Essays in Monetary Theory

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Introduction

There is some dispute over what constitutes money. A general consensus, however, is that an object which is widely used as a medium of exchange is referred to as money. Historically, objects such as shells, whales' teeth and stones but also productive objects such as grain and cattle circulated as money according to Einzig (1966). In this dissertation, however, money exclusively refers to fiat money, i.e. intrinsically worthless pieces of paper which are provided by the government or a central bank.

The central issue of monetary theory is to explain why agents value an intrinsically worthless object and why they are willing to trade goods and services in exchange for it. It seems quite intuitive that money serves as a medium of exchange. However, it might not be so clear why a medium of exchange is needed for trade to occur in the first place. Historically, for example, primitive tribes used gift-giving arrangements instead of a medium of exchange according to Fusfeld (1957). But even if we accept that a medium of exchange is necessary to facilitate trade, the question remains why money takes this role if other assets are available. A medium of exchange is only accepted against goods because it enables consumption in the future. Thus, an object which serves as a medium of exchange needs to be durable. In contrast to other objects, however, money does not seem to qualify as a good store of value. Given a positive rate of inflation money loses value each period, whereas other assets obtain a positive return. Brunner and Meltzer (1971) report that money circulated as a medium of exchange in many countries even in periods of high inflation. Thus, money must be better suited as a medium of exchange than other assets, i.e. it must be more marketable, to overcome its inferiority as a store of value. This, finally, raises the question why money is more marketable than other assets.

There are different ways of generating a demand for money in the economic literature. An analysis of the demand for money in partial equilibrium is given by the theory of transaction demand proposed by Baumol (1952) and Tobin (1956). They study agents' investment behavior within a period. Agents can invest their given resources in either interest-bearing bonds or money which yields no interest. The demand for money is generated because money is necessary to carry out transactions. The agents' objective is to spend their entire endowment evenly during the course of the period. Thus, agents optimally choose how often to convert their bond holdings into money. The model, however, does not explain who the counterpart of the agents' transactions is [compare Hellwig (1993)].

An early general equilibrium version of the demand for money is Alchian (1977). In his model, trade is decentralized and agents need a medium of exchange. There are several goods, all of which can be used as media of exchange. When purchasing a good, agents need to spend resources to verify its quality. Government issued money, however, does not require an inspection due to the government's high reputation. That is why money is used as a medium of exchange in Alchian (1977).

A popular way to incorporate money into macroeconomic Walrasian models was proposed by Sidrauski (1967) and is called the *Money in Utility (MIU)* approach. As the name suggests, an agent's money holdings is an argument of his utility function in MIU models. Agents receive direct utility from holding useless pieces of paper and they receive consumption utility when they spend it in the future. Thus, the motivation to hold money in MIU models differs from our understanding that people hold money because they can use it as a medium of exchange.

Another approach to implement money into the Walrasian framework goes back to Clower (1967). It is called the *Cash in Advance (CIA)* approach. In CIA models, consumption goods can only be purchased using money as the only means of payment. That is, CIA models generate a demand for money because of its role as a medium of exchange. The CIA constraint is rather extreme, however, because it requires money to be the only asset that can be used to acquire consumption goods. Furthermore, Hellwig (1993) points out that money is not necessary to guarantee "that agents actually pay for what they buy" because of "the simultaneity of all exchanges under the Walrasian budget constraint" [Hellwig (1993), p. 221]. Thus, it seems artificial to impose a CIA constraint within the Walrasian framework.

Finally, the framework which is used to study money in this dissertation is called *Monetary Search Theory* as proposed by Kiyotaki and Wright (1989, 1993). In contrast to the MIU and the CIA approach, Monetary Search Theory departs from the assumption of a frictionless Walrasian market. A key feature of monetary search models is a lack of what Jevons (1910) calls a *double coincidence of wants*. This is, for example, achieved in a model with bilateral trade where only one of the agents wants to consume the other's good. An alternative way of constructing a *coincidence of wants* problem would be a multilateral meeting in which only a fraction of the agents can produce the good which the rest of the agents wants to consume.

Kocherlakota (1998) shows that a lack of a *double coincidence of wants* is not sufficient, however, to generate a demand for money. Consider an economy populated by an infinite number of infinitely-lived agents who are randomly matched in pairs each period. In each match, there is a lack of a double coincidence of wants: Agent i wants to consume agent j's good but not the other way around. Trade occurs without a medium of exchange if agent j delivers the good to agent i free of charge (gift-giving). Such a mechanism can be sustained among utility maximizing agents by a simple trigger strategy: Agent j delivers his goods to agents i only if agent i has never deviated from the gift-giving strategy in the past. Therefore, agent j chooses to deliver goods to agent i if the punishment of never consuming in the future again is severe enough (depending on parameters). Two conditions need to be fulfilled to implement this trigger. First, it must be possible to verify whether or not agents are able to produce their trade partners' good and second, all agents' trade histories must be recorded. Kocherlakota (1998) shows that money replaces the need for monitoring and record-keeping. Assume now that agent i holds a medium of exchange (money). This is a signal to agent j that agent i has worked to obtain the money which is why Kocherlakota (1998) calls money 'memory'. Most money search models do not mention the lack of record-keeping and monitoring, instead, they assume agents to be completely anonymous. Anonymity, however, implies a lack of record-keeping and monitoring.

In general, the money search literature can be divided into three generations. The key features of money search models of the first generation are exposed by examining Kiyotaki and Wright (1993). The economy consists of a continuum of agents. A fraction of them are endowed with a unit of money and the rest holds a unit of a real commodity. There is an infinite number of real commodities with measure one. Each agent receives a positive utility only from a fraction $x \in (0, 1)$ of the available commodities. All goods, that yield a positive consumption utility to an agent, are called his *consumption* goods. Each good is equally popular, i.e. a given good is the consumption good of a fraction x of the population. Agents can produce a unit of a real commodity only after having consumed one of their consumption goods. In contrast to goods, money cannot be produced. Agents are randomly matched in pairs and decide whether or not to trade. An agent is called a *money trader* if he holds a unit of money and he is called a *commodity trader* if he holds a unit of a commodity. Agents are anonymous which prevents trades against credit and, thus, makes a medium of exchange necessary in *single coincidence of wants* meetings.

As common to generation one models, neither goods nor money are divisible which guarantees that trade always occurs one for one. This assumption is crucial as it keeps the distribution of money and goods tractable: At each point in time, an agent holds either a unit of money or a unit of a real commodity. Furthermore, the fraction of agents holding a unit of money is the same as the fraction of agents who were initially endowed with a unit of money.

There are three different equilibria in their model. In the *pure-monetary* equilibrium, agents accept money as a payment against goods because they believe that money will always be accepted in the future. In contrast, money is not accepted to purchase real commodities in the *non-monetary* equilibrium because agents assume that money will not be accepted by other agents in future periods. Finally, there is a *mixed-monetary* equilibrium, where agents are indifferent between accepting money or not: Agents believe that money and real commodities will be accepted with the same probability x. Regardless of whether or not the equilibrium is monetary or non-monetary, agents always swap goods in a *double coincidence of wants* meeting. Each good is equally likely to be the consumption good of future trade partners and agents suffer a transaction cost from receiving a real commodity. Thus, agents do not accept real commodities which are not their consumption goods and trade does not occur in a *single coincidence of wants* meeting in the *non-monetary* equilibrium. In the *pure-monetary* equilibrium, agents trade one unit of the real commodity for one unit of money. Finally, in the *mixed-monetary* equilibrium, only a fraction of agents accepts money.

In contrast to generation one, generation two models allow for the real commodity to be divisible. Money, however, remains indivisible in generation two. Consider Trejos and Wright (1995) as an example for generation two models. As in Kiyotaki and Wright (1993) a fraction of all agents is endowed with a unit of money. Denote agents that hold money as buyers and the rest as sellers. To construct a coincidence of wants problem, each agent can produce a different perishable good. Again, each good yields positive utility to a certain fraction of agents and is equally popular. An agent never receives positive utility from consuming the good he produced. As mentioned above, goods are divisible in generation two models and agents can produce any amount of their distinct good. Finally, disutility from production depends positively on the amount of production.

Double coincidence of wants meetings are excluded by assuming that buyers search for sellers who are fixed in place. Consider a *single coincidence of wants* meeting between a buyer and a seller. In contrast to generation one, the price of money in terms of goods is not fixed at one anymore. Instead, agents need to determine how many goods the seller has to produce in exchange for one unit of money. Thus, the price of money in terms of the good is determined endogenously through bargaining. The outcome is derived by the Nash (1950) bargaining solution. The seller produces the amount of goods agreed upon and gives them to the buyer in exchange for the unit of money. In the next period, the seller becomes a buyer and the buyer becomes a seller. As in generation one, the distribution of money is tractable because the fraction of agents holding money at each point in time is equal to the fraction of agents endowed with money. In contrast to generation one prices are endogenously determined in generation two. There is only a *(pure)-monetary* and a *non-monetary* equilibrium in this model depending on whether or not agents believe that money will be accepted in the future.

Money remains indivisible to guarantee tractability in generation two models. To see this point, assume that money was divisible in Trejos and Wright (1995). The price determined in each match would then depend on the amount of the agents' money holdings which would be a function of all their previous trades. Thus, it would be necessary to keep track of all agents' complete trade histories and this model would not be solvable.

Generation three models allow for money and goods to be divisible. There are three different approaches to deal with the distribution of money when money is divisible. They are introduced in the following order: Shi (1997), Menzio, Shi, and Sun (2011) and Lagos and Wright (2005). We conclude this list with Lagos and Wright (2005) because the models in the following chapters of this dissertation build on the Lagos and Wright (2005) framework.

There is an infinite number of households in Shi (1997). Each household again consists of an infinite number of members whose objective is to maximize household utility, rather than their individual utility. A fraction of each household's members is called money traders and the rest is called producers. Initially, all households are endowed with the same amount of money which they divide evenly among their money traders. Producers can produce a good which is unique to their household. Furthermore, all members of a household only wish to consume a fraction of all available goods. Each period, all members of a household are randomly matched in pairs with members of other households. In a *coincidence of wants* meeting, they determine the terms of trade through Nash bargaining. Subsequently, they return to their respective households. The household consumes the goods and divides the amount of money, which its members brought back, evenly among its money traders, again. Although each single member of a household returns to his household with a stochastic amount of money, each household holds the same amount of money at the end of each period which is due to the law of large numbers.

Assume members maximized their individual utility instead of their household's utility. Members who return to their household with their consumption good would have an incentive to claim that they were not in a *coincidence of wants meeting*. For the same reason, individual utility maximizing members would never produce in a *single coincidence of wants meeting*. Households would need to be able to monitor their members and enforce punishment to induce their desired behavior.

In Menzio, Shi, and Sun (2011) the economy is populated by a continuum of agents who are divided into at least three types. Each type is able to produce a distinct good which is only desired by a fraction of agents of a different type. Depending on their current money holdings, agents choose to be buyers or sellers at the beginning of a period. All sellers of the same type are pooled together in firms. Firms employ the sellers as workers and choose how many goods to produce and where to sell them. The goods market consists of a continuum of submarkets. The terms of trade and the matching probabilities in each submarket are common knowledge. Given that information, firms choose where to open trading posts and buyers choose the submarket they wish to enter. Menzio, Shi, and Sun (2011) use a constant returns to scale matching function to match buyers and trading posts in each submarket. Once a buyer and a trading post are matched, they trade according to the terms of trade specified by their particular submarket.

In Menzio, Shi, and Sun (2011), directed search and free entry of firms guarantee tractability: Only buyers who hold the same amount of money visit the same submarket. Buyers only care about the tradeoff between the matching probability and the terms of trade in their respective submarket but not about the distribution of buyers among other submarkets. On the other hand, the firm knows the money balance of buyers that enter each specific submarket in equilibrium, for sure. Due to free entry, the matching probability in equilibrium delivers the optimal tradeoff for buyers in each submarket. Thus, the model becomes tractable even with non-degenerate money distributions in Menzio, Shi, and Sun (2011).

In Lagos and Wright (2005), agents trade in two distinct markets each period. During the day, agents enter a market which is similar to the one in Trejos and Wright (1995), except that money now is divisible. There is an infinite number of agents and each of them can produce a unique perishable good. To construct a *coincidence of wants* problem, agents only derive positive utility from a fraction of all goods and each good is equally popular, i.e. it is desired by the same amount of agents. In contrast to Trejos and Wright (1995), all agents are endowed with money initially. Again, agents are matched in pairs and determine the terms of trade through Nash bargaining. In a *double coincidence of wants meeting* both agents swap goods without the use of money. In a *single coincidence of wants* meeting, only one agent (buyer) wants to consume the other one's (seller) good. A medium of exchange is necessary for trade to occur because of the prevailing anonymity. Thus, the buyer receives goods from the seller and pays him with money which is the only durable object. After trade in the day market has occurred, agents hold different amounts of money.

At night, trade takes place in a frictionless Walrasian market where agents can produce perishable goods. In contrast to the distinct goods in the day market, all agents derive positive utility from consuming the night market goods. Thus, agents choose consumption, labor and investment in money in the night market. Agents have quasi-linear preferences in the night market. As a consequence, they all choose to leave the night market with the same amount of money, irrespective of their money holdings when entering the night market. Trade histories do not need to be considered to determine the terms of trade in the day market because all agents enter the day market with the same money holdings each period.

The Lagos and Wright (2005) trick comes at a price, however. First, the Lagos and Wright (2005) framework does not generate a wealth effect, i.e. the marginal value of money in the night market is the same for all agents irrespective of their money holdings. Note that there would be a wealth effect in the day market if agents entered it with different amounts of money. Second, the framework is not suited to study price dynamics because the night market sets the economy back to its initial state at the end of each period.

At this point we would like to define the terms *liquidity*, *liquidity value* and *desired level of consumption* as they are essential concepts in the following chapters of this dissertation. Hahn (1990) defines liquidity as the ready convertibility of an object into commodities. Compare three objects A, B and C. Assume that object A can always be instantaneously converted into commodities. In this case, we say that object A is perfectly liquid. In contrast, assume object B is only accepted as a means of payment by some agents. Finally object C which is never accepted as a medium of exchange is said to be *not liquid*. Thus, we can rank the objects according to their liquidity: Object A is more liquid than object B which, in turn, is more liquid than object C.

The *liquidity value* of an object describes the part of the buyer's payoff from trade which is generated by the marginal (last) unit of the object. If an object is not accepted as a medium of exchange, its liquidity value equals zero. Furthermore, an object's liquidity value is zero if a buyer does not spend his entire holdings of that object. In this case, its marginal (last) unit is not used to trade and thus, does not generate trade surplus for the buyer.

Finally, we define the buyer's *desired level of consumption* as the amount where his marginal utility of consumption equals his marginal cost of acquiring it. In other words, the desired level of consumption maximizes the buyer's utility, i.e. it is the unconstrained solution to the buyer's optimization problem.

The models in the following chapters build on the Lagos and Wright (2005) framework. They all include two storable assets, money and capital. Productive capital, which can be most easily thought of as grain, competes with money as a medium of exchange. In chapter one, money and capital are both perfectly liquid. Capital serves as an input in production in both, the day and in the night market. Lagos and Rocheteau (2008) study a similar model but capital is only used as an input in night market production. Thus, the focus of chapter one lies on studying the effects of capital as a productive input in both markets. The marginal unit of money and capital, both offset the loss from discounting in the monetary equilibrium. Both assets yield the same liquidity value because they are both perfectly liquid. As a consequence, the change in the real value of the marginal unit of money between periods must be equal to the productive marginal return on capital in the both markets which is strictly positive. Thus, the marginal unit of money must become more valuable over time, i.e. there must be a deflation, in the monetary equilibrium.

We study different mechanisms to determine the terms of trade in the day market. The Friedman rule is not the optimal monetary policy if buyers make a *take-it-or-leave-it* offer to sellers (*buyer-take-all* bargaining). Sellers do not value capital as an input in day market production because they do not receive a payoff in the day market. As a consequence, agents hold strictly less than the optimal amount of capital. At the Friedman rule, an increase of inflation induces agents to substitute out of money and into capital, thus improving welfare. Comparing this model to a version where capital is only a productive input in the Walrasian night market, we find that agents hold the same amount of capital regardless of whether or not capital is used as an input in the day market. The agents' capital investment decision is independent of capital being an input in day market production because sellers do not value it as a productive input.

Another pricing mechanism is called *price-taking*. Agents are divided into buyers and sellers, buyers can only consume and sellers can only produce. They meet in a competitive market where they are completely anonymous, however. A medium of exchange is necessary to facilitate trade due to the *single coincidence of wants* problem and the prevailing anonymity. The Friedman rule is the optimal monetary policy in *price-taking*: In contrast to *buyer-take-all* bargaining, sellers receive the payoff generated from their capital holdings and, thus, value capital for its role as a productive input in the day market. Furthermore, agents hold more capital if it is used as an input in both markets as opposed to capital only being used in night market production because sellers value capital as a DM production input in *price-taking*.

In the second chapter of this dissertation, capital is only used as a productive input in the night market. Buyers and sellers are matched in pairs and buyers make a *take-it-or-leave-it* offer to sellers. As usual, buyers need a medium of exchange to facilitate trade. Money is more liquid than capital: Money can always be used as a medium of exchange. Capital, on the other hand, can only be used as a payment in a fraction of all transactions. Thus, money and capital only yield the same liquidity value in transactions where both can be used as a medium of exchange (type 1 transaction). In transactions where money is the only payment option (type 2 transactions), the liquidity value of capital is zero and (outside the Friedman rule) smaller than the liquidity value of money.

The monetary equilibrium can be divided into three parts depending on the rate of inflation. At the Friedman rule, money is held at no cost. Thus, agents choose to hold enough assets to purchase their desired level of day market consumption if they become buyers. Both, the marginal unit of money and the marginal unit of capital always yield a liquidity value of zero. For higher rates of inflation which are still below a certain threshold, buyers hold enough assets to afford their desired day market consumption only in transactions where both assets can be used as media of exchange. Thus, money yields a liquidity value greater than zero in type 2 transactions only. Finally above the threshold level of inflation, buyers can never afford their desired day market consumption. In this case, money always yields a liquidity value greater than zero and capital's liquidity value exceeds zero in type 1 transactions where capital is a permissible medium of exchange.

An increase of the rate of inflation does not impact capital accumulation if the rate of inflation is below the threshold level. The marginal unit of capital is only used as an input in production in the night market but it is not used in the day market because buyers can afford their desired level of consumption in type 1 transactions without spending all of their assets. Thus, the marginal unit of capital has no effect on day market allocations if the rate of inflation is below the threshold. The marginal unit of money, on the other hand, is used as a medium of exchange in transactions if capital cannot be used as payment (type 2 transactions). Thus, there is no link between agents' money and capital holdings at the margin. As a consequence, inflation has no effect on agents' capital holdings.

Above the threshold, the marginal unit of money is always used as a means of payment and the marginal unit of capital is used as a medium of exchange in type 1 transactions. In response to money becoming more expensive, agents choosing their investment in money and capital when they are in the night market face the following trade-off: On the one hand, money can always be used as a medium of exchange but it loses value each period. On the other hand, capital earns a positive return each period but it can only be used as a medium of exchange in some transactions. In the end, agents choose to substitute out of money and into capital in response to an increase of inflation.

Finally, in the model of chapter three, capital is used as a productive input in the day market and in the night market. This stands in contrast to chapter two where capital is only used as a productive input in the night market. As in chapter two, money is more liquid than capital. The pricing mechanism employed is *price-taking*. Each period, agents are evenly divided into buyers and sellers. Half of the buyers and half of the sellers enter a competitive market where money and capital can be used as a medium of exchange (market 1). The rest enters a competitive market where money is the only permissible payment (market 2).

As in chapter two, the monetary equilibrium can be divided into three parts depending on the rate of inflation. Buyers can afford their desired level of day market consumption in both markets at the Friedman rule. For higher rates of inflation which are still below a certain threshold, only buyers in market 1 can afford their desired level of day market consumption and, finally, for rates of inflation above the threshold level, buyers can never afford their desired consumption.

Inflation impacts on capital accumulation in a different way than in chapter two. There is always a link between the agents' decisions regarding capital and money investment because capital acts as a productive input in the day market. An increase of inflation reduces capital investment if the rate of inflation is below the threshold level: Due to its productive use in the day market, the marginal unit of capital impacts allocations in market 2 where the marginal unit of money is used as a medium of exchange. Thus, market 2 links an agent's decision on how much money and how much capital to hold. Note that this link was not provided in chapter two. As a consequence the increase of inflation acts as a tax on both, money and capital. We denote this as income effect.

Above the threshold level of inflation, there is an incentive for agents to substitute out of money and into capital (substitution effect) because both are used as media of exchange in market 1 at the margin. If the probability of entering market 1 is sufficiently high, the substitution effect outweighs the income effect and an increase of inflation leads to an increase in capital investment (Tobin effect). Otherwise, the income effect dominates and an increase of inflation leads to a reduction in capital investment (Stockman effect).

Chapter 1

Capital as a Medium of Exchange in a Monetary Framework

1.1 Introduction

Monetary search theory as introduced by Kiyotaki and Wright (1989, 1993) argues that fiat money has positive value in terms of real goods because it is used as a medium of exchange. In monetary models anonymous buyers and sellers need a medium of exchange to facilitate trade. Sellers do not use capital as an input in their production process in most monetary models. In this chapter, capital is not only used as a productive input by sellers but it is also used as a medium of exchange by buyers. We study the impact of capital as a production input and a medium of exchange on monetary equilibria.

We consider several different ways to determine the terms of trade, i.e. implicit or explicit prices, in this paper. First, a buyer and a seller bargain bilaterally where the terms of trade are obtained by generalized Nash bargaining. Second, we consider generalized Nash bargaining where the buyer has all the bargaining power (*buyer-take-all* bargaining). Third, anonymous buyers and sellers trade in a centralized market where they choose consumption given the market price (*price-taking*). After trade in the market where agents are anonymous has concluded, agents trade in a frictionless Walrasian market. Lagos and Wright (2005) were the first to introduce the interplay of the two markets into monetary theory. All agents use capital as an input in production in the Walrasian market. Additionally, sellers use capital in production for a second time in the anonymous market. Money is supplied by the government in the frictionless market.

Whether monetary equilibria exist depends on the rate of inflation. The maximum (gross) rate of inflation which is consistent with a monetary equilibrium is strictly smaller than one, i.e. monetary equilibria exist at deflation only. If deflation is high enough to offset the loss from discounting (Friedman rule) buyers hold enough assets to afford their desired consumption and thus, they value neither the marginal unit of money nor the marginal unit of capital for their roles as media of exchange. The buyers' marginal willingness to pay is zero at the Friedman rule as an additional unit of money or capital does not increase their payoff from trade anymore. Away from the Friedman rule, however, buyers cannot afford their desired level of consumption and an additional unit of money or capital raise their payoff. Thus, their marginal willingness to pay is greater than zero.

The Friedman rule is the optimal policy in the *buyer-take-all* scenario. The resulting monetary equilibrium constitutes the first best from a welfare perspective. Buyers can afford their desired consumption and capital is valued as an input in both markets. At the margin, neither money nor capital are valued for their roles as media of exchange.

In *price-taking* the Friedman rule is not the optimal policy. Sellers never value capital for its role as a medium of exchange because they do not receive any trade surplus. That is why capital is undervalued at the Friedman rule and the equilibrium capital stock is inefficiently low. Moving away from the Friedman rule, buyers cannot afford their desired consumption anymore which is why they assign a positive liquidity value to the marginal unit of capital. As a consequence, they hold more capital in the new stationary equilibrium and welfare is higher than under the Friedman rule.

Comparing this model to a version in which sellers do not use capital as an input in production shows that the results depend on the mechanism determining the terms of trade. With *price-taking* agents hold more capital if sellers use it as an input in production: Sellers fully value capital for its role as a productive input. Consequently, capital is valued for one more role if it is an input in the sellers' production process than if it is not. That is why agents hold more capital if capital is used as an input in production. In the *buyer-take-all* version of this model, buyers receive the entire trade surplus which is why sellers do not value capital for its role as an input in their production process. It follows that the equilibrium capital stock is independent of whether or not capital is used as an input in the sellers' production in *buyer-take-all* bargaining.

Lagos and Rocheteau (2008) study two competing media of exchange in an economy with three assets, two of which can be used as media of exchange. Agents can store their resources in money, illiquid or liquid capital. The storage technologies depend on individual savings. In contrast to illiquid capital, money and liquid capital can be used as media of exchange in the matching market where anonymous buyers and sellers randomly meet in pairs. Lagos and Rocheteau (2008) show that money and liquid capital have to yield the same return and the same liquidity value in any monetary equilibrium. In contrast to the model in this chapter where capital is used as a productive input in both markets, capital in Lagos and Rocheteau (2008) is only used for production in the Walrasian market but not in the matching market.

Aruoba, Waller, and Wright (2011) introduce capital into the Lagos and Wright (2005) framework. In their model capital is used as a productive input in both markets. It is not used as a medium of exchange, however. Anonymous buyers and sellers are matched in pairs. In a certain fraction of all matches, buyers are allowed to trade against credit. In all other meetings, buyers need money to trade. In 'credit meetings', buyers can always afford their desired consumption. In meetings without credit, buyers can only afford their desired consumption at the Friedman rule. Away from the Friedman rule, buyers value the marginal unit of money for the liquidity it provides.

1.2 Model

The economy consists of a $[0, 1]$ continuum of agents who live forever. Each period is divided into three stages (subperiods) namely a production stage, the day market and the night market. Agents do not discount between subperiods but they do so between periods.

Each period begins with the production stage where agents individually produce homogeneous durable goods. One unit of the homogeneous good can be costlessly converted into one unit of capital at an instant. Agents' technology f uses capital k as its sole input for production with $f'(k) > 0$ and $f''(k) < 0$. In addition to capital there is a second asset in the economy called fiat money. It is provided by the government and cannot be counterfeited.

After the production stage, the day market opens. Agents are anonymous in the DM and there is no technology to verify an agent's identity. At the beginning of this stage, agents are hit by a shock determining their status in the day market. An agent either becomes a buyer, a seller or does not participate in DM trade. The probability of becoming a buyer is σ and the probability of becoming a seller is σ, as well. Sellers can produce perishable DM goods which only buyers can consume. The prevailing anonymity in the day market prevents trades against credit. Thus, buyers need a medium of exchange for trade to occur. Potential candidates for this role are the two assets, money and capital. In the remainder, we use different mechanisms to determine the terms of trade in the day market.

In the day market, a buyer's utility from consuming q units of the DM good is $u(q)$ where u is a concave function. The seller uses the amount of capital which he brings into the match as a production input. His disutility from producing q units of the DM good is $c(q, k)$ if he uses k units of capital in production. The function c satisfies $c_q(q, k) > 0$, $c_{qq}(q, k) > 0$, $c_k(q, k) < 0$ and $c_{kk}(q, k) > 0$ where c_i $(i = q, k)$ denotes the partial derivative of c with respect to i and c_{ij} $(j = q, k)$ denotes the second derivative with respect to i and j.

In the third subperiod (at night), agents enter a standard Walrasian market in which trade is centralized and frictionless. Given their endowment of money and capital, agents choose net consumption of the homogeneous good and an asset portfolio which they wish to enter next period with. They demand intrinsically useless money because it can be used as a medium of exchange if they become buyers in the day market. Net consumption of x units of the homogeneous good yields x utils.

Consider a central planner (CP) who maximizes welfare. Assume that he can monitor agents in the day market and force sellers to produce. He cannot change the matching process, however. The central planner maximizes the cross-section average of average discounted present values of expected utility over all infinite future.

$$\begin{aligned} \mathbb{W} = & \max_{\{q_t, x_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left\{ \sigma \left[u(q_t) - c\left(q_t, F(k_t)\right) \right] + x_t \right\} \\ \text{s.t.} \quad & x_t + k_{t+1} = F(k_t) \end{aligned} \tag{1.1}$$

where $F(k_t) \equiv k_t + f(k_t)$. In the day market, buyers and sellers are randomly matched in pairs. Their measure is σ respectively. The central planner induces sellers to produce goods and to give them to buyers at no cost. Thus, the representative buyer receives $u(q_t)$ utils from consumption and the representative seller bears a cost of $c(q_t, F(k_t))$ utils. Note, that sellers use their entire capital holdings in DM production. An agent who enters period t with k_t capital holdings enters the day market which occurs after the production stage with $F(k_t)$ units of capital. The budget constraint in maximization problem (1.1) states that net consumption and capital investment are financed by income. The first order conditions of maximization problem (1.1) are

$$u'(q_t^{**}) = c_q(q_t^{**}, F(k_t)) \tag{1.2}$$

$$1 = \beta F'(k_{t+1}) \left[1 - \sigma c_k \left(q_{t+1}^{**}, F(k_{t+1})\right)\right] \tag{1.3}$$

where F' denotes the first derivative of F. The central planner's equilibrium satisfies the first order conditions (1.2) - (1.3) and the budget constraint of maximization problem (1.1). It denotes the socially optimal equilibrium and is the benchmark for subsequent welfare comparisons.

Equation (1.2) defines the socially optimal amount of the DM good q^{**} as a function of k_t because the seller uses his capital holdings in DM production. Equation (1.3) shows that capital is not only valued for its NM return but also for its role as a DM production input. Notice that capital does not yield a liquidity value because the central planner induces sellers to deliver goods without repayment. That is why intrinsically useless money is not needed in the central planner's solution. Finally, per-capita net consumption x_t is obtained by the budget constraint of maximization problem (1.1).

