BPS state counting using wall-crossing, holomorphic anomalies and modularity

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Abstract

In this thesis we examine the counting of BPS states using wall-crossing, holomorphic anomalies and modularity. We count BPS states that arise in two setups: multiple M5-branes wrapping $P \times T^2$, where P denotes a divisor inside a Calabi-Yau threefold and topological string theory on elliptic Calabi-Yau threefolds. The first setup has a dual description as type IIA string theory via a D4-D2-D0 brane system. Furthermore it leads to two descriptions depending on the size of P and T^2 relative to each other. For the case of a small divisor P this setup is described by the $(0,4)$ Maldacena-Strominger-Witten conformal field theory of a black hole in M-theory and for the case of small T^2 the setup can by described by $\mathcal{N} = 4$ topological Yang-Mills theory on P. The BPS states are counted by the modified elliptic genus, which can be decomposed into a vector-valued modular form that provides the generating function for the BPS invariants and a Siegel-Narain theta function. In the first part we discuss the holomorphic anomaly of the modified elliptic genus for the case of two M5-branes and divisors with $b_2^+(P) = 1$. Due to the wall-crossing effect the change in the generating function is captured by an indefinite theta function, which is a mock modular form. We use the Kontsevich-Soibelman wall-crossing formula to determine the jumps in the modified elliptic genus. Using the regularisation procedure for mock modular forms of Zwegers, modularity can be restored at the cost of holomorphicity. We show that the non-holomorphic completion is due to bound states of single M5-branes. At the attractor point in the moduli space we prove the holomorphic anomaly equation, which is compatible with the holomorphic anomaly equations observed in the context of $\mathcal{N} = 4$ Yang-Mills theory on $\mathbb{P}^2$ and E-strings on a del Pezzo surface. We calculate the generating functions of BPS invariants for the divisors $\mathbb{P}^2, \mathbb{F}_0, \mathbb{F}_1$ and the del Pezzo surface dP_8 and dP_9 $\left(\frac{1}{2}\text{K3}\right)$.

In the second part we study the quantum geometry of elliptic Calabi-Yau threefolds and examine topological string theory on these spaces. We find a holomorphic anomaly equation for the topological amplitudes with respect to the base that is recursive in the genus and in the base class. The topological amplitudes with respect to the base can be expressed in terms of quasi-modular forms, which resembles the holomorphic anomaly. In particular this generalises a holomorphic anomaly discovered for the $\frac{1}{2}K3$. For genus zero and base $\mathbb{F}_1$ we prove the holomorphic anomaly by using mirror symmetry and we motivate our holomorphic anomaly equation by establishing the connection to the holomorphic anomaly equations of Bershadsky-Cecotti-Ooguri-Vafa. Using T-duality allows us to relate this anomaly to the anomaly for the case of D4-D2-D0 BPS state counting. We calculate the generating function of BPS invariants of $\frac{1}{2}K3$ for higher rank branes in topological string theory and by using algebraic-geometric techniques that were developed in the context of stability of sheaves.

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Walk on walk on with hope in your heart
And you'll never walk alone

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Contents

CHAPTER 1

Introduction and Motivation

The aim of this thesis is to present the progress [1, 2] in counting BPS states and to demonstrate their relation to (mock) modular forms via the wall-crossing effect which gives rise to holomorphic and modular anomalies. At the beginning of this thesis we want to give a general introduction to the topic and in particular emphasize the bigger picture these results fit in. We start by reviewing the current state of theoretical physics. In particular we outline why we live in exciting times for questions regarding fundamental physics. In the next section we present some open questions which so far lack an answer. A theory that answers all these questions needs in particular to include a quantum theory of gravity. String theory is a candidate for a theory of quantum gravity, that provides answers and insights to these questions and deepens our fundamental understanding of nature also beyond our current mostly perturbative description of physics. Besides this, it has also brought new ideas to mathematics and lead to an intensive exchange between the two fields. We give an outline of some of these astonishing results. The results of this thesis are of interest both from a physics and from a mathematical perspective. The physics question is connected to black holes and a microscopic resolution of the entropy problem. These microstates are provided by Bogomol'nyi–Prasad–Sommerfield (BPS) states that are protected, non-perturbative objects in the context of supersymmetry and can be counted in the framework of string theory by calculating certain partition functions. These partition functions are the generating functions of BPS invariants and provide information about their microscopic degeneracy. From the mathematical point of view these partition functions allow to calculate topological invariants of instanton moduli spaces and furthermore some duality symmetries from physics can be described by using modularity. In the setups we consider modularity is restored at the cost of holomorphy and leads to the expansion of the partition function in terms of quasi modular and mock modular forms. In particular this is studied via the stability of BPS states. The change in the partition function can be calculated by using wall-crossing formulae and using techniques from algebraic geometry.

The current state of theoretical physics

Theoretical physics in the 21st century is build on two strong pillars of 20th century physics: quantum field theory and general relativity [3]. Together they prove the success of the reduction approach to our description of nature. General relativity unifies Newton's theory of gravity with Einstein's theory of special relativity and provides the beautiful idea of geometrisation of physics. Quantum field theory provides the basic building block of our nature in form of the standard model of particle physics.

Again, nature shows its beautiful structure as the standard model can be formulated in terms of gauge theory which is the mathematical formulation of certain symmetries in physics. We will discuss both theories and their experimental implications, starting with quantum field theory and the standard model of particle physics in particular.

The standard model of particle physics

The standard model of particle physics [4, 5] is described as a gauge theory with gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ and describes the fundamental building blocks of nature and their interactions. It consists out of three families of quarks and leptons. Additionally the bosons play the role of the mediator for the corresponding interactions. For the electro-magnetic force we have the photon. The weak interaction is mediated by the $W^{\pm}$ and the Z^0 bosons and the strong interaction by the gluons. The tremendous success of the standard model is the observation of all these particles and the measurement of their properties [6]. However, until recently one building block was missing – the Higgs boson, which is responsible for the masses in the standard model by spontaneous symmetry breaking via the Higgs mechanism [7–10]. The associated Higgs boson seems to be discovered recently at the LHC, though a complete verification and measurement of its properties is still ongoing [11, 12]. This would complete the success of the standard model of particle physics.

The standard model of cosmology

Einstein's theory of general relativity geometrises physics in the sense that gravity is understood as the curvature of space and time and hence gravity becomes a mathematical property. The theory predicted for example the precision of mercury, the effect of gravitational lensing and gravitational time dilation, which allows a more precise usage of GPS.

In recent years a second standard model has been established in cosmology, the so called ΛCDM model. Starting with Hubble's observation that the universe is expanding [13], Einstein's theory of general relativity provides us with a framework to ask and answer questions about the past, present and future of the universe. These models use the cosmological principle, which states that the universe is statistically homogeneous and isotropic, which fits with observations on large scales as can be seen for example from the Sloan Digital Sky survey and the cosmic microwave background. The corresponding geometries of these models are the so called Friedmann-Lemaitre-Robertson-Walker geometries, see for a review [14]. It contains various parameters that have to be measured like the mass density Ω_m, the radiation density Ω_r, the dark energy density Ω_Λ and the curvature. It turns out e.g. by observation of the cosmic microwave background, that our universe is flat and dark energy dominated. In particular the latest results by the Planck satellite provide new precise values of these parameters [15].

Open questions

It seems that we have two successful descriptions of nature, one at the small scale and another one at larger scales. However, both theories have open questions that we want to present here. We start with the standard model of particle physics. Since the discovery of the Higgs boson it seems to be complete. It depends on many parameters, that have to be measured and plugged into the standard model and of course one would like to know if there is a deeper reason for these values. We can also ask the question why our space-time is four dimensional? Why do we have three families of quarks and leptons etc.? Also the question of hierarchy within the standard model is an interesting question. This question is based on the large difference between the electroweak symmetry breaking scale at $\sim$100 GeV and the

Planck scale at 1.22×10^{19} GeV, where all four forces are relevant. It is possible to solve this problem by a lot of fine tuning. However one would expect to find a deeper reason why this should be the case.

From the observational data of cosmology we know that the matter density is measured to be around 0.3 and the baryonic part is around 0.04 and this implies, that most of the contribution to the matter density is due to so-called dark matter. This also has been first observed in the rotational curves of galaxies, which do not follow Kepler's law. However, since it does not have a particle explanation within the standard model, this raises the question what this dark matter is. Furthermore dark energy constitutes around 70% of our universe and there exists no explanation from particle physics what this mysterious dark energy is. Another interesting problem in cosmology is related to the horizon and flatness problem, which are also in the context of fine tuning problems. These two problems deal with the question why the universe is so homogeneous and isotropic even though the corresponding regions have not been in causal contact with each other and why the current density of the universe is so close to the critical density. In order to overcome such problems the concept of inflation [16] was introduced, which states that for a short period of time at the beginning of the universe, it is subject to an exponential expansion. Also for inflation a complete particle physics understanding is still lacking. Another problem in general relativity is the so called black hole entropy problem. Hawking and Bekenstein [17, 18] showed, that black holes can be considered as thermodynamic objects. In particular the horizon area of the black hole is proportional to its entropy. From Boltzmann's law it is known that a macroscopic entropy has a microscopic origin in terms of degeneracies of microstates. Due to the singular structure of a black hole the classical theories break down and demand a new theory.

Furthermore there is an additional tempting task for theoretical physics. The idea of unification has proven to be very successful and of course it is tempting to try to find a unified description of nature, which combines quantum field theory and general relativity in a unified framework. It turns out that general relativity is non-renormalisable and this makes a ad hoc unification difficult and requires new ideas to construct a theory of quantum gravity.

String theory as a theory of quantum gravity

A promising attempt to overcome the quest for a unified theory is string theory [19–26]. In some sense string theory is the crown of reductionism, as it states that the fundamental building block of nature is not a point particle but instead a one dimensional object – the string. With this approach one introduces a minimal length, the string length l_s which provides a Lorentz covariant UV cut-off. We give a short review on the history of string theory. String theory first entered the physics stage within the S-matrix approach of QCD and the Veneziano amplitude [27], which was soon realised to be based on strings instead of particles [28–34]. The incorporation of fermions into string theory included a new symmetry between bosons and fermions – supersymmetry [35–37]. We discuss some of the so far theoretical implications of supersymmetry.

Supersymmetry is a theory that relates bosons to fermions and vice versa [38–42]. It was shown by Coleman and Mandula that the symmetry group of a meaningful physical theory is the direct product of the Poincaré group and some internal symmetry group [43]. Though at first supersymmetry seems to permitted by the Coleman-Mandula theorem, it was shown in [44] that supersymmetry is the only possible extension of the Poincaré symmetry as it is subject to a graded Lie algebra, i.e. a Lie superalgebra which is not subject to the Coleman-Mandula theorem. Supersymmetry itself provides an answer to the hierarchy problem as it cancels quadratic divergences in the one loop corrections to the Higgs mass, see e.g. [45]. Furthermore it also provides candidates for the dark matter particle. This of course would in principle raise the question, if a supersymmetric extension might just be enough to answer some of the problems, we outlined before. However, gravity is still missing in our unified picture. This can be achieved by considering local supersymmetry which gives rise to supergravity, that contains a spin two particle: the graviton. Nevertheless, it can be shown that supergravity does not provide a complete solution as it is non-renormalisable and does not provide chiral fermions and hence a new theory – like string theory – has to be studied. Though string theory was proposed as a theory of quantum gravity quite early [46, 47] it took until 1981 when superstring theory was invented [48–51]. In 1984 the first superstring revolution started, when it was shown that the superstring can be formulated anomaly free in ten dimensions and the gauge group is either SO(32) or $E_8 \times E_8$ which is realised as the heterotic string theory and furthermore Calabi-Yau manifolds can be used to compactify the extra dimensions [52–55]. The idea of compactification dates back to the work of Kaluza and Klein [56–58] who tried to unify gravity with electromagnetism. It turns out that Calabi-Yau manifolds provide good compactification backgrounds as supersymmetry is ensured by the Calabi-Yau condition [59, 60]. The idea of geometrisation of physics and electric-magnetic duality was used in 1994 by Seiberg and Witten to calculate instanton corrections in $\mathcal{N} = 2$ gauge theory [61, 62]. By the time more and more formulations of string theory were discovered and in 1995 it was shown by Witten, that they can all be understood as the limit of one theory, the so called M-theory [63]. Today's picture of string theory consists of five different theories: type I, type IIA, type IIB, heterotic $E_8 \times E_8$ and heterotic SO(32). The various theories are related to each other by a new kind of symmetry, duality symmetries [64–67], which provides a different description of the same theory. This includes for example S-duality which allows to map a strongly coupled theory to a weakly coupled theory. Though classical examples of dualities have already been observed for example in the Ising model, string theory provided insights into the quantum nature of these dualities.

Together with Polchinski's discovery of D-branes the second superstring revolution started [68]. D-branes provide non-perturbative objects and in particular they provide a new perspective on gauge theories in the context of string theory. In 1997 Maldacena discovered the AdS/CFT duality [69], which has lead to interactions with condensed matter physics and provide a theoretical prediction which might be experimentally measured [70].

There are two possible directions in the study of string theory. On the one hand, one can study string phenomenology and try to construct supersymmetric extensions of the standard model or cosmological models from first principles of string theory. On the other hand one might be interested in understanding string theory on a more conceptual level and also to study its mathematical implications. We want to give a short account on these two approaches to study string theory, starting with string phenomenology.

String phenomology

As already mentioned, the aim of string phenomenology is to make contact with the known results from the standard model of elementary particle physics. Of course, a large gap between the Planck scale at 1.22×10^{19} GeV to energies accessible via particle accelerators like the LHC at 14 TeV has to be addressed correctly. Nevertheless, to test string theory one should be able to make predictions of model building beyond the standard model that hopefully will be discovered within the LHC data or cosmological data. One such hint would be the discovery of supersymmetry. Within supersymmetry unification of the gauge couplings is possible and the study of a grand unified theory (GUT) has been of interested since the first GUT with SU(5) symmetry [71]. To obtain supersymmetric extensions of the standard model various methods within string theory have been developed. We want to comment on three of these, namely intersecting D-branes, F-theory and heterotic orbifolds.

Model building via intersecting D-branes is based on the observation, that the intersection of two D-branes supports chiral fermions [72]. The corresponding backgrounds for these kinds of constructions are orientifolds which are Calabi-Yau manifolds modded out by a discrete symmetry. Due to the use of D-branes supporting gauge fields it is possible to construct gauge groups and in particular physical quantities like the Yukawa couplings can be understood in terms of geometry. Hence these models provide an interesting background for phenomenological questions [73–75].

F-theory has been proposed as a non-perturbative version of type IIB string theory [76]. In particular the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of type IIB is geometrised to an elliptic curve/torus such that F-theory becomes twelve dimensional. To make contact with the four-dimensional world the compactification space is now a complex four dimensional Calabi-Yau manifold, where locally at each point of the three dimensional base manifold there is a torus. However, this torus can degenerate and this notes the presence of D7-branes. At the singularities of the elliptic curve one can use an A-D-E classification and therefore the construction of GUT gauge groups is possible [77–82].

The idea of heterotic orbifold model building relies on the fact that the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string theory already provides a gauge group. Via the inclusion

$$\mathrm{SU}(3)_C\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y\subset\mathrm{SU}(5)\subset\mathrm{SO}(10)\subset\mathrm{E}_6\subset\mathrm{E}_8, \tag{1.1}$$

one can see how the gauge group of the standard model is included in one the E_8 gauge groups. This breaking pattern is achieved by first compactifying the heterotic string on a T^6 and then modding out the so called point group $\mathbb{Z}_n$ or $\mathbb{Z}_n\times\mathbb{Z}_m$ on the torus. Due to modular invariance this symmetry has to be embedded in the gauge degrees of freedom and this results into a breaking of the gauge group [83–88]. In recent years by using methods of toric geometry and blow ups, the orbifold singularities have been smoothened.

Mathematics and string theory

Though it's experimental verification is still on the way, string theory provides many interesting insights into mathematics and vice versa. We want to present three examples for this fruitful partnership: mirror symmetry, Chern-Simons theory and knot theory as well as modular forms as a third example.

We start with mirror symmetry. As we already mentioned, to make contact with the four dimensional world the ten dimensional theory must be compactified on a Calabi-Yau threefold, i.e. a six real dimensional manifold. The statement of mirror symmetry is, that type IIA string theory compactified on a Calabi-Yau threefold M gives the same theory as type IIB string theory on a Calabi-Yau threefold W [89, 90]. The two manifolds M and W form a mirror pair and in particular the Hodge numbers get exchanged, i.e. $h^{2,1}(M) = h^{1,1}(W)$ and $h^{1,1}(M) = h^{2,1}(W)$. Mirror symmetry allows to perform calculations in a simpler setup and then via mirror symmetry to obtain the result on the other side of the duality. In particular the use of mirror symmetry allowed to calculate the number of rational curves in the quintic Calabi-Yau threefold [91, 92]. These results were proven to be mathematical correct by Giventhal and Lian, Liu and Yau [93, 94]. It turns out that Gromov-Witten (GW) invariants can be calculated in the framework of topological string theory [95, 96], where the GW invariants appear in the free energy of the A-model. Topological string theory calculates the free energy $F(g_s,t)$ of a Riemann surface $\Sigma_{g,h}$ of genus g and h holes corresponding to open topological string theory or a compact Riemann surface Σ_g corresponding to closed topological string theory and its embedding into the Calabi-Yau target space. We denote by g_s the string coupling and by t background moduli. The free energy is subject to a genus expansion of the form

$$F(g_s,t)=\sum_{g=0}^{\infty}g_s^{2g-2}F^{(g)}(t), \tag{1.2}$$

where the $F^{(g)}(t)$ are called topological amplitudes. Topological string theory comes in two versions, the A- and the B-model [97]. The calculation of the free energy of the A-model is performed by doing the calculation on the B-model side, and then using mirror symmetry to map back to the A-model. In physics it is conjectured in the Strominger-Yau-Zaslow (SYZ) conjecture, that mirror symmetry is a special kind of T-duality [98]. From a mathematics point of view a full understanding of mirror symmetry is still lacking, though in the context of homological mirror symmetry the idea of equivalent categories is formulated [99] and has been checked for various classes. The calculation of the topological amplitudes $F^{(g)}(t)$ has been performed up to genus 52 [100] which is possible due to the fact, that the topological amplitudes are subject to a holomorphic anomaly equation [101] which provides a recursive structure in the genus

$$\bar{\partial}_{\bar{i}} F^{(g)}(t) = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\left(\sum_{n=1}^{g-1} D_j F^{(n)}(t) D_k F^{(g-n)}(t) + D_j D_k F^{(g-1)}(t)\right). \tag{1.3}$$

This recursive structure can be understood as the degeneration of a genus g surface via splitting and pinching. In this context the background independence of topological string theory has also been observed [102].

Furthermore topological string theory is by a chain of dualities connected to Chern-Simons theory. Chern-Simons theory itself can be used to calculate knot invariants, like for example the Jones polynomial. It was Edward Witten, who won the fields medal in 1990 for his work on the Jones polynomial and its relation to Chern Simons theory [103].

In the last example we wish to address modular forms. The definition of a modular form is that of a function $f : \mathbb{H} \to \mathbb{C}$ that transforms in a covariant way under transformations of $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \mathrm{SL}(2,\mathbb{Z}). \tag{1.4}$$

In particular a modular form is periodic and can be expanded in a Fourier series

$$f(\tau) = \sum_n a_n q^n, \quad q = e^{2\pi i \tau}, \tag{1.5}$$

which is a useful property for counting problems to express certain generating functions. In the context of topological string theory, the topological amplitudes turn out to be either modular, quasi-modular or almost-holomorphic modular forms under a subgroup of $\mathrm{Sp}(b_3(X),\mathbb{Z})$ of the target space Calabi-Yau manifold [104]. Monstrous moonshine is another example for the interplay of modular forms between mathematics and string theory. It was noted that the Fourier coefficients of the j-function

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots \tag{1.6}$$

carry information about the dimensions of representations of the Monster group. This was proved by using the language of vertex operator algebras and an explicit representation is given by the bosonic string compactified on the Leech lattice [105–107]. In a nutshell we see that the impact of mathematics on string theory and vice versa is very strong.

Black holes in string theory

We have explained that string theory is capable to provide a solution to the problems of the standard model of particle physics and those of cosmology. One very important intrinsic test of string theory as a theory of quantum gravity is its ability to solve the black hole entropy problem, which combines macroscopic and microscopic questions. In fact, for extremal supersymmetric Reissner-Nordström black holes Strominger and Vafa showed that string theory provides a microscopic explanation of the macroscopic entropy [108]. An extremal Reissner-Nordström black hole has the property, that its mass is determined by the charges of the black hole. The setup of Strominger and Vafa is that of type II string theory on $\mathrm{K3} \times S^1$ giving rise to a black hole with axionic charge p and electric charge q. From the low energy action it is possible to identify this setup with an Reissner-Nordström black hole in five dimensions and the macroscopic area is given by

$$S_{\text{macro}} = 2\pi \sqrt{\frac{pq^2}{2}}. \tag{1.7}$$

The key idea in their solution is to realise that the microstates of an extremal supersymmetric black hole are in one to one correspondence with BPS states, whose mass is determined by the magnitude of the central charge. For the class of BPS states some operators of the supersymmetry algebra become trivial, which makes BPS states interesting as they represent a protected quantity in case variations of the parameter space of the setup considered. It is possible to perform a precise counting of these BPS states as they correspond to D-brane charges [109]. Taking the limit of small K3 compared to S^1 leads to a description as a supersymmetric sigma model where the target space is the symmetric product of $\frac{1}{2}(q^2+1)$ K3's [110]. By conformal field theory (CFT) arguments it is now possible to calculate the microscopic entropy which reads

$$S_{\text{micro}} = 2\pi \sqrt{p\left(\frac{1}{2}q^2+1\right)}, \tag{1.8}$$

and is equal to the macroscopic value for the case of large charges.

With this remarkable result the study of microstates and black holes is continued. In particular the task to write down a generating function which counts the microstates is of particular interest. For the case of $\mathcal{N}=4$ extremal black holes arising from compactifications of type II on $\mathrm{K3}\times T^2$ the generating function is given as a Siegel modular form $\Phi_{10}(\Omega)$ [111], where Ω can be understood as the period matrix of a genus two surface. It can be shown, that in the case of a degeneration of the genus two surface into two genus one surfaces the heterotic result is recovered, which is given for either the magnetic or the electric charges by

$$\frac{1}{\eta^{\chi(K3)}} = \frac{1}{\eta^{24}}. \tag{1.9}$$

This is again an impressive example where modular forms and string theory meet. In particular the mathematical origin as the Hilbert scheme of K3 was proven by Göttsche [112].

Another way to calculate the microscopic degeneracies is by using topological string theory, where the ideas have been first applied for black holes in M-theory. Here one considers an M-theory compactification on a Calabi-Yau threefold such that the corresponding BPS black hole is in five dimensions. The corresponding BPS states carry charge due to the reduction of the M-theory 3-form. Furthermore the BPS states also carry spin (j_L, j_R) due to the little group $\mathrm{SO}(4) = \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ but the BPS condition forces one of them to vanish. The counting of BPS states is then achieved by the so called elliptic genus Z

$$Z = \mathrm{Tr}\,(-1)^{F_R} q^{L_0} \bar{q}^{\bar{L}_0}, \tag{1.10}$$

where F_R denotes the right moving fermion number and $L_0, \bar{L}_0$ the zero modes of the Virasoro algebra. The image of holomorphic maps provide cycles that can be wrapped by branes to give a BPS state and these holomorphic maps are counted by the A-model. The topological amplitude $F^{(g)}$ represents in the four-dimensional effective action of type IIA on a Calabi-Yau manifold the coefficient of the gravitational correction to the scattering of $2g-2$ graviphotons

$$\int d^4x\, F^{(g)} R_+^2 F_+^{2g-2}. \tag{1.11}$$

Using a Schwinger loop computation and integrating out the BPS states it can be shown, that the elliptic genus exactly captures these R_+^2 corrections and hence topological string theory can be used in five dimensions to count black hole microstates [113,114]. It is also possible to count black hole microstates in four dimensions. The entropy is simply determined by the volume of the Calabi-Yau manifold,

$$S(\Omega) = \frac{i\pi}{4} \int_X \Omega \wedge \bar{\Omega}, \tag{1.12}$$

with Ω the holomorphic $(3,0)$ form. By using the attractor mechanism it is possible to show that the moduli, i.e. vector multiplet scalars, of the compactification at the horizon are locally independent of the moduli at infinity and therefore only depend on the charges of the black hole, which allows to fix Ω [115–119]. An interesting observation between the partition function of the black hole and the topological string partition function was made [120] as the entropy of the black hole almost looks like the imaginary part of the B-model prepotential and in fact by applying a Legendre transform of $\operatorname{Im} F^{(0)}$ it is possible to state the Ooguri-Strominger-Vafa (OSV) conjecture

$$Z_{\mathrm{BH}} = |Z_{\mathrm{top}}|^2. \tag{1.13}$$

Stability of BPS states

Having outlined the bigger picture, we want make contact with the topic of this thesis which deals with the stability of BPS states and provide further details. Since the BPS states depend on the background moduli of the theory, one might wonder how they change if the moduli are varied. Denef studied the correspondence between low-energy effective supergravity solutions and BPS states from type II string theory on a Calabi-Yau manifold [121]. Multi-centred black holes solutions exists, e.g. two-centred solutions with central charge

$$Z(\Gamma, t) = Z(\Gamma_1, t) + Z(\Gamma_2, t). \tag{1.14}$$

When moving in the moduli space, there might happen a decay in the charges of the form $\Gamma \to \Gamma_1 + \Gamma_2$ at the wall of marginal stability. By looking at the binding energy, it is possible to calculate the position, where the two-centred solution becomes marginal stable. Of course the question is, if it is possible to calculate the microscopic degeneracy when crossing a wall of marginal stability. The related phenomenon is the so called wall-crossing effect, which from a physics point of view describes the decay of a BPS state. This decay happens if the central charges of the decay products align, i.e. charge and energy conservation hold. The change in the degeneracy is calculated by wall-crossing formulas which were discussed in physics and in mathematics [122–135].

In mathematics it has been observed that for surfaces with $b_2^+(P) = 1$ the Donaldson-Thomas invariants change [136]. The generating function of black hole microstates can often be expressed in terms of modular forms. However, if we encounter a decay then of course the question is how this is resembled by the generating function. In particular this implies holomorphic anomalies in the generating function. For example these have been observed in the context of multiple M5 branes and the modified elliptic genus, where the M5 branes wrap a divisor[1] P inside a Calabi-Yau threefold [137]. This setup has a CFT description for one M5 brane [138]. For the case of r M5 branes it was shown to be dual to $\mathcal{N} = 4$ $U(r)$

[1] We will use the expression divisor and surface interchangeable.

topological super Yang-Mills (SYM) theory [139, 140]. In particular the non-holomorphic contributions arise from reducible connections

$$\mathrm{U}(r) \to \mathrm{U}(m)\mathrm{U}(r-m), \tag{1.15}$$

which corresponds to an anomaly in the context of E-strings [141]. In the gauge theory setup the generating functions of Euler numbers of instanton moduli spaces are calculated, which were also studied in mathematics [142–144]. In this thesis we study the relation between wall-crossing and non-holomorphicity and relate them to each other. In particular we use the wall-crossing formula by Göttsche, where the moduli dependence of the BPS generating function is captured by an indefinite theta function. It turns out that for the case $b_2^+(P) = 1$ and $r = 2$ the different sectors of the partition function need a non-holomorphic completion to assure invariance under S-duality. These functions were established in [145]. The origin of the holomorphic anomaly is due to the formation of bound states. These non-holomorphic completions can be described by mock modular forms that were developed by Zwegers [146–148]. Göttsche's indefinite theta function is also an example of these functions. Mock modularity was also used in a physical context studying the wall-crossing of degeneracies of $\mathcal{N} = 4$ dyons [149,150]. Also for theories with $\mathcal{N} = 2$ supersymmetry mock modularity was discussed [151–160].

For studying D4 brane charges greater than two, we want to study the relation to topological string theory in the context of elliptically fibred Calabi-Yau threefolds. Topological string theory on local Calabi-Yau manifolds has been a remarkable success story. It counts the open and closed instantons corrections to topological numbers, which can be seen as an extension from classical geometry to quantum geometry. By now we can solve it in very different ways, namely by localisation, by direct integration of the holomorphic anomaly equations, by the topological vertex [161] or by the matrix model techniques in the remodelled B-model [162]. Topological string theory on local Calabi-Yau manifolds gives deep insights in the interplay between large N gauge theory/string theory duality, mirror duality, the theory of modular forms and knot theory and is by geometric engineering [163] intimately related to the construction of effective $\mathcal{N} = 2$ and $\mathcal{N} = 1$ rigid supersymmetric theories in four dimensions.

On global, i.e. compact, Calabi-Yau manifolds, which give rise to $\mathcal{N} = 2$ and $\mathcal{N} = 1$ effective supergravity theories in four dimensions, the situation is less understood. Direct integration extends the theory of modular objects to the Calabi-Yau spaces and establishes that closed topological string amplitudes can be written as a polynomial in modular objects, but the boundary conditions for the integration are in contrast to the local case not completely known.

In [164] mirror symmetry was made local in the decompactification limit of fibred Calabi-Yau threefolds. Here we want to do the opposite and study how the quantum geometry extends from the local to the global case, when a class of local Calabi-Yau geometries is canonically compactified by an elliptic Calabi-Yau fibration with projection $\pi : M \to B$. This easy class of local to global pairs, will be described to a large extend by complete intersections in explicit toric realizations. If the elliptic fibration has only I_1 fibres the classical cohomology of M is completely determined by the classical intersection of the base B and the number of sections, which depends on the Mordell Weyl group of the elliptic family.

The decisive question to which extend this holds for the quantum geometry is addressed in this thesis using mirror symmetry. The instanton numbers are counted by (quasi)-modular forms of congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ capturing curves with a fixed degree in the base for all degrees in the fibre. The weights of the forms depend on the genus and the base class. This structure has been discovered for elliptically fibred surfaces in [165] and for elliptically fibred threefolds in [166]. We establish here a holomorphic anomaly equation based on the non-holomorphic modular completion of the quasi-modular forms which is iterative in the genus, as in [101], and also in the base classes generalising [139, 167].

Our construction can be viewed also as a step to a better understanding of periods and instanton corrections in F-theory compactifications and a preliminary study using the data of [168–170] reveals that the structure at the relevant genera $g = 0, 1$ extends.

On elliptic Calabi-Yau fibrations, double T-duality on the elliptic fibre (Fourier-Mukai transform) [139, 171–173] transforms D2 branes wrapped on base classes into D4 branes which also wrap the elliptic fibre and vice versa. The D4 brane holomorphic anomaly is therefore related to the one of GW theory for these geometries. Moreover, the mirror periods provide predictions for D4 brane BPS invariants which correspond to those of (small) black holes in supergravity.

Outline of this thesis

In chapter 2 we present an overview on various methods to count BPS states in string theory. We start by introducing the BPS property for $\mathcal{N} = 2$ supersymmetry and present two classes of $\mathcal{N} = 2$ SUSY theories: gauge theories, where we discuss the Seiberg-Witten solution and supergravity with black holes. The next section is devoted to stability conditions of BPS states. We review the Kontsevich-Soibelmann wall-crossing formula and give a brief account on the mathematical notion of stability conditions. We also summarise some implications of wall-crossing from the literature. For the example of SU(2) Seiberg-Witten theory we illustrate some of the aspects discussed before. Modular forms and their properties are addressed in the following section. This is followed by a review of topological string theory, mirror symmetry and the holomorphic anomaly equations of Bershadsky-Cecotti-Ooguri-Vafa (BCOV). We finish this chapter with a review of the setup that allows us to count D4-D2-D0 states and we review the effective description of multiple M5 branes.

In chapter 3 we discuss the construction of Calabi-Yau manifolds via toric geometry. As we will deal with complex surfaces that appear as a divisor embedded in our Calabi-Yau manifold, we give a short review of their classification. A review on elliptic curves stating Kodaira's classification and some properties of elliptically fibred Calabi-Yau threefolds finish this chapter.

In chapter 4 we discuss our results on the wall-crossing holomorphic anomaly of multiple M5 branes and their mock modularity. We use the Kontsevich-Soibelmann wall-crossing formula to derive the wall-crossing formula for the D4-D2-D0 system and state the relation to Göttsche's wall-crossing formula that is captured by an indefinite theta function, i.e. a mock modular form. Using the regularisation procedure of Zwegers restores the modular invariance of the theory but lacks holomorphy. For the case of two M5 branes we proof the holomorphic anomaly at the attractor point. We calculate the generating function of D4-D2-D0 states with at most two D4 branes for various surfaces at the end of this chapter. We speculate on possible extensions to higher D4 brane charge. The results of this chapter were published in [1].

In chapter 5 we discuss the quantum geometry of elliptic fibrations. We start by reviewing the B-model approach to these compactification spaces and present our finding of a new holomorphic anomaly equation. Then we discuss the origin of modularity by a study of the monodromy group. We derive our holomorphic anomaly equation by using mirror symmetry for the example of the base $\mathbb{F}_1$ at genus zero. A second derivation of the holomorphic anomaly equation is based on a derivation of the BCOV holomorphic anomaly equations. Then we use T-duality to relate the D4 brane charge to D2 brane charge in order to establish the relation between the anomaly for D4-D2-D0 systems and the topological string anomaly. We discuss higher rank branes on elliptic Calabi-Yau fibrations and compare the predictions from the periods for D4-brane BPS invariants with existing methods in the literature for the computation of small charge BPS invariants. The predictions of the periods are in many cases compatible with these methods. We specialise to the elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$. The periods of its mirror geometry provide the BPS invariants of D4-branes on the rational elliptic surface (also known as $\frac{1}{2}K_3$)

as proposed originally in [139]. We revisit and extend the verification of this proposal for ≤ 3 D4 branes using algebraic-geometric techniques. The results of this chapter were published in [2].

Publications

This thesis is based on the following publications of the author:

- M. Alim, B. Haghighat, M. Hecht, A. Klemm, M. Rauch and *T. Wotschke*, "Wall-crossing holomorphic anomaly and mock modularity of multiple M5-branes," arXiv:1012.1608 [hep-th]. Accepted for publication by *Communications in Mathematical Physics*
- A. Klemm, J. Manschot and *T. Wotschke*,"Quantum geometry of elliptic Calabi-Yau manifolds," arXiv:1205.1795 [hep-th]. Published in *Communications in Number Theory and Physics*, Volume 6, Number 4.

CHAPTER 2

Counting BPS states in string theory

In this chapter we want to give some preliminary background material for the central results about BPS states and their invariants in this thesis. We want to study the dependence of background parameters, like moduli fields and coupling constants, of theories. Moduli fields arise for example in the compactification of string theory on a Calabi-Yau manifold as vector or hyper multiplets. The understanding how correlators of the theory change if the background parameters change helps us to understand and maybe solve the theory. An example of such an understanding is in the case of topological field theories [97]. More generally supersymmetric field theories allow for an intensive study of this question. This is because they imply for example duality symmetries, like electric magnetic or strong to weak coupling symmetries and we have a holomorphic structure that helps us to apply ideas from complex analysis [174]. In particular theories with $\mathcal{N} = 2$ supersymmetry in four dimensions are at the border between being solvable and trivial as its features can be captured by a holomorphic quantity: the prepotential $\mathcal{F}$. As such, it is between the self-dual $\mathcal{N} = 4$ theory and $\mathcal{N} = 1$ theories where only at some regions it is possible to apply the powerful tools of complex analysis. BPS states are well-defined all over moduli space and provide a powerful tool into non-perturbative physics. However, a variation of the moduli of the theory might lead to certain decays of these states and are captured by the wall-crossing effect. We give an introduction to the BPS condition in the case of $\mathcal{N} = 2$ supersymmetry. Next we show the application of BPS in the context of two $\mathcal{N} = 2$ theories: gauge theories and extremal black holes. In particular the counting of black hole microstates unifies ideas from these two examples. Then we discuss the stability of BPS states and the formulas that allow to describe the change of the BPS degeneracies. In the case of $\mathcal{N} = 2\,\mathrm{SU}(2)$ Seiberg Witten theory some of these ideas can be illustrated. The generating function of BPS states is expressed by modular forms, which are discussed in section 2.6. A powerful method to count BPS states is topological string theory. We review the construction and the holomorphic anomaly equations. The chapter ends with a discussion of D4-D2-D0 BPS states, which are the central object of our study. The corresponding setup with multiple M5 branes giving rise to D4-D2-D0 BPS states has two interesting limits: the Maldacena-Strominger-Witten (MSW) CFT of a black hole [138] and as a $\mathcal{N} = 4$ U(r) topological SYM theory [140].

2.1 BPS states and the supersymmetry algebra

In the following we consider $\mathcal{N}=2$ supersymmetry (SUSY)[1] in four dimensions and discuss the representation theory of the SUSY algebra $\mathfrak{g}$. We mainly follow [175, 176] and further useful references about supersymmetry include [45, 177–180]. The SUSY algebra $\mathfrak{g}$ splits into a bosonic part $\mathfrak{g}_0$ and a fermionic part $\mathfrak{g}_1$. The fermionic part consists out of the SUSY charges which are denoted by Q_α^A, where $\alpha=1,2$ and $\dot{\alpha}=1,2$ denote the spinor indices and $A=1,2$ the number of supersymmetries. Since the SUSY charge is a fermionic operator it squares to zero. The conjugated SUSY charge is denoted by

$$(Q_\alpha^A)^\dagger = \bar{Q}_{\dot{\alpha}A}. \tag{2.1}$$

In addition we raise and lower indices by ϵ_{AB} with $\epsilon^{12}=-\epsilon^{21}=\epsilon_{21}=-\epsilon_{12}=1$, such that

$$\bar{Q}_{\dot{\alpha}A} = \epsilon_{AB}\bar{Q}^B_{\dot{\alpha}}. \tag{2.2}$$

We denote the four-momentum by $P_\mu=(E,\vec{p})^T$ and the Pauli-matrices by $\sigma^\mu_{\alpha\dot{\beta}}$. The Pauli matrices are given as follows

$$\begin{aligned}\sigma^0 &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\\ \sigma^2 &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.\end{aligned} \tag{2.3}$$

Since we have extended SUSY we also have a central charge $Z\in\mathbb{C}$ and we often write

$$Z = |Z|e^{i\alpha}. \tag{2.4}$$

We summarise the building blocks of the $\mathcal{N}=2$ SUSY algebra in table 2.1.

