Dual weights in the theory of harmonic Siegel modular forms

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To Sven

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Summary

We define harmonic Siegel modular forms based on a completely new approach using vector-valued covariant operators. The Fourier expansions of such forms are investigated for two distinct slash actions. Two very different reasons are given why these slash actions are natural. We prove that they are related by ξ-operators that generalize the ξ-operator for elliptic modular forms. We call them dual slash actions or dual weights, a name which is suggested by the many properties that parallel the elliptic case.

Based on Kohnen's limit process for real-analytic Siegel Eisenstein series, we show that, under mild assumptions, Jacobi forms can be obtained from harmonic Siegel modular forms, generalizing the classical Fourier-Jacobi expansion. The resulting Fourier-Jacobi coefficients are harmonic Maaß-Jacobi forms, which are defined in full generality in this work. A compatibility between the various ξ-operators for Siegel modular forms, Jacobi forms, and elliptic modular forms is deduced, relating all three kinds of modular forms.

Zusammenfassung

Fußend auf einem vollständig neuen Ansatz, dem vektorwertige kovariante Operatoren zu Grunde liegen, definieren wir den Begriff der harmonischen Siegelschen Modulform. Dieser Definition schließt sich eine Untersuchung der für zwei verschiedene Strichoperationen auftretenden Fourier-Entwicklungen an. Die besagten Operationen sind natürlich in zweierlei Hinsicht, auf die wir beide näher eingehen. Darüber hinaus besteht eine Verbindung zwischen diesen beide Strichoperatoren, die durch zwei ξ-Operatoren, die wiederum den elliptischen ξ-Operator verallgemeinern, vermittelt wird. Die bemerkenswerte Ähnlichkeit zum Verhalten von elliptischen Modulformen dual Gewichts legt die Verwendung dieses Begriffs auch für die hier untersuchten Gewichte Siegelscher Modulformen nahe.

Eine Verallgemeinerung der klassischen Fourier-Jacobi-Entwicklung kann aufbauend auf Kohnens Grenzwertprozess für reell-analytische Siegelsche Eisensteinreihen für eine große Klasse von harmonischen Siegelschen Modulformen hergeleitet werden. Die herbei auftretenden Fourier-Jacobi-Entwicklungen stellen sich als Maaß-Jacobiformen heraus, die in voller Allgemeinheit in dieser Arbeit definiert werden. Wir zeigen schließlich, dass die verschiedenen ξ-Operatoren für Siegelsche Modulformen, Jacobiformen und elliptische Modulformen miteinander verträglich sind und stellen so einen Zusammenhang zwischen diesen drei Arten von Modulformen her.

CHAPTER 1

Introduction

This work aims at extending the concept of dual weights that is defined for harmonic elliptic modular forms to Siegel modular forms of degree 2. We will define harmonic Siegel modular forms and investigate the properties of two ξ-operators that relate the associated dual weights. Ultimately, we establish a connection between harmonic Siegel modular forms and harmonic Maaß-Jacobi forms, which we define for general Jacobi indices.

Siegel modular forms are modular forms for the integral symplectic group $\mathrm{Sp}_n(\mathbb{Z})$ (see **[Sie51]** for the definition of such modular forms in a more general context). The latter is the group of all integral matrices in the real symplectic group, which can be obtained as the stabilizer of the standard symplectic form

$$J^{(n)} := \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix};$$
$$\mathrm{Sp}_n(\mathbb{R}) := \big\{ g \in \mathrm{M}_{2n}(\mathbb{R}) \,:\, g^{\mathrm{T}} J^{(n)} g = J^{(n)} \big\},$$

where $\mathrm{M}_{2n}(\mathbb{R})$ is the space of $2n \times 2n$ matrices that have entries in $\mathbb{R}$. We write $\mathrm{M}^{\mathrm{T}}_n(\mathbb{R})$ for the space of symmetric $n \times n$ matrices with entries in $\mathbb{R}$. A matrix $Y \in \mathrm{M}^{\mathrm{T}}_n(\mathbb{R})$ is positive definite, $Y > 0$, if all eigenvalues of Y are positive. The Siegel upper half space

$$\mathbb{H}_n := \big\{ Z = X + iY \in \mathrm{M}^{\mathrm{T}}_n(\mathbb{C}) \,:\, Y > 0 \big\}$$

is a homogeneous space for $\mathrm{Sp}_n(\mathbb{R})$. Denoting a typical element g of $\mathrm{Sp}_n(\mathbb{R})$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathrm{M}_n(\mathbb{R})$, the action of $\mathrm{Sp}_n(\mathbb{R})$ on $\mathbb{H}_n$ is given by

$$gZ \mapsto (aZ + b)(cZ + d)^{-1}.$$

Classically, one uses the factor of automorphy

$$\boldsymbol{\alpha}^{(n)}_{k,0}(g, Z) := \det(cZ + d)^{-k} \tag{0.1}$$

to define Siegel modular forms: For $n > 1$, a (holomorphic) Siegel modular form is a holomorphic function $f : \mathbb{H}_n \to \mathbb{C}$ such that

$$\big(f|^{(n)}_k \, g \big)(Z) := \boldsymbol{\alpha}^{(n)}_{k,0}(g, Z)\, f(gZ) = f(Z)$$

for all $g \in \Gamma^{(n)} := \mathrm{Sp}_n(\mathbb{Z})$. We write $M^{(n)}_k$ for the space of such functions. This is the space of so-called classical or, equivalently, scalar-valued Siegel modular forms. We use this nomenclature to indicate that there are also vector-valued Siegel modular forms, treated, e.g., in **[Fre83, vdG08]**. The corresponding factors of automorphy originate in higher dimensional representations of $K \simeq \mathrm{U}_n(\mathbb{C})$, the stabilizer of $iI_n \in \mathbb{H}_n$ in $\mathrm{Sp}_n(\mathbb{R})$. Vector-valued modular forms will show only up indirectly in Chapter 2, and in no other place

The definition of Siegel modular forms in the case of $n = 1$, which in this case are the same as elliptic modular forms, involves an extra condition at the cusps.

Writing $e(x) := e^{2\pi i\, x}$, we require that an elliptic modular form f has a Fourier expansion of the form

$$f(\tau) = \sum_{n \geq 0} a_f(n)\, e(n\tau)$$

with $a_f(n) \in \mathbb{C}$ and $\tau = x + iy = Z \in \mathbb{H}_1$. This condition can be rephrased using a bound on the growth towards infinity:

$$|f(\tau)| < c\, y^a$$

for some $a, c \in \mathbb{R}$ as $y \to \infty$. The analogous condition in the case of $n > 1$ is satisfied automatically due to the Köcher principle.

There is a notion of harmonic elliptic modular forms, studied by Bruinier and Funke in [**BF04**]. They consider functions that vanish under the weight k hyperbolic Laplacian

$$\Delta_k := 4y^2 \partial_\tau \partial_{\overline{\tau}} - 2iky \partial_{\overline{\tau}}.$$

Since it factors, more precisely, since we have $\Delta_k = (4y^2\partial_\tau - 2iky)\partial_{\overline{\tau}}$, this notion includes holomorphic elliptic modular forms as a special case. But it does not allow, however, for many additional examples as long as the above growth condition is not relaxed. Weak harmonic Maaß forms grow, by definition, at most as fast as $c\, e^{ay}$ for some $a, c \in \mathbb{R}$. A multitude of nonholomorphic weak harmonic Maaß forms exists.

The concept of weak harmonic Maaß forms turned out to be related to the notion of mock modular forms. More specifically, mock modular forms are the holomorphic parts of harmonic weak Maaß forms. A first completely understood example, predating the discovery of the complete theory, was given by Zagier in [**HZ76**]. Zwegers [**Zwe02**] completed the mock theta functions communicated by Ramanujan in his 1913's letter to Hardy. Although many tried, only Zwegers succeeded in providing a framework for the study of these mock theta functions. He added certain simple, but nonholomorphic terms, restoring modularity, that is, invariance under so-called congruence subgroups of $\Gamma^{(1)}$. These completions later turned out to be examples of harmonic weak Maaß forms [**Zag07, Ono09**], uniting the researchers in both areas. The shadows of mock theta function are, by definition, unary theta series. To define the shadow of a harmonic Maaß form, and thus of mock modular forms, factor the Laplacian as follows:

$$\Delta_k = 4\xi_{2-k}\xi_k \quad \text{with} \quad \xi_k := y^{k-2}\overline{\partial_{\overline{\tau}}\,\cdot}.$$

The shadow of a mock modular form is the image of its completion under ξ_k. To ease the discussion, we will also call the image of a harmonic weak Maaß form under ξ_k its shadow. Clearly, the kernel of ξ_{2-k} consists of elliptic modular forms holomorphic on $\mathbb{H}_1$, and hence the shadows of weak harmonic Maaß forms are contained in $M^{(1)!}_{2-k}$, the space of weakly holomorphic elliptic modular forms of weight $2-k$. This justifies to say that k and $2-k$ are dual weights.

We have seen that in the elliptic case one can equivalently require harmonicity or impose the condition that the image under ξ_k is holomorphic on $\mathbb{H}_1$. The theory of harmonicity and the theory of ξ-operators differ, if $n \geq 2$. In Chapter 4, we will define harmonic Siegel modular forms of degree 2. There are two types of slash actions $|^{(2)}_k$ and $|^{(2),\mathrm{sk}}_k := |^{(2)}_{\frac{1}{2}, k-\frac{1}{2}}$, defined in Chapter 2, that are natural in a sense to be specified in the last section of Chapter 2. They are defined based on the factors of automorphy

$$\boldsymbol{\alpha}^{(n)}_{\alpha,\beta}(g, Z) := \det(cZ + d)^{-\alpha} \det(c\overline{Z} + d)^{-\beta}$$

with $\alpha = k$, $\beta = 0$ or $\alpha = \frac{1}{2}$, $\beta = k - \frac{1}{2}$. We will define harmonicity based on the matrix-valued Laplacian attached to those slash actions. We set

$$\begin{aligned} \Omega_k &:= -4Y(Y\partial_{\overline{Z}})^{\mathrm{T}}\partial_Z + 2ikY\partial_{\overline{Z}} \quad \text{and} \\ \Omega^{\mathrm{sk}}_k &:= -4Y(Y\partial_{\overline{Z}})^{\mathrm{T}}\partial_Z - i(2k-1)Y\partial_Z + iY\partial_{\overline{Z}}, \end{aligned}$$

where

$$\partial_Z := \begin{pmatrix} \partial_\tau & \frac{1}{2}\partial_z \\ \frac{1}{2}\partial_z & \partial_{\tau'} \end{pmatrix}, \quad \partial_{\overline{Z}} := \begin{pmatrix} \partial_{\overline{\tau}} & \frac{1}{2}\partial_{\overline{z}} \\ \frac{1}{2}\partial_{\overline{z}} & \partial_{\overline{\tau}'} \end{pmatrix}, \quad \text{and} \quad Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right).$$

The two operators differ after conjugating the second with $\det(Y)^{k-\frac{1}{2}}$ by a multiple of the identity. Thus the notion of natural slash actions is equivalent to the choice of natural eigenvalues of the trace of one of the two considered matrix-valued Laplacians. For readers with roots in the theory of automorphic representations, it is important to note that harmonicity of a function on $\mathbb{H}_2$ implies that it is an eigenfunction of all Casimir operators.

Whereas harmonicity is defined based on matrix-valued operators, the dual slash actions $|^{(2)}_k$ and $|^{(2),\mathrm{sk}}_{3-k}$, or $|^{(2),\mathrm{sk}}_k$ and $|^{(2)}_{3-k}$ are related by ξ-operators that are necessarily scalar-valued. In contrast to the elliptic case, they are order 2 operators:

$$\begin{aligned} \xi^{(2)}_k &:= -\det(Y)^{k-\frac{3}{2}} \Big(i\big(y\partial_{\overline{\tau}} + v\partial_{\overline{z}} + y'\partial_{\overline{\tau}'}\big) - 4\det(Y)\,(\partial_{\overline{\tau}}\partial_{\overline{\tau}'} - \tfrac{1}{4}\partial^2_{\overline{z}}) \Big) \quad \text{and} \\ \xi^{(2),\mathrm{sk}}_k &:= -4\det(Y)^{k-\frac{1}{2}}\,(\partial_\tau\partial_{\tau'} - \tfrac{1}{4}\partial_z^2). \end{aligned}$$

In fact, it turns out that there is no scalar-valued lowering or raising operator of order 1.

The matrix-valued Laplacian and the ξ-operators are only loosely related, a fact that originates in the more complicated representation theory of $\mathrm{U}_2(\mathbb{C}) \hookrightarrow \mathrm{Sp}_2(\mathbb{R})$. Nevertheless, in Chapter 2, we will provide a full explanation of their interaction, culminating in the statement: If $\Omega_k\, f = 0$ for $f \in C^\infty(\mathbb{H}_2)$, then $\Omega^{\mathrm{sk}}_{3-k}\, \xi^{(2)}_k f = 0$; if $\Omega^{\mathrm{sk}}_k\, f = 0$, then $\Omega_{3-k}\, \xi^{(2),\mathrm{sk}}_k f = 0$. In other words, the notions of harmonicity and dual weights presented in this work are compatible.

For many applications, it is crucial to know the Fourier expansion of Siegel modular forms. In [**BRR11a**], possible Fourier coefficients of harmonic Siegel modular forms were studied. To obtain satisfactory results a quite technical condition was imposed. In Chapter 4, we remove this condition and extend the considerations to holomorphic slash actions. We prove that for rank 2 indices T that are not negative definite and for all but two weights the space of possible Fourier coefficients is one-dimensional.

Jacobi forms are an intermediate construction between Siegel modular forms and elliptic modular forms. They are automorphic forms for the nonreductive, centrally extended real Jacobi group

$$\big(\mathrm{Sp}_n(\mathbb{R}) \ltimes \mathrm{M}_{n,N}(\mathbb{R})\big)\,\tilde{\times}\mathrm{M}^{\mathrm{T}}_n(\mathbb{R}), \tag{0.2}$$

where $\mathrm{M}_{n,N}(\mathbb{R})$ is the space of $n \times N$ matrices and, as before, $\mathrm{M}^{\mathrm{T}}_n(\mathbb{R}) \subseteq \mathrm{M}_n(\mathbb{R})$ is the subspace of symmetric matrices. Our investigation in Chapter 3 will focus on the case $n = 1$, that we need to study degree 2 Siegel modular forms.

We write $\widetilde{\mathrm{M}}_n^{\mathrm{T}}(\frac{1}{2}\mathbb{Z}) \subseteq \mathrm{M}_n^{\mathrm{T}}(\frac{1}{2}\mathbb{Z})$ for the submodule of matrices with integral diagonal entries. Every holomorphic Siegel modular form f of degree $n + N$ has a Fourier-Jacobi expansion

$$f(Z) = \sum_{L \in \widetilde{\mathrm{M}}_n^{\mathrm{T}}(\frac{1}{2}\mathbb{Z})} \phi_L(\tau, z)\, e(\mathrm{tr}(L\tau')),$$

where the ϕ_L are Jacobi forms. To actually relate harmonic Siegel modular forms and Jacobi forms we need to define harmonicity for the latter. Following the approach taken in [**Pit09, BR10**], we only impose a vanishing condition under the Jacobi Casimir operator. Since the centrally extended Jacobi group is not reductive and for reasons that are explained in Chapter 3, this condition is too weak. Semi-holomorphicity, that is, holomorphicity with respect to the elliptic variable z, is a suitable further restriction, that can be justified representation theoretically. Semi-holomorphic Jacobi forms are sufficient to extend the theory of Fourier-Jacobi expansions to harmonic Siegel modular forms. But such important examples like Zwegers's μ-function cannot be subsumed. For this reason, we suggest the notion of higher analytic orders in the Heisenberg part. Like sesquiharmonic Maaß forms defined in [**BDR11**], they are forms satisfying a relaxed vanishing condition, formulated in terms of covariant operators. Zwegers's μ-function is a Maaß-Jacobi form (with singularities) of analytic order 1 in the Heisenberg part. The investigation of these forms of higher analytic order in the Heisenberg part, as it is not needed to work with harmonic Siegel modular forms, is not pursued in this work. Certainly, further efforts should be made to clarify the role that these forms play in a general theory of harmonic Jacobi forms.

Kohnen's limit process is a generalization of the usual Fourier-Jacobi expansion of holomorphic Siegel modular forms to real-analytic Eisenstein series, which has been suggested in [**Koh93**]. Until very recently, it was not known whether it can be applied to any larger set of Siegel modular forms. We first extend Kohnen's work to Fourier-Jacobi expansions of Eisenstein series with $n = 1$ and arbitrary N. The result will, in particular, justify the definitions made in Chapter 3. Second, we extend the range of functions it can be applied to if $n = N = 1$. In [**BRR11a**], the author and his collaborators proved that, under mild assumptions, Kohnen's limit process converges for all harmonic Siegel modular forms of degree 2 that are associated to the skew slash action. We prove a version that also holds for the holomorphic slash action in Chapter 5. This enables us to state a compatibility result for all major types of modular forms defined in this work. The quintessence is that the Fourier-Jacobi expansion based on Kohnen's limit process and the ξ-operator for Siegel, Jacobi and elliptic modular forms commute.

The most noteworthy fact about harmonic Siegel modular forms is the following: While in the holomorphic case their Fourier expansions are indexed by positive definite quadratic forms, in the case of skew slash actions they seem to be mainly indexed by indefinite quadratic forms. We provide methods to study their Fourier expansion by means of Fourier-Jacobi expansions. This enables us to carry out detailed studies in the future, at least for positive weights. The situation is less satisfactory, however, in the case of negative weights. To obtain results this work restricted to investigations of harmonic Siegel modular forms with moderate growth. Since Kohnen's limit process leads to harmonic Maaß-Jacobi forms of negative weight that have moderate growth, we do not expect many examples apart from the Eisenstein series, defined in Chapter 4. Clearly, Kohnen's limit process cannot be trivially extended to Siegel modular forms with exponential growth, since it depends on taking a limit towards infinity. The author's future effort will concentrate on investigating this more delicate situation, and he will also aim at providing constructions for Siegel modular forms of skew weight. Only with these example at hand one can finally decide how useful this newly emerging theory is.

Work by the author in the joint publications [CR11] and [BRR11a]

This thesis is partially cumulative. In order to meet the university's requirements, we will discuss in detail which parts of this work originate in which preprint, and which parts had not been written up before this thesis was written. In general, results and even some formulations were adopted without changes from [**CR11**] and [**BRR11a**]. After this thesis was completed, results given in Section 3 of Chapter 4 were partially added to [**BRR11a**].

Chapter 2 is solely due to the author, although Section 1, which revisits known theories, adopts great parts of [**BCR07, BCR11**], varying the formulations only slightly when appropriate. The representation theoretic interpretation of the matrix-valued Laplace operator, presented in Section 4 was already given in a preliminary version of [**BRR11a**], but all results are due to the author.

Chapter 3 is almost completely based on Section 2 to 4 of [**CR11**]. The later work, written jointly with Charles Conley, can be easily divided into three parts. While Section 5 was written completely by Charles Conley, Section 3 and 4 are the author's work. Section 2 of [**CR11**] is the result of truly joint work. The Casimir operator was investigated by Charles Conley. The generators for the algebra of all covariant differential operators were first given by the author, and the actual statements given in [**CR11**], including their relations as well, were then proved by Charles Conley. The definition of harmonic Jacobi forms was given by the author and so was the remark relating them to automorphic representations. We only cite [**CR11**, Section 5], whereas we reproduce all other parts of [**CR11**]. Section 6 of Chapter 3 is completely new. The Jacobi skew slash action has not been dealt with in [**CR11**], either, but a special case was introduced in [**BRR11a**]. The observation that made necessary the introduction of skew Maaß-Jacobi forms in [**BRR11a**] can be attributed to a joint effort of Olav Richter and the author during a lively discussion.

Chapter 4 and 5 are based on [**BRR11a**], but have been largely extended. In particular, the holomorphic slash action has not been dealt with before. The results on the Fourier expansions of harmonic Siegel modular forms were much weaker in the preliminary version of [**BRR11a**]. The idea to define a space of harmonic Siegel modular forms based on the matrix-valued Laplace operator emerged immediately after the author had provided the representation theoretic interpretation of its covariance and after Olav Richter pointed out to the author that Maaß had already obtained results on the Fourier expansion of what we call harmonic functions on $\mathbb{H}_2$. One should mention that already at least two years ago Özlem Imamoğlu speculated that the matrix-valued Laplace operator "should play some role". The aim of [**BRR11a**] was to prove convergence of Kohnen's limit process for a reasonable space of real-analytic Siegel modular forms. The strategy to analyze the Fourier expansion of harmonic Siegel modular forms and to prove that only those that already occur in the Fourier expansion of Poincaré-Eisenstein series contribute is due to the author. So are the investigations of harmonic Fourier expansions and the Fourier expansions of harmonic Siegel modular forms contained in Section 2 and Section 3 of Chapter 4. The proof of Theorem 2.5 in Chapter 5 depends on a brilliant idea by Olav Richter, who suggested to restrict to functions, that "lie above holomorphic ones", that is, $\xi_k^{(2),\mathrm{sk}} f \in M_{3-k}^{(2)}$. A reinterpretation of this restriction in terms of the support of the Fourier expansion of f that reveals how deeply they are connected to properties of Fourier indices was given by the author. It led to the definition of $M_k^{(2),\mathrm{sk}}$, which is essential to the generalization of Theorem 2.5 to holomorphic slash actions. The results in Section 1 and 3 of Chapter 5, unless they are marked as citations, are completely due to the author.

CHAPTER 2

Invariant and covariant differential operators

In this chapter, we discuss invariant and covariant differential operators for the symplectic group. As is well-known, the Siegel upper half space $\mathbb{H}_n$ is isomorphic to the quotient of $\mathrm{Sp}_n(\mathbb{R})$ by K the stabilizer of iI_n, which is a compact subgroup isomorphic to $\mathrm{U}_n(\mathbb{R})$, as an $\mathrm{Sp}_n(\mathbb{R})$-homogeneous space. The isomorphism between K and $\mathrm{U}_n(\mathbb{R})$ is given by the map

$$\mathrm{U}_n(\mathbb{R}) \ni a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{0.3}$$

This structure can be used to interpret any Siegel modular form or, more generally, any function on $\mathbb{H}_n$ as a section of an $\mathrm{Sp}_n(\mathbb{R})$-bundle $\mathrm{Sp}_n(\mathbb{R}) \times_K V$ for some K-module V. We will use the theory of differential operators for such bundles to compute invariant differential operators for Siegel modular forms of genus 2. Helgason's survey [**Hel77**] is a good reference for the concepts used in this chapter. It contains a discussion of most topics that play a roll in the studies of classical aspects of automorphic forms.

We first revise the theory well-known to representation theorists. Section 2 and 3 contain computations special to $\mathrm{Sp}_n(\mathbb{R})$ and its Lie-algebra, performed in preparation for the considerations in the subsequent section. Several types of covariant operators, which we will need later, are introduced in Section 4. In the last section, we will discuss natural slash actions. Based on the degeneration of the so-called matrix-valued Laplace operator, we relate covariant operators for $\mathrm{Sp}_2(\mathbb{R})$ and those for the centrally extended Jacobi group, defined in Chapter 3.

1. Differential operators for Lie groups

The way we present the general theory of differential operators in this section is largely based on Helgason's work [**Hel77**] and two articles by Bringmann, Conley and Richter [**BCR07, BCR11**]. Since an introduction as clear as in the last two articles is available nowhere else, we have adopted it with minor modifications only.

For the time being, fix a real Lie group G, a closed subgroup K and a complex, finite dimensional K-module (σ_V, V). We will usually omit σ_V when referring to the action of K on V. We write $[g, v] = [gk^{-1}, kv]$ for the elements of the complex G-vector bundle $G \times_K V$. This bundle can be interpreted as a G-bundle over the homogeneous space G/K with projection $G \times_K V \to G/K,\ [g, v] \mapsto gK$. The structure as a G-bundle is given by $(g', [g, v]) \mapsto [g'g, v]$.

We denote the space of smooth sections of $G \times_K V$ by $C^\infty(G/K, G \times_K V)$. With another complex, finite dimensional K-module (σ_W, W) we want to describe smooth and covariant differential operators from $C^\infty(G/K, G \times_K V)$ to $C^\infty(G/K, G \times_K W)$.

DEFINITION 1.1. *A differential operator T from $C^\infty(G/K, G\times_K V)$ to $C^\infty(G/K, G\times_K W)$ is called* covariant *if*

$$T(gf) = gT(f)$$

for all $g \in G$, where $(gf)(h) = f(g^{-1}h)$.

The space of such operators will be denoted by $\mathbb{D}(G/K, V, W)$.

The space of smooth differential operators from $C^\infty(G/K,\, G\times_K V)$ to $C^\infty(G/K,\, G\times_K W)$ form themselves a space of smooth sections of a vector bundle over G/K. To define this bundle, we denote the (real) Lie algebra of G by $\mathfrak{g}_0$ and its complexification by $\mathfrak{g}$. The corresponding Lie algebras for K are denoted by $\mathfrak{k}_0$ and $\mathfrak{k}$. We write $U(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$. This algebra is filtered by the degree of its elements, and we write $U(\mathfrak{g})_d$ for the corresponding (finite dimensional) spaces. The following space will serve as differential operators at $\mathrm{id} \in G$:

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{k}} V := (\mathfrak{U}(\mathfrak{g}) \otimes V)/\big\langle ZY \otimes v - Z \otimes Yv \,:\, Z \in \mathfrak{U}(\mathfrak{g}), Y \in \mathfrak{k}, v \in V\big\rangle.$$

Under left multiplication this space is a $\mathfrak{g}$-module. The restriction of this module structure to $\mathfrak{k}$ yields a filtered $\mathfrak{k}$-module, that thus arises from a filtered K-algebra.

Central to our investigation are the following proposition and its corollary. In order to state it, let V^* denote the dual of a G-module V.

PROPOSITION 1.2 ([**BCR11**, Proposition 4.1]). *For any two complex finite dimensional representations V and W of K, there is a G-covariant linear isomorphism from the space of sections*

$$C^\infty\big(G/K,\, G \times_K \big(W \otimes (\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{k}} V^*)\big)\big)$$

to the space of smooth W-valued differential operators on $G \times_K V$. It carries the degree filtration of $\mathfrak{U}(\mathfrak{g})$ to the order filtration of the differential operators, and it respects composition up to symbol.

COROLLARY 1.3 ([**BCR11**, Corollary 4.2]). *There is a linear isomorphism from*

$$\big(W \otimes (\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{k}} V^*)\big)^K$$

to $\mathbb{D}(G/K, V, W)$. It carries the degree filtration of $\mathfrak{U}(\mathfrak{g})$ to the order filtration of $\mathbb{D}(G/K, V, W)$ and respects composition up to symbol.

For a real Lie group G as above with closed subgroup K the homogeneous space G/K is called hermitian, if it admits a complex structure such that G acts by holomorphic maps. This is the case for $G = \mathrm{Sp}_n(\mathbb{R})$ and the corresponding $K = \mathrm{U}_n(\mathbb{R})$. We write $\mathfrak{c}$ for the center of $\mathfrak{k}$. An argument by Harish-Chandra shows that G/K is hermitian if and only if the centralizer $Z_{\mathfrak{g}}(\mathfrak{c})$ equals $\mathfrak{k}$. In this case, we have a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ of the complexified Lie algebra. Since $\mathfrak{p} := \mathfrak{p}^+ \oplus \mathfrak{p}^-$ is $\mathfrak{k}$-invariant, this provides a $\mathfrak{k}$-splitting of $\mathfrak{g}$. Further, if K is connected, the splitting is K-invariant. In the light of this fact, the next corollary is of outstanding importance to our investigation. We write $\mathcal{S}(\mathfrak{p})$ for the symmetric algebra of the $\mathfrak{k}$-module $\mathfrak{p}$ to state it.

COROLLARY 1.4 ([**BCR11**, Corollary 4.3]). *Suppose that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a K-splitting. Then there is a linear isomorphism form $\big(\mathcal{S}(\mathfrak{p}) \otimes V^* \otimes W\big)^K$ to*

$\mathbb{D}(G/K,V,W)$ which carries the degree filtration of $\mathcal{D}(\mathfrak{p})$ to the order filtration of $\mathbb{D}(G/K,V,W)$ and respects composition up to symbol.

Further, if K is connected, then

$$\big(\mathcal{S}(\mathfrak{p})\otimes V^*\otimes W\big)^K = \big(\mathcal{S}(\mathfrak{p})\otimes V^*\otimes W\big)^{\mathfrak{k}}.$$

2. Invariants in the case $\mathfrak{g}=\mathfrak{sp}_2$

To apply Corollary 1.4, we need only calculate invariant vectors in the $\mathfrak{k}$-module $\mathcal{S}(\mathfrak{p})\otimes V^*\otimes W$. We will need the corresponding differential operators in the case $G=\mathrm{Sp}_2(\mathbb{R})$. Thus we assume that $\mathfrak{g}=\mathfrak{sp}_2$ throughout the rest of this chapter. The precise structure and the decomposition of $\mathfrak{g}$ is given in the next proposition.

PROPOSITION 2.1. *We have $\mathfrak{sp}_2=\mathfrak{k}\oplus\mathfrak{p}$ with*

$$\begin{aligned}\mathfrak{k} &= \left\{\begin{pmatrix}A & B\\ -B & A\end{pmatrix} : A \text{ skew symmetric, } B \text{ symmetric}\right\}\\ &\simeq \mathbb{C}\oplus\mathfrak{sl}_2,\\ \mathfrak{p} &= \left\{\begin{pmatrix}A & B\\ B & -A\end{pmatrix} : A, B \text{ symmetric}\right\}.\end{aligned}$$

The center $\mathfrak{c}$ of $\mathfrak{k}$ is spanned by $h_{\mathfrak{c}} := \left(\begin{smallmatrix} & -I_2\\ I_2 & \end{smallmatrix}\right)$, and the Lie subalgebra of $\mathfrak{k}$ which is isomorphic to $\mathfrak{sl}_2$ is spanned by

$$e_{\mathfrak{k}} := \frac{1}{2}\left(\begin{array}{cc|cc} & i & & 1\\ -i & & 1 & \\ \hline & -1 & & i\\ -1 & & -i & \end{array}\right),\qquad h_{\mathfrak{k}} := \left(\begin{array}{cc|cc} & & -i & \\ & & & i\\ \hline i & & & \\ & -i & & \end{array}\right)\qquad \text{and}$$

$$f_{\mathfrak{k}} := \frac{1}{2}\left(\begin{array}{cc|cc} & i & & -1\\ -i & & -1 & \\ \hline & 1 & & i\\ 1 & & -i & \end{array}\right).$$

The commutation relations are $[e_{\mathfrak{k}},f_{\mathfrak{k}}]=h_{\mathfrak{k}}$, $[h_{\mathfrak{k}},e_{\mathfrak{k}}]=2e_{\mathfrak{k}}$, and $[h_{\mathfrak{k}},f_{\mathfrak{k}}]=-2f_{\mathfrak{k}}$.

PROOF. The decomposition of the Lie algebra can be easily verified. To see that $\exp(\mathfrak{k})$ generates the subgroup $\mathrm{U}_2(\mathbb{R})\simeq K\subset\mathrm{Sp}_2(\mathbb{R})$, it is sufficient to note that $\exp(t\,e_{\mathfrak{k}})$, $\exp(t\,h_{\mathfrak{k}})$, and $\exp(t\,f_{\mathfrak{k}})$ are elements of K for all $t\in\mathbb{R}$, which is immediate. $\square$

We will write $L_l(k)$ for one fixed $(l+1)$-dimensional, irreducible $\mathfrak{k}$-module that $h_{\mathfrak{c}}$ acts on by multiplication with $-2i\,k$.

PROPOSITION 2.2. *The complexified Lie algebra $\mathfrak{sp}_2$ admits a decomposition*

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}^+\oplus\mathfrak{p}^-.$$

The positive part $\mathfrak{p}^+$ is isomorphic to $L_2(1)$ as a $\mathfrak{k}$-module, and it is spanned by

$$e_{\mathfrak{p}}^+ := \left(\begin{array}{cc|cc} & & & \\ & 1 & & -i \\ \hline & & & \\ & -i & & -1 \end{array}\right), \qquad h_{\mathfrak{p}}^+ := \left(\begin{array}{cc|cc} & -i & & -1 \\ -i & & -1 & \\ \hline & -1 & & i \\ -1 & & i & \end{array}\right),$$

$$f_{\mathfrak{p}}^+ := \left(\begin{array}{cc|cc} 1 & & -i & \\ & & & \\ \hline -i & & -1 & \\ & & & \end{array}\right).$$

The action of $\mathfrak{sl}_2 \subset \mathfrak{k}$ is given by

$$\begin{aligned} &[e_{\mathfrak{k}}, e_{\mathfrak{p}}^+] = 0, && [e_{\mathfrak{k}}, h_{\mathfrak{p}}^+] = -2e_{\mathfrak{p}}^+, && [e_{\mathfrak{k}}, f_{\mathfrak{p}}^+] = h_{\mathfrak{p}}^+, \\ &[h_{\mathfrak{k}}, e_{\mathfrak{p}}^+] = 2e_{\mathfrak{p}}^+, && [h_{\mathfrak{k}}, e_{\mathfrak{p}}^+] = 0, && [h_{\mathfrak{k}}, f_{\mathfrak{p}}^+] = -2f_{\mathfrak{p}}^+, \\ &[f_{\mathfrak{k}}, e_{\mathfrak{p}}^+] = -h_{\mathfrak{p}}^+, && [f_{\mathfrak{k}}, h_{\mathfrak{p}}^+] = 2f_{\mathfrak{p}}^+, && [f_{\mathfrak{k}}, f_{\mathfrak{p}}^+] = 0. \end{aligned}$$

The negative part $\mathfrak{p}^-$, which is isomorphic to $L_2(-1)$, is spanned by

$$e_{\mathfrak{p}}^- := \left(\begin{array}{cc|cc} 1 & & i & \\ & & & \\ \hline i & & -1 & \\ & & & \end{array}\right), \qquad h_{\mathfrak{p}}^- := \left(\begin{array}{cc|cc} & i & & -1 \\ i & & -1 & \\ \hline & -1 & & -i \\ -1 & & -i & \end{array}\right),$$

$$f_{\mathfrak{p}}^- := \left(\begin{array}{cc|cc} & & & \\ & 1 & & i \\ \hline & & & \\ & i & & -1 \end{array}\right).$$

The action of $\mathfrak{sl}_2 \subset \mathfrak{k}$ on $\mathfrak{p}^-$ is given by the same relations as above, with the superscript $+$ replaced by $-$.

Proof. A direct verification gives the generators and their relations. Since $\mathrm{Z}_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$, there is a decomposition $\mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ of $\mathfrak{g}$. Because $\mathfrak{p}^\pm$ are irreducible as $\mathfrak{k}$-modules, this decomposition coincides with the decomposition in the statement after a suitable choice of real, positive roots. $\square$

Based on Proposition 2.1 and 2.2, we can investigate the $\mathfrak{k}$-invariants of $\mathcal{S}(\mathfrak{p}) \otimes V^* \otimes W$. We will not compute the corresponding differential operators using Corollary 1.4. Instead, we will prove several uniqueness results, postponing the construction of the covariant operators to Section 4. Recall that $\mathbb{D}(G/K, V, W)^d$ is the space of order d covariant operators from $C^\infty(G/K, G \times_K V)$ to $C^\infty(G/K, G \times_K W)$.

Proposition 2.3. *Suppose that $k \in \mathbb{Z}$. The spaces*

$$\begin{aligned} &\mathbb{D}(G/K, L_0(k), L_2(k))^2, \\ &\mathbb{D}(G/K, L_0(k), L_2(k \pm 1))^2, \\ &\mathbb{D}(G/K, L_2(k), L_0(k \pm 1))^2, \quad \textit{and} \\ &\mathbb{D}(G/K, L_0(k), L_2(k \pm 2))^4 \end{aligned}$$

are one-dimensional.

PROOF. We denote the d^{th} symmetric power of $\mathfrak{p}$ by $\mathcal{S}^d(\mathfrak{p}) \subset \mathcal{S}(\mathfrak{p})$. By Corollary 1.4 it suffices to prove that $(\mathcal{S}^d(\mathfrak{p}) \otimes V^* \otimes W)^{\mathfrak{k}}$ is one-dimensional, where (d, V, W) is $\big(2, L_0(k), L_2(k)\big)$, $\big(2, L_0(k), L_0(k \pm 1)\big)$, $\big(2, L_2(k), L_0(k \pm 1)\big)$, or $\big(4, L_0(k), L_2(k \pm 2)\big)$. In each case one can prove along the same line that $(\mathcal{S}^{\tilde{d}}(\mathfrak{p}) \otimes V^* \otimes W)^{\mathfrak{k}}$ is trivial if $\tilde{d} < d$.

We will use the Clebsch-Gordon formulas [**GW09**]:

$$\begin{aligned} \mathcal{S}^n(L_2) &\simeq L_{2n} \oplus L_{2n-4} \oplus \cdots \oplus L_{2\mathrm{res}_2(n)} \quad \text{and} \\ L_n \otimes L_m &\simeq L_{n+m} \oplus L_{n+m-2} \oplus \cdots \oplus L_{|n-m|}, \end{aligned} \tag{2.1}$$

where $\mathrm{res}_2(n)$ is the residue 0 or 1 of n modulo 2. The first isomorphism gives

$$\begin{aligned} \mathcal{S}^2(\mathfrak{p}) &\simeq \mathcal{S}^2\big(L_2(-1) \oplus L_2(1)\big) \\ &\simeq L_4(2) \oplus L_0(2) \oplus L_4(0) \oplus L_2(0) \oplus L_0(0) \oplus L_4(-2) \oplus L_0(-2). \end{aligned}$$

On the other hand, we have

$$L_0(-k) \otimes L_2(k) \simeq L_2(0).$$

Hence $L_0(0)$ has multiplicity one in $\mathcal{S}^2(\mathfrak{p}) \otimes L_0(-k) \otimes L_2(k)$, that is, the space of $\mathfrak{k}$-invariants has dimension one.

The computations for second and third case are similar. In the fourth case, the second and third factor of $\mathcal{S}^4(\mathfrak{p}) \otimes L_0(-k) \otimes L_2(k \pm 2)$ simplify to

$$L_0(-k) \otimes L_2(k \pm 2) \simeq L_2(\pm 2).$$

Thus it is sufficient to compute the multiplicities of modules $L_l(\mp 2)$, $l \in \mathbb{Z}_{\geq 0}$ in $\mathcal{S}^4(\mathfrak{p})$. The corresponding submodule is

$$\begin{aligned} \mathcal{S}^4(L_2(1) \oplus L_2(-1)) \supset \mathcal{S}^3(L_2(\pm 1)) \otimes L_2(\mp 1) \\ \simeq \big(L_6(\pm 3) \oplus L_2(\pm 3)\big) \otimes L_2(\mp 1). \end{aligned}$$

By (2.1), the tensor product with $L_2(\pm 2)$ contains as many copies of $L_0(0)$ as there are copies of $L_2(\mp 2)$ in the above module. Since the $\mathfrak{sl}_2$-module $L_6 \otimes L_2$ does not contain L_2 and since $L_2 \otimes L_2$ contains exactly one copy of L_2, the fourth case is proved. $\square$

3. Cocycles for $\mathrm{Sp}_n(\mathbb{R})$

Cocycles of $\mathrm{Sp}_n(\mathbb{R})$ are functions $\boldsymbol{\alpha} : \mathrm{Sp}_n(\mathbb{R}) \times \mathbb{H}_n \to \mathrm{GL}_l(\mathbb{R})$ that satisfy $\boldsymbol{\alpha}(gg', \tau) = \boldsymbol{\alpha}(g, g'\tau) \cdot \boldsymbol{\alpha}(g', \tau)$. Any such cocycle defines a representation of $K \subseteq \mathrm{Sp}_n(\mathbb{R})$, and we will say that two cocycles are equivalent if these representations are isomorphic. A cocycle defines an $\mathrm{Sp}_n(\mathbb{R})$-vector bundle on any quotient of $\mathbb{H}_n$ by a discrete subgroup of $\mathrm{Sp}_n(\mathbb{R})$.

We will give a family of scalar cocycles for $\mathrm{Sp}_n(\mathbb{R})$, and for $\mathrm{Sp}_2(\mathbb{R})$ we will give additional noncommutative cocycles. The former correspond to line bundles over $\mathrm{Sp}_n(\mathbb{R})/K$, whereas the latter originate in higher dimensional representations of K.

Recall that the structure of $\mathbb{H}_n$ as an $\mathrm{Sp}_n(\mathbb{R})$-homogeneous space is given by Möbius transformations

$$gZ = (aZ + b)(cZ + d)^{-1},$$

with $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$.

For $\alpha, \beta \in \mathbb{C}$ with $\alpha - \beta \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$ set

$$\boldsymbol{\alpha}^{(2)}_{\alpha,\beta;l}(g, \tau) := \det(c\tau + d)^{-\alpha} \det(c\overline{\tau} + d)^{-\beta} \cdot \sigma_l((c\tau + d)^{-\mathrm{T}}), \tag{3.1}$$

where σ_l is the natural representation on the symmetric power $\mathcal{S}^l(\mathbb{C}^2)$. We will write $\boldsymbol{\alpha}^{(2)}_{\alpha,\beta}$ for $\boldsymbol{\alpha}^{(2)}_{\alpha,\beta;0}$. The next proposition shows that this family exhausts the cocycles of $\mathrm{Sp}_2(\mathbb{R})/K$ up to equivalence.

