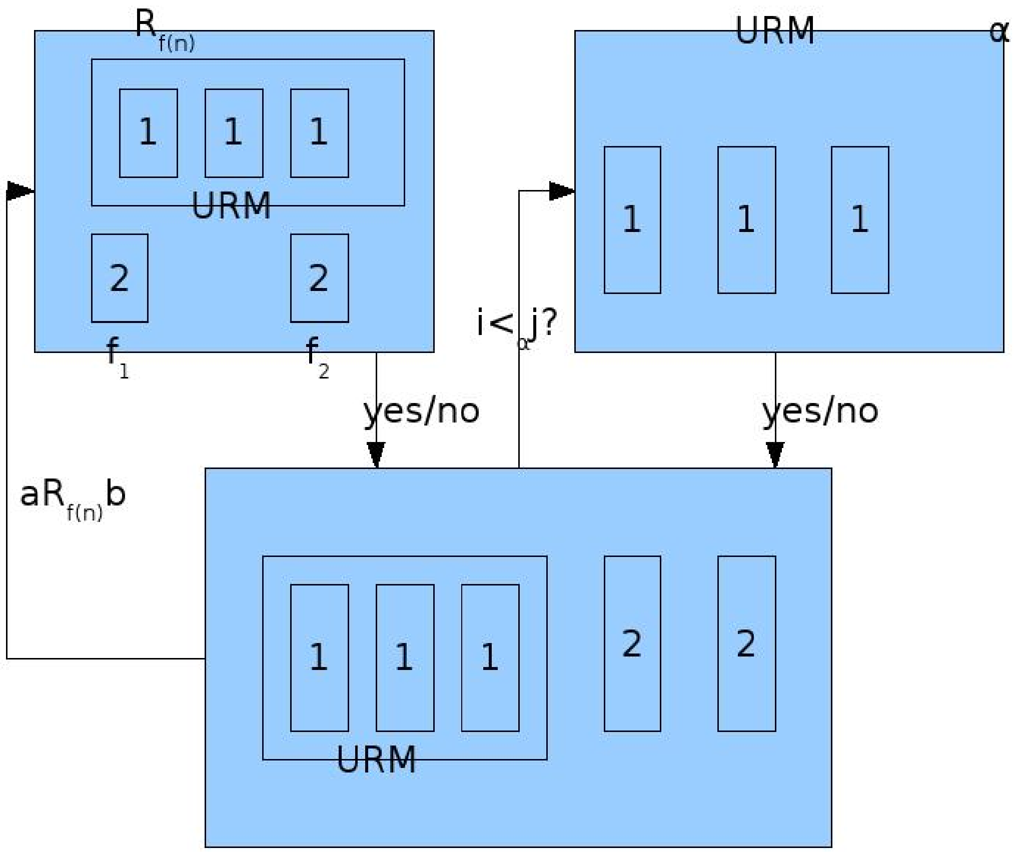
$n \in x \leftrightarrow R_{f(n)}$ is well-founded

for a recursive function $f : \omega \to \omega$. But x is also Σ^1_1, and standard results on Σ^1_1-bounding hence imply the existence of $\alpha < \omega_1^{CK}$ such that:

$n \in x \leftrightarrow R_{f(n)}$ is well-founded and $otp(R_{f(n)}) < \alpha$.

We argue that the RHS can be checked in type $\langle 9, 4, 0\rangle$.
This is done by carefully reconsidering the algorithm from [5]. First, $R_{f(n)}$ can be computed in type $\langle 3, 2, 0\rangle$: Three type 1-registers are sufficient to implement the algorithm for semi-deciding $R_{f(n)}$, the two additional type 2-registers f_1, f_2 are used as flags to solve the halting problem for this machine. If it halts, we output 1, otherwise 0.
Second, a relation of order type α can be decided by another 3 classical registers since $\alpha < \omega_1^{CK}$.
Now, using two type 2-registers R_1, R_2 for storing stacks and three extra type 1-registers for organizational purposes, we can carry out the algorithm which is given in [5] along with its correctness proof, for deciding whether $R_{f(n)}$ embedds in the relation of order type α. The extra type 3-registers serve the purpose to take care of the macros.

The overall organization is then as follows:



Hence any real computable by a $\langle ?, ?, 0\rangle$-machine is computable in type $\langle 9, 4, 0\rangle$, which is what we wanted to show. $\square$

12.5 More on recognizability

Having introduced the notion of a recognizable real, the question is natural to consider the structure of recognizable reals.

Proposition 48: *Every real x computable in type $\langle ?, ?, ? \rangle$ is also recognizable in this type. In fact, if x is computable in type $\langle a, b, c \rangle$, then it is recognizable in type $\langle a+3, b, c+1 \rangle$.*

Proof: We can recognize x simply by computing its bits one after the other, each time checking by an oracle call whether it matches the number in the oracle. In a separate overflowing register R, we store the position of the bit we are currently checking. If it doesn't match, we halt with negative result. Otherwise, R will overflow, and we halt with positive result. This recognizes x. The complexity bound is immediate from the construction. □

Proposition 49: *There is a recognizable real $x \subset \omega$ such that $x \notin L_\mu$.*

Proof: The real r defined in the section on the lost melody theorem is recognizable, but not $ITRM$-computable, hence $\notin L_\mu$ by the results of section 12.3. □

The set of reals recognizable by type $\langle ?, ?, ? \rangle$-programs is hence strictly larger than the class of reals computable in this type. A closer inspection of the proof reveals that in fact only one overflowing register was needed to recognize the number r defined there. Hence, $\langle ?, ?, k \rangle$-recognizability is a strictly weaker property than $\langle ?, ?, k \rangle$-computability for $1 \leq k < \omega$.

Definition 50: *If τ is a (generalized) type, denote by $RECOG_\tau$ the class of reals recognizable in type τ. $RECOG$ is simply $RECOG_{\langle ?,?,? \rangle}$. $RECOG_\tau(x)$ abbreviates the statement that $\omega \supset x \in RECOG_\tau$.*

Over a transitive ϵ-structure M, $RECOG(x)$ can be expressed as a Σ_3-statement claiming that there is a program P such that for all reals z in M there is a computation according to P that outputs 1 if and only if $z = x$. Denote this formal statement by FRECOG(x).

Lemma 51: *If $M \models ZF^-$ is transitive, then for $x \in M \cap 2^\omega$, we have $M \models$ FRECOG(x) iff $RECOG(x)$.*

Proof: Recall (e.g. from [21]) that $ITRM$-computations are absolute between transitive ZF^--models. So all we have to show is that, for an $ITRM$-program P, the computation of P on input 0 in oracle x will halt in M if it halts at all. As M is a ZF^--model, it follows from $x \in M$ that $\omega_i^x \in M$ for $i \in \omega$. Suppose that P uses n registers in total. The lemma about halting times of $ITRM$s with k overflowing registers easily relativizes, so any computation of P in oracle x must stop after at most ω_{n+1}^x many steps if it stops at all. (Recall that we drop the CK-superscript.) So, if the computation halts at all, its halting time is in M. By the absoluteness of computations, the computation within M is the same as in V. So, if there is a program Q that recognizes x, this fact will be reflected by M. $\square$

Now, each recognizable real uniquely corresponds to a (typed) $ITRM$-program that recognizes it. Hence $RECOG$ is countable. In particular, $sup\{RECOG\} < \omega_1$. On the other hand, $\{\alpha < \omega_1 | L_\alpha \models ZF^-\} \uparrow_{\omega_1}$. Thus, we can prove the following:

Lemma 52: *The set of recognizable reals RECOG has gaps in the constructible order, i.e. there are reals $r_1 <_L r_2 <_L r_3$ such that $\{r_1, r_3\} \subset RECOG$, but $r_2 \notin RECOG$.*

Proof: As $\{\alpha < \omega_1 | L_\alpha \models ZF^-\}$ is cofinal in ω_1 but $card(RECOG) = \omega$, there must be a minimal γ such that $L_\gamma \models ZF^-+$'There is an unrecognizable real'. Let r_3 be a minimal code of L_γ as in the proof of the lost melody theorem. By the same argument as there, r_3 is recognizable. However, let x be an unrecognizable (in the sense of L_γ) element of L_γ, then $x <_L r_3$ and, by the last lemma, x really is unrecognizable. So $0 <_L x <_L r_3$ witnesses our claim. $\square$

One can also consider a notion of relative recognizability. If $x, y \subset \omega$, $x \uplus y$ denotes $z \subset \omega$ such that $2i \in z$ iff $i \in x$ and $2i + 1 \in z$ iff $i \in y$, the join of x and y.

Definition 53: *$x \subset \omega$ is recognizable in $y \subset \omega$ iff there is an $ITRM$-program P such that $P^{y \uplus x}(0) = 1$ iff $z = x$. Write $x \leq_{recog} y$ in this case.*

Proposition 54: *If $y \leq_{ITRM} z$ and $x \leq_{recog} y$, then $x <_{recog} z$.*

Proof: All information about y relevant for identifying x can be computed from z by the first assumption. $\square$

It might be tempting to consider e.g. degrees of recognizability. Unfortunately, this does not seem to make sense, as $\leq_{RECOG}$ is not transitive, and hence no equivalence classes can be formed.

Lemma 55: $\leq_{RECOG}$ *is not transitive.*

Proof: Let x and r_3 be as in the proof of the last lemma. We then have $x \leq_{recog} r_3 \leq_{recog} \emptyset$: The second was shown above, the first follows from the last proposition, as x is as an element of the structure coded by r_3 easily $ITRM$-computable from r_3. But $x \not\leq_{recog} \emptyset$ by the choice of x. $\square$

Another question is how large the 'gaps' between recognizable ordinals can become.

Definition 56: *Let δ be a countable ordinal. A δ-gap in RECOG is a set $S = \{x \in L \cap 2^{\omega} | r_1 \leq_L x <_L r_2 \text{ for some } r_1, r_2 \in L \cap 2^{\omega}\}$ such that $r_2 \in RECOG$, $S \cap RECOG = \emptyset$ and $opt(S) = \delta$.*

Theorem 57: *For every $\delta < \omega_1^{CK}$, there is a δ-gap in RECOG.*

Proof: For each such δ, there is $\gamma(\delta)$ such that $L_{\gamma(\delta)} \models ZF^{-} + V = L$ and $L_{\gamma(\delta)}$ contains a $<_L$-intervall of reals with order type δ and all elements unrecognizable. This is again because there are cofinally many α with $L_{\alpha} \models ZF^{-} + V = L$ in ω_1, but only boundedly many recognizable reals. As $\delta < \omega_1^{CK}$, there is a code $x \subset \omega$ for δ. Also, let r' be a $<_L$-minimal code for $L_{\gamma(\delta)}$ as in the proof of the lost melody theorem.
We claim that $x \uplus r'$ is recognizable. This suffices, for it then follows that $x \uplus r'$ is $>_L$ all elements of the δ-intervall of unrecognizables in $L_{\gamma(\delta)}$. Now for the claim:
Clearly, $a \uplus b$ is recognizable if a and b are. (The converse is false.) As $\delta < \omega_1^{CK}$, it is computable, hence recognizable. Let ${r'}^{-1}(a)$ denote the element of $L_{\gamma(\delta)}$ coded by a for $a \in \omega$. For each pair $\langle a, b \rangle \in \omega^2$, the set $\{j \in \omega | {r'}^{-1}(j) \in 2^{\omega} \cap L \wedge {r'}^{-1}(a) <_L {r'}^{-1}(j) <_L {r'}^{-1}(b)\}$ is computable from r'. Now we only need to check for each of these pairs whether all elements in between are unrecognizable and whether the order type of the intervall is $\geq \delta$. The first can be done using the implementation of the truth-predicate

given in the proof of the lost melody theorem: We just test for each natural number i whether i belongs to the intervall and represents an unrecognizable real number. If no recognizable real in the intervall is found, we return a positive answer, otherwise a negative answer. For the second, we use again the algorithm by Koepke given in [5] for testing the embeddability of recursive order types. □

In fact, we can do much better.