Name	comment
SUSY algebra $\mathfrak{g}$	$\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$.
bosonic algebra $\mathfrak{g}_0$	$\mathfrak{g}_0=\mathrm{poin}(3,1)\oplus\mathfrak{su}(2)_R\oplus\mathfrak{u}(1)_R\oplus\mathbb{C}$
poin(3, 1)	four-dimensional Poincaré algebra
$\mathfrak{su}(2)_\mathrm{R}$	R-symmetry rotates the SUSY charges. This is often not present in the case of supergravity.
$\mathfrak{u}(1)_\mathrm{R}$	Q_α^A has charge $+1$ and $\bar{Q}^A_{\dot{\alpha}}$ has charge -1. The $\mathfrak{u}(1)_\mathrm{R}$ can be broken
$\mathbb{C}$	central charge Z
fermionic algebra $\mathfrak{g}_1$	generated by Q_α^A and $\bar{Q}^A_{\dot{\alpha}}$

Table 2.1: The $\mathcal{N}=2$ SUSY algebra $\mathfrak{g}$ and its components.

[1] For $\mathcal{N}>2$ the discussions follows a similar pattern as we will get $\frac{\mathcal{N}}{2}$ central charges Z_i that arise from the antisymmetric central charge matrix Z^{IJ}. For brevity we focus on $\mathcal{N}=2$.

The supersymmetry generators Q are subject to the following anti-commutation relations

$$\begin{aligned} \left\{Q^A_\alpha, Q_{\dot\beta B}\right\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^A_B, \\ \left\{Q^A_\alpha, Q^B_\beta\right\} &= 2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z}, \\ \left\{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\right\} &= -2\epsilon_{\dot\alpha\dot\beta}\epsilon_{AB} Z. \end{aligned} \tag{2.5}$$

We want to derive the BPS bound satisfied by SUSY states, i.e. their mass is bounded from below by the magnitude of the central charge Z of the supersymmetry algebra

$$M \geq |Z|. \tag{2.6}$$

For this we discuss the representation theory of the SUSY algebra. Within the representation theory we distinguish between massive and massless representations. Since we are interested in massive representations, i.e. representations satisfying $P^2 = M^2$, we can always choose a rest frame such that the only non-trivial component of the four-momentum is P^0. For deriving the BPS bound we define the following set of operators R^A_α and T^B_β which will lead to a split of the fermionic part of the SUSY algebra $\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$. They are defined as follows

$$\begin{aligned} R^A_\alpha &= \kappa^{-1} Q^A_\alpha + \kappa\sigma^0_{\alpha\dot\beta}\bar{Q}^{\dot\beta A}, \\ T^A_\alpha &= \kappa^{-1} Q^A_\alpha - \kappa\sigma^0_{\alpha\dot\beta}\bar{Q}^{\dot\beta A}. \end{aligned} \tag{2.7}$$

In the above equations we have introduced a phase κ that is related to the action of the R-symmetry $\mathfrak{su}(2)_\mathrm{R}$, which rotates the supersymmetry charges among each other with rotation $\mathcal{R} = \kappa^2$. In the following we set $\mathcal{R} = -e^{-i\alpha}$ as this will give us the the bound (2.6). This is also the sharpest bound as otherwise we would obtain $\mathrm{Re}\left(\frac{Z}{\mathcal{R}}\right)$ as a bound instead of $|Z|$. Obviously this is not the lowest possible bound and we can use the R-symmetry appropriately. In a next step, we evaluate the anti-commutators of R^A_α and T^a_α by using (2.5) and we obtain

$$\begin{aligned} \{R^A_\alpha, R^B_\beta\} &= 4(M - |Z|)\epsilon_{\alpha\beta}\epsilon^{AB}, \\ \{T^A_\alpha, T^B_\beta\} &= -4(M + |Z|))\epsilon_{\alpha\beta}\epsilon^{AB}. \end{aligned} \tag{2.8}$$

A short calculation shows that

$$(R^1_1 + (R^1_1)^\dagger)^2 = 4(M - |Z|), \tag{2.9}$$

and hence since the left-hand side of (2.9) is the square of a hermitian operator, this implies that the right-hand side is bounded to be non-negative. This implies the BPS bound (2.6), that we wanted to show.

With the BPS bound at hand, the representation theory of the massive representation can be further distinguished into two classes. If the bound is satisfied, i.e. $M = |Z|$, we consider so called *short* or *BPS representations* and for the case that $M > |Z|$ we have *long representations*[2].

[2] In the case of $\mathcal{N} > 2$ extended supersymmetry we have more central charges which allow a similar classification of the representation theory. Depending on how many of the operators become trivial, the multiplets are shortened accordingly.

Long representations

We first give a brief discussion on the long representations. In this case R^A_α and T^A_α form two seperate Clifford algebras $\mathfrak{g}^+_1$ and $\mathfrak{g}^-_1$. We comment on the representation theory of $\mathfrak{g}^+_1$ since the representation theory for $\mathfrak{g}^-_1$ is similiar. For $\mathfrak{g}^+_1$ it is possible to construct a four dimensional representation ρ_{hh} with a heighest weight state $|\Omega\rangle$ such that $R^A_1|\Omega\rangle = 0$. Then it is possible to act with each of the remaining operators on $|\Omega\rangle$ cf. table 2.2 for details.

level	state
0	$|\Omega\rangle$ with $R^1_1|\Omega\rangle = 0$
1	$R^1_2|\Omega\rangle, R^2_2|\Omega\rangle$
2	$R^1_2 R^2_2|\Omega\rangle$

Table 2.2: The states of the four dimensional representation ρ_{hh} for the case that R^1_1 annihilates the ground state $|\Omega\rangle$.

Therefore, the general representation theory for the long representations is of the form

$$\rho_{\text{long}} = \rho_{hh} \otimes \rho_{hh} \otimes \mathfrak{h}, \tag{2.10}$$

where $\mathfrak{h}$ denotes a representation of the bosonic little group $\mathfrak{so}(3) \oplus \mathfrak{su}(2)_R$.

Short representations

In the case that the BPS bound (2.6) is satisfied we obtain the BPS or short representations. Note, that in this case from (2.8) the relation between the R^A_α becomes trivial, which leads to a shortening of the corresponding multiplets. This shortening is the decisive property of BPS states, since the mass now equals the central charge, which is a conserved property all over moduli space. Hence, the R^A_α are called preserved supersymmetries and the T^A_α are broken supersymmetries. The non-trivial part of the short representation follows from the representation theory of the broken supersymmetries T^A_α and has the following form

$$\rho_{\text{short}} = \rho_{hh} \otimes \mathfrak{h}, \tag{2.11}$$

where again $\mathfrak{h}$ is a representation of $\mathfrak{so}(3) \oplus \mathfrak{su}(2)_R$. We summarise some of the choices for $\mathfrak{h}$ and the resulting representations in table 2.3. At this point we end our discussion of the supersymmetry multiplets and refer the reader to [45, 175] for more details on supersymmetry.

2.2 BPS states and gauge theories

We want to study BPS states in the context of $\mathcal{N} = 2$ gauge theory with a gauge group G of rank r. The moduli space can be distinguished between the Higgs branch where the gauge group is completely broken, and the Coulomb branch $\mathcal{B}$ of the vector multiplets, were the gauge group G breaks down to $\mathrm{U}(1)^r$. We focus in the following on the Coulomb branch where we can distinguish between electric charges q and magnetic charges p which are combined in a charge vector $\Gamma = (p, q)^T$. The Coulomb

long representations	
half-hypermultiplet	$\rho_{\mathrm{hh}} = \left(0;\frac{1}{2}\right)\oplus\left(\frac{1}{2};0\right)$
$\mathfrak{h} = (0,0)$	$\rho_{\mathrm{long}} = 2\,(0;0)\oplus(1;0)\oplus(0;1)\oplus 2\left(\frac{1}{2};\frac{1}{2}\right)$
short representations	
$\mathfrak{h} = (0,0)$	ρ_{hh}
$\mathfrak{h} = \left(\frac{1}{2};0\right)$ vector multiplet	$\rho_{\mathrm{vm}} = (0;0)\oplus\left(\frac{1}{2};\frac{1}{2}\right)\oplus(1,0)$

Table 2.3: Representations for different choices of the algebra $\mathfrak{h}$.

branch has the structure of a special Kähler manifold [181, 182]. The Dirac quantisation condition [183, 184] states that the symplectic charge product has to be an integer number

$$\langle\Gamma_1,\Gamma_2\rangle = \Gamma_1^T\cdot\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}\cdot\Gamma_2 = p_1\cdot q_2 - p_2\cdot q_1 = -\langle\Gamma_2,\Gamma_1\rangle\in\mathbb{Z}. \tag{2.12}$$

This leads to the fact that the charges Γ belong to a charge lattice Λ. We introduce a Darboux basis for this charge lattice $\{\alpha_I,\beta^I\}$ with $I = 1,\dots,r$ with the only non-trivial symplectic product being

$$\langle\alpha_I,\beta^J\rangle = \delta_I^J. \tag{2.13}$$

In order to determine the BPS spectrum, we have to determine the central charge $Z = Z(\Gamma,t)$, which depends on the charges Γ and the position on the Coulomb branch $t\in\mathcal{B}$. It was shown that the central charge Z is holomorphic on $\mathcal{B}$ [174], which allows for an explicit determination of the central charge Z.

We introduce the corresponding one-particle Hilbert space $\mathcal{H}_t$ of charges Γ at t in $\mathcal{B}$ by

$$\mathcal{H}_t = \bigoplus_{\Gamma\in\Lambda}\mathcal{H}_{t,\Gamma}. \tag{2.14}$$

The corresponding central charge function can then be understood as

$$Z(\Gamma,t):\mathcal{H}_{t,\Gamma}\to\mathbb{C} \tag{2.15}$$

and it is linear in the charges, i.e.

$$Z(\Gamma_1+\Gamma_2,t) = Z(\Gamma_1,t)+Z(\Gamma_2,t). \tag{2.16}$$

Since the charges $\Gamma = p^I\alpha_I - q_I\beta^I$ are elements of a lattice Λ, it is sufficient to evaluate the central charge $Z(\Gamma,t)$ on the basis elements $\{\alpha_I,\beta^I\}$ of this lattice

$$Z(\alpha_I,t) = a^I,\quad Z(\beta^I,t) = a_{D,I}, \tag{2.17}$$

which allows us to express the central charge as follows

$$Z(\Gamma,t) = q_I a^I + p^I a_{D,I}. \tag{2.18}$$

Seiberg and Witten showed [61, 62] that the low-energy abelian gauge theory consists out of a self-dual gauge theory, where the self-dual two-form field strength F has the following form, where

$$F = \alpha_I F^I - \beta^I G_I \in \Omega^2(\mathbb{R}^{1,3}) \otimes \Lambda_{\mathbb{R}}, \quad \Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}, \tag{2.19}$$

and it is subject to

$$dF = 0. \tag{2.20}$$

Given a Darboux basis, it is possible to split the charge lattice Λ into two components, which upon complexification are spanned by f_I and $\bar{f}_I$, which can be expressed as follows

$$f_I = \alpha_I + \tau_{IJ}\beta^J, \quad \bar{f}_I = \alpha_I + \bar{\tau}_{IJ}\beta^J, \quad \tau_{IJ} = \mathrm{Re}(\tau_{IJ}) + i\,\mathrm{Im}(\tau_{IJ}) = X_{IJ} + iY_{IJ}. \tag{2.21}$$

The symplectic charge product of the f_I's implies that $\tau_{IJ} = \tau_{JI}$. By use of $\mathcal{N} = 2$ supersymmetry it is possible to determine τ_{IJ} as derivatives of the holomorphic prepotential $\mathcal{F}(a)$ with respect to a^I

$$\tau_{IJ} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} = \frac{\partial a_{D,I}}{\partial a^J}, \tag{2.22}$$

where we have used the supersymmetry constrained $a_{D,I} = \frac{\partial \mathcal{F}}{\partial a^I}$. Hence, the bosonic low-energy effective action is given as

$$S = \frac{1}{4\pi}\left(\int_{M^4} -\mathrm{Im}\tau_{IJ}\left(da^I \star d\bar{a}^J + F^I \star F^J\right) + \mathrm{Re}\tau_{IJ} F^I F^J\right), \tag{2.23}$$

where we denote by M^4 the four-dimensional space-time. We will discuss these results for the explicit example of $\mathcal{N} = 2$ SU(2) Seiberg-Witten theory in section 2.5.

Type II string theory on a Calabi-Yau threefold

We finish this chapter with some general remarks about the compactification of type II on a Calabi-Yau threefold and its moduli. This theory has $\mathcal{N} = 2$ supersymmetry in four dimensions due to the SU(3) holonomy of X so that from the original 32 supercharges only eight remain. Depending on whether we take type IIA or type IIB we find the following number of hyper and vector multiplets in table 2.4.

theory	# hyper multiplets	# vector multiplets	# gravity multiplets
IIA	$h^{2,1}(X) + 1$	$h^{1,1}(X)$	1
IIB	$h^{1,1}(X) + 1$	$h^{2,1}(X)$	1

Table 2.4: The $\mathcal{N} = 2$ super multiplets of type II string theory compactified on a Calabi-Yau threefold X.

For the case of type IIA we get $h^{1,1}$ vector multiplets where the bosonic content is given by vectors from the RR threeform $C_{\mu i\bar{j}}$, scalars from the reduction of the metric $G_{i\bar{j}}$ and the B-field $B_{i\bar{j}}$. The bosonic content of the $h^{2,1}+1$ hyper multiplets arise from $C_{ij\bar{k}}$ and G_{ij} and the complex scalar given by the axion dilaton $S = a + ie^{-\Phi}$. The bosonic content of the $h^{1,1}+1$ vector multiplets is due to the RR 4 form $C_{\mu ij\bar{k}}$ and the metric G_{ij} and the axion dilaton. In the $h^{2,1}$ hyper multiplets we find contributions from the RR 4-form $C_{\mu\nu i\bar{j}}$, the metric $G_{i\bar{j}}$, the B-field $B_{i\bar{j}}$ and the RR 2-form $C_{i\bar{j}}$.

Locally the moduli space $\mathcal{M}$ splits into the vector multiplet moduli space $\mathcal{M}_{\text{vector}}$ and the hyper multiplet moduli space $\mathcal{M}_{\text{hyper}}$.

Geometric engineering

It is possible to construct $\mathcal{N} = 2$ gauge theories via so called geometric engineering of type II string theory compactified on a local Calabi-Yau threefold X with gravity decoupled [163, 185]. Let Ω be the holomorphic $(3,0)$ form on the Calabi-Yau manifold, then the central charge $Z(\Gamma, t)$ is expressed as follows

$$Z(\Gamma, t) = e^{\frac{K}{2}} \int_C \Omega, \tag{2.24}$$

with the three-cycle given as $C = p^I \alpha_I - q_I \beta^I$ and K denoting the Kähler potential see appendix A.3. In such a compactification setup we have the following identifications of table 2.5 for the charge lattice and the rank r of the gauge group.

string theory	rank	charge lattice
IIA	$r = h^{1,1}(X)$	$\Lambda = H^{\text{even}}(X, \mathbb{Z})$
IIB	$r = h^{2,1}(X)$	$\Lambda = H^{\text{odd}}(X, \mathbb{Z})$

Table 2.5: The rank of the charge lattice and the identification of the charge lattice with the corresponding co-homology groups of the Calabi-Yau manifold.

In this geometric setup the charges are the Poincaré duals to the cycles C. For further details about geometric engineering, we refer to the literature mentioned above.

2.3 BPS states and black holes

BPS states provide a significant test of string theory within the context of black hole physics and the resolution of the entropy problem. It was shown by Bekenstein and Hawking that black holes behave like thermodynamical systems and therefore are subject to three laws of black hole thermodynamics [17, 18, 186]. The second law of black hole thermodynamics states that the area of the event horizon of a black hole is an increasing function with respect to time. Therefore it is possible to associate to the area of the event horizon the black hole entropy, which follows the Hawking area law

$$S_{\text{BH}} \propto \frac{A}{4}. \tag{2.25}$$

This provides a significant test of string theory as a theory of quantum gravity, as Boltzmann's law asks for a microscopic interpretation of the macroscopic entropy in (2.25). So the task is to identify black hole microstates with degeneracies $\Omega(\Gamma, t)$ such that

$$S_{\text{BH}} = S_{\text{micro}} = k_{\text{B}} \log \Omega(\Gamma, t). \tag{2.26}$$

BPS states provide a microscopic realisation of black hole microstates and it was first shown by Strominger and Vafa that in the large charge limit using Cardy's formula [187] the microscopic and the macroscopic results agree [108]. We want to give a short recap of general aspects of BPS black holes. For further details we refer to the literature [24, 121, 123, 188–191].

Reissner-Nordström black holes

Reissner-Nordström black holes are black holes with charges. The corresponding action of Einstein-Maxwell theory reads

$$S = \int d^4x \sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{2.27}$$

and electric charges q and magnetic charges p can be obtained as surface integrals of the field strength and its dual

$$q = \frac{1}{4\pi}\oint \star F, \quad p = \frac{1}{4\pi}\oint F. \tag{2.28}$$

Birkhoff's theorem states the existence of a unique spherically symmetric solution of the corresponding Einstein field equations, the Reissner-Nordström solution with

$$ds^2 = -e^{2f(r)}dt^2 + e^{-2f(r)}dr^2 + r^2 d\Omega^2, \quad e^{2f(r)} = 1 - \frac{2M}{r} + \frac{q^2 + p^2}{r^2}, \tag{2.29}$$

with M the mass of the black hole. A Reissner-Nordström black hole has two horizons r_+ and r_- as can be seen by rewriting

$$e^{2f(r)} = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right), \quad r_\pm = M \pm \sqrt{M^2 - q^2 - p^2} \quad . \tag{2.30}$$

Depending on the square root one can distinguish between three cases, where the extremal case is $M^2 = p^2 + q^2$, i.e. the mass is determined by the charges of the black hole. This can be embedded into the construction of extremal supersymmetric Reissner-Nordström black holes, where one dimensionally reduces p-branes, i.e. supersymmetric solutions of supergravity that are the higher dimensional analogue to Reissner-Nordström black holes. Using then the language of D-branes it is possible to calculate black hole microstates, for a review see [188, 189].

The attractor equations

Next we discuss BPS black holes in terms of $\mathcal{N} = 2$ supergravity and the question of stability which will lead to the *attractor equations* which state that the moduli fields at the horizon are only fixed by the charges of the black hole [115–119, 121]. We follow the exposition in [24]. Black holes in $\mathcal{N} = 2$ supergravity can be constructed by compactifying type II string theory on a Calabi-Yau threefold, where the entropy can be determined from the vector multiplets.

The metric for a multi-centred black hole takes the following form

$$ds^2 = -e^{2U(x)}(dt^2 + \omega)^2 + e^{-2U(x)}dx^2, \tag{2.31}$$

with $\omega = \omega_i(x)dx^i$ and U, ω vanish at spatial infinity $r = |x| \to \infty$. We denote the center of the black holes by x_i and the corresponding charges by Γ_i. We first consider the case, that we have a single centre and $\omega = 0$. Demanding that supersymmetry is unbroken in this setup, i.e. the variations with respect to the gravitino ψ_μ and the gauginos λ^α vanish

$$\delta\psi_\mu = \delta\lambda^\alpha = 0. \tag{2.32}$$

This leads to the attractor equations for $U(\tau)$ with[3] $\tau = \frac{1}{r}$ and the moduli

$$t^a = \frac{X^a}{X^0} \tag{2.33}$$

with

$$X^I = e^{\frac{K}{2}} \int_{A^I} \Omega, \quad F_I = e^{\frac{K}{2}} \int_{B_I} \Omega, \tag{2.34}$$

where (A^I, B_J) form a symplectic basis of three cycles. They read [121, 123]

$$\begin{aligned} \frac{dU(\tau)}{d\tau} &= -e^{U(\tau)}|Z|, \\ \frac{dt^a}{d\tau} &= -2e^{U(\tau)}\partial^\alpha |Z|. \end{aligned} \tag{2.35}$$

From this it can be shown, that $|Z|$ is a monotonically decreasing function of τ with a minimum. In the near horizon limit we assume that the central charge $Z = Z_{\text{hor}} \neq 0$ and the attractor equations imply that the near horizon geometry is given by $AdS_2 \times S^2$ and the horizon area A_{hor} equals

$$A_{\text{hor}} = 4\pi |Z_{\text{hor}}|. \tag{2.36}$$

Independent of the value of the initial conditions the system will evolve to the horizon. This is sometimes refered to as the attractor flow. If the charge is given as

$$\Gamma = p^I F_I - q_I X^I \tag{2.37}$$

then the attractor equations imply for the charges near the horizon

$$p^I = -2\,\text{Im}\,(\bar{Z}X^I), \quad q_I = -2\,\text{Im}(\bar{Z}F_I). \tag{2.38}$$

Also in the context of counting microstates for $\mathcal{N} = 4$ black holes there are interesting results form the attractor flow and Borcherds-Kac-Moody algebras, see e.g. [192]. For the multi centred metric the attractor equations read

$$\begin{aligned} H &= 2e^{-U}\text{Im}(e^{-i\alpha}e^{\frac{K}{2}}\Omega), \\ \star d\omega &= \int_X dH \wedge H. \end{aligned} \tag{2.39}$$

At $r = \infty$ the factor H is given by the following expression

$$H = -\sum_{i=1}^{N} \frac{\Gamma_i}{|x - x_i|} + 2\text{Im}(e^{-i\alpha}e^{\frac{K}{2}}\Omega)\Big|_{r=\infty}. \tag{2.40}$$

Denef showed, that BPS particles can form a bound state via the attractor mechanism. The non-singular solution reads in this case

$$\sum_{i\neq j} \frac{\langle \Gamma_i, \Gamma_j \rangle}{|x_i - x_j|} = 2\text{Im}e^{-i\alpha}Z(\Gamma_i, u)|_{r=\infty}. \tag{2.41}$$

[3] τ is the flow parameter of the attractor flow towards the horizon. For the case of multi-centred black holes it reads $\tau = \sum_i \frac{1}{|x-x_i|}$.

For the special case of a two-centred solution with charges $\Gamma_i, i = 1,2$ and corresponding central charges $Z(\Gamma_i, t)$ and α the phase of the total central charge $Z(\Gamma_1;t)+Z(\Gamma_2;t)$, Denef showed the distance between the two centres to be

$$R_{12} = |x_1 - x_2| = \frac{1}{2}\langle\Gamma_1,\Gamma_2\rangle\frac{1}{\mathrm{Im}e^{-i\alpha}Z(\Gamma_1;t)}. \tag{2.42}$$

By demanding charge and energy conservation we obtain two conditions for a decay of the bound state with associated central charge $Z(\Gamma_1+\Gamma_2,t)$

$$\begin{aligned} &\text{charge conservation:} && Z(\Gamma_1+\Gamma_2,t) = Z(\Gamma_1,t)+Z(\Gamma_2,t),\\ &\text{energy consveration:} && |Z(\Gamma_1+\Gamma_2,t)| = |Z(\Gamma_1,t)|+|Z(\Gamma_2,t)|. \end{aligned} \tag{2.43}$$

In this case the central charges $Z(\Gamma_1,t)$ and $Z(\Gamma_2,t)$ align see figure 2.1. This implies by using Denef's bound state formula that we have the folllowing condition as the distance has to be a positive number

$$\langle\Gamma_1,\Gamma_2\rangle\mathrm{Im}(Z(\Gamma_1,t)\bar{Z}(\Gamma_2,t))|_{r=\infty} > 0. \tag{2.44}$$

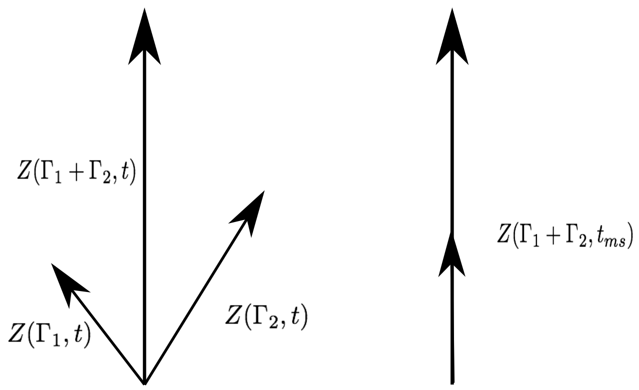

Figure 2.1: The alignment of central charges at a wall of marginal stability.

Therefore we can conclude that in the case of a decay the distance between the two centres tends to infinity giving rise to two single-centred black holes, just as expected.

2.4 Stability of BPS states

The mass of a BPS particle depends on the central charge. However, the central charge $Z(\Gamma,t)$ itself depends on various background moduli, which so far we have denoted by t. The variation of these moduli can cause a decay of the bounded BPS state, which is the wall-crossing effect. There exists a precise formula, the Kontsevich-Soibelman wall-crossing formula (KSWCF), that allows to calculate the change in the BPS spectrum. Before we can discuss the KSWCF, we have to define an index $\Omega(\Gamma,t)$, that we use to count BPS states. We present various indices that are used in the literature. However, we will mainly be interested in counting D4-D2-D0 bound states and therefore most of our discussion will focus on indices, that are appropriate to count these BPS states. The change in the index can then be calculated by the KSWCF, which we discuss in detail and present some examples. Furthermore we present various stability conditions, which are generalisations of the decay picture of the central charge we introduced in the previous section. This is done by introducing the language of sheaves which allow to give a mathematical description of BPS states.

2.4.1 Walls of marginal stability and BPS indices

The point in moduli space where a decay as described in (2.43) appears, is characterised by an alignment of the central charges. This happens at a co-dimension one object, the *wall of marginal stability* $\mathrm{MS}(\Gamma_1, \Gamma_2)$ which is given by the following points in moduli space

$$\mathrm{MS}(\Gamma_1, \Gamma_2) = \left\{ t \left| \frac{Z(\Gamma_1, t)}{Z(\Gamma_2, t)} > 0 \right. \right\}. \tag{2.45}$$

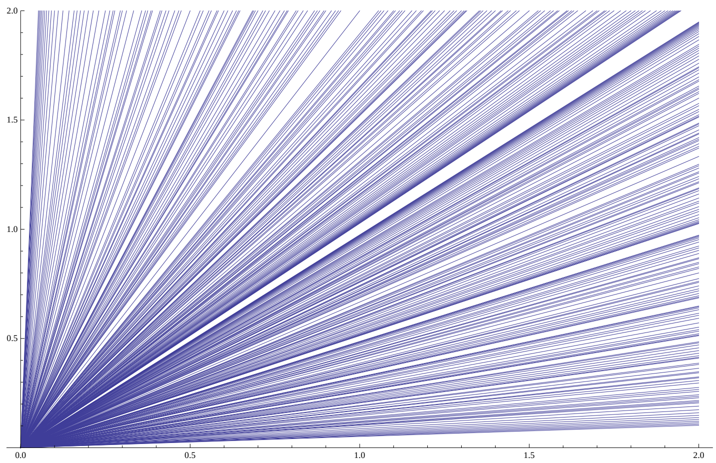

Figure 2.2: An example for the wall structure of the moduli space for the surface $\mathbb{P}^1 \times \mathbb{P}^1$. The coordinate axes are with respect to the two $\mathbb{P}^1$'s.

In order to count BPS states we have to define an index $\Omega(\Gamma, t)$ which counts the BPS states. Remembering that our BPS Hilbert space is graded with respect to the charges Γ. We define the BPS index of interests as follows

$$\Omega(\Gamma, t) = \frac{1}{2} \mathrm{Tr}_{\mathcal{H}^{\mathrm{BPS}}} (2J_3)^2 (-1)^{2J_3}. \tag{2.46}$$

This index is referred to as the *second helicity supertrace*, with J_3 being the generator of the spatial rotations $\mathfrak{so}(3)$. This index has the advantage, that it does vanish on fake, i.e. long representations. By this, we describe a long representation such that $M(t) > |Z(\Gamma, t)|$ but for certain values of t it might happen, that $M(t) = |Z(t)|$ which leads to a fake BPS representation of the form

$$\rho_{\mathrm{long}} \to \rho_{\text{fake-BPS}} = \rho_{hh} \otimes \mathfrak{h}' = \rho_{hh} \otimes \rho_{hh} \otimes \mathfrak{h}. \tag{2.47}$$

In the following we elaborate on the index (2.46) to clarify the issue of fake-BPS states. The character $\chi(\rho)$ of a representation ρ with respect to the Cartan elements J_3 of $\mathfrak{so}(3)$ and I_3 of $\mathfrak{su}(2)_R$ would take the following form

$$\chi(\rho) = \mathrm{Tr}_\rho\, q_1^{2J_3} q_2^{2I_3}. \tag{2.48}$$

We can see from table 2.6 that the character (2.48) is not a good quantity to distinguish between fake and true BPS states, as the characters of the long and short representations only differ by one factor. So if we consider a fake-representation of the form (2.47) then this index does not distinguish between fake and BPS states. This issue is resolved by taking derivatives and setting $q_1 = -q_2 = y$ leading to a character that vanishes on long representations. Taking one time a derivative w.r.t. $q_1 \partial_{q_1}$ gives the

representation	character $\chi(\rho)$
ρ_{long}	$(q_1 + q_1^{-1} + q_2 + q_2^{-1})^2 \chi(\mathfrak{h})$
ρ_{short}	$(q_1 + q_1^{-1} + q_2 + q_2^{-1})\chi(\mathfrak{h})$

Table 2.6: The characters of the long and short representations.

protected spin character $\tilde{\Omega}(\Gamma, u, y)$ which is given as [123, 193, 194]

$$\tilde{\Omega}(\Gamma, u, y) = \mathrm{Tr}_{\mathfrak{h}}(-1)^{2J_3}(-y)^{\mathcal{J}_3}. \tag{2.49}$$

Of course it is also possible to take two derivatives and set parameters accordingly such that we obtain the second helicity supertrace in (2.46). In section 2.8 we introduce the modified elliptic genus, which has similar properties and provides the correct index.

2.4.2 The Kontsevich-Soibelmann wall-crossing formula

If we cross a wall of marginal stability, the BPS index (2.46) will change due to the decay of the BPS states. We denote the moduli on the two sides of the wall by t_+ and t_- However, the change $\Delta\Omega(\Gamma)$ is given by

$$\Delta\Omega(\Gamma) = \Omega(\Gamma, t_+) - \Omega(\Gamma, t_-) \tag{2.50}$$

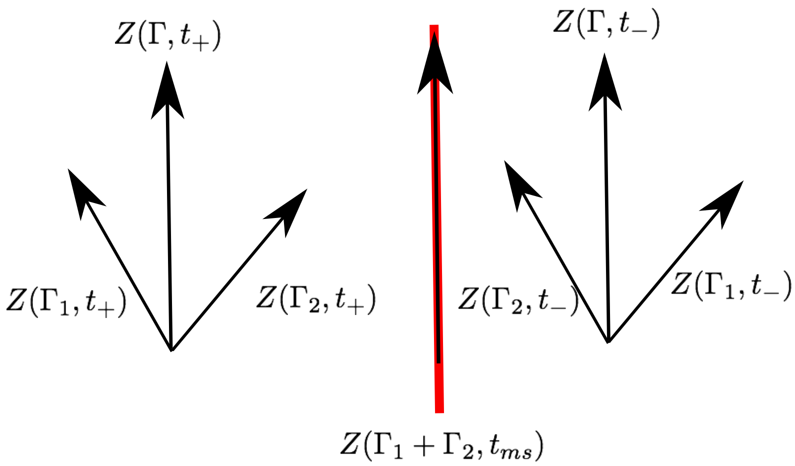

Figure 2.3: Crossing a wall of marginal stability and the central charges at t_+, t_{ms} and t_-.

The KSWCF calculates the change in the index [126]. We first state the result of the formula, before we discuss some more formal issues. The KSWCF reads as follows

$$\prod_{\Gamma: Z(\Gamma;t)\in V}^{\curvearrowright} T_\Gamma^{\Omega(\Gamma;t_+)} = \prod_{\Gamma: Z(\Gamma;t)\in V}^{\curvearrowright} T_\Gamma^{\Omega(\Gamma;t_-)}, \tag{2.51}$$

where V is a region in $\mathbb{R}^2$ that is bounded by two rays starting at the origin and $\curvearrowright$ denotes clockwise ordering of the factors in the product with resect to the central charges $Z(\Gamma,t)$ where the symplectomorphisms T_Γ are given as

$$T_\Gamma = \exp\left(-\sum_{n=1}^{\infty}\frac{e_{n\Gamma}}{n^2}\right), \tag{2.52}$$

and the e_Γ satisfy the following Lie algebra commutation relation

$$[e_{\Gamma_1}, e_{\Gamma_2}] = (-1)^{\langle\Gamma_1,\Gamma_2\rangle}\langle\Gamma_1,\Gamma_2\rangle e_{\Gamma_1+\Gamma_2}. \tag{2.53}$$

We want to comment on the fact that T_Γ represent symplectomorphisms. We denote the charge lattice by Λ and the dual lattice by Λ^*. Complexification of the dual lattice gives a r-dimensional torus

$$\mathbb{T}^r = \Lambda^* \otimes \mathbb{C}^*. \tag{2.54}$$

Now one defines functions X_Γ on $\mathbb{T}^r$ such that it acts on a basis $\{\Gamma_i^*\}_i$ via

$$\begin{aligned} X_\Gamma : \mathbb{T}^r &\to \mathbb{C} \\ \sum_i a_i\Gamma_i^* &\mapsto \exp\sum_i a_i\Gamma_i^*(\Gamma). \end{aligned} \tag{2.55}$$

From the definition it can be easily seen that

$$X_{\Gamma_1}X_{\Gamma_2} = X_{\Gamma_1+\Gamma_2}. \tag{2.56}$$

The complex torus $\mathbb{T}^r$ can be equipped with a symplectic structure ω

$$\omega = \frac{1}{2}\langle\Gamma_i,\Gamma_j\rangle^{-1}\frac{dX_{\Gamma_i}}{X_{\Gamma_i}} \wedge \frac{dX_{\Gamma_j}}{X_{\Gamma_j}}. \tag{2.57}$$

Now it is possible to state the action of the symplectomorphisms T_Γ as follows

$$\begin{aligned} T_\Gamma : \mathbb{T}^r &\to \mathbb{T}^r \\ X_{\Gamma'} &\mapsto X_{\Gamma'}(1-\sigma(\Gamma)X_\Gamma)^{\langle\Gamma',\Gamma\rangle}, \end{aligned} \tag{2.58}$$

with $\sigma(\Gamma) = (-1)^{\langle\Gamma_e,\Gamma_m\rangle}$ with $\Gamma = \Gamma_e + \Gamma_m$. The e_Γ can then be interpreted as infinitesimal symplectomorphisms of the Hamiltonian $\sigma(\Gamma)X_\Gamma$.

We illustrate the KSWCF for the cases of a two-particle bound state $\Gamma \to \Gamma_1 + \Gamma_2$, which leads to the primitive wall-crossing formula [123]. We have the following elements in the Lie algebra: $e_{\Gamma_1}, e_{\Gamma_2}$ and $e_{\Gamma_1+\Gamma_2}$ and we assume all other elements to be zero. Using the Kontsevich-Soibelmann wall-crossing formula we have the following identity

$$T_{\Gamma_1}^{\Omega(\Gamma_1,t_+)}T_{\Gamma_1+\Gamma_2}^{\Omega(\Gamma_1+\Gamma_2,t_+)}T_{\Gamma_2}^{\Omega(\Gamma_2,t_+)} = T_{\Gamma_2}^{\Omega(\Gamma_2,t_-)}T_{\Gamma_1+\Gamma_2}^{\Omega(\Gamma_1+\Gamma_2,t_-)}T_{\Gamma_1}^{\Omega(\Gamma_1,t_-)} \tag{2.59}$$

Using the Lie-Algebra relations and permuting the symplectomorphisms we arrive at the following formula for the change in the indices $\Delta\Omega(\Gamma_1+\Gamma_2)$:

$$\Delta\Omega(\Gamma_1+\Gamma_2, t_{\mathrm{ms}}) = (-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\langle\Gamma_1,\Gamma_2\rangle\Omega(\Gamma_1)\Omega(\Gamma_2). \tag{2.60}$$

From the perspective of Denef's bound state formula, we can interpret the primitive wall-crossing formula in terms of the two dyons for each center. First of all the change in the index can be expressed as

$$\Delta\Omega(\Gamma_1 + \Gamma_2, t) = \pm\chi_{|\langle\Gamma_1,\Gamma_2\rangle|}\Omega(\Gamma_1, t)\Omega(\Gamma_2, t), \tag{2.61}$$

and the associated electromagnetic field of the dyon is subject to a representation of $\mathfrak{so}(3)$ with dimension $|\langle\Gamma_1, \Gamma_2\rangle|$ that has spin $\frac{1}{2}(|\langle\Gamma_1, \Gamma_2\rangle| - 1)$, where the subtraction of -1 is due to quantum corrections. The index then resembles the decay of the two centres.