In the following, the time subscript t is dropped, a variable in period $t+1$ is denoted by the subscript $+1$. Money M is supplied by the government. Initially, all agents are endowed with the same amount of money. The government can change the supply of money in the night market of each period. The evolution of money supply is given by $M_{+1} = (1+v)M$ where v expresses the government's decision in the night market of period t. The government sets $v > 0$ if it decides to inject new money in period t and $v \leq 0$ otherwise. Its real budget constraint is given by

$$T = v\frac{M}{\mathbb{P}} \tag{1.4}$$

where $\mathbb{P}$ is the price of one unit of NM output, or equivalently one unit of capital, in terms of money at time t. If the government injects new money ($v > 0$), it generates seignorage from selling $M_{+1} - M$ units of new money in the night market at the price $\mathbb{P}^{-1}$ per unit. The seignorage is transfered to the agents in the night market via a lump sum payment ($T > 0$). If $v < 0$, the government reduces money supply by purchasing $M_{+1} - M$ units of money at the price of $\mathbb{P}^{-1}$ per unit. To finance its expenditure of $(M_{+1} - M)/p$ in terms of capital, the government raises a lump sum tax which can be paid in money or in capital. Define real money and gross inflation as $Z \equiv M/\mathbb{P}$ and $\pi_{+1} \equiv \mathbb{P}_{+1}/\mathbb{P}$, respectively.

An agent who enters the night market with a portfolio (z, k) maximizes

$$\begin{aligned} W(z,k) = & \max_{x,k_{+1},z_{+1}} \{x + \beta V(z_{+1}, F(k_{+1}))\} \\ \text{s.t.} \quad x + z_{+1}\pi_{+1} + k_{+1} = & k + z + T \end{aligned} \tag{1.5}$$

The objective function in maximization problem (1.5) shows the value W of an agent who enters the night market with an asset portfolio (z, k). He receives x utils from net consumption in the night market. Net consumption $x \in \mathbb{R}$ combines an agent's consumption and labor choice which are both not explicitly modeled. Thus, $x < 0$ implies that an agent's disutility from working exceeds his consumption utility. His continuation value in the day market is given by the value function V which depends on the asset portfolio that he chooses to leave the NM with. It is discounted at the rate β because the day market takes place in the next period. Recall that agents enter the production stage before entering the day market in the beginning of next period. Hence, an agent leaving the night market with k_{+1} units of capital enters the day market with $F(k_{+1})$. The budget constraint of maximization problem (1.5) states that an agent's net consumption and new asset portfolio have to be financed by the assets he brought into the NM plus government transfers.

According to his maximization problem (1.5), an agent chooses (z_{+1}, k_{+1}) which satisfy

$$z_{+1} \quad : \quad -\pi_{+1} + \beta \frac{\partial V(z_{+1}, F(k_{+1}))}{\partial z_{+1}} \geq 0 \tag{1.6}$$

$$k_{+1} \quad : \quad 1 = \beta \frac{\partial V(z_{+1}, F(k_{+1}))}{\partial k_{+1}} \tag{1.7}$$

The left-hand side of equation (1.6) shows the effect of a unit of money on an agent's utility. He spends π_{+1} units of the night market good to have one unit of (real) money in the next period, i.e. z_{+1}, which lowers his utility in the night market by π_{+1}. In the next period, the unit of (real) money provides him with some payoff given by the second term on the left-hand side of equation (1.6). An agent decides to buy a unit of money (z_{+1}) if its benefit exceeds its loss. Therefore, he holds money if and only if condition (1.6) is satisfied for positive values of z_{+1}. In the remainder of this paper, condition (1.6) is referred to as a first-order-condition or money Euler equation if it is satisfied at equality. In that case, the optimal solution for z_{+1} solves the money Euler equation (1.6).

Equation (1.7) is the capital Euler equation. A unit of capital can be purchased at the price of one unit of the NM good. The present value of the benefit it provides in the next period is shown on the right-hand side of the capital Euler equation. The utility gain of the marginal unit of capital equals its utility loss at the optimal k_{+1}.

Net consumption x balances the budget constraint of maximization problem (1.5). Note that an agent can always afford his choices of z_{+1} and k_{+1} regardless of his asset holdings (z, k) because net consumption can be negative. Thus, $x \in \mathbb{R}$ guarantees that the constraint in maximization problem (1.5) is always satisfied. Reconsider a government intervention in the night market. If the government reduces the money supply, $x \in \mathbb{R}$, furthermore, guarantees that agents can afford to make the necessary lump sum payment ($T < 0$) regardless of their asset holdings.

At the beginning of the DM subperiod, agents are hit by a shock which determines whether they act as buyers, sellers or if they do not trade in the day market. Consider an agent entering the day market with (z, k). His lifetime utility before the shock has realized is given by

$$V(z, k) = \sigma V^b(z, k) + \sigma V^s(z, k) + W(z, k) \tag{1.8}$$

The agent's value V before being matched is his expected payoff from either becoming a buyer, a seller or not trading in the DM. The probability of him becoming a buyer is σ. In this case, his lifetime utility is given by $V^b(z, k) + W(z, k)$. If he becomes a seller which happens with the same probability, his value is $V^s(z, k) + W(z, k)$. Finally, if he does not trade (probability $1 - 2\sigma$), he receives a payoff of $W(z, k)$.

1.3 Bargaining

In this section, buyers and sellers are randomly matched in pairs and use generalized Nash bargaining to determine the terms of trade. Before the matching process (but after types have realized), buyers and sellers have the chance to optimize their portfolios: They choose whether or not to bring all of their assets into their respective matches. Denote the optimization after types have realized as 're-optimization'. Lagos and Rocheteau (2008) show that there is a hold-up problem on buyers' asset holdings if they do not get a chance to re-optimize their portfolios. This can be explained as follows. The amount of assets which agents purchase in the night market might be too large from the perspective of a buyer in the day market. If it is, a buyer receives more than his desired amount of the DM good due to generalized Nash bargaining, and assigns a negative liquidity value to the marginal unit of money and capital. To avoid this, buyers can re-optimize their asset portfolios before being matched with sellers.

There is a holdup problem on the side of the seller, as well. Recall that the seller uses his capital holdings as an input in day market production. He reduces his production cost by bringing more capital into the day market, thereby increasing the DM trade surplus. Due to Nash bargaining, however, he has to share the trade surplus with the buyer and, thus, undervalues capital. Aruoba, Waller, and Wright (2011) call this the hold-up problem on the seller's capital. Allowing the seller to re-optimize, does not solve the hold-up problem but it guarantees that the seller always benefits from the marginal unit of his capital holdings, i.e. the payoff from his marginal unit of capital is nonnegative.

A buyer with an asset portfolio (z, k) chooses to bring $k^b \leq k$ and $z^b \leq z$ into his match and a seller with the same portfolio brings $k^s \leq k$ and $z^s \leq z$. As a tie-breaking rule, we assume that an agent takes all of his assets with him if he is indifferent. The value functions V^b and V^s are given by

$$V^b(z,k) = \max_{k^b \leq k, z^b \leq z} u(q^b) - d^b_z - d^b_k \tag{1.9}$$

$$V^s(z,k) = \max_{k^s \leq k, z^s \leq z} -c(q^s, k^s) + d^s_z + d^s_k \tag{1.10}$$

The terms of trade in our agent's match are denoted q^b, d^b_z and d^b_k if he becomes a buyer and by q^s, d^s_z and d^s_k if he becomes a seller. In either case, they depend on the amount of money and capital which our agent and his random trade partner bring into the match. Equation (1.9) shows the buyer's payoff in the day market. His lifetime utility is given by $V^b(z,k)+W(z,k)$. He chooses k^b and z^b to maximize his DM payoff and receives positive utility from consuming q^b units of the DM good. In exchange, he pays d^b_z units of money and d^b_k units of capital. Similarly, a seller chooses k^s and z^s optimally with regard to equation (1.10). He bears a cost of $c(q^s, k^s)$ from producing q^s goods. In return he receives a stochastic payment of $d^s_z + d^s_k$.

Buyers and sellers are randomly matched in pairs. In each match, they use Nash bargaining to determine how many goods the seller delivers (q) and how much money and capital the buyer pays in return (d_z, d_k). Consider a match between a buyer with an asset portfolio (z^b, k^b) and a seller with the portfolio (z^s, k^s). Both agents know their partner's asset holdings in the match. Recall that buyers and sellers only bring a fraction of their assets into the match. The buyer's and the seller's total assets are given by the pairs $(\bar{z}^b, \bar{k}^b)$ and $(\bar{z}^s, \bar{k}^s)$ respectively. The buyer's payoff from trade is given by $u(q) - d_z - d_k + W(\bar{z}^b, \bar{k}^b)$ and his threat point is $W(\bar{z}^b, \bar{k}^b)$. Consequently, his surplus from trade is $u(q) - d_z - d_k$. Similarly, the seller's surplus from trade is $-c(q, k^s) + d_z + d_k$. The terms of trade (q, d_z, d_k) maximize

$$\begin{aligned} &\max_{q, d_z, d_k} [u(q) - d_z - d_k]^{\theta} \left[-c(q, k^s) + d_z + d_k\right]^{1-\theta} \\ &\text{s.t} \qquad d_z \leq z^b, \qquad d_k \leq k^b \end{aligned} \tag{1.11}$$

where θ denotes the buyer's bargaining power and $1 - \theta$ the seller's. The terms of trade maximize the product of the buyer's and seller's trade surplus weighted by their bargaining powers. Due to anonymity, sellers do not extend credit to buyers, and buyers need a medium of exchange to facilitate trade. That is why buyers are constrained by the amount of money and capital which they brought into their respective match. This is captured by the two constraints. The solution to maximization problem (1.11) is

$$q = \begin{cases} q^{**}, & \text{if } z^b + k^b \geq g(q^{**}, k^s) \\ h(z^b + k^b, k^s), & \text{otherwise} \end{cases} \tag{1.12}$$

$$d_z + d_k = g(q, k^s) \tag{1.13}$$

where $g(q, k^s) = \frac{\theta u'(q)c(q,k^s)+(1-\theta)c_1(q,k^s)u(q)}{\theta u'(q)+(1-\theta)c_1(q,k^s)}$ is the value of the payment which is necessary to purchase q units of the DM good from a seller with capital holdings k^s. Conversely, $h(a, b) < q^{**}$ denotes the amount of DM goods that a buyer receives from a seller in exchange for his assets a, given the seller uses b units of capital in production. Recall that the portfolios (z^b, k^b) and (z^s, k^s) are known to both agents. In the unconstrained solution to maximization problem (1.11), the buyer spends less than his total asset holdings and receives q^{**} DM goods which solve $u'(q) = g_q(q, k^s)$. The value of his payment is given by $g(q^{**}, k^s)$ as shown in equation (1.13). Note that equation (1.13) only pins down the total value of money and capital that the buyer has to pay but not its composition. That is, the buyer's monetary payment is expressed as a function of his payment in capital and vice versa. In the constrained solution, the buyer spends all of the assets he brought into the match $(z^b + k^b)$ and receives $q = h(z^b + k^b, k^s) < q^{**}$ DM goods in return. Equation (1.12) shows that the amount of DM goods traded in a match is a function of the seller's capital holdings and the buyer's money and capital holdings. It is, however, independent of the seller's money holdings.

Consider the seller's optimization problem (1.10) after types have realized. Since q is independent of his money holdings, the seller is indifferent on how much money to bring into the match. Thus, the tie-breaking rule applies and he brings all of his money holdings into the match, i.e. $z^s = \bar{z}^s$. Note that the results do not depend on the choice of the tie-breaking rule. Sellers use their capital holdings in production and therefore, q depends on the amount of capital that the seller brings into the match. The unconstrained solution $k^s < \bar{k}^s$ to his maximization problem (1.10) solves

$$c_q(q, k^s) \frac{g_k(q, k^s)}{g_q(q, k^s)} - c_k(q, k^s) = 0 \tag{1.14}$$

The seller takes the following trade-off into account. If he brings an additional unit of capital into the match, he reduces his marginal production cost, thereby increasing his trade surplus for a given q. This effect is captured by $-c_k(q, k^s) > 0$. The seller, however, has to share this extra trade surplus with the buyer because of generalized Nash bargaining with $\theta \in (0, 1)$. To do so, he produces more goods. At the margin, he increases q by $-g_k(q, k^s)/g_q(q, k^s) > 0$ units which leads to a marginal cost of $c_q(q, k^s)g_k(q, k^s)/g_q(q, k^s)$. At the optimal choice of k^s, the marginal benefit equals the marginal cost as shown by equation (1.14).

If the amount of capital which satisfies equation (1.14) is greater than $\bar{k}^s$ (constrained solution), then the seller brings all of his capital holdings, $k^s = \bar{k}^s$, into the match. In this case, the left-hand side of equation (1.14) is greater than zero, i.e. $c_q \frac{g_k}{g_q} - c_k > 0$. To sum up, the chance to re-optimize guarantees that the seller's benefit from the marginal unit of his capital holdings is nonnegative.

Next, consider the buyer's maximization problem (1.9). The buyer's unconstrained solution solves

$$u'(q^*) = g_q(q^*, k^s) \tag{1.15}$$

where q that solves equation (1.15) is denoted as q^*. The buyer chooses to bring as many assets as he needs to purchase q^* into the match. Thus, the buyer chooses q^* such that his marginal utility of consumption equals the marginal price of the DM good (g_q) and not the seller's marginal cost (c_q) as in the central planner's solution. Thus, q^* is referred to as the buyer's desired level of consumption. For given k^s and $\theta < 1$, the buyer chooses $q^* < q^{**}$. At $\theta = 1$, sellers are compensated by their production cost, i.e. $g(q, k^s) = c(q, k^s)$ and, consequently, $q^* = q^{**}$.

In the constrained solution to maximization problem (1.9), the buyer brings all of his assets into the match and receives $q < q^*$ units of the DM good in return.

Note that the buyers' and sellers' value functions in the day market do not depend on their capital holdings from the previous period. Thus, the first order conditions (1.6) and (1.7) reveal that an agent's choice of money and capital in the night market is independent of his asset portfolio at the beginning of the night market. Consequently, all agents leave the night market with the same asset portfolio and the distributions of money and capital holdings are degenerate at the end of each period. This is the same result as in Lagos and Wright (2005). In this model, it is generated because $W(z, k)$ is linear in x and $V(z_{+1}, k_{+1})$ is independent of z and k.

Recall the buyer's and the seller's value functions in the day market, i.e. equations (1.9) and (1.10). The uncertainty over their trade partner's asset holdings vanishes as all agents enter the day market with the same portfolio regardless of their trade histories. To stress this point, denote an agent's money and capital holdings when leaving the night market by upper-case letters (Z_{+1}, K_{+1}). In the following computation it is necessary to differentiate between buyers' and sellers' money and capital holdings, however. That is why a buyer's and seller's asset portfolios when entering the day market are denoted by $(\bar{Z}^b, F(\bar{K}^b))$ and $(\bar{Z}^s, F(\bar{K}^s))$, respectively.

There are four exhaustive cases to be considered, denoted by (i)-(iv). In case (i), buyers do not hold enough assets to afford their desired amount of DM goods and sellers choose not to bring all of their asset holdings into their matches, i.e. $\bar{Z}^b + F(\bar{K}^b) < g(q^*, K^s)$ and $K^s < F(\bar{K}^s)$. In this case the buyer spends all of his assets, i.e. $Z^b + K^b = \bar{Z}^b + F(\bar{K}^b)$ and receives $q < q^*$ which solves

$$\bar{Z}^b + F(\bar{K}^b) = g(q, K^s) \tag{1.16}$$

Equation (1.16) implicitly defines q as a function of the buyers asset holdings ($\bar{Z}^b$, $F(\bar{K}^b)$) and the amount of capital that sellers bring into their matches K^s. Note that q does not depend on the sellers total capital holdings $F(\bar{K}^s)$ in case (i).

The derivatives of the day market's value function can now be computed, using the information above and setting $\bar{K}^b = \bar{K}^s = K$, $\bar{Z}^b = \bar{Z}^s = Z$. They take the form

$$\frac{\partial V(Z, F(K))}{\partial z} = 1 + \sigma\left(\frac{u'(q)}{g_q(q, K^s)} - 1\right) \tag{1.17}$$

$$\frac{\partial V(Z, F(K))}{\partial k} = F'(K)\left\{1 + \sigma\left(\frac{u'(q)}{g_q(q, K^s)} - 1\right)\right\} \tag{1.18}$$

Inserting both partial derivatives, we can rewrite the first order conditions (1.6) and (1.7) as

$$i \equiv \pi\beta^{-1} - 1 = \sigma\left(\frac{u'(q)}{g_q(q, K^s)} - 1\right) \tag{1.19}$$

$$\beta^{-1} = F'(K)\left\{1 + \sigma\left(\frac{u'(q)}{g_q(q, K^s)} - 1\right)\right\} \tag{1.20}$$

where $i \geq 0$ is the nominal interest rate and the liquidity value is denoted by $[u'(q)/g_q(q, K^s)] - 1$. Equation (1.19) is the money- and equation (1.20) is the capital Euler equation in an equilibrium which satisfies case (i). Both first order conditions are static and money and capital yield the same liquidity value because they are both perfectly liquid.

According to the money Euler equation (1.19), the nominal interest rate equals money's expected liquidity value which can be explained as follows: An agent in the night market buys the marginal unit of money if he is compensated for the loss from discounting. He will use the marginal unit of money to purchase goods if he becomes a buyer in the day market which happens with probability σ. In this case, the marginal unit of money increases his DM consumption by $1/c_q$ goods and raises his DM payoff by $[u'(q) - g_q(q, K^s)]/g_q(q, K^s)$. Hence, his expected liquidity value is depicted on the right hand side of equation (1.19). Notice that q is a function of inflation and the amount of capital that sellers bring into the match K^s. According to the capital Euler equation (1.20) the gross real interest rate β^{-1} equals capital's marginal return from the production of the homogeneous good multiplied by one plus its expected liquidity value. At the margin, capital K is valued for its roles as a productive NM input and as medium of exchange. It is not, however, valued as a DM input at the margin because sellers do not bring all of their capital holdings into their matches.

Combining the two Euler equations yields

$$\pi^{-1} = F'(K) \tag{1.21}$$

According to equation (1.21), capital's marginal rate of return in the production stage equals money's marginal NM return which is just the inverse of inflation. Notice that equation (1.21) pins down the equilibrium value of K as a function of only inflation.

Recall that agents choose K^s such that equation (1.14) is satisfied. Given their asset holdings, they trade q which solves equation (1.16). Both equations are repeated here for convenience.

$$c_q(q, K^s)\frac{g_k(q, K^s)}{g_q(q, K^s)} = c_k(q, K^s) \tag{1.22}$$

$$Z + F(K) = g(q, K^s) \tag{1.23}$$

$$X = F(K) - K_{+1} \tag{1.24}$$

where $F(K) = K + f(K)$. Thus, the resource constraint (1.24) states that net consumption is the difference between output $f(K)$ and capital investment. Equations

(1.22) and (1.19) determine the equilibrium values for q and K^s. Given K, K^s and q equations (1.23) and (1.24) determine Z and X, respectively.

In case (ii) buyers still cannot afford q^* but in contrast to case (i) sellers bring all of their capital holdings into the matches, i.e. $K^s = F(\bar{K}^s)$. Buyers again spend all of their assets $Z^b + K^b = \bar{Z}^b + F(\bar{K}^b)$. They receive $q < q^*$ DM goods which is the solution to

$$\bar{Z}^b + F(\bar{K}^b) = g(q, F(\bar{K}^s)) \tag{1.25}$$

Equation (1.25) describes q as a function of the buyer's asset holdings and the seller's total capital holdings. Recall that q did not depend on $\bar{K}^s$ in case (i). In equilibrium, all agents enter the day market with the same assets. Thus, $\bar{K}^s = \bar{K}^b \equiv K$, $\bar{Z}^s = \bar{Z}^b \equiv Z$ and the derivatives of the DM value function take the form

$$\frac{\partial V(Z, F(K))}{\partial z} = 1 + \sigma\left(\frac{u'(q)}{g_q(q, F(K))} - 1\right) \tag{1.26}$$

$$\begin{aligned}\frac{\partial V(Z, F(K))}{\partial k} =& F'(K)\left\{1 + \sigma\left(\frac{u'(q)}{g_q(q, F(K))} - 1\right)\right\} \\ &+ F'(K)\sigma\left[c_q(q, F(K))\frac{g_k(q, F(K))}{g_q(q, F(K))} - c_k(q, F(K))\right]\end{aligned} \tag{1.27}$$

Inserting equations (1.26) and (1.27) into the FOCs of z and k gives us the equilibrium conditions for case (ii).

$$i \equiv \pi\beta^{-1} - 1 = \sigma\left(\frac{u'(q)}{g_q(q, F(K))} - 1\right) \tag{1.28}$$

$$\begin{aligned}\beta^{-1} =& F'(K)\left\{1 + \sigma\left(\frac{u'(q)}{g_q(q, F(K))} - 1\right)\right\} \\ &+ F'(K)\sigma\left[c_q(q, F(K))\frac{g_k(q, F(K))}{g_q(q, F(K))} - c_k(q, F(K))\right]\end{aligned} \tag{1.29}$$

The money Euler equation (1.28) defines q as a function of inflation and capital K. In case (ii), the seller brings all of his capital $F(K)$ into the match and uses it to produce q. This stands in contrast to case (i) where the seller only used a fraction of his capital holdings in DM production. The capital Euler equation (1.29) shows that the marginal unit of capital is now valued for its roles as NM input, medium of exchange and DM input. At the margin, capital increases the trade surplus in the day market by $-c_k(q, F(K))F'(K)$ because it lowers the marginal production cost. Due to Nash bargaining, however, the seller cannot keep the entire trade surplus which is generated through his capital holdings; instead he has to share it with the buyer. The seller's share of the marginal trade surplus is captured by the last term in equation (1.29).

Combining both Euler equations yields

$$\pi = F'(K)^{-1} - \beta\sigma \left[c_q(q, F(K)) \frac{g_k(q, F(K))}{g_q(q, F(K))} - c_k(q, F(K)) \right] \tag{1.30}$$

Depending on the rate of inflation, the value of a unit of (real) money changes each period. Thus, one unit of current period (real) money is worth π^{-1} units of next period real money which is why we can interpret the inverse of inflation as money's return in the night market. In contrast, capital's total return consists of its return in the production stage and in the day market.

The second term on the right-hand side of equation (1.30) is greater than zero. Consequently, the return on money exceeds the return that capital generates in the production stage and is strictly greater than 1, i.e. $\pi^{-1} > F'(K) > 1$ since $F'(K) = 1+f'(K)$. Note that $\pi^{-1} > 1$ requires a deflation, i.e. $\pi < 1$.

The Euler equations (1.28) and (1.29) determine the solutions for K and q. Agents hold Z units of money which solves (1.25) with $\bar{Z}^b = Z$ and $\bar{K}^b = \bar{K}^s = K$. Finally net consumption is obtained from the budget constraint (1.24).

In case (iii) buyers have sufficient assets to afford their desired consumption and sellers only bring a fraction of their asset holdings into the DM, i.e. $\bar{Z}^b + F(\bar{K}^b) > g(q^*, K^s)$ and $K^s < F(\bar{K}^s)$. At the margin buyers do not value either asset for its liquidity because they only spend parts of it. That is why the liquidity value vanishes and the FOCs of the night market value function [equation (1.5)] take the form

$$\pi = \beta \tag{1.31}$$

$$\beta^{-1} = F'(K) \tag{1.32}$$

Recall that the marginal unit of money is only held if condition (1.6) is satisfied. Condition (1.6) becomes $\pi \leq \beta$ in case (iii) because the marginal unit of money is only used as a store of value and not as a medium of exchange. Since π cannot be smaller than β in equilibrium, the marginal unit of money is only held if it generates a return which compensates for the loss from discounting [equation (1.31)]. This is achieved at the Friedman (1969) rule, i.e. $\pi = \beta$. Equation (1.32) shows that the marginal unit of capital is only valued as an input in the production of the homogeneous good. Thus, its marginal return offsets the loss from discounting as well.

The solutions for q and the amount of capital that sellers bring into their matches K^s are obtained from the solution to the sellers' DM maximization problem (1.22) and bargaining, i.e. $u'(q^*) = g_q(q^*, K^s)$. Given K which is obtained from equation (1.32) net consumption is obtained from the budget constraint (1.24). The amount of money holdings is indetermined in case (iii) because money can be held costlessly at the Friedman rule.

Finally, in case (iv) buyers can afford q^* and sellers bring all of their capital into their respective matches, i.e. $\bar{Z}^b + F(\bar{K}^b) > g(q^*, F(\bar{K}^s))$ and $K^s = F(\bar{K}^s)$. The marginal unit of capital is used in two production processes. Therefore, it generates a return as a production input in the day market and in the production stage, respectively. The FOC of capital [equation (1.7)] takes the following form in case (iv)

$$\beta^{-1} = F'(K)\left[1 + \sigma\left(c_q(q, F(K))\frac{g_k(q, F(K))}{g_q(q, F(K))} - c_k(q, F(K))\right)\right] \tag{1.33}$$

The money Euler equation (1.31) is the same as in case (iii) because money still only generates a return through deflation. Notice that equation (1.33) and the money Euler equation imply that the return on money exceeds the return that capital generates in the production of homogeneous goods, i.e. $\pi^{-1} > F'(K)$ which can be explained as follows. The marginal unit of money and the marginal unit of capital yield the same total return. In the case of money, its return is entirely generated through deflation, i.e. π^{-1}. Capital's total return consists of its return in the production stage and its return in day market production. Since both are strictly positive, it follows that $\beta^{-1} = \pi^{-1} > F'(K)$.

The solutions for K and q are obtained form the bargaining solution $g_q(q^*, F(K)) = u'(q^*)$ and the capital Euler equation (1.33). Again, net consumption is determined by the budget constraint (1.24) and Z is indetermined.

As mentioned earlier, the chance to re-optimize after types have realized prevents a hold-up problem on buyers. Buyers would receive more than their desired level of DM consumption in cases (iii) and (iv) if they were not allowed to re-optimize their portfolios. In terms of the model, the re-optimization guarantees that the liquidity value is nonnegative as it would not be for $q > q^*$. At the same time this implies that the amount of DM goods traded is always inefficiently low for $\theta \in (0, 1)$ because $q^* < q^{**}$. In contrast to Lagos and Rocheteau (2008), the terms of trade in the DM do not only depend on the buyers' asset holdings but they also depend on the sellers' capital which is used as an input in DM production. Thus, the re-optimization guarantees that sellers always benefit from the marginal unit of capital in the day market, i.e. $c_q g_k / g_q - c_q \geq 0$.

We can interpret $\partial q / \partial Z^b$ as the buyers' marginal willingness to pay. If buyers cannot afford their desired level of consumption, i.e. $Z^b + F(K^b) < g(q^*, F(K^s))$, their marginal willingness to pay is given by $1/g_q$ and it is strictly decreasing in their asset holdings which can be explained as follows. Buyers spend all of their assets to purchase as much q as possible because $q < q^*$. Thus, their marginal payoff in the day market, which is decreasing in q, decreases in the amount of their asset holdings, as well. Buyers' marginal willingness to pay is zero if they hold enough assets to purchase their desired level of consumption, i.e. $Z^b + F(K^b) > g(q^*, F(K^s))$. In this case, their marginal willingness to pay is independent of the amount of their assets as they do not wish to increase their consumption beyond q^*. Note that the marginal willingness to pay does not exist at the point where $Z^b + F(K^b) = g(q^*, F(K^s))$. At this point, q is not differentiable: An increase of money holdings does not impact q but a reduction lowers q.

1.4 Buyer-take-all bargaining ($\theta = 1$)

In this section the DM terms of trade are determined by *buyer-take-all* bargaining (BTA). That is, buyers have all the bargaining power in a bilateral meeting with a seller. The BTA solution coincides with the buyer making a *take-it-or-leave-it* offer to the seller. Buyers reimburse sellers for their production costs to guarantee that they are willing to participate in DM trade. Regardless of their asset holdings, buyers receive all and sellers receive none of the DM trade surplus. Consequently agents cannot influence their payoffs after types have realized which makes the re-optimization stage redundant at $\theta = 1$.

Buyers and sellers enter the DM with a portfolio $(Z^b, F(K^b))$ and $(Z^s, F(K^s))$ respectively. Only two of the four exhaustive cases from the previous section remain at $\theta = 1$, denoted as case (i) and (ii). In case (i), buyers do not have enough assets to purchase their desired amount of DM goods, i.e. $Z^b + F(K^b) < c(q^*, K^s)$. Note that the necessary payment to acquire q units of the DM good reduces to $c(q, K^s)$ at $\theta = 1$ because the buyer reimburses the seller in the amount of his production cost.

The equilibrium conditions in case (i) are given by

$$i \equiv \pi\beta^{-1} - 1 = \sigma\left(\frac{u'(q)}{c_q(q, F(K))} - 1\right) \tag{1.34}$$

$$\beta^{-1} = F'(K)\left\{1 + \sigma\left(\frac{u'(q)}{c_q(q, F(K))} - 1\right)\right\} \tag{1.35}$$

$$Z + F(K) = c(q, F(K)) \tag{1.36}$$

Equations (1.34) and (1.35) are the money and the capital Euler equation. Both, money and capital are valued as media of exchange at the margin because $q < q^*$. Thus, the marginal unit of money or capital in the day market raises q by $1/c_q(q, F(K))$ units. The accompanying increase in the buyers' trade surplus is given by the liquidity value $[u'(q) - c_q(q, F(K))]/c_q(q, F(K))$. In contrast to $\theta < 1$ sellers do not value capital as a DM input at $\theta = 1$ [equation (1.35)] because they never participate from the DM trade surplus. Equation (1.36) shows that buyers compensate sellers in the amount of their production cost in exchange for q. Combining the money and capital Euler equations (1.34) and (1.35) yields

$$\pi^{-1} = F'(K) \tag{1.37}$$

Both, money and capital are valued for two reasons. Buyers value them equally as media of exchange. Thus, they have to yield the same value for their respective second function which is captured by equation (1.37). The marginal return that capital generates as an input in the production stage equals the return on money which is the inverse of inflation. Since $F'(K) > 1$ this implies that deflation is necessary in case (i), i.e. $\pi < 1$. Equation (1.37) determines K as a function of π. Given K and π, equation (1.34) provides the solution for q. Finally, Z and X are obtained by equation (1.36) and the budget constraint (1.24), respectively.