PROPOSITION 3.1. *The K-representation $\boldsymbol{\alpha}^{(2)}_{\alpha,\beta;l}(\cdot, iI_2)$ corresponds to the $\mathfrak{k}$-module $L_l(\alpha-\beta+\frac{1}{2}l)$ defined in Section 2.*

PROOF. The representation $\boldsymbol{\alpha}^{(2)}_{\alpha,\beta;l}(\cdot, iI_2) \,:\, K \rightarrow \mathrm{GL}_{l+1}(\mathbb{C})$ is irreducible. Thus the induced $\mathfrak{k}$-module is isomorphic to $L_l(\tilde{k})$ for some $\tilde{k}$. To understand the action of the center $\mathfrak{c}$ of $\mathfrak{k}$ it suffices to consider the derivative of the action of $\exp(t\,h_{\mathfrak{c}})$ with $h_{\mathfrak{c}}$ defined in Proposition 2.1. For $v \in \mathbb{C}^{l+1}$, we find

$$\partial_t \sigma_l\Big(\begin{pmatrix} e^{it} & \\ & e^{it} \end{pmatrix}\Big) v\Big|_{t=0} = -il\, v \quad \text{and}$$

$$\partial_t \det \begin{pmatrix} e^{it} & \\ & e^{it} \end{pmatrix}^{-\alpha} \det \begin{pmatrix} e^{-it} & \\ & e^{-it} \end{pmatrix}^{-\beta} v\Big|_{t=0} = -2i(\alpha-\beta)\, v.$$

This proves the claim. □

COROLLARY 3.2. *Every irreducible cocycle of $\mathrm{Sp}_2(\mathbb{R})/K$ is equivalent to $\alpha^{(2)}_{k,0;l}$ for exactly one choice of $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$.*

PROOF. The center of $K \simeq \mathrm{U}_2(\mathbb{R})$ is S^1, and hence its irreducible cocycles are indexed by integers, that correspond to indices $k \in \frac{1}{2}\mathbb{Z}$. The representations with $k \notin \mathbb{Z}$ do not extend to cocycles for the whole symplectic group. This can be seen by considering the image of $\left(\begin{smallmatrix} e^{it} & \\ & 1 \end{smallmatrix}\right)$ at iI_2. The vector-valued weights l correspond to the irreducible representations of $\mathrm{SU}_2(\mathbb{C})$. □

In analogy to the family $\alpha^{(2)}_{\alpha,\beta;l}$ of cocycles, we define a family of slash actions $|^{(2)}_{\alpha,\beta;l}$ for $\mathrm{Sp}_2(\mathbb{R})$ on functions $\mathbb{H} \rightarrow \mathcal{S}^l(\mathbb{C}^2)$. The indices α and β run through $\mathbb{C}$ with $\alpha - \beta \in \mathbb{Z}$, and l runs through $\mathbb{Z}_{\geq 0}$ as before.

$$\big(f|^{(2)}_{\alpha,\beta;l}g\big)(Z) = \det(cZ+d)^{-\alpha} \det(c\overline{Z}+d)^{-\beta} \sigma_l\big((cZ+d)^{-\mathrm{T}}\big) f(gZ). \tag{3.2}$$

If $\beta = 0$ or $l = 0$, we suppress the second or third index. The slash actions $|^{(2)}_{\alpha,\beta;l}$ and $|^{(2)}_{\alpha-\beta;l}$ are equivalent, and this equivalence is realized by multiplication with $\det(Y)^{\beta}$. We will call k the scalar weight of $|^{(2)}_{k;l}$ and l its vector-valued weight. The slash action $|^{(2)}_{\frac{1}{2},k-\frac{1}{2}}$, that will play an important role in Chapter 4, will be denoted by $|^{(2),\mathrm{sk}}_{k}$. It is the weight k skew slash action. Care must be taken with this notion, since the representation theoretic weight of the weight k skew slash action is $1-k$.

A family of scalar cocycles for $\mathrm{Sp}_n(\mathbb{R})$ is given by

$$\boldsymbol{\alpha}^{(n)}_{\alpha,\beta}(g, Z) = \det(cZ+d)^{-\alpha} \det(c\overline{Z}+d)^{-\beta}. \tag{3.3}$$

PROPOSITION 3.3. *Every scalar cocycle of $\mathrm{Sp}_n(\mathbb{R})/K$ is equivalent to $\boldsymbol{\alpha}^{(n)}_{k,0}$ for exactly one $k \in \mathbb{Z}$.*

PROOF. The center of $K \simeq \mathrm{U}_n(\mathbb{R})$ is S^1, and thus its representations are indexed by $k \in \frac{1}{n}\mathbb{Z}$. Only representations with $k \in \mathbb{Z}$ extend to cocycles for the whole symplectic group, yielding the claim. □

REMARK 3.4. *For general n, the representations of $\mathrm{GL}_n(\mathbb{C})$ give rise to the cocycles of $\mathrm{Sp}_n(\mathbb{R})/K$. The reader is referred to* [**GW09**] *for the representation theory of the general linear group.*

The slash actions corresponding to $\boldsymbol{\alpha}^{(n)}_{\alpha,\beta}$ will be denoted by $|^{(n)}_{\alpha,\beta}$. The equivalence of $|^{(n)}_{\alpha,\beta}$ and $|^{(n)}_{\alpha-\beta,0}$ is induced by multiplication with $\det(Y)^{\beta}$.

4. Covariant differential operators on $\mathbb{H}_2$

We will deduce expressions for covariant operators on $\mathbb{H}_2$ with respect to the slash actions defined in the preceding section.

DEFINITION 4.1. *A differential operator T on $\mathbb{H}_n$ is* covariant from $|^{(n)}_{\alpha,\beta}$ to $|^{(n)}_{\alpha',\beta'}$ *if for all $g \in \mathrm{Sp}_n(\mathbb{R})$ and $f \in C^\infty(\mathbb{H}_n)$, we have*

$$T\big(f|^{(n)}_{\alpha,\beta}\, g\big) = (Tf)\big|^{(n)}_{\alpha',\beta'}\, g.$$

Similarly, a differential operator T on $\mathbb{H}_2$ is covariant from $|^{(2)}_{\alpha,\beta;l}$ to $|^{(2)}_{\alpha',\beta';l'}$ *if for all $g \in \mathrm{Sp}_2(\mathbb{R})$ and $f \in C^\infty(\mathbb{H}_2 \to \mathcal{S}^l(\mathbb{C}^2))$, we have*

$$T\big(f|^{(2)}_{\alpha,\beta;l}\, g\big) = (Tf)\big|^{(2)}_{\alpha',\beta';l'}\, g.$$

We call a covariant operator invariant, if the slash action of its domain and codomain coincide.

We will only treat differential operators with values in $\mathbb{C}$ or $\mathcal{S}^2(\mathbb{C}^2)$. As a model for the second space we choose $\mathrm{M}_2^{\mathrm{T}}(\mathbb{C})$ and the action of $\mathrm{GL}_2(\mathbb{C})$ on this space will be given by

$$(g, v) \mapsto g v g^{\mathrm{T}}.$$

The next theorem is central to the theory of invariant operators on $\mathbb{H}_n$. A detailed proof by means of analytic methods can be found in [**Maa71**, Chapter 8]. In Maaß's book, the reader can also find an explicit set of generators.

THEOREM 4.2. *The algebra of $\mathrm{Sp}_n(\mathbb{R})$-invariant differential operators on scalar-valued functions on $\mathbb{H}_n$ is generated by n elements of degrees $2i$ for $1 \le i \le n$.*

We will give the generators of the algebra of invariant differential operators in the case $n = 2$. Define

$$\partial_Z := \begin{pmatrix} \partial_\tau & \frac{1}{2}\partial_z \\ \frac{1}{2}\partial_z & \partial_{\tau'} \end{pmatrix} \quad \text{and} \quad \partial_{\overline{Z}} := \begin{pmatrix} \partial_{\overline{\tau}} & \frac{1}{2}\partial_{\overline{z}} \\ \frac{1}{2}\partial_{\overline{z}} & \partial_{\overline{\tau}'} \end{pmatrix}.$$

In what follows, we will multiply these matrices. The corresponding product is the natural product coming from composition of operators. Maaß defines

$$\Lambda_\beta := -\beta I_2 + 2iY\partial_{\overline{Z}}, \quad K_\alpha := \alpha I_2 + 2iY\partial_Z, \quad \text{and} \quad A^{(1)}_{\alpha,\beta} = \Lambda_{\beta-\frac{3}{2}} K_\alpha.$$

As a special case of the main theorem in [**Maa71**, Chapter 8], we formulate

THEOREM 4.3. *The differential operators*

$$H_1^{(\alpha,\beta)} := \mathrm{tr}\big(A^{(1)}_{\alpha,\beta}\big) \qquad \textit{and} \tag{4.1}$$

$$H_2^{(\alpha,\beta)} := \mathrm{tr}\big(A^{(1)}_{\alpha,\beta}\, A^{(1)}_{\alpha,\beta}\big) - \mathrm{tr}\big(\Lambda_\beta\, A^{(1)}_{\alpha,\beta}\big) + \tfrac{1}{2}\mathrm{tr}\big(\Lambda_\beta\big)\mathrm{tr}\big(A^{(1)}_{\alpha,\beta}\big), \tag{4.2}$$

are invariant for the slash action $|^{(2)}_{\alpha,\beta}$. They generate the algebra of $|^{(2)}_{\alpha,\beta}$-invariant differential operators on $\mathbb{H}_2$.

There is a further, covariant operator, which Maaß introduced in [**Maa53**]:

$$\begin{aligned}\Omega_{\alpha,\beta} &:= \Lambda_{\beta-\frac{3}{2}} K_\alpha + \alpha(\beta - \tfrac{3}{2}) I_2 \\ &= -4Y(Y\partial_{\overline{Z}})^{\mathrm{T}}\partial_Z - 2i\beta Y\partial_Z + 2i\alpha Y\partial_{\overline{Z}}.\end{aligned} \tag{4.3}$$

Maaß called this operator the vector-valued Laplace operator. To avoid confusion with the covariant operators for vector-valued slash actions, we will call it the matrix-valued Laplace operator. If $\beta = 0$ we will suppress the second index. We will write Ω^{sk}_k for $\Omega_{\frac{1}{2},k-\frac{1}{2}}$. In order to state the covariance of $\Omega_{\alpha,\beta}$, we need the following slash action for functions $f : \mathbb{H}_2 \rightarrow \mathrm{M}_2(\mathbb{C})$:

$$\big(f|^{(\mathrm{M})}_{\alpha,\beta}\, g\big)(Z) = \det(cZ+d)^{-\alpha} \det(c\overline{Z}+d)^{-\beta}\; (cZ+d)^{-\mathrm{T}} f(gZ)\, (cZ+d)^{\mathrm{T}}.$$

In his book, Maaß gave a clear proof of the covariance properties of this operators.

Theorem 4.4 ([**Maa71**, Chapter 8]). *The operator $\Omega_{\alpha,\beta}$ is covariant from $|_{\alpha,\beta}$ to $|^{(\mathrm{M})}_{\alpha,\beta}$.*

To understand the operator $\Omega_{\alpha,\beta}$ in terms of modern, representation theoretic language, we need the next proposition.

Proposition 4.5. *The cocycle associated to $|^{(\mathrm{M})}_{\alpha,\beta}$ is equivalent to the direct sum*

$$\boldsymbol{\alpha}_{\alpha-\beta;0} \oplus \boldsymbol{\alpha}_{\alpha-\beta-1;2}.$$

Proof. We need to analyze the action of $h_{\mathfrak{c}}$ and $h_{\mathfrak{k}}$ defined in Proposition 2.1. For $v \in \mathrm{M}_2(\mathbb{C})$ and $Z = iI_2$, we find

$$\begin{aligned}&\partial_t\; v|^{(\mathrm{M})}_{\alpha,\beta} \exp(t\, h_{\mathfrak{c}})\Big|_{t=0} \\ &= \partial_t\; \det\begin{pmatrix} e^{-it} & \\ & e^{-it}\end{pmatrix}^{-\alpha} \det\begin{pmatrix} e^{it} & \\ & e^{it}\end{pmatrix}^{-\beta} \begin{pmatrix} e^{-it} & \\ & e^{-it}\end{pmatrix} v \begin{pmatrix} e^{it} & \\ & e^{it}\end{pmatrix}\Big|_{t=0} \\ &= -2i(\alpha-\beta)\, v\end{aligned}$$

and

$$\begin{aligned}&i\,\partial_t\; v|^{(\mathrm{M})}_{\alpha,\beta} \exp(-it\, h_{\mathfrak{k}})\Big|_{t=0} \\ &= i\,\partial_t\; \det\begin{pmatrix} e^{it} & \\ & e^{-it}\end{pmatrix}^{-\alpha} \det\begin{pmatrix} e^{-it} & \\ & e^{it}\end{pmatrix}^{-\beta} \begin{pmatrix} e^{-it} & \\ & e^{it}\end{pmatrix} v \begin{pmatrix} e^{it} & \\ & e^{-it}\end{pmatrix}\Big|_{t=0} \\ &= \begin{pmatrix} & 2v_{12} \\ -2v_{21} & \end{pmatrix}.\end{aligned}$$

From the second equality, we deduce that the representation at $Z = iI_2$ is the direct sum of a one-dimensional and a 3-dimensional irreducible representation. Using the first equation, we conclude that the scalar weights are the desired ones. ☐

The next proposition makes a connection between the matrix-valued Laplace operator and the invariant operators for $|_{\alpha,\beta}$.

Proposition 4.6 ([**Maa53**]). *If $f : \mathbb{H}_2 \rightarrow \mathbb{C}$ satisfies $\Omega_{\alpha,\beta}(f) = 0$, then f is an eigenfunction of all scalar-valued invariant differential operators. Furthermore, f vanishes under the Laplace operator $H^{(1)}_{\alpha,\beta} - 2\alpha(\beta - \frac{3}{2})$.*

PROOF. We outline a proof based on representation theory, which is different from Maaß's argument in [**Maa53**].

Note that $\operatorname{tr}(\Omega_{\alpha,\beta})$ is an invariant differential operator of order 2. Hence it suffices to prove that G is an eigenfunction of an invariant differential operator of order 4 that is not of the form $c_1\big(H_1^{(\alpha,\beta)}\big)^2 + c_2 H_1^{(\alpha,\beta)}$ for some $c_1, c_2 \in \mathbb{C}$. Helgason's treatment of covariant differential operators in [**Hel77, Hel92**] shows that the $|_{(\alpha-1,\beta;2)}$-component of $\Omega_{\alpha,\beta}$ composed with an appropriate covariant differential operator, the existence of which is clear, yields an invariant differential operator of order 4. Any function vanishing under $\Omega_{\alpha,\beta}$ will also vanish under this operator. Finally, this composed operator annihilates $\det(Y)^s$ for any $s \in \mathbb{C}$ and $H_1^{(\alpha,\beta)}\,(\det(Y)^s) = (3 - 2\beta - 2s)(\alpha + s)\det(Y)^s$, which yields the claim. $\square$

The preceding proposition provides evidence for the importance of the matrix-valued Laplace operator. We will call a function $f : \mathbb{H}_2 \to \mathbb{C}$ that vanishes under $\Omega_{\alpha,\beta}$ *harmonic* of type (α,β). Usually, the type of harmonicity will be clear from the context.

Besides the operators that leave the scalar weight invariant, we will need a raising operator for functions $\mathbb{H}_2 \to \mathbb{C}$. Define

$$\begin{aligned} M_\alpha = \quad & \alpha(\alpha - \tfrac{1}{2}) + 2i(\alpha - \tfrac{1}{2})\big(y\partial_\tau + v\partial_z + y'\partial_{\tau'}\big) \\ & - 4\det(Y)(\partial_\tau\partial_{\tau'} - \tfrac{1}{4}\partial_z^2) \end{aligned} \tag{4.4}$$

and $N_\beta = \mathrm{i}\, M_\beta\, \mathrm{i}$ with $(\mathrm{i}f)(Z) := f(-\overline{Z})$ for any $f : \mathbb{H}_2 \to \mathbb{C}$. In [**Maa71**, Chapter 19], Maaß studied the action of these operators on Eisenstein series.

Anticipating the outstanding role of $|_k^{(2)}$ and $|_k^{(2),\mathrm{sk}}$, we define two corresponding ξ-operators, which establish a connection between these two slash actions. Set

$$\xi_k^{(2)} := \det(Y)^{k-\frac{3}{2}}\, N_0 \quad \text{and} \quad \xi_k^{(2),\mathrm{sk}} := \det(Y)^{k-\frac{3}{2}}\, M_{\frac{1}{2}}.$$

The first ξ-operator is covariant from $|_k^{(2)}$ to $|_{3-k}^{(2),\mathrm{sk}}$, and the latter is covariant from $|_k^{(2),\mathrm{sk}}$ to $|_{3-k}^{(2)}$.

REMARK 4.7. *From a representation theoretic point of view $\xi_k^{(2)}$ is a lowering operator and $\xi_k^{(2),\mathrm{sk}}$ is a raising operator.*

Using the results obtained in Section 2 it is easy to show that these ξ-operators are unique.

PROPOSITION 4.8. *The raising and lowering operators $\xi_k^{(2)}$ and $\xi_k^{(2),\mathrm{sk}}$ are unique up to scalar multiples.*

PROOF. This follows from Proposition 2.3 and Corollary 1.4, since there are no scalar-valued raising and lowering operators of degree less than 2. $\square$

The above ξ-operators connect the dual holomorphic and skew slash actions $|_k^{(2)}$ and $|_{3-k}^{(2),\mathrm{sk}}$, and $|_k^{(2),\mathrm{sk}}$ and $|_{3-k}^{(2)}$. The next proposition shows that they preserve harmonicity.

PROPOSITION 4.9. *Suppose that $\Omega_k^{\mathrm{sk}} f = 0$; then $\Omega_k\, \xi_k^{(2),\mathrm{sk}} f = 0$. Vice versa, suppose that $\Omega_k f = 0$; then $\Omega_k^{\mathrm{sk}}\, \xi_k^{(2)} f = 0$.*

PROOF. We prove the first case. The second follows along the same lines, using raising operators instead of lowering operators.

It suffices to prove that $\Omega^{\mathrm{sk}}_{3-k}\xi^{(2)}_k$ equals $\check{\xi}^{(2)}_k\Omega_k$ for a suitable order 2 operator $\check{\xi}^{(2)}_k$ that is covariant from $|^{(\mathrm{M})}_{\frac{1}{2},k-\frac{1}{2}}$ to $|^{(\mathrm{M})}_{3-k,0}$. Because the trace of Ω_k is the usual Laplace operator, it is clear which operator that the scalar valued ξ-operator must be chosen for the scalar component. In order to find the right operator for the 3-dimensional part, we will apply Proposition 2.3 and Corollary 1.4 several times. Since there are no lowering operators of degree less than 2, there is, up to multiplicative scalars, exactly one operator $\check{\xi}^{(2)}_k$ with the desired covariance. There is no operator of order less than 4 that is covariant from $|^{(2)}_k$ to the slash action associated to $\boldsymbol{\alpha}^{(2)}_{\frac{1}{2}-1,3-k-\frac{1}{2};2}$, and there is, up to multiplicative scalars, exactly one such operator of order 4. Consequently, after suitable normalization, $\Omega^{\mathrm{sk}}_{3-k}\xi^{(2)}_k$ and $\check{\xi}^{(2)}_k\Omega_k$ coincide. □

For the initial discussion in Section 2 of Chapter 4, we will need the following considerations. Since $\mathrm{SO}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{R}) \hookrightarrow \mathrm{Sp}_2(\mathbb{R})$ via the block diagonal embedding, it is natural to consider the following coordinates

$$Y = \begin{pmatrix} t & \\ & t' \end{pmatrix} \left[\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \right], \tag{4.5}$$

where $0 < t, t' \in \mathbb{R}$ and $\theta \in \mathbb{R}$. We will express the differentials with respect to y, y' and v by those with respect to t, t' and θ.

LEMMA 4.10. *If $t \neq t'$, we have*

$$\begin{pmatrix} \partial_y \\ \partial_{y'} \\ \partial_v \end{pmatrix} = \begin{pmatrix} \cos^2(\theta) & \sin^2(\theta) & \sin(2\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -\sin(2\theta) \\ -\sin(\theta)\cos(\theta) & \sin(\theta)\cos(\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_{t'} \\ (t-t')^{-1}\,\partial_\theta \end{pmatrix} \tag{4.6}$$

PROOF. It is sufficient to express the entries of Y in terms of t, t' and θ.

$$Y = \begin{pmatrix} t\cos^2(\theta) + t'\sin^2(\theta) & (t-t')\cos(\theta)\sin(\theta) \\ (t-t')\cos(\theta)\sin(\theta) & t\sin^2(\theta) + t'\cos^2(\theta) \end{pmatrix}.$$

Computing the Jacobian and taking its inverse yields the result. □

Crucial to this system of coordinates is the following property:

PROPOSITION 4.11. *Let $a : \{Y \in \mathrm{M}^{\mathrm{T}}_2(\mathbb{R}) : Y > 0\} \rightarrow \mathbb{C}$ be a real-analytic function. For $i = 1, 2$, write*

$$\big(H^{(i)}_{\alpha,\beta} a(Y) e(x + x')\big) = \sum_{m \in \mathbb{Z}} b^{(i)}_m \Big(t, t', \big(\partial_t^r \partial_{t'}^{r'} \partial_\theta^s a(t, t', \theta)\big)\Big)\, e^{im\theta},$$

where the last argument of $b^{(i)}_m$ means that $b^{(i)}_m$ depends on arbitrary but finitely many derivatives of a. Then $b^{(i)}_m = 0$, whenever $m \neq 0$.

PROOF. Set $f(Z) = a(Y)e(x + x')$. We abbreviate

$$\mathrm{rot}(\theta) := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad \text{and} \quad l_{\mathrm{rot}(\theta)} := \begin{pmatrix} \mathrm{rot}(\theta) & \\ & \mathrm{rot}(\theta) \end{pmatrix}.$$

Then

$$f|^{(2)}_{\alpha,\beta}\, l_{\mathrm{rot}(\hat{\theta})} = a\big(Y[\mathrm{rot}(-\hat{\theta})]\big).$$

Since $H^{(i)}_{\alpha,\beta}$ is covariant, we find

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} b^{(i)}_m\Big(t,t',\big(\partial_t^r\partial_{t'}^{r'}\partial_\theta^s a(t,t',\theta)\big)\Big)\,e^{im\theta}\\
&= H^{(i)}_{\alpha,\beta} f = \Big(H^{(i)}_{\alpha,\beta}\big(f\big|^{(2)}_{\alpha,\beta}\, l_{\mathrm{rot}(\hat\theta)}\big)\Big)\Big|^{(2)}_{\alpha,\beta}\, l_{\mathrm{rot}(-\hat\theta)}\\
&= \sum_{m\in\mathbb{Z}} b^{(i)}_m\Big(t,t',\big(\partial_t^r\partial_{t'}^{r'}\partial_\theta^s a(t,t',\theta)\big)\Big)\,e^{im(\theta+\hat\theta)}
\end{aligned}$$

for all $\hat\theta \in \mathbb{R}$. This proves the statement. □

5. Natural weights

In this section, we will argue that the slash actions $|^{(2)}_k$ and $|^{(2),\mathrm{sk}}_k$, up to complex conjugation, are the only natural slash actions for degree 2 Siegel modular forms. Note that, representation theoretically, these families of slash actions are equivalent when k runs through $\mathbb{Z}$. Because we will later restrict to harmonic functions, and the the matrix-valued Laplace operators for these slash actions differ by a multiple of I_2, it makes sense to distinguish them. The discussion of natural weights could be phrased equivalently in terms of eigenvalues of $H^{(1)}_{k,0}$.

In [**Maa71**, Chapter 19], Maaß remarked that for given n, there are exactly n distinct values of α such that the Siegel Eisenstein series

$$E^{(n)}_{\alpha,\beta} = \sum_{g\,:\,\Gamma^{(2)}_\infty\backslash\Gamma^{(2)}} 1|_{\alpha,\beta}\, g.$$

vanishes under the raising operator M_α defined in (4.4) for $n=2$ and in [**Maa71**, Chapter 19] for general n. These α are $0, \frac{1}{2}, \ldots, \frac{n-1}{2}$. Based on this observation Imamoḡlu and Richter reasoned in [**IR10**] that there are n distinct natural slash actions

$$|^{(n)}_{0,k}, \ldots, |^{(n)}_{\frac{n-1}{2},k-\frac{n-1}{2}} \quad\text{or}\quad |^{(n)}_{k,0}, \ldots, |^{(n)}_{k-\frac{n-1}{2},\frac{n-1}{2}}.$$

for degree n Siegel modular forms. Complex conjugation relates $|^{(n)}_{\alpha,\beta}$ to $|^{(n)}_{\beta,\alpha}$ for any α and β, so that these $2n$ slash actions should be thought of as n truly distinct ones.

In the elliptic case, that is, if $n=1$, the holomorphic slash action $|^{(1)}_{k,0}$ is the only natural one. In the case of $n=2$, which we are mainly concerned with, the holomorphic slash action and the skew slash action $|^{(2),\mathrm{sk}}_k = |^{(2)}_{\frac{1}{2},k-\frac{1}{2}}$ are natural. The skew slash action has no analog in the elliptic case, and thus promises to lead to new phenomenons.

When defining a space of harmonic modular forms it should be characterized by covariant operators to guarantee compatibility with the invariance properties that modular forms satisfy. Further, to promise to be useful for applications, it is indispensable to include Eisenstein series. That is, the covariant operators should have vanishing constant coefficient. We combine this fact with the above observation to a fundamental conclusion, that cannot possibly be made more precise, but should be guiding, whenever one considers real-analytic Siegel modular forms.

CONCLUSION 5.1. *A natural definition of harmonic Siegel modular forms is based on covariant differential operators that, under the natural slash actions* $|^{(n)}_{0,k}, \ldots, |^{(n)}_{\frac{n-1}{2},k-\frac{n-1}{2}}$, *have vanishing constant coefficient.*

There is much more to say about how a good definition should be motivated, but we will not go into details. We understand, however, why Ω_k and Ω_k^{sk} are the right operators to use when defining a well-behaved space of harmonic Siegel modular forms of degree 2.

Vice versa, starting with $\Omega_{\alpha,\beta}$, based on the theory of Jacobi forms, we can argue why $|_k^{(2)}$ and $|_k^{(2),\mathrm{sk}}$ are, indeed, natural slash actions.

We say that a function $f \in C^\infty(\mathbb{H}_2)$ converges smoothly and C^∞ as $y' \to \infty$, if all derivatives of f with respect to τ, $\overline{\tau}$, z and $\overline{z}$ converge to the derivatives of the limit and any derivative involving y' converges to 0.

THEOREM 5.2. *Fix $m \in \mathbb{Z}$ and let $f(Z) = a(\tau, z, y')\, e(mx')$. Suppose that $f(Z)\, e^{2\pi m y'}$ converges smoothly and C^∞ as $y' \to \infty$. If $f(Z)$ vanishes under $\Omega_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{R}$, then the limit*

$$\lim_{y' \to \infty} f(Z) e^{2\pi m y'}$$

vanishes under $\partial_z \partial_{\overline{z}}$.

PROOF. We will compute the limit (5.1) in two ways. Note that, since $f(Z)\, e^{2\pi m y'}$ converges C^∞, all derivatives are bounded as $y' \to \infty$. In particular, after division by $y'Y$, only the highest order term of $\Omega_{\alpha,\beta}\, f(Z)\, e^{2\pi m y'}$ does not tend to zero as $y' \to \infty$. Further, we have

$$\lim_{y' \to \infty} \partial_{\tau'} f(Z)\, e^{2\pi m y'} = \lim_{y' \to \infty} \partial_{\overline{\tau}'} f(Z)\, e^{2\pi m y'} = \pi m f(Z)\, e^{2\pi m y'}.$$

Consequently,

$$\lim_{y' \to \infty} y'^{-1} Y^{-1} \Omega_{\alpha,\beta}\, f(Z)\, e^{2\pi m y'} \tag{5.1}$$

$$= -4\Big(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_{\overline{\tau}} & \frac{1}{2}\partial_{\overline{z}} \\ \frac{1}{2}\partial_{\overline{z}} & \pi m \end{pmatrix} \Big)^{\mathrm{T}} \begin{pmatrix} \partial_{\tau} & \frac{1}{2}\partial_{z} \\ \frac{1}{2}\partial_{z} & \pi m \end{pmatrix} \lim_{y' \to \infty} f(Z)\, e^{2\pi m y'}.$$

The top left entry of this equals $-\partial_z \partial_{\overline{z}} \lim_{y' \to \infty} f(Z)\, e^{2\pi m y'}$.

Next, we use the vanishing of $\Omega_{\alpha,\beta} f(Z)$:

$$\lim_{y' \to \infty} y'^{-1} Y^{-1} \Omega_{\alpha,\beta}\, f(Z)\, e^{2\pi m y'}$$

$$= -4\Bigg(\Big(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \pi i m \end{pmatrix} \Big)^{\mathrm{T}} \big(\partial_z f(Z)\big)\, e^{2\pi m y'}$$

$$+ \Big(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \big(\partial_{\overline{z}} f(Z)\big) \Big)^{\mathrm{T}} \begin{pmatrix} 0 & 0 \\ 0 & -\pi i m \end{pmatrix} e^{2\pi m y'}$$

$$+ \Big(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \pi i m \end{pmatrix} \Big)^{\mathrm{T}} \begin{pmatrix} 0 & 0 \\ 0 & -\pi i m \end{pmatrix} f(Z)\, e^{2\pi m y'} \Bigg).$$

The top left entry of this matrix is zero, yielding the claim. □

The next corollary can only be proved at the end of Section 3 in Chapter 3. We also anticipate the notation introduced there.

COROLLARY 5.3. *Fix $m \in \mathbb{Z}$ and let $f(Z) = a(\tau, z, y')\, e(mx')$ be invariant under either of the slash actions $|_k^{(2)}$ or $|_k^{(2),\mathrm{sk}}$ of the full Jacobi group embedded into*

$\mathrm{Sp}_2(\mathbb{Z})$. *Suppose that* $f(Z)\, e^{2\pi m y'}$ *converges smoothly and* C^∞ *for* $y' \to \infty$. *Then the limit is holomorphic in* z.

Anticipating the results of Chapter 5, one can hope to construct Fourier-Jacobi coefficients of a real-analytic Siegel modular form of degree 2 that are independent of y', employing the limit in Theorem 5.2. If the convergence is sufficiently good, in the light of the results of Chapter 3, the resulting Jacobi form has a θ-decomposition, since it is semi-holomorphic. There are two natural slash actions for Jacobi forms on $\mathbb{H}_1 \times \mathbb{C}$. The first, $|^{\mathrm{J}}_{k,m}$, corresponds to multiplying θ-series with Maaß forms for $|^{(1)}_{k-\frac{1}{2},0}$, and the second, $|^{\mathrm{J,sk}}_{k}$, corresponds to multiplying θ-series with complex conjugates of Maaß form. Hence for Siegel modular forms of degree 2 the slash actions $|^{(2)}_{k}$ and $|^{(2),\mathrm{sk}}_{k}$ are natural.

REMARK 5.4. *These considerations can be generalized to arbitrary degrees* n. *The slash actions* $|^{(n)}_{k,0}$ *and* $|^{(n)}_{\frac{n-1}{2},k-\frac{n-1}{2}}$ *can be obtained by means of Jacobi forms studied in Chapter 3. To obtain the remaining natural slash actions one needs to consider Jacobi forms on* $\mathbb{H}_{n-1} \times \mathbb{C}^{n-1}$. *The remaining issue, from the point of view of natural weights, is to find the differential operators that generalize* $\Omega_{\alpha,\beta}$ *in the spirit of Theorem 5.2.*

CHAPTER 3

Harmonic Jacobi forms

In this chapter, we will discuss harmonic Jacobi forms. This discussion is mostly based on [**CR11**] and [**BRR11a**, Section 3]. We summarize the results that we will obtain. As in the case of degree 2 Siegel modular forms, we will find two natural slash actions $|^{\mathrm{J}}_{k,L}$ and $|^{\mathrm{J,sk}}_{k,L}$, which are justified based on the holomorphic and skew-holomorphic theta decomposition (see [**EZ85, Sko90**] and Theorem 5.5).

The Casimir operator, of degree 3 or 4, for the centrally extended real Jacobi group that we will deduce does not suffice to force the Fourier addends of harmonic forms into a finite dimensional space. Instead, we will use an additional, invariant operator that originates in the Heisenberg part of the real Jacobi group. This is possible, because neither the Jacobi group nor its central extension are reductive. In accordance to the result that we will obtain in Chapter 5, we will focus on semi-holomorphic forms. For fixed Fourier index, the space of possible Fourier coefficients of such a form has dimension 2. This suffices to relate them to harmonic weak Maaß forms, which are known by the work of Bruinier and Funke [**BF04**]. This face gives rise to a rich but manageable arithmetic structure of harmonic semi-holomorphic Maaß-Jacobi forms. Despite the outstanding importance of semi-holomorphic Maaß-Jacobi forms, we will also discuss an alternative approach, that subsumes the multivariable Appell sums presented in [**Zwe10**]. This discussion is contained in Section 6. A modification of the definitions given in Section 4, that allows for the definition of mixed mock Jacobi forms, was presented in [**CR11**, Section 3]. Although we will later mostly make use of Jacobi forms with scalar Jacobi indices, we will present the theory for matrix-valued indices in full generality. A strong reason for this is the fact that some interesting phenomenons only occur in this more general setting. It enables us to formulate a striking generalization of the work contained in [**Koh94**]. This generalization will be deduced in Section 1 of Chapter 5.

In this chapter we need further notation, which is adopted from [**CR11**]. Regarding elements of R^m as column vectors, we will freely identify $R^m \otimes R^n$ with $\mathrm{M}_{m,n}(R)$ via $v \otimes w \mapsto vw^{\mathrm{T}}$. Write ϵ_i for the i^{th} standard basis vector of R^m and ϵ_{ij} for the elementary matrix with $(i,j)^{\mathrm{th}}$ entry 1 and other entries 0, the sizes of ϵ_i and ϵ_{ij} being determined by the context. For any $N \times N$ matrix A and any N-vector w, set

$$A[w] := w^{\mathrm{T}} A w.$$

Since we will not be concerned with any Siegel modular forms, we drop the notation $Z = X + iY \in \mathbb{H}_n$ throughout the whole chapter.

1. The centrally extended Jacobi group

The real Jacobi group G^J_N for rank N indices and its subgroup Γ^J_N, the full Jacobi group, are

$$G^J_N := \mathrm{SL}_2(\mathbb{R}) \ltimes (\mathbb{R}^N \otimes \mathbb{R}^2) \quad \text{and} \quad \Gamma^J_N := \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^N \otimes \mathbb{Z}^2). \tag{1.1}$$

The product in G^J_N arises from the natural right action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$. It can be written most simply using the above identification of $\mathbb{R}^N \otimes \mathbb{R}^2$ with $\mathrm{M}_{N,2}(\mathbb{R})$: for $g,\, \check{g} \in \mathrm{SL}_2(\mathbb{R})$ and $X,\, \check{X} \in \mathrm{M}_{N,2}(\mathbb{R})$,

$$(g, X)(\check{g}, \check{X}) = (g\check{g}, X\check{g} + \check{X}).$$

Maintaining the $\mathrm{M}_{N,2}(\mathbb{R})$ identification, the centrally extended real Jacobi group $\tilde{G}^J_N$ for rank N indices and its product are

$$\tilde{G}^J_N := \big\{(g, X, \kappa) \,:\, (g, X) \in G^J_N,\ \kappa \in \mathrm{M}_N(\mathbb{R}),\ \kappa + \tfrac{1}{2} X J_2 X^{\mathrm{T}} \in \mathrm{M}^{\mathrm{T}}_N(\mathbb{R})\big\}, \tag{1.2}$$

$$(g, X, \kappa)(\check{g}, \check{X}, \check{\kappa}) := (g\check{g}, X\check{g} + \check{X}, \kappa + \check{\kappa} - X\check{g} J_2 \check{X}^{\mathrm{T}}). \tag{1.3}$$

Note that G^J_N is centerless, and the center of $\tilde{G}^J_N$ is $\mathrm{M}^{\mathrm{T}}_N(\mathbb{R})$. In [**CR11**, Section 5], the fact was used that $\tilde{G}^J_N$ is a subgroup of $\mathrm{Sp}_{N+1}(\mathbb{R})$. To give a concrete embedding, fix an element $g := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{R})$. The usual embedding is

$$\tilde{G}^J_N \to \mathrm{Sp}_{N+1}(\mathbb{R}),$$
$$\big(g, (\mu, \lambda), \kappa\big) \mapsto \begin{pmatrix} I_N & \lambda & \kappa & \mu \\ & a & * & b \\ & & I_N & \\ & c & * & d \end{pmatrix}.$$

Henceforth write μ and λ for the columns of any element X of $\mathrm{M}_{N,2}(\mathbb{R})$. The extended Jacobi group acts on the Jacobi upper half plane

$$\mathbb{H}_{1,N} := \mathbb{H}_1 \times \mathbb{C}^N$$

by an extension of the usual elliptic slash action: For $\tau \in \mathbb{H}_1$, as a special case of (3.3), we have

$$g\tau := (a\tau + b)(c\tau + d)^{-1}, \qquad \boldsymbol{\alpha}^{(1)}_{\alpha,\beta}(g, \tau) = (c\tau + d)^{-\alpha}(c\overline{\tau} + d)^{-\beta}.$$

Recall that the associated slash action of $\mathrm{SL}_2(\mathbb{R})$ on $C^\infty(\mathbb{H}_1)$ is written:

$$f\big|_{\alpha,\beta}\, g(\tau) = \boldsymbol{\alpha}^{(1)}_{\alpha,\beta}(g, \tau)\, f(g\tau).$$

For future reference and as a special case of [**Maa71**, Chapter 6] and Theorem 4.2 of Chapter 2, let us mention that the algebra of differential operators on $C^\infty(\mathbb{H}_1)$ invariant with respect to the $|_{\alpha,\beta}$-action is the polynomial algebra on one variable generated by the $|_{\alpha,\beta}$-Casimir operator of $\mathrm{SL}_2(\mathbb{R})$, which, in the case of $\alpha = k, \beta = 0$, differs by an additive constant from the weight k hyperbolic Laplacian

$$\Delta_k := 4y^2 \partial_\tau \partial_{\overline{\tau}} - 2iky\partial_{\overline{\tau}}. \tag{1.4}$$

By Section 3 of Chapter 2, we know that $\{\boldsymbol{\alpha}^{(1)}_{k,0} \,:\, k \in \mathbb{Z}\}$ exhausts the cocycles of the action under consideration up to equivalence. The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}_1$ generalizes to the following well-known left action of G^J_N on $\mathbb{H}_{1,N}$:

$$(g, X)(\tau, z) := \big(g\tau, \boldsymbol{\alpha}^{(1)}_{1,0}(g, \tau)(z + \lambda\tau + \mu)\big). \tag{1.5}$$

Regard this as an action of $\tilde{G}^J_N$. As such, the stabilizer of the element $(i, 0)$ of $\mathbb{H}_{1,N}$ is $\tilde{K}^J_N := \mathrm{SO}_2 \times \{0\} \times \mathrm{M}^{\mathrm{T}}_N(\mathbb{R})$, and the equivalence classes of the scalar cocycles of the action are in bijection with the representations of $\tilde{K}^J_N$ on $\mathbb{C}$.

In order to describe a complete family of cocycles, define a matrix-valued function $a : \tilde{G}^J_N \times \mathbb{H}_{1,N} \to \mathrm{M}^{\mathrm{T}}_N(\mathbb{C})$ by

$$a\big((g, X, \kappa), (\tau, z)\big) := \kappa + \mu\lambda^{\mathrm{T}} + X_1 z^{\mathrm{T}} + z\lambda^{\mathrm{T}} + \lambda\lambda^{\mathrm{T}}\tau \\ - c\boldsymbol{\alpha}^{(1)}_{1,0}(M, \tau)(z + \lambda\tau + \mu)(z + \lambda\tau + \mu)^{\mathrm{T}}.$$

For $L \in \mathrm{M}^{\mathrm{T}}_N(\mathbb{C})$, define $\boldsymbol{\alpha}^{\mathrm{J}}_L : \tilde{G}^J_N \times \mathbb{H}_{1,N} \to \mathbb{C}$ by

$$\boldsymbol{\alpha}^{\mathrm{J}}_L\big((g, (X, \kappa), (\tau, z)\big) := \exp\big(2\pi i \operatorname{tr}\big(La\big((g, X, \kappa), (\tau, z)\big)\big)\big).$$

LEMMA 1.1 ([**CR11**, Lemma 2.2]). *For all $k \in \mathbb{Z}$ and $L \in \mathrm{M}^{\mathrm{T}}_N(\mathbb{C})$, $\boldsymbol{\alpha}^{(1)}_{k,0}\,\boldsymbol{\alpha}^{\mathrm{J}}_L$ is a scalar cocycle with respect to the action (1.5) on $\mathbb{H}_{1,N}$ of the centrally extended Jacobi group $\tilde{G}^J_N$. Moreover, any scalar cocycle of this action is equivalent to exactly one of these cocycles.*

PROOF. The proof that $\boldsymbol{\alpha}^{(1)}_{k,0}$ is a cocycle of the action of $\tilde{G}^J_N$ on $\mathbb{H}_{1,N}$ is the same as the proof that it is a cocycle of the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$. The proof that α^{J}_L is a cocycle is standard in the case $N = 1$ and proceeds along the same lines in general. One must prove that $a(g\check{g}, x) = a(g, \check{g}x) + a(\check{g}, x)$. First check that it suffices to prove this for both g and $\check{g}$ in either the semisimple or the nilpotent part of $\tilde{G}^J_N$, and then check each of the resulting four cases directly. The second sentence follows immediately from the classification of representations of $\tilde{K}^J_N$. $\square$

As a consequence of this lemma, we have the following family of slash actions of $\tilde{G}^J_N$ on $C^\infty(\mathbb{H}_{1,N})$: for α, $\beta \in \mathbb{Z}$ and $L \in \mathrm{M}^{\mathrm{T}}_N(\mathbb{C})$,

$$\phi\big|^{\mathrm{J}}_{\alpha,\beta,L}(g, X, \kappa)(\tau, z) := \boldsymbol{\alpha}^{(1)}_{\alpha,\beta}(M, \tau)\,\boldsymbol{\alpha}^{\mathrm{J}}_L\big((g, X, \kappa), (\tau, z)\big) \\ \cdot\ \phi\big((g, X, \kappa)(\tau, z)\big).$$

Observe that since $\boldsymbol{\alpha}^{(1)}_{\beta,\beta}$ is positive, $|^{\mathrm{J}}_{\alpha,\beta,L}$ makes sense for all α, $\beta \in \mathbb{C}$ with $\alpha - \beta \in \mathbb{Z}$. We write $|^{\mathrm{J}}_{k,L}$ for $|^{\mathrm{J}}_{k,0,L}$ and $|^{\mathrm{J,sk}}_{k,L}$ for $|^{\mathrm{J}}_{\frac{1}{2},k-\frac{1}{2},L}$. By Lemma 1.1, any slash action is equivalent to exactly one of the actions $|^{\mathrm{J}}_{k,L}$; As we have mentioned, $|^{\mathrm{J}}_{\alpha,\beta,L}$ is equivalent to $|^{\mathrm{J}}_{\alpha-\beta,L}$. Similarly, any slash action is equivalent to exactly one of the actions $|^{\mathrm{J,sk}}_{k,L}$. The difference between both slash actions, as argued in Section 5 of Chapter 2, originates in the fact that we normalize all Casimir operators such that their constant term vanishes.

In analogy to the $\mathrm{Sp}_2(\mathbb{R})$-case, we will say that $|^{\mathrm{J}}_{k,L}$ is the weight k holomorphic slash action and $|^{\mathrm{J,sk}}_{k,L}$ is the weight k skew slash action. As in the symplectic case, the representation theoretic weight of the latter is $1 - k$.

2. Classical definitions of Jacobi forms

The next definitions are almost classical. The first can be found in [**EZ85**], and the second can be found in [**Sko90**], both with stronger growth conditions. Weakly skew-holomorphic forms were, in particular, defined in [**BR10**]. Elements of $C^\infty(\mathbb{H}_{1,N})$ holomorphic in $\mathbb{C}^N$ will be called semi-holomorphic.