2.4.3 Stability conditions

We give short review on stability conditions of D-brane bound states and a short overview of the more general results in the field of mathematics and follow [195]. The theory of stability conditions for triangulated categories was developed in [125]. The connection between D-branes and derived categories can be found in [196–203]. For more details on the relation to the KSWCF see [126].

Mathematics of stability conditions

The strategy to define a stability condition is to generalise the central charge function to a stability function and then define the notion of a stability condition. We denote by $\mathcal{A}$ an abelian category and the Grothendieck group by $K(\mathcal{A})$.

Definition: A stability function on an abelian category $\mathcal{A}$ is a linear map $Z : K(\mathcal{A}) \to \mathbb{C}$ such that for all non-trivial $\mathcal{E} \in \mathcal{A}$ we have $Z(\mathcal{E}) \in \mathbb{H} \cup \mathbb{R}_{<0}$. ◇

From this two observations can be made. First of all, it is possible to express the stability function $Z(\mathcal{E})$ as

$$Z(\mathcal{E}) = m(\mathcal{E}) \exp(i\pi\phi(\mathcal{E})), \quad \text{with} \quad m(\mathcal{E}) \in \mathbb{R}_{>0}, \phi(\mathcal{E}) \in (0, 1], \tag{2.62}$$

where we refer to $\phi(\mathcal{E})$ as the phase, which can be determined from (2.62). Secondly, the linearity condition used in (2.16) generalises in the sense that for a short exact sequence

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0 \tag{2.63}$$

the stability condition is linear

$$Z(\mathcal{E}) = Z(\mathcal{E}_1) + Z(\mathcal{E}_2). \tag{2.64}$$

We state the definition of a *stability condition*.

Definition: A stability condition on an abelian category $\mathcal{A}$ consists out of a stability function Z and a slicing $P = \{P(\phi)\}$ with the subcategories of semistable objects $P(\phi) \subset \mathcal{A}$ and $\phi \in (0, 1]$ such that

i) $\mathrm{Hom}(P(\phi_1), P(\phi_2)) = 0$ if $\phi_1 > \phi_2$ and

ii) for all non-trivial $\mathcal{E} \in \mathcal{A}$ there exists a Harder-Narasimhan (HN) filtration, i.e. there exists a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E} \tag{2.65}$$

and the semistable factors $A_i = \mathcal{E}_i/\mathcal{E}_{i-1} \in P(\phi_i)$ of $\mathcal{E}$ and $\phi_1 > \cdots > \phi_n$.

◇

It is possible to extend the definition to triangulated categories, but we refer to the literature for more details [125]. Furthermore note that in the case of extended HN filtration the condition on the ϕ_i is extended to $\phi_i \leq \phi_j$. We continue our discussion with stability conditions of surfaces and their relation to D-branes such that we can proceed with our discussion of the necessary tools for D4-D2-D0 BPS states. However, we will switch to a perspective, that is more physical and we will see explicit notions of stability that are used in physics. For a precise mathematical discussion of stability conditions on surfaces we refer to [195].

Stability conditions on surfaces – D-branes and sheaves

In order to clarify our notation we collect some facts about D-brane charges and the stability conditions for a bound-state system of D4-D2-D0 branes, wrapped around a divisor $i : P \hookrightarrow X$ inside a Calabi–Yau three-fold X. We start by presenting the relation between D-brane charges and mathematical properties of sheaves. Then we present three stability conditions, namely Π-stability, μ-stability and Gieseker stability, see e.g. [202–204] for a review. The D4-D2-D0 brane-system is specified by a (coherent) sheaf $\mathcal{E}$ on P. The image of the K-theory charge of the sheaf $\mathcal{E}$ in $H^{\text{even}}(X, \mathbb{Q})$ is given by the Mukai vector [198, 205, 206]

$$\Gamma = \mathrm{ch}(i_*\mathcal{E})\,\sqrt{\mathrm{Td}(X)}, \tag{2.66}$$

where $i_*\mathcal{E}$ denotes the extension-sheaf to X. Using the Grothendieck-Riemann-Roch-theorem

$$i_*(\mathrm{ch}(\mathcal{E})\,\mathrm{Td}(P)) = \mathrm{ch}(i_*\mathcal{E})\,\mathrm{Td}(X), \tag{2.67}$$

and the expressions

$$\mathrm{Td}(Y)^a = 1 + \frac{a}{2}c_1(Y) + \frac{(3a^2 - a)c_1(Y)^2 + 2a\,c_2(Y)}{24}, \tag{2.68}$$

$$\mathrm{ch}(Y) = \sum_{i=0}^{3}\mathrm{ch}_i(Y) = \mathrm{rk}(Y) + c_1(Y) + \frac{1}{2}c_1(Y)^2 - c_2(Y), \tag{2.69}$$

$$\mathrm{ch}(Y^*) = \sum_{i=0}^{3}\mathrm{ch}_i(Y^*) = \mathrm{rk}(Y) - c_1(Y) + \frac{1}{2}c_1(Y)^2 - c_2(Y), \tag{2.70}$$

where Y^* denotes the dual sheaf, one obtains [207]:

$$\begin{aligned}\Gamma = {}& r[P] + r\,i_*\left(\frac{c_1(\mathcal{E})}{r} + \frac{c_1(P)}{2}\right)\\ &+ r i_*\left(\frac{c_1(P)^2 + c_2(P)}{12} + \frac{\frac{1}{2}(c_1(P)c_1(\mathcal{E}) + c_1(\mathcal{E})^2) - c_2(\mathcal{E})}{r}\right) - \frac{c_2(X)\cdot[P]}{24},\end{aligned} \tag{2.71}$$

where r is the rank of the sheaf $\mathcal{E}$ and one has to note, that $c_1(X) = 0$ as X is a Calabi-Yau manifold. Using the adjunction formula we arrive at

$$\Gamma = (Q_6\,,\, Q_4\,,\, Q_2\,,\, Q_0) = r\left(0\,,\,[P]\,,\,i_*F\,,\left[\frac{\chi(P)}{24} + \int_P \frac{1}{2}F^2 - \Delta\right]\right). \tag{2.72}$$

Here we introduced

$$F = \frac{c_1(\mathcal{E})}{r} + \frac{c_1(P)}{2}, \tag{2.73}$$

$$\Delta = \frac{1}{2r^2}\left(2r\,c_2(\mathcal{E}) - (r-1)\,c_1(\mathcal{E})^2\right). \tag{2.74}$$

The quantity Δ is called the discriminant.

Π-stability

Given the K-theory charges in (2.72) the expression for the central charge is subject to instanton corrections. Those can be obtained from mirror symmetry on the A-model side see section 2.7

$$\begin{aligned} Z(\mathcal{E}) &= -\int e^{-(B+iJ)}\,\Gamma(\mathcal{E}) + (\text{instanton} - \text{corrections}) \\ &= -\frac{r}{2}[P]\cdot t^2 + t(i_*c_1(\mathcal{E}) + \frac{r}{2} i_* c_1(P)) - \mathrm{ch}_2(\mathcal{E}) \\ &\quad - \frac{1}{2}c_1(\mathcal{E})c_1(P) - \frac{r}{8}c_1(P)^2 - \frac{r}{24}c_2(P) + O(e^{-t}), \end{aligned} \tag{2.75}$$

where J is the Kähler form of X and $t = B + iJ$. As we have seen in the general discussion in 2.4.3 in (2.62) We now denote the phase of $Z(\mathcal{E})$ by

$$\phi(\mathcal{E}) = \frac{1}{\pi}\,\mathrm{Arg}\, Z(\mathcal{E}) = \frac{1}{\pi}\,\mathrm{Im}\log Z(\mathcal{E}). \tag{2.76}$$

A sheaf $\mathcal{E}$ is called Π-(semi)-stable [208, 209] iff for every (well-behaved) subsheaf $\mathcal{F}$:

$$\phi(\mathcal{F}) \leq \phi(\mathcal{E}), \tag{2.77}$$

where the strict inequality amounts to stability. If the inequality is strictly fulfilled (a stable sheaf) a decay is impossible by charge and energy conservation.

μ-stability

In a large volume phase ($t \to \infty$) of the Calabi-Yau manifold the instanton-corrections are suppressed by $O(e^{-t})$ and the classical expressions become exact. In this limit we are left with [124]:

$$\varphi(\mathcal{E}) = \frac{1}{\pi}\,\mathrm{Im}\log\left(-\frac{r}{2}J^2\cdot[P]\right) + 2\,\frac{J\cdot\hat{\mu}}{J^2\cdot[P]} + O\left(\frac{1}{J^2}\right). \tag{2.78}$$

Π-stability now amounts to the definition

$$(i^*J)\cdot\frac{c_1(\mathcal{F})}{r(\mathcal{F})} \leq (i^*J)\cdot\frac{c_1(\mathcal{E})}{r(\mathcal{E})} \quad \text{for any nice subsheaf} \quad \mathcal{F}\subseteq\mathcal{E}, \tag{2.79}$$

where i^*J denotes the pullback of the Kähler form of the Calabi-Yau to P and all expressions are understood on P. The quantity appearing in the above definition is called slope and denoted by $\mu(\mathcal{E})$, i.e.

$$\mu(\mathcal{E}) := (i^*J)\cdot\frac{c_1(\mathcal{E})}{r(\mathcal{E})}. \tag{2.80}$$

The above condition is called μ-(semi-)stability and the classical notion of the stringy Π-stability. Note also, that μ-stability is not sensitive to how the lower dimensional charges are distributed among decay products. This is in contrast to Π-stability, where quantum corrections change this insensitivity.

Gieseker stability

Gieseker stability uses the Hilbert polynomials $p_J(\mathcal{E},n) = \frac{\chi(\mathcal{E}\otimes J^n)}{r(\mathcal{E})}$ and reads

$$p_J(\mathcal{E},n) = \frac{J^2n^2}{2} + \left(\frac{c_1(\mathcal{E})\cdot J}{r(\mathcal{E})} - \frac{[P]\cdot J}{2}\right)n + \frac{1}{r(\mathcal{E})}\left(\frac{c_1(\mathcal{E})^2 - [P]\cdot c_1(\mathcal{E})}{2} - c_2(\mathcal{E})\right) + \chi(\mathcal{O}_S) \tag{2.81}$$

We can give a interpretation from physics. Recall, that the central charge $Z(\Gamma,t)$ is the object of interests concerning stability issues. The Hilbert polynomial resembles the terms in the central charge. A sheaf is Gieseker stable, if for every subsheaf $\mathcal{E}' \subseteq \mathcal{E}$ the Hilbert polynomial satisfies

$$p_J(\mathcal{E}',n) \prec p_J(\mathcal{E},n), \tag{2.82}$$

where we denote by $\prec$ the lexicographic ordering, i.e. one compares in decreasing power of n the coefficients in the corresponding Hilbert polynomials. We list the procedure for $p_J(\mathcal{E}',n) \prec p_J(\mathcal{E},n)$ in table 2.7. This procedure terminates, if the corresponding coefficient for $\mathcal{E}'$ is smaller than the coefficient for $\mathcal{E}$. Note, that the coefficient of n^2 is always the same for both sheaves as J is the same for both Hilbert polynomials.

deg n	term
1	$\left(\frac{c_1(\mathcal{E}')\cdot J}{r(\mathcal{E}')} - \frac{[P]\cdot J}{2}\right) < \left(\frac{c_1(\mathcal{E})\cdot J}{r(\mathcal{E})} - \frac{[P]\cdot J}{2}\right)$
0	$\frac{1}{r(\mathcal{E}')}\left(\frac{c_1(\mathcal{E}')^2 - [P]\cdot c_1(\mathcal{E}')}{2} - c_2(\mathcal{E}')\right) < \frac{1}{r(\mathcal{E})}\left(\frac{c_1(\mathcal{E})^2 - [P]\cdot c_1(\mathcal{E})}{2} - c_2(\mathcal{E})\right)$

Table 2.7: The lexicographic ordering of Gieseker stability $p_J(\mathcal{E}',n) \prec p_J(\mathcal{E},n)$.

In order to describe possible decays we will use the HN filtration, as we already described in (2.65).

Dimension of moduli space

On general grounds the moduli space of a D-brane modelled by a sheaf $\mathcal{E}$ is given by $\mathrm{Ext}^1(\mathcal{E},\mathcal{E})$. The elements of this group count the number of marginal open string operators in the spectrum of the BCFT describing the B-brane. We assume, that P is a rational surface and further that the sheaf $\mathcal{E}$ is μ-stable and that $(i^*J)\cdot[P] \leq 0$. Under these assumptions the moduli space is smooth and the following formula for its dimension holds [210]

$$\dim \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) = 1 + r^2(2\Delta - 1). \tag{2.83}$$

A consequence is that for a slope-stable sheaf one has

$$\Delta \geq 0, \tag{2.84}$$

which is a condition on the stable bundle's Chern classes.

2.4.4 Verification of the Kontsevich-Soibelman wall-crossing formula from physics

In this section we want to give a short review of some implications of wall-crossing that have been discussed in the recent literature. However, this selection is far from being complete and concerning issues of modularity we refer to the following sections of this thesis. We already discussed in section

2.2 BPS states in the context of wall-crossing. We will give an explicit example for SU(2) Seiberg-Witten theory in section 2.5, but we give a short review of the idea to prove wall-crossing from physical arguments [128].

KSWCF from three dimensional field theory

We already discussed the effective, four dimensional action of Seiberg-Witten theories. The idea of [128] is to explain the KSWCF from a three dimensional reduction of the four dimensional theory to $\mathbb{R}^3 \times S^1_R$. It can be shown, that the low energy theory on $\mathbb{R}^3$ corresponds to a sigma model description, where the target space is given by a $2r$ torus fibration $\mathcal{M}$, where one obtains over any point of the Coulomb branch $\mathcal{B}$ a $2r$ torus, obtained from the electric and magnetic gauge fields in the fourth dimension. The three dimensional version of the Lagrangian (2.23) has the following form in the limit of $R \to \infty$ and after dualising the gauge field

$$\mathcal{L}^{(3)} = -\frac{R}{2}\mathrm{Im}\, \tau |da|^2 - \frac{1}{8\pi^2 R}(\mathrm{Im}(\tau))^{-1}\, |dz|^2, \tag{2.85}$$

from which it is possible to read off locally the semi-flat metric g^{sf} of the sigma model

$$g^{\mathrm{sf}} = R\, \mathrm{Im}\tau |da|^2 + \frac{1}{4\pi^2 R}(\mathrm{Im}(\tau))^{-1}|dz|^2. \tag{2.86}$$

This is referred to as the semi-flat metric, since the fibres of $\mathcal{M}$ are flat. However, one would like to determine the metric g which is valid all over the moduli space $\mathcal{B}$. For g^{sf} this is problematic at its singularities. However, this issue is resolved by corrections obtained from 4d BPS instanton corrections. To determine the corrected g, the authors of [128] use Stokes' phenomena of asymptotic series and provide a method to construct g by applying twistorial methods, which for brevity we do not review here. The instanton corrections can also change, as the degeneracies $\Omega(\Gamma, t)$ of the BPS states change in the moduli space due to the wall-crossing effect. It turns out, that the smoothness of the metric g is correlated to the fact that the KSWCF is valid. Therefore the KSWCF can be interpreted as a consistency condition. This is shown, by constructing an appropriate Riemann-Hilbert problem and determination of the asymptotic behaviour of certain objects $\mathcal{X}_\Gamma$, that are holomorphic Darboux coordinates on $\mathcal{M}$, which is understood then as a symplectic manifold [129]. Of course, this finding is a powerful physics consistency check of the KSWCF. Note furthermore that in this case the Stokes' phenomenon plays an important role and the product of the symplectomorphisms $T_\Gamma^{\Omega(\Gamma,t)}$ can be interpreted as Stokes' factor $S^{+,-}$

$$S^{+,-} = \prod_\Gamma T_\Gamma^{\Omega(\Gamma,t)}, \tag{2.87}$$

such that a transition between a stokes line seperating $\mathcal{X}_\Gamma^+$ and $\mathcal{X}_\Gamma^-$ is given schematically as

$$\mathcal{X}_\Gamma^+ = S^{+,-}\mathcal{X}_\Gamma^-, \tag{2.88}$$

which reveals the connection between the wall-crossing effect and the Stokes' phenomenon. For more details on Stokes' phenomena we refer to [211, 212]. In subsequent work [129] the authors provide additional verifications of the KSWCF in the case of $(2,0)$ SCFTs on Riemann surfaces, where the asymptotic behaviour is obtained by using a WKB approximation. In [134] the relation to BPS halos and line operators is clarified and it is shown, that the KSWCF follows from properties of these. However, there has also been an interesting development in the context of 2d-4d systems [132, 133, 135].

2.5 An example: SU(2) Seiberg-Witten theory

In this section we want to explore some of the aspects about stability conditions of BPS states in the context of Seiberg-Witten theory for the gauge group SU(2) [61, 62]. This section first gives a short overview of the ingredients of SU(2) Seiberg-Witten theory and then states the explicit results for this theory and therefore giving an example of the gauge theories, that we discussed in 2.2. We also give a short overview of some recent results using quivers and their mutations [213, 214], that use ideas of wall-crossing. Useful reviews on Seiberg-Witten theory are [185, 215, 216]. This section is based on [185, 215].

2.5.1 The setup and the solution

Let the gauge group be SU(2) and we consider $\mathcal{N} = 2$ SYM theory. In the adjoint of SU(2) we have an $\mathcal{N} = 2$ vector multiplet or called chiral multiplet. It has the structure in table 2.8 and contains gauge fields A^i_μ, two Weyl fermions λ^i_α and $\psi^{i\beta}$ and scalars ϕ^i. Furthermore, we might have a hyper multiplet, which is depicted in the table 2.9 and contains two Weyl fermions $\psi_q, \psi^\dagger_{\tilde{q}}$ and two complex bosons $q, \tilde{q}^\dagger$. For our discussion we do not incorporate hyper multiplets at this point. In $\mathcal{N} = 1$ language we have a

			spin
	A^i_μ		1
λ^i_α		$\psi^{\beta i}$	$\frac{1}{2}$
	ϕ^i		0

Table 2.8: The components of the $\mathcal{N} = 2$ vector multiplet

	ψ_q	
q		$\tilde{q}^\dagger$
	$\psi^\dagger_{\tilde{q}}$	

Table 2.9: The components of the $\mathcal{N} = 2$ hyper multiplet

vector multiplet $W^i_\alpha = (A^i_\mu, \lambda^i_\alpha)$ and a chiral multiplet $\Phi^i = (\phi^i, \psi^{i\beta})$. The classical potential is caused by the scalar field $\phi = \phi^i \sigma^i$ and reads

$$V(\phi) = \frac{1}{g^2}\mathrm{Tr}[\phi, \phi^\dagger]^2. \tag{2.89}$$

As we discussed earlier, we want to study these theories at the Coulomb branch $\mathcal{B}$. The flat directions with $V(\phi) = 0$ of (2.89) can be parametrised by

$$\phi = \frac{1}{2}a\sigma^3, \tag{2.90}$$

where the complex number a labels the different vacua of our theory. Due to the Weyl group of SU(2) there exists a $\mathbb{Z}_2$ symmetry that maps $a \to -a$. A gauge invariant formulation is given by $u(a)$

$$u(a) = \frac{1}{2}a^2. \tag{2.91}$$

Indeed, this parameterises the Coulomb branch[4] of our theory because for non-trivial u the gauge group breaks from SU(2) to U(1). As u can be any complex number we see that the Coulomb branch $\mathcal{B} = \mathbb{C}$. It can be shown [217], that the region $u = \infty$ corresponds to the weak coupling region and for the case that $u = 0$ we have the strongly coupled region and the full gauge symmetry of SU(2) is restored giving rise to two massless gauge bosons $W^{\pm}$. As we discussed in section 2.2 the low-energy effective Lagrangian is determined by the prepotential $\mathcal{F}$. Classically it reads

$$\mathcal{F} = \frac{1}{2}\tau_0 a^2, \tag{2.92}$$

with the effective gauge coupling τ_0 given as

$$\tau_0 = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{2.93}$$

However, the prepotential is subject to quantum corrections from one loop corrections and instanton corrections [218] and takes the following form

$$\mathcal{F} = \frac{1}{2}\tau_0 a^2 + \frac{1}{2\pi i} a^2 \log\frac{a^2}{\Lambda^2} + a^2 \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k}. \tag{2.94}$$

Since we can determine $\tau(a)$ by deriving $\mathcal{F}$ we obtain

$$\tau(u) = \text{const.} + \frac{2i}{\pi}\log\frac{u}{\Lambda^2} + \dots. \tag{2.95}$$

If we now consider the moduli space at $u = \infty$, from (2.95) it can be deduced by looping around $u = \infty$ that we have $\tau \mapsto \tau - 4$. One of the key insights of Seiberg and Witten was that the quantum moduli space has only two singularities at

$$u = \pm\Lambda^2 \tag{2.96}$$

instead of the strong coupling singularity at $u = 0$. At $u = 0$ we had the massless $W^{\pm}$ bosons and for the singularity at $u = \Lambda^2$ the excitation can be obtained from a study of BPS states. The central charge reads

$$Z = qa + pa_D \tag{2.97}$$

with electric charges q and magnetic charges p. At the singularity at $u = \Lambda^2$ we have $a \neq 0$ and $a_D = 0$ which implies that at this singularity the magnetic monopole $(p, q) = (\pm 1, 0)$ is massless. So whereas the weak coupling region is best described by a at $u = \infty$, and near the singularity at $u = \Lambda^2$ one uses the dual description by a_D, which has a dual $\mathcal{F}_D$ with a in (2.94) replaced by a_D and $\mathcal{F}_k$ by $\mathcal{F}_{k,D}$. This implies that in the variable a_D the originally strongly coupled theory in a becomes weakly coupled. The second singularity $u = -\Lambda^2$ contains similar information due to the $\mathbb{Z}_2$ symmetry in u and reads $\mathcal{F}_D(a_D - 2a)$. At this point a dyon $(1, -2)$ becomes massless. Patching the three different regions together and using in each the weak coupling description allows for a solution of the instanton corrections. The 1-loop term comes with a logarithmic term which can be determined by monodromy arguments. The corresponding monodromy matrices act on $(a_D, a)^T$ and can be determined as

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \quad M_{+\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad M_{-\Lambda^2} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}. \tag{2.98}$$

[4] Here we change our notation from $t \in \mathcal{B}$ to u, which is used in this context in the literature.

These also satisfy the global consistency condition

$$M_{\Lambda^2} M_{-\Lambda^2} = M_\infty, \tag{2.99}$$

which must hold, since the monodromy around infinity includes the two poles. It turns out, that two of the matrices in (2.98) generate the modular subgroup $\Gamma_0(4)$ and hence the quantum moduli space is given by $\mathbb{H}/\Gamma_0(4)$. Note in particular, that in the quantum version the S-duality transformation $\tau \mapsto -\frac{1}{\tau}$ is not part of the symmetries. Conjugation by M_∞ leads to a change in the charges as

$$(p,q) \to (-p, -q-4p) \tag{2.100}$$

and hence the electric charge of the dyon is not uniquely defined leading to the conjugated dyons $\pm(1,2)$. The modular group gives rise to an elliptic curve which encodes the information about the quantum moduli space. It turns out that the moduli space of the gauge theory corresponds to the moduli space of the elliptic curve, which itself takes the following form

$$y^2(x,u) = (x^2-u)^2 - \Lambda^4, \tag{2.101}$$

and the gauge coupling τ is given by the relation of the periods ω_D and ω as

$$\tau = \frac{\omega_D}{\omega}. \tag{2.102}$$

It is possible to collect the information about the periods in the Seiberg-Witten differential λ

$$\lambda = \frac{1}{\sqrt{2}\pi} x^2 \frac{dx}{y(x,u)} \tag{2.103}$$

and via period integrals it is possible to determine a and a_D

$$a_D(u) = \oint_\beta \lambda, \quad a(u) = \oint_\alpha \lambda. \tag{2.104}$$

The BPS spectrum can be separated into two regions: the strong coupling region $\mathcal{M}_{\text{strong}}$ and the weak coupling region $\mathcal{M}_{\text{weak}}$. These two regions consist of the following BPS states starting with$\mathcal{M}_{\text{strong}}$

- the magnetic monopole with charge $(p,q) = \pm(1,0)$,
- a dyon with charge $(p,q) = \pm(1,-2)$ in the lower u plane, and $\pm(1,2)$ in the upper u plane.

In the weak coupling region $\mathcal{M}_{\text{weak}}$ the BPS spectrum has the following components

- dyons with charge $\pm(1,2k), k \in \mathbb{Z}$
- and the $W^\pm$ gauge bosons with charge $(0,\pm 2)$.

These two regions are separated by a line of marginal stability which is given by

$$\text{MS} = \left\{ t : \frac{a_D}{a} \in \mathbb{R} \right\} \tag{2.105}$$

and when crossing it one can observe a possible decay, see also fig 2.4. Note, that this definition of a wall of marginal stability is of course equivalent to the definition stated above. In particular when crossing the wall of marginal stability the $W^\pm$ can decay into the monopole-dyon pair.

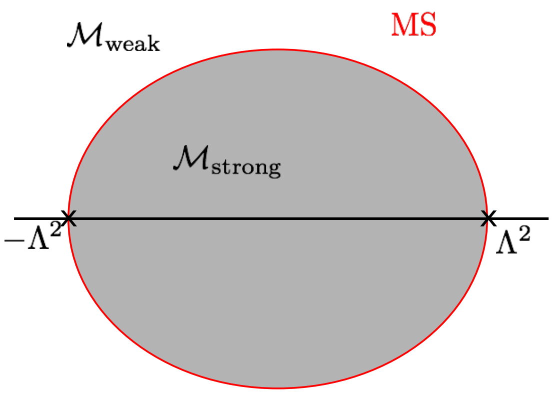

Figure 2.4: The moduli space of SU(2) Seiberg-Witten theory in the u-plane. The strong coupling region $\mathcal{M}_{\text{strong}}$ is separated from the weak coupling region $\mathcal{M}_{\text{weak}}$ by a line of marginal stability MS.

The periods can be evaluated by determining the Picard-Fuchs equations and with that at hand it is possible to calculate $\mathcal{F}$ and $\mathcal{F}_D$. We refer to the literature for details [219] and for a general overview and other gauge groups [185].

2.5.2 Quiver description of SU(2) Seiberg-Witten theory

As we have just seen, the BPS states can be obtained from monodromy considerations. It is also possible to apply newly developed tools in [213,214], which we want to review here as they make use of stability conditions. For more technical details we refer to the literature. The mutation of the gauge theory quiver generates the complete BPS spectrum. Given an $\mathcal{N} = 2$ gauge theory with a rank r gauge group G and flavor group G_f, at the Coulomb branch, the charge lattice Λ of electric, magnetic and flavour charges is of rank $2r + f$. It is possible to find a positive integral basis $\{\Gamma_i\}_{i=1}^{2r+f}$ of the BPS hypermultiplets. Each basis element Γ_i gives rise to a node of the quiver. The different nodes are connected by arrows, if $\langle\Gamma_i, \Gamma_j\rangle > 0$ and the number of arrows corresponds to the value of the symplectic charge product and the direction is from the node for Γ_i to the node for Γ_j. For the case of SU(2) Seiberg-Witten theory the basis elements are the monopole $\Gamma_1 = (1, 0)$ and the dyon $\Gamma_2 = (-1, 2)$ with charge product $\langle\Gamma_1, \Gamma_2\rangle = 2$ which leads to the following quiver

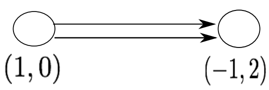

Figure 2.5: The quiver for SU(2) Seiberg-Witten theory.

The idea of BPS quivers is to know the BPS spectrum at one point $t \in \mathcal{B}$ and then via mutations of the quiver to construct the spectrum at other positions in the moduli space. A quiver representation R associates to each node a vector space $\mathbb{C}^{n_i}$ and for each arrow a linear maps $B_{ij}^a : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j}$. A subrepresentation $S \subset R$ consists out of vector spaces $\mathbb{C}^{m_i} \subset \mathbb{C}^{n_i}$ and maps $b_{ij}^a : \mathbb{C}^{m_i} \to \mathbb{C}^{m_j}$ such that the resulting diagram of embedding S into R commutes. A mutation of a fixed Γ_i has the following action on the nodes

$$\begin{aligned} &\Gamma_i \mapsto -\Gamma_i \\ &\Gamma_j \mapsto \begin{cases} \Gamma_j + \langle\Gamma_i, \Gamma_j\rangle\Gamma_i, & \langle\Gamma_i, \Gamma_j\rangle > 0 \\ \Gamma_j, & \langle\Gamma_i, \Gamma_j\rangle \leq 0. \end{cases}, \quad j \neq i \end{aligned} \tag{2.106}$$

Furthermore the mutation transformation rotates the associated central charges out of the corresponding upper half plane. The stability condition for quivers is Π-stability with respect to arguments of the central charges of the representation and the subrepresentations. The representation R is stable against all subrepresentations S, if

$$\arg Z_u(\Gamma_S) < \arg Z_u(\Gamma_R). \tag{2.107}$$

When we perform the mutation in the strong coupling region, we have for the phases of the central charges $\arg Z_1 < \arg Z_2$ with $Z_k = Z(\Gamma_k)$. Therefore the sequence of mutations starts with Γ_2. After two mutations one arrives at the anti-particle quiver, thus as we have discussed before, only the monopole and the dyon survive. In contrast to the weak coupling region, where we have $\arg Z_1 > \arg Z_2$. Therefore the first mutation is with Γ_1 and then followed by the second node. Note, that this leads to dyons of charge $\pm(1, 2n)$ as well as the $W^{\pm}$-boson $(0, \pm 2)$ which is the composite of Γ_1 and Γ_2. We have summarised the mutations in figure 2.6 So using the language of quivers and mutations, we have a nice way to calculate the BPS spectrum of the $\mathcal{N} = 2\,\mathrm{SU}(2)$ gauge theory.

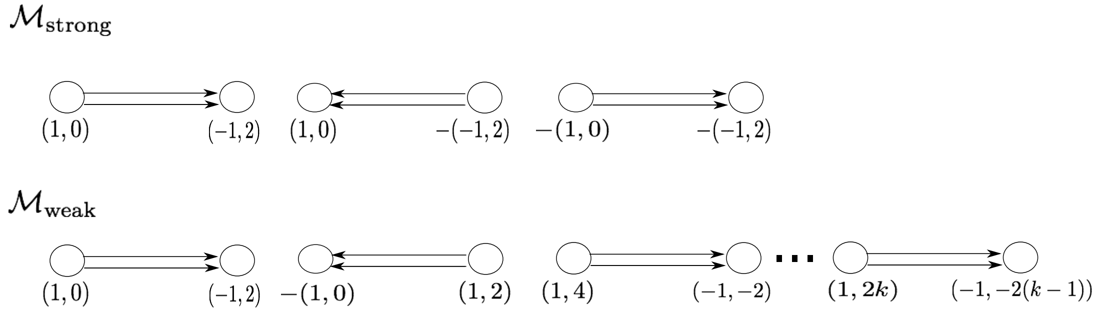

Figure 2.6: The mutations of the Seiberg-Witten quiver in the strong and weak coupling region.

2.6 Modular forms

Modularity plays an important role in mathematics and physics. In mathematics modular forms have applications in number theory, representation theory and algebraic geometry for example. In string theory, the 1-loop partition function of the closed string can be expressed via modular forms, due to the invariance of the torus under transformations of the modular group. In the context of S-duality, which makes it possible to relate a strongly coupled theory to a weekly coupled theory, modularity is one of the key aspects. In addition, the counting of black hole microstates can be expressed via the Fourier coefficients of modular forms. We give a brief introduction to elliptic, Siegel and mock modular forms following [220].

2.6.1 Elliptic and Siegel modular forms

The complex upper half plane $\mathbb{H}$ is the set of all complex numbers with positive imaginary part

$$\mathbb{H} = \{\tau \in \mathbb{C} | \operatorname{Im}(\tau) > 0\}. \tag{2.108}$$

We look at the action of the special linear group $\mathrm{SL}(2, \mathbb{R})$ consisting out of matrices of the following form

$$\mathrm{SL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{R},\; ad - bc = 1 \right\}. \tag{2.109}$$

On an element $\tau \in \mathbb{H}$ of the upper half plane it acts freely and transitively as the Möbius transformation

$$\tau \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma \in \mathrm{SL}(2,\mathbb{R}). \tag{2.110}$$

This transformation is only unique up to an overall factor of ± 1. Therefore it is sufficient to work with the group

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2. \tag{2.111}$$

The upper half plane $\mathbb{H}$ can also be characterised as

$$\mathbb{H} = \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}. \tag{2.112}$$

Consider the construction of the torus T^2 by means of two periods w_1 and w_2, $w_i \in \mathbb{C}^*, i = 1,2$ spanning a lattice Λ

$$\Lambda = \{aw_1 + bw_2 \,|\, a, b \in \mathbb{Z}\}. \tag{2.113}$$

Furthermore we assume that $w_2 \neq 0$ and $\mathrm{Im}\frac{w_1}{w_2} > 0$. Then the torus T^2 is given by taking the quotient of $\mathbb{C}$ with the lattice Λ

$$T^2 = \mathbb{C}/\Lambda. \tag{2.114}$$

This is simply the identification of the opposite sides of the parallelogram spanned by w_1 and w_2, see also figure 2.7. The lattice is invariant under transformations of the modular group $\Gamma_1 = \mathrm{SL}(2,\mathbb{Z})$

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z},\ ad - bc = 1. \tag{2.115}$$

Due to the assumptions about w_1 and w_2, this allows us to transform to a lattice spanned by

$$\Lambda_\tau = \{a\tau + b1,\ a, b \in \mathbb{Z}\}. \tag{2.116}$$

The parameter τ is called modular parameter and it is given by the ratio of w_1 and w_2. It is a parameter

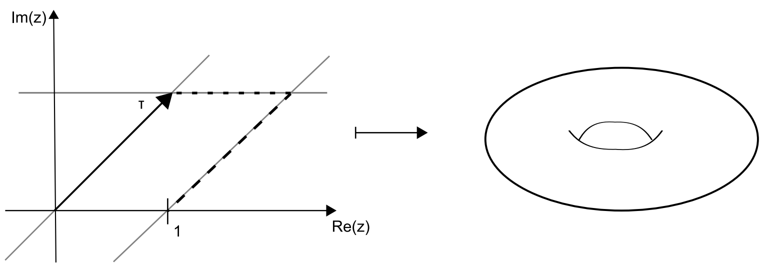

Figure 2.7: The construction of a torus $T^2 = \mathbb{C}/\Lambda_\tau$ with modular parameter τ.

of the moduli space of complex structures of the torus T^2. The torus T^2 is a genus one Riemann surface and for a more general treatment of Riemann surfaces see for example [221].

We can identify the generators of the modular group Γ_1 as the transformation $-\mathbb{1}_2$ and the so called T-

and S-transformations. They are explicitly given by

$$T:\tau \quad \mapsto \tau+1, \quad T \quad = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.117}$$

$$S:\tau \quad \mapsto -\tfrac{1}{\tau}, \quad S \quad = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.118}$$

These two operations can also be understood from the geometry of the torus by looking at the corresponding lattice Λ_τ. The T-transformation simply changes the basis vectors of the lattice from which the torus is constructed, but they still span the same lattice and therefore we obtain the same torus. The S-transformation exchanges the two cycles under a rescaling of 1 and τ. The moduli space of complex structures is the fundamental domain $\mathcal{F} = \mathbb{H}/\Gamma_1$ of the modular group $\mathrm{SL}(2,\mathbb{Z})$

$$\mathcal{F} = \left\{ z \in \mathbb{H} \,\middle|\, |z| > 1, -\frac{1}{2} \le \mathrm{Re}\, z \le \frac{1}{2} \right\}. \tag{2.119}$$

Different points in the fundamental domain $\mathcal{F}$ are not equivalent under the action of Γ_1. We visualize $\mathcal{F}$ in figure 2.8. For a proof that the fundamental domain $\mathcal{F}$ is given by (2.119) see for example [220].

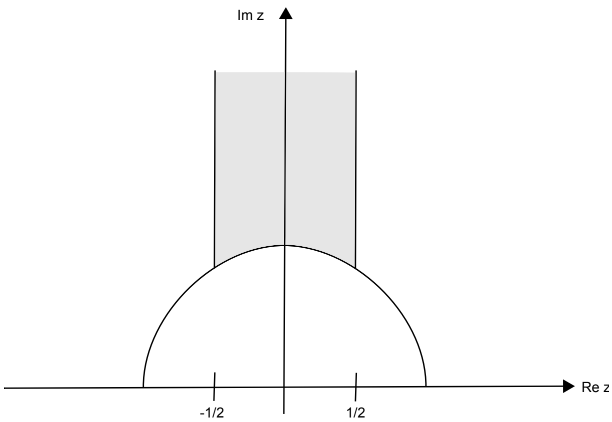

Figure 2.8: The fundamental domain $\mathcal{F} = \mathbb{H}/\Gamma_1$ represented by the grey shaded area.

Under $\tau \mapsto \tau + 1$ the fundamental domain $\mathcal{F}$ gets translated to the right. Under $\tau \mapsto -\frac{1}{\tau}$ the fundamental domain $\mathcal{F}$ is mapped to the interior of the circle and under the action of the full modular group Γ_1 we obtain a tessellation of the upper half plane $\mathbb{H}$. We now give the definition of a modular function followed by the definition of a modular form, which provides a different transformation behaviour, that is more suitable for applications.