In case (ii) buyers have sufficient assets to purchase their desired DM consumption, i.e. $Z + F(K) > c(q^*, F(K))$. Thus, the equilibrium conditions at $\theta = 1$ are

$$\pi = \beta \tag{1.38}$$

$$\beta^{-1} = F'(K) \tag{1.39}$$

$$u'(q) = c_q(q, F(K)) \tag{1.40}$$

According to equation (1.38) the marginal unit of money is only held if $\pi = \beta$. That is, if deflation is high enough to offset the loss from discounting (Friedman rule). The marginal return on capital in the production stage $F'(K)$ offsets the loss from discounting, as well. The equilibrium amount of capital in case (ii) is obtained from equation (1.39). Given K, the desired amount of DM goods q solves equation (1.40). The solution for Z is indetermined because agents are indifferent on how much money to hold at the Friedman rule. Net consumption is again obtained from the budget constraint (1.24).

Proposition 1 proves the existence and uniqueness (if applicable) of a monetary equilibrium. It shows how the equilibrium depends on the rate of inflation π.

Proposition 1. *There exists a monetary equilibrium for $\pi \in [\beta, \bar{\pi}]$ where $\bar{\pi} < 1$.*

- *if $\pi \in (\beta, \bar{\pi}]$, the equilibrium is unique and satisfies* (1.24) *and* (1.34) - (1.36)
- *if $\pi = \beta$, K, q and X are uniquely determined by equations* (1.24) *and* (1.38) - (1.40)

In *buyer-take-all* bargaining, sellers are always reimbursed for their production cost but they never participate in the trade surplus. That is why they do not value capital as an input in DM production. If $\pi = \beta$, capital is valued as an input in the production stage and money is valued by the inverse of inflation. If $\pi > \beta$, both additionally yield the same liquidity value. Thus, in any monetary equilibrium, money and capital generate the same total utility which implies $F'(K) = \pi^{-1}$.

Again, consider the buyer's marginal willingness to pay, $\partial q / \partial Z^b$. If $\pi > \beta$, his marginal willingness to pay is given by $1/c_q$ and it is strictly decreasing in his asset holdings. The buyer cannot afford his desired level of DM consumption as $Z^b +$

$F(K^b) < c(q^*, F(K^s))$ and he spends his entire assets to purchase as much DM consumption as possible. Therefore, his marginal DM payoff, which depends negatively on q, is decreasing in his asset holdings. If $\pi = \beta$, the buyer's marginal willingness to pay is zero and it is independent of his asset holdings because he does not spend all of his assets to purchase q^*. Note that the marginal willingness to pay does not exist at $Z^b + F(K^b) = c(q^*, F(K^s))$ as q is not differentiable at this point.

Consider the central planner's solution. He values capital as NM and DM input. The socially optimal amounts of DM goods q^{CP} and capital K^{CP} are obtained by his FOCs (1.2) and (1.3). Given K^{CP} the socially optimal amount of DM goods q^{CP} is obtained from equation (1.2). Note that the central planner's capital Euler equation (1.3) implies that $F'(K^{CP}) < \beta^{-1}$.

Proposition 2 makes a statement on the optimal policy.

Proposition 2. *The Friedman (1969) rule is not the welfare-maximizing monetary policy in buyer-take-all bargaining, i.e. the optimal rate of inflation satisfies $\pi > \beta$.*

Consider *buyer-take-all* bargaining at the Friedman rule, i.e. $\pi = \beta$. According to equation (1.39) the amount of capital held is obtained from $F'(K) = \beta^{-1}$. Thus, agents hold strictly less than the welfare-optimal capital stock at the Friedman rule, i.e. $K < K^{CP}$. Equation (1.40) determines q^{**} as a function of K. Even though it has the same functional form as in the central planner's solution, the solution for $q^{**}(K)$ in BTA does not coincide with q^{CP}. Since agents hold less capital than the central planner, we have $q^{**}(K) < q^{CP}$. Increasing inflation raises the equilibrium capital stock and leads to higher welfare.

Note that this result depends crucially on capital being used as a medium of exchange and as a productive input in the day market. First, the Friedman rule is not the optimal policy because of capital's role as a DM input: Sellers do not value capital as a productive DM input because they do not receive any DM trade surplus in *buyer-take-all* bargaining. Second, welfare increases by raising π because of capital's role as a medium of exchange. At $\pi > \beta$, buyers cannot afford their desired level of DM consumption anymore and, thus, they value the marginal unit of money and the marginal unit of capital as media of exchange. As a consequence, agents hold more capital than they did at the Friedman rule.

Proposition 3 compares this model (model A) with a version in which capital is not used as a DM input (model B). The disutility from production in model B is given by $c(q)$.

Proposition 3. *Agents hold the same amount of capital whether or not it is used as a DM input.*

In model A and B, the marginal unit of capital earns a return in the production stage and, if $\pi > \beta$, as a medium of exchange. In both models capital and money yield the same return, i.e. $F'(K) = \pi^{-1}$. Thus, the amount of capital held in both models is the same.

Recall that the desired amount of DM goods traded q^* is the solution to $u'(q) = c_q(q, F(K))$. Lemma 1 shows the effect that capital has on the production cost of q^*

Lemma 1. The functions $q^*(K)$ and $c(q^*, F(K))$ are strictly increasing in K.

The derivative of $c(q^*(K), F(K))$ with respect to K in the proof of lemma 1 in the appendix is obtained using the general functional form $c(q, F(K)) = q^{\psi} F(K)^{1-\psi}$ where $\psi > 1$. Capital has two effects if used as an input in DM production. On the one hand, it reduces the cost of producing a given amount q^* and on the other hand, it increases the desired amount q^*. Lemma 1 shows that the second effect outweighs the first one. Consequently the cost from producing q^* increases if k is an input in DM production.

Again, proposition 4 compares this model (model A) to its version without capital as a DM input (model B). Consider the following general production cost functions given by $c(q, F(K)) = q^{\psi} F(K)^{1-\psi}$ for model A and $c(q) = q^{\psi}$ for model B.

Proposition 4. *The cost of producing q^* units of the DM good is higher if capital is used as a productive DM input.*

Proposition 4 describes a level effect which comes into play if capital is used as a DM input. From lemma 1 we know that the production cost of the desired amount of DM goods is strictly increasing in K.

Outside the Friedman rule, monetary equilibria exist only if money is assigned a liquidity value according to condition (1.6). Thus, monetary equilibria exist only if capital by itself does not provide enough liquidity to purchase the desired amount of DM goods. Whether or not this condition is fulfilled depends on the parameterization, especially on the DM utility of consumption and the disutility from production. Proposition 4 shows that the cost of q^* is higher if capital is a productive input for given parameters. Thus, the set of parameters allowing for the existence of a monetary equilibrium is larger if capital is used as a DM input.

1.5 Price Taking

In this section the terms of trade in the day market are determined by 'price taking'. When entering the day market, agents are hit by a shock determining their status in the DM. They either become buyers, sellers or they do not trade in the day market. The probability of an agent becoming a buyer or a seller is given by σ, respectively. As before, only sellers can produce the DM good q and only buyers can consume it. In this section the day market is not decentralized but centralized. Buyers and sellers trade DM goods given the market clearing price P. In contrast to the night market, agents are still anonymous in the DM which prevents credit from being accepted. Thus, a medium of exchange is still necessary for trade to occur in the day market.

An agent's lifetime utility before the shock has realized is still given by equation (1.8). Consider a buyer with an asset portfolio (z^b, k^b). He solves the following optimization in the day market.

$$
\begin{aligned}
V^b(z^b, k^b) = \max_q \quad & u(q) - d_k^b - d_z^b \\
\text{s.t.} \quad & Pq = d_z + d_k \\
& d_z^b \le z^b \quad ; \quad d_k^b \le k^b
\end{aligned}
\tag{1.41}
$$

where P is the price of day market goods in terms of money and capital. The objective function of maximization problem (1.41) is the buyer's payoff in the day market. He receives utility $u(q)$ from consuming q units of the DM good. Given the price-level

P in the day market, the first constraint specifies the amount of money and capital which the buyer has to spend in order to receive q. Consequently, he enters the night market with $k^b - d_k^b$ units of capital and $z^b - d_z^b$ units of money. Due to the anonymity, the buyer is constrained in his payment by his asset holdings which is captured by the two final constraints.

The unconstrained solution for q in maximization problem (1.41) solves

$$u'(q) = P \tag{1.42}$$

If the buyer is not constrained by his asset holdings, he chooses to buy q such that his marginal utility from consumption equals his marginal cost which is given by P, i.e. the price of a unit of the DM good. According to the first constraint in maximization problem (1.41) he pays Pq units of money and capital in return. In the constrained solution to maximization problem (1.41), the buyer spends all of his assets, i.e. $d_z = z^b$ and $d_k = k^b$, and receives $q = (z^b + k^b)/P$ units of the DM good in return.

The marginal unit of money or capital does not affect the amount of goods traded in the day market if the buyer is not constrained by his asset holdings. That is why the partial derivatives of $V^b(z^b, k^b)$ with respect to his capital or money holdings equal zero. The partial derivatives for a buyer who is asset constrained are

$$\frac{\partial V^b(z^b, k^b)}{\partial z^b} = \frac{u'(q)}{P} - 1 \tag{1.43}$$

$$\frac{\partial V^b(z^b, k^b)}{\partial k^b} = \frac{u'(q)}{P} - 1 \tag{1.44}$$

A marginal unit of money or capital increases the buyer's demand for DM goods by $1/P$ which raises his utility in the day market by $u'(q)/P$. In the night market, the buyer suffers a utility loss of one because the marginal unit of either asset, spent in the DM, lowers his NM net consumption by one unit.

Consider a seller who enters the day market with portfolio (z^s, k^s). His optimization problem is given by

$$\begin{aligned} V^s(z^s, k^s) = \max_q \quad & - c(q, k^s) + d_z^s + d_k^s \\ \text{s.t.} \;\; & Pq = d_z^s + d_k^s \end{aligned} \tag{1.45}$$

A seller in the day market chooses his production q to maximize his DM payoff V^s. He suffers a utility loss of $c(q, k^s)$ from producing q units of the DM good. In return he receives a payment which guarantees him a higher utility in the night market. The constraint in maximization problem (1.45) reveals the price of q goods in terms of money and capital. Note that the total payment of $d_z^s + d_k^s$, which he receives, can be the sum of many individual buyers' payments. The solution to his maximization problem satisfies

$$c_q(q, k^s) = P \tag{1.46}$$

The solution q to maximization problem (1.45) solves equation (1.46) given his capital holdings k^s and the price level P. Sellers choose to produce the amount q which equates marginal cost and price. The payment which they receive in return is given by the constraint of maximization problem (1.45).

Using the information above, the derivative of the seller's payoff in the day market with respect to his capital and his money holdings can be computed as

$$\frac{\partial V^s(z, k^s)}{\partial k^s} = -c_k(q, k^s) - \frac{c_{qk}(q, k^s)}{c_{qq}(q, k^s)} [P - c_q(q, k^s)] \tag{1.47}$$

$$\frac{\partial V^s(z, k^s)}{\partial z^s} = 0 \tag{1.48}$$

Equation (1.47) shows that the seller's capital holdings influence his payoff in the day market in two ways. First, a seller produces a given amount of goods at a lower cost if he holds more capital. At the margin, his capital holdings reduce his production cost for a given q by $-c_k > 0$. Second, the seller's capital holdings k^s increase his production of q by $-c_{qk}/c_{qq} > 0$ units according to equation (1.46). The profit generated from this increase in q is captured by the second term on the right-hand side of equation (1.47). The marginal profit is zero, however, because the seller optimally chooses his production such that his marginal production cost equals his marginal revenue, i.e. $P = c_q$. The seller's money holdings do not influence his DM payoff because money has no productive use.

In equilibrium, each agent enters the day market with the same asset portfolio (Z, $F(K)$). Furthermore, the price P in the day market clears the market, i.e. given P total supply of the DM good (Q^{supply}) equals total demand (Q^{demand}). All buyers and all sellers choose the same q respectively because they all enter the DM with the same asset holdings. Thus, total supply and total demand of the DM good can be written as $Q^{supply} = \sigma q_s$ and $Q^{demand} = \sigma q_b$ where q_s and q_b denote the amount of DM goods traded by an individual seller or buyer, respectively. Equating total supply and total demand yields $q_b = q_s \equiv q$. Consequently, the price-level P is given by $c_q(q, F(K))$ in the constrained and in the unconstrained equilibrium.

There are two exhaustive cases to consider, denoted by (i) and (ii). Buyers are constrained by their asset holdings in case (i). Inserting the value functions' partial derivatives above into the night market's FOCs (1.6) and (1.7) and setting $k^b = k^s = F(K)$ and $z^b = z^s = Z$ yields

$$i \equiv \pi\beta^{-1} - 1 = \sigma \left(\frac{u'(q)}{c_q(q, F(K))} - 1 \right) \tag{1.49}$$

$$\beta^{-1} = F'(K) \left\{ 1 + \sigma \left(\frac{u'(q)}{c_q(q, F(K))} - 1 \right) \right\} - \sigma F'(K) c_k(q, F(K)) \tag{1.50}$$

Equation (1.49) is derived from the money Euler equation. It shows that the nominal interest rate equals the product of the liquidity value and the probability of an agent becoming a buyer σ. Notice that the liquidity value has the same functional form as in *buyer-take-all* bargaining. Equation (1.50) stems from the capital Euler equation. Capital is valued for its role as a NM input by all agents. Furthermore, buyers value it as a medium of exchange and sellers as a DM input. In contrast to Nash bargaining, sellers value the marginal unit of capital for its entire productivity increase in DM production (welfare-optimal). In total, capital is still overvalued because of its role as a medium of exchange.

Combining equations (1.49) and (1.50) yields

$$\pi = F'(K)^{-1} + \beta\sigma c_k(q, F(K)) \tag{1.51}$$

In case (i), the marginal unit of capital impacts the agents' utility for three reasons and the marginal unit of money influences utility for two reasons. At the margin, capital is used as a medium of exchange, as a productive input in the day and as a productive input in the night market. Agents receive positive utility from all of its roles. Money is used as a medium of exchange, as well. Furthermore, money's value in terms of the night market good potentially changes each period due to inflation π. Money earns a positive return in the night market if its real value increases from one period to the next. That is, if $\pi < 1 \Leftrightarrow \pi^{-1} > 1$. Denote π^{-1} as money's return in the night market.

Equation (1.51) is derived by combining equations (1.49) and (1.50), eliminating money's and capital's liquidity value. It implies $1 < F'(K) < \pi^{-1}$ since $c_k < 0$. Thus, an equilibrium in case (i) requires money to earn a positive return in the night market, i.e. deflation $\pi < 1$.

The values for K and q are determined by equations (1.49) and (1.50). In case (i), the price-level P is given by

$$P = c_q(q, F(K)) = \frac{Z + F(K)}{q} \tag{1.52}$$

Given the solutions for K and q, equation (1.52) pins down Z. Finally, the budget constraint (1.24) provides the solution for net consumption X.

In case (ii) buyers are not asset constrained and thus, can afford to buy q^* which solves $u'(q) = c_q(q, F(K)) = P$. Imposing equilibrium and inserting the partial derivatives of V into the FOCs (1.6) and (1.7) yields

$$\pi = \beta \tag{1.53}$$

$$\beta^{-1} = F'(K)\left[1 - \sigma c_k(q^*, F(K))\right] \tag{1.54}$$

Equation (1.53) is derived from the money Euler equation. The marginal unit of money is not used as a medium of exchange since $Z + F(K) > c(q^*, F(K))$. Thus, agents are only willing to hold money at the margin if the loss from discounting is offset by deflation which corresponds to the Friedman rule. According to the capital Euler equation (1.54) all agents value capital as NM input. Additionally, sellers value it as a DM production input.

Combining both Euler equations (1.53) and (1.54) yields

$$\pi^{-1} = F'(K)[1 - \sigma c_k(q^*, F(K))] \tag{1.55}$$

According to equation (1.55), $\pi^{-1} > F'(K)$ as in case (i). Again, this is due to the fact that capital is valued for two and money only for one function. Note that $\pi = \beta$ in case (ii), however.

The solutions for K and q are determined by $u'(q) = c_q(q, F(K))$ and equation (1.54).

Money holdings are indetermined in case (ii) because money can be costlessly held at the Friedman rule. A lower bound on money holdings can be given by $Z > Pq^* - F(K)$, however. Again, net consumption is determined by the budget constraint (1.24).

Proposition 5 shows that the existence of a monetary equilibrium depends on the rate of inflation π. If a monetary equilibrium exists, π determines whether or not buyers are constrained by their asset holdings in the DM.

Proposition 5. *There exists a monetary equilibrium for* $\pi \in [\beta, \bar{\pi}]$ *where* $\bar{\pi} < 1$.

- *The equilibrium is unique for* $\pi \in (\beta, \bar{\pi}]$ *and satisfies equations* (1.23) *and* (1.49) - (1.52)
- *For* $\pi = \beta$*, the equilibrium values for* K, q *and* X *are uniquely determined by* $u'(q) = c_q(q, F(K))$ *and equations* (1.24) *and* (1.54)*. The equilibrium value for* Z *is indetermined.*

Buyers cannot afford q^* if $\pi > \beta$. Consequently, they use their entire assets to purchase as much q as possible. In this case, their marginal willingness to pay, i.e. $\partial q / \partial Z^b$, is given by $1/P$ and it decreases as the price of the DM good goes up. The buyers' marginal willingness to pay is zero if $\pi = \beta$: Buyers hold enough assets to purchase their desired level of DM consumption and thus, they cannot increase their DM payoff anymore. As a consequence, they do not use the marginal unit of their money or capital holdings as payment. If follows that the buyers' marginal willingness to pay is only strictly greater than zero for $\pi > \beta$.

The optimal policy is again determined by comparing the *price-taking* to the central planner's solution. The central planner chooses K and q according to (1.2) and (1.3). Recall that the central planner fully values capital for its roles as NM and DM input.

Proposition 6. *The Friedman rule, i.e.* $\pi = \beta$*, is the optimal monetary policy.*

With *price-taking* agents fully value capital as NM and DM inputs which coincides with the central planner. If $\pi > \beta$ agents value capital for the liquidity it provides. At the Friedman rule, however, the liquidity value vanishes and agents hold the same amount of capital and DM goods as the central planner. Thus, $\pi = \beta$ replicates the first-best.

Proposition 7 compares this model (model A) to a version in which capital is not used as a DM input (model B). A seller who produces q units of the DM good suffers a disutility $c(q)$ which is independent of K in model B.

Proposition 7. *In a monetary equilibrium (with price-taking), agents hold strictly more capital if it is used as an input in DM production.*

In this section we have seen that $F'(K) < \pi^{-1}$ if capital is used as a productive input in the DM (model A). In model B both money and capital are valued for two roles. At the margin, the total return on either asset offsets the loss from discounting, that is, the marginal unit of money yields the same total return (in terms of utility) as the marginal unit of capital. Since they yield the same liquidity value, they also have to generate the same return for their second role respectively, i.e. $F'(K) = \pi^{-1}$ in model B. Call K^i the capital stock in model i. Consequently, for a given rate of inflation the marginal return on capital in the production stage is greater in model B than in model A, i.e. $F'(K^A) < F'(K^B)$. Since F is concave, agents hold more capital if it is used as a productive DM input (model A).

1.6 Concluding Remarks

In any monetary equilibrium the amount of goods traded in the day market depends on inflation. If $\pi = \beta$ buyers have sufficient assets to purchase their desired amount of DM goods. In this case, buyers do not use the marginal unit of either money or capital as a medium of exchange. If $\bar{\pi} > \pi > \beta$ buyers do not hold enough assets to allow for their desired DM consumption. Buyers use the marginal unit of money and capital as payment to purchase DM goods. Thus, they value the marginal unit of money and capital for the liquidity it provides.

Throughout this chapter we consider several ways to determine the terms of trade in the day market. In all of them, capital generates a return in the production stage and money yields a return in the night market due to government intervention. Money's (net) return is positive if the government reduces the money supply. In *price-taking*, sellers generate a return from capital in the day market because they use it as an input in DM production. Additionally, buyers use the marginal unit of both, money and capital to increase their utility of consumption in the day market if $\pi > \beta$. Regardless of the rate of inflation, capital serves one more function than money at the margin. It follows that $\pi^{-1} > F'(K) > 1$ because the marginal unit of money and the marginal unit of capital have to yield the same total return in a monetary equilibrium.

In *buyer-take-all* bargaining, buyers extract the entire trade surplus in the day market from sellers which is why sellers do not value capital for its role as a productive DM input anymore. Again, depending on the rate of inflation, buyers value the marginal unit of money and capital as media of exchange. Thus, the marginal unit of money and the marginal unit of capital are valued for an equal number of roles and $\pi^{-1} = F'(K) > 1$. Finally, generalized Nash bargaining with $\theta \in (0, 1)$ is a hybrid of the other two cases: Sellers do not fully value the marginal unit of capital as a productive DM input because they have to share the trade surplus generated from their capital holdings with buyers. With all three protocols, it follows that monetary equilibria exist only for deflation, i.e. $\pi < 1$.

The Friedman rule, i.e. $\pi = \beta$, is the optimal monetary policy in the *price-taking* version of this model. At the Friedman rule *price-taking* generates the *first best* as given by the central planner' solution: Sellers fully value the marginal unit of capital for its roles as NM and as DM input in production. The marginal unit of capital is not valued as a medium of exchange because buyers can already afford their desired level of DM consumption. Furthermore, agents hold the same amount of capital as the central planner.

In contrast, the Friedman rule is not the optimal policy in *buyer-take-all* bargaining. In BTA, sellers do not value capital as a DM input because they do not benefit from the DM trade surplus. That is why agents hold less than the welfare-optimal amount of capital at $\pi = \beta$. For $\pi > \beta$, buyers cannot afford their desired level of DM consumption anymore. This induces agents in the night market to purchase more capital because they can use capital to increase their DM consumption if they become buyers in the day market. Thus, welfare rises if π is slightly greater than β as it brings the equilibrium capital holdings closer to the central planner's choice - even though this increase is generated through a different motif than in the central planner's solution.

An effect of capital as a DM input can be seen by comparing this model to a version in which capital is not used as an input in DM production. With *price-taking* agents hold more capital in any monetary equilibrium if it is used as an input in DM production. As mentioned above, sellers fully value capital as a DM input and thus, $\pi^{-1} > F'(K)$. If capital is not used as an input in production the NM return on money and capital have to be equal in equilibrium, i.e. $\pi^{-1} = F'(K)$. This implies that the capital stock is greater if capital is used as a DM input. This result is also true for generalized bargaining with $\theta \in (0, 1)$ if sellers bring all of their capital into the DM. The equilibrium capital stock in *buyer-take-all* bargaining is independent of capital's role as a DM input because sellers do not receive any trade surplus in the day market. Thus, $\pi^{-1} = F'(K)$ regardless of whether or not capital is an input in DM production.

Outside the Friedman rule, i.e. $\pi > \beta$, agents hold money only if it is used as a medium of exchange. Thus, monetary equilibria do not exist if buyers hold enough capital to afford q^*. The cost to acquire q^* is greater if capital is used as an input in DM production. Consequently, the set of parameters which support a monetary equilibrium is larger if capital is used as an input in DM production.

Finally, the buyers' marginal willingness to pay is strictly greater than zero if $\pi > \beta$ as buyers cannot afford their desired level of consumption. Thus, buyers' DM payoffs are strictly increasing in the amount of assets which they bring into the day market. In contrast, buyers' marginal willingness to pay is zero at the Friedman rule, i.e. $\pi = \beta$. In this case, buyers can already afford their desired level of DM consumption. Consequently, they cannot increase their DM payoff by bringing additional assets into the day market anymore.

1.7 Appendix

Proof of Proposition 1. In a monetary equilibrium with $\pi = \beta$, K is uniquely determined by $\beta^{-1} = F'(K)$. Given the unique solution for K, q is uniquely determined by $u'(q) = c_q(q, F(K))$ and X is the unique solution to the budget constraint (1.24). Z is indetermined at the Friedman rule.

For $\pi > \beta$, K is uniquely determined by equation (1.37), i.e. $\pi^{-1} = F'(K)$. Note that $0 \leq K < \infty \Rightarrow \pi < 1$. Given K, the money Euler equation uniquely determines q. Finally Z and X are uniquely determined by equation (1.36) and the budget constrained (1.24), respectively. Note that as $\pi \to \beta$, $u'(q) = c_q(q, F(K))$. □

Proof of Proposition 2. At the Friedman rule ($\pi = \beta$), K and q are lower than in the central planner's solution. The capital stock held by agents at the $\pi = \beta$ K^{FR} is uniquely determined by $\beta^{-1} = F'(K)$ and q^{FR} solves $u'(q) = c_q(q, F(K))$ given K. In the central planner's solution, K^{CP} and q^{CP} solve equations (1.2) and (1.3) which are repeated for convenience:

$$u'(q) = c_q(q, F(k)) \tag{1.56}$$

$$1 = \beta F'(k) \left[1 - \sigma c_k \left(q, F(k)\right)\right] \tag{1.57}$$

Equation (1.57) implies $\beta^{-1} > F'(K^{CP})$ because $c_k < 0$. Given K^{CP} equation (1.56) determines q^{CP} as a function of K^{CP}. Note that agents at $\pi = \beta$ derive q from the same functional form. Comparing the equilibrium capital stock at $\pi = \beta$ to the central planner's yields $K^{CP} > K^{FR}$. Since the central planner holds more capital, his equilibrium value of $q^{**}(K)$ is also greater than at the Friedman rule, i.e. $q^{FR} < q^{CP}$.

At $\pi > \beta$, K and q solve the money and capital Euler equations (1.34) and (1.35) which implies $\pi^{-1} = F'(K)$. Thus, the equilibrium capital stock is increasing in π and consequently agents hold more capital at $\pi > \beta$ than at $\pi = \beta$. Per-capita net consumption equals output in a stationary equilibrium, i.e. $X = f(K)$. Consequently, an increase in capital leads to a new stationary equilibrium with higher net consumption.

Consider the economy-wide welfare in a stationary equilibrium.

$$\mathbb{W} = \frac{1}{1-\beta} \left\{\sigma \left[u(q) - c\left(q, F(k)\right)\right] + x\right\} \tag{1.58}$$

The derivative of $\mathbb{W}$ with respect to π is given by

$$\frac{d\mathbb{W}}{d\pi} = \frac{1}{1-\beta}\frac{\partial K}{\partial \pi}\left[\frac{\partial X}{\partial K} - \sigma c_k F' + \frac{\partial q}{\partial K}\sigma(u' - c_q)\right] \tag{1.59}$$

where $-c_k F' > 0$, $\frac{\partial K}{\partial \pi} > 0$ and $\frac{\partial X}{\partial K} > 0$. As $\pi \to \beta$, $u' = c_q$. Thus, the third term in the bracket vanishes and $\frac{d\mathbb{W}}{d\pi} > 0$. This implies that the welfare-maximizing rate of inflation is strictly greater than β. □

Proof of Lemma 1. q is implicitly defined as a function of K by $u'(q) = c_q(q, F(K))$. Thus, $\frac{\partial q^{**}(K)}{\partial K} = \frac{-c_{qk}F'}{c_{qq} - u''} > 0$ because $c_{qk}, u'' < 0$ and $F', c_{qq} > 0$.

Using the general functional form $c(q^{**}(K), F(K)) = q^{\psi} F(K)^{1-\psi}$ where $\psi > 1$, $q = q^{**}(K)$ can be written as $q = \left(\frac{\gamma}{\psi}\right)^{\frac{1}{\psi-\gamma}} F(K)^{\frac{\psi-1}{\psi-\gamma}}$. The cost of acquiring q is given by $c(q^{**}(K), F(K)) = \left(\frac{\gamma}{\psi}\right)^{\frac{\psi}{\psi-\gamma}} F(K)^{\frac{(\psi-1)\gamma}{\psi-\gamma}}$. Since the exponent of $F(K)$ is greater than zero, it follows that $\frac{\partial c(q^{**}(K), F(K))}{\partial K} > 0$. □

Proof of Proposition 4. Lemma 1 shows that the cost of acquiring $q^*(K)$ is strictly increasing in K. Recall that K is obtained from $\beta^{-1} = F'(K)$. For reasonable parameter values, [i.e. $\alpha > \beta^{-1} - 1$ which implies $K > 1$, given $f(k) = k^{\alpha}$], it follows that $c(q, F(K)) = q^{\psi} F(K)^{1-\psi} > c(q^B) = (q^B)^{\psi}$ where q^B is obtained from $u'(q) = c_q(q)$. □

Proof of Proposition 5. For $\pi > \beta$, q and K are obtained from the money and capital Euler equations (1.34) and (1.35). To proove existence and uniqueness, we use the following very general functional forms, i.e. $u(q) = q^{\gamma}$, $c(q, K) = q^{\psi} K^{1-\psi}$ and $F(K) = K + K^{\alpha}$ where $\gamma, \alpha < 1$ and $\psi > 1$. Solving the money Euler equation for q as a function of K and inserting it into the capital Euler equation yields

$$1 = \beta(1 + \alpha\frac{1}{K^{1-\alpha}})\left\{\beta^{-1}\pi + (\psi - 1)\sigma\left[(1 + \frac{i}{\sigma})\frac{\psi}{\gamma}\right]^{\frac{\psi}{\gamma-\psi}} \frac{1}{K^{\frac{\psi(1-\gamma)}{\psi-\gamma}}}\right\} \tag{1.60}$$

Notice in equation (1.60) that the exponents of K are positive. In the following denote the right-hand side of equation (1.60) by RHS. Therefore, $K = 0 \Rightarrow RHS \to \infty$. Conversely, $K \to \infty$ yields $RHS = \pi$ because both terms that include K vanish. If $\pi < 1$, RHS is greater than 1 for small values of K and smaller than K for large values. Thus, an equilibrium exists.

