Homology Computations for Mapping Class Groups, in particular for $\Gamma^0_{3,1}$

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Rui Wang
aus
Beijing, China

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1. Gutachter: Prof. Dr. Carl-Friedrich Bödigheimer

2. Gutachter: Prof. Dr. Jens Franke

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Summary

In this thesis we compute the homology of mapping class groups of orientable and non-orientable surfaces. The surfaces we consider are of genus g, have one boundary curve and m permutable punctures. The corresponding moduli spaces $\mathfrak{M}^m_{g,1}$ in the orientable and $\mathfrak{N}^m_{g,1}$ in the non-orientable case are classifying spaces for the mapping class groups.

We are able to compute the integral homology of the moduli spaces $\mathfrak{M}^m_{g,1}$ for $h = 2g+m < 6$ and of $\mathfrak{N}^m_{g,1}$ for $h = g+m+1 < 5$ (Note that we give a non-orientable surface the genus g if it is the connected sum of $g+1$ projective planes). For $h = 6$ in the orientable case and $h = 5$ in the non-orientable case (these are the cases $\mathfrak{M}^0_{3,1}$, $\mathfrak{M}^2_{2,1}$ and $\mathfrak{M}^4_{1,1}$ resp. $\mathfrak{N}^0_{4,1}$, $\mathfrak{N}^1_{3,1}$, $\mathfrak{N}^2_{2,1}$ and $\mathfrak{N}^3_{1,1}$) we can compute some p-torsion in the homology and the mod-p Betti numbers for several primes. But this is enough evidence to conjecture that we have indeed the entire integral homology in these cases, too.

The computations are based on a cell structure of the moduli spaces. This cell structure is bi-simplicial and the associated chain complex $\mathbb{Q}_{\bullet\bullet}(h,m)$ resp. $\mathbb{NQ}_{\bullet\bullet}(h,m)$ can be described by parts of the classifying spaces of symmetric groups $\mathfrak{S}_2, \dots, \mathfrak{S}_{2h}$ resp. by parts of the classifying space of a category of pairings.

Motivated by B. Visy's Dissertation, we investigate ways to simplify the homology computation for $\mathfrak{M}^m_{g,1}$ and $\mathfrak{N}^m_{g,1}$. On the one hand, we extend the notion of factorable groups to factorable categories and study the homology of the norm complex associated to a factorable category; moreover, similar to the fact that a symmetric group is factorable, we prove that the category of pairings is a factorable category. On the other hand, from the cell structures of $\mathfrak{M}^m_{g,1}$ and $\mathfrak{N}^m_{g,1}$ with their orientation systems, we construct the double complexes $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$ and $\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h,m)$ and study their homology.

For the actual computations, we implemented the new algorithms in a C++ program.

Contents

Introduction

The main aim of this thesis is to do new computations on the homology of moduli spaces of surfaces, or equivalently, of mapping class groups of surfaces.

We consider the moduli space $\mathfrak{M}^m_{g,1}$ of conformal equivalence classes of Riemann surfaces $F = F^m_{g,1}$ of genus $g \geq 0$ with one boundary curve and $m \geq 0$ permutable punctures. Denote by $\Gamma^m_{g,1}$ the corresponding mapping class group, i.e. the isotropy classes of orientation-preserving diffeomorphisms fixing the boundary point-wise and permuting the punctures. Since the diffeomorphisms are required to fix the boundary curve, $\Gamma^m_{g,1}$ is torsion free. Therefore $\Gamma^m_{g,1}$ acts freely on the contractible Teichmüller space $\mathfrak{Teich}^m_{g,1}$ and the manifold $\mathfrak{M}^m_{g,1} = \mathfrak{Teich}^m_{g,1}/\Gamma^m_{g,1}$ is a classifying space of $\Gamma^m_{g,1}$.

A non-orientable surface together with a dianalytic (i.e. the coordinate changes are holomorphic or antiholomorphic) structure is called a Kleinian surface. Analogously to the case of a Riemann surface, let $\mathfrak{N}^m_{g,1}$ be the moduli space of dianalytic equivalence classes of Kleinian surfaces $NF = NF^m_{g,1}$ of genus $g \geq 0$ [1] with one boundary curve and $m \geq 0$ permutable punctures. Let $N\Gamma^m_{g,1}$ denote the corresponding mapping class group. Again, $N\Gamma^m_{g,1}$ is torsion free, acts freely on the contractible Teichmüller space $\mathfrak{NTeich}^m_{g,1}$ and the manifold $\mathfrak{N}^m_{g,1} = \mathfrak{NTeich}^m_{g,1}/N\Gamma^m_{g,1}$ is the classifying space of $N\Gamma^m_{g,1}$.

In order to compute the homology groups, it is helpful to find a suitable cell structure of these moduli spaces. In [B1], Bödigheimer first introduced an affine vector bundle $\mathfrak{Harm}^m_{g,1}$ over $\mathfrak{M}^m_{g,1}$. The fiber of $\mathfrak{Harm}^m_{g,1}$ over a point $F \in \mathfrak{M}^m_{g,1}$ consists of harmonic functions on F with certain prescribed singularities. He then analysed the gradient flow of a point u in $\mathfrak{Harm}^m_{g,1}$ and associated to u a so called stable critical graph $\mathcal{K}$. The graph $\mathcal{K}$ in turn produces a *parallel slit domain*, which is the complex plane with $h = 2m + g$ pairs of parallel horizontal slits. This method is called Hilbert uniformization. In this way, Bödigheimer obtained the space of parallel slit domains $\mathfrak{Par}^m_{g,1} = \mathrm{Par}(h, m) \smallsetminus \mathrm{Par}'(h, m)$, which is a manifold. Here $\mathrm{Par}(h, m)$ is a finite cell complex and $\mathrm{Par}(h, m)'$ is a subcomplex (consisting of "degenerate" surfaces). The main result in [B1] is that $\mathfrak{Par}^m_{g,1}$ is homeomorphic to $\mathfrak{Harm}^m_{g,1}$. On the other hand, $\mathfrak{Harm}^m_{g,1}$ is a flat fiber bundle with contractible fiber over $\mathfrak{M}^m_{g,1}$ and therefore homotopy equivalent to $\mathfrak{M}^m_{g,1}$, thus $\mathfrak{Par}^m_{g,1}$ has the some homotopy type as $\mathfrak{M}^m_{g,1}$:

$$\begin{tikzcd}
\mathfrak{Harm}^m_{g,1} \arrow[r, "\cong"'] \arrow[d, "\simeq"'] & \mathfrak{Par}^m_{g,1} = \mathrm{Par}(h,m) \smallsetminus \mathrm{Par}'(h,m) \\
\mathfrak{M}^m_{g,1} &
\end{tikzcd}$$

[1] The genus of a non-orientable surface is g, if it is a connected sum of $g+1$ real projective planes; thus we differ from the usual convention by one.

The cell structure on $\mathrm{Par}(h,m)$ is bi-simplicial, therefore the cellular chain complex $\mathbb{Q}_{\bullet\bullet}(h,m)$ of the relative manifold $(\mathrm{Par}(h,m), \mathrm{Par}'(h,m))$ is a double chain complex. $\mathbb{Q}_{\bullet\bullet}(h,m)$ also has a more combinatorial description using symmetric groups. We will only sketch this approach here, more details can be found in section 2.1.

Denote by $\mathbb{P}_{p,q}(h)$ the free abelian group generated by all $(q+1)$-tuples $\Sigma = (\sigma_q, \dots, \sigma_0)$ with σ_i in the symmetric group $\mathfrak{S}_{p+1}$ such that

$$N(\Sigma) := N(\sigma_q\sigma_{q-1}^{-1}) + \cdots + N(\sigma_1\sigma_0^{-1}) \leq h.$$

Here $N(\alpha)$ is the word length norm of α with respect to the generating set of $\mathfrak{S}_{p+1}$ which consists of all transpositions.

Define a double complex $\mathbb{P}_{\bullet\bullet}(h) := \bigoplus \mathbb{P}_{p,q}(h)$, $0 \leq p \leq 2h$, $q \leq h$, whose vertical and horizontal face operators are $\partial_i'(\Sigma) = (\sigma_q, \dots, \widehat{\sigma}_i, \dots, \sigma_0)$ and $\partial_j''(\Sigma) = (D_j(\sigma_q), \dots, D_j(\sigma_0))$ respectively, where D_j is defined on page 30. The subcomplex $\mathbb{P}'_{\bullet\bullet}(h,m)$ of $\mathbb{P}_{\bullet\bullet}(h)$ is generated by those cells of $\mathbb{P}_{\bullet\bullet}(h)$ which violate one of the conditions on page 30, for example

(1) $N(\Sigma) = h$

(2) σ_q has $m+1$ cycles

(3) σ_0 is the rotation $(0\ 1 \dots p)$

Then $\mathbb{Q}_{\bullet\bullet}(h,m) \cong \mathbb{P}_{\bullet\bullet}(h)/\mathbb{P}'_{\bullet\bullet}(h,m)$ is the double complex we are looking for.

The Hilbert uniformization method can also be applied to moduli spaces of Kleinian surfaces, see [Z1] and [E] for more details. Like in the orientable case, there is an affine vector bundle $\mathfrak{N}\mathfrak{harm}^m_{g,1}$ over $\mathfrak{N}^m_{g,1}$ whose fiber over a point $NF \in \mathfrak{N}^m_{g,1}$ consists of dianalytic functions on NF with certain prescribed singularities. Again by analysing the gradient flows of the points in $\mathfrak{N}\mathfrak{harm}^m_{g,1}$, the space of parallel slits domains $\mathfrak{N}\mathfrak{Par}^m_{g,1} = \mathrm{NPar}(h,m) \smallsetminus \mathrm{NPar}'(h,m)$, which is also a manifold, can be obtained. Here $\mathrm{NPar}(h,m)$ is a finite cell complex and $\mathrm{NPar}(h,m)'$ is a subcomplex (consisting of "degenerate" surfaces). The main result in [E] is that $\mathfrak{N}\mathfrak{Par}^m_{g,1}$ is homeomorphic to $\mathfrak{N}\mathfrak{harm}^m_{g,1}$. Moreover, since $\mathfrak{N}\mathfrak{harm}^m_{g,1}$ is a flat fiber bundle with contractible fiber over $\mathfrak{N}^m_{g,1}$ and therefore homotopy equivalent to $\mathfrak{N}^m_{g,1}$, $\mathfrak{N}\mathfrak{Par}^m_{g,1}$ has the some homotopy type as $\mathfrak{N}^m_{g,1}$.

The cell structure on $\mathrm{NPar}(h,m)$ is bi-simplicial, therefore the cellular chain complex $\mathbb{NQ}_{\bullet\bullet}(h,m)$ of the relative manifold $(\mathrm{NPar}(h,m), \mathrm{NPar}'(h,m))$ is a double chain complex. Again there is a more combinatorial description of $\mathbb{NQ}_{\bullet\bullet}(h,m)$, using pairings. We will give a brief review of this here, more details can be found in section 3.1.

Denote by $\Lambda_p \subset \mathfrak{S}_{2p}$ the set of fixed-point free involutions – so-called pairings – on $2p$ letters. Let $\mathbb{NP}_{p,q}(h)$ be the free abelian group generated by all $(q+1)$-tuples $\Lambda = (\lambda_q, \dots, \lambda_0)$ with $\lambda_i \in \Lambda_p$, such that

$$N_{\Lambda_p}(\Lambda) = \frac{1}{2}(N_{\mathfrak{S}_{2p}}(\lambda_q\lambda_{q-1}^{-1}) + \ldots + N_{\mathfrak{S}_{2p}}(\lambda_1\lambda_0^{-1})) \leq h,$$

where $N_{\mathfrak{S}_{2p}}(\alpha)$ is the word length norm of α with respect to the generating set of $\mathfrak{S}_{2p}$ which consists of all transpositions.

Define a double complex $\mathbb{NP}_{\bullet\bullet}(h) := \bigoplus \mathbb{NP}_{p,q}(h)$, $0 \leq p \leq 2h$, $q \leq h$, whose vertical and horizontal face operators are $\partial'_i(\Lambda) = (\lambda_q, \ldots, \widehat{\lambda_i}, \ldots, \lambda_0)$ and $\partial''_j(\Lambda) = (D_j(\lambda_q), \ldots, D_j(\lambda_0))$ respectively, where D_j is defined on page 51. The subcomplex $\mathbb{NP}'_{\bullet\bullet}(h, m)$ of $\mathbb{NP}_{\bullet\bullet}(h)$ is generated by the cells of $\mathbb{NP}_{\bullet\bullet}(h)$ which violate any of conditions listed on page 51. Some of these conditions are:

(1) $N_{\mathbf{\Lambda}_p}(\Lambda) = h$

(2) $\lambda_q \circ J$ has $2(m+1)$ cycles, where the special element $J \in \Lambda_p$ is defined in (3.1.5)

(3) λ_0 is given by (3.1.6)

Then $\mathbb{NQ}_{\bullet\bullet}(h, m) \cong \mathbb{NP}_{\bullet\bullet}(h)/\mathbb{NP}'_{\bullet\bullet}(h, m)$ is the desired double complex.

Computations on the homology of $\mathfrak{M}^m_{g,1}$ already have been done using the spectral sequences of the double complexes $\mathbb{Q}_{\bullet\bullet}(h, m)$ and $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h, m)$ – which will be described later – in the series of works [Eh], [A] and [ABE]. In [ABE], the results up to $h = 5$ for $\mathfrak{M}^m_{g,1}$ are obtained; this article gives also a good overview of the homology computations that have been done by other authors at this time. A special feature of the computational results in [ABE] is that they do not lie in the stable range, and very little information is known in this situation as h becomes bigger. In the non-orientable situation, in [Z2], mod-2 homology of $\mathfrak{N}^m_{g,1}$ was computed for $h = 2, 3$ via the double complex $\mathbb{NQ}_{\bullet\bullet}(h, m)$.

During the computations using the double complex $\mathbb{Q}_{\bullet\bullet}(h, m)$, Ehrenfried, Abhau and Bödigheimer realized that the spectral sequence of $\mathbb{Q}_{\bullet\bullet}(h, m)$ has the property that its E^1-term concentrates on the top degree and thus it converges at E^2. This phenomenon later was fully explained by Visy in [V]. He introduced the concept of factorable groups while studying the norm complex $\mathcal{N}_*(G)[h]$ of a normed group G and proved the important result that the homology of $\mathcal{N}_*(G)[h]$ concentrates on the top degree h. The behavior of the spectral sequence of $\mathbb{Q}_{\bullet\bullet}(h, m)$ can then be explained from the facts that every symmetric group $\mathfrak{S}_p$ is factorable and that $\mathbb{Q}_{p,*}(h, m)$ is isomorphic to a direct summand of $\mathcal{N}_*(\mathfrak{S}_p)[h]$.

The theory developed by Visy also allows one to construct the E^1-term of the spectral sequence of $\mathbb{Q}_{\bullet\bullet}(h, m)$ directly. The E^1-term is equivalent to a chain complex $(W_*(h, m), d)$. This motivated us to do more homology computations. For example, theoretically the homology of $\Gamma^0_{g,1}$ and $\Gamma^1_{g,1}$ can be computed using $(W_*(h, m), d)$, since the spaces $\mathfrak{M}^0_{g,1}$ and $\mathfrak{M}^1_{g,1}$ are orientable.

In the present work, we have extended the theory of factorable groups in two aspects to make full use of the idea of factorability.

The first one is, in order to simplify the homology computation about $\mathfrak{N}^m_{g,1}$ in the same manner as that of $\mathfrak{M}^m_{g,1}$, we generalize the notion of a factorable group to a factorable category. We do this, because the categories of pairings are involved in the double complex $\mathbb{NQ}_{\bullet\bullet}(h, m)$ in the same way as symmetric groups are involved in $\mathbb{Q}_{\bullet\bullet}(h, m)$.

The other aspect is, since the double complexes $\mathbb{Q}_{\bullet\bullet}(h, m)$ and $\mathbb{NQ}_{\bullet\bullet}(h, m)$ are the cell structure of relative manifolds, Poincaré duality is needed to connect their cohomology to the homology of the moduli spaces $\mathfrak{M}^m_{g,1}$ and $\mathfrak{N}^m_{g,1}$:

$$H^*(\mathrm{Par}(h,m),\mathrm{Par}'(h,m);\mathcal{O}) \cong H_{3h-*}(\mathfrak{M}^m_{g,1};\mathbb{Z}),$$
$$H^*(\mathrm{NPar}(h,m),\mathrm{NPar}'(h,m);\mathcal{O}) \cong H_{3h-*}(\mathfrak{N}^m_{g,1};\mathbb{Z}).$$

Hence when the moduli space is non-orientable (which is the case for $\mathfrak{M}^m_{g,1}$ when $m \geq 2$ and for $\mathfrak{N}^m_{g,1}$ for all m) the orientation system $\mathcal{O}$ on the relative manifold is involved in the computation of its integral homology. Therefore, we construct the double complexes $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$ and $\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h,m)$, which are the cell complexes of the relative manifolds with orientation system. They have the same $\mathbb{Z}$-modules as $\mathbb{Q}_{\bullet\bullet}(h,m)$ and $\mathbb{NQ}_{\bullet\bullet}(h,m)$, but different boundary operators. These double complexes also turn out to have the properties that the E^1-terms of their spectral sequences concentrate on the top degree and converge at E^2. We obtain these results by arguments very similar to those used in proving the corresponding properties of factorable groups.

After these theoretical preparations, we were able to do the homology computations for the mapping class groups $\Gamma^m_{g,1}$ with $h \leq 6$ and for $N\Gamma^m_{g,1}$ with $h \leq 5$. The computations were carried out with the help of a computer program written in C++. But for $\Gamma^m_{g,1}$ with $h = 6$ and $m = 0, 2, 4$, and for $N\Gamma^m_{g,1}$ when $h = 5$ and $m = 0, 1, 2, 3$, we only get partial information about their homology groups, because some of matrices involved were so huge, that it was not possible to compute the Smith normal form on standard computers. However, we conjecture that we have actually obtained the full information.

Among the mapping class groups we considered, $\Gamma^0_{3,1}$ is a particularly interesting example. Based on our computation, we conjecture:

$$H_n(\mathfrak{M}^0_{3,1}) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n=1 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & n=2 \\ \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4\oplus\mathbb{Z}_7 & n=3 \\ \mathbb{Z}_2^2\oplus\mathbb{Z}_3^2 & n=4 \\ \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_3 & n=5 \\ \mathbb{Z}\oplus\mathbb{Z}_2^3 & n=6 \\ \mathbb{Z}_2 & n=7 \\ 0 & n=8 \\ \mathbb{Z} & n=9 \\ 0 & n\geq 10 \end{cases}$$

For $n = 0, 1, 4, 7, 8, 9$ and $n \geq 10$ in this list, the homology groups have been verified by the computational results and the universal coefficient theorem. It is known that $H_1(\Gamma^0_{3,1})$ lies in the stable range and should be 0 according to the theory of the stable homology of mapping class groups, which is consistent with our result. Moreover, $H_2(\Gamma^0_{3,1})$ recently was computed with completely different methods by Sakasai ([S]) to be $\mathbb{Z}\oplus\mathbb{Z}_2$, which is also consistent with our conjecture. If we take this result into consideration, then $H_3(\Gamma^0_{3,1}) = \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4\oplus\mathbb{Z}_7$ is also verified. Thus only $H_5(\Gamma^0_{3,1})$ and $H_6(\Gamma^0_{3,1})$ are not completely determined and remain as conjecture.

Apart from this, our computational results for $h \leq 5$ coincide with the results in [ABE]. Comparing the mod-2 homology computations of $N\Gamma^m_{g,1}$ in [Z2] for $h = 2, 3$

with our computations, we find that most of the results are consistent, but there is a discrepancy for $N\Gamma^1_{1,1}$.

The contents and structure of this thesis are as follows:

In **chapter 1**, the notion of factorable groups is generalized to factorable categories and the homology of the norm complex $\mathcal{N}_*(\mathscr{C})[h]$ associated to a factorable category $\mathscr{C}$ is studied parallel to that of a factorable group. This chapter owes a lot to [V]. The main result in this chapter is:

> **Theorem 1.2.11.** *If $\mathscr{C}$ is a factorable small category with respect to the norm N, then the homology of the complex $\mathcal{N}_*(\mathscr{C})[h]$ is concentrated in the top degree h:*
> $$H_q(\mathcal{N}_*(\mathscr{C})[h]) = 0, \qquad \text{if} \quad q < h.$$

Moreover, the generators of the homology group $H_h(\mathcal{N}_*(\mathscr{C})[h])$ can be found systematically by introducing a homomorphism of modules $\kappa : V_h(\mathscr{C}) \to H_h(\mathcal{N}_*(\mathscr{C})[h])$, where $V_h(\mathscr{C})$ is a direct summand of the module $\mathcal{N}_h(\mathscr{C})[h]$.

> **Theorem 1.3.3.** *Let $\mathscr{C}$ be a factorable normed category. If $\mathscr{C}$ satisfies the right cancellation property and has finitely many morphisms with norm one, then $\kappa : V_h(\mathscr{C}) \to H_h(\mathcal{N}_*(\mathscr{C})[h])$ is an isomorphism.*

This is a generalization of Theorem 5.4.1 in [V], where Visy proved that κ is an isomorphism for symmetric groups in the framework of factorable groups.

In the beginning of **chapter 2** the relation between moduli spaces $\mathfrak{M}^m_{g,1}$ and symmetric groups is presented. Then the construction of the double complex $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$ of the relative manifold $(\mathrm{Par}(h,m), \mathrm{Par}'(h,m))$ with the orientation system $\mathcal{O}$ is given and its homology is studied. The main result of this part is:

> **Theorem 2.3.15.** *The homology of the complex $\widetilde{\mathbb{Q}}_{p,*}(h,m)$ is concentrated in the top degree h:*
> $$H_q(\widetilde{\mathbb{Q}}_{p,*}(h,m)) = 0, \qquad \text{if} \quad q < h.$$

This leads to the simplification of the homology computation of $\Gamma^m_{g,1}$.

In **chapter 3** first the relation between moduli spaces $\mathfrak{N}^m_{g,1}$ and the category of pairings is recalled. Then the fact that the category of pairings is a factorable category is proved:

> **Theorem 3.2.2.** *Λ_p is a factorable category.*

In the end, we construct the double complex $\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h,m)$ of the relative manifold $(\mathrm{NPar}(h,m), \mathrm{NPar}'(h,m))$ with the orientation system $\mathcal{O}$ and study its homology. We have the following result:

Theorem 3.3.13. *The homology of the complex* $\widetilde{\mathbb{NQ}}_{p,*}(h,m)$ *is concentrated in the top degree* h*:*

$$H_q(\widetilde{\mathbb{NQ}}_{p,*}(h,m)) = 0, \qquad \textit{if} \quad q < h.$$

This allows a simplified computation of the homology of $N\Gamma^m_{g,1}$.

In **chapter 4**, we list the results of the homology computations and state the Conjecture 4.2.1. The main part of this conjecture concerns $\Gamma^0_{3,1}$ and was discussed above.

In the **appendix** we give an overview of the computer program, with which the homology computations have been performed.

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Chapter 1

Factorable Normed Categories

In this chapter we will generalize the notion of normed and factorable groups introduced in [V] to the case of small categories. This generalization is very straightforward; and the known methods for factorable groups remain almost unchanged, when we study the homology properties of the norm complex associated to a factorable category.

Note that because all the categories considered in this thesis are small categories, we will call them categories for short.

1.1 Norm Filtration

First we define the notion of a *norm* on a category. Let $\mathscr{C}$ be a category, and denote its set of morphisms $\mathrm{Mor}(\mathscr{C})$ by $\mathscr{M}$. For simplicity of notation, we will write a morphism $g : X \to Y \in \mathscr{M}$ as g, without mentioning the source X and target Y explicitely.

Definition 1.1.1. A *norm* on $\mathscr{C}$ is a function $N : \mathscr{M} \longrightarrow \mathbb{N}$ satisfying the following properties:

(1) $N(g) = 0 \Leftrightarrow g = \mathrm{id}_X$ for some object X of $\mathscr{C}$.

(2) $N(g_2 \circ g_1) \leq N(g_1) + N(g_2)$, if g_1 and g_2 are composable.

The simplest example of a norm is the following:

Example 1.1.2. *The constant norm.* The *constant norm* N_c on a category $\mathscr{C}$ assigns to each morphism $g \in \mathscr{M}$ which is not an identity morphism, a fixed value $c \in \mathbb{N}^+$.

Example 1.1.3. *Normed Groups.* To a group G, one can associate two categories.

One is $\mathscr{E}G$, whose set of object is the set of group elements of G and for any two objects g_1, g_2, there is exactly one morphism from g_1 to g_2, i.e. $\mathrm{Mor}(g_1, g_2) = \{(g_2 g_1^{-1}, g_1)\}$. The composition of morphisms in $\mathscr{E}G$ is $(g_3 g_2^{-1}, g_2) \circ (g_2 g_1^{-1}, g_1) = (g_3 g_1^{-1}, g_1)$.

The other one is $\mathscr{B}G$, which has one object, and the set of morphisms is the set of group elements. The composition of morphisms in $\mathscr{B}G$ is the group multiplication.

Assume that G has a norm N, that is, $N : G \to \mathbb{N}$ is a function satisfying

(1) $N(g) = 0 \Leftrightarrow g = 1$

(2) $N(g_2g_1) \leq N(g_1) + N(g_2)$.

Then N induces a norm N^e on $\mathscr{E}G$ by $N^e((g_2g_1^{-1}), g_1) := N(g_2g_1^{-1})$ as well as a norm N^b on $\mathscr{B}G$ by $N^b(g) := N(g)$.

Let $\pi : \mathscr{E}G \to \mathscr{B}G$ be the functor which sends every object of $\mathscr{E}G$ to the unique object of $\mathscr{B}G$ and sends a morphism $(g_2g_1^{-1}, g_1) \in \mathscr{E}G$ to $g_2g_1^{-1} \in \mathscr{B}G$. It is easy to see that the functor π preserve the norms, i.e. $N^b(\pi((g_2g_1^{-1}, g_1))) = N^e((g_2g_1^{-1}, g_1))$.

A broad class of norms, which also provide a way to construct norms on categories, are:

Example 1.1.4. *The word length norm.* A set of morphisms $S = \{g_i \mid i \in I\} \subseteq \mathscr{M}$ is a *generating set*, if any morphism $g \in \mathscr{M}$ can be written as a composition of finitely many elements in S. The word length norm on $\mathscr{C}$ with respect to the generating set S of $\mathscr{C}$ is defined for each $g \in \mathscr{M}$ to be the minimal number of generators (with multiplicity) from the set S needed to present g.

A norm N on $\mathscr{C}$ induces a filtration on the set of morphisms $\mathscr{M}$:

$$\mathcal{F}_0\mathscr{M} \subseteq \mathcal{F}_1\mathscr{M} \subseteq \ldots \subseteq \mathcal{F}_h\mathscr{M} \subseteq \ldots$$

by defining

$$\mathcal{F}_h\mathscr{M} := \{g \in \mathscr{M} \mid N(g) \leq h\}.$$

Now we introduce more notations for the sake of the discussions later. First, $T_h(\mathscr{M}) := \mathcal{F}_h\mathscr{M} \smallsetminus \mathcal{F}_{h-1}\mathscr{M}$ and $T(\mathscr{M}) := T_m(\mathscr{M})$, where m is the smallest non-zero value of the norm. Furthermore, the set of composable n-tuples of morphisms is

$$\mathscr{M}(n) := \{(g_n, \ldots g_1) \mid g_n \circ \ldots \circ g_1 \text{ exists }\}.$$

In particular, $\mathscr{M}(0)$ is the set of all identity morphisms and $\mathscr{M}(1) = \mathscr{M}$.

The norm N on $\mathscr{C}$ now induces a norm (still denoted by N) on $\mathscr{M}(n)$:

$$N((g_n, \ldots, g_1)) := \sum_{i=1}^{n} N(g_i), \text{ for } (g_n, \ldots, g_1) \in \mathscr{M}(n).$$

This in turn induces a norm filtration on $\mathscr{M}(n)$:

$$\mathcal{F}_0(\mathscr{M}(n)) \subseteq \mathcal{F}_1(\mathscr{M}(n)) \subseteq \ldots \subseteq \mathcal{F}_h(\mathscr{M}(n)) \subseteq \ldots$$

by defining

$$\mathcal{F}_h(\mathscr{M}(n)) := \{(g_n, \ldots, g_1) \in \mathscr{M}(n) \mid N((g_n, \ldots, g_1)) \leq h\}.$$

Consider the *normalized bar complex* $(B_*(\mathscr{C}), d)$ of $\mathscr{C}$:

$$\cdots \xrightarrow{d} B_{q+1}(\mathscr{C}) \xrightarrow{d} B_q(\mathscr{C}) \xrightarrow{d} B_{q-1}(\mathscr{C}) \xrightarrow{d} \cdots$$

Here $B_q(\mathscr{C})$ is the free $\mathbb{Z}$-module generated by all $(g_q, \dots, g_1) \in \mathscr{M}(q)$ in which g_i is not an identity morphism for $i = 1, \dots, q$. We use the notation $(g_q|\dots|g_1)$ for such a generator, also call it a q-simplex. The boundary d of a generator $(g_q|\dots|g_1)$ is defined to be $d = \sum_{i=0}^{q}(-1)^i d_i$, where

$$d_i((g_q|\dots|g_1)) = \begin{cases} (g_q|\dots|g_2) & i = 0 \\ (g_q|\dots|g_{i+1} \circ g_i|\dots|g_1) & 0 < i < q \\ (g_{q-1}|\dots|g_1) & i = q \end{cases}$$

As with $\mathscr{M}(n)$, the norm N induces a norm (still denoted by N) on the set of simplices of $B_q(\mathscr{C})$ for each $q \geq 0$

$$N(g_q|\dots|g_1) = \sum_{i=1}^{q} N(g_i), \text{ for } (g_q|\dots|g_1) \in B_q(\mathscr{C}).$$

N can in turn induce a filtration on $B_q(\mathscr{C})$

$$\mathcal{F}_0(B_q(\mathscr{C})) \subseteq \mathcal{F}_1(B_q(\mathscr{C})) \subseteq \dots \subseteq \mathcal{F}_h(B_q(\mathscr{C})) \subseteq \dots$$

where $\mathcal{F}_h B_q(\mathscr{C})$ is the submodule of $B_q(\mathscr{C})$ generated by the simplices with norm at most h. Due to the inequality (2) in the definition of a norm on $\mathscr{C}$, $d : \mathcal{F}_h B_q(\mathscr{C}) \to F_h B_{q-1}(\mathscr{C})$, hence $\mathcal{F}_h B_*(\mathscr{C})$ is a subcomplex of $B_*(\mathscr{C})$ and there is an increasing filtration of $B_*(\mathscr{C})$.

$$\mathcal{F}_0 B_*(\mathscr{C}) \subseteq \mathcal{F}_1 B_*(\mathscr{C}) \subseteq \dots \subseteq \mathcal{F}_h B_*(\mathscr{C}) \subseteq \dots$$

Finally, the *norm complex* $(\mathcal{N}_*(\mathscr{C})[h], d)$ is defined to be quotient complex:

$$\mathcal{N}_*(\mathscr{C})[h] := \mathcal{F}_h B_*(\mathscr{C}) / \mathcal{F}_{h-1} B_*(\mathscr{C}), \tag{1.1.1}$$

where d is the induced boundary. As in the case of factorable groups, $(\mathcal{N}_*(\mathscr{C})[h], d)$ is our main object of interest, when we study factorable categories in the following sections.

Following is a more concrete description of the i-th face operator of $(\mathcal{N}_*(\mathscr{C})[h], d)$.

$$d_i((g_q|\dots|g_1)) = \begin{cases} (g_q|\dots|g_{i+1} \circ g_i|\dots|g_1) & \text{if } N(g_{i+1} \circ g_i) = N(g_{i+1}) + N(g_i) \\ 0 & \text{if } N(g_{i+1} \circ g_i) < N(g_{i+1}) + N(g_i) \end{cases}$$

Note that d_0 and d_q are always zero in the $(\mathcal{N}_*(\mathscr{C})[h], d)$, because they lower the norm as in the second line of formula above.

1.2 Factorability and Homology of the Norm Complex

In this section we will define factorable categories and prove the theorem that the homology of the norm complex $\mathcal{N}_*(\mathscr{C})[h]$ concentrates on the top degree for a factorable category $\mathscr{C}$. Although the way to develop the theory is almost the same as in [V], we choose to write it down in detail for the completeness of the text.

Without loss of generality, from now on we always assume that m, the smallest non-zero value of the norm, is 1, unless otherwise specified.

First we introduce the notion of a *graded object*, which later helps to make the definition of factorability in a concise form. Let S be a filtered set

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_h \subseteq \ldots, \quad S = \bigcup_{h \geq 0} S_h,$$

the *graded object* $\mathcal{G}r_*(S)$ associated to S is defined as the wedge sum of the filtration quotients

$$\mathcal{G}r_*(S) := \bigvee_{h \geq 0} S_h / S_{h-1} = \{+\} \sqcup \bigsqcup_{h \geq 0} S_h \smallsetminus S_{h-1},$$

where $+$ is an extra basepoint and S_{-1} is the empty set. If a map $\varphi : S \to S'$ between filtered sets is filtration-preserving, i.e., $\varphi(S_h) \subseteq S'_h$ for every h, then φ induces a graded map $\varphi_* : \mathcal{G}r_*(S) \to \mathcal{G}r_*(S')$ between the associated graded objects. More precisely, $\varphi_*(+) := +$; and for $s \in S_h \smallsetminus S_{h-1}$, if $\varphi(s) \in S'_h \smallsetminus S'_{h-1}$, then $\varphi_*(s) := \varphi(s)$, otherwise $\varphi_*(s) := +$.

Recall that we have the following filtration of $\mathscr{M}(n)$ for a normed category $\mathscr{C}$:

$$\mathcal{F}_0(\mathscr{M}(n)) \subseteq \mathcal{F}_1(\mathscr{M}(n)) \subseteq \ldots \subseteq \mathcal{F}_h(\mathscr{M}(n)) \subseteq \ldots,$$

the associated graded object is then:

$$\mathcal{G}r_*(\mathscr{M}(n)) := \bigvee_{h \geq 0} \mathcal{F}_h(\mathscr{M}(n)) / \mathcal{F}_{h-1}(\mathscr{M}(n)) = \{+\} \sqcup \bigsqcup_{h \geq 0} \mathcal{F}_h(\mathscr{M}(n)) \smallsetminus \mathcal{F}_{h-1}(\mathscr{M}(n)).$$

In particular, when $n = 1$, the graded object is:

$$\mathcal{G}r_*(\mathscr{M}) := \bigvee_{h \geq 0} \mathcal{F}_h(\mathscr{M}) / \mathcal{F}_{h-1}(\mathscr{M}) = \{+\} \sqcup \bigsqcup_{h \geq 0} T_h(\mathscr{M}).$$

Since composition of morphisms $\mu : \mathscr{M}(n) \to \mathscr{M}$ is filtration-preserving, it induces a graded map $\mu_* : \mathcal{G}r_*(\mathscr{M}(n)) \to \mathcal{G}r_*(\mathscr{M})$. That is, if $N(g_n \circ \ldots \circ g_1)) = N(\mu((g_n, \ldots, g_1))) = N((g_n, \ldots, g_1)) = N(g_n) + \ldots + N(g_1)$, then $\mu_*((g_n, \ldots, g_1)) = g_n \circ \ldots \circ g_1$; otherwise $\mu_*((g_n, \ldots, g_1)) = +$.