Definition: A modular function is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that is invariant under the action of the modular group $\Gamma_1 = \mathrm{SL}(2,\mathbb{Z})$

$$f(\gamma\tau) = f(\tau), \qquad \forall \gamma \in \Gamma_1, \tau \in \mathbb{H}. \tag{2.120}$$

◇

Definition: A modular form of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that under

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$ the function f transforms as

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau). \tag{2.121}$$

In addition a modular form has subexponential growth at infinity, i.e. it has polynomial growth $f(\tau) = O(1)$ as $\tau_2 \to \infty$ and $f(\tau) = O(\tau_2^{-k})$ as $\tau_2 \to 0$ with $\tau = \tau_1 + i\tau_2$. ◇

We denote by $M_k(\Gamma)$ the space of all holomorphic modular forms of weight k on the discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$. The weight k can take values either in $\mathbb{Z}$ or in $\mathbb{Z}+\frac{1}{2}$. Every modular form can be expanded in a Fourier series as it is periodic under $f(\tau) = f(\tau + 1)$ given by a T-transformation,

$$f(\tau) = \sum_{n=0}^{\infty} a_n\, q^n, \quad q = e^{2\pi i\tau}. \tag{2.122}$$

For the case that $a_0 = 0$ the modular form $f(\tau)$ is called a cusp form.
We are now going to discuss a few examples.

Definition: The Eisenstein series E_k of weight k is given by

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z},\\ (c,d)=1}} \frac{1}{(c\tau + d)^k}. \tag{2.123}$$

◇

The weight k has to be even as otherwise it can be shown that the only modular form with odd weight is zero. For the case that $k > 2$ and even the Eisenstein series are modular forms. Let us consider the case of $k = 2$. Under a modular transformation $\gamma \in \Gamma_1$ the Eisenstein series $E_2(\tau)$ transforms as

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) - \frac{12}{2\pi i} c(c\tau + d). \tag{2.124}$$

From this we see that we get an additional term which spoils the demanded transformation behaviour. This problem can be solved by including an additional term containing τ_2, which turns $E_2(\tau)$ into a non-holomorphic modular form of weight two by

$$\hat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi\tau_2}. \tag{2.125}$$

Note that $E_2(\tau)$ falls in the class of quasi-modular forms and more generally almost holomorphic modular forms.

Definition: A function $\hat{f} : \mathbb{H} \to \mathbb{C}$ is called an almost holomorphic modular form of weight k on $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ if it transforms modular covariant under Γ and is subject to the same growth condition as modular forms but can be expanded as

$$\hat{f}(\tau, \bar{\tau}) = \sum_{k=0}^{N} f_k(\tau)\, \tau_2^{-k}, \quad N \geq 0. \tag{2.126}$$

◇

A quasi-modular form is given by $f_0(\tau)$ which itself can be expanded in terms of $E_2(\tau)$ as

$$f_0(\tau) = \sum_{k=0}^{N} h_k(\tau) E_2(\tau)^k, \tag{2.127}$$

with $h_k(\tau)$ a modular form.

The ring of modular forms $M_*(\Gamma_1) = \bigoplus_k M_k(\Gamma_1)$ is generated by E_4 and E_6

$$\begin{aligned} M_*(\Gamma_1) &= \mathbb{C}[E_4, E_6], \\ f(\tau) &= \sum_{4i+6j=k} c_{ij} E_4^i E_6^j, \end{aligned} \tag{2.128}$$

where $f(\tau)$ is any modular form of even weight k and c_{ij} are expansion coefficients. For this it is crucial to note that E_4^3 and E_6^2 are algebraically independent and therefore E_4 and E_6 as well.
The Fourier expansion of the Eisenstein series is given by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1}, \tag{2.129}$$

where B_k denotes the k-th Bernoulli number. The complex dimensions of the the corresponding space of modular forms can be shown to be

$$\dim M_k(\Gamma_1) = \begin{cases} 0 & k < 0,\ k \,\text{odd}, \\ \left\lfloor \frac{k}{12} \right\rfloor & k \equiv 2 \mod 12, \\ \left\lfloor \frac{k}{12} \right\rfloor + 1 & k \not\equiv 2 \mod 12\,. \end{cases} \tag{2.130}$$

From this we see that there are no modular forms for Γ_1 with negative weight.

Definition: The discrimant function $\Delta(\tau)$ is a modular form of weight 12 and we can write it in terms of E_4 and E_6 as

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \tag{2.131}$$

$$\Delta(\tau) = \frac{1}{1728} (E_4^3(\tau) - E_6^2(\tau)). \tag{2.132}$$

◇

Moreover the discriminant function $\Delta(\tau)$ is related to the Dedekind $\eta(\tau)$ function which appears quite often as its Fourier coefficient a_n is equal to the partitions $p(n)$ of n

$$\begin{aligned} \eta(\tau) &= \Delta(\tau)^{\frac{1}{24}} = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \\ \frac{1}{\eta(\tau)} &= q^{\frac{1}{24}} \sum_{n=0}^{\infty} p(n) q^n \\ &= q^{\frac{1}{24}} \left(1 + q + 2q^2 + 3q^3 + 5q^4 + \dots\right). \end{aligned} \tag{2.133}$$

It has the following transformation properties under S- and T-transformation

$$
\begin{aligned}
\eta(\tau) &\mapsto \eta(\tau+1) = e^{\frac{\pi i}{12}}\,\eta(\tau),\\
\eta(\tau) &\mapsto \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau).
\end{aligned}
\tag{2.134}
$$

Theta Series

We now present the notion of theta series. In particular they appear in the context of lattices see [222].

Definition: A theta series $\Theta_Q(\tau)$ is a modular form of weight $\frac{m}{2}$ associated to a positive-definite integer valued quadratic form

$$
Q : \mathbb{Z}^m \to \mathbb{Z},
\tag{2.135}
$$

and is given by

$$
\Theta_Q(\tau) = \sum_{x\in\mathbb{Z}^m} q^{Q(x)} = \sum_{n=0}^{\infty} R_Q(n) q^n.
\tag{2.136}
$$

The n-th Fourier coefficient $R_Q(n)$ with $n \geq 0$ of a theta series Θ_Q denotes the number of vectors $x \in \mathbb{Z}^m$ such that $Q(x) = n$. ⋄

If m is even we can write the quadratic form as

$$
Q(x) = \frac{1}{2}\, xAx,
\tag{2.137}
$$

where $A = (a_{ij})$, $i, j = 1, \ldots, m$ is a symmetric matrix. The entries a_{ij} are integers as Q is integer valued and for the diagonal elements we have $a_{ii} \in 2\mathbb{Z}$. Furthermore, as A should be positive definite we have $\det A > 0$ and for all of its minors. The transformation under a general modular transformation is given by a theorem of Hecke and Schoenberg, stating that

$$
\Theta_Q\left(\frac{a\tau+b}{c\tau+d}\right) = \chi_\Delta(a)(c\tau+d)^k\Theta_Q(\tau).
\tag{2.138}
$$

We denote by Δ the discriminant of Q and by $\chi_\Delta(p)$ the character with

$$
\Delta = (-1)^m \det A, \quad \chi_\Delta(p) = \left(\frac{\Delta}{p}\right).
\tag{2.139}
$$

We have introduced the Legendre symbol $(\cdot)$ defined as

$$
\left(\frac{a}{p}\right) = \begin{cases} 0 & a \equiv 0 \mod p,\\ 1 & a \not\equiv 2 \mod p,\ \exists\, x\in\mathbb{Z} : x^2 \equiv a \mod p,\\ -1 & \text{otherwise.}\end{cases}
\tag{2.140}
$$

We can also understand the notion of a theta series in terms of lattices Λ. The length $|p|$ of a lattice vector $p \in \Lambda$ can be calculated by the quadratic form $Q(p) = \frac{1}{2}pAp$. For the case that $\det A = 1$ we have a unimodular lattice and furthermore from the above properties one can show that the quadratic form Q is positive-definite, even and unimodular with rank $m \equiv 0 \mod 8$. An example for an even, unimodular lattice[5] Λ^8 of rank eight is given by the root lattice of the Lie algebra E_8. It is spanned by

[5] In fact it is also self-dual, i.e. $\Lambda^8 = \Lambda^{8*}$

the following set of simple roots

$$\begin{aligned}
\alpha_1 &= (0,1,1,0,0,0,0,0),\\
\alpha_2 &= (0,0,-1,1,0,0,0,0),\\
\alpha_3 &= (0,0,0,-1,1,0,0,0),\\
\alpha_4 &= (0,0,0,0,-1,1,0,0),\\
\alpha_5 &= (0,0,0,0,0,-1,-1,0),\\
\alpha_6 &= (0,0,0,0,0,0,1,1),\\
\alpha_7 &= \left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right),\\
\alpha_8 &= (0,0,0,0,0,0,1,-1).
\end{aligned} \tag{2.141}$$

We can check that this choice of simple root has the following properties

$$\alpha_i \cdot \alpha_j = A_{ij}, \tag{2.142}$$

where A_{ij} denotes the entries the Cartan matrix (A_{ij}) of E_8

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1\\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}. \tag{2.143}$$

The corresponding theta series is of weight $\frac{8}{2} = 4$ and as the first Fourier coefficient is one the corresponding theta series $\Theta_{\Lambda^{E_8}}$ is

$$\Theta_{\Lambda^{E_8}} = E_4\,. \tag{2.144}$$

For an even, self-dual and unimodular lattice of rank 16 we have two possibilities. On the one hand we can construct the direct sum $\Lambda^{16} = \Lambda^{E_8} \oplus \Lambda^{E_8}$ on the other hand there is a non decomposable lattice Λ^{16} which is related to the group $\mathrm{Spin}(32)/\mathbb{Z}_2$. The associated theta series however are the same as the Fourier series always starts with 1 and hence we get

$$\Theta_{\Lambda^{16}} = (\Theta_{\Lambda^{E_8}})^2. \tag{2.145}$$

After having discussed the relation of theta series to lattices we can focus on three examples in one dimension giving rise to the Jacobi theta series. The Jacobi theta series $\tilde{\vartheta}_3(\tau)$ is the associated series to the quadratic form $Q : x \mapsto x^2$

$$\tilde{\vartheta}_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}. \tag{2.146}$$

A function in one variable and of weight $\frac{1}{2}$ is called a unary theta series. For the case of $\tilde{\vartheta}_3(\tau)$ this can be checked by looking at the transformation properties under modular transformations

$$\begin{aligned} \tilde{\vartheta}_3(\tau+1) &= \tilde{\vartheta}_3(\tau), \\ \tilde{\vartheta}_3(-\frac{1}{4\tau}) &= \sqrt{-2i\tau}\,\tilde{\vartheta}_3(z). \end{aligned} \tag{2.147}$$

Additional unary theta series can be easily constructed by slightly modifying $\tilde{\vartheta}_3$ and are given by

$$\begin{aligned} \vartheta_m(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}, \\ \vartheta_f(\tau) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{n^2}. \end{aligned} \tag{2.148}$$

These three theta functions fulfill the Jacobi identity

$$\tilde{\vartheta}_3^4 - \vartheta_m^4 - \vartheta_f^4 = 0. \tag{2.149}$$

The theta functions that we will make use of in our calculation of partition functions are ϑ_2, ϑ_3 and ϑ_4. They are related to the theta series above by changing the argument τ to $\tau/2$

$$\tilde{\vartheta}_3\left(\frac{\tau}{2}\right) = \vartheta_3(\tau), \quad \vartheta_m\left(\frac{\tau}{2}\right) = \vartheta_4(\tau), \quad \vartheta_f\left(\frac{\tau}{2}\right) = \vartheta_2(\tau). \tag{2.150}$$

We will also make use of theta functions with rational characteristics a and b [223]. These are given by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau, z) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-a)^2} e^{2\pi i(v-b)(n-a)}. \tag{2.151}$$

Note that there are different conventions in the literature on how the characteristics enter in the above definition. In the context of $\mathbb{Z}_2$ orbifolds for example one uses $\frac{a}{2}$ and $\frac{b}{2}$ instead of a and b as this gives integer values. The theta functions with rational characteristics are examples of Jacobi forms, which satisfy a certain transformation behavior in z and τ.

Definition: A Jacobi form $\phi_m(\tau, z)$ of weight k and index m has the following properties:

$$\begin{aligned} \phi_m\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) &= (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \phi_m(\tau, z), \quad q = e^{2\pi i\tau}, y = e^{2\pi i z}, \\ \phi_m(\tau, z+\lambda\tau+\mu) &= e^{-2\pi i m(\lambda^2\tau+2\lambda z)} \phi_m(\tau, z), \quad \lambda, \mu \in \mathbb{Z}, \\ \phi_m(\tau, z) &= \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathbb{Z}, \\ r^2\leq 4mn}} c(n, r)\, e^{2\pi(n\tau+rz)}. \end{aligned} \tag{2.152}$$

◇

The Fourier expansion of a Jacobi form is given by

$$\phi_m(\tau, z) = \sum_{n,l\in\mathbb{Z}} c(n, l) q^n y^l, \quad c(n, l) = 0, \quad \text{for} \quad 4mn - l^2 \geq 0. \tag{2.153}$$

If we set z to zero we see, that we obtain the usual transformation behaviour of a modular form. A weak Jacobi form fulfils $c(n, l) = 0$ for $n \geq 0$. For more details on Jacobi forms see [224].

The relation to the Jacobi theta functions above is given by the following identification:

$$\vartheta_1(\tau) = \vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\tau,0) = 0, \qquad \vartheta_2(\tau) = \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(\tau,0), \tag{2.154}$$

$$\vartheta_3(\tau) = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau,0), \qquad \vartheta_4(\tau) = \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(\tau,0). \tag{2.155}$$

If $z = 0$ we simply write $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(0,\tau) = \vartheta\begin{bmatrix}a\\b\end{bmatrix}$. Under modular transformations the theta functions with characteristics (a, b) transform as

$$\begin{aligned}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau+1,z) &= e^{i\pi a(1-a)}\vartheta\begin{bmatrix}a\\a+b-1/2\end{bmatrix}(\tau,z),\\ \vartheta\begin{bmatrix}a\\b\end{bmatrix}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) &= \sqrt{-i\tau}\,e^{2i\pi ab+i\pi\frac{z^2}{\tau}}\,\vartheta\begin{bmatrix}b\\-a\end{bmatrix}(\tau,z).\end{aligned} \tag{2.156}$$

Instead of using the sum representation of the theta functions, they also admit a representation by products, which turns out to be useful in applications

$$\begin{aligned}\vartheta_1(\tau,z) = \vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\tau,z) &= 2q^{\frac{1}{8}}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1-q^ne^{2\pi iz})(1-q^ne^{-2\pi iz}),\\ \vartheta_2(\tau,z) = \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(\tau,z) &= 2q^{\frac{1}{8}}\cos(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1+q^ne^{2\pi iz})(1+q^ne^{-2\pi iz}),\\ \vartheta_3(\tau,z) = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau,z) &= \prod_{n=1}^{\infty}(1-q^n)(1+q^{n+\frac{1}{2}}e^{2\pi iz})(1+q^{n+\frac{1}{2}}e^{-2\pi iz}),\\ \vartheta_4(\tau,z) = \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(\tau,z) &= \prod_{n=1}^{\infty}(1-q^n)(1-q^{n+\frac{1}{2}}e^{2\pi iz})(1-q^{n+\frac{1}{2}}e^{-2\pi iz}).\end{aligned} \tag{2.157}$$

Hecke theory

We want to give short review on Hecke theory, since it will appear in the context of E-strings when determining the modified elliptic genus. We start by introducing the notion of the Hecke operator T_m.

Definition: The Hecke operator $T_m : M_k(\Gamma_1) \to M_k(\Gamma_1)$ with an integer $m \geq 1$ acts on a modular form $f(\tau) \in M_k(\Gamma_1)$ as follows

$$T_m f(\tau) = m^{k-1} \sum_{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_1\backslash\mathcal{M}_n} (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right), \tag{2.158}$$

with $\mathcal{M}_n = \{\mathrm{Mat}(2,\mathbb{Z}) : \det M = n\}$. ◇

The summation runs over $\Gamma_1\backslash\mathcal{M}_n$ and this can be reformulated to matrices of the form

$$\begin{pmatrix}a & b\\ 0 & d\end{pmatrix} \quad \text{with}\, ad = m, 0 \leq b < d. \tag{2.159}$$

such that the action of the Hecke operator becomes

$$T_m f(\tau) = m^{k-1} \sum_{\substack{ad=m,\\ a,d>0}} \frac{1}{d^k} \sum_{b \mod d} f\left(\frac{a\tau + b}{d}\right). \tag{2.160}$$

It can in fact be shown, that all the Hecke operators commute. As we have seen before, an important property of modular forms is that they are subject to a Fourier expansion and since the Hecke operator maps modular forms again to modular forms, we can ask for the Fourier expansion. It reads as follows

$$T_m f(\tau) = \sum_{n\geq 0} \left(\sum_{\substack{r|(m,n)\\ r>0}} r^{k-1} a_{\frac{mn}{r^2}} \right) q^n. \tag{2.161}$$

Siegel Modular Forms

Siegel modular forms provide a generalisation of the concept of modular forms for the modular group $SL(2,\mathbb{Z})$ to the symplectic group $Sp(2g,\mathbb{Z})$. This also includes replacing the upper half plane $\mathbb{H}$ by the so called Siegel upper half plane $\mathbb{H}_g$. Siegel modular forms appear in the context of physics for example in black hole microstate counting, where the Fourier coefficients of the corresponding Siegel modular form give the number of states see section 2.6.2. A good introduction to Siegel modular forms can be found in the article by van der Geer in [220].

Definition: The Siegel upper half plane $\mathbb{H}_g$ is defined as the set of $g \times g$ complex symmetric matrices with positive imaginary part (i.e. every entry has a positive imaginary part)

$$\mathbb{H}_g = \left\{ \Omega \in \mathrm{Mat}(g \times g, \mathbb{C}) | \, \Omega^T = \Omega, \ \mathrm{Im}\, \Omega > 0 \right\}. \tag{2.162}$$

For the case $g = 1$ we obtain the complex upper half plane $\mathbb{H}$ with $\Omega = \tau$. For the case that $g = 2$ we obtain the period matrix of a genus two Riemann surface

$$\Omega = \begin{pmatrix} \tau & \nu \\ \nu & \rho \end{pmatrix}. \tag{2.163}$$

We consider the lattice $\mathbb{Z}^{2g}$ with the symplectic form

$$\Sigma = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}. \tag{2.164}$$

The group $Sp(2g,\mathbb{Z})$ is the automorphism group with respect to this symplectic form

$$Sp(2g,\mathbb{Z}) = \{\gamma \in GL(2g,\mathbb{Z}) | \gamma^T \Sigma \gamma = \Sigma\}. \tag{2.165}$$

If we write for $\gamma \in Sp(2g,\mathbb{Z})$ a matrix representation

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.166}$$

where A, B, C, D are $g \times g$ matrices, then we can rewrite the definition (2.165) into conditions on the matrices A, B, C, D

$$AD^T - BC^T = \mathbb{1}_g, \quad AB^T = BA^T, \quad CD^T = DC^T. \tag{2.167}$$

On an element Ω of the Siegel upper half plane $\mathbb{H}_g$ the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ acts similarly as the modular group $\mathrm{SL}(2, \mathbb{Z})$ via

$$\Omega \mapsto \gamma\Omega = (A\Omega + B)(C\Omega + D)^{-1}, \quad \gamma \in \mathrm{Sp}(2g, \mathbb{Z}). \tag{2.168}$$

Denote a subgroup $\Gamma_g(N)$ of $\mathrm{Sp}(2g, \mathbb{Z})$ via

$$\Gamma_g(N) = \{\gamma \in \mathrm{Sp}(2g, \mathbb{Z}) | \gamma \equiv \mathbb{1}_{2g} \mathrm{mod}\, N\}. \tag{2.169}$$

This subgroup acts freely for the case that $N \geq 3$. Now we have all the notions we need in order to define a scalar valued/classical Siegel modular form.

Definition: A scalar valued/classical Siegel modular form of weight k is a holomorphic function

$$f : \mathbb{H}_g \to \mathbb{C}, \tag{2.170}$$

such that for all $\Omega \in \mathbb{H}_g$ and $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$:

$$f(\gamma\Omega) = \det(C\Omega + D)^k f(\Omega). \tag{2.171}$$

◇

This does not include the fact that the modular form is holomorphic at infinity which is only true for $g \geq 2$ due to the Koecher principle, see [220] which states that Siegel modular forms with the above conditions are bounded on subsets of the Siegel upper half plane $\mathbb{H}_g$. This ensures holomorphicity and allows for a Fourier expansion.

Denote by $\rho : \mathrm{GL}(g, \mathbb{C}) \to \mathrm{GL}(V)$ a representation where V denotes a finite-dimensional complex vector space.

Definition: A Siegel modular form of weight ρ is a holomorphic function $f : \mathbb{H}_g \to V$ if

$$f(\gamma\Omega) = \rho(C\Omega + D) f(\Omega), \quad \forall\gamma \in \mathrm{Sp}(2g, \mathbb{Z}), \Omega \in \mathbb{H}_g, \tag{2.172}$$

and in the case of $g = 1$ the function is holomorphic at infinity. ◇

The next step is to expand a Siegel modular form in a Fourier series, which can be performed as $f(\Omega + N) = f(\Omega)$ where N is an integral symmetric $g \times g$ matrix. Then we can write for $f(\Omega) : \mathbb{H}_g \to \mathbb{C}$

$$f(\Omega) = \sum_s a(s) e^{2\pi i \mathrm{Tr}(s\tau)}, \quad s \,\text{half integral}\, g \times g \,\text{matrix}, \quad a(s) \in V. \tag{2.173}$$

A matrix s is called half integral if $s \in \mathrm{GL}(g, \mathbb{Q})$ and $2s$ is an integral matrix with even diagonal entries. The trace in (2.173) can be evaluated

$$\mathrm{Tr}(s\Omega) = \sum_{i=1}^{g} s_{ii}\Omega_{ii} + 2 \sum_{1 \leq i < j \leq g} n_{ij}\Omega_{ij}. \tag{2.174}$$

Let $u \in \mathrm{GL}(g,\mathbb{Z})$ be a $g \times g$ matrix. Then the Fourier coefficients satisfy the following equation

$$a(u^T su) = \rho(u^T)a(s), \tag{2.175}$$

and from this one concludes that a scalar valued Siegel modular form of weight k with $kg \equiv 1 \bmod 2$ vanishes and furthermore due to the Koecher principle a Siegel modular form of negative weight vanishes. We want to discuss the Fourier expansion and the notion of theta series for Siegel modular forms. Before we can do this, we have to introduce some notions.

Definition: We define the Siegel operator $\Phi : M_\rho(\Gamma_g) \to M'_\rho(\Gamma_{g-1})$ on $M_\rho(\Gamma_g)$ for $f \in M_\rho(\Gamma_g)$ via

$$\Phi f = \lim_{t\to\infty} f\begin{pmatrix}\Omega' & 0\\ 0 & it\end{pmatrix}, \quad \Omega' \in \mathbb{H}_{g-1}, t \in \mathbb{R}. \tag{2.176}$$

◇

The Fourier expansion of Φf reads

$$(\Phi f)(\Omega') = \sum_{n'\geq 0} a\begin{pmatrix}n' & 0\\ 0 & 0\end{pmatrix} e^{2\pi i \mathrm{Tr}(n'\tau')}. \tag{2.177}$$

A Siegel modular form f is called cusp form, if $\Phi f = 0$ and we denote the corresponding set by

$$S_\rho(\Gamma_g) = \{f \in M_\rho(\Gamma_g) | \Phi f = 0\}\,. \tag{2.178}$$

For the construction of scalar valued Siegel forms one can introduce the Klingen Eisenstein series $E_{g,r,k}(f)$.

Definition: Let $0 \leq r \leq g$, $f \in S_k(\Gamma_r)$, $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and for $\Omega = \begin{pmatrix}\tau_1 & z\\ z & \tau_2\end{pmatrix}$, $\tau_1 \in \mathbb{H}_r$ and $\tau_2 \in \mathbb{H}_{g-r}$ we write $\tau^* = \tau_1$ and then the Klingen Eisenstein series $E_{g,r,k}$ is defined as

$$E_{g,r,k}(f) = \sum_{A\in P_r/\Gamma_g} f\left(\left(\frac{a\Omega+b}{c\Omega+d}\right)^*\right)\det(c\Omega+d)^{-k}. \tag{2.179}$$

The subgroup P_r is given by

$$P_r = \left\{\begin{pmatrix}a' & 0 & b' & *\\ * & u & * & *\\ c' & 0 & d' & *\\ 0 & 0 & 0 & u^{-t}\end{pmatrix} \in \Gamma_g \middle| \begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix} \in \Gamma_r,\ u \in \mathrm{GL}(g-r,\mathbb{Z})\right\}. \tag{2.180}$$

◇

This reduces to the known Eisenstein series for $r = 0$ and $f = 1$.

Theta series

A theta series of weight $\frac{1}{2}$ on $\Gamma_{2g}(2) \subset \mathrm{Sp}(2g, \mathbb{Z})$ is given by the following expression

$$\theta[\epsilon] = \sum_{m \in \mathbb{Z}^g} e^{2\pi i\left[\left(m+\frac{1}{2}\epsilon'\right)^T \tau\left(m+\frac{1}{2}\epsilon'\right)+\frac{1}{2}\left(m+\frac{1}{2}\epsilon'\right)^T \epsilon''\right]}, \tag{2.181}$$

where $\epsilon = \begin{pmatrix} \epsilon' \\ \epsilon'' \end{pmatrix}$ denotes the theta-constants and $\epsilon', \epsilon'' \in \{0,1\}^g$. We call ϵ odd if $\epsilon' \epsilon''^T$ is odd.

For the case that $g = 2$ one can construct the so called Igusa cusp form Φ_{10} of weight ten on $\mathrm{Sp}(4, \mathbb{Z})$

$$\Phi_{10} = -2^{14} \prod_{\epsilon \text{ even}} \theta[\epsilon]^2. \tag{2.182}$$

In contrast to the Fourier expansion of a Siegel modular form, there exists the Fourier-Jacobi development, which relates a scalar valued Siegel modular form to a Jacobi form. This expansion will be used in the context of mock modular forms. Let $\Omega \in \mathbb{H}_g$ and let it be explicitly given as

$$\Omega = \begin{pmatrix} \tau' & z \\ z^T & \tau'' \end{pmatrix}, \quad \tau' \in \mathbb{H}_1, z \in \mathbb{C}^{g-1}, \tau'' \in \mathbb{H}_{g-1}, \tag{2.183}$$

Then one uses the invariance under $\tau' \mapsto \tau' + b$ with $b \in \mathbb{Z}$ and that a modular form f can be expanded by a Fourier-Jacobi expansion as

$$f(\Omega) = \sum_{m=0}^{\infty} \phi_m(\tau'', z) e^{2\pi i m \tau'}. \tag{2.184}$$

and ϕ_m is holomorphic. For the case that $g = 2$ one finds that $\phi_m(\tau'', z)$ is a Jacobi form of weight k and index m and the corresponding set is denoted by $J_{k,m}$.

We finish this section on Siegel modular forms by stating the theorem by Igusa [225] telling us:

Theorem: The graded ring $M = \oplus_k M_k(\Gamma_2)$ of scalar valued modular forms of genus 2 is generated by $E_4, E_6, \Phi_{10}, \Phi_{12}, \Phi_{35}$ where one mods out $\Phi_{35}^2 = R$

$$M = \mathbb{C}[E_4, E_6, \Phi_{10}, \Phi_{12}, \Phi_{35}]/\{\Phi_{35}^2 = R\} \tag{2.185}$$

In this context R is a polynomial in E_4, E_6, Φ_{10} and Φ_{12}, which can be found in the original paper by Igusa [225]. Note furthermore that Φ_{12} denotes the weight 12 Siegel modular form.

2.6.2 Mock modularity

This section reviews the definition and some basic properties of mock modular forms. The remarkable fact about mock modular forms, besides their mathematical beauty, is the fact that 17 examples were discovered in Ramanujan's last letter to Hardy in 1920. However, it took till 2002 when a definition was given by Sander Zwegers. Various applications were discovered in the following [226, 227]. Starting with the general definition of a mock modular form, we give an overview of the three possible realisations of mock modular forms: Appell-Lerch sums, indefinite theta series and as Fourier coefficients of meromorphic Jacobi forms. For more details on mock modular forms we refer to [146–148, 228].

Mock modular forms

Following [147], we denote the space of mock modular forms of weight k by $\mathbb{M}_k$ and the space of modular forms by M_k. Mock modular forms are holomorphic functions of $\tau \in \mathbb{H}$, but do not transform in a modular covariant way. However, to every mock modular form h of weight k there exists a shadow $g \in M_{2-k}$ such that the function $\hat{h}$, given by

$$\hat{h}(\tau) = h(\tau) + g^*(\tau) \tag{2.186}$$

transforms as of weight k. Denoting by $g^c(z) = \overline{g(-\bar{z})}$, the completion $g^*(\tau)$ is defined by

$$g^*(\tau) = -(2i)^k \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-k} g^c(z)\, dz. \tag{2.187}$$

Thus, $\hat{h}$ is modular but has a non-holomorphic dependence. The corresponding space containing forms of type (2.186) is denoted by $\widehat{\mathbb{M}}_k$. Given g as the expansion $g(\tau) = \sum_{n\geq 0} b_n q^n$, the completion $g^*(\tau)$ can also be written as

$$g^*(\tau) = \sum_{n\geq 0} n^{k-1} \bar{b}_n \beta_k(4n\tau_2)\, q^{-n}, \tag{2.188}$$

with $\tau_2 = \mathrm{Im}\,(\tau)$ and β_k defined by

$$\beta_k(t) = \int_t^{\infty} u^{-k} e^{-\pi u} du. \tag{2.189}$$

Conversely, given $\hat{h}$, one determines the shadow g by taking the derivative of $\hat{h}$ with respect to $\bar{\tau}$. One easily sees that

$$\frac{\partial \hat{h}}{\partial \bar{\tau}} = \frac{\partial g^*}{\partial \bar{\tau}} = \tau_2^{-k} \overline{g(\tau)}. \tag{2.190}$$

This viewpoint opens another characterisation of $\widehat{\mathbb{M}}_k$ as the set of real-analytic functions F that fulfil a certain differential equation. To be precise, let us define the space $\mathfrak{M}_k$ as the space of real-analytic functions F in the upper half-plane $\mathbb{H}$ transforming as a modular form under $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$, i.e.

$$F(\gamma\tau) = \rho(\gamma)(c\tau + d)^k F(\tau),$$

where $\rho(\gamma)$ denotes some character of Γ and we demand exponential growth at the cusps. Hence, the space of completed mock modular forms $\widehat{\mathbb{M}}_k$ can now be characterised by

$$\widehat{\mathbb{M}}_k = \left\{ F \in \mathfrak{M}_k \,\Big|\, \frac{\partial}{\partial \tau}\left(\tau_2^k \frac{\partial F}{\partial \bar{\tau}}\right) = 0 \right\}. \tag{2.191}$$

This definition induces the following maps[6]

$$\mathfrak{M}_k = \mathfrak{M}_{k,0} \xrightarrow{\tau_2^k \partial_{\bar{\tau}}} \mathfrak{M}_{0,2-k} \xrightarrow{\tau_2^{2-k} \partial_{\tau}} \mathfrak{M}_{k,0} = \mathfrak{M}_k, \tag{2.192}$$

so that the composition can be converted to the Laplace operator in weight k. Hence, mock modular forms in $\widehat{\mathbb{M}}_k$ have the special eigenvalue $\frac{k}{2}\left(1 - \frac{k}{2}\right)$ and are sometimes also called harmonic weak Maass forms.

[6] A function $f \in \mathfrak{M}_{k,l}$ transforms under modular transformations $\gamma \in \Gamma$ with bi-weight (k,l) and character ρ, i.e. $f(\gamma\tau) = \rho(\gamma)(c\tau + d)^k (c\bar{\tau} + d)^l f(\tau)$.

Example: E_2 as a mock modular form

In the following a simple example of a mock modular form is presented. The modular completion of the holomorphic Eisenstein series E_2 has the form

$$\widehat{E_2}(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}\,.$$

From $\partial_{\bar{\tau}}\widehat{E}_2 = \tau_2^{-2}\frac{3i}{2\pi}$ we get $\overline{g} = \frac{3i}{2\pi}$, a constant shadow. Doing the integral indeed yields

$$g^*(\tau) = -(2i)^2\int_{-\bar{\tau}}^{\infty}(z+\tau)^{-2}\frac{3i}{2\pi}dz = -\frac{6i}{\pi}\left[\frac{-1}{z+\tau}\right]_{-\bar{\tau}}^{\infty} = -\frac{3}{\pi\tau_2}. \tag{2.193}$$

Further, there is a notion of mixed mock modular forms, which are functions that transform in the tensor space of mock modular forms and modular forms. However, we will call them simply mock modular forms as well.

Appell-Lerch sums

We define a Appell-Lerch sum $\mu(u, v, \tau)$ via the following expression

$$\mu(u,v,\tau) = \frac{a^{\frac{1}{2}}}{\vartheta_1(\tau,v)}\sum_{n\in\mathbb{Z}}\frac{(-b)^n q^{\frac{1}{2}n(n+1)}}{1-aq^n}, \quad a = e^{2\pi i u}, b = e^{2\pi i v}. \tag{2.194}$$

We summarise the modular properties under the generators S and T of the modular group

$$\begin{aligned} T: &\quad \mu(u,v,\tau+1) = e^{-\frac{2\pi i}{8}}\mu(u,v,\tau) \\ S: &\quad \left(\frac{\tau}{i}\right)^{-\frac{1}{2}} e^{\frac{\pi i(u-v)^2}{\tau}}\mu\left(\frac{u}{\tau},\frac{v}{\tau},-\frac{1}{\tau}\right) = -\mu(u,v,\tau) + \frac{1}{2}h(u-v,\tau), \end{aligned} \tag{2.195}$$

with the Mordell integral $h(z, \tau)$ given as

$$h(z,\tau) = \int_{-\infty}^{\infty}\frac{e^{\pi i x^2\tau - 2\pi x z}}{\cosh(\pi x)}\,dx \tag{2.196}$$

From (2.195) we can see that the Appell-Lerch sum fails to transform covariantly under modular transformations, though it almost transforms like a Jacobi form of weight $\frac{1}{2}$. The Appell-Lerch series can be turned into a modular object $\hat{\mu}$ by using the modular completion $R(z, \tau)$

$$\hat{\mu}(u,v;\tau) = \mu(u,v,\tau) - \frac{R(u-v,\tau)}{2}, \tag{2.197}$$

which is given as

$$R(z,\tau) = \sum_{v\in\mathbb{Z}+\frac{1}{2}}(-1)^{v-\frac{1}{2}}\left(\operatorname{sgn}(v) - E\left((v+\frac{z_2}{\tau_2})\sqrt{2\tau_2}\right)\right)e^{-2\pi i v z}q^{-\frac{v^2}{2}}. \tag{2.198}$$

The idea here is that the failure in modularity is compensated by the shadow in $\hat{\mu}$. For the proof we refer the reader to the literature [146]. The modular properties of $\hat{\mu}(u, v, \tau)$ read as follows

$$\begin{aligned} \hat{\mu}(u, v, \tau+1) &= e^{-\frac{2\pi i}{8}} \hat{\mu}(u, v, \tau), \\ \hat{\mu}\left(\frac{u}{\tau}, \frac{v}{\tau}, \frac{-1}{\tau}\right) &= -\left(\frac{\tau}{i}\right)^{\frac{1}{2}} e^{-\frac{\pi i (u-v)^2}{\tau}} \hat{\mu}(u, v, \tau) . \end{aligned} \tag{2.199}$$

This concludes our discussion about Appell-Lerch sums.

Indefinite theta-series

We already discussed ordinary theta series as vector valued modular forms in section 2.6.1. A general theta function $\Theta_{a,b}(\tau)$ is defined as follows

$$\Theta_{a,b}(\tau) = \sum_{n \in \Lambda + a} e^{2\pi i \langle b, n\rangle} q^{Q(n)}, \tag{2.200}$$

with Λ a lattice and Q a quadratic form. Now we assume, that the quadratic form Q has a signature $(r-1, 1)$. With this quadratic form Q we have to take care of the convergence of the theta series, which is spoiled by $v \in \Lambda$ that have a negative quadratic form. To restore convergence, we remove the vectors with negative quadratic form by introducing a cone C via

$$C = \{x \in \mathbb{R}^r : Q(x) < 0\}. \tag{2.201}$$

This leads to the indefinite theta series $\Theta_{a,b}^{c,c'}$ which is defined via

$$\Theta_{a,b}^{c,c'}(\tau) = \sum_{n \in \Lambda + a} \left(\operatorname{sgn}(\langle c, n\rangle) - \operatorname{sgn}(\langle c', n\rangle)\right) e^{2\pi i \langle b, n\rangle} q^{Q(n)}, \quad c, c' \in C. \tag{2.202}$$

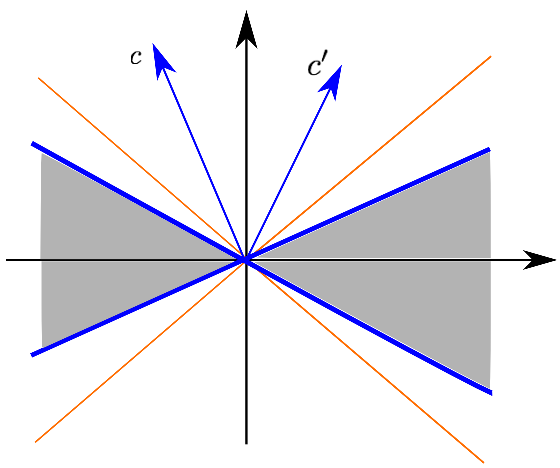

Figure 2.9: An indefinite theta function for a lattice of signature $(1, 1)$. The choice of c, c' and the region the indefinite theta function sums over is coloured in grey.