DEFINITION 2.1 (Weakly holomorphic Jacobi forms). *A weakly holomorphic Jacobi form of weight k and index L is a holomorphic function $\phi : \mathbb{H}_{1,N} \to \mathbb{C}$ satisfying the equation $\phi|^{\mathrm{J}}_{k,L}\, g = \phi$ for all $g \in \Gamma^J_N$ and the growth condition $|\phi(\tau, z)| < e^{ay} e^{2\pi L[v]/y}$ for some $a > 0$ as $y \to \infty$. We write $J_{k,L}$ for the space of all such forms.*

For brevity, write $Ł := 2\pi i L$. For L invertible, define the heat operator

$$\mathbb{L}_L := 2\partial_\tau + (2Ł)^{-1}[\partial_z]. \tag{2.1}$$

It plays an important role in the theory of Jacobi forms, since it annihilates theta series.

DEFINITION 2.2 (Weakly skew-holomorphic Jacobi forms). *A skew-holomorphic Jacobi form of weight k and index L is a semi-holomorphic function $\phi \in C^\infty(\mathbb{H}_{1,N})$ satisfying the following conditions. First, for all $g \in \Gamma^J_N$ we have $\phi|^{\mathrm{J,sk}}_{k,L}\, g = \phi$. Second, ϕ is in the kernel of the heat operator $\mathbb{L}_L$. Third, $|\phi(\tau, z)| < e^{ay} e^{2\pi L[v]/y}$ for some $a > 0$ as $y \to \infty$. We write $J^{\mathrm{sk}}_{k,L}$ for the space of all such forms.*

REMARK 2.3. *Skew-holomorphic Jacobi forms were first introduced by Skoruppa in* [**Sko90**]. *There are several articles treating a slightly more general notion than the one we have given. See, in particular,* [**Hay06**].

The Fourier expansion of skew-holomorphic Jacobi forms is also classical. In order to state it, write

$$D := D_L(n, r) := |L|(4n - L^{-1}[r])$$

for the negative discriminant of a Fourier index (n, r).

PROPOSITION 2.4. *The Fourier expansion of $\phi \in J^{\mathrm{sk}}_{k,L}$ has the form*

$$\phi(\tau, z) = \sum_{\substack{n \in Z, r \in \mathbb{Z}^N \\ s.t.\ D \gg -\infty}} c(n, r)\, \exp\big(\tfrac{\pi D}{|L|} y\big)\, q^n \zeta^r.$$

PROOF. By the semi-holomorphicity of ϕ a addend in the Fourier expansion has the form $a(y; n, r) e(nx) \zeta^r$. Imposing the differential equation $\mathbb{L}_L \phi = 0$ shows that there is at most one nonzero $a(y; n, r)$ that can occur. Finally, it is easy to check that, indeed, the above Fourier expansion vanishes under $\mathbb{L}_L$. The claim follows. □

3. Covariant operators

At this point, we state the main results of [**CR11**, Section 5]. They were given for the holomorphic slash action, and we generalize them to the skew slash action. All statements for the skew slash action follow from the ones for the holomorphic slash action after conjugation by $y^{k - \frac{1}{2}}$.

DEFINITION 3.1. *A differential operator T on $\mathbb{H}_{1,N}$ is covariant from $|^{\mathrm{J}}_{k,L}$ to $|^{\mathrm{J}}_{k',L'}$ if for all $g \in \tilde{G}^J_N$ and $f \in C^\infty(\mathbb{H}_{1,N})$, we have*

$$T\big(f\big|^{\mathrm{J}}_{k,L} g\big) = (Tf)\big|^{\mathrm{J}}_{k',L'} g.$$

Let $\mathbb{D}^{\mathrm{J}}(k, L;\, k', L')$ be the space of covariant operators from $|^{\mathrm{J}}_{k,L}$ to $|^{\mathrm{J}}_{k',L'}$, and let $\mathbb{D}^{\mathrm{J},r}(k, L;\, k', L')$ be the space of those of order $\leq r$. When $k' = k$ and $L' = L$, we refer to such operators as $|^{\mathrm{J}}_{k,L}$-invariant and write simply $\mathbb{D}^{\mathrm{J}}_{k,L}$ and $\mathbb{D}^{\mathrm{J},r}_{k,L}$.

A differential operator T on $\mathbb{H}_{1,N}$ is covariant from $|^{\mathrm{J,sk}}_{k,L}$ to $|^{\mathrm{J,sk}}_{k',L'}$ if for all $g \in \tilde{G}^J_N$ and $f \in C^\infty(\mathbb{H}_{1,N})$, we have

$$T\big(f\big|^{\mathrm{J,sk}}_{k,L}\, g\big) = (Tf)\big|^{\mathrm{J,sk}}_{k',L'}\, g.$$

The spaces of differential operators $\mathbb{D}^{\mathrm{J,sk}}(k,L;\, k',L')$, $\mathbb{D}^{\mathrm{J,sk},r}(k,L;\, k',L')$, $\mathbb{D}^{\mathrm{J,sk}}_{k,L}$ and $\mathbb{D}^{\mathrm{J,sk},r}_{k,L}$ are defined analogously.

For $\mu \in \mathbb{Z}^N$ and $L \in \widetilde{\mathrm{M}}^{\mathrm{T}}_N(\frac{1}{2}\mathbb{Z})$ define

$$\theta_{L,\mu}(\tau,z) := \sum_{r \in \mathbb{Z}^N,\, r \equiv \mu (L\mathbb{Z}^N)} q^{\frac{L^{-1}[r]}{4}} \zeta^r. \tag{3.1}$$

PROPOSITION 3.2. *For $\mu \in \mathbb{Z}^N$ and $L \in \widetilde{\mathrm{M}}^{\mathrm{T}}_N(\frac{1}{2}\mathbb{Z})$ we have*

$$\mathbb{L}_L \theta_{L,\mu} = 0.$$

PROOF. The proof is standard. The claim follows immediately when applying $\mathbb{L}_L$ to the individual terms of the right hand side of (3.1). □

Recall the Laplacian (1.4) and our notation $\tau := x + iy \in \mathbb{C}$, and set $z := u + iv \in \mathbb{C}^N$. Define

$$\begin{aligned}\mathcal{C}^{\mathrm{J}}_{k,L} :=& -2\Delta_{k-N/2} + 2y^2\big(\partial_{\overline{\tau}} \L^{-1}[\partial_z] + \partial_\tau \L^{-1}[\partial_{\overline{z}}]\big) - 8y\partial_\tau v^{\mathrm{T}}\partial_{\overline{z}} \\ &- \tfrac{1}{2}y^2\big(\L^{-1}[\partial_{\overline{z}}]\L^{-1}[\partial_z] - (\partial_{\overline{z}}^{\mathrm{T}}\L^{-1}\partial_z)^2\big) + 2y(v^{\mathrm{T}}\partial_{\overline{z}})\partial_z^{\mathrm{T}}\L^{-1}\partial_u \\ &- \tfrac{1}{2}(2k-N+1)iy\partial_{\overline{z}}^{\mathrm{T}}\L^{-1}\partial_u + 2v^{\mathrm{T}}(v^{\mathrm{T}}\partial_{\overline{z}})\partial_z + (2k-N-1)iv^{\mathrm{T}}\partial_{\overline{z}}\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\mathcal{C}^{\mathrm{J,sk}}_{k,L} :=& -2\overline{\Delta}_{k-\frac{N}{2}} + 2y^2\big(\partial_{\overline{\tau}} \L^{-1}[\partial_z] + \partial_\tau \L^{-1}[\partial_{\overline{z}}]\big) - 8y\partial_\tau v^{\mathrm{T}}\partial_{\overline{z}} \\ &- \tfrac{1}{2}y^2\big(\L^{-1}[\partial_{\overline{z}}]\L^{-1}[\partial_z] - (\partial_{\overline{z}}^{\mathrm{T}}\L^{-1}\partial_z)^2\big) + 2y(v^{\mathrm{T}}\partial_{\overline{z}})\partial_z^{\mathrm{T}}\L^{-1}\partial_u \\ &- \tfrac{1}{2}iy\partial_{\overline{z}}^{\mathrm{T}}\L^{-1}\partial_u + \tfrac{1}{4}(2k-N)iy\L^{-1}[\partial_u] + 2v^{\mathrm{T}}(v^{\mathrm{T}}\partial_{\overline{z}})\partial_z \\ &+ (2k-N-1)iv^{\mathrm{T}}\partial_{\overline{z}}.\end{aligned} \tag{3.3}$$

THEOREM 3.3. *For L invertible, the operators $\mathcal{C}^{\mathrm{J}}_{k,L}$ and $\mathcal{C}^{\mathrm{J,sk}}_{k,L}$ are, up to additive and multiplicative scalars, the Casimir operator of $\tilde{G}^J_N$ with respect to the $|^{\mathrm{J}}_{k,L}$ and the $|^{\mathrm{J,sk}}_{k,L}$-action. They generate the images of the $|_{k,L}$ and the $|^{\mathrm{J,sk}}_{k,L}$-action of the center of the universal enveloping algebra of $\tilde{G}^J_N$. In particular, they lie in the center of $\mathbb{D}^{\mathrm{J}}_{k,L}$ and $\mathbb{D}^{\mathrm{J,sk}}_{k,L}$, respectively. The actions of $\mathcal{C}^{\mathrm{J}}_{k,L}$ and $\mathcal{C}^{\mathrm{J,sk}}_{k,L}$ on semi-holomorphic functions are*

$$-2\Delta_{k-N/2} + 2y^2\partial_{\overline{\tau}}\L^{-1}[\partial_z] \quad \textit{and} \quad \big(4y^2\partial_{\overline{\tau}} + \tfrac{i}{2}y(2k-N)\big)\,\mathbb{L}_L. \tag{3.4}$$

Note that for $N > 1$, (3.2) and (3.3) are of order 4. For $N = 1$, they have order 3 and reduce to the operator $C^{k,m}$ given in [**BR10**] and the operator $\mathcal{C}^{sk}_{k,m}$ given in [**BRR11a**] with $L = m$. (There is a misprint in [**BR10**]: the term $k(z-\overline{z})\partial_{\overline{z}}$ should be $(1-k)(z-\overline{z})\partial_{\overline{z}}$. This stems in part from a similar misprint in (8) of [**Pit09**], where the term $(z-\overline{z})\partial_{\overline{z}}$ coming from (6) of [**Pit09**] is missing.)

DEFINITION 3.4. *The raising operators, X_+, X_+^{sk}, Y_+ and Y_+^{sk}, and the lowering operators, X_-, X_-^{sk}, Y_- and Y_-^{sk}, are*

$$X_+^{k,L} := 2i\big(\partial_\tau + y^{-1}v^{\mathrm{T}}\partial_z + y^{-2}Ł[v]\big) + ky^{-1}, \qquad X_-^{k,L} := -2iy\big(y\partial_{\overline{\tau}} + v^{\mathrm{T}}\partial_{\overline{z}}\big),$$

$$Y_+^{k,L} := i\partial_z + 2iy^{-1}Łv, \qquad Y_-^{k,L} := -iy\partial_{\overline{z}},$$

$$X_+^{\mathrm{sk};k,L} := 2i\big(y^2\partial_\tau + yv^{\mathrm{T}}\partial_z + Ł[v]\big) + \tfrac{N}{2}y,$$

$$X_-^{\mathrm{sk};k,L} := -2i\big(\partial_{\overline{\tau}} + y^{-1}v^{\mathrm{T}}\partial_{\overline{z}}\big) + \big(k - \tfrac{N}{2}\big)y^{-1},$$

$$Y_+^{\mathrm{sk};k,L} := iy\partial_z + 2iŁv, \qquad Y_-^{\mathrm{sk};k,L} := -i\partial_{\overline{z}}.$$

REMARK 3.5. *We will call X_-^{sk} and Y_-^{sk}, and X_+^{sk} and Y_+^{sk} lowering and raising operators, respectively. They lower and raise the* representation theoretic weight, *but on the weight of the skew slash action they act differently. E.g., for $g \in \tilde{G}_N^J$ and $f \in C^\infty(\mathbb{H}_{1,N})$*

$$X_-^{\mathrm{sk};k,L}\big(f|_{k,L}^{\mathrm{J,sk}}\, g\big) = \big(X_-^{\mathrm{sk};k,L} f\big)|_{k+1,L}^{\mathrm{J,sk}}\, g\big).$$

For $N = 1$ and $L = m$, the operators for the holomorphic slash action are the operators given on page 59 of [**BS98**]. (There is a misprint in their formula for Y_-: the expression $\frac{1}{2}(\tau - \overline{\tau})f_{\overline{z}}$ on the far right should be multiplied by -1.) Since $Y_\pm^{k,L}$ do not contain derivatives with respect to τ or $\overline{\tau}$, they stay the same, up to multiplication by powers of y, for the skew slash actions. Note that $Y_\pm^{k,L}$ are actually N-vector operators. We write $Y_{\pm,j}^{k,L}$ for their entries.

Frequently, we will suppress the superscript (k, L). Care must be taken with this abbreviation, as for example X_+Y_+ means $X_+^{k+1,L}Y_+^{k,L}$. In contrast, we will always write the superscript sk, when we refer to the operators $X_\pm^{\mathrm{sk}}$.

PROPOSITION 3.6. *The spaces $\mathbb{D}^{\mathrm{J},1}(k, L;\, k \pm 2, L)$ are 1-dimensional, and the spaces $\mathbb{D}^{\mathrm{J},1}(k, L;\, k \pm 1, L)$ are N-dimensional. They have bases given by*

$$\mathbb{D}^{\mathrm{J},1}(k, L;\, k \pm 2, L) = \mathrm{span}\big\{X_\pm^{k,L}\big\},$$

$$\mathbb{D}^{\mathrm{J},1}(k, L;\, k \pm 1, L) = \mathrm{span}\big\{Y_{\pm,j}^{k,L} \,:\, 1 \le j \le N\big\}.$$

The spaces $\mathbb{D}_{k,L}^{\mathrm{J},1}$ are equal to $\mathbb{D}_{k,L}^{\mathrm{J},0} = \mathbb{C}$. All other $\mathbb{D}^{\mathrm{J},1}(k, L;\, k', L')$ are zero.

An analog result holds for the skew-holomorphic slash action: The spaces $\mathbb{D}^{\mathrm{J,sk},1}(k, L;\, k \pm 2, L)$ are 1-dimensional, and the spaces $\mathbb{D}^{\mathrm{J,sk},1}(k, L;\, k \pm 1, L)$ are N-dimensional. They have bases given by

$$\mathbb{D}^{\mathrm{J,sk},1}(k, L;\, k \pm 2, L) = \mathrm{span}\big\{X_\mp^{\mathrm{sk},k,L}\big\},$$

$$\mathbb{D}^{\mathrm{J,sk},1}(k, L;\, k \pm 1, L) = \mathrm{span}\big\{Y_{\mp,j}^{\mathrm{sk},k,L} \,:\, 1 \le j \le N\big\}.$$

The spaces $\mathbb{D}_{k,L}^{\mathrm{J,sk},1}$ are equal to $\mathbb{D}_{k,L}^{\mathrm{J,sk},0} = \mathbb{C}$. All other $\mathbb{D}^{\mathrm{J,sk},1}(k, L;\, k', L')$ are zero.

The raising operators for the holomorphic slash action commute with one another, as do the lowering operators for the holomorphic slash action (but keep in mind that, for example, $X_+Y_+ = Y_+X_+$ means $X_+^{k+1,L}Y_+^{k,L} = Y_+^{k+2,L}X_+^{k,L}$). The same holds for the raising and lowering operators for the skew slash action. The *commutators between all other operators are*

$$[X_-,\, X_+] = -k, \quad [Y_{-,j},\, Y_{+,j'}] = i\L_{jj'}, \quad [X_-,\, Y_+] = -Y_-, \quad [Y_-,\, X_+] = Y_+$$

$$[X_-^{\mathrm{sk}},\, X_+^{\mathrm{sk}}] = -k, \quad [Y_{-,j}^{\mathrm{sk}},\, Y_{+,j'}^{\mathrm{sk}}] = i\L_{jj'}, \quad [X_-^{\mathrm{sk}},\, Y_+^{\mathrm{sk}}] = -Y_-^{\mathrm{sk}}, \quad [Y_-^{\mathrm{sk}},\, X_+^{\mathrm{sk}}] = Y_+^{\mathrm{sk}}.$$

PROPOSITION 3.7. *Any covariant differential operator of order r may be expressed as a linear combination of products of up to r raising and lowering operators. There is a unique such expression in which the raising operators are all to the left of the lowering operators.*

The expression of this form for the holomorphic Casimir operator is

$$\begin{aligned}\mathcal{C}^{\mathrm{J}}_{k,L} &= -2X_+X_- + i\big(X_+\L^{-1}[Y_-] - \L^{-1}[Y_+]X_-\big)\\ &\quad - \tfrac{1}{2}\big(\L^{-1}[Y_+]\L^{-1}[Y_-] - Y_+^{\mathrm{T}}(Y_+^{\mathrm{T}}\L^{-1}Y_-)\L^{-1}Y_-\big)\\ &\quad - \tfrac{1}{2}(2k-N-3)iY_+^{\mathrm{T}}\L^{-1}Y_-.\end{aligned}$$

The corresponding expression for the skew Casimir operator is obtained from this by adding superscripts sk *where applicable and subtracting the constant term* $(k-\frac{N}{2})(2k-N-1)$.

PROPOSITION 3.8. *The algebra $\mathbb{D}^{\mathrm{J}}_{k,L}$ is generated by $\mathbb{D}^{\mathrm{J},3}_{k,L}$. The spaces $\mathbb{D}^{\mathrm{J},3}_{k,L}$ and $\mathbb{D}^{\mathrm{J},2}_{k,L}$ are of dimensions $2N^2+N+2$ and N^2+2, respectively. Bases for them are given by the following equations:*

$$\begin{aligned}\mathbb{D}^{\mathrm{J},3}_{k,L} &= \mathrm{span}\big\{X_+Y_{-,i}Y_{-,j},\ Y_{+,i}Y_{+,j}X_- \,:\, 1\le i\le j\le N\big\} \oplus \mathbb{D}^2_{k,L},\\ \mathbb{D}^{\mathrm{J},2}_{k,L} &= \mathrm{span}\big\{1,\ X_+X_-,\ Y_{+,i}Y_{-,j} \,:\, 1\le i,j\le N\big\}.\end{aligned}$$

The corresponding result for the skew slash action is obtained by adding superscripts sk *where applicable.*

We end this section with a postponed proof, that we can complete with the help of covariant operators.

PROOF OF COROLLARY 5.3 IN CHAPTER 2. The limit $\lim_{y'\to\infty} f(Z)\, e^{-2\pi i m\tau'}$ is invariant under the full Jacobi group and has Jacobi index m. More precisely, it vanishes under $|^{\mathrm{J}}_{k,m}\,(1-g)$ or $|^{\mathrm{J,sk}}_{k,m}\,(1-g)$ for all $g\in\Gamma^J_1$. We only consider the first case; The second follows from a completely analogous calculation. By Theorem 5.2, the above limit also vanishes under

$$\partial_z\partial_{\overline{z}} = y^{-1}\big(i\partial_z - 4\pi m y^{-1}v + 4\pi m y^{-1}v\big)Y_-^{k,m}.$$

Hence it vanishes under $Y_+^{k,m}Y_-^{k,m} + 4\pi m y^{-1}vY_-^{k,m}$. The commutator of this operator and $1-(I_2,(0,1))\in\mathbb{Z}[\Gamma^J_1]$ is a nonzero multiple of $Y_-^{k,m}$, hence the result. $\square$

4. Harmonic Maaß-Jacobi forms

The focus of this chapter are the spaces of harmonic Maaß-Jacobi forms and harmonic skew-Maaß-Jacobi forms of index L and weight k. In order to define them, fix $k\in\mathbb{Z}$ and a positive definite integral even lattice L of rank N. We will identify L with its Gram matrix with respect to a fixed basis divided by 2, a positive definite symmetric matrix with entries in $\frac{1}{2}\mathbb{Z}$ and diagonal entries in $\mathbb{Z}$. Write $|L|$ for the covolume of the lattice, the determinant of the Gram matrix.

The full Jacobi group Γ^J_N defined in (1.1) clearly has a central extension by $\mathrm{M}^{\mathrm{T}}_N(\mathbb{Z})$ that is a subgroup of $\tilde{G}^J_N$. It is easy to check that when L is a Gram matrix of an integral lattice, the cocycle $\boldsymbol{\alpha}^{\mathrm{J}}_L$ is trivial on $\mathrm{M}^{\mathrm{T}}_N(\mathbb{Z})$. Therefore the $|^{\mathrm{J}}_{k,L}$ and the $|^{\mathrm{J,sk}}_{k,L}$-actions factor through to actions of Γ^J_N, which we will also denote by $|^{\mathrm{J}}_{k,L}$ and $|^{\mathrm{J,sk}}_{k,L}$.

The next definitions are inspired by a direct adoption of the definition of automorphic forms in the case of reductive groups (see [**Bor66**]). The definition for holomorphic slash actions and $N = 1$ was suggested in [**BR10**], the one for the skew slash action and $N = 1$ in [**BRR11a**].

DEFINITION 4.1 (Maaß-Jacobi forms [**CR11**]). *A Maaß-Jacobi form of weight k and index L is a real-analytic function $\phi : \mathbb{H}_{1,N} \to \mathbb{C}$ satisfying the following conditions:*

(i) *For all $A \in \Gamma^J_N$, we have $\phi|^{\mathrm{J}}_{k,L} A = \phi$.*
(ii) *ϕ is an eigenfunction of $\mathcal{C}^{\mathrm{J}}_{k,L}$.*
(iii) *For some $a > 0$, $\phi(\tau, z) = O\left(e^{ay} e^{2\pi \frac{L[v]}{y}}\right)$ as $y \to \infty$.*

If ϕ is annihilated by the Casimir operator $\mathcal{C}^{k,L}$, it is said to be a harmonic Maaß-Jacobi form. We denote the space of all harmonic Maaß-Jacobi forms of fixed weight k and index L by $\mathbb{J}_{k,L}$.

DEFINITION 4.2 (skew Maaß-Jacobi forms). *A skew Maaß-Jacobi form of weight k and index L is a real-analytic function $\phi : \mathbb{H}_{1,N} \to \mathbb{C}$ satisfying the following conditions:*

(i) *For all $A \in \Gamma^J_N$, we have $\phi|^{\mathrm{J,sk}}_{k,L} A = \phi$.*
(ii) *ϕ is an eigenfunction of $\mathcal{C}^{\mathrm{J,sk}}_{k,L}$.*
(iii) *For some $a > 0$, $\phi(\tau, z) = O\left(e^{ay} e^{2\pi \frac{L[v]}{y}}\right)$ as $y \to \infty$.*

If ϕ is annihilated by the Casimir operator $\mathcal{C}^{\mathrm{J,sk}}_{k,L}$, it is said to be a harmonic skew Maaß-Jacobi form. We denote the space of all harmonic skew Maaß-Jacobi forms of fixed weight k and index L by $\mathbb{J}^{\mathrm{sk}}_{k,L}$.

In general, the space of functions $f(\tau, z) = a(y, v; n, r) e(nx) e(r^{\mathrm{T}} v)$ satisfying either $\mathcal{C}^{\mathrm{J}}_{k,L} f = 0$ or $\mathcal{C}^{\mathrm{J,sk}}_{k,L} f = 0$ is infinite dimensional. That is, a single differential equation is imposed on an $N + 1$ variable function. This is no problem from the representation theoretic point of view, but for applications it is impractical. One theorem in [**BS98**] is particularly relevant to the discussion what the right definition of real-analytic Jacobi forms should be. It only concerns automorphic forms for $\tilde{G}^J_1$ that satisfy a stronger version of (iii) in Definition 4.1. Since we do not give any details and the growth condition that we impose is weaker, we formulate this theorem and its generalization as a remark.

REMARK 4.3. *Adapting the proof in* [**BS98**, Section 2.6]*, which is based on* [**LV80**, Section 1.3] *and* [**MVW87**, Section 2.I.2]*, we see that any automorphic representation of $\tilde{G}^J_N$ is a tensor product $\tilde{\pi} \otimes \pi^L_{\mathrm{SW}}$. Here $\tilde{\pi}$ is a genuine representation of the metaplectic cover of SL_2, and π^L_{SW} is the Schrödinger-Weil representation of central character L. The latter is the extension to the metaplectic cover of the Jacobi group of the Schrödinger representation of the Heisenberg group, which is induced from the character $e^{2\pi i \mathrm{tr}(L\kappa)}$ of its center. Thus, as in* [**Pit09**]*, semi-holomorphic forms play an important role in the representation-theoretic treatment of harmonic Maaß-Jacobi forms and skew Maaß-Jacobi forms.*

Using the operators $Y_\pm$ one can define a filtration of subspaces of all harmonic Maaß-Jacobi forms and skew Maaß-Jacobi forms. If we disregard singularities, these subspaces contain Zwegers's μ-function [**Zwe02**] and the multivariable Appell sums [**Zwe10**], after the definition is extended to half-integral weights. Those filtrations seem the most promising restriction of Definition 4.1 and 4.2, that cuts out spaces of Jacobi forms that are relevant to applications. In [**CR11**, Section 3] and [**BRR11b**], this approach is discussed in greater detail.

DEFINITION 4.4. *Let $0 < l \in \frac{1}{2}\mathbb{Z}$. A real-analytic function $\phi : \mathbb{H}_{1,N} \rightarrow \mathbb{C}$ satisfying either $\mathcal{C}^{\mathrm{J}}_{k,L} f = 0$ or $\mathcal{C}^{\mathrm{J,sk}}_{k,L} f = 0$ is said to have analytic order l in the Heisenberg part, if*

$$\begin{aligned} (Y_+ Y_-)^l f = 0, &\quad \text{if } l \in \mathbb{Z}; \\ Y_- (Y_+ Y_-)^{\lfloor l \rfloor} f = 0, &\quad \text{if } l \notin \mathbb{Z}. \end{aligned}$$

REMARKS 4.5.

(i) *Semi-holomorphic, harmonic functions form a special case of the above definition. They have analytic order $\frac{1}{2}$ in the Heisenberg part.*
(ii) *We will see that Zwegers's μ-function and the multivariable Appell function, ignoring singularities and the resulting problems with the growth condition, have analytic order 1 in the Heisenberg part. This is discussed in Section 7.*
(iii) *The conditions in the above definition are Hecke equivariant, so that it makes sense to look for Hecke eigenforms in the space of semi-holomorphic forms and space of forms of the kind that Zwegers has considered.*
(iv) *No example of forms with analytic order greater than 1 is known to the author. The considerations in* [**BRR11b**] *show that such a function must have truly real-analytic, i.e., nonholomorphic singularities.*

The next theorem illustrates how rigid the Fourier expansions of finite analytic order in the Heisenberg part are.

THEOREM 4.6. *The space of functions $f(\tau, z) = a(y, v; n, r)\, e(nx + r^{\mathrm{T}} u) \in C^\infty(\mathbb{H}_{1,N})$ with $\mathcal{C}^{\mathrm{J}}_{k,L} f = 0$ or $\mathcal{C}^{\mathrm{J,sk}}_{k,L} f = 0$ and analytic order l in the Heisenberg part has dimension less than $4l$.*

PROOF. It suffices to prove that the intersections of the kernel $\ker Y_\pm$ and $\ker \mathcal{C}^{\mathrm{J}}_{k,L}$, and $\ker Y_\pm$ and $\ker \mathcal{C}^{\mathrm{J}}_{k,L}$ on the space of functions $f(\tau, z) = a(y, v; n, r) e(nx + r^{\mathrm{T}} u)$ have dimension at most 2. The kernel of $Y_\pm$ on functions $\breve{a}(v; r) e(r^{\mathrm{T}} u)$ has dimension 1. Indeed,

$$Y_+\, \breve{a}(v; r)\, e(r^{\mathrm{T}} u) = y\big(\pi r \breve{a}(v; r) + \tfrac{1}{2} \partial_v \breve{a}(v; r)\big)\, e(r^{\mathrm{T}} u)$$

and

$$Y_-\, \breve{a}(v; r)\, e(r^{\mathrm{T}} u) = \big(-\pi r \breve{a}(v; r) + \tfrac{1}{2} \partial_v \breve{a}(v; r) + 2iy^{-1} \L v \breve{a}(v; r)\big)\, e(r^{\mathrm{T}} u).$$

This leads to an order 1 ordinary differential equation for $\breve{a}$. We conclude that for fixed τ there is at most one $a(y, v; n, r)$ such that $f(\tau, z)$ lies in the kernel of $Y_\pm$. More precisely, any such a splits as a product $\tilde{a}(y; n) \breve{a}(v; r)$. Applying $\mathcal{C}^{\mathrm{J}}_{k,L}$ and $\mathcal{C}^{\mathrm{J,sk}}_{k,L}$ to the corresponding f gives rise to an order 2 differential equation for $\tilde{a}$. This proves the claim. $\square$

5. Semi-holomorphic Maaß-Jacobi forms

Recall that elements of $C^\infty(\mathbb{H}_{1,N})$ holomorphic in $\mathbb{C}^N$ are called semi-holomorphic. We will denote the space of semi-holomorphic harmonic Maaß-Jacobi forms by $\mathbb{J}^{\mathrm{z}}_{k,L}$, and we will write $\mathbb{J}^{\mathrm{sk,z}}_{k,L}$ for the space of semi-holomorphic harmonic skew-Maaß-Jacobi forms. Semi-holomorphic forms vanish under Y_- and Y_-^{sk}.

The theory of semi-holomorphic forms essentially mimics that of harmonic weak Maaß forms. Indeed, in Theorem 5.5 we will see that the θ-decomposition gives a well-behaved bijection between vector-valued weak harmonic Maaß forms and harmonic semi-holomorphic (skew-)Maaß-Jacobi forms.

We first discuss semi-holomorphic Fourier expansions of (skew-)Maaß-Jacobi forms. Recall that the negative discriminant of a Fourier index (n,r) is denoted by

$$D := D_L(n,r) := |L|(4n - L^{-1}[r]).$$

By analogy with [**BF04**, page 55], define a function

$$H(y) := e^{-y} \int_{-2y}^{\infty} e^{-t} t^{-k+\frac{N}{2}}\, dt.$$

Proposition 5.1. *Any semi-holomorphic harmonic Maaß-Jacobi form has a Fourier expansion of the form*

$$\begin{aligned} y^{\frac{2+N}{2}-k} \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^N\\ s.t.\ D=0}} c^0(n,r)\, q^n\zeta^r + \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^N\\ s.t.\ D\gg-\infty}} c^+(n,r)\, q^n\zeta^r \\ + \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^N\\ s.t.\ D\ll\infty}} c^-(n,r)\, H\big(\pi \tfrac{D}{2|L|} y\big) \exp\big(\tfrac{\pi D}{2|L|} y\big)\, q^n\zeta^r. \end{aligned}$$

Any semi-holomorphic harmonic skew-Maaß-Jacobi forms has a Fourier expansion of the form

$$\begin{aligned} y^{\frac{2+N}{2}-k} \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^N\\ s.t.\ D=0}} c^0(n,r)\, q^n\zeta^r + \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^N\\ s.t.\ D\ll\infty}} c^+(n,r)\, e^{-\pi y \frac{D}{|L|}} q^n\zeta^r \\ + \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^N\\ s.t.\ D\gg-\infty}} c^-(n,r)\, H\big(-\pi \tfrac{D}{2|L|} y\big) \exp\big(\tfrac{\pi D}{2|L|} y\big)\, q^n\zeta^r. \end{aligned}$$

Proof. This can be proved as in the case of rank 1 lattices, by solving the differential equation for the coefficients coming from the Casimir operator and then imposing the growth condition. □

Our investigation will concentrate on semi-holomorphic harmonic (skew-)Maaß-Jacobi forms, and in particular their relation to (skew-)holomorphic forms. To state this relation we must define two ξ-operators. Proceeding as in [**BR10**, Section 4], we first define the lowering operators

$$D_-^{(L)} := -2iy\big(y\,\partial_{\overline{\tau}} + v^{\mathrm{T}}\partial_{\overline{z}} - \tfrac{1}{4} y \L^{-1}[\partial_{\overline{z}}]\big) \;=\; X_- - \tfrac{i}{2}\L^{-1}[Y_-], \tag{5.1}$$

$$D_+^{\mathrm{sk},(L)} := iy^2(2\partial_\tau + \tfrac{1}{2}\L^{-1}[\partial_z]) = X_+^{\mathrm{sk}} - \tfrac{i}{2}\L^{-1}[Y_+] = iy^2\mathbb{L}_L. \tag{5.2}$$

Using these operator, we define the ξ-operators by

$$\xi^{\mathrm{J}}_{k,L} := y^{k-\frac{N}{2}-2} D_-^{(L)} \quad\text{and}\quad \xi^{\mathrm{J,sk}}_{k,L} := y^{k-\frac{N}{2}-2} D_+^{\mathrm{sk},(L)}. \tag{5.3}$$

This is an analog of the ξ-operator in [**Maa52**], which sends Maaß forms to their holomorphic shadows. In our setting skew-holomorphic forms take the place of holomorphic ones when ξ^{J} is applied.

PROPOSITION 5.2. *If* $\phi \in \mathbb{J}^{\mathrm{z}}_{k,L}$*, then* $\xi^{\mathrm{J}}_{k,L}\phi$ *is an element of* $J^{\mathrm{sk}}_{2+N-k,L}$*. If* $\phi \in \mathbb{J}^{\mathrm{sk,z}}_{k,L}$*, then* $\xi^{\mathrm{J,sk}}_{k,L}\phi$ *is an element of* $J_{2+N-k,L}$*.*

PROOF. By Proposition 3.6, $D^{\mathrm{J},(L)}_{-}$ is a covariant operator from $|^{\mathrm{J}}_{k,L}$ to $|^{\mathrm{J}}_{k-2,L}$. Applying $\xi^{\mathrm{J}}_{k,L}$ to the Fourier expansion of a Maaß-Jacobi form as in Proposition 5.1 shows that the Fourier expansion of $\xi_{k,L}\phi$ has the correct form. The analog argument works for skew-Maaß-Jacobi forms, again using Proposition 5.1 to verify that the Fourier expansion of the image has the correct form. □

The ξ-operator is compatible with the theta decomposition. To state this precisely, let $\mathrm{M}\Gamma$ be the full elliptic metaplectic group. Denote the spaces of vector-valued harmonic Maaß forms for the Weil representation ρ_L by $[\mathrm{M}\Gamma, k - \frac{N}{2}, \rho_L]^{\mathrm{M}}$. For weakly holomorphic vector-valued modular forms change the superscript to !. The ξ-operator $\xi_{k-\frac{N}{2}} f = y^{k-\frac{N}{2}}\, \overline{\partial_{\overline{\tau}} f}$ maps the space of harmonic Maaß forms to the space of weakly holomorphic forms.

Recall that $\theta_{L,\mu}$ is the a theta series for L:

$$\theta_{L,\mu}(\tau, z) := \sum_{r \in \mathbb{Z}^N,\, r \equiv \mu (L\mathbb{Z}^N)} q^{\frac{L^{-1}[r]}{4}} \zeta^r.$$

It is well-known to be a modular form in $[\mathrm{M}\Gamma, k - \frac{N}{2}, \overline{\rho_L}]^{\mathrm{M}}$.

DEFINITION 5.3 (Theta decomposition). *The Maaß-Jacobi and the skew-Maaß-Jacobi theta decompositions are the maps*

$$\begin{aligned} \theta^{\mathrm{z}}_L &: \mathbb{J}^{\mathrm{z}}_{k,L} \to [\mathrm{M}\Gamma, k - \tfrac{N}{2}, \rho_L]^{\mathrm{M}} \quad \textit{and} \\ \theta^{\mathrm{sk,z}}_L &: \mathbb{J}^{\mathrm{sk,z}}_{k,L} \to [\mathrm{M}\Gamma, k - \tfrac{N}{2}, \rho_L]^{\mathrm{M}} \end{aligned}$$

defined by

$$\begin{aligned} f(\tau, z) &= \sum_{\mu (\mathbb{Z}^N / L\mathbb{Z}^N)} \theta^{\mathrm{z}}_L(f)_\mu(\tau)\, \theta_{L,\mu}(\tau, z) \\ f(\tau, z) &= \sum_{\mu (\mathbb{Z}^N / L\mathbb{Z}^N)} \overline{\theta^{\mathrm{sk,z}}_L(f)_\mu(\tau)}\, \theta_{L,\mu}(\tau, z). \end{aligned}$$

The holomorphic and the skew-holomorphic theta decomposition maps

$$\begin{aligned} \theta_L &: J_{k,L} \to [\mathrm{M}\Gamma, k - \tfrac{N}{2}, \rho_L] \quad \textit{and} \\ \theta^{\mathrm{sk}}_L &: J^{\mathrm{sk}}_{k,L} \to [\mathrm{M}\Gamma, k - \tfrac{N}{2}, \rho_L] \end{aligned}$$

are defined by

$$\begin{aligned} f(\tau, z) &= \sum_{\mu (\mathbb{Z}^N / L\mathbb{Z}^N)} \theta_L(f)_\mu(\tau)\, \theta_{L,\mu}(\tau, z) \quad \textit{and} \\ f(\tau, z) &= \sum_{\mu (\mathbb{Z}^N / L\mathbb{Z}^N)} \overline{\theta^{\mathrm{sk}}_L(f)_\mu(\tau)}\, \theta_{L,\mu}(\tau, z). \end{aligned}$$

REMARK 5.4. *A harmonic (skew-)Maaß form admits a theta decomposition if and only if it is semi-holomorphic.*

THEOREM 5.5. *The θ-decomposition of forms in $\mathbb{J}^{\mathrm{z}}_{k,L}$ and $\mathbb{J}^{\mathrm{sk}}_{2+N-k,L}$ commutes with the ξ-operators $\xi^{\mathrm{J}}_{k,L}$ and $\xi^{\mathrm{J}}_{k-\frac{N}{2}}$. More precisely, the following diagram is commutative:*

$$\begin{tikzcd}
\mathbb{J}^{\mathrm{z}}_{k,L} \arrow[r, "\xi^{\mathrm{J}}_{k,L}"] \arrow[d, "\theta^{\mathrm{z}}_L"] & \mathbb{J}^{\mathrm{sk}}_{2+N-k,L} \arrow[d, "\theta^{\mathrm{sk}}_L"] \\
{[\mathrm{M}\Gamma, k-\frac{N}{2}, \rho_L]^{\mathrm{M}}} \arrow[r, "\xi_{k-\frac{N}{2}}"] & {[\mathrm{M}\Gamma, 2+\frac{N}{2}-k, \rho_L]^{!}}
\end{tikzcd}.$$

The analog diagram for skew-Maaß-Jacobi forms is commutative as well:

$$\begin{tikzcd}
\mathbb{J}^{\mathrm{sk,z}}_{k,L} \arrow[r, "\xi^{\mathrm{J,sk}}_{k,L}"] \arrow[d, "\theta^{\mathrm{sk,z}}_L"] & \mathbb{J}_{2+N-k,L} \arrow[d, "\theta_L"] \\
{[\mathrm{M}\Gamma, k-\frac{N}{2}, \rho_L]^{\mathrm{M}}} \arrow[r, "\xi_{k-\frac{N}{2}}"] & {[\mathrm{M}\Gamma, 2+\frac{N}{2}-k, \rho_L]^{!}}
\end{tikzcd}.$$

PROOF. This is a calculation completely analogous to that in [**BR10**, Section 6]. $\square$

6. Higher analytic order in the Heisenberg part

We briefly treat forms of analytic order greater than $\frac{1}{2}$, which we defined in 4.4. The next proposition focuses on Hecke operators. For $\phi \in \mathbb{J}_{k,L}$ and $l \in \mathbb{Z}_{\geq 1}$ recall the Hecke operators

$$\phi\big|^{\mathrm{J}}_{k,L} T_l := l^{k-4} \sum_{\substack{g \in \Gamma^J_N \backslash \mathrm{M}_2(\mathbb{Z}) \\ \det(M)=l^2 \\ \gcd(M)=\square}} \; \sum_{X \in l\mathrm{M}_{2,N}(\mathbb{Z}) \backslash \mathrm{M}_{2,N}(\mathbb{Z})} \phi\big|^{\mathrm{J}}_{k,L} (g, X),$$

where $\gcd(M) = \square$ means that the greatest common divisor of the entries of M is a square. In the case of $N = 1$, this operators has been defined in [**EZ85**]. The natural analog for skew weights is

$$\phi\big|^{\mathrm{J,sk}}_{k,L} T_l := l^{k-4} \sum_{\substack{g \in \Gamma^J_N \backslash \mathrm{M}_2(\mathbb{Z}) \\ \det(M)=l^2 \\ \gcd(M)=\square}} \; \sum_{X \in l\mathrm{M}_{2,N}(\mathbb{Z}) \backslash \mathrm{M}_{2,N}(\mathbb{Z})} \phi\big|^{\mathrm{J,sk}}_{k,L} (g, X).$$

The former maps $\mathbb{J}_{k,L}$ to $\mathbb{J}_{k,L}$ and the latter maps $\mathbb{J}^{\mathrm{sk}}_{k,L}$ to $\mathbb{J}^{\mathrm{sk}}_{k,L}$. A (skew-)Maaß-Jacobi forms that is an eigenvector of all the T_l, $l \in \mathbb{Z}_{\geq 1}$ is called a Hecke eigenform.

By Remark 4.3, it follows that for an automorphic form of $\tilde{G}^J_N$, that is, a harmonic Maaß-Jacobi form with an automorphic representation attached to it, there is always a semi-holomorphic one that has the same eigenvalues. This is not necessarily true for all harmonic (skew-)Maaß-Jacobi forms, defined in 4.1 and 4.2, since the growth conditions imposed are too weak. The next proposition tells us that at least for (skew-)Maaß-Jacobi forms of finite analytic order in the Heisenberg part a similar reduction theorem holds.

PROPOSITION 6.1. *Suppose that a nonzero Hecke eigenform $\phi \in \mathbb{J}_{k,L}$ has analytic order $0 < l \in \frac{1}{2}\mathbb{Z}$ in the Heisenberg part. Then there is a nonvanishing Hecke eigenform $\tilde{\phi} \in \mathbb{J}_{k,L}$ with the same Hecke eigenvalues that has analytic order $0 < l \leq 1$ in the Heisenberg part.*

The analog statement holds for skew-Maaß-Jacobi forms $\phi \in \mathbb{J}^{\mathrm{sk}}_{k,L}$.

Proof. We may assume that $0 < l \in \frac{1}{2}\mathbb{Z}$ is minimal. By definition, the harmonic Jacobi form $\tilde{\phi} := (Y_+ Y_-)^{\lfloor l \rfloor}\, \phi$ is nonzero if $l \notin \mathbb{Z}$, and $\tilde{\phi} := (Y_+ Y_-)^{l-1}\, \phi$ is nonzero if $l \in \mathbb{Z}$. In either case, $\tilde{\phi}$ has the same eigenvalues like ϕ, since it is the image under covariant operators. Thus the theorem is proved. $\square$

Remarks 6.2.