Definition 1.2.1. A map $\eta : \mathscr{M} \to \mathscr{M}(2)$ sending $g \in \mathscr{M}$ to a composable pair $(\bar{\eta}(g), \eta'(g)) \in \mathscr{M}(2)$ is called a *factorization map* on $\mathscr{C}$, if it satisfies the following properties:

(1) $\bar{\eta}(g) \circ \eta'(g) = g$

(2) $N(\bar{\eta}(g)) + N(\eta'(g)) = N(g)$

(3) $\eta'(g) \in T(\mathscr{M})$ for any $g \neq \mathrm{id}$.

$$\begin{tikzcd} X \arrow[r, "g'"'] \arrow[rr, bend left, "g"] & Z \arrow[r, "\bar{g}"'] & Y \end{tikzcd}$$

In particular, it follows from property (2) that $\eta(\mathrm{id}) = (\mathrm{id}, \mathrm{id})$ and $\eta(t) = (\mathrm{id}, t)$ for any $t \in T(\mathscr{M})$.

We call $\eta'(g)$ the *prefix* of g and $\bar{\eta}(g)$ the *remainder* of g. The prefix of g will also be denoted by g' and the remainder of g by $\bar{g}$ in the following text.

Definition 1.2.2. A normed category $\mathscr{C}$ with norm N is called *factorable*, or *norm-factorable* if it admits a factorization map η such that the following diagram of graded objects and graded maps commutes:

$$\begin{tikzcd}
\mathcal{G}r_*(\mathscr{M}(2)) \arrow[r, "(\eta\times id)_*"] \arrow[dd, "\mu_*"] & \mathcal{G}r_*(\mathscr{M}(3)) \arrow[d, "(id\times\mu)_*"'] & \\
 & \mathcal{G}r_*(\mathscr{M}(2)) \arrow[r, "(id\times\eta)_*"] & \mathcal{G}r_*(\mathscr{M}(3)) \arrow[d, "(\mu\times id)_*"] \\
\mathcal{G}r_*(\mathscr{M}) \arrow[rr, "\eta_*"] & & \mathcal{G}r_*(\mathscr{M}(2))
\end{tikzcd}$$

By using the language of graded objects and graded maps, Definition 1.2.2 is formulated in a concise way. However, to see more clearly what the commutativity of diagram 1.2.2 means, we would like to translate the requirements into the ungraded, thus a more concrete version. Consider the two filtration-preserving maps α_u, α_l, which induce the upper composition and lower composition in the diagram respectively. That is,

$$\alpha_u = (\mu \times id) \circ (id \times \eta) \circ (id \times \mu) \circ (\eta \times id) : \mathscr{M}(2) \to \mathscr{M}(2)$$
$$\alpha_u : (g, h) \mapsto (\bar{g}, g', h) \mapsto (\bar{g}, g' \circ h) \mapsto (\bar{g}, \overline{g' \circ h}, (g' \circ h)') \mapsto (\bar{g} \circ \overline{g' \circ h}, (g' \circ h)')$$

and

$$\alpha_l = \eta \circ \mu : \mathscr{M}(2) \longrightarrow \mathscr{M}(2)$$
$$\alpha_l : (g, h) \mapsto g \circ h \mapsto (\overline{g \circ h}, (g \circ h)').$$

Then the commutativity of the diagram in Definition 1.2.2 is equivalent to the following requirements:

(A) The map α_u is norm-preserving if and only if the map α_l is norm-preserving. That is, for $(g, h) \in \mathscr{M}(2)$

$$N(\alpha_u((g, h))) = N(g) + N(h) \Longleftrightarrow N(\alpha_l((g, h))) = N(g) + N(h).$$

(B) if $N(\alpha_l((g, h))) = N(g) + N(h)$ (and hence $N(\alpha_u((g, h))) = N(g) + N(h)$ due to (A)), then $\alpha_u((g, h)) = \alpha_l((g, h))$ holds. That is, for $(g, h) \in \mathscr{M}(2)$

$$N(\alpha_l((g, h))) = N(g) + N(h) \Longrightarrow (g \circ h)' = (g' \circ h)' \text{ and } \bar{g} \circ \overline{g' \circ h} = \overline{g \circ h}.$$

To verify whether a normed category $\mathscr{C}$ is factorable with the factorization map η, one needs to check if the requirements (A) and (B) are satisfied by all pairs $(g,h) \in \mathscr{M}(2)$. However, it will turn out that it is sufficient to check all pairs $(g,t) \in \mathscr{M}(2)$ with $t \in T(\mathscr{M})$.

Proposition 1.2.3. *Assume that $\eta : \mathscr{M} \to \mathscr{M}(2)$ is a factorization map on $\mathscr{C}$. If conditions (A) and (B) are satisfied by all pairs $(g,t) \in \mathscr{M}(2)$ with $t \in T(\mathscr{M})$, then they are satisfied by all pairs $(g,h) \in \mathscr{M}(2)$.*

For the proof, it is useful to introduce the following

Definition 1.2.4. Assume that $(\mathscr{C}, N)$ is a normed category. A tuple $(a_n, \ldots, a_1)$ of composable morphisms in $\mathscr{C}$ is called a *geodesic tuple* if $N(a_n \circ \ldots \circ a_1) = N(a_n) + \ldots + N(a_1)$. We write $a_n // \ldots // a_1$ to denote that $(a_n, \ldots, a_1)$ is a geodesic tuple. In particular, a geodesic tuple (a_2, a_1) is called a *geodesic pair*.

Note that by definition $(\bar{g}, g')$ is a geodesic pair. Moreover, one can observe that $\alpha_u((g,h))$ is norm-preserving if and only if (g', h) and $(\bar{g}, \overline{g' \circ h})$ are geodesic pairs and that $\alpha_l((g,h))$ is norm-preserving if and only if (g,h) is a geodesic pair.

We will make frequent use of the following

Lemma 1.2.5.

$$c//b//a \iff (b//a \text{ and } c//b \circ a) \iff (c//b \text{ and } c \circ b//a). \tag{1.2.1}$$

Proof. We will prove only the first equivalence, because the equivalence

$$c//b//a \iff (c//b \text{ and } c \circ b//a)$$

can be proved in the same way.

Let $c//b//a$. If (b,a) or $(c, b \circ a)$ is not a geodesic pair, we get the contradiction

$$N(c \circ b \circ a) < N(a) + N(b) + N(c).$$

From $c//b \circ a$ and $a//b$, it follows that

$$N(c \circ b \circ a) = N(c) + N(b \circ a) = N(a) + N(b) + N(c),$$

hence $c//b//a$. ☐

Proof of Proposition 1.2.3. We use induction on the norm of h.

When $N(h) = 0$, i.e., $h = \mathrm{id}$,

$$\alpha_u((g,h)) = (\bar{g}, g') = \alpha_u((g,h)),$$

so (A) and (B) are satisfied.

When $N(h) = 1$, i.e. $h \in T(\mathscr{M})$, (A) and (B) are satisfied by assumption.

Now assume that (A) and (B) are satisfied by all pairs $(g,h) \in \mathscr{M}(2)$ with $N(h) < n$ ($n \geq 2$), we want to show that (A) and (B) are satisfied by each pair $(g,h) \in \mathscr{M}(2)$ with $N(h) = n$.

First we show this for (A).

(I) If $g/\!/h$, then by (1.2.1), we conclude

$$\begin{array}{lcccccc} & \bar{g}/\!/g' \text{ and } g/\!/h & \Longrightarrow & \bar{g}/\!/g'/\!/h & \Longrightarrow & g'/\!/h \text{ and } \bar{g}/\!/g' \circ h \\ \text{and} & (\overline{g' \circ h}/\!/(g' \circ h)' \text{ and } \bar{g}/\!/g' \circ h) & \Longrightarrow & \bar{g}/\!/\overline{g' \circ h}/\!/(g'/\!/h)' & \Longrightarrow & \bar{g}/\!/\overline{g' \circ h}. \end{array}$$

Thus we have proved $g'/\!/h$ and $\bar{g}/\!/\overline{g' \circ h}$.

(II) If $g'/\!/h$ and $\bar{g}/\!/\overline{g' \circ h}$, we want to show that $g/\!/h$. The proof will be in three steps. The first step is to show that $g/\!/\bar{h}$. By (1.2.1), we have

$$\bar{h}/\!/h' \text{ and } g'/\!/h \Longrightarrow g'/\!/\bar{h}/\!/h' \Longrightarrow g'/\!/\bar{h} \text{ and } g' \circ \bar{h}/\!/h'.$$

Since $h' \in T(\mathscr{M})$, applying the assumption of the proposition to the pair $(g' \circ \bar{h}, h')$, it follows from condition (A) that $\overline{g' \circ \bar{h}}/\!/\overline{(g' \circ \bar{h})' \circ h'}$, and from condition (B) that

$$\overline{g' \circ \bar{h}} \circ \overline{(g' \circ \bar{h})' \circ h'} = \overline{g' \circ \bar{h} \circ h'} = \overline{g' \circ h}.$$

By (1.2.1), we have that

$$\overline{g' \circ \bar{h}}/\!/\overline{(g' \circ \bar{h})' \circ h'} \text{ and } \bar{g}/\!/\overline{g' \circ h} \Longrightarrow \bar{g}/\!/\overline{g' \circ \bar{h}}/\!/\overline{(g' \circ \bar{h})' \circ h'} \Longrightarrow \bar{g}/\!/\overline{g' \circ \bar{h}}.$$

Since we have shown $g'/\!/\bar{h}$ and $\bar{g}/\!/\overline{g' \circ \bar{h}}$, it follows by induction (because $N(\bar{h}) < n$) from condition (A) that $g/\!/\bar{h}$.

The second step is to show $g \circ \bar{h}/\!/h'$. Since $g/\!/\bar{h}$, it follows by induction from condition (B) that $(g' \circ \bar{h})' = (g \circ \bar{h})'$ and $\bar{g} \circ \overline{g' \circ \bar{h}} = \overline{g \circ \bar{h}}$. Recall that we have shown in the first step $\bar{g}/\!/\overline{g' \circ \bar{h}}/\!/\overline{(g' \circ \bar{h})' \circ h'}$, therefore

$$\bar{g} \circ \overline{g' \circ \bar{h}}/\!/\overline{(g' \circ \bar{h})' \circ h'}, \quad \text{i.e.} \quad \overline{g \circ \bar{h}}/\!/\overline{(g' \circ \bar{h})' \circ h'}. \tag{$*$}$$

By (1.2.1)

$$\overline{g' \circ \bar{h}}/\!/(g' \circ \bar{h})' \text{ and } g' \circ \bar{h}/\!/h' \quad \Longrightarrow \quad \overline{g' \circ \bar{h}}/\!/(g' \circ \bar{h})'/\!/h' \quad \Longrightarrow \quad (g' \circ \bar{h})'/\!/h',$$

that is $(g \circ \bar{h})'/\!/h'$. This together with $(*)$ and $h' \in T(\mathscr{M})$, and applying the assumption of the proposition to the pair $(g \circ \bar{h}, h')$, it follows from condition (A) that $g \circ \bar{h}/\!/h'$.

The third step is to show $g/\!/h$. This again follows from (1.2.1):

$$g/\!/\bar{h} \text{ and } g \circ \bar{h}/\!/h' \Longrightarrow g/\!/\bar{h}/\!/h' \Longrightarrow g/\!/h.$$

Next we show that (B) is satisfied by each pair $(g, h) \in \mathscr{M}(2)$ with $N(h) = n$. If $\alpha_l((g, h))$ is norm-preserving, i.e, if $g/\!/h$, then condition (B) is satisfied by (g, h), which can be shown by the following equations:

$$\begin{array}{rcl} (g \circ h)' & = & (g \circ \bar{h} \circ h')' \overset{(*_1)}{=} ((g \circ \bar{h})' \circ h')' \overset{(*_2)}{=} ((g' \circ \bar{h})' \circ h')' \\ & \overset{(*_3)}{=} & (g' \circ \bar{h} \circ h')' = (g' \circ h)' \\ \overline{g \circ h} & = & \overline{g \circ \bar{h} \circ h'} \overset{(*_1)}{=} \overline{g \circ \bar{h}} \circ \overline{(g \circ \bar{h})' \circ h'} \overset{(*_2)}{=} \bar{g} \circ \overline{g' \circ \bar{h}} \circ \overline{(g \circ \bar{h})' \circ h'} \\ & \overset{(*_3)}{=} & \bar{g} \circ \overline{g' \circ \bar{h} \circ h'} = \bar{g} \circ \overline{g' \circ h} \end{array}$$

The reasons why the labeled equalities hold are explained in the following.

($*_1$): By (1.2.1)

$$\bar{h}//h' \text{ and } g//h \Longrightarrow g//\bar{h}//h' \Longrightarrow g//\bar{h} \text{ and } g \circ \bar{h}//h'.$$

Applying the assumption of the proposition to the pair $(g \circ \bar{h}, h')$, the equalities follow from condition (B).

($*_2$): Since $g//\bar{h}$, using induction ($N(\bar{h}) < n$) on the pair $(g, \bar{h})$, the equalities follow from condition (B).

($*_3$): By (1.2.1), we have

$$\bar{g}//g' \text{ and } g//\bar{h} \Longrightarrow \bar{g}//g'//\bar{h} \Longrightarrow g'//h$$
$$\bar{h}//h' \text{ and } g'//h \Longrightarrow g'//\bar{h}//h' \Longrightarrow g' \circ \bar{h}//h'.$$

Applying the assumption of the proposition to the pair $(g' \circ \bar{h}, h')$, the equalities follow from condition (B).

□

Now we will see some examples of factorable categories. The category of pairings, which is the motivating and the most important new example in this thesis because of its role in moduli spaces of Kleinian surfaces, will be shown in Chapter 3.

Example 1.2.6. *The constant norm.* Assume N is a constant norm on the category $\mathscr{C}$: $N(f) = m > 0$, for any $f \in \mathscr{M}$ which is not an identity morphism. Define a map $\eta : \mathscr{M} \to \mathscr{M} \times \mathscr{M}$ by $\eta(f) := (\mathrm{id}, f)$. Then η is a factorization map and $\mathscr{C}$ is a factorable category with norm N and the factorization map η.

Example 1.2.7. *Factorable groups.* Suppose G is a factorable group (see [V]) with norm N and factorization map η. In Example 1.1.3, we have introduced two categories $\mathscr{E}G$ and $\mathscr{B}G$ associated to G with the respective norms N^e and N^b. The map η defines a factorization map η^e on $\mathscr{E}G$ by

$$\eta^e((g_2g_1^{-1}, g_1)) := ((\bar{\eta}(g_2g_1^{-1}), \eta'(g_2g_1^{-1})g_1), (\eta'(g_2g_1^{-1}), g_1))$$

as well as a factorization map η^b on $\mathscr{B}G$ by $\eta^b(g) := (\bar{\eta}(g), \eta'(g))$.

It is easy to see that $\mathscr{E}G$ is a factorable category with norm N^e and the factorization map η^e; and that $\mathscr{B}G$ is a factorable category with norm N^b and the factorization map η^b. Moreover, the functor π commutes with the factorable structures, i.e. $\pi \circ \eta^e = \eta^b \circ \pi$.

Example 1.2.8. *Free category generated by a quiver.* A *quiver* $Q = (V, A)$ is a directed graph with V the set of vertices and A the set of arrows, where loops and multiple arrows between two vertices are allowed. The *free category* $F(Q)$ generated by Q is defined in the following way: The set of objects of $F(Q)$ is V. The set of morphisms of $F(Q)$ consists of finite sequences of arrows in A, that fit together head-to-tail. And the composition in $F(Q)$ is the concatenation of sequences that fit together head-to-tail. The identity morphisms are the empty sequence.

A is a generating set of the category $F(Q)$. Let N be the word length norm on $F(Q)$ with respect to A. A morphism $f \neq \mathrm{id}$ in $F(Q)$ is of the form $f = a_n \circ a_{n-1} \circ \ldots \circ a_1$, where $a_i \in A$ are arrows. Define a map η on $F(Q)$ as follows:

$$\eta(f) := (a_n \circ a_{n-1} \circ \ldots \circ a_2, a_1)$$

for $f = a_n \circ a_{n-1} \circ \ldots \circ a_1$ and $\eta(\mathrm{id}) = (\mathrm{id}, \mathrm{id})$. Then $F(Q)$ is a factorable category with η as the factorization map.

Example 1.2.9. *Free groupoid generated by a quiver.* The *free groupoid* $G(Q)$ generated by Q is defined by: The set of objects of $G(Q)$ is V. The set of morphisms of $G(Q)$ consists of finite *reduced sequences* of arrows and their (formal) inverses in A, that fit together head-to-tail. Here by a *reduced sequence*, we mean that there exists no arrow a, such that a is next to its inverse a^{-1} in the sequence. The composition in $G(Q)$ is the concatenation of sequences that fit together head-to-tail, and if an arrow a is next to its inverse a^{-1} in a sequence, they are omitted.

Let A^{-1} denote the set of (formal) inverses of the arrows in A, then $A \cup A^{-1}$ is a generating set of the category $G(Q)$. Let N be the word length norm on $G(Q)$ with respect to $A \cup A^{-1}$. A morphism $f \neq \mathrm{id}$ in $F(Q)$ is of the form $f = \tilde{f} \circ a$, where a is an arrow or the inverse of an arrow and $\tilde{f}$ is a reduced sequence. Define a map η on $G(Q)$ by $\eta(f) = (\tilde{f}, a)$ for $f = \tilde{f} \circ a$ and $\eta(\mathrm{id}) = (\mathrm{id}, \mathrm{id})$. Then $G(Q)$ is a factorable category with η as the factorization map.

In particular, if Q has one vertex, then $G(Q)$ is a free group with generating set A. The definitions of the norm N and factorization map η on $G(Q)$ here then coincide with those defined for free groups in [V].

A first property of factorability is the existence of a *normal form* on a factorable category $\mathscr{C}$.

Definition 1.2.10. Let $g \in \mathscr{M}$. If $N(g) = n > 0$, then g has the decomposition $g = g_n \circ \ldots \circ g_1$ with $g_i \in T(\mathscr{M})$, which is determined by the following iterative process:

$$\begin{array}{ll}
\gamma_0 = g & \\
g_1 = \gamma_0', & \gamma_1 = \bar{\gamma}_0 \\
\vdots & \vdots \\
g_{k+1} = \gamma_k', & \gamma_{k+1} = \bar{\gamma}_k \\
\vdots & \vdots \\
g_n = \gamma_{n-1}', & \gamma_n = \bar{\gamma}_{n-1}
\end{array}$$

We call the tuple $\mathrm{nF}(g) := (g_n, \ldots, g_1)$ the *normal form* of g. In particular, since $g_i \in T(\mathscr{M})$, $T(\mathscr{M})$ is a generating set for $\mathscr{M}$.

The most important property of factorability studied in this thesis is its impact on the homology of the norm complex $\mathcal{N}_*(\mathscr{C})[h]$ defined by (1.1.1). We have the following

Theorem 1.2.11. *If $\mathscr{C}$ is a factorable small category with respect to the norm N, then the homology of the complex $\mathcal{N}_*(\mathscr{C})[h]$ is concentrated in the top degree h:*

$$H_q(\mathcal{N}_*(\mathscr{C})[h]) = 0, \qquad \textit{if} \quad q < h.$$

The proof of this theorem will be given at the end of this section. Before that, we will make some preparation for the proof. First the *partition-type filtration* will be introduced.

The *partition-type* of a generator $\Sigma = (g_q|\dots|g_1) \in \mathcal{N}_*(\mathscr{C})[h]$ is defined to be the tuple of positive integers $\mathrm{pt}(\Sigma) := (N(g_q), \dots, N(g_1))$. The set of ordered, positive partitions of h with length q is denoted by

$$\mathrm{Part}_q(h) = \Big\{ L = (l_q, \dots, l_1) \Big| \sum_{i=1}^{q} l_i = h,\ l_i \in \mathbb{N}^+ \text{ for all } i \Big\}$$

Let $\mathrm{Part}(h)$ be the set of all ordered positive partitions of h, i.e.

$$\mathrm{Part}(h) = \bigsqcup_{j=1}^{h} \mathrm{Part}_j(h).$$

$\mathrm{Part}(h)$ has 2^{h-1} elements and there is a total ordering, namely the lexicographic ordering on $\mathrm{Part}(h)$: For any two elements $L = (l_q, \dots, l_1)$ and $L' = (l'_r, \dots, l'_1)$ of $\mathrm{Part}(h)$, we define $L \lhd L'$ if there exists an index $j \geq 0$, such that $l_{q-i} = l'_{r-i}$ for $i = 0, 1, \dots, j-1$ and $l_{q-j} > l'_{r-j}$. The relation induces a total order on the set $\mathrm{Part}(h)$ and we call this order the *partition-type order* of $\mathrm{Part}(h)$. Note that in this ordering (h) is the minimal element and $(1, 1, \dots, 1)$ is the maximal element.

For example, the partition-type order of $\mathrm{Part}(4)$ is:

$$(4) \lhd (3,1) \lhd (2,2) \lhd (2,1,1) \lhd (1,3) \lhd (1,2,1) \lhd (1,1,2) \lhd (1,1,1,1)$$

For $L \in \mathrm{Part}(h)$, let $\mathcal{N}_*(\mathscr{C})[L]$ denote the submodule of $\mathcal{N}_*(\mathscr{C})[h]$ generated by all tuples Σ with the partition type $\mathrm{pt}(\Sigma) = L$. We number the partition-types in $\mathrm{Part}(h)$ as $L_1 \lhd L_2 \lhd \dots \lhd L_{2^{h-1}}$ and define

$$\mathcal{P}_i\mathcal{N}_*(\mathscr{C})[h] = \bigoplus_{j=1}^{i} \mathcal{N}_*(\mathscr{C})[L_j]$$

The *partition-type filtration* is then the following increasing filtration of the complex $\mathcal{N}_*(\mathscr{C})[h]$ of length 2^{h-1}:

$$0 \subseteq \mathcal{P}_1\mathcal{N}_*(\mathscr{C})[h] \subseteq \cdots \subseteq \mathcal{P}_{2^{h-1}}\mathcal{N}_*(\mathscr{C})[h] = \mathcal{N}_*(\mathscr{C})[h]$$

We summarize some useful properties concerning partition-type in the following Lemma. A consequence of Lemma 1.2.12 *(i)* is that the boundary operator d strictly lowers the filtration degree, thus $d : \mathcal{P}_i\mathcal{N}_*(\mathscr{C})[h] \to \mathcal{P}_{i-1}\mathcal{N}_{*-1}(\mathscr{C})[h]$. In particular, this shows that the partition-type filtration is well-defined.

Lemma 1.2.12.

(i) *Let Σ be a generator of $\mathcal{N}_*(\mathscr{C})[h]$. Assume that for some $1 \leq i < j \leq q-1$, the two faces $d_i(\Sigma)$ and $d_j(\Sigma)$ are non-zero, then* $\mathrm{pt}(d_j(\Sigma)) \lhd \mathrm{pt}(d_i(\Sigma)) \lhd \mathrm{pt}(\Sigma)$.

(ii) *Let Σ and Σ' be two generators of $\mathcal{N}_q(\mathscr{C})[h]$. Assume that* $\mathrm{pt}(\Sigma') \lhd \mathrm{pt}(\Sigma)$ *and* $\mathrm{pt}(\Sigma) = (l_q, \ldots, l_r, 1, \ldots, 1)$. *If $d_i(\Sigma')$ and $d_j(\Sigma)$ are non-zero for some $1 \leq i \leq q-1$, $1 \leq j \leq r-1$, then* $\mathrm{pt}(d_i(\Sigma')) \lhd \mathrm{pt}(d_j(\Sigma))$.

Proof. *(i)* follows from the definition of partition-type.

(ii) Assume that $\mathrm{pt}(\Sigma') = (l'_q, \ldots, l'_1)$. Since $\mathrm{pt}(\Sigma') \lhd \mathrm{pt}(\Sigma)$, there exists $k \geq 0$, such that $l_{q-i} = l'_{q-i}$ for $i = 1, \ldots, k-1$ and $l_{q-k} < l'_{q-k}$.

Now we claim that $q - k > r$. If this is not the case, i.e., $q - k \leq r$, we have then

$$\begin{aligned} &l_{q-i} = l'_{q-i} \quad \text{for } i = 1, \ldots, k-1 \\ &l_{q-k} < l'_{q-k} \\ &l_i = 1 \leq l'_i \quad \text{for } 1 \leq i \leq q-k-1 \leq r-1. \end{aligned}$$

It follows that $N(\Sigma) < N(\Sigma')$, a contradiction to $N(\Sigma) = N(\Sigma') = h$.

Since $q - k > r$, we have

$$\mathrm{pt}(d_i(\Sigma')) \lhd \mathrm{pt}(d_j(\Sigma)) \quad \text{if } i, j \leq r-1.$$

This together with *(i)* of this Lemma shows that

$$\mathrm{pt}(d_i(\Sigma')) \lhd \mathrm{pt}(d_j(\Sigma)) \quad \text{if } 1 \leq i \leq q-1, j \leq r-1.$$

□

By conditions (2) and (3) from Definition 1.2.1, η induces maps

$$\eta_i = \eta_i^q : \mathcal{N}_q(\mathscr{C})[l_q, \ldots, l_i, \ldots, l_1] \longrightarrow \mathcal{N}_{q+1}(\mathscr{C})[l_q, \ldots, l_i - 1, 1, \ldots, l_1]$$

by acting on the the i-th entry for any $q < h$ and all i with $l_i \geq 2$. More precisely, η_i is defined as

$$\eta_i : (g_q|\ldots|g_i|\ldots|g_1) \mapsto (g_q|\ldots|\bar{\eta}(g_i)|\eta'(g_i)|\ldots|g_1)$$

for a generator $(g_q|\ldots|g_i|\ldots|g_1)$ of $\mathcal{N}_q(\mathscr{C})[l_q, \ldots, l_i, \ldots, l_1]$ with $N(g_i) \geq 2$ and extended linearly to a map of modules.

Furthermore, we define maps

$$f_i = \eta_i d_i : \mathcal{N}_q(\mathscr{C})[l_q, \ldots, l_{i+1}, l_i, \ldots, l_1] \to \mathcal{N}_q(\mathscr{C})[l_q, \ldots, l_{i+1} + l_i - 1, 1, \ldots, l_1]$$

for $1 \leq i \leq q-1$.

We need the following properties of the maps f_i.

Lemma 1.2.13.

(1) $d_i f_i = d_i$ (1.2.2a)

(2) $d_j f_i = f_i d_{j-1}$ *for* $i + 2 \leq j \leq q$ (1.2.2b)

(3) $d_j f_i = f_{i-1} d_j$ *for* $1 \leq j \leq i-2$ (1.2.2c)

(4) $d_{i+1} f_i f_{i+1} = f_i d_i$ (1.2.2d)

Proof. We have the following equalities

$$d_i\eta_i = \mathrm{id} \tag{1.2.3a}$$
$$d_j\eta_i = \eta_i d_{j-1} \text{ for } i+2 \leq j \leq q \tag{1.2.3b}$$
$$d_j\eta_i = \eta_{i-1} d_j \text{ for } 1 \leq j \leq i-2 \tag{1.2.3c}$$
$$d_{i+1}\eta_i d_i \eta_{i+1} = \eta_i d_i \tag{1.2.3d}$$

Here (1.2.3a) follows from property (1) of the definition 1.2.1 of factorization maps. (1.2.3b) and (1.2.3c) are immediate from the definitions. (1.2.3d) follows from the definition of factorable categories, or equivalently, from conditions (A) and (B) on page 11. More precisely, $d_{i+1}\eta_i d_i \eta_{i+1}(\Sigma) = 0$ if and only if $\eta_i d_i(\Sigma) = 0$ for $\Sigma \in \mathcal{N}_*(\mathscr{C})[h]$; and if $\eta_i d_i(\Sigma) \neq 0$, $d_{i+1}\eta_i d_i \eta_{i+1}(\Sigma) = \eta_i d_i(\Sigma)$.

The above equalities (1.2.3a)-(1.2.3d) of the map η_i together with the simplicial equalities $d_i d_j = d_{j-1} d_i$ for $i < j$ give rise to the equalities (1)-(4) in the Lemma respectively. □

Now we introduce the map

$$F_r : \mathcal{N}_q(\mathscr{C})[l_q, \ldots, l_r, 1, \ldots, 1] \to \mathcal{N}_{q+1}(\mathscr{C})[l_q, \ldots, l_r - 1, 1, \ldots, 1]$$

defined as the composition

$$(-1)^r \left(\mathrm{id} - f_{r-1} + \cdots + (-1)^{r-i} f_i f_{i+1} \cdots f_{r-1} + \cdots + (-1)^{r-1} f_1 f_2 \cdots f_{r-1}\right) \eta_r$$

and the signed sum of the first r face operators

$$d_{(r)} = \sum_{i=1}^{r} (-1)^i d_i.$$

The following properties involving F_r and $d_{(r)}$ will be essential in the proof of Theorem 1.2.11.

Lemma 1.2.14.

(i) $d_{(r)} F_r = \mathrm{id} - F_{r-1} d_{(r-1)}$

(ii) $d_j F_r = F_r d_{j-1}$ *for* $r+2 \leq j \leq q$

Proof. For *(i)*, we will compute the terms of $d_{(r)}F_r$ by computing $d_j F_r$ for each $j \leq r$. There are two different cases.

(1) When $j = r$, we have

$$d_r \eta_r \overset{(1.2.3a)}{=} \mathrm{id}$$

and for $i \leq r-1$,

$$\begin{aligned} d_r f_i f_{i+1} \ldots f_{r-1} \eta_r \overset{(1.2.2b)}{=} \ldots \overset{(1.2.2b)}{=} & f_i f_{i+1} \ldots f_{r-2} d_r f_{r-1} \eta_r \\ \overset{(1.2.3d)}{=} & f_i f_{i+1} \ldots f_{r-2} \eta_{r-1} d_{r-1} \end{aligned}$$

(2) When $j < r$ with j fixed, we have:

If $i \geq j+2$, then

$$\begin{aligned} d_j f_i f_{i+1} \dots f_{r-1} \eta_r \overset{(1.2.2c)}{=} \dots &\overset{(1.2.2c)}{=} f_{i-1} \dots f_{r-2} d_j \eta_r \\ &\overset{(1.2.3c)}{=} f_{i-1} \dots f_{r-2} \eta_{r-1} d_j \end{aligned}$$

For $i = j$ and $i = j+1$, by (1.2.2a) the terms $d_j f_{j+1} \dots f_{r-1}\eta_r$ and $d_j f_j f_{j+1} \dots f_{r-1}\eta_r$ are equal, so they cancel each other in the sum $d_j F_r$, because they have opposite signs.

If $i \leq j-1$, we have

$$\begin{aligned} d_j f_i f_{i+1} \dots f_{r-1} \eta_r \overset{(1.2.2b)}{=} \dots &\overset{(1.2.2b)}{=} f_i f_{i+1} \dots f_{j-2} d_j f_{j-1} f_j \dots f_{r-1} \eta_r \\ &\overset{(1.2.2d)}{=} f_i \dots f_{j-2} f_{j-1} d_{j-1} f_{j+1} \dots f_{r-1} \eta_r \\ \overset{(1.2.2c)}{=} \dots &\overset{(1.2.2c)}{=} f_i \dots f_{r-2} \eta_{r-1} d_{j-1} \end{aligned}$$

Building the signed sum $d_{(r)} F_r$, we obtain the formula *(i)*.

For *(ii)*, since $j \geq r+2$, we have

$$\begin{aligned} d_j f_i f_{i+1} \dots f_{r-1} \eta_r \overset{(1.2.2b)}{=} \dots &\overset{(1.2.2b)}{=} f_i f_{i+1} \dots f_{r-1} d_j \eta_r \\ &\overset{(1.2.3b)}{=} f_i \dots f_{r-2} \eta_{r-1} d_{j-1} \end{aligned}$$

Building the signed sum $d_j F_r$, we obtain the formula *(ii)*. □

Proof of Theorem 1.2.11.

We use the spectral sequence associated to the partition-type filtration of $\mathcal{N}_*(\mathscr{C})[h]$ for the proof. The E^0-terms of this spectral sequence are

$$E^0_{p,q} = \mathcal{P}_p \mathcal{N}_q(\mathscr{C})[h] / \mathcal{P}_{p-1} \mathcal{N}_q(\mathscr{C})[h].$$

In other words, $E^0_{p,q}$ is the free $\mathbb{Z}$-module generated by all $\Sigma \in \mathcal{N}_q(\mathscr{C})[h]$ with $\mathrm{pt}(\Sigma) = L_p$, where L_p is the p-th partition-type in the total order of $\mathrm{Part}(h)$. Because L_p determines the degree $q = q(L_p)$, q is a function of p. So in each column of the spectral sequence, there is at most one non-zero term $E^0_{p,q(p)}$. Moreover, the spectral sequence has finite terms since $E_{p,q} = 0$ for $p > 2^{h-1}$ or $q > h$.

Theorem 1.2.11 follows if we show that

$$E^\infty_{p,q} = Z^\infty_{p,q} / (Z^\infty_{p-1,q} + B^\infty_{p,q}) = 0, \quad \text{if} \quad q < h. \tag{$*$}$$

Here

$$\begin{aligned} Z^\infty_{p,q} &= \{ c \in \mathcal{P}_p \mathcal{N}(\mathscr{C})[h] \mid d(c) = 0 \} \\ \text{and} \quad B^\infty_{p,q} &= \{ c \in \mathcal{P}_p \mathcal{N}(\mathscr{C})[h] \mid c \in \mathrm{Im}(d : \mathcal{N}_{q+1}(\mathscr{C})[h] \to \mathcal{N}_q(\mathscr{C})[h]) \}. \end{aligned}$$

Now we prove $(*)$. Assume that $q < h$ and $c \in Z^{\infty}_{p,q} \subseteq \mathcal{P}_p\mathcal{N}(\mathscr{C})[h] = \bigoplus_{i=1}^{p} \mathcal{N}_q(\mathscr{C})[L_i]$, where L_i is the i-th partition in $\mathrm{Part}(h)$ according to the partition-type order. c has a unique decomposition

$$c = c_1 + \ldots + c_p, \quad c_i \in \mathcal{N}_q(\mathscr{C})[L_i].$$

Our aim is to show that $c \in (Z^{\infty}_{p-1,q} + B^{\infty}_{p,q})$. If $c_p = 0$, this is true since $c \in Z^{\infty}_{p-1,q}$. If $c_p \neq 0$, assume $L_p = (l_q, \ldots, l_r, 1, \ldots, 1)$ with $l_r \geq 2$. First we claim that $d_j(c_p) = 0$ for all $j \leq r-1$. This is because on the one hand

$$d(c_p) = d(c - (c_1 + \ldots + c_{p-1})) = -d(c_1 + \ldots + c_{p-1});$$

on the other hand, all terms in $d(c_1 + \ldots + c_{p-1})$ have smaller partition-types than that of $d_j(c_p)$, $j \leq r-1$, since Lemma 1.2.12*(ii)* implies that

$$\mathrm{pt}(d_i(c_k)) \lhd \mathrm{pt}(d_j(c_p)), \quad \text{for } k < p, 1 \leq i \leq k-1, j \leq r-1.$$

Since $d_j(c_p) = 0$ for all $j \leq r-1$, we have $d_{(r-1)}(c_p) = 0$. It follows from Lemma 1.2.14*(ii)* that $d_{(r)}F_r(c_p) = c_p - F_{r-1}d_{(r-1)}c_p = c_p$, from this and the fact that $\mathrm{pt}(F_r(c_p)) = (l_q, \ldots, l_{r-1}, 1, \ldots, 1) \in \mathrm{Part}_{q+1}(h)$ we have

$$dF_r(c_p) - c_p = \sum_{i=r+1}^{q} (-1)^i d_i(F_r(c_p)) \in \mathcal{P}_{p-1}\mathcal{N}_q(\mathscr{C})[h].$$

Since $d(c - dF_r(c_p)) = 0$ and $c - dF_r(c_p) = (c - c_p) - (dF_r(c_p) - c_p) \in \mathcal{P}_{p-1}\mathcal{N}_q(\mathscr{C})[h]$, by definition,

$$c - dF_r(c_p) \in Z^{\infty}_{p-1,q}. \tag{1.2.4}$$

Moreover, since $dF_r(c_p) = (dF_r(c_p) - c_p) + c_p \in \mathcal{P}_p\mathcal{N}_q(\mathscr{C})[h]$, we have $dF_r(c_p) \in B^{\infty}_{p,q}$. Therefore, $c = (c - dF_r(c_p)) + dF_r(c_p) \in Z^{\infty}_{p-1,q} + B^{\infty}_{p,q}$. □

1.3 Homology Classes

A natural question in view of Theorem 1.2.11 is: How can we find a set of generators for the free $\mathbb{Z}$-module $H_h(\mathcal{N}_*(\mathscr{C})[h])$? In this section we will describe generators for the homology of the norm complex. The method used here is again taken from [V]; however, we will prove the stronger result Theorem 1.3.3, which answers the question above completely if $\mathscr{C}$ satisfies the right cancellation property and has finitely many norm-one morphisms.