However, the indefinite theta function in (2.202) does not transform modular covariant. In this case the modular completion is achieved by the following procedure:

$$\hat{\Theta}^{c,c'}_{a,b}(\tau) = \sum_{n\in\Lambda+a}\left(E\left(\frac{\langle c,n\rangle\sqrt{\tau_2}}{\sqrt{-Q(c)}}\right) - E\left(\frac{\langle c',n\rangle\sqrt{\tau_2}}{\sqrt{-Q(c')}}\right)\right)e^{2\pi i\langle b,n\rangle}q^{Q(n)}, \tag{2.203}$$

where $E(x) = \mathrm{sgn}(x)(1-\beta_{\frac{1}{2}}(x^2))$ and this allows to write the modular completion of the indefinite theta function as

$$\hat{\Theta}^{c,c'}_{a,b}(\tau) = \Theta^{c,c'}_{a,b}(\tau) - \Phi^{c}_{a,b}(\tau) - \Phi^{c'}_{a,b}(\tau). \tag{2.204}$$

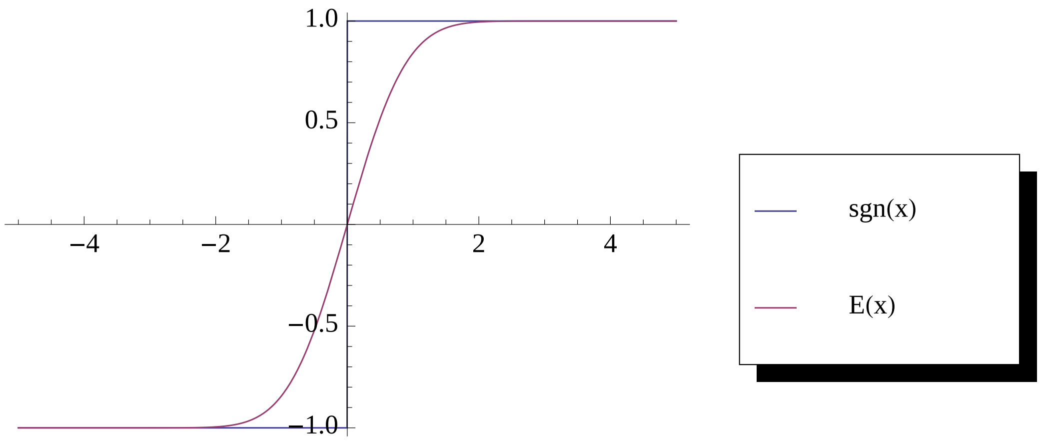

Figure 2.10: The sign and the error function

Note that we basically replaced the sgn function by the smooth error function.

Fourier coefficients of meromorphic Jacobi forms

The last representation of mock modular forms is via Fourier coefficients of meromorphic Jacobi forms. We give a brief overview of this and especially we give a short overview of this in the context of $\mathcal{N}=4$ black hole microstates [146, 150]. Given a Jacobi-form $\phi(\tau,z)$ it is subject to a Fourier expansion

$$\phi(\tau,z) = \sum_{l \bmod N} h_l(\tau)\Theta_{N,l}(\tau,z), \tag{2.205}$$

where $h_l(\tau)$ is the Fourier coefficient of the Jacobi form $\phi(\tau,z)$

$$h^{z_0}_l(\tau) = e^{-\frac{\pi i l^2\tau}{N}}\int_{z_0}^{z_0+1} e^{-2\pi i l z}\phi(\tau,z)\,dz, \quad z_0\in\mathbb{C}. \tag{2.206}$$

We assume the Jacobi form $\phi(\tau,z)$ to be meromorphic with respect to z. However, this raises the question on how to take care of the poles of $\phi(\tau,z)$ when calculating $h_l(\tau)$ via a Fourier integral. Due to the pole structure the evaluation of $h_l(\tau)$ now depends on the choice of z_0 and $\phi(\tau,z)$ is not modular and also not periodic in l. In the following it is assumed, that z_0 is not a pole of $\phi(\tau,z)$ and the integral is calculated along the line that connects z_0 and z_0+1 without crossing a pole. Concerning the periodicity in l, it can be shown that

$$h^{z_0+\tau}_l(\tau) = h^{z_0}_{l+2m}(\tau) \tag{2.207}$$

which implies that the l-th Fourier coefficient is given by the expansion of the finite part of ϕ denoted by ϕ^F

$$\phi^F(\tau, z) = \sum_{l \in \mathbb{Z}/2m\mathbb{Z}} h_l^{-\frac{l\tau}{2m}}(\tau)\Theta_{m,l}(\tau, z). \tag{2.208}$$

For the case that ϕ is holomorphic we have $\phi = \phi^F$. It was shown in [150] that for a meromorphic Jacobi-form $\phi(\tau, z)$ with simple poles $z = z_s = \alpha\tau + \beta$, $(\alpha, \beta) \in S \subset \mathbb{Q}^2$ we capture the information about the poles in $D_s(\tau)$

$$D_s(\tau) = 2\pi i e^{2\pi i m \alpha z_s} \mathrm{Res}_{z=z_s} \phi(\tau, z). \tag{2.209}$$

Then the Jacobi form $\phi(\tau, z)$ is subject to the following decomposition into a finite part $\phi^F(\tau, z)$ and a polar part $\phi^P(\tau, z)$

$$\phi(\tau, z) = \phi^F(\tau, z) + \phi^P(\tau, z), \tag{2.210}$$

where the polar part $\phi^P(\tau, z)$ is given as

$$\phi^P(\tau, z) = \sum_{s \in S/\mathbb{Z}^2} D_s(\tau) A_m^s(\tau, z), \tag{2.211}$$

with $A_m^s(\tau, z)$ being the universal Appell-Lerch sum. If $\phi(\tau, z)$ is an ordinary Jacobi form, then $\phi(\tau, z) = \phi(\tau, z)^F$. The universal Appell-Lerch sum is given by

$$A_m^s(\tau, z) = e^{-2\pi i m \alpha z_s} Av^m \left[R_{-2m\alpha} \left(\frac{e^{2\pi i z}}{e^{2\pi i z_s}} \right) \right]. \tag{2.212}$$

We define the components of the universal Appell-Lerch sum. First, we introduce the averaging operator $Av^m(F(y))$ as

$$Av^m(F(y)) = \sum_{\lambda \in \mathbb{Z}} q^{m\lambda^2} y^{2m\lambda} F(q^\lambda y), \tag{2.213}$$

which maps any polynomial in y to a function in z with a transformation behaviour like a Jacobi form of index m. Furthermore we introduce the rational function $R_c(y)$

$$R_c(y) = \frac{1}{2} \frac{y^{\lfloor c \rfloor + 1} + y^{\lceil c \rceil}}{y - 1}, \tag{2.214}$$

and it can be shown, that each $h_l(\tau)$ is a mixed mock modular form of weight $k - \frac{1}{2}|k - 1$ which has a modular completion $\hat{h}_l(\tau)$ given by

$$\hat{h}_l(\tau) = h_l(\tau) - \sum_{s \in S/\mathbb{Z}^2} D_s(\tau) \Theta_{m,l}^{s^*}(\tau), \quad \Theta_{m,l}^{s^*}(\tau) = \frac{e^{2\pi i m \alpha \beta}}{2} \sum_{\lambda \in \mathbb{Z} + \alpha + l/2m} \mathrm{sgn}(\lambda) e^{-4\pi i m \beta \lambda} \mathrm{E}(2|\lambda| \sqrt{\pi m \tau_2}) q^{-m\lambda^2}, \tag{2.215}$$

where $\Theta_{m,l}^{s^*}$ is known as the Eichler integral. The completion $\hat{\phi}^F$ then takes the following form

$$\sum_{l \bmod 2m} \hat{h}_l(\tau) \theta_{m,l}(\tau, z). \tag{2.216}$$

For a further discussion we refer to the literature [150]. Instead we want to review the appearance of mock Jacobi-forms in the context of $\mathcal{N} = 4$ black hole microstate counting. The idea that the counting function is described by a Siegel-Modular form goes back to [111] and has been developed futher in [229]. The moduli space and the wall-crossing phenomena are discussed in [149, 230, 231].

The relation to mock modular forms is first discussed in [150]. In general useful reviews can be found in [232].

The underlying physical setup is that of Cadhuri-Hockney-Lykken (CHL) compactifications. In the type II picture we compactify on $(K3\times T^2)/\mathbb{Z}_n$ which is dual to a compactification of the heterotic string on $T^6/\mathbb{Z}_n$, where the $\mathbb{Z}_n$ action is the CHL orbifold. The orbifold acts in such a way that we obtain $\mathcal{N}=4$ supersymmetry and for $n>1$ the gauge group is reduced. For more details on the action of these orbifolds see [233, 234]. We focus on the case that we have a trivial orbifold action and for CHL orbifolds the results can be generalized [229, 235–239] The S-duality group is in this setup[7] $\mathrm{SL}(2,\mathbb{Z})$ and the T-duality group is $\mathrm{O}(22,6,\mathbb{Z})$. We have charges living in a lattice $\Lambda^{22,6}$ where we have a charge vector $\Gamma=(p^I,q^I)$ with $I=1,\ldots,28$. It is possible to count BPS states and in fact they can be captured by the unique Igusa cusp form $\Phi_{10}(\Omega)$ [111,230,240,241]. The degeneracies $\Omega(\Gamma)$ are usually expressed in terms of T-duality invariants[8]

$$\Omega(\Gamma)=\Omega\left(\frac{1}{2}p^2,\frac{1}{2}q^2,q\cdot p\right). \tag{2.217}$$

These are encoded in the Fourier coefficients of Φ_{10}^{-1}

$$\frac{1}{\Phi_{10}}=\sum_{\substack{m\geq -1\\ n\geq -1\\ l}} e^{2\pi i(m\tau+n\rho+l\nu)}\Omega(m,n,l). \tag{2.218}$$

The Igusa cusp form has several representations and we collect some of them in table 2.10.

name	representation
product representation	$\Phi_{10}(\Omega)=qyp\prod_{n,l,m}(1-q^n y^l p^m)^{c(4mn-l^2)}$, $c(4mn-l^2)$ are determined form the elliptic genus of $K3$ $$\begin{aligned} Z_{\mathrm{K3}}(\tau,z)&=8\left(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau)^2}+\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau)^2}+\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau)^2}\right)\\ &=\sum_{\substack{n\geq 0\\ r\in\mathbb{Z}}} c(4n-r^2)q^n y^r. \end{aligned} \tag{2.219}$$
Determinant representation	$\Phi_{10}(\Omega)=2^{-12}\prod_{\substack{a,b\\ 4a\cdot b\in 2\mathbb{Z}}}\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(\Omega)$
Maass lift of ϕ_{10}	$\phi_{10}(\rho,\nu)=\eta^{24}(\rho)\frac{\vartheta_1^2(\rho,\nu)}{\eta^6(\rho)}$.

Table 2.10: Representations of the Igusa cusp form Φ_{10}.

[7] For the other CHL orbifolds, we get a congruence subgroup $\Gamma_1(n)$

[8] Note that in here we denote the charges by (p,q) as well as the formal variables $q=e^{2\pi i\tau}$ and $p=e^{2\pi i\rho}$. The meaning should be clear from the context.

Formula (2.218) can be inverted such, that it is possible to calculate the $\Omega(\Gamma)$ via

$$\Omega(\Gamma) = \int_C d^3\Omega \, e^{-i\pi\Gamma^T\cdot\Omega\cdot\Gamma} \frac{1}{\Phi_{10}}, \tag{2.220}$$

where C denotes the contour [149]. However, the contour depends on the pole structure of Φ_{10}^{-1} which is reflected in its wall-crossing behaviour. Also note that we can study the limit, where the period $\nu \to 0$, which is the limit in which the genus two surface splits into two genus one surfaces. In this limit the dyon partition function behaves in the following way

$$\lim_{\nu\to 0} \frac{1}{\Phi_{10}} = \frac{1}{4\pi\nu^2} \frac{1}{\eta^{24}(\tau)\eta^{24}(\rho)}, \tag{2.221}$$

which reproduces the expected result for a $K3$ surface as $\eta^{-\chi(\mathrm{K3})}$. To make contact with our previous discussion about meromorphic Jacobi forms, we consider the following Fourier-Jacobi expansion

$$\frac{1}{\Phi_{10}} = \sum_{m=-1}^{\infty} \psi_m(\tau,\nu) p^m, \quad p = e^{2\pi i\rho}, \tag{2.222}$$

where $\psi_m(\tau,z)$ becomes a meromorphic Jacobi form of weight 2 with a double pole at $\nu = 0$. The polar part of $\psi_m(\tau,\nu)$ has the following structure

$$\hat{\psi}_m^P(\tau,\nu) = \frac{\mathrm{Coeff}_{q^m}(\Delta^{-1}(\tau))}{\Delta(\tau)} A_{2,m}(\tau,\nu), \quad A_{2,m}(\tau,\nu) = \sum_{n\in\mathbb{Z}} \frac{q^{mn^2+s} y^{2mn+1}}{(1-q^n y)^2}, \tag{2.223}$$

The modular completion in this case is due to two-centred black holes, and is subject to wall-crossing. The anti-holomorphic derivative of the mock modular form calculates the shadow. We will encounter this phenomenon also in this thesis, when we discuss holomorphic anomalies.

Mathieu Moonshine

Another place, where mock modular forms lead to an interesting new phenomenon is in the context of Mathieu moonshine and the elliptic genus of K3. Monstrous moonshine leads to an interesting relation between the Monster group, modular forms and conformal field theory. Mathieu moonshine seems to follow a similar pattern for mock modular forms and the Mathieu group M24. If course it is very tempting to perform an analysis similar to the case of Monstrous moonshine. This also represents another example where ideas from physics and mathematics lead to new and interesting results as well as a new understanding. The observation was made in [157, 158, 242–251].

The elliptic genus of $K3$ is defined as

$$Z_{K3}(\tau,z) = \mathrm{Tr}_{RR}(-1)^{F+\bar{F}} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} e^{2\pi i z J_0}, \tag{2.224}$$

where the trace is taken over the RR sector of the theory and due to spectral flow symmetry the elliptic genus does not depend on $\bar{q}$. We denote by J_0 the zero mode of the $\mathfrak{su}(2)$ algebra. Futhermore, spectral flow symmetry implies that $Z_{K3}(\tau,z)$ is a Jacobi form of weight zero and index $m = \frac{c}{6} = 0$. The corresponding space of Jacobi forms is one-dimensional and therefore takes the following form

$$Z_{K3}(\tau,z) = 8\left[\left(\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau,0)}\right)^2 + \left(\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau,0)}\right)^2 + \left(\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau,0)}\right)^2\right]. \tag{2.225}$$

This also satisfies some consistency conditions, like $\chi(K3) = Z_{K3}(\tau, z = 0) = 24$ and the signature $Z_{K3}(\tau, z = \frac{1}{2}) = 16 + O(q)$. The above result can be rewritten in terms of characters for the short representations of the Ramond sector $\chi_{\text{BPS}} = \chi^R_{h=\frac{1}{4},l=0}(\tau, z)$ and the non-BPS characters $\chi_{\text{non-BPS}}$.

$$
\begin{aligned}
Z_{K3}(\tau, z) &= 24\chi_{\text{BPS}} + \Sigma(\tau)\chi_{\text{non-BPS}} \\
&= 24\chi^R_{h=\frac{1}{4},l=0}(\tau, z) + \Sigma(\tau)\frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^3},
\end{aligned}
\tag{2.226}
$$

with

$$
\begin{aligned}
\chi_{\text{non-BPS}} &= \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^3}, \\
\chi_{\text{BPS}} &= \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^3}\mu(\tau, z), \\
\mu(\tau, z) &= \frac{-ie^{\pi i z}}{\vartheta_1(\tau, z)} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} e^{2\pi i n z}}{1 - q^n e^{2\pi i z}}, \\
\Sigma(\tau) &= -8\left[\mu(\tau, z = \frac{1}{2}) + \mu(\tau, z = \frac{1+\tau}{2} + \mu(\tau, z = \frac{\tau}{2}))\right] \\
&= -2q^{-\frac{1}{8}}\left(1 - \sum_{n=1}^{\infty} A_n q^n\right).
\end{aligned}
\tag{2.227}
$$

Again, we encounter the Appell-Lerch sum in the character of the BPS states, which is the link to mock modular forms. The coefficients A_n have the following asymptotic behaviour, as was shown by use of the Rademacher expansion [151, 153]

$$
A_n \approx \frac{2}{\sqrt{8n-1}} e^{2\pi\sqrt{\frac{1}{2}(n-\frac{1}{8})}}
\tag{2.228}
$$

The first few numbers are collected in table 2.11.

n	1	2	3	4	5	6	7	8	9	...
A_n	45	231	770	2277	5769	13915	30843	65550	132825	...

Table 2.11: The coefficients A_n that can be decomposed into dimensions of irreducible representations of the Mathieu group.

The crucial observation is that the first five coefficients are equal to the dimension of representations of M_{24} and A_6 and A_7 can be decomposed into the sum of irreducible representations, for higher n this is still possible but not unique. In the following section we collect some facts about the Mathieu group M_{24}.

The Mathieu group M_{24}

In this section we collect some facts about the Mathieu group M_{24}. It can be thought of as a subgroup of the permutation group S_{24} and it contains 244823040 elements. It contains 26 conjugacy classes and 26 irreducible representations. However, M_{24} can also be understood in terms of a rank 24 even self-dual lattice. The Mathieu group M_{24} is the subgroup of permutations S_{24} of the coordinates of $\mathbb{Z}_2^{24}$ which preserve $\mathcal{G}$, where $\mathcal{G} = N/A_1^{24}$, N being an even-self dual lattice, such that

$$A_1^{24} \subset N \subset A_1^{24*}, \tag{2.229}$$

where A_1^{24} is the rank 24 lattice obtained from the root lattice of A_1 and A_1^{24*} the corresponding dual lattice and $\mathbb{Z}_2^{24} \simeq A_1^{24*}/A_1^{24}$. Furthermore $\mathcal{G}$ has to satisfy other properties as well, which fix it uniquely:

- An element of $\mathcal{G}$ is represented by 24 entries being either 0 or 1 as it is a subgroup of $\mathbb{Z}_2^{24}$.
- As N is even self-dual, $\mathcal{G}$ is 12-dimensional and the weight[9] of every element of $\mathcal{G}$ is 4.
- Identifying those elements of N with length squared equal to two are the roots of A_1^{24}.

Recall, that the cohomology lattice $H^*(K3,\mathbb{Z})$ of $K3$ is also a 24-dimensional even, self-dual lattice. In physics notation this is sometimes called the quantum lattice

$$H^*(K3,\mathbb{Z}) = H^0(K3,\mathbb{Z}) \oplus H^2(K3,\mathbb{Z}) \oplus H^4(K3,\mathbb{Z}) = \Lambda^{\mathrm{E}_8} \oplus \Lambda^{\mathrm{E}_8} \oplus \Lambda^{1,1} \oplus \Lambda^{1,1} \oplus \Lambda^{1,1} \oplus \Lambda^{1,1} \tag{2.230}$$

It was shown by Nikulin and others, that the symmetry group preserving the holomorphic two-form is a strict subgroup of M_{24}.

Twisted characters and relation to $\mathcal{N} = 4$ dyons

As we saw, $\Sigma(\tau)$ contains informations about the dimensions of the irreducible representations of M_{24}. These are graded with respect to the zero mode of the Virasoro algebra L_0 and therefore we have [158]

$$\begin{aligned} R(M_{24}) &= \bigoplus_{n=1}^{\infty} R_n(M_{24}) \\ \Sigma(\tau) &= \sum_{n=1}^{\infty} \dim(R_n) q^n \end{aligned} \tag{2.231}$$

However, this result may be generalised to twisted $K3/\mathbb{Z}_p$ surfaces, where the twist is generated by a $\mathbb{Z}_p$ action with $p = 2, 3, 5, 7$ and the associated twisted character reads

$$\Sigma_g(\tau) = \sum_{n=1}^{\infty} \mathrm{Tr}_{R_n(M_{24})}(g) q^n, \quad g \in M_{24}. \tag{2.232}$$

These quotients have been classified in Nikuhlin's list [252]. There is a version of moonshine for M_{24}, which works as follows. Let g be an element of M_{24} than the corresponding partition function reads [158]

$$g \,\text{with cycle shape} \quad 1^{i_1} 2^{i_2} \cdots r^{i_r} \mapsto \eta_g(\tau) = \prod_{l=1}^{r} \eta(l\tau)^{i_l}. \tag{2.233}$$

These two versions of the Monstrous moonshine for M_{24} are related by a generalised Borcherds-Kac Moody algebra, i.e. an infinite dimensional algebra, with non-positive definite Cartan matrix which in

[9] which is defined by the number 1's in the representation of an element

addition contains imaginary roots. The character of such an algebra is subject to the so-called denominator identity

$$e^{\rho}\prod_{\alpha\in\Lambda_+}(1-e^{\alpha})^{\mathrm{mult}(\alpha)}=\sum_{w\in W}\epsilon(w)w(e^{\rho},\Sigma) \tag{2.234}$$

where we denote by ρ the Weyl vector, Λ_+ the set of positive roots and Σ a combination of imaginary roots. The Weyl group controls actually the wall-crossing phenomena [231] as can be seen by using the product representation of the generating function Φ_p of the CHL compactification $(\mathrm{K3}\times T^2)/\mathbb{Z}_p$ and hence this provides a link to $\mathcal{N}=4$ black hole results. It is possible to interpret the subalgebra of positive roots as an infinite-dimensional representation of M_{24}. The η_g products are recovered by making use of the behaviour at the pole $\nu\to 0$.

2.7 Topological string theory

Topological string theory can be described by a two dimensional non-linear sigma model with target space being a Calabi-Yau manifold. We start with a short review of the $\mathcal{N}=2$ superconformal algebra and then we switch to $\mathcal{N}=(2,2)$ theories. We review the construction of the A and the B model and how they are related by mirror symmetry. We finish this section with a discussion of the holomorphic anomaly equations, their interpretation and how they can be used to solve for the higher genus topological amplitudes. Reviews on topological string theory include [89, 95, 96, 194, 253–257] and we mainly follow [96, 257].

2.7.1 The chiral ring structure of topological string theory

The $\mathcal{N}=2$ superconformal algebra consists of the energy-momentum tensor $T(z)$, two currents $G^{\pm}(z)$ and a U(1) current $J(z)$. We collect their properties in table 2.12. The currents $G^{\pm}(z)$ are fermionic and are therefore subject to periodic or antiperiodic boundary conditions

$$G^{\pm}(e^{2\pi i}z)=-e^{\mp 2\pi i a}G^{\pm}(z). \tag{2.235}$$

From the operator product expansion one shows the following relations for the modes

$$\begin{aligned}
[L_m,L_n]&=(m-n)L_{m+n}+\frac{c}{12}m(m^2-1)\delta_{m+n,0},\\
[J_m,J_n]&=\frac{c}{3}m\delta_{m+n,0},\\
[L_n,J_m]&=-mJ_{m+n},\\
[L_n,G^{\pm}_{m\pm a}]&=\left(\frac{n}{2}-(m\pm a)\right)G^{\pm}_{m+n\pm a},\\
\{G^{+}_{n+a},G^{-}_{m-a}\}&=2L_{m+n}+(n-m+2a)J_{n+m}+\frac{c}{3}\left((n+a)^2-\frac{1}{4}\right)\delta_{m+n,0}.
\end{aligned} \tag{2.236}$$

From these relations we recognise the Virasoro algebra of the energy-momentum modes. Note, that for different values of a the algebras are isomorphic to each other due to the existence of a spectral flow symmetry. The representation theory of this algebra is as follows: a highest weight state $|\phi\rangle$ is an eigenstate under the Cartan elements L_0 and G_0 and is annihilated by the positive modes

$$\begin{aligned}
&L_0|\phi\rangle=h_\phi|\phi\rangle,\quad J_0|\phi\rangle=q_\phi|\phi\rangle\\
&L_n|\phi\rangle=0,\quad G^{\pm}_r|\phi\rangle=0,\quad J_m|\phi\rangle=0,\qquad \forall n,m,r>0.
\end{aligned} \tag{2.237}$$

In order to construct chiral rings, we define a chiral field to be a primary field, i.e. a field ϕ such that

name	conformal weight h	expansion
energy-momentum tensor $T(z)$	2	$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}$
currents $G^\pm(z)$	$\frac{3}{2}$	$G^\pm = \sum_{n\in\mathbb{Z}} \frac{G^\pm_{n\pm a}}{z^{n\pm a+\frac{3}{2}}}$
$U(1)$ current $J(z)$	1	$J(z) = \sum_{n\in\mathbb{Z}} \frac{J_n}{z^{n+1}}$

Table 2.12: Operators of the $\mathcal{N} = 2$ SCFT, their conformal weight and their expansion.

$\phi|0\rangle = |\phi\rangle$ that in addition is annihilated by $G^+_{-\frac{1}{2}}$

$$G^+_{-\frac{1}{2}}|\phi\rangle = 0. \tag{2.238}$$

The notion of anti-chiral fields relates to the fact, that these chiral fields are annihilated by $G^-_{-\frac{1}{2}}$. If one furthermore includes the anti-holomorphic generators $\bar{G}^\pm$, then one can construct several pairs of rings that are collected in table 2.13. The ring structure is ensured by the following relation, where we denote

Ring	annihilation operators
(c, c)	$(G^+_{-\frac{1}{2}}, \bar{G}^+_{-\frac{1}{2}})$
(a, c)	$(G^-_{-\frac{1}{2}}, \bar{G}^+_{-\frac{1}{2}})$
(c, a)	$(G^+_{-\frac{1}{2}}, \bar{G}^-_{-\frac{1}{2}})$
(a, a)	$(G^-_{-\frac{1}{2}}, \bar{G}^-_{-\frac{1}{2}})$

Table 2.13: The ring structure of chiral primary fields and the corresponding annihilation operators of $|\phi\rangle$.

the set of chiral primaries by ϕ_i

$$\phi_i\phi_j = C^k_{ij}\phi_k, \tag{2.239}$$

with C^k_{ij} being the three-point function on the sphere $C_{ijk} = \langle\phi_i\phi_j\phi_k\rangle$ and the indices are raised with respect to the topological metric $g_{ij} = \langle\phi_i\phi_j\rangle$. By using the relations in (2.236) it is possible to show the following relation between the weights h_ϕ, q_ϕ and the central charge c

$$h_\phi \geq \frac{q_\phi}{2}, \quad h_\phi \geq \frac{c}{6}. \tag{2.240}$$

2.7.2 Deformations

In a next step, we discuss deformations of the underlying theory, i.e. we add marginal operators having weight $h + \bar{h} = 2$ to the original action. Thus, this allows us to consider a flow from one CFT to another and these deformation operators span the moduli space of the CFT. We denote by Σ_g the world sheet of the CFT

$$\delta S = z^i \int_{\Sigma_g} \phi_i^{(2)} + \bar{z}^{\bar{i}} \int_{\Sigma_g} \bar{\phi}_{\bar{i}}^{(2)}. \tag{2.241}$$

These operators can be created as follows, where we first start with the (c, c) ring. We start with the following set of operators[10] $\phi_i^{(0)}$ and deform it to $\phi_i^{(1)}$ using the commutator

$$\phi_i^{(1)} = \left[G^-(z), \phi_i^{(0)}(w, \bar{w})\right] = \oint dz\, G^-(z)\, \phi(w, \bar{w}). \tag{2.242}$$

In a second step we construct $\phi_i^{(2)}$ as follows

$$\phi_i^{(2)} = \left\{\bar{G}^-(z), \phi_i^{(1)}(w, \bar{w})\right\}. \tag{2.243}$$

In table 2.14 we summarise the corresponding weights and charges and we can see, that $\phi_i^{(2)}$ is a marginal operator.

field	$(h, \bar{h})$	$(q, \bar{q})$
$\phi_i^{(0)}$	$(\frac{1}{2}, \frac{1}{2})$	$(1, 1)$
$\phi_i^{(1)}$	$(1, \frac{1}{2})$	$(0, 1)$
$\phi_i^{(2)}$	$(1, 1)$	$(0, 0)$

Table 2.14: The deformation fields and their weights.

For the (a, c) ring the construction of fields follows a similar way but by using first $\bar{G}^-$ followed by G^+.

2.7.3 Non-linear sigma model realisation

Next, we consider the interpretation of the $\mathcal{N} = (2, 2)$ CFT as a non-linear sigma model from the genus g Riemann surface Σ_g into the target space X. The action of the non-linear sigma model includes bosonic fields $\phi : \Sigma_g \to X$ and fermionic fields $\psi \in \Gamma(K^{\frac{1}{2}} \otimes \phi^* T^{(1,0)} X), \chi \in \Gamma(\bar{K}^{\frac{1}{2}} \otimes \phi^* T^{(1,0)} X)$ and in a similar fashion for the anti-holomorphic fields. In total the action reads

$$\begin{aligned} S = \int_{\Sigma_g} d^2 z \Big(& \frac{1}{2} g_{i\bar{j}} \partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} + B_{i\bar{j}} (\partial_z \phi^i \bar{\partial}_{\bar{z}} \phi^{\bar{j}} - \bar{\partial}_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}}) + \frac{i}{2} g_{i\bar{j}} \psi^i D_z \psi^{\bar{j}} + \frac{i}{2} g_{i\bar{j}} \chi^i D_{\bar{z}} \chi^{\bar{j}} \\ & + R_{i\bar{k}j\bar{l}} \psi^i \psi^{\bar{k}} \chi^j \chi^{\bar{l}} \Big) \end{aligned} \tag{2.244}$$

[10] again we only concentrate on the holomorphic part. The anti-holomorphic part can be obtained in a similar fashion

It turns out that the target space X is subject to the Calabi-Yau condition[11]. Conformal invariance is ensured, if the beta function vanishes. It can be shown, that the beta function is proportional to the Ricci tensor and hence the Calabi-Yau condition is satisfied. Furthermore it is possible to determine the supersymmetry variation of the various fields in (2.244) and the operators of the $\mathcal{N} = 2$ SCFT. We collect the operators in table 2.15.

name	expression
energy momentum tensor	$T(z) = -g_{i\bar{j}}\partial_z\phi^i\partial_z\phi^{\bar{j}} + \frac{1}{2}g_{i\bar{j}}\psi^i\partial_z\psi^{\bar{j}} + \frac{1}{2}g_{i\bar{j}}\psi^{\bar{j}}\partial_z\psi^i$
currents	$G^+(z) = \frac{1}{2}g_{i\bar{j}}\psi^i\partial_z\phi^{\bar{j}}$ $G^-(z) = \frac{1}{2}g_{i\bar{j}}\psi^{\bar{j}}\partial_z\phi^i$
U(1) current	$J(z) = \frac{1}{4}g_{i\bar{j}}\psi^i\psi^{\bar{j}}$

Table 2.15: The operators of the chiral ring of the $\mathcal{N} = 2$ SCFT. For the anti-holomorphic operators similar expressions hold.

2.7.4 Topological field theories

We distinguish between two types of models: the A-model corresponding to the (a, c) ring and the B-model, which is the (c, c) ring. However, to get these theories one has to perform a topological twist in the algebra. To understand this, we first review some general facts about topological theories. In general, one can distinguish between two types of topological field theories, namely those of the Schwarz type [258] and those of the Witten type [97], where the later will be of interest for our discussion. For a general overview see [259]. A topological theory is characterised by the fact, that there exists a scalar symmetry Q which is nilpotent, i.e.

$$Q^2 = 0, \tag{2.245}$$

and the action S is Q exact, i.e. introducing the gauge fermion V the action can be written as

$$S = \{Q, V\} \tag{2.246}$$

and the energy-momentum tensor is also Q exact

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\} \tag{2.247}$$

with $G_{\mu\nu} = \frac{\delta V}{\delta g^{\mu\nu}}$. The topological property of these field theories is that the correlation functions of some operators $\{O_i\}_i$ do not depend on the metric, i.e.

$$\frac{\delta}{\delta g^{\mu\nu}}\langle O_{i_1}O_{i_2}\ldots O_{i_n}\rangle = 0. \tag{2.248}$$

However, the Q exactness also implies that for any operator O we have

$$\{Q, O\} = 0. \tag{2.249}$$

From the previous discussion it is clear, that the operator Q is a BRST operator and the corresponding physical states of the theory are in one-to-one correspondence with the cohomology classes of Q.

[11] in this case we refer to the notion of Ricci-flatness

The topological twist and the A- and the B-model

As we have already introduced, we have the (a, c) ring giving rise to the A-model and the (c, c) ring giving rise to the B-model[12]. We state the corresponding topological charges for the corresponding chiral states, which get annihilated by the corresponding currents $G^{\pm}(z)$ and their corresponding anti-holomorphic counterparts. The A- and B- model $Q_{A/B}$ look as follows

$$Q_A = G^-_{-\frac{1}{2}} + \bar{G}^+_{-\frac{1}{2}}, \quad Q_B = G^+_{-\frac{1}{2}} + \bar{G}^+_{-\frac{1}{2}}, \tag{2.250}$$

which can be checked to square to zero. In addition one has to perform a topological twist of the energy momentum tensors as presented in table 2.16. The topological twist allows for a globally well defined topological charge, such that $Q_{A/B}$ become Grassmann valued scalars and due to the shift in the energy momentum tensor the conformal weight is modified as well as the energy-momentum tensor becomes $Q_{A/B}$ exact. In the A-model the cohomology of the operators corresponds to the de Rahm cohomology

A model twist	B model twist
$T \to T + \frac{1}{2}\partial J$	$T \to T - \frac{1}{2}\partial J$
$\bar{T} \to \bar{T} - \frac{1}{2}\bar{\partial}\bar{J}$	$\bar{T} \to \bar{T} - \frac{1}{2}\bar{\partial}\bar{J}$

Table 2.16: The topological twist for the A- and the B-model

of the target space X. The corresponding moduli space is given by the Kähler moduli space. Note, that in the A-Model one has instanton corrections. These instanton corrections are not present in the B-model and the corresponding moduli space is the complex structure moduli space of the underlying Calabi-Yau manifold.

If we couple the A- or the B-model to $2d$ gravity, we obtain topological string theory. The coupling to gravity is achieved like in the bosonic string by integrating correlation functions over the moduli space of the Riemann surface Σ_g. The aim of topological string theory is to calculate the free energy $F(t, g_s)$ which depends on the background moduli t^i and the string coupling g_s. We perform a genus expansion of the free energy in terms of the string coupling constant

$$F(t, g_s) = \sum_{g=0}^{\infty} g_s^{2g-2} F^{(g)}(t). \tag{2.251}$$

For reasons of convergence it is sometimes useful to study the partition function $Z = \exp(F)$ instead of the free energy, since it is subject to an asymptotic expansion in the string coupling constant. From the point of view of deformations in the B-model, the topological amplitudes $F^{(g)}$ are defined as follows for $g > 1$

$$F^{(g)} = \int_{\Sigma_g} [dm\, d\bar{m}] \left\langle \prod_{a=1}^{3g-3} \left(\int_{\Sigma_g} \mu_a G^- \right) \left(\int_{\Sigma_g} \mu_{\bar{a}} \bar{G}^- \right) \right\rangle_{\Sigma_g}, \tag{2.252}$$

where we denote by $\mu_a \in H^{0,1}(\Sigma_g, T\Sigma_g)$ the Beltrami differentials which described the $3g-3$ dimensional moduli space of Σ_g and $dm, d\bar{m}$ are the corresponding duals to the Beltrami differentials.

[12] The other rings correspond to conjugated models.

2.7.5 Mirror symmetry

In this section we give a brief overview on mirror symmetry. Mirror symmetry provides a duality between the A- and the B-model. The origin of this duality is the $U(1)$ symmetry flip $J \leftrightarrow -J$. We summarise some facts about the different models in table 2.17. Mirror symmetry now exchanges these

Model	A	B
Calabi-Yau manifold	X	Y
counted objects	holomorphic maps	constant maps
deformations	Kähler deformations t^i	Complex structure deformations z^i
moduli space	$\dim \mathcal{M}_A = h^{1,1}(X)$	$\dim \mathcal{M}_B = h^{2,1}(Y)$

Table 2.17: Comparison of the A- and the B-model.

two and in particular the free energies in the A-model can be related to those in the B-model via the so called mirror map, which relates the coordinates on the two sides of the duality i.e. $t = t(z)$. We give a more detailed discussion of the variations in the A- and in the B-model.

A-Model

In the A-model we study deformations of the Kähler form. Let B denote the B-field, J the Kähler class on X and $\beta^i \in H_2(X, \mathbb{Z})$, since the path integral localises to holomorphic maps which depend on the homology classes in $H_2(X, \mathbb{Z})$. The Kähler parameters t^a are given as

$$t^a = \int_{\beta^a} (B + iJ). \tag{2.253}$$

To this we associate the parameters

$$q_\beta = \exp\left(2\pi i \int_\beta (B + iJ)\right). \tag{2.254}$$

The topological twist also changes the fermions, which are elements of the following sections

$$\begin{aligned} \psi^i &\in \Gamma(\phi^* T^{1,0} X), \quad \psi^{\bar{i}} \in \Gamma(K \otimes \phi^* T^{0,1} X), \\ \chi^i &\in \Gamma(\bar{K} \otimes \phi^* T^{1,0} X), \quad \chi^{\bar{i}} \in \Gamma(\phi^* T^{0,1} X). \end{aligned} \tag{2.255}$$

Determining the supersymmetry variation implies, that the operators in the A-model are equivalent to the de Rham cohomology of the Calabi-Yau manifold X. In addition it is shown, that the path integral localises to holomorphic maps. Therefore, the $F^{(g)}(t)$ count on the A-model side holomorphic curves of genus g and class $\beta \in H_2(X, \mathbb{Z})$. These are the GW invariants $N_{g,\beta}$

$$F^{(g)}(t) = \sum_{\beta \in H_2(X,\mathbb{Z})} N_{g,\beta}\, q^\beta. \tag{2.256}$$

In the A-model the path integral is obtained by summing over all instanton sectors and therefore we no longer deal with ordinary geometry but with so called quantum geometry, where corrections to the classical geometry are taken into account. In the A-model the deformation is given by the even cohomology which is related via the connection ∇_A

$$H^0(X,\mathbb{C}) \xrightarrow{\nabla_A} H^2(X,\mathbb{C}) \xrightarrow{\nabla_A} H^4(X,\mathbb{C}) \xrightarrow{\nabla_A} H^6(X,\mathbb{C}). \tag{2.257}$$

In the defintion of the connection on the elements of $D_i \in H^2(X,\mathbb{Z}), i = 1,\dots,h^{1,1}(X)$ we have

$$\nabla_A D_k = \sum_{i,j=1}^{h^{1,1}} C_{ijk} C_j \otimes \frac{dq_i}{q_i} \tag{2.258}$$

with $C_j \in H^4(X,\mathbb{Z})$ being the dual to D_i. The three point function reads in this case

$$C_{ijk} = \langle O_i O_j O_k \rangle = D_i \cap D_j \cap D_k + \sum_{\beta \in H_2(X,\mathbb{Z})} N_{0,\beta} \frac{q^\beta}{1-q^\beta}, \tag{2.259}$$

where $N_{0,\beta}$ are the genus zero GW invariants.