Differentiating RHS with respect to K shows that it is strictly decreasing in K. Thus, $RHS(K)$ equals 1 at a unique K. Given the unique solution for K q is uniquely determined by the money Euler equation. Finally, Z and X are uniquely determined by equations (1.52) and (1.24), respectively.

Similarly, it can be shown that a monetary equilibrium for $\pi = \beta$ exists and that it is (apart from the amount of money held) unique. $\square$

Chapter 2

Media of Exchange with Differing Liqudity

2.1 Introduction

This paper analyzes how inflation affects investment and factor prices in a monetary model with two media of exchange, money and claims to capital. The assets differ in two dimensions. Capital earns a greater return than money which (generally) loses value due to inflation. Furthermore, money is more liquid than capital.

Following Lagos and Wright (2005), trade takes place in two distinct markets each period. In the morning of each period, agents enter a decentralized market where they are randomly matched in pairs. Agents are completely anonymous in the morning which makes a medium of exchange necessary for trade to occur in *single coincidence of wants* meetings. In the evening of each period, agents trade in a frictionless Walrasian market.

There is a part of the economy where both, capital and money are accepted as media of exchange and there is another part of the economy where only money is accepted as medium of exchange. There are two motives to hold capital: It generates a return as a productive input in the frictionless market and it can be used as a medium of exchange in a fraction of all transactions in the decentralized market. In contrast to capital, money can always be used as a medium of exchange. It, however, loses value each period if there is positive (net) inflation. Money's higher liquidity compared to capital's compensates for its inferiority in the rate of return.

There are three types of stationary equilibria. If inflation is greater than some critical value, buyers' money and capital holdings do not provide enough liquidity to afford their desired level of consumption. Consequently, they value the marginal unit of both assets for its role as a medium of exchange. In contrast to money, capital is only valued as a medium of exchange in a fraction of all transactions.

For rates of inflation below the critical value, buyers can only afford their desired level of consumption if both, money and capital are accepted as media of exchange. As a consequence, neither money nor capital are valued as a medium of exchange at the margin if both assets can be used as payment. Consider the part of the economy where money is the only permissible medium of exchange: Buyers' money holdings alone do not suffice to purchase their desired level of consumption. Thus, the marginal unit of money is still valued for its role as a medium of exchange if money is the only permissible means of payment. Finally, the return of money (i.e. deflation) offsets the loss from discounting at the Friedman rule. Consequently, agents hold enough money to purchase their desired level of consumption and neither money nor capital are valued as a medium of exchange at the margin.

This model generates a Tobin effect if the marginal unit of capital and the marginal unit of money are both valued as media of exchange. In other words, an increase of inflation raises capital investment if the rate of inflation is above the critical value. A rise in inflation makes holding money costlier and agents substitute out of money and into capital in response. The higher capital stock leads to a decrease of the marginal product of capital and an increase of the marginal product of labor. Thus, wages go up and the rental rate of capital decreases. Due to higher wages, agents work and consume more in the evening, i.e. in the Walrasian market. The total value of their assets in terms of the decentralized market's good is lower for higher rates of inflation. Hence, consumption in the decentralized market decreases. Note that agents always hold a positive amount of money regardless of the rate of inflation as money is the only possible medium of exchange in some transactions. On the other hand, inflation does not impact capital investment or factor prices if the rate of inflation is below the critical value, i.e. if agents can afford their desired consumption in transactions where both assets are accepted as media of exchange.

This model nests Aruoba and Wright (2003) and Lagos and Rocheteau (2008) as special cases. It resembles Lagos and Rocheteau (2008) if capital can always be used as a medium of exchange. In this case, money and capital have the same liquidity value.

They can only coexist if they are perfect substitutes, i.e. if they yield the same return. Thus, inflation pins down the gross return on capital and the capital stock accordingly.

If, on the other hand, capital can never be used as a medium of exchange, money is the only asset generating a liquidity value as in Aruoba and Wright (2003). In this case, inflation does not have an effect on investment and factor prices. This is what Aruoba and Wright (2003) call "neoclassical dichotomy".

Aruoba, Waller, and Wright (2011) break the "neoclassical dichotomy" by introducing capital as an input in both markets. They consider two types of matches in the market with frictions. In monitored transactions sellers extend credit to buyers who, then, can always afford their desired level of consumption. In unmonitored transactions buyers can only use money as a medium of exchange and they always receive less than their desired amount of goods from sellers. In their model, an increase in inflation leads to a reduction in capital investment for all levels of inflation. Thus, it generates a Stockman effect where inflation acts as a tax on consumption and on capital investment.

Lester, Postlewaite, and Wright (2009) study money and capital as competing media of exchange. As in this analysis, they assume that capital can only be used as a medium of exchange in some transactions whereas money can always take this role. In contrast to this paper, capital in their model are Lucas' trees. The supply of capital is fixed and each 'tree' yields a constant return. That is why they cannot study the effects of inflation on investment and factor prices.

2.2 Environment

Time is discrete and continues forever. Each period is divided into two subperiods, called morning and evening. The economy is populated by a unit measure of agents who only discount between periods.

In the morning, each agent is able to produce a unique perishable good. Thus, the set of distinct goods available in the morning is of measure one. Agents discriminate between the different goods and only receive a positive utility from a fraction of all goods. A good is called an agent's consumption good if he derives a positive utility from consuming it. The measure of each agent's consumption goods is $\sigma < 0.5$. Agents receive $u(q)$ utils from consuming q units of any one of their consumption goods. Furthermore, there are no two agents with the same set of consumption goods. To prevent autarky, agents never receive positive utility from the good which they can produce themselves. They bear a cost $c(q)$ from producing q units of their unique good. As usual, u is concave and c is convex. In contrast, goods in the evening market are homogeneous and durable. Consumption of x goods yields utility $U(x)$ where U is concave, as well. Additionally, agents work h hours at a firm and bear a cost of h utils from work.

A representative firm produces the homogeneous goods in the evening. Its production technology F uses aggregate capital K and aggregate labor H as inputs. The firm rents labor and capital from the agents and pays them a wage w and a rental rate r which, in equilibrium, equal labor's and capital's marginal products, respectively. Note that capital and the homogeneous consumption good are the same. If agents use the homogeneous good to save, it is referred to as capital. That is why the price for capital in terms of the consumption good is always one. There is another storage technology available. Agents can purchase fiat money which is provided by the government. In contrast to capital, it has no productive use, however.

In the morning, trade is decentralized. Agents are randomly matched in pairs and trade bilaterally. That is why the morning market is also referred to as the decentralized market (DM). Agents are completely anonymous and there is no technology available to verify one's identity. This implies that all trades have to be quid pro quo. In the evening, agents trade in a standard Walrasian market in which trade is centralized and frictionless. This market is called the centralized market (CM).

Consider two agents matched in a pair in the decentralized market. Assume that two agents, who can both produce the other's consumption good, never meet, excluding a 'double coincidence of wants' [Jevons (1910)] meeting. Thus, there remain two cases to consider. If neither agent can produce the other agent's consumption good, both leave the decentralized market without having traded. In the other case only one agent (seller) can produce the other agent's (buyer) consumption good. Due to the anonymity, the buyer has to offer something in exchange for the seller's goods. If the buyer has a medium of exchange, he makes a 'take it or leave it' offer, which the seller either accepts or rejects. If the offer is accepted, they trade according to the negotiated terms of trade, otherwise, they leave the DM without having traded.

Both assets, money and claims to capital, can be used as media of exchange in a fraction λ of all transactions. Otherwise, only money can be used as payment. Lester, Postlewaite, and Wright (2009) motivate this strict refusal of capital in some transactions by assuming that claims to capital can be instantaneously counterfeited at no cost. The authenticity of a claim to capital can only be verified in a fraction of all transactions, called 'informed' transactions. In all other transactions ('uninformed' transactions), sellers anticipate that buyers will always produce counterfeits on the spot to pay with. Consequently, sellers only accept money in uninformed transactions whereas both, money and capital are accepted as payment in informed transactions.

Money, M, is supplied by the government at zero cost. Initially, each agent is endowed with an equal amount of money. Each period in the centralized market, the government can inject new money or remove money from the system. Thus, the total supply of money at the end of the centralized market in period t, M_{t+1}, can be expressed recursively as $M_{t+1} = (1+v_t)M_t$ where v_t is the government's choice variable in period t. If the government decides to inject new money ($v_t > 0$), it sells $M_{t+1} - M_t$ units of new money to the agents in the centralized market and generates a real revenue (seignorage) of $(M_{t+1} - M)/\mathbb{P}_t$ where $\mathbb{P}_t$ denotes the price of one unit of capital in terms of money. On the other hand, the government pays $(M_{t+1} - M_t)/\mathbb{P}_t$ units in terms of the CM good to reduce the supply of money by $M_{t+1} - M_t$ units if $v_t < 0$. Finally the government does not intervene in the centralized market if $v_t = 0$. Its budget constraint is given by

$$T_t = v_t \frac{M_t}{\mathbb{P}_t} \tag{2.1}$$

The right-hand side of equation (2.1) shows the real value of the change in money supply. The government raises lump-sum taxes, i.e. $T < 0$, from the agents to finance its expenditures if it reduces the supply of money. That is, it collects assets (money and capital) for a total value of $v_t M_t/\mathbb{P}_t$ from the agents lump-sum. Otherwise, the government transfers its seignorage to the agents lump-sum. Instead of nominal money balances, we consider real money balances, $Z_t \equiv M_t/\mathbb{P}_t$, in the remainder of this chapter. Furthermore, (gross) inflation is given by the relative change in prices, i.e. $\pi_t \equiv \mathbb{P}_{t+1}/\mathbb{P}_t$.

2.3 Central Planner

Consider the central planner's solution as a reference for the welfare analysis of the subsequent competitive equilibrium. In a *single coincidence of wants* meeting, the central planner can induce producers to produce and give their goods to buyers free of charge. The central planner maximizes the cross-section average of average discounted present values of expected utility over all infinite future.

$$\begin{aligned} \mathbb{W} = \max_{\{X_t, H_t, K_{t+1}, q_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \left[U(X_t) - H_t + \sigma \left[u(q_t) - c(q_t) \right] \right] \\ F(K_t, H_t) = & X_t + K_{t+1} - (1-\delta)K_t \end{aligned} \tag{2.2}$$

In the centralized market, agents receive $U(X_t)$ utils from the consumption of X_t units of the homogeneous good and they suffer a utility loss from supplying H_t hours of work. The probability of an agent becoming a buyer or a seller in the decentralized market is given by σ respectively. In the DM buyers (measure σ) consume q_t units of their consumption good which yields a utility $u(q_t)$. Sellers (measure σ) produce q_t units of their production good at a cost of $c(q_t)$ utils. The constraint in maximization problem (2.2) shows that CM output $F(K_t, H_t)$ is either consumed or invested in capital $K_{t+1} - (1-\delta)K_t$ where δ denotes the rate at which capital depreciates.

The solution to maximization problem (2.2) satisfies

$$u'(q_t) = c'(q_t) \tag{2.3}$$

$$U'(X_t) = \frac{1}{F_H(K_t, H_t)} \tag{2.4}$$

$$U'(X_t) = U'(X_{t+1})\beta \left[1 + F_K(K_{t+1}, H_{t+1}) - \delta\right] \tag{2.5}$$

$$F(K_t, H_t) = X_t + K_{t+1} - (1-\delta)K_t \tag{2.6}$$

The amount of DM goods traded in a match q is uniquely determined by equation (2.3). At the optimum, the buyer's marginal utility from consumption equals the seller's marginal disutility from production. Equation (2.4) shows the representative agent's intratemporal trade-off in the centralized market. In equilibrium, the marginal utility from CM consumption equals his marginal disutility from working in the centralized market. In contrast, the capital Euler equation (2.5) reveals his intertemporal trade-off. An increase of his capital investment in period t reduces his CM consumption in period t but increases his CM consumption in period $t+1$. In equilibrium, both effects are equal at the margin according to the capital Euler equation. The resource constraint (2.6) states that output is either consumed or invested. Note that money is not valued because agents do not need a medium of exchange to facilitate DM trade in the central planner's solution.

2.4 Individual Problem

The agent's maximization problem in the centralized market is given by

$$\begin{aligned} W(z,k) = \max_{x,h,k',z'} & \; U(x) - h + \beta V(z',k') \\ \text{s.t.} \quad x + k' + \pi z' =& (1 + r - \delta)k + wh + z + T \end{aligned} \tag{2.7}$$

where the time-subscript is dropped for variables in the current period and next period's variables are denoted by a prime. The objective function of maximization problem (2.7) shows an agent's value function upon entering the centralized market with z units of real money and k units of capital. In the centralized market an agent derives positive utility from consuming x units of the CM good and bears disutility from working h hours. In the next morning, he enters the decentralized market. His continuation value is given by the decentralized market value function V and depends on his new asset portfolio (z', k'). The constraint in maximization problem (2.7) is the agent's budget constraint. Consumption, investment in capital , $k' - (1-\delta)k$, and investment in real money, $\pi z' - z$, are financed by wage and labor income, wh and rk, respectively.

Combining the objective function and the constraint of maximization problem (2.7) yields

$$\begin{aligned} W(z,k) =& \frac{1}{w}\left[(1 + r - \delta)k + z + T\right] \\ &+ \max_{x,k',z'} \left\{ U(x) - \frac{1}{w}\left[x + k' + \pi z'\right] + \beta V(z',k') \right\} \end{aligned} \tag{2.8}$$

The maximization problem as given by the last term on the right-hand side of equation (2.8) provides the optimal choices of (x, k', z'). An agent's individual labor supply,

$h \in \mathbb{R}$, is determined by his individual budget constraint in maximization problem (2.7) given his choices of (x, k', z'). If $h < 0$, he buys labor services (e.g. back rubs) from other agents at the market clearing wage w. If $h > 0$, he works for the firm and possibly provides labor services to other agents. Corner solutions to maximization problem (2.7) are excluded because h is not bounded. Notice from equation (2.8) that an agent's individual choice variables x, k' and z' are independent of his asset portfolio (z, k) [given $V(z', k')$ does not depend on z or k]. It is possible to separate state and choice variables in maximization problem (2.7) because of the quasi-linear preferences in the centralized market. As a consequence, the distribution of money and capital holdings is degenerate at the end of each period as all agents leave the CM with the same asset portfolio regardless of their portfolio when entering the centralized market. This is the same result as in Lagos and Wright (2005). Finally note, that $W(z + a, k + b)$ can be written as $W(z, k) + \frac{a}{w} + \frac{(1+r-\delta)b}{w}$ for arbitrary a and b.

Differentiating maximization problem (2.8) yields

$$x: \qquad U'(x) = \frac{1}{w} \tag{2.9}$$

$$k': \qquad U'(x) = \beta \frac{\partial V(z', k')}{\partial k'} \tag{2.10}$$

$$z': \qquad -U'(x)\pi + \beta \frac{\partial V(z', k')}{\partial z'} \geq 0 \tag{2.11}$$

Equation (2.9) is the standard intratemporal first order condition between consumption and labor. An agent needs to provide $1/w$ units of labor to afford a unit of consumption. At the optimum, he chooses x such that his marginal utility from consumption equals his marginal disutility from providing $1/w$ units of labor which are necessary to purchase the last unit of consumption. Equations (2.10) is the capital Euler equations. As with consumption, the price of capital in terms of labor is $1/w$. Agents hold capital to increase their expected payoff in the future. At the optimum, the marginal disutility from providing $1/w$ units of labor in the centralized market is offset by the increase in future payoffs provided by the marginal unit of capital.

The left-hand side of inequality (2.11) depicts the partial derivative of maximization problem (2.7) with respect to next period's money holdings z'. An agent needs to provide π/w units of labor to afford a unit of money. Money, on the other hand, increases his continuation value. Agents decide to purchase a unit of money if the utility it provides in the future exceeds the utility loss from acquiring it. Thus, money is held if there exists a $z' > 0$ such that condition (2.11) is satisfied. Condition (2.11) is called money Euler equation if it is fulfilled at equality. In this case, marginal (future) utility equals marginal cost at the optimal choice of z'.

Next, consider an agent entering the decentralized market with k units of capital and z units of real money. His DM value function V (before the matching) can be decomposed into three parts.

$$V(z,k) = \sigma V^b(z,k) + \sigma V^s(z,k) + W(z,k) \tag{2.12}$$

With probability σ, he is matched with an agent who can produce his consumption good. In this case, he is the buyer and receives a surplus of $V^b(z,k)$ from trade in the decentralized market. His lifetime-utility is, therefore, given by $V^b(z,k) + W(z,k)$. With the same probability, he becomes the seller, i.e. he can produce the other's consumption good. This provides him with a trade surplus of $V^s(z,k)$ in the decentralized market. Finally, neither he nor his partner can produce the other's consumption good. In this case, which happens with the residual probability, $1 - 2\sigma$, both agents leave the decentralized market without trading.

The buyer's payoff from trade in the decentralized market V^b takes the form

$$V^b(z,k) = \lambda \left[u(q_1^b) - \frac{1}{w} d_z^{1,b} - \frac{1}{w}(1 + r - \delta) d_k^{1,b} \right] + (1 - \lambda) \left[u(q_2^b) - \frac{1}{w} d_z^{2,b} \right] \tag{2.13}$$

where q_1^b, q_2^b, $d_k^{1,b}$, $d_z^{1,b}$ and $d_z^{2,b}$ denote the terms of trade which are endogenously determined in each match. As a consequence, the terms of trade in equation (2.13) are functions of our agent's and his random trading partner's asset holdings. There are two cases to be distinguished. The buyer is in an informed match with probability λ and in an uninformed match with probability $1 - \lambda$. If he is in an informed match (case 1), he gets q_1^b units of the seller's good in exchange for $d_z^{1,b}$ units of money and $d_k^{1,b}$ units of capital. In an uninformed match (case 2), the seller only accepts money as medium of exchange. Consequently the terms of trade are given by the pair $(q_2^b, d_z^{2,b})$.

Similarly, V^s is defined as

$$V^s(z,k) = \lambda \left[-c(q_1^s) + \frac{1}{w} d_z^{1,s} + \frac{1}{w}(1 + r - \delta) d_k^{1,s} \right] + (1 - \lambda) \left[-c(q_2^s) + \frac{1}{w} d_z^{2,s} \right] \tag{2.14}$$

where q_1^s, q_2^s, $d_k^{1,s}$, $d_z^{1,s}$ and $d_z^{2,s}$ depend on k, z and the random trading partner's asset portfolio. A seller produces q_1^s goods at a cost of $c(q_1^s)$ utils in an informed transaction (probability λ). In return he receives $d_z^{1,s}$ units of money and $d_k^{1,s}$ units of capital. If he is in an uninformed transaction, he produces q_2^s goods, suffers a utility loss of $c(q_2^s)$ and is compensated with $d_z^{2,s}$ units of money. It is necessary to distinguish between the terms of trade in equations (2.13) and (2.14) because they are match-specific.

Inserting (2.13) and (2.14) into (2.12) yields the DM value function in its full form

$$\begin{aligned} V(z,k) =& \sigma \left\{ \lambda \left[u(q_1^b) - \frac{1}{w} d_z^{1,b} - \frac{1}{w}(1+r-\delta) d_k^{1,b} \right] + (1-\lambda) \left[u(q_2^b) - \frac{1}{w} d_z^{2,b} \right] \right\} \\ &+ \sigma \left\{ \lambda \left[-c(q_1^s) + \frac{1}{w} d_z^{1,s} + \frac{1}{w}(1+r-\delta) d_k^{1,s} \right] + (1-\lambda) \left[-c(q_2^s) + \frac{1}{w} d_z^{2,s} \right] \right\} \\ &+ W(z,k) \end{aligned} \tag{2.15}$$

Consider a match between a buyer and a seller with asset portfolios (z^b, k^b) and (z^s, k^s), respectively. Agents' asset holdings are common knowledge within each match. The buyer makes a take-it-or-leave-it offer to the seller and the seller accepts or rejects it. If it is accepted, they trade as agreed upon. Otherwise, they leave the decentralized market without having traded. In an informed match, an offer is a triple (q_1, d_z^1, d_k^1). The buyer makes the offer which maximizes his payoff subject to the seller's participation constraint.

$$\begin{aligned} &\max_{q_1, d_z^1, d_k^1} u(q_1) - \frac{1}{w} d_z^1 - \frac{1}{w}(1+r-\delta) d_k^1 + W(z,k) \\ &\text{s.t.} \ -c(q_1) + \frac{1}{w} d_z^1 + \frac{1}{w}(1+r-\delta) d_k^1 \geq 0 \\ &\quad d_z^1 \leq z^b \ \text{ and } \ d_k^1 \leq k^b \end{aligned} \tag{2.16}$$

The objective function in maximization problem (2.16) shows the buyer's DM payoff from his proposed offer (q_1, d_z^1, d_k^1). He receives the DM good from the seller and pays money and capital in exchange which decreases his continuation value to $W(z - d_z^1, k - d_k^1)$. Due to the linearity of w, the CM continuation value can be written as in the objective function of maximization problem (2.16). The seller only accepts the buyer's offer if it makes him at least as well off as he would be without trading in the DM. Therefore, his threat point is given by $W(z,k)$. If he accepts the offer, he loses $c(q_1)$ utils from production but advances to the CM with additional assets. Again, due to the linearity of w, the seller's participation constraint can be expressed as in maximization problem (2.16). Finally, the buyer cannot write IOUs and is, thus, constrained by his asset holdings.

The solution to maximization problem (2.16) is given by

$$q_1 = \begin{cases} q^* & \text{if } \frac{1}{w}z^b + \frac{1}{w}(1+r-\delta)k^b \geq c(q^*) \\ c^{-1}\left(\frac{1}{w}z^b + \frac{1}{w}(1+r-\delta)k^b\right) & \text{otherwise} \end{cases} \tag{2.17}$$

$$\frac{1}{w}d_z^1 + \frac{1+r-\delta}{w}d_k^1 = c(q_1) \tag{2.18}$$

In the unconstrained solution to maximization problem (2.16), the buyer receives q^* units of the DM consumption good from the seller, where q^* solves $u'(q_1) = c'(q_1)$. In this case, equation (2.18) determines the real value of the buyer's payment $d_z < z^b$ and $d_k < k^b$. Note that the composition of his payment into money and capital remains indetermined. In the constrained solution, the buyer spends all of his money and capital holdings and receives q_1 which is obtained from equation (2.18) with $d_z^1 = z^b$ and $d_k^1 = k^b$.

An offer in an uninformed match is a couple (q_2, d_z^2). Again, consider a match between a buyer with assets (z^b, k^b) and a seller with (z^s, k^s). Similarly to the informed match, the buyer makes an offer (q_2, d_z^2) which maximizes his DM payoff subject to the seller's participation and his asset constraint.

$$\begin{aligned} &\max_{q_2, d_z^2} \; u(q_2) - \frac{1}{w}d_z^2 + W(z,k) \\ &\text{s.t.} \; -c(q_2) + \frac{1}{w}d_z^2 \geq 0 \quad \text{and} \quad d_z^2 \leq z^b \end{aligned} \tag{2.19}$$

The solution to the maximization problem (2.19) is given by

$$q_2 = \begin{cases} q^* & \text{if } \frac{1}{w}z^b \geq c(q^*) \\ c^{-1}\left(\frac{1}{w}z^b\right) & \text{otherwise} \end{cases} \tag{2.20}$$

$$d_z^2 = \begin{cases} wc(q^*) & \text{if } \frac{1}{w}z^b \geq c(q^*) \\ z^b & \text{otherwise} \end{cases} \tag{2.21}$$

The buyer receives q^* units of the seller's good if he has enough money to compensate the seller for his production cost $c(q^*)$. In this case, the seller receives $wc(q^*) \leq z_b$ units of the buyer's money holdings. Otherwise, the buyer trades all of his money z^b in exchange for $c^{-1}\left(\frac{1}{w}z^b\right)$ units of the DM consumption good.

Comparing equations (2.17) and (2.20) reveals two important relations between q_1 and q_2. First, $q_2 = q^* \Rightarrow q_1 = q_2 = q^*$. Thus, a buyer, who holds enough money to afford q^* in uninformed matches, can afford q^* in informed matches as well. Second, $q_1 < q^* \Rightarrow q_2 < q_1 < q^*$. If a buyer's money and capital holdings do not suffice to trade q^*, he will get more q if he is allowed to use both money and capital as media of exchange, as compared to only money. In general, we can write $q_2 \leq q_1 \leq q^*$.

2.5 Stationary Equilibrium

Recall that, due to the quasilinear CM preferences, each agent chooses to exit the CM with the same (per-capita) amount of money, capital and consumption, i.e. we can write $x = X$, $k' = K'$ and $z' = Z'$, where capitalized letters denote per-capita variables. Furthermore, remember that labor and capital are compensated with their marginal products $w = F_H(K, H)$ and $r = F_K(K, H)$, respectively.

We now consider three exhaustive cases, denoted by (i), (ii) and (iii). In case (i), $\frac{1}{w}Z' + \frac{1}{w}(1 + r - \delta)K' \leq c(q^*)$. According to equations (2.17) - (2.21), this implies that buyers spend all their assets ($d_z^1 = Z$, $d_k^1 = K$) in informed matches and all of their money $d_z^2 = Z$ in uninformed matches. The solutions to q_1 and q_2 are obtained by

$$c(q_1') = \frac{1}{F_H(K', H')}Z' + \frac{1}{F_H(K', H')}(1 + F_K(K', H') - \delta)K' \tag{2.22}$$

$$c(q_2') = \frac{1}{F_H(K', H')}Z' \tag{2.23}$$

where $q_2' < q_1' < q^*$. Given these terms of trade, the derivatives of the decentralized market value function are

$$\frac{\partial V(z',k')}{\partial z'} =\frac{1}{F_H(K',H')}\left\{1+\sigma\left[\lambda\left(\frac{u'(q_1')}{c'(q_1')}-1\right)+(1-\lambda)\left(\frac{u'(q_2')}{c'(q_2')}-1\right)\right]\right\} \tag{2.24}$$

$$\frac{\partial V(z',k')}{\partial k'} =\frac{1}{F_H(K',H')}(1+F_K(K',H')-\delta)\left\{1+\sigma\lambda\left(\frac{u'(q_1')}{c'(q_1')}-1\right)\right\} \tag{2.25}$$

The marginal unit of money increases an agent's DM payoff because it is used to purchase goods in informed and in uninformed transactions. The partial derivative of the DM value function with respect to money [equation (2.24)] equals one plus money's expected liquidity value in terms of labor, i.e. divided by F_H: Money's expected liquidity value is the sum of its liquidity value in informed transactions, i.e. $u'(q_1')/c'(q_1')-1$, and in uninformed transactions, i.e. $u'(q_2')/c'(q_2')-1$, weighted by the probabilities of becoming a buyer in informed and in uninformed transactions. Similarly, equation (2.25) expresses the marginal value of capital brought into the decentralized market in terms of labor. At the margin, capital is valued for its role as a productive CM input and as a medium of exchange in informed matches. Thus, its expected liquidity value is given by $\sigma\lambda(u'(q_1')/c'(q_1')-1)$.

Combining equations (2.9) - (2.11), (2.24) and (2.25) yields the centralized market's first order conditions

$$U'(X) =\frac{1}{F_H(K,H)} \tag{2.26}$$

$$U'(X)\pi =\beta U'(X')\left\{1+\sigma\left[\lambda\left(\frac{u'(q_1')}{c'(q_1')}-1\right)+(1-\lambda)\left(\frac{u'(q_2')}{c'(q_2')}-1\right)\right]\right\} \tag{2.27}$$

$$U'(X) =\beta U'(X')(1+F_K(K',H')-\delta)\left\{1+\sigma\lambda\left(\frac{u'(q_1')}{c'(q_1')}-1\right)\right\} \tag{2.28}$$

Equation (2.26) shows the intratemporal trade-off: Marginal utility of consumption equals marginal disutility form working in the centralized market. According to the money and capital Euler equations (2.27) and (2.28) the benefit from bringing an additional unit of money or captial into the decentralized market equals its opportunity cost of consuming less in the centralized market. Note that $U'(X)$ is multiplied by π on the left-hand side of the money Euler equation (2.27) because inflation represents the cost of holding money. The right-hand side of equation (2.28) is multiplied by $(1+F_K(K',H')-\delta)$ because capital generates a return $F_K(K',H')$ and depreciates at the rate δ in next period's centralized market.

Finally, aggregate labor H is determined by the economy's resource constraint and the evolution of money in real terms pins down π.

$$F(K,H) = X + K' - (1-\delta)K \tag{2.29}$$
$$Z' = \frac{1+v}{\pi}Z \tag{2.30}$$

where v is the exogenous government's decision variable. Note that Z does not enter the resource constraint (2.29) because the government transfers its entire seignorage to the agents lump-sum.