We begin by dividing $\mathcal{N}_q(\mathscr{C})[h]$ into a direct sum of two submodules with special properties. Let $Q_q(\mathscr{C})[h]$ be the set of q-tuples in $\mathcal{N}_q(\mathscr{C})[h]$ which are in the image of some $\eta_i : \mathcal{N}_{q-1}(\mathscr{C})[h] \to \mathcal{N}_q(\mathscr{C})[h], 1 \leq i \leq q-1$. From property (1) of Definition 1.2.1 of factorization maps, a q-tuple $\Sigma \in \mathcal{N}_q(\mathscr{C})[h]$ is in $Q_q(\mathscr{C})[h]$ if and only if $f_i(\Sigma) = \Sigma$ for some $1 \leq i \leq q-1$, i.e.

$$Q_q(\mathscr{C})[h] = \{\Sigma = (g_q|\ldots|g_1) \in \mathcal{N}_q(\mathscr{C})[h] \mid f_i(\Sigma) = \Sigma \text{ for some } 1 \leq i \leq q-1\}.$$

Let $R_q(\mathscr{C})[h]$ be the set of q-tuples in $\mathcal{N}_q(\mathscr{C})[h]$ which are not in $Q_q(\mathscr{C})[h]$. That is,

$$R_q(\mathscr{C})[h] = \{\Sigma = (g_q|\dots|g_1) \in \mathcal{N}_q(\mathscr{C})[h] \mid f_i(\Sigma) \neq \Sigma \text{ for all } 1 \leq i \leq q-1\}.$$

Denote by $\mathcal{Q}_q(\mathscr{C})[h]$ and $\mathcal{R}_q(\mathscr{C})[h]$ the submodules of $\mathcal{N}_q(\mathscr{C})[h]$ generated by $Q_q(\mathscr{C})[h]$ and $R_q(\mathscr{C})[h]$ respectively, then we have

$$\mathcal{N}_q(\mathscr{C})[h] = \mathcal{Q}_q(\mathscr{C})[h] \oplus \mathcal{R}_q(\mathscr{C})[h].$$

Theorem 1.2.11 says that if $c \in \mathcal{N}_q(\mathscr{C})[h]$ is a cycle and $q < h$, then c is a boundary, i.e., $c = d(b)$ for some $b \in \mathcal{N}_{q+1}(\mathscr{C})[h]$. In fact, we can require more for b.

Lemma 1.3.1. *If c is a cycle in $\mathcal{N}_q(\mathscr{C})[h]$ and $q < h$, then $c = d(b)$ for some $b \in \mathcal{Q}_{q+1}(\mathscr{C})[h]$.*

Proof. Assume that $c \in \mathcal{P}_p\mathcal{N}_q(\mathscr{C})[h]$. Decompose c as $c = c_1 + \dots + c_p$, where $c_i \in \mathcal{N}_q(\mathscr{C})[L_i]$. We will use induction on the filtration degree p.

If L_p is the smallest partition in $\mathrm{Part}_q(h)$ according to the partition-type order of $\mathrm{Part}(h)$, i.e. $L_p = (h-q+1, 1, \dots, 1)$. Then $c = c_p$. It follows from $d(c) = 0$ and Lemma 1.2.12 *(i)* that $d_i(c) = 0, i = 1, \dots, p-1$. By Lemma 1.2.14*(ii)*, we have

$$dF_q(c) = d_{(q)}F_q(c) = c - F_{q-1}d_{(q-1)}(c) = c.$$

By definition, $F_q(c) \in \mathcal{Q}_{q+1}(\mathscr{C})[h]$, hence the Lemma is proved in this case.

Now let p be arbitrary. Following the proof of Theorem 1.2.11, we see from (1.2.4) that $c - dF_r(c_p)$ is a cycle in $\mathcal{P}_{p-1}\mathcal{N}_q(\mathscr{C})[h]$, where r is determined by L_p. Using induction, there exists $b \in \mathcal{Q}_{q+1}(\mathscr{C})[h]$, such that $c - dF_r(c_p) = d(b)$. Hence $c = d(F_r(c_p) + d)$ with $F_r(c_p) + d \in \mathcal{Q}_{q+1}(\mathscr{C})[h]$. ☐

The inductive procedure shown in the proof above leads to an algorithm, which finds for each cycle c a chain $b \in \mathcal{Q}_{q+1}(\mathscr{C})[h]$, such that $c = d(b)$. Before describing this algorithm, we will take a closer look at the decomposition of c. Assume that $c \in \mathcal{P}_p\mathcal{N}_q(\mathscr{C})[h]$, let $L_{q_1} \lhd L_{q_2} \lhd \dots$ be the subsequence of all partition-types in $\mathrm{Part}_q(h) \subset \mathrm{Part}(h)$. Suppose that L_{q_n} is the biggest partition-type in this sequence with the property that $q_n \leq p$. Then $c \in \mathcal{P}_p\mathcal{N}_q(\mathscr{C})[h]$ has the unique decomposition

$$c = c_{q_1} + \dots + c_{q_n}, \quad c_{q_i} \in \mathcal{N}_q(\mathscr{C})[L_{q_i}] \tag{1.3.1}$$

The algorithm consists of n steps:

Algorithm 1.3.2.

- *Set* $b^{(0)} = 0$.
- *For* $1 \leq i \leq n$*:*

$$\begin{aligned} c^{(i)} &:= c - d(b^{(i-1)}) = c^{(i)}_{q_1} + \dots + c^{(i)}_{q_{n-i+1}}, \quad c^{(i)}_{q_i} \in \mathcal{N}_q(\mathscr{C})[L_{q_i}]; \\ b^{(i)} &:= F_{r_i}(c^{(i)}_{q_{n-i+1}}) + b^{(i-1)}, \end{aligned}$$

where r_i is determined by $L_{q_{n-i+1}}$ as in the proof of Theorem 1.2.11.

- *Since* $c^{(n+1)} := c - d(b^{(n)}) = 0$, *the result is given by* $b := b^{(n)}$.

In the following we will denote by $V_h(\mathscr{C})$ the free $\mathbb{Z}$-module generated by $R_h(\mathscr{C})[h]$, i.e. $V_h(\mathscr{C})$ is isomorphic to $\mathcal{R}_h(\mathscr{C})[h]$ as $\mathbb{Z}$-modules. Now we construct a homomorphism

$$\kappa : V_h(\mathscr{C}) \to H_h(\mathcal{N}_*(\mathscr{C})[h])$$

which hence provides a way to find homology classes in $H_h(\mathcal{N}_*(\mathscr{C})[h])$ systematically. κ is first defined on $R_h(\mathscr{C})[h]$, and then extended linearly to a homomorphism between free $\mathbb{Z}$-modules. For $X \in R_h(\mathscr{C})[h]$, $d(X)$ is a cycle in $\mathcal{N}_{h-1}(\mathscr{C})[h]$; hence we can find $b \in \mathcal{Q}_h(\mathscr{C})[h]$ by the algorithm above such that $d(X) = d(b)$. $\kappa(X)$ is then defined to be the cycle $X - b$.

To get a explicit formula for $\kappa(X)$, we examine the algorithm in the case $c = d(X)$ more closely. As the initial cycle, $d(X)$ decomposes into $n = h - 1$ terms, hence there will be $h - 1$ steps to find b. For the i-th step, there is an explicit formula for $c^{(i)}_{q_{n-i+1}} = c^{(i)}_{q_{h-i}}$,

$$c^{(i)}_{q_{h-i}} = (-1)^i d_i(X - b^{(i-1)})$$

This is because $c^{(i)} = d(X - b^{(i-1)})$ and every non-zero term in $X - b^{(i-1)}$ has partition-type $L_{2^{h-1}} = (1, \dots, 1)$. Moreover $r_i = i$ in the i-th step. It follows that

$$\begin{aligned} X - b^{(i)} &= X - b^{(i-1)} - F_{r_i}(c^{(i)}_{q_{h-i}}) = X - b^{(i-1)} - (-1)^i F_i \circ d_i(X - b^{(i-1)}) \\ &= (\mathrm{id} - (-1)^i F_i \circ d_i)(X - b^{(i-1)}). \end{aligned} \tag{1.3.2}$$

Since $b = b^{(h-1)}$, by definition $\kappa(X) = X - b = X - b^{(h-1)}$. Hence the following formula for $\kappa(X)$ can be derived from (1.3.2):

$$\begin{aligned} \kappa(X) = (\mathrm{id} - (-1)^{h-1} F_{h-1} \circ d_{h-1}) \dots \\ \dots (\mathrm{id} - (-1)^i F_i \circ d_i) \dots (\mathrm{id} - F_2 \circ d_2)(\mathrm{id} + F_1 \circ d_1)(X) \end{aligned}$$

or equivalently

$$\begin{aligned} \kappa(X) \quad = \quad & (1 - f_{h-1} + f_{h-2} f_{h-1} \dots + (-1)^{h-1} f_1 \cdots f_{h-1}) \\ & \vdots \\ \circ \; & (1 - f_i + f_{i-1} f_i \dots + (-1)^i f_1 \cdots f_i) \\ & \vdots \\ \circ \; & (1 - f_2 + f_1 f_2) \\ \circ \; & (1 - f_1)(X). \end{aligned}$$

We can see that $\kappa(X)$ in this formula has $h!$ terms, but some can cancel out in the sum.

The following theorem shows that κ is an isomorphism under certain assumptions on $\mathscr{C}$. As a consequence, the set $\{\kappa(X) \mid X \in R_h(\mathscr{C})[h]\}$ is a set of generators for $H_h(\mathcal{N}_*(\mathscr{C})[h])$ in this case.

Recall that a category $\mathscr{C}$ is said to satisfy the *right cancellation property* if for any morphisms a, b, c of $\mathscr{C}$, $b \circ a = c \circ a$ implies $b = c$.

Theorem 1.3.3. *Let $\mathscr{C}$ be a factorable normed category. If $\mathscr{C}$ satisfies the right cancellation property and has finitely many morphisms with norm one, then $\kappa : V_h(\mathscr{C}) \to H_h(\mathcal{N}_*(\mathscr{C})[h])$ is an isomorphism.*

Proof. First we show that κ is injective. Define a map $\pi : H_h(\mathcal{N}_*(\mathscr{C})[h]) \to V_h(\mathscr{C})$ as follows: For $[c] \in H_h(\mathcal{N}_*(\mathscr{C})[h])$, where $c \in \mathcal{N}_h(\mathscr{C})[h]$ is a cycle with the unique decomposition: $c = c_v + c_q$, where $c_v \in V_h(\mathscr{C})$, $c_q \in \mathcal{Q}_h(\mathscr{C})[h]$, we define $\pi([c]) := c_v$. π is well defined, because if $[c] = [c'] \in H_h(\mathcal{N}_*(\mathscr{C})[h]$, then $c = c'$. It is easy to check that $\pi \circ \kappa = \mathrm{id}_{V_h(\mathscr{C})}$, thus κ is injective.

Next we show the surjectivity of κ. For simplicity, we denote $\mathcal{N}_*(\mathscr{C})[h]$, $R_*(\mathscr{C})[h]$ and $Q_*(\mathscr{C})[h]$ by $\mathcal{N}_*$, R_* and Q_* respectively from now on. Let r_h be the number of elements in R_h. Because κ is injective, to prove that κ is surjective, it is enough to show that $r_h = \mathrm{rank}(H_h(\mathcal{N}_*))$. Since $\mathscr{C}$ has finitely many morphisms with norm one, $\mathcal{N}_*$ has finitely many tuples. We know that $H_i(\mathcal{N}_*) = 0$, for $i < h$, so $\chi(\mathcal{N}_*) = (-1)^h \mathrm{rank}(H_h(\mathcal{N}_*))$, where $\chi(\mathcal{N}_*)$ is the Euler-characteristic of the complex $\mathcal{N}_*$. Therefore it is enough to show that $(-1)^h r_h = \chi(\mathcal{N}_*)$.

Let $X = (g_q|\dots|g_1)$ be a tuple in $\mathcal{N}_q, q \leq h$. As in Definition 1.2.10, each g_i $(i = 1, \dots, q)$ has a normal form $\mathrm{nF}(g_i) = (g_{i,n_i}, \dots, g_{i,2}, g_{i,1})$, where $n_i = N(g_i)$. Then $\Phi(X)$ is defined as the tuple of concatenated normal forms

$$\begin{aligned}\Phi(X) :&= (\mathrm{nF}(g_q), \dots, \mathrm{nF}(g_1)) \\ &= (g_{q,n_q}|\dots|g_{q,1}|\dots|g_{1,n_1}|\dots|g_{1,1}) \in \mathcal{N}_h.\end{aligned}$$

Two tuples in $\mathcal{N}_*$ will be called *related*, if they are mapped to the same image by Φ. Denote by $[X]$ the equivalence class, which contains the tuple $X \in \mathcal{N}_*$.

We call a tuple of norm-one morphisms $(g_n|\dots|g_1)$ a *stable string*, if

$$\eta'(g_{j+1} \circ g_j) = g_j \quad \text{for every} \quad j < n.$$

Because $\mathscr{C}$ is factorable with the factorization map η and satisfies the right cancellation property, $(g_n|\dots|g_1)$ is a stable string if and only if $\mathrm{nF}(g_n \circ \dots \circ g_1) = (g_n, \dots, g_1)$.

For any tuple X in $\mathcal{N}_*$ we want to describe the equivalence class $[X]$. Since $\Phi(X) \in [X]$ and $\Phi(X) \in \mathcal{N}_h$, it is enough to consider $[\Gamma]$ with tuple Γ in $\mathcal{N}_h$. From the definition of Φ, if Γ, Γ' are tuples in $\mathcal{N}_h$ and $\Gamma \neq \Gamma'$, then $[\Gamma] \neq [\Gamma']$. Any tuple $\Gamma = (g_h|\dots|g_1) \in \mathcal{N}_h$ can be considered as a sequence of concatenated stable strings

$$\Gamma = (\overset{\substack{g_h \\ \parallel}}{g_{q,n_q}}\ |\dots|g_{q,1}|\dots|g_{i,n_i}|\dots|g_{i,1}|\dots|g_{1,n_1}|\dots|\ \overset{\substack{g_1 \\ \parallel}}{g_{1,1}}) \in \mathcal{N}_h.$$

Here for each i, $(g_{i,n_i}|\dots|g_{i,1})$ is a stable string and $\eta'(g_{i+1,1} \circ g_{i,n_i}) \neq g_{i,n_i}$.

Now we show that the class $[\Gamma]$ is the set of tuples in $\mathcal{N}_*$ which are obtained from Γ by composing any neighboured morphisms within stable strings of Γ:

- Due to the property of a stable string, a tuple obtained in this way is in $[\Gamma]$.
- Any $X \in [\Gamma]$ can be obtained from Γ by composing certain neighboured morphisms within stable strings, because $\Phi(X) = \Gamma$.

- Different ways of composing give rise to different tuples in $\mathcal{N}_*$. This can be shown by looking at the partition type (i.e. the norm of each morphism) of the tuples.

There are $h-q$ pairs of neighbours in Γ that can be composed, and composing j (with $0 \leq j \leq h-q$) of them will produce a tuple in $\mathcal{N}_{h-j}$. So when $q = h$, i.e., when $\Gamma \in R_h$, we have $[\Gamma] = \{\Gamma\}$. For a tuple $X \in \mathcal{N}_i$ (i.e. $X \in R_i \cup Q_i$), let $\operatorname{sign}(X) := (-1)^i$. When $q < h$

$$\sum_{X \in [\Gamma]} \operatorname{sign}(X) = \sum_{j=0}^{h-p} \tbinom{h-q}{j} (-1)^{h-j} = (-1)^h \sum_{j=0}^{h-q} \tbinom{h-q}{j} (-1)^j = 0.$$

Finally,

$$\begin{aligned}\chi(\mathcal{N}_*) &= \sum_{X \in R_* \cup Q_*} \operatorname{sign}(X) = \sum_{X \in R_h} \operatorname{sign}(X) + \sum_{\substack{[\Gamma] \\ \Gamma \in Q_h}} \sum_{X \in [\Gamma]} \operatorname{sign}(X) \\ &= \sum_{X \in R_h} \operatorname{sign}(X) = (-1)^h r_h.\end{aligned}$$

□

Remark 1.3.4.

1) Because groupoids satisfy the right cancellation property, κ is an isomorphism for any factorable groupoid that has finitely many norm-one morphisms. Hence for symmetric groups and the category of pairings, which are the main subjects of interest in this thesis, κ is an isomorphism.

2) The free category $F(Q)$ generated by a quiver Q also satisfies the right cancellation property. Therefore if Q has finitely many arrows, κ is an isomorphism for $F(Q)$, whose factorable structure is described in Example 1.2.8.

◇

Chapter 2

Application to Moduli Spaces of Riemann Surfaces

In this chapter we present the connection between symmetric groups and moduli spaces of Riemann surfaces. It is shown in [V] that symmetric groups are factorable groups with respect to a certain word length norm; we discuss the application of this fact to the computation of homology groups of these moduli spaces. When an orientation system is needed for the computation, although the factorability of symmetric groups is not directly applicable, the methods developed for the theory of factorable groups can be adapted to this case.

2.1 Moduli Spaces of Riemann Surfaces

Let $F^m_{g,1}$ denote a Riemann surface of genus $g \geq 0$ with one boundary curve and $m \geq 0$ permutable punctures and $\mathfrak{M}^m_{g,1}$ denote the moduli space of conformal equivalence classes of all such surfaces. Let $\Gamma^m_{g,1}$ be the mapping class group of $F^m_{g,1}$, i.e., the isotropy classes of orientation-preserving diffeomorphism fixing the boundary pointwise and permuting the punctures. $\mathfrak{M}^m_{g,1}$ is the classifying space of $\Gamma^m_{g,1}$, because $\Gamma^m_{g,1}$ acts freely on the contractible Teichmüller space $\mathfrak{Teich}^m_{g,1}$ and $\mathfrak{M}^m_{g,1} = \mathfrak{Teich}^m_{g,1}/\Gamma^m_{g,1}$.

Using the Hilbert uniformization method, Bödigheimer found a finite cell complex $\mathrm{Par}(h,m)$ with a subcomplex $\mathrm{Par}'(h,m)$ such that $\mathrm{Par}(h,m) \smallsetminus \mathrm{Par}'(h,m)$ is an open manifold of dimension $3h$ and homotopy equivalent to $\mathfrak{M}^m_{g,1}$. Here the new parameter h is determined as $h = 2g + m$. $\mathrm{Par}(h,m)$ is the space of parallel slit domains and $\mathrm{Par}'(h,m)$ is the subspace of $\mathrm{Par}(h,m)$ consisting of degenerate slit configurations.

We will recall some main facts about parallel slit domains first. More detailed material can be found in [B2], [Eh] and [B3]. There is also a more combinatorial description of $\mathrm{Par}(h,m)$ using symmetric groups, which we will give later. The description of $\mathrm{Par}(h,m)$ through slit configurations provides geometric pictures, which is important for the construction of an orientation system in section 2.3.

Definition 2.1.1. A *slit* is a subset of the complex plane $\mathbb{C}$, which is of the form

$$\{z = (x,y) \in \mathbb{C} | x \leq x_0, y = y_0\}$$

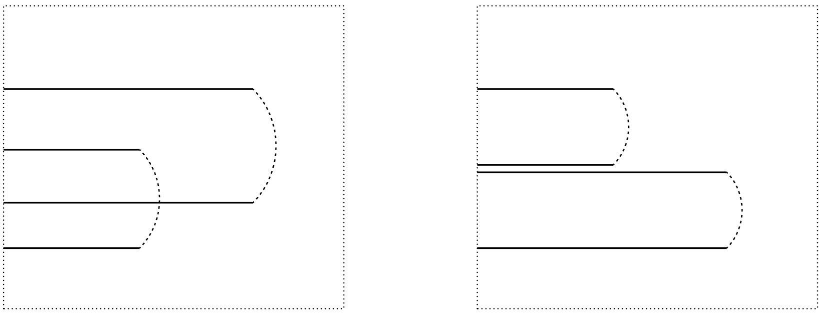
for a point $z_0 = (x_0, y_0) \in \mathbb{C}$. Z_0 is called the *endpoint* of this slit.

Definition 2.1.2. A *slit configuration* $L = (L_1, \ldots, L_{2h}|\lambda)$ consists of the following data

(1) $2h$ slits $L_1, \ldots, L_{2h}$ with endpoints $z_1 = (x_1, y_1), \ldots, z_{2h} = (x_{2h}, y_{2h})$, respectively

(2) a pairing (fixed point free involution) $\lambda \in \mathfrak{S}_{2h}$,

such that $y_1 \leq \ldots \leq y_{2h}$ and $x_k = x_{\lambda(k)}$ for $k = 1, \ldots, 2h$.



$L = (L_1, L_2, L_3, L_4 \mid \lambda)$
$\lambda = (1\ 3)(2\ 4)$

$L = (L_1, L_2, L_3, L_4 \mid \lambda)$
$\lambda = (1\ 2)(3\ 4)$

Figure 2.1: Examples of slit configurations

In many situations, it would be helpful for the intuition if one draws pictures for slit configurations as in Figure 2.1. Note that if two slits L_i, L_j in a slit configuration L have the same y-level and $i > j$ (resp. $i < j$), then L_i is put "above" (resp. "below") L_j in the picture.

To each slit configuration L, one can associate a two-dimensional space $F(L)$. Namely, one first cuts the complex plane $\mathbb{C}$ along each slit L_i, $i = 1, \ldots, 2h$; then glues the upper (resp. lower) bank of the slit L_i to the lower (resp. upper) bank of the slit $L_{\lambda(i)}$, $i = 1, \ldots, 2h$.

Definition 2.1.3. A slit configuration L is called *non-degenerate*, if the associated space $F(L)$ is a smooth surface; otherwise L is called *degenerate*.

If L is non-degenerate, $F(L)$ can be compactified by adding a point Q at infinity to the complex plane $\mathbb{C}$ and $m \geq 0$ points $P_1, \ldots, P_m$ at the end of the "tubes" created by the gluing. Each point P_i is called a *puncture* of $F(L)$. There is an obvious complex structure on $F(L)$ such that $\mathbb{C} \setminus (\cup_{i=1}^{2h} L_i) \hookrightarrow F(L)$ is a holomorphic inclusion. Moreover, specifying a point with a tangent direction is equivalent to specifying a boundary curve, thus $F(L)$ is a Riemann surface with one boundary curve (Q, X) and m punctures $P_1, \ldots, P_m$, where X corresponds to the horizontal direction on $\mathbb{C}$. By computing the Euler characteristic, the genus of $F(L)$ is determined to be $g = \frac{1}{2}(h - m)$.

Definition 2.1.4. The *puncture-number* of a non-degenerate slit configuration L is defined to be m, if the associated surface $F(L)$ has m punctures.

Remark 2.1.5. Assume the puncture-number of $L = (L_1, \dots, L_{2h}|\lambda)$ is m. Let $\omega_{2h} = (1\ 2 \dots 2h) \in \mathfrak{S}_{2h}$. Then the number of cycles of $\lambda \circ \omega_{2h}$ is $m + 1$. $\diamond$

There is an equivalence relation between slit configurations which is defined by *Rauzy jumps*. Before going to the definition, we fix a notation about permutation first. For $a, b \in \{1, \dots, 2h\}$, $a \leq b$, let $s_{a,b}$ denote the cycle $(a\ a+1 \dots b)$, the inverse of $s_{a,b}$ is then the cycle $s_{a,b}^{-1} = (b\ b-1 \dots a)$.

Definition 2.1.6. If a slit configuration $L = (L_1, \dots, L_{2h}|\lambda)$ contains a slit L_i, $i \in \{1, \dots, 2h\}$, such that $L_i \subseteq L_{i+1}$ or $L_i \subseteq L_{i-1}$. Define a permutation $\alpha \in \mathfrak{S}_{2h}$ in the following four possible cases:

(1) $\alpha := s_{i,\lambda(i+1)}^{-1}$, if $L_i \subseteq L_{i+1}, \lambda(i+1) > i+1$.

(2) $\alpha := s_{\lambda(i+1)+1,i}$, if $L_i \subseteq L_{i+1}, \lambda(i+1) < i$.

(3) $\alpha := s_{i,\lambda(i-1)-1}^{-1}$, if $L_i \subseteq L_{i-1}, \lambda(i-1) > i$.

(4) $\alpha := s_{\lambda(i-1),i}$, if $L_i \subseteq L_{i-1}, \lambda(i-1) < i-1$.

Define a new slit configuration $\tilde{L} = (\tilde{L}_1, \dots, \tilde{L}_{2h}|\tilde{\lambda})$ as follows:

$$\tilde{L}_{\alpha(j)} = L_j, j \in \{1, \dots, 2h\} \backslash \{i\}$$

and

$$\tilde{L}_{\alpha(i)} := \{(x, y) \in \mathbb{C} | x \leq x_i, y = y_{\lambda(i+1)}\}$$

in the cases (1) and (2), and

$$\tilde{L}_{\alpha(i)} := \{(x, y) \in \mathbb{C} | x \leq x_i, y = y_{\lambda(i-1)}\}$$

in the cases (3) and (4).

The pairing $\tilde{\lambda}$ is defined by

$$\tilde{\lambda} = \alpha \circ \lambda \circ \alpha^{-1}.$$

We say that $\tilde{L}$ is obtained from L via a *Rauzy jump* of the slit L_i over the slit pair $L_{i+1}, L_{\lambda(i+1)}$ (in the cases (1) and (2)) or $L_{i-1}, L_{\lambda(i-1)}$ (in the cases (3) and (4)).

(1) (4) (2) (3)

Figure 2.2: Rauzy jumps

Rauzy jumps generate an equivalence relation: two slit configurations are equivalent, if they can transform into each other by Rauzy jumps. We denote the equivalence class of a slit configuration L by $[L]$.

Remark 2.1.7. If two slit configurations L, L' are equivalent under Rauzy jumps and L is non-degenerate, then the associated surfaces $F(L)$ and $F(L')$ are conformally equivalent. $\diamond$

Let $L = (L_1, \dots, L_{2h}|\lambda)$ be a slit configuration. Assume that the slits of L lie at p-distinct y-levels

$$-\infty < v_1 < v_2 \ldots < v_p < \infty, \tag{2.1.1}$$

and that a_i is the number of slits lying on level $y = v_i$. Then

$$0 < a_i < 2h, \quad \sum_{i=1}^{p} a_i = 2h, \quad 2 \le p \le 2h. \tag{2.1.2}$$

Further assume that the endpoints of L lie at q-distinct x-levels

$$-\infty < u_q < \ldots < u_2 < u_1 < \infty, \tag{2.1.3}$$

Denote the set of indices of the slits over $x = u_j$ by $B_j \subseteq \{1, \dots, 2h\}$. Then

$$B_1 \cup B_2 \cup \ldots \cup B_q = \{1, \dots, 2h\}, \quad 1 \le q \le h. \tag{2.1.4}$$

We summarize all these data into a *symbol* E

$$E := (a_1, \dots, a_p|\lambda|B_1, \dots, B_q).$$

Let

$$f : \mathbb{R} \to (0,1), \quad x \mapsto f(x) = \frac{\arctan(x)}{\pi} + \frac{1}{2} =: \tilde{x}$$

f is a strictly monotonically increasing homeomorphism between $\mathbb{R}$ and $(0,1)$.

Set $s_0 = 1 - \tilde{u}_1$, $s_i = \tilde{u}_i - \tilde{u}_{i+1}$ for $1 \le i \le q-1$ and $s_q = \tilde{u}_q$ for the x-coordinate and $t_0 = \tilde{v}_1, t_j = \tilde{b}_{j+1} - \tilde{b}_j$ for $1 \le j \le p-1$ for the y-coordinate. Then

$$\sum_{i=0}^{q} s_i = 1 = \sum_{j=0}^{p} t_i.$$

Therefore $(s_0, s_1, \dots, s_q)$ and $(t_0, t_1, \dots, t_p)$ are barycentric coordinates in the open bi-simplex $\mathring{\Delta}^q \times \mathring{\Delta}^p$. By varying the coordinates s_i and t_j in $\mathring{\Delta}^q$ and $\mathring{\Delta}^p$ respectively, we get all the slit configurations with the same symbol E.

The notions *degenerate*, *Rauzy jump* and *puncture-number* can also be defined for symbols. Denote the equivalence class of a symbol E under Rauzy jumps by $[E]$.

Let $\mathfrak{Par}(h,m) := \{[L] = [L_1, \dots, L_{2h}|\lambda] | L$ is non-degenerate and has puncture-number $m\}$. It turns out that $\mathfrak{Par}(h,m)$ is a manifold of dimension $3h$ and there is a homotopy equivalence

$$\mathfrak{Par}(h,m) \simeq \mathfrak{M}^m_{g,1} \text{ with } h = 2g + m.$$

Let $\mathrm{Par}(h,m)$ be the closure of $\mathfrak{Par}(h,m)$ and define

$$\mathrm{Par}'(h,m) := \mathrm{Par}(h,m) \smallsetminus \mathfrak{Par}(h,m).$$

The relative manifold $(\mathrm{Par}(h,m), \mathrm{Par}'(h,m))$ is a bi-simplicial cell complex; the associated chain complex $\mathbb{Q}_{\bullet\bullet}(h,m)$ is therefore a double complex. Every (p,q)-cell of $\mathbb{Q}_{\bullet\bullet}(h,m)$ is an equivalence class of symbols $[E] = [a_1, \dots, a_p|\lambda|B_1, \dots, B_q]$, where E is non-degenerate, has puncture-number m and satisfies (2.1.2), (2.1.4).

The i-th *vertical* face operator on E and hence the i-th *vertical* face operator on $[E]$ are given by

$$\partial_i'(E) := (a_1, \dots, a_p|\lambda|B_0, \dots, B_j \cup B_{j+1}, \dots, B_q),\ i = 1, \dots, q-1$$
$$\partial_i'([E]) := [\partial_i'(E)] = [a_1, \dots, a_p|\lambda|B_0, \dots, B_j \cup B_{j+1}, \dots, B_q],\ i = 1, \dots, q-1$$

and the j-th *horizontal* face operator on E and hence the j-th *horizontal* face operator on $[E]$ are

$$\partial_j''(E) := (a_1, \dots, a_i + a_{i+1}, \dots, a_p|\lambda|B_0, \dots, B_q),\ j = 1, \dots, p-1$$
$$\partial_j''([E]) := [\partial_j''(E)] = [a_1, \dots, a_i + a_{i+1}, \dots, a_p|\lambda|B_0, \dots, B_q],\ j = 1, \dots, p-1,$$

in which a face is defined to be zero, if it is not a cell in $\mathbb{Q}_{\bullet\bullet}(h,m)$. Let

$$\partial'([E]) := \sum_{i=1}^{q-1} (-1)^i \partial_i'([E])$$
$$\partial''([E]) := \sum_{j=1}^{p-1} (-1)^j \partial_j''([E]),$$

then the boundary operator ∂ of the complex $\mathbb{Q}_{\bullet\bullet}(h,m)$ is given by

$$\partial([E]) := \partial'([E]) + (-1)^q \partial''([E]).$$

Description of $\mathbb{Q}_{\bullet\bullet}(h,m)$ via Symmetric Groups

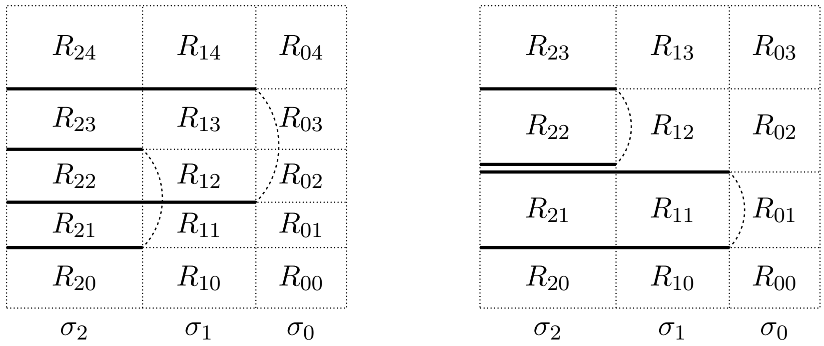
Every symbol $E = (a_1, \dots, a_p|\lambda|B_1, \dots, B_q)$ can be characterized by a *grid picture*, whose grid is given by the horizontals of the slits and the verticals at the slit ends. We number the columns $0, 1, \dots, q$ from right to left and rows $0, 1, \dots, p$ from bottom to top. Let $R_{i,j}$ denote the j-th rectangle in the i-th column.

For the i-th column let $\sigma_i \in \mathfrak{S}_{p+1}$ denote the permutation describing the re-gluing of the slit plane: the upper edge of the rectangle $R_{i,j}$ is glued to the lower edge of $R_{i,\sigma_i(j)}$, for $i = 0, 1, \dots, q; j = 0, 1, \dots, p-1$. Furthermore, set $\sigma_i(p) = 0$. Denote by $\mathrm{ncyc}(\sigma_q)$ the number of cycles of σ_q. Since $\sigma_q \in \mathfrak{S}_{p+1}$ has the same number of cycles as the permutation $\lambda \circ \omega_{2h} \in \mathfrak{S}_{2h}$ in Remark 2.1, the puncture-number m of E is $\mathrm{ncyc}(\sigma_q) - 1$. These observations lead to the following description of the double complex $\mathbb{Q}_{\bullet\bullet}(h,m)$ via symmetric groups.

Let $\mathbb{P}_{p,q}(h)$ be the free abelian group generated by all $(q+1)$-tuples $\Sigma = (\sigma_q, \dots, \sigma_0)$ with $\sigma_i \in \mathfrak{S}_{p+1} = \mathfrak{S}(0, \dots, p)$-the symmetric group on the letters $0, 1, \dots, p$, such that

$$N(\Sigma) := N(\sigma_q \sigma_{q-1}^{-1}) + \cdots + N(\sigma_1 \sigma_0^{-1}) \leq h.$$

Here $N(\alpha)$ denotes the word length norm of α with respect to the generating set of $\mathfrak{S}_{p+1}$ which consists of all transpositions. In other words, $N(\alpha) = p + 1 - \mathrm{ncyc}(\alpha)$.



$\sigma_0 = (0\ 1\ 2\ 3\ 4)$
$\sigma_1 = (0\ 1\ 4)(2\ 3)$
$\sigma_2 = (0\ 3\ 2\ 1\ 4)$

$\sigma_0 = (0\ 1\ 2\ 3)$
$\sigma_1 = (0\ 2\ 3)(1)$
$\sigma_2 = (0\ 3)(1)(2)$

Figure 2.3: Grid pictures

Define a double complex $\mathbb{P}_{\bullet\bullet}(h) := \bigoplus \mathbb{P}_{p,q}(h)$, $0 \le p \le 2h$, $q \le h$. The vertical and horizontal boundary operators on the double complex $\mathbb{P}_{\bullet\bullet}(h)$ are:

$$\partial' = \sum_{i=0}^{q} (-1)^i \partial'_i \qquad \text{with} \qquad \partial'_i(\Sigma) = (\sigma_q, \dots, \widehat{\sigma}_i, \dots, \sigma_0)$$

$$\partial'' = \sum_{j=0}^{p} (-1)^j \partial''_j \qquad \text{with} \qquad \partial''_j(\Sigma) = (D_j(\sigma_q), \dots, D_j(\sigma_0)),$$

and $\partial = \partial' + (-1)^q \partial''$ is the boundary operator on $\mathbb{P}_{\bullet\bullet}(h)$. Here $D_j : \mathfrak{S}_{p+1} \to \mathfrak{S}_p$ is defined as:

$$D_j(\alpha) = s_j \circ (\alpha(j)\ j) \circ \alpha \circ d_j$$

where $d_j : [p-1] \to [p]$ is the simplicial degeneracy function which avoids the value j, and $s_i : [p] \to [p-1]$ is the simplicial face function which repeats the value i. In other words, to get $D_j(\alpha)$, one deletes the letter j from the cycle of α where it occurs in and renormalizes the indices.