B-Model

In the B-model the moduli space of the SCFT corresponds to the moduli space of complex structures on Y. Good coordinates are now provided by periods of the holomorphic three-form Ω and $\alpha^a \in H_3(Y)$

$$z^a = \int_{\alpha^a} \Omega, \quad a = 1,\dots,h^{2,1}(Y). \tag{2.260}$$

In the B-model the fermions are elements of the following spaces

$$\begin{aligned} \psi^i &\in \Gamma(K \otimes \phi^* T^{1,0} Y), \quad \psi^{\bar{i}} \in \Gamma(\phi^* T^{0,1} Y), \\ \chi^i &\in \Gamma(\bar{K} \otimes \phi^* T^{1,0} Y), \quad \chi^{\bar{i}} \in \Gamma(\phi^* T^{0,1} Y). \end{aligned} \tag{2.261}$$

Studying the supersymmetry variation implies that the operators are identified with $(0,p)$ forms in $\Lambda^p TY$. The path integral localises to constant maps and in particular the B-model is not subject to instanton corrections. So all the calculations can be done in the classical setting, which is one of the advantages, to perform the calculation in the B-model first and then map it via mirror symmetry to the A-model. The variations in the B-model correspond to variations of the complex structure of Y. We therefore study the cohomology $H^3(Y,\mathbb{Z})$ and split it such that we have a holomorphic variation of the complex structure, which is given by the Hodge filtration $\{F^p(Y)\}_p$

$$F^p(Y) = \bigoplus_{n \geq p} H^{n,3-n}(Y). \tag{2.262}$$

In this context Griffith transversality applies, which states that the Gauss-Manin connection $\nabla_B F^p \subset F^{p-1}$ which gives rise to the following variation [260, 261]

$$F^3 \xrightarrow{\nabla_B} F^2 \xrightarrow{\nabla_B} F^1 \xrightarrow{\nabla_B} F^0. \tag{2.263}$$

Denote by $\theta_i = z^i\partial_{z^i}, i = 1,\dots,h^{2,1}(Y)$. Using (2.263), we consider the following spaces

$$(F^3, F^2/F^3, F^1/F^2, F^0/F^1). \tag{2.264}$$

Elements of these spaces are obtained by taking derivatives of the holomorphic $(3,0)$ form $\Omega^{3,0}$ and can be collected in the vector $\omega(z)$ of dimension $2h^{2,1}+2$ with the derivatives chosen accordingly

$$\omega(z) = \begin{pmatrix} \Omega^{3,0}(z) \\ \theta_i\Omega^{3,0}(z) \\ \theta_j\theta_i\Omega^{3,0}(z) \\ \theta_k\theta_j\theta_i\Omega^{3,0}(z) \end{pmatrix}. \tag{2.265}$$

Choosing $C^\alpha \in H_3(Y)$ we can define the period matrix $\Pi(z)$ via

$$\Pi^\alpha = \int_{C^\alpha} \omega(z). \tag{2.266}$$

In order to make progress, we introduce a symplectic bases (α_I, β^I) for $H^3(Y,\mathbb{C})$ such that

$$\Omega(z) = X^I(z)\alpha_I + F_I(z)\beta^I. \tag{2.267}$$

Now we can state the coordinates (X^I, F_I), which form the period vector Π as

$$\begin{aligned} \Pi &= \begin{pmatrix} F_I \\ X^I \end{pmatrix}, \\ &= \begin{pmatrix} \int_{B_I} \Omega \\ \int_{A^I} \Omega \end{pmatrix}. \end{aligned} \tag{2.268}$$

The period vector is subject to the *Picard-Fuchs* differential equations with respect to the differential operator $\mathcal{L}_a$

$$\mathcal{L}_a\Pi = 0, \quad a = 1,\dots,h^{2,1}(Y), \tag{2.269}$$

which can be obtained e.g. by the Griffith-Dwork reduction method or by using symmetries of the ambient space. We will often make use of homogeneous coordinates z^i which are given as follows

$$z^i = \frac{X^i}{X^0}, \quad i = 1,\dots,h^{2,1}(Y). \tag{2.270}$$

In terms of these coordinates the prepotential $F^{(0)}(z)$ is calculated as

$$F^{(0)}(z) = \frac{1}{2(X^0)^2} \sum_{I=0}^{h^{2,1}(Y)} X^I F_I. \tag{2.271}$$

The structure of the moduli space, i.e. the space of complex structure deformations, turns out to be a special Kähler manifold $\mathcal{M}$ with line bundle being the Hodge bundle $\mathcal{L}$ and the free energies $F^{(g)} \in \Gamma(\mathcal{L}^{2-2g})$. Since we have a Kähler manifold, the metric $G_{i\bar{j}}$ can be computed from a Kähler potential K

which is given in terms of the top-form as

$$K = -\log\left(i\int_Y \Omega\wedge\bar{\Omega}\right) = -\log i\Pi^\dagger\Sigma\Pi. \tag{2.272}$$

To determine the Yukawa couplings and the curvature tensor, we observe that there is a induced connection $(D_i)^k_j$ on $T^*\mathcal{M}\otimes\mathcal{L}^n$ which reads

$$(D_i)^k_j = \delta^k_j(\partial_i + n\partial_i K) - \Gamma^k_{ij}. \tag{2.273}$$

This allows to write the Yukawa couplings C_{ijk} as derivatives of the prepotential $F^{(0)}$

$$C_{ijk} = D_i D_j D_k F^{(0)}, \tag{2.274}$$

which are subject to

$$D_i C_{jkl} = D_j C_{ikl}. \tag{2.275}$$

The curvature tensor $R^l_{i\bar{j}k}$ is given as

$$R^l_{i\bar{j}k} = -\left[\bar{\partial}_{\bar{j}}, D_i\right]^l_k = G_{i\bar{j}}\delta^{\bar{l}}_k + G^{k\bar{j}}\delta_{i\bar{l}} - C_{ikm}\bar{C}^{lm}_{\bar{j}}, \qquad \bar{C}^{lm}_{\bar{i}} = \bar{C}_{\bar{i}\bar{l}\bar{m}}G^{i\bar{i}}G^{l\bar{l}}e^{2K}. \tag{2.276}$$

With the help of the mirror map it is possible to introduce coordinates t^i on the A-model side as

$$t^i(z) = \frac{X^i(z)}{X^0(z)} \tag{2.277}$$

and the relation between the period vectors reads

$$\begin{aligned}\Pi(z(t)) = \begin{pmatrix} F_0 \\ F_i \\ X^0 \\ X^i \end{pmatrix}_B &= X^0\begin{pmatrix} 2F_0 - t^i\partial_{t^i}F_0 \\ \partial_{t^i}F_0 \\ 1 \\ t^i \end{pmatrix} \\ &= X^0\begin{pmatrix} \frac{d_{ijk}t^it^jt^k}{3!} + c_it^i - i\chi\frac{\zeta(3)}{(2\pi)^3} + 2f(q) - t^i\partial_{t^i}f(q) \\ -\frac{d_{ijk}t^jt^k}{2!} + A_{ij}t^j + c_i + \partial_{t^i}f(q) \\ 1 \\ t^i \end{pmatrix},\end{aligned} \tag{2.278}$$

where d_{ijk} denotes the triple intersection number, $c_i = \frac{1}{24}\int_X c_2 J_i$ and J_i are elements of the Kähler cone. We will use these results for a B-model approach to elliptic Calabi-Yau manifolds in section5.2. From this we see why mirror symmetry is such a powerful tool, since it allows to perform a geometrical calculation on the B-model side and then via the mirror map to obtain results on the A-model side. However, in the next section we discuss a useful tool in calculating the topological amplitudes, namely the holomorphic anomaly equations.

2.7.6 Holomorphic anomaly equations

The topological amplitudes $F^{(g)}$ are subject to the so called BCOV holomorphic anomaly equations [101], which provide a recursion in the genus of the $F^{(g)}$. They take the following form for $g > 1$

$$\bar{\partial}_{\bar{i}} F^{(g)} = \frac{1}{2}\bar{C}^{jk}_{\bar{i}}\left(D_j D_k F^{(g-1)} + \sum_{s=1}^{g-1} D_j F^{(s)} D_k F^{(g-s)}\right), \tag{2.279}$$

and for $g = 1$ we obtain the following equation

$$\bar{\partial}_{\bar{i}}\partial_j F_1 = \frac{1}{2}\bar{C}^{kl}_{\bar{i}} C_{jkl} - G_{i\bar{j}}\left(\frac{\chi}{24} - 1\right). \tag{2.280}$$

This can be derived by starting with the definition of the topological amplitudes (2.252) in terms of deformations and taking the $\bar{t}^{\bar{i}}$ derivative and realising that this leads to the insertion of an anti-chiral field $\bar{\phi}_{\bar{i}}$.

$$\bar{\partial}_{\bar{t}^j} F^{(g)} = \int_{\mathcal{M}_g} [dm\, d\bar{m}] \int d^2z \left\langle \oint_{C_z} G^+ \oint_{C'_z} \bar{G}^+ \bar{\phi}^{(2)}_j(z) \prod_{i=1}^{3g-3} \int_\Sigma \mu_i G^- \int_\sigma \bar{\mu}_i \bar{G}^- \right\rangle \tag{2.281}$$

by commuting $G^+, \bar{G}^+$ with G^- and $\bar{G}^-$ we obtain by using the SCFT algebra a term which is proportional to the energy-momentum tensor, which itself is obtained as a variation with respect to the metric and therefore it is possible to rewrite the expression as

$$\bar{\partial}_{\bar{t}^j} F^{(g)} = \int_{\mathcal{M}_g} [dm\, d\bar{m}] \sum_{k,\bar{k}=1}^{3g-3} \frac{\partial^2}{\partial m_k \partial \bar{m}_{\bar{k}}} \left\langle \phi^{(2)}_j(z) \prod_{i=1}^{3g-3} \int_\Sigma \mu_i G^- \int_\sigma \bar{\mu}_i \bar{G}^- \right\rangle. \tag{2.282}$$

This integral vanishes expect for possible boundary terms of the moduli space of Riemann surfaces $\mathcal{M}_g$. The moduli space $\mathcal{M}_g$ can degenerate in two ways:

- either the genus g can split into two components such that $g = g_1 + g_2$,
- the genus g moduli space can degenerate to a genus $g - 1$ moduli space.

The holomorphic anomaly equation reflects these degenerations in its two terms.

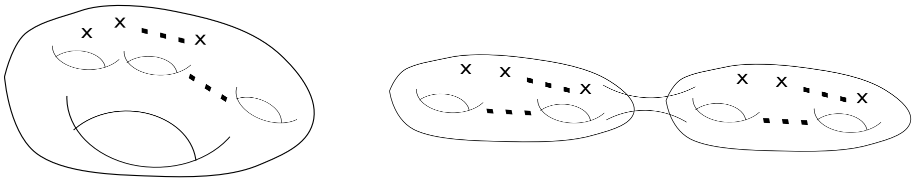

Figure 2.11: Graphical interpretation of the holomorphic anomaly. The figure to the left shows the pinching of the genus g moduli space leading to genus $g - 1$. The figure to the right shows the split of the genus g into g_1 and g_2. Furthermore both figures show punctures which are due to insertions by taking derivatives of the topological amplitudes $C^{(g)}_{i_1 \ldots i_n} = D_{i_1} \ldots D_{i_n} F^{(g)}$

Also for the case of n-point functions $C^{(g)}_{i_1 \ldots i_n}$ an anomaly equation was discovered, where $C^{(g)}_{i_1 \ldots i_n}$ are given as covariant derivatives of the topological amplitudes $F^{(g)}$ for $g \geq 1$

$$C^{(g)}_{i_1 \ldots i_n} = D_{i_1} \ldots D_{i_n} F^{(g)} \tag{2.283}$$

and for the case that $g = 0$ we have that the number of insertions $n \geq 4$. Furthermore it is also possible to define them as follows with $g > 1$ and for all n or $n > 0$ if $g > 1$ and $n \geq 3$ if $g = 0$.

$$C^{(g)}_{i_1\ldots i_n} = \int_{\Sigma_g} [dm\, d\bar{m}] \left\langle \prod_{r=1}^{n} \int_{\Sigma_g} \phi^{(2)}_{i_r} \prod_{a=1}^{3g-3} \left(\int_{\Sigma_g} \mu_a G^- \right) \left(\int_{\Sigma_g} \mu_{\bar{a}} \bar{G}^- \right) \right\rangle_{\Sigma_g} . \tag{2.284}$$

The holomorphic anomaly equation reads in this case

$$\begin{aligned} \bar{\partial}_{\bar{i}} C^{(g)}_{i_1\ldots i_n} =& \frac{1}{2} \bar{C}^{jk}_{\bar{i}} \left(C^{(g-1)}_{jki_1\ldots i_n} + \sum_{r=0}^{g} \sum_{s=0}^{n} \frac{1}{s!(n-s)!} \sum_{\sigma\in S_n} C^{(r)}_{ji_{\sigma(1)}\ldots i_{\sigma(s)}} C^{(g-r)}_{ki_{\sigma(s+1)}\ldots i_{\sigma(n)}} \right) \\ & - (2g-2+n-1) \sum_{s=1}^{n} G_{\bar{i}i_s} C^{(g)}_{i_1\ldots i_{s-1} i_{s+1}\ldots i_n} . \end{aligned} \tag{2.285}$$

We observe that this also provides a recursive structure in the genus as before as well as that the insertions are permuted and the last term corresponds to insertions that meet. The holomorphic anomaly equations provide a recursive structure in the topological amplitudes, which allows to calculate them. Since most of the time we just deal with genus zero and one in this thesis we just want to give a small recap on these solution techniques and refer to the literature for further details. In general topological string theory can be solved by direct integration of the holomorphic anomaly equations [262, 263], by localisation, by the topological vertex [161] or by the matrix model techniques in the remodelled B-model [162].

Direct integration of the holomorphic anomaly equations

The general idea is to re-express both sides of the holomorphic anomaly equation in terms of anti-holomorphic derivates, such that the expressions the derivative acts on only differ by a holomorphic term, the so called holomorphic ambiguity $f_g(z)$. This is achieved in several steps, which we outline in the following [264, 265].

1. Introduce the propagators

$$(S, S^i, S^{ij}) \in (\mathcal{L}^{-2}, \mathcal{L}^{-2} \otimes T^*\mathcal{M}, \mathcal{L}^{-2} \otimes \mathrm{Sym}^2(T^*\mathcal{M})). \tag{2.286}$$

   The correlation functions are then interpreted as vertices.

2. Note, the following relations for (S, S^i, S^{ij})

$$\bar{C}_{\bar{i}\bar{j}\bar{k}} = e^{-2K} D_{\bar{i}} D_{\bar{j}} \bar{\partial}_{\bar{k}} S, \quad \partial_{\bar{i}} S^{ij} = \bar{C}^{ij}_{\bar{i}}, \quad \partial_{\bar{i}} S^j = G_{i\bar{i}} S^{ij}, \quad \partial_{\bar{i}} S = G_{i\bar{i}} S^i. \tag{2.287}$$

3. Rewrite the holomorphic anomaly equations, provide the data for lower genus and boundary conditions for fixing the holomorphic ambiguity.

Though this procedure works, it depends on a large number of iterations and therefore becomes unfavourable. Instead it is more effective to use modularity and express the topological amplitudes as a polynomial in a ring of non-holomorphic generators, which are given by

$$(K_i = \partial_i K, S^{ij}, S^i, S) \tag{2.288}$$

with weights $(1, 1, 2, 3)$ and the $F^{(g)}$ are of weight $3g-3$. We modify the generators in order to rewrite the anti-holomorphic derivative of the $F^{(g)}$ in terms of derivatives w.r.t. the generators, i.e.

$$\bar{\partial}_{\bar{i}} F^{(g)} = \bar{C}_{\bar{i}}^{jk} \left(\frac{\partial F^{(g)}}{\partial \hat{S}^{jk}} - \frac{1}{2} \frac{\partial F^{(g)}}{\partial \hat{S}^{k}} \hat{K}_j - \frac{1}{2} \frac{\partial F^{(g)}}{\partial \hat{S}^{j}} \hat{K}_k + \frac{1}{2} \frac{\partial F^{(g)}}{\partial \hat{S}} \hat{K}_j \hat{K}_k \right) + G_{i\bar{j}} \frac{\partial F^{(g)}}{\partial \hat{K}_j}. \tag{2.289}$$

In order to derive this expression, we introduced the new generators $(\hat{K}_i, \hat{S}^{ij}, \hat{S}^i, \hat{S})$ which are defined via [264]

$$\hat{K}_i = K_i, \quad \hat{S}^{ij} = S^{ij}, \quad \hat{S}^i = S^i - S^{ij} K_j, \quad \hat{S} = S - S^i K_i + \frac{1}{2} S^{ij} K_i K_j. \tag{2.290}$$

This allows to rewrite the connection term Γ^l_{ij} as

$$\Gamma^l_{ij} = \delta^l_i K_j + \delta^l_j K_i - C_{ijk} S^{kl} + s^l_{ij}, \tag{2.291}$$

with s^l_{ij} being an unfixed holomorphic function. An important reason why this method works is due to the fact, that the generators close under the anti-holomorphic derivative as can be checked explicitly

$$\begin{aligned}
\partial_i \hat{S}^{jk} &= C_{imn} \hat{S}^{mj} \hat{S}^{nk} + \delta^j_i \hat{S}^k + \delta^k_i \hat{S}^j - s^j_{im} \hat{S}^{mk} - s^k_{im} \hat{S}^{mj} + h^{jk}_i, \\
\partial_i \hat{S}^j &= C_{imn} \hat{S}^{mj} S^n + 2\delta^j_i \hat{S} - s^j_m \hat{S}^m - h_{ik} \hat{S}^{kj} + h^j_i, \\
\partial_i \hat{S} &= \frac{1}{2} C_{imn} \hat{S}^m \hat{S}^n - h_{ij} \hat{S}^j + h_i, \\
\partial_i \hat{K}_j &= K_i K_j - C_{ijn} \hat{S}^{mn} K_m + s^m_{ij} \hat{K}_m - C_{ijk} \hat{S}^k + h_{ij},
\end{aligned} \tag{2.292}$$

where again $h^{jk}_i, h^j_i, h_i, h_{ij}$ are holomorphic functions, which can be determined. By going to a certain point in moduli space, where the Yukawa coupling becomes invertible, it is is possible to determine S^{ij}, S^i and S and with this at hand it is possible to determine the $F^{(g)}$ [166]. For example for the case of genus one the topological amplitude $F^{(1)}$ reads as follows

$$F^{(1)} = \frac{1}{2} \left(3 + h^{1,1}(X) - \frac{\chi}{12} \right) K + \frac{1}{2} \log\det G^{-1} + \sum_{i=1}^{h^{1,1}(X)} s_i \log z_i + \sum_j r_j \log \Delta_j, \tag{2.293}$$

where the last summation is over the number of discriminant components. It is still an open question how to fix the coefficients s_i and r_j via boundary conditions. The singular behaviour of $F^{(1)}$ is given by the following behaviour

$$F^{(1)} \sim -\frac{1}{24} \sum_i \log z_i \int_X c_2 J_i. \tag{2.294}$$

In particular if we encounter a conifold singularity for the discriminant Δ we have

$$F^{(1)} \sim -\frac{1}{12} \log \Delta. \tag{2.295}$$

In table 2.18 we collect some boundary conditions that can be used to fix the holomorphic ambiguity f_g, for which in general the following ansatz near a singularity Δ is used

$$f_g \sim \frac{p(\bar{z}_i)}{\Delta^{2g-2}}. \tag{2.296}$$

With this it is possible to fix the holomorphic ambiguity and integrate the holomorphic anomaly.

point in moduli space	behaviour of $F^{(g)}$
large complex structure limit	$F^{(g)}\|_{q^a=0} = (-1)^g \frac{\chi}{2} \frac{\|B_{2g}B_{2g-2}\|}{2g(2g-2)(2g-2)!}, \quad g > 1$
conifold locus	$F^{(g)}(t_c) = b\frac{B_{2g}}{2g(2g-2)} t_c^{2-2g} + O(t_c^0), \quad g > 1.$ $t_c = \Delta^{\frac{1}{m}}$, for a conifold we have $b = 1, m = 1$.

Table 2.18: The behaviour of the topological amplitudes $F^{(g)}$ at various points in moduli space.

Background independence

The holomorphic anomaly equation of BCOV has also been investigated in the context of background independence [102]. Given a reference point in the moduli space of a theory, background independence refers to the dependence of correlators on the chosen reference point. The holomorphic anomaly equations refer to this dependence but it turns out that they ensure background independence in topological string theory [102]. The idea is to interpret the full topological string partition function

$$Z(g_s, t) = \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2} F^{(g)}(t)\right) \tag{2.297}$$

as a wave function living in a Hilbert space that is constructed by geometric quantisation of $H^3(X, \mathbb{R})$. Note that the holomorphic anomaly equation for $Z(g_s, t)$ takes the following form

$$\left(\bar{\partial}_{\bar{i}} - \frac{1}{4} g_s^2 \bar{C}_{\bar{i}}^{jk} D_j D_k\right) Z(g_s, t) = 0. \tag{2.298}$$

For the geometric quantisation $H^3(X, \mathbb{Z})$ is interpreted as a symplectic phase space $\mathcal{W}$ with symplectic structure ω and its quantisation requires the choice of a polarisation. Let J denote a complex structure of X, then the complex structure on $\mathcal{W}$ also depends on J and the Hilbert space $\mathcal{H}_J$ is build from sections of a holomorphic line bundle over the symplectic phase space $\mathcal{W}$. The wave functions $\psi(t^i, z^i)$ depend on t^i, which are coordinates on the moduli space $\mathcal{M}$ and hence parameterise J, and complex coordinates z^i of $\mathcal{W}$. Background independence now makes a statement on the dependence of the wave function $\psi(t^i, z^i)$ via the following equation

$$\left(\frac{\partial}{\partial t^i} - \frac{1}{4}\left[\frac{\partial J}{\partial t^i}\omega^{-1}\right]^{kl} \frac{D}{Dz^k}\frac{D}{Dz^l}\right)\psi = 0, \tag{2.299}$$

which is equivalent to (2.298). The idea behind this equation is to identify the different $\mathcal{H}_J$ by using a flat connection ∇ such that a variation of J induces a change of ψ by a Bogoliubov transformation. Background independence is then reformulated to demanding invariance of ψ under parallel transport with respect to ∇.

2.7.7 Stable pairs and enumerative invariants

We finish our discussion of topological string theory by giving a short summary of enumerative invariants, since these were calculated by methods of topological string theory, see e.g. [266]. We have already discussed the GW invariants. In this section we discuss the Pandharipande-Thomas and Gopakumar-Vafa invariants. In Pandharipande-Thomas theory one counts D6-D2-D0 bound states via the method of stable pairs.

Definition: A stable pair [267–269] on a Calabi-Yau threefold consists of sheaf $\mathcal{F}$ on X and a section $s \in H^0(\mathcal{F})$ such that

i) $\mathcal{F}$ is pure of dimension 1 and

ii) s generates $\mathcal{F}$ outside a finite set of points.

◇

The moduli space of stable pairs is denoted by $\mathcal{M}_n(X,\beta)$ with $n = \chi(\mathcal{F})$ and $\beta = \mathrm{ch}_2(\mathcal{F})$. We denote the degree of the virtual fundamental class of $\mathcal{M}_n(X,\beta)$ by $P_{n,\beta}$ and introduce the generating function for Pandharipande-Thomas invariants Z_{PT} by

$$Z_{\mathrm{PT}} = \sum_{n,\beta} P_{n,\beta} q^n Q^\beta. \tag{2.300}$$

It is conjectured, that Z_{PT} is equal to the generating function for disconnected GW invariants by a change of variables, i.e. if we write

$$Z_{\mathrm{GW}} = \exp(F_{\mathrm{GW}}(\lambda, Q)) = \exp \sum_{\beta\neq 0} \sum_{g} N_{g,\beta} \lambda^{2g-2} Q^\beta, \tag{2.301}$$

then the identification is achieved by $q = -e^{-i\lambda}$.
Another set of invariants of Calabi-Yau threefolds are the Gopakumar-Vafa invariants n^g_β [114]. They are related by the following expression to the GW invariants

$$\sum_{\beta,g} N_{g,\beta} \lambda^{2g-2} Q^\beta = \sum_{\substack{\beta,g,k\\ \beta\neq 0}} n^g_\beta \frac{1}{k} \left(2 \sin \frac{k\lambda}{2}\right)^{2g-2} Q^{k\beta}. \tag{2.302}$$

For more details about the relations and applications of these invariants, we refer to the literature [266, 270–275].

2.8 Counting D4-D2-D0 BPS states

In this section we want to provide some background on the main counting objective of this thesis, namely D4-D2-D0 BPS states. These are obtained by wrapping multiple M5 branes on a divisor in a Calabi-Yau manifold. Depending on the size of the divisor different descriptions are available. For the case of a small divisor this gives the MSW CFT which provides a microscopic description in the case, that one M5 brane is present. We start this section by a discussion of this CFT and in particular we introduce the modified elliptic genus, which allows for a counting of the BPS states. The counting of D4-D2-D0 can also be performed by geometric and split attractor flow methods. The method we will make extensive use of is via sheaves and stability conditions. We finish our discussion with another limit of the setup such that the description becomes an $\mathcal{N} = 4$ U(n) topological SYM theory, with n the number of M5 branes. In this setup a holomorphic anomaly of the generating function was observed.

2.8.1 The Maldacena-Strominger-Witten conformal field theory

Let X be a Calabi-Yau threefold and we study M-theory compactifications, that give rise to extremal black holes. We choose r M5 branes that wrap

$$P \times S^1_M, \tag{2.303}$$

where $P = p^A \Sigma_A$ is a four cycle in the Calabi-Yau manifold X with $\Sigma_A \in H_4(X, \mathbb{Z})$ and S^1_M denotes the M-theory cycle. For the case that the divisor P is small relative to S^1_M the setup can be described by a $(0,4)$ CFT[13], which is called the MSW CFT [138]. The MSW CFT is understood for the case of one M5 brane. We give short review on the MSW CFT and follow [277, 278]. Before starting our discussion, we compactify the time direction to a circle S^1_t such that together with the M-theory circle they form a torus T^2 and in total the world-volume theory of the M5-brane is given as [139]

$$P \times T^2. \tag{2.304}$$

It can be shown that form a type IIA string theory point of view this corresponds to a D4-D2-D0 brane configuration. In particular we introduce the charge vector Γ

$$\Gamma = (Q_6, Q_4, Q_2, Q_0) = r(0, p^A, q_A, q_0), \tag{2.305}$$

where the Q_p are the Dp-brane charges and r is the number of coincident M5-branes wrapping the divisor specified by p^A. The induced M2/D2 brane charge from the M5 brane flux is denoted by q_A and by q_0 the Kaluza-Klein momentum or D0 brane charge along the S^1_M. Note, that the first entry in the charge vector Γ equals zero and corresponds to the D6 brane charge.

This CFT on T^2 has $(0,4)$ supersymmetry as it is inherited from the $(0,2)$ theory of the M5 world volume theory. We obtain the field content of this CFT by dimensional reducing the M5 brane on P in the following. The starting point is the $(0,2)$ supersymmetric theory. There are 3 scalars X^a describing the position of the brane in space time. Then there is a two form field $b_{\mu\nu}$, such that we get a self-dual field strength h via $h = db$. We decompose the h field as follows

$$h = d\phi^A \wedge \alpha_A, \quad \alpha_A \in H^2(P, \mathbb{Z}), \tag{2.306}$$

and see that this gives rise to self- and anti-self-dual fields on the divisor P because of the self-duality of h. We denote their corresponding numbers b_2^+ and b_2^-. Fermions are constructed in a similar way by first reducing the six-dimensional fermions ψ into fermionic zero modes ψ_I^P

$$\psi = \sum_I \psi_2^I \otimes \psi_I^P. \tag{2.307}$$

In total we have $4h^{2,0}$ right-moving fermions as the zero modes on the divisor correspond to the harmonic $(0,2)$ forms. In addition we have the $\mathcal{N} = 4$ centre of mass multiplet, which we summarise in table 2.19. Before we can state the central charges of the MSW CFT, we need to introduce some further notion from

[13] The target space sigma model description of which was given in ref. [276], for more details see ref. [277] and references therein. In the following we will be concerned with the natural extension of the analysis of the degrees of freedom to r M5-branes.

field	description
$X^i, i = 1, 2, 3$	massless scalars for motion of black hole in space
$\varphi = p^A \phi_A$	unique self-dual two-form on P
$\tilde{\psi}^{\pm\pm}$	goldstinos from the broken supersymmetries

Table 2.19: The content of the centre of mass multiplet.

geometry. We introduce the triple intersection number D_{ABC} by

$$D_{ABC} = \int_X J_A \wedge J_B \wedge J_C, \tag{2.308}$$

and construct the following objects

$$\begin{aligned} D_{AB} &= -D_{ABC} p^C \\ D^{AB} D_{BC} &= \delta^A{}_C \\ 6D = D_{ABC} p^A p^B p^C &= P^3. \end{aligned} \tag{2.309}$$

The Euler characteristic $\chi(P)$ and signature $\sigma(P)$ of P are given by

$$\begin{aligned} \chi(P) &= P^3 + c_2(X) \cdot P, \\ \sigma(P) &= -\frac{1}{3} P^3 - \frac{2}{3} c_2(X) \cdot P. \end{aligned} \tag{2.310}$$

They can be expressed in terms of $b_2^\pm$ by

$$\begin{aligned} b_2^+ &= \frac{1}{3} P^3 + \frac{1}{6} P \cdot c_2(X) - 1, \\ b_2^- &= \frac{2}{3} P^3 + \frac{5}{6} P \cdot c_2(X) - 1. \end{aligned} \tag{2.311}$$

The central charges of the MSW CFT are given by

$$c_{\mathrm{L}} = 6D + c_2 \cdot P, \quad c_{\mathrm{R}} = 6D + \frac{1}{2} c_2 \cdot P. \tag{2.312}$$

The entropy of the black hole with D4-D2-D0 charges reads

$$S_{\mathrm{BH}} = 2\pi \sqrt{D \hat{q}_0}, \tag{2.313}$$

where we introduced the induced charge $\hat{q}_0$. The induced charge $\hat{q}_0$ arises, when adding D2 charge to the system. This itself contributes to the momentum along the S^1 and implies the shift

$$q_0 \to \hat{q}_0 = q_0 + \frac{1}{12} D^{AB} q_A q_B. \tag{2.314}$$

This matches the macroscopic entropy of the corresponding black hole. Note, that for a general D6-D4-

D2-D0 black hole with charge vector $\Gamma = (p^0, p^A, q_A, q_0)$ and with corresponding prepotential [279]

$$\mathcal{F} = \frac{D_{ABC}X^A X^B X^C}{X^0}, \tag{2.315}$$

the entropy is given by

$$S = 2\pi\sqrt{Q^3 p^0 - J^2(p^0)}, \quad J = -\frac{q_0}{2} + \frac{P^3}{{p^0}^2} + \frac{p^A q_A}{2p^0}. \tag{2.316}$$

where one determines Q by solving the following set of equations for y^A

$$3D_{ABC}y^A y^B = q_A + \frac{3D_{ABC}p^B p^C}{p^0}, \tag{2.317}$$

and then determines Q via

$$Q^{\frac{3}{2}} = D_{ABC}y^A y^B y^C. \tag{2.318}$$

Note, that not necessarily there exists a solution for all possible charges.

The modified elliptic genus

We now return to our discussion of the charge vector. A priori the set of all possible induced D2-brane charges, or equivalently of $U(1)$ fluxes of the world-volume of the M5-brane would be in one-to-one correspondence with $\Lambda_P = H^2(P, \mathbb{Z})$ which is generically a larger lattice than $\Lambda = i^* H^2(X, \mathbb{Z})$, where $i : P \hookrightarrow X$. The physical BPS states are always labeled by the smaller lattice Λ. The metric d_{AB} on Λ is given by

$$d_{AB} = -\int_P \alpha_A \wedge \alpha_B, \tag{2.319}$$

where α_A is a basis of two-forms in Λ, which is the dual basis to Σ_A of $H_4(X, \mathbb{Z})$. In order to obtain a generating series of the degeneracies of those BPS states one has to sum over directions along $\Lambda^\perp$ which is the orthogonal complement to Λ in Λ_P w.r.t. d_{AB} [137]. [14]

The partition function of the MSW CFT counting the BPS states is given by the modified elliptic genus[15] [139, 280]

$$Z_P'^{(r)}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}} (-1)^{F_{\mathrm{R}}} F_{\mathrm{R}}^2\, q^{L_0' - \frac{c_{\mathrm{L}}}{24}}\, \bar{q}^{\bar{L}_0' - \frac{c_{\mathrm{R}}}{24}}\, e^{2\pi i z \cdot Q_2}, \tag{2.320}$$

where the trace is taken over the RR Hilbert space. Furthermore, vectors are contracted w.r.t. the metric d_{AB}, i.e. $x \cdot y = x^A y_A = d_{AB} x^A y^B$. For a single M5-brane it was shown in ref. [278] that $Z_P'^{(1)}(\tau, z)$ transforms like a $\mathrm{SL}(2, \mathbb{Z})$ Jacobi form of bi-weight $(0, 2)$ due to the insertion of F_{R}^2, we demand that the same is true for all r.

Following ref. [278] the center of mass momentum $\vec{p}_{\mathrm{cm}}$ for the system of r M5-branes can be integ-

[14] In general, the lattice $\Lambda \oplus \Lambda^\perp$ is only a sublattice of $H^2(P, \mathbb{Z})$, because $\det d_{AB} \neq 1$ in general, see for example ref. [276] and ref. [123] for a more recent exposition. However, we will only be concerned with divisors P with $b_2^+(P) = 1$, such that $\det d_{AB} = 1$.

[15] We follow the mathematics convention of not writing out explicitly the dependence on $\bar{\tau}$ which will be clear in the context. Moreover, we denote $q = e^{2\pi i \tau}$ and $\tau = \tau_1 + i\tau_2$. To avoid confusion without introducing new notation we will denote the charge vector of D2-brane charges by $\underline{q}$, its components by q_A.

rated out. In this way L_0' and $\bar{L}_0'$ can be written in the form

$$L_0' = \frac{1}{2}\vec{p}_{\rm cm}^{\,2} + L_0, \quad \bar{L}_0' = \frac{1}{2}\vec{p}_{\rm cm}^{\,2} + \bar{L}_0. \tag{2.321}$$

This allows one to split up the center of mass contribution and rewrite formula (2.320) as

$$\begin{aligned} Z_P'^{(r)}(\tau,z) &= \int d^3p_{\rm cm}(q\bar{q})^{\frac{1}{2}\vec{p}_{\rm cm}^{\,2}} Z_P^{(r)}(\tau,z) \\ &\sim (\tau_2)^{-\frac{3}{2}}\, Z_P^{(r)}(\tau,z), \end{aligned} \tag{2.322}$$

where $Z_P^{(r)}(\tau,z)$ is now a Jacobi form of weight $(-\frac{3}{2},\frac{1}{2})$ which we simply call elliptic genus for short in the following. For more details on the modular properties see C.2. We comment a bit on the counting of the elliptic genus from the perspective of the MSW CFT, which resembles our discussion of the supersymmetric indices. The ordinary Witten index vanishes because the field content is that of a small $\mathcal{N}=4$ SCFT plus the center of mass multiplet, which gives a vanishing Witten index as the bosonic fields cancel the fermionic ones. In the R sector we have the following commutation relations with the $\mathcal{N}=4$ supercurrents $\tilde{G}^{\pm\pm}$ and the bosonic currents J_R^i, $i=1,\dots,3$ and $\tilde{J}^\phi = \bar{\partial}\phi$.

$$\{\tilde{G}_0^{\alpha a},\tilde{\psi}_0^{\beta b}\} = \epsilon^{\alpha\beta}\epsilon^{ab}J_0^\phi, \quad \{\tilde{G}_0^{\alpha a},\tilde{J}_0^\phi\} = \tilde{\psi}_0^{\alpha a}. \tag{2.323}$$

The highest weight state $|\Omega\rangle$ satisfies

$$\tilde{\psi}_0^{-\pm}|\Omega\rangle = 0. \tag{2.324}$$

It is easy to see, that the multiplet $\{|\Omega\rangle, \tilde{\psi}_0^{+\pm}|\Omega\rangle, \tilde{\psi}_0^{++}\tilde{\psi}_0^{+-}|\Omega\rangle\}$ gives a non-trivial contribution to the modified elliptic genus as can be seen from table 2.20 [277].