In case (ii), buyers can afford q^* in informed but not in uninformed matches. In this case, equations (2.27) and (2.28) become

$$U'(X)\pi = \beta U'(X')\left\{1 + \sigma(1-\lambda)\left(\frac{u'(q_2')}{c'(q_2')} - 1\right)\right\} \tag{2.31}$$
$$U'(X) = \beta U'(X')(1 + F_K(K',H') - \delta) \tag{2.32}$$

Thus, buyers in informed transactions do not value the marginal unit of any asset as a medium of exchange, whereas the marginal unit of money still offers a positive liquidity value in uninformed transactions. In exchange for q^*, buyers (in informed transactions) compensate sellers for their production costs according to

$$c(q^*) = \frac{1}{F_H(K',H')}d_z' + \frac{1}{F_H(K',H')}(1 + F_K(K',H') - \delta)d_k' \tag{2.33}$$

Note that equation (2.33) uniquely determines the value of the compensation but its composition into money and capital remains indeterminate.

Finally, in case (iii), buyers have enough money to trade for q^*, equations (2.31) and (2.23) become

$$U'(X)\pi = \beta U'(X') \tag{2.34}$$
$$c(q^*) = \frac{1}{F_H(K',H')}d_z' \tag{2.35}$$

At the margin, buyers never value any asset for its liquidity because they can always afford q^*. They compensate sellers for their production cost according to (2.35). The capital Euler equation is given by equation (2.32) because, as in case (ii), capital does not yield a positive liquidity value in case (iii).

So far, there are three possible cases (i), (ii) and (iii). In the first case, buyers can never afford q^*. In the second case, buyers can only afford q^* in informed matches but not in uninformed ones and, in the third case, buyers can always afford q^*. Definition 1 characterizes a stationary equilibrium. In a stationary equilibrium, all aggregate variables are constant over time. That is why, the primes are dropped in the remainder of this section.

Definition 1. A stationary equilibrium is a list $(K, Z, X, H, q_1, q_2, d_k^1, d_z^1, d_z^2, \pi)$ with prices $w = F_H(K, H)$ and $r = F_K(K, H)$ which always satisfies

$$U'(X) = \frac{1}{w} \tag{2.36}$$

$$F(K, H) = X + \delta K \tag{2.37}$$

$$\pi = 1 + v \tag{2.38}$$

Additionally, the following conditions are met in a stationary equilibrium where q^* is the solution to $u'(q) = c'(q)$:

(i) if $c(q^*) > \frac{1}{w}Z + \frac{1}{w}(1 + r - \delta)K$:

$$\pi = \beta \left\{ 1 + \sigma \left[\lambda \left(\frac{u'(q_1)}{c'(q_1)} - 1 \right) + (1 - \lambda) \left(\frac{u'(q_2)}{c'(q_2)} - 1 \right) \right] \right\} \tag{2.39}$$

$$\beta^{-1} = (1 + r - \delta) \left\{ 1 + \sigma\lambda \left(\frac{u'(q_1)}{c'(q_1)} - 1 \right) \right\} \tag{2.40}$$

$$c(q_1) = \frac{1}{w}Z + \frac{1 + r - \delta}{w}K \tag{2.41}$$

$$c(q_2) = \frac{1}{w}Z \tag{2.42}$$

$$d_z^1 = d_z^2 = Z \quad \text{and} \quad d_k^1 = K \tag{2.43}$$

(ii) if $\frac{1}{w}(1+r-\delta)K \geq c(q^*) - \frac{1}{w}Z > 0$:

$$\pi = \beta\left\{1 + \sigma(1-\lambda)\left(\frac{u'(q_2)}{c'(q_2)} - 1\right)\right\} \tag{2.44}$$
$$\beta^{-1} = 1 + r - \delta \tag{2.45}$$
$$c(q^*) = \frac{1}{w}d_z^1 + \frac{1+r-\delta}{w}d_k^1 \tag{2.46}$$
$$c(q_2) = \frac{1}{w}Z \tag{2.47}$$
$$q_1 = q^* \quad \text{and} \quad d_z^2 = Z \tag{2.48}$$

(iii) otherwise:

$$\pi = \beta \tag{2.49}$$
$$\beta^{-1} = 1 + r - \delta \tag{2.50}$$
$$c(q^*) = \frac{1}{w}d_z^1 + \frac{1+r-\delta}{w}d_k^1 \tag{2.51}$$
$$c(q^*) = \frac{1}{w}d_z^2 \tag{2.52}$$
$$q_1 = q_2 = q^* \tag{2.53}$$

Assigning functional forms allows us to analyze the comparative statics of the system. First, however, we characterize the stationary equilibrium using functional forms. For the utility from consumption in the centralized market, we choose $U(X) = log(X)$. The representative firm's production function is Cobb-Douglas, $F(K,H) = K^{\alpha}H^{1-\alpha}$ where $\alpha \in (0,1)$. The utility of consumption in the decentralized market is given by $u(q) = Cq^{\gamma}$ where C is an arbitrary coefficient and $0 < \gamma < 1$ to get concavity. Finally, the cost of DM production is $c(q) = q$.

Again, consider case (i). Using the functional forms, equations (2.36), (2.37) and (2.39) - (2.42) can be reduced to the following system of two equations in two unknowns.

$$q_2^{LW} = (\gamma C(1-\lambda))^{\frac{1}{1-\gamma}}\left[\frac{i}{\sigma} + 1 - \lambda\gamma C q_1^{\gamma-1}\right]^{\frac{1}{\gamma-1}} \tag{2.54}$$
$$q_2^{EE} = q_1 - \frac{\alpha}{1 - (1-\delta(1-\alpha))\beta(1+\sigma\lambda(\gamma C q_1^{\gamma-1} - 1))} \tag{2.55}$$

where $i \equiv \pi\beta^{-1}-1$ is the net nominal interest rate. Equations (2.54) and (2.55), both express q_2 as a function of q_1. Call equation (2.54) the 'LagosWright' equation and equation (2.55) the 'Euler' equation. Due to these labels, q_2 carries the superscript LW in equation (2.54) and EE in equation (2.55). Equation (2.54) is derived from the money Euler equation in case (i). It is called the 'LagosWright' equation because it relates DM output to the nominal interest rate, similar to an equation in Lagos and Wright (2005). The second equation does not contain the nominal interest rate. It is derived from the capital Euler equation and is thus referred to as 'Euler' equation. Proposition 8 discusses the LagosWright equation in more detail.

Proposition 8. *The LagosWright function $q_2^{LW}(q_1)$ has the following characteristics:*

(i) $\lim_{q_1 \to q_1^0} q_2^{LW}(q_1) = +\infty$

(ii) $q_2^{LW}(q_j) < q_2^{LW}(q_i)$*, for all* $q_j > q_i > q_1^0$

(iii) $q_2^{LW}(q_1^{LW,min}) = q_1^{LW,min}$ *where* $q_1^{LW,min} \equiv q^* \left(\frac{\sigma}{i+\sigma}\right)^{\frac{1}{1-\gamma}}$

(iv) $\lim_{q_1 \to +\infty} q_2^{LW}(q_1) = q^* \left[\frac{(1-\lambda)\sigma}{i+\sigma}\right]^{\frac{1}{1-\gamma}} > 0$

The proof of Proposition 8 can be found in the appendix. Since the 'LagosWright' function is strictly decreasing, it uniquely crosses the 45° line at $q_1^{LW,min}$ and approaches zero from above. Its shape is shown in figure 2.1.

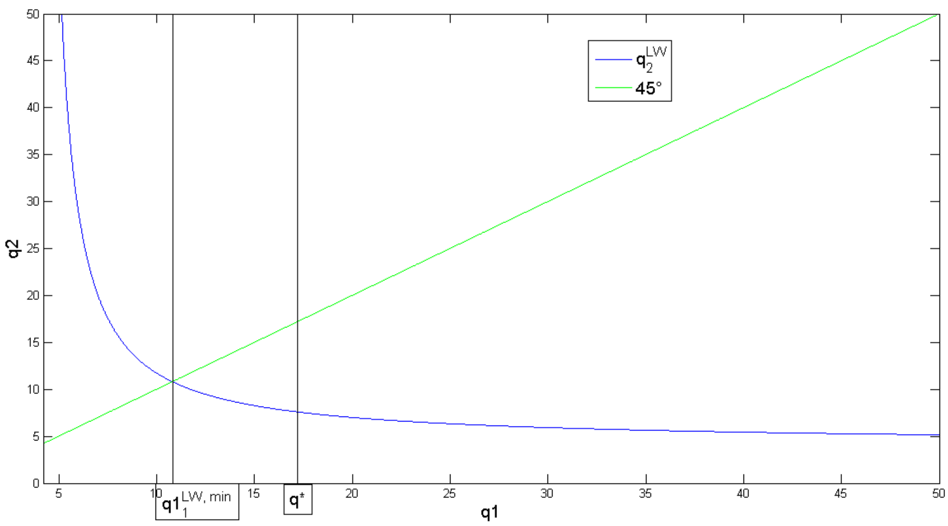

Figure 2.1: Course of the 'LagosWright' function

Figure 2.1 shows values of q_2 on its ordinate and values of q_1 on its abscissa. The blue line represents the 'LagosWright' function $q_2^{LW}(q_1)$. It uniquely maps values of q_1 into q_2 where q_2 is strictly decreasing in q_1. The 'LagosWright' function crosses the 45° line (green line) at the point $q_1^{LW,min}$. Thus, $q_1 > q_2^{LW}(q_1)$ for $q_1 > q_1^{LW,min}$ only. Recall that the 'LagosWright' function was derived for case (i) which implies $q_2 < q_1 < q^*$. Consequently, an equilibrium in case (i) never exists for $q_1 < q_1^{LW,min}$.

The 'LagosWright' function is derived by rearanging the money Euler equation in case (i). Its negative slope can be explained as follows: An agent chooses whether or not to buy money in the centralized market. He does not know his status in the decentralized market when making this decision. Thus, he buys a unit of money in the centralized market if his expected utility gain in next period's decentralized market due to money (i.e. money's expected liquidity value) compensates him for the loss from discounting and the devaluation of money due to inflation. Money's expected liquidity value is the sum of its liquidity values in informed and in uniformed transactions (weighted by their respective probabilities). It is pinned down by the nominal rate of inflation ($\pi\beta^{-1} - 1$) according to the money Euler equation (2.39). In case (i), the marginal unit of money is always used as a medium of exchange and thus, its liquidity value in both, informed and uninformed transactions is strictly greater than zero. Thus, there are countless combinations of liquidity values in informed and in uniformed transactions which generate a given expected liquidity value. Note that the liquidity value always depends negatively on the amount of goods traded in the respective match as marginal utility of consumption is decreasing in q. Thus, a large q_1 leads to a small liquidity value in informed transactions and (given the nominal rate of inflation) requires a large liquidity value in uninformed transactions, i.e. a small q_2. This explains the negative relationship between q_1 and q_2 as shown by the negative slope of the 'LagosWright' function.

Proposition 9 characterizes the Euler function.

Proposition 9. *The properties of the Euler function $q_2^{EE}(q_1)$ are*

(i) $\lim_{q_1 \to 0^+} q_2^{EE}(q_1) = 0$

(ii) $\lim_{q_1 \to \underline{q}^-} q_2^{EE}(q_1) = +\infty$

(iii) $\lim_{q_1 \to \underline{q}^+} q_2^{EE}(q_1) = -\infty$

(iv) $\lim_{q_1 \to +\infty} q_2^{EE}(q_1) = q_1$

(v) $q_2^{EE}(q_i) > q_2^{EE}(q_j)$, *for all* $\underline{q} > q_i > q_j$

(vi) $q_2^{EE}(q_i) > q_2^{EE}(q_j)$, *for all* $q_i > q_j > \underline{q}$

Proposition 9 is proven in the appendix. The course of the 'Euler' function is shown in figure 2.2.

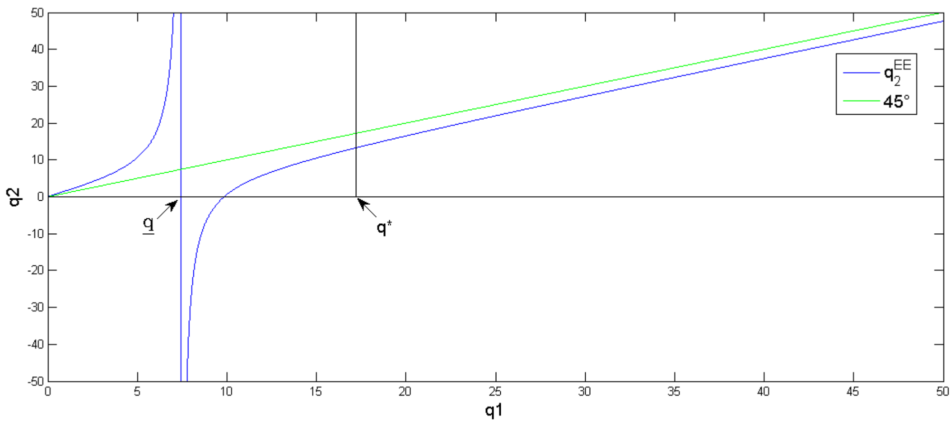

Figure 2.2: Course of the 'Euler' function

The 'Euler' function (blue line) lies above the 45° line (green line) and is strictly increasing in q_1 for values of $q_1 < \underline{q}$. For $q_1 > \underline{q}$, $q_2^{EE}(q_1)$ is also strictly increasing in q_1 but it lies below the 45° line. The Euler function is obtained from the equilibrium conditions of case (i) where $q_2 < q_1 < q^*$. Thus, if an equilibrium in case (i) exists, it must satisfy $q_1 > \underline{q}$.

The positive slope of the 'Euler' function can be explained as follows: The 'Euler' function is derived from the capital Euler equation, the resource constraint and the equations determining the terms of trade in informed and in uninformed matches [equations (2.41) and (2.42)]. Combining the latter two equations yields

$$q_1 = q_2 + \frac{(1 + \alpha K^{\alpha-1} H^{1-\alpha} - \delta)}{(1-\alpha) K^{\alpha} H^{-\alpha}} \tag{2.56}$$

According to equation (2.56) the amount of goods traded in an informed transaction q_1 consists of the amount of goods traded in an uninformed transaction plus the amount of goods which capital buys. This explains the positive relation between q_1

and q_2 and thus, the positive slope of the 'Euler' function $q_2^{EE}(q_1)$.

Theorem 1. *There exists a unique stationary equilibrium with*

(i) $q_2 < q_1 < q^*$ *if* $\pi > \bar{\pi} > \beta$

(ii) $q_2 < q_1 = q^*$ *if* $\bar{\pi} > \pi > \beta$

(iii) $q_2 = q_1 = q^*$ *if* $\pi = \beta$

The proof of Theorem 1 can be found in the appendix.

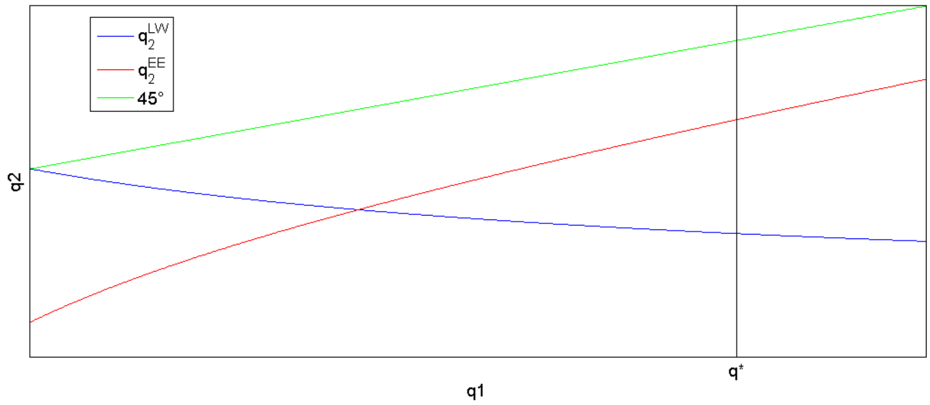

Figure 2.3: A stationary equilibrium with $\pi > \bar{\pi} > \beta$

The 'LagosWright' (blue line) and the 'Euler' function (red line) are plotted in figure 2.3 where q_1 is depicted on the abscissa and q_2 on the ordinate. Recall that both functions are derived under the assumption of case (i), i.e. $\frac{1}{w}Z + \frac{1}{w}(1 + r - \delta)K \leq c(q^*)$. Thus, their intercept only constitutes a stationary equilibrium if it satisfies $0 < q_2 < q_1 \leq q^*$.

This condition has a few implications. First, only the part of the 'Euler' equation which lies to the right of $q_1 = \underline{q}$ (see figure 2.2) is relevant for determining a stationary equilibrium. Second, any intercept between the relevant part of the 'Euler' function and the 'LagosWright' function lies below the 45° line (green line) and satisfies $0 < q_2 < q_1$. Thus, it remains to be checked whether $q_1 \leq q^*$ at the intercept. As stated in Theorem 1, the intercept lies to the left of $q_1 = q^*$ if the rate of inflation π is greater than some critical value $\bar{\pi}$. Intuitively, a decrease in π shifts the 'LagosWright' function upwards but does not influence the 'Euler' function. Consequently, q_1 (and q_2) increase in response to a drop in π.

Figure 2.3 shows the unique stationary equilibrium where $q_2 < q_1 < q^*$ [case (i) in Theorem 1]. If the intercept of the 'LagosWright' and 'Euler' function lies to the right of $q_1 = q^*$, there exists a unique stationary equilibrium with $q_2 \leq q_1 = q^*$. As long as money is held at a cost, i.e. $\pi > \beta$, the inequality is strict and the unique stationary equilibrium solves the equations of case (ii) in Theorem 1. If $\pi = \beta$, then money provides enough liquidity for agents to trade q^* in uninformed matches, i.e. $q_2 = q_1 = q^*$.

Consider an increase of the rate of inflation. It shifts the 'LagosWright' function downwards. The 'Euler' function remains unchanged because it does not contain π. Therefore, the new intersection between the two curves lies to the southwest of the old one. If it lies to the left of $q_1 = q^*$, the new equilibrium trades lower q_1 and q_2 than the old one. Otherwise, the equilibrium q_2 decreases and q_1 is not effected by the increase.

Theorem 2. *The comparatives statics with respect to monetary policy can be summarized by*

- *for* $\beta < \pi < \bar{\pi}$:
  $\frac{\partial q_2}{\partial \pi} < 0,\ \frac{\partial Z}{\partial \pi} < 0,\ \frac{\partial q_1}{\partial \pi} = 0,\ \frac{\partial K}{\partial \pi} = 0,\ \frac{\partial H}{\partial \pi} = 0,\ \frac{\partial X}{\partial \pi} = 0,\ \frac{\partial r}{\partial \pi} = 0,\ \frac{\partial w}{\partial \pi} = 0.$
- *for* $\pi > \bar{\pi}$*:*
  $\frac{\partial q_2}{\partial \pi} < 0,\ \frac{\partial q_1}{\partial \pi} < 0,\ \left|\frac{\partial q_2}{\partial \pi}\right| > \left|\frac{\partial q_1}{\partial \pi}\right|,\ \frac{\partial K}{\partial \pi} > 0,\ \frac{\partial H}{\partial \pi} > 0,\ \frac{\partial X}{\partial \pi} > 0,\ \frac{\partial Z}{\partial \pi}$ *cannot be signed,* $\frac{\partial r}{\partial \pi} < 0,\ \frac{\partial w}{\partial \pi} > 0.$

Consider the first part of Theorem 2. At $\beta < \pi < \bar{\pi}$ buyers can afford q^* in informed transactions only. In this case the CM variables X, K and H are determined independently of the DM variables $q_1 = q^*$ and q_2. This is what Aruoba and Wright (2003) call a 'neoclassical dichotomy'. It can be explained as follows: At the margin, agents value money and capital for their respective roles in different markets: The marginal unit of capital is only used as a productive input in the centralized market as buyers in informed matches can already afford their desired level of consumption. The marginal unit of money, on the other hand, is used as a medium of exchange in uninformed transactions where buyers can only afford $q_2 < q^*$. Thus, there is no asset linking the activities in both markets together and agents choose K, H and X independently of q_1, q_2 and Z. Note that q_1 which solves $u'(q_1) = c'(q_1)$ is independently determined from all other variables for $\pi < \bar{\pi}$.

The value of a unit of (real) money (Z) in terms of CM consumption in the next period is given by $1/\pi$. Thus, an increase in inflation lowers the value of a given amount of money in the next period: Buyers can afford less DM consumption, q_2, in exchange for a unit of money in the next period if π increases. Furthermore, agents hold less money due to the increased inflation. Note that the factor prices w and r do not change in response to a change in inflation because they only depend on K and H.

This 'neoclassical dichotomy' vanishes, however, if $\pi > \bar{\pi}$. Buyers in informed transactions cannot afford q^*, anymore. In this case, the marginal unit of capital provides the link between centralized and decentralized market activity: Capital yields a positive liquidity value in informed transactions. Thus, from the perspective of an agent in the centralized market, capital yields a strictly positive expected liquidity value. Note that it only depends on q_1 and not q_2 as capital's liquidity value in uniformed transactions is always zero. An agent choosing his capital investment in the centralized market considers capital's rate of return as well as its expected liquidity value. Consequently, inflation which impacts q_1 has an effect on CM variables. The effect of an increase of π for $\pi > \bar{\pi}$ is shown in figure 2.4.

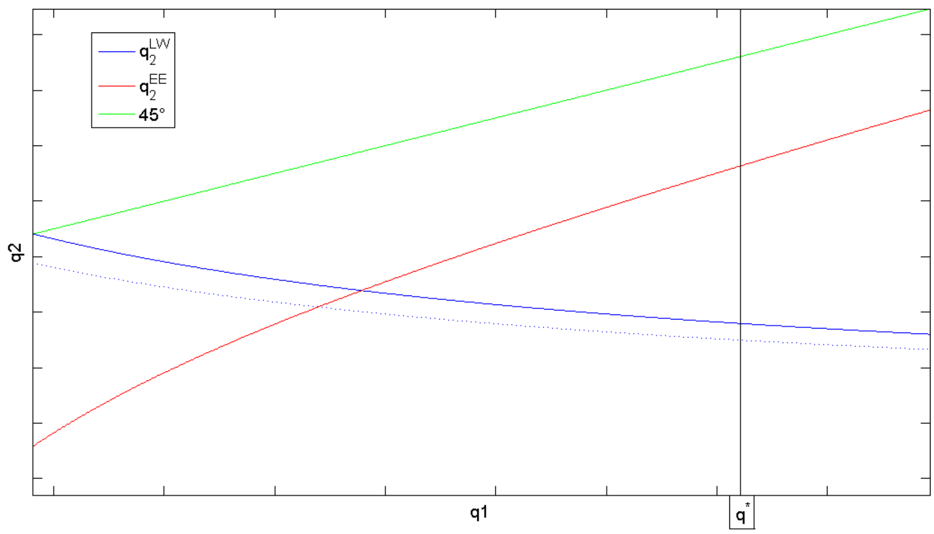

Figure 2.4: Effect of a marginal increase in π for $\pi > \bar{\pi} > \beta$

The second part of Theorem 2 reveals the effects of an increase in π if the rate of inflation is above the threshold level $\bar{\pi}$. The 'LagosWright' function shifts downwards in response to a rise in π as shown by the dotted blue line in figure 2.4. The increase of the nominal interest rate makes holding money costlier. Thus, agents substitute out of money and into capital in the new stationary equilibrium which increases investment δK (Tobin effect). The higher capital stock leads to a decrease in the rental rate of capital, $F_K(K, H)$, and an increase of the wage rate, $F_H(K, H)$. Due to the higher wages, agents choose to work and consume more in the centralized market. Figure 2.4 shows that the new q_1 and q_2 in the decentralized market are lower than before the increase in π. Recall that the amount of DM goods traded in uninformed matches is determined by $q_2 = Z/w$. Since the wage w increases, the value of the agents' money holdings Z does not necessarily have to decrease for q_2 to decline.

Recall the solution to the central planner's maximization problem (2.2) as a benchmark for optimal welfare. The central planner does not assign a liquidity value to either asset. Furthermore, agents in a *single coincidence of wants* meeting trade q^* units of the DM good which equates marginal utility of consumption and marginal disutility from production, i.e. $u'(q) = c'(q)$. Theorem 3 states the welfare-maximizing rate of inflation.

Theorem 3. *The Friedman rule, i.e.* $\pi = \beta$*, is the welfare-maximizing monetary policy.*

At $\pi = \beta$ agents do not value the marginal unit of capital or money for its role as medium of exchange because they can afford q^* both in informed and in uninformed transactions. Furthermore, the equilibrium values of the capital stock, CM consumption and labor as determined by equations (2.36), (2.37) and (2.50) coincide with the central planner's choices. Thus, the level of welfare at $\pi = \beta$ is the same as in the central planner's solution. Note, that the amount of money held in a stationary equilibrium with $\pi = \beta$ is indetermined but strictly greater than zero.

If, on the other hand, $\beta < \pi \leq \bar{\pi}$ buyers can afford q^* in informed transactions. Thus, the marginal unit of capital is not valued for its liquidity. All CM variables (X, K, H) are independent of inflation and coincide with the central planner's solution. However, buyers cannot purchase their desired DM consumption in uninformed transactions. That is why the average amount of DM consumption $\lambda q^* + (1 - \lambda)q_2$ is strictly lower than in the central planner's solution. Consider $\pi > \bar{\pi}$. In this case agents can never afford q^* and capital is assigned a liquidity value in informed matches. Average DM consumption is lower than at $\beta < \pi \leq \bar{\pi}$. Due to capital's liquidity value, CM variables are not independent of inflation anymore and the capital stock is strictly greater than in the *first-best*.

The preceding analysis has considered the effects of a change in the rate of inflation. In the following, the impact of a real shock on equilibrium allocations is examined. To do so, include a *total factor productivity component (TFP)*, S, in the CM production function, i.e. $F(K, H) = SK^{\alpha}H^{1-\alpha}$. Note that $S = 1$ in the former analysis. The effects of a permanent change of S are depicted in Theorem 4.

Theorem 4. *The comparative statics of an increase in the TFP component in production are given by*
$\frac{\partial K}{\partial S} > 0,\ \frac{\partial w}{\partial S} > 0,\ \frac{\partial X}{\partial S} > 0,\ \frac{\partial Z}{\partial S} > 0,\ \frac{\partial q_1}{\partial S} = 0,\ \frac{\partial q_2}{\partial S} = 0,\ \frac{\partial H}{\partial S} = 0,\ \frac{\partial r}{\partial S} = 0$

The signs of the effects of a real shock on the production side do not depend on the level of inflation. Furthermore, a TFP shock never influences the amount of goods traded in the decentralized market q_1 and q_2.

An increase in the TFP component leads to an efficiency gain in production. There is an interesting equilibrium effect, however. In the new stationary equilibrium, firms employ more capital but the amount of labor remains unchanged. Wages rise as a consequence to an increased capital stock and an unchanged amount of labor. The increase in S and the resulting increase in K offset each other, such that the rental rate of capital remains unchanged. According to equation (2.40), q_1 does not change in response to a change in S because the derivative of $r = F_K(K, H)$ with respect to S is zero. Therefore, equation (2.39) yields $\partial q_2/\partial S = 0$.

A constant q_2 implies that the value of the payment which the buyer makes to the seller in an uninformed transaction, i.e. Z/w, is the same as in the initial stationary equilibrium. Thus, the real value of money, Z, has to increase to offset the increase in $w = F_H(K, H)$. Finally, equation (2.26) implies that $\partial X/\partial S > 0$.

2.6 Concluding Remarks

This paper analyzes the effect of inflation on capital accumulation and factor prices. The rate of inflation determines whether agents achieve their desired consumption in the decentralized market. At the Friedman rule ($\pi = \beta$) the value of buyers' assets is sufficient to guarantee their desired level of consumption in all transactions. They never assign a positive liquidity value to either asset. At a level of inflation π such that $\beta < \pi < \bar{\pi}$ buyers can only afford q^* in informed transactions where they can use money and capital as media of exchange. Consequently, only buyers in uninformed transactions value the marginal unit of money for its liquidity and capital's liquidity value is always zero. Finally, if $\pi > \bar{\pi}$ buyers' assets are never sufficient to purchase q^*. At the margin, money and capital are both used as media of exchange.

The Friedman rule is the optimal monetary policy in this model as it replicates the central planner's solution. Neither money nor capital yield a liquidity value at the Friedman rule. As in the *first best* agents value capital for its role as an input in CM production only.

Whether a change of inflation has an impact on capital accumulation and factor prices depends on the level of inflation. It does not have an effect if inflation is below $\bar{\pi}$: An increase of inflation increases the cost of holding money and agents bring less money into the decentralized market. Consequently, buyers receive less DM goods in uninformed transactions, i.e. $\partial q_2/\partial \pi < 0$. The amount of DM goods traded in informed matches is not altered by this, however, because agent's money and capital holdings still provide enough liquidity to afford the desired DM consumption. In this case ($\pi < \bar{\pi}$), there is no link between centralized and decentralized market activity because, at the margin, capital is only valued for its role as a productive input in the centralized market and money provides a value in the decentralized market as medium of exchange only. Therefore, a change in the rate of inflation does not influence CM variables such as CM consumption, capital accumulation, labor and factor prices.

For $\pi > \bar{\pi}$, centralized and decentralized market activity are not independent of each other anymore because the marginal unit of capital is now valued in both markets. It is used as a medium of exchange in the decentralized and as an input in production in the centralized market. An increase of inflation raises the cost of holding money. In response, agents hold more of their wealth in capital in the new stationary equilibrium. The surge of the equilibrium capital stock induces the marginal productivity of capital to fall and the marginal product of labor to rise. This increases the labor wage and lowers the rental rate of capital because both inputs are paid by their marginal products. Higher wages induce agents to work and consume more in the centralized market. The value of buyers' money holdings in terms of the DM consumption good declines in response to an increase of inflation. Thus, buyers receive less DM goods in uninformed and in informed transactions than in the old stationary equilibrium.