(i) *In the next section, we will see that Zweger's μ function has analytic order* 1 *in the Heisenberg part. Under Y_- it is mapped to a meromorphic Jacobi form. The same holds true for the more general Appell sum divided by a suitable theta series.*

(ii) *Assuming that indeed semi-holomorphic forms provide all Hecke eigensystems of harmonic (skew-)Maaß-Jacobi forms of finite analytic order in the Heisenberg part, the theory of such Hecke eigensystems should be governed by the one corresponding theory for weak harmonic vector-valued elliptic modular forms. Such a theory was initiated in* [**BS10**]*, but the full picture is not yet complete.*

(iii) *The results obtained in* [**BRR11b**] *show that there cannot be any non-singular Maaß-Jacobi form of analytic order* 1 *in the Heisenberg part.*

7. Examples

7.1. Eisenstein series. There are two harmonic Jacobi Eisenstein series. To define them, denote by $\Gamma^J_{N,\infty}$ the parabolic subgroup of the full Jacobi group Γ^J_N:

$$\Gamma^J_{N,\infty} := \Big\{ \big(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), (\mu, 0) \big) \in \Gamma^J_N \Big\}. \tag{7.1}$$

Define the Jacobi Eisenstein series

$$E^{\mathrm{J}}_{\alpha,\beta,L} := \sum_{g \in \Gamma^J_{N,\infty} \backslash \Gamma^J_N} 1 |^{\mathrm{J}}_{\alpha,\beta,L}\, g. \tag{7.2}$$

The right hand side converges locally absolutely uniformly, if $\alpha + \beta > 2 + N$.

The Eisenstein series can be generalized to Poincaré-Eisenstein series for the holomorphic and skew slash action, which feature an addition y power:

$$P^{\mathrm{J}}_{k,s,L} := \sum_{g \in \Gamma^J_{N,\infty} \backslash \Gamma^J_N} y^s |^{\mathrm{J}}_{k,L}\, g = y^s E^{\mathrm{J}}_{k+s,s,L} \tag{7.3}$$

and

$$P^{\mathrm{J,sk}}_{k,s,L} := \sum_{g \in \Gamma^J_{N,\infty} \backslash \Gamma^J_N} y^s |^{\mathrm{J,sk}}_{k,L}\, g = y^s E^{\mathrm{J}}_{\frac{1}{2}+s, k-\frac{1}{2}+s, L}. \tag{7.4}$$

Both are harmonic for $s = 0, k > 2+N$ and $s = \frac{2+N}{2} - k, k < 0$. Note that Arakawa considered various Eisenstein series in [**Ara90**].

For later use, we give the precise image of the Poincaré-Eisenstein series under the ξ-operators.

Proposition 7.1. *We have*

$$\xi^{\mathrm{J}}_{k,L} P^{\mathrm{J}}_{k,\frac{2+N}{2}-k,L} = \big(\tfrac{2+N}{2} - k\big) P^{\mathrm{J,sk}}_{2+N-k,0,L} \quad \textit{and}$$
$$\xi^{\mathrm{J,sk}}_{k,L} P^{\mathrm{J,sk}}_{k,\frac{2+N}{2}-k,L} = \big(\tfrac{2+N}{2} - k\big) P^{\mathrm{J}}_{2+N-k,0,L}.$$

Proof. It suffices to apply $\xi^{\mathrm{J}}_{k,L}$ and $\xi^{\mathrm{J,sk}}_{k,L}$ to $y^{\frac{2+N}{2}-k}$, yielding the result. $\square$

7.2. Poincaré series. In [**BR10**, Section 5], the authors define Maaß-Poincaré series for the Jacobi group. They restrict to Jacobi indices of rank one. Their considerations were generalized in [**CR11**, Section 4.1] to arbitrary lattice indices. The results obtained there are easily carried over to skew-Maaß-Jacobi forms.

Throughout this section, n will be an integer and r will be in $\mathbb{Z}^N$. Maintain D as above and set h as follows:

$$D = D_L(n, r) = |L|(4n - L^{-1}[r]), \qquad h = h_L(r) := |L|\, L^{-1}[r].$$

Using the M-Whittaker function $M_{\nu,\mu}$ (see [**WW96**]), we define

$$\mathcal{M}_{s,\kappa}(t) := |t|^{-\frac{\kappa}{2}} M_{\mathrm{sgn}(t)\frac{\kappa}{2},\, s-1/2}(|t|), \tag{7.5}$$

$$\phi^{(n,r)}_{k,L,s}(\tau, z) := \mathcal{M}_{s,k-N/2}(\pi \tfrac{D}{|L|} y)\; e(r^{\mathrm{T}} z + i \tfrac{L^{-1}[r]}{4} y + nx), \tag{7.6}$$

$$\phi^{\mathrm{sk},(n,r)}_{k,L,s}(\tau, z) := \mathcal{M}_{s,k-N/2}(\pi \tfrac{D}{|L|} y)\; e(r^{\mathrm{T}} z + \tfrac{L^{-1}[r]}{4} \tau - \tfrac{D}{4|L|} x). \tag{7.7}$$

Lemma 7.2. *The function $\phi^{(n,r)}_{k,L,s}$ defined in* (7.6) *is an eigenfunction of the Casimir operator $\mathcal{C}^{\mathrm{J}}_{k,L}$ in* (3.2)*, with eigenvalue*

$$-2s(1-s) - \tfrac{1}{2}\Big(k^2 - k(N+2) + \tfrac{1}{4} N(N+4)\Big). \tag{7.8}$$

The function $\phi^{\mathrm{sk},(n,r)}_{k,L,s}$ defined in (7.7) *is an eigenfunction of the Casimir operator $\mathcal{C}^{\mathrm{J,sk}}_{k,L}$ in* (3.3)*, with eigenvalue*

$$-2\overline{s}(1-\overline{s}) - \tfrac{1}{2}\Big(k^2 - k(N+2) + \tfrac{1}{4} N(N+4)\Big). \tag{7.9}$$

Proof. Factor ϕ as follows:

$$\phi^{(n,r)}_{k,L,s}(\tau, z) = e(r^{\mathrm{T}} z + \tfrac{L^{-1}[r]}{4} \tau) \;\cdot\; e(\tfrac{D}{4|L|} x)\, \mathcal{M}_{s,k-\frac{N}{2}}(-\pi \tfrac{D}{|L|} y). \tag{7.10}$$

The first factor is holomorphic in τ and the second is constant in z. Hence when applying $\mathcal{C}^{\mathrm{J}}_{k,L}$ the contribution of the first factor cancels. We need only consider $-2\Delta_{k-\frac{N}{2}}$, yielding (7.8).

The analog consideration applied to the factorization

$$\phi^{\mathrm{sk},(n,r)}_{k,L,s}(\tau, z) = e(r^{\mathrm{T}} z + \tfrac{L^{-1}[r]}{4} \tau) \;\cdot\; e(-\tfrac{D}{4|L|} x)\, \mathcal{M}_{s,k-\frac{N}{2}}(-\pi \tfrac{D}{|L|} y)$$

yields (7.9). □

The Poincaré series

$$P^{(n,r)}_{k,L,s} := \sum_{g \in \Gamma^J_{N,\infty} \backslash \Gamma^J_N} \phi^{(n,r)}_{k,L,s}\Big|^{\mathrm{J}}_{k,L}\, g \tag{7.11}$$

for the holomorphic slash action was introduced in [**CR11**], and it is easily seen to be semi-holomorphic. We will also consider the skew Poincaré series

$$P^{\mathrm{sk},(n,r)}_{k,L,s} := \sum_{g \in \Gamma^J_{N,\infty} \backslash \Gamma^J_N} \phi^{\mathrm{sk},(n,r)}_{k,L,s}\Big|^{\mathrm{J,sk}}_{k,L}\, g. \tag{7.12}$$

The usual estimate

$$\mathcal{M}_{s,k-N/2}(y) \ll y^{\mathfrak{Re}(s) - \frac{2k-N}{4}} \quad \text{as} \quad y \to 0$$

ensures absolute and uniform convergence for $\mathfrak{Re}(s) > 1+\frac{N}{4}$. Of particular interest is the case $s \in \{\frac{k}{2}-\frac{N}{4},\, 1+\frac{N}{4}-\frac{k}{2}\}$, where the Poincaré series is annihilated by the Casimir operator. In particular, we have proved

THEOREM 7.3. *For $k<0$, the maps*

$$\begin{aligned} \xi^{\mathrm{J}}_{k,L} &: \mathbb{J}^{\mathrm{z}}_{k,L} \to J^{\mathrm{sk}}_{2+N-k,L} \quad \text{and} \\ \xi^{\mathrm{J,sk}}_{k,L} &: \mathbb{J}^{\mathrm{sk,z}}_{k,L} \to J_{2+N-k,L} \end{aligned}$$

are surjective.

PROOF. The Poincaré-Eisenstein series $P^{\mathrm{J}}_{k,\frac{2+N}{2}-k,L}$ and $P^{\mathrm{J,sk}}_{k,\frac{2+N}{2}-k,L}$ are mapped to the holomorphic and skew-holomorphic Eisenstein series. Hence it is sufficient to consider cusp forms in $J_{2+N-k,L}$ and $J^{\mathrm{sk}}_{2+N-k,L}$.

We will use the well-known identity:

$$\mathcal{M}_{1+\frac{N}{4}-\frac{k}{2},k-\frac{N}{2}}(-y) = (k-\tfrac{N}{2}-1)e^{y/2}\big(\Gamma(1+\tfrac{N}{2}-k,y) - \Gamma(1+\tfrac{N}{2}-k)\big).$$

The operator $\xi^{\mathrm{J}}_{k,L}$ is covariant. For $s = 1+\frac{N}{4}-\frac{k}{2}$ the Poincaré series $P^{(n,r)}_{k,L,s}$ is locally absolutely convergent, if $k<0$. Thus, in this case, we may compute the images of the Poincaré series under the ξ-operators using (7.10).

$$\begin{aligned} \xi^{\mathrm{J}}_{k,L} P^{(n,r)}_{k,L,s} = &\sum_{g\in\Gamma^J_{N,\infty}\backslash\Gamma^J_N} e(rz + \tfrac{L^{-1}[r]}{4}\tau) y^{k-2-\frac{N}{2}} \\ &\qquad \cdot \Big(-2iy^2\partial_{\overline{\tau}}\, e(\tfrac{-Dx}{4|L|})\,(k-\tfrac{N}{2}-1)e^{\pi\frac{D}{2|L|}y} \\ &\qquad\quad \cdot \big(\Gamma(1+\tfrac{N}{2}-k, \tfrac{\pi D y}{|L|}) - \Gamma(1+\tfrac{N}{2}-k)\big)\Big) \\ = &\sum_{g\in\Gamma^J_{N,\infty}\backslash\Gamma^J_N} e(rz+\tfrac{L^{-1}[r]}{4}\tau) \\ &\qquad \cdot (-2)ie(\tfrac{-D\tau}{4|L|})\,(k-\tfrac{N}{2}-1)\big(\tfrac{\pi D}{|L|}\big)^{1+\frac{N}{2}-k} e^{-\pi\frac{D}{|L|}y}. \end{aligned}$$

This shows that up to multiplicative scalars, the image is the skew-holomorphic Poincaré series

$$\xi^{\mathrm{J}}_{k,L} P^{(n,r)}_{k,L,s} = \sum_{g\in\Gamma^J_{N,\infty}\backslash\Gamma^J_N} e(rz+\tfrac{L^{-1}[r]}{4}\tau)\, e(-\tfrac{D}{4|L|}\tau)\, e^{-\pi\frac{D}{|L|}y}.$$

By a standard argument, that involves the Petersson scalar product, one can show that these series span the space of cuspforms in $J^{\mathrm{sk}}_{k,L}$. This proves the surjectivity of the first map.

To prove the second part apply (3.4) to the factorization (7.2) and use the fact that the heat operator (2.1) annihilates theta series. □

REMARK 7.4. *A Zagier type duality holds for the coefficients of the Poincaré series for the holomorphic slash action as was proved in* [**BR10, CR11**]. *It is not hard to see that the same duality holds for the Poincaré series for the skew slash action as well; See also Remark (d) on page 15 of* [**BRR11a**].

7.3. Higher Appell functions. In [**Zwe10**], Zwegers generalized his investigation of the so called μ-function (see [**Zwe02**]) to a more general set of functions. Changing slightly his notation, we write Q be a positive quadratic form on $\mathbb{R}^N$ with Gram matrix L. Let $B(l, l') := Q(l + l') - Q(l) - Q(l')$ be the associated bilinear form and $\lambda \in L^{-1}\mathbb{Z}$. For $\tilde{v} \in \mathbb{C}$ define

$$A_{Q,\lambda}(\tilde{v}, v; \tau) := \sum_{l \in \mathbb{Z}^N} \frac{q^{Q(l)} e\big(B(l, v)\big)}{1 - q^{B(l,\lambda)} e(\tilde{v})}. \tag{7.13}$$

Fixing $u \in \mathbb{C} \setminus \mathbb{Z} + \tau\mathbb{Z}$ and dividing by the theta series

$$\Theta_Q(v; \tau) := \sum_{l \in \mathbb{Z}^N} q^{Q(l)} e\big(B(l, v)\big),$$

we obtain the holomorphic part of a harmonic Jacobi-Maaßform with poles of weight $\frac{1}{2}$ and index L. The corresponding completion is given in Definition 1.5 of [**Zwe10**], and Theorem 1.7 of [**Zwe10**] shows that this completion is modular with respect to a subgroup of Γ^J_N. From the considerations to be found there, it becomes also clear that $A_{Q,\lambda}$ has analytic order 1 in the Heisenberg part. It does not fall under Definition 4.1, however, because it has singularities.

CHAPTER 4

Siegel modular forms

This section contains the definition of real-analytic Siegel modular forms and of harmonic real-analytic Siegel modular forms. The spirit of the former definition dates back to, for example, Borel [**Bor66**], whose primary motivation originated in automorphic representations. The representations at infinity of automorphic representations over $\mathbb{Q}$ are usually described as $(\mathfrak{g}, K)$-modules [**Wal88**]. This naturally leads to the condition that a real-analytic automorphic form must be an eigenfunction of all Casimir operators. From the strong approximation theorems (see [**Shi64**] for the symplectic group), it follows that real-analytic automorphic forms generate an automorphic representation.

For arithmetic applications one needs to impose more conditions. In the elliptic case, that is, for the reductive group SL_2, harmonicity is usually required. A function on $\mathbb{H}_1$ is harmonic of weight k if it is in the kernel of the Laplace operator Δ_k, given in (1.4) of Chapter 3. For fixed $n \in \mathbb{Z}$, the space of possible Fourier coefficients $a(y,n)$ of a smooth function $\sum_{n\in\mathbb{Z}} a(y,n)e(nx) \in \ker \Delta_k$ is two-dimensional over $\mathbb{C}$.

In the case of Siegel modular forms the Casimir operators are $H^{[1]}_{\alpha,\beta}$ and $H^{[2]}_{\alpha,\beta}$, which have been defined in (4.1) of Chapter 2. By the results of Section 2, it is insufficient to consider functions in the kernel of these operators. A further analytic condition for harmonic Siegel modular forms that promise to be useful for applications is needed. Vanishing under the matrix-valued Laplace operators Ω_k or Ω_k^{sk} that are defined in (4.3) of Chapter 2, we will show, is the right condition.

1. Harmonic Siegel modular forms

DEFINITION 1.1. *A function $f \in C^\infty(\mathbb{H}_2)$ is a real-analytic Siegel modular form of degree 2 for the full Siegel modular group of weight (α,β) if the following conditions are satisfied:*

(i) $H^{[1]}_{\alpha,\beta} f = d_1 f$ *and* $H^{[2]}_{\alpha,\beta} f = d_2 f$ *for some* $d_1, d_2 \in \mathbb{C}$.
(ii) $f|_{\alpha,\beta}\, g = f$ *for all* $g \in \mathrm{Sp}_2(\mathbb{Z})$.
(iii) $|f(Z)| < c(\mathrm{tr}(Y))^a$ *as* $\mathrm{tr}(Y) \to \infty$ *for some* $a, c \in \mathbb{R}$.

DEFINITION 1.2. *A real-analytic Siegel modular form of weight (α,β) is called harmonic if $\Omega_{\alpha,\beta}\, f = 0$.*

We will write $\mathbb{M}^{(2)}_k$ for the space of harmonic Siegel modular forms of holomorphic weight $(k, 0)$, and we will denote the space of harmonic Siegel modular forms of skew weight $(\frac{1}{2}, k-\frac{1}{2})$ by $\mathbb{M}^{(2),\mathrm{sk}}_k$.

We say that a function $f : \mathbb{R} \to \mathbb{C}$ grows rapidly towards infinity if its absolute value cannot be bounded by any polynomial as the argument tends to infinity. An analog definition can be made for functions $f : \mathbb{C} \to \mathbb{C}$. Condition (iii) of Definition 1.1 can be rephrased like this: A real-analytic Siegel modular forms must not grow rapidly towards the boundary of $\mathbb{H}_2$.

REMARKS 1.3.

(i) *Since* $H^{[1]}_{\alpha,\beta}$ *is the Laplace-Beltrami operator for* $\mathbb{H}_2$*, it is elliptic. Hence, by the elliptic regularity theorem, all eigenfunctions and, in particular, all harmonic Siegel modular forms are real-analytic.*

(ii) *The growth condition in Definition 1.1 does not become obsolete by the Koecher principle. Indeed, the Eisenstein series* $\det(Y)^{\frac{3}{2}-k}E_{2-k,1}$*, that is a harmonic Siegel modular form, grows towards infinity.*

(iii) *Since* $\mathbb{M}^{(2)}_k$ *and* $\mathbb{M}^{(2),\mathrm{sk}}_k$ *are defined using covariant operators, they are invariant under the usual Hecke action. Details on Hecke operator for Siegel modular forms can be found in* [**Fre94**] *and* [**Kri90**].

(iv) *We will show that*

$$E^{(2)}_k \textit{ (for } k > 3) \quad \textit{and} \quad \det Y^{\frac{3}{2}-k}E^{(2)}_{\frac{3}{2},\frac{3}{2}-k} \textit{ (for } k < 0)$$

belong to $\mathbb{M}^{(2)}_k$*. In the skew case,*

$$E^{(2)}_{\frac{1}{2},k-\frac{1}{2}} \textit{ (for } k > 3) \quad \textit{and} \quad \det(Y)^{\frac{3}{2}-k}E^{(2)}_{2-k,1} \textit{ (for } k < 0)$$

belong to $\mathbb{M}^{(2),\mathrm{sk}}_k$.

(v) *By Proposition 4.6 in Chapter 2, we know that vanishing under* $\Omega_{\alpha,\beta}$ *implies vanishing under* $H^{[1]}_{\alpha,\beta}$ *and* $H^{[2]}_{\alpha,\beta}$*. Thus it is sufficient to check harmonicity to ensure condition (i) of Definition 1.1.*

For reference, we mention that the Fourier expansion of a harmonic Siegel modular form for the full modular group is indexed by matrices $T \in \mathrm{M}^{\mathrm{T}}_2(\frac{1}{2}\mathbb{Z})$ that have integral diagonal entries. For general n, we denote the set of such matrices by $\widetilde{\mathrm{M}}^{\mathrm{T}}_n(\frac{1}{2}\mathbb{Z})$.

It is easily verified that $\Omega_k\, e(\mathrm{tr}(TZ)) = 0$ for all $T \in \mathrm{M}^{\mathrm{T}}_2(\mathbb{R})$. Hence all holomorphic modular forms are examples of harmonic Siegel modular forms. Further examples can be provided by means of nonholomorphic Eisenstein series. To define them write

$$\Gamma^{(n)} := \mathrm{Sp}_n(\mathbb{Z}) \quad \text{and} \quad \Gamma^{(n)}_\infty := \left\{\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) \in \Gamma^{(n)}\right\}. \tag{1.1}$$

The degree n Siegel Eisenstein series are

$$E^{(n)}_{\alpha,\beta} := \sum_{g\in\Gamma^{(n)}_\infty\backslash\Gamma^{(n)}} 1|^{(n)}_{\alpha,\beta}\, g. \tag{1.2}$$

These Eisenstein series converge if $\alpha + \beta > n + 1$. They can be generalized to Poincaré-Eisenstein series, which we only define in the case $n = 2$:

$$P^{(2)}_{k,s} := \sum_{g\in\Gamma^{(2)}_\infty\backslash\Gamma^{(2)}} \det(Y)^s\big|_k\, g = \det(Y)^s E^{(2)}_{k+s,s} \tag{1.3}$$

and

$$P^{(2),\mathrm{sk}}_{k,s} := \sum_{g\in\Gamma^{(2)}_\infty\backslash\Gamma^{(2)}} \det(Y)^s\big|^{\mathrm{sk}}_k\, g = \det(Y)^s E^{(2)}_{\frac{1}{2}+s,k+s-\frac{1}{2}}. \tag{1.4}$$

We find that $P^{(2)}_{k,s}$ and $P^{(2),\mathrm{sk}}_{k,s}$ converge absolutely if $2\,\mathfrak{Re}(s) + k > 3$.

PROPOSITION 1.4. *If* $s = 0$ *and* $k > 3$ *or* $s = \frac{3}{2} - k$ *and* $k < 0$*, then* $P^{(2)}_{k,s} \in \mathbb{M}^{(2)}_k$ *and* $P^{(2),\mathrm{sk}}_{k,s} \in \mathbb{M}^{\mathrm{sk},(2)}_k$.

PROOF. A direct computation shows that

$$\begin{aligned}\Omega_k \det(Y)^s &= \Big(\big(\tfrac{3}{2}I_2 + 2iY\partial_{\overline{Z}}\big)\big(kI_2 + 2iY\partial_Z\big) - \tfrac{3}{2}k\Big)\det(Y)^s\\ &= \big((\tfrac{3}{2}-s)(k+s) - \tfrac{3}{2}k\big)\det(Y)^s\\ &= -s(s+k-\tfrac{3}{2})\det(Y)^s\end{aligned}$$

and

$$\begin{aligned}\Omega_k^{\mathrm{sk}} \det(Y)^s &= \Big(\big(-(k-2)I_2 + 2iY\partial_{\overline{Z}}\big)\big(\tfrac{1}{2}I_2 + 2iY\partial_Z\big) + \tfrac{k}{2} - 1\Big)\det(Y)^s\\ &= \big((-k+2-s)(\tfrac{1}{2}+s) + \tfrac{k}{2} - 1\big)\det(Y)^s\\ &= -s(s+k-\tfrac{3}{2})\det(Y)^s.\end{aligned}$$

The covariance of Ω_k and Ω_k^{sk} proves that $\Omega_k\, P^{(2)}_{k,s} = 0$ and $\Omega_k^{\mathrm{sk}}\, P^{(2),\mathrm{sk}}_{k,s} = 0$ for $s = 0$ and $s = \frac{3}{2} - k$. Finally, (iii) of Definition 1.1 is satisfied for the Eisenstein series $E^{(2)}_{\alpha,\beta}$, if they converge, and hence also for $P^{(2)}_{k,s}$ and $P^{(2),\mathrm{sk}}_{k,s}$. □

For later use, we state the next proposition.

PROPOSITION 1.5. *We have*

$$\begin{aligned}\xi_k^{(2)} P^{(2)}_{k,\frac{3}{2}-k} &= (\tfrac{3}{2}-k)(1-k)\, P^{(2),\mathrm{sk}}_{3-k,0} \quad \text{and}\\ \xi_k^{(2),\mathrm{sk}} P^{(2),\mathrm{sk}}_{k,\frac{3}{2}-k} &= (\tfrac{3}{2}-k)(2-k)\, P^{(2)}_{3-k,0}.\end{aligned}$$

PROOF. It suffices to apply the ξ-operators to $\det(Y)^{\frac{3}{2}-k}$. □

2. Real-analytic and harmonic Fourier expansions

We first argue that there are too many possible Fourier expansions of general real-analytic Siegel modular forms. In [**Niw91**], Niwa calculated eigenfunctions of $H^{[1]}_{0,0}$ and $\widetilde{H}^{[2]}_{0,0}$. The operator $\widetilde{H}^{[2]}_{0,0}$ was given by Nakajima in [**Nak82**]. He claimed it was an order 4 invariant operator for $\mathrm{O}_{2,3}(\mathbb{R})$ on the usual homogeneous space. This would immediately lead to an invariant operator for $\mathrm{Sp}_2(\mathbb{R})$. Unfortunately, Nakajima considered the action of $\mathrm{O}_{2,3}(\mathbb{R})$ on a domain that was not the usual one. This invalidates his result for any application to our situation. Indeed, using the Sage script 1, that can be found in Appendix A, one checks that $\widetilde{H}^{[2]}_{0,0}$ used by Niwa is not $\mathrm{Sp}_2(\mathbb{R})$-invariant. Nevertheless, Proposition 4.11 in Chapter 2 can be used to prove the next corollary. In order to state it, recall the coordinates (t, t', θ) defined in (4.5) in Chapter 2.

COROLLARY 2.1. *Let*

$$a(Y,T) = \sum_{m\in\mathbb{Z}} b_m(t,t')e^{im\theta}. \tag{2.1}$$

Then $a(Y,T)\, e(\mathrm{tr}(TX))$ is an eigenfunction of $H^{[1]}_{\alpha,\beta}$ and $H^{[2]}_{\alpha,\beta}$ with eigenvalues d_1 and d_2 if and only if all $b_m(t,t')e^{im\theta}\, e(\mathrm{tr}(TX))$ are eigenfunctions with the same eigenvalues d_1 and d_2.

PROOF. By Proposition 4.11 in Chapter 2, the operators $H^{[1]}_{\alpha,\beta}$ and $H^{[2]}_{\alpha,\beta}$ can be expressed in terms of the derivatives ∂_t, $\partial_{t'}$ and ∂_θ and the variables t and t'. Using the uniqueness of Fourier expansions with respect to the variable θ, the claim follows. □

When computing possible Fourier expansions of Siegel modular forms, Niwa used an analog of the above corollary that holds for $H^{[1]}_{\alpha,\beta}$ and $\widetilde{H}^{[2]}_{\alpha,\beta}$ to restrict his considerations to functions $b_m(t,t')$ defined as above. In the case $T = I_2$ he expressed them as power series in $t - t'$, and finally solved the resulting differential equations. Although his calculations do not apply to our operators, we believe the essence of his results still holds true. In other words, for every $m \in \mathbb{Z}$ and almost all pairs of eigenvalues $(d_1,\, d_2) \in \mathbb{C}^2$ there is at least one nonzero common eigenfunction of the operators $H^{[1]}_{\alpha,\beta}$ and $H^{[2]}_{\alpha,\beta}$ that has the shape $b_m(t,t')e^{im\theta}\, e(\mathrm{tr}(TX))$.

In principle this conjecture is accessible by computer calculations — Also Niwa used computer support, as he clarified in private correspondence. More precisely, in a power series expansion $b_m(t,t') = \sum_{l=0}^{\infty} c_l(y) x^l$, where $x = t - t'$ and $y = t + t'$, the first nonvanishing coefficient c_{l_0} will roughly behave like an exponential function. Then it suffices to prove that the power series expansion for b_m resulting from the analog of Recursion (1.9.2) in [**Niw91**] converges. In Niwa's case this follows directly by estimates for the derivatives of c_{l_0} and the structure of the recursion.

We content ourselves with this very incomplete discussion. Much more can be said about the Fourier expansions of harmonic Siegel modular forms. A theorem by Maaß [**Maa53**] shows that harmonicity with respect to $\Omega_{\alpha,\beta}$ is a strong restriction on Fourier expansions. To state a precise result, we use two systems of coordinates, both introduced by Maaß to facilitate his calculations. Define

$$Y =: \sqrt{\det Y} \begin{pmatrix} (\mathtt{x}^2 + \mathtt{y}^2)\mathtt{y}^{-1} & \mathtt{x}\mathtt{y}^{-1} \\ \mathtt{x}\mathtt{y}^{-1} & \mathtt{y}^{-1} \end{pmatrix}$$

and

$$\mathtt{u} := \mathrm{tr}(YT), \quad \mathtt{v} := (\mathrm{tr}(YT))^2 - 4\det(YT).$$

We will write $\mathrm{rk}(T)$ for the rank of a matrix T.

THEOREM 2.2 ([**Maa53**]). *Let $f(Z) = a(Y,T)\, e^{i\,\mathrm{tr}(TX)}$, where $T \in \mathrm{M}_2^{\mathrm{T}}(\mathbb{R})$, and suppose $\Omega_{\alpha,\beta}(f) = 0$ where $\alpha + \beta \neq 1,\, \frac{3}{2},\, 2$. Then $a(Y,T)$ is given as follows:*

(i) *If $T = 0$, then*

$$a(Y,0) = \phi(\mathtt{x},\mathtt{y}) \det Y^{\frac{1}{2}(1-\alpha-\beta)} + c_1 \det Y^{\frac{3}{2}-\alpha-\beta} + c_2, \tag{2.2}$$

where $c_1, c_2 \in \mathbb{C}$ and $\phi(\mathtt{x},\mathtt{y})$ is an arbitrary solution (analytic for $\mathtt{y} > 0$) of the wave equation

$$\mathtt{y}^2(\partial_{\mathtt{x}}^2\, \phi + \partial_{\mathtt{y}}^2\, \phi) - (\alpha+\beta-1)(\alpha+\beta-2)\, \phi = 0.$$

(ii) *If $\mathrm{rk}(T) = 1$, $T \geq 0$, then*

$$a(Y,T) = \phi(\mathtt{u}) \det Y^{\frac{3}{2}-\alpha-\beta} + \psi(\mathtt{u}), \tag{2.3}$$

where ϕ and ψ are confluent hypergeometric functions that satisfy the differential equations

$$\begin{aligned} \mathtt{u}\,\phi'' + (3-\alpha-\beta)\,\phi' + (\alpha-\beta-\mathtt{u})\,\phi = 0 \quad \textit{and} \\ \mathtt{u}\,\psi'' + (\alpha+\beta)\,\psi' + (\alpha-\beta-\mathtt{u})\psi = 0. \end{aligned}$$

In particular, there are four linear independent solutions $a(Y,T)$ in this case.

(iii) *If* $\mathrm{rk}(T) = 2$, $T > 0$, *then*

$$a(Y,T) = \sum_{n=0}^{\infty} g_n(\mathtt{u})\, \mathtt{v}^n \qquad (|\mathtt{v}| < \mathtt{u}^2), \tag{2.4}$$

where the functions $g_n(\mathtt{u})$ *are recursively defined by*

$$4(n+1)^2 \mathtt{u}\, g_{n+1} + \mathtt{u}\, g_n'' + 2(2n + \alpha + \beta)\, g_n' + (2(\alpha - \beta) - \mathtt{u})\, g_n = 0$$

and

$$g_0(\mathtt{u}) = \mathtt{u}^{1-\alpha-\beta}\psi(\mathtt{u}), \quad \textit{with} \quad \psi'(\mathtt{u}) = \mathtt{u}^{-1}\phi(\mathtt{u}) \quad \textit{and}$$
$$\phi'' = \left(1 + \frac{2(\beta - \alpha)}{\mathtt{u}} + \frac{(\alpha + \beta - 1)(\alpha + \beta - 2)}{\mathtt{u}^2}\right) \phi.$$

In particular, there are three linear independent solutions $a(Y,T)$ *in this case.*

(iv) *If* $\mathrm{rk}(T) = 2$, T *indefinite, then*

$$a(Y,T) = \sum_{n=0}^{\infty} h_n(\mathtt{v})\, \mathtt{u}^n \quad (\mathtt{u}^2 < \mathtt{v}), \tag{2.5}$$

where the functions $h_n(\mathtt{v})$ *are recursively defined by*

$$(n+2)(n+1)\, h_{n+2} + 4\mathtt{v}\, h_n'' + 4(\alpha + \beta + n)\, h_n' - h_n = 0$$

and

$$\begin{aligned}(\alpha - \beta)\, h_1 = {} & 8\mathtt{v}^2\, h_0''' + 4(2 + 3\alpha + 3\beta)\mathtt{v}\, h_0'' \qquad (2.6)\\ & + (4(\alpha+\beta)^2 + 2(\alpha + \beta - 1) - 2\mathtt{v})\, h_0' - (\alpha + \beta)\, h_0,\\ (\beta - \alpha)\, h_0 = {} & 2\mathtt{v}\, h_1' + (\alpha + \beta)\, h_1. \qquad (2.7)\end{aligned}$$

In particular, there are four linear independent solutions $a(Y,T)$ *in this case.*

Finally, any solution $a(Y,T)e^{i\,\mathrm{tr}(TX)}$ *to the operator* $\Omega_{\alpha,\beta}$ *gives rise to a solution* $a(Y,T)e^{i\,\mathrm{tr}(-TX)}$ *to the operator* $\Omega_{\beta,\alpha}$.

REMARK 2.3. *The theorem says that for* $m \neq 0$, *we have* $b_m = 0$ *for any expansion* (2.1) *of a Fourier coefficient that is annihilated by* $\Omega_{\alpha,\beta}$.

To solve the differential equations that show up in Theorem 2.2, we need the next proposition. The generalized hypergeometric function

$${}_p\mathrm{F}_q(a_1, \dots, a_p;\, b_1, \dots, b_q;\, \mathtt{v}) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{\mathtt{v}^n}{n!}, \tag{2.8}$$

that will show up, is treated, for example, in [**Luk69**].

PROPOSITION 2.4. *Suppose that* h_1 *satisfies the differential equation in (2.5) for some* h_0 *and* h_1. *Then* h_1 *solves the differential equation*

$$\begin{aligned}0 = {} & 16\mathtt{v}^3\, h_1^{(4)} \qquad (2.9)\\ & + (32(\alpha+\beta) + 64)\, \mathtt{v}^2\, h_1^{(3)}\\ & + \big((20(\alpha+\beta)^2 + 60(\alpha+\beta) + 28)\, \mathtt{v} - 4\,\mathtt{v}^2\big)\, h_1''\\ & + \big(4(\alpha+\beta)^3 + 10(\alpha+\beta)^2 - 4 + 2(\alpha+\beta) - (4(\alpha+\beta) + 4)\, \mathtt{v}\big)\, h_1'\\ & + \big((\alpha-\beta)^2 - (\alpha+\beta)^2\big)\, h_1.\end{aligned}$$

PROOF. Using (2.7), we can express the derivatives of h_0 in terms of those of h_1. For $l \in \mathbb{Z}_{\geq 0}$ we have

$$(\beta-\alpha)\, h_0 = 2\mathtt{v}\, h_1^{(l+1)} + (\alpha+\beta)\, h_1^{(l)}. \tag{2.10}$$

We insert this into the $(\alpha-\beta)$-multiple of (2.6) and obtain

$$\begin{aligned} -(\alpha-\beta)^2\, h_1 = \;& 8\mathtt{v}^2\big(2\mathtt{v}\, h_1^{(4)} + (\alpha+\beta+6)\, h_1^{(3)}\big) \\ & + 4\big(2+3(\alpha+\beta)\big)\, \mathtt{v}\big(2\mathtt{v}\, h_1^{(3)} + (\alpha+\beta+4)\, h_1''\big) \\ & + \big(4(\alpha+\beta)^2 + 2(\alpha+\beta-1) - 2\mathtt{v}\big)\big(2\mathtt{v}\, h_1'' + (\alpha+\beta+2)\, h_1'\big) \\ & - (\alpha+\beta)\big(2\mathtt{v}\, h_1' + (\alpha+\beta)\, h_1\big), \end{aligned}$$

yielding the claim. □

LEMMA 2.5. *For $\alpha = k \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$ and $\beta = 0$ the differential equation (2.9) has the four fundamental solutions*

$$\begin{aligned} &\mathtt{v}^{\frac{-k}{2}}\, {}_2\mathrm{F}_3\big(\tfrac{k}{2}, \tfrac{-k}{2}; \tfrac{1}{2}, \tfrac{k-1}{2}, 1-\tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\big), && \mathtt{v}^{\frac{1-k}{2}}\, {}_2\mathrm{F}_3\big(\tfrac{k+1}{2}, \tfrac{1-k}{2}; \tfrac{3}{2}, \tfrac{k}{2}, \tfrac{3-k}{2}; \tfrac{\mathtt{v}}{4}\big), \\ &1, && \textit{and}\quad \mathtt{v}^{\frac{3}{2}-k}\, {}_2\mathrm{F}_3\big(\tfrac{3}{2}, \tfrac{3}{2}-k; 2-\tfrac{k}{2}, \tfrac{5-k}{2}, \tfrac{5}{2}-k; \tfrac{\mathtt{v}}{4}\big). \end{aligned} \tag{2.11}$$

For $\alpha = \frac{1}{2}$ and $\beta = k - \frac{1}{2} \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$ the differential equation (2.9) has the four fundamental solutions

$$\begin{aligned} &{}_1\mathrm{F}_2\big(\tfrac{1}{2}; \tfrac{1+k}{2}, 1+\tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\big), && \mathtt{v}^{-\frac{k}{2}}\, {}_1\mathrm{F}_2\big(\tfrac{1-k}{2}; \tfrac{1}{2}, 1-\tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\big), \\ &\mathtt{v}^{\frac{1-k}{2}}\, {}_1\mathrm{F}_2\big(1-\tfrac{k}{2}; \tfrac{3}{2}, \tfrac{3-k}{2}; \tfrac{\mathtt{v}}{4}\big), && \textit{and}\quad \mathtt{v}^{\frac{3}{2}-k}\, {}_2\mathrm{F}_3\big(1, 2-k; \tfrac{5}{2}-k, 2-\tfrac{k}{2}, \tfrac{5-k}{2}; \tfrac{\mathtt{v}}{4}\big). \end{aligned} \tag{2.12}$$

The proof of this lemma will be computer based. We need the next lemma to prove the correctness of the according script. Write ${}_p\mathrm{F}_q(\mathbf{a}; \mathbf{b}; \mathtt{v})$ for the hypergeometric series with parameters $\mathbf{a} = a_1, \ldots, a_p$ and $\mathbf{b} = b_1, \ldots, b_q$. Given $t \in \mathbb{C}$, we write $\mathbf{a} + t$ for $a_1 + t, \ldots, a_p + t$ and $\mathbf{b} + t$ for $b_1 + t, \ldots, b_q + t$.

LEMMA 2.6. *Suppose $\mathcal{D}$ is an order D linear differential operator on smooth functions of $\mathtt{v}$. Assume that $\mathcal{D}$ has coefficients in $\mathbb{C}[\mathtt{v}, k]$, and that these coefficients have maximal degree $m_{\mathtt{v}}$ in $\mathtt{v}$. If $l \in \mathbb{Z}$ and none of the b_j's are nonpositive integers, then*

$$\mathcal{D}\, \mathtt{v}^l\, {}_p\mathrm{F}_q(\mathbf{a}; \mathbf{b}; \mathtt{v}) = 0,$$

if and only if the t-th coefficients ($l - D \leq t \leq l + D + m_{\mathtt{v}}$) of $\mathcal{D}\, \mathtt{v}^l\, {}_p\mathrm{F}_q(\mathbf{a}; \mathbf{b}; \mathtt{v})$ vanish as functions of k.

PROOF. It suffices to prove that

$$\mathcal{D}\, \mathtt{v}^l\, {}_p\mathrm{F}_q(\mathbf{a}; \mathbf{b}; \mathtt{v}) = \mathtt{v}^{l-D}\big(p_1\, {}_p\mathrm{F}_q(\mathbf{a}+D; \mathbf{b}+D; \mathtt{v}) + p_2\big)$$

for some $p_1, p_2 \in \mathbb{C}(k)[\mathtt{v}]$ of degree at most $2D + m_{\mathtt{v}}$. Without loss of generality let $\mathcal{D} = \partial_{\mathtt{v}}^i$ with $i \in \{0, \ldots, D\}$ and, in particular, $m_{\mathtt{v}} = 0$.

We proceed by mathematical induction on D. The case $D = 0$ is clear. Suppose $\mathcal{D} = c_1 \partial_{\mathtt{v}} \widetilde{\mathcal{D}} + c_2$ for some constants c_1, c_2 and an order $D - 1$ operator $\widetilde{\mathcal{D}}$. By induction hypothesis, we have

$$\widetilde{\mathcal{D}}\, \mathtt{v}^l\, {}_p\mathrm{F}_q(\mathbf{a}; \mathbf{b}; \mathtt{v}) = \mathtt{v}^{l-D+1}\big(\tilde{p}_1\, {}_p\mathrm{F}_q(\mathbf{a}+D-1; \mathbf{b}+D-1; \mathtt{v}) + \tilde{p}_2\big),$$

where $\tilde{p}_1, \tilde{p}_2$ have maximal degree $2D - 2$. The definition of the hypergeometric functions implies the relations

$$\mathtt{v}^l\, {}_p\mathrm{F}_q(\mathbf{a}; \mathbf{b}; \mathtt{v}) = \mathtt{v}^{l-1} \prod_i a_i \prod_j b_j^{-1}\, \big(\mathtt{v} + \mathtt{v}^2\, {}_p\mathrm{F}_q(\mathbf{a}+1; \mathbf{b}+1; \mathtt{v})\big)$$

and

$$\partial_{\mathtt{v}}\, \mathtt{v}^l\, {}_p\mathrm{F}_q(\mathbf{a};\mathbf{b};\mathtt{v}) = \mathtt{v}^{l-1}\Big(\prod_i a_i \prod_j b_j^{-1}\, \mathtt{v}\, {}_p\mathrm{F}_q(\mathbf{a}+1;\mathbf{b}+1;\mathtt{v}) + l\, {}_p\mathrm{F}_q(\mathbf{a};\mathbf{b};\mathtt{v}) \Big),$$

which yield the claim. □

PROOF OF LEMMA 2.5. It is clear that for all k under consideration the hypergeometric functions in (2.11) and (2.12) are well-defined.

We can use Lemma 2.6 to reduce the proof to the computation of finitely many coefficients in a Laurent expansion with respect to $\mathtt{v}$. More precisely, we have $D = 4$ and $m_{\mathtt{v}} = 3$. If $\tilde{h}_1$ is any of the potential solutions in (2.11) or (2.12) to the differential equation (2.9), it suffices to check that the first $D + 2m_{\mathtt{v}} = 11$ coefficients of

$$\begin{aligned}
&16\mathtt{v}^3\, \tilde{h}_1^{(4)} + \big((8(\alpha+\beta)+48)\,\mathtt{v}^2 + (12(\alpha+\beta)+8)\,\mathtt{v}\big)\,\tilde{h}_1^{(3)} \\
&+ \big((20(\alpha+\beta)^2 + 60(\alpha+\beta)+28)\,\mathtt{v} - 4\,\mathtt{v}^2\big)\,\tilde{h}_1'' \\
&+ \big(4(\alpha+\beta)^3 + 10(\alpha+\beta)^2 + 2(\alpha+\beta) - (4(\alpha+\beta)+4)\,\mathtt{v}\big)\,\tilde{h}_1' \\
&+ \big(((\alpha-\beta)^2 - (\alpha+\beta)^2\big)\,\tilde{h}_1
\end{aligned}$$

vanish as rational functions of k. In principle, this calculation could be carried out directly, but since it is extremely long, we prefer giving a computer assisted proof. The according Sage [**S$^+$11**] script can be found in Section 2 of Appendix A. A numerical double check is performed using the Sage script in Section 3 of the same chapter. Both scripts are written in such a way that they can be directly loaded in Sage (using the command `load "filename"`). The absence of assertion errors raised by Sage during the computations then proves the claim. □

In the next lemma, we list regularized solutions to (2.9). We will need the Pochhammer symbol

$$(a)_n := \prod_{i=0}^{n-1} (a+i), \tag{2.13}$$

defined for $n \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{C}$.