	$\lvert\Omega\rangle$	$\tilde{\psi}_0^{+\pm}\lvert\Omega\rangle$	$\tilde{\psi}_0^{++}\tilde{\psi}_0^{+-}\lvert\Omega\rangle$	index contribution
Witten index	1	2×-1	1	0
elliptic genus	0	-1	2	1

Table 2.20: Contributions of the multiplet $\{|\Omega\rangle, \tilde{\psi}_0^{+\pm}|\Omega\rangle, \tilde{\psi}_0^{++}\tilde{\psi}_0^{+-}|\Omega\rangle\}$ to the Witten index and the elliptic genus.

Given a state with charge q_A the following identity holds

$$\left(\tilde{G}_0^{\pm\pm} - p^A q_A \tilde{\psi}_0^{\pm\pm}\right)|q\rangle = 0, \tag{2.325}$$

from which it can be inferred that supersymmetries are preserved non-linearly.

The decomposition of the elliptic genus

The elliptic genus $Z_P^{(r)}(\tau,z)$ and equivalently the generating function of D4-D2-D0 BPS degeneracies is subject to a theta-function decomposition, which has been studied in many places, see for example refs. [123,137,193,278,281]. This is ensured by two features of the superconformal algebra of the (0,4) CFT. One of these is that the $\overline{\tau}$ contribution entirely comes from BPS states $|\underline{q}\rangle$ satisfying

$$\left(\overline{L}_0 - \frac{c_{\rm R}}{24} - \frac{r}{2}q_{\rm R}^2\right)|\underline{q}\rangle = 0, \tag{2.326}$$

the other one is the spectral flow isomorphism of the $\mathcal{N} = (0,4)$ superconformal algebra, which we want to recall for r M5-branes here, building on refs. [278, 282], see also [193]. Proposition 2.9 of ref. [282] describes the spectral flow symmetry by an isomorphism between moduli spaces of vector bundles on complex surfaces. The complex surface here is the divisor P and the vector bundle configuration describes the bound-states of D4-D2-D0 branes. Within this setup the result of [282] translates for arbitrary r to a symmetry under the transformations

$$\begin{aligned} q_0 &\mapsto q_0 - k\cdot \underline{q} - \frac{1}{2}k\cdot k, \\ \underline{q} &\mapsto \underline{q} + k, \end{aligned} \tag{2.327}$$

where $k \in \Lambda$. Physically these transformations correspond to monodromies around the large radius point in the moduli-space of the Calabi-Yau manifold [193]. Denote by Λ^* the dual lattice of Λ with respect to the metric rd_{AB}. Keeping only the holomorphic degrees of freedom one can write

$$\begin{aligned} Z_P^{(r)}(\tau, z) &= \sum_{Q_0;Q_A} d(Q, Q_0)\, e^{-2\pi i\tau Q_0}\, e^{2\pi i z\cdot Q_2} \\ &= \sum_{q_0;\underline{q}\in\Lambda^*+\frac{[P]}{2}} d(r, \underline{q}, -q_0)\, e^{-2\pi i\tau r q_0}\, e^{2\pi i r z\cdot \underline{q}}, \end{aligned} \tag{2.328}$$

where $d(r, \underline{q}, -q_0)$ are the BPS degeneracies and the shift[16] $\frac{[P]}{2}$ originates from an anomaly [206, 283]. Now, spectral flow symmetry predicts [278]

$$d(r, \underline{q}, -q_0) = (-1)^{rp\cdot k} d(r, \underline{q} + k, -q_0 + k\cdot \underline{q} + \frac{k^2}{2}). \tag{2.329}$$

Making use of this symmetry and the following definition

$$\underline{q} = k + \mu + \frac{[P]}{2}, \qquad \mu \in \Lambda^*/\Lambda, \qquad k \in \Lambda, \tag{2.330}$$

one is led to the conclusion that the elliptic genus can be decomposed in the form

$$Z_P^{(r)}(\tau, z) = \sum_{\mu\in\Lambda^*/\Lambda} f_{\mu,J}^{(r)}(\tau)\theta_{\mu,J}^{(r)}(\tau, z), \tag{2.331}$$

$$f_{\mu,J}^{(r)}(\tau) = \sum_{r\hat{q}_0 \geq -\frac{c_L}{24}} d_\mu^{(r)}(\hat{q}_0) e^{2\pi i\tau r\hat{q}_0}, \tag{2.332}$$

$$\theta_{\mu,J}^{(r)}(\tau, z) = \sum_{k\in\Lambda+\frac{[P]}{2}} (-1)^{rp\cdot(k+\mu)} e^{2\pi i\bar{\tau} r\frac{(k+\mu)_+^2}{2}} e^{2\pi i\tau r\frac{(k+\mu)_-^2}{2}} e^{2\pi i r z\cdot(k+\mu)}, \tag{2.333}$$

where $J \in \mathcal{C}(P)$ and $\mathcal{C}(P)$ denotes the Kähler cone of P restricted to $\Lambda\otimes\mathbb{R}$ and $\hat{q}_0 = -q_0 - \frac{1}{2}\underline{q}^2$ is invariant under the spectral flow symmetry. The subscript $+$ refers to projection onto the sublattice generated by the Kähler form J and $-$ is the projection to its orthogonal complement, i.e.

$$k_+^2 = \frac{(k\cdot J)^2}{J\cdot J}, \quad k_-^2 = k^2 - k_+^2. \tag{2.334}$$

[16] In components, $[P]$ is given by $d_{AB}p^A$.

The modular weight of $f^{(r)}_{\mu,J}(\tau)$ is $(-1-\frac{b_2(X)}{2}, 0)$ and the weight of $\Theta^{(r)}_{\mu,J}(\tau,z)$ is $(\frac{b_2(X)-1}{2}, \frac{1}{2})$. There are two issues here for the case of rigid divisors with $b_2^+(P) = 1$ on which we want to comment as this class of divisors is the focus of our work. First of all note, that q_0 contains a contribution of the form[17] $\frac{1}{2}\int_P F\wedge F$ where $F \in \Lambda_P$. Now, F can be decomposed into $F = \underline{q} + \underline{q}_\perp$ with $\underline{q}_\perp \in \Lambda^\perp$, which allows us to write

$$\hat{q}_0 = \tilde{q}_0 + \frac{1}{2}\underline{q}_\perp^2. \tag{2.335}$$

For $b_2^+(P) = 1$ and $r = 1$, the degeneracies $d(r,\mu,\tilde{q}_0)$ are independent of the choice of $\underline{q}_\perp$ and moreover it was shown by Göttsche [112] that

$$\sum_{\tilde{q}_0} d(1,\mu,\tilde{q}_0)\, e^{2\pi i\tau\tilde{q}_0} = \frac{1}{\eta^{\chi(P)}}. \tag{2.336}$$

Then, for $r = 1$ (2.332) becomes

$$f^{(1)}_{\mu,J}(\tau) = \frac{\vartheta_{\Lambda^\perp}(\tau)}{\eta^{\chi(P)}(\tau)}, \qquad \vartheta_{\Lambda^\perp}(\tau) = \sum_{\underline{q}_\perp\in\Lambda^\perp} e^{i\pi\tau\underline{q}_\perp^2}. \tag{2.337}$$

The second subtlety is concerned with the dependence on a Kähler class J. Due to wall-crossing phenomena we will find that $f^{(r)}_{\mu,J}(\tau)$ also depends on J. We expect that it has the following expansion ($\tilde{q}_0 = \frac{d}{r} - \frac{c_L}{24}$)

$$f^{(r)}_{\mu,J}(\tau) = (-1)^{rp\cdot\mu} \sum_{d\geq 0} \bar{\Omega}(\Gamma;J)\, q^{d-\frac{r\chi(P)}{24}}. \tag{2.338}$$

Here, the factor $(-1)^{rp\cdot\mu}$ is inserted to cancel its counterpart in the definition of $\theta^{(r)}_{\mu,J}$, which was only included to make the theta-functions transform well under modular transformations. The invariants $\bar{\Omega}(\Gamma;J)$ are rational invariants first introduced by Joyce [284, 285] and are defined as follows

$$\bar{\Omega}(\Gamma;J) = \sum_{m|\Gamma} \frac{\Omega(\Gamma/m;J)}{m^2}, \tag{2.339}$$

where $\Omega(\Gamma,J)$ is an integer-valued index of BPS degeneracies, given by [286]

$$\Omega(\Gamma,J) = \frac{1}{2}\mathrm{Tr}(2J_3)^2(-1)^{2J_3}, \tag{2.340}$$

where J_3 is a generator of the rotation group Spin(3). Note, that for a single M5-brane $\bar{\Omega}$ and Ω become identical and independent of J. Note, that the multi-cover contributions come here with a factor m^{-2}, whereas the generating function $F^{(0)}(t^a)$ (5.1) of genus zero GW invariants $\bar{n}^{(0)}_\gamma$ weights the Gopakumar-Vafa invariants $n^{(0)}_{\gamma/m,}$ by m^{-3}

$$\bar{n}^{(0)}_\gamma = \sum_{m|\gamma, m\geq 1} n^{(0)}_{\gamma/m}/m^3. \tag{2.341}$$

[17] See appendix 2.4.3 for details.

2.8.2 Counting BPS states a la Gaiotto-Strominger-Yin and using split attractor flows

Gaiotto-Strominger-Yin (GSY) proposed a geometric counting method for the modified elliptic genus of the M5 brane [137, 287] , i.e. the charge vector takes the following form

$$\Gamma = (0, 1, q_1, q_0). \tag{2.342}$$

The idea of GSY is to describe the moduli space of the D4-D2-D0 BPS setup by first fixing the number of D0 branes and then add D2 brane charge Δq_1 as well as D0 brane charge Δq_0. For these setups, one considers the moduli space of D4 branes, such that they pass through the D0 branes. We state the results for the following pairs Δq_0 and Δq_1. For the case of this section GSY we modify the definition of $\hat{q}_0$ a bit by the change in the D0 brane charge

$$\hat{q}_0 = q_{0,\text{ind}} + \Delta q_0 - \frac{1}{2}\underline{q}^2, \tag{2.343}$$

where $q_{0,\text{ind}}$ is the induced D0 charge due to the pure D4-brane

$$q_{0,\text{ind}} = -\frac{c_{\text{L}}}{24} - \frac{P^3}{8} \tag{2.344}$$

and in addition one has to take into account the induced D0 charge due to flux F that is admitted by the D4 branes

$$\Delta q_0 = -\frac{1}{2}\int_P \left(F + \frac{P}{2}\right)^2 + \frac{P^3}{8}. \tag{2.345}$$

Additional D2-brane charge Δq_1 due to the flux is given by

$$\Delta q_{1,a} = \int_P F \wedge J_a. \tag{2.346}$$

There are different possibilities how we can realise the corresponding fluxes. For the case that we realise the flux by $F = [C] - [C']$ we obtain for the change in the charges

$$\begin{aligned} \Delta q_1 &= d - d', \\ \Delta q_0 &= [C] \cdot [C'] - (g + g') + 2 - d'. \end{aligned} \tag{2.347}$$

For the case that $F = J - [C]$ we obtain

$$\begin{aligned} \Delta q_1 &= J^2 - d, \\ \Delta q_0 &= -J^2 + 2d - g + 1. \end{aligned} \tag{2.348}$$

Realizing the flux by a curve $F = [C]$ of degree d and genus g we obtain

$$\begin{aligned} \Delta q_1 &= d, \\ \Delta q_0 &= 1 - g. \end{aligned} \tag{2.349}$$

Results of this counting method for 1-parameter models are presented in appendix D.2.

D4-D2-D0 degeneracies using split flows

Besides the geometric counting method it is also possible to perform the counting by using split flows [191, 288]. The D4-D2-D0 system has the following charges

$$\begin{aligned} p^0 &= 0 \\ p &= 1 \\ q &= \int_P (F + \frac{P}{2}) \wedge J \\ q_0 &= \frac{1}{2} \int_P (F + \frac{P}{2})^2 + \frac{\chi(P)}{24} - N, \end{aligned} \tag{2.350}$$

where N denotes the number of D0-instantons. Assuming that the flux F takes the following form:

$$F = J + C - C', \tag{2.351}$$

where the intersection number between the curves is $C \cdot C' = 0$, the charge vector Γ, decaying into $\Gamma = \Gamma_1 + \Gamma_2$ we can write [123]

$$\Gamma = \left(0, \tilde{P}, \beta_2 - \beta_1 + \tilde{P}\tilde{S}, \frac{P^3 + c_2 P}{24} + \frac{1}{2}\tilde{P}\tilde{S}^2 + \tilde{S}(\beta_2 - \beta_1) - n_2 - \frac{\tilde{P}}{2}\beta_2 + n_1 - \frac{\tilde{P}}{2}\beta_1\right) \tag{2.352}$$

The following identifications hold:

$$\begin{aligned} \tilde{P} = P, \beta_1 = [C'], \beta_2 = [C], \tilde{S} = S = J + \frac{P}{2}, \\ n_1 = -\chi_h(C') - N_1, n_2 = \chi_h(C) + N_2, N = N_1 + N_2. \end{aligned} \tag{2.353}$$

D4-D2-D0 states are counted by using $D6/\overline{D6}$ tachyon condensation picture.

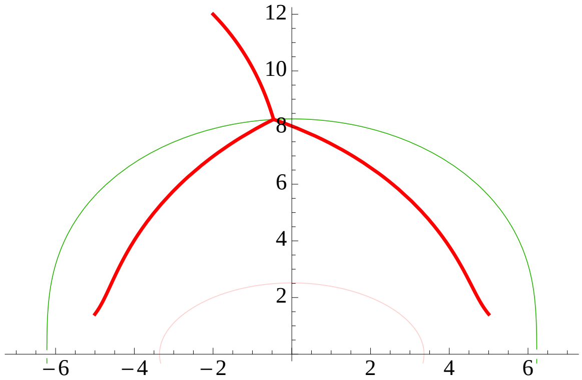

Figure 2.12: The split of the attractor flow at a wall of marginal stability.

The corresponding index is then just given by using the primitive wall-crossing formula [127, 154, 288, 289]

$$\begin{aligned}\Omega(\Gamma_{D4}) &= \sum_{\Gamma\to\Gamma_1+\Gamma_2} (-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\langle\Gamma_1,\Gamma_2\rangle\Omega(\Gamma_1)\Omega(\Gamma_2), \\ &= (-1)^{\langle\Gamma_{D6},\Gamma_{\overline{D6}}\rangle-1}\langle\Gamma_1,\Gamma_2\rangle N_{\mathrm{DT}}(\beta_1,n_1)N_{\mathrm{DT}}(\beta_2,n_2),\end{aligned} \tag{2.354}$$

where we denote by $N_{\mathrm{DT}}(\beta,n)$ the Donaldson-Thomas invariants. The Donaldson-Thomas invariants for the considered one-parameter models can be obtained from [100].

2.8.3 $\mathcal{N} = 4$ Super Yang-Mills, E-strings and bound-states

In the following we recall the relation [139] of the elliptic genus of M5-branes to the $\mathcal{N} = 4$ topological SYM theory of Vafa and Witten [140]. Our goal is to relate the holomorphic anomaly equation which we will derive from wall-crossing in the next section to the anomalies appearing in the $\mathcal{N} = 4$ context. We review moreover the connection of the anomaly to the formation of bound-states given in ref. [139].

The $\mathcal{N} = 4$ topological SYM arises by taking a different perspective on the world-volume theory of n M5-branes on $P \times T^2$ considering the theory living on P which is the $\mathcal{N} = 4$ topological SYM theory described in ref. [140]. The gauge coupling of this theory is given by

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}, \tag{2.355}$$

and is geometrically realised by the complex structure modulus of the T^2. The partition function of this theory counts instanton configurations by computing the generating functions of the Euler numbers of moduli spaces of gauge instantons [140]. S-duality translates to the modular transformation properties of the partition function. The analogues of D4-D2-D0 charges are the rank of the gauge group, different flux sectors and the instanton number.

In ref. [139] the relation is made between this theory and the geometrical counting of BPS states of exceptional strings obtained by wrapping M5-branes around a del Pezzo surface dP_9, also called $\frac{1}{2}$K3. This string is dual to the heterotic string with an E_8 instanton of zero size [290, 291] and is therefore called E-string. In F-theory this corresponds to a $\mathbb{P}^1$ shrinking to zero size [292–294]. The geometrical study of the BPS states of this non-critical string was initiated in ref. [165] and further pursued in refs. [141, 295, 296]. In ref. [139] the counting of BPS states of the exceptional string with increasing winding n was related to the $\mathcal{N} = 4$ U(n) SYM partition functions.

In the following we will use the geometry of ref. [165] which is an elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$, which in turn is a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$.[18] We will denote by t_E, t_F and t_D the Kähler parameters of the elliptic fiber, the fiber and the base of $\mathbb{F}_1$, respectively and enumerate these by 1, 2, 3 in this order. We further introduce $\tilde{q}_a = e^{2\pi i \tilde{t}_a}$, $a = 1, 2, 3$ the exponentiated Kähler parameters appearing in the instanton expansion of the A-model at large radius, which are also the counting parameters of the BPS states.

Within this geometry we will be interested in the elliptic genus of M5-branes wrapping two different surfaces, one is a K3 corresponding to wrapping the elliptic fibre and the fibre of $\mathbb{F}_1$, the resulting string is the heterotic string. The other possibility is to wrap the base of $\mathbb{F}_1$ and the elliptic fibre corresponding to $\frac{1}{2}$K3 and leading to the E-string studied in refs. [139, 141, 165, 295, 296]. The two possibilities are realised by taking the limits $t_D, t_F \to i\infty$, respectively. The resulting surface in both cases is still elliptically fibered which allows one to identify the D4-D0 charges n and p with counting

[18] The toric data of this geometry is summarized in appendix B.2.

curves wrapping n-times the base and p-times the fibre of the elliptic fibration [139]. The multiple wrapping is hence encoded in the expansion of the prepotential $F^{(0)}(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$ of the geometry. In order to get a parameterisation inside the Kähler cone of the K3 in which the corresponding curves in $H_2(\mathrm{K3}, \mathbb{Z})$ intersect with the standard metric of the hyperbolic lattice $\Gamma^{1,1}$, we define $t_1 = \tilde{t}_1$, $t_2 = \tilde{t}_2 - \tilde{t}_1$ and $t_3 = \tilde{t}_3$ as well as the corresponding $q_1 = \tilde{q}_1$, $q_2 = \tilde{q}_2/\tilde{q}_1$ and $q_3 = \tilde{q}_3$. Taking q_2 or $q_3 \to 0$, the multiple wrapping of the base is expressed by

$$F^{(0)}(t_1, t_a) = \sum_{n\geq 1} Z^{(n)}(t_1) q_a^n, \quad a = 2 \text{ or } 3. \tag{2.356}$$

The $Z^{(n)}$ can be identified with the elliptic genus of n M5-branes wrapping the corresponding surface after taking a small elliptic fibre limit [139]. In this limit the contribution coming from the theta-functions (2.333) reduce to $\tau_2^{-3/2}\left(\tau_2^{-1/2}\right)$ for the K3($\frac{1}{2}$K3) cases, these are the contributions of 3(1) copies of the lattice $\Gamma^{1,1}$ appearing in the decomposition of the lattices of K3($\frac{1}{2}$K3). Omitting these factors gives the $Z^{(n)}$ of weight $(-2, 0)$ in both cases. The elliptic genera of wrapping n M5-branes corresponding to n strings are in both cases related recursively to the lower wrapping. The nature of the recursion depends crucially on the ability of the strings to form bound-states.

The heterotic string, no bound-states

The heterotic string is obtained from wrapping an M5-brane on the K3 by taking the $q_3 \to 0$ limit. The heterotic string does not form bound-states and the recursion giving the higher wrappings in this case is the Hecke transformation of $Z^{(1)}$ as proposed in ref. [139]. The formula for the Hecke transformation in this case is given by

$$Z^{(n)}(t) = n^{w_L - 1} \sum_{a,b,d} d^{-w_L} Z^{(1)}\left(\frac{at+b}{d}\right), \tag{2.357}$$

with $ad = n$ and $b < d$ and $a, b, d \geq 0$. Which specialises for $w_L = -2$ and $n = p$, where p is prime to

$$Z^{(p)}(t) = \frac{1}{p^3} Z^{(1)}(pt) + \frac{1}{p}\left[Z^{(1)}\left(\frac{t}{p}\right) + Z^{(1)}\left(\frac{t}{p} + \frac{1}{p}\right) + \cdots + Z^{(1)}\left(\frac{t}{p} + \frac{p-1}{p}\right)\right]. \tag{2.358}$$

For example the partition functions for $n = 1, 2$ obtained from the instanton part of the prepotential of the geometry read

$$Z^{(1)} = -\frac{2E_4E_6}{\eta^{24}}, \quad Z^{(2)} = -\frac{E_4E_6\left(17E_4^3 + 7E_6^2\right)}{96\eta^{48}}, \tag{2.359}$$

and are related by the Hecke transformation. Further examples of higher wrapping are given in the appendix D.1. The fact that the partition functions of higher wrappings of the M5-brane on the K3, which correspond to multiple heterotic strings, are given by the Hecke transformation was interpreted [139] by the absence of bound-states. Geometrically, multiple M5-branes on a K3 can be holomorphically deformed off one another. This argument fails for surfaces with $b_2^+ = 1$ and in particular for $\frac{1}{2}$K3.

One reason that the higher $Z^{(n)}$ can be determined in such a simple way from $Z^{(1)}$ can be understood in topological string theory from the fact that the BPS numbers on K3 depend only on the intersection of a curve $C^2 = 2g - 2$ [297], and not on their class in $H_2(\mathrm{K3}, \mathbb{Z})$. This allows to prove (2.357) to all orders in the limit of the topological string partition function under consideration by slightly modifying the proof in [298]. Using the Picard-Fuchs system of the elliptic fibration one shows in the limit $q_3 \to 0$

the first equality in the identity

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial}{\partial t_2}\right)^3 F^{(0)}|_{q_3\to 0} &= \frac{E_4(t_1)E_6(t_1)E_4(t_2)}{\eta(t_1)^{24}(j(t_1)-j(t_2))} \\ &= \frac{q_1}{q_1-q_2} + E_4(t_2) - \sum_{d,l,k>0} l^3 c(kl) q_1^{kl} q_2^{ld}\,,\end{aligned} \tag{2.360}$$

where $j = E_4^3/\eta^{24}$ and $c(n)$ are defined as

$$-\frac{1}{2}Z^{(1)} = \sum_n c(n)q^n. \tag{2.361}$$

This equations shows two things. The BPS numbers inside the Kähler cone of K3 depend only on $C^2 = kl$ and all $Z^{(n)}$ are given by one modular form. The second fact can be used as in [298] to establish that

$$\frac{1}{2}\left(\frac{\partial}{\partial t_2}\right)^3 F^{(0)}|_{q_3\to 0} = \sum_{n=0}^{\infty} F_n(t_1)q_2^n, \tag{2.362}$$

where F_n is the Hecke transform of F_1, i.e. $n^3 F_n = F_1|T_n$. Using Bol's identity and restoring the n^3 factors yields (2.357).

E-strings and bound-states

The recursion relating the higher windings of the E-strings to lower winding, developed in [139, 141, 296] in contrast reads

$$\frac{\partial Z^{(n)}}{\partial E_2} = \frac{1}{24}\sum_{s=1}^{n-1} s(n-s)Z^{(s)}\,Z^{(n-s)}\,, \tag{2.363}$$

which becomes an anomaly equation, when E_2 is completed into a modular object $\widehat{E}_2$ by introducing a non-holomorphic part (see appendix 2.6.2). The anomaly reads:

$$\partial_{\bar{t}_1}\widehat{Z}^{(n)} = \frac{i(\mathrm{Im}\, t_1)^{-2}}{16\pi}\sum_{s=1}^{n-1} s(n-s)\widehat{Z}^{(s)}\widehat{Z}^{(n-s)}\,, \tag{2.364}$$

and was given the interpretation [139] of taking into account the contributions from bound-states. Starting from [165]

$$Z^{(1)} = \frac{E_4\sqrt{q}}{\eta^{12}}, \tag{2.365}$$

and using the vanishing of BPS states of certain charges one obtains recursively all $Z^{(n)}$ [139, 141, 296]. E.g. the $n = 2$ the contribution reads:

$$\widehat{Z}^{(2)} = \frac{qE_4E_6}{12\eta^{24}} + \frac{q\widehat{E_2}E_4^2}{24\eta^{24}}\,, \tag{2.366}$$

where the second summand has the form $\widehat{E}_2\left(Z^{(1)}\right)^2$ and takes into account the contribution from bound-states of singly wrapped M5-branes.

A relation to the anomaly equations appearing in topological string theory [101] was pointed out in ref. [139] and proposed for arbitrary genus in refs. [167, 299]. The higher genus generalization reads [167, 299]:

$$\frac{\partial Z_g^{(n)}}{\partial E_2} = \frac{1}{24} \sum_{g_1+g_2=g} \sum_{s=1}^{n-1} s(n-s) Z_{g_1}^{(s)} Z_{g_2}^{(n-s)} + \frac{n(n+1)}{24} Z_{g-1}^{(n)} , \qquad (2.367)$$

where the instanton part of the A-model free energies at genus g is denoted by $F^{(g)}(q_1, q_2, q_3)$, and $F^{(g)}(q_1, q_2 \to 0, q_3) = \sum_{n\geq 1} Z_g^{(n)} q_3^n$. The $Z_g^{(n)}$ have the form [299]

$$Z_g^{(n)} = P_g^{(n)}(E_2, E_4, E_6) \frac{q_1^{n/2}}{\eta^{12n}} , \qquad (2.368)$$

where $P_g^{(n)}$ denotes a quasi-modular form of weight $2g + 6n - 2$. We will explore this relation on more general grounds in the context of elliptic Calabi-Yau manifolds in chapter 5.

2.8.4 Counting BPS states via wall-crossing and sheaves

In this section we want to give a short review on counting BPS states by using the techniques of sheaves. We have already discussed the connection between D-brane charges and sheaves as well as stability conditions. Of course the question arises, how this can be helpful for calculating BPS invariants. This section provides the necessary notions and techniques to answer this question. These techniques have been discussed and developed in different places in the literature [152, 155, 300]. We follow [301].

In the following we denote the moduli space by $M_J(\Gamma)$ of the sheaf $\mathcal{E}$ with charge vector Γ at J. For the case, that the charge vector reads

$$\Gamma = (1, q_a, n), \qquad (2.369)$$

it is possible to show, that $M_J(\Gamma)$ corresponds to the Hilbert scheme of points $P^{[n]}$ on P. The generating function for the topological Euler number has been proven to be [112, 140]

$$\sum_{n\geq 0} \chi(S^{[n]}) q^n = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^{\chi(P)} . \qquad (2.370)$$

We introduce the χ_y genus of a smooth, projective complex d-dimensional variety X via

$$\chi_y(X) = \sum_{p,q=0}^{\dim_{\mathbb{C}} X} (-1)^{p-q} y^p h^{p,q}(X), \qquad (2.371)$$

with $h^{p,q}(X) = \dim H^{p,q}(X, \mathbb{Z})$. If the only non-trivial cohomology of the moduli space is of type (p, p), then the χ_y becomes the Poincaré polynomial which is the generating function for the Betti numbers $b_i(X) = \sum_{q+p=1} h^{p,q}(X)$

$$p(X, y) = \sum_{i=0}^{2\dim_{\mathbb{C}} X} b_i(X) y^i . \qquad (2.372)$$

So far we have only been interested in calculating the BPS numbers, which correspond to the Euler-numbers of moduli spaces. However, it is possible to perform a refinement in the following way

$$\Omega(\Gamma, w, J) = \frac{w^{-\dim \mathcal{M}_J}}{w - w^{-1}} \chi_{w^2}(M_J(\Gamma)), \quad w = e^{2\pi i z} . \qquad (2.373)$$

Defintions for $\bar{\Omega}(\Gamma, w, J)$ are given by

$$\bar{\Omega}(\Gamma, w, J) = \sum_{m|\Gamma} \frac{\Omega(\Gamma/m, -(-w)^m; J)}{m}. \tag{2.374}$$

The corresponding general generating function $f^{(r)}_{\mu,J}(\tau, z)$ is defined by

$$f^{(r)}_{\mu,J}(\tau, z) = \sum_n \bar{\Omega}(\Gamma, w, J) q^{r\Delta(\Gamma) - \frac{r\chi}{24}}. \tag{2.375}$$

We will be interested to calculate the BPS invariants for divisors, that are rational surfaces, i.e. either we have $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$. Note, that $\mathbb{P}^2$ is the blow down of $\mathbb{F}_1$ and can therefore we can concentrate on the case of $\mathbb{F}_n$ since we can use blow-up formulas for the generating function. In order to calculate the generating function $f^{(r)}_{c_1,J}(\tau, z)$ we choose the parameter J to be a suitable polarization J_*. In the following we denote the fibre of these surfaces by F.

Definition: A suitable polarisation J_* for the charge vector $\Gamma = (r, \mathrm{ch}_1, \mathrm{ch}_2)$ is subject to the following conditions

i) J_* does not lie on a wall for Γ and

ii) for a semi-stable subsheaf $\mathcal{E}' \subset \mathcal{E}$ one of the two conditions holds either

$$(\mu(\mathcal{E}') - \mu(\mathcal{E})) \cdot F = 0 \tag{2.376}$$

or

$$(\mu(\mathcal{E}') - \mu(\mathcal{E})) \cdot F \qquad \text{and} \qquad (\mu(\mathcal{E}') - \mu(\mathcal{E})) \cdot J_* \tag{2.377}$$

have the same sign. ◇

The suitable polarisation we will use for the mentioned surface is close to the fibre and denoted by $J_{\epsilon,1}$. For the case that $c_1 \cdot F = 0 \mod r$ the generating function $f^{(r)}_{c_1,J_{\epsilon,1}}(\tau, z)$ is given by formula (5.9) in [301]

$$f^{(r)}_{c_1,J_{\epsilon,1}}(\tau, z) = H_{r,c_1}(z, \tau, F) - \sum_{\mathrm{ch}_2} \sum_{\substack{\Gamma_1 + \ldots \Gamma_l = \Gamma, l>1 \\ p_J(\Gamma_i, n) \geq p_J(\Gamma_{i+1}, n)}} \bar{\Omega}(\{\Gamma_i\}, w, J_{\epsilon,1}) q^{r\Delta(\Gamma) - \frac{r\chi(S)}{24}}, \tag{2.378}$$

where we have introduced

$$H_{r,c_1}(z, \tau, F) = \frac{i(-1)^{r-1}\eta(\tau)^{2r-3}}{\vartheta_1(2z, \tau)^2 \vartheta_1(4z, \tau)^2 \cdot \cdots \cdot \vartheta_1((2r-2)z, \tau)^2 \vartheta_1(2rz, \tau)^2} \tag{2.379}$$

and the invariants [302, 303]

$$\bar{\Omega}(\{\Gamma_i\}, w, J) = \frac{1}{|\mathrm{Aut}(\{\Gamma_i\}, J)|} w^{-\sum_{i<j} r_i r_j (\mu_i - \mu_j) \cdot K_P} \prod_{i=1}^n \bar{\Omega}(\Gamma_i, w, J). \tag{2.380}$$

In here we denote by

$$|\mathrm{Aut}(\{\Gamma_i\}, J)| = \prod_a m_a! \tag{2.381}$$

the product over all quotients E_i with equal reduced Hilbert polynomial $p_J(E_i, n)$. It is possible to carry out the discussion more generally using the language of motivic invariants [301].

CHAPTER 3

Geometries and their construction

In this chapter we want to give a short review on geometries and constructions that appear frequently in this thesis. In particular we give a short review on the construction of Calabi-Yau manifolds as hypersurfaces and complete intersections where we use in particular methods from toric geometry. For some generalities of complex geometry and Calabi-Yau manifolds we refer to the appendix A. Then we present the classification of complex surfaces by Enriques and Kodaira and Kodaira's classification of singular elliptic fibres. This allows us to bring these topics together in the discussion of the classical geometry of elliptically fibred Calabi-Yau spaces.

3.1 Calabi-Yau manifolds as hypersurfaces and complete intersections in weighted projective space

In this section we give a short recap of toric geometry in order to obtain Calabi-Yau manifolds. We follow [89, 304, 305]. We start with a weighted projective space $\mathbb{P}^n(w_1,\dots,w_{n+1})$. A Calabi-Yau manifold $X_{d_1,\dots,d_m}[w_1,\dots,w_{n+1}]$ can be obtained by the zero locus of polynomials $\{P_i\}_{i=1}^m$ with degree $\deg P_i = d_i$, i.e.

$$X_{d_1,\dots,d_m}[w_1,\dots,w_{n+1}] = \{[z_1,\dots,z_{n+1}] \in \mathbb{P}^n(w_1,\dots,w_{n+1})\big| \\ \forall i \in \{1,\dots,m\} : P_i(z_1,\dots,z_{n+1}) = 0\}\,, \tag{3.1}$$

if the following condition is satisfied

$$\sum_{i=1}^{m} d_i = \sum_{i=1}^{n+1} w_i. \tag{3.2}$$

It can be easily checked, that (3.2) ensures the Calabi-Yau condition, e.g. that the first Chern class $c_1(X_{d_1,\dots,d_m}[w_1,\dots,w_{n+1}]) = 0$. We can distinguish between two types of singularities, namely singular points which are locally of the form $\mathbb{C}^3/\mathbb{Z}_n$ and singular curves which locally can be described as $\mathbb{C}^2/\mathbb{Z}_n$, which are subject to means of toric geometry [89, 306, 307].

Recall that a toric variety X is a complex algebraic variety containing as an open subset an algebraic torus $\mathbb{T}^r \subset X$, which is accompanied with an action of $\mathbb{T}^r$ on X such that the restriction of this action to $\mathbb{T}^r$ is the usual multiplication on $\mathbb{T}^r$. In physics this can be easily realised by gauge linear sigma models.

Explicitly, we study a $U(1)^s$ gauge theory with n chiral super fields X_i that are charged under $U(1)^s$ with charge vector $Q_i = \{Q_{i,1}, \ldots, Q_{i,s}\}$. The potential energy of the gauge theory now reads

$$U(x_i) = \sum_{k=1}^{s} \frac{e_k^2}{2} \left(\sum_{i=1}^{n} Q_{i,k}|x_i|^2 - r_k \right), \tag{3.3}$$

where the e_k denote gauge couplings, x_i the scalar components of X_i and r_k correspond to the Fayet-Iliopoulos (FI) parameters. The classical ground states $\mathcal{M}$ can now be obtained by looking at the zero locus of (3.3) modulo gauge equivalent configurations

$$\mathcal{M} = \left\{ x \in \mathbb{C}^n | \sum_{i=1}^{n} Q_{i,k}|x_i|^2 = r_k \right\} / U(1)^s. \tag{3.4}$$

For given FI-parameters that allow a solution of the condition in (3.4) and corresponding charges, $\mathcal{M}$ can be described as a $(n-s)$ dimensional toric variety with a fan with n edges. We present the mathematical approach to this setup.

We denote by Λ a rank r lattice and we denote $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$. For building a fan Σ we first define a strongly convex rational polyhedral cone.

Definition: A strongly convex polyhedral cone $\sigma \subset \Lambda_{\mathbb{R}}$ is generated by the set of vectors $v_1, \ldots, v_k$ such that

i) $\sigma = \left\{ \sum_{i=1}^{k} a_i v_i | \forall i : a_i \geq 0 \right\}$ and

ii) $\sigma \cap (-\sigma) = \{0\}$.

◇

Definition: Σ is called a fan, if it is a collection of strongly convex rational polyhedral cones in $\Lambda_{\mathbb{R}}$ such that

i) each face of a cone in Σ is also a cone in Σ and

ii) the intersection of two cones σ and σ' in Σ is a face of each of the two cones.

◇

A toric variety X_Σ from the fan Σ is obtained by the following quotient

$$X_\Sigma = (\mathbb{C}^n - Z(\Sigma))/G, \tag{3.5}$$

with the torus $\mathbb{T}^r$ given by $\mathbb{C}^n/G$. We explain this construction in the following by clarifying $Z(\Sigma)$ and G. We denote by $\Sigma(1)$ the set of one-dimensional cones, i.e. the edges of the fan and set $n = |\Sigma(1)|$. In a next step we associate to each edge $\sigma_i^{(1)} \in \Sigma(1)$ a coordinate $x_{\sigma_i^{(1)}}$. Let $\mathcal{S}$ be the subset $\mathcal{S} \subset \Sigma(1)$ that does not span a cone of Σ and denote by

$$V(\mathcal{S}) = \left\{ x_{\sigma_i^{(1)}} = 0, \forall \sigma_i^{(1)} \in \mathcal{S} \right\}. \tag{3.6}$$

Then consider the union of all these sets

$$Z(\Sigma) = \bigcup_{\sigma \in \mathcal{S}} V(\mathcal{S}). \tag{3.7}$$

The only thing that is left to clarify is the group G, which is given as the kernel of the map

$$\phi : \mathrm{Hom}(\Sigma(1), \mathbb{C}^*) \to \mathrm{Hom}(M, \mathbb{C}^*), \tag{3.8}$$

where $M = \mathrm{Hom}(\Lambda, \mathbb{Z})$. In a convenient basis ϕ can be expressed

$$\begin{aligned} \phi : (\mathbb{C}^*)^n &\to (\mathbb{C}^*)^r \\ (t_1, \dots, t_n) &\mapsto \left(\prod_{j=1}^n t_j^{\nu_{j,1}}, \dots, \prod_{j=1}^n t_j^{\nu_{j,r}} \right). \end{aligned} \tag{3.9}$$

This completes the construction of a toric variety X_Σ.