Theorem 58: *For every $\delta < \mu$, there is a δ-gap in RECOG.*

Proof: It suffices to see that there is a ω_n^{CK}-gap for each $n \in \omega$. Basically, we can re-use the above proof, now taking $ITRM$-computable rather than recursive well-orderings. These are exactly the order types below μ. The only place in the above proof where this matters is in the algorithm that tests for embeddability of one well-order $<_1$ into another $<_2$, say, where we need to decide $<_1$ and $<_2$. But this can be achieved by using some extra registers. So we can again test whether an intervall as above has length $\geq \delta$ and proceed as above to see that the $<_L$-smallest code for an L-stage modeling ZF^- and having a δ-gap in $RECOG$ is recognizable. □

Remark/Open Questions:
(1) If it weren't for the well-foundedness test, we could carry out all steps necessary for recognizing r on a $\langle ?, 1, 0\rangle$-machine. However, this step is crucial. At this time, the question remains open: Are there lost melodies in type $\langle ?, ?, 0\rangle$?.
(2) Also, it is unclear how recognizability depends on the number of resetting registers: Are there e.g. reals recognizable in type $\langle ?, ?, n\rangle$ but not in type $\langle ?, ?, n+1\rangle$ for some $n > 0$?

12.6 Some Reverse Mathematics

In this short section, we analyze the complexity of the halting problem for $\langle ?, ?, n\rangle$-machines in terms of subsystems of second order arithmetic. We adapt ideas of unpublished notes from Koepke and Welch [9], where this is, in our language, done for $\langle 0, 0, n\rangle$-machines. Focusing on typed machines instead, we can improve some of the bounds.

First of all, we review some definitions and basic results. All of these can be found in the book by Simpson [13].

The language $\mathtt{L}_2$ of second-order arithmetic consists of the quantifiers $\exists$, $\forall$, the logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\leftrightarrow$, the relation symbols $<$, ϵ and $=$, the function symbols $+$ and $*$, the constant symbols 0 and 1, all of which are intended to have their obvious meaning, along with two types of variables v_i and X_i ($i \in \omega$), intended to range over ω and $\mathfrak{P}(\omega)$, respectively. Formulas, sentences etc. are defined in the obvious way. If ϕ is a formula such that all quantifiers of ϕ range over ω, ϕ is called arithmetical. If ϕ is of the form $\exists X\psi$ with ψ arithmetical, ψ is a Σ^1_1-formula; the other stages of the Levy hierarchy, Σ^1_n, Π^1_n and Δ^1_n are then defined as usual.

Definition 59: *The axioms of second order arithmetic are the following:*

- *(1) Basic Axioms*
  - *(a)* $\forall n(n+1 \neq 0)$
  - *(b)* $\forall m, n(m+1 = n+1 \rightarrow m = n)$
  - *(c)* $\forall m(m+0 = m)$
  - *(d)* $\forall m, n(m+(n+1) = (m+n)+1)$
  - *(e)* $\forall m(m * 0 = 0)$
  - *(f)* $\forall m, n(m*(n+1) = (m*n)+m)$
  - *(g)* $\forall m \neg m < 0$
  - *(h)* $\forall m, n(m < n+1 \leftrightarrow (m < n \vee m = n))$
- *(2) The Induction Axiom*
  $\forall X(0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X))$

- *(3) The Comprehension Scheme*
  For any $\mathtt{L}_2$*-formula* $\phi(n)$ *in which* X *does not occur as a free variable:* $\exists X \forall n(n \in X) \leftrightarrow \phi(n))$.

Subsystems of second order arithmetic are now obtained by restricting (3) to some subclass of $\mathtt{L}_2$-formulas. If Π is a class of $\mathtt{L}_2$-formulas, $\Pi - CA_0$ denotes the axiom system consisting of (1), (2) and $\exists X \forall n(n \in X) \leftrightarrow \phi(n))$ for all $\mathtt{L}_2 \in \Pi$ such that X does not occur as a free variable in ϕ.
The only subsystems we will need here are the ones given in the next definition.

Definition 60: *All of the following systems include the basic axioms.*

- RCA_0 *(Axiom of Recursive Comprehension): This is the induction axiom restricted to* Σ_1^0*-classes, i.e. the statement* $(\phi(0) \wedge \forall n(\phi(n) \rightarrow \phi(n+1))) \rightarrow \forall n \phi(n)$ *for* Σ_1^0*-formulas* ϕ*, together with the statement that, for* $\phi(n)$ *a* Σ_1^0*-formula,* ψ *a* Π_1^0*-formula, we have* $\forall n(\phi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \phi(n))$*. Here,* n *is a number variable and* X *a set variable not contained in* $free(\phi)$.

- ACA_0*: Call an* $\mathtt{L}_2$*-formula arithmetical if it contains no quantification over set variables occurs in it. (Note that this does not preclude the occurence of free set variables.)* ACA_0 *is then the statement that for all arithmetical formulas* ϕ*, there is* $X \subset \omega$ *such that* $\forall n(n \in X \leftrightarrow \phi(n))$ *together with the induction axiom.*

- $\Pi_1^1 - CA_0$ *is the statement that, for all* Π_1^1*-formulas* ψ*, there is* $X \subset \omega$ *such that*
  $\forall n(n \in X \leftrightarrow \psi(n))$ *together with the induction axiom.*

Theorem 61: *Denote by* $\boldsymbol{T}(\Phi)$ *the theory of* Φ*, where* Φ *is a set of* $\mathtt{L}_2$*-sentences. Then* $\boldsymbol{T}(RCA_0) \subset \boldsymbol{T}(ACA_0) \subset \boldsymbol{T}(\Pi_1^1 - CA_0)$*, and both inclusions are proper.*

Proof: See [13]. □

Also, we will need to consider the following set existence axiom.

Definition 62: *ATR_0: Let $\phi(n, X, \vec{p})$ be an arithmetical formula, where n is a number variable, X is a set variable and $\vec{p}$ is a (possibly empty) vector of number and set variables serving as parameters. Define a function $\Phi : \mathfrak{P}(\omega) \to \mathfrak{P}(\omega)$ by $\Phi(X) = \{n \in \omega | \phi(n, X)\}$ for $X \subset \omega$. Suppose $\mathtt{I}$ is a countable index set and $<_{\mathtt{I}}$ is a well-ordering of $\mathtt{I}$. For each $i \in \mathtt{I}$, we define $Y \subset \omega \times \mathtt{I}$ by $Y_i := \Phi(Y^i) = \{m | (m, a) \in Y\}$, $Y^i = \{(n, b) | n \in Y_j \wedge j <_{\mathtt{I}} i\}$. Intuitively, this amounts to iterating Φ along $<_{\mathtt{I}}$.*
Now, ATR_0 (axiom of transfinite recursion) is ACA_0 together with the statement that, for any ϕ, $(\mathtt{I}, <_{\mathtt{I}})$ as above, the set Y exists.

Lemma 63: *Over RCA_0, the following statements are equivalent:*

1. $\Pi^1_1 - CA_0$
2. *If $\langle T_k | k \in \omega\rangle$, $T_k \subset \omega^{<\omega}$ is a countable sequence of countable trees, then the set $\{k | T_k$ has a path$\}$ exists.*

Proof: See [13] 6.VI.1.1. □

We will use these principles to classify the strength of the existence of halting sets. For this, we assume that the typed programs are effectively enumerated in a convenient way, so that sets of typed programs canonically correspond to real numbers.

Definition 64:

- *$ITRM_n$ denotes the statement:*
  'The halting set $H_{\langle ?,?,n\rangle} := \{P | P$ is an $ITRM$-program of type $\langle ?, ?, n\rangle$ and $P \downarrow\}$ exists.'
- *Likewise, $ITRM_\omega$ denotes the statement: 'For any $X \subset \omega$ and any type $\tau = \langle a, b, c\rangle$, the halting set*
  $H^Z_\tau := \{P | P$ is an $ITRM$-program of type τ and $P^Z \downarrow\}$ exists.'

Theorem 65:

1. *KP+'ω^{CK}_n exists'$\vdash ITRM_n$*
2. $\Pi^1_1 - CA_0 \vdash ITRM_\omega$

Proof: (1) We show that a program of type $\langle ?, ?, n\rangle$ either halts after less than ω_n^{CK} many steps or does not halt. From this, it follows that $ITRM_n$ is Σ_1-definable by $n \in ITRM_n \leftrightarrow \exists\alpha < \omega_n^{CK}(P_n$ is in halting state after α many steps). Then the existence of $ITRM_n$ is implied by KP.
Now we prove the halting criterium by induction on n. First suppose that $n = 0$, so there are no overflowing registers in the typed program P. Suppose P uses m registers in total and denote by r_i^ι the content of the ith register at time $\iota \in On$, I_ι the command line at time ι, and $s_\iota = \langle r_1^\iota, ..., r_m^\iota, I_\iota\rangle$. Assume for a contradiction that P stops, but not before ω_1^{CK}. Then each component of $s_{\omega_1^{CK}}$ will be the $\liminf$ of the sequence of the corresponding component of s_ζ for $\zeta < \omega_1^{CK}$. In particular, each component of $s_{\omega_1^{CK}}$ must have occured cofinally often below ω_1^{CK}. Hence, by the $\liminf$-rule, $\{\zeta < \omega_1^{CK} | r_i^\zeta = r_i^{\omega_1^{CK}}\}$ is club in ω_1^{CK} for any $i \leq n$. By admissibility, the intersection of these finitely many clubs is again a club, so $\{\zeta < \omega_1^{CK} | s_\zeta = s_{\omega_1^{CK}}\}$ is also a club in ω_1^{CK}. But this means in particular that the program state at time ω_1^{CK} has already occured at some time $\iota < \omega_1^{CK}$. By a simple induction, $s_{\iota+\zeta} = s_{\omega_1^{CK}+\zeta}$ for every $\zeta < \omega_1^{CK}$, so the computation cycles, which contradicts the assumption that it halts.
If $n = k + 1$, we proceed similarly: If the computation does not halt before ω_n^{CK}, then consider the state $s := s_{\omega_n^{CK}}$. By the case $n = 0$, a certain number i of the n type 3-registers must have overflown and contain 0, let this be the registers $R'_1, ..., R'_i$. Hence, there will be some $\tau < \omega_n^{CK}$ such that none of the registers R'_1 to R'_i contains a 0 at a stage of computation between τ and ω_n^{CK}. By basic ordinal arithmetic, there exists $\tau < \delta < \omega_n^{CK}$ that is of the form $\delta = \gamma + \omega_k^{CK}$. δ is a limit ordinal, and at stage δ, the number of registers having overflown must be less than n. Since initial segments of computations can always be emulated by suitable choice of the initial state, we get from the inductive assumption that the program cycles, again a contradiction.
(2) $\Pi_1^1 - CA_0$ is sufficient to obtain the existence of Z^+ for any $Z \subset \omega$. But then, we have also $Z^{(n)}$, the nth hyperjump of Z. From this, it follows that ω_n^{CK} exists. So we get that ω_k^{CK} exists for every $k \in \omega$, which is sufficient by (1).
□

Conversely, there are also lower bounds for these principles:

Theorem 66:

1. $ATR_0 + ITRM \vdash \Pi^1_1 - CA_0$
2. $ATR_0 + ITRM_{n+1} \vdash$ *'*$\mathcal{O}^n$ *exists'*

Proof: (1) First note that, under $ITRM_\omega$, if we have an $ITRM$-program P and a set X, then the set $\{y \in X | P(x) \downarrow 1\}$ exists: To see this, change P to a program P' that halts on input n if P returns 1 on this output and loops otherwise. $ITRM_\omega$ implies the existence of
$ITRM'_\omega := \{(Q, x) | x \subset \omega \wedge$ 'Q is an $ITRM$-program such that $Q^x \downarrow\}$. The desired set is now obtained by intersecting and projecting.
Recall that ATR_0 is stronger than ACR_0 and that over ACR_0, $\Pi^1_1 - CA_0$ was equivalent with the statement that, for any countable sequence of trees, the subsequence consisting of those trees having a branch exists. Testing a tree for the existence of a branch can be done by the same algorithm that checks partial orders coded by reals for well-foundedness. Hence there is an $ITRM$-program P deciding for a coded tree whether it contains a branch, which, for $\langle T_i | i \in \omega \rangle$ a sequence of trees, implies the existence of
$\{i \in \omega | T_i$ contains a branch$\}$, and hence, in turn, $\Pi^1_1 - CA_0$.
(2) We have shown above how the nth hyperjump can be computed in type $\langle ?, ?, n \rangle$ by some $ITRM$-program P. By $ITRM_{n+1}$ and ACA_0, the set of initial values $n \in \omega$ such that $P(n) = 1$ exists. But this is exactly $ITRM_\omega$.
□

This concludes our analysis of $ITRM$s.