LEMMA 2.7. *Let $\alpha = k$ and $\beta = 0$ for some $k \in \mathbb{Z}$. The differential equation (2.9) has the fundamental solution*

$$1 \tag{2.14}$$

and other fundamental solutions depending on the sign and the parity of k.

If $k \geq 4$ and k is even, further three fundamental solutions are

$$\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{1-k}{2}}\, {}_2\mathrm{F}_3\big(\tfrac{k+1}{2}, \tfrac{1-k}{2}; \tfrac{3}{2}, \tfrac{k}{2}, \tfrac{3-k}{2}; \tfrac{\mathtt{v}}{4}\big) \tag{2.15}$$

and the regularized hypergeometric series, that can be analytically continued to $\tilde{k} = k$,

$$\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{-k}{2}}\, {}_2\mathrm{F}_3\big(\tfrac{k}{2}, \tfrac{-\tilde{k}}{2}; \tfrac{1}{2}, \tfrac{k-1}{2}, 1-\tfrac{\tilde{k}}{2}; \tfrac{\mathtt{v}}{4}\big) \;-\; \frac{\big(\frac{-k}{2}\big)_{\frac{k}{2}}\, \big(\frac{k}{2}\big)_{\frac{k}{2}}}{\big(\frac{1}{2}\big)_{\frac{k}{2}}\, \big(\frac{k-1}{2}\big)_{\frac{k}{2}}\, \big(\frac{k}{2}\big)!}\,\Gamma\big(1-\tfrac{\tilde{k}}{2}\big) \tag{2.16}$$

and

$$\left(\tfrac{\mathtt{v}}{4}\right)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\left(\tfrac{3}{2}, \tfrac{3}{2}-k; 2-\tfrac{\tilde{k}}{2}, \tfrac{5-k}{2}, \tfrac{5}{2}-k; \tfrac{\mathtt{v}}{4}\right) \tag{2.17}$$
$$- \frac{\left(\frac{3}{2}\right)_{\frac{k}{2}-1} \left(\frac{3}{2}-k\right)_{\frac{k}{2}-1} \Gamma\left(2-\frac{\tilde{k}}{2}\right)}{\left(2-\frac{k}{2}\right)_{\frac{k}{2}-1} \left(\frac{5-k}{2}\right)_{\frac{k}{2}-1} \left(\frac{5}{2}-k\right)_{\frac{k}{2}-1} \left(\frac{k}{2}-1\right)!} \left(\tfrac{\mathtt{v}}{4}\right)^{\frac{1-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{k+1}{2}, \tfrac{1-k}{2}; \tfrac{3}{2}, \tfrac{k}{2}, \tfrac{3-k}{2}; \tfrac{\mathtt{v}}{4}\right).$$

If $k \geq 3$ and k is odd, further three fundamental solutions are

$$\left(\tfrac{\mathtt{v}}{4}\right)^{\frac{-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{k}{2}, \tfrac{-k}{2}; \tfrac{1}{2}, \tfrac{k-1}{2}, 1-\tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\right) \tag{2.18}$$

and the regularized hypergeometric series, that can be analytically continued to $\tilde{k} = k$,

$$\left(\tfrac{\mathtt{v}}{4}\right)^{\frac{1-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{k+1}{2}, \tfrac{1-\tilde{k}}{2}; \tfrac{3}{2}, \tfrac{k}{2}, \tfrac{3-\tilde{k}}{2}; \tfrac{\mathtt{v}}{4}\right) - \frac{\left(\frac{k+1}{2}\right)_{\frac{k-1}{2}} \left(\frac{1-k}{2}\right)_{\frac{k-1}{2}}}{\left(\frac{3}{2}\right)_{\frac{k-1}{2}} \left(\frac{k}{2}\right)_{\frac{k-1}{2}} \left(\frac{k-1}{2}\right)!} \Gamma\left(\tfrac{3-\tilde{k}}{2}\right) \tag{2.19}$$

and

$$\left(\tfrac{\mathtt{v}}{4}\right)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\left(\tfrac{3}{2}, \tfrac{3}{2}-k; 2-\tfrac{k}{2}, \tfrac{5-\tilde{k}}{2}, \tfrac{5}{2}-k; \tfrac{\mathtt{v}}{4}\right) \tag{2.20}$$
$$- \frac{\left(\frac{3}{2}\right)_{\frac{k-3}{2}} \left(\frac{3}{2}-k\right)_{\frac{k-3}{2}} \Gamma\left(\frac{5-\tilde{k}}{2}\right)}{\left(2-\frac{k}{2}\right)_{\frac{k-3}{2}} \left(\frac{5}{2}-k\right)_{\frac{k-3}{2}} \left(\frac{k-3}{2}\right)!} \left(\tfrac{\mathtt{v}}{4}\right)^{\frac{-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{k}{2}, \tfrac{-k}{2}; \tfrac{1}{2}, \tfrac{k-1}{2}, 1-\tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\right).$$

The last regularization is a sum of two well-defined hypergeometric series, if $k = 3$.

If $k < 0$, then a further fundamental solution is

$$\left(\tfrac{\mathtt{v}}{4}\right)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\left(\tfrac{3}{2}, \tfrac{3}{2}-k; 2-\tfrac{k}{2}, \tfrac{5-k}{2}, \tfrac{5}{2}-k; \tfrac{\mathtt{v}}{4}\right). \tag{2.21}$$

If, in addition, k is even, then two further fundamental solutions are

$$\left(\tfrac{\mathtt{v}}{4}\right)^{\frac{-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{k}{2}, \tfrac{-k}{2}; \tfrac{1}{2}, \tfrac{k-1}{2}, 1-\tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\right) \tag{2.22}$$

and the regularized hypergeometric series, that can be analytically continued to $\tilde{k} = k$,

$$\left(\tfrac{\mathtt{v}}{4}\right)^{\frac{1-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{k+1}{2}, \tfrac{1-k}{2}; \tfrac{3}{2}, \tfrac{\tilde{k}}{2}, \tfrac{3-k}{2}; \tfrac{\mathtt{v}}{4}\right) \tag{2.23}$$
$$- \frac{\left(\frac{k+1}{2}\right)_{1-\frac{k}{2}} \left(\frac{1-k}{2}\right)_{1-\frac{k}{2}} \Gamma\left(\frac{\tilde{k}}{2}\right)}{\left(\frac{3}{2}\right)_{1-\frac{k}{2}} \left(\frac{3-k}{2}\right)_{1-\frac{k}{2}} \left(1-\frac{k}{2}\right)!} \left(\tfrac{\mathtt{v}}{4}\right)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\left(\tfrac{3}{2}, \tfrac{3}{2}-k; 2-\tfrac{k}{2}, \tfrac{5-k}{2}, \tfrac{5}{2}-k; \tfrac{\mathtt{v}}{4}\right).$$

If $k < 0$ is odd, two further fundamental solutions are

$$\left(\tfrac{\mathtt{v}}{4}\right)^{\frac{1-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{k+1}{2}, \tfrac{1-k}{2}; \tfrac{3}{2}, \tfrac{k}{2}, \tfrac{3-k}{2}; \tfrac{\mathtt{v}}{4}\right) \tag{2.24}$$

and the regularized hypergeometric series, that can be analytically continued to $\tilde{k} = k$,

$$\left(\tfrac{\mathtt{v}}{4}\right)^{\frac{-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{k}{2}, \tfrac{-k}{2}; \tfrac{1}{2}, \tfrac{\tilde{k}-1}{2}, 1-\tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\right) \tag{2.25}$$
$$- \frac{\left(\frac{-k}{2}\right)_{\frac{3-k}{2}} \left(\frac{k}{2}\right)_{\frac{3-k}{2}} \Gamma\left(\frac{\tilde{k}-1}{2}\right)}{\left(\frac{1}{2}\right)_{\frac{3-k}{2}} \left(1-\frac{k}{2}\right)_{\frac{3-k}{2}} \left(\frac{1-k}{2}\right)!} \left(\tfrac{\mathtt{v}}{4}\right)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\left(\tfrac{3}{2}, \tfrac{3}{2}-k; 2-\tfrac{k}{2}, \tfrac{5-k}{2}, \tfrac{5}{2}-k; \tfrac{\mathtt{v}}{4}\right).$$

If $k = 0$, then three fundamental solutions are given by (2.21), (2.23) and the regularized hypergeometric series, that can be analytically continued to $\tilde{k} = k$,

$$\Gamma\left(\tfrac{\tilde{k}}{2}\right) \Gamma\left(\tfrac{-\tilde{k}}{2}\right) \left(\tfrac{\mathtt{v}}{4}\right)^{\frac{-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{\tilde{k}}{2}, \tfrac{-\tilde{k}}{2}; \tfrac{1}{2}, \tfrac{k-1}{2}, 1-\tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\right). \tag{2.26}$$

Let $\alpha = \frac{1}{2}$ and $\beta = k - \frac{1}{2}$ for some $k \in \mathbb{Z}$. If $k \geq 3$ then the differential equation (2.9) has the fundamental solution

$$ {}_1\mathrm{F}_2\big(\tfrac{1}{2}; \tfrac{k+1}{2}, 1 + \tfrac{k}{2}; \tfrac{\mathrm{v}}{4}\big). \tag{2.27}$$

Depending on the parity of k, for even k further three fundamental solutions are

$$ \big(\tfrac{\mathrm{v}}{4}\big)^{\frac{1-k}{2}} {}_1\mathrm{F}_2\big(1 - \tfrac{k}{2}; \tfrac{3}{2}, \tfrac{3-k}{2}; \tfrac{\mathrm{v}}{4}\big) \tag{2.28}$$

and the two regularized hypergeometric series, that can be analytically continued to $\tilde{k} = k$,

$$ \big(\tfrac{\mathrm{v}}{4}\big)^{\frac{-k}{2}} {}_1\mathrm{F}_2\big(\tfrac{1-k}{2}; \tfrac{1}{2}, 1 - \tfrac{\tilde{k}}{2}; \tfrac{\mathrm{v}}{4}\big) - \frac{\big(\frac{1-k}{2}\big)_{\frac{k}{2}}\,\Gamma\big(1 - \frac{\tilde{k}}{2}\big)}{\big(\frac{1}{2}\big)_{\frac{k}{2}}\,\big(\frac{k}{2}\big)!}\, {}_1\mathrm{F}_2\big(\tfrac{1}{2}; \tfrac{k+1}{2}, 1 + \tfrac{k}{2}; \tfrac{\mathrm{v}}{4}\big) \tag{2.29}$$

and

$$ \begin{aligned} &\big(\tfrac{\mathrm{v}}{4}\big)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\big(1, 2 - \tilde{k}; \tfrac{5}{2} - k, 2 - \tfrac{\tilde{k}}{2}, \tfrac{5-k}{2}; \tfrac{\mathrm{v}}{4}\big) \\ &\quad - \frac{(2-k)_{\frac{k}{2}-1}\,\Gamma\big(2 - \frac{\tilde{k}}{2}\big)}{\big(\frac{5}{2} - k\big)_{\frac{k}{2}-1}\,\big(\frac{5-k}{2}\big)_{\frac{k}{2}-1}} \big(\tfrac{\mathrm{v}}{4}\big)^{\frac{1-k}{2}} {}_1\mathrm{F}_2\big(1 - \tfrac{k}{2}; \tfrac{3}{2}, \tfrac{3-k}{2}; \tfrac{\mathrm{v}}{4}\big). \end{aligned} \tag{2.30}$$

For odd k, further three fundamental solutions are

$$ \big(\tfrac{\mathrm{v}}{4}\big)^{\frac{-k}{2}} {}_1\mathrm{F}_2\big(\tfrac{1-k}{2}; \tfrac{1}{2}, 1 - \tfrac{k}{2}; \tfrac{\mathrm{v}}{4}\big) \tag{2.31}$$

and the two regularized hypergeometric series, that can be analytically continued to $\tilde{k} = k$,

$$ \big(\tfrac{\mathrm{v}}{4}\big)^{\frac{1-k}{2}} {}_1\mathrm{F}_2\big(1 - \tfrac{k}{2}; \tfrac{3}{2}, \tfrac{3-\tilde{k}}{2}; \tfrac{\mathrm{v}}{4}\big) - \frac{\big(1 - \frac{k}{2}\big)_{\frac{k-1}{2}}\,\Gamma\big(\frac{3-\tilde{k}}{2}\big)}{\big(\frac{3}{2}\big)_{\frac{k-1}{2}}\,\big(\frac{k-1}{2}\big)!}\, {}_1\mathrm{F}_2\big(\tfrac{1}{2}; \tfrac{k+1}{2}, 1 + \tfrac{k}{2}; \tfrac{\mathrm{v}}{4}\big). \tag{2.32}$$

and

$$ \begin{aligned} &\big(\tfrac{\mathrm{v}}{4}\big)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\big(1, 2 - \tilde{k}; \tfrac{5}{2} - k, 2 - \tfrac{k}{2}, \tfrac{5-k}{2}; \tfrac{\mathrm{v}}{4}\big) \\ &\quad - \frac{(2-k)_{\frac{k-3}{2}}\,\Gamma\big(\frac{5-\tilde{k}}{2}\big)}{\big(\frac{5}{2} - k\big)_{\frac{k-3}{2}}\,\big(2 - \frac{k}{2}\big)_{\frac{k-3}{2}}} \big(\tfrac{\mathrm{v}}{4}\big)^{-\frac{k}{2}} {}_1\mathrm{F}_2\big(\tfrac{1-k}{2}; \tfrac{1}{2}, 1 - \tfrac{k}{2}; \tfrac{\mathrm{v}}{4}\big). \end{aligned} \tag{2.33}$$

The last regularization is a linear combination of well-defined hypergeometric series, if $k = 3$.

If $k \leq 0$, then three fundamental solutions to the differential equation (2.9) are

$$ \begin{gathered} \big(\tfrac{\mathrm{v}}{4}\big)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\big(1, 2 - k; \tfrac{5}{2} - k, 2 - \tfrac{k}{2}, \tfrac{5-k}{2}; \tfrac{\mathrm{v}}{4}\big), \\ \big(\tfrac{\mathrm{v}}{4}\big)^{-\frac{k}{2}} {}_1\mathrm{F}_2\big(\tfrac{1-k}{2}; \tfrac{1}{2}, 1 - \tfrac{k}{2}; \tfrac{\mathrm{v}}{4}\big) \quad \textit{and} \quad \big(\tfrac{\mathrm{v}}{4}\big)^{\frac{1-k}{2}} {}_1\mathrm{F}_2\big(1 - \tfrac{k}{2}; \tfrac{3}{2}, \tfrac{3-k}{2}; \tfrac{\mathrm{v}}{4}\big). \end{gathered} \tag{2.34}$$

Depending on the parity of k, for even k, a further fundamental solution is the linear combination of hypergeometric series, that can be analytically continued to $\tilde{k} = k$,

$$ {}_1\mathrm{F}_2\big(\tfrac{1}{2}; \tfrac{k+1}{2}, 1 + \tfrac{\tilde{k}}{2}; \tfrac{\mathrm{v}}{4}\big) \; - \; \frac{\big(\frac{1}{2}\big)_{\frac{-k}{2}}\,\Gamma\big(1 + \frac{\tilde{k}}{2}\big)}{\big(\frac{k+1}{2}\big)_{\frac{-k}{2}}\,\big(\frac{-k}{2}\big)!} \big(\tfrac{\mathrm{v}}{4}\big)^{\frac{-k}{2}} {}_1\mathrm{F}_2\big(\tfrac{1-k}{2}; \tfrac{1}{2}, 1 - \tfrac{k}{2}; \tfrac{\mathrm{v}}{4}\big). \tag{2.35}$$

The last regularization is a sum of two well-defined hypergeometric series, if $k = 0$.

For odd k a, further fundamental solution is the linear combination of hypergeometric series, that can be analytically continued to $\tilde{k} = k$,

$$ {}_1\mathrm{F}_2\big(\tfrac{1}{2}; \tfrac{\tilde{k}+1}{2}, 1 + \tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\big) \;-\; \frac{(\frac{1}{2})_{\frac{1-k}{2}}\, \Gamma(\frac{\tilde{k}+1}{2})}{(1+\frac{k}{2})_{\frac{1-k}{2}}\, (\frac{1-k}{2})!} \big(\tfrac{\mathtt{v}}{4}\big)^{\frac{1-k}{2}}\; {}_1\mathrm{F}_2\big(1 - \tfrac{k}{2}; \tfrac{3}{2}, \tfrac{3-k}{2}; \tfrac{\mathtt{v}}{4}\big). \tag{2.36} $$

REMARK 2.8. *The cases $k = 1$ and $k = 2$ are not treated in Lemma 2.7, because they are already excluded by the assumptions of Theorem 2.2.*

REMARK 2.9. *It is possible to express some of the linear combinations of hypergeometric functions as Meijer $\mathrm{G}^{m\,n}_{p\,q}$-functions, explained in* [**Luk69**]. *Examples are the solutions (2.29) and (2.32), which can be expressed in terms of Meijer $\mathrm{G}^{2\,1}_{2\,4}$-functions. By* [**Luk69**, Section 6.5 (1)], *they equal*

$$ \mathrm{G}^{2\,1}_{2\,4}\Big(-\tfrac{\mathtt{v}}{4}\Big|\, \begin{matrix} \frac{3}{2}-k, \frac{1}{2} \\ \frac{3}{2}-k, \frac{1-k}{2}, \frac{-k}{2}, 0 \end{matrix} \Big) \quad \textit{and} \quad \mathrm{G}^{2\,1}_{2\,4}\Big(-\tfrac{\mathtt{v}}{4}\Big|\, \begin{matrix} \frac{3}{2}-k, \frac{1}{2} \\ \frac{3}{2}-k, \frac{-k}{2}, \frac{1-k}{2}, 0 \end{matrix} \Big). $$

PROOF OF LEMMA 2.7. If we can prove that all functions given in the statement of the lemma are well-defined and that they are solutions to the differential equation (2.9), we are left with proving their linear independence. In that case, linear independence follows from the fact that the initial exponent of Laurent expansion of the given solutions for fixed k are pairwise distinct.

We suppose that we have already proved that all regularizations of hypergeometric functions that occur in the statement of the lemma are well-defined for the corresponding values of k. Under this assumption, we are left with proving that they are solutions to the differential equation (2.9) for h_1. The differential operators attached to equation (2.9) is analytic for $\alpha = k$, $\beta = 0$ and $\alpha = \frac{1}{2}$, $\beta = k - \frac{1}{2}$. By Lemma 2.5, the hypergeometric series that occur are solutions for all $k \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$. In other words, the functions vanish under the said differential operators for these k. Since the functions and the operators are analytic, it follows that they are solutions for all k that we consider.

It is obvious that all (nonregularized) hypergeometric series that occur in the statement are well defined for the corresponding k. Thus we are left with proving that the regularized hypergeometric series are also well-defined. We will consider the coefficients of the Laurent expansion with respect to $\mathtt{v}$. It is sufficient to show that the poles of each such coefficient considered as a rational function of $k \in \mathbb{C}$ cancel at $\tilde{k} = k$.

We consider (2.16). Thus we assume that $k > 3$ and $2 \mid k$, and we will be concerned with functions of $\tilde{k}$. We claim that the limit $\tilde{k} \to k$ of the following function exists:

$$ \Gamma(1 - \tfrac{\tilde{k}}{2})^{-1}\, \big(\tfrac{\mathtt{v}}{4}\big)^{\frac{-k}{2}}\; {}_2\mathrm{F}_3\big(\tfrac{k}{2}, \tfrac{-\tilde{k}}{2}; \tfrac{1}{2}, \tfrac{k-1}{2}, 1 - \tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\big). \tag{2.37} $$

To compute the limit, consider the Laurent expansion with respect to $\mathtt{v}$:

$$ \Gamma(1 - \tfrac{\tilde{k}}{2})^{-1}\, \big(\tfrac{\mathtt{v}}{4}\big)^{\frac{-k}{2}} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{k}{2})\,\Gamma(n+\frac{-\tilde{k}}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{k-1}{2})\,\Gamma(1-\frac{\tilde{k}}{2})}{\Gamma(\frac{k}{2})\,\Gamma(\frac{-\tilde{k}}{2})\,\Gamma(n+\frac{1}{2})\,\Gamma(n+\frac{k-1}{2})\,\Gamma(n+1-\frac{\tilde{k}}{2})\,\Gamma(n+1)} \big(\frac{\mathtt{v}}{4}\big)^n. $$

The gamma factor in front of the above expression cancels the one in the numerator of each addend, and we can employ the limit $\tilde{k} \to k$. The poles of $\Gamma(n - \frac{\tilde{k}}{2})$, which occurs in the numerator, and $\Gamma(\frac{-\tilde{k}}{2})$, which occurs in the denominator, cancel only if $n < 1 - \frac{k}{2}$. On the other hand, $\Gamma(n + 1 - \frac{\tilde{k}}{2})$, which occurs in the denominator, has a pole for all $n < \frac{k}{2}$. Consequently, after taking the limit we are left with the coefficient for $n = \frac{k}{2}$. It equals

$$\left(\tfrac{k}{2}\right)_{\frac{k}{2}} \left(\tfrac{-k}{2}\right)_{\frac{k}{2}} \left(\tfrac{1}{2}\right)^{-1}_{\frac{k}{2}} \left(\tfrac{k-1}{2}\right)^{-1}_{\frac{k}{2}} \left(\tfrac{k}{2}\right)!^{-1}. \tag{2.38}$$

We conclude that the limit of (2.37) exits and that the pols in (2.16) cancel.

One can prove that all other linear combinations of hypergeometric series are well-defined by exactly the same method. For completeness we list the Laurent expansions and the limits that occur.

Consider (2.17), thus $k > 3$ and $2 \mid k$. The regularization

$$\Gamma(2 - \tfrac{\tilde{k}}{2})^{-1} \left(\tfrac{\mathrm{v}}{4}\right)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\left(\tfrac{3}{2}, \tfrac{3}{2} - k; 2 - \tfrac{\tilde{k}}{2}, \tfrac{5-k}{2}, \tfrac{5}{2} - k; \tfrac{\mathrm{v}}{4}\right)$$

has the Laurent expansion

$$\begin{aligned}&\Gamma(2 - \tfrac{\tilde{k}}{2})^{-1} \left(\tfrac{\mathrm{v}}{4}\right)^{\frac{3}{2}-k} \\ &\quad\cdot \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{3}{2})\,\Gamma(n + \frac{3}{2} - k)\,\Gamma(2 - \frac{\tilde{k}}{2})\,\Gamma(\frac{5-k}{2})\,\Gamma(\frac{5}{2} - k)}{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2} - k)\,\Gamma(n + 2 - \frac{\tilde{k}}{2})\,\Gamma(n + \frac{5-k}{2})\,\Gamma(n + \frac{5}{2} - k)\,\Gamma(n+1)} \left(\frac{\mathrm{v}}{4}\right)^n.\end{aligned}$$

As $\tilde{k} \to k$ the coefficients with $n < \frac{k}{2} - 1$ vanish. Hence we replace n by $n + \frac{k}{2} - 1$. We conclude that the limit equals

$$\left(\tfrac{\mathrm{v}}{4}\right)^{\frac{3}{2}-k} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{k+1}{2})\,\Gamma(n + \frac{1-k}{2})\,\Gamma(2 - \frac{k}{2})\,\Gamma(\frac{5-k}{2)}\,\Gamma(\frac{5}{2} - k)}{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2} - k)\,\Gamma(n+1)\,\Gamma(n + \frac{3}{2})\,\Gamma(n + \frac{3-k}{2})\,\Gamma(n + \frac{k}{2})} \left(\frac{\mathrm{v}}{4}\right)^n.$$

By sorting out the correct gamma factors we obtain

$$\begin{aligned}&\left(\tfrac{3}{2}\right)_{\frac{k}{2}-1} \left(\tfrac{3}{2} - k\right)_{\frac{k}{2}-1} \left(2 - \tfrac{k}{2}\right)^{-1}_{\frac{k}{2}-1} \left(\tfrac{5-k}{2}\right)^{-1}_{\frac{k}{2}-1} \left(\tfrac{5}{2} - k\right)^{-1}_{\frac{k}{2}-1} \left(\tfrac{k}{2} - 1\right)!^{-1} \\ &\quad\cdot \left(\tfrac{\mathrm{v}}{4}\right)^{\frac{1-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{k+1}{2}, \tfrac{1-k}{2}; \tfrac{3}{2}, \tfrac{k}{2}, \tfrac{3-k}{2}; \tfrac{\mathrm{v}}{4}\right).\end{aligned}$$

Consider (2.19), thus $k > 3$ and $2 \nmid k$. The regularization

$$\Gamma(\tfrac{3-\tilde{k}}{2})^{-1} \left(\tfrac{\mathrm{v}}{4}\right)^{\frac{-k}{2}} {}_2\mathrm{F}_3\left(\tfrac{k+1}{2}, \tfrac{1-\tilde{k}}{2}; \tfrac{3}{2}, \tfrac{k}{2}, \tfrac{3-\tilde{k}}{2}; \tfrac{\mathrm{v}}{4}\right)$$

has the Laurent expansion

$$\begin{aligned}&\Gamma(\tfrac{3-\tilde{k}}{2})^{-1} \left(\tfrac{\mathrm{v}}{4}\right)^{\frac{-k}{2}} \\ &\quad\cdot \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{k+1}{2})\Gamma(n + \frac{1-\tilde{k}}{2})\Gamma(\frac{3}{2})\Gamma(\frac{k}{2})\Gamma(\frac{3-\tilde{k}}{2})}{\Gamma(\frac{k+1}{2})\Gamma(\frac{1-\tilde{k}}{2})\Gamma(n + \frac{3}{2})\Gamma(n + \frac{k}{2})\Gamma(n + \frac{3-\tilde{k}}{2})\Gamma(n+1)} \left(\frac{\mathrm{v}}{4}\right)^n.\end{aligned}$$

In analogy with (2.37), the gamma factor $\Gamma(n + \frac{1-\tilde{k}}{2})\,\Gamma(\frac{1-\tilde{k}}{2})^{-1}$ vanishes as $\tilde{k} \to k$. if $n > \frac{k-1}{2}$, and $\Gamma(n + \frac{3-\tilde{k}}{2})^{-1}$ vanishes, if $n < \frac{k-1}{2}$. Consequently, the limit of the above expansion equals

$$\left(\tfrac{k+1}{2}\right)_{\frac{k-1}{2}} \left(\tfrac{1-k}{2}\right)_{\frac{k-1}{2}} \left(\tfrac{3}{2}\right)^{-1}_{\frac{k-1}{2}} \left(\tfrac{k}{2}\right)^{-1}_{\frac{k-1}{2}} \left(\tfrac{k-1}{2}\right)!^{-1}.$$

It follows that the pols in (2.19) cancel.

Consider (2.20), thus $k > 3$ and $2 \nmid k$. The regularization

$$\Gamma(\tfrac{5-\tilde{k}}{2})^{-1} \left(\tfrac{\mathrm{v}}{4}\right)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\left(\tfrac{3}{2}, \tfrac{3}{2} - k; 2 - \tfrac{k}{2}, \tfrac{5-\tilde{k}}{2}, \tfrac{5}{2} - k; \tfrac{\mathrm{v}}{4}\right)$$

has the Laurent expansion

$$\Gamma\big(\tfrac{5-\tilde{k}}{2}\big)^{-1}\,\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{3}{2}-k}$$
$$\cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{2})\,\Gamma(n+\frac{3}{2}-k)\,\Gamma(2-\frac{k}{2})\,\Gamma(\frac{5-\tilde{k}}{2})\,\Gamma(\frac{5}{2}-k)}{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2}-k)\,\Gamma(n+2-\frac{k}{2})\,\Gamma(n+\frac{5-\tilde{k}}{2})\,\Gamma(n+\frac{5}{2}-k)\,\Gamma(n+1)}\Big(\frac{\mathtt{v}}{4}\Big)^n.$$

Since as $\tilde{k} \to k$ the coefficients with $n < \frac{k-3}{2}$ vanish, we shift the sum accordingly, and we obtain

$$\big(\tfrac{3}{2}\big)_{\frac{k-3}{2}}\,\big(\tfrac{3}{2}-k\big)_{\frac{k-3}{2}}\,\big(2-\tfrac{k}{2}\big)^{-1}_{\frac{k-3}{2}}\,\big(\tfrac{k-3}{2}\big)!^{-1}\,\big(\tfrac{5}{2}-k\big)^{-1}_{\frac{k-3}{2}}$$
$$\cdot\,\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{-k}{2}}\;{}_2\mathrm{F}_3\big(\tfrac{k}{2},\tfrac{-k}{2};\tfrac{1}{2},\tfrac{k-1}{2},1-\tfrac{k}{2};\tfrac{\mathtt{v}}{4}\big).$$

We conclude that the poles in (2.20) cancel.

Consider (2.23), thus $k < 0$ and $2 \mid k$. The regularization

$$\Gamma\big(\tfrac{\tilde{k}}{2}\big)^{-1}\,\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{1-k}{2}}\;{}_2\mathrm{F}_3\big(\tfrac{k+1}{2},\tfrac{1-k}{2};\tfrac{3}{2},\tfrac{\tilde{k}}{2},\tfrac{3-k}{2};\tfrac{\mathtt{v}}{4}\big)$$

has the Laurent expansion

$$\Gamma\big(\tfrac{\tilde{k}}{2}\big)^{-1}\,\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{1-k}{2}}$$
$$\cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{k+1}{2})\,\Gamma(n+\frac{1-k}{2})\,\Gamma(\frac{3}{2})\,\Gamma(\frac{\tilde{k}}{2})\,\Gamma(\frac{3-k}{2})}{\Gamma(\frac{k+1}{2)}\,\Gamma(\frac{1-k}{2})\,\Gamma(n+\frac{3}{2})\,\Gamma(n+\frac{\tilde{k}}{2})\,\Gamma(n+\frac{3-\tilde{k}}{2})\,\Gamma(n+1)}\Big(\frac{\mathtt{v}}{4}\Big)^n.$$

The coefficients with $n < 1-\frac{k}{2}$ vanish as $\tilde{k} \to k$. Hence the limit equals

$$\big(\tfrac{k+1}{2}\big)_{1-\frac{k}{2}}\,\big(\tfrac{1-k}{2}\big)_{1-\frac{k}{2}}\,\big(\tfrac{3}{2}\big)^{-1}_{1-\frac{k}{2}}\,\big(1-\tfrac{k}{2}\big)!^{-1}\,\big(\tfrac{3-k}{2}\big)^{-1}_{1-\frac{k}{2}}$$
$$\cdot\,\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{3}{2}-k}\;{}_2\mathrm{F}_3\big(\tfrac{3}{2},\tfrac{3}{2}-k;2-\tfrac{k}{2},\tfrac{5-k}{2},\tfrac{5}{2}-k;\tfrac{\mathtt{v}}{4}\big).$$

Consider (2.25), thus $k < 0$ and $2 \nmid k$. The regularization

$$\Gamma\big(\tfrac{\tilde{k}-1}{2}\big)\,\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{-\tilde{k}}{2}}\;{}_2\mathrm{F}_3\big(\tfrac{k}{2},\tfrac{-k}{2};\tfrac{1}{2},\tfrac{\tilde{k}-1}{2},1-\tfrac{k}{2};\tfrac{\mathtt{v}}{4}\big)$$

has the Laurent expansion

$$\Gamma\big(\tfrac{\tilde{k}-1}{2}\big)^{-1}\,\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{-k}{2}}$$
$$\cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{k}{2})\,\Gamma(n+\frac{-k}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{\tilde{k}-1}{2})\,\Gamma(1-\frac{k}{2})}{\Gamma(\frac{k}{2})\,\Gamma(\frac{-k}{2})\,\Gamma(n+\frac{1}{2})\,\Gamma(n+\frac{\tilde{k}-1}{2})\,\Gamma(n+1-\frac{k}{2})\,\Gamma(n+1)}\Big(\frac{\mathtt{v}}{4}\Big)^n.$$

The coefficients with $n < \frac{3-k}{2}$ vanish as $\tilde{k} \to k$. Hence the limit equals

$$\big(\tfrac{-k}{2}\big)_{\frac{3-k}{2}}\,\big(\tfrac{k}{2}\big)_{\frac{3-k}{2}}\,\big(\tfrac{1}{2}\big)^{-1}_{\frac{3-k}{2}}\,\big(1-\tfrac{k}{2}\big)^{-1}_{\frac{3-k}{2}}\,\big(\tfrac{1-k}{2}\big)!^{-1}$$
$$\cdot\,\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{3}{2}-k}\;{}_2\mathrm{F}_3\big(\tfrac{3}{2},\tfrac{3}{2}-k;2-\tfrac{k}{2},\tfrac{5-k}{2},\tfrac{5}{2}-k;\tfrac{\mathtt{v}}{4}\big).$$

We conclude that the poles in (2.25) cancel.

Consider (2.26), thus $k = 0$. The regularized hypergeometric series in (2.26) has the Laurent expansion

$$\Gamma\big(\tfrac{\tilde{k}}{2}\big)\,\Gamma\big(\tfrac{-\tilde{k}}{2}\big)\,\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{-k}{2}}$$
$$\cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{\tilde{k}}{2})\,\Gamma(n+\frac{-\tilde{k}}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{k-1}{2})\,\Gamma(1-\frac{k}{2})}{\Gamma(\frac{\tilde{k}}{2})\,\Gamma(\frac{-\tilde{k}}{2})\,\Gamma(n+\frac{1}{2})\,\Gamma(n+\frac{k-1}{2})\,\Gamma(n+1-\frac{k}{2})}\Big(\frac{\mathtt{v}}{4}\Big)^n.$$

The limit $\tilde{k} \to k$ is well-defined, since the poles of the gamma factors $\Gamma(\frac{\tilde{k}}{2})$ and $\Gamma(\frac{-\tilde{k}}{2})$ cancel.

Consider (2.29), thus $k>3$ and $2\mid k$. The regularization

$$\Gamma(1-\tfrac{\tilde{k}}{2})^{-1}\,(\tfrac{\mathrm{v}}{4})^{\frac{-k}{2}}\;{}_1\mathrm{F}_2\big(\tfrac{1-k}{2};\tfrac{1}{2},1-\tfrac{\tilde{k}}{2};\tfrac{\mathrm{v}}{4}\big)$$

has the Laurent expansion

$$\Gamma(1-\tfrac{\tilde{k}}{2})^{-1}\,(\tfrac{\mathrm{v}}{4})^{\frac{-k}{2}} \cdot\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1-k}{2})\,\Gamma(\frac{1}{2})\,\Gamma(1-\frac{\tilde{k}}{2})}{\Gamma(\frac{1-k}{2})\,\Gamma(n+\frac{1}{2})\,\Gamma(n+1-\frac{\tilde{k}}{2})\,\Gamma(n+1)}\big(\frac{\mathrm{v}}{4}\big)^{n}.$$

As $\tilde{k}\rightarrow k$ the coefficients vanish, if $n<\frac{k}{2}$. Hence employing the limit $\tilde{k}\rightarrow k$, we obtain

$$(\tfrac{1-k}{2})_{\frac{k}{2}}\,(\tfrac{1}{2})_{\frac{k}{2}}^{-1}\,(\tfrac{k}{2})!^{-1}\;{}_1\mathrm{F}_2\big(\tfrac{1}{2};\tfrac{1+k}{2},1+\tfrac{k}{2};\tfrac{\mathrm{v}}{4}\big).$$

It follows that the poles in (2.29) cancel.

Consider (2.30), thus $k>3$ and $2\mid k$. The regularization

$$\Gamma(2-\tfrac{\tilde{k}}{2})^{-1}\,(\tfrac{\mathrm{v}}{4})^{\frac{3}{2}-k}\;{}_2\mathrm{F}_3\big(1,2-\tilde{k};\tfrac{5}{2}-k,2-\tfrac{\tilde{k}}{2},\tfrac{5-k}{2};\tfrac{\mathrm{v}}{4}\big)$$

has the Laurent series expansion

$$\Gamma(2-\tfrac{\tilde{k}}{2})^{-1}\,(\tfrac{\mathrm{v}}{4})^{\frac{3}{2}-k} \cdot\sum_{n=0}^{\infty}\frac{\Gamma(n+2-\tilde{k})\,\Gamma(\frac{5}{2}-k)\Gamma(2-\frac{\tilde{k}}{2})\,\Gamma(\frac{5-k}{2})}{\Gamma(2-\tilde{k})\,\Gamma(n+\frac{5}{2}-k)\,\Gamma(2-\frac{\tilde{k}}{2})\,\Gamma(n+\frac{5-k}{2})}\big(\frac{\mathrm{v}}{4}\big)^{n}.$$

As before, we conclude that all coefficients with $n<\frac{k}{2}-1$ vanish as $\tilde{k}\rightarrow k$. If $n>2-k$ the poles of $\Gamma(2-\frac{\tilde{k}}{2})$ and $\Gamma(2-\tilde{k})$ cancel. Hence taking the limit $\tilde{k}\rightarrow k$, we get

$$(2-k)_{\frac{k}{2}-1}\,(\tfrac{5}{2}-k)_{\frac{k}{2}-1}^{-1}\,(\tfrac{5-k}{2})_{\frac{k}{2}-1}^{-1}\,(\tfrac{\mathrm{v}}{4})^{\frac{1-k}{2}}\;{}_1\mathrm{F}_2\big(1-\tfrac{k}{2};\tfrac{3}{2},\tfrac{3-k}{2};\tfrac{\mathrm{v}}{4}\big).$$

Consequently, the poles in (2.30) cancel.

Consider (2.32), thus $k>3$ and $2\nmid k$. The regularization

$$\Gamma(\tfrac{3-\tilde{k}}{2})^{-1}(\tfrac{\mathrm{v}}{4})^{\frac{1-k}{2}}\,{}_1\mathrm{F}_2\big(1-\tfrac{k}{2};\tfrac{3}{2},\tfrac{3-\tilde{k}}{2};\tfrac{\mathrm{v}}{4}\big)$$

has the Laurent expansion

$$\Gamma(\tfrac{3-\tilde{k}}{2})^{-1}\,(\tfrac{\mathrm{v}}{4})^{\frac{1-k}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-\frac{k}{2})\,\Gamma(\frac{3}{2})\,\Gamma(\frac{3-\tilde{k}}{2})}{\Gamma(1-\frac{k}{2})\,\Gamma(n+\frac{3}{2})\,\Gamma(n+\frac{3-\tilde{k}}{2})\Gamma(n+1)}\big(\frac{\mathrm{v}}{4}\big)^{n}.$$

The coefficients for $n<\frac{k-1}{2}$ vanish as $\tilde{k}\rightarrow k$. Hence taking the limit of the above expansion, we obtain

$$(1-\tfrac{k}{2})_{\frac{k-1}{2}}\,(\tfrac{3}{2})_{\frac{k-1}{2}}^{-1}\,(\tfrac{k-1}{2})!^{-1}\;{}_1\mathrm{F}_2\big(\tfrac{1}{2};\tfrac{1+k}{2},1+\tfrac{k}{2};\tfrac{\mathrm{v}}{4}\big).$$

Consequently, the poles in (2.32) cancel.

Consider (2.33), thus $k>3$ and $2\nmid k$. The regularization

$$\Gamma(\tfrac{5-\tilde{k}}{2})^{-1}\,(\tfrac{\mathrm{v}}{4})^{\frac{3}{2}-k}\;{}_2\mathrm{F}_3\big(1,2-k;\tfrac{5}{2}-k,2-\tfrac{k}{2},\tfrac{5-\tilde{k}}{2};\tfrac{\mathrm{v}}{4}\big)$$

has the Laurent expansion

$$\Gamma(\tfrac{5-\tilde{k}}{2})^{-1}\,(\tfrac{\mathrm{v}}{4})^{\frac{3}{2}-\tilde{k}} \cdot\sum_{n=0}^{\infty}\frac{\Gamma(n+2-\tilde{k})\,\Gamma(\frac{5}{2}-k)\,\Gamma(2-\frac{k}{2})\,\Gamma(\frac{5-\tilde{k}}{2})}{\Gamma(2-\tilde{k})\,\Gamma(n+\frac{5}{2}-k)\,\Gamma(n+2-\frac{k}{2})\,\Gamma(n+\frac{5-\tilde{k}}{2})}\big(\frac{\mathrm{v}}{4}\big)^{n}.$$

The coefficients with $n < \frac{k-3}{2}$ vanish as $\tilde{k} \to k$. If $n > 2-k$, the poles of $\Gamma(2-\tilde{k})$ and $\Gamma(\frac{5-\tilde{k}}{2})$ cancel. Hence the limit of the above series equals

$$(2-k)_{\frac{k-3}{2}}\,(\tfrac{5}{2}-k)^{-1}_{\frac{k-3}{2}}\,(2-\tfrac{k}{2})^{-1}_{\frac{k-3}{2}}\,(\tfrac{\mathrm{v}}{4})^{-\frac{k}{2}}\,{}_1\mathrm{F}_2\big(\tfrac{1-k}{2};\tfrac{1}{2},1-\tfrac{k}{2};\tfrac{\mathrm{v}}{4}\big)$$

We conclude that the poles in (2.33) cancel.

Consider (2.35), thus $k < 0$ and $2 \mid k$. The regularization

$$\Gamma(1+\tfrac{\tilde{k}}{2})^{-1}\,{}_1\mathrm{F}_2\big(\tfrac{1}{2};\tfrac{1+k}{2},1+\tfrac{\tilde{k}}{2};\tfrac{\mathrm{v}}{4}\big)$$

has the Laurent expansion

$$\Gamma(1+\tfrac{\tilde{k}}{2})^{-1}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1+k}{2})\,\Gamma(1+\frac{\tilde{k}}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1+k}{2})\,\Gamma(n+1+\frac{\tilde{k}}{2})}\big(\frac{\mathrm{v}}{4}\big)^n.$$

The coefficients with $n < \frac{k}{2}$ vanish as $\tilde{k} \to k$. Hence the limit of the above series equals

$$(\tfrac{1}{2})_{\frac{-k}{2}}\,(\tfrac{k+1}{2})^{-1}_{\frac{-k}{2}}\,(\tfrac{-k}{2})!^{-1}\,(\tfrac{\mathrm{v}}{4})^{\frac{-k}{2}}\,{}_1\mathrm{F}_2\big(\tfrac{1-k}{2};\tfrac{1}{2},1-\tfrac{k}{2};\tfrac{\mathrm{v}}{4}\big).$$

It follows that the poles in (2.35) cancel.