A permanent *total factor productivity* shock in CM production does not have an impact on DM variables q_1 and q_2. An increase of the TFP component leads to an efficiency gain in production. In response, agents hold more capital but choose to work the same amount in the new stationary equilibrium which leads to an increase in wages. The rental rate of capital, however, remains constant as the increase in the TFP component and in capital holdings balance each other out. The resulting rise in CM output leads to an increase of CM consumption and money holdings. The value of a buyer's asset portfolio in the decentralized market in terms of the DM consumption good remains unchanged as the increase in capital and money holdings is balanced out by the increase of wages. Thus, q_1 and q_2 are not affected by an increase of the TFP component.

To sum up, this paper showed that the effect of a marginal increase of the rate of inflation depends on the prevailing level of inflation. A marginal increase of inflation raises capital investment (Tobin effect) for rates of inflation which are above the threshold. The impact of inflation on investment vanishes for rates of inflation below the threshold. Furthermore, a change in factor productivity (a real shock) never impacts DM variables irrespective of the rate of inflation.

2.7 Appendix

Proof of Proposition 8. The 'LagosWright' equation is repeated here for convenience:

$$q_2^{LW}(q_1) = q^*(1-\lambda)^{\frac{1}{1-\gamma}} \left[\frac{i}{\sigma} + 1 - \lambda\gamma C q_1^{\gamma-1}\right]^{\frac{1}{\gamma-1}} \tag{2.57}$$

where $q^* = (\gamma C)^{\frac{1}{1-\gamma}}$ is the solution to $u'(q) = c'(q)$.

Part (i): $\lim_{q_1 \to q_1^0} q_2^{LW}(q_1) = +\infty$

The exponent of the square bracket $\frac{1}{\gamma-1} < 0$ in equation (2.57). Thus, q_2^{LW} approaches ∞ if the term in the square bracket approaches 0^+ (from the positive side). This yields:

$$0 = \frac{i}{\sigma} + 1 - \lambda\gamma C q_1^{\gamma-1}$$

$$\Leftrightarrow q_1^0 = q^* \left(\frac{\lambda\sigma}{i+\sigma}\right)^{\frac{1}{1-\gamma}}$$

Note that $0 < q_1^0 < q^*$ since $\lambda < 1$.

Part (ii): $q_2^{LW}(q_j) < q_2^{LW}(q_i)$, for all $q_j > q_i > q_1^0$

The derivative of $q_1^{LW}(q_1)$ is strictly decreasing:

$$\frac{\partial q_2^{LW}(q_1)}{\partial q_1} = -(\gamma C 1(1-\lambda))^{\frac{1}{1-\gamma}} \lambda\gamma C \left[\frac{i}{\sigma} + 1 - \lambda\gamma C q_1^{\gamma-1}\right]^{\frac{2-\gamma}{\gamma-1}} q_1^{\gamma-2} < 0$$

since the term in the square bracket is strictly positive for $q_1 > q_1^0$.

Part (iii): $q_2^{LW}(q_1^{LW,min}) = q_1^{LW,min}$ where $q_1^{LW,min} \equiv q^* \left(\frac{\sigma}{i+\sigma}\right)^{\frac{1}{1-\gamma}}$

The strictly decreasing 'LagosWright' function crosses the 45° line uniquely at $q_1^{LW,min} = q^* \left(\frac{\sigma}{1+\sigma}\right)^{\frac{1}{1-\gamma}}$. Note that $q_1^0 < q_1^{LW,min} \leq q^*$. The first part ($q_1^0 < q_1^{LW,min}$) follows from the fact that $\lambda < 1$ [λ is contained in q_1^0 as shown in part (i)]. For all $i > 0$, $q_1^{LW,min} < q^*$. Only for $i = 0 \Leftrightarrow \pi = \beta$, we get $q_1^{LW,min} < q^*$.

Part (iv): $\lim_{q_1 \to +\infty} q_2^{LW}(q_1) = q^* \left[\frac{(1-\lambda)\sigma}{i+\sigma}\right]^{\frac{1}{1-\gamma}} > 0$

As $q_1 \to \infty$, $\lambda\gamma C q_1^{\gamma-1} \to 0$ because $\gamma - 1 < 0$ and the term in the square bracket in equation (2.57) reduces to $\frac{i}{\sigma} + 1$. Hence, $q_2(\infty) = q^* \left[\frac{(1-\lambda)\sigma}{i+\sigma}\right]^{\frac{1}{1-\gamma}}$ $\square$

Proof of Proposition 9. The 'Euler' function is given by

$$q_2^{EE} = q_1 - \frac{\alpha}{1 - (1 - \delta(1-\alpha))\beta(1 + \sigma\lambda(\gamma C q_1^{\gamma-1} - 1))} \tag{2.58}$$

Define the second term of equation (2.58) as $f(q_1) = \frac{\alpha}{1-(1-\delta(1-\alpha))\beta(1+\sigma\lambda(\gamma C q_1^{\gamma-1}-1))}$.
Part (i): $\lim_{q_1 \to 0^+} q_2^{EE}(q_1) = 0$

As $q_1 \to 0^+$, $f(q_1) \to 0^-$, i.e. f approaches zero from the left. Consequently, q_2^{EE} approaches 0 from the right, i.e. $q_2^{EE}(q_1) = 0^+$.

Part (ii): $\lim_{q_1 \to \underline{q}^-} q_2^{EE}(q_1) = +\infty$

The function f is not defined at $q_1 = \underline{q}$ which solves $1-(1-\delta(1-\alpha))\beta(1+\sigma\lambda(\gamma C q_1^{\gamma-1} - 1))$. It is given as $\underline{q} = \left[\frac{(1-\delta(1-\alpha))\beta\sigma\lambda\gamma C}{1-(1-\delta(1-\alpha))\beta(1-\sigma\lambda)}\right]^{\frac{1}{1-\gamma}} > 0$.
As $q_1 \to \underline{q}^-$, $f(q_1)$ approaches $-\infty$. Thus, $q_2^{EE}(q_1) = q_1 - f(q_1) = +\infty$.

Part (iii): $\lim_{q_1 \to \underline{q}^+} q_2^{EE}(q_1) = -\infty$

Similar to part (iii), as $q_1 \to \underline{q}^+$, $f(q_1) \to +\infty$. Thus, $q_2^{EE}(q_1) = q_1 - f(q_1) = -\infty$.

Part (iv): $\lim_{q_1 \to +\infty} q_2^{EE}(q_1) = q_1$

As $q_1 \to +\infty$, $f(q_1)$ approaches $\frac{\alpha}{1-(1-\delta(1-\alpha))\beta(1-\sigma\lambda)}$. Consequently, $q_2^{EE} = q_1 - f(q_1) = q_1$.

Parts (v) and (vi): $q_2^{EE}(q_i) > q_2^{EE}(q_j)$, for all $\underline{q} > q_i > q_j > 0$ and $q_2^{EE}(q_i) > q_2^{EE}(q_j)$, for all $q_i > q_j > \underline{q}$

The function $q_2^{EE}(q_1)$ is strictly increasing for all $q_1 > 0$. The derivative of $f(q_1)$ takes the form

$$f'(q_1) = -\alpha \left[1 - (1 - \delta(1-\alpha))\beta(1 + \sigma\lambda(\gamma C q_1^{\gamma-1} - 1))\right]^{-2} (1-\gamma)(1-\delta(1-\alpha))\beta\sigma\lambda\gamma C q_1^{\gamma-2} < 0$$

The derivative is strictly smaller than zero for all $q_1 > 0$. Note however, that a derivative does not exist at $q_1 = \underline{q}$. Given this, $\frac{\partial q_2^{EE}(q_1)}{\partial q_1} = 1 - f'(q_1) > 0$. $\square$

Proof of Theorem 1. A stationary equilibrium in case (i) with $q_2 < q_1 < q^*$ exists if the 'LagosWright' and the 'Euler' function cross at $q_1 < q^*$. In this case at $q_1 = q^*$ we have $q_2^{EE}(q^*) \geq q_2^{LW}(q^*)$ which is equivalent to

$$\pi > \bar{\pi} \equiv \beta \left\{ 1 + \sigma(1-\lambda) \left[\left(1 - \frac{\alpha}{\left[1-(1-\delta(1-\alpha))\beta\right] q^*} \right)^{\lambda-1} - 1 \right] \right\} > \beta$$

Furthermore, if the stationary equilibrium is unique if $\pi > \bar{\pi}$ since the 'LagosWright' function is strictly decreasing and the 'Euler' function is strictly increasing in q_1.

For $\beta < \pi < \bar{\pi}$ there exists a stationary equilibrium with $q_2 < q_1 = q^*$ where q_1 is uniquely obtained from $u'(q_1) = c'(q_1)$. The remaining variables are obtained from the following system of equations.

$$\pi = \beta \left\{ 1 + \sigma(1-\lambda) \left(\frac{u'(q_2)}{c'(q_2)} - 1 \right) \right\} \tag{2.59}$$

$$\beta^{-1} = 1 + F_K(K,H) - \delta \tag{2.60}$$

$$U'(x) = \frac{1}{F_H(K,H)} \tag{2.61}$$

$$F(K,H) = X - \delta K \tag{2.62}$$

The solution for q_2 is uniquely obtained from the money Euler equation (2.59). K, X and H are the unique solutions to equations (2.60) - (2.62).

At $\pi = \beta$, the money Euler equation (2.59) implies $u'(q_2) = c'(q_2)$. Thus, $q_1 = q_2 = q^*$. The solution for money Z is indetermined since money is held at no cost. Again, K, X and H are uniquely obtained from (2.60) - (2.62). □

Proof of Theorem 2. The proof is divided into two parts.

Part 1: $\beta < \pi < \bar{\pi}$

The variables K, H, X, q_2 are obtained from equations (2.59)-(2.62), $q_1 = q^*$ solves $u'(q_1) = c'(q_1)$ and Z is given by $c(q_2) = \frac{Z}{w}$. Note that K, X, H and q_1 are independent of a marginal change in π. So are w and r. From equation (2.59), it follows that $\frac{\partial q_2}{\partial \pi} < 0$ which implies $\frac{\partial Z}{\partial \pi} < 0$ since $\frac{\partial w}{\partial \pi} = 0$.

Part 2: $\pi > \bar{\pi}$

Notice that only the 'LagosWright' and not the 'Euler' function contains π since $i = \beta^{-1}\pi - 1$. Differentiating the 'LagosWright' function with respect to inflation yields

$$\frac{\partial q_2^{LW}(q_1)}{\partial \pi} = -(1-\gamma)\beta^{-1}(\gamma C(1-\lambda))^{\frac{1}{1-\gamma}} \left[\frac{i}{\sigma} + 1 - \lambda\gamma C q_1^{\gamma-1}\right]^{\frac{2-\gamma}{\gamma-1}} < 0$$

Thus, $q_2^{LW}(q_1)$ decreases if π increases. Consequently, the 'LagosWright' curve moves to the left and crosses the 'Euler' function at a lower q_1 and q_2, i.e. $\frac{\partial q_1}{\partial \pi} < 0$ and $\frac{\partial q_2}{\partial \pi} < 0$. Since $\frac{\partial q_2^{EE}(q_1)}{\partial q_1} > 1$, q_2 decreases more from a marginal change in π than q_1, i.e. $\left|\frac{\partial q_2}{\partial \pi}\right| > \left|\frac{\partial q_1}{\partial \pi}\right|$.

The effects of a marginal increase of π on H, K are given by

$$\begin{aligned}\frac{\partial H}{\partial \pi} =& \frac{\delta(1-\alpha)}{A(1-\delta(1-\alpha))}\left[\frac{\partial q_1}{\partial \pi} - \frac{\partial q_2}{\partial \pi}\right] > 0 \\ \frac{\partial K}{\partial \pi} =& \frac{1}{1-\alpha} K^{\alpha} \left(\frac{1}{\delta}(1-\alpha)H^{-\alpha} + \frac{1-\alpha}{A\delta}\alpha H^{-\alpha-1}\right)\frac{\partial H}{\partial \pi} > 0\end{aligned}$$

The rental rate of capital $r = \alpha\left(\frac{H}{K}\right)^{1-\alpha}$ can be expressed in a equilibrium as $r = \alpha\delta\left(1 - \frac{1-\alpha}{A}H^{-1}\right)^{-1}$. It follows that

$$\frac{\partial r}{\partial \pi} = -\alpha\delta\left(1 - \frac{1-\alpha}{A}H^{-1}\right)^{-2}\left(\frac{1-\alpha}{A}H^{-2}\right)\frac{\partial H}{\partial \pi} < 0$$

since $\frac{\partial H}{\partial \pi} > 0$. Given this inequality, this implies for $r = \alpha\left(\frac{H}{K}\right)^{1-\alpha}$ where the equilibrium K is not inserted:

$$\begin{aligned}\frac{\partial r}{\partial \pi} =& (1-\alpha)\alpha\left(\frac{H}{K}\right)^{\alpha}\frac{K\frac{\partial H}{\partial \pi} - H\frac{\partial K}{\partial \pi}}{K^2} < 0 \\ \Leftrightarrow& \frac{\frac{\partial H}{\partial \pi}}{H} < \frac{\frac{\partial K}{\partial \pi}}{K}\end{aligned}$$

That is, in a stationary equilibrium the percentage change in K from a marginal change in π is greater than the percentage change in H.

$$\frac{\partial w}{\partial \pi} = \alpha(1-\alpha)\left(\frac{K}{H}\right)^{\alpha-1}\frac{H\frac{\partial K}{\partial \pi} - K\frac{\partial H}{\partial \pi}}{H^2}$$

Using $\frac{\frac{\partial H}{\partial \pi}}{H} < \frac{\frac{\partial K}{\partial \pi}}{K}$, the last term turns out to be positive, implying $\frac{w}{\pi} > 0$. X is obtained from $x = w$. Thus, $\frac{\partial X}{\partial \pi} > 0$. Finally, Z solves $c(q_2) = \frac{Z}{w}$. Note that $\frac{\partial Z}{\partial \pi} = c(q_2)\frac{\partial w}{\partial \pi} + wc'(q_2)\frac{\partial q_2}{\partial \pi}$ cannot be signed since $\frac{\partial w}{\partial \pi} > 0$ and $\frac{\partial q_2}{\partial \pi} < 0$. □

Proof of Theorem 4. The proof is divided into two parts.

Part 1: $\beta < \pi < \bar{\pi}$

$q_1 = q^*$ which is determined from $u'(q_1) = c'(q_1)$ and q_2 from equation (2.59) are independently determined of the TFP shock S. K, X and H are determined from equations (2.60) - (2.62) with $F(K, H) = SK^{\alpha}H^{1-\alpha}$. Given K and H, $w = S(1-\alpha)\left(\frac{K}{H}\right)^{\alpha}$ and $r = S\alpha\left(\frac{H}{K}\right)^{1-\alpha}$. The stationary equilibrium values for H and r are independent of S. The remaining partials turn out to be $\frac{\partial K}{\partial S} > 0$, $\frac{\partial X}{\partial S} > 0$ and $\frac{\partial w}{\partial S} > 0$. Finally, Z is determined from $Z = wc(q_2)$. Since $\frac{\partial q_2}{\partial S} = 0$, this yields $\frac{\partial Z}{\partial S} > 0$.

Part 2: $\pi > \bar{\pi}$

The stationary equilibrium values of q_2, q_1, H and r are independent of S. The remaining four variables K, w, X, Z depend positively on the TFP component. Thus, all partials have the same sign as for $\beta < \pi < \bar{\pi}$. □

Chapter 3

On the Impact of Inflation on Investment in a Monetary Model

3.1 Introduction

Does inflation have an impact on investment? This question is a matter of dispute in the theoretical literature. Models such as Tobin (1965) and Fischer (1979) predict a positive impact of inflation on investment because agents substitute out of money and into capital if inflation increases. This response is called Tobin effect. Models following Stockman (1981) display the opposite effect which we call Stockman effect. Agents hold money because money is necessary to acquire consumption. Therefore, an increase of inflation reduces capital investment because inflation acts as a tax. Finally, models following Sidrauski (1967) anticipate no effect of inflation on capital investment in which case money is called superneutral.

The empirical literature is unclear on the subject matter, as well. There are studies supporting the Tobin effect [e.g. Ahmed and Rogers (2000)], the Stockman effect [e.g Barro (1995)] or the superneutrality of money [Bullard and Keating (1995)].

The model in this chapter generates the Stockman or the Tobin effect depending on the rate of inflation and the liquidity of the assets available in the economy. Capital investment always declines in response to an increase of inflation (Stockman effect), if inflation is below a certain threshold. If inflation is above that threshold, the response of capital investment depends on the acceptability of capital as a medium of exchange: The model generates the Stockman effect if capital is insufficiently liquid. Otherwise, an increase of inflation leads to a rise in capital investment (Tobin Effect).

This chapter adopts the Lagos and Wright (2005) framework where agents trade in two distinct markets (called day- and night market) each period. In contrast to the model in chapter two, capital is used for production in both markets which is analogous to Aruoba, Waller, and Wright (2011). The night market is a standard frictionless Walrasian market. The day market contains a friction, namely anonymity amongst agents. Therefore, trade in the day market can only occur if buyers have a medium of exchange. The two assets in the economy (money and capital) can fulfill this role. They differ in their liquidity, however. Money is always accepted as a medium of exchange whereas capital is only accepted in a fraction of all transactions. During the day market agents either enter market 1 where they can use both assets as media of exchange or they enter market 2 where money is the only permissible means of payment.

The equilibria in this model depend on the rate of inflation. At the Friedman (1969) rule, agents are willing to hold money up to a level where they can afford their desired level of consumption in both markets because opportunity costs of holding money are zero. In other words, agents' money holdings are sufficiently high to purchase their desired level of consumption. For higher rates of inflation, there is an opportunity cost associated with holding money as negative inflation does not offset the loss from discounting anymore. Thus, agents choose to purchase the marginal unit of money in the night market only if they expect to use it as a medium of exchange in the day market. If the rate of inflation is below a certain threshold, agents hold enough money to purchase their desired level of consumption in market 1 where both, money and capital can be used as media of exchange. As a consequence, buyers in market 1 do not spend the marginal unit of their money or capital holdings. In market 2, on the other hand, buyers cannot afford their desired level of consumption and use the marginal unit of money as a medium of exchange. For rates of inflation above the threshold, neither buyers in market 1 nor buyers in market 2 can afford their desired level of consumption.

Consider an agent's response to a marginal increase of inflation. If the rate of inflation is below the threshold, agents can afford their desired level of consumption in market 1 only. That is, they use all of their money as a medium of exchange in market 2, whereas they only spend parts of their capital and money holdings in market 1. The amount of goods traded in market 2 depends positively on both, the amount of money brought into the market by buyers and the sellers' capital which is used as an input in production. A rise in inflation increases the cost of holding money and agents choose to purchase less money in the night market. Furthermore, agents choose their capital investment in the night market. When doing so, they know that they will sell less goods if they become sellers in market 2 because they will face buyers with less money holdings as compared to before the increase of inflation. As a consequence, they optimally choose to bring less capital into the day market, as well. This negative response of capital holdings to an increase in inflation can be interpreted as an income effect. It constitutes the total effect if the rate of inflation is below the threshold and, thus, the model generates the Stockman effect.

In addition to the income effect described above, a marginal increase of inflation leads to a substitution effect as well if the rate of inflation is above the threshold level: Neither buyers in market 1 nor buyers in market 2 can afford to purchase their desired level of consumption. Thus, buyers in market 1 use all of their money and capital holdings and buyers in market 2 use all of their money holdings to buy as much consumption as possible. An agent in the night market substitutes out of money and into capital in response to an increase in inflation. This substitution has two consequences. First, by reducing his exposure to more expensive money, he can afford a higher level of consumption in market 1 than if he had not substituted. Second, his consumption in market 2 decrease because he cannot use his capital as a medium of exchange in market 2. Notice that there is no substitution effect if the rate of inflation is below the threshold because agents can already afford their desired level of consumption in market 1 and therefore, consumption in market 1 is not affected by a change in inflation. We say that the substitution effect dominates the income effect if the rise in inflation increases capital holdings (Tobin effect). This is the case if the probability of entering market 1 is sufficiently high: Agents are willing to substitute more heavily if the probability of them sacrificing consumption (i.e. entering market 2 as a buyer) is low. Otherwise, we observe the Stockman effect, i.e. capital holdings decrease.

As mentioned above, capital is used as a production input in both markets in Aruoba, Waller, and Wright (2011). In a fraction of all transactions, agents can use money as a medium of exchange. In all other transactions, they can use credit. Agents can always afford their desired level of consumption in credit transactions but they can never afford it in money transactions. Thus, an increase of inflation in Aruoba, Waller, and Wright (2011) always leads to the Stockman effect: Agents hold money because only money increases their consumption at the margin. Consequently, inflation acts as a tax on consumption and capital investment.

3.2 Environment

The economy consists of a measure one of agents who live forever. Each period is divided into two subperiods, called day and night. Agents discount between periods but not between subperiods. In the first subperiod agents enter the day market (DM) and in the second subperiod they enter the night market (NM).

At night there is a good which is referred to as night market good or NM good. It is not perfectly durable as it depreciates at the rate δ each period and it can either be consumed or stored. If it is stored, it is denoted as capital. The night market good is produced by a firm. Its technology uses aggregate capital K as its sole input in production. As usually the production function $F(K)$ satisfies $F'(K) > 0$ and $F''(K) < 0$. Agents rent their capital holdings to the firm and receive a compensation r after production has occurred. The firm sells its output to the agents and generates a profit of $P = F(K) - rK$. Besides capital agents can use fiat money as a store of value in the night market. Fiat money is an intrinsically useless asset provided by the government.

At the beginning of each period agents are randomly hit by a shock determining their type in the day market. Half of them become sellers and the other half become buyers. Sellers are able to produce a perishable good which only buyers can consume. The perishable good is called the day market good or DM good. In contrast to the night market good, the day market good is individually produced by sellers. A seller uses his own capital k as an input in production. He suffers a utility loss of $c(q, k)$ from producing q units of the DM good. Buyers receive utility $u(q)$ from consuming q units of the day market good. Thus, the sellers' capital is used as an input in both, night and day market production, whereas buyers' capital is used for production in the night market only.

The night market is a standard Walrasian market. Agents choose their consumption and investment in capital and money and the Walrasian auctioneer matches supply and demand. Trade in the day market is centralized as well. In contrast to the frictionless night market, however, there is a friction apparent in the day market: Agents are anonymous and there is no technology to identify one's identity. This friction hinders trade in the day market as sellers do not extend credit to buyers and therefore, buyers need a medium of exchange to purchase DM goods.

At the beginning of the day market agents are hit by a second shock determining their location. With probability λ, they enter a market (market 1) where they are able to use their money and capital as media of exchange, and, with probability $1-\lambda$, they enter another market (market 2), where money is the only permissible means of payment. Thus, the set of agents in the first subperiod (day) is decomposed into four subsets: (i) sellers in market 1, (ii) buyers in market 1, (iii) sellers in market 2 and (iv) buyers in market 2. Subsets (i) and (ii) are of measure $\lambda/2$ and subsets (iii) and (iv) have measure $(1-\lambda)/2$.

3.3 Central Planner

Consider the central planner's problem first. In contrast to individual agents, the central planner can observe each agents' type (buyer or seller) in the day market. Furthermore, he can induce them to act as he pleases. The central planner maximizes the cross-section average of average discounted present values of expected utility over all infinite future.

$$\begin{aligned} \mathbb{W} = \max_{K_{t+1}, X_t, q_t} \sum_{t=0}^{\infty} \beta^t \left\{ X_t + 0.5 \left[u(q_t) - c(q_t, K_t) \right] \right\} \\ \text{s.t. } F(K_t) = X_t + K_{t+1} - (1-\delta) K_t \end{aligned} \tag{3.1}$$

According to the objective function in maximization problem (3.1), per-capita consumption of X_t units of the night market good in period t yields X_t utils. In the day market, one half of the agents (measure 0.5) become buyers and the other half become sellers. The central planner induces the representative seller to produce q_t goods at a utility cost of $c(q_t, K_t)$ and allocates the goods to the buyers. The average buyer's utility of consumption is given by $u(q_t)$. Due to the central planner's monitoring and enforcement, trade in the day market is conducted without the use of a medium of exchange. The economy's resource constraint in maximization problem (3.1) shows that investment into capital $K_{t+1} - \delta K_t$ and consumption at night X_t are financed by output $F(K_t)$.

The solution to maximization problem (3.1) satisfies

$$u'(q_t) = c_q(q_t, K_t) \tag{3.2}$$
$$\beta^{-1} = 1 + F'(K_{t+1}) - \delta - 0.5c_k(q_{t+1}, K_{t+1}) \tag{3.3}$$

The central planner chooses the socially optimal level of consumption in the day market q_t such that the buyer's marginal utility of consumption equals the seller's marginal cost from production. Note that equation (3.2) describes the optimal level of consumption (*first best*) during the day as a function of the current capital stock. To clarify notation, the functional relationship as depicted by equation (3.2) is called $q^* \equiv q(K)$. Thus, in the central planner's solution, q^* denotes the *first best* level of consumption because K is chosen optimally, as well. In the remainder of this paper, q^* does not necessarily equal the *first best* level of consumption, however.

Next period's capital stock K_{t+1} solves equation (3.3) given the solution for q_{t+1}. Capital generates a return $F'(K_t)$ and depreciates at the rate δ in the night market. The marginal unit of capital, therefore, buys $1+F'(K)-\delta$ units of consumption in the night market for all agents. Sellers profit from the marginal unit of capital in the day market, as well, as it lowers their production cost of a given amount of DM goods. At the optimum, the total utility generated by the marginal (last) unit of capital offsets the loss from discounting. Finally, the solution for per-capita consumption X_t is obtained from the constraint in maximization problem (3.1). The central planner's solution describes the *first best* allocation and is the benchmark for the subsequent welfare comparison.

3.4 Individual Problem

The behavior of individual agents given market prices is derived in this section. Before doing so, however, we introduce money which can be used as a medium of exchange in the day market. Money is provided by the government. Initially, each agent is endowed with the same amount of money. In the night market of each period the government chooses whether or not to change the supply of money M. Its evolution is given by $M_{t+1} = (1 + v_t)M_t$ where v_t summarizes the government's decision. If $v_t > 0$, the government injects $M_{t+1} - M_t$ units of new money into the system: In equilibrium, agents purchase the entire $M_{t+1} - M_t$ units of money in exchange for

$(M_{t+1} - M_t)/\mathbb{P}_t$ units of capital (or equivalently, the night market good) where $\mathbb{P}_t$ denotes the price of a unit of capital in terms of money in period t. If the government reduces the money supply, i.e. $v_t < 0$, it buys money from the agents in the night market. Finally, $v_t = 0$ implies no change of the money supply and requires no government intervention in NM trade. The government communicates its decision variable v_t to the agents before the night market opens, thus, eliminating uncertainty about next period's money supply.

The government's budget constraint in real terms is given by

$$T_t = v_t \frac{M_t}{\mathbb{P}_t} \tag{3.4}$$

Consider an increase of the money supply, i.e. $v_t > 0$. In this case, the right-hand side of equation (3.4) shows the government's revenue (seignorage) in the night market of period t. It sells $M_{t+1} - M_t$ units of new money at the price $1/\mathbb{P}_t$ per unit. The entire seignorage is transfered to the agents in a lump-sum payment $T > 0$. If $v_t < 0$, the government raises lump-sum taxes $T < 0$ to finance the reduction of the money supply. Thus, the government collects a total lump-sum tax of $(M_{t+1} - M_t)/\mathbb{P}_t$ which can be paid by the agents either in money or in capital.

Define real money as $Z_t \equiv M_t/\mathbb{P}_t$. The evolution of money can be restated in terms of real money as

$$\pi_{t+1} Z_{t+1} = (1 + v_t) Z_t \tag{3.5}$$

where $\pi_{t+1} \equiv \mathbb{P}_{t+1}/\mathbb{P}_t$ denotes the rate of inflation between periods t and $t+1$. The right-hand side of equation (3.5) shows the supply of real money in the night market of period t, i.e. $(1 + v_t)Z_t = M_{t+1}/\mathbb{P}_t$. It is equal to the real money supply at the beginning of period $t+1$, i.e. $Z_{t+1} = M_{t+1}/\mathbb{P}_{t+1}$, if and only if gross inflation π_{t+1} equals one.

To shorten notation, the time-subscript is dropped for variables in the current period and next period's variables are denoted by a prime. In the night market, agents receive utility x from net consumption which has the property that it can take on positive as well as negative values, i.e. $x \in \mathbb{R}$. Since an agent's consumption and labor effort are not explicitly modeled, we can interpret net consumption as a combination of the two. Thus, $x > 0$ if an agent's utility of consumption exceeds his disutility from labor and $x < 0$ otherwise. An agent with an asset portfolio (z, k) chooses next period's asset holdings (z', k') and NM net consumption x to maximize his lifetime utility.

$$\begin{aligned} W(z,k) &= \max_{x,z',k'} \; x + \beta V(z',k') \\ \text{s.t. } x + k' + z'\pi' &= (1 + r - \delta)k + z + T + P \end{aligned} \tag{3.6}$$

The objective function of maximization problem (3.6) represents the lifetime-utility of an agent when entering the night market with an asset portfolio (z, k). Besides net consumption, an agent chooses next period's asset portfolio (z', k') to maximize his objective function subject to his budget constraint. His expenditures in the night market consist of his net consumption x and his investments in money $z'\pi' - z$ and capital $k' - (1 - \delta)k$ where δ is the rate of capital depreciation and π' is the rate of inflation. An agent pays $z'\pi'$ units of the NM good to enter next period with z' units of money. According to the budget constraint in maximization problem (3.6) he finances his expenditures by his return on capital rk and the transfers received from the government T and from the firm P.