13 Fragments of a Hyperfine Core Model Theory

13.1 Preliminary Remarks

Through the use of alternative finestructures, the technical complexity of proofs in L could be considerably decreased. Therefore, one should expect that similar effects could be achieved when considering relativized versions of the constructible hierarchy, providing us with a simplified fine structure for core models. However, all my attempts at this have failed so far at a rather early stage even for small core models like K^{DJ}. These various ways to analyze L through the use of extra operations do not seem to be compatible with the formation of ultrapowers: In particular, the preservation of even the simplest fine-structural parameters with a two-line proof in Jensen's approach poses unexpected difficulties. This section is a brief and sometimes sketchy summary of possible attempts and their problems. My hope is that this can be useful as a starting point for further efforts in this area, as a simplified fine structure for core models would certainly be appreciated by students and researchers alike. The idea of building up the model L^A in such a fine way that every object can be described by a formula of low complexity over a level of the hierarchy seems to be the most promising. After all, also in Jensen's work, the Σ_1-case can usually be dealt with in an elegant and canonical manner.

There are already many different approaches in this spirit: Using the F-hierarchy, using hyperings, combined with extensions or several notions of fine ultrapower. However, they ultimately all face the same difficulties, so we focus here on the arguably most common one, namely the Friedman-Koepke hyperfine structure in connection with lifted directed systems. Our exposition is based on [41], where a similar strategy was attempted. It seems to be the most promising candidate for a theory meeting the following requirements:
(1) It should be possible to define and construct the core model K^{DJ} in this framework, which includes a theory of iterations, coiterations, parameter preservation etc.
(2) Important structural results like rigidity or the covering lemma should have natural proofs in it.

(3) It should simplify the current approach at least in the sense that reference to formula complexity is avoided and a uniform concept of ultrapower is used, circumventing the concept of Σ_n-ultrapowers and its relatives.
(4) For applications of the core model, the theory has to be good enough to allow deeper combinatorics like proofs of square or morass in the core model.
(5) Preferably, generalizations to larger core models should be possible.
Hyperfine methods, either in the sense of Friedman-Koepke or hyperings, are more likely to meet (4). Virtually everything in the following exposition carries over to the F-hierarchy in a canonical way.
It has been stated in several places (see e.g. [43]) that hyperfine structure theory could be used to structure the Dodd-Jensen core model, focusing on questions of iteration strategies at higher levels. It turned out, however, that, much earlier, fundamental difficulties arise with the representability of the target model in the source model of an extension.
We shall develop the concepts of premice and iterations in the hyperfine structure and their theory as far as possible. Then, we give an account on the problems with this approach, sketching how one should go on if one could solve these. Finally, we discuss several alternative notions of ultrapowers and general ideas for ways out.

13.2 Hyperfine Structures and their Iterations

In this section, we introduce the basic notions and theorems of an iteration theory for hyperfine structures: We define premice and iterations and prove basic preservation theorems for hyperfine extension maps. Familiarity with the Gödels L_α-hierarchy and their relativizations is assumed. Apart from that, the most definitions easily carry over from the context of hyperings, and we do not repeat them here.

13.2.1 Hyperfine Extensions

The basic construction for the following is a method to extend a map $E : L^A_\alpha \to L^B_\beta$ to a larger domain. This extension process can then be iterated. The construction we use here is very similar to the one we gave for the relativized F-hierarchy earlier, presenting the source structure as a limit of a directed system that we can map over and unfold. Hence, we merely sketch the method. Details can be found in [41].

In the context of the L-hierarchy, the possible choices for a hypering are restricted. This is mainly due to the fact that, once the formulas under consideration become more complex than mere $\mathbf{S}_0$-formulas, more caution is necessary to allow a helpful condensation lemma. Here, we use a relativized version of the hyperfine structure of Friedman and Koepke. We state some definitions and lemmas, most of which are straightforward to prove. Details can be found in [16].
Now, fix an enumeration $\langle\phi_i|i \in \omega\rangle$ of the first-order ϵ-formulas with an extra predicate with subformulas appearing earlier. To approach $L^A_{\alpha+1} = \{I(L^A_\alpha, \phi, q)|\phi$ is a formula and $q \subset_{fin} L^A_\alpha\}$ from L^A_α, we use structures equipped with restricted Skolem functions for some initial segment of the formulas.

Definition 1: *For a class A, a hyperfine L^A-level is a structure of the form $\langle L^A_\alpha, \epsilon, <_A, I, N, S, A, S_{\phi_1}, S_{\phi_2}, ..., S_{\phi_n}|\vec{x}\rangle$, where $\vec{x} \subset_{fin} L^A_\alpha$. Locations are hence of the form c^n_α as in the classical hypering, their ordering $<_{loc}$ etc. are as in the chapter on hyperings.*

Definition 2: *An extender is an elementary map between relativized levels of the L-hierarchy. Let L^A_s be a hyperfine level. Then E is a local extender on L^A_s iff there are $\gamma < \alpha(s)$, δ and B such that $E : L^A_\gamma \to L^B_\delta$ is elementary.*

Definition 3: *Let $M = L^A_s$ be a hyperfine level, E an extender on M. Suppose that $Ext(M, E)$ is well-founded with transitive collapse $N = L^F_t$ and denote by π the extension map. Then E is a measure iff*
$N = L^F_t\{rng(\pi) \cup \{\kappa\}\}$.

Definition 4: *A sequence of extenders $\vec{F}$ is a set of pairs of the form $\langle\iota, F_\iota\rangle$, such that F_ι is either an extender with $sup\{rng(F_\iota) \cap On\} = \iota$ or $\emptyset$ for any ordinal ι.*
If $\vec{F}$ is a sequence of extenders, then $X \subset L^{\vec{F}}$ is $\vec{F}$-constructibly closed ($\vec{F}$-cc) iff X is closed under I, N, S and extenders on $\vec{F}$ whose indices are elements of X: That is, if x and ι are in X, then so is $(\vec{F}(\iota))(x)$. The corresponding hull is denoted by $F^{\vec{F}}_s\{X\}$.

Definition 5: *Let* $M = \langle L^E_\alpha, \in, E|(L^F_\alpha)^2, E_\alpha\rangle$*, where* E *is a sequence of extenders. Then* M *is a coarse measure structure iff:*

- *For each* $\iota \in On$*,* E_ι *is either* $\emptyset$ *or a measure with* $E_\iota : L^{E|\lambda}_\lambda \rightarrow_{el} L^{E|\iota}_\iota$ *cofinal. This* λ *is denoted by* $\lambda(\iota)$*.*

- *If* $\iota < \alpha$ *is such that* $E_\iota \neq \emptyset$*, then* $\langle L^{E|\iota}_\iota, \in, E|\iota\rangle \models ZF^-(E)$*, i.e. all axioms of* ZF^- *where the underlying set of formulas comprises all expressions using the extra predicate symbol* E*.*

- *If* $\iota < \alpha$ *is such that* $E_\iota \neq \emptyset$*,* $\kappa = crit(E_\iota)$*, then* $L^{E|\iota}_\iota \models card(\kappa)$*. Furthermore we have* $\lambda(\iota) = (\kappa^+)^{L^{E|\iota}_\iota}$ *(i.e. there are no* $L^{E|\iota}_\iota$*-cardinals between* κ *and* $\lambda(\iota)$*) and* $H^{L^{E|\iota}_\iota}_{\kappa^+} = L^{E|\lambda}_\lambda$*.*

Definition 6: *If* E *is a sequence of measures,* $s = \langle\alpha, \phi, \vec{p}\rangle$ *a location, then* L^E_s *is a fine measure structure if* L^E_γ *is coarse measure structure for* $\gamma \leq \alpha$*.*

Definition 7: *An extender* F *on* $M = L^E_s$ *with* $crit(E) := \kappa$ *is active, iff, for all* $r <_{loc} s$*,* $p \subset_{fin} M$*, we have* $coll[L^E_r\{\kappa \cup p\}] \in dom(F)$*, we say* $dom(F)$ *is a base for* M*. For* M *a fine measure structure, if there is* $\beta < \alpha(s)$ *such that* $E(\beta)$ *is an active extender for* M*, then* M *is called active. Otherwise,* M *is passive.*

We will restrict ourselves to fine measure structures with at most one active measure. Analogous to the condensation lemmata for the F-hierarchy and general hyperings, we can now show:

Theorem 8: *Let* $M = L^E_s$ *be a fine measure structure, and suppose that* X *is a substructure of* M *that includes the critical point* κ_M *of the unique active extender of* M *if it exists. Let* $\pi : \bar{M} \rightarrow_{coll^{-1}} X$*. Then* $\bar{M} = L^{\bar{E}}_{\bar{s}}$ *is itself a fine measure structure.*

Proof: $\square$

The other properties of hyperings, like finiteness, continuity, monotony and compactness are also easily proved.

Definition 9: *If* $M = L^E_s$*, then* M *is called sound if, whenever* $\alpha < On^M$*,* $q \subset_{fin} M$ *are such that* $coll(L^E_s\{\alpha \cup q\}) \notin M$*, we have* $L^E_s\{\alpha \cup q\} = L^E_s$*.*
If $M||\beta$ *is sound for every* $\beta < On^M$*, then* M *is called initially sound.*

For $M = L^E_s$ an active fine measure structure, s a limit location, F the active measure on M, $crit(F) = \kappa$, we consider a directed system with underlying index set $\{\langle r, q\rangle | r <_{loc} s \wedge q \subset_{fin} M\}$, ordered by $\langle r_1, q_1\rangle \leq \langle r_2, q_2\rangle$ iff $r_1 \leq_{loc} r_2$, $q_1 \subseteq q_2$, and $\alpha(q_1) \in q_2$ if $\alpha(q_1) < \alpha(q_2)$. The structure associated with $\langle r, q\rangle$ is $L^{E(r,q)}_{s(r,q)} := coll[L^E_r\{\kappa \cup q\}]$. By the monotonicity property, $\langle r_1, q_1\rangle \leq \langle r_2, q_2\rangle$ implies $L^E_{r_1}\{\kappa \cup q_1\} \subset L^E_{r_2}\{\kappa \cup q_2\}$, so we can define system maps $\pi_{r_1 q_1 r_2 q_2}$ as in section 6. By definition of an active extender, all components of this system will be elements of $dom(F)$, so they can be mapped over by F. The direct limit of the mapped system is then the extension of L^E_s by F, denoted $Ext(L^E_s, F)$. It is routine similar to theorem 4.3.28 to show that this structure, if well-founded, is isomorphic to some active fine measure structure $L^{\hat{E}}_{\hat{s}}$. Also, we can define a map $\pi_F : L^E_s \to L^{\hat{E}}_{\hat{s}}$ extending F. Its preservation properties will be given by theorem 12.

Definition 10: *The fine measure structure $M = \langle L^E_s, E, G\rangle$ is a premouse iff:*

- *M is initially sound*
- *If $F = E(\beta)$ is an extender and $N \simeq Ext(M, F)$ is well-founded, then $E^N|\beta + 1 = E^M|\beta$*
- *For each $\eta < \alpha(s)$, there is at most one active extender for $M||\eta$ on E*
- *G is the extender with largest index on E*

Convention: For an active premouse $M = L^{\vec{E}}_s$, the unique active measure on E is called the top extender of M, denoted E^M_{top}.