Consider (2.36), thus $k < 0$ and $2 \nmid k$. The regularization

$$\Gamma(\tfrac{\tilde{k}+1}{2})^{-1}\,{}_1\mathrm{F}_2\big(\tfrac{1}{2};\tfrac{\tilde{k}+1}{2},1+\tfrac{k}{2};\tfrac{\mathrm{v}}{4}\big)$$

has the Laurent expansion

$$\Gamma(\tfrac{\tilde{k}+1}{2})^{-1}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1+\tilde{k}}{2})\,\Gamma(1+\frac{k}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1+\tilde{k}}{2})\,\Gamma(n+1+\frac{k}{2})}\big(\frac{\mathrm{v}}{4}\big)^n.$$

The coefficients with $n < \frac{k-1}{2}$ vanish as $\tilde{k} \to k$. Consequently, the limit equals

$$(\tfrac{1}{2})_{\frac{1-k}{2}}\,(1+\tfrac{k}{2})^{-1}_{\frac{1-k}{2}}\,(\tfrac{1-k}{2})!^{-1}\,(\tfrac{\mathrm{v}}{4})^{\frac{-k}{2}}\,{}_1\mathrm{F}_2\big(\tfrac{1-k}{2};\tfrac{1}{2},1-\tfrac{k}{2};\tfrac{\mathrm{v}}{4}\big).$$

We conclude that the poles in (2.36) cancel. □

From this proof we immediately conclude:

COROLLARY 2.10. *The function (2.17), up to addition of a polynomial, is a multiple of (2.15). The same holds for the pairs of functions (2.20) and (2.18), (2.23) and (2.21), (2.25) and (2.21), (2.29) and (2.27), (2.30) and (2.28), (2.32) and (2.27), (2.33) and (2.31), (2.35) and the second function in (2.34), and (2.36) and the third function in (2.34).*

Of cause, we need the solutions to all other differential equations in Theorem 2.2, that turn out to be much easier to solve. We need the Whittaker functions $W_{\kappa,\mu}$ and $M_{\kappa,\mu}$, which, by definition, solve the differential equation

$$\phi'' + \Big(-\tfrac{1}{4} + \kappa\mathrm{u}^{-1} + \big(\tfrac{1}{4}-\mu^2\big)^2\mathrm{u}^{-2}\Big)\,\phi = 0. \tag{2.39}$$

We have $W_{\kappa,\mu}(\mathrm{u}) \to 0$ as $\mathrm{u} \to 0$, whereas $M_{\kappa,\mu}$ grows rapidly towards infinity.

LEMMA 2.11. *Let $\alpha = k$ and $\beta = 0$. The space of functions $\phi(\mathrm{u})$ and $\psi(\mathrm{u})$ in (2.3) is spanned by*

$$\mathrm{u}^{\frac{k-3}{2}}W_{\frac{k}{2},1-\frac{k}{2}}(2\mathrm{u}),\qquad \mathrm{u}^{\frac{k-3}{2}}M_{\frac{k}{2},1-\frac{k}{2}}(2\mathrm{u}), \tag{2.40}$$

and

$$\mathrm{u}^{\frac{-k}{2}}W_{\frac{k}{2},\frac{1-k}{2}}(2\mathrm{u}),\qquad \mathrm{u}^{\frac{-k}{2}}M_{\frac{k}{2},\frac{1-k}{2}}(2\mathrm{u}), \tag{2.41}$$

respectively.

The space of functions $g_0(\mathtt{u})$ *in (2.4) is spanned by*

$$\mathtt{u}^{1-k}\int_{\mathtt{u}}^{\infty}\tilde{\mathtt{u}}^{-1}W_{k,\frac{3}{2}-k}(2\tilde{\mathtt{u}})\,d\tilde{\mathtt{u}},\qquad \mathtt{u}^{1-k}\int_{\mathtt{u}}^{1}\tilde{\mathtt{u}}^{-1}W_{k,\frac{3}{2}-k}(2\tilde{\mathtt{u}})\,d\tilde{\mathtt{u}},\quad \textit{and} \tag{2.42}$$

$$\mathtt{u}^{1-k}\int_{\mathtt{u}}^{1}\tilde{\mathtt{u}}^{-1}M_{k,\frac{3}{2}-k}(2\tilde{\mathtt{u}})\,d\tilde{\mathtt{u}}.$$

Let $\alpha=\frac{1}{2}$ *and* $\beta=k-\frac{1}{2}$*. The space of functions* $\phi(\mathtt{u})$ *and* $\psi(\mathtt{u})$ *in (2.3) is spanned by*

$$\mathtt{u}^{\frac{k-3}{2}}W_{\frac{1-k}{2},1-\frac{k}{2}}(2\mathtt{u}),\qquad \mathtt{u}^{\frac{k-3}{2}}M_{\frac{1-k}{2},1-\frac{k}{2}}(2\mathtt{u}), \tag{2.43}$$

and

$$\mathtt{u}^{\frac{-\alpha-\beta}{2}}W_{\frac{1-k}{2},\frac{1-k}{2}}(2\mathtt{u}),\qquad \mathtt{u}^{\frac{-\alpha-\beta}{2}}M_{\frac{1-k}{2},\frac{1-k}{2}}(2\mathtt{u}), \tag{2.44}$$

respectively.

The space of functions $g_0(\mathtt{u})$ *in (2.4) is spanned by*

$$\mathtt{u}^{1-k}\int_{\mathtt{u}}^{\infty}\tilde{\mathtt{u}}^{-1}W_{1-k,\frac{3}{2}-k}(2\tilde{\mathtt{u}})\,d\tilde{\mathtt{u}},\quad \mathtt{u}^{1-k}\int_{\mathtt{u}}^{1}\tilde{\mathtt{u}}^{-1}W_{1-k,\frac{3}{2}-k}(2\tilde{\mathtt{u}})\,d\tilde{\mathtt{u}},\quad \textit{and} \tag{2.45}$$

$$\mathtt{u}^{1-k}\int_{\mathtt{u}}^{1}\tilde{\mathtt{u}}^{-1}M_{1-k,\frac{3}{2}-k}(2\tilde{\mathtt{u}})\,d\tilde{\mathtt{u}}.$$

REMARK 2.12. *Some of the Whittaker functions can be expressed in terms of incomplete gamma functions or exponentials. See* [**BRR11a**] *for details in the skew case.*

PROOF. We start reformulating the differential equations for ϕ and ψ in (2.3). Set $\phi(\mathtt{u})=\mathtt{u}^{\frac{\alpha+\beta-3}{2}}\tilde{\phi}(2\mathtt{u})$ and $\psi(\mathtt{u})=\mathtt{u}^{\frac{-\alpha-\beta}{2}}\tilde{\psi}(2\mathtt{u})$. In order to treat $\tilde{\phi}$, we abbreviate $l=\frac{\alpha+\beta-3}{2}$, and deduce from the differential equation for ϕ that

$$\begin{aligned}0=\;&\mathtt{u}\big(l(l-1)\mathtt{u}^{l-2}\,\tilde{\phi}(2\mathtt{u})+2l\mathtt{u}^{l-1}\,\tilde{\phi}'(2\mathtt{u})+\mathtt{u}^{l}\,\tilde{\phi}''(2\mathtt{u})\big)\\&+(3-\alpha-\beta)\big(l\mathtt{u}^{l-1}\,\tilde{\phi}(2\mathtt{u})+\mathtt{u}^{l}\,\tilde{\phi}'(2\mathtt{u})\big)\\&+(\alpha-\beta-\mathtt{u})\,\tilde{\phi}(2\mathtt{u}).\end{aligned}$$

Since we have $\mathtt{u}\neq 0$, we can deduce that

$$0=\tilde{\phi}''(2\mathtt{u})+\Big(-\tfrac{1}{4}+\tfrac{\alpha-\beta}{2}(2\mathtt{u})^{-1}+\big(\tfrac{1}{4}-(1-\tfrac{\alpha+\beta}{2})^2\big)(2\mathtt{u})^{-2}\Big)\,\tilde{\phi}(2\mathtt{u}). \tag{2.46}$$

Let now $l=\frac{-\alpha-\beta}{2}$, and consider the differential equation for ψ:

$$\begin{aligned}0=\;&\mathtt{u}\big(l(l-1)\mathtt{u}^{l-2}\,\tilde{\psi}(2\mathtt{u})+2l\mathtt{u}^{l-1}\,\tilde{\psi}'(2\mathtt{u})+u^{l}\,\tilde{\psi}''(2\mathtt{u})\big)\\&+(\alpha+\beta)\big(l\mathtt{u}^{l-1}\tilde{\psi}(2\mathtt{u})+\mathtt{u}^{l}\tilde{\psi}'(2\mathtt{u})\big)\\&+(\alpha-\beta-\mathtt{u})\,\tilde{\psi}(2\mathtt{u}).\end{aligned}$$

Reordering the terms, we obtain

$$0=\tilde{\psi}''(2\mathtt{u})+\Big(-\tfrac{1}{4}+\tfrac{\alpha-\beta}{2}(2\mathtt{u})^{-1}+\big(\tfrac{1}{4}-(\tfrac{1}{2}-\tfrac{\alpha+\beta}{2})^2\big)(2\mathtt{u})^{-2}\Big)\,\tilde{\psi}(2\mathtt{u}). \tag{2.47}$$

The differential equation for ϕ in (2.4) can be easily manipulated to yield

$$0=\tilde{\phi}''(2\mathtt{u})+\Big(-\tfrac{1}{4}+(\alpha-\beta)(2\mathtt{u})^{-1}\big(\tfrac{1}{4}-(\tfrac{3}{2}-(\alpha+\beta))^2\big)(2\mathtt{u})^{-2}\Big)\,\tilde{\phi}(2\mathtt{u}). \tag{2.48}$$

From (2.46), (2.47) and (2.48), we recognize the Whittaker differential equation (2.39). The parameters are $\kappa=\frac{\alpha-\beta}{2}$, $\mu=\pm(1-\frac{\alpha+\beta}{2})$ in the first case,

$\kappa = \frac{\alpha-\beta}{2}$, $\mu = \pm(\frac{1}{2} - \frac{\alpha+\beta}{2})$ in the second case, and $\kappa = \alpha - \beta$, $\mu = \pm(\frac{3}{2} - (\alpha+\beta))$ in the last case. This proves all claims concerning Whittaker functions.

Finally, one obtains the solution for g_0 directly from solutions above, since the equation $\psi'(\mathtt{u}) = \mathtt{u}^{-1}\phi(\mathtt{u})$ involves only the derivative of ψ. □

3. Fourier expansions of harmonic Siegel modular forms

So far, we have concentrated on arbitrary Fourier expansions that are harmonic in the sense that they are in the kernel of either Ω_k or Ω_k^{sk}. The space of solutions is still quite large and difficult to work with. In particular, the case of indefinite Fourier indices, that was treated in Lemma 2.7, has turned out to be complicated. On the other hand, we have only made use of one and a half properties that harmonic Siegel modular forms have by definition. Fourier expansions occur in the theory of Siegel modular forms as a consequence of invariance under the unipotent part of $\mathrm{Sp}_2(\mathbb{Z})$. It is striking that we will need invariance under the full modular group to exclude the solution to the wave equation that occurred in Maaß's theorem (see (2.2)). A third property shared by all harmonic Siegel modular forms is the growth condition in Definition 1.1. It is a surprisingly difficult to determine the growth of the Fourier coefficients that the solutions to (2.4) and (2.5) give rise to. Already Maaß [**Maa53**] asked what the properties of these Fourier coefficients were as Z approaches the boundary of the Siegel upper half space. The proof of Theorem 3.1 gives a satisfying answer to this question.

A weak form of Theorem 3.1 was already proved in [**BRR11a**]. In that work, the author and his collaborators not only restricted their attention to the skew slash action, but they also needed to impose a further, technical condition on $\xi_k^{(2),\mathrm{sk}} f$. The version that we present does not depend on this condition anymore.

The next theorem sharpens Theorem 2.2 in the case of harmonic Siegel-Maaß forms. Note that the exponentials of the Fourier series expansions in Theorem 2.2 and 3.1 differ by 2π.

THEOREM 3.1. *Suppose that $k \neq 1, 2$. Let*

$$f(Z) = \sum_T a(Y,T)\, e^{2\pi i\,\mathrm{tr}(TX)} \in \mathbb{M}_k.$$

(i) *If $T = 0$, then $a(Y,0)$ is contained in the two dimensional space spanned by*

$$\det Y^{\frac{3}{2}-k} \quad \textit{and} \quad 1. \tag{3.1}$$

(ii) *If $\mathrm{rk}(T) = 1$ and $T \geq 0$ then $a(Y,T)$ is contained in a two dimension space spanned by*

$$\det(Y)^{\frac{3}{2}-k} \mathtt{u}^{\frac{k-3}{2}} W_{\frac{k}{2},1-\frac{k}{2}}(4\pi\mathtt{u}) \quad \textit{and} \quad \mathtt{u}^{\frac{-k}{2}} W_{\frac{k}{2},\frac{1-k}{2}}(4\pi\mathtt{u}). \tag{3.2}$$

(iii) *If $\mathrm{rk}(T) = 2$ and $T > 0$, then $a(Y,T)$ is a multiple of*

$$\sum_{n=0}^{\infty} g_n(2\pi\mathtt{u})\,(4\pi^2\mathtt{v})^n,$$

where g_n is defined by the recursion in (iii) of Theorem 2.2 and

$$g_0(\mathtt{u}) = \mathtt{u}^{1-k} \int_{\mathtt{u}}^{\infty} \tilde{\mathtt{u}}^{-1} W_{k,\frac{3}{2}-k}(2\tilde{\mathtt{u}})\, d\tilde{\mathtt{u}}.$$

(iv) *If* $\mathrm{rk}(T) = 2$ *and* T *is indefinite, then* $a(Y,T)$ *is contained in a one-dimensional space depending on* T *only. If* $k < 0$ *is even, this space is spanned by the corresponding Fourier coefficients of* $P^{(2)}_{k,\frac{3}{2}-k}$.

Let

$$f(Z) = \sum_T a(Y,T)\, e^{2\pi i\, \mathrm{tr}(TX)} \in \mathbb{M}^{\mathrm{sk}}_k.$$

(i) *If* $T = 0$*, a statement like in the above case (i) holds. That is,* $a(Y,0)$ *is contained in the two dimensional space spanned by*

$$\det Y^{\frac{3}{2}-k} \quad \textit{and} \quad 1. \tag{3.3}$$

(ii) *If* $\mathrm{rk}(T) = 1$ *and* $T \geq 0$ *then* $a(Y,T)$ *is contained in a two dimension space spanned by*

$$\det(Y)^{\frac{3}{2}-k} \mathtt{u}^{\frac{k-3}{2}} W_{\frac{1-k}{2},1-\frac{k}{2}}(2\mathtt{u}) \quad \textit{and} \quad \mathtt{u}^{\frac{-\alpha-\beta}{2}} W_{\frac{1-k}{2},\frac{1-k}{2}}(2\mathtt{u}). \tag{3.4}$$

(iii) *If* $\mathrm{rk}(T) = 2$ *and* $T > 0$*, then* $a(Y,T)$ *is a multiple of*

$$\sum_{n=0}^{\infty} g_n(2\pi\mathtt{u})\,(4\pi^2\mathtt{v})^n,$$

where g_n *is defined by the recursion in (iii) of Theorem 2.2 and*

$$g_0(\mathtt{u}) = \mathtt{u}^{1-k} \int_{\mathtt{u}}^{\infty} \tilde{\mathtt{u}}^{-1} W_{1-k,\frac{3}{2}-k}(2\tilde{\mathtt{u}})\, d\tilde{\mathtt{u}}.$$

(iv) *If* $\mathrm{rk}(T) = 2$ *and* T *is indefinite, then* $a(Y,T)$ *is contained in a one-dimensional space depending on* T *only. If* $k \neq 1,3$ *is odd, this space is spanned by the corresponding Fourier coefficients of* $P^{(2),\mathrm{sk}}_{k,0}$ *(for* $k > 3$*) and* $P^{(2),\mathrm{sk}}_{k,\frac{3}{2}-k}$ *(for* $k < 0$*).*

To prove this theorem, we will need the next lemmas. We write $(p)_j$ for the j^{th} coefficient of a polynomial p.

LEMMA 3.2. *Suppose that a sequence of Laurent polynomials* l_n *in* $\mathtt{u}$ *satisfies a recursion of the form*

$$l_{n+1} = \sum_{d=0}^{D} p_{n,d}\, l_n^{(d)},$$

where $D \in \mathbb{Z}_{\geq 0}$ *and the* $p_{n,d}$ *are Laurent polynomials in* $\mathtt{u}$*. Furthermore, suppose that* $\deg_{\mathtt{u}} p_{n,0} = 0$ *and* $\deg_{\mathtt{u}} p_{n,d} < d$ *for* $d \neq 0$*. Assume that the valuation of all* $p_{n,d}$*, denoted by* $\mathrm{val}_{\mathtt{u}}(p_{n,d})$*, is uniformly bounded, and let* V *be a lower bound on* $\mathrm{val}_{\mathtt{u}}(p_{n,d}) - d$*. Suppose that* $(n|V|)^d \cdot p_{n,d}$ *has uniformly bounded coefficients as* $n \to \infty$*. If the leading coefficients of* l_0 *and* $p_{n,0}$ *are positive, then there is a constant* κ *such that the series*

$$\sum_{n=0}^{\infty} l_n \cdot \left(\frac{\mathtt{u}}{\kappa}\right)^n \tag{3.5}$$

is well-defined as a formal Laurent series and it has bounded coefficients

If, in addition,

$$\binom{n-i+\#\{(d,j):(p_{\tilde n,d})_j\neq 0 \text{ for some } \tilde n\}-1}{n-i}$$

$$\cdot\Big(|V|+|\deg_{\mathbf{u}} l_0|+(\deg_{\mathbf{u}} l_0-\operatorname{val}_{\mathbf{u}} l_0)\Big)^{n-i}\Bigg(\prod_{n'=j+1}^{n+j}(p_{n',0})_0\Bigg)\Big(\max_{(d,j)\neq(0,0)} h_{d,j}\Big)^{n-i} \tag{3.6}$$

is bounded for $n\geq 1$, $j\geq \deg_{\mathbf{u}} l_0$, *and* $0\leq i\leq \min\{n,\deg_{\mathbf{u}} l_0-\operatorname{val}_{\mathbf{u}} l_0\}$, *where the first factor is the usual binomial coefficient and*

$$h_{d,j}:=\left(\sum_{n'=0}^{n+j}\frac{(n'+1)^d\big|(p_{n',d})_j\big|}{(p_{n',0})_0}\right)^{\frac{1}{d-j}},$$

then κ *can be chosen such that, in addition, all coefficients of* $\mathbf{u}^j$ *with* $j>\deg_{\mathbf{u}} l_0$ *in* (3.5) *are positive.*

Proof. Set $D_l:=\deg_{\mathbf{u}} l_0$ and $V_l:=\operatorname{val}_{\mathbf{u}} l_0$. By assumption on the degrees of the $p_{n,d}$, all l_n have degree less than or equal to D_l. We can deduce by induction from the assumption on the leading coefficient of $p_{n,0}$ that the leading coefficients of the l_n are all positive. Let $B_1>1$ be a bound on the absolute values of the coefficients of l_0 such that $B_1^{-1}<(l_0)_{D_l}$. Let $B_2>1$ be a bound on the absolute value of $\big(|V_l|+|D_l|+n|V|\big)\sum_{d,j}\big|(p_{n,d})_j\big|$ for all n. Note that the valuation of l_n is bounded by V_l+nV. Using mathematical induction, we can prove that the coefficients of l_n have absolute values bounded by $B_1B_2^n$. Set $\kappa=2B_2$. The absolute value of the j^{th} coefficient in (3.5) is given by

$$\Big|\sum_{n=0}^{\infty}\frac{(l_n)_{j-n}}{\kappa^n}\Big|\leq\sum_{n=0}^{\infty}\frac{B_1B_2^n}{\kappa^n}\leq 2B_1.$$

Hence the series (3.5) is, indeed, well-defined, and the absolute values of its coefficients are bounded by $2B_1$.

To prove the positivity of the coefficients for $\mathbf{u}^j$ with sufficiently large j suppose that (3.6) is bounded. Let $B_3>1$ be a bound for (3.6) valid for all $n\geq 1$ and $j\geq D_l$.

We first bound $(l_{n+j})_{D_l-n}$ for all $n\geq 1$ and $j\geq D_l$. Recall that a multiset is a set, where elements can occur with multiplicity different from 1. In particular, sums ranging over multisets respect these multiplicities. Set

$$DJ(\tilde n):=\Big\{\text{multiset } S \text{ of pairs } (d,j)\neq(0,0)\ :\ \sum_{(d,j)\in S}(d-j)=\tilde n,\ \forall(d,j)\in S:(p_{n,d})_j\neq 0 \text{ for some } n\Big\}.$$

We find that $\big|(l_{n+j})_{D_l-n}\big|$ is bounded by

$$\sum_{i=0}^{\min\{n,D_l-V_l\}}(l_0)_{D_l-i}\Bigg(\prod_{n'=0}^{n+j}(p_{n',0})_0\Bigg)\cdot\sum_{S\in DJ(n-i)}\prod_{(d,j)\in S}\sum_{n'=0}^{n+j}\frac{\big(|V|(n'+1)+i+|D_l|\big)^d\big|(p_{n',d})_j\big|}{(p_{n',0})_0}. \tag{3.7}$$

We explain how to obtain this estimate. The first factor, $(l_0)_{D_l - i}$, is the coefficient of l_0 that the contribution to $(l_{n+j})_{D_l-n}$ originates in. The second factor is the product of the leading coefficients of $p_{n',0}$'s. It originates in the fact that the recursion formula for l_{n+1} will result in either multiplication of the coefficients of l_n by $(p_{n',0})_0$, or differentiation and multiplication by negative power of $\mathtt{u}$. The last factor captures the latter contribution. The elements of $DJ(n-i)$ reflect all possible ways to lower the exponent of $\mathtt{u}$ in $(l_0)_{D_l-i}\,\mathtt{u}^{D_l-i}$ to $D_l - n$, which is the exponent of $\mathtt{u}$ showing up in $(l_{n+j})_{D_l-n}\,\mathtt{u}^{D_l-n}$. The most inner sum reflects the fact that operations lowering the power of $\mathtt{u}$ can occur in any step of the recursion.

The sum over $DJ(n-i)$ in (3.7) can be estimated as follows: The inner sum is replaces by its $\big(1/(d-j)\big)^{\mathrm{th}}$ power, yielding $h_{d,j}$, and the product over the $(d,j) \in S$ is replaced by the $(n-i)^{\mathrm{th}}$ power of the maximum of all $h_{d,j}$'s. This gives rise to the next estimate:

$$
\begin{aligned}
(l_{n+j})_{D_l-n} \leq \sum_{i=0}^{\min\{n,\, D_l - V_l\}} &(l_0)_{D_l-i} \Big(\prod_{n'=0}^{n+j} (p_{n',0})_0 \Big) \\
&\cdot \big(|V| + i + |D_l|\big)^{n-i} \sum_{S \in DJ(n-i)} \Big(\max_{d,j} h_{d,j} \Big)^{n-i}.
\end{aligned}
$$

The cardinality of $DJ(n-i)$ is bounded by:

$$
\binom{n - i + \#\{(d,j) : (p_{\tilde{n},d})_j \neq 0 \text{ for some } \tilde{n}\}}{n-i}.
$$

In other words, $\big|(l_{n+j})_{D_l-n}\big| \leq (D_l - V_l) B_1 B_3 \prod_{n'=0}^{j} (p_{n',0})_0$.

We replace κ by $\max\{\kappa,\, 3(D_l - V_l) B_1^2 B_3\}$, so that

$$
(l_{n+j})_{D_l-n} \kappa^{-n} \leq 3^{-n} (l_0)_{D_l} \prod_{n'=0}^{j} (p_{n',0})_0 = 3^{-n} (l_j)_{D_l}
$$

for all $n > 0$. With this κ, the positivity of all the coefficients of $\mathtt{u}^j$ with $j \geq D_j$ in (3.5) follows from

$$
\begin{aligned}
&\frac{(l_{j-D_l})_{D_l}}{\kappa^{j-D_l}} - \Big| \sum_{n=j-D_l+1}^{\infty} \frac{(l_n)_{j-n}}{\kappa^n} \Big| \\
\geq{}& \Big(\frac{1}{\kappa}\Big)^{j-D_l} \Big((l_{j-D_l})_{D_l} - \sum_{n=1}^{\infty} \frac{|(l_{j-D_l+n})_{D_l-n}|}{\kappa^n} \Big) \\
\geq{}& \Big(\frac{1}{\kappa}\Big)^{j-D_l} (l_{j-D_l})_{D_l} \frac{2}{3} \geq 0.
\end{aligned}
$$

□

LEMMA 3.3. *For l_0 with $\deg_{\mathtt{u}} l_0 = 0$ and $(l_0) > 0$, the recursions in (iii) and (iv) of Theorem 2.2 satisfy the assumptions of Lemma 3.2.*

PROOF. Is suffices to show that

$$
\frac{(n-i)^3 2^n (n+j)^{\frac{3}{2}(n-i)}}{(j+1)_n^2} \leq \frac{n^3 2^n (n+j)^{\frac{3}{2}n}}{(j+1)_n^2}
$$

is bounded for $n, j \geq 1$. When writing the Pochhammer symbol as a quotient of factorials, this is immediate from Stirling's formula. □

LEMMA 3.4. *Up to multiplicative scalars, almost all coefficients of the Laurent expansion of* (2.35) *and the second function in* (2.34)*, and* (2.36) *and the third function in* (2.34) *are equal.*

PROOF. The poles of the coefficients of one series are canceled by another one, that is multiplied with a suitable gamma factor. The claim follows, since the Taylor expansions of the gamma factors around $\tilde{k} = k$ have a nonvanishing constant term. $\square$

LEMMA 3.5. *If $k \le 0$ is even, the quotient of the coefficient of $\mathtt{v}^n$ of the Laurent expansion of*

$$\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\big(1, 2-k; \tfrac{5}{2}-k, 2-\tfrac{k}{2}, \tfrac{5-k}{2}; \tfrac{\mathtt{v}}{4}\big)$$

and the coefficient of $\mathtt{v}^n$ of the Laurent expansion of

$$\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{1-k}{2}} {}_1\mathrm{F}_2\big(1-\tfrac{k}{2}; \tfrac{3}{2}, \tfrac{3-k}{2}; \tfrac{\mathtt{v}}{4}\big)$$

tends to zero as $n \to \infty$ through half-integral numbers.

If $k \le 0$ is odd, the quotient of the coefficient of $\mathtt{v}^n$ of the Laurent expansion of

$$\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{3}{2}-k} {}_2\mathrm{F}_3\big(1, 2-k; \tfrac{5}{2}-k, 2-\tfrac{k}{2}, \tfrac{5-k}{2}; \tfrac{\mathtt{v}}{4}\big)$$

and the coefficient of $\mathtt{v}^n$ of the Laurent expansion of

$$\big(\tfrac{\mathtt{v}}{4}\big)^{-\frac{k}{2}} {}_1\mathrm{F}_2\big(\tfrac{1-k}{2}; \tfrac{1}{2}, 1-\tfrac{k}{2}; \tfrac{\mathtt{v}}{4}\big)$$

tends to zero as $n \to \infty$ through half-integral numbers.

In particular, any linear combination of the first and the second, or the third and the fourth hypergeometric series grows rapidly as $\mathtt{v} \to \infty$.

PROOF. Assume that k is even. Up to addition of a polynomial, the second hypergeometric function equals

$$\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{3}{2}-k} {}_1\mathrm{F}_2\big(2-k; \tfrac{5-k}{2}, \tfrac{5}{2}-k; \tfrac{\mathtt{v}}{4}\big).$$

Consequently, the quotient of the coefficient of $\mathtt{v}^{\tilde{n}}$ with $\tilde{n} = n + \frac{3}{2} - k$ asymptotically equals

$$\Gamma(\tilde{n}+1)\,\Gamma(\tilde{n}+2-\tfrac{k}{2})^{-1} = (\tilde{n}+2-\tfrac{k}{2})^{-1}_{1-\frac{k}{2}},$$

which tends to zero as $\tilde{n} \to \infty$.

In the case of $2 \nmid k$, a similar argument works, since the second hypergeometric series equals, up to a polynomial,

$$\big(\tfrac{\mathtt{v}}{4}\big)^{\frac{3}{2}-k} {}_1\mathrm{F}_2\big(2-k; 2-\tfrac{k}{2}, \tfrac{5-k}{2}; \tfrac{\mathtt{v}}{4}\big).$$

The rapid growth of the linear combinations of the hypergeometric series in the lemma follows, since the coefficients of said linear combinations are ultimately all positive or all negative. $\square$

PROOF OF THEOREM 3.1. The statements (i) are based on a computer assisted proof. The Sage script, which makes use of Singular [**DGPS10**] and Plural [**LS03**], can be found in Section 4 of Appendix A. The script is written in such a way that it can be loaded directly in Sage (using the command `load "filename"`). The absence of assertion errors raised by Sage during the computations proves the claim.

To prove the other statements we will analyze the growth of potential Fourier coefficients. The Fourier coefficients of a Siegel modular form f of degree 2 are given by the integral

$$a(Y,T) = \int_{\mathbb{R}^3} f(Z)e(-\mathrm{tr}(TX))\, dX.$$

Thus any $a(Y,T)$ that occurs in the Fourier expansion of a harmonic Siegel modular form is bounded from above by $a\,\mathrm{tr}(Y)^c$ for some $a, c \in \mathbb{R}$ for $\mathrm{tr}(Y) \to \infty$, that is, it does not grow rapidly.

The statements (ii) follow directly from the growth of Whittaker functions. The M-Whittaker function grows rapidly as $\mathtt{u} \to \infty$ as was stated after the defining differential equation (2.39). Hence it does not occur in the Fourier expansion of a harmonic Siegel modular form.

For the same reason, in the cases (iii), the solutions in (2.42) and (2.45) that include the M-Whittaker function do not occur. Indeed, in the series expansion (2.4) we may set $\mathtt{v} = 0$. In this special case, the growth condition for harmonic Siegel modular form reduces to a growth condition for g_0.

Suppose that both of the functions in (2.4) that involve the W-Whittaker function occur as functions g_0 for coefficients $a(Y,T)$ of a harmonic Siegel modular form. Then, in particular, their difference, which is a nonzero multiple of $\mathtt{u}^{1-k}$, occurs. For $l_0 = \mathtt{u}^{1-k}$ and the recursion in (iii) of Theorem 2.2, choose κ according to the second part of Lemma 3.2 and Lemma 3.3. We may set $\mathtt{v} = \mathtt{u}\kappa^{-1}$ in the series

$$\sum_{n=0}^{\infty} l_n(\mathtt{u})\mathtt{v}^n,$$

that still converges locally absolutely by Theorem 2.2. By the choice of κ, all coefficients c_j of this series for j sufficiently large are positive. Thus this specialization grows rapidly for $\mathtt{u} \to \infty$. This contradicts the assumption that neither of the initial functions g_0 leads to a rapidly growing Fourier coefficient. Thus both cases (iii) are proved.

Consider the cases (iv). We first argue that for every $k \in \mathbb{Z}$, three fundamental solutions to (2.9) that are listed in Lemma 2.7 lead to rapidly growing Fourier coefficients $a(T,Y)$. This follows, if the solution is a polynomial using Lemma 3.2 and 3.3. If it is a nonpolynomial hypergeometric series or it differs from such a series by a polynomial only, it follows by setting $\mathtt{u} = 0$ and the fact that nonpolynomial hypergeometric series grow rapidly towards infinity.

By Corollary 2.10, in the case $\beta = 0$, the following sets span a space of functions, that either grow rapidly or are polynomials:

- If $k \geq 4$ and k is even, the fundamental solutions (2.14), (2.15), and (2.17) form such a set.
- If $k \geq 3$ and k is odd, the fundamental solutions (2.14), (2.18), and (2.20) form such a set.
- If $k = 0$, the fundamental solutions (2.14), (2.21), and (2.23) form such a set.
- If $k < 0$ and k is even, the fundamental solutions (2.14), (2.21), and (2.23) form such a set.
- If $k < 0$ and k is odd, the fundamental solutions (2.14), (2.21), and (2.25) form such a set.

Using Lemma 3.2 and 3.3 as indicated above, the theorem follows in the case $\beta = 0$.

The same argument works if $\alpha = \frac{1}{2}$ in the following cases:

- If $k \geq 4$ and k is even, a set as above is formed by the fundamental solutions (2.27), (2.28), and (2.29).
- If $k \geq 3$ and k is odd, a set as above is formed by the fundamental solutions (2.27), (2.31), and (2.32).

We have to use a different argument if $k \leq 0$. Suppose that k is even. By Lemma 3.5, any nonzero linear combination of the first and third solution in (2.34) grows rapidly. Using Corollary 2.10, we find that the third solution in (2.34) and (2.35) differ by a polynomial. If k is odd, the same argument works with the first and second solution in (2.34) and (2.36).

Thus all cases of the theorem are proved. $\square$

CHAPTER 5

Fourier-Jacobi expansions

In [**Koh94**], Kohnen analyzed a family of Jacobi Poincaré series by relating them to Siegel Eisenstein series. His main motivation was to obtain information about the former. In [**Koh93**], he gave a reinterpretation of his results, generalizing the notion of Fourier-Jacobi expansions of holomorphic Siegel modular forms to real-analytic Siegel Eisenstein series. A fundamental question that he asked was whether this provides a possibility to define Fourier-Jacobi expansions for arbitrary real-analytic Siegel modular forms. Kohnen left open what a real-analytic Siegel modular form should be. In Chapter 4, we have given a definition of harmonic Siegel modular forms, and we will now prove that under mild assumptions, we can employ the method suggested by Kohnen to obtain Fourier-Jacobi expansions of harmonic Siegel modular forms. At the time [**Koh94**] was published only harmonic elliptic modular forms were known, and no other type of harmonic modular forms. In particular, Kohnen could not realize that the Jacobi forms that show up as Fourier-Jacobi coefficients of degree 2 Siegel Eisenstein series are harmonic Maaß-Jacobi forms, which we have defined in Chapter 3. We will prove that this is the case. This discovery provides a link between harmonic Siegel modular forms and Jacobi forms and justifies the notion of harmonicity that we have introduced.

Furthermore, in Section 1, we will generalize Kohnen's result to Fourier-Jacobi expansions with matrix indices. The procedure that we suggest gives semi-holomorphic (skew-)Maaß-Jacobi forms, substantiating our claim concerning their outstanding role in the theory of all harmonic Maaß-Jacobi forms.

1. Fourier-Jacobi expansions of Eisenstein series

Given any function $f \in C^\infty(\mathbb{H}_n)$ that is invariant under the slash action of the modular group $\Gamma^{(n)}$, we can form a nonholomorphic Fourier Jacobi expansion

$$f(Z) = \sum_{m \in \mathbb{Z}} e^{2\pi i\, m x'}\, \phi_m(\tau, z, y'), \tag{1.1}$$

where $x' + iy' = \tau' \in \mathbb{H}_1$, $\tau \in \mathbb{H}_{n-1}$ and $z \in \mathbb{C}^{n-1}$ are the entries of $Z \in \mathbb{H}_n$. For later use, we define $\mathrm{FJ}_m(f) := \phi_m$. In the classical, that is, holomorphic, case the ϕ_m split as products

$$\phi_m(\tau, z, y') = e^{-2\pi m y'}\, \tilde{\phi}_m(\tau, z),$$

leading to the desirable expansion

$$f(Z) = \sum_{m \in \mathbb{Z}} e^{2\pi i\, m \tau'}\, \tilde{\phi}_m(\tau, z),$$

where the $\tilde{\phi}_m$ do not depend on τ' and hence can be considered as easier than f.

The quintessence of Kohnen's work, from our perspective, is given in the next theorem. Recall that $E^{\mathrm{J}}_{\alpha,\beta,m}$ denotes the real-analytic Jacobi Eisenstein series defined in (7.2) of Chapter 3. The following linear combination of these Eisenstein series will show up:

$$\tilde{E}^{\mathrm{J}}_{\alpha,\beta,m} := \sum_{\substack{t^2 | m \\ t>0}} \sigma_{\alpha+\beta-1}\big(\tfrac{m}{t^2}\big) \sum_{\substack{e | t \\ e>0}} \mu(e)\, \big(\tfrac{e}{t}\big)^{2\beta}\, E^{\mathrm{J}}_{\alpha,\beta,e^2 t^{-2} m}\big(\tfrac{t^2}{e^2}\tau,\, \tfrac{t}{e} z\big). \tag{1.2}$$

Further, we write ζ for the Riemann ζ-function.

THEOREM 1.1 ([**Koh94**]). *Let*

$$\det(Y)^{\beta} E^{(n)}_{\alpha,\beta}(Z) = \sum_{m \in \mathbb{Z}} e^{2\pi i\, m x'} \phi_m(\tau, z, y')$$

be the nonholomorphic Fourier-Jacobi expansion of the modified Siegel Eisenstein series with $\alpha + \beta > n+1$, and assume that $2 \mid \alpha - \beta$. Then

$$\lim_{y' \to \infty} e^{2\pi m y'} \phi_m(\tau, z, y')$$

exits. It equals

$$\frac{(-1)^{\frac{\alpha-\beta}{2}} (2\pi)^{\alpha+\beta}}{\Gamma(\alpha)} \zeta(\alpha+\beta)^{-1} \det(y)^{\beta} \tilde{E}^{\mathrm{J}}_{\alpha,\beta,m}. \tag{1.3}$$

PROOF. This is (18) and Proposition 1 in [**Koh94**]. $\square$

We will call the process of taking the nonholomorphic Fourier-Jacobi expansion and then employing the limit given in Theorem 1.1 the Kohnen limit process. In all cases under consideration it yields a semi-holomorphic (skew-)Maaß-Jacobi form. For this reason, we will call the limit, multiplied by $\det(Y)^{-\beta}$, the m^{th} Fourier-Jacobi coefficient, even though it does not occur in the nonholomorphic Fourier-Jacobi expansion of the original function. Nevertheless, it can be interpreted as a Fourier-Jacobi coefficient in an infinitesimal neighborhood of the Satake boundary, and it preserves most of the information about f.

REMARK 1.2. *In Section 7 of Chapter 3, we have seen that the Poincaré Eisenstein series $P^{\mathrm{J}}_{k,s,m} = \det(y)^s\, E^{\mathrm{J}}_{k+s,s,m}$ is a harmonic Maaß-Jacobi form and that $P^{\mathrm{J,sk}}_{k,s,m} = \det(y)^s\, E^{\mathrm{J}}_{\frac{1}{2}+s,k-\frac{1}{2}+s,m}$ is a harmonic skew-Maaß-Jacobi form, if $s = 0$ or $s = \frac{3}{2} - k$. Thus the above theorem shows that semi-holomorphic harmonic (skew-)Maaß-Jacobi forms occur as Fourier-Jacobi coefficients of Siegel modular forms.*

To justify the definition of harmonic Jacobi forms of arbitrary index, and to emphasize the significance of semi-holomorphic forms, we will generalize the above result to Fourier-Jacobi expansions with matrix indices. The essential ingredient will be the neat analysis of the analytic part of the Fourier coefficients of Eisenstein series carried out in [**Shi82**]. Care must be taken when applying the result. Although we will produce real-analytic Jacobi forms, we will not prove that they are nonzero. Such a statement would be equivalent to the bounds obtained by Shimura being asymptotically sharp. This is commonly believed, but no proof is available.

In the rest of this section, we need to vary slightly the notation that we used in Chapter 4. For $n > 1$, let $Z = \big(\begin{smallmatrix} \tau & z \\ z^{\mathrm{T}} & \tau' \end{smallmatrix}\big)$ with $\tau \in \mathbb{H}_1$, $z^{\mathrm{T}} \in \mathbb{C}^{n-1}$ and $x' + iy' = \tau' \in \mathbb{H}_{n-1}$ be a typical element of $\mathbb{H}_n$. Recall that $\widetilde{\mathrm{M}}^{\mathrm{T}}_{n-1}(\frac{1}{2}\mathbb{Z})$ denotes the set of symmetric matrices with entries in $\frac{1}{2}\mathbb{Z}$ that have integral diagonal entries.

The nonholomorphic Fourier-Jacobi expansion with $(n-1)\times(n-1)$ indices of a function $f \in C^\infty(\mathbb{H}_n)$ that is invariant under the modular group is

$$f(Z) = \sum_{L \in \widetilde{\mathrm{M}}^{\mathrm{T}}_{n-1}(\frac{1}{2}\mathbb{Z})} e^{2\pi i\,\mathrm{tr}(Lx')}\, \phi_L(\tau, z, y').$$

We will provide a formula for ϕ_L in the case that f is a real-analytic Siegel Eisenstein series. It can be proved along the lines of the proof of [**Koh94**, Theorem 1] and [**Böc83**]. The Fourier coefficients $c^{\mathrm{E}}_{\alpha,\beta}(L, Y)$ of the real-analytic degree $n-1$ Siegel Eisenstein series

$$E^{(n-1)}_{\alpha,\beta}(\tau') = \sum_{L \in \widetilde{\mathrm{M}}^{\mathrm{T}}_{n-1}(\frac{1}{2}\mathbb{Z})} c^{\mathrm{E}}_{\alpha,\beta}(L, y')\, e^{2\pi i\,\mathrm{tr}(Lx')}$$

occur in that formula.