Differentiating maximization problem (3.6) after eliminating x yields

$$z' : \quad \beta \frac{\partial V(z', k')}{\partial z'} - \pi' \geq 0 \tag{3.7}$$

$$k' : \quad \beta \frac{\partial V(z', k')}{\partial k'} = 1 \tag{3.8}$$

The left-hand side of equation (3.7) shows the derivative of maximization problem (3.6) with respect to next period's money holdings z'. A marginal increase of z' has two effects. On the one hand, it lowers the agent's net consumption in the night market by π' and, on the other hand, it increases his continuation value as he enters next period with more money. A unit of money is only held if its benefit in next period's day market exceeds its utility loss from lowering net consumption in the current night market. Thus, inequality (3.7) gives a condition for money to be held. The agent does not hold money if there is no $z' > 0$ which satisfies condition (3.7). We refer to condition (3.7) as money Euler equation or first order condition if it is satisfied at equality for some nonnegative z'. In this case, the marginal benefit from money equals its marginal loss at the optimal z'. The first order condition of k' is given by equation (3.8). At the optimal k', the utility loss from a marginal decrease of net consumption is offset by the increase of next period's lifetime utility.

Consider an agent entering the day market with an asset portfolio (z, k). Before trade occurs, two shocks realize which determine the agent's status and location. First, he becomes a buyer or a seller with a fifty percent probability, respectively. Second, he enters market 1 where money and capital can be used as media of exchange with probability λ. Otherwise (probability $1-\lambda$), he enters market 2 where money is the only means of payment. His expected lifetime utility $V(z, k)$ before the shocks have realized is

$$V(z,k) = .5\left(\lambda V_1^b(z,k) + (1-\lambda)V_2^b(z,k)\right) + .5\left(\lambda V_1^s(z,k) + (1-\lambda)V_2^s(z,k)\right) \quad (3.9)$$

Equation (3.9) decomposes an agent's expected lifetime utility in the beginning of the day into his values in each state of nature. The value of a buyer [seller] in market $i = 1, 2$ is given by $V_i^b(z,k)$ $[V_i^s(z,k)]$.

The partial derivatives of the day market's value function $V(z, k)$ must be computed for the night market's first order conditions (3.7) and (3.8). They take the form

$$\begin{aligned} \frac{\partial V(z,k)}{\partial z} =& .5\left(\lambda\frac{\partial V_1^b(z,k)}{\partial z} + (1-\lambda)\frac{\partial V_2^b(z,k)}{\partial z}\right) \\ &+ .5\left(\lambda\frac{\partial V_1^s(z,k)}{\partial z} + (1-\lambda)\frac{\partial V_2^s(z,k)}{\partial z}\right) \end{aligned} \quad (3.10)$$

$$\begin{aligned} \frac{\partial V(z,k)}{\partial k} =& .5\left(\lambda\frac{\partial V_1^b(z,k)}{\partial k} + (1-\lambda)\frac{\partial V_2^b(z,k)}{\partial k}\right) \\ &+ .5\left(\lambda\frac{\partial V_1^s(z,k)}{\partial k} + (1-\lambda)\frac{\partial V_2^s(z,k)}{\partial k}\right) \end{aligned} \quad (3.11)$$

Equation (3.10) shows the change of an agent's expected lifetime utility due to his marginal unit of money. It is a weighted sum of its impact in each possible state. The partial derivative of an agent's value function with respect to his capital holdings is decomposed in a similar way according to (3.11). In the following section we consider buyers and sellers in markets 1 and 2 in order to compute equations (3.10) and (3.11).

3.4.1 Market 1

This section analyzes the optimization problems faced by buyers and sellers in market 1. Recall that money and capital can be used as media of exchange in market 1 whereas money is the only permissible means of payment in market 2.

A buyer who enters market 1 with assets (z, k) faces the following optimization problem

$$\begin{aligned} V_1^b(z,k) = \max_{q_1} & \, u(q_1) - (1+r-\delta)d_k^1 - d_z^1 + W(z,k) \\ \text{s.t. } & q_1\tilde{p} = d_z^1 + d_k(1+r-\delta) \\ & d_z^1 \leq z \\ & d_k^1 \leq k \end{aligned} \tag{3.12}$$

The objective function of maximization problem (3.12) represents his lifetime utility. He receives $u(q_1)$ utils from consuming q_1 units of the perishable day market good. The first constraint in maximization problem (3.12) determines the amount of money and capital which he has to spend to buy q_1 units of consumption at the given market price $\tilde{p}$. After trade has occurred, he enters the night market with $z - d_z^1$ units of money and $k - d_k^1$ units of capital. Thus, his lifetime utility in the night market is given by $W(z - d_z^1, k - d_k^1)$. According to equation (3.6) it can be restated as $W(z - d_z^1, k - d_k^1) = W(z,k) - (1+r-\delta)d_k^1 - d_z^1$ which is the last component in the buyer's objective function in maximization problem (3.12). The prevailing anonymity in the day market precludes trades against credit. The final two constraints in maximization problem (3.12) assure trades to be *quid pro quo*. They guarantee that the buyer does not spend more money or more capital than he owns.

In the unconstrained solution to maximization problem (3.12), q_1 solves

$$u'(q_1) = \tilde{p} \tag{3.13}$$

At the optimal level of consumption q_1, the buyer's marginal utility equals the market price $\tilde{p}$. According to the first constraint in maximization problem (3.12), the buyer spends $q_1\tilde{p}$ units of money and capital to finance his consumption. Notice that, even though the total value of the payment is determined, its composition into money and capital remains indetermined, i.e. the first constraint in maximization problem (3.12) expresses d_z^1 as a function of d_k^1 or vice versa.

If the buyer is asset constrained he cannot afford the level of consumption which solves equation (3.13). Thus, he spends all of his money and capital holdings to purchase as much q_1 as possible, i.e. $d_z^1 = z, d_k^1 = k$ and $q_1 = [z + (1 + r - \delta)k]/\tilde{p}$.

Next, we compute the partial derivatives of the buyer's value function $V_1^b(z,k)$ with respect to z and k which can then be inserted into equations (3.10) and (3.11). They take the form

$$\frac{\partial V^b(z,k)}{\partial z} = u'(q_1)\frac{\partial q_1}{\partial z} - (1+r-\delta)\frac{\partial d_k^1}{\partial z} - \frac{\partial d_z^1}{\partial z} + \frac{\partial W(z,k)}{\partial z} \tag{3.14}$$

$$\frac{\partial V^b(z,k)}{\partial k} = u'(q_1)\frac{\partial q_1}{\partial k} - (1+r-\delta)\frac{\partial d_k^1}{\partial k} - \frac{\partial d_z^1}{\partial k} + \frac{\partial W(z,k)}{\partial k} \tag{3.15}$$

The partial derivatives of q_1, d_k^1 and d_z^1 with respect to z and k in equations (3.14) and (3.15) can be computed using the solutions to the buyer's maximization problem (3.12) which were derived above. Again, we have to distinguish between an unconstrained and a constrained buyer.

Consider the unconstrained solution first. Buyers do not use the marginal unit of money or capital as payment because they can already afford their desired level of consumption. Consequently, q_1, d_z^1 and d_k^1 are independent of the marginal unit of k and z. Thus, equations (3.14) and (3.15) reduce to

$$\frac{\partial V^b(z,k)}{\partial z} = \frac{\partial W(z,k)}{\partial z} \tag{3.16}$$

$$\frac{\partial V^b(z,k)}{\partial k} = \frac{\partial W(z,k)}{\partial k} \tag{3.17}$$

At the margin a buyer who is not asset constrained values neither money nor capital as medium of exchange. Thus, the marginal unit of either asset does not affect his payoff during the day but only at night.

In the constrained solution to maximization problem (3.12), the buyer does not have sufficient assets to afford his desired level of consumption. He spends all of his assets to purchase as much q_1 as possible. In the constrained solution, the partial derivatives of q_1, d_k^1 and d_z^1 with respect to z and k take the form

$$\frac{\partial d_z^1}{\partial z_b} = 1 \quad ; \quad \frac{\partial d_k^1}{\partial z_b} = 0 \quad ; \quad \frac{\partial q_1}{\partial z_b} = \frac{1}{\tilde{p}} \tag{3.18}$$

$$\frac{\partial d_z^1}{\partial k_b} = 0 \quad ; \quad \frac{\partial d_k^1}{\partial k_b} = 1 \quad ; \quad \frac{\partial q_1}{\partial k_b} = \frac{1 + r - \delta}{\tilde{p}} \tag{3.19}$$

The partials in lines (3.18) and (3.19) show the effects of a marginal unit of money and capital on the buyer's decisions. A buyer who is constrained by his asset holdings uses the entire marginal unit of money to increase his consumption by $1/\tilde{p}$ where $\tilde{p}$ is the price of the consumption good q_1 in terms of the NM good. Similarly, he spends his entire marginal unit of capital to raise q_1. In contrast to money, the marginal unit of capital buys $(1+r-\delta)/\tilde{p}$ units of consumption as capital generates a return r and depreciates at the rate δ in the night market. A marginal increase of either asset is used entirely to increase consumption and not to substitute between the two payment options. Thus, the marginal unit of his money holdings does not influence his capital expenditures and vice versa. Inserting the partials in lines (3.18) and (3.19) into the derivatives of the night market's value function (3.14) and (3.15) yields

$$\frac{\partial V^b(z,k)}{\partial z} = \frac{u'(q_1)}{\tilde{p}} \tag{3.20}$$

$$\frac{\partial V^b(z,k)}{\partial k} = u'(q_1)\frac{1 + r - \delta}{\tilde{p}} \tag{3.21}$$

The buyer receives a payoff from the marginal unit of money according to equation (3.20). As mentioned above, the marginal unit of money raises his consumption by $1/\tilde{p}$ units which leads to an increase in his marginal utility by $u'(q_1)/\tilde{p}$ in the day market. The marginal unit of money does not generate a payoff in future (sub)periods because it is spent in the day market. Similarly, the marginal unit of capital increases q_1 by $(1+r-\delta)/\tilde{p}$, raising his marginal lifetime-utility by $u'(q_1)(1+r-\delta)/\tilde{p}$ as shown in equation (3.21).

Next, consider a seller who enters market 1 with the portfolio (z, k). His optimization problem is given by

$$\begin{aligned} V_1^s(z,k) = &\max_{q_1} -c(q_1,k) + (1+r-\delta)D_k^1 + D_z^1 + W(z,k) \\ &\text{s.t. } q_1\tilde{p} = D_z^1 + D_k^1(1+r-\delta) \end{aligned} \tag{3.22}$$

The seller chooses to produce the amount of goods q_1 which solves maximization problem (3.22). He suffers a utility cost of $c(q_1, k)$ from producing q_1 units of the day market good. Notice that the cost depends on his own capital holdings as it is used in his production process. In return he receives a payment of money D_z^1 and capital D_k^1 which increases his payoff in the night market. According to the night market's value function (3.6), $W(z+D_z^1, k+D_k^1) = D_z^1 + (1+r-\delta)D_k^1 + W(z,k)$. The seller's revenue in the day market is determined by the constraint of maximization problem (3.22) given the price level $\tilde{p}$. The amount of capital which the seller receives D_k^1 is multiplied by $(1+r-\delta)$ to account for capital's return and depreciation in the night market. Note that the amount of money (capital) that an individual seller receives does not need to equal the amount of money (capital) that an individual buyer spends because trade is multilateral rather than bilateral. In contrast to the buyer who is constrained by his asset holdings, the seller does not face any additional constraints.

The solution to the seller's maximization problem, q_1, solves

$$c_q(q_1, k) = \tilde{p} \tag{3.23}$$

The seller optimally chooses q_1 such that his marginal disutility from production equals his marginal benefit, i.e. the good's price. Finally, the constraint of maximization problem (3.22) determines the payment which the seller receives in return for his chosen amount of goods q_1. Note that the constraint determines only the value of the payment but not its composition into money and capital. That is, the amount of money received by the seller D_z^1 is a function of the amount of capital received D_k^1 and vice versa.

The partial derivatives of the seller's value function $V^s(z,k)$ with respect to z and k

take the form

$$\frac{\partial V_1^s(z,k)}{\partial z} = -c_q(q_1,k)\frac{\partial q_1}{\partial z} + (1+r-\delta)\frac{\partial D_k^1}{\partial z} + \frac{\partial D_z^1}{\partial z} + \frac{\partial W(z,k)}{\partial z} \tag{3.24}$$

$$\frac{\partial V_1^s(z,k)}{\partial k} = -c_k(q_1,k) - c_q(q_1,k)\frac{\partial q_1}{\partial k} + (1+r-\delta)\frac{\partial D_k^1}{\partial k} + \frac{\partial D_z^1}{\partial k} + \frac{\partial W(z,k)}{\partial k} \tag{3.25}$$

The partial derivatives of q_1 with respect to z and k on the right-hand side of equations (3.24) and (3.25) can be computed using the solution to the seller's maximization problem (3.22). Equation (3.23) which determines q_1 yields

$$\frac{\partial q_1}{\partial k} = -\frac{c_{qk}(q_1,k)}{c_{qq}(q_1,k)} > 0 \tag{3.26}$$

$$\frac{\partial q_1}{\partial z} = 0 \tag{3.27}$$

The response of production to a marginal increase of the seller's capital and money holdings is shown in equations (3.26) and (3.27). Since the seller uses his capital holdings as an input in production, his optimal amount of production q_1 is a function of the market price $\tilde{p}$ and k. The positive response of q_1 to an increase in k as depicted in equation (3.26) can be explained as follows: Assume that equation (3.23) is satisfied at (q_1^0, k^0) initially. A marginal increase of the seller's capital holdings to k^1 induces him to produce a given amount of goods at a lower marginal cost, i.e. $c_q(q_1^0, k^1) < \tilde{p}$. Thus, the seller increases his production to q_1^1 such that equation (3.23) holds at equality at k^1. According to equation (3.27), an increase of his money holdings has no effect on his production in the day market because money is not used in his production process.

According to the constraint of maximization problem (3.22), the amount of money and capital which the seller receives depends on his production of q_1. Therefore, an increase of his capital holdings has an indirect effect on the payment he receives because q_1 depends on k as shown by equation (3.26). The constraint of maximization problem (3.22) delivers the final partial derivative

$$\frac{\partial D_z^1}{\partial q_1} = \tilde{p} - (1+r-\delta)\frac{\partial D_k^1}{\partial q_1} \tag{3.28}$$

Recall that the total amount of the seller's compensation is $q_1\tilde{p}$. A marginal increase of q_1 increases the total value of the compensation by the market price $\tilde{p}$. It can either be paid in money or in capital. Due to this degree of freedom, the additional amount of money which the seller receives is a function of the additional amount of capital received and vice versa.

Using the information above, equations (3.24) and (3.25) can be expressed as

$$\frac{\partial V_1^s(z,k)}{\partial z} = \frac{\partial W(z,k)}{\partial z} \tag{3.29}$$

$$\frac{\partial V_1^s(z,k)}{\partial k} = -\, c_k(q_1,k) + \frac{c_{qk}(q_1,k)}{c_{qq}(q_1,k)}\left[c_q(q_1,k) - \tilde{p}\right] + \frac{\partial W(z,k)}{\partial k} \tag{3.30}$$

As shown in equation (3.29) the marginal unit of money only impacts the seller's night market payoff. In contrast to money, capital is used as an input in production. An increase of the seller's capital holdings lowers his cost of producing a given amount of goods which is captured by the first term on the right-hand side of equation (3.30). Furthermore, the seller raises his production by the amount given in equation (3.26) which is multiplied by the marginal profit from selling the additional goods. Note that the seller always chooses his production of q_1 such that his marginal production cost equals the market price, i.e. $c_q(q_1, k) = \tilde{p}$. Consequently, the additional profit generated by the marginal unit of q_1 is always zero at the optimal choice of q_1, i.e. the term in the square bracket in equation (3.29) is zero. Like money, the marginal unit of capital increases the seller's continuation value in the night market.

3.4.2 Market 2

In contrast to market 1, money is the only permissible medium of exchange in market 2. Consider a buyer with k units of capital and z units of money. His maximization problem in market 2 is given by

$$\begin{aligned} V_2^b(z,k) = & \max_{q_2} u(q_2) - d_z^2 + W(z,k) \\ \text{s.t. } & q_2\hat{p} = d_z^2 \\ & d_z^2 \leq z \end{aligned} \tag{3.31}$$

As in market 1, the buyer receives $u(q_2)$ utils from the consumption of q_2 units of the day market good. The price of q_2 units of the day market good in terms of the night market good is $q_2\hat{p}$ where $\hat{p}$ is the price level in market 2. Given q_2 and $\hat{p}$, the first constraint in maximization problem (3.31) uniquely determines the amount of the buyer's monetary payment d_z^2 because money is the only medium of exchange in market 2. Due to the anonymity in the day market, trades have to be *quid pro quo* and the buyer's monetary payment d_z^2 cannot exceed his money holdings z. After DM trade, he enters the night market with $z - d_z^2$ units of money and an unchanged k units of capital. According to equation (3.6) his continuation value $W(z - d_z^2, k)$ can be rewritten as $W(z, k) - d_z^2$.

In the unconstrained solution to maximization problem (3.31), q_2 solves

$$u'(q_2) = \hat{p} \tag{3.32}$$

At the optimal q_2, the buyer's marginal utility of consumption equals his marginal cost which is given by the market price $\hat{p}$. The amount of money necessary to purchase q_2 is depicted in the constraint of maximization problem (3.31). At the margin buyers do not value either asset as a means of payment in the unconstrained solution. Thus, they do not use it until the night market. The partial derivatives of an unconstrained buyer's value function $V_2^b(z, k)$ with respect to his money and capital holdings are given by

$$\frac{\partial V_2^b}{\partial z} = \frac{\partial W(z, k)}{\partial z} \tag{3.33}$$

$$\frac{\partial V_2^b}{\partial k} = \frac{\partial W(z, k)}{\partial k} \tag{3.34}$$

In the constrained solution of maximization problem (3.31), the buyer's money holdings are not sufficient to purchase q_1 which solves equation (3.32). Thus, he spends his entire money holdings $d_z^2 = z$ to purchase as much consumption as possible, i.e. $q_2 = z/\hat{p}$. The partial derivatives of $V_2^b(z, k)$ with respect to z and k in the constrained solution take the form

$$\frac{\partial V_2^b(z, k)}{\partial z} = \frac{u'(q_2)}{\hat{p}} \tag{3.35}$$

$$\frac{\partial V_2^b(z, k)}{\partial k} = \frac{\partial W(z, k)}{\partial k} \tag{3.36}$$

Equation (3.35) is the partial derivative of the buyer's value function with respect to z. A marginal increase of his money holdings raises his consumption by $1/\hat{p}$ and yields a marginal utility of $u'(q_2)/\hat{p}$. He enters the night market with zero money holdings because he spent all of his money to purchase consumption goods in the day market. Notice that the derivative of $V_2^b(z,k)$ with respect to k is the same in the constrained as in the unconstrained solution because capital cannot be used as a medium of exchange in market 2.

A seller with asset holdings (z,k) chooses the level of consumption q_2 which optimizes

$$\begin{aligned} V_2^s(z,k) &= \max_{q_2} -c(q_2,k) + d_z^2 + W(z,k) \\ \text{s.t. } q_2\hat{p} &= D_z^2 \end{aligned} \tag{3.37}$$

He produces q_2 goods at a cost of $c(q_2,k)$ utils. His revenue from producing q_2 units of the day market good is $q_2\hat{p}$ given the market price $\hat{p}$. The payment D_z^2, which he receives, is entirely monetary as expressed by the constraint in maximization problem (3.37). He, therefore, enters the night market with $z + D_z^2$ units of money and k units of capital.

The solution to the seller's maximization problem in market 2 is given by

$$c_q(q_2,k) = \hat{p} \tag{3.38}$$

The seller chooses his production q_2 according to equation (3.38). His marginal production cost equals the market price $\hat{p}$ at the optimum. Notice that q_2 is a function of his capital but not of his money holdings because only capital - and not money - is used as a production input.

The partial derivatives of the seller's value function $V_2^s(z,k)$ with respect to his money and capital holdings take the form

$$\frac{\partial V_2^s(z,k)}{\partial z} = \frac{\partial W(z,k)}{\partial z} \tag{3.39}$$

$$\frac{\partial V_2^s(z,k)}{\partial k} = -c_k(q_2,k) + \frac{c_{qk}(q_2,k)}{c_q q(q_2,k)}\left[c_q(q_2,k) - \hat{p}\right] + \frac{\partial W(z,k)}{\partial k} \tag{3.40}$$

The seller does not use his money holdings in the day market. That is why his marginal lifetime utility in the day market with respect to money equals his marginal lifetime utility in the night market as expressed in equation (3.39). Equation (3.40) depicts the partial derivative of $V_2^s(z,k)$ with respect to k. The marginal unit of capital lowers his production cost for a given level of production q_2 according to the first term on the right-hand side of equation (3.40). In response to a marginal increase of k, the seller's marginal production cost $c_q(q_2, k)$ decreases which induces him to produce more. The rise in production is given by $c_{qk}(q_2, k)/c_{qq}(q_2, k)$ and raises his marginal income by $\hat{p}$ and his marginal cost by $c_q(q_2, k)$ per unit. Recall that the seller chooses q_2 such that his marginal cost equals the market price. Thus, the second term vanishes at the optimal choice of q_2. Finally, the seller uses his marginal unit of capital in the night market which is depicted by $\partial W(z, k)/\partial k$.

3.5 Equilibrium

Given the individual agents' behavior derived in the previous section, this section determines the equilibrium prices to close the model.

Recall the night market's first order conditions (3.7) and (3.8). An agent's decision on how much money and capital to take into the next period does not depend on his current asset portfolio, i.e. z' and k' are independent of z and k. This is true for the following two reasons. First, the night market's value function $W(z, k)$ is linear in x and, second, (z, k) does not enter $V(z', k')$. Therefore, any agent chooses the same amount of money z' and capital k' regardless of his trade history. To make this point explicit, denote an agent's asset holdings when leaving the night market by upper-case letters, i.e. $z = Z$ and $k = K$.

All buyers in market 1 choose the same level of consumption q_1^d because they all hold the same asset portfolio when entering the day market. Since the measure of buyers in market 1 equals 0.5λ, the total demand for the good in market 1 can be written as $Q_1^d = 0.5\lambda q_1^d$. Similarly, all sellers in market 1 hold the same asset portfolio (Z, K) in the beginning of a period. Consequently, they all choose the same amount of production q_1^s and total supply of the good in market 1 amounts to $Q_1^s = 0.5\lambda q_1^s$.

In equilibrium the price level $\tilde{p}$ clears market 1, i.e. $Q_1^d = Q_1^s$ at $\tilde{p}$. Thus, the amount consumed by each single buyer equals the amount produced by each single seller, i.e. $q_1^d = q_1^s \equiv q_1$. The equilibrium price level in market 1 is always given by $\tilde{p} = c_q(q_1, K)$ according to equation (3.23). If buyers are not constrained by their asset holdings the price level can also be expressed as $\tilde{p} = c_q(q_1, K) = u'(q_1)$. Otherwise, it can be written as $\tilde{p} = c_q(q_1, K) = [Z + (1 + r - \delta)K]/q_1$.

The equilibrium price level in market 2, $\hat{p}$ is derived in a similar fashion. The measure of buyers and sellers in market 2 is $0.5(1 - \lambda)$, respectively. Thus, total supply and total demand for the good in market 2 are $Q_2^s = 0.5(1 - \lambda)q_2^s$ and $Q_2^d = 0.5(1 - \lambda)q_2^d$. In equilibrium, supply equals demand which implies $q_2^s = q_2^d \equiv q_2$. The equilibrium price level which clears market 2 is given by $\hat{p} = c_q(q_2, K)$.

The money (capital) holdings are generally not the same across all sellers in market 1 after trade has occurred. According to the constraint in maximization problem (3.22) only the total value of the compensation is equal for all sellers in market 1 but its composition is random. The same holds true for the buyers' payment if they are not constrained by their asset holdings. Otherwise, buyers spend their entire asset holdings. In market 2, however, the amount of money spent by each single buyer equals the amount of money received by each single seller because money is the only means of payment.

Take a look at the optimization problem in the night market (3.6). The envelope conditions yield

$$\frac{\partial W(Z,K)}{\partial z} = 1 \tag{3.41}$$

$$\frac{\partial W(Z,K)}{\partial k} = 1 + r - \delta \tag{3.42}$$

Equations (3.41) and (3.42) show how the marginal unit of money and capital affects an agent's lifetime utility in the beginning of the night market. Note that both partials are evaluated at $z = Z$ and $k = K$. According to equation (3.41) the marginal unit of money brought into the night market buys a marginal unit of net consumption x, thereby increasing the agent's payoff at night by one. The marginal unit of capital is worth $1 + r - \delta$ units of net consumption because capital earns a return r and depreciates at the rate δ at the start of the night market.

In equilibrium the return on capital equals its marginal product, i.e. $r = F'(K)$. Inserting $z = Z$, $k = K$, the equilibrium price in market 1 [$\tilde{p} = c_q(q_1, K)$], $r = F'(K)$ and the derivatives of $W(Z, K)$ with respect to z and k [equations (3.41) and (3.42)] into the partial derivatives of the seller's value function in market 1, i.e. equations

(3.29) and (3.30), yields

$$\frac{\partial V_1^s(Z,K)}{\partial z} = 1 \tag{3.43}$$
$$\frac{\partial V_1^s(Z,K)}{\partial k} = -c_k(q_1,K) + 1 + F'(K) - \delta \tag{3.44}$$

The seller uses money in the night market but not during the day. Therefore, the marginal unit of money raises his lifetime utility by one as he uses it to purchase one unit of net consumption at night. Capital, on the other hand, is used as a production input in both, the day and the night market. In the day market the marginal unit of capital reduces his production cost, i.e. $c_k(q_1, K) < 0$, and in the night market it is used to buy $1 + F'(K) - \delta$ units of net consumption.

Next, consider a buyer in market 1. If he is not constrained by his asset holdings, the partial derivatives of his value function with respect to z and k take the form

$$\frac{\partial V_1^b(Z,K)}{\partial z} = 1 \tag{3.45}$$
$$\frac{\partial V_1^b(Z,K)}{\partial k} = 1 + F'(K) - \delta \tag{3.46}$$

In the unconstrained case, the buyer does not use the marginal unit of either asset as medium of exchange. Instead, he uses it to obtain net consumption in the night market. One unit of money yields one unit of net consumption whereas one unit of capital buys $1 + F'(K) - \delta$ units of x.

Finally, if the buyer is constrained by his asset holdings, the partial derivatives turn out to be

$$\frac{\partial V_1^b(Z,K)}{\partial z} = \frac{u'(q_1)}{c_q(q_1,K)} \tag{3.47}$$
$$\frac{\partial V_1^b(Z,K)}{\partial k} = (1 + F'(K) - \delta)\frac{u'(q_1)}{c_q(q_1,K)} \tag{3.48}$$

The buyer who is constrained by his asset holdings uses all of his assets as a means of payment in the day market. He receives $u'(q_1)$ utils from the marginal unit of day market consumption. In equilibrium, he obtains $1/c_q(q_1, K)$ units of q_1 from the marginal unit of money and $(1+F'(K)-\delta)/c_q(q_1,K)$ units of consumption from the marginal unit of capital.

The equilibrium versions of equations (3.33) - (3.36), (3.39) and (3.40), i.e. the partial derivatives of the buyer's and seller's value functions in market 2, can be expressed in a similar fashion. In contrast to market 1, the price level in market 2 is given by $\hat{p} = c_q(q_2, K)$.

In the following, the partial derivatives of next period's DM value function $V(z', k')$ are inserted into the first order conditions of the NM value function $W(z, k)$ [equations (3.7) and (3.8)]. Note that the buyer's and seller's value functions in both markets are combined in the day market's value function $V(z', k')$, i.e. equation (3.9). Due to the buyer's asset constraint there are three exhaustive cases to be considered, denoted by (i), (ii) and (iii).

In case (i) buyers in both markets are not constrained by their asset holdings. Thus, buyers have sufficient assets to purchase their desired level of consumption in market 1 and 2. Inserting the market 1 and market 2 price level, i.e. $\tilde{p} = c_q(q_1, K)$ and $\hat{p} = c_q(q_2, K)$, the unconstrained solutions to the buyers' maximization problem in market 1 and 2 [equations (3.13) and (3.32)] take the form

$$u'(q) = c_q(q, K) \tag{3.49}$$

where both market 1 and market 2 consumption solve equation (3.49), i.e. $q_1(K) = q_2(K) = q^*$. As shown in equation (3.49), q_1 and q_2 follow the same functional form as the central planner's optimal level of consumption. Given the *first best* capital stock, the solutions for q_1 and q_2 are socially optimal. Furthermore, equations (3.7) and (3.8) yield

$$\beta^{-1} = 1 + F'(K) - \delta - 0.5\left[\lambda c_k(q_1, K) + (1-\lambda)c_k(q_2, K)\right] \tag{3.50}$$

$$\pi = \beta \tag{3.51}$$

Note that equations (3.50) and (3.51) are static. Primes indicating next period's variables are dropped in the two equations above and in all static equations in the remainder of this paper. Equation (3.50) is the capital Euler equation. All agents use the marginal unit of capital to buy $1 + F'(K) - \delta$ units of net consumption in the night market. Additionally sellers which constitute fifty percent of the population use capital as an input in their production process. The marginal unit of capital reduces their disutility from production by $c_k(q_1, K)$ if they are located in market 1 which applies to a fraction λ of all sellers. The remaining $1 - \lambda$ sellers in market 2 experience a reduction of their production cost by $c_k(q_2, K)$ utils. Capital's total marginal return offsets the loss from discounting, i.e. it equals β^{-1}. According to equation (3.51), agents hold the marginal unit of money only at the Friedman (1969) rule, i.e. if deflation compensates for the loss from discounting. Finally, aggregate net consumption X is obtained by the resource constraint which is the only dynamic equilibrium condition.

$$X = F(K) - K' + (1 - \delta)K \tag{3.52}$$

The resource constraint is derived from the night market's budget constraint [the constraint in maximization problem (3.6)] after inserting the firm's profits P, the government's taxes T and using aggregate consumption, capital and money holdings instead of individual quantities. Equation (3.52) states that aggregate net consumption X is the the difference between output $F(K)$ and investment $K' - (1 - \delta)K$. If $\pi = \pi'$, then next period's capital stock equals this period's and investment is given by δK.