The interesting stages in constructing a model relative to an extender sequence are the $<_{loc}$-minimal locations where the base property is lost:

Definition 11: *A premouse $M = L^E_s$ is critical for an extender F if, for all $t <_{loc} s$, F is an extender for L^E_t and there is $p \subset_{fin} L^E_s$ such that $coll[L^E_s\{\kappa \cup p\}] \notin dom(F)$, where $\kappa = crit(F)$.*

Critical premice are hence the maximal structures M to which an extender can be applied, i.e. for which $dom(F)$ is a base for M. By the finiteness property, such an s must be a limit location. (Otherwise, the same hull could be formed using the predecessor of s, adding one Skolem value to the finite parameter p.) At these levels, also a maximum of information is preserved by the extension maps. We have the following preservation properties:

Theorem 12: *Let L_s^E be a critical premouse, $\pi : L_s^E \to Ext(L_s^E, E_{top}) =: L_{\hat{s}}^{\hat{E}}$ be an extension map with critical point κ. Then π has the following properties:*

- *(a) $\pi : L_s^E \to L_{\hat{s}}^{\hat{E}}$ preserves the basic operations of s*
- *(b) π is Σ_1^s-preserving*
- *(c) The set of locations in $rng(\pi)$ is $<_{loc}$-cofinal in $\hat{s}$*
- *(d) The powerset of κ is preserved, i.e. $\mathfrak{P}(\kappa) \cap L_s^E = \mathfrak{P}(\kappa) \cap L_{\hat{s}}^{\hat{E}}$*

Proof:

- (a) Let $M := L_s^E$, $N := L_{\hat{s}}^{\hat{E}}$, $\vec{r} \subset_{fin} M$, $S_{\phi_i}^M(\vec{r}) = x \in M$, where $s = \langle \alpha, m, \vec{p} \rangle$, $t := \langle \alpha, i, \vec{r} \rangle \leq_{loc} s$. Set $t^+ := \langle \alpha, i, \vec{r}^+ \rangle$, where $\vec{r}^+$ denotes the $<_{lex}$-successor of $\vec{r}$ as usual. As s is a limit location, $t^+ <_{loc} s$. Ergo $coll[L_{s^+}^E\{\kappa \cup \vec{r}^\wedge\{x\}\}] \in M||(\kappa^+)^M$ by definition of s. Write H_1 for $coll[L_{t^+}^E\{\kappa \cup \vec{r}^\wedge\{x\}\}] =: L_{t_1}^{E_1}$ and denote by σ the collapsing map. We have $S_{\phi_i}^{H_1}(\sigma(\vec{r})) = \sigma(x)$. Set $\sigma(\vec{r}) = \bar{\vec{r}}$, $\sigma(x) = \bar{x}$. This is a expressible as a PL_1-statement ψ over H_1, so $H_1 \models \psi(x, \vec{r})$. As E was elementary, we must have $H_2 := E(H_1) \models \psi(E(\bar{\vec{x}}), E(\bar{\vec{r}}))$. Let $\hat{\sigma}_{H_2}$ be the limit map from component H_2 to N. As the system maps in the lifted direct system are homomorphisms of measure structures, we must, with $\hat{\sigma}_{H_2}(E(\bar{x})) = x'$, $\hat{\sigma}_{H_2}(E(\bar{\vec{r}})) = \vec{r}'$, have $S_{\phi_i}^N(\vec{r}') = x'$. But $x' = \hat{\sigma}_{H_2}(E(\bar{x}))$, where $\bar{x} = \sigma_{H_1}^{-1}(x)$, thus $x' = \hat{\sigma}_{H_2} \circ E \circ \sigma_{H_1}^{-1}(x) = \pi_{E_{top}^M}(x)$, similarly for $\vec{r}'$. Therefore $\pi_{E_{top}^M}(S_{\phi_i}^M(\vec{r})) = S_{\phi_i}^N(\pi_{E_{top}^M}(\vec{r}))$, as desired.
  The same reasoning applies to $I, N, S, \epsilon, <, =$.

- (b) (1) The upwards preservation is immediate by (a): If u is a witness for $\exists x \psi(x, \vec{v})$, ψ a Σ_1^s-formula, $\vec{v} \subset_{fin} L_s^E$, then $\pi_{E_{top}^M}(u)$ is a witness for $\exists x \psi_{\pi_{E_{top}^M}}(x, \pi_{E_{top}^M}(\vec{v}))$, where $\psi_{\pi_{E_{top}^M}}$ denotes the mapped formula as

usual.
(2) Now for downwards preservation: Let $L^{\hat{E}}_{\hat{s}} \models \exists x\psi(x, \vec{v}')$, $\vec{v}' = \pi_{E^M_{top}}(\vec{v})$, $\vec{v}' \subset_{fin} L^{\hat{E}}_{\hat{s}}$. Suppose $u \in L^{\hat{E}}_{\hat{s}}$ is a witness, so $L^{\hat{E}}_{\hat{s}} \models \psi(u, \vec{v}')$. For the truth of the Σ^s_0-formula $\psi(u, \vec{v}')$, only a finite number of Skolem values can be relevant. Thus, there must be $\bar{s} <_{loc} s$ such that $\psi(u, \vec{v}')$ is a $\Sigma^{\bar{s}}_0$-formula. Consider $H_2 := E^M_{top}(H_1)$ from the mapped directed system with the property that $\hat{u}, \hat{\vec{v}}' \in H_2$ with $\sigma_{\hat{H}_2}(\hat{u}) = u$, $\sigma_{\hat{H}_2}(\hat{\vec{v}}') = \vec{v}'$. Then $H_2 \models \exists x\psi(x, \hat{\vec{v}}')$. As E^M_{top} is elementary, we get $H_1 \models \exists x\psi(x, \hat{\vec{v}}')$, where $\sigma_{H_1 H_2}(\bar{\vec{v}}') = \hat{\vec{v}}'$. If $\bar{u} \in H_1$ is such that $H_1 \models \psi(\bar{u}, \bar{\vec{v}}')$, then, since the system maps a homomorphisms and setting $u = \sigma_{H_1}(\bar{u})$, we get $L^E_s \models \psi(u, \vec{v})$, and finally $L^E_s \models \exists s\psi(x, \vec{v})$.

- (c) By construction of $L^{\hat{E}}_{\hat{s}}$, every location l in the target structure has a pre-image in some component $\pi(C)$ of the mapped directed system. Suppose $C = coll'' L^E_r\{\mu \cup q\}$, then $l <_{loc} r$ and r has a π-preimage $\bar{r}$ in L^E_s. Hence $l <_{loc} \pi(\bar{r})$.

- (d) As $\kappa = crit(\pi)$, we have $\pi|\kappa = id|\kappa$. If $x \in \mathfrak{P}(\kappa) \cap L^E_s$, then π will preserve any statement of the form $\zeta \in x$, $\zeta \in L^E_s$: so $\zeta \in x$ iff $\pi(\zeta) \in \pi(x)$, but $\pi(\zeta) = \zeta$, and hence $x = \pi(x) \cap \kappa$, which is an element of $L^{\hat{E}}_{\hat{s}}$. Hence $\mathfrak{P}(\kappa) \cap L^E_s = \mathfrak{P}(\kappa) \cap L^{\hat{E}}_{\hat{s}}$.
On the other hand, let $x \in L^{\hat{E}}_{\hat{s}} \cap \mathfrak{P}(\kappa)$. Then there must be some component $C = \pi(\bar{C})$ of the mapped directed system along with some $\hat{x} \in C$ such that $x = \pi_C(\hat{x})$. We can assume without loss of generality that $\kappa \subset \bar{C}$ (and hence $\subset C$). From the construction of the extension, it follows easily that π_C is the identity on κ, as $\zeta \in \kappa$ is neither moved by the extender used nor by the collapsing map. Hence $\bar{x} = x$. But $\bar{x} \in C \in L^E_s$, so $x \in L^E_s$. Thus, also $L^{\hat{E}}_{\hat{s}} \cap \mathfrak{P}(\kappa) \subset L^E_s \cap \mathfrak{P}(\kappa)$.

□

13.2.2 Iterations

Definition 13: *Let M be a premouse. A pre-iteration of M of length θ is a sequence $\langle M_\iota, \nu_\iota, s_\iota, \pi_{\iota\gamma} | 0 \leq \iota < \gamma < \theta \rangle$ such that for $\iota < \theta$:*

- *M_ι is a premouse, $\nu \in On^{M_\iota}$, s_ι is an M_ι-location*
- *If ι is a limit ordinal, then there is $\delta(\iota) < \iota$ such that s_ζ is the top location of M_ζ for $\delta < \zeta < \iota$*
- *If ι is a limit ordinal, then $M_\iota = \mathrm{dirlim}\langle M_\zeta, \pi_{\zeta\eta} | \delta(\iota) < \zeta < \iota\rangle$ and $\pi_{\zeta\iota}$ is the direct limit map for $\delta(\iota) < \zeta < \iota$ and undefined for all other values of ζ*
- *If $\iota = \delta + 1$, then $M_\iota = Ext(M_\delta || s_\delta, E^{M_\delta}_{\nu_\delta})$*
- *If $\iota = \delta+1$ and s_δ is the top location of M_δ, then $\pi_{\delta\iota}$ is the corresponding extension map. Otherwise, it is undefined.*
- *If $\iota = \delta+1$, $\zeta < \iota$, then $\pi_{\zeta\iota} := \pi_{\delta\iota} \circ \pi_{\zeta\delta}$ if both of these maps are defined, and undefined otherwise.*

From now on, we write $lt(\mathtt{I})$ for the length of the iteration $\mathtt{I}$.

Definition 14: *If $\mathtt{I}$ is a pre-iteration of a premouse M, $\theta := lt(\mathtt{I})$ and s_ι is not the top location of M_ι, then ι is called a drop of $\mathtt{I}$. The set of drops of $\mathtt{I}$ is denoted by $D(\mathtt{I})$. Furthermore, if $\alpha(s_\iota) = On^{M_\iota}$, we call ι a finestructural drop. Otherwise, if $\alpha(s_\iota) < On^{M_\iota}$, ι is a level drop. The set of level drops is abbreviated by $LD(\mathtt{I})$. If $D(\mathtt{I}) = \emptyset$, then $\mathtt{I}$ is called simple.*

Remark: For the study of premice with at most one active measure, finestructural drops can be avoided.

For a pre-iteration $\mathtt{I}$ and an ordinal $\iota < lt(\mathtt{I})$, $\mathtt{I}|\iota$ denotes the iteration $\mathtt{I}$ up to the index ι.

Proposition 15: *If $\mathtt{I}$ is a pre-iteration of a premouse M, $\iota < lt(\mathtt{I})$, then $D(\mathtt{I}|\iota)$ is finite.*

Proof: Otherwise, there would be some limit ordinal $\bar\iota \leq \iota$ such that $D(\mathtt{I}|\bar\iota)$ is infinite, contradicting the rule for limit steps in the definition of pre-iterations.□

Corollary 16: *If $\mathtt{I}$ is a pre-iteration, $otp(D(\mathtt{I})) \leq \omega$.*

Proof: Clear from the preceeding proposition.□

Our main interest lies in the cases where there are only finitely many drops.

Definition 17: *A pre-iteration $\mathcal{I}$ is an iteration if $D(\mathcal{I})$ is finite. A pre-iteration that is not an iteration is called degenerate.*

Definition 18: *A premouse M is a mouse if it is active, critical, iterable and does not have degenerate iterations.*

Corollary 19: *If M is a mouse, then any simple iterate N of M is also a mouse.*

Proof: (Sketch) If N was not iterable or had degenerate iterations, the same would apply to M. □

13.3 The Copy process and the Dodd-Jensen Lemma

In this section, we formulate and prove one of the most important lemmata of core model theory, the Dodd-Jensen-Lemma, for hyperfine levels. A proof sufficient for building K^{DJ} can be found in [30], a more general formulation in [35]. Roughly, it states that there can be no map π from $M = L_s^A$ preserving $\Sigma_1^{s,A}$-formulas to an iterate N of M unless the iteration from M to N is simple. This implies in particular that being a simple or non-simple iterate of a certain premouse is a stable property and does not depend on the choice of the iteration.

Lemma 20: *(The Copy Construction): Let $\bar{M} := L_{\bar{s}}^{\bar{\boldsymbol{E}}}$ and $M := L_s^{\boldsymbol{E}}$ be premice, $\sigma : (\bar{M}, \bar{\boldsymbol{E}}) \to_{\Sigma_1^{\bar{s}}} (M, \boldsymbol{E})$ an embedding and $\mathcal{I}^{\bar{M}} := \langle \bar{M}_i, \bar{\nu}_i, \bar{\alpha}_i, \bar{\pi}_{ij} | i < j < \theta \rangle$ an iteration of $\bar{M}$ of length θ. Then there is an iteration $\mathcal{I}^M := \langle M_i, \nu_i, \alpha_i \pi_{ij} | i < j < \theta \rangle$ of the same length and embeddings $\langle \sigma_i | i < \theta \rangle$ such that if $\langle \bar{M}_i | i < \theta \rangle =: \langle L_{\bar{s}_i}^{\bar{\boldsymbol{E}}_i} | i < \theta \rangle$ and $\langle M_i | i < \theta \rangle =: \langle L_{s_i}^{\boldsymbol{E}_i} | i < \theta \rangle$ are the structures on the $\bar{M}$- and the M-side, respectively, we have $\sigma_i : \bar{M}_i \to_{\Sigma_1^{\bar{s}_i}} M$ such that the σ_i commute with the iteration maps $\bar{\pi}_{ij}$, π_{ij}.*

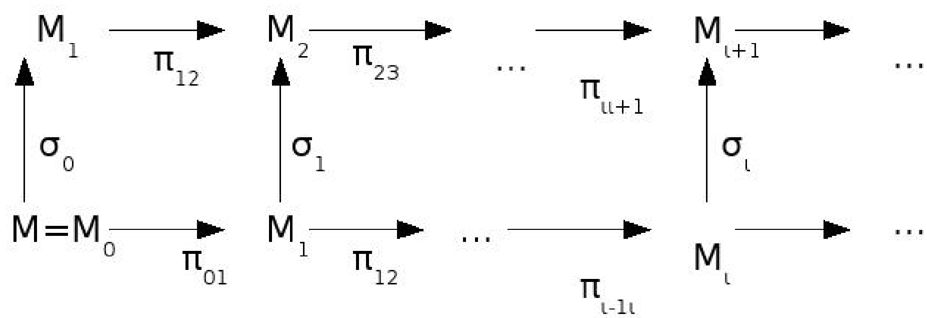
Proof: We prove this by induction on θ. For $\theta = 0$, there is nothing to show.
Suppose first $\theta = \eta + 1$ is a successor. Let $\langle \sigma_\iota | \iota \leq \eta \rangle$ be the embeddings from $\bar{M}_\iota$ to M_ι by induction. Set $\bar{M}_\theta =: M$, $M_\theta =: N$ for convenience. Let $\hat{M} = Ext(M||\alpha, E_\nu^M)$ be the next structure on the M-side. Then we define the next step on the N-side to be $\hat{N} := Ext(N||\sigma_\eta(\alpha), E^N_{\sigma_\eta(\nu)})$.