The subcomplex $\mathbb{P}'_{\bullet\bullet}(h,m)$ of $\mathbb{P}_{\bullet\bullet}(h)$ is generated by the *degenerate* cells of $\mathbb{P}_{\bullet\bullet}(h)$, these are the cells which violate any of the following conditions:

1) $N(\Sigma) = h$

2) $\mathrm{ncyc}(\sigma_q) = m + 1$

3) $\sigma_i(p) = 0$, for $i = 0, \dots q$

4) σ_0 is the rotation $\omega_{p+1} = (0\ 1 \dots p)$

5) $\sigma_{i+1} \ne \sigma_i$ for $i = 0, \dots q-1$

6) There is no $0 \le k \le p-1$, such that $\sigma_i(k) = k+1$ for all $i = 0, \dots q$.

Finally $\mathbb{Q}_{\bullet\bullet}(h,m) \cong \mathbb{P}_{\bullet\bullet}(h)/\mathbb{P}'_{\bullet\bullet}(h,m)$ is the desired double complex. In particular, the face operators ∂'_0, ∂'_q, ∂''_0 and ∂''_p are always zero on $\mathbb{Q}_{p,q}(h,m)$.

From above, a cell Σ in $\mathbb{Q}_{\bullet\bullet}(h,m)$ is represented by a $(q+1)$-tuple $\Sigma = (\sigma_q, \ldots, \sigma_0)$ with $\sigma_i \in \mathfrak{S}_{p+1}$. This is called the *homogeneous notation*. The *inhomogeneous notation*, which we will define now, is more important for later use. Let $\tau_i := \sigma_i \circ \sigma_{i-1}^{-1} \in \mathfrak{S}_p = \mathfrak{S}(1,\ldots,p), i = 1,\ldots,q$, then $\Sigma = (\tau_q|\ldots|\tau_1)$ is called the *inhomogeneous notation*. It follows from definition that $\sigma_i = \tau_i \cdots \tau_1 \circ \omega_{p+1}$. The conditions and boundary operators on Σ can then be translated into the language using τ_i.

The equivalence of the two ways to define $\mathbb{Q}_{\bullet\bullet}(h,m)$-namely by equivalence classes of symbols and by tuples of permutations-was given on page 21 of [Eh]. We give some remarks about the bijection here.

Remark 2.1.8.

1) Equivalent symbols E and E' in $\mathbb{Q}_{p,q}(h,m)$ give rise to the same permutations $\sigma_i \in \mathfrak{S}_p, i = 0,1,\ldots,q$, since they determine the same way of gluing.

2) Given a tuple $\Sigma = (\tau_q|\ldots|\tau_1) \in \mathbb{Q}_{p,q}(h,m)$ in inhomogeneous notation, we can construct a slit configuration L as follows. Choose real numbers $x_1 > x_2 > \ldots > x_q$ and $y_1 < y_2 < \ldots < y_p$. Starting from $i = 1$, we do the following for $i = 1,\ldots,q$ in sequence. Write τ_i into a product of $N(\tau_i)$ transpositions.

$$\tau_i = (a_{N(\tau_i)}\ b_{N(\tau_i)}) \ldots (a_1\ b_1)$$

Then for $j = 1,\ldots,N(\tau_i)$ in sequence, we put two paired slits with endpoints (x_i, y_{a_j}) and (x_i, y_{b_j}) on the complex plain $\mathbb{C}$; and if slits with the same y-level y_{a_j}, y_{b_j} already exist, then put the new one above all the existing ones.

In the end, there will be $N(\tau_q) + \cdots + N(\tau_1) = h$ pairs of slits. We name the slits $L_1,\ldots,L_{2h}$ from bottom to top respectively and let λ records how the slits are paired. Then $L = (L_1,\ldots,L_{2h}|\lambda)$ is a slit configuration, whose symbol E corresponds to $\Sigma = (\tau_q|\ldots|\tau_1)$ in the sense described before.

$\diamond$

2.2 Symmetric Groups

We have seen that symmetric groups play an important role in the cell structure of moduli spaces of Riemann surfaces. Besides, they are also the main examples of factorable groups as shown in [V]. The properties of symmetric groups presented in this section are taken from [V], chapter 5.

Think of the symmetric group $\mathfrak{S}_p$ to be the group of permutations of the set $I_p = \{1,2,\ldots,p\}$. Let $T = T_p = \{(i,j) \mid 1 \le i < j \le p\}$ be the set of transpositions. The norm $N = N_{\mathfrak{S}_p}$ on $\mathfrak{S}_p$ is taken to be the word length norm with respect to T. It turns out that, for $\sigma \in \mathfrak{S}_p$, $N(\sigma) = p - \mathrm{ncyc}(\sigma)$, where $\mathrm{ncyc}(\sigma)$ denotes the number of cycles of σ.

Let $H : \mathfrak{S}_p \to I_p$ be a function, which assigns to a permutation σ the largest element of the set I_p which is not a fixed point of σ, i.e. $H(\sigma) = \max\{j \in I_p \mid \sigma(j) \ne j\}$. Define a function $\eta = \eta_{\mathfrak{S}_p} : \mathfrak{S}_p \to \mathfrak{S}_p \times \mathfrak{S}_p$ as follows. For $1 \ne \sigma \in \mathfrak{S}_p$, $\eta(\sigma) =$

$(\bar{\eta}(\sigma), \eta'(\sigma)$, whereas the two factors are given by:

$$\eta'(\sigma) = (i \;\; \sigma^{-1}(i))$$
$$\bar{\eta}(\sigma) = \sigma \cdot \eta'(\sigma)$$

with $i = H(\sigma)$. It can then be proved that $\mathfrak{S}_p$ is a factorable group, η being the factorization map.

Denote $R_h(\mathfrak{S}_p)[h]$ simply by $R_p(h)$. For any h-tuple of transpositions $\tau = (\tau_h, \dots, \tau_1)$, we have

$$\tau \in R_p(h) \iff H(\tau_h) \geq \dots \geq H(\tau_1)$$

Hence an h-tuple in $R_p(h)$ is also called a *monotone* tuple. Let $r_p(h)$ be the number of elements in $R_p(h)$, then $r_p(h)$ satisfies the following recursive formula:

$$r_p(h) = \begin{cases} (p-1)r_p(h-1) + r_{p-1}(h) & p \geq 3, h \geq 1 \\ 1 & p = 2 \\ 1 & h = 0 \end{cases}$$

Visy pointed out in [V] that the bi-sequences of this form are actually re-indexed Stirling numbers of the second kind. He used this fact to prove that κ is an isomorphism for $\mathfrak{S}_p$ by showing that the bi-sequence of the absolute values of the Euler characteristics satisfies the same formula.

Since $\mathfrak{S}_p$ is a factorable normed group, we know from the general theory in [V] that the homology of the norm complex $(\mathcal{N}_*(\mathfrak{S}_p)[h], d)$ concentrates on the top degree h, and a basis of $H_h(\mathcal{N}_*(\mathfrak{S}_p)[h], d)$ can also be constructed.

Recall that $\mathbb{Q}_{\bullet\bullet}(h, m)$ is a double complex with vertical boundary operator $\partial' = \sum_{i=1}^{q-1}(-1)^i \partial'_i$ and horizontal boundary operator $\partial'' = \sum_{i=1}^{p-1}(-1)^i \partial''_i$. For a fixed p, the following Lemma shows the relation between the chain complex $(\mathbb{Q}_{p,*}(h, m), \partial')$ and the norm complex $(\mathcal{N}_*(\mathfrak{S}_p)[h], d)$. This thus provides an application of the theory of factorable normed groups to the study of homology of moduli spaces of Riemann surfaces.

Lemma 2.2.1. *The p-th vertical complex $\mathbb{Q}_{p,*}(h, m)$ with differential ∂' is a direct summand of $\mathcal{N}_*(\mathfrak{S}_p)[h]$ with differential d.*

Proof. We define a homomorphism $\imath : \mathbb{Q}_{p,q}(h, m) \to \mathcal{N}_*(\mathfrak{S}_p)[h]$ by letting

$$\imath((\sigma_q, \dots, \sigma_0)) := (\sigma_q \sigma_{q-1}^{-1} | \dots | \sigma_1 \sigma_0^{-1})$$

for each $\Sigma = (\sigma_q, \dots, \sigma_0) \in \mathbb{Q}_{p,q}(h, m)$ and extending linearly to a homomorphism between $\mathbb{Z}$-modules. Since any tuple $\Sigma = (\sigma_q, \dots, \sigma_0) \in \mathbb{Q}_{p,q}(h, m)$ satisfies $\sigma_i(p) = 0$ for $i = 0, \dots, q$ and σ_0 is the rotation $\omega_{p+1} = (0\ 1 \dots p)$, $\imath$ is an injective homomorphism.

The statement that $(\mathbb{Q}_{p,*}(h, m), \partial')$ is a direct summand of $(\mathcal{N}_*(\mathfrak{S}_p)[h], d)$ is then equivalent to: For any $0 \leq i \leq q$, the following properties 1) and 2) hold.

1) $d_i \circ \imath = \imath \circ \partial'_i$, i.e. $(\mathbb{Q}_{p,*}(h, m), \partial')$ is a subcomplex of $(\mathcal{N}_*(\mathfrak{S}_p)[h], d)$.

2) $(\Sigma \in \mathcal{N}_*(\mathfrak{S}_p)[h] \setminus \imath(\mathbb{Q}_{p,*-1}(h,m))$ and $d_i(\Sigma) \in \imath(\mathbb{Q}_{p,*-1}(h,m))) \Longrightarrow d_i(\Sigma) = 0$.

This can be checked directly using the definition of $\mathbb{Q}_{\bullet\bullet}(h,m)$ on page 30, hence we leave out the details here. □

Remark 2.2.2. The maps $\imath : \mathbb{Q}_{p,*}(h,m) \to \mathcal{N}_*(\mathfrak{S}_p)[h]$ for different m together induce the injective homomorphism

$$\imath : \bigoplus_{\substack{0 \le m \le h \\ h-m \text{ even}}} \mathbb{Q}_{p,*}(h,m) \hookrightarrow \mathcal{N}_*(\mathfrak{S}_p)[h].$$

However, this map is not surjective for $p > 2$, because any tuple $\Sigma = (\sigma_q, \dots, \sigma_0) \in \mathbb{Q}_{p,q}(h,m)$ must satisfy the extra condition that there exists no $0 \le k \le p-1$ such that k is fixed by τ_i for all $i = 1, \dots q$. ◇

Let $\{E^r_{p,q}, d_r\}_r$ be the spectral sequence associated to the double complex $\mathbb{Q}_{\bullet\bullet}(h,m)$. Then we have $\{E^0_{p,q} = \mathbb{Q}_{p,q}(h,m)\}$ and $d_0 = \partial' : \mathbb{Q}_{p,q}(h,m) \to \mathbb{Q}_{p,q-1}(h,m)$. Moreover, $E^1_{p,q} = H_q(\mathbb{Q}_{p,*}(h,m))$ and the differential $d_1 : E^1_{p,q} \to E^1_{p-1,q}$ is induced by ∂''.

An immediate consequence from Theorem 1.2.11 and Lemma 2.2.1 is the following

Corollary 2.2.3. *The vertical homology $E^1_{p,q} = H_q(\mathbb{Q}_{p,*}(h,m), \partial')$ is concentrated in the top degree $q = h$. Thus the E^1-term is a chain complex with differential induced by ∂'', and the spectral sequence collapses with $E^2 = E^\infty$.*

We define a complex $(W_*(h,m), d)$ by $W_p(h,m) := \kappa^{-1}(H_h(\mathbb{Q}_{p,*}(h,m))) \subseteq V_h(\mathfrak{S}_p)$ and $d := \kappa^{-1} \circ \partial'' \circ \kappa$. It is equivalent to the E^1-term of the spectral sequence. A closer examination shows that $W_p(h,m)$ is generated by those tuples $(\tau_h|\dots|\tau_1) \in R_h(\mathfrak{S}_p)[h]$, which satisfy the following extra conditions:

1) There exists no $1 \le k \le p$, such that k is fixed by τ_i for all $i = 1, \dots h$.

2) The number of cycles of $\tau_h \circ \dots \circ \tau_1 \circ \omega_p$ is $m+1$, where $\omega_p = (1\ 2 \dots p)$.

It is proved in [M] that $\mathfrak{M}^m_{g,1}$ is orientable if and only if $m = 0$ or $m = 1$. The double complex $\mathbb{Q}_{\bullet\bullet}(h,m)$ and hence the complex $(W_*(h,m), d)$ can be used to compute the integral homology of $\mathfrak{M}^0_{g,1}$ and $\mathfrak{M}^1_{g,1}$ due to the Poincaré duality

$$H^*(\mathbb{Q}_{\bullet\bullet}(h,m)) = H^*(\mathrm{Par}(h,m), \mathrm{Par}'(h,m); \mathbb{Z}) \cong H_{3h-*}(\mathfrak{M}^m_{g,1}; \mathbb{Z}),\ m = 0, 1.$$

For arbitrary m, one can compute the $\mathbb{Z}_2$-homology of $\mathfrak{M}^m_{g,1}$ using the mod-2 version of $(W_*(h,m), d)$ and the Poincaré duality

$$H^*(\mathbb{Q}_{\bullet\bullet}(h,m), \mathbb{Z}_2) = H^*(\mathrm{Par}(h,m), \mathrm{Par}'(h,m); \mathbb{Z}_2) \cong H_{3h-*}(\mathfrak{M}^m_{g,1}; \mathbb{Z}_2).$$

2.3 Orientation System

Since the manifold $\mathfrak{M}^m_{g,1}$ is non-orientable for $m \geq 2$, to compute the integral homology of $\mathfrak{M}^m_{g,1}$ when $m \geq 2$, we need to consider the orientation system $\mathcal{O}$ on the relative manifold $(\mathrm{Par}(h,m), \mathrm{Par}'(h,m))$ and use the Poincaré duality

$$H^*(\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)) = H^*(\mathrm{Par}(h,m), \mathrm{Par}'(h,m); \mathcal{O}) \cong H_{3h-*}(\mathfrak{M}^m_{g,1}; \mathbb{Z}).$$

Here $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$ is the chain complex of the relative manifold with the orientation system $\mathcal{O}$. In this section, we will give the construction of $\widetilde{\mathbb{Q}}_{p,q}(h,m)$ and study its homology, which turns out to have analogous properties to that of $\mathbb{Q}_{\bullet\bullet}(h,m)$.

$\widetilde{\mathbb{Q}}_{p,q}(h,m)$ has the same generators as $\mathbb{Q}_{p,q}(h,m)$, but the boundary operators differ. In $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$, the vertical and horizontal face operators on a cell $e \in \widetilde{\mathbb{Q}}_{p,q}(h,m)$ are of the form

$$\widetilde{\partial}'_i(e) = \epsilon'_i(e)\, \partial'_i(e), \quad i = 1, \ldots, q-1 \tag{2.3.1}$$

$$\widetilde{\partial}''_i(e) = \epsilon''_i(e)\, \partial''_i(e), \quad i = 1, \ldots, p-1 \tag{2.3.2}$$

where the sign $\epsilon'_i(e),\ \epsilon''_i(e) \in \{\pm 1\}$ depends on e and i, not only on i.

Before going to the definition of $\epsilon'_i(e),\ \epsilon''_j(e)$ and hence of the double complex $\widetilde{\mathbb{Q}}_{p,q}(h,m)$, we need to study the cells of $\mathbb{Q}_{\bullet\bullet}(h,m)$ in more detail. The main reference for the construction of of $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$ is [A].

Recall that a cell $e \in \mathbb{Q}_{p,q}(h,m)$ can be considered as an equivalence class of symbols under Rauzy jumps or as a tuple of permutations $e = (\tau_q | \ldots | \tau_1)$ with $\tau_i \in \mathfrak{S}_p$. For a top-dimensional cell $e \in \mathbb{Q}_{2h,h}(h,m)$, i.e. $p = 2h$, $q = h$, there is a unique symbol E which is a representative of $e = [E]$. We call the cells in $\mathbb{Q}_{2h,h}(h,m)$ *generic*. If e is not a generic cell, there can be more than one representative for e, and by definition, any two of them can be transformed into each other by Rauzy jumps.

We need some terminology to go further with the discussion.

Definition 2.3.1. Let $D = \{i_1, \ldots, i_n\} \subset \mathbb{N}$ with $i_1 < \ldots < i_n$. We define

$$\partial'_D := \partial'_{i_1} \cdots \partial'_{i_n} \qquad \text{and} \qquad \partial''_D := \partial''_{i_1} \cdots \partial''_{i_n}.$$

In the following discussion, D_1 and D_2 always satisfy

$$D_1 \subseteq \{1, \ldots, h-1\} \quad \text{and} \quad D_2 \subseteq \{1, \ldots, 2h-1\}.$$

Definition 2.3.2. We call $(\hat{e}, D_1, D_2)$ a *generification* of the cell e, if $\hat{e}$ is a cell in $\mathbb{Q}_{2h,h}(h,m)$ and $\partial''_{D_2}\partial'_{D_1}(\hat{e}) = e$.

Denote the set of top-dimensional cells which have e as a face by

$$\mathcal{H}(e) = \{\hat{e} \mid \hat{e} \text{ is a generic cell and } \partial''_{D_2}\partial'_{D_1}(\hat{e}) = e \text{ for some sets } D_1, D_2\}.$$

We know that e can have more than one symbol as representatives and for each representative E, there exists a cell $\hat{e} \in \mathcal{H}(e)$ and sets D_1, D_2, such that

$$\partial''_{D_2}\partial'_{D_1}(\hat{e}) = E. \tag{2.3.3}$$

Here we note that since the symbol E appears at the right-hand side of the equation (2.3.3), the boundary operators are meant to be in the sense of symbols, not in the sense of equivalence classes of symbols. In the rest of this section, the boundary operators are often meant for symbols, but this will always be clear from the context.

The following definition plays an central role in the determination of the face operators on $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$. It comes from the idea that we want to choose a "standard" representative for each equivalence class of symbol $[E]$.

Definition 2.3.3. The *normal form* of a cell $e = (\tau_q|\dots|\tau_1) \in \mathbb{Q}_{p,q}(h,m)$ is the generification $(\mathrm{NF}(e), D_1, D_2)$ of e which is determined by the following procedure:

1) For each i, assume the normal form of τ_i is $\mathrm{nF}(\tau_i) = (\tau_{i,N(\tau_i)}, \dots, \tau_{i,1})$, where $N(\tau_i)$ is the norm of τ_i. Define a cell $e^{(0)}$ in $\mathbb{Q}_{p,h}(h,m)$ as

$$e^{(0)} := (\tau_{q,N(\tau_q)}|\dots|\tau_{q,1}|\dots|\tau_{1,N(\tau_1)}|\dots|\tau_{1,1}) := (\tau_h^{(0)}|\dots|\tau_1^{(0)})$$

and $D_1 := \{1,\dots,h-1\} \smallsetminus \{\sum_{k=1}^{i} N(\tau_k) \mid i = 1,\dots,q-1\}$.

2) First we define a map $S_j : \mathfrak{S}_p \to \mathfrak{S}_{p+1}$ for each $j = 1,\dots,p$, which sends a permutation $\alpha \in \mathfrak{S}_p$ to the permutation $S_j(\alpha) \in \mathfrak{S}_{p+1}$ defined as follows:

$$S_j(\alpha)(k) := \begin{cases} d_j \circ \alpha \circ s_j(k) & \text{if} \quad k \neq j \\ j & \text{if} \quad k = j \end{cases}$$

To get $\mathrm{NF}(e)$ and D_2, we perform the following algorithm on $e^{(0)}$:

Algorithm 2.3.4.

- $D_2^{(0)} := \emptyset$.
- *For* $1 \le n \le h-p$:
  Let k *be the lowest index such that there are more than one transposition of* $e^{(n-1)} = (\tau_h^{(n-1)}|\dots|\tau_1^{(n-1)})$ *which do not fix* k *and assume* $i_0 = \min\{1 \le i \le h \mid \tau_i^{(n-1)}(k) \neq k\}$. *Define*

  $$e^{(n)} := (\tau_h^{(n)}|\dots|\tau_1^{(n)}) \in \mathbb{Q}_{p+n,h}(h,m) \qquad \textit{and} \qquad D_2^{(n)} := D_2^{(n)} \cup \{k\}.$$

  where

  $$\tau_i^{(n)} := \begin{cases} S_{k+1}(\tau_i^{(n-1)}) & \textit{if} \quad i = i_0 \\ S_k(\tau_i^{(n-1)}) & \textit{if} \quad i \neq i_0 \end{cases}$$

- $\mathrm{NF}(e) := e^{(h-p)}$ *and* $D_2 := D_2^{(h-p)}$.

Remark 2.3.5.

1) It follows from the procedure that $\mathrm{NF}(e) \in \mathbb{Q}_{2h,h}(h,m)$ and $\partial''_{D_2}\partial'_{D_1}(\mathrm{NF}(e)) = e$, so $(\mathrm{NF}(e), D_1, D_2)$ is a generification of e as claimed in the definition.

2) For any generification $(\hat{e}, D_1, D_2)$ of e, D_1 is uniquely determined by the partition-type of e (hence uniquely determined by e).

3) The second step can be understood more intuitively from the perspective of slit configurations: Let $E^{(0)}$ be the symbol which is the representative of $e^{(0)}$ chosen as in 2) of Remark 2.1. Whenever two slits touch each other in $E^{(0)}$, we will add a stripe between them, so that in the end $E^{(0)}$ is transformed into a top dimensional cell $\mathrm{NF}(e)$.

$\diamond$

Now we will explain two kinds of operators-namely swaps and Rauzy jumps-on a generification $(\hat{e}, D_1, D_2)$ of e. We fix the assumption that

$$\hat{e} = (\tau_h|\ldots|\tau_1) \in \mathcal{H}(e),$$

where τ_i is a transposition for $1 \le i \le h$.

Definition 2.3.6. If $i \in D_1$, the *swap* of $(\hat{e}, D_1, D_2)$ at i is defined to be the generification $(\mathrm{swap}_i(\hat{e}), D_1, D_2))$, where

$$\mathrm{swap}_i(\hat{e}) := (\tau_h|\ldots|\tau_{i+2}|\tau_i|\tau_{i+1}|\tau_{i-1}|\ldots|\tau_1).$$

Remark 2.3.7. Clearly $\mathrm{swap}_i(\hat{e}) \in \mathcal{H}(e)$ and $\partial''_{D_2}\partial'_{D_1}(\mathrm{swap}_i(\hat{e})) = \partial''_{D_2}\partial'_{D_1}(\hat{e})$ as symbols, hence $(\mathrm{swap}_i(\hat{e}), D_1, D_2)$ is a generification of e as claimed in the definition. $\diamond$

For a fixed representative E of e, there can be more than one $\hat{e}$ satisfying (2.3.3). However, from the definition of the face operators, D_1 is uniquely determined by e (hence also by E) and D_2 is uniquely determined by E. Moreover, $\hat{e}$ is determined by E up to swaps as shown in the following Lemma.

Lemma 2.3.8. *Suppose $(\hat{e}, D_1, D_2)$ and $(\check{e}, D_1, D_2)$ are generifications of e such that*

$$\partial''_{D_2}\partial'_{D_1}(\hat{e}) = E = \partial''_{D_2}\partial'_{D_1}(\check{e}),$$

where E is a symbol representing e. Then $(\hat{e}, D_1, D_2)$ can be transformed into $(\check{e}, D_1, D_2)$ by swaps.

Proof. From the definition of the horizontal face operators we have

$$\partial'_{D_1}(\hat{e}) = \partial'_{D_1}(\check{e}) =: \tilde{e} \in \mathbb{Q}_{2h,q}(h, m).$$

Assume that $\hat{e} = (\tau_h|\ldots|\tau_1)$, $\check{e} = (\check{\tau}_h|\ldots|\check{\tau}_1)$ and $\tilde{e} = (\tilde{\tau}_q|\ldots|\tilde{\tau}_1)$. Let $n_0 = 1$ and

$$n_i = \sum_{k=1}^{i} N(\tilde{\tau}_k) \quad i = 1, \ldots, q.$$

Then we have

$$\tilde{\tau}_i = \tau_{n_i} \circ \ldots \circ \tau_{n_{i-1}+1} = \check{\tau}_{n_i} \circ \ldots \circ \check{\tau}_{n_{i-1}+1}.$$

Since $\hat{e}$ and $\check{e}$ have vertical dimension $2h$, the indices not fixed by τ_i $(i = 1, \ldots, h)$ are pairwise distinct and the indices not fixed by $\check{\tau}_i$ $(i = 1, \ldots, h)$ are pairwise distinct. Therefore the tuple $(\check{\tau}_{n_i}|\ldots|\check{\tau}_{n_{i-1}+1})$ is obtained from $(\tau_{n_i}|\ldots|\tau_{n_{i-1}+1})$ by permuting its transpositions. Hence $\check{e} = (\check{\tau}_h|\ldots|\check{\tau}_1)$ is obtained from $\hat{e} = (\tau_h|\ldots|\tau_1)$ by permuting its transpositions.

Because a swap exchanges two neighbored transpositions and any permutation $\sigma \in \mathfrak{S}_h$ is a composition of elements in $\{(i\ \ i{+}1) \mid i = 1, \ldots, h{-}1\}$, $\hat{e}$ can be transformed into $\check{e}$ by swaps. $\square$

Before defining Rauzy jumps for the generification $(\hat{e}, D_1, D_2)$ of e, we fix some notations first. For each $i \in D_2$, there exist unique $k, l \in \{1, \ldots, h\}$, such that $\tau_k(i) \neq i, \tau_l(i+1) \neq i+1$. Furthermore $k \neq l$, since otherwise $e \in \mathbb{P}'_{\bullet\bullet}(h, m)$.

Definition 2.3.9. The *Rauzy jump* of $(\hat{e}, D_1, D_2)$ at $i \in D_2$ is defined to be the generification $(\mathrm{Rauzy}_i(\hat{e}), D_1, \tilde{D}_2)$, where $\mathrm{Rauzy}_i(\hat{e})$ and $\tilde{D}_2$ are defined as follows. First define

$$\alpha := \begin{cases} s_{\tau_l(i+1)+1,i} & \text{if} \quad l < k \quad \text{and} \quad \tau_l(i+1) < i \\ s^{-1}_{i,\tau_l(i+1)} & \text{if} \quad l < k \quad \text{and} \quad \tau_l(i+1) > i+1 \\ s_{\tau_k(i),i+1} & \text{if} \quad l > k \quad \text{and} \quad \tau_k(i) < i \\ s^{-1}_{i+1,\tau_k(i)-1} & \text{if} \quad l > k \quad \text{and} \quad \tau_k(i) > i+1 \end{cases}$$

and

$$\alpha_{D_2} := \begin{cases} s_{\tau_l(i+1),i} & \text{if} \quad l < k \quad \text{and} \quad \tau_l(i+1) < i \\ s^{-1}_{i,\tau_l(i+1)-1} & \text{if} \quad l < k \quad \text{and} \quad \tau_l(i+1) > i+1 \\ s_{\tau_k(i),i} & \text{if} \quad l > k \quad \text{and} \quad \tau_k(i) < i \\ s^{-1}_{i,\tau_k(i)-1} & \text{if} \quad l > k \quad \text{and} \quad \tau_k(i) > i+1, \end{cases}$$

then

$$\begin{aligned} \tilde{\tau}_j &:= \alpha \circ \tau_j \circ \alpha \quad \text{for} \quad 1 \leq j \leq h \\ \tilde{D}_2 &:= \alpha_{D_2}(D_2) \\ \mathrm{Rauzy}_i(\hat{e}) &:= (\tilde{\tau}_h|\ldots|\tilde{\tau}_1). \end{aligned}$$

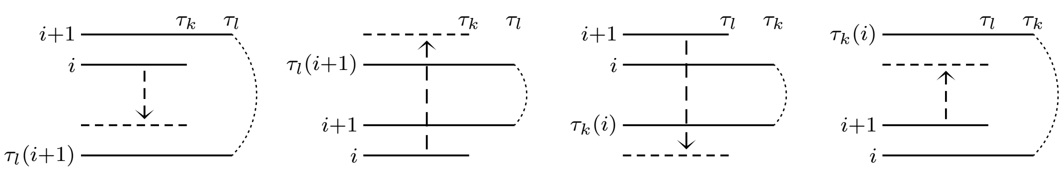


Figure 2.4: Rauzy jumps

Remark 2.3.10. It turns out that $\mathrm{Rauzy}_i(\hat{e}) \in \mathcal{H}(e)$ and $\partial''_{\tilde{D}_2}\partial'_{D_1}(\mathrm{Rauzy}_i(\hat{e})) = \partial''_{D_2}\partial'_{D_1}(\hat{e}) = e$, hence $(\mathrm{Rauzy}_i(\hat{e}), D_1, \tilde{D}_2)$ is a generification of e as claimed in the definition. Moreover, as symbols, $\partial''_{\tilde{D}_2}\partial'_{D_1}(\mathrm{Rauzy}_i(\hat{e}))$ and $\partial''_{D_2}\partial'_{D_1}(\hat{e})$ are different from each other by a Rauzy jump in the sense of slit configurations as defined in

Definition 2.1.6. This is why here we call the operator Rauzy_i on a generification also Rauzy jump. $\diamond$

The following Lemma contains the essential idea and technique for the determination of the face operators on $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$

Lemma 2.3.11. *If $(\hat{e}, D_1, D_2)$ and $(\check{e}, D_1, \check{D}_2)$ are generifications of e, then there exists a series of swaps and Rauzy jumps, by which $(\hat{e}, D_1, D_2)$ can be transformed into $(\check{e}, D_1, \check{D}_2)$.*

Proof. Suppose $\partial''_{D_2}\partial'_{D_1}(\hat{e}) = E$, $\partial''_{\check{D}_2}\partial'_{D_1}(\check{e}) = \check{E}$, where E and $\check{E}$ are symbols representing e. Then E and E' are different from each other by Rauzy jumps.

We will first prove that if $E \neq \check{E}$, then there exists a generification $(\tilde{e}, D_1, \tilde{D}_2)$ of e, such that $\partial''_{\tilde{D}_2}\partial'_{D_1}(\tilde{e}) = \check{E}$ and $(\hat{e}, D_1, D_2)$ can be transformed into $(\tilde{e}, D_1, \tilde{D}_2)$ by Rauzy jumps and swaps.

We name the slits of E by $L_1, \dots, L_{2h}$ from the bottom up as in Definition 2.1.2 and write $\hat{e} = (\tau_h|\dots|\tau_1)$.

First assume that $\check{E}$ can be obtained from E by one Rauzy jump. Then there exists some $i \in D_2$, such that the Rauzy jump is either that L_i jumps over L_{i+1} or that L_{i+1} jumps over L_i. Let $k, l \in \{1, \dots, h\}$ be the unique indices satisfying $\tau_k(i) \neq i, \tau_l(i+1) \neq i+1$. There are two possible cases:

(1) If (L_i jumps over L_{i+1} and $l < k$) or (L_{i+1} jumps over L_i and $l > k$). Then take $(\tilde{e}, D_1, \tilde{D}_2)$ to be the Rauzy jump of $(\hat{e}, D_1, D_2)$ at i.

(2) If (L_i jumps over L_{i+1} and $l > k$) or (L_{i+1} jumps over L_i and $l < k$). Then first transform $(\hat{e}, D_1, D_2)$ into $(\mathring{e}, D_1, D_2)$ with $\mathring{e} = (\tau_h|\dots|\tau_k|\dots|\tau_l|\dots|\tau_1)$ by swaps; and then take $(\tilde{e}, D_1, \tilde{D}_2)$ to be the Rauzy jump of $(\mathring{e}, D_1, D_2)$ at i.

If E is transformed into $\check{E}$ by a series of Rauzy jumps, then for each Rauzy jump in sequence, the step above is performed and in the end we get a generification $(\tilde{e}, D_1, \tilde{D}_2)$ of e with $\partial''_{\tilde{D}_2}\partial'_{D_1}(\tilde{e}) = \check{E}$.

Finally, since $\partial''_{\tilde{D}_2}\partial'_{D_1}(\tilde{e}) = \check{E} = \partial''_{\check{D}_2}\partial'_{D_1}(\check{e})$, we know from Lemma 2.3.8 that $(\tilde{e}, D_1, \tilde{D}_2)$ can be transformed into $(\check{e}, D_1, \check{D}_2)$ by swaps. $\square$

As a special case of Lemma 2.3.11, any generification $(\hat{e}, D_1, D_2)$ of e can be transformed into the normal form of e by swaps and Rauzy jumps.

Now we are ready to define the face operators $\widetilde{\partial}'_i$ and $\widetilde{\partial}''_j$ of $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$, which reduces to defining $\epsilon'_i(e)$ and $\epsilon''_j(e)$ as in (2.3.1) and (2.3.2).

First we consider the case of a vertical face operator. Assume that e is a cell in $\widetilde{\mathbb{Q}}_{p,q}(h,m)$. We only need to consider the case when $\partial'_i(e)$ is not degenerate, since otherwise $\widetilde{\partial}'_i(e) = 0$ by definition. Denote $\partial'_i(e)$ by $\bar{e}$ for simplicity of notation. There are three steps to determine $\epsilon'_i(e)$.

Algorithm 2.3.12.

1) Let $(\mathrm{NF}(e), D_1^e, D_2^e)$ *be the normal form of* e*. Let* $\bar{i}$ *be the index which satisfies*

$$\bar{i} = i + \#\{j \in D_1 \mid j < \bar{i}\}.$$

Define $\bar{D}_1 := D_1 \cup \{\bar{i}\}$*, then* $(\mathrm{NF}(e), \bar{D}_1, D_2)$ *is a generification of* $\bar{e}$*.*

2) $(\mathrm{NF}(e), \bar{D}_1, D_2)$ *can be transformed into the normal form* $(\mathrm{NF}(\bar{e}), D_1^{\bar{e}}, D_2^{\bar{e}})$ *of* $\bar{e}$ *by swaps and Rauzy jumps. This gives rise to a sequence of generifications* $\{(e^i, D_1^i, D_2^i)\}_{i=0}^n$ *of* $\bar{e}$ *such that* $(e^0, D_1^0, D_2^0) = (\mathrm{NF}(e), \bar{D}_1, \bar{D}_2)$*,* $(e^n, D_1^n, D_2^n) = (\mathrm{NF}(\bar{e}), D_1^{\bar{e}}, D_2^{\bar{e}})$ *and* $(e^{i+1}, D_1^{i+1}, D_2^{i+1})$ *is obtained from* (e^i, D_1^i, D_2^i) *by a swap or a Rauzy jump for* $0 \leq i \leq n-1$*.*

3) For $0 \leq i \leq n-1$*: Denote* $e^i = (\tau_h, \ldots, \tau_1)$*.*

- *If* $(e^{i+1}, D_1^{i+1}, D_2^{i+1})$ *is obtained from* (e^i, D_1^i, D_2^i) *by a swap, then define* $\epsilon(i) := -1$*.*
- *If* $(e^{i+1}, D_1^{i+1}, D_2^{i+1})$ *is obtained from* (e^i, D_1^i, D_2^i) *by a Rauzy jump, assume at* j*, then define* $\epsilon(i) := (-1)^{s+t+1}$*, where* s, t *are determined as follows: Let* k, l *be the unique indices satisfying* $\tau_k(j) \neq j$*,* $\tau_l(j+1) \neq j+1$*, then define*

$$(s,t) := \begin{cases} (j, \tau_l(j+1)) & \text{if } \ l < k,\ \tau_l(j+1) < j \\ (j, \tau_l(j+1) - 1) & \text{if } \ l < k,\ \tau_l(j+1) > j+1 \\ (j, \tau_k(j)) & \text{if } \ l > k,\ \tau_k(j) < j \\ (j, \tau_k(j) - 1) & \text{if } \ l > k,\ \tau_k(j) > j+1. \end{cases}$$

Finally $\epsilon_i'(e) := \prod_{i=0}^{n-1} \epsilon(i)$*.*

In the case of a horizontal face operator, the algorithm is very similar. Assume that e is a cell in $\widetilde{\mathbb{Q}}_{p,q}(h,m)$ and that $\partial_j''(e)$ is not degenerate. Denote $\partial_j''(e)$ by $\bar{e}$. The definition of $\epsilon_j''(e)$ follows the same three steps as that of $\epsilon_i'(e)$, only with some modification in the first step:

1) Let $(\mathrm{NF}(e), D_1^e, D_2^e)$ be the normal form of e. Let $\bar{j}$ be the index which satisfies

$$\bar{j} = j + \#\{i \in D_2 \mid i < \bar{j}\}.$$

Define $\bar{D}_2 := D_2 \cup \{\bar{j}\}$, then $(\mathrm{NF}(e), D_1, \bar{D}_2)$ is a generification of $\bar{e}$.