THEOREM 1.3. *Fix $0 < L \in \widetilde{\mathrm{M}}^{\mathrm{T}}_{n-1}(\frac{1}{2}\mathbb{Z})$. The L^{th} nonholomorphic Fourier-Jacobi coefficient of the degree n Siegel Eisenstein series $E_{\alpha,\beta}$ equals*

$$\sum_{\substack{\mu^{\mathrm{T}} \in \mathbb{Z}^{n-1} \\ t : \left(\mathrm{M}_{n-1}(\mathbb{Z}) \cap \mathrm{GL}_{n-1}(\mathbb{Q})\right)/\mathrm{GL}_{n-1} \\ \binom{\mu}{t}\ \text{primitive}}} \;\; \sum_{\substack{L' \in \widetilde{\mathrm{M}}^{\mathrm{T}}_{n-1}(\frac{1}{2}\mathbb{Z}) \\ L'[t^{\mathrm{T}}] = L}} \;\; \sum_{g : \Gamma^{(1)}_\infty \backslash \Gamma^{(1)}} \boldsymbol{\alpha}^{(1)}_{\alpha,\beta}(g, \tau)$$

$$\cdot\, e\Big(\mathrm{tr}\big(L'(\mathfrak{Re}(g\tau \cdot \mu^{\mathrm{T}}\mu - c\boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\,(zt)^{\mathrm{T}} zt + 2\boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\,\mu^{\mathrm{T}} zt)\big)\Big)$$

$$\cdot\, c^{\mathrm{E}}_{\alpha,\beta}\big(L', y'[t] + \mathfrak{Im}(g\tau \cdot \mu^{\mathrm{T}}\mu - c\boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\,(zt)^{\mathrm{T}} zt$$

$$+\, \boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\,(\mu^{\mathrm{T}} zt + t^{\mathrm{T}} z^{\mathrm{T}} \mu))\big).$$

PROOF. From [**Koh94**], we adopt the notation $g^\uparrow$, $h^\downarrow$, and l_u for the images under the embeddings

$$\mathrm{Sp}_1(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{Sp}_n(\mathbb{Z}), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b & \\ & I_{n-1} & & \\ c & & d & \\ & & & I_{n-1} \end{pmatrix},$$

$$\mathrm{Sp}_{n-1}(\mathbb{Z}) \hookrightarrow \mathrm{Sp}_n(\mathbb{Z}), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix}, \quad \text{and}$$

$$\mathrm{GL}_n(\mathbb{Z}) \hookrightarrow \mathrm{Sp}_n(\mathbb{Z}), \qquad u \mapsto \begin{pmatrix} u^{-\mathrm{T}} & \\ & u \end{pmatrix}.$$

Define

$$\Gamma^{(n)}_{\mathrm{rk}\,n-1} := \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^{(n)} \,:\, \mathrm{rk}\,(c_{ij})_{\substack{i=2,\ldots,n \\ j=2,\ldots,n}} = n-1 \Big\},$$

which is a set that $\Gamma^{(n)}_\infty$ acts on. Define

$$\mathrm{GL}_n(\mathbb{Z})_{\mathrm{rk}\,n-1} := \big\{ g \in \mathrm{GL}_n(\mathbb{Z}) \,:\, \mathrm{rk}\,(g^{-1}_{ij})_{\substack{i=2,\ldots,n \\ j=2,\ldots,n}} = n-1 \big\}.$$

This set is acted on by

$$\mathrm{GL}_n(\mathbb{Z})_\infty := \big\{ g \in \mathrm{GL}_n(\mathbb{Z}) \,:\, (g_{1,j})_{j=2,\ldots,n} = 0 \big\}.$$

Since $\operatorname{rk} L = n-1$, we may restrict our consideration to the Fourier coefficients of $E_{\alpha,\beta}$ for indices T such that the bottom right $(n-1)\times(n-1)$ block is invertible. By a standard argument that can be found in [**Maa71**, Chapter 18], we may consider the restricted Siegel Eisenstein series

$$\tilde{E}^{(n)}_{\alpha,\beta}(Z) := \sum_{g\,:\,\Gamma^{(n)}_\infty \backslash \Gamma^{(n)}_{\operatorname{rk} n-1}} 1|^{(n)}_{\alpha,\beta}\, g,$$

that has the same Fourier coefficients for indices that satisfy

$$\operatorname{rk}(T_{ij})_{\substack{i=2,\ldots,n\\ j=2,\ldots,n}} = n-1.$$

In [**Böc83**, Proposition 5] it is proved that $h^{\downarrow} l_u g^{\uparrow}$ runs through a set of representatives of $\Gamma^{(n)}_\infty \backslash \Gamma^{(n)}_{\operatorname{rk} n-1}$, if g, u, and h run through sets of representatives of $\Gamma^{(1)}_\infty \backslash \Gamma^{(1)}$, $\mathrm{GL}_n(\mathbb{Z})_\infty \backslash \mathrm{GL}_n(\mathbb{Z})_{\operatorname{rk} n-1}$, and $\Gamma^{(n-1)}_\infty \backslash \Gamma^{(n-1)}_{\operatorname{rk} n-1}$, respectively. The equality

$$\boldsymbol{\alpha}^{(n)}_{\alpha,\beta}(h^{\downarrow} l_u g^{\uparrow}, Z) = \boldsymbol{\alpha}^{(n)}_{\alpha,\beta}(h^{\downarrow}, l_u g^{\uparrow} Z)\, \boldsymbol{\alpha}^{(n)}_{\alpha,\beta}(g^{\uparrow}, Z) \det(u)$$

follows from the cocycle relation that $\boldsymbol{\alpha}^{(n)}_{\alpha,\beta}$ satisfies and from the shape of l_u.

Combining the decomposition of $\Gamma^{(n)}_\infty \backslash \Gamma^{(n)}_{\operatorname{rk} n-1}$ with the formula for the cocycle, we can compute the Fourier expansion of $\tilde{E}_{\alpha,\beta}(Z)$.

$$\begin{aligned}\tilde{E}_{\alpha,\beta}(Z) = \sum_{u\,:\,\mathrm{GL}_n(\mathbb{Z})_\infty \backslash \mathrm{GL}_n(\mathbb{Z})_{\operatorname{rk} n-1}} &\det(u) \sum_{g\,:\,\Gamma^{(1)}_\infty \backslash \Gamma^{(1)}} \boldsymbol{\alpha}^{(n)}_{\alpha,\beta}(g^{\uparrow}, Z)\\ &\cdot \sum_{h\,:\,\Gamma^{(n)}_\infty \backslash \Gamma^{(n)}_{\operatorname{rk} n-1}} \boldsymbol{\alpha}^{(n)}_{\alpha,\beta}(h^{\downarrow}, l_u g^{\uparrow} Z).\end{aligned}$$

The cocylces that occur can be simplified. We have

$$\boldsymbol{\alpha}^{(n)}_{\alpha,\beta}(h^{\downarrow}, Z) = \boldsymbol{\alpha}^{(n-1)}_{\alpha,\beta}\big(h, Z\big[\begin{pmatrix} 0\cdots 0\\ I_{n-1}\end{pmatrix}\big]\big)$$

and

$$\boldsymbol{\alpha}^{(n)}_{\alpha,\beta}(g^{\uparrow}, Z) = \boldsymbol{\alpha}^{(1)}_{\alpha,\beta}\big(g, Z\big[\begin{pmatrix}1 & 0\cdots 0\end{pmatrix}^{\mathrm{T}}\big]\big).$$

From the former relation and by the same argument in [**Maa71**, Chapter 18] that we have used above, the inner sum equals

$$\sum_{0 < L \in \widetilde{\mathrm{M}}^{\mathrm{T}}_{n-1}(\frac{1}{2}\mathbb{Z})} c^{\mathrm{E}}_{\alpha,\beta}\big(L,\, (l_u g^{\uparrow} Z)\big[\begin{pmatrix} 0\cdots 0\\ I_{n-1}\end{pmatrix}\big]\big).$$

By [**Böc83**, Lemma 6], a system of representatives of $\mathrm{GL}_n(\mathbb{Z})_\infty \backslash \mathrm{GL}_n(\mathbb{Z})_{\operatorname{rk} n-1}$ is given by a set of matrices u, where the last $n-1$ columns of u^{-1} run through

$$\big\{ \left(\begin{smallmatrix}\mu\\ t\end{smallmatrix}\right) \,:\, \mu^{\mathrm{T}} \in \mathbb{Z}^{n-1},\ t : \big(\mathrm{M}_{n-1}(\mathbb{Z}) \cap \mathrm{GL}_{n-1}(\mathbb{Q})\big)/\mathrm{GL}_{n-1}(\mathbb{Z}),\ \left(\begin{smallmatrix}\mu\\ t\end{smallmatrix}\right) \text{ primitive} \big\}.$$

Combining this and the equality

$$\begin{aligned}&(l_u g^{\uparrow} Z)\big[\begin{pmatrix} 0\cdots 0\\ I_{n-1}\end{pmatrix}\big]\\ &= \tau'[t] + g\tau \cdot \mu^{\mathrm{T}}\mu - c\boldsymbol{\alpha}_{1,0}(g,\tau)(zt)^{\mathrm{T}} zt + \boldsymbol{\alpha}_{1,0}(g,\tau)\,(\mu^{\mathrm{T}} zt + t^{\mathrm{T}} z^{\mathrm{T}} \mu)\end{aligned}$$

yields the result.

$\square$

Corollary 1.4. *Let ϕ_L be the L^{th} nonholomorphic Fourier-Jacobi coefficient of* $\det(Y)^{\beta} E^{(n)}_{\alpha,\beta}$. *Then*

$$\limsup_{\delta \to \infty} e^{2\pi i \operatorname{tr}(Ly')} \, \phi_L(\tau, z, y')$$

with $y' = \delta I_{n-1} + 2y^{-1} z^{\mathrm{T}} z$ exists and is a semi-holomorphic Maaß-Jacobi form. If $\beta = 0$ or $\alpha = \frac{3}{2}$, it is, up to a power of y, harmonic for $|^{\mathrm{J}}_k$, and if $\alpha = \frac{1}{2}$ or $\beta = 1$, it is, up to a power of y, harmonic for $|^{\mathrm{J,sk}}_k$.

Proof. We can rewrite the addends of the right hand side in Theorem 1.3:

$$\begin{aligned}
&\boldsymbol{\alpha}^{(1)}_{\alpha,\beta}(g,\tau) \\
&\quad \cdot e\big(\operatorname{tr}\big(L(g\tau \cdot (\mu t^{-1})^{\mathrm{T}}(\mu t^{-1}) - c\boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\, z^{\mathrm{T}} z + 2\boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\, (\mu t^{-1})^{\mathrm{T}} z\big)\big) \\
&\quad \cdot \Big(e\big(-i \operatorname{tr}\big(L' \,\mathfrak{Im}(g\tau \cdot \mu^{\mathrm{T}}\mu - c\boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\, (zt)^{\mathrm{T}} zt + \boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\, (\mu^{\mathrm{T}} zt + t^{\mathrm{T}} z^{\mathrm{T}} \mu)\big)\big)\big) \\
&\qquad c^{\mathrm{E}}_{\alpha,\beta}\big(L', y'[t] + \mathfrak{Im}(g\tau \cdot \mu^{\mathrm{T}}\mu - c\boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\, (zt)^{\mathrm{T}} zt \\
&\qquad\qquad + \boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\, (\mu^{\mathrm{T}} zt + t^{\mathrm{T}} z^{\mathrm{T}} \mu))\big)\Big)
\end{aligned}$$

We use the bounds for $c^{\mathrm{E}}_{\alpha,\beta}$ that are given in [**Shi82**]. The following estimate can be found in the introduction of Shimura's paper:

$$|c^{\mathrm{E}}_{\alpha,\beta}(L, y')| \le A \det(y')^{\beta}\, e^{-2\pi \operatorname{tr}(Ly')},$$

for some $A > 0$. By the calculations in [**Maa71**, Chapter 18], $c^{\mathrm{E}}_{\alpha}(L, Y)$ is real. That is, the function

$$\delta^{(n-1)\beta} e^{-2\pi \operatorname{tr}(Ly')} \phi_L(\tau, z, y')$$

with y' as above is bounded from above and below as $\delta \to \infty$. Consequently, the limes superior exists and equals up to a multiplicative constant

$$\boldsymbol{\alpha}^{(1)}_{\alpha,\beta}(g,\tau)\, e\big(\operatorname{tr}\big(L(g\tau \cdot (\mu t^{-1})^{\mathrm{T}}(\mu t^{-1}) - c\boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\, z^{\mathrm{T}} z + 2\boldsymbol{\alpha}^{(1)}_{1,0}(g,\tau)\, (\mu t^{-1})^{\mathrm{T}} z\big)\big).$$

We can rewrite this in terms of the Jacobi slash action, yielding

$$1|^{\mathrm{J}}_{\alpha,\beta,L}(g, (\mu t^{-1}, 0) g),$$

and hence it follows that the limes superior is an eigenfunction of the Jacobi Casimir operators given in (3.2) and (3.3) of Chapter 3. Since $\mathcal{C}^{\mathrm{J}}_{k,m}\, y^s = 0$ and $\mathcal{C}^{\mathrm{J,sk}}_{k,m}\, y^s = 0$ if and only if $s = 0$ or $s = \frac{3}{2} - k$, the statement follows. □

2. Harmonic Siegel modular forms of degree 2

In this section, we restrict our attention to degree 2 Siegel modular forms. From Theorem 1.1 and the results of Section 3 in Chapter 4, we will deduce that the Kohnen limit process works for all harmonic Siegel modular forms that satisfy a relatively mild condition.

Given a function ϕ that occurs as a coefficient in the nonholomorphic Fourier-Jacobi expansion of a real-analytic Siegel modular form we define

$$(\mathcal{L}\phi)(\tau, z) := \lim_{y' \to \infty} \phi(\tau, z, y') \tag{2.1}$$

and

$$(\mathcal{L}^{\mathrm{sk}}\phi)(\tau, z) := y^{\frac{1}{2}-k} \lim_{y' \to \infty} \det(Y)^{k-\frac{1}{2}} \phi(\tau, z, y') \tag{2.2}$$

if the limits exist.

We reformulate Theorem 1.1 using these operators.

COROLLARY 2.1. *Let $\phi_m(\tau, z, y')$ be the m^{th} coefficient of the nonholomorphic Fourier-Jacobi expansion of $P^{(2)}_{k,s}(Z)$. If $m > 0$, then the limit $\mathcal{L}\phi_m$ exists for $s = 0$, $k > 3$ and for $s = \frac{3}{2} - k$, $k < 0$, and we have:*

(i) *If $s = 0$ and $k > 3$, then $\mathcal{L}\phi_m \in J_{k,m}$ is holomorphic.*

(ii) *If $s = \frac{3}{2} - k$ and $k < 0$, then $\mathcal{L}\phi_m \in \mathbb{J}_{k,m}$ is harmonic.*

Let $\phi_m(\tau, z, y')$ be the m^{th} coefficient of the nonholomorphic Fourier-Jacobi expansion of $P^{(2),\mathrm{sk}}_{k,s}(Z)$. If $m > 0$, then the limit $\mathcal{L}^{\mathrm{sk}}\phi_m$ exists for $s = 0$, $k > 3$ and for $s = \frac{3}{2} - k$, $k < 0$, and we have:

(i) *If $s = 0$ and $k > 3$, then $\mathcal{L}^{\mathrm{sk}}\phi_m \in J^{\mathrm{sk}}_{k,m}$ is skew-holomorphic.*

(ii) *If $s = \frac{3}{2} - k$ and $k < 0$, then $\mathcal{L}^{\mathrm{sk}}\phi_m \in \mathbb{J}^{\mathrm{sk}}_{k,m}$ is harmonic.*

In particular, the Fourier coefficients for indefinite indices do not vanish.

PROOF. By Theorem 1.1, the functions $\mathcal{L}\phi_m$, in the first case, and $\mathcal{L}^{\mathrm{sk}}\phi_m$, in the second case, exist, and they equal

$$y^s \tilde{E}^{\mathrm{J}}_{k+s,k,m} \quad \text{and} \quad y^s \tilde{E}^{\mathrm{J}}_{\frac{1}{2}+s,k-\frac{1}{2}+s,m}.$$

The rescaling by $\frac{t}{e}$ employed in (1.2) can be expressed in terms of the slash actions $|^{\mathrm{J}}_{k,e^2t^{-2}m}\, g$ and $|^{\mathrm{J,sk}}_{k,e^2t^{-2}m}\, g$ with the matrix $g = \sqrt{te}^{-1} \left(\begin{smallmatrix} t & 0 \\ 0 & e \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Thus it suffices to show that the Jacobi-Eisenstein series $y^s E^{\mathrm{J}}_{k+s,s,m}$ and $y^s E^{\mathrm{J}}_{\frac{1}{2}+s,k-\frac{1}{2}+s,m}$ vanish under $\mathcal{C}^{\mathrm{J}}_{k,m}$ and $\mathcal{C}^{\mathrm{J,sk}}_{k,m}$ if $s = 0$ and $s = \frac{3}{2} - k$. This is a special case of the Eisenstein series presented in Section 7 of Chapter 3. $\square$

The technical condition that we need to treat the Kohnen limit process for more general real-analytic Siegel modular forms will be formulated in terms of the ξ-operators introduced in Section 4 of Chapter 2. The next proposition gives information about the vanishing of the Fourier expansion of a harmonic Siegel modular form when either of them is applied.

PROPOSITION 2.2. *Consider the Fourier coefficients given in Theorem 3.1 of Chapter 4. That is, let*

$$f(Z) = \sum_T a(Y,T) e^{2\pi i\, \mathrm{tr}(TX)} \in \mathbb{M}^{(2)}_k \cup \mathbb{M}^{(2),\mathrm{sk}}_k.$$

Suppose that $k < 0$ or $k > 3$.

(i) *If $T = 0$, then the kernel of $\xi^{(2)}_k(\,\cdot\, e^{2\pi i \mathrm{tr}(TX)})$ on the space of Fourier coefficients $a(Y,T)$ is spanned by 1.*

(ii) *If $\mathrm{rk}(T) = 1$ and $T \geq 0$, then the kernel of $\xi^{(2)}_k(\,\cdot\, e^{2\pi i \mathrm{tr}(TX)})$ on the space of Fourier coefficients $a(Y,T)$ is spanned by the second function in (ii) of Theorem 3.1.*

(iii) *Suppose that $\mathrm{rk}(T) = 2$ and $T > 0$. If $f \in \mathbb{M}^{(2)}_k$, then $a(Y,T)$ lies in the kernel of $\xi^{(2)}_k(\,\cdot\, e^{2\pi i \mathrm{tr}(TX)})$. If $f \in \mathbb{M}^{(2),\mathrm{sk}}_k$, then any nonvanishing $a(Y,T)$ is not contained in the kernel of $\xi^{(2)}_k(\,\cdot\, e^{2\pi i \mathrm{tr}(TX)})$.*

(iv) *Suppose that $\mathrm{rk}(T) = 2$ and T is indefinite. If $f \in \mathbb{M}^{(2)}_k$, then any nonvanishing $a(Y,T)$ is not contained in the kernel of $\xi^{(2)}_k(\,\cdot\, e^{2\pi i \mathrm{tr}(TX)})$. If $f \in \mathbb{M}^{(2),\mathrm{sk}}_k$, then $a(Y,T)$ lies in the kernel of $\xi^{(2)}_k(\,\cdot\, e^{2\pi i \mathrm{tr}(TX)})$.*

Proof. The statements concerning the ξ-operators in (i) follows from the fact that $\xi^{(2)}$ and $\xi^{(2),\mathrm{sk}}$ are multiples of $\det(\partial_{\overline{Z}})$ and $\det(\partial_Z)$. To prove statement (ii) we first consider the images of the Fourier coefficients under $\det(\partial_{\overline{Z}})$ and $\det(\partial_Z)$. We have

$$\det(\partial_{\overline{Z}}) = \tfrac{1}{4}\big(\partial_x\partial_{x'} + i\partial_x\partial_{y'} + i\partial_{x'}\partial_y - \partial_y\partial_{y'} - \tfrac{1}{4}\partial_u^2 - \tfrac{i}{2}\partial_u\partial_v + \tfrac{1}{4}\partial_v^2\big).$$

Since both ξ-operators are $\mathrm{Sp}_2(\mathbb{R})$-invariant, it is sufficient to consider the case $T = \left(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix}\right)$. Then the first function in (3.2) equals

$$(yy' - v^2)^{\frac{3}{2}-k} y^{\frac{k-3}{2}}\, W_{\frac{k}{2},1-\frac{k}{2}}(2y)\, e^{2\pi i x}.$$

Applying the above operator and restricting to $v = 0$, we obtain

$$\begin{aligned} &\tfrac{1}{4}(\tfrac{3}{2} - k)(-2\pi y - \partial_y y - \tfrac{1}{2}\partial_v v)(yy' - v^2)^{\frac{1}{2}-k} y^{\frac{k-3}{2}}\, W_{\frac{k}{2},1-\frac{k}{2}}(2y)\, e^{2\pi i x}\big|_{v=0} \\ &= \tfrac{1}{4}(\tfrac{3}{2} - k)(yy')^{\frac{1}{2}-k}(-2\pi y + 1 + 1 + \partial_y - \tfrac{1}{2})y^{\frac{k-3}{2}}\, W_{\frac{k}{2},1-\frac{k}{2}}(2y)\, e^{2\pi i x}. \end{aligned}$$

Considering the first coefficient of the resulting power series expansion with respect to $y^{\frac{1}{2}}$, coming from the derivative with respect to y, we see that this function does not vanish. Up to a nonzero factor, $\xi_k^{(2),\mathrm{sk}}$ equals $\det(\partial_Z)$ so that the nonvanishing under $\xi_k^{(2),\mathrm{sk}}(\,\cdot\, e^{2\pi i x})$ is proved.

Since $\det(\partial_Z)$ equals

$$\tfrac{1}{4}\big(\partial_x\partial_{x'} - i\partial_x\partial_{y'} - i\partial_{x'}\partial_y - \partial_y\partial_{y'} - \tfrac{1}{4}\partial_u^2 + \tfrac{i}{2}\partial_u\partial_v + \tfrac{1}{4}\partial_v^2\big),$$

and the first function in (3.4) equals

$$(yy' - v^2)^{\frac{3}{2}-k} y^{\frac{k-3}{2}}\, W_{\frac{1-k}{2},1-\frac{k}{2}}(2y)\, e^{2\pi i x},$$

the very same calculations yields

$$\tfrac{1}{4}(\tfrac{3}{2} - k)(yy')^{\frac{1}{2}-k}(2\pi y + 1 + 1 + \partial_y - \tfrac{1}{2})y^{\frac{k-3}{2}}\, W_{\frac{1-k}{2},1-\frac{k}{2}}(2y)\, e^{2\pi i x}.$$

We multiply this by $\det(Z - \overline{Z}) = -4yy'$. The ξ-operator for the holomorphic slash action features an additional addend $2i(y\partial_\tau + v\partial_z + y'\partial_{\tau'})$, which leads to the contribution

$$\big(-2\pi y + (\tfrac{3}{2} - k) + y\partial_y + (\tfrac{3}{2} - k)\big).$$

The first coefficient, which comes from the term ∂_y, of the resulting power series expansion with respect to $y^{\frac{1}{2}}$ does not vanish. Since the second function in (3.4) only depends on x and y, it clearly vanishes under $\det(\partial_{\overline{Z}})$. This completes the skew case.

In the case of holomorphic weights, recall that the second function in (3.2) coincides with $e^{-2\pi i y}$, that up to multiplicative scalars, occurs in the Fourier expansion of holomorphic Eisenstein series. Consequently, it is annihilated by the antiholomorphic derivatives in $\xi_k^{(2)}$.

Consider the case (iii). For holomorphic weights the coefficient $a(Y, T)$ coincides, up to scalar multiples, with $e^{-2\pi \mathrm{tr}(TY)}$. Hence it is annihilated by the antiholomorphic derivatives in $\xi_k^{(2)}(\,\cdot\, e^{2\pi i \mathrm{tr}(TX)})$. In the case of skew weights, we analyze the one-sided Taylor expansion of the image under the ξ-operator. We may assume that $T = I_2$, and we will use the notation $\mathtt{w} = y - y'$. We find

$$\begin{pmatrix} \partial_y \\ \partial_{y'} \\ \partial_v \end{pmatrix} = \begin{pmatrix} 1 & 2\mathtt{w} & 1 \\ 1 & -2\mathtt{w} & -1 \\ 0 & 4\sqrt{\mathtt{v} - \mathtt{w}^2} & 0 \end{pmatrix} \begin{pmatrix} \partial_{\mathtt{u}} \\ \partial_{\mathtt{v}} \\ \partial_{\mathtt{w}} \end{pmatrix}.$$

To determine the action of $\xi_k^{(2),\mathrm{sk}} = -4\det(Y)\det(\partial_Z)$ on

$$f(\mathtt{u},\mathtt{v},X) = a(Y,T)\, e^{2\pi i\, \mathrm{tr}(TX)} = \sum_{n=0}^{\infty} g_n(2\pi \mathtt{u})\, (4\pi^2 \mathtt{v})^n e^{2\pi i\, \mathrm{tr}(X)},$$

we use the fact that f is independent of $\mathtt{w}$, proved in [**Maa53**]. Suppressing the argument X, we find

$$\begin{aligned}
&4\det(\partial_Z) f(\mathtt{u},\mathtt{v})\Big|_{\mathtt{v}=0} \\
&= \big(\partial_x\partial_{x'} - i\partial_x(\partial_\mathtt{u} - 2\mathtt{w}\partial_\mathtt{v} - \partial_\mathtt{w}) - i\partial_{x'}(\partial_\mathtt{u} + 2\mathtt{w}\partial_\mathtt{v} + \partial_\mathtt{w}) \\
&\quad - (\partial_\mathtt{u} - 2\mathtt{w}\partial_\mathtt{v} - \partial_\mathtt{w})(\partial_\mathtt{u} + 2\mathtt{w}\partial_\mathtt{v} + \partial_\mathtt{w}) \\
&\quad - \tfrac{1}{4}\partial_u^2 + \tfrac{i}{2}\partial_u 4\sqrt{\mathtt{v}-\mathtt{w}^2}\partial_\mathtt{v} + \tfrac{1}{4}4\sqrt{\mathtt{v}-\mathtt{w}^2}\partial_\mathtt{v} 4\sqrt{\mathtt{v}-\mathtt{w}^2}\partial_\mathtt{v}\big) f(\mathtt{u},\mathtt{v})\Big|_{\mathtt{v}=0} \\
&= \big(-4\pi^2 + 2\pi 2\partial_\mathtt{u} - \partial_\mathtt{u}^2 + 2\partial_\mathtt{v} + 4\mathtt{w}^2\partial_\mathtt{v}^2 \\
&\quad + 4\sqrt{\mathtt{v}-\mathtt{w}^2}^{\,2}\partial_\mathtt{v}^2 + 4\sqrt{\mathtt{v}-\mathtt{w}^2}\tfrac{1}{2}\sqrt{\mathtt{v}-\mathtt{w}^2}^{\,-1}\partial_\mathtt{v}\big) f(\mathtt{u},\mathtt{v})\Big|_{\mathtt{v}=0} \\
&= \Big(\big(-4\pi^2 + 4\pi\partial_\mathtt{u} - \partial_\mathtt{u}^2 + 4\partial_\mathtt{v}\big) f\Big)(\mathtt{u},0) \\
&= -8\pi^2\big(\tfrac{k-1}{u} g_0(2\pi\mathtt{u}) + (1 - \tfrac{k}{u}) g_0'(2\pi\mathtt{u}) - g_0''(2\pi\mathtt{u})\big) e^{2\pi i\, \mathrm{tr}(X)}.
\end{aligned}$$

Passing to one-sided Taylor expansions of $a(Y,T)$, it is sufficient to prove that the last expression has a nonvanishing Laurent expansion with respect to $\mathtt{u}$. We neglect the factor $e^{2\pi i\, \mathrm{tr}(X)}$, which does not play a role in our considerations.

We analyze the initial exponent of the candidates for g_0, that is,

$$\mathtt{u}^{1-k}\int_\mathtt{u}^\infty \tilde{\mathtt{u}}^{-1} W_{1-k,\frac{3}{2}-k}(2\tilde{\mathtt{u}})\, d\tilde{\mathtt{u}} \quad\text{and}\quad \mathtt{u}^{1-k}\int_\mathtt{u}^1 \tilde{\mathtt{u}}^{-1} W_{1-k,\frac{3}{2}-k}(2\tilde{\mathtt{u}})\, d\tilde{\mathtt{u}},$$

They differ by a multiple of $\mathtt{u}^{1-k}$, and the initial term of the Laurent series of the first function is a multiple of $\mathtt{u}^{\frac{1}{2}\pm(\frac{3}{2}-k)}$. Consequently, the initial term of the Taylor expansion of the image under $4\det(\partial_{\overline{Z}})$ is the one coming from the second derivative g_0''. This proves the case (iii).

Consider statement (iv). In the case of skew weights, we deduce the statement from Corollary 2.1 and the fact that holomorphic Siegel Eisenstein series have vanishing coefficients for indefinite Fourier indices. We are reduced to the holomorphic case. As before we will compute the action of the ξ-operator on one-sided Taylor expansions. We may assume that $T = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, and we write $\mathtt{w}$ for $y + y'$. As above, we compute

$$\begin{pmatrix} \partial_y \\ \partial_{y'} \\ \partial_v \end{pmatrix} = \begin{pmatrix} 1 & 2\mathtt{w} & 1 \\ -1 & 2\mathtt{w} & 1 \\ 0 & -4\sqrt{\mathtt{w}^2 - \mathtt{v}} & 0 \end{pmatrix} \begin{pmatrix} \partial_\mathtt{u} \\ \partial_\mathtt{v} \\ \partial_\mathtt{w} \end{pmatrix}.$$

We will apply $\xi_k^{(2)}$ to

$$f(\mathtt{u},\mathtt{v},X) = \sum_{n=0}^{\infty} h_n(4\pi^2\mathtt{v})\,(2\pi\mathtt{u})^n\, e^{2\pi i\, \mathrm{tr}(TX)}$$

and restrict the image to $\mathtt{u} = 0$. In this situation, suppressing the argument X, we find that the image under $\xi_k^{(2)}$ yields up to the factor $\det(Y)^{k-\frac{3}{2}}$:

$$\begin{aligned}
&\Big(\tfrac{2i}{2}\big(y(\partial_x + i\partial_y) + v(\partial_u + i\partial_v) + y'(\partial_{x'} + i\partial_{y'})\big)\\
&\qquad - 4\det(Y)\big((\partial_x + i\partial_y)(\partial_{x'} + i\partial_{y'}) - \tfrac{1}{4}(\partial_u + i\partial_v)^2\big)\Big)\, f(\mathtt{u},\mathtt{v})\Big|_{\mathtt{u}=0}\\
&= \Big(i\big(\tfrac{\mathtt{w}}{2}(2\pi i + i(\partial_{\mathtt{u}} + 2\mathtt{w}\partial_{\mathtt{v}} + \partial_{\mathtt{w}})) + \tfrac{1}{2}\sqrt{\mathtt{w}^2 - \mathtt{v}}(i(-4\sqrt{\mathtt{w}^2-\mathtt{v}}\partial_{\mathtt{v}}))\\
&\qquad + \tfrac{\mathtt{w}}{2}(-2\pi i + i(-\partial_{\mathtt{u}} + 2\mathtt{w}\partial_{\mathtt{v}} + \partial_{\mathtt{w}}))\big)\\
&\qquad - \tfrac{\mathtt{v}}{4}\big(4\pi^2 - 4\pi\partial_{\mathtt{u}} + \partial_{\mathtt{u}}^2 - 2\partial_{\mathtt{v}} - 4\mathtt{w}^2\partial_{\mathtt{v}}^2\\
&\qquad + 4\sqrt{\mathtt{w}^2-\mathtt{v}}\tfrac{-1}{2}\sqrt{\mathtt{w}^2-\mathtt{v}}^{\,-1}\partial_{\mathtt{v}} + 4(\mathtt{w}^2 - \mathtt{v})\partial_{\mathtt{v}}^2\big)\Big) f(\mathtt{u},\mathtt{v})\Big|_{\mathtt{u}=0}\\
&= \Big(2\mathtt{v}\partial_{\mathtt{v}} - \tfrac{\mathtt{v}}{4}(4\pi^2 + 4\pi\partial_{\mathtt{u}} + \partial_{\mathtt{u}}^2 - 4\partial_{\mathtt{v}} - 4\mathtt{v}\partial_{\mathtt{v}}^2)\Big) f(0,\mathtt{v})\\
&= \tfrac{-\mathtt{v}}{4}\Big(\big(4\pi^2 + 4\pi\partial_{\mathtt{u}} + \partial_{\mathtt{u}}^2 - 12\partial_{\mathtt{v}} - 4\mathtt{v}\partial_{\mathtt{v}}^2\big) f\Big)(0,\mathtt{v}).
\end{aligned}$$

We use the recurrence in (iv) of Theorem 2.2 to express this in terms of derivatives of h_0 and h_1. This yields

$$\begin{aligned}
&\tfrac{-\mathtt{v}}{4}\Big(4\pi^2 h_0(4\pi^2\mathtt{v}) - 4\pi 2\pi h_1(4\pi^2\mathtt{v})\\
&\qquad + 4\pi^2\big(-2\cdot 16\pi^4\mathtt{v}h_0''(4\pi^2\mathtt{v}) - 2k4\pi^2 h_0'(4\pi^2\mathtt{v}) + \tfrac{1}{2}h_0(4\pi^2\mathtt{v})\big)\\
&\qquad - 12\cdot 4\pi^2 h_0'(4\pi^2\mathtt{v}) - 4\mathtt{v}16\pi^4 h_0''(4\pi^2\mathtt{v})\Big).
\end{aligned}$$

Assume that the initial term of the Taylor expansion of h_1 is $\mathtt{v}^l$ for some l. Then the initial term of the Taylor expansion of the image is

$$\mathtt{v}^l\Big(-32\pi^6 l(l-1)(2l+k) + 8\pi^4 l(k(2l+k) + 2(l-1)) + 12\pi^2 l\Big).$$

Since k and l are rational, we deduce that this does not vanish except if $k = -2$ and $l = 1$. The corresponding solution given in Lemma 2.7 in Chapter 4 is a polynomial and hence does not occur as the Fourier coefficient of a Siegel modular form by Lemma 3.2 in Chapter 4. Hence none of the nonzero Fourier coefficients for indefinite indices that occur in the Fourier expansion of harmonic Siegel modular form of holomorphic weight vanish under $\xi_k^{(2)}$. □

Theorem 1.1 and Proposition 2.2 show that the Fourier coefficients of Eisenstein series for the skew slash action and positive weight vanish, if the Fourier index is positive definite. This justifies the next definition, mimicking the space of holomorphic Siegel modular forms, the elements of which are supported on positive semi-definite indices.

For $k > 0$, set

$$M_k^{(2),\mathrm{sk}} = \Big\{ \sum_{T \in \widetilde{\mathrm{M}}_2^{\mathrm{T}}(\frac{1}{2}\mathbb{Z})} a(Y,T)\, e(\mathrm{tr}(TX)) \in \mathbb{M}_k^{(2),\mathrm{sk}} \;:\; a(Y,T) = 0 \text{ if } T > 0\Big\}, \quad (2.3)$$

and for $k \le 0$, set

$$M_k^{(2),\mathrm{sk}} = \{0\} \subseteq \mathbb{M}_k^{(2),\mathrm{sk}}. \quad (2.4)$$

Like in the case of holomorphic weights, we say that $f \in \mathbb{M}_k^{(2)} \cup \mathbb{M}_k^{(2),\mathrm{sk}}$ is a cusp form if $a(Y,T) = 0$ for all T with $\mathrm{rk}(T) \neq 2$.

To prove the convergence of the Kohnen limit process, we need

COROLLARY 2.3. *Suppose that $f \in \mathbb{M}_k^{(2)}$ with Fourier expansion as in Proposition 2.2 and $\xi_k^{(2)} f = 0$. Then $a(Y,T) = 0$ for T indefinite. Similarly, if $f \in \mathbb{M}_k^{(2),\mathrm{sk}}$ and $\xi_k^{(2),\mathrm{sk}} f = 0$, then $a(Y,T) = 0$ for $T > 0$.*

LEMMA 2.4. *Suppose that $f \in \mathbb{M}_k^{(2)}$ with Fourier expansion as in Proposition 2.2 and $\xi_k^{(2)} f = 0$. Then $\mathcal{L} a(Y,T)$ exits, if $\mathrm{rk}(T) = 1$ and $T \geq 0$. If $f \in \mathbb{M}_k^{(2),\mathrm{sk}}$ and $\xi_k^{(2),\mathrm{sk}} f = 0$, then $\mathcal{L}^{\mathrm{sk}} a(Y,T)$ exits, if $\mathrm{rk}(T) = 1$ and $T \geq 0$.*

PROOF. The asymptotic of the Whittaker function $W_{\kappa,\mu}(\mathrm{u})$ as $\mathrm{u} \to \infty$ is $\mathrm{u}^{\kappa} e^{-\frac{\mathrm{u}}{2}}$. Consequently, the asymptotic of the functions given in (ii) of Theorem 3.1 in Chapter 4 are $e^{-2\pi \mathrm{u}}$ and $\mathrm{u}^{\frac{1}{2}-k}\, e^{-2\pi \mathrm{u}}$ in the holomorphic and skew case. This proves the statement. $\square$

THEOREM 2.5. *Let $k \in \mathbb{Z}$ with $k < 0$ or $k > 3$. Assume that $f \in \mathbb{M}_k^{(2)} \cup \mathbb{M}_k^{(2),\mathrm{sk}}$ with Fourier-Jacobi expansion as in (1.1). If $f \in \mathbb{M}_k$, k is even, and $\xi_k^{(2)} f \in M_{3-k}^{(2),\mathrm{sk}}$ then*

(i) *If $k \geq 4$, then $\mathcal{L} \circ \mathrm{FJ}_m\, f \in J_{k,m}$ exists and is holomorphic.*
(ii) *If $k \leq -2$, then $\mathcal{L} \circ \mathrm{FJ}_m\, f \in \mathbb{J}_{k,m}$ exists and is harmonic.*

If $f \in \mathbb{M}_k^{\mathrm{sk}}$, k is odd and $\xi_k^{(2),\mathrm{sk}} f \in M_{3-k}^{(2)}$ then

(i) *If $k \geq 5$, then $\mathcal{L}^{\mathrm{sk}} \circ \mathrm{FJ}_m\, f \in J_{k,m}^{\mathrm{sk}}$ exists and is skew-holomorphic.*
(ii) *If $k \leq -1$, then $\mathcal{L}^{\mathrm{sk}} \circ \mathrm{FJ}_m\, f \in \mathbb{J}_{k,m}^{\mathrm{sk}}$ exists and is harmonic.*

REMARK 2.6. *For $k < 0$ the condition on the image under $\xi_k^{(2)}$ and $\xi_k^{(2),\mathrm{sk}}$ is automatically satisfied.*

PROOF. From Corollary 2.3, it follows that only Fourier coefficients of Eisenstein series may occur in the Fourier expansion of f. By Corollary 2.1, the operators $\mathcal{L}$ and $\mathcal{L}^{\mathrm{sk}}$ applied to such Fourier coefficients lead to Fourier coefficients that are holomorphic, skew-holomorphic or harmonic, respectively.

We can interchange the limit that occurs in the definition of $\mathcal{L}$ and $\mathcal{L}^{\mathrm{sk}}$ and the sum in the Fourier expansion of ϕ_m, since f is an automorphic form with polynomial growth. $\square$

3. Siegel modular forms and Jacobi forms

THEOREM 3.1. *Let $0, 2 \neq k \in 2\mathbb{Z}$, and suppose that $f \in \mathbb{M}_k^{(2)}$ is a cusp form or an Eisenstein series satisfying $\xi_k^{(2)} f \in M_{3-k}^{(2),\mathrm{sk}}$. Then*

$$\mathcal{L}\big(\mathrm{FJ}_m\big(\,\xi_k^{(2)} f\,\big)\big) = \tfrac{3}{2}(k-1)\,\xi_{k,m}^{\mathrm{J}}\big(\,\mathcal{L}^{\mathrm{sk}}\big(\mathrm{FJ}_m f\big)\,\big).$$

Similarly, let $1, 3 \neq k \in 2\mathbb{Z}+1$, and suppose that $f \in \mathbb{M}_k^{(2),\mathrm{sk}}$ is a cusp form or an Eisenstein series satisfying $\xi_k^{(2),\mathrm{sk}} f \in M_{3-k}^{(2)}$. Then

$$\mathcal{L}\big(\mathrm{FJ}_m\big(\,\xi_k^{(2)\mathrm{sk}} f\,\big)\big) = -\,\xi_{k,m}^{\mathrm{J},\mathrm{sk}}\big(\,\mathcal{L}^{\mathrm{sk}}\big(\mathrm{FJ}_m f\big)\,\big)$$

PROOF. It suffices to compare the left and the right hand side for Poincaré-Eisenstein series, since, by Theorem 3.1 in Chapter 4, by Proposition 2.2 and by Theorem 1.1, all Fourier coefficients that occur in the Fourier expansion of

Siegel modular forms that we consider occur in the Fourier expansion of Poincaré-Eisenstein series.

By Proposition 1.5 in Chapter 3 and Theorem 1.1, we have

$$\mathcal{L}\big(\mathrm{FJ}_m\big(\,\xi_k^{(2)} f\,\big)\big) = (\tfrac{3}{2} - k)(1 - k)\frac{(-1)^{\frac{k-2}{2}}(2\pi)^{3-k}}{\Gamma(\frac{1}{2})}\zeta(3-k)^{-1} y^{k-\frac{3}{2}}\,\tilde{E}^{\mathrm{J}}_{k-1,1,m}$$

in the holomorphic case, and

$$\mathcal{L}\big(\mathrm{FJ}_m\big(\,\xi_k^{(2)\mathrm{sk}} f\,\big)\big) = (\tfrac{3}{2} - k)(2 - k)\frac{(-1)^{\frac{3-k}{2}}(2\pi)^{3-k}}{\Gamma(3-k)}\zeta(3-k)^{-1}\,\tilde{E}^{\mathrm{J}}_{3-k,0,m},$$

otherwise.

On the other hand, by Proposition 7.1 in Chapter 3, we have

$$\xi^{\mathrm{J}}_{k,m}\big(\,\mathcal{L}^{\mathrm{sk}}\big(\mathrm{FJ}_m f\big)\,\big) = (\tfrac{3}{2} - k)\frac{(-1)^{\frac{-k}{2}}(2\pi)^{3-k}}{\Gamma(\frac{3}{2})}\zeta(3-k)^{-1} y^{k-\frac{3}{2}}\,\tilde{E}^{\mathrm{J}}_{k-1,1,m}$$

in the holomorphic case, and

$$\xi^{\mathrm{J,sk}}_{k,m}\big(\,\mathcal{L}^{\mathrm{sk}}\big(\mathrm{FJ}_m f\big)\,\big) = (\tfrac{3}{2} - k)\frac{(-1)^{\frac{1-k}{2}}(2\pi)^{3-k}}{\Gamma(2-k)}\zeta(3-k)^{-1}\,\tilde{E}^{\mathrm{J}}_{3-k,0,m}.$$

This yields the result. $\square$

We summarize the results of this chapter and Chapter 3 in the next diagram.

Corollary 3.2. *The following diagram commutes up to multiplicative scalars that only depend on k.*

$$\begin{tikzcd}
\mathbb{M}^{(2)}_{3-k},\, \mathbb{M}^{(2)}_{k} \arrow[r, leftrightarrow, "\xi_k^{(2)}\qquad \xi_k^{(2),\mathrm{sk}}"] \arrow[d, "\mathcal{L}\circ\mathrm{FJ}_m"] & \mathbb{M}^{(2),\mathrm{sk}}_{k},\, \mathbb{M}^{(2),\mathrm{sk}}_{3-k} \arrow[d, "\mathcal{L}^{\mathrm{sk}}\circ\mathrm{FJ}_m"] \\
\mathbb{J}_{3-k,m},\, \mathbb{J}_{k,m} \arrow[r, leftrightarrow, "\xi_k^{\mathrm{J}}\qquad \xi_k^{\mathrm{J,sk}}"] \arrow[d, "\theta_m^{\mathrm{z}}"] & \mathbb{J}^{\mathrm{sk}}_{k,m},\, \mathbb{J}^{\mathrm{sk}}_{3-k,m} \arrow[d, "\theta_m^{\mathrm{sk,z}}"] \\
\mathbb{M}_{\frac{5}{2}-k},\, \mathbb{M}_{k-\frac{1}{2}} \arrow[r, leftrightarrow, "\xi^{(1)}_{k-\frac{1}{2}}"] & \mathbb{M}_{k-\frac{1}{2}},\, \mathbb{M}_{\frac{5}{2}-k}
\end{tikzcd}$$

APPENDIX A

Sage scripts

1. Nakajima's order 4 operator

This code is written in Sage [**S$^+$11**], but it directly calls Plural [**LS03**] to perform computations in the algebra of differential operators on $C^\infty(\mathbb{H}_2)$. Plural implements algorthims to obtain standard forms of noncommutative polynomials. We refrain from describing exactly the syntax and basic semantics, but rather refer the reader to the online manual.

The code can be run as is, except that within the Plural code there are line-breaks introduced to fit the lines into the page. These must be removed before running the code.

Great parts of the code are dedicated to sanity tests, computing several substitutions on basic examples and comparing them with either the obvious result or a second result obtained by means of alternative methods. To run these tests, run

- `test_initialization()` or
- `test_rewrite()`.

The absence of assertion error then shows that all tests have passed.