To obtain σ_θ, we proceed as follows, where π_M and π_N denote extension maps, while $\pi_C^{\hat{M}}$, $\pi_C^{\hat{N}}$ are the limit map for the component C of the mapped directed system used to obtain $\hat{M}$ and $\hat{N}$, respectively:
Let $x \in \hat{M}$ be arbitrary, and pick C from the mapped directed system for M along with $\bar{x} \in C$ such that $\pi_C(\bar{x}) = x$. Then $\sigma_\theta(x) := \pi_{\sigma_\eta(C)} \circ \sigma_\eta(\bar{x})$. From the properties of directed systems, it follows that this definition does not depend on the choice of the representative $\bar{x}$, so that σ_θ is well-defined. As a composition of maps preserving Σ_1-formulas for their respective structures, it is also $\Sigma_1^{\hat{M}}$-preserving.
If θ is a limit ordinal, $\bar{M}_\theta$ is the direct limit of $\langle \bar{M}_\gamma, \pi_{\gamma\delta}^{\bar{M}} | 0 \leq \gamma < \delta < \theta \rangle$, while $M_\theta = \text{dirlim}\langle M_\gamma, \pi^M_{\gamma\delta} | 0 \leq \gamma < \delta < \theta \rangle$. For $x \in \bar{M}_\theta$, let $\lambda < \theta$ be large enough such that x has a pre-image $\bar{x}$ in $\bar{M}_\lambda$ under the iteration map $\pi_{\lambda\theta}$. Then, set $\sigma_\theta(x) := \pi^M_{\lambda\theta} \circ \sigma_\lambda(\bar{x})$. The requirements for this map are again easily checked.
$\square$

Lemma 21: *(The Dodd-Jensen Lemma for hyperfine levels) Let $M = L_s^E$ be a mouse, $\mathtt{I}$ an iteration of M with final mouse $\hat{M} = L_{\hat{s}}^{\hat{E}}$. Assume there is $\sigma : (M, E) \to_{\Sigma_1^{s,E}} (\hat{M}, \hat{E})$. Then:*

1. *$\mathtt{I}$ is simple.*
2. *By (1), let $\pi : M \to \hat{M}$ be the iteration map. Then π is minimal in the sense that $\pi(\zeta) \leq \sigma(\zeta)$ for all $\zeta \in On^M$.*

Proof: (1) Assume otherwise, i.e. $\mathtt{I}$ contains a drop. Letting $M_0 := M$, $\hat{M}_0 := \hat{M}$, we define iterations $\mathtt{I}^M$, $\mathtt{I}^{\hat{M}}$ of M and $\hat{M}$ along with maps $\sigma_\iota : (M_\iota, E^{M_\iota}) \to_{\Sigma_1^{M_\iota}} (\hat{M}_\iota, E^{\hat{M}_\iota})$ as follows:



$M_{\iota+1} := \hat{M}_\iota$ and $\hat{M}_{\iota+1}$ is the result of copying the iteration from M_ι to $M_{\iota+1}$ via σ_ι to an iteration of $\hat{M}_\iota$, using the copy construction from the last lemma. Obviously, all these iterations will be non-simple and hence together form a degenerate iteration of M, which contradicts the assumption that M is a mouse.

(2) Assume otherwise, and pick $\zeta \in On^M$ such that $\sigma(\zeta) < \pi(\zeta)$. For $0 \leq i < j < \omega$, denote by π_{ij} the iteration map from M_i to M_j in the iteration just defined. (The iteration map from $\hat{M}_i$ to $\hat{M}_j$ is then $\pi_{(i+1)(j+1)}$.) Since the iteration maps commute with the σ_i, we have $\sigma_{i+1} \circ \pi_{i(i+1)} = \pi_{(i+1)(i+2)} \circ \sigma_i$. Consider now $\langle \zeta_i | i \in \omega \rangle := \langle \sigma_k \circ \sigma_{k-1} \circ ... \circ \sigma_0(\zeta) | k \in \omega \rangle$. Since $\pi(\zeta) = \pi_{01}(\zeta) > \sigma(\zeta)$ and by repeatedly using commutativity, $\zeta_i < \sigma_k \circ ... \circ \sigma_1 \circ \pi_{01}(\zeta) = \pi_{k(k+1)} \circ \sigma_{k-1} \circ ... \circ \sigma_0(\zeta)$.
Now consider $\text{dirlim}\langle M_i, \pi_{ij} | 0 \leq i < j < \omega \rangle =: M^*$. As M is a mouse, M^* is a simple, well-founded iterate of M. Denoting by π^*_k the limit map from M_i to M^* and applying π^*_{k+1} to the above inequality, it follows that $\langle \pi^*_{k+1} \circ \sigma_k \circ ... \circ \sigma_0(\zeta) < \pi^*_k \circ \sigma_{k-1} \circ ... \circ \sigma_0(\zeta) | k \in \omega \rangle$ is an infinite decreasing sequence of ordinals in M^*, which contradicts the well-foundedness of M^*. □

Corollary 22: *Let M be a premouse, N an iterate of M. Then either all iterations of M resulting in N are simple or all are non-simple.*

Proof: Suppose there is a simple iteration from M to N, and let π_{MN} be the iteration map. $\pi_{MN} : M \to N$ is at least Σ_1^M. If N was also a non-simple iterate of M, we would have a Σ_1^M-preserving map from M to a non-simple iterate of itself, contradicting the Dodd-Jensen-Lemma.□

Corollary 23: *Let M be a premouse, $\alpha \in On^M$. Then there is no $\pi : M \rightarrow_{\Sigma_1^M} M||\alpha$.*

Proof: Immediate from the last corollary.□

Another important consequence of the Dodd-Jensen lemma is the independence of the iteration map from the actual iteration:

Corollary 24: *Let M be a premouse, $\mathtt{I}_1$, $\mathtt{I}_2$ iterations of M with final premouse N with iteration maps π_1 and π_2, respectively. Then $\pi_1 = \pi_2$.*

Proof: By the minimality property of iteration maps, we have $\pi_1(\zeta) \leq \pi_2(\zeta)$ and $\pi_2(\zeta) \leq \pi_1(\zeta)$ for each $\zeta \in On^M$. Hence π_1 and π_2 agree on the ordinals, and, since they preserve Σ_1, hence on M. □

We draw another important conclusion from the copy construction, which allows to restrict well-foundedness considerations of iterations to iterations of countable length. In slight abuse of notation, we call the structure obtained from some iterate of M by the final use of a measure a pre-iterate of M, even if this structure is not well-founded.

Lemma 25: *Let M be a premouse such that any countable pre-iterate of M is well-founded and such that there are no countable degenerate pre-iterations of M. Then M is a mouse.*

Proof: Assume otherwise, and let $\mathtt{I}$ be a pre-iteration of M of length θ such that M_θ is ill-founded or such that $D(\mathtt{I})$ is infinite. Take γ large enough such that $\mathtt{I} \in H_\gamma$ and let $\bar{H}$ be the transitive collapse of the elementary hull of $\mathtt{I}$ in H_γ, σ the uncollapse. $\bar{H}$ is transitive and countable and contains pre-images $\bar{\mathtt{I}}$ of $\mathtt{I}$ and $\bar{M}$ of M. As the relevant notions are absolute, $\bar{\mathtt{I}}$ is a countable pre-iteration of $\bar{M}$ that is not an iteration. Use the copy construction to obtain from $\bar{M}$, $\mathtt{I}$ and σ a countable pre-iteration $\mathtt{J}$ of M that is not an iteration. This contradicts the assumption about countable pre-iterations of M.□

13.4 Finestructural parameters

Definition 26: *For $M = L_s^E$, we denote by $\rho_M = \rho_s^E$, the projectum of M, i.e. the smallest ordinal δ such that there is $q \subset_{fin} M$ with the property that $coll(L_s^E\{\delta \cup \{q\}) \notin M$.*

Definition 27: *For M as above, $\rho = \rho_M$, $p_M = p_s^E$ is the standard parameter of M, the $<_{lex}$-smallest $q \subset_{fin} M$ such that $coll(L_s^E\{\rho_M \cup q\}) \notin M$.*

Lemma 28: *ρ_M is a cardinal in M.*

Proof: Otherwise, there is $\upsilon < \rho_s^E$, $g \in M$ such that $g : \upsilon \to_{surj} \rho_s^E$. Let $\alpha = On^M$ be the ordinal height of M. Since $g \in M$, there is $N(g) \in M$, say $N(g) = \langle \beta, \phi_i, q\rangle$, $q \subset_{fin} M||\beta$, $\beta < \alpha$, so $g(\iota) = \gamma \leftrightarrow M||\beta \models \phi_i(\iota, \gamma, q)$. It follows that $g(\iota) = S_{\phi_i}^{M||\beta}(\{\iota\}^\wedge q)$. Consequently $\rho_s^E \subset L_s^E\{\rho \cup \{\beta\}^\wedge q\}$, so $L_s^E\{\rho_s^E \cup p_s^E\} \subset L_s^E\{\upsilon \cup \{\beta\}^\wedge q^\wedge p_s^E\}$. If we had indeed
$a := L_s^E\{\upsilon \cup \{\beta\}^\wedge q^\wedge p_s^E\} \in M$, then $L_s^E\{\rho_s^E \cup p_s^E\}$ would be definable over a and hence an element of M as well, which it isn't by definition. But this contradicts the minimality of ρ_s^E. □

In fact, we have shown more:

Corollary 29: *Let $\upsilon < \rho_s^E$, then $\rho_s^E \not\subset L_s^E\{\upsilon \cup q\}$ for all $q \subset_{fin} L_s^E$.*

Proof: Immediate by the proof of the last lemma. □

If we want to characterize mice by properties of their iterates, we need some notion of preservation of definability. One of the main ideas of hyperfine structure is to replace the study of definability by the study of hulls. The preservation of definitions should hence corresponds to the isomorphy of hulls. This is given by the following results.