Steps 2) and 3) remain the same as in Algorithm 2.3.12; one only needs to adapt the notations when necessary.

In the following we will explain why the face operators $\widetilde{\partial}_i'$, $\widetilde{\partial}_j''$ are well defined in this way. There are two points to show:

(a) The definition of $\epsilon_i'(e)$ and $\epsilon_j''(e)$ does not depend on the specific swaps and Rauzy jumps chosen in the second step.

(b) The simplicial identities hold:

$$\begin{aligned} \widetilde{\partial}'_i\widetilde{\partial}'_j &= \widetilde{\partial}'_{j-1}\widetilde{\partial}'_i && \text{if} \quad i<j \\ \widetilde{\partial}''_i\widetilde{\partial}''_j &= \widetilde{\partial}''_{j-1}\widetilde{\partial}''_i && \text{if} \quad i<j \\ \widetilde{\partial}'_i\widetilde{\partial}''_j &= \widetilde{\partial}''_j\widetilde{\partial}'_i && \forall \quad i,\ j \end{aligned}$$

The validity of these two points is due to the same reason. Before going to that, we want to characterize $\epsilon'_i(e)$ and $\epsilon''_j(e)$ in another perspective.

Suppose $S=\{(e^i,D_1^i,D_2^i)\}_{i=0}^{n(S)}$ is a sequence of generifications of a cell $e\in\widetilde{\mathbb{Q}}_{\bullet\bullet}$ such that $(e^{i+1},D_1^{i+1},D_2^{i+1})$ is obtained from (e^i,D_1^i,D_2^i) by a swap or a Rauzy jump for $0\le i\le n(S)-1$. For each i, if $(e^{i+1},D_1^{i+1},D_2^{i+1})$ is obtained from (e^i,D_1^i,D_2^i) by a swap at i_0, the we have $\partial'_{i_0}(e^i)=\partial'_{i_0}(e^{i+1})$; if $(e^{i+1},D_1^{i+1},D_2^{i+1})$ is obtained from (e^i,D_1^i,D_2^i) by a Rauzy jump at j, we have $\partial''_s(e^i)=\partial''_t(e^{i+1})$, where s and t are determined as in step *3)*. In both cases, the cells e^i and e^{i+1} has a common face of dimension $3h-1$, which we denote by $e^{i,i+1}$.

Denote the barycenter of a cell $\varrho\in\widetilde{\mathbb{Q}}_{\bullet\bullet}$ by $c(\varrho)$. Let γ_i be the segment (considered as a path) from $c(e^i)$ to $c(e^{i+1})$ which crosses the common face $e^{i,i+1}$ for each $0\le i\le n(S)-1$, then $\gamma:=\gamma_{n-1}\dots\gamma_2\gamma_1$ is a path from $c(e^0)$ to $c(e^{n(S)})$. Furthermore, for each i define

$$\epsilon(i):=\begin{cases}(-1)^{i_0+i_0+1}=-1 & \text{if} \quad \partial'_{i_0}(e^i)=e^{i,i+1}=\partial'_{i_0}(e^{i+1})\\ (-1)^{s+t+1} & \text{if} \quad \partial''_s(e^i)=e^{i,i+1}=\partial''_t(e^{i+1})\end{cases}$$

and finally define $\epsilon(S):=\prod_{i=1}^{n(S)-1}\epsilon(i)$.

Since the path γ lies in the manifold $\mathfrak{Par}(h,m)$, we consider the two-sheeted covering of $\mathfrak{Par}(h,m)$, which is defined as in Page 234 of [H]:

$$\widetilde{\mathfrak{Par}}(h,m):=\{\mu_x\mid x\in\mathfrak{Par}(h,m) \text{ and } x \text{ is a local orientation of } \mathfrak{Par}(h,m) \text{ at } x\}.$$

For each $0\le i\le n$, let $[e^i]\in H_{3h}(\Delta^p\times\Delta^q,\partial(\Delta^p\times\Delta^q))$ be the orientation of the simplex $\Delta^p\times\Delta^q$ representing e^i which is determined by the ordering of the vertices, $[e^i]$ is then a local orientation of $\mathfrak{Par}(h,m)$ at $c(e^i)$. The path γ has a unique lift $\widetilde{\gamma}$ in $\widetilde{\mathfrak{Par}}(h,m)$ with fixed starting point $\widetilde{\gamma}(0)=[e^0]$, and it follows from the definition of $\epsilon(S)$ that $\widetilde{\gamma}(1)=\epsilon(S)[e^n]$.

By definition $\epsilon'_i(e)$ (resp. $\epsilon''_i(e)$) is the same as $\epsilon(S)$, where S is a sequence of generifications of the cell $\partial'_i(e)$ (resp. $\partial''_j(e)$) chosen in Algorithm 2.3.12. Thus we have given a characterization of $\epsilon'_i(e)$ and $\epsilon''_i(e)$ from the geometric perspective, and this also explains why $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$ is the double complex of $(\mathrm{Par}(h,m),\mathrm{Par}'(h,m))$ under the orientation system $\mathcal{O}$.

Using the geometric meaning of $\epsilon(S)$, we will show that $\epsilon(S)$ defined above depends only on the initial generification (e^0,D_1^0,D_2^0) and the terminal generification $(e^{n(S)},D_1^{n(S)},D_2^{n(S)})$ of the sequence S. For each $0\le i\le n(S)$, let ω_i be the segment (considered as a path) from $c(e^i)$ to $c(\partial''_{D_2^i}\partial'_{D_1^i}(e^i))$ in the simplex $\Delta^p\times\Delta^q$ representing e^i. Since e^i,e^{i+1} and $e^{i,i+1}$ all have $\partial''_{D_2^i}\partial'_{D_1^i}(e^i)$ as a face, the path $\omega_{i+1}\gamma_i\omega_i^{-1}$ is null-homotopic. Therefore the path $\omega_{n(S)}\gamma_i\omega_0^{-1}$ as a composition of null-homotopic paths is also null-homotopic. Suppose S' is another sequence of generifications with the same initial and terminal triples as S, we get another path γ' from $c(e^0)$ to $c(e^{n(S)})$ in the same manner as γ. Moreover, $\omega_{n(S)}\gamma_i'\omega_0^{-1}$ is null-homotopic due to the same reasoning as for $\omega_n\gamma_i\omega_0^{-1}$. Therefore γ' is homotopic to γ'. Let $\widetilde{\gamma}'$ be the lift of γ' in $\widetilde{\mathfrak{Par}}(h,m)$ with $\widetilde{\gamma}'(0) = [e^0] = \widetilde{\gamma}(0)$. It follows from the homotopy lifting property that $\widetilde{\gamma}'(1) = \widetilde{\gamma}(1)$ and hence $\epsilon(S') = \epsilon(S)$.

Now we come to the validation of (a) and (b). (a) follows from the fact that any choice of S for $\epsilon_i'(e)$ (resp. $\epsilon_i''(e)$) in Algorithm 2.3.12 has the same initial triple $(\mathrm{NF}(e), \bar{D}_1, \bar{D}_2)$ and the same terminal triple $(\mathrm{NF}(\bar{e}), D_1^{\bar{e}}, D_2^{\bar{e}})$.

Since the simplicial identities hold for the face operators ∂_i' and ∂_j'', (b) is equivalent to

$$\epsilon_i'(\partial_j'(e))\epsilon_j'(e) = \epsilon_{j-1}'(\partial_i'(e))\epsilon_i'(e) \qquad \text{if} \quad i<j \tag{2.3.4}$$
$$\epsilon_i''(\partial_j''(e))\epsilon_j''(e) = \epsilon_{j-1}''(\partial_i''(e))\epsilon_i''(e) \qquad \text{if} \quad i<j \tag{2.3.5}$$
$$\epsilon_i'(\partial_j''(e))\epsilon_j''(e) = \epsilon_j''(\partial_i'(e))\epsilon_i'(e) \qquad \forall \quad i,j \tag{2.3.6}$$

provided that the cells involved are not degenerate. We will show that (2.3.4) is true. (2.3.5) and (2.3.6) can be verified by the same argument with minor modifications.

Assume that $e \in \widetilde{\mathbb{Q}}_{p,q}(h,m)$, denote $\partial_j'(e)$ by $\bar{e}$ and $\partial_i'\partial_j'(e)$ by ϱ. Suppose $S = \{(e^k, D_1^k, D_2^k)\}_{k=0}^{n(S)}$ is a sequence of generifications of $\bar{e}$ chosen in Algorithm 2.3.12 to compute $\epsilon_j'(e)$ and $\bar{S} = \{(\bar{e}^k, \bar{D}_1^k, \bar{D}_2^k)\}_{k=0}^{n(\bar{S})}$ is a sequence of generifications of ϱ chosen in Algorithm 2.3.12 to compute $\epsilon_i'(\bar{e})$. We can construct a sequence of generifications $\tilde{S} = \{(e^k, \tilde{D}_1^k, D_2^k)\}_{k=0}^{n(S)}$ of ϱ out of S, where $\tilde{D}_1^k$ is defined as follows: Let $\tilde{i}$ be the index which satisfies

$$\tilde{i} = i + \#\{j \in D_1^k \mid j < \tilde{i}\},$$

then $\tilde{D}_1^k := D_1^k \cup \{\tilde{i}\}$, In particular, the terminal triple of $\tilde{S}$ is the same as the initial triple of $\bar{S}$, hence $\tilde{S}$ and S can be joined together at this position to form a sequence of generifications T of ϱ. It follows from definition that $\epsilon(T) = \epsilon(\bar{S})\epsilon(\tilde{S}) = \epsilon_i'(\partial_j'(e))\epsilon_j'(e)$.

The above procedure applying to $\partial_i'(e)$ (instead of $\partial_j'(e)$) and $\partial_{j-1}\partial_i'(e) = \varrho$ will produce another sequence of generifications T' of ϱ with $\epsilon(T') = \epsilon_{j-1}'(\partial_i'(e))\epsilon_i'(e)$. Direct checking shows that T and T' have the same initial and terminal triples, hence $\epsilon(T') = \epsilon(T)$ and (2.3.4) holds.

Now we will study the homology of the double complex $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$. The methods and results are very similar with that in [V], and hence are also very similar to the contents in the first chapter of this thesis. Therefore we will not provide every detail here.

For a fixed p, consider the complex $\widetilde{\mathbb{Q}}_{p,*}(h,m)$. As a $\mathbb{Z}$-module, $\widetilde{\mathbb{Q}}_{p,q}(h,m)$ is the same as $\mathbb{Q}_{p,q}(h,m)$, a direct summand of $\mathcal{N}_q(\mathfrak{S}_p)$, hence $\widetilde{\mathbb{Q}}_{p,*}(h,m)$ inherits a partition-type filtration from $\mathcal{N}_*(\mathfrak{S}_p)$, which is defined by

$$\mathcal{P}_i\widetilde{\mathbb{Q}}_{p,*}(h,m) = \bigoplus_{j=1}^{i} \widetilde{\mathbb{Q}}_{p,*}(h,m)[L_j],$$

where $\widetilde{\mathbb{Q}}_{p,*}(h,m)[L]$ denote the submodule of $\widetilde{\mathbb{Q}}_{p,*}(h,m)$ generated by all tuples Σ with the partition type $\mathrm{pt}(\Sigma) = L$.

Since the face operators of $\widetilde{\mathbb{Q}}_{p,*}(h,m)$ differ from that of $\mathcal{N}_*(\mathfrak{S}_p)$ only by a sign, Lemma 1.2.12 is still applicable here; and in particular, the partition-type filtration of $\widetilde{\mathbb{Q}}_{p,*}(h,m)$ is well defined.

Define the map $\eta_i = \eta_i^q : \widetilde{\mathbb{Q}}_{p,q}(h,m) \longrightarrow \widetilde{\mathbb{Q}}_{p,q+1}(h,m)$ to be the restriction of the map $\eta_i = \eta_i^q : \mathcal{N}_q(\mathfrak{S}_p) \longrightarrow \mathcal{N}_{q+1}(\mathfrak{S}_p)$ on $\widetilde{\mathbb{Q}}_{p,q}(h,m)$. Furthermore, define

$$\widetilde{f}_i = \eta_i \widetilde{\partial}'_i : \widetilde{\mathbb{Q}}_{p,q}(h,m)[l_q,\ldots,l_{i+1},l_i,\ldots,l_1] \to \widetilde{\mathbb{Q}}_{p,q}(h,m)[l_q,\ldots,l_{i+1}+l_i-1,1,\ldots,l_1]$$

for $1 \le i \le q-1$.

We have the following Lemma concerning $\widetilde{f}_i$, which has the same structure as Lemma 1.2.13.

Lemma 2.3.13.

(1) $\widetilde{\partial}'_i \widetilde{f}_i = \widetilde{\partial}'_i$ (2.3.7a)

(2) $\widetilde{\partial}'_j \widetilde{f}_i = \widetilde{f}_i \widetilde{\partial}'_{j-1}$ *for* $i+2 \le j \le q$ (2.3.7b)

(3) $\widetilde{\partial}'_j \widetilde{f}_i = \widetilde{f}_{i-1} \widetilde{\partial}'_j$ *for* $1 \le j \le i-2$ (2.3.7c)

(4) $\widetilde{\partial}'_{i+1} \widetilde{f}_i \widetilde{f}_{i+1} = \widetilde{f}_i \widetilde{\partial}'_i$ (2.3.7d)

Proof. First we want to prove the following equalities concerning the map η_i:

$$\widetilde{\partial}'_i \eta_i = \mathrm{id} \tag{2.3.8a}$$

$$\widetilde{\partial}'_j \eta_i = \eta_i \widetilde{\partial}'_{j-1} \text{ for } i+2 \le j \le q \tag{2.3.8b}$$

$$\widetilde{\partial}'_j \eta_i = \eta_{i-1} \widetilde{\partial}'_j \text{ for } 1 \le j \le i-2 \tag{2.3.8c}$$

$$\widetilde{\partial}'_{i+1} \eta_i \widetilde{\partial}'_i \eta_{i+1} = \eta_i \widetilde{\partial}'_i \tag{2.3.8d}$$

Because of the equalities (1.2.3a) - (1.2.3d) in the proof of Lemma 1.2.13, we only need to show the following equalities, provided that the cells involved are non-degenerate.

$$\epsilon'_i(\eta_i(e)) = 1 \tag{2.3.9a}$$

$$\epsilon'_j(\eta_i(e)) = \epsilon'_{j-1}(e) \text{ for } i+2 \le j \le q \tag{2.3.9b}$$

$$\epsilon'_j(\eta_i(e)) = \epsilon'_j(e) \text{ for } 1 \le j \le i-2 \tag{2.3.9c}$$

$$\epsilon'_{i+1}(\eta_i \partial'_i \eta_{i+1}(e)) \epsilon'_i(\eta_{i+1}(e)) = \epsilon'_i(e) \tag{2.3.9d}$$

In the following proof, we always assume that $e \in \widetilde{\mathbb{Q}}_{p,q}(h,m)$. To (2.3.9a): Assume the normal form of $\eta_i(e)$ is $(\mathrm{NF}(\eta_i(e)), D_1, D_2)$. Let $\bar{D}_1$ be the set constructed from D_1 and the index i as in step 1) of Algorithm 2.3.12. Then it follows from definition that $(\mathrm{NF}(\eta_i(e)), \bar{D}_1, D_2)$ is the normal form of $\widetilde{\partial}'_i(\eta_i(e))$. Thus the sequence S in step 2) of Algorithm 2.3.12 can be chosen to have only one element $(\mathrm{NF}(\eta_i(e)), \bar{D}_1, D_2)$, hence $\epsilon'_i(\eta_i(e)) = 1$.

To (2.3.9b): Let $S = \{e^i, D_1^k, D_2^k\}_{i=0}^{n(S)}$ be a sequence chosen in Algorithm 2.3.12 for computing $\epsilon'_j(\eta_i(e))$. We can construct out of S a sequence of generifications

$\tilde{S} = \{(e^k, \tilde{D}_1^k, D_2^k)\}_{k=0}^{n(S)}$ of $\widetilde{\partial}'_{j-1}(e)$, where $\tilde{D}_1^k$ is defined as follows: Let $\tilde{i}$ be the index which satisfies

$$\tilde{i} = i + \#\{j \in D_1^k \mid j < \tilde{i}\},$$

then $\tilde{D}_1^k := D_1^k \cup \{\tilde{i}\}$. By definition of normal forms, a direct checking shows that the initial and terminal triples of $\tilde{S}$ satisfy the conditions in Algorithm 2.3.12 for computing $\epsilon'_{j-1}(e)$. Hence it follows from $\epsilon(S) = \epsilon(\tilde{S})$ that $\epsilon'_j(\eta_i(e)) = \epsilon'_{j-1}(e)$.

To (2.3.9c): The proof is similar to that of (2.3.9b).

To (2.3.9d): Since $\partial'_i\partial'_i\eta_{i+1}(e) = \partial'_i\partial'_{i+1}\eta_{i+1}(e)$, we have

$$\epsilon'_i(\partial'_i\eta_{i+1}(e))\epsilon'_i(\eta_{i+1}(e)) = \epsilon'_i(\partial'_{i+1}\eta_{i+1}(e))\epsilon'_{i+1}(\eta_{i+1}(e)).$$

The right-hand side of this equality can be simplified to $\epsilon'_i(e)$ since $\partial'_{i+1}\eta_{i+1}(e) = e$ and $\epsilon'_{i+1}(\eta_{i+1}(e)) = 1$. Moreover, an argument like in the proof of (2.3.9b) shows that $\epsilon'_i(\partial'_i\eta_{i+1}(e)) = \epsilon'_{i+1}(\eta_i\partial'_i\eta_{i+1}(e))$. Therefore (2.3.9d) holds.

The equalities (2.3.8a) - (2.3.8d) together with the simplicial equalities $\widetilde{\partial}'_i\widetilde{\partial}'_j = \widetilde{\partial}'_{j-1}\widetilde{\partial}'_i$ for $i < j$ give rise to the equalities (2.3.7a) - (2.3.7d). □

Now we introduce the map

$$\widetilde{F}_r : \widetilde{\mathbb{Q}}_{p,q}(h,m)[l_q, \ldots, l_r, 1, \ldots, 1] \to \widetilde{\mathbb{Q}}_{p,q+1}(h,m)[l_q, \ldots, l_r - 1, 1, \ldots, 1]$$

defined as the composition

$$(-1)^r \left(\mathrm{id} - \widetilde{f}_{r-1} + \cdots + (-1)^{r-i}\widetilde{f}_i\widetilde{f}_{i+1}\cdots\widetilde{f}_{r-1} + \cdots + (-1)^{r-1}\widetilde{f}_1\widetilde{f}_2\cdots\widetilde{f}_{r-1}\right)\eta_r$$

and the signed sum of the first r face operators

$$\widetilde{\partial}'_{(r)} = \sum_{i=1}^{r}(-1)^i\widetilde{\partial}'_i.$$

Applying the methods in Section 1.2, we have the following results.

Lemma 2.3.14.

(i) $\partial'_{(r)}\widetilde{F}_r = \mathrm{id} - \widetilde{F}_{r-1}\partial'_{(r-1)}$

(ii) $\partial'_j\widetilde{F}_r = \widetilde{F}_r\partial'_{j-1}$ *for* $r + 2 \leq j \leq q$

Proof. With adaption of notations, the proof is the same as that of Lemma 1.2.14. □

Theorem 2.3.15. *The homology of the complex $\widetilde{\mathbb{Q}}_{p,*}(h,m)$ is concentrated in the top degree h:*

$$H_q(\widetilde{\mathbb{Q}}_{p,*}(h,m)) = 0, \qquad \textit{if} \quad q < h.$$

Proof. With adaption of notations, the proof is the same as that of Theorem 1.2.11. □

Furthermore, like in Section 1.3, we can construct a homomorphism

$$\widetilde{\kappa} : V_h(\mathfrak{S}_p) \cap \widetilde{\mathbb{Q}}_{p,h}(h,m) \to H_h(\widetilde{\mathbb{Q}}_{p,*}(h,m))$$

by

$$\begin{aligned} \widetilde{\kappa}(\Sigma) := (\mathrm{id} - (-1)^{h-1}\widetilde{F}_{h-1} \circ \widetilde{\partial}'_{h-1}) \, \ldots \\ \ldots \, (\mathrm{id} - (-1)^{i}\widetilde{F}_{i} \circ \widetilde{\partial}'_{i}) \, \ldots \, (\mathrm{id} - \widetilde{F}_{2} \circ \widetilde{\partial}'_{2})(\mathrm{id} + \widetilde{F}_{1} \circ \widetilde{\partial}'_{1})(\Sigma) \end{aligned}$$

and have the following

Proposition 2.3.16. *$\widetilde{\kappa}$ is an isomorphism.*

Proof. Considering tuples in $\widetilde{\mathbb{Q}}_{p,*}(h,m)$ instead of in $\mathcal{N}_*(\mathfrak{S}_p)$, the proof follows the same way as that of Theorem 1.3.3. □

Let $\{E^r_{p,q}, d_r\}_r$ be the spectral sequence associated to the double complex $\widetilde{\mathbb{Q}}_{p,q}(h,m)$. Then $\{E^0_{p,q} = \widetilde{\mathbb{Q}}_{p,q}(h,m)\}$ and $d_0 = \widetilde{\partial}' : \widetilde{\mathbb{Q}}_{p,q}(h,m) \to \widetilde{\mathbb{Q}}_{p,q-1}(h,m)$. Theorem 2.3.15 says that, the vertical homology $E^1_{p,q} = H_q(\widetilde{\mathbb{Q}}_{p,*}(h,m), \widetilde{\partial}')$ is concentrated in the top degree $q = h$. Thus the E^1-term is a chain complex with differential $d_1 : E^1_{p,q} \to E^1_{p-1,q}$ induced by $\widetilde{\partial}''$ and the spectral sequence collapses with $E^2 = E^\infty$.

We can define a complex $(\widetilde{W}_*(h,m), \widetilde{d})$ in the same manner as the $(W_*(h,m), d)$ by letting

$$\widetilde{W}_p(h,m) := \widetilde{\kappa}^{-1}(H_h(\widetilde{\mathbb{Q}}_{p,*}(h,m))) \subseteq V_h(\mathfrak{S}_p)$$

and $\widetilde{d} := \widetilde{\kappa}^{-1} \circ \widetilde{\partial}'' \circ \widetilde{\kappa}$. This complex is equivalent to the E^1-term of the spectral sequence of $\widetilde{\mathbb{Q}}_{p,q}(h,m)$ and will be used to compute the homology groups of $\mathfrak{M}^m_{g,1}$.

From the definition of κ and $\widetilde{\kappa}$, it is not difficult to see that $\widetilde{W}_p(h,m)$ and $W_p(h,m)$ are generated by the same subset of tuples in $R_h(\mathfrak{S}_p)[h]$, hence they are the same $\mathbb{Z}$-modules.

Chapter 3

Application to Moduli Spaces of Kleinian Surfaces

In this chapter we introduce the category of pairings as a new example of factorable categories and discuss the application of it to the homology computation of moduli spaces of Kleinian surfaces. Since an orientation system is needed for the computation of the integral homology groups of these moduli spaces, the factorability of the category of pairings is not directly applicable in this case. However, the methods developed for the theory of factorable categories apply to this case.

3.1 Moduli Space of Kleinian Surfaces

Let $NF^m_{g,1}$ denote a Kleinian surface of genus $g \geq 0$ with one boundary curve and $m \geq 0$ permutable punctures and $\mathfrak{N}^m_{g,1}$ denote the moduli space of dianalytic equivalence classes of such surfaces. Here we note that the genus of a non-orientable surface is g, if it is a connected sum of $g+1$ real projective planes. Let $N\Gamma^m_{g,1}$ be the mapping class group of $NF^m_{g,1}$, i.e., the isotropy classes of orientation-preserving diffeomorphism fixing the boundary point-wise and permuting the punctures. $\mathfrak{N}^m_{g,1}$ is the classifying space of $N\Gamma^m_{g,1}$, because $N\Gamma^m_{g,1}$ acts freely on the contractible Teichmüller space $\mathfrak{NTeich}^m_{g,1}$ and $\mathfrak{N}^m_{g,1} = \mathfrak{NTeich}^m_{g,1}/N\Gamma^m_{g,1}$.

Using the Hilbert uniformization method, Bödigheimer found a finite cell complex $\mathrm{NPar}(h,m)$ with a subcomplex $\mathrm{NPar}'(h,m)$ such that $\mathrm{NPar}(h,m) \smallsetminus \mathrm{NPar}'(h,m)$ is an open manifold of dimension $3h$ and homotopy equivalent to $\mathfrak{N}^m_{g,1}$. Here the parameter h is determined as $h = g+m+1$. $\mathrm{NPar}(h,m)$ is the space of parallel slit domains and $\mathrm{NPar}'(h,m)$ is the subspace of $\mathrm{NPar}(h,m)$ consisting of degenerate slit configurations.

As with the case of moduli spaces of Riemann surfaces, we recall some main facts on parallel slit domains first. More detailed information can be found in [Z1] and [E]. There is also a more combinatorial description of $\mathrm{NPar}(h,m)$ using pairings, which will be given later.

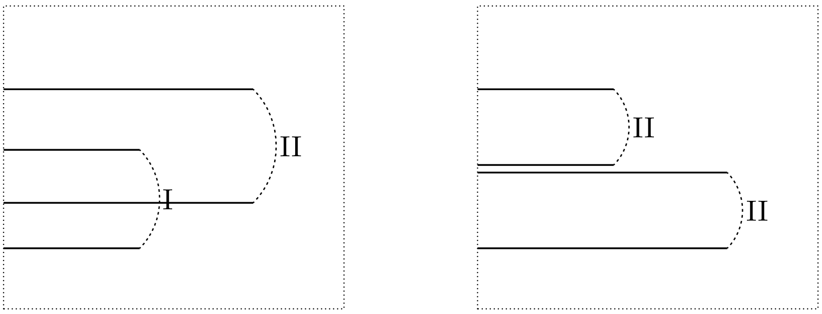
The definition of a *slit* remains the same as in Definition 2.1.1, but a *slit configuration* will contain more information than in the case of Riemann surfaces.

Definition 3.1.1. A *slit configuration* $L = (L_1, \ldots, L_{2h}|\lambda|T)$ consists of the following data

(1) $2h$ slits $L_1, \ldots, L_{2h}$ with endpoints $z_1 = (x_1, y_1), \ldots, z_{2h} = (x_{2h}, y_{2h})$, such that $y_1 \leq \ldots \leq y_{2h}$ and $x_k = x_{\lambda(k)}$ for $k = 1, \ldots, 2h$,

(2) a pairing (fixed point free involution) $\lambda \in \mathfrak{S}_{2h}$,

(3) the type sequence $T = (t_1, \ldots, t_{2h})$, with $t_i = \mathrm{I}$ or II, $t_i = t_{\lambda(i)}$.



$L = (L_1, L_2, L_3, L_4 \mid \lambda \mid T)$
$\lambda = (1\ 3)(2\ 4)$
$T = (\mathrm{I}, \mathrm{II}, \mathrm{I}, \mathrm{II})$

$L = (L_1, L_2, L_3, L_4 \mid \lambda \mid T)$
$\lambda = (1\ 2)(3\ 4)$
$T = (\mathrm{II}, \mathrm{II}, \mathrm{II}, \mathrm{II})$

Figure 3.1: Examples of slit configurations

To each slit configuration L, one can associate a two dimensional space $F(L)$ as follow: First one cuts the complex plane $\mathbb{C}$ along each slit L_i, $i = 1, \ldots, 2h$; then for $i = 1, \ldots, 2h$, if $t_i = \mathrm{I}$, the upper (resp. lower) bank of the slit L_i is glued to the lower (resp. upper) bank of the slit $L_{\lambda(i)}$ and if $t_i = \mathrm{II}$, the upper (resp. lower) bank of L_i is glued to upper (resp. lower) bank of L_i.

Definition 3.1.2. A slit configuration L is called *non-degenerate*, if the associated space $F(L)$ is a smooth surface; otherwise L is called *degenerate*.

A slit pair L_i and $L_{\lambda(i)}$ is said to be of *type* I (resp. *type* II) if $t_i = \mathrm{I}$ (resp. $t_i = \mathrm{II}$). For a non-degenerate slit configuration L, the associated space $F(L)$ is non-orientable if and only if L contains a slit pair of type II. In this case, we call L *non-orientable*. We will only consider non-orientable slit configurations in this chapter.

$F(L)$ can be compactified by adding a point Q at infinity to the complex plane $\mathbb{C}$ and $m \geq 0$ points $P_1, \ldots, P_m$ at the end of the "tubes" created by the gluing. Each point P_i is called a *puncture* of $F(L)$. There exists a dianalytic structure on $F(L)$ such that $\mathbb{C} \setminus (\cup_{i=1}^{2h} L_i) \hookrightarrow F(L)$ is a dianalytic inclusion. Moreover, specifying a point with a tangent direction is equivalent to specifying a boundary curve, thus $F(L)$ is a Kleinian surface with one boundary curve (Q, X) and m punctures $P_1, \ldots, P_m$, where X corresponds to the horizontal direction on $\mathbb{C}$. By computing the Euler characteristic, the genus of $F(L)$ is determined to be $g = h - m - 1$.

Definition 3.1.3. The *puncture-number* of a non-degenerate slit configuration L is defined to be m, if the associated surface $F(L)$ has m punctures.

Remark 3.1.4. Assume the puncture-number of $L = (L_1, \dots, L_{2h}|\lambda)$ is m. Define a permutation σ on $4h$ indices $i^+, i^-, i = 1, \dots 2h$ by

$$\sigma(i^+) = \begin{cases} (\lambda(i+1))^+ & \text{if } t_{i+1} = \mathrm{I} \\ (\lambda(i+1))^- & \text{if } t_{i+1} = \mathrm{II} \end{cases}$$

$$\sigma(i^-) = \begin{cases} (\lambda(i-1))^- & \text{if } t_{i+1} = \mathrm{I} \\ (\lambda(i-1))^+ & \text{if } t_{i+1} = \mathrm{II} \end{cases}$$

Then the number of cycles of σ is $2(m+1)$. ◇

There is an equivalence relation between slit configurations which is defined by *Rauzy jumps*. Recall that for $a,\ b \in \{1, \dots, 2h\},\ a \le b$, $s_{a,b}$ denotes the cycle $(a\ a+1 \dots b)$ and its inverse $s_{a,b}^{-1}$ is $(b\ b-1 \dots a)$.

Definition 3.1.5. If a slit configuration $L = (L_1, \dots, L_{2h}|\lambda|T)$ contains a slit L_i, $i \in \{1, \dots, 2h\}$, such that $L_i \subseteq L_{i+1}$ or $L_i \subseteq L_{i-1}$. Define a permutation $\alpha \in \mathfrak{S}_{2h}$ in the following eight possible cases:

(1)	$\alpha := s_{i,\lambda(i+1)}^{-1}$	if $L_i \subseteq L_{i+1}$,	$\lambda(i+1) > i+1$,	$t_{i+1} = \mathrm{I}$.
(2)	$\alpha := s_{\lambda(i+1)+1,i}$	if $L_i \subseteq L_{i+1}$,	$\lambda(i+1) < i$,	$t_{i+1} = \mathrm{I}$.
(3)	$\alpha := s_{i,\lambda(i-1)-1}^{-1}$	if $L_i \subseteq L_{i-1}$,	$\lambda(i-1) > i$,	$t_{i-1} = \mathrm{I}$.
(4)	$\alpha := s_{\lambda(i-1),i}$	if $L_i \subseteq L_{i-1}$,	$\lambda(i-1) < i-1$,	$t_{i-1} = \mathrm{I}$.
(5)	$\alpha := s_{i,\lambda(i+1)-1}^{-1}$	if $L_i \subseteq L_{i+1}$,	$\lambda(i+1) > i+1$,	$t_{i+1} = \mathrm{II}$.
(6)	$\alpha := s_{\lambda(i+1),i}$	if $L_i \subseteq L_{i+1}$,	$\lambda(i+1) < i$,	$t_{i+1} = \mathrm{II}$.
(7)	$\alpha := s_{i,\lambda(i-1)}^{-1}$	if $L_i \subseteq L_{i-1}$,	$\lambda(i-1) > i$,	$t_{i-1} = \mathrm{II}$.
(8)	$\alpha := s_{\lambda(i-1)+1,i}$	if $L_i \subseteq L_{i-1}$,	$\lambda(i-1) < i-1$,	$t_{i-1} = \mathrm{II}$.

Define a new slit configuration $\tilde{L} = (\tilde{L}_1, \dots, \tilde{L}_{2h}|\tilde{\lambda}|\tilde{T})$ as follows:

$$\tilde{L}_{\alpha(j)} = L_j, j \in \{1, \dots, 2h\} \backslash \{i\}$$

and

$$\tilde{L}_{\alpha(i)} := \{(x,y) \in \mathbb{C} | x \le x_i, y = y_{\lambda(i+1)}\}$$

in the cases (1),(2),(5),(6) and

$$\tilde{L}_{\alpha(i)} := \{(x,y) \in \mathbb{C} | x \le x_i, y = y_{\lambda(i-1)}\}$$

in the cases (3),(4),(7),(8).

The pairing $\tilde{\lambda}$ is defined by

$$\tilde{\lambda} = \alpha \circ \lambda \circ \alpha^{-1}.$$

$\tilde{T} = (\tilde{t}_1, \dots, \tilde{t}_{2h})$ is defined as follows: In the cases (1)-(4), $\tilde{t}_{\alpha(j)} := t_j$ for $j \in \{1, \dots, 2h\}$. In the cases (5)-(8), $\tilde{t}_{\alpha(j)} := t_j$ if $j \in \{1, \dots, 2h\} \backslash \{\alpha(i), \alpha(\lambda(i))\}$ and $\tilde{t}_{\alpha(i)} := \mathrm{I}$, $\tilde{t}_{\alpha(\lambda(i))} := \mathrm{I}$ (resp. $\tilde{t}_{\alpha(i)} := \mathrm{II}$, $\tilde{t}_{\alpha(\lambda(i))} := \mathrm{II}$) if $t_i = \mathrm{II}$ (resp. $t_i = \mathrm{I}$).

We say that $\tilde{L}$ is obtained from L via a *Rauzy jump* of the slit L_i over the slit pair $L_{i+1}, L_{\lambda(i+1)}$ (in the cases (1),(2),(5),(6)) or $L_{i-1}, L_{\lambda(i-1)}$ (in the cases (3),(4),(7),(8)).