The basic idea behind the script is the following: Any invariant operator T satisfies

$$(Tf)(Z) = \boldsymbol{\alpha}_{\alpha,\beta}(g,Z)^{-1}\big(T\,\boldsymbol{\alpha}_{\alpha,\beta}(g,Z)f(g\,\cdot)\big)(g^{-1}Z).$$

This induces an automorphism on the algebra of differential operators. We compute the images of the generators. This way, to prove or disprove $\mathrm{Sp}_n(\mathbb{R})$-invariance it suffices to express every differential operator in terms of a given set of polynomial variables and elementary differential operators, to apply the automorphism, and to check for equality with the original term.

```
#==============================================================================
#
# Structure:
# -- Initialization of Singular
# -- Rewrite and application
# -- Tests
# -- Invariance tests
#
# Usage:
#     Run
#       sage: test_translation()
#     to test for translation invariance.
#     Run
#       sage: test_involution()
#     to test for invariance under the involution.
#
#==============================================================================

##############################################################################
#### Initialization of Singular
##############################################################################
```

```
#==========================================================================
#
# We will use Singular and more specifically Plural to compute the
# differential operators.
# -- 'zb' refers to \bar z, the complex conjugate of z.
# -- The operators A_i are defined in Section 8 of Maass' book "Siegel's
#    modular forms and Dirichlet series".
# -- The operators 'M' and 'Mb' are raising and lowering operators, that
#    up to conjugation and multiplication with appropriate powers of
#    det Y are the xi-operators defined in the paper.
# -- 'Niwa1' and 'Niwa2' are the operators that Niwa uses when computing
#    the Fourier expansion of Siegel wave forms. He obtained these directly
#    from Nakajimas work.
#
#==========================================================================

sing = Singular()

initialization = \
"""
ring R = 0, (i, pi,
             z11, z22, z12, zdetinv,
             zb11, zb22, zb12, zbdetinv, ydetinv,
             dz11, dz22, dz12,
             dzb11, dzb22, dzb12, alpha, beta),
             rp;
```

```
def y11 =-i/2 * (z11 - zb11);
def y22 =-i/2 * (z22 - zb22);
def y12 =-i/2 * (z12 - zb12);

matrix D[19][19];
D[6, 12] =-zdetinv^2 * z22;
D[6, 13] =-zdetinv^2 * z11;
D[6, 14] =-zdetinv^2 * (-2 * z12);
D[10, 15] =-zbdetinv^2 * zb22;
D[10, 16] =-zbdetinv^2 * zb11;
D[10, 17] =-zbdetinv^2 * (-2 * zb12);

D[11, 12] = -ydetinv^2 * (-i)/2  * y22;
D[11, 13] = -ydetinv^2 * (-i)/2  * y11;
D[11, 14] = -ydetinv^2 * i * y12;

D[11, 15] = -ydetinv^2 * i/2 * y22;
D[11, 16] = -ydetinv^2 * i/2 * y11;
D[11, 17] = -ydetinv^2 * (-i)  * y12;

D[3, 12] = 1;
D[4, 13] = 1;
D[5, 14] = 1;
```

```
D[7, 15] = 1;
D[8, 16] = 1;
D[9, 17] = 1;

def wa = nc_algebra(1, D);
setring wa;

def x11 = 1/2 * (z11 + zb11);
def x22 = 1/2 * (z22 + zb22);
def x12 = 1/2 * (z12 + zb12);
def y11 =-i/2 * (z11 - zb11);
def y22 =-i/2 * (z22 - zb22);
def y12 =-i/2 * (z12 - zb12);
def zdet = z11 * z22 - z12^2;
def zbdet = zb11 * zb22 - zb12^2;
matrix yinv[2][2] = y22, -y12, -y12, y11;
yinv = yinv * ydetinv;
def ydet = y11 * y22 - y12^2;

ideal rels = groebner(ideal( i^2 + 1, zdetinv*(z11*z22 - z12^2) - 1,
                             zbdetinv*(zb11*zb22 - zb12^2) - 1,
                             ydetinv*(y11*y22 - y12^2) - 1));
```

```
matrix imat[2][2] = i,0,0,i;
matrix ymat[2][2] = y11, y12, y12, y22;
matrix dz[2][2] = dz11, 1/2 * dz12, 1/2 * dz12, dz22;
matrix dzb[2][2] = dzb11, 1/2 * dzb12, 1/2 * dzb12, dzb22;


matrix K[2][2] = alpha + 2 * imat * ymat * dz;
matrix Lambda[2][2] =-beta + 2 * imat * ymat * dzb;

matrix A1[2][2] = Lambda * K + 3/2 * K;
matrix A2[2][2] =   A1 * A1 - 3/2 * Lambda * A1 + 1/2 * Lambda * trace(A1)
                  + imat * ymat
                    * transpose(- 1/2 * imat * yinv * transpose(transpose(Lambda) * transpose(A1)));

def H1 = trace(A1);
def H2 = trace(A2);

def M =    alpha * (alpha - 1/2) + (alpha - 1/2)*(2*i*y11*dz11 + 2*i*y12*dz12 + 2*i*y22*dz22)
         - 4 * (y11*y22 - y12^2) * (dz11 * dz22 - 1/4 * dz12 * dz12);
def Mb =    alpha * (alpha - 1/2) + (alpha - 1/2)*( - 2*i*y11*dzb11 - 2*i*y12*dzb12 - 2*i*y22*dzb22)
          - 4 * (y11*y22 - y12^2) * (dzb11 * dzb22 - 1/4 * dzb12 * dzb12);

def C = 2 * subst(Mb, alpha, beta-1) * subst(M, alpha, alpha);

list t = list(y11, dz11), list(y22, dz22), list(y12, dz12);
list tb = list(y11, dzb11), list(y22, dzb22), list(y12, dzb12);
```

```
def Niwa1 = - (y11 * y22 - y12^2) * (dz11 * dzb22 + dzb11 * dz22 - 1/2 * dz12 * dzb12);
for (int l = 1; l <= 3; l = l + 1) {
  for (int j = 1; j <= 3; j = j + 1) {
    Niwa1 = Niwa1 + t[l][1] * tb[j][1] * t[l][2] * tb[j][2];
  }}

def Niwa2 = (y11 * y22 - y12^2)^2 * (dz11 * dz22 - 1/4 * dz12^2)
                                   * (dzb11 * dzb22 - 1/4 * dzb12^2)
            + i * 1/4 * (y11 * y22 - y12^2) * (y11 * dz11 + y12 * dz12 + y22 * dz22)
                                   * (dzb11 * dzb22 - 1/4 * dzb12^2)
            + i * 1/4 * (y11 * y22 - y12^2) * (y11 * dzb11 + y12 * dzb12 + y22 * dzb22)
                                   * (dz11 * dz22 - 1/4 * dz12^2)
            + 1/16 * (y11 * y22 - y12^2) * (dz11 * dzb22 + dzb11 * dz22 - 1/2 * dz12 * dzb12);
"""
initialization = initialization.replace("\n", "")
sing.eval(initialization);
```

```
#===========================================================================
#
# We define substitutions. The first set of substitution's corresponds to
# conjugation with the substitution $Z \mapsto -Z^{-1}$.  More precisely,
# if v is a variable and substv is the substition defined below, we have
# $(v f(-Z^{-1}))(-Z^{-1}) = substv f(Z)$.
# The second set of substititions with prefix 'substslash' corresponds to
# conjugation with $\det Z^{-\alpha} \det \bar Z^{-\beta}$.  More precisely, with
# the same notation as above, we have
# $\det Z^{\alpha} \det \bar Z^{\beta} v \det Z^{-\alpha} \det \bar Z^{-\beta} f
#  = substslashv f$.
#
#===========================================================================

substitution_initialization = \
"""
def substzdetinv = z11 * z22 - z12^2;
def substzbdetinv = zb11 * zb22 - zb12^2;
def substydetinv = ydetinv * zdet * zbdet;
def substzdet = zdetinv;
def substzbdet = zbdetinv;
def substz11  = zdetinv  * (-z22);
def substz22  = zdetinv  * (-z11);
def substz12  = zdetinv  * z12;
```

```
def substzb11 = zbdetinv * (-zb22);
def substzb22 = zbdetinv * (-zb11);
def substzb12 = zbdetinv * zb12;


def substdz11 = z11^2 * dz11 + z12^2 * dz22 + z11 * z12 * dz12;
def substdz22 = z12^2 * dz11 + z22^2 * dz22 + z22 * z12 * dz12;
def substdz12 = 2 * z11 * z12 * dz11 + 2 * z22 * z12 * dz22 + (z11 * z22 + z12^2) * dz12;
def substdzb11 = zb11^2 * dzb11 + zb12^2 * dzb22 + zb11 * zb12 * dzb12;
def substdzb22 = zb12^2 * dzb11 + zb22^2 * dzb22 + zb22 * zb12 * dzb12;
def substdzb12 = 2 * zb11 * zb12 * dzb11 + 2 * zb22 * zb12 * dzb22 + (zb11 * zb22 + zb12^2) * dzb12;

def substslashdz11 = dz11 - alpha * zdetinv * z22;
def substslashdz22 = dz22 - alpha * zdetinv * z11;
def substslashdz12 = dz12 + alpha * zdetinv * 2 * z12;
def substslashdzb11 = dzb11 - beta * zbdetinv * zb22;
def substslashdzb22 = dzb22 - beta * zbdetinv * zb11;
def substslashdzb12 = dzb12 + beta * zbdetinv * 2 * zb12;
"""
substitution_initialization = substitution_initialization.replace("\n", "")
sing.eval(substitution_initialization);
```

```
##############################################################################
#### Rewrite and application
##############################################################################

#==============================================================================
#
# By rewriting we mean applying the substitions defined above to a differential
# operator. This happens within Singular. Instead of Singular maps we use
# Python string processing.
# By application we mean applying an operator to an element of the polynomial
# ring 'P' defined below. In this case we always assume that 'alpha' and
# 'beta' do not occur in the operator.
#
#==============================================================================

rewrite_dict = { "z11" : "substz11", "z22" : "substz22", "z12" : "substz12",
                 "zb11" : "substzb11", "zb22" : "substzb22", "zb12" : "substzb12",
                 "ydetinv" : "substydetinv", "zdetinv" : "substzdetinv",
                 "zbdetinv" : "substzbdetinv", "dz11" : "substdz11",
                 "dz22" : "substdz22", "dz12" : "substdz12",
                 "dzb11" : "substdzb11", "dzb22" : "substdzb22",
                 "dzb12" : "substdzb12" }

rewrite_slash_dict = { "dz11" : "substslashdz11", "dz22" : "substslashdz22",
                       "dz12" : "substslashdz12", "dzb11" : "substslashdzb11",
                       "dzb22" : "substslashdzb22", "dzb12" : "substslashdzb12" }
```

```python
def rewrite(h) :
    """
    We first apply the substitution corresponding to the
    `\det Z^{-\alpha} \det \bar Z^{-\beta}` part of the slash action.
    Second, we apply the substitution corresponding to `Z \mapsto -Z^{-1}`.

    INPUT:
        - `h` -- A string.

    OUTPUT:
        A string.
    """
    mons = map(split_monomial, get_monomials(sing.eval(h)))
    slash_rew = join( ["(" + join(map(rewrite_slash_term, m), '*') + ")" for m in mons], '+' )
    slash_rew = sing.eval("reduce(%s, rels);" % (slash_rew,))

    mons = map(split_monomial, get_monomials(slash_rew))
    res = join( ["(" + join(map(rewrite_invol_term, m), '*') + ")" for m in mons], '+' )

    return sing.eval("reduce(%s, rels);" % (res,))
```

```python
def rewrite_invol_term(t):
    """
    Apply the substitution corresponding to an involution to a singular variable.
    """
    try:
        return rewrite_dict[t]
    except KeyError:
        return t

def rewrite_slash_term(t):
    """
    Apply the substitution corresponding to the
    `\det Z^{-\alpha} \det \bar Z^{-\beta}` part of the slash action
    to a singular variable.
    """
    try:
        return rewrite_slash_dict[t]
    except KeyError:
        return t
```

```python
def get_monomials(t):
    """
    Split a Singular expression into monomials.
    """
    t = t.replace("-", "+(-1)*")
    if t[0] == "+":
        t = t[1:]

    return t.split("+")

def split_monomial(m):
    """
    Split a Singular monomial into a list of variables and constants.
    """
    m = m.split("*")
    return flatten( [v
                     if v.find("^") == -1
                     else [v[:v.find("^")] for _ in range(Integer(v[v.find("^") + 1:])) ]
                     for v in m],
                    list )
```

```
### Application of operators

K.<i> = QuadraticField(-1)
P.<z11, z22, z12, zb11, zb22, zb12> = K[]
zdet = (z11 * z22 - z12^2)
zbdet = (zb11 * zb22 - zb12^2)

apply_dict = { "dz11" : lambda e: diff(e, z11),
               "dz22" : lambda e: diff(e, z22),
               "dz12" : lambda e: diff(e, z12),
               "dzb11" : lambda e: diff(e, zb11),
               "dzb22" : lambda e: diff(e, zb22),
               "dzb12" : lambda e: diff(e, zb12),
               "z11" : lambda e: z11 * e,
               "z22" : lambda e: z22 * e,
               "z12" : lambda e: z12 * e,
               "zb11" : lambda e: zb11 * e,
               "zb22" : lambda e: zb22 * e,
               "zb12" : lambda e: zb12 * e,
               "zdetinv" : lambda e: e / (z11 * z22 - z12^2),
               "zbdetinv" : lambda e: e / (zb11 * zb22 - zb12^2),
               "(-1)" : lambda e: -e,
               "i" : lambda e: i*e }
```

```python
def apply_term(t, e):
    """
    Apply a term to an expression in `P`.

    INPUT:
        - `t` -- A sting corresponding to a Singular expression.
        - `e` -- An element of `P`.

    OUTPUT:
        An element of `P`.
    """
    mons = map(split_monomial, get_monomials(sing.eval(t)))

    return sum(apply_monomial(m, e) for m in mons)

def apply_monomial(m, e):
    """
    Apply a monomial to an expression in `P`.
    """
    for v in reversed(m):
        try:
            e = apply_dict[v](e)
        except KeyError:
            e = QQ(v) * e

    return e
```

```
##################################################################################
#### Tests
##################################################################################

#=================================================================================
#
# To ensure that all definitions above are correct we run a series of tests,
# that are designed to find potential mistakes in the implementation.
#
#=================================================================================

def test_initialization():
    """
    Test the initialization of Singular (without the substitution part).
    """
    ### Commutators

    # Tests for holomorphic variables. The left hand side is executed and has to
    # yield the right hand side
    print( "homogeneous commutator tests" )
    tests = [
     ("dz11 * z11", "z11*dz11+1"),
     ("dz22 * z22", "z22*dz22+1"),
     ("dz12 * z12", "z12*dz12+1"),
     ("dz11 * z12", "z12*dz11"),
     ("dz11 * z22", "z22*dz11"),
```

```
 ("dz12 * z11", "z11*dz12"),
 ("dz12 * z22", "z22*dz12"),
 ("dz22 * z11", "z11*dz22"),
 ("dz22 * z12", "z12*dz22"),
 ("z11 * z12", "z11*z12"),
 ("z11 * z22", "z11*z22"),
 ("z12 * z22", "z22*z12") ]
for (t,r) in tests :
    tt = t
    rr = r
    if sing.eval(tt + ";") != rr :
        raise AssertionError( tt + "::" + sing.eval(tt + ";") + ";;" + rr )

    tt = t.replace("z", "zb")
    rr = r.replace("z", "zb")
    if sing.eval(tt + ";") != rr :
        raise AssertionError( tt + "::" + sing.eval(tt + ";") + ";;" + rr )

# Misc tests
print( "misc commutator tests" )
tests = [
 ("zf11 * zg11", "z11*zb11"),
 ("zf22 * zg22", "z22*zb22"),
 ("zf12 * zg12", "z12*zb12"),
 ("dzf11 * zg11", "zg11*dzf11"),
 ("dzf22 * zg22", "zg22*dzf22"),
```

```
 ("dzf12 * zg12", "zg12*dzf12"),
 ("dzf11 * zg12", "zg12*dzf11"),
 ("dzf11 * zg22", "zg22*dzf11"),
 ("dzf12 * zg11", "zg11*dzf12"),
 ("dzf12 * zg22", "zg22*dzf12"),
 ("dzf22 * zg11", "zg11*dzf22"),
 ("dzf22 * zg12", "zg12*dzf22") ]
for (t,r) in tests :
    tt = t.replace("zf", "z").replace("zg", "zb")
    rr = r.replace("zf", "z").replace("zg", "zb")
    if sing.eval(tt + ";") != rr :
        raise AssertionError( tt + "::" + sing.eval(tt + ";") + ";;" + rt )

    tt = t.replace("zf", "zb").replace("zg", "z")
    rr = r.replace("zf", "zb").replace("zg", "z")
    if sing.eval(tt + ";") != rr :
        raise AssertionError( tt + "::" + sing.eval(tt + ";") + ";;" + rt )

### Reductions
print( "homogenous reduction tests" )
tests = [
 ("i * i", "-1"),
 ("i2", "-1"),
 ("zdetinv * zdet", "1"),
 ("zbdetinv * zbdet", "1"),
 ("ydetinv * ydet", "1"),
```

```
  ("(ymat␣*␣yinv)[1,1]", "1"),
  ("(ymat␣*␣yinv)[1,2]", "0"),
  ("(ymat␣*␣yinv)[2,1]", "0"),
  ("(ymat␣*␣yinv)[2,2]", "1"),
  ("(imat␣*␣imat)[1,1]", "-1"),
  ("(imat␣*␣imat)[2,1]", "0"),
  ("(imat␣*␣imat)[1,2]", "0"),
  ("(imat␣*␣imat)[2,2]", "-1"),
  ("x11␣+␣i*y11", "z11"),
  ("x22␣+␣i*y22", "z22"),
  ("x12␣+␣i*y12", "z12"),
  ("x11␣-␣i*y11", "zb11"),
  ("x22␣-␣i*y22", "zb22"),
  ("x12␣-␣i*y12", "zb12")
]
for (tt,rr) in tests :
    if sing.eval("reduce(" + tt + ",␣rels);") != rr :
        raise AssertionError( tt + "::" + sing.eval("reduce(" + tt + ",␣rels);") + ";;" + rr )

### Relations of C, Niwa1 and H*
if sing.eval("reduce(subst(4␣*␣Niwa1␣+␣H1,␣alpha,␣0,␣beta,␣0),␣rels);") != "0" :
    raise AssertionError( "4␣*␣Niwa1␣!=␣-H1" )
if sing.eval("reduce(C␣-␣(H1^2␣-␣H2)␣+␣1/2␣*␣(␣1␣+␣alpha␣-␣beta)␣*␣H1,␣rels);") != "0" :
    raise AssertionError( "C␣!=␣(H1^2␣-␣H2)␣-␣1/2␣*␣(␣1␣+␣alpha␣-␣beta)␣*␣H1" )
```

```
    ### Leading terms of the operators H*
    if sing.eval("reduce(subst(H1, alpha, 0, beta, 0), rels);")[:21] != "1/2*zb12^2*dz12*dzb12" :
        raise AssertionError( "Leading term of H1 incorrect" )
    if sing.eval("reduce(subst(H2, alpha, 0, beta, 0), rels);")[:25] != "1/8*zb12^4*dz12^2*dzb12^2" :
        raise AssertionError( "Leading term of H2 incorrect" )

def test_rewrite() :
    """
    Test the substitution part of the initialization of Singular and the rewrite.
    """
    ### get_monomials
    print "get_monomials"
    tests = [
     ("-z22*zdetinv", ["(-1)*z22*zdetinv"]),
     ("dz11+z11", ["dz11","z11"]),
     ("z12^2*dz11+z11^3", ["z12^2*dz11","z11^3"]),
     ("-z12^2*dz11-z11^3", ["(-1)*z12^2*dz11","(-1)*z11^3"])
    ]
    for (tt,rr) in tests :
        if get_monomials(tt) != rr :
            raise AssertionError( tt + "::" + repr(get_monomials(tt)) + ";;" + repr(rr) )
```

```
### split_monomial
print "split_monomial"
tests = [
 ("(-1)*z22*zdetinv", ["(-1)", "z22", "zdetinv"]),
 ("dz11", ["dz11"]),
 ("z11^3", ["z11", "z11", "z11"]),
 ("z12^2*dz11", ["z12", "z12", "dz11"]),
 ("(-1)*1/2*z12^2*dz11", ["(-1)", "1/2", "z12", "z12", "dz11"])
]
for (tt,rr) in tests :
    if split_monomial(tt) != rr :
        raise AssertionError( tt + "::" + repr(split_monomial(tt)) + ";;" + repr(rr) )

### apply_monomial
print "apply_monomial"
p = z11^2 + zb12*z22^7 + i * z12*zb22 + 7/23 * zb11^5
tests = [
 ("i*dz11", 2*i*z11),
 ("z11^3", z11^3*p),
 ("z12^2*dzb11", 35/23*z12^2*zb11^4),
 ("(-1)*1/2*z12^2*dz22", -7/2*z12^2*z22^6*zb12),
 ("4/5*dz12*dzb22*zb22", i*8/5*zb22)
]
```

```
for (tt, rr) in tests :
    if apply_monomial(split_monomial(tt), p) != rr :
        raise AssertionError( tt + "::" + repr(apply_monomial(split_monomial(tt), p))
                                                           + ";;" + repr(rr) )

### apply_term
print "apply␣term"
p = z11^2 + zb12*z22^7 + i * z12*zb22 + 7/23 * zb11^5
tests = [
 ("dz11+z11", diff(p, z11) + z11*p),
 ("-z12^2*dz11-z11^3", -z12^2 * diff(p,z11) - z11^3 * p),
 ("-i*zb22^3*dz11*z11^2-2/3*z11^3", -i*zb22^3 * diff(z11^2 * p, z11) - 2/3 * z11^3*p)
]
for (tt, rr) in tests :
    if apply_term(tt, p) != rr :
        raise AssertionError( tt + "::" + repr(apply_term(tt, p)) + ";;" + repr(rr) )

### elementary substitutions
print "elementary␣substitutions"
tests = [
 ("z11", "-z22*zdetinv"),
 ("z22", "-z11*zdetinv"),
 ("z12", "z12*zdetinv"),
]
```

```
for (t,r) in tests :
    tt = t
    rr = r
    if rewrite(tt) != rr :
        raise AssertionError( tt + "::" + sing.eval(tt + ";") + ";;" + rr )

    tt = t.replace("z", "zb")
    rr = r.replace("z", "zb")
    if rewrite(tt) != rr :
        raise AssertionError( tt + "::" + sing.eval(tt + ";") + ";;" + rr )


### product substitutions
print "product␣substitutions"
tests = [
 ("zdet", "zdetinv", "1"),
 ("ydet", "ydetinv", "1")
]
for (t1, t2, r) in tests :
    tt1 = t1
    tt2 = t2
    rr = r
    if sing.eval("reduce((" + rewrite(tt1) + ")*(" + rewrite(tt2) + "),␣rels);") != rr :
        raise AssertionError( "rewrite:" + tt1 + ",," + tt2
                + "::" + sing.eval("reduce((" + rewrite(tt1) + ")*(" + rewrite(tt2) + "),␣rels);")
                + ";;" + rr )
```

```
    tt1 = t1.replace("z", "zb")
    tt2 = t2.replace("z", "zb")
    rr = r
    if sing.eval("reduce((" + rewrite(tt1) + ")*(" + rewrite(tt2) + "),␣rels);") != rr :
        raise AssertionError( "rewrite:" + tt1 + ",," + tt2
                  + "::" + sing.eval("reduce((" + rewrite(tt1) + ")*(" + rewrite(tt2) + "),␣rels);")
                  + ";;" + rr )

### application of rewrites
print "application␣of␣rewrites"
zdet = z11*z22 - z12^2
zbdet = zb11*zb22 - zb12^2
varsubs_dict = { z11 : -z22 / zdet, z22 : -z11 / zdet, z12 : z12 / zdet,
                 zb11 : -zb22 / zbdet, zb22 : -zb11 / zbdet, zb12 : zb12 / zbdet }

p = z11 * z22
tests = [
  "z11",
  "z22",
  "z12",
  "zb11",
  "zb22",
  "zb12"
]
```

```
for tt in tests :
    for alpha, beta in mrange([3,3], tuple) :
        if    apply_term(sing.eval( "subst(%s, alpha, %s, beta, %s)"
                                   % (rewrite(tt), alpha, beta) ), p) \
           != (    zdet**alpha * zbdet**beta * apply_term(tt, zdet**(-alpha) * zbdet**(-beta)
                * p.subs(varsubs_dict)) ).subs(varsubs_dict) :
            raise AssertionError( tt + "::" + repr(alpha) + "::" + repr(beta) + "::"
                + repr(apply_term(sing.eval( "subst(%s, alpha, %s, beta, %s)"
                                            % (rewrite(tt), alpha, beta) ), p))
                + ";;"
                + repr((    zdet**alpha * zbdet**beta * apply_term(tt, zdet**(-alpha) * zbdet**(-beta)
                         * p.subs(varsubs_dict)) ).subs(varsubs_dict)) )

p = z11^2 + zb12*z22^7 + i * z12*zb22 + 7/23 * zb11^5
tests = [
 "dz11",
 "dz11^2",
 "dzb12^2",
 "dz11^2+i*dzb22",
 "1/2*z11*dzb12-i*zb22^2*dz22"
]
```

```
    for tt in tests:
        for alpha, beta in mrange([3,3], tuple):
            if    apply_term(sing.eval( "subst(%s, alpha, %s, beta, %s)"
                                       % (rewrite(tt), alpha, beta) ), p) \
              != (  zdet**alpha * zbdet**beta
                  * apply_term(tt, zdet**(-alpha) * zbdet**(-beta) * p.subs(varsubs_dict))) \\
                 .subs(varsubs_dict) :
               raise AssertionError( tt + "::" + repr(alpha) + "::" + repr(beta) + "::"
                   + repr(apply_term(sing.eval( "subst(%s, alpha, %s, beta, %s)"
                                              % (rewrite(tt), alpha, beta) ), p))
                   + ";;"
                   + repr((  zdet**alpha * zbdet**beta
                           * apply_term(tt, zdet**(-alpha) * zbdet**(-beta) * p.subs(varsubs_dict))) \\
                          .subs(varsubs_dict)) )

################################################################################
#### Invariants tests
################################################################################

#===============================================================================
#
# We test several operators on invariance. Note that invariance under translation
# is obvious, since all coefficients are trivial in $X$. Testing the involution
# $|_{\alpha, \beta} J$ reveals that Nakajima's second operator is not invariant.
#
#===============================================================================
```

```
def test_translation() :
    tests = \
      [ "subst(Niwa1, z11, z11 + 1, zb11, zb11 + 1) - Niwa1;",
        "subst(Niwa1, z22, z22 + 1, zb22, zb22 + 1) - Niwa1;",
        "subst(Niwa1, z12, z12 + 1, zb12, zb12 + 1) - Niwa1;",
        "subst(Niwa2, z11, z11 + 1, zb11, zb11 + 1) - Niwa2;",
        "subst(Niwa2, z22, z22 + 1, zb22, zb22 + 1) - Niwa2;",
        "subst(Niwa2, z12, z12 + 1, zb12, zb12 + 1) - Niwa2;",
        "subst(H1, z11, z11 + 1, zb11, zb11 + 1) - H1;",
        "subst(H1, z22, z22 + 1, zb22, zb22 + 1) - H1;",
        "subst(H1, z12, z12 + 1, zb12, zb12 + 1) - H1;",
        "subst(H2, z11, z11 + 1, zb11, zb11 + 1) - H2;",
        "subst(H2, z22, z22 + 1, zb22, zb22 + 1) - H2;",
        "subst(H2, z12, z12 + 1, zb12, zb12 + 1) - H2;" ]

    return map(sing.eval, tests)

def test_involution() :
    tests = [ "reduce(subst(%s - %s, alpha, 0, beta, 0), rels);" % (rewrite('Niwa1'), 'Niwa1'),
              "reduce(subst(%s - %s, alpha, 0, beta, 0), rels);" % (rewrite('Niwa2'), 'Niwa2'),
              "reduce(%s - %s, rels);" % (rewrite('C'), 'C'),
              "reduce(%s - %s, rels);" % (rewrite('H1'), 'H1'),
              "reduce(%s - %s, rels);" % (rewrite('H2'), 'H2') ]

    return map(sing.eval, tests)
```

2. Lemma 2.5 in Chapter 4

This script is completely written in Sage [**S**$^+$**11**], but it makes indirect use of PyNac [**BS**$^+$**11**], a library for symbolic calculations, that is based on GiNaC [**BFK02**]. We derive the Laurent expansion of the the potential solutions given in Lemma 2.5 of Chapter 4, and check whether sufficiently many coefficients of the image under the differential operators assigned to equation (2.9) in Chapter 4 vanish.

```
## A function computing the Pochhammer symbol (a)_n
pochhammer = lambda a, n: prod(a + k for k in range(n))
## A functin computing the power series expansion of the
## generalized hypergeometric series
## {}_p {\rm F}_q (ass; bss; v) up to O(v^ord)
hyperexpansion = lambda ass, bss, ord: \
  [    prod(pochhammer(a, n) for a in ass)
    / prod(pochhammer(b, n)
    for b in bss) / factorial(n) for n in range(ord) ]

## The variables v and k, that we will use below
var('v k')

## To compute the coefficients of the functions below up
## to O(v^11) we need to expand the hypergeometric
## series up to O(v^15).
hyperord = 15

## The parameters for the solutions in the holomorphic
## and skew case (excluding 1, that is an obvious solution).
## The first entrie is the exponent of v that the
## hypergemetric function is multiplied with.
## The second is the list of all a's and the
## third is the list of all b's.
```

```
h1s_hol = [ ( 3/2 - k    , [3/2, 3/2 - k],
              [2 - k/2, (5 - k)/2, 5/2 - k] ),
            ( -k/2       , [k/2, -k/2],
              [1/2, (k - 1)/2, 1 - k/2] ),
            ( (1 - k)/2, [(1 + k)/2, (1 - k)/2],
              [3/2, k/2, (3 - k)/2] ) ]

h1s_skew = [( 0          , [1/2],
              [(1 + k)/2, 1 + k/2] ),
            ( -k/2       , [(1 - k)/2],
              [1/2, 1 - k/2] ),
            ( (1 - k)/2, [1 - k/2],
              [3/2, (3 - k)/2] ),
            ( 3/2 - k    , [1, 2 - k],
              [5/2 - k, 2 - k/2, (5 - k)/2] )]


## the expansions of the solutions.
h1exps_hol = \
  [ v**e * sum(map( operator.mul,
                    hyperexpansion(ass, bss, hyperord),
                    [ (v/4)**n for n in range(hyperord) ]))
    for (e, ass, bss) in h1s_hol ]
```

```
h1exps_skew = \
    [ v**e * sum(map( operator.mul,
                      hyperexpansion(ass, bss, hyperord),
                      [ (v/4)**n for n in range(hyperord) ]))
      for (e, ass, bss) in h1s_skew ]

## the differential operator that is associated to the
## differential equation for h1
diffop = lambda alpha, beta: ( lambda h1:
  16 * v**3 * diff(h1, v, 4)
  + ( 32 * (alpha + beta) + 64 ) * v**2 * diff(h1, v, 3)
  + (    (    20 * (alpha + beta)**2
            + 60 * (alpha + beta) + 28) * v
       - 4 * v**2 ) * diff(h1, v, 2)
  + ( - (4 * (alpha + beta) + 4) * v
       + 4 * (alpha + beta)**3 + 10 * (alpha + beta)**2
       + 2 * (alpha + beta) - 4) * diff(h1, v)
  + ( (alpha - beta)**2 - (alpha + beta)**2 ) * h1  )

diffop_hol = diffop(k, 0)
diffop_skew = diffop(1/2, k - 1/2)

## The expansion of the images of potential solution under
## the above differential operators
h1ims_hol = map(diffop_hol, h1exps_hol)
h1ims_skew = map(diffop_skew, h1exps_skew)
```

```
## Sage returns the coefficients of a power series or
## polynomial in increasing order with respect to the
## exponents.  Thus it is sufficient to check the
## first 11 elements of h1ims.coefficients()
assert all( [ e.simplify_rational()
                for (e,_) in h1im.coefficients(v)[:11] ]
              == 11*[0]
              for h1im in h1ims_hol )
assert all( [ e.simplify_rational()
                for (e,_) in h1im.coefficients(v)[:11] ]
               == 11*[0]
              for h1im in h1ims_skew )
```

3. Numerical double checks for Lemma 2.5 in Chapter 4

This script, though written in Sage [**S**$^+$**11**], is largely based on mpmath [**J**$^+$**11**], a libary for arbitrary precision calculations, completely written in Python. We numerically double check the results obtained in Section 2. The calculations performed, even though they are implemented mostly naively, are challanging, since, in particular, the differential operator applied is badly conditioned for the solutions in the holomorphic case. We compensate for this by using 500 digits precision. The reader using the code will notice that this needs to be increased drastically when evaluating the occurring expressions at larger values.

```python
import mpmath

mpmath.mp.dps = 500
mpd = mpmath.diff

## The differential operators in the holomorphic or skew case.  We
## define functions that for each k provide a function, that evaluates
## the image of a function f under the differential operator at v.
diffop_hol = lambda k: ( lambda f, v:
                     4 * k**3 * mpd(f, v) + 20 * k**2 * v * mpd(f, v, 2)
                   + 32 * k * v**2 * mpd(f, v, 3) + 16 * v**3 * mpd(f, v, 4)
                   + 10 * k**2 * mpd(f, v) - 4 * k * v * mpd(f, v)
                   + 60 * k * v * mpd(f, v, 2) - 4 * v**2 * mpd(f, v, 2)
                   + 64 * v**2 * mpd(f, v, 3)
                   + 2 * k * mpd(f, v) - 4 * v * mpd(f, v)
                   + 28 * v * mpd(f, v, 2)
                   - 4 * mpd(f, v) )
diffop_skew = lambda k: ( lambda f, v:
                   - 4 * k**3 * mpd(f, v) - 20 * k**2 * v * mpd(f, v, 2)
                   - 32 * k * v**2 * mpd(f, v, 3) - 16 * v**3 * mpd(f, v, 4)
                   - 10 * k**2 * mpd(f, v) + 4 * k * v * mpd(f, v)
                   - 60 * k * v * mpd(f, v, 2) + 4 * v**2 * mpd(f, v, 2)
                   - 64 * v**2 * mpd(f, v, 3) + 2 * k * f(v)
                   - 2 * k * mpd(f, v) + 4 * v * mpd(f, v)
                   - 28 * v * mpd(f, v, 2) - f(v)
                   + 4 * mpd(f, v) )
```

```
## The solutions in the holomorphic case expressed as functions that use the
## arithmetic implemented in the mpmath library.
h1s_hol = [ lambda k: ( lambda v: 1 ),
            lambda k: ( lambda v: v^(3/2 - k)
                 * mpmath.hyper([3/2, 3/2 - k], [2 - k/2, (5 - k)/2, 5/2 - k], v/4) ),
            lambda k: ( lambda v: v^(-k/2)
                 * mpmath.hyper([k/2, -k/2], [1/2, (k - 1)/2, 1 - k/2], v/4) ),
            lambda k: ( lambda v: v^((1-k)/2)
                 * mpmath.hyper([(1 + k)/2, (1 - k)/2], [3/2, k/2, (3 - k)/2], v/4) ) ]

## The solution in the skew case.
h1s_skew = [ lambda k: (lambda v: mpmath.hyper([1/2], [(1 + k)/2, 1 + k/2], v/4)),
             lambda k: (lambda v: v**(-k/2)
                  * mpmath.hyper([(1 - k)/2], [1/2, 1 - k/2], v/4)),
             lambda k: (lambda v: v**((1 - k)/2)
                  * mpmath.hyper([1 - k/2], [3/2, (3 - k)/2], v/4)),
             lambda k: (lambda v: v**(3/2 - k)
                  * mpmath.hyper([1, 2 - k], [5/2 - k, 2 - k/2, (5 - k)/2], v/4)) ]


## We need to cut off the results. We use one third of the internal precision
## to test the vanishing of the results. This is necessary, because the
## hypergeometric functions for large values of v behave numerically very
## badly.
cut = lambda v: mpmath.absmax(v) < mpmath.mpf('1e-%s' % (mpmath.mp.dps // 2) )
```

```
## We test the vanishing of the solutions under the differential operators
## for a set of four "weights" k and three values of v.
for k in map(mpmath.mpc, [4 + i, -2 - 5*i, -3 + 7*i, 2 - i]) :
    for v in map(mpmath.mpf, [1000, 10000, 100000]) :
        assert all( cut( diffop_hol(k)( h1(k), v ) ) for h1 in h1s_hol )
        assert all( cut( diffop_skew(k)( h1(k), v ) ) for h1 in h1s_skew )
```

4. Theorem 3.1 in Chapter 4

The following script, written in Sage [**S$^+$11**], makes use of Singular [**DGPS10**] and its implementation of (commutative) Gröbner bases. The script, even though it is short, is somewhat involved. The basic idea is to use the (projective) invariance under $\mathrm{GL}_2(\mathbb{Z})$ that any Fourier coefficient $a(Y, 0)$ of a Siegel modular form must satisfy. Given a differential equation that such a coefficient satisfies, its pullback under all $\mathrm{GL}_2(\mathbb{Z})$-transforms is satisfied the same coefficient. We deduce five such different equations by applying $\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$ once and $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ several times. The resulting equations for the potential solution, that we call ϕ, and its derivatives form an ideal. We compute a Gröbner basis of this ideal, that contains ϕ. Consequently, ϕ must vanish, which was the claim in Theorem 3.1 of Chapter 4.

```
#==============================================================================
#
# The symmetry arising from GL_2(ZZ) leads to periodicity and
# another symmetry, that we want to express here. It leads to additional
# differential equations for phi.
#
#==============================================================================

# 'x' and 'y' will be as in the paper
P.<x,y> = QQ[]

## The basic substitution corresponding to [[0,-1],[0,1]] \in \GL{2}(\ZZ)
## leads to new coordinates \tilde x and \tilde y
xtilde = -x * y**-2 * (1 + x**2 * y**-2)**-1
ytilde = (y*(x**2 * y**-2 + 1))**-1

## The function phi satisfies a differential equation
## y^2(phi_xx + phi_yy) + phi = 0
## Hence the function psi(x, y) = phi(xtilde, ytilde) satisfies
## an additional differential equation. Since we assume that phi is \GL{2}(\ZZ)
## symmetric, we have phi = \pm psi. In particular, the differential equation for
## psi also holds for phi.
```

```
## The prefix v refers to symbolic variables, distinguishing all variables
## defined here from polynomial expressions, that we will use below. The
## suffixes 'x' and 'y' stand for differentials.
vphi = function('phi', x,y)
vpsi = phi(xtilde, ytilde)

## Later we will use the differentials of psi to
## reduce the differential equation for phi.
vdpsix = diff(vpsi, x)
vdpsixx = diff(vdpsix, x)
vdpsiy = diff(vpsi, y)
vdpsiyy = diff(vdpsiy, y)
vdpsixy = diff(vdpsix, y)

## We insert \tilde x and \tilde y into the differentials of phi.
vdphix = diff(vphi, x)(xtilde, ytilde)
vdphixx = diff(vphi, x, x)(xtilde, ytilde)
vdphiy = diff(vphi, y)(xtilde, ytilde)
vdphiyy = diff(vphi, y, y)(xtilde, ytilde)
vdphixy = diff(vphi, x, y)(xtilde, ytilde)
```

```
## Rconv is a formal trick to convert the symbolic expressions to polynomials.
## The term order that we define is not necessary and we carry it through for clarity.
Rconv = PolynomialRing(QQ, ['x', 'y', 'dphix', 'dphixx', 'dphixy', 'dphiy', 'dphiyy',
                             'dpsixx', 'dpsix', 'dpsixy', 'dpsiyy', 'dpsiy', 'psi'],
                            order = TermOrder('dp', 2) + TermOrder('dp', 5) + TermOrder('dp', 6))
Rconv.inject_variables()

## We need the following substitutions to express the differentials of psi
## in terms of differentials of phi.
psisubs = {vphi(xtilde, ytilde): psi, vdphix: dphix,
           vdphixx: dphixx, vdphixy: dphixy, vdphiy: dphiy,
           vdphiyy: dphiyy}

## We will Groebner reduce the following relations. Note that multiplication with
## (x**2 + y**2)**4 * y**6 is only done to allow conversion into 'Rconv'.
relations = map( lambda expr, evar: \
                      (x**2 + y**2)**-4 * y**-6 \
                    * Rconv( ( (x**2 + y**2)**4 * y**6 * expr.subs(psisubs) ) \
                             .simplify_rational().factor() )
                    - evar,
                  [vdpsix, vdpsixx, vdpsixy, vdpsiy, vdpsiyy],
                  [dpsix, dpsixx, dpsixy, dpsiy, dpsiyy] )
```

```
## We use the fraction field of QQ[x, y] as a base ring to simplify the computation of
## the substitution terms
RR.<x, y> = QQ[]
RR = RR.fraction_field()
R = PolynomialRing(RR, ['dphix', 'dphixx', 'dphixy', 'dphiy', 'dphiyy',
                        'dpsixx', 'dpsix', 'dpsixy', 'dpsiyy', 'dpsiy', 'psi'],
                       order = TermOrder('dp', 5) + TermOrder('dp', 6))

## We have to separate the numerator and the denominator to allow conversion
## into 'R'.
relations = [R(r.numerator()) / RR(r.denominator()) for r in relations]

## We will reduce the wave equation deq to obtain additional differential
## equations for phi
deq = y**2 * (dphixx + dphiyy) + psi

nI = R.ideal(relations + [deq])
gI = nI.groebner_basis(algorithm = 'toy:buchberger')

## Suppose that phi = \pm psi. The first differential equation that phi then must
## satisfy is deq1.  We deduce futher relations from the Groebner basis gI
deq1 = y**2 * (dpsixx + dpsiyy) + psi
deq2 = gI[-1]
## The assertion checks that we have substituted all occurences of phi
assert set(deq2.monomials()).intersection(set([dphix, dphixx, dphixy, dphiy, dphiyy])) == set()
```

```
## The differential equation deq2 contains x. We assume that phi is 1-periodic
## in x, and we need only add shifts of deq2 to the set of equations satisfied
## by psi to show that there are no solutions but the trivial.
deqs = [deq1, deq2] + [ R(dict( (e, c.numerator().subs(x = x + n)/c.denominator().subs(x = x + n))
                                  for (e,c) in deq2.dict().iteritems()) )
                        for n in range(1, 5) ]

## We use a new polynomial ring to reduce the number of variables that we have to
## handle and to impose another term order, that is fast in our situation.
RR.<x, y> = QQ[]
RR = RR.fraction_field()
R.<dpsixx, dpsix, dpsixy, dpsiyy, dpsiy, psi> = PolynomialRing(RR, order = TermOrder('lex', 6))

deqs = [R(dict((e[5:], p[e]) for e in p.exponents())) for p in deqs]

dI = R.ideal(deqs)
gb = dI.groebner_basis(algorithm = 'toy:buchberger')
## gb contains psi, proving that there is no solution to the above system of
## differential equations except 0.
assert psi in gb
```

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