Lemma 30: *Let E be an M-Extender, $M = L_s^F$, $\rho \leq crit(E) =: \kappa$ be an ordinal, $p \subset_{fin} M$, $N := L_{\hat{s}}^{\hat{F}} = Ext(M, E)$, $\pi_E : M \to N$ the extended embedding, $\hat{p} := \pi_E(p)$, $s = \langle \alpha, n, \vec{r}\rangle$ a limit M-location, $\hat{s} =: \langle \hat{\alpha}, \hat{n}, \hat{\vec{r}}\rangle$ likewise. Suppose $\langle \alpha, i, p\rangle <_{loc} s$. Then $S_{\phi_i}^M(p) \downarrow \leftrightarrow S_{\phi_i}^N(\hat{p}) \downarrow$.*

Proof: The direction from left to right is clear, as π_E is a homomorphism of measure structures and hence preserves the Skolem value.
For the other direction, suppose $S_{\phi_i}^N(\hat{p}) \downarrow = \hat{x}$ and choose a component $\hat{K}$ of the mapped directed system, so that its pre-image $E^{-1}(\hat{K}) =: K$ contains pre-images of $\hat{p}', \hat{x}'$ for $\hat{p}, \hat{x}$ and is formed using a location $\bar{s}$ with $\pi_E(\bar{s}) >_{loc} \langle \hat{\alpha}, i, \hat{p}\rangle$. Then $\hat{K} \models S_{\phi_i}^{\hat{K}}(\hat{p}') = \hat{x}'$, which can be expressed as an ϵ-formula $\phi(\hat{x}', \hat{p}')$ over $\hat{K}$. So $\hat{K} \models \exists y \phi(y, \hat{p}')$. As E is elementary and $\hat{p} \in rng(\pi_E)$,

we can pull this back to $K \models \exists y\phi(y,\bar{p})$, where $\bar{p}$ is the pre-image of p under the system map. Thus there exists $\bar{x} \in K$ with $K \models \phi(\bar{x},\bar{p}) \leftrightarrow S^K_{\phi_i}(\bar{p}) = \bar{x}$. As K belongs to a location $>_{loc} \langle \alpha, i, p \rangle$, it reflects $S^M_{\phi_i}$ at $\bar{p}$. So $S^M_{\phi_i}(p) = x$, in particular $S^M_{\phi_i}(p) \downarrow$. $\Box$

Lemma 31: *Let $M, N, E, s, \hat{s}, F, \hat{F}, \pi_E, \alpha, \hat{\alpha}, n, \hat{n}, \vec{r}, \hat{\vec{r}}, p, \hat{p}$ be as in the last lemma.*
Then we have $L^F_s\{\rho \cup p\} \simeq L^{\hat{F}}_{\hat{s}}\{\rho \cup \hat{p}\}$.

Proof: We already know that π_E is a homomorphism of measure structures. Therefore, $\sigma : L^F_s\{\rho \cup p\} \rightarrow L^{\hat{F}}_{\hat{s}}\{\rho \cup \hat{p}\}$, given by $t^F_s(\zeta_1, ..., \zeta_k, p) \rightarrow t^{\hat{F}}_{\hat{s}}(\zeta_1, ..., \zeta_k, \hat{p})$ for s-terms t, $\zeta_1, ..., \zeta_k \in \rho$ will be structure-preserving as well. We have to show that σ is surjectiv.
So assume for a contradiction that this is not the case and pick $x \in L^{\hat{F}}_{\hat{s}}\{\rho \cup \hat{p}\} - rng(\sigma)$. As $s \in L^{\hat{F}}_{\hat{s}}\{\rho \cup \hat{p}\}$, there exists an $\hat{s}$-term t and $\zeta_1, ..., \zeta_k \in \rho$ such that $x = t^{\hat{F}}_{\hat{s}}(\zeta_1, ..., \zeta_k, \hat{p})$. t is a combination of $I, N, S, S^N_{\phi_1}, ..., S^N_{\phi_{\hat{n}}} | \hat{\vec{r}}$. Without loss of generality, we may assume that t is chosen in such a way that no subterm of t generates an object outside of $rng(\sigma)$.
We now proceed by induction on the structure of t; t will be of the form $\Theta(x_1, ..., x_l)$ with $x_1, ..., x_l \in rng(\sigma)$ and Θ an $\hat{s}$-operation. If Θ is I, N or S, then $x = \sigma(\Theta(\sigma^{-1}(x_1), ..., \sigma^{-1}(x_l))) \in rng(\sigma)$, which contradicts the choice of x. The same is true for $\Theta = S^N_{\phi_i}$ in case that $S^M_{\phi_i}(\sigma^{-1}(x_1), ..., \sigma^{-1}(x_l))$ is defined. (Note that by our wlog assumption on t no parameters from $N - M$ can become relevant.) So we must have $S^M_{\phi_i}(\sigma^{-1}(x_1), ..., \sigma^{-1}(x_l)) \uparrow$ but $S^N_{\phi_i}(x_1, ..., x_l) \downarrow = x$. But this is excluded by the last lemma. $\Box$

13.5 On Parameter Preservation

"Es ist nichts! Sucht das Heil woanders!"
Nietzsche, Zur Genealogie der Moral

The core model below a measurable cardinal is a structure of the form L^E, where E is a canonical sequence of local measures. The sequence E is constructed recursively, chosing measures one after another in such a way that the corresponding structures remain sound. It turns out that, unless there is an inner model with a measurable cardinal, this recursion works and offers a unique choice in each step. For proving this and in applications of core models, it is important that iterations maps preserve the central fine

structural information: If N is an iterate of M, κ the critical point of the iteration map π and $\rho_M \leq \kappa$, then we should have $\rho_M = \rho_N$ and $\pi(p_M) = p_N$. In Jensen's fine structure, one basically reduces everything to Σ_1 by introducing sufficiently strong extra predicates, then using the very nice special properties of Σ_1 along with some version of the Los theorem to describe the target structure in the ground structure and show that the first projectum below the critical point does not change.
Unfortunately, the situation is very different for the hyperfine theory. Nothing analogous to the Los theorem is in sight so far, and the preservation of the finestructural information is basically an open question. The problem seems to be that the extensions of hyperfine mice, although very uniform with respect to formula complexity, does not allow the formulation of relevant properties of the target structure in the ground structure. In this section, we explicate this difficulty in some detail.

First, some positive results:

Lemma 32: *Let $M = L_s^F$ be a premouse, E an M-extender, $L_{\hat{s}}^{\hat{F}} = N = Ext(M, E)$, $\kappa := crit(E)$, $\rho \leq \kappa$, $p \subset_{fin} M$ such that $coll(L_s^F\{\rho \cup p\} \notin M$ and hence $\notin (H_{\kappa^+})^M$ (as there is a surjection from ρ onto $coll(L_s^F\{\rho \cup p\})$). Then $coll(L_{\hat{s}}^{\hat{F}}\{\rho \cup \pi_E(p)\}) \notin (H_{\kappa^+})^N$, and hence $\notin N$.*

Proof: Set $\hat{h} := coll(L_{\hat{s}}^{\hat{F}}\{\rho \cup \pi_E(p)\}) \in N$. As $\rho \leq \kappa$ and there is a surjection $g : \kappa \to_{surj} \hat{h}$ definable over $\hat{h}$ from κ, we would have $\hat{h} \in (H_{\kappa^+})^N$. E is an M-extender, which implies $(H_{\kappa^+})^M = (H_{\kappa^+})^N$. Hence $\hat{h} \in (H_{\kappa^+})^M$. But we already saw that $L_s^F\{\rho \cup p\} \simeq L_{\hat{s}}^{\hat{F}}\{\rho \cup \pi_E(p)\}$, ergo $h := coll(L_s^F\{\rho \cup p\} = coll(L_{\hat{s}}^{\hat{F}}\{\rho \cup \pi_E(p)\}) = \hat{h}$, implying $coll(L_s^F\{\rho \cup p\} \in (H_{\kappa^+})^M$, a contradiction. □

Corollary 33: *In the above situation, if additionally $\rho_s^F \leq \kappa$, then $\rho_{\hat{s}}^{\hat{F}} \leq \rho_s^F$.*

Proof: By the above argument, there is a hull of the form $F_{\hat{s}}^{\hat{F}}\{\rho \cup q\}$, $q \subset_{fin} L_{\hat{s}}^F$ such that its collapse is $\notin L_{\hat{s}}^{\hat{F}}$. (Namely we can take $q = \pi_E(p_s^F)$.) This suffices. □

Corollary 34: *If, in the above situation, we additionally have $\rho_s^F = \rho_{\hat{s}}^{\hat{F}}$, then also $p_{\hat{s}}^{\hat{F}} \leq_{lex} \pi_E(p_s^F)$.*

Proof: We know that $coll(L^{\hat{F}}_{\hat{s}}\{\rho^F_s \cup \pi_E(p^F_s)\}) \notin L^{\hat{F}}_{\hat{s}}$. As $\rho^F_s = \rho^{\hat{F}}_{\hat{s}}$, $\pi_E(p^F_s)$ defines a new collapsed hull of the form $coll(L^{\hat{F}}_{\hat{s}}\{\rho^{\hat{F}}_{\hat{s}} \cup q\})$. But $p^{\hat{F}}_{\hat{s}}$ was defined to be the $<_{lex}$-smallest such q, so we must indeed have $p^{\hat{F}}_{\hat{s}} \leq_{lex} \pi_E(p^F_s)$. □

The important next step would be to demonstrate the preservation of the projectum. However, unexpected difficulties arise when attempting to use the techniques from classical fine-structure here. To give an impression what happens, let us consider the proof for the preservation of the Σ_1-projectum under ultrapowers in Jensen's fine structure.

Fact: Denote by $\rho_{J^{\mathtt{U}}_\alpha}$ the smallest ρ such that there is a Σ_1-formula ψ and a finite set $p \subset_{fin} J^{\mathtt{U}}_\alpha$ with the property that $\{\iota | J^{\mathtt{U}}_\alpha \models \psi(\iota, p)\} \cap \rho \notin J^{\mathtt{U}}_\alpha$. Suppose κ is a cardinal in $J^{\mathtt{U}}_\alpha$, and that for some $\beta < \alpha$, $U := \mathtt{U}(\beta) = \{x | (\beta, x) \in \mathtt{U}\}$ is an ultrafilter on κ in $J^{\mathtt{U}}_\alpha$. (This means that U is an ultrafilter on $\mathfrak{P}(\kappa) \cap J^{\mathtt{U}}_\alpha$.) Further, assume that $Ult(J^{\mathtt{U}}_\alpha, U)$ is well-founded, hence isomorphic to some $J^{\hat{\mathtt{U}}}_{\hat{\alpha}}$ and let π be the ultrapower map. Set $\rho = \rho_{J^{\mathtt{U}}_\alpha}$, $\hat{\rho} = \rho_{J^{\hat{\mathtt{U}}}_{\hat{\alpha}}}$. Then $\rho = \hat{\rho}$ if $\rho \leq \kappa$.

Proof: First, note that $\mathfrak{P}(\kappa) \cap J^{\mathtt{U}}_\alpha = \mathfrak{P}(\kappa) \cap J^{\hat{\mathtt{U}}}_{\hat{\alpha}}$. $\subset$ is clear. If $\kappa \supset x \in J^{\hat{\mathtt{U}}}_{\hat{\alpha}}$, then there is a function $J^{\mathtt{U}}_\alpha \ni f_x : \kappa \to J^{\mathtt{U}}_\alpha$ such that
$\iota \in x \leftrightarrow \{\eta < \kappa | \iota \in f_x(\eta)\} \in U$. The RHS is expressable as a Σ_0-formula over $J^{\mathtt{U}}_\alpha$ (there is only one implicite existential quantifier, which is bounded to U), hence $x \in J^{\mathtt{U}}_\alpha$ by general closure properties of relativized J-stages.
It is now easy to see that $\hat{\rho} \leq \rho$: For let ϕ be Σ_1 and $p \subset_{fin} J^{\mathtt{U}}_\alpha$ such that $x := \{\iota < \rho | J^{\mathtt{U}}_\alpha \models \phi(\iota, p)\} \notin J^{\mathtt{U}}_\alpha$. Then $x = \{\iota < \rho | J^{\hat{\mathtt{U}}}_{\hat{\alpha}} \models \phi(\iota, \pi(p))\}$ is Σ_1-definable over $J^{\hat{\mathtt{U}}}_{\hat{\alpha}}$, but is, by our initial remark, not an element of this structure. Hence $\hat{\rho} \leq \rho$.
For the other direction, we show that for any Σ_1-formula ϕ and finite vector $\hat{p} \subset_{fin} J^{\hat{\mathtt{U}}}_{\hat{\alpha}}$ there are a Σ_1-formula ψ and $p \subset_{fin} J^{\mathtt{U}}_\alpha$ such that:
(*) $\{\iota < \kappa | J^{\hat{\mathtt{U}}}_{\hat{\alpha}} \models \phi(\iota, \hat{p})\} = \{\iota < \kappa | J^{\mathtt{U}}_\alpha \models \psi(\iota, p)\}$.
From this, $\rho \leq \hat{\rho}$ is immediate: Since $\hat{\rho} \leq \rho$, we have $\hat{\rho} \leq \kappa$. So any subset of $\hat{\rho}$ that is Σ_1-definable over $J^{\hat{\mathtt{U}}}_{\hat{\alpha}}$ is also Σ_1-definable over $J^{\mathtt{U}}_\alpha$. Since the power sets agree, we must have $\rho \leq \hat{\rho}$.
To finish the proof, let $x = \{\iota < \kappa | J^{\hat{\mathtt{U}}}_{\hat{\alpha}} \models \exists y \phi(\iota, y, \hat{p})\}$, where ϕ is Σ_0, $\hat{p} \subset_{fin} J^{\hat{\mathtt{U}}}_{\hat{\alpha}}$. Pick $f_{\hat{p}} : \kappa \to J^{\mathtt{U}}_\alpha$ from $J^{\mathtt{U}}_\alpha$ that represents $\hat{p}$ in the ultrapower,

i.e. $[f_{\hat{p}}]_U = \hat{p}$. Now, using that $rng(\pi) \uparrow_{\hat{\alpha}}$ and the theorem of Los:
$J^{\hat{\mathbf{U}}}_{\hat{\alpha}} \models \exists y \phi(\iota, y, \hat{p}) \leftrightarrow \exists \zeta < \omega\alpha \exists y \in F^{\hat{\mathbf{U}}}_{\pi(\zeta)} \phi(\iota, y, \hat{p})$
$\leftrightarrow \exists \zeta < \omega\alpha [\{\gamma < \kappa | \exists f_y \in F^{\mathbf{U}}_{\zeta} \phi(\iota, f_y(\gamma), f_{\hat{p}}(\gamma))\} \in U]$.
The last is a Σ_1-definition of x over $J^{\mathbf{U}}_{\alpha}$, so we are done. $\Box$