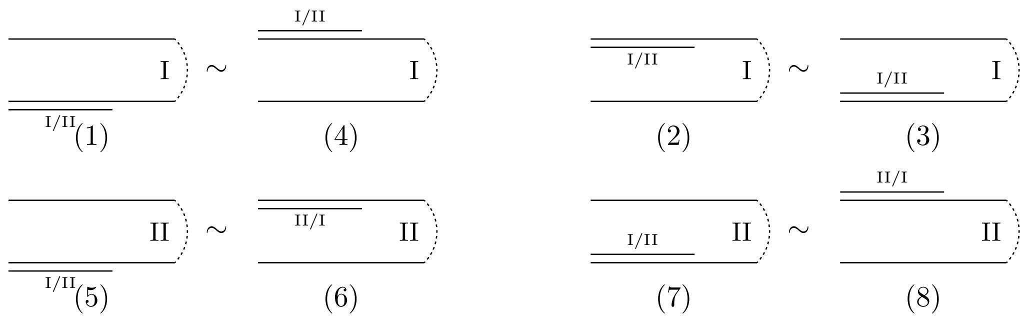


Figure 3.2: Rauzy jumps

Rauzy jumps generate an equivalence relation: two slit configurations are equivalent, if they can transform into each other by Rauzy jumps. We denote the equivalence class of a slit configuration L by $[L]$.

Remark 3.1.6. If two slit configurations L, L' are equivalent under Rauzy jumps and L is non-degenerate, then the associated surfaces $F(L)$ and $F(L')$ are dianalytically equivalent. $\diamond$

Let $L = (L_1, \dots, L_{2h}|\lambda)$ be a slit configuration. Assume that the slits of L lie at p-distinct y-levels

$$-\infty < v_1 < v_2 \dots < v_p < \infty, \tag{3.1.1}$$

and that a_i is the number of slits lying on level $y = v_i$. Then

$$0 < a_i < 2h, \quad \sum_{i=1}^{p} a_i = 2h, \quad 2 \leq p \leq 2h. \tag{3.1.2}$$

Further assume that the endpoints of L lie at q-distinct x-levels

$$-\infty < u_q < \dots < u_2 < u_1 < \infty, \tag{3.1.3}$$

Denote the set of indices of the slits over $x = u_j$ by $B_j \subseteq \{1, \dots, 2h\}$. Then

$$B_1 \cup B_2 \cup \dots \cup B_q = \{1, \dots, 2h\}, \quad 1 \leq q \leq h. \tag{3.1.4}$$

We summarize all these data into a *symbol* E

$$E := (a_1, \dots, a_p|\lambda|B_1, \dots, B_q|T).$$

Recall the strictly monotonically increasing homeomorphism f from $\mathbb{R}$ to $(0,1)$:

$$f : x \mapsto \frac{\arctan(x)}{\pi} + \frac{1}{2} =: \tilde{x}$$

Set $s_0 = 1 - \tilde{u}_1$, $s_i = \tilde{u}_i - \tilde{u}_{i+1}$ for $1 \leq i \leq q-1$ and $s_q = \tilde{u}_q$ for the x-coordinate and $t_0 = \tilde{v}_1, t_j = \tilde{b}_{j+1} - \tilde{b}_j$ for $1 \leq j \leq p-1$ for the y-coordinate. Then

$$\sum_{i=0}^{q} s_i = 1 = \sum_{j=0}^{p} t_i.$$

Therefore $(s_0, s_1, \ldots, s_q)$ and $(t_0, t_1, \ldots, t_p)$ are barycentric coordinates in the open bi-simplex $\mathring{\Delta}^q \times \mathring{\Delta}^p$. By varying the coordinates s_i and t_j in $\mathring{\Delta}^q$ and $\mathring{\Delta}^p$ respectively, we get all the slit configurations with the same symbol E.

The notions *degenerate*, *Rauzy jump*, *puncture-number* and *non-orientable* can also be defined for symbols. Denote the equivalence class of a symbol E under Rauzy jumps by $[E]$.

Let $\mathfrak{NPar}(h,m) := \{[L] = [L_1, \ldots, L_{2h}|\lambda|T] | L$ is non-degenerate, non-orientable and has puncture-number $m\}$. It turns out that $\mathfrak{NPar}(h,m)$ is a manifold of dimension $3h$ and there is a homotopy equivalence

$$\mathfrak{NPar}(h,m) \simeq \mathfrak{N}_{g,1}^m \text{ with } h = g + m + 1.$$

Let $\mathrm{NPar}(h,m)$ be the closure of $\mathfrak{NPar}(h,m)$ and define

$$\mathrm{NPar}'(h,m) := \mathrm{NPar}(h,m) \setminus \mathfrak{NPar}(h,m).$$

The relative manifold $(\mathrm{NPar}(h,m), \mathrm{NPar}'(h,m))$ is a bi-simplicial cell complex, thus the associated chain complex $\mathbb{NQ}_{\bullet\bullet}(h,m)$ is a double complex. Every (p,q)-cell of $\mathbb{NQ}_{\bullet\bullet}(h,m)$ is an equivalence class of symbols $[E] = [a_1, \ldots, a_p|\lambda|B_1, \ldots, B_q|T]$, where E is non-degenerate, non-orientable, has puncture-number m and satisfies (3.1.2), (3.1.4).

The i-th *vertical* face operator on E and hence the i-th *vertical* face operator on $[E]$ are given by

$$\partial_i'(E) := (a_1, \ldots, a_p|\lambda|B_0, \ldots, B_j \cup B_{j+1}, \ldots, B_q|T),\ i = 1, \ldots, q-1$$
$$\partial_i'([E]) := [\partial_i'(E)] = [a_1, \ldots, a_p|\lambda|B_0, \ldots, B_j \cup B_{j+1}, \ldots, B_q|T],\ i = 1, \ldots, q-1$$

and the j-th *horizontal* face operator on E and hence the j-th *horizontal* face operator on $[E]$ are

$$\partial_j''(E) := (a_1, \ldots, a_i + a_{i+1}, \ldots, a_p|\lambda|B_0, \ldots, B_q|T),\ j = 1, \ldots, p-1$$
$$\partial_j''([E]) := [\partial_j''(E)] = [a_1, \ldots, a_i + a_{i+1}, \ldots, a_p|\lambda|B_0, \ldots, B_q|T],\ j = 1, \ldots, p-1$$

in which a face is defined to be zero, if it is not a cell in $\mathbb{NQ}_{\bullet\bullet}(h,m)$. Let

$$\partial'([E]) := \sum_{i=1}^{q-1} (-1)^i \partial_i'([E])$$
$$\partial''([E]) := \sum_{j=1}^{p-1} (-1)^j \partial_j''([E]),$$

then the boundary operator ∂ of the complex $\mathbb{NQ}_{\bullet\bullet}(h,m)$ is given by

$$\partial([E]) := \partial'([E]) + (-1)^q \partial''([E]).$$

Description of $\mathbb{NQ}_{\bullet\bullet}(h,m)$ via pairings

Let

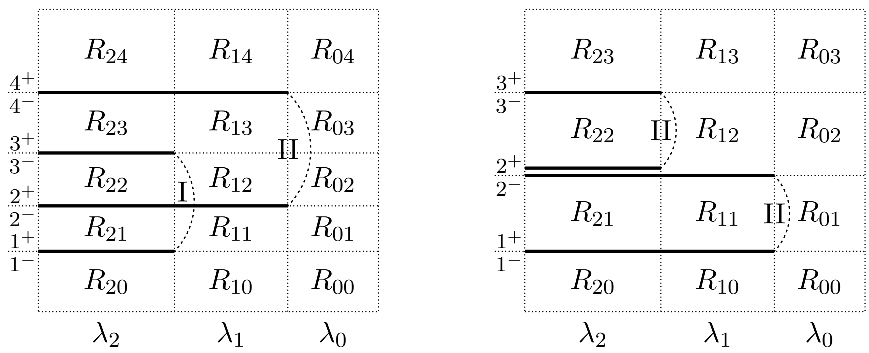
$$\begin{aligned}\mathbf{\Lambda}_p &= \{\text{pairings of } 2p \text{ points}\}\\ &\cong \Big\{\lambda \in \mathfrak{S}_{2p} = \mathfrak{S}(1^-,1^+,\ldots,p^-,p^+) \mid \lambda \text{ is fixed point free and } \lambda^2 = \mathrm{id}\Big\}\end{aligned}$$

$\mathbf{\Lambda}_p$ is a subset of $\mathfrak{S}_{2p}$ and forms a conjugation class. We give notations to two special elements in $\mathbf{\Lambda}_p$, namely

$$J = (1^+ \ \ 2^-)\cdots(k^+ \ \ (k+1)^-)\cdots(p^+ \ \ 1^-) \tag{3.1.5}$$

$$\lambda_0 = (1^- \ \ 1^+)\cdots(k^- \ \ k^+)\cdots(p^- \ \ p^+) \tag{3.1.6}$$

Every symbol $E = (a_1,\ldots,a_p|\lambda|B_1,\ldots,B_q|T)$ can be characterized by a *grid picture*, whose grid is given by the horizontals of the slits and the verticals at the slit ends. We number the columns $0,1,\ldots,q$ from right to left and rows $0,1,\ldots,p$ from bottom to top. Let $R_{i,j}$ denote the j-th rectangle in the i-th column.



$\lambda_0 = (1^- \ 1^+)(2^- \ 2^+)(3^- \ 3^+)(4^- \ 4^+)$

$\lambda_1 = (1^- \ 1^+)(2^- \ 4^-)(2^+ \ 4^+)(3^- \ 3^+)$

$\lambda_2 = (1^- \ 3^+)(1^+ \ 3^-)(2^- \ 4^-)(2^+ \ 4^+)$

$\lambda_0 = (1^- \ 1^+)(2^- \ 2^+)(3^- \ 3^+)$

$\lambda_1 = (1^- \ 2^-)(1^+ \ 2^+)(3^- \ 3^+)$

$\lambda_2 = (1^- \ 2^-)(1^+ \ 3^-)(2^+ \ 3^+)$

Figure 3.3: Grid pictures

For the i-th column, denote the upper (resp. lower) bank of the common edge of $R_{i-1,j}$ and $R_{i,j}$ by j^+ (resp. j^-), $j = 1,\ldots,p$. Let $\lambda_i \in \mathfrak{S}_{2p} = \mathfrak{S}(1^-,1^+,\ldots,p^-,p^+)$ be the pairing which describes the re-gluing of the slit plane in the i-th column: the banks $\star$ and $\lambda(\star)$ are glued together. Note that $\lambda_q \circ J \in S_{2p}$ has the same number of cycles as the permutation $\sigma \in \mathfrak{S}_{2h}$ defined in Remark 3.1, thus the puncture-number m of E is $\frac{1}{2}\operatorname{ncyc}(\lambda_q \circ J) - 1$. These observations lead to the following description of the double complex $\mathbb{NQ}_{\bullet\bullet}(h,m)$ via pairings.

Let $\mathbb{NP}_{p,q}(h)$ be the free abelian group generated by all $(q+1)$-tuples $\Lambda = (\lambda_q,\ldots,\lambda_0)$ with $\lambda_i \in \mathbf{\Lambda}_p$, such that

$$N_{\mathbf{\Lambda}_p}(\Lambda) = \frac{1}{2}(N_{\mathfrak{S}_{2p}}(\lambda_q\lambda_{q-1}^{-1}) + \ldots + N_{\mathfrak{S}_{2p}}(\lambda_1\lambda_0^{-1})) \le h,$$

where $N_{\mathfrak{S}_{2p}}(\alpha)$ is the word length norm of α with respect to the generating set of $\mathfrak{S}_{2p}$ which consists of all transpositions.

Define a double complex $\mathbb{NP}_{\bullet\bullet}(h) := \bigoplus \mathbb{NP}_{p,q}(h)$, $0 \le p \le 2h$, $q \le h$. The vertical and horizontal boundary operators on the double complex $\mathbb{NP}_{\bullet\bullet}(h)$ are:

$$\partial' = \sum_{i=0}^{q}(-1)^i \partial'_i \qquad \text{with} \qquad \partial'_i(\Lambda) = (\lambda_q, \dots, \widehat{\lambda}_i, \dots, \lambda_0)$$
$$\partial'' = \sum_{j=0}^{p}(-1)^j \partial''_j \qquad \text{with} \qquad \partial''_j(\Lambda) = (D_j(\lambda_q), \dots, D_j(\lambda_0)).$$

and $\partial = \partial' + (-1)^q \partial''$ is the boundary operator on $\mathbb{NP}_{\bullet\bullet}(h)$. Here $D_j : \mathbf{\Lambda}_p \to \mathbf{\Lambda}_{p-1}$ is defined as:

$$D_j(\lambda) = s_{(j+1)^-} \circ s_{j^+} \circ (\lambda(j^+)\ (j+1)^-) \circ (\lambda((j+1)^-)\ j^+) \circ \lambda \circ d_{(j+1)^-} \circ d_{j^+}$$

where $d_\star : [2p-1] \to [2p]$ is the simplicial degeneracy function which avoids the letter $\star$, and $s_\star : [2p] \to [2p-1]$ is the simplicial face function which repeats the letter $\star$. Note that here $[2p]$ and $[2p-1]$ denote the strings of ordinal indices $1^- \to 1^+ \to \dots \to p^- \to p^+$ and $1^- \to 1^+ \to \dots \to p^-$ respectively. In other words, to get the pairing $D_j(\lambda)$, one deletes the letters j^+, $(j+1)^-$ in λ; and if $\lambda(j^+) \neq (j+1)^-$, then $D_j(\lambda)$ maps $\lambda(j^+)$ and $\lambda((j+1)^-)$ to each other; finally one renormalizes the indices.

The subcomplex $\mathbb{NP}'_{\bullet\bullet}(h,m)$ of $\mathbb{NP}_{\bullet\bullet}(h)$ is generated by the *degenerate* cells of $\mathbb{NP}_{\bullet\bullet}(h)$, these are the cells which violate any of the following conditions:

1) $N_{\mathbf{\Lambda}_p}(\Lambda) = h$
2) $\mathrm{ncyc}(\lambda_q \circ J) = 2(m+1)$
3) λ_0 is as defined in (3.1.6)
4) $\lambda_{i+1} \neq \lambda_i$ for $i = 0, \dots q$.
5) There is no $1 \le k \le p$, such that $\lambda_i(k^-) = k^+$ for all $i = 0, \dots q$.
6) there exists $1 \le j,\ k \le p$, $0 \le i \le q$ such that $\lambda_i(j^+) = \lambda_i(k^+)$

Here the last condition guarantees that the slit configuration is non-orientable.

Finally $\mathbb{NQ}_{\bullet\bullet}(h,m) \cong \mathbb{NP}_{\bullet\bullet}(h)/\mathbb{NP}'_{\bullet\bullet}(h,m)$ is the desired double complex. In particular, the face operators ∂'_0, ∂'_q, ∂'_0 and ∂'_p are always zero in $\mathbb{NQ}_{\bullet\bullet}(h,m)$.

From above, a cell Λ in $\mathbb{NQ}_{\bullet\bullet}(h,m)$ is represented by a $(q+1)$-tuple $\Lambda = (\lambda_q, \dots, \lambda_0)$ with $\lambda_i \in \mathbf{\Lambda}_p$. This is called the *homogeneous notation.* The *inhomogeneous notation*, which we will define now, is more important for later use. Let $\tau_i := \lambda_i \lambda_{i-1}^{-1} = \lambda_i \lambda_{i-1}, i = 1, \dots, q$, then $\Lambda = (\tau_q | \dots | \tau_1)$ is called the *inhomogeneous notation.* It follows from definition that $\lambda_i = \tau_i \cdots \tau_1 \circ \lambda_0$. The conditions and boundary operators on Λ can then be translated into the language using τ_i.

The equivalence of the two ways to define $\mathbb{NQ}_{\bullet\bullet}(h,m)$-namely by equivalence classes of symbols and by tuples of pairings-was given in [E]. We give some remarks about the bijection here.

Remark 3.1.7.

1) Equivalent symbols E and E' in $\mathbb{NQ}_{p,q}(h,m)$ give rise to the same pairings $\lambda_i \in \Lambda_p$, $i = 0, 1, \ldots, q$, since they determine the same way of gluing.

2) Given a tuple $\Lambda = (\tau_q|\ldots|\tau_1) \in \mathbb{NQ}_{p,q}(h,m)$ in inhomogeneous notation, we can construct a slit configuration L as follows. Choose real numbers $x_1 > x_2 > \ldots > x_q$ and $y_1 < y_2 < \ldots < y_p$. Starting from $i = 1$, we do the following for $i = 1, \ldots, q$ in sequence.

   It will be proved in the next section that the cycles of τ_i appear in pairs. Choose one cycle from each pair to form a permutation $\zeta_i \in \mathfrak{S}_{2p}$. Write ζ_i into a product of $n_i = N_{\mathbf{\Lambda}_p}(\tau_i) = \frac{1}{2} N_{\mathfrak{S}_{2p}}(\tau_i)$ transpositions.

$$\zeta_i = (a_{n_i}\ b_{n_i}) \ldots (a_1\ b_1)$$

   where $a_j, b_j \in \{1^-, 1^+, \ldots, p^-, p^+\}$ for $1 \le j \le n_i$. Then for $j = 1, \ldots, n_i$ in sequence, we put two paired slits on the complex plain $\mathbb{C}$: One slit has endpoint (x_i, y_k), if $a_j = k^+$ (resp. $a_j = k^-$); and if slits with the same y-level y_k already exist, then put the new one above (resp. below) all the existing ones. The other slit has (x_i, y_l) as endpoint, if $b_j = l^+$ (resp. $b_j = l^-$); and if slits with the same y-level y_l already exist, then put the new one above (resp. below) all the existing ones.

   In the end, there will be $N_{\mathbf{\Lambda}_p}(\tau_q) + \cdots + N_{\mathbf{\Lambda}_p}(\tau_1) = h$ pairs of slits. We name the slits $L_1, \ldots, L_{2h}$ from bottom to top respectively. Let λ be the pairing which records how the slits are paired and $T = (t_1, \ldots, t_{2h})$ is defined as follows: For $1 \le i \le 2h$, suppose $(a\ b)$ is the transposition, with which the slit pair L_i and $L_{\lambda(i)}$ are added, then $t_i = t_{\lambda(i)} :=$ I if a and b have the same superscripts $+$ or $-$, $t_i = t_{\lambda(i)} :=$ II if the superscripts of a and b are different. Then $L = (L_1, \ldots, L_{2h}|\lambda|T)$ is a slit configuration, whose symbol E corresponds to $\Lambda = (\tau_q|\ldots|\tau_1)$ in the sense described before.

$\diamond$

3.2 The Category of Pairings

We have seen that, like symmetric groups in moduli spaces of Riemann surfaces, pairings play an important role in the cell structure of moduli spaces of Kleinian surfaces. Besides, the categories of pairings are also the main examples of factorable categories as we will explain in this section.

The category of pairings Λ_p has all the pairings of the set $I_{2p} = \{1, 2, \ldots, 2p\}$ as its set of objects; and for any two objects $\lambda_1, \lambda_2 \in \Lambda_p$, the set of morphisms

$$\mathrm{Mor}(\lambda_1, \lambda_2) := \{(\lambda_2\lambda_1^{-1}, \lambda_1)\} = \{(\lambda_2\lambda_1, \lambda_1)\}.$$

Since there is a unique morphism from λ_1 to λ_2, Λ_p is a groupoid. In the following we will denote the set of morphisms of the category Λ_p by $\mathscr{M}(\Lambda_p)$ and the set of composable n-tuples of morphisms by $\mathscr{M}(\Lambda_p)(n)$. Assume $((\tau_n, \lambda_n), \ldots, (\tau_1, \lambda_1))$

is an element in $\mathscr{M}(\Lambda_p)(n)$, then it follows from the definition of $\mathscr{M}(\Lambda_p)(n)$ that $\tau_i\lambda_i = \lambda_{i+1}$ for $i = 1, \ldots, n-1$. Thus without loss of information, we will denote $((\tau_n, \lambda_n), \ldots, (\tau_1, \lambda_1))$ simply by $(\tau_n, \cdots, \tau_1, \lambda_1)$.

The norm $N = N_{\mathbf{\Lambda}_p}$ on Λ_p is defined as follows: For any morphism $(\tau, \lambda) \in \mathscr{M}(\Lambda_p)$, i.e. $\tau\lambda$, $\lambda \in \Lambda_p$,

$$N((\tau, \lambda)) := N(\tau) := \frac{1}{2} N_{\mathfrak{S}_{2p}}(\tau). \tag{3.2.1}$$

This is to say, the norm of (τ, λ) in the category Λ_p is defined to be half of the norm of τ as an element in the symmetric group $\mathfrak{S}_{2p}$, where the norm on $\mathfrak{S}_{2p}$ is taken to be the word length norm with respects to its set of transpositions.

In the remaining part of this chapter, unless otherwise specified, the notation $N(\tau)$ is always meant to be as in (3.2.1), not $N_{\mathfrak{S}_{2p}}(\tau)$.

The set of morphisms in Λ_p with norm one is denoted as usual by $T(\mathscr{M}(\Lambda_p))$. We will see more clearly about how these morphisms look like, after the following Lemma.

Lemma 3.2.1. *Let (τ, λ) be a morphism in Λ_p, i.e. $\tau\lambda$, $\lambda \in \Lambda_p$. Assume that $(a_1, \ldots, a_n)$ is a cycle of τ, then $(\lambda_1(a_n), \ldots, \lambda_1(a_1))$ is also a cycle of τ which is different from $(a_1, \ldots, a_n)$.*

Proof. First we have $\tau(a) = b \Rightarrow \tau(\lambda(b)) = \lambda(a)$. It remains to prove that for any a, a and $\lambda_1(a)$ are not in the same cycle of τ. If this is not true, let $k \geq 0$ be the smallest non-negative integer, such that there exists an element a with $\tau^k(a) = \lambda(a)$. Then $k \neq 0$, since $a \neq \lambda(a)$. Moreover $k \neq 1$, since otherwise $k = 1 \Rightarrow \tau\lambda(\lambda(a)) = \tau\lambda^2(a) = \tau(a) = \lambda(a)$, a contradiction to that $\tau\lambda$ has no fixed points. Therefore $k \geq 2$. however, since $\tau\lambda\tau(a) = \lambda(a)$, we have $\tau^k(a) = \lambda(a) \Rightarrow \tau^k(a) = \tau\lambda\tau(a) \Rightarrow \tau^{k-1}(a) = \lambda\tau(a)$, i.e. $\tau^{k-2}(\tau(a)) = \lambda(\tau(a))$, which is a contradiction to the choice of k. ☐

A consequence of this Lemma is that for any morphism (τ, λ) in Λ_p, the cycles of τ appear "canonically" in pairs.

In particular, let $\lambda \in \Lambda_p$ be a pairing and $(i\ j) \in \mathfrak{S}_{2p}$ a transposition. Then if $\lambda(i) \neq j$, we have i, j, $\lambda(i)$, $\lambda(j)$ are pairwise different and $(i\ j)(\lambda(i)\,\lambda(j))\,\lambda$ is a pairing; moreover, $((i\ j)(\lambda(i)\,\lambda(j)), \lambda)$ is the unique norm-one morphism which starts from λ and has i, j in the same cycle (in fact, taking the norm-one condition and Lemma 3.2.1 into consideration, this cycle must be $(i\ j)$). If $\lambda(i) = j$, by Lemma 3.2.1, there exists no norm-one morphism staring from λ and with i, j in the same cycle. At this point we can also see that, if $(t, \lambda) \in T(\mathscr{M}(\Lambda_p))$, then $t \cdot t = \mathrm{id}$.

Recall that $H : \mathfrak{S}_{2p} \to I_{2p}$ is a function which assigns to a permutation σ the largest element of the set I_{2p} that is not a fixed point of σ: $H(\sigma) = \max\{j \in I_{2p} \mid \sigma(j) \neq j\}$. Define a function $\eta = \eta_{\mathbf{\Lambda}_p} : \mathscr{M}(\Lambda_p) \to \mathscr{M}(\Lambda_p)(2)$ as follows. For $(\tau, \lambda) \in \mathscr{M}(\Lambda_p)$, i.e., $\lambda, \tau\lambda \in \Lambda_p$, if $\tau \neq \mathrm{id}$, then

$$\eta((\tau, \lambda)) := (\bar{\eta}(\tau), \eta'(\tau), \lambda) \tag{3.2.2}$$

where the $\eta'(\tau)$ and $\bar{\eta}(\tau)$ are given by:

$$\begin{aligned} \eta'(\tau) &:= (\lambda(i)\ \ \lambda\tau^{-1}(i)) \cdot (i\ \ \tau^{-1}(i)) \\ \bar{\eta}(\tau) &:= \tau \cdot \eta'(\tau) \end{aligned}$$

where $i = H(\tau)$. Because $i, \tau^{-1}(i)$ are in the same cycle of τ, it follows from Lemma 3.2.1 that $\lambda(i) \neq \tau^{-1}(i)$. Hence as we have noted before $i, \tau^{-1}(i), \lambda(i), \lambda\tau^{-1}(i)$ are pairwise different and $((\lambda(i)\ \ \lambda\tau^{-1}(i)) \cdot (i\ \ \tau^{-1}(i)),\ \lambda)$ is the unique norm-one morphism from λ which has $i, \tau^{-1}(i)$ in the same cycle.

For $(\mathrm{id}, \lambda) \in \mathscr{M}(\Lambda_p)$, we define $\eta((\mathrm{id}, \lambda)) = (\mathrm{id}, \mathrm{id}, \lambda)$.

Note that (3.2.2) is equivalent to the more explicit form:

$$\eta'((\tau, \lambda)) := (\eta'(\tau), \lambda) \quad \text{and} \quad \bar{\eta}((\tau, \lambda)) := (\bar{\eta}(\tau), \eta'(\tau)\lambda)$$

Lemma 3.2.2. *η is a factorization map.*

Proof. We need to show that η satisfies the conditions (1), (2) and (3) in Definition 1.2.1. If $\tau = \mathrm{id}$, all the conditions are fulfilled. And that η satisfies (3) follows from the definition. If $\tau \neq \mathrm{id}$, let $i = H(\tau)$.

Because $i, \tau^{-1}(i), \lambda(i)$ and $\lambda\tau^{-1}(i)$ are pairwise different, $\eta'(\tau)\eta'(\tau) = \mathrm{id}$, hence $\bar{\eta}(\tau)\eta'(\tau) = \tau\eta'(\tau)\eta'(\tau) = \tau$, which implies $\bar{\eta}((\tau, \lambda)) \circ \eta'((\tau, \lambda)) = (\tau, \lambda)$, hence condition (1) holds.

It remains to show condition (2). Multiplying τ with the transposition $(i, \tau^{-1}(i))$ will affect only the cycle $c_i = (\ldots, \tau^{-1}(i), i, \tau(i), \ldots)$ of τ. More precisely, in $\tau(i, \tau^{-1}(i))$, the cycle c_i decomposes into the cycles $(\ldots, \tau^{-1}(i), i, \tau(i), \ldots)$ and (i), while all other cycles of τ remain the same. In the same way, multiplying τ with the transposition $(\lambda(i), \lambda\tau^{-1}(i))$ will affect only the cycle $c_{\lambda(i)} = (\ldots, \lambda\tau(i), \lambda(i), \lambda\tau^{-1}(i), \ldots)$ of τ, which decompose $c_{\lambda(i)}$ into the cycles $(\ldots, \lambda\tau(i), \lambda\tau^{-1}(i), \ldots)$ and $(\lambda(i))$, while all other cycles of τ remain the same. Since $i, \tau^{-1}(i), \lambda(i), \lambda\tau^{-1}(i)$ are pairwise different, these two multiplication will not affect each other. As a result, $\bar{\eta}(\tau) = \tau(\lambda(i), \lambda\tau^{-1}(i))(i, \tau^{-1}(i))$ has two more cycles than τ. Hence $N(\bar{\eta}(\tau)) = N(\tau) - 1 = N((\tau, \lambda)) - 1$, which implies $N(\bar{\eta}((\tau, \lambda))) = N((\tau, \lambda)) - 1$. $\square$

Theorem 3.2.3. *Λ_p is a factorable category.*

Proof. More precisely, we will prove that Λ_p is a factorable category with the norm defined in (3.2.1) and the factorization map η.

Due to Proposition 1.2.3, we only need to prove that conditions (A) and (B) in section 1.2 are satisfied by all pairs $(\tau, t, \lambda) \in \mathscr{M}(\Lambda_p)(2)$ with $(t, \lambda) \in T(\mathscr{M}(\Lambda_p))$.

If $\tau = \mathrm{id}$, it can be checked easily that the conditions are fulfilled.

If $\tau \neq \mathrm{id}$, from the analysis about elements in $T(\mathscr{M}(\Lambda_p))$ before, we can assume that $t = (i\ \ j)(\lambda(i)\ \ \lambda(j))$, where $i = \max\{i, j, \lambda(i), \lambda(j)\}$ and $\eta'(\tau) = (k\ \ l)(t\lambda(k)\ \ t\lambda(l))$, where $k = H(\tau) = \max\{k, l, t\lambda(k), t\lambda(l)\}$.

First we consider the special case $\eta'(\tau) = t$. The two compositions in question are then:

$$\alpha_u : (\tau, \eta'(\tau), \lambda) \mapsto (\bar{\eta}(\tau), \eta'(\tau), \eta'(\tau), \lambda) \mapsto (\bar{\eta}(\tau), \mathrm{id}, \lambda) \mapsto (\bar{\eta}(\tau), \mathrm{id}, \mathrm{id}, \lambda) \mapsto (\bar{\eta}(\tau), \mathrm{id}, \lambda)$$

and

$$\alpha_l : (\tau, \eta'(\tau), \lambda) \mapsto (\bar{\eta}(\tau), \lambda) \mapsto (\bar{\eta}(\bar{\eta}(\tau)), \eta'(\bar{\eta}(\tau)), \lambda).$$

None of the two compositions are norm-preserving: $N((\tau, \eta'(\tau), \lambda)) = N(\tau) + 1$, while $N(\alpha_u((\tau, \eta'(\tau), \lambda))) = N((\bar{\eta}(\tau), \mathrm{id}, \lambda)) = N(\tau) - 1$ and $N(\alpha_l((\tau, \eta'(\tau), \lambda))) = N((\bar{\eta}(\tau), \lambda)) = N(\tau) - 1$. Hence the conditions are satisfied in this case.

Now consider the case when $\eta'(\tau) \neq t$. We have the following

Lemma 3.2.4. *If $\eta'(\tau) \neq t$, then $\eta'(\tau \cdot t) = \eta'(\eta'(\tau) \cdot t)$.*

Proof. If $k = H(\tau) \geq H(t) = i$, then the fixed points of τ larger than k are also fixed points of $\tau \cdot t$ and $\eta'(\tau) \cdot t$. Moreover, $(\tau \cdot t)^{-1}(k) = t^{-1} \cdot \tau^{-1}(k) = t^{-1} \cdot (\eta'(\tau))^{-1}(k) = t^{-1}(l) \neq k$, since if $t(k) = l$, it would follow that $t = \eta'(\tau)$. Hence $H(\tau \cdot t) = H(\eta'(\tau) \cdot t) = H(\tau) = k$ and $\eta'(\tau \cdot t) = \eta'(\eta'(\tau) \cdot t) = (k\ \ t^{-1}(l)) \cdot (\lambda(k)\ \lambda t^{-1}(l)) = (k\ \ t(l)) \cdot (\lambda(k)\ \lambda t(l))$.

Similarly, if $k = H(\tau) < H(t) = i$, then $H(\tau \cdot t) = H(\eta'(\tau) \cdot t) = H(t) = i$ and $(\tau \cdot t)^{-1}(i) = (\eta'(\tau) \cdot t)^{-1}(i) = t^{-1}(i) = j$, since i is a fixed point of τ and $\eta'(\tau)$. Hence $\eta'(\tau \cdot t) = \eta'(\eta'(\tau) \cdot t) = (i\ \ j) \cdot (\lambda(i)\ \lambda(j)) = t$.

□

Now we finish the proof of the Theorem. If $\eta'(\tau) \neq t$, from the Lemma above we have

$$\alpha_u((\tau, t, \lambda)) = (\tau \cdot t \cdot \eta'(\eta'(\tau) \cdot t), \eta'(\eta'(\tau) \cdot t), \lambda) = (\tau \cdot t \cdot \eta'(\tau \cdot t), \eta'(\tau \cdot t), \lambda) = \alpha_l((\tau, t, \lambda)).$$

Hence $\alpha_u((\tau, t, \lambda)) = \alpha_l((\tau, t, \lambda))$ holds regardless of whether α_u and α_l are norm-preserving or not, which implies that conditions (A) and (B) are fulfilled. □

Therefore the general results on factorable categories can be applied to the category of pairings. Hence we obtain that the homology of the norm complex $\mathcal{N}(\Lambda_p)[h]$ is concentrated on the top degree h, and $\kappa : V_h(\Lambda_p) \to H_h(\mathcal{N}_h(\Lambda_p)[h])$ is an isomorphism. Furthermore, we have that for any h-tuple of norm-one morphisms $\tau = (\tau_h, \ldots, \tau_1)$,

$$\tau \in R_h(\Lambda_p)(h) \Longleftrightarrow H(\tau_h) \geq \cdots \geq H(\tau_1)$$

Hence we also call an element in $R_h(\Lambda_p)(h)$ a *monotone* tuple.

Recall that $\mathbb{NQ}_{\bullet\bullet}(h, m)$ is a double complex with vertical boundary operator $\partial' = \sum_{i=1}^{q-1}(-1)^i \partial'_i$ and horizontal boundary operator $\partial'' = \sum_{i=1}^{p-1}(-1)^i \partial''_i$. For a fixed p, the following Lemma shows the relation between the chain complex $(\mathbb{NQ}_{p,*}(h, m), \partial')$ and the norm complex $(\mathcal{N}_*(\Lambda_p)[h], d)$. This thus provides an application of the theory of factorable categories to the study of homology of moduli spaces of Kleinian surfaces.

Lemma 3.2.5. *The p-th vertical complex $\mathbb{NQ}_{p,*}(h, m)$ with differential ∂' is a direct summand of $\mathcal{N}_*(\Lambda_p)[h]$ with differential d.*

Proof. We define a homomorphism $\imath : \mathbb{NQ}_{p,q}(h, m) \to \mathcal{N}_*(\Lambda_p)[h]$ by letting

$$\imath((\lambda_q, \ldots, \lambda_0)) := ((\lambda_q \lambda_{q-1}, \lambda_{q-1}) | \ldots | (\lambda_1 \lambda_0, \lambda_0))$$

for each $\Lambda = (\lambda_q, \ldots, \lambda_0) \in \mathbb{NQ}_{p,q}(h, m)$ and extending linearly to a homomorphism between $\mathbb{Z}$-modules. Note that here we consider the objects of Λ_p to be pairings of the set $\{1^-, 1^+, \ldots, p^-, p^+\}$ instead of $\{1, 2, \ldots, 2p\}$. Since λ_0 is the fixed pairing defined in (3.1.6) for any tuple $\Lambda = (\lambda_q, \ldots, \lambda_0) \in \mathbb{NQ}_{p,q}(h,m)$, $\imath$ is an injective homomorphism.

The statement that $(\mathbb{NQ}_{p,*}(h,m), \partial')$ is a direct summand of $(\mathcal{N}_*(\Lambda_p)[h], d)$ is then equivalent to: For any $0 \le i \le q$, the following properties 1) and 2) hold.

1) $d_i \circ \imath = \imath \circ \partial'_i$, i.e. $(\mathbb{NQ}_{p,*}(h,m), \partial')$ is a subcomplex of $(\mathcal{N}_*(\Lambda_p)[h], d)$.

2) $(\Lambda \in \mathcal{N}_*(\Lambda_p)[h] \smallsetminus \imath(\mathbb{NQ}_{p,*-1}(h,m))$ and $d_i(\Lambda) \in \imath(\mathbb{NQ}_{p,*-1}(h,m))) \Longrightarrow d_i(\Sigma) = 0$.