In case (ii) only buyers in market 2 are constrained by their asset holdings. They use their entire money holdings to purchase as much q_2 as possible. Thus, q_2 solves

$$Z = q_2 c_q(q_2, K) \tag{3.53}$$

where $c_q(q_2, K)$ is the equilibrium price level in market 2. Buyers in market 1 can still afford their desired level of DM consumption. That is, q_1 solves equation (3.49) as in case (i). Equations (3.7) and (3.8) in case (ii) are given by

$$\beta^{-1} = 1 + F'(K) - \delta - 0.5\left[\lambda c_k(q_1, K) + (1 - \lambda)c_k(q_2, K)\right] \tag{3.54}$$

$$\pi = \beta\left\{1 + 0.5(1 - \lambda)\left(\frac{u'(q_2)}{c_q(q_2, K)} - 1\right)\right\} \tag{3.55}$$

The capital Euler equation (3.54) in case (ii) is the same as in case (i) because agents cannot use capital as a medium of exchange in market 2. The marginal unit of capital generates a return $r = F'(K)$, depreciates at the rate δ in the night market and reduces the sellers' production cost in markets 1 and 2. The money Euler equation in case (ii) [equation (3.55)] differs from its counterpart in case (i), however. Buyers in market 2 who are of measure $0.5(1-\lambda)$ use the marginal unit of money as a means of payment because they cannot afford their desired level of consumption, i.e. $q_2 < q^*$. Therefore, they assign a positive liquidity value to the marginal unit of money. The liquidity value , denoted by $u'(q_2)/c_q(q_2, K) - 1 > 0$, can be explained in the following way. The marginal unit of money increases buyers' utility in the day market by $u'(q_2)/c_q(q_2, K)$. At the same time, it reduces their night market utility because it cannot be used to purchase one unit of net consumption in the night market anymore.

Finally, in case (iii) buyers in market 1 and market 2 are constrained by their asset holdings. As in case (ii) buyers in market 2 spend their entire money holdings to purchase as much day market consumption as possible. Buyers in market 1 use all of their money and capital holdings and receive q_1 units of the day market good in return where q_1 solves

$$Z + [1 + F'(K) - \delta]K = q_1 c_q(q_1, K) \tag{3.56}$$

The market 1 price level is given by $c_q(q_1, K)$ in equation (3.56). Thus, the price of q_1 goods in terms of the night market good is given by the right-hand side of equation (3.56). Recall that capital K is multiplied by $1 + F'(K) - \delta$ because it earns a return $r = F'(K)$ and depreciates at the rate δ in the subsequent night market. In case (iii), equations (3.7) and (3.8) take the form

$$\begin{aligned} \beta^{-1} =& [1 + F'(K) - \delta] \left\{ 1 + 0.5\lambda \left(\frac{u'(q_1)}{c_q(q_1, K)} - 1 \right) \right\} \\ & - 0.5 \left[\lambda c_k(q_1, K) + (1-\lambda) c_k(q_2, K) \right] \end{aligned} \tag{3.57}$$

$$\pi = \beta \left\{ 1 + 0.5 \left[\lambda \left(\frac{u'(q_1)}{c_q(q_1, K)} - 1 \right) + (1-\lambda) \left(\frac{u'(q_2)}{c_q(q_2, K)} - 1 \right) \right] \right\} \tag{3.58}$$

Equation (3.57) differs from the capital Euler equations in cases (i) and (ii): Buyers in market 1 (measure 0.5λ) use the marginal unit of capital as a medium of exchange which increases their utility in the day market by $[1 + F'(K) - \delta]u'(q_1)/c_q(q_1, K)$. On the other hand their net consumption in the night market decreases by $1 + F'(K) - \delta$ units. As before, the marginal unit of capital reduces sellers' production costs in the day market and increases all agents' net consumption at night by $1 + F'(K) - \delta$ units. According to equation (3.58) buyers in both markets assign a positive liquidity value to the marginal unit of money as it is used to purchase more consumption in both markets.

Lemma 2. Consumption in market 1 is always (weakly) greater than consumption in market 2, i.e. $q_1 \geq q_2$. The inequality is strict in cases (ii) and (iii).

The proof of Lemma 2 can be found in the appendix. The intuition of Lemma 2 is straightforward. At the Friedman rule money alone generates enough liquidity to purchase the desired level of DM consumption. Thus, buyers in both markets receive q^*. For $\pi > \beta$ money does not suffice to purchase q^* and buyers in market 1 who can spend their money and capital holdings can afford more consumption than buyers in market 2 who can only use their money holdings as a means of payment.

Proposition 10. *A monetary equilibrium has the property*

- $q_2 = q_1 = q^*$ *if* $\pi = \beta$
- $q_2 < q_1 = q^*$ *if* $\bar{\pi} > \pi > \beta$
- $q_2 < q_1 < q^*$ *if* $\pi > \bar{\pi} > \beta$

The proof of Proposition 10 can be found in the appendix. Agents in both markets can afford their desired level of consumption if $\pi = \beta$, i.e. $q_1 = q_2 = q$, which constitutes case (i). Note that the liquidity value is zero in both markets if $\pi = \beta$ because buyers do not use the marginal unit of money or capital as a medium of exchange in either market. The capital stock K and day market consumption $q_1 = q_2$ are simultaneously obtained by equations (3.49) and (3.50). The equilibrium money holdings Z are indetermined if $\pi = \beta$ as money can be held at no cost, i.e. equation (3.51) is fulfilled independently of Z. Aggregate net consumption is the residual of output and capital investment according to equation (3.52).

Only buyers in market 1 can afford their desired level of day market consumption ($q^* = q_1 > q_2$) if $\bar{\pi} > \pi > \beta$ which coincides with case (ii). Thus, buyers in market 2 assign a liquidity value to the marginal unit of money as shown by the money Euler equation (3.55). It describes the equilibrium value for q_2 as a function of the equilibrium capital stock K. The capital Euler equation (3.54) determines K given the solutions to q_1 and q_2 and equation (3.49) establishes q_1 as a function of K. Thus, the equilibrium solutions to q_1, q_2 and K are simultaneously obtained from equations (3.49), (3.54) and (3.55). Equation (3.53) reveals the equilibrium money holdings Z given q_2 and K and the resource constraint (3.52) determines X.

Buyers in both markets cannot afford their desired level of DM consumption if $\pi > \bar{\pi}$ [case (iii)]. Lemma 2 shows that $q_1 > q_2$. Buyers in market 1 use the marginal unit of money and capital as a means of payment, whereas buyers in market 2 can only use the marginal unit of money as a medium of exchange. Equation (3.56) reveals the amount goods q_1 that buyers receive in exchange for all of their assets. Note that the right-hand side of equation (3.56) depicts the value of q_1 goods in terms of the night market good. The equilibrium values of q_1, q_2, K and Z are simultaneously determined by equations (3.53), (3.56) - (3.58). Finally, aggregate consumption is obtained from the resource constraint (3.52).

Recall the central planner's problem (3.1) which generates the socially optimal (*first-best*) allocation. Proposition 11 determines the socially optimal rate of inflation, using the central planner's solution as the benchmark.

Proposition 11. *The Friedman (1969) rule, i.e.* $\pi = \beta$*, is the welfare-maximizing policy.*

The government runs the Friedman rule in order to maximize welfare. That is, it implements a deflation which offsets the loss from discounting. At $\pi = \beta$, all buyers can afford their desired amount of DM consumption. All agents value the marginal unit of capital for its role as an input in the night market production. Furthermore, sellers value it as a productive input in the day market. Buyers do not assign a liquidity value to the marginal unit of any asset because they can already afford their desired DM consumption in both markets. At the margin all assets are valued as in the *first-best* solution if $\pi = \beta$. Consequently, agents hold the *first best* amount of capital which, according to equations (3.49) and (3.50), implies the *first best* amount of day market and night market consumption.

In the remainder of this chapter, we analyze how a marginal increase of the rate of inflation affects capital investment. To do so, functional forms are assigned as follows: $u(q) = Cq^\gamma$, $c(q, K) = q^\psi K^{1-\psi}$ and $F(K) = K^\alpha$ with parameters C, γ, ψ and α.

Recall that the utility of consumption and the production function are concave in their arguments. To guarantee this, the parameters α and γ must lie in the open interval $(0, 1)$. The parameter ψ can be restricted in a similar way. The function $c(q, K)$ must satisfy $c_q > 0$, $c_k < 0$ $c_{qq} > 0$, $c_{kk} > 0$. The preceding conditions are met if $\psi > 1$.

Proposition 12. *A marginal increase of inflation leads to a reduction in capital investment if $\beta < \pi < \bar{\pi}$ and $\lambda \in (0, 1)$.*

Only buyers in market 1 can afford their desired level of DM consumption, i.e. $q_2 < q_1 = q^*$, if $\beta < \pi < \bar{\pi}$. Consequently, buyers in market 2 spend all of their money to purchase as much DM consumption as possible. An increase of inflation has a negative income effect as it raises the cost of holding money. The income effect carries over to an agent's capital investment decision because the amount of goods traded in market 2 depends on the sellers' capital and the buyers' money holdings at the margin: A seller who uses capital as an input in DM production generates less revenue in market 2 after the increase in inflation because he faces buyers with lower money holdings. As a consequence, agents in the night market decide to purchase less capital.

There is no substitution out of money and into capital in response to an increase of inflation, however. Assume agents substituted: Their expected payoff in market 1 would not be altered by this decision because buyers in market 1 can already afford q^*. In market 2, however, buyers cannot afford their desired level of consumption and spend their entire money holdings to purchase as much consumption as possible. A substitution out of money and into capital would, therefore, lower their payoff in market 2 without changing their payoff in market 1. As a consequence, agents do not substitute, and, the total effect of an increase of inflation is given by the income effect: Capital investment decreases which we call the Stockman effect.

Next, consider case (iii) where $\pi > \bar{\pi}$. Using the functional forms shown above, the equilibrium equations of case (iii) can be reduced to the following two equations in two unknowns q_1 and K (see the appendix for derivations)

$$\begin{aligned}\beta^{-1} =&(1+\alpha K^{\alpha-1}-\delta)\left\{1+0.5\lambda\left(\frac{\gamma C}{\psi}q_1^{\gamma-\psi}K^{\psi-1}-1\right)\right\}\\ &-0.5\left[(1-\psi)K^{-\psi}q_1^{\psi}-(1-\lambda)(1-\psi)\frac{1+\alpha K^{\alpha-1}-\delta}{\psi}\right]\end{aligned} \tag{3.59}$$

$$1+2(\beta^{-1}\pi-1) =K^{\psi-1}\frac{\gamma C}{\psi}\left(\lambda q_1^{\gamma-\psi}+(1-\lambda)\left[q_1^{\psi}-\frac{1+\alpha K^{\alpha-1}-\delta}{\psi}K^{\psi}\right]^{\frac{\gamma-\psi}{\psi}}\right) \tag{3.60}$$

Figure 3.1 is drawn using the following parameter values. We set $\alpha = 0.3$, $\beta = 0.9756$, $\delta = 0.07$, $\psi = 1.1$, $\gamma = 0.5$ and $C = 30$. Note that the parameter C in the utility of day market consumption has to be chosen quite large to guarantee a low value of $\bar{\pi}$. The intuition is as follows: A large C increases the utility of day market consumption and raises the buyers' desired level of DM consumption q^*. Thus, the level of inflation (depreciation of money) at which buyers in market 1 cannot afford their desired day market consumption anymore is lower for larger values of C.

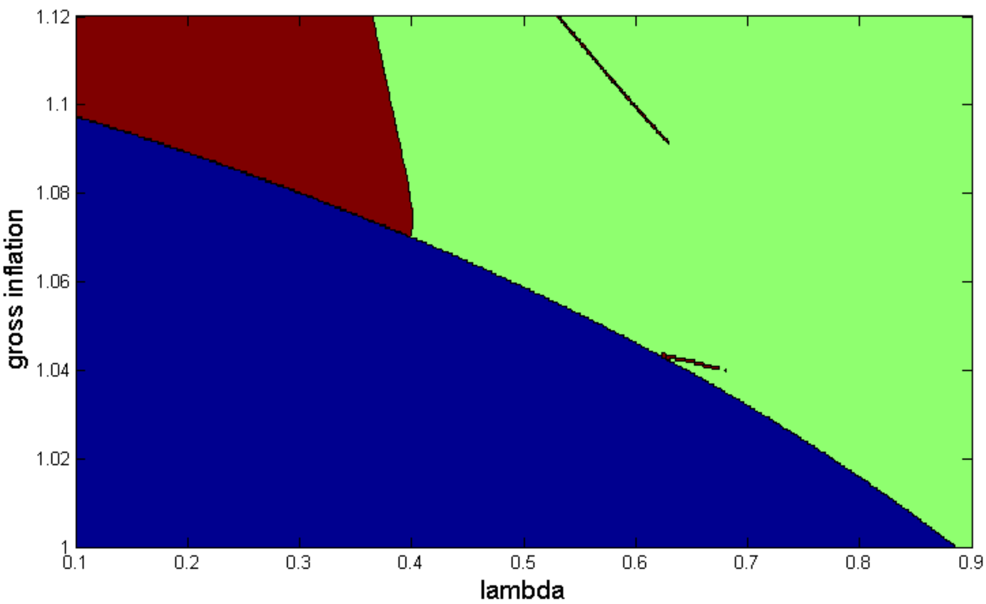

Figure 3.1: Effect of an increase in π on capital investment

The ordinate of figure 3.1 shows values of π in the interval $[1, 1.12]$ and the abscissa depicts values of λ in $[0.1, 0.9]$. Given the parameterization above, we compute K and q_1 as the solution to equations (3.59) and (3.60) for each pair (λ, π). The blue area in figure 3.1 denotes combinations of λ and π which lead to an equilibrium with $q_2 < q_1 = q^*$, i.e. case (ii). The green and red areas are an equilibrium with $q_2 < q_1 < q^*$ [case (iii)]. For a given λ, the value of π at which the blue area turns into red or green is called $\bar{\pi}$. According to Proposition 10, rates of inflation below $\bar{\pi}$ generate an equilibrium with $q_2 < q_1 = q^*$. At $\pi > \bar{\pi}$, buyers in market 1 do not hold enough assets to purchase their desired DM consumption, i.e. $q_2 < q_1 < q^*$.

Figure 3.1 shows that $\bar{\pi}$ is a function of λ. The negative dependence of $\bar{\pi}$ on λ can be explained as follows: Agents hold money because it can be used as a medium of exchange in market 1 and in market 2. Recall that the additional DM payoff generated by the marginal unit of money is always greater in market 2 than in market 1 as $q_2 < q_1$ for $\pi > \beta$. In other words, money's liquidity value in market 2 exceeds its liquidity value in market 1. Consider the money Euler equation (3.55) in an equilibrium with $q_2 < q_1 = q^*$. The expected payoff from the marginal unit of money is pinned down by the nominal interest rate $i \equiv \pi\beta^{-1} - 1$. Assume the probability of entering market 2 decreases, i.e. λ increases. At the new λ, the payoff from the marginal unit of money (money's liquidity value) needs to increase for the expected payoff (money's expected liquidity value) to remain constant. Thus, a buyer raises money's liquidity value by bringing less money into the day market as this lowers his market 2 consumption.

Intuitively, the negative dependence of Z on λ can be explained as follows: A part of an agent's money holdings can be interpreted as an insurance against becoming a buyer in market 2 where only money can be used to buy consumption. If entering market 2 becomes less likely, i.e. λ increases, agents purchase less insurance (money). Finally, $\bar{\pi}$ depends negatively on λ because an agent with more money holdings can afford his desired consumption q^* in market 1 for higher values of π than an agent who holds less money.

In the red area of figure 3.1, an increase of the rate of inflation inflation π leads to a reduction of capital investment ('Stockman effect') whereas an increase of π raises capital investment in the green area ('Tobin effect'). An increase of the rate of inflation has two effects for $\beta < \bar{\pi} < \pi$. We call the first effect 'income effect'. The increase of inflation makes holding money more expensive and agents reduce their money holdings, accordingly. The income effect impacts an agent's capital investment decision, as well: Sellers generate less revenue in the day market because they face buyers with lower money holdings than before the increase in π. Anticipating this, agents decide to purchase less capital in the night market. We refer to the second effect as the 'substitution effect': Agents substitute out of money and into capital in response to an increase of inflation. This substitution has two consequences: On the one hand, agents can afford more consumption if they become buyers in market 1 than if they had not substituted. On the other hand, they receive less consumption if they become buyers in market 2 because only money can be used as a medium of exchange in market 2.

Whether or not the substitution effect dominates the income effect depends on the probability of entering market 1. For high probabilities of entering market 1 (large λ), agents substitute more heavily out of money and into capital in response to an increase of inflation than for low λ because the probability of suffering the negative consequences of the substitution, i.e. entering market 2 as a buyer, is low for high λ. If the probability of entering market 1 is sufficiently high (large λ), the substitution effect dominates the income effect and capital investment is larger in the new steady state than in the old one (Tobin effect). Otherwise, the income effect dominates the substitution effect and capital investment decreases in response to an increase in π (Stockman effect).

3.6 Concluding Remarks

This paper showed that there are three types of equilibria. First, if $\pi = \beta$ agents hold enough money to afford their desired level of DM consumption, i.e. $q_1 = q_2 = q^*$. Second, agents' money holdings do not suffice to purchase q^* if $\beta < \pi < \bar{\pi}$. Thus, only buyers in market 1 where money and capital can be used as media of exchange can afford q^* and $q_2 < q_1 = q^*$. Third, if π is greater than $\bar{\pi}$, even buyers in market 1 cannot afford their desired DM consumption anymore. Consequently, $q_2 < q_1 < q^*$.

The Friedman rule is the optimal policy as it replicates the central planner's solution. At $\pi = \beta$, agents value the marginal unit of capital for its productive uses in the day- and in the night market which coincides with the central planner's solution. Furthermore, they do not value either asset as a medium of exchange at the margin. They choose the *first best* capital stock and the first best levels of day- and night market consumption.

The numerical exercise revealed that $\bar{\pi}$ depends negatively on the acceptance probability of capital λ. Agents insure themselves against becoming buyers in the day market by purchasing money. They purchase more 'insurance' if the probability of the worst case, namely that they become buyers in market 2, is high than if it is low. Thus, they hold more assets if λ is low than if it is high. In market 1, buyers can afford q^* for higher values of inflation if they hold more assets which explains the negative dependence of $\bar{\pi}$ on λ.

A marginal increase of inflation always leads to a reduction of capital investment (Stockman effect) if $\beta < \pi < \bar{\pi}$: Buyers hold enough assets (money and capital) to purchase their desired level of consumption in market 1. In market 2, however, buyers' money holdings are not sufficient to purchase q^*. An agents' capital and money investment decisions are linked because the amount of goods traded in market 2 depends on the marginal unit of money and on the marginal unit of capital. Consequently, inflation acts as a tax on money and on capital and an increase of inflation lowers capital investment (income effect). The income effect constitutes the total effect if the rate of inflation is below $\bar{\pi}$.

In addition to the income effect, there is a substitution effect if inflation is above $\bar{\pi}$: Agents choose to substitute out of money and into capital in response to an increase in inflation. The substitution increases consumption in market 1 but it decreases consumption in market 2 where only money can be used as a medium of exchange. Thus, agents are willing to substitute more heavily if it is more likely that they enter market 1 and the substitution effect dominates the income effect if λ is sufficiently large. In this case, capital investment increases in response to an increase of inflation (Tobin effect). Otherwise, the income effect dominates and capital investment decreases in response to an increase of inflation (Stockman effect).

To sum up, an increase of inflation leads to the Stockman effect if inflation is below $\bar{\pi}$ and if inflation is above $\bar{\pi}$ and λ is low. Otherwise, if inflation is above $\bar{\pi}$ and λ is sufficiently high, an increase of inflation induces the Tobin effect.

According to this model, the effect of monetary policy on investment depends on the availability of media of exchange other than money. Interpret $\lambda \in (0, 1)$ which describes the liquidity of assets other than money as a measure of the development of an economy's capital market. A large λ suggests a highly developed capital market as other assets are very liquid. Thus, this model suggests a negative effect of monetary policy on investment in economies with underdeveloped capital markets (small λ). In economies with sufficiently developed capital markets (large λ), the effect of monetary policy depends on the prevailing rate of inflation. Expansionary monetary policy can stimulate capital investment if inflation is sufficiently high.

3.7 Appendix

Proof of Lemma 2. Equations (3.53) and (3.56), which are repeated here for convenience, determine q_1 and q_2 in case (iii), i.e. if both are smaller than q^*.

$$Z = q_2 c_q(q_2, K) \tag{3.61}$$

$$Z + [1 + F'(K) - \delta]K = q_1 c_q(q_1, K) \tag{3.62}$$

Inserting equation (3.61) into equation (3.62) yields $[1+F'(K)-\delta]K = q_1 c_q(q_1, K) - q_2 c_q(q_2, K)$ where the left-hand side is positive as $K > 0$. Thus, $q_1 c_q(q_1, K) > q_2 c_q(q_2, K)$ which can be rearranged as

$$\frac{q_1}{q_2} > \frac{c_q(q_2, K)}{c_q(q_1, K)} \tag{3.63}$$

Assume that $q_1 > q_2$. In this case the left-hand side of condition (3.63) is larger than 1. Recall that $c_{qq}(q, K) > 0$ which implies $c_q(q_1, K) > c_q(q_2, K)$ given our assumption. Thus, the right-hand side of inequality (3.63) is smaller than 1 and condition (3.63) is satisfied. The condition is not satisfied for $q_1 < q_2$. Thus, consumption in market 1 exceeds consumption in market 2, i.e. $q_1 > q_2$, in case (iii).

In case (ii), money provides a liquidity value in market 2 only, i.e. $q_1 > q_2$. Neither asset yields a liquidity value in case (i) which implies $q_1 = q_2 = q^*$. □

Proof of Proposition 10. First, consider $\pi = \beta$. According to the money Euler equation (3.51), the marginal unit of money does not yield a liquidity value, i.e. $q_1 = q_2 = q^*$.

If π is slightly increased, the marginal unit of money must be valued for its liquidity in a monetary equilibrium. That is either $q_1 < q^*$ or $q_2 < q^*$. Lemma 2 showed that $q_1 > q_2$ if $\pi > \beta$. Thus the monetary equilibrium for $\bar{\pi} > \pi > \beta$ has the property $q_2 < q_1 = q^*$ and solves equations (3.49), (3.52) and (3.53) - (3.55).

The money Euler equation (3.55) in case (ii) is repeated here for convenience:

$$\pi = \beta \left\{ 1 + 0.5(1-\lambda) \left(\frac{u'(q_2)}{c_q(q_2, K)} - 1 \right) \right\} \tag{3.64}$$

The marginal unit of money reveals a liquidity value since $q_2 < q^*$. A buyer in market 1 can afford $q_1 = q^*$ and a buyer in market 2 is constrained by his money holdings. Thus,

$$
\begin{aligned}
q_1 c_q(q_1, K) &< Z + [1 + F'(K) - \delta]K \\
q_2 c_q(q_2, K) &= Z
\end{aligned}
$$

Combining the two equations above yields

$$q_1 c_q(q_1, K) < q_2 c_q(q_2, K) + (1 + F'(K) - \delta)K \tag{3.65}$$

The liquidity value of the marginal unit of money increases in response to an increase of π, i.e. $u'(q_2)/c_q(q_2, K)$ rises. It increases if q_2 decreases or K increases as $u'' < 0$, $c_{qq} > 0$ and $c_{qk} < 0$. In any case, $c_q(q_2, K)$ decreases if π increases. Consider equation (3.65). As π increases the impact of $q_2 c_q(q_2, K)$ decreases and vanishes as π approaches ∞. Condition (3.65) is binding at some $\pi \equiv \bar{\pi}$ because the agent's capital stock by itself is not sufficient to purchase q^* which is a condition for the existence of monetary equilibria. Thus, for $\pi > \bar{\pi}$, we have $q_2 < q_1 < q^*$. □

Proof of Proposition 12. Using the functional forms equations (3.49) and (3.55) express q_1 and q_2 as functions of K, respectively. They take the form:

$$q_1 = \left(\frac{\gamma C}{\psi}\right)^{\frac{1}{\psi-\gamma}} K^{\frac{\psi-1}{\psi-\gamma}} \equiv q^* \tag{3.66}$$

$$q_2 = \left[\left(\frac{1-\lambda}{2(\pi\beta^{-1}-1)+1-\lambda}\right)\frac{\gamma C}{\psi}\right]^{\frac{1}{\psi-\gamma}} K^{\frac{\psi-1}{\psi-\gamma}} \tag{3.67}$$

Inserting equations (3.66) and (3.67) into equation (3.54) yields

$$
\begin{aligned}
\beta^{-1} =& 1 + \alpha K^{\alpha-1} - \delta + 0.5(\psi-1)K^{\frac{\psi(\gamma-1)}{\psi-\gamma}} \left[\lambda\left(\frac{\gamma C}{\psi}\right)^{\frac{\psi}{\psi-\gamma}}\right. \\
&\left. +(1-\lambda)\left(\left(\frac{1-\lambda}{2(\pi\beta^{-1}-1)+1-\lambda}\right)\frac{\gamma C}{\psi}\right)^{\frac{\psi}{\psi-\gamma}}\right]
\end{aligned}
$$

Equation (3.7) determines the solution for K. Note that it depends on the rate of inflation π. Differentiating reveals the solution

$$\frac{\partial K}{\partial \pi} = -\frac{A_3}{A_1 + A_2} < 0$$

where $A_1 \equiv -\alpha(1-\alpha)K^{\alpha-2} < 0$,
$A_2 = -\frac{\psi-1}{2}\frac{\psi(1-\gamma)}{\psi-\gamma}\left[\lambda\left(\frac{\gamma C}{\psi}\right)^{\frac{\psi}{\psi-\gamma}} + (1-\lambda)\left(\left(\frac{1-\lambda}{2(\pi\beta^{-1}-1)+1-\lambda}\right)\frac{\gamma C}{\psi}\right)^{\frac{\psi}{\psi-\gamma}}\right] K^{\frac{\gamma(\psi-1)}{\psi-\gamma}-1} < 0$
and $A_3 \equiv -\frac{\gamma C}{\psi-\gamma}\frac{(\psi-1)(1-\lambda)^2}{2(2(\beta^{-1}\pi-1)+1-\lambda)^2}\left(\left(\frac{1-\lambda}{2(\pi\beta^{-1}-1)+1-\lambda}\right)\frac{\gamma C}{\psi}\right)^{\frac{\gamma}{\psi-\gamma}} K^{\frac{\psi(\gamma-1)}{\psi-\gamma}} < 0.$ □

Functional Forms. Using the functional forms: $u(q) = Cq^\gamma$, $c(q, K) = q^\psi K^{1-\psi}$ and $F(K) = K^\alpha$ the equilibrium conditions in case (iii), i.e. equations (3.53) and (3.56) - (3.58), take the form

$$\begin{aligned}\beta^{-1} =&(1+\alpha K^{\alpha-1}-\delta)\left\{1+0.5\lambda\left(\frac{\gamma C}{\psi}q_1^{\gamma-\psi}K^{\psi-1}-1\right)\right\} \\ &-0.5\left[\lambda(1-\psi)q_1^\psi K^{-\psi}+(1-\lambda)(1-\psi)q_2^\psi K^{-\psi}\right]\end{aligned} \tag{3.68}$$

$$\pi =\beta\left\{1+0.5\left[\lambda\left(\frac{\gamma C}{\psi}q_1^{\gamma-\psi}K^{\psi-1}-1\right)+(1-\lambda)\left(\frac{\gamma C}{\psi}q_2^{\gamma-\psi}K^{\psi-1}-1\right)\right]\right\} \tag{3.69}$$

$$q_2^\psi =q_1^\psi - \frac{1+\alpha K^{\alpha-1}-\delta}{\psi}K^\psi \tag{3.70}$$

Equations (3.68) and (3.69) are the capital- and the money Euler equation. Equation (3.70) is derived by inserting (3.53) into (3.56). Using equation (3.70) to eliminate q_2 in equations (3.68) and (3.69) yields two equations in two unknowns q_1 and K.

$$\begin{aligned}\beta^{-1} =&(1+\alpha K^{\alpha-1}-\delta)\left\{1+0.5\lambda\left(\frac{\gamma C}{\psi}q_1^{\gamma-\psi}K^{\psi-1}-1\right)\right\} \\ &-0.5\left[\lambda(1-\psi)q_1^\psi K^{-\psi}+(1-\lambda)(1-\psi)K^{-\psi}q_1^\psi-(1-\lambda)(1-\psi)\frac{1+\alpha K^{\alpha-1}-\delta}{\psi}\right]\end{aligned} \tag{3.71}$$

$$1+2(\beta^{-1}\pi-1) = K^{\psi-1}\frac{\gamma C}{\psi}\left(\lambda q_1^{\gamma-\psi}+(1-\lambda)\left[q_1^\psi-\frac{1+\alpha K^{\alpha-1}-\delta}{\psi}K^\psi\right]^{\frac{\gamma-\psi}{\psi}}\right) \tag{3.72}$$

Equations (3.71) and (3.72) are used to construct figure 1. □

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