In our setting, the inequality $\hat{\rho} \leq \rho$ was unproblematic. The central step for the reverse direction is (*). This would require a variant of the Los theorem for the *Ext*-construction: The truth of some quantifier-free formula $\hat{\phi}$ with some parameter $\hat{p}$ for the target structure should be expressable in a quantifier-free way over the source structure in some parameter p. The difficulty with this is that the target N is presented in the source M in an inappropriate manner: The source can't talk about components of the mapped directed system, let alone by using quantifier-free formulas. The directed systems used to represent the structure N in M is an outer object to M. This crucial difference between the common ultrapower construction and the use of *Ext* accounts for the non-representability of the target structure in the source structure, hence for the failure to show parameter preservation, and thereby for the failure of the approach.

It might be tempting to combine hyperfine levels with ultrapowers by ultrafilters and use an extension construction like $'Ult(L^{\mathbf{U}}_s, U)'$, where U is an $L^{\mathbf{U}}_s$-ultrafilter on $\kappa \in L^{\mathbf{U}}_s$ and the ultrapower is build in the usual way from U-equivalence classes of functions in $L^{\mathbf{U}}_s$. This fails badly for the L^A_t-hierarchy due to the lack of closure in general relativized L_α-levels: If $x \in L_{\alpha+1} - L_\alpha$, then $const_x \notin L_{\alpha+1}$, where $const_x$ denotes the function $const_x(\iota) = x$ for $\iota < \kappa$. Thus, no embedding from the original structure into the 'ultrapower' can be defined. This problem can be circumvented by applying the idea of hyperfine structure to J-stages instead and considering structures of the form $J^{\mathbf{U}}_s$. Now the embedding is definable, but does in general not even respect the s-operations like the restricted Skolem functions for formulas of complexity Σ_2 and above. This was achieved in the *Ext*-construction by forming and mapping s-hulls, a feature which is absent when the extension is build as an ultrapower.

Attempts to achieve preservation in either concept of extension by adding extra information to the structures like truth predicates, codes for the whole directed system etc. so far all faced the problem that they enhance the expressive power on both sides in such a way that the problem remains.

If the problem of parameter preservation could be solved, i.e. the extender construction was set up in such a way that, for critical mice, projecta and standard parameters are preserved, the construction of the core model would go through smoothly. For a rough sketch, assign to each mouse $M = L_s^E$ the measure structure $\mathtt{C}(M) := coll'' L_s^E\{\rho_M \cup p_M\}$, the core of M. Taking the preservation of fine structure for granted, the comparison process for mice can be defined and proved to converge as in the classical setting. It then follows that $\mathtt{C}(M)$ is a mouse if M is, and that M is a simple iterate of $\mathtt{C}(M)$. A core mouse is a mouse M that is the core of some mouse, which is equivalent to $\mathtt{C}(M) = M$ as well as to the soundness of M. Information on mice can thus be reduced to information on sound mice. Using the preservation of fine structure, one deduces that in a seemingly restricted, but in fact sufficient situation, the comparison process is trivial: I.e., for any two critical core mice $M = L_s^E$, $N = L_t^F$ whose extender sequences agree up to the maximum of their projecta are compatible in the sense that one is an initial segment of the other. Once this is known, all is available to show that the following recursion works, which defines the Dodd-Jensen core model K^{DJ}:

Definition 35: *(The Dodd-Jensen Core Model) We define an extender sequence $\mathtt{E}^{DJ}$ recursively:*

- $E_0^{DJ} = \emptyset$
- *If $\alpha = \beta + 1$ for some β, then $E_\alpha^{DJ} = \emptyset$*
- *Suppose $\mathtt{E}^{DJ}|\kappa$ is defined. Let $\mathtt{M}_\kappa$ be the set of sound mice M such that $\rho_M = \kappa$ and $\mathtt{E}^M|\kappa = \mathtt{E}^{DJ}|\kappa$. Let $\lambda = sup\{On^M | M \in \mathtt{M}\}$. Then $\mathtt{E}^{DJ}|\lambda = \mathtt{E}^{\bigcup_{M \in \mathtt{M}_\kappa} \mathtt{E}^M}$.*

The preservation of fine structure under extensions is hence the one, but crucial barrier remaining for a hyperfine core model. In the next section, we consider several possible ways to attack this difficulty by modifying the extension process.

13.6 What might be done

13.6.1 Alternative Concepts of Fine Ultrapowers and their difficulties

Compared with the fine structure based on reducts and standard codes, the hyperfine approach can be seen as attempt to represent the structure by Skolem functions rather than truth predicates. A typical structure occuring in the construction fine ultrapowers in e.g. [30] can be equivalently represented in the form $\langle J^U_{\rho^U_{\alpha,n}}, A_1, ..., A_n \rangle$, where $\rho^U_{\alpha,n}$ is the nth projectum of J^U_α. Here, A_{i+1} is the Σ_1 truth predicate over the ith reduct. Now, Σ_1 truth predicates and the canonical Skolem functions S_ϕ used in hyperfine structure are obviously quite resemblant to each other. The reason for the difficulties of the hyperfine approach is then the fact that, while the truth predicates give information on a complexity class of formulas with convenient closure properties, these properties are absent when adding Skolem functions for formulas one after another. For example, if the truth predicate for ϕ and ψ in certain parameters is given, then so is the one for $\phi \wedge \psi$. The same is not true in the hyperfine case: Having S_ϕ and S_ψ available does not allow one to search for witnesses for $\phi \wedge \psi$.
When seen from this angle, a solution would be to introduce the Skolem functions in appropriate 'portions' as well, say, for all formulas at a certain level of the Levy hierarchy at once. This would correspond to a generalization of locations where the underlying ordering of the formulas is no longer necessarily of order type ω. One could, for example, let $<_i$ be a canonical well-ordering of the Σ_i-formulas with subformulas coming earlier and then define $<$ as the concatenation of $\langle <_i \mid i \in \omega \rangle$. This is an ordering in order type ω^2. Allowing critical locations to be only of the form $\langle \alpha, \phi, \omega \cdot n, \emptyset \rangle$, everything would work exactly as in Jensen's fine structure. However, at the same time, the uniformity and all other advantages of hyperfine structure theory would be lost and one would end up with a mere reformulation of classical core model theory. Even worse, by adding infinitely many operators at once, a crucial feature of hyperfine structures used in applications, the finiteness property, is lost.

13.6.2 A More Algebraic Approach

A possible way out might be the 'model-theoretical' ultrapowers introduced by Magidor in [11]. The basic idea is to take the complete theory $\mathbf{T}$ of the source structure and code the desired fine structure into that theory to arrive at $\mathbf{T}' \supset \mathbf{T}$. The ultrapower is then a certain (minimal) model of $\mathbf{T}$ together with an extra ω-sequence of indiscernibles. Magidor works with the L_α-hierarchy. An attempt to carry this over to the F-hierarchy would probably start as indicated below.

The structure to work with are of the form $\mathbf{M} := \langle F^U_\alpha, \in, I, N, S, <, U\rangle$, where U is a measure in M on some M-cardinal κ. We will call these F-measure structures. To take an ultrapower of M, expand the language $\mathbf{S}$ for the fine hierarchy by constant symbols c_x for each element x of M to get the expanded language $\mathbf{S}^M$. $\mathbf{T}(M)$ is then the complete $\mathbf{S}_0^M$-theory of M. Introduce another ω many constant symbols $\langle \iota_i | i < \omega\rangle$, intended to denote a sequence of indiscernibles below the image of the critical point κ.
Let $\mathbf{T}(\mathbf{M})$ be the complete $\mathbf{S}_0^M$-theory of M. Let
$\mathbf{T}^* := \mathbf{T}(\mathbf{M}) \cup \{\iota_i < \iota_j | 0 < i < j < \omega\} \cup \{\iota_i \in c_\kappa | i \in \omega\} \cup \{\phi(\iota_{i_1}, ..., \iota_{i_n}) \leftrightarrow \phi(\iota_1, ..., \iota_n) | \phi \text{ an } \mathbf{S}_0\text{-formula}\} \cup \{t(\iota_{i_1}, ..., \iota_{i_n}) = t(\iota_1, ..., \iota_n) | t \text{ an } \mathbf{S}_0\text{-term}\} \cup$
$\cup \{U(t(\iota_1, ..., \iota_n)) \leftrightarrow t(\iota_{i_1}, ..., \iota_{i_n}) \in c_\kappa \wedge \forall i > i_n(\iota_i \in t(\iota_{i_1}, ..., \iota_{i_n})) | t \text{ an } \mathbf{S}_0\text{-term}\}$.

Now for each ordinal α, the αth iterate of M is obtained by taking the constructible closure of an α-sequence of indiscernibles of type $\mathbf{T}^*$. If this structure is well-founded for each α, then M is called iterable. K^{DJ} can then be defined as $\{x | \exists M = F^U_\gamma \exists \kappa \in On | M \text{ is iterable } \wedge crit(U) = \kappa \wedge x \in M||\kappa\}$.

As in [30], one can now show that, for $(\mathtt{A}, T)$ iterable and λ a cardinal greater than $card(\mathtt{A})$, we have $\mathtt{A}^T_\lambda$ of the form $L_\alpha[\mathbb{F}_\lambda]$, where $\mathbb{F}_\lambda$ is the closed unbounded filter on λ. From this it follows that, for any two iterable structures $(\mathtt{A}, T)$, $(\mathtt{B}, S)$, there is δ such that one of $\mathtt{A}^T_\delta$, $\mathtt{B}^S_\delta$ is an initial segment of the other. (Just let δ be a cardinal greater than $max\{card(\mathtt{A}), card(\mathtt{B})\}$.) This allows a comparison and hence a well-ordering of iterable structures.

The key to obtaining representability of the extension in the source structure is the concept of a realized structure: An iterable structure $(\mathtt{A}, T)$ with critical point κ is realized if there is an infinite set $C \subset \kappa$ such that the

following holds for any $\mathbf{S}_0$-formula ϕ:
$\phi(\iota_1, ..., \iota_k) \in T \leftrightarrow$ There is $C' \subset C$ cofinite such that $\forall \langle \zeta_1, ..., \zeta_n \rangle \subset_{fin} C' \mathtt{A} \models \phi(\zeta_1, ..., \zeta_n)$.

If a structure is realized, terms on the RHS of an ultrapower can often be mirrored on the LHS by using elements of C' in place of indiscernibles. In particular, the ultrapower map will be fine up to the target structure in the case of a realized structure. The necessary variant of a Los theorem is thereby directly implemented in the construction.

A rough sketch of this construction of the Dodd-Jensen core model can be found in the appendix to [11]. Some experiments with this indicate that it would indeed be compatible with the F-hierarchy, and would result in a gain in terms of simplicity as the necessity of splitting the ultrapower construction into ω many cases according to the minimal n such that a Σ_n-hull collapses the critical point (the measurable cardinal) vanishes. This would meet requirements (1) and (2) from the preliminary remarks. By simplified proofs of the covering lemma for L and the proof of an approximation lemma for the core model given in [11], it is very likely that (3) will hold as well. (4) is at least doubtful, as methods in the spirit of these Σ_n-hulls have yet not led to proofs for any of the more involved combinatorial facts about L (such as square or morasses) to the best of our knowledge. (5) almost certainly fails for models much bigger than K^{DJ} when iterations and the corresponding indiscernibles become more complicated.