This can be checked directly using the definition of $\mathbb{NQ}_{\bullet\bullet}(h,m)$ on page 51, hence we leave out the details here. $\square$

Remark 3.2.6. The maps $\imath : \mathbb{NQ}_{p,*}(h,m) \to \mathcal{N}_*(\Lambda_p)[h]$ for different m together induce the injective homomorphism

$$\imath : \bigoplus_{0 \le m \le h-1} \mathbb{NQ}_{p,*}(h,m) \hookrightarrow \mathcal{N}_*(\Lambda_p)[h].$$

However, this map is not surjective, because any tuple $\Lambda = (\lambda_q, \ldots, \lambda_0) \in \mathbb{NQ}_{p,q}(h,m)$ must satisfy the extra condition 5) and 6) on page 51. $\diamond$

Because $\lambda_0 \in \Lambda_p$ is fixed, we will use some simplified notation in the remaining part of the thesis. A tuple $((\lambda_q\lambda_{q-1}, \lambda_{q-1})| \ldots |(\lambda_1\lambda_0, \lambda_0))$ in the image of $\imath$ will be denoted simply by $(\tau_q| \ldots |\tau_1)$, where $\tau_i = \lambda_i\lambda_{i-1} \in \mathfrak{S}_{2p}$. This is the inhomogeneous notation already introduced on page 51. And we will abbreviate the morphism (τ_i, λ_{i-1}) to τ_i in this case.

Let $\{E^r_{p,q}, d_r\}_r$ be the spectral sequence of the double complex $\mathbb{NQ}_{\bullet\bullet}(h,m)$. Then we have $\{E^0_{p,q} = \mathbb{NQ}_{p,q}(h,m)\}$ and $d_0 = \partial' : \mathbb{NQ}_{p,q}(h,m) \to \mathbb{NQ}_{p,q-1}(h,m)$. Moreover, $\{E^1_{p,q} = H_q(\mathbb{NQ}_{p,*}(h,m))\}$ and the differential $d_1 : E^1_{p,q} \to E^1_{p-1,q}$ is induced by ∂''.

An immediate consequence of Theorem 1.2.11 and Lemma 3.2.5 is the following

Corollary 3.2.7. *The vertical homology $E^1_{p,q} = H_q(\mathbb{NQ}_{p,*}(h,m), \partial')$ is concentrated in the top degree $q = h$. Thus the E^1-term is a chain complex with differential induced by ∂'', and the spectral sequence collapses with $E^2 = E^\infty$.*

The complex $(W^N_*(h,m), d)$ is defined by $W^N_p(h,m) := \kappa^{-1}(H_h(\mathbb{NQ}_{p,*}(h,m), \partial')) \subseteq V_h(\Lambda_p)$ and $d := \kappa^{-1} \circ \partial'' \circ \kappa$. It is equivalent to the E^1-term of the spectral sequence. A closer examine shows that $W^N_p(h,m)$ is generated by those tuples $(\tau_h| \ldots |\tau_1) \in R_h(\Lambda_p)[h]$, which satisfy the following extra conditions:

1) There exists no $k \in \{1^-, 1^+, \ldots, p^-, p^+\}$, such that k is fixed by τ_i for all $i = 1, \ldots h$.

2) The number of cycles of $\tau_h \circ \ldots \circ \tau_1 \circ \lambda_0 \circ J$ is $2(m+1)$.

3) There exists $1 \le j,\ k \le p$, $0 \le i \le h$ such that $\lambda_i(j^+) = \lambda_i(k^-)$.

It is proved in [Z1] that $\mathfrak{N}_{g,1}^m$ is always non-orientable. As a result, the double complex $\mathbb{NQ}_{\bullet\bullet}(h,m)$ and hence the complex $(W_*^N(h,m),d)$ cannot be used to compute the integral homology of $\mathbb{NQ}_{\bullet\bullet}(h,m)$. However, one can compute the $\mathbb{Z}_2$-homology of $\mathfrak{N}_{g,1}^m$ using the mod-2 version of the complex $(W_*^N(h,m),d)$ and the Poincaré duality

$$H^*(\mathbb{NQ}_{\bullet\bullet}(h,m),\mathbb{Z}_2) = H^*(\mathrm{NPar}(h,m),\mathrm{NPar}'(h,m);\mathbb{Z}_2) \cong H_{3h-*}(\mathfrak{N}_{g,1}^m;\mathbb{Z}_2).$$

3.3 Orientation System

Since the manifold $\mathfrak{N}_{g,1}^m$ is non-orientable, to compute the integral homology of $\mathfrak{N}_{g,1}^m$, we need to consider the orientation system $\mathcal{O}$ on the relative manifold and use the Poincaré duality

$$H^*(\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h,m)) = H^*(\mathrm{NPar}(h,m),\mathrm{NPar}'(h,m);\mathcal{O}) \cong H_{3h-*}(\mathfrak{N}_{g,1}^m;\mathbb{Z}).$$

Here $\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h,m)$ is the chain complex of the relative manifold with the orientation system $\mathcal{O}$. In this section we will give the construction of $\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h,m)$ and study its homology, which turns out to have analogous properties to that of $\mathbb{NQ}_{\bullet\bullet}(h,m)$. The discussion here will follow the same line as in Section 2.3.

$\widetilde{\mathbb{NQ}}_{p,q}(h,m)$ has the same generators as $\mathbb{NQ}_{p,q}(h,m)$, but the boundary operators differ. In $\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h,m)$, the vertical and horizontal face operators on a cell $e \in \widetilde{\mathbb{NQ}}_{p,q}(h,m)$ are of the form

$$\widetilde{\partial}'_i(e) = \epsilon'_i(e)\,\partial'_i(e), \quad i = 1,\dots,q-1 \tag{3.3.1}$$

$$\widetilde{\partial}''_i(e) = \epsilon''_i(e)\,\partial''_i(e), \quad i = 1,\dots,p-1 \tag{3.3.2}$$

where the sign $\epsilon'_i(e)$, $\epsilon''_i(e) \in \{\pm 1\}$ depends on e and i, not only on i.

Before going to the definition of $\epsilon'_i(e)$, $\epsilon''_j(e)$ and hence of the double complex $\widetilde{\mathbb{NQ}}_{p,q}(h,m)$, we need to study the cells of $\mathbb{NQ}_{\bullet\bullet}(h,m)$ in more detail.

Recall that a cell $e \in \mathbb{NQ}_{p,q}(h,m)$ can be considered as an equivalence class of symbols under Rauzy jumps or as a tuple of pairings $e = (\tau_q|\dots|\tau_1)$ with $\tau_i \in \Lambda_p$. For a top-dimensional cell $e \in \mathbb{NQ}_{2h,h}(h,m)$, i.e. $p = 2h$, $q = h$, there is a unique symbol E, which is a representative of $e = [E]$. We call the cells in $\mathbb{Q}_{2h,h}(h,m)$ *generic*. If e is not a generic cell, there can be more than one representative for e, and by definition, any two of them can be transformed into each other by Rauzy jumps.

We need some terminology from Section 2.3. The notion of ∂''_{D_2} and ∂'_{D_1} and the definition of a *generification* remain the same here. Recall that $(\hat{e}, D_1, D_2)$ is a *generification* of the cell e, if $\hat{e}$ is a cell in $\mathbb{Q}_{2h,h}(h,m)$ and $\partial''_{D_2}\partial'_{D_1}(\hat{e}) = e$. Moreover, the set of top-dimensional cells which have e as a face is still denoted by

$$\mathcal{H}(e) = \{\hat{e} \mid \hat{e} \text{ is a generic cell and } \partial''_{D_2}\partial'_{D_1}(\hat{e}) = e \text{ for some sets } D_1, D_2\}.$$

We know that e can have more than one symbols as representatives and for each representative E, there exists a cell $\hat{e} \in \mathcal{H}(e)$ and sets D_1, D_2, such that

$$\partial''_{D_2}\partial'_{D_1}(\hat{e}) = E. \tag{3.3.3}$$

Note that the boundary operators are meant to be in the sense of symbols, not in the sense of equivalence classes of symbols, since the symbol E appears at the right-hand side of the equation (3.3.3).

The definition of *normal form* plays an central role in defining the face operators on $\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h,m)$. It comes from the idea that one would like to choose a "standard" representative for each class of symbol $[E]$. Before giving this definition, we need to introduce some terminologies concerning Λ_p first.

We will use set of indices $\{1^-,1^+,\dots,p^-,p^+\}$ instead of $\{1,\dots,2p\}$ for Λ_p, whose geometric meaning is shown in Section 3.1. Moreover, on $\{1^-,1^+,\dots,p^-,p^+\}$ the order $1^- < 1^+ < \dots < p^- < p^+$ is assumed. For a index k^+ (resp. k^-), we call k the *number part* of k^+ (resp. k^-) and denote it by $|k^+| = k$ (resp. $|k^-| = k$). To denote the superscript, we define $\mathrm{sgn}(k^+) := +$ and $\mathrm{sgn}(k^-) := -$. Furthermore, define $|\tau| := \{|i| \mid \tau(i) \neq i\}$ for $\tau \in \mathfrak{S}_{2p}$.

It's known from last section that if $N(\tau) = 1$, $(\tau,\lambda) \in \mathscr{M}(\Lambda_p)$, then τ is a composition of two transpositions; write $\tau = (i_1\, j_1)(i_2\, j_2)$, then i_1, j_1, i_2, j_2 are pair-wise different. We define $\tau^{(H)}$ to be the transposition of τ, which contains the biggest non-fixed index $H(\tau)$ according to the order above; and define $\tau^{(L)}$ to be the other transposition.

For $(\tau,\lambda) \in \mathscr{M}(\Lambda_p)$, the cycles of τ appear in pairs; and if λ as well as one cycle from each such pair are known, we can then determine τ. Therefore for a cell $e = (\tau_q|\dots|\tau_1) \in \mathbb{NQ}_{p,q}(h,m)$, each $\tau_i\,(i = 1,\dots,q)$ can be recovered from partial information in the above sense, we call this process *complete.* For example, for $e = (\tau_h|\dots|\tau_1) \in \mathbb{NQ}_{2h,h}(h,m)$, if a transposition from each τ_i is given, we can complete the given tuple of transpositions to get e.

Definition 3.3.1. The *normal form* of a cell $e = (\tau_q|\dots|\tau_1) \in \mathbb{NQ}_{p,q}(h,m)$ is the generification $(\mathrm{NF}(e), D_1, D_2)$ of e which is determined by the following procedure:

1) For each i, assume the normal form of the morphism τ_i in the category Λ_p is $\mathrm{nF}(\tau_i) = (\tau_{i,n_i},\dots,\tau_{i,1})$, where $n_i = N(\tau_i)$. Pick out the transposition $\tau_{i,j}^{(H)}$ from each $\tau_{i,j}$ and arrange them into a tuple of transpositions

$$e^{(0)} := (\tau_{q,n_q}^{(H)}|\dots|\tau_{q,1}^{(H)}|\dots|\tau_{1,n_1}^{(H)}|\dots|\tau_{1,1}^{(H)}) := (\tau_h^{(0)}|\dots|\tau_1^{(0)})$$

   and define $D_1 := \{1,\dots,h-1\} \smallsetminus \{\sum_{k=1}^{i} n_k \mid i = 1,\dots,q-1\}$.

2) First we define a map $S_j : \mathfrak{S}_{2p} \to \mathfrak{S}_{2(p+1)}$ for each $j = 1,\dots,p$, which sends a permutation $\alpha \in \mathfrak{S}_{2p}$ to the permutation $S_j(\alpha) \in \mathfrak{S}_{2(p+1)}$ as follows:

$$S_j(\alpha)(k) := \begin{cases} d_{j^-} \circ d_{j^+} \circ \alpha \circ s_{j^-} \circ s_{j^+} & \text{if} \quad |k| \neq j \\ k & \text{if} \quad |k| = j \end{cases}$$

   To get $\mathrm{NF}(e)$ and D_2, we perform the following algorithm on $e^{(0)}$:

   Algorithm 3.3.2.

   - $D_2^{(0)} := \emptyset$.

- *For* $1 \le n \le h-p$*:*
  Let k *be the lowest index such that there are more than one transposition of* $e^{(n-1)} = (\tau_h^{(n-1)}|\dots|\tau_1^{(n-1)})$ *which do not fix* k *and set*
  $$i_0 := \begin{cases} \min\{1 \le i \le h \mid \tau_i^{(n-1)}(k) \neq k\} & \textit{if} \quad \mathrm{sgn}(k) = + \\ \max\{1 \le i \le h \mid \tau_i^{(n-1)}(k) \neq k\} & \textit{if} \quad \mathrm{sgn}(k) = - \end{cases}$$
  Define
  $$e^{(n)} := (\tau_h^{(n)}|\dots|\tau_1^{(n)}) \qquad \textit{and} \qquad D_2^{(n)} := D_2^{(n)} \cup \{|k|\}.$$
  where
  $$\tau_i^{(n)} := \begin{cases} S_{|k|+1}(\tau_i^{(n-1)}) & \textit{if} \quad i = i_0 \\ S_{|k|}(\tau_i^{(n-1)}) & \textit{if} \quad i \neq i_0 \end{cases}$$
- *Complete* $e^{(h-p)}$*, we obtain* $\mathrm{NF}(e)$ *and* $D_2 := D_2^{(h-p)}$.

Remark 3.3.3.

1) It follows from the procedure that $\mathrm{NF}(e) \in \mathbb{Q}_{2h,h}(h,m)$ and $\partial''_{D_2}\partial'_{D_1}(\mathrm{NF}(e)) = e$, so $(\mathrm{NF}(e), D_1, D_2)$ is a generification of e as claimed in the definition.
2) For any generification $(\hat{e}, D_1, D_2)$ of e, D_1 is uniquely determined by the partition-type of e (hence uniquely determined by e).
3) The second step can be understood more intuitively from the perspective of slit configurations: Let $E^{(0)}$ be the symbol which is the representative of
   $$e^{(0)} = (\tau_{q,n_q}|\dots|\tau_{q,1}|\dots|\tau_{1,n_1}|\dots|\tau_{1,1})$$
   chosen as in 2) of Remark 3.1. Whenever two slits touch each other in $E^{(0)}$, we will add a stripe between them, so that in the end $E^{(0)}$ is transformed into a top dimensional cell $\mathrm{NF}(e)$.

$\diamond$

Now we will explain two kinds of operators-namely swaps and Rauzy jumps-on a generification $(\hat{e}, D_1, D_2)$ of e. We fix the assumption that
$$\hat{e} = (\tau_h|\dots|\tau_1) \in \mathcal{H}(e).$$

Definition 3.3.4. If $i \in D_1$, the *swap* of $(\hat{e}, D_1, D_2)$ at i is defined to be the generification $(\mathrm{swap}_i(\hat{e}), D_1, D_2)$, where
$$\mathrm{swap}_i(\hat{e}) := (\tau_h|\dots|\tau_{i+2}|\tau_i|\tau_{i+1}|\tau_{i-1}|\dots|\tau_1).$$

Remark 3.3.5. Since $\hat{e}$ is a top-dimensional cell, it is easy to check that $\mathrm{swap}_i(\hat{e}) \in \mathcal{H}(e)$ and $\partial''_{D_2}\partial'_{D_1}(\mathrm{swap}_i(\hat{e})) = \partial''_{D_2}\partial'_{D_1}(\hat{e})$ as symbols, hence $(\mathrm{swap}_i(\hat{e}), D_1, D_2)$ is a generification of e as claimed in the definition. $\diamond$

For a fixed representative E of e, there can be more than one $\hat{e}$ satisfying (2.3.3). However, from the definition of the face operators, D_1 is uniquely determined by e (hence also by E) and D_2 is uniquely determined by E. Moreover, $\hat{e}$ is determined by E up to swaps as shown in the following Lemma.

Lemma 3.3.6. *Suppose* $(\hat{e}, D_1, D_2)$ *and* $(\check{e}, D_1, D_2)$ *are generifications of* e *such that*
$$\partial''_{D_2}\partial'_{D_1}(\hat{e}) = E = \partial''_{D_2}\partial'_{D_1}(\check{e}),$$
where E *is a symbol representing* e. *Then* $(\hat{e}, D_1, D_2)$ *can be transformed into* $(\check{e}, D_1, D_2)$ *by swaps.*

Proof. From the definition of the horizontal face operators we have
$$\partial'_{D_1}(\hat{e}) = \partial'_{D_1}(\check{e}) =: \tilde{e} \in \mathbb{NQ}_{2h,q}(h,m).$$

First we have an observation for any cell $(\tau_h|\dots|\tau_1) \in \mathbb{NQ}_{2h,h}(h,m)$: If $\tau_i = t_i^{(2)}t_i^{(1)}$, where $t_i^{(1)}, t_i^{(2)}$ are permutations, then $|t_i^{(1)}| = |t_i^{(2)}|$ and they both contain two elements. This is due to $\lambda_0 = (p^+\ p^-)\dots(1^+\ 1^-)$ and $\bigcup_{i=1}^h |\tau_i| = \{1, \dots, 2h\}$.

Now assume that $\hat{e} = (\tau_h|\dots|\tau_1)$, $\check{e} = (\check{\tau}_h|\dots|\check{\tau}_1)$ and $\tilde{e} = (\tilde{\tau}_q|\dots|\tilde{\tau}_1)$. Let $n_0 = 1$ and
$$n_i = \sum_{k=1}^{i} N(\tilde{\tau}_k) \quad i = 1, \dots, q.$$
Then we have
$$\tilde{\tau}_i = \tau_{n_i} \circ \dots \circ \tau_{n_{i-1}+1} = \check{\tau}_{n_i} \circ \dots \circ \check{\tau}_{n_{i-1}+1}.$$

Because $\hat{e}$ and $\check{e}$ are top-dimensional cells, the indices not fixed by $\tau_i\,(i = 1, \dots, h)$ are pairwise distinct and the indices not fixed by $\check{\tau}_i\,(i = 1, \dots, h)$ are pairwise distinct. Therefore the tuple $(\check{\tau}_{n_i}^{(H)}|\check{\tau}_{n_i}^{(L)}|\dots|\check{\tau}_{n_{i-1}+1}^{(H)}|\check{\tau}_{n_{i-1}+1}^{(L)})$ is obtained from the tuple $(\tau_{n_i}^{(H)}|\tau_{n_i}^{(L)}|\dots|\tau_{n_{i-1}+1}^{(H)}|\tau_{n_{i-1}+1}^{(L)})$ by permuting its transpositions. Using the observation above, we can see that $\check{e} = (\check{\tau}_h|\dots|\check{\tau}_1)$ is obtained from $\hat{e} = (\tau_h|\dots|\tau_1)$ by permuting its entries.

Because a swap exchanges two neighbored transpositions and any permutation $\sigma \in \mathfrak{S}_h$ is a composition of elements in $\{(i\ \ i+1) \mid i = 1, \dots, h-1\}$, $\hat{e}$ can be transformed into $\check{e}$ by swaps. $\square$

Before going to define the Rauzy jumps, we give some more notations first. Assume that (τ_1, λ) with $\tau_1 = (i\ j)(\lambda(i)\ \lambda(j))$ is a morphism in Λ_p, then (τ_2, λ) with $\tau_2 = (i\ \lambda(j))(j\ \lambda(i))$ is also a morphism in Λ_p. We call (τ_2, λ) (resp. (τ_1, λ)) the *alternate* of (τ_1, λ) (resp. (τ_2, λ)) and denote this by $\mathrm{alt}((\tau_1, \lambda)) = (\tau_2, \lambda)$ (resp. $\mathrm{alt}((\tau_2, \lambda)) = (\tau_1, \lambda)$). For $a, b \in \{1, \dots, 2h\}$, $a \le b$, let $\beta_{a,b}$ denote the permutation
$$(a^+\ (a+1)^+ \dots b^+)(a^-\ (a+1)^- \dots b^-),$$
then $\beta_{a,b}^{-1}$, the inverse of $\beta_{a,b}$, is the permutation
$$(b^+\ (b-1)^+ \dots a^+)(b^-\ (b-1)^- \dots a^-).$$

Now consider the generification $(\hat{e}, D_1, D_2)$ of e. For each $i \in D_2$, there exist unique $k, l \in \{1, \dots, h\}$, such that $|\tau_k(i^+)| \ne i, |\tau_l((i+1)^-)| \ne i+1$. Furthermore $k \ne l$, since otherwise $e \in \mathbb{NP}'_{\bullet\bullet}(h,m)$.

Definition 3.3.7. The *Rauzy jump* of $(\hat{e}, D_1, D_2)$ at $i \in D_2$ is defined to be the generification $(\mathrm{Rauzy}_i(\hat{e}), D_1, \tilde{D}_2)$, where $\mathrm{Rauzy}_i(\hat{e})$ and $\tilde{D}_2$ are defined as follows. First define

$$\alpha_{D_2} := \begin{cases} s_{|\tau_l((i+1)^-)|,i} & \text{if} \quad l < k \quad \text{and} \quad |\tau_l((i+1)^-)| < i \\ s^{-1}_{i,|\tau_l((i+1)^-)|-1} & \text{if} \quad l < k \quad \text{and} \quad |\tau_l((i+1)^-)| > i+1 \\ s_{|\tau_k(i^+)|,i} & \text{if} \quad l > k \quad \text{and} \quad |\tau_k(i^+)| < i \\ s^{-1}_{i,|\tau_k(i^+)|-1} & \text{if} \quad l > k \quad \text{and} \quad |\tau_k(i^+)| > i+1, \end{cases}$$

then

$$\tilde{D}_2 := \alpha_{D_2}(D_2).$$

The definition of $\mathrm{Rauzy}_i(\hat{e})$ falls into two types:

- The first type. Let

$$\beta := \begin{cases} \beta_{|\tau_l((i+1)^-)|+1,i} & \text{if } l < k,\ |\tau_l((i+1)^-)| < i,\ \mathrm{sgn}(\tau_l((i+1)^-)) = - \\ \beta^{-1}_{i,|\tau_l((i+1)^-)|} & \text{if } l < k,\ |\tau_l((i+1)^-)| > i+1,\ \mathrm{sgn}(\tau_l((i+1)^-)) = - \\ \beta_{|\tau_k(i^+)|,i+1} & \text{if } l > k,\ |\tau_k(i^+)| < i,\ \mathrm{sgn}(\tau_k(i^+)) = + \\ \beta^{-1}_{i+1,|\tau_k(i^+)|-1} & \text{if } l > k,\ |\tau_k(i^+)| > i+1,\ \mathrm{sgn}(\tau_k(i^+)) = +, \end{cases}$$

then

$$\begin{aligned} \tilde{\tau}_j &:= \beta \circ \tau_j \circ \beta \quad \text{for} \quad 1 \le j \le h \\ \mathrm{Rauzy}_i(\hat{e}) &:= (\tilde{\tau}_h | \dots | \tilde{\tau}_1). \end{aligned}$$

- The second type. Let

$$\beta := \begin{cases} \beta_{|\tau_l((i+1)^-)|,i} & \text{if } l < k,\ |\tau_l((i+1)^-)| < i,\ \mathrm{sgn}(\tau_l((i+1)^-)) = + \\ \beta^{-1}_{i,|\tau_l((i+1)^-)|-1} & \text{if } l < k,\ |\tau_l((i+1)^-)| > i+1,\ \mathrm{sgn}(\tau_l((i+1)^-)) = + \\ \beta_{|\tau_k(i^+)|+1,i+1} & \text{if } l > k,\ |\tau_k(i^+)| < i,\ \mathrm{sgn}(\tau_k(i^+)) = - \\ \beta^{-1}_{i+1,|\tau_k(i^+)|} & \text{if } l > k,\ |\tau_k(i^+)| > i+1,\ \mathrm{sgn}(\tau_k(i^+)) = -, \end{cases}$$

then

$$\begin{aligned} \tilde{\tau}_j &:= \begin{cases} \beta \circ \tau_j \circ \beta & \text{if} \quad 1 \le j \le h,\ j \ne \max\{k, l\} \\ \mathrm{alt}(\beta \circ \tau_j \circ \beta) & \text{if} \quad j = \max\{k, l\} \end{cases} \\ \mathrm{Rauzy}_i(\hat{e}) &:= (\tilde{\tau}_h | \dots | \tilde{\tau}_1). \end{aligned}$$

Remark 3.3.8. It turns out that $\mathrm{Rauzy}_i(\hat{e}) \in \mathcal{H}(e)$ and $\partial''_{\tilde{D}_2}\partial'_{D_1}(\mathrm{Rauzy}_i(\hat{e})) = \partial''_{D_2}\partial'_{D_1}(\hat{e}) = e$, hence $(\mathrm{Rauzy}_i(\hat{e}), D_1, \tilde{D}_2)$ is a generification of e as claimed in the definition. Moreover, as symbols, $\partial''_{\tilde{D}_2}\partial'_{D_1}(\mathrm{Rauzy}_i(\hat{e}))$ and $\partial''_{D_2}\partial'_{D_1}(\hat{e})$ are different from each other by a Rauzy jump in the sense of slit configurations as defined in Definition 3.1.5. This is why here we call the operator Rauzy_i on a generification also Rauzy jump. $\diamond$

As in Section 2.3, the following Lemma contains the essential idea and technique to determine the face operators on $\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h, m)$.

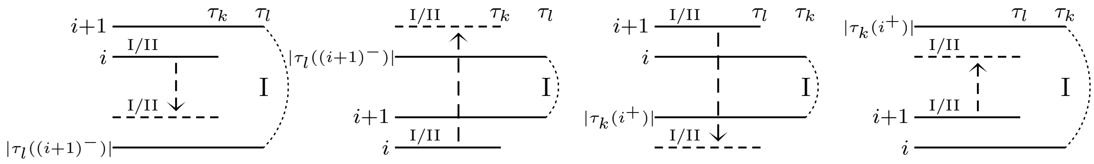


Figure 3.4: Rauzy jumps – the first type

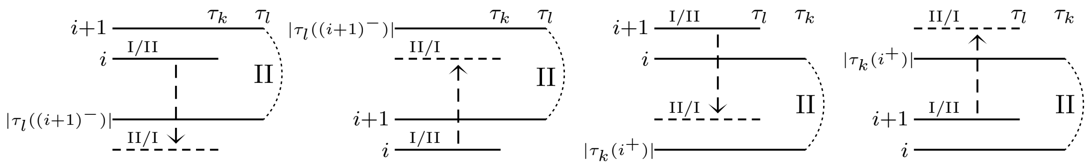


Figure 3.5: Rauzy jumps – the second type

Lemma 3.3.9. *If $(\hat{e}, D_1, D_2)$ and $(\check{e}, D_1, \check{D}_2)$ are generifications of e, then there exists a series of swaps and Rauzy jumps, by which $(\hat{e}, D_1, D_2)$ can be transformed into $(\check{e}, D_1, \check{D}_2)$.*

The proof of Lemma 3.3.9 is similar to that of Lemma 2.3.11, hence we will leave it out here. As a special case of Lemma 3.3.9, any generification $(\hat{e}, D_1, D_2)$ of e can be transformed into the normal form of e by swaps and Rauzy jumps.

Now we are ready to define the face operators $\widetilde{\partial}'_i$ and $\widetilde{\partial}''_j$ of $\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h,m)$, which reduces to defining $\epsilon'_i(e)$ and $\epsilon''_j(e)$ as in (3.3.1) and (3.3.2).

First we consider the case of a vertical face operator. Assume that e is a cell in $\widetilde{\mathbb{NQ}}_{p,q}(h,m)$. We only need to consider the case when $\partial'_i(e)$ is not degenerate, since otherwise $\widetilde{\partial}'_i(e) = 0$ by definition. Denote $\partial'_i(e)$ by $\bar{e}$ for simplicity of notation. There are three steps to determine $\epsilon'_i(e)$.

Algorithm 3.3.10.

1) Let $(\mathrm{NF}(e), D_1^e, D_2^e)$ be the normal form of e. Let $\bar{i}$ be the index which satisfies

$$\bar{i} = i + \#\{j \in D_1 \mid j < \bar{i}\}.$$

Define $\bar{D}_1 := D_1 \cup \{\bar{i}\}$, then $(\mathrm{NF}(e), \bar{D}_1, D_2)$ is a generification of $\bar{e}$.

2) $(\mathrm{NF}(e), \bar{D}_1, D_2)$ can be transformed into the normal form $(\mathrm{NF}(\bar{e}), D_1^{\bar{e}}, D_2^{\bar{e}})$ of $\bar{e}$ by swaps and Rauzy jumps. This gives rise to a sequence of generifications $\{(e^i, D_1^i, D_2^i)\}_{i=0}^n$ of $\bar{e}$ such that $(e^0, D_1^0, D_2^0) = (\mathrm{NF}(e), \bar{D}_1, \bar{D}_2)$, $(e^n, D_1^n, D_2^n) = (\mathrm{NF}(\bar{e}), D_1^{\bar{e}}, D_2^{\bar{e}})$ and $(e^{i+1}, D_1^{i+1}, D_2^{i+1})$ is obtained from (e^i, D_1^i, D_2^i) by a swap or a Rauzy jump for $0 \le i \le n-1$.

3) For $0 \le i \le n-1$: Denote $e^i = (\tau_h, \ldots, \tau_1)$.

- *If $(e^{i+1}, D_1^{i+1}, D_2^{i+1})$ is obtained from (e^i, D_1^i, D_2^i) by a swap, then define $\epsilon(i) := -1$.*

- *If $(e^{i+1}, D_1^{i+1}, D_2^{i+1})$ is obtained from (e^i, D_1^i, D_2^i) by a Rauzy jump, assume at j, then define $\epsilon(i) := (-1)^{s+t+1}$, where s, t are determined as follows: Let k, l be the unique indices satisfying $|\tau_k(j^+)| \neq j$, $|\tau_l((j+1)^-)| \neq j+1$, then define*

$$(s,t) := \begin{cases} (j, |\tau_l((j+1)^-)|) & \text{if } \; l < k, \; |\tau_l((j+1)^-)| < j \\ (j, |\tau_l((j+1)^-)| - 1) & \text{if } \; l < k, \; |\tau_l((j+1)^-)| > j+1 \\ (j, |\tau_k(j^+)|) & \text{if } \; l > k, \; |\tau_k(j^+)| < j \\ (j, |\tau_k(j^+)| - 1) & \text{if } \; l > k, \; |\tau_k(j^+)| > j+1. \end{cases}$$

Finally $\epsilon'_i(e) := \prod_{i=0}^{n-1} \epsilon(i)$.

In the case of a horizontal face operator, the algorithm is very similar. Assume that e is a cell in $\widetilde{\mathbb{NQ}}_{p,q}(h,m)$ and that $\partial''_j(e)$ is not degenerate. Denote $\partial''_j(e)$ by $\bar{e}$. The definition of $\epsilon''_j(e)$ follows the same three steps as that of $\epsilon'_i(e)$, only with some modification in the first step:

1) Let $(\mathrm{NF}(e), D_1^e, D_2^e)$ be the normal form of e. Let $\bar{j}$ be the index which satisfies

$$\bar{j} = j + \#\{i \in D_2 \mid i < \bar{j}\}.$$

Define $\bar{D}_2 := D_2 \cup \{\bar{j}\}$, then $(\mathrm{NF}(e), D_1, \bar{D}_2)$ is a generification of $\bar{e}$.

Steps 2) and 3) remain the same as in Algorithm 3.3.10; one only needs to adapt the notations when necessary.

Following the same argument as in Section 2.3, it can be shown that the face operators $\widetilde{\partial}'_i$, $\widetilde{\partial}''_j$ are well defined and $\widetilde{\mathbb{NQ}}_{\bullet\bullet}(h,m)$ is the double complex of the relative manifold $(\mathrm{NPar}(h,m), \mathrm{NPar}'(h,m))$ under the orientation system $\mathcal{O}$.

Now we will study the homology of the double complex $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$. The methods and results are parallel to that in Section 2.3, so we will provide even less details here. In particular, all the results will be stated without proof, since each of them can be proved similarly as in Section 2.3.

For a fixed p, consider the complex $\widetilde{\mathbb{NQ}}_{p,*}(h,m)$. As a $\mathbb{Z}$-module, $\widetilde{\mathbb{NQ}}_{p,q}(h,m)$ is the same as $\mathbb{NQ}_{p,q}(h,m)$, a direct summand of $\mathcal{N}_q(\mathfrak{S}_p)$, hence $\widetilde{\mathbb{NQ}}_{p,*}(h,m)$ inherits a partition-type filtration from $\mathcal{N}_*(\Lambda_p)$, which is defined by

$$\mathcal{P}_i\widetilde{\mathbb{NQ}}_{p,*}(h,m) = \bigoplus_{j=1}^{i} \widetilde{\mathbb{NQ}}_{p,*}(h,m)[L_j],$$

where $\widetilde{\mathbb{NQ}}_{p,*}(h,m)[L]$ denote the submodule of $\widetilde{\mathbb{NQ}}_{p,*}(h,m)$ generated by all tuples Σ with the partition type $\mathrm{pt}(\Sigma) = L$.

Since the face operators of $\widetilde{\mathbb{NQ}}_{p,*}(h,m)$ differ from that of $\mathcal{N}_*(\mathfrak{S}_p)$ only by a sign, Lemma 1.2.12 is still applicable here; and in particular, the partition-type filtration of $\widetilde{\mathbb{NQ}}_{p,*}(h,m)$ is well defined.

Define the map $\eta_i = \eta_i^q : \widetilde{\mathbb{NQ}}_{p,q}(h,m) \longrightarrow \widetilde{\mathbb{NQ}}_{p,q+1}(h,m)$ to be the restriction of the map $\eta_i = \eta_i^q : \mathcal{N}_q(\Lambda_p) \longrightarrow \mathcal{N}_{q+1}(\Lambda_p)$ on $\widetilde{\mathbb{NQ}}_{p,q}(h,m)$. Furthermore, define

$$\begin{aligned}\widetilde{f}_i = \eta_i \widetilde{\partial}'_i : \widetilde{\mathbb{NQ}}_{p,q}(h,m)[l_q, \ldots, l_{i+1}, l_i, \ldots, l_1] \\ \to \widetilde{\mathbb{NQ}}_{p,q}(h,m)[l_q, \ldots, l_{i+1} + l_i - 1, 1, \ldots, l_1]\end{aligned}$$

for $1 \leq i \leq q-1$.

We have the following Lemma concerning $\widetilde{f}_i$.

Lemma 3.3.11.

(1) $\widetilde{\partial}'_i \widetilde{f}_i = \widetilde{\partial}'_i$ (3.3.4a)

(2) $\widetilde{\partial}'_j \widetilde{f}_i = \widetilde{f}_i \widetilde{\partial}'_{j-1}$ *for* $i+2 \leq j \leq q$ (3.3.4b)

(3) $\widetilde{\partial}'_j \widetilde{f}_i = \widetilde{f}_{i-1} \widetilde{\partial}'_j$ *for* $1 \leq j \leq i-2$ (3.3.4c)

(4) $\widetilde{\partial}'_{i+1} \widetilde{f}_i \widetilde{f}_{i+1} = \widetilde{f}_i \widetilde{\partial}'_i$ (3.3.4d)

Now we introduce the map

$$\widetilde{F}_r : \widetilde{\mathbb{NQ}}_{p,q}(h,m)[l_q, \ldots, l_r, 1, \ldots, 1] \to \widetilde{\mathbb{NQ}}_{p,q+1}(h,m)[l_q, \ldots, l_r - 1, 1, \ldots, 1]$$

defined as the composition

$$(-1)^r \left(\mathrm{id} - \widetilde{f}_{r-1} + \cdots + (-1)^{r-i} \widetilde{f}_i \widetilde{f}_{i+1} \cdots \widetilde{f}_{r-1} + \cdots + (-1)^{r-1} \widetilde{f}_1 \widetilde{f}_2 \cdots \widetilde{f}_{r-1} \right) \eta_r$$

and the signed sum of the first r face operators

$$\widetilde{\partial}'_{(r)} = \sum_{i=1}^{r} (-1)^i \widetilde{\partial}'_i.$$

We have the following results.

Lemma 3.3.12.