We can, of course, not exclude that appropriate concepts of definability/hull operators and extensions can be found that bring together ultrapowers and alternative fine structure. At this point, however, this remains a big, open question.

14 Concluding Remarks and Future Goals

The use of the F-hierachy circumvents the necessity of considering formula complexity in finestructural arguments. It is a flexible setting and well suited for working in L. Many case distinctions of classical finestructure are made unnecessary, the techniques for extending embeddings become shorter and more transparent. This leads to slicker and more accessible proofs for the covering lemma and what is called here the approximation lemma. Notions from hyperfine structure theory can be incorporated in this setting, leading to hyperings, a natural refinement of the hierarchy, which allows proofs of central combinatorial properties of L. The use of the horizontal hypering achieves the independence of the desired objects from the ordering of formulas and other deliberate properties of the underlying language. One might consider carrying out proofs for other principles like the existence of gap-2-morasses or Sy Friedman's morass with square.

In the context of generalized computability theory, such subleties of structuring the levels are mostly irrelevant, as the levels under considerations have sufficiently strong closure properties anyway to appear in any of the common hierarchies. Through the introduction of typed machines, the analysis of computational strength of $ITRM$'s could be recasted in a conceptionally more stable and interesting way and in large parts be finished. Open questions are here for example whether a lost melody theorem can hold for weak $ITRM$s and whether there is a universal weak $ITRM$. Interesting new questions would certainly arise by enriching the variety of register types: For example, type 3 registers containing a 0 could also pass on information on the question whether this is a regular limes inferior or due to an overflow.

Relatiziations of the F-hierarchy and hyperfine structure theory are possible, but working with them poses difficulties. When approaching core models, the preservation of finestructural parameters seems incompatible with the extra operations. Intuitively, these structures enriched with all kinds of tools seem to lack 'self-consciousness', i.e. the ability to represent the extension in the source structure. The development of Magidor with F- or hyperfine levels seems to have a realistic chance of leading to a fine structure for core models independent of formula complexities and hence simpler. The next step should be to elaborate this for very decent core models like e.g. Welch's core model up to a Σ_2-mouse [18]. This should already be sufficient to see whether the central problems present themselves again. If they can be solved, the route to K^{DJ} is probably free. However, for higher core models, the possibility of finestructural drops in iterations would still remain a challenge.

15 Literature

References

[1] K. Devlin. Constructibility. Springer 1984

[2] J.D. Hamkins and A. Lewis. Infinite Time Turing Machines. J. Symbolic Logic, 65(2), 567–604 (2000)

[3] K. Devlin. Aspects of constructibility. Lecture Notes in Mathematics, vol. 345, Springer 1973

[4] K. Devlin. The axiom of constructibility. Springer 1984

[5] P. Koepke. Infinite time register machines. In A. Beckmann et al. (eds.) Logical approaches to computational barriers, LNCS, vol. 3988, pp. 257–266. Springer, Heidelberg (2006)

[6] R. B. Jensen. The fine structure of the constructible hierarchy. Ann. Math. Logic (1972)

[7] P. Koepke and R. Miller. An enhanced theory of infinite time register machines. In Arnold Beckmann et al. (eds.) Logic and Theory of Algorithms, LNCS, vol. 5028, pp. 306-315 (2008)

[8] S.D. Friedman, P. Koepke, B. Piwinger. Hyperfine structure and gap-1-morasses. In Journal of Symbolic Logic, Vol. 71, No. 2, 2006, pp. 480-490

[9] P. Koepke, P. Welch. A Generalized Dynamical System, Infinite Time Register Machines, and $\Pi^1_1 - CA_0$. Unpublished notes. (2010)

[10] P. Welch. Representing sets of ordinals as countable unions of simple sets in the core model. Journal of Symbolic Logic, Vol. 61, 1996

[11] M. Magidor. Representing sets of ordinals as countable unions of sets in the core model. Transactions of the American Mathematical Society, Vol 317, No. 1, 1990

[12] T. Jech. Set Theory. 3rd Millennium edition, revisited and expanded. Springer (2002)

[13] S.G. Simpson. Subsystems of Second Order Arithmetic. Springer 1999

[14] P Koepke: A new finestructural hierarchy for the constructible universe. CRM preprints, Barcelona 554 (2003)

[15] M. van Eijmeren, P. Koepke. A refinement of Jensen's constructible hierarchy. In Foundations of the Formal Sciences III, B. Löwe et al., eds., Kluwer (2004), 159-169

[16] S.D. Friedman, P. Koepke. An elementary approach to the fine structure of L. Bulletin of Symbolic Logic 4, 453-468

[17] P. Koepke. Simplified constructibility theory: minicourse Helsinki, March 2005. Graduate texts in mathematics, University of Helsinki 5 (2005)

[18] P. Welch. An Introduction to inner model theory. Lecture Notes from Kobe University, 1997

[19] J. Silver: How to eliminate the fine structure from the work of Jensen. Handwritten Notes from 199x

[20] Th. L. Richardson. Silver Machine approach to the constructible universe. Ph.D. thesis, University of California, Berkeley, 1979

[21] M. Carl, T. Fischbach, P. Koepke, R. Miller, M. Nasfi, G. Weckbecker. The basic theory of infinite time register machines. Archive for Mathematical Logic 49 (2010) 2, 249-273

[22] P. Koepke: Ordinal Computability. In Mathematical Theory and Computational Practice. K. Ambos-Spies et al, eds., Lecture Notes in Computer Science 5635 (2009), 280-289

[23] K.I. Devlin, R.B. Jensen. Marginalia to a Theorem of Silver. Lecture Notes in Mathematics, 1975, Vol. 499, 115-142

[24] A. Kanamori. The higher infinite. Springer 2nd edition (November 28, 2008)

[25] S.D. Friedman. Finestructure and Class Forcing. Walter de Gruyter 2000

[26] K. Kunen. Some applications of iterated ultrapowers in set theory. Ann. Math. Logic 1. (1970), 179-227

[27] G.E. Sacks. Higher Recursion Theory. Perspectives in Mathematical Logic, Volume 2, Springer 1990

[28] S.D. Friedman. Σ^* approach to finestructure of L. Fundamenta Mathematicae Vol 154, 1997

[29] S.D. Friedman. Fine Structure Theory and its Applications, Proceedings of the AMS Summer Institute at Cornell, 1985, pp. 259-269

[30] A.J. Dodd. The core model. Cambridge University Press 1982

[31] W. Mitchell. The Covering Lemma. In Handbook of Set Theory, Springer, 2010, pp. 1497-1594

[32] E. Schimmerling. The ABC's of Mice. The Bulletin of Symbolic Logic, Vol. 7, No. 4, Dec. 2001

[33] W. Mitchell. Beginning inner model theory. In Handbook of Set Theory, Springer 2010

[34] R. Schindler, M. Zeman. Fine structure theory. In Handbook of Set Theory, Springer 2010

[35] M. Zeman. Inner models and large cardinals. de Gruyter 2002

[36] A. Dodd, R. Jensen. The core model. Annals of Mathematical Logic 20 (1981), 43-75

[37] R. Jensen. Inner model and large cardinals. The Bulletin of Symbolic Logic, Vol. 1, No. 4, Dec. 1995

[38] N. Cutland. Computability: An Introduction to Recursive Function Theory. Cambridge University Press, 1980

[39] K. Gödel. The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis with the Axioms of Set Theory. in: Annals of Mathematics Studies. Princeton University Press, Princeton NJ 3.1940.

[40] B. Piwinger. Mind the gap: Hyperfine Structure Theory and Morasses. PhD thesis. Vienna 2004.

[41] P. Koepke. Basic Ideas of Core Model Theory. Unpublished seminar notes from 2004.

[42] P. Koepke. A new Finestructure for Relativized Constructibility. Unpublished Notes, 2008

[43] P. Welch. Σ^* Fine Structure. In Handbook of Set Theory, vol. 1, Springer 2009. 657-745

16 Zusammenfassung

Die Arbeit befasst sich mit alternativen Methoden zur Analyse von Gödels konstruktiblem Universum L, dem $\subseteq$-minimalen klassenmächtigen Modell von ZFC und anderer konstruktibler Strukturen.
Im ersten Teil werden F-Strukturen eingeführt, ein Ansatz von Koepke zur Vereinfachung der Feinstrukturtheorie von Kernmodellen. Wir gewinnen einige Vorteile für die weitere Entwicklung aus der Einführung einer Namensfunktion N unter die Basisfunktionen und kleinerer Modifikationen des Hüllenoperators. Es wird demonstriert, dass die F-Hierarchie ein geeignetes Instrument zum Beweis wichtiger Eigenschaften von L ist, wie etwa die Hausdorffsche verallgemeinerte Kontinuumshypothese GCH oder das kombinatorische Prinzip $\diamondsuit$. Dann wird eine Methode zur Erweiterung strukturerhaltender Funktionen, sogenannter feiner Abbildungen, angegeben, Koepkes vereinfachter Beweis des Überdeckungssatzes für L erläutert und ein Approximationssatz für L gezeigt:
Unter $\neg 0^\sharp$ ist jedes $X \subset On$ von überabzählbarer Konfinalität, das unter den Basisfunktionen der F-Hierarchie abgeschlossen ist, Vereinigung von abzählbar vielen Elementen von L.
Gegenüber dem Beweis des Approximationssatzes von Magidor, der die Abgeschlossenheit unter primitiv-rekursiven Mengenfunktionen voraussetzt, gewinnen wir deutlich an Kürze und Einfachheit.
Anschließend ergänzen wir die Basisfunktionen der F-Hierarchie durch Ansätze aus der Hyperfeinstrukturtheorie von Friedman und Koepke. Im Kontext der F-Hierarchie ergibt sich daraus das allgemeinere Konzept des Hyperings, das wir ausführen und benutzen, um die hyperfeinstrukturellen Beweise des Quadrat- und Morastprinzips in die F-Hierarchie zu übertragen. Das hierbei vornehmlich benutzte horizontale Hypering H_2 sorgt dabei für eine Unabhängigkeit der konstruierten Objekte von der gewählten Aufzählung der Formeln.
Anschliessend betrachten wir Infinite Time Register Machines ($ITRM$s) sowohl als Anwendung von wie auch als weiteren Zugang zu konstruktiblen Methoden. $ITRM$s sind Registermaschinen, deren Laufzeiten beliebige Ordinalzahlen sein können. Wir beweisen das Lost-Melody-Theorem für ITRMs, d.h. die Existenz einer reellen Zahl, die durch eine $ITRM$ als Orakelzahl erkannt, aber nicht berechnet werden kann. Wir führen getypte Maschinen ein, die Register mit verschiedenem Limesverhalten parallel verwenden und klassifizieren die Maschinentypen hinsichtlich ihrer Berechnungsstärke. Ins-

besondere zeigen wir, dass $ITRM$s mit $n+1$ überlaufenden Registern und einigen schwächeren Hilfsregistern den n-ten Hypersprung berechnen und das Halteproblem für $ITRM$s mit n überlaufenden Registern lösen können. Wir beweisen, dass die Menge der $ITRM$-erkennbaren reellen Zahlen in der Ordnung von L Lücken aufweist, und zwar mindestens von der Größe $\sup\{\omega_i^{CK}|i \in \omega\}$, wobei ω_i^{CK} die i-te zulässige Ordinalzahl bezeichnet. Außerdem zeigen wir, dass die beweistheoretischen Analysen von Welch bezüglich der Existenz der Haltezahlen für verschiedene Maschinen im Kontext der getypten Maschinen zu präziseren Schranken führen.
Im letzten Teil skizzieren wir Ansätze zu einer Übertragung alternativer Feinstrukturen auf allgemeinere konstruktible Strukturen, sogenannte Kernmodelle. Wir übertragen zentrale Konzepte, zeigen einige Erhaltungseigenschaften für die Ultrapotenzkonstruktion und ein Dodd-Jensen-Lemma für das entsprechende Iterationskonzept. Die Bewahrung der feinstrukturellen Information in Iterationen hingegen scheitert. Daran anschließend erläutern wir kurz die Gründe dieser Schwierigkeiten und diskutieren mögliche Auswege.