(i) $\partial'_{(r)} \widetilde{F}_r = \mathrm{id} - \widetilde{F}_{r-1} \partial'_{(r-1)}$

(ii) $\partial'_j \widetilde{F}_r = \widetilde{F}_r \partial'_{j-1}$ *for* $r+2 \leq j \leq q$

Theorem 3.3.13. *The homology of the complex* $\widetilde{\mathbb{NQ}}_{p,*}(h,m)$ *is concentrated in the top degree h:*

$$H_q(\widetilde{\mathbb{NQ}}_{p,*}(h,m)) = 0, \qquad \textit{if} \quad q < h.$$

Furthermore, we can construct a homomorphism

$$\widetilde{\kappa} : V_h(\Lambda_p) \cap \widetilde{\mathbb{NQ}}_{p,h}(h,m) \to H_h(\widetilde{\mathbb{NQ}}_{p,*}(h,m))$$

by

$$\begin{aligned}\widetilde{\kappa}(\Lambda) := (\mathrm{id} - (-1)^{h-1} \widetilde{F}_{h-1} \circ \widetilde{\partial}'_{h-1}) \ \ldots \\ \ldots \ (\mathrm{id} - (-1)^i \widetilde{F}_i \circ \widetilde{\partial}'_i) \ \ldots \ (\mathrm{id} - \widetilde{F}_2 \circ \widetilde{\partial}'_2)(\mathrm{id} + \widetilde{F}_1 \circ \widetilde{\partial}'_1)(\Lambda)\end{aligned}$$

and have the following

Proposition 3.3.14. *$\widetilde{\kappa}$ is an isomorphism.*

Let $\{E^r_{p,q}, d_r\}_r$ be the spectral sequence associated to the double complex $\widetilde{\mathbb{NQ}}_{p,q}(h,m)$. Then $\{E^0_{p,q} = \widetilde{\mathbb{NQ}}_{p,q}(h,m)\}$ and $d_0 = \widetilde{\partial}' : \widetilde{\mathbb{NQ}}_{p,q}(h,m) \to \widetilde{\mathbb{NQ}}_{p,q-1}(h,m)$. Theorem 3.3.13 says that, the vertical homology $E^1_{p,q} = H_q(\widetilde{\mathbb{NQ}}_{p,*}(h,m), \widetilde{\partial}')$ is concentrated in the top degree $q = h$. Thus the E^1-term is a chain complex with differential $d_1 : E^1_{p,q} \to E^1_{p-1,q}$ induced by $\widetilde{\partial}''$ and the spectral sequence collapses with $E^2 = E^\infty$.

We can define a complex $(\widetilde{W}^N_*(h,m), \widetilde{d})$ by letting

$$\widetilde{W}^N_p(h,m) := \widetilde{\kappa}^{-1}(H_h(\widetilde{\mathbb{NQ}}_{p,*}(h,m))) \subseteq V_h(\Lambda_p)$$

and $\widetilde{d} := \widetilde{\kappa}^{-1} \circ \widetilde{\partial}'' \circ \widetilde{\kappa}$. This complex is equivalent to the E^1-term of the spectral sequence of $\widetilde{\mathbb{NQ}}_{p,q}(h,m)$ and will be used to compute the homology groups of $\mathfrak{N}^m_{g,1}$.

From the definition of κ and $\widetilde{\kappa}$, it is not difficult to see that $\widetilde{W}^N_p(h,m)$ and $W^N_p(h,m)$ are generated by the same subset of tuples in $R_h(\Lambda_p)[h]$, hence they are the same $\mathbb{Z}$-modules.

Chapter 4

Computational Results

In this chapter we will present the results of the homology computations. The theoretical background for this computation has been already presented in the first three chapters and the computation is done using computer programs.

4.1 The Riemann Surface Case

Integral homology of $\mathfrak{M}$ for $h \leq 5$ was computed by Ehrenfried and Abhau (see [Eh], [A] or [ABE]), where the double complexes $\mathbb{Q}_{\bullet\bullet}(h,m)$ and $\widetilde{\mathbb{Q}}_{\bullet\bullet}(h,m)$ were used.

The complexes $(W_*(h,m),d)$ and $(\widetilde{W}_*(h,m),\widetilde{d})$ introduced in chapter 2 can make the computation easier, because it reduces the procedure from two main steps – i.e., the E_0- and E_1-terms – to one step; and the construction of generators in $\widetilde{W}_p(h,m)$ is also more direct than in the E_1-term. This enables us to compute the homology groups for the next cases when $h = 6$. However, $\widetilde{W}_p(h,m)$ still has a large number of cells for some p unless $m = 6$, hence the incidence matrices are huge in this case, which makes the computation of the integral Smith normal form a heavy task for the computer. As a result, we cannot compute the integral homology of $\mathfrak{M}$ for $h = 6$ under reasonable condition and in reasonable time.

However, we are able to get partial information on the homology. Namely, we compute the $\mathbb{Z}_{l^k}$ torsion parts of the homology groups, when l is a small prime and k a small integer. This is done by computing the Smith normal form (SNF) over the ring $\mathbb{Z}_{l^k}$.

The following table shows the number of generators for $\widetilde{W}_p(h,m)$, when $h = 6$, $m = 0,2,4,6$ respectively.

$m \backslash p$	$0,1$	2	3	4	5	6	7	8	9	10	11	12
0	0	1	82	1221	7640	26150	54756	73582	63976	34905	10890	1485
2	0	0	42	1302	12180	54390	138348	216342	212772	128730	43890	6468
4	0	0	0	0	420	4410	18060	38850	48300	35070	13860	2310
6	0	0	0	0	0	0	132	660	1320	1320	660	132

For the case $h = 6, m = 6$ ($\mathfrak{M}^6_{0,1}$), we can determine the homology group completely:

$$H_n(\mathfrak{M}^6_{0,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} & n = 1 \\ \mathbb{Z}_2 & n = 2 \\ \mathbb{Z}_2 & n = 3 \\ \mathbb{Z}_3 & n = 4 \\ 0 & n \geq 5 \end{cases}$$

In the following, we list the partial information we get for $h = 6$, $m = 0, 2, 4$ respectively. In the tables, "torsion" contains the complete list of torsion summands of the forms $\mathbb{Z}_{2^k}(1 \leq k \leq 6)$, $\mathbb{Z}_{3^k}(1 \leq k \leq 4)$, $\mathbb{Z}_{5^k}(k = 1, 2)$, $\mathbb{Z}_7$, $\mathbb{Z}_{11}$ and $\mathbb{Z}_{13}$ in the n-th homology group. $\beta_n(l)$ denotes the n-th mod-l Betti numbers and $* \leq 23$ is any prime number, which is bigger than the l in the previous line in the corresponding table.

The case $h = 6, m = 0$ ($\mathfrak{M}^0_{3,1}$):

$n =$	0	1	2	3	4	5	6	7	8	9
Torsion			$\mathbb{Z}_2$	$\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_3, \mathbb{Z}_7$	$\mathbb{Z}_2^2, \mathbb{Z}_3^2$	$\mathbb{Z}_2, \mathbb{Z}_3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$		
$\beta_n(2)$	1	0	2	4	4	4	5	4	1	1
$\beta_n(3)$	1	0	1	2	3	4	2	0	0	1
$\beta_n(5)$	1	0	1	1	0	1	1	0	0	1
$\beta_n(7)$	1	0	1	2	1	1	1	0	0	1
$\beta_n(*)$	1	0	1	1	0	1	1	0	0	1

The case $h = 6, m = 2$ ($\mathfrak{M}^2_{2,1}$):

$n =$	0	1	2	3	4	5	6	7	8
Torsion		$\mathbb{Z}_2^2, \mathbb{Z}_5$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5, \mathbb{Z}_3^3$	$\mathbb{Z}_2^4, \mathbb{Z}_3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	
$\beta_n(2)$	1	2	5	9	10	11	9	4	1
$\beta_n(3)$	1	0	1	3	4	6	3	0	0
$\beta_n(5)$	1	1	2	3	1	2	2	0	0
$\beta_n(*)$	1	0	1	3	1	2	2	0	0

The case $h = 6, m = 4$ ($\mathfrak{M}^4_{1,1}$):

$n =$	0	1	2	3	4	5	6
Torsion		$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	
$\beta_n(2)$	1	2	4	8	8	5	2
$\beta_n(*)$	1	1	0	2	3	2	1

4.2 The Kleinian Surface Case

In [Z1], the mod-2 homology of $\mathfrak{N}^m_{g,1}$ was computed for $h = g + m + 1 = 2, 3$, where the double complex $\mathbb{NQ}_{\bullet\bullet}(h, m)$ was used. Here we note again that the genus of a non-orientable surface is g, if it is a connected sum of $g + 1$ real projective planes.

The complex $(\widetilde{W}^N_*(h, m), \widetilde{d})$ allows us to compute the integral homology of these moduli spaces and more-namely when $h = 4, 5$. In the cases $h = 5, m < 4$, $\widetilde{W}^N_p(h, m)$ has again a large number of generators, hence we will compute the $\mathbb{Z}_{l^k}$ torsion parts of the homology groups, when l is a small prime and k a small integer.

The following table shows the number of generators for $\widetilde{W}^N_p(h, m)$, when $2 \le h \le 4$.

	$p =$	$0,1$	2	3	4	5	6	7	8
$h = 2$	$m = 0$	0	2	6	4	-	-	-	-
	$m = 1$	0	1	6	5	-	-	-	-
$h = 3$	$m = 0$	0	5	48	122	120	41	-	-
	$m = 1$	0	2	30	96	110	42	-	-
	$m = 2$	0	0	6	34	50	22	-	-
$h = 4$	$m = 0$	0	10	222	1228	2940	3522	2086	488
	$m = 1$	0	5	174	1208	3320	4377	2786	690
	$m = 2$	0	0	24	280	1000	1560	1120	304
	$m = 3$	0	0	0	29	180	366	308	93

For all the cases with $2 \le h \le 4$, we can compute the integral homology groups completely.

The case $h = 2, m = 0$ ($\mathfrak{N}^0_{1,1}$):

$$H_n(\mathfrak{N}^0_{1,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 1 \\ 0 & n \ge 2 \end{cases}$$

The case $h = 2, m = 1$ ($\mathfrak{N}^1_{0,1}$):

$$H_n(\mathfrak{N}^1_{0,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} & n = 1 \\ 0 & n \ge 2 \end{cases}$$

The case $h = 3, m = 0$ ($\mathfrak{N}^0_{2,1}$):

$$H_n(\mathfrak{N}^0_{2,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2^2 & n = 1 \\ \mathbb{Z}_2 & n = 2 \\ \mathbb{Z} & n = 3 \\ 0 & n \ge 4 \end{cases}$$

The case $h = 3, m = 1$ ($\mathfrak{N}^1_{1,1}$):

$$H_n(\mathfrak{N}^1_{1,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2^2 & n = 1 \\ \mathbb{Z}_2 & n = 2 \\ 0 & n \geq 3 \end{cases}$$

The case $h = 3, m = 2$ ($\mathfrak{N}^2_{0,1}$):

$$H_n(\mathfrak{N}^2_{0,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Here we want to point out that, the above results for $h = 2, 3$ are consistent with the mod-2 homology computation in [Z1], except for the homology groups $H_1(\mathfrak{N}^1_{1,1}, \mathbb{Z}_2)$ and $H_2(\mathfrak{N}^1_{1,1}, \mathbb{Z}_2)$. According to our computation, $H_1(\mathfrak{N}^1_{1,1}, \mathbb{Z}_2) = H_2(\mathfrak{N}^1_{1,1}, \mathbb{Z}_2) = \mathbb{Z}_2^3$, whereas the result in [Z1] is $H_1(\mathfrak{N}^1_{1,1}, \mathbb{Z}_2) = H_2(\mathfrak{N}^1_{1,1}, \mathbb{Z}_2) = \mathbb{Z}_2^2$.

The case $h = 4, m = 0$ ($\mathfrak{N}^0_{3,1}$):

$$H_n(\mathfrak{N}^0_{3,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2^3 & n = 1 \\ \mathbb{Z}_2^5 & n = 2 \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_4^2 & n = 3 \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2^2 & n = 4 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 5 \\ 0 & n \geq 6 \end{cases}$$

The case $h = 4, m = 1$ ($\mathfrak{N}^1_{2,1}$):

$$H_n(\mathfrak{N}^1_{2,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2^3 & n = 1 \\ \mathbb{Z}_2^4 & n = 2 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_2^2 & n = 3 \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2 & n = 4 \\ 0 & n \geq 5 \end{cases}$$

The case $h = 4, m = 2$ ($\mathfrak{N}^2_{1,1}$):

$$H_n(\mathfrak{N}^2_{1,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2^3 & n = 1 \\ \mathbb{Z}_2^4 & n = 2 \\ \mathbb{Z}^2 & n = 3 \\ \mathbb{Z}^2 & n = 4 \\ 0 & n \geq 5 \end{cases}$$

The case $h = 4, m = 3$ ($\mathfrak{N}^3_{0,1}$):

$$H_n(\mathfrak{N}^3_{0,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 1 \\ \mathbb{Z}_2 & n = 2 \\ 0 & n \geq 3 \end{cases}$$

The following table gives the number of generators for $\widetilde{W}_p^N(h,m)$, when $h=5$.

$m\backslash p$	$0,1$	2	3	4	5	6	7	8	9	10
0	0	21	1008	10284	44880	104334	140112	109452	46320	8229
1	0	10	726	9172	45880	117630	170310	141336	62892	11660
2	0	0	126	2676	17360	52350	84910	76736	36516	7150
3	0	0	0	188	2110	8430	16380	16840	8838	1870
4	0	0	0	0	130	906	2324	2836	1674	386

For the case $h=5, m=4$ ($\mathfrak{N}_{0,1}^4$), we can determine its homology group completely:

$$H_n(\mathfrak{N}_{0,1}^4) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & n=1 \\ \mathbb{Z}_2^2 & n=2 \\ \mathbb{Z}_4 & n=3 \\ 0 & n\geq 4 \end{cases}$$

In the following, we list the partial information we get for $h=5$, $m=0,1,2,3$ respectively. As in the last section, "torsion" contains the complete list of torsion summands of the forms $\mathbb{Z}_{2^k}(1\leq k\leq 6)$, $\mathbb{Z}_{3^k}(1\leq k\leq 4)$, $\mathbb{Z}_{5^k}(k=1,2)$, $\mathbb{Z}_7$, $\mathbb{Z}_{11}$ and $\mathbb{Z}_{13}$ in the n-th homology group. And $\beta_n(l)$ denotes the n-th mod-l Betti numbers; $*\leq 23$ is any prime number, which is bigger than the l in the previous line in the corresponding table.

The case $h=5, m=0$ ($\mathfrak{N}_{4,1}^0$):

$n=$	0	1	2	3	4	5	6	7
Torsion		$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4,\mathbb{Z}_4^3,\mathbb{Z}_5$	$\mathbb{Z}_2^8,\mathbb{Z}_4,\mathbb{Z}_3$	$\mathbb{Z}_2^8,\mathbb{Z}_3$	$\mathbb{Z}_2^2$	
$\beta_n(2)$	1	2	6	12	17	17	10	3
$\beta_n(3)$	1	0	0	1	2	2	1	1
$\beta_n(5)$	1	0	0	2	2	0	0	1
$\beta_n(*)$	1	0	0	1	1	0	0	1

The case $h=5, m=1$ ($\mathfrak{N}_{3,1}^1$):

$n=$	0	1	2	3	4	5	6	7
Torsion		$\mathbb{Z}_2^4$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8,\mathbb{Z}_4^3$	$\mathbb{Z}_2^{11},\mathbb{Z}_4$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2$	
$\beta_n(2)$	1	4	12	23	29	23	9	1
$\beta_n(*)$	1	0	0	4	6	3	0	0

The case $h=5, m=2$ ($\mathfrak{N}_{2,1}^2$):

$n=$	0	1	2	3	4	5	6
Torsion		$\mathbb{Z}_2^4$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^7,\mathbb{Z}_4$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^2$	
$\beta_n(2)$	1	4	11	20	21	11	2
$\beta_n(*)$	1	0	0	5	6	2	0

The case $h = 5, m = 3$ ($\mathfrak{N}^3_{1,1}$):

$n =$	0	1	2	3	4	5
Torsion		$\mathbb{Z}_2^3$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	
$\beta_n(2)$	1	4	8	10	7	2
$\beta_n(*)$	1	1	0	3	4	1

From the last line of each table for the homology groups, we see that when $*$ is bigger than a certain prime, $\beta_n(*)$ remains the same until $* = 23$—the biggest prime we have tested. This "stabilization" phenomenon leads us to the conjecture that the torsion terms we have listed for $\mathfrak{M}^m_{g,1}$ when $h = 6, m = 0, 2, 4$ and for $\mathfrak{N}^m_{g,1}$ when $h = 5, m = 0, 1, 2, 3$ are *all* the torsion terms in the respective homology groups, and therefore $\beta_n(*)$ denotes the dimension of the n-th rational homology groups. Besides, some of the homology groups can be determined from the computation with the help of the universal coefficient theorem. We summarize these in the following:

Conjecture 4.2.1.

$$H_n(\mathfrak{M}^0_{3,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7 & n = 3 \\ \mathbb{Z}_2^2 \oplus \mathbb{Z}_3^2 & n = 4 \\ \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 & n = 5 \\ \mathbb{Z} \oplus \mathbb{Z}_2^3 & n = 6 \\ \mathbb{Z}_2 & n = 7 \\ 0 & n = 8 \\ \mathbb{Z} & n = 9 \\ 0 & n \geq 10 \end{cases}$$

When $n = 0, 1, 4, 7, 8, 9$ or $n \geq 10$, the above results for $H_n(\mathfrak{M}^0_{3,1})$ are verified. And if we take the result $H_2(\mathfrak{M}^0_{3,1}) = \mathbb{Z} \oplus \mathbb{Z}_2$ from [S], then $H_3(\mathfrak{M}^0_{3,1})$ is also confirmed.

$$H_n(\mathfrak{M}^2_{2,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2^2 \oplus \mathbb{Z}_5 & n = 1 \\ \mathbb{Z} \oplus \mathbb{Z}_2^2 & n = 2 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_2^4 & n = 3 \\ \mathbb{Z} \oplus \mathbb{Z}_2^5 \oplus \mathbb{Z}_3^3 & n = 4 \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_3 & n = 5 \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2^3 & n = 6 \\ \mathbb{Z}_2 & n = 7 \\ 0 & n \geq 8 \end{cases}$$

When $n = 0, 1, 7$ or $n \geq 8$, the above results for $H_n(\mathfrak{M}^2_{2,1})$ are verified.

$$H_n(\mathfrak{M}^4_{1,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 1 \\ \mathbb{Z}_2^3 & n = 2 \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2^3 & n = 3 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_2^2 & n = 4 \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2 & n = 5 \\ \mathbb{Z} & n = 6 \\ 0 & n \geq 7 \end{cases}$$

When $n = 0, 1, 2$ or $n \geq 7$, the above results for $H_n(\mathfrak{M}^4_{1,1})$ are verified.

$$H_n(\mathfrak{N}^0_{4,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2^2 & n = 1 \\ \mathbb{Z}_2^4 & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_5 & n = 3 \\ \mathbb{Z} \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 & n = 4 \\ \mathbb{Z}_2^8 \oplus \mathbb{Z}_3 & n = 5 \\ \mathbb{Z}_2^2 & n = 6 \\ \mathbb{Z} & n = 7 \\ 0 & n \geq 8 \end{cases}$$

When $n = 0, 1, 2, 5, 6, 7$ or $n \geq 8$, the above results for $H_n(\mathfrak{N}^0_{4,1})$ are verified.

$$H_n(\mathfrak{N}^1_{3,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2^4 & n = 1 \\ \mathbb{Z}_2^8 & n = 2 \\ \mathbb{Z}^4 \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_4^3 & n = 3 \\ \mathbb{Z}^6 \oplus \mathbb{Z}_2^{11} \oplus \mathbb{Z}_4 & n = 4 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_2^8 & n = 5 \\ \mathbb{Z}_2 & n = 6 \\ 0 & n \geq 7 \end{cases}$$

When $n = 0, 1, 2, 6$ or $n \geq 7$, the above results for $H_n(\mathfrak{N}^1_{3,1})$ are verified.

$$H_n(\mathfrak{N}^2_{2,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2^4 & n = 1 \\ \mathbb{Z}_2^7 & n = 2 \\ \mathbb{Z}^5 \oplus \mathbb{Z}_2^7 \oplus \mathbb{Z}_4 & n = 3 \\ \mathbb{Z}^6 \oplus \mathbb{Z}_2^7 & n = 4 \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2^2 & n = 5 \\ 0 & n \geq 6 \end{cases}$$

When $n = 0, 1, 2$ or $n \geq 6$, the above results for $H_n(\mathfrak{N}^2_{2,1})$ are verified.

$$H_n(\mathfrak{N}^3_{1,1}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2^3 & n = 1 \\ \mathbb{Z}_2^5 & n = 2 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_2^2 & n = 3 \\ \mathbb{Z}^4 \oplus \mathbb{Z}_2 & n = 4 \\ \mathbb{Z} & n = 5 \\ 0 & n \geq 6 \end{cases}$$

When $n = 0, 1$ *or* $n \geq 6$*, the above results for* $H_n(\mathfrak{N}^3_{1,1})$ *are verified.*

Appendix: The computer program

In this appendix we explain the structure and some important functions of the computer program, with which the homology computations are done. The program is written in C++. It is attached on a CD to this dissertation.

The chain complexes

The files `chain_complex_Z.h` and `chain_complex_Z.cpp` provide the implementation of the complex $(\widetilde{W}_*(h,m),\widetilde{d})$, while the files `chain_complex_non_Z.h` and `chain_complex_non_Z.cpp` provide the implementation of $(\widetilde{W}_*^N(h,m),\widetilde{d})$. And as the basis for both complexes, transpositions are implemented in the file `transposition.h` and `transposition.cpp`.

The files `transposition.h` and `transposition.cpp` contain the class `transposition`, which gives the construction of and functions on transpositions. A transposition t is constructed as an ordered pair of numbers (h,l), where $h>l$; and the functions on a transposition t include: to check whether a given number s is not fixed by t (`contain(s)`); to find the image of s, if s is not fixed by t (`partner(s)`).

First we give an overview of the classes in the program files `chain_complex_Z.h` and `chain_complex_Z.cpp`.

- Class `htupel` implements a tuple of transpositions. A q-tuple of transpositions ht is represented by a vector $(t_0,\ldots,t_{q-1})$, where $t_i, 1\le i\le q-1$ is the representation of a transposition according to the class `transposition`. The important functions on an `htupel` ht include:
  - `type()`: To check if ht is a monotone tuple.
  - `f(i)`: Computes $\widetilde{f}_i(ht)$ defined in Section 2.3. Or equivalently, to get $f_i(ht)$, which is defined in Section 1.2 and to give the sign $\epsilon'_i(ht)$ defined in 2.3 simultaneously. If the return value is -1, $\partial'_i(ht)$ is degenerate; if the return value is 0, the sign $\epsilon'_i(ht)$ is 1. If the return value is 1, $\epsilon'_i(ht)$ is -1.
  - `hboundary(k)`: Computes $\widetilde{\partial}''_k(ht)$. Or equivalently, to get $\partial''_k(ht)$ and give the sign $\epsilon''_k(ht)$ defined in 2.3 simultaneously. If the return value is -1, $\partial''_k(ht)$ is degenerate; if the return value is 0, the sign $\epsilon''_k(ht)$ is 1. If the return value is 1, $\epsilon''_k(ht)$ is -1.

- `puncture(p)`: Counts the number of punctures of ht, under the assumption that every transposition in ht is in $\mathfrak{S}_p$.
- `operator==(ht2)` and `operator<=(ht2)`: Implements an ordering of tuples.

Asides from these, the function `height(k,i)` is used in the functions `f(i)` and `hboundary(k)` during the computation of the orientation system $\mathcal{O}$.

- Class `generator_list` gives the construction of– and functions on–the complete lists of generators of the complex $(\widetilde{W}_*(h,m),\widetilde{d})$. These generators are tuples of transpositions described in Section 2.2. A list is represented by a vector of `htupel`s. Functions in the class `generator_list` include:

  - `generator_list(int h, int p)`: This constructs the list of generators of

  $$\bigoplus_m(\widetilde{W}_p(h,m),\widetilde{d}),$$

  where the direct sum is over $m=0,2,\dots,h$ if h is even and over $m=1,3,\dots,h$ if h is odd.
  - `search(ht)`: Finding a tuple ht in the list.

- Class `htupel_with_multiplicity` is a subclass of `htupel`. A `htupel_with-_multiplicity` htp is a `htupel` ht with an additional datum mp, which is an integer standing for "multiplicity". This is meant to represent the elements of the free abelian group generated by ht. Functions in `htupel_with_multiplicity` include:

  - `sum(htm)`: If htm has the same `htupel` ht as htp, then add the multiplicity of htm to the multiplicity of htp. This is the addition operation in the free abelian group generated by ht.
  - `negative()`: Change the mp of htp to $-mp$. This is the inverse operation in the free abelian group generated by ht.

- Class `linear_combination_Z` is the implementation of linear combinations of `htupel`s with coefficients in $\mathbb{Z}$. This is meant to represent the elements of the free abelian group generated by `htupel`s. A `linear_combination_Z` is realized as a set of `htupel_with_multiplicity`s. The important functions on a `linear_combination_Z` lc include:

  - `add(p)`: Add another `linear_combination_Z` p to lc. This is the sum operator in the free abelian group generated by `htupel`s.
  - `hboundary(p)`: To compute the horizontal operator on lc:

  $$\widetilde{\partial}''(lc)=\sum_{i=1}^{p-1}(-1)^i\widetilde{\partial}''_i(lc)$$

  - `f_combined(r)`: To compute $(\mathrm{id}-\widetilde{f}_r+\widetilde{f}_{r-1}\widetilde{f}_r\dots+(-1)^r\widetilde{f}_1\cdots\widetilde{f}_r)(lc)$. This is the r-th inductive step in computing $\tilde{\kappa}$, which is defined in Section 2.3.

- `matrix_construction (h,p,c)`: To compute the incidence matrix for the boundary operator: $\widetilde{W}_p(h,c) \xrightarrow{\widetilde{d}_p} \widetilde{W}_{p-1}(h,c)$. The matrix is written to a file named `matrix_Z_h_p_c.txt`.

Now we are going to describe the classes in `chain_complex_non_Z.h` and `chain_complex_non_Z.cpp`.

- Class `transposition_pair_rf` implements norm-one morphisms in the category of pairings. As discussed in Section 3.2, such a morphism t from the pairing λ is of the form $t = (h_2, l_2)(h_1, l_1)$, where $h_1 > l_1, h_2 > l_2$ are four disjoint numbers and $h_1 = \max\{h_1, l_1, h_2, l_2\}$. The `transposition_pair_rf` t is constructed as a ordered pair of `transpositions` (t_L, t_H) and a logical data CR, where $t_H = (h_1, l_1)$, $t_L = (h_2, l_2)$; and $CR =$`true` if $\lambda(h_1) = h_2$, $CR =$`false` if $\lambda(h_1) = l_2$. The important functions on `transposition_pair_rf` t include:
  - `contain(s)`: To check whether a given number s is not fixed by t.
  - `partner(s)`: to find the image of number s, if s is not fixed by t.

- Class `htupel_non` implements a tuple of norm-one morphisms in the category of pairings. A q-tuple of of norm-one morphisms ht is represented by a vector $(t_0, \ldots, t_{q-1})$, where $t_i, 1 \le i \le q-1$ is the representation of a norm-one morphism according to the class `transposition_pair_rf`.
  - `type()`: To check if ht is a monotone tuple.
  - `non_orientable()`: To check if the surface associated to ht is non-orientable.
  - `f(i)`: To compute $\widetilde{f}_i(ht)$ defined in Section 3.3. Or equivalently, to get $f_i(ht)$, which is defined in 1.2 and to give the sign $\epsilon'_i(ht)$ defined in Section 3.3 simultaneously. If the return value is -1, $\partial'_i(ht)$ is degenerate; If the return value is 0, the sign $\epsilon'_i(ht)$ is 1; If the return value is 1, $\epsilon'_i(ht)$ is -1.
  - `hboundary(k)`: To compute $\widetilde{\partial}''_k(ht)$ defined in Section 3.3. Or equivalently, to get $\partial''_k(ht)$ and give the sign $\epsilon''_k(ht)$ defined in Section 3.3 simultaneously. If the return value is -1, $\partial''_k(ht)$ is degenerate; If the return value is 0, the sign $\epsilon''_k(ht)$ is 1; If the return value is 1, $\epsilon''_k(ht)$ is -1.
  - `puncture(p)`: To compute the number of punctures of ht, under the assumption that every norm-one morphism in ht is in Λ_p.
  - `operator==(ht2)` and `operator<=(ht2)`: Implements an ordering of tuples.

  Aside from these, there are functions, which serve as technically intermediate steps in computing the orientation system $\mathcal{O}$ in the functions in `f(i)` and `hboundary(k)`. Such functions include `complete(i, t)`, `height(i, a, b, c)` and `alternate(i, c)`. Among these, `complete(i, t)` implements the process "complete" and `alternate(i, c)` implements the operation "alternate"; both concepts are defined in Section 3.3.

- Class `generator_list_non` gives the construction of and functions on the complete lists of generators of the complex $(\widetilde{W}^N_*(h,m), \widetilde{d})$. These generators are tuples of norm-one morphisms in the category of pairings described in Section 3.2. A list is represented by a vector of `htupel_nons`.

- `generator_list_non(int h, int p)`: This constructs the list of generators of

$$\bigoplus_{m=0}^{h-1}(\widetilde{W}_p^N(h,m),\widetilde{d}).$$

- `search(ht)`: To search a tuple ht in the list.

- Class `htupel_non_with_multiplicity` is a subclass of `htupel_non`. A `htupel-_non_with_multiplicity` *htp* is a `htupel_non` *ht* with an additional datum *mp*, which is an integer standing for "multiplicity". This is meant to represent the elements of the free abelian group generated by *ht*. The important functions on an `htupel_non_with_multiplicity` *htp* include:

  - `sum(htm)`: If *htm* has the same `htupel_non` *ht* as *htp*, then add the multiplicity of *htm* to the multiplicity of *htp*. This is the sum operation in the free abelian group generated by *ht*.
  - `negative()`: Change the *mp* of *htp* to $-mp$. This is the inverse operation in the free abelian group generated by *ht*.

- Class `linear_combination_non_Z` is the implementation of linear combinations of `htupel_non`s with coefficients in $\mathbb{Z}$. This is meant to represent the elements of the free abelian group generated by `htupel_non`s. A `linear_combination_non_Z` is realized as a set of `htupel_non_with_multiplicity`s. The important functions on a `linear_combination_non_Z` *lc* include:

  - `add(p)`: Add another `linear_combination_non_Z` p to lc. This is the sum operator in the free abelian group generated by `htupel_non`s.
  - `hboundary(p)`: To compute the horizontal operator on lc:

$$\widetilde{\partial}''(lc) = \sum_{i=1}^{p-1}(-1)^i\widetilde{\partial}_i''(lc)$$

  - `f_combined(r)`: To compute $(\mathrm{id} - \widetilde{f}_r + \widetilde{f}_{r-1}\widetilde{f}_r \ldots + (-1)^r\widetilde{f}_1\cdots\widetilde{f}_r)(lc)$. This is the r-th inductive step in computing $\tilde{\kappa}$, which is defined in Section 3.3.
  - `matrix_construction (h,p,c)`: To compute the incidence matrix for the boundary operator: $\widetilde{W}_p^N(h,c) \xrightarrow{\widetilde{d}_p} \widetilde{W}_{p-1}^N(h,c)$. The matrix is written to a file named `matrix_non_Z_h_p_c.txt`.

In addition, we also have the files `chain_complex_Z_2.h`, `chain_complex_Z_2.cpp`, `chain_complex_non_Z_2.h` and `chain_complex_non_Z_2.cpp`, which are implementing the mod-2 version of the complex $(\widetilde{W}_*(h,m),\widetilde{d})$ and $(\widetilde{W}_*^N(h,m),\widetilde{d})$ respectively. Since in the mod-2 case one does not need to compute the coefficient system, some important functions in these files are simpler and faster. However, to obtain the incidence matrices is not the most demanding part in the homology computation; thus these files are not necessary for the computations done in this thesis, but are added on the accompanying CD for completeness.

About the matrix computations

To determine the integral homology groups of $(\widetilde{W}_*(h,m),\widetilde{d})$ and $\widetilde{W}_*^N(h,m),\widetilde{d})$, one needs to compute the Smith normal form (SNF) of each incidence matrix. There are many programs available to compute SNF, if a matrix is not too large. For example, we have applied the computer program by Abhau ([A]) for SNF computation. It took only a few hours on a standard 64-bit machine to confirm the computations Abhau had done before.

However, it is still not possible to compute the SNF of a very large matrix in reasonable time on standard workstations; this is the computationally most demanding part for our homology computation. In the computations we have made, it was not possible to obtain the SNF for some incidence matrices from $(\widetilde{W}_*(h,m),\widetilde{d})$, when $h=6, m=0,2,4$ and from $(\widetilde{W}_*^N(h,m),\widetilde{d})$, when $h=5, m=0,1,2,3,4$. At present, the limitation seems to be mostly due to available memory. For example, the matrix `matrix_non_Z_6_7_0.txt` required more than 8GB of RAM after a few hours of computation, though the initial matrix needs less than 10 MB.

Because we wanted to use deterministic algorithms only, we have written programs to compute the Smith normal form over $\mathbb{Z}_{l^k}$, where l is a prime and k an integer. In this way, torsion terms of the form $\mathbb{Z}_{l^i}, i=1,\ldots,k-1$ of the homology groups can be computed.

The files `sparse_matrix.h`, `sparse_matrix.cpp`, `ppower_matrix.h` and `ppower_matrix.cpp` deal with the computation of SNF over $\mathbb{Z}_{l^k}$. Now we give a brief overview of these, and in particular, some methods behind making the computation less demanding with respect to memory consumption and running time.

Because the incidence matrices involved are sparse matrices, we use a sparse form to represent a mod-P matrix, with $P>1$ an integer. This is implemented in the files `sparse_matrix.h` and `sparse_matrix.cpp`. Suppose a matrix M has r non-zero rows, then the data of M has r arrays, one for each row; and each array records the non-zero entries and their corresponding column indices in pairs (value, index), in increasing order by the index-entry. Moreover, we pack every pair of data (value, index) into a single machine integer to save memory.

The files `ppower_matrix.h` and `ppower_matrix.cpp` contain the class `ppower_matrix`, which is a subclass of `sparse_matrix`. The important functions here are:

`matrix_read(s,int P)`: This function reads the contents of the file named s into a mod-P `ppower_matrix`. The file s should be in the same format as a incidence matrix `matrix_Z_h_p_c.txt` constructed by `matrix_construction (h,p,c)`. Examples for matrix files can be found on the CD.

`SR_Rank()`: This computes the SNF over the ring $\mathbb{Z}_{l^k}$, where, with l a prime and k a positive integer. The result is a k-vector v, where $v[0]$ is the mod-l rank of the matrix, and each $v[i], 1\le i\le k-1$ is the number of l^i terms in the SNF. The reference for the algorithm is [DSV]. Moreover, we adopt also the "reordering" technique described in [DV] to save memory and computation time, thus making the computation possible for the large matrices occurring in this thesis.

In addition, the files `fp_matrix.h` and `fp_matrix.cpp` are about the class `fp_matrix`, also a subclass of `sparse_matrix`. It deals with the matrices over Z_l, with l a prime, a special case of the `ppower_matrix`. The function `SR_Rank()` here computes the mod-l rank of a matrix. Though this is already covered in the corresponding function in `ppower_matrix`, the algorithm here is more direct and faster.

The files `sparse_f2_matrix.h`, `sparse_f2_matrix.cpp`, `f2_matrix.h`, `f2_matrix.cpp` are especially written for the computation over $\mathbb{Z}_2$. That is to say, for computing the mod-2 rank of matrices and hence mod-2 Betti numbers. Because of the simplicity of $\mathbb{Z}_2$, the algorithms in these files are more efficient both in memory and running time than the general case described above.

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