The charges introduce relation between the edges $\nu_i \in \Sigma(1)$

$$\sum_{i=1}^n Q_{i,a} \nu_i = 0. \tag{3.10}$$

This argument can also be turned around, i.e. given a set of edges, it is possible to determine the corresponding charges. In table 3.1 we collect some data for the fans of $\mathbb{P}^2$ and $\mathbb{F}_n$. The charges $Q_{i,a}$

geometry	edges $\Sigma(1)$	charges
$\mathbb{P}^2$	$(-1,-1),(1,0),(0,1)$	$l^{(1)} = (1,1,1)$
$\mathbb{F}_n$	$(1,0),(-1,-n),(0,1),(0,-1)$	$l^{(1)} = (1,1,n,0)$ $l^{(2)} = (0,0,1,1)$

Table 3.1: The toric data for $\mathbb{P}^2$ and $\mathbb{F}_n$

correspond to the intersection numbers of divisors $\{D_i\}_{i=1}^n$, that are invariant under the torus action, with curves $\{C_k\}_{k=1}^s$ that span $H^2(X, \mathbb{Z})$

$$Q_{i,a} = D_i \cdot C_a. \tag{3.11}$$

Usually the curves C_a form a generating set of the Mori cone. In particular it can be shown, that the Mori cone is spanned by curves that correspond to $r-1$ dimensional cones. The corresponding Mori vectors are denoted by $l^{(i)}$.

Blow ups

We give a short introduction to the blow-up procedure by using toric geometry. We will use these methods for calculation the generating functions of D4-D2-D0 states for various surfaces embedded into the Calabi-Yau manifold, that are related to each other by blow ups. The easiest example is that $\mathbb{F}_1$ is the blow up of $\mathbb{P}^2$, see figure 3.1. The blow up procedure involves the following steps:

1. Given a fan Σ. We say that another fan Σ' subdivides Σ if the edges of $\Sigma(1)$ are contained in the set of edges of $\Sigma'(1)$ and if each cone $\sigma' \in \Sigma'$ is contained in a cone $\sigma \in \Sigma$.

2. Given a point $p \in X_\Sigma$ that we want to blow up, we first find the corresponding cone $\sigma \in \Sigma$ with primitive generators $\{\nu_1, \dots, \nu_r\}$.

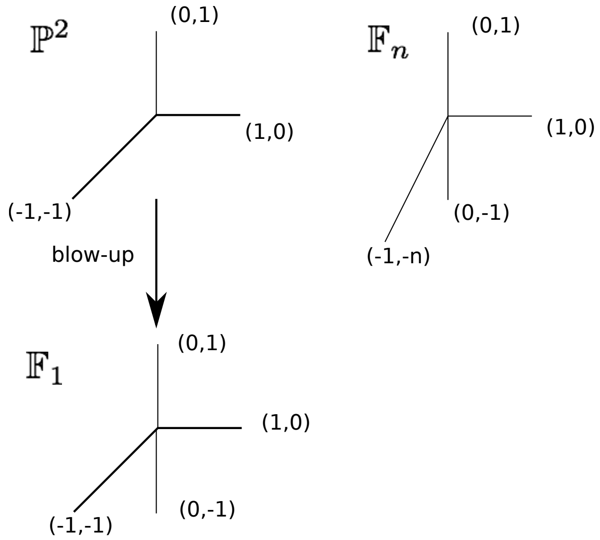

Figure 3.1: The fans for $\mathbb{P}^2$ and $\mathbb{F}_n$. Furthermore the blow up of $\mathbb{P}^2$ is equivalent to $\mathbb{F}_1$.

3. We introduce a new edge v_{r+1} that is obtained by adding the primitive generators of σ

$$v_{r+1} = \sum_{i=1}^{r} v_i. \tag{3.12}$$

4. A subdivision of σ and combining the cones of Σ with the new cones leads a subdividing fan Σ', that is the blow up of Σ in p.

It is easy to see from (3.12) that the blow-up procedure introduces a new charge vector $l^{(r)} = (1, \dots, 1, -1, 0, \dots)$. Given an toric algebraic variety X_Σ it is always possible to find a resolution of its singularities by finding a subdividing fan $\tilde{\Sigma}$ such that $X_{\tilde{\Sigma}}$ is the resolved toric algebraic variety.

Toric varieties and polyhedrons

In order to describe projective toric varieties $\mathbb{P}_\Delta$, we give a short discussion on polyhedra $\Delta \subset \mathbb{R}^n$ that is based on Batyrev's construction [308, 309], we follow [304, 305].

Definition: An integral polyhedron Δ is a polyhedron with integral vertices. ⋄

We denote the integral points of Δ in the following by v_i.

Definition: An integral polyhedron Δ is called reflexive, if the dual polyhedron Δ^*

$$\Delta^* = \{(x_1, \dots, x_n) : \sum_{i=1}^{n} x_i y_i \geq -1, \forall (y_1, \dots, y_n) \in \Delta\}, \tag{3.13}$$

is also an integral polyhedron. ⋄

In order to proceed in our construction of the toric variety $\mathbb{P}_\Delta$, we define a the complete rational fan $\Sigma(\Delta)$ and the toric variety will be realised with respect to this fan.

Definition: The complete rational fan $\Sigma(\Delta)$ is the collection of all n-l-dimensional dual cones $\sigma^*(F_l)$ with $l = 0, \dots, n$, where for each l-dimensional face $F_l \subset \Delta$ the n-dimensional cone $\sigma(F_l)$ is defined via

$$\sigma(F_l) = \{\lambda(p - p') : \lambda \in \mathbb{R}_+, p \in \Delta, p' \in F_l\}. \tag{3.14}$$

◇

For constructing Calabi-Yau hypersurfaces it is necessary to look at the vanishing locus $Z_p \subset (\mathbb{C}^*)^n \subset \mathbb{P}_\Delta$ of the Laurent polynomial with coefficients $\{a_i\} \in \mathbb{C}^{s+1}$

$$p(a,X) = \sum_{i=0}^{s} a_i X^{\nu_i} \in \mathbb{C}[X_1^{\pm 1},\dots,X_n^{\pm 1}], \qquad X^\nu = X_1^{\nu_1}\cdot\dots\cdot X_n^{\nu_n}, \tag{3.15}$$

In the following we denote by $p = p_\Delta$ the Laurent polynomial from 3.15 with respect to the integral points in Δ.

Definition: The pair (p, Z_p) is called Δ-regular if for all faces $F_l \subset \Delta$ the two quantities p_{F_l} and $X_i\partial_{X_i}p_{F_l}$ do not vanish simultaneously. ◇

If Δ is reflexiv, it is possible to resolve the closure $\bar{Z}_p$ to a Calabi-Yau manifold $\hat{Z}_p$ and a variation of the moduli $\{a_i\}_{i=0}^{0}$ leads to a family of Calabi-Yau manifolds. According to Batyrev [308] it is then possible to calculate the Hodge numbers of the two reflexive polyhedra (Δ,Δ^*) corresponding to mirror Calabi-Yau manifolds. We denote by $l(F)$ the number of integral points on a face $F \subset \Delta$ and by $l'(F)$ the number of points in the interior of the face. Then the Hodge numbers can be calculated as[1]

$$\begin{aligned} h^{1,1}(\hat{Z}_{p,\Delta}) = h^{2,1}(\hat{Z}_{p,\Delta^*}) &= l(\Delta^*) - (n+1) - \sum_{\mathrm{codim}\,F^*=1} l'(F^*) + \sum_{\mathrm{codim}\,F^*=2} l'(F^*)l'(F), \\ h^{1,1}(\hat{Z}_{p,\Delta^*}) = h^{1,1}(\hat{Z}_{p,\Delta}) &= l(\Delta) - (n+1) - \sum_{\mathrm{codim}\,F=1} l'(F) + \sum_{\mathrm{codim}\,F=2} l'(F)l'(F^*). \end{aligned} \tag{3.16}$$

For Calabi-Yau manifolds realised as complete complete intersection, we consider l hypersurfaces in k projective spaces and we want to collect some useful formulae for the calculation of topological data following [305]

$$\left(\begin{array}{c||c} \mathbb{P}^{n_1}[w_1^{(1)},\dots,w_{n_1+1}^{(1)}] & d_1^{(1)},\dots,d_l^{(1)} \\ \vdots & \vdots \\ \mathbb{P}^{n_k}[w_1^{(k)},\dots,w_{n_1+1}^{(k)}] & d_1^{(k)},\dots,d_l^{(k)} \end{array}\right) \tag{3.17}$$

Again, the Calabi-Yau condition is satisfied, if $c_1^m = 0$, i.e.

$$c_1^m = \sum_{i=0}^{n_m+1} w_i^{(m)} - \sum_{i=1}^{l} d_i^{(m)} \quad \forall m = 1,\dots,k. \tag{3.18}$$

We denote the i-th Kähler form with respect to the i-th projective space by J_i and define the map

$$\Pi(J_m) = \left(\prod_{r=1}^{k} \frac{\partial_{J=r}^{n_r}}{n_r!}\right)\left(\frac{\prod_{i=1}^{k}\prod_{j=1}^{n_i+1}(1+w_j^{(i)}J_i)}{\prod_{j=1}^{l}(1+\sum_{i=1}^{k} d_j^{(i)}J_i)}\right)\left(\frac{\prod_{j=1}^{l}\sum_{i=1}^{k} d_j^{(i)}J_i}{\prod_{i=1}^{k}\prod_{j=1}^{n_i+1} w_j^{(i)}}\right) J_m\Bigg|_{J_1=\dots=J_k=0}, \tag{3.19}$$

which for the case that the Calabi-Yau manifold has no singularities allows to determine the topological data as

$$\chi(X) = \Pi(1), \int_X c_2 \wedge J_m = \Pi(J_m), D_{ijk} = \int_X \Pi(J_iJ_jJ_k). \tag{3.20}$$

[1] We restrict ourselves to the case of Calabi-Yau threefolds.

3.2 Complex surfaces

As we want to determine D4-D2-D0 states that arise from wrapping a complex surface in a Calabi-Yau manifold, we give a give a short review on complex surfaces P following [310].

Definition: A rational surface is a surface (i.e. complex two-dimensional) such that it is birationally isomorphic to $\mathbb{P}^2$. ⋄

This statement is by Noether's lemma equivalent to the fact, that P contains an irreducible rational curve C with $\dim|C| \geq 1$. In the following we denote by C_i a curve in P.

Definition: A rational ruled surface is a surface $\pi : P \to \tilde{P}$ such that the curves $\pi(C_i)$ form a pencil of irreducible disjoint rational curves. ⋄

The class of rational ruled surface is equivalent to $\mathbb{P}(E)$ with E a holomorphic vector bundle of rank two over $\mathbb{P}^1$. The Hirzebruch surface $\mathbb{F}_n$ represents the unique $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ with an irreducible curve of self-intersection $-n$. These surfaces can be obtained as blow-ups from the rational surfaces $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ see also 3.1. A general rational surface can be obtained as the blow-up of either $\mathbb{P}^2$ or $\mathbb{F}_n$. For a detailed proof of this theorem, we refer to [310].

There exists a classification theorem by Enriques and Kodaira [310], which allows for a classification according to the Kodaira number $\kappa(P)$ which depends on the so called plurigenera $P_n(P) = h^0(S, \mathcal{O}(K_P^n))$ with $n > 0$, that are invariant under the blow-up procedure. The Kodaira number $\kappa(S)$ can then be obtained as presented in table 3.2.

κ	P_n	Enriques-Kodaira classification
-1	$\forall n : P_n(S) = 0$	minimal surfaces are either $\mathbb{P}^2$ or a ruled surface
0	$\exists M : \forall P_n(S) < M : P_n(S) \in \{0, 1\}$	1. $h^{1,0} = 0, h^{2,0} = 1$: K3 surface 2. $h^{1,0} = 0, h^{2,0} = 0$: Enriques surface 3. $h^{1,0} = 1$: hyperelliptic surfaces 4. $h^{1,0} = 2$: abelian surface
1	$\exists c : \forall n P_n(S) \leq nc$	elliptic surfaces
2	$\frac{P_n(S)}{n}$ unbounded	surfaces of general type

Table 3.2: The Enriques-Kodaira classification of complex surfaces.

3.3 Classical geometry of elliptically fibred Calabi-Yau spaces

In this section we study the classical geometry of elliptically fibered Calabi-Yau threefolds M with base B and projection map $\pi : M \to B$. We provide expressions for the Chern classes as well as the construction of such Calabi-Yau three manifolds by means of toric geometry. Elliptic fibrations are locally described by a Weierstrass form

$$y^2 = 4x^3 - xw^4 g_2(\underline{u}) - g_3(\underline{u})w^6, \tag{3.21}$$

where $\underline{u}$ are coordinates on the base B. The $j(u)$ function can be obtained from

$$j(u) = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2}, \tag{3.22}$$

which in turn can be used to calculate $j(\tau)$ via

$$j(\tau) = 1728 \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2}, \tag{3.23}$$

as j is an invariant of the corresponding elliptic curve. Let us return to (3.21). A global description can be

fibre	ord(g_2)	ord(g_3)	ord(Δ)	$j(\tau)$	group	monodromy
I_0	≥ 0	≥ 0	0	$\mathbb{R}$	-	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
I_1	0	0	1	∞	U(1)	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
I_n	0	0	$n > 1$	∞	A_{n-1}	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
II	≥ 1	1	2	0	-	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
III	1	≥ 2	3	1	A_1	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
IV	≥ 2	2	4	0	A_2	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
I_n^*	$(2, \geq 2)$	$(\geq 3, 3)$	$n + 6$	∞	D_{n+4}	$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$
IV^*	≥ 3	4	8	0	E_6	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$
III^*	3	≥ 5	9	1	E_7	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
II^*	≥ 4	5	10	0	E_8	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

Table 3.3: Kodaira's classification of singular fibres

defined by an embedding as a hypersurface or complete intersection in an ambient space W. Explicitly we consider cases, which allow a representation as a hypersurface or complete intersection in a toric ambient space. We restrict our attention to the case where the fiber degenerations are only of Kodaira type I_1, which means that the discriminant $\Delta = g_2^3 - 27g_3^2$ of (3.21) has only simple zeros on B, which are not simultaneously zeros of g_2 and g_3, see also table 3.3. It was observed in [169] that such tame fibrations can be constructed torically over toric bases, which are given themselves defined by reflexive polyhedra. These tame fibrations are not enough to address immediately phenomenological interesting models in F-theory, due to the lack of non-abelian gauge symmetry in the effective four-dimensional physics, which come precisely from more singular fibres in the Kodaira classification. However, we note that the examples discussed here have a particular large number of complex moduli. Adjusting the latter and blowing up the singularities, not necessarily torically, is a more local operation, at least of co-dimension one in the base, which can be addressed in a second step.

3.3.1 The classical geometrical data of elliptic fibrations

Let $W \to B$ be a fibre bundle whose fibre is an $r-1$ dimensional weighted projective space $\mathbb{P}(w_1, \dots, w_r)$ and B an almost toric Fano surface. We define elliptically fibred Calabi-Yau threefolds $M \to B$ as hypersurfaces or complete intersections in W. We consider the following choices of weights

$$(w_1, \dots, w_r) = \{(1,2,3), (1,1,2), (1,1,1), (1,1,1,1)\}\,. \tag{3.24}$$

In particular the elliptic fibres are degree 6, 4, 3 hypersurfaces and a bidegree $(2,2)$ complete intersection in the coordinates of the given weighted projective space. In the case of rational elliptic surfaces these fibres lead to E_8, E_7, E_6, and D_5 del Pezzo surfaces, named so as the integers cohomology lattice of the surface contains the intersection form of the Cartan-matrix of the corresponding Lie algebras. In the following we keep these names for the fibration types.

Let us discuss the first case. This leads canonically to an embedding with a single section, however most of the discussion below applies to the other cases with minor modifications. Denote by $\alpha = c_1(\mathcal{O}(1))$ with $\mathcal{O}(1)$ the line bundle on W induced by the hyperplane class of the projective fibre and $K = -c_1$ the canonical bundle of the base.

The coordinates w, x, y are sections of $\mathcal{O}(1)$, $\mathcal{O}(1)^2 \otimes K^{-2}$ and $\mathcal{O}(1)^3 \otimes K^{-3}$ while g_2 and g_3 are sections of K^{-4} and K^{-6} respectively so that (3.21) is a section of $\mathcal{O}(1)^6 \otimes K^{-6}$. The corresponding divisors $w = 0, x = 0, y = 0$ have no intersection, i.e. $\alpha(\alpha + c_1)(\alpha + c_1) = 0$ in the cohomology ring of W and

$$\alpha(\alpha + c_1) = 0 \tag{3.25}$$

in the cohomology ring of M. Let us assume that the discriminant Δ vanishes for generic complex moduli only to first order in the coordinates of B at loci, which are not simultaneously zeros of g_2 and g_3. In this case its class must satisfy

$$[\Delta] = c_1(B) = -K \tag{3.26}$$

to obey the Calabi-Yau condition and the fibre over the vanishing locus of the discriminant is of Kodaira type I_1. For this generic fibration, the properties of M depend only on the properties of B.

For example using the adjunction formula and the relation (3.25) to reduce to linear terms in α allows to write the total Chern class as[2]

$$C = \left(1 + \sum_{i=1}^{d_M-1} c_i\right) \frac{(1+\alpha)(1 + w_2\alpha + w_2 c_1)(1 + w_3\alpha + w_3 c_1)}{1 + d\alpha + dc_1}\,. \tag{3.27}$$

The Chern forms C_k of M are the coefficients in the formal expansion of (3.27) of the degree k in terms

[2] In the $D5$ complete intersection case $d_1 = d_2 = 2$. One has to add a factor $(1 + \alpha + c_1)$ in the numerator and a factor $(1 + 2\alpha + 2c_1)$ in the denominator.

Fibre	C_2	C_3	C_4
$E8$	$12\alpha c_1 + (11c_1^2 + c_2)$	$-60\alpha c_1^2 - (60c_1^3 + c_2c_1 - c_3)$	$12\alpha c_1(30c_1^2 + c_2)$
$E7$	$6\alpha c_1 + (5c_1^2 + c_2)$	$-18\alpha c_1^2 - (18c_1^3 + c_2c_1 - c_3)$	$6\alpha c_1(12c_1^2 + c_2)$
$E6$	$4\alpha c_1 + (3c_1^2 + c_2)$	$-8\alpha c_1^2 - (8c_1^3 + c_2c_1 - c_3)$	$4\alpha c_1(6c_1^2 + c_2)$
$D5$	$3\alpha c_1 + (2c_1^2 + c_2)$	$-4\alpha c_1^2 - (4c_1^3 + c_2c_1 - c_3)$	$3\alpha c_1(3c_1^2 + c_2)$

Table 3.4: Chern classes C_i of regular elliptic Calabi-Yau manifolds. Integrating α over the fibre yields a factor $a = \frac{\prod_i d_i}{\prod_i w_i}$, i.e. the number of sections $1, 2, 3, 4$ for the three fibrations in turn.

of a and the monomials of the Chern forms c_i of base B. The formulas (3.25) and (3.27) apply for all projectivisations.

In the following the results for various dimensions d_M are presented. For $d_M = 2$ one gets from Table 3.4 by integrating over the fibre in all cases $\chi(M) = 12\int_B c_1$ and $\mathbb{P}^1$ is the only admissible base. Similar for $d_M = 3$ one gets for the different projectivisations $\chi(M) = -60\int_B c_1^2$, $\chi(M) = -36\int_B c_1^2$, $\chi(M) = -24\int_B c_1^2$ and $\chi(M) = -16\int_B c_1^2$.

The following discussion extends to all dimensions but for the sake of brevity we specialise to Calabi-Yau threefolds. Let K_i, $i = 1, \ldots, b_2(B)$, span the Kähler (or ample) cone of B with intersection numbers $K_iK_j = c_{ij}$. Moreover, let C^i be a basis for the dual Kähler cone. We expand the canonical class of B in terms of K_i and C^i as:

$$K = -c_1 = -\sum_i a^i K_i = -\sum a_i C^i\ , \tag{3.28}$$

with a_i and a^i in $\mathbb{Z}$. We denote by $\mathcal{K}_a$, $a = 1, \ldots, h^{1,1}(M)$, the divisors of the total space of the elliptic fibration and distinguish between $\mathcal{K}_e$ the divisor dual to the elliptic fibre curve and $\mathcal{K}_i$, $i = 1, \ldots, b = b_2(B)$, which are $\pi^*(C^i)$

$$\begin{aligned} \mathcal{K}_e^3 &= a\int_B c_1^2,\\ \mathcal{K}_e^2\mathcal{K}_i &= aa_i,\\ \mathcal{K}_e\mathcal{K}_i\mathcal{K}_j &= ac_{ij}\ . \end{aligned} \tag{3.29}$$

Here a denotes the number of sections, see table 3.4. The intersection with the second Chern class of the total space can be calculated using table 3.4 for the elliptic and other fibres as

$$\begin{aligned} \int_M c_2 J_e &= \begin{cases} \int_B (11c_1^2 + c_2) & E_8,\\ 2\int_B (5c_1^2 + c_2) & E_7,\\ 3\int_B (3c_1^2 + c_2) & E_6,\\ 4\int_B (2c_1^2 + c_2) & D_5, \end{cases} \\ \int_M c_2 J_i &= 12a_i. \end{aligned} \tag{3.30}$$

Here we denoted by J_i the basis of harmonic $(1,1)$ forms dual to the $\mathcal{K}_i$.

Let us note two properties about the intersection numbers. These properties can be established using the properties of the toric almost Fano bases B and (3.29), which follows from the construction of the elliptic fibration summarised in (3.42). To start, define the matrix

$$C_e = \begin{pmatrix} \int_B c_1^2 & a_1 & \dots & a_b \\ a_1 & & & \\ \vdots & & c_{ij} & \\ a_b & & & \end{pmatrix}, \tag{3.31}$$

then we can conclude from properties of the intersection numbers and the canonical class that

$$\det(C_e) = 0. \tag{3.32}$$

A further property concerns a decoupling limit between base and fibre in the Kähler moduli space. Generally we can make a linear change in the basis of Mori vectors l_i, which results in corresponding linear change of the basis in dual spaces of the Kähler moduli t^i and the divisors D_i

$$\tilde{l}_i = \sum_j m_{ij} l_j, \qquad \tilde{t}^i = \sum (m^T)_{ij} t^j . \tag{3.33}$$

To realise a decoupling between the base and the fibre we want to find a not necessarily integer basis change, which eliminates the couplings $\tilde{\mathcal{K}}_e^2 \tilde{\mathcal{K}}_i$ and leaves the couplings $\tilde{\mathcal{K}}_e \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}_j$ invariant. It follows from (3.28, 3.29) and the obvious transformation of the triple intersections that there is a unique solution

$$m = \begin{pmatrix} 1 & \frac{a^1}{2} & \dots & \frac{a^b}{2} \\ 0 & 1 & 0\dots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \dots 0 & 1 \end{pmatrix}, \tag{3.34}$$

such that

$$\begin{aligned} \tilde{\mathcal{K}}_e^3 &= a\left(\int_B c_1^2 - \tfrac{3}{2} a_i a^i + \tfrac{3}{4} c_{ij} a^i a^j\right), \\ \tilde{\mathcal{K}}_e^2 \tilde{\mathcal{K}}_i &= 0, \\ \tilde{\mathcal{K}}_e \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}_j &= a c_{ij} . \end{aligned} \tag{3.35}$$

As we have seen the classical topological data of the total space of the elliptic fibration follows from simple properties of the fibre and the topology of the base. We want to extend these results in the next section to the quantum cohomology of the elliptic fibration. We focus again on the Calabi-Yau threefold case, where the instanton contributions to the quantum cohomology is richest. To actually calculate quantum cohomology we need an explicit realisation of a class of examples, which we discuss in the next subsection.

3.3.2 Realisations in toric ambient spaces

In this subsection we discuss the toric bases B leading to the above described tame elliptically fibered Calabi-Yau d_M-folds with only I_1 fibre singularities. It was observed in examples in [169], that they can be defined over toric bases defined themselves by reflexive polyhedra [308] Δ_B in $d_M - 1$ dimensions. Here we explore a class of elliptic Calabi-Yau fibrations, which are defined from a reflexive polyhedra Δ_B as the canonical hypersurface in the toric ambient space defined by the reflexive polyhedra (3.42) following Batyrev's work [308]. Note that for each Δ_B, one has the choice of the elliptic fibre as discussed in the previous section. We provide the toric data, including a basis for the Mori cone for this class of elliptic Calabi-Yau fibrations. The construction of the Mori cone from the star triangulation and the associated secondary fan follows the discussion in [304,311]. Throughout the subsection we assume some familarity with the construction of the toric ambient spaces from polyhedra as described in section 3.1 and in [306,309].

For the threefold case one has the following possibilities of 2-dimensional reflexive base polyhedra in 3.2 [308] .

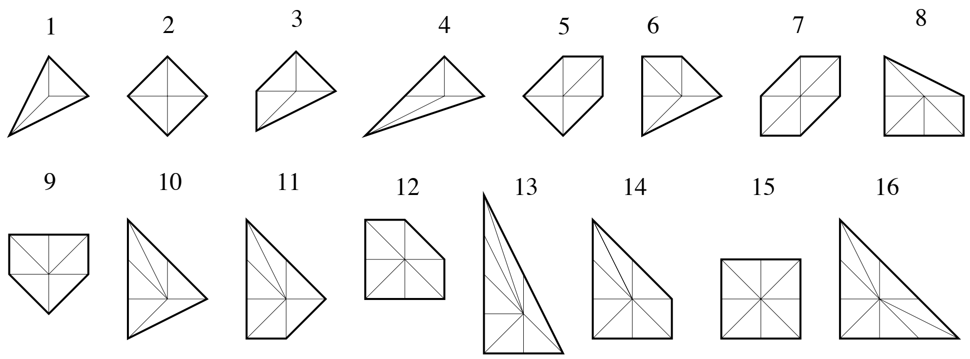

Figure 3.2: These are the 16 reflexive polyhedra Δ_B in two dimensions, which build 11 dual pairs (Δ_B, Δ_B^*). Polyhedron k is dual to polyhedron $17-k$ for $k = 1, \ldots, 5$. The polyhedra $6, \ldots, 11$ are selfdual.

The toric ambient spaces, which allow for smooth Calabi-Yau hypersurfaces of complex dimension d_M as section of the canonical bundle, can be described by pairs of reflexive polyhedra (Δ, Δ^*) of real dimension $d_M - 1$. Together with a complete star triangulation of Δ, they define a complex family of Calabi-Yau threefolds. The mirror family is given by exchanging the role of Δ and Δ^*. A complete triangulation divides Δ in simplices of volume 1. In a star triangulation all simplices contain the unique interior lattice point of the reflexive polyhedron. Let us give first two examples for toric smooth ambient spaces in which the canonical hypersurface leads to the E_8 elliptic fibration over $\mathbb{P}^2$ and over the Hirzebruch surface $\mathbb{F}_1$. The polyhedron Δ for the E_8 elliptic fibration over $\mathbb{P}^2$ with $\chi(M) = -540$ is given by the following data

		ν_i				$l^{(e)}$	$l^{(1)}$
D_0	1	0	0	0	0	−6	0
D_1	1	1	0	−2	−3	0	1
D_2	1	0	1	−2	−3	0	1
D_3	1	−1	−1	−2	−3	0	1
D_z	1	0	0	−2	−3	1	−3
D_x	1	0	0	1	0	2	0
D_y	1	0	0	0	1	3	0

. (3.36)

Here we give the relevant points ν_i of the four dimensional convex reflexive polyhedron Δ embedded into a hyperplane in a five dimensional space and the linear relations $l^{(i)}$ spanning the Mori cone. This model has an unique star triangulation, given in (3.44). We calculate the intersection ring as follows from (3.29) with $a = 1$

$$\mathcal{R} = 9J_e^3 + 3J_e^2 J_1 + J_e J_1^2 \,. \tag{3.37}$$

The evaluation of c_2 on the basis of the Kähler cone follows from (3.30) as $\int_M c_2 J_e = 102$ and $\int_M c_2 J_1 = 36$.

The polyhedron Δ for the E_8 elliptic fibration over $\mathbb{F}_1$ with $\chi = -480$ reads

$$\begin{array}{c|ccccc|ccc|ccc} & & & \nu_i & & & l^{(e)} & l^{(1)} & l^{(2)} & l^{(e)}+l^{(2)} & l^{(1)}+l^{(2)} & -l^{(2)} \\ D_0 & 1 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & -6 & 0 & 0 \\ D_1 & 1 & 1 & 0 & -2 & -3 & 0 & 0 & 1 & 1 & 1 & -1 \\ D_2 & 1 & 0 & 1 & -2 & -3 & 0 & 1 & 0 & 0 & 1 & 0 \\ D_3 & 1 & -1 & -1 & -2 & -3 & 0 & 0 & 1 & 1 & 1 & -1 \\ D_4 & 1 & 0 & -1 & -2 & -3 & 0 & 1 & -1 & -1 & 0 & 1 \\ D_z & 1 & 0 & 0 & -2 & -3 & 1 & -2 & -1 & 0 & -3 & 1 \\ D_x & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ D_y & 1 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 3 & 0 & 0 \end{array} . \tag{3.38}$$

This example shows that there are two Calabi-Yau phases possible over $\mathbb{F}_1$, which are related by flopping a $\mathbb{P}^1$ represented by $l^{(2)}$. This transforms the half K3 to a del Pezzo eight surface, which can be shrunken to a point. In the first phase, the triangulation is described by (3.44) the intersection ring and $\int_M c_2 J_i$ follows by (3.29, 3.30) as

$$\mathcal{R} = 8J_e^3 + 3J_e^2 J_1 + J_e J_1^2 + 2J_e^2 J_2 + J_1 J_2 J_3 \,. \tag{3.39}$$

and $\int_M c_2 J_e = 92$, $\int_M c_2 J_1 = 36$ and $\int_M c_2 J_3 = 24$. For the second phase we flop the $\mathbb{P}^1$ that corresponds to the Mori cone element $l^{(2)}$. Generally, if we flop the curve C this changes the triple intersection of the divisors $\mathcal{K}_i \mathcal{K}_j \mathcal{K}_k$ [294] by

$$\Delta_{ijk} = -(C \cdot \mathcal{K}_i)(C \cdot \mathcal{K}_j)(C \cdot \mathcal{K}_k) \,. \tag{3.40}$$

Now the intersection of the curves C_i which correspond to the Mori cone vector $l^{(i)}$ with the toric divisors D_k is given by $(C_i \cdot D_k) = l^{(i)}_k$. On the other hand the $\mathcal{K}_k$ are combinations of D_k restricted to the hypersurface so that $(\mathcal{K}^k \cdot C_i) = \delta^k_i$.

In addition one has to change the basis in order to maintain positive intersection numbers[3] $\tilde{l}^{(e)} = l^{(e)} + l^{(2)}$, $\tilde{l}^{(1)} = l^{(1)} + l^{(2)}$ and $\tilde{l}^{(2)} = -l^{(2)}$. For the $(1,1)$ forms J_i, which transform dual to the curves, we get then the intersection ring in the new basis of the Kähler cone

$$\mathcal{R} = 8\tilde{J}_e^3 + 3\tilde{J}_e^2 \tilde{J}_1 + \tilde{J}_e \tilde{J}_1^2 + 9\tilde{J}_e^2 \tilde{J}_2 + 3\tilde{J}_e \tilde{J}_1 \tilde{J}_2 + \tilde{J}_1^2 \tilde{J}_2 + 9\tilde{J}_e \tilde{J}_2^2 + 3\tilde{J}_1 \tilde{J}_2^2 + 9\tilde{J}_2^3 \,. \tag{3.41}$$

The intersections with c_2 are not affected by the flop, only the basis change has to be taken into account. In the second phase the triangulation of the base is given in the the middle of figure 2 and the triangulation of Δ is specified by (3.43). In this phase a E_8 del Pezzo surface can be shrunken to get to the elliptic fibration over $\mathbb{P}^2$. This identifies the classes of the latter example as $J_e = \tilde{J}_2$, $J_1 = \tilde{J}_1$, while the divisor dual to $\tilde{J}_e^3$ is shrunken.

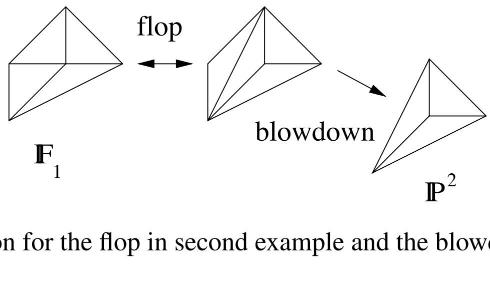

Figure 3.3: The base triangulation for the flop in second example and the blowdown of an E_8 del Pezzo surface.

[3] This is one criterion that holds in a simplicial Kähler cone. The full specification is that $\int_C J > 0$, $\int_{\mathcal{D}} J \wedge J > 0$ and $\int_M J \wedge J \wedge J > 0$ for J in the Kähler cone and C, $\mathcal{D}$ curves and divisors. If the latter is simplicial and generated by J_i then $J = \sum d_i J_i$ with $d_i > 0$.

We denote by Δ_B the toric polyhedron for the base and specifying by

$$\{(e_1, e_2)\} = \{(-2,-3), (-1,-2), (-1,-1)\}$$

the toric data for the E_8, E_7, E_6 fibre respectively. It is easy to see that all toric hypersurfaces with the required fibration have the following general form of the polyhedron Δ

$$\begin{array}{cccccc|cccc} & & & \nu_i & & & l^{(e)} & l^{(1)} & \dots & l^{(b)} \\ D_0 & 1 & 0 & 0 & 0 & 0 & \sum_i e_i - 1 & 0 & \dots & 0 \\ D_1 & 1 & & & e_1 & e_2 & 0 & * & \dots & * \\ \vdots & 1 & & \Delta_B & \vdots & \vdots & \vdots & * & \dots & * \\ D_r & 1 & & & e_1 & e_2 & 0 & * & \dots & * \\ D_z & 1 & 0 & 0 & e_1 & e_2 & 1 & -\sum * & \dots & -\sum * \\ D_x & 1 & 0 & 0 & 1 & 0 & -e_1 & 0 & \dots & 0 \\ D_y & 1 & 0 & 0 & 0 & 1 & -e_2 & 0 & \dots & 0 \end{array} \quad . \tag{3.42}$$

We note that the fibre elliptic curve is realized in a two dimensional toric variety, which can be defined also by a reflexive 2 dimensional polyhedron Δ_F. It is embedded into Δ so that the inner of Δ_F is also the origin of Δ. Its corners are given by

$$\{(0, 0, e_1, e_2), (0, 0, 1, 0), (0, 0, 0, 1)\} .$$

The E_6, E_7 and E_8 fibre types correspond to the polyhedra in Figure 1 with numbers 1, 4 and 10. To check the latter equivalence requires an change of coordinates in $SL(2,\mathbb{Z})$. The dual reflexive polyhedron Δ^* contains Δ^*_F embedding likewise in the coordinate plane spanned by the 3rd and 4th axis.

A triangulation of Δ_B as in Figure 1 or 2 lifts in a universal way to a star triangulation of Δ as follows. To set the conventions denote by (ν^B_i, e_1, e_2) the points of the embedded base polyhedron Δ_B and label them as the points of Δ_B starting with the positive x-axis, which points to the right in the figures, and label points of Δ_B counter clockwise from $1, \dots, r$. The inner point in Δ_B, $(0, 0, e_1, e_2)$ is labelled z. The two remaining points of Δ; $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ are labelled by x and y.

Denote the k-th d-dimensional simplex in Δ_B by the labels of its vertices, i.e.

$$\text{sim}^{(d)}_k := (\lambda^k_1, \dots, \lambda^k_{d+1})$$

and in particular denote the outer edges of Δ_B by

$$\{\text{ed}_k | k = 1, \dots, r\} := \{(1, 2), \dots, (r, 1)\} .$$

Any triangulation of Δ_B is lifted to a star triangulation of Δ, which is spanned by the simplices containing besides the inner point $(0, 0, 0, 0)$ of Δ the points with the labels

$$\text{Tr}_\Delta = \{(\text{sim}^{(2)}_k, x), (\text{sim}^{(2)}_k, y) | k = 1, \dots, p\} \cup \{(\text{ed}_k, x, y) | k = 1, \dots r\} . \tag{3.43}$$

In particular for star triangulations of Δ_B one has

$$\text{Tr}_\Delta = \{(\text{ed}_k, z, x), (\text{ed}_k, z, y), (\text{ed}_k, x, y) | k = 1, \dots r\} \tag{3.44}$$

and generators of the Mori cone for the elliptic phase contain the Mori cone generators $l^{(1)}, \dots, l^{(b)}$, which correspond to a star triangulation of the base polyhedron, which is the one in Figure 1. We list here the Mori cones of the first seven cases,

Δ_B	1(1)	2(2)		3(2)		4(3)		5(3)			6(3)			7(4)					
ν_i^B	$l^{(1)}$	$l^{(1)}$	$l^{(2)}$	$l^{(1)}$	$l^{(2)}$	$l^{(1)}$	$l^{(2)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$
z	−3	−2	−2	−2	−1	0	−2	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	−1
1	1	1	0	1	0	0	1	−1	1	0	1	0	0	−1	1	0	0	0	1
2	1	0	1	0	1	1	0	1	−1	1	−1	1	0	1	−1	1	0	0	0
3	1	1	0	1	−1	−2	1	0	1	−1	1	−1	1	0	1	−1	1	0	0
4		0	1	0	1	1	0	0	0	1	0	1	−2	0	0	1	−1	1	0
5								1	0	0	0	0	1	0	0	0	1	−1	1
6														1	0	0	0	1	−1
ex	–		–		1		–			4			3					17	

the remaining 9 cases are given in the appendix E. We indicate in the brackets behind the model the number of Kähler moduli. If the latter is smaller then the number of Mori cone generators and the dual Kähler cone are non-simplicial. This is the case for the models 7,9 and for 11-16. In the last column we list the number of extra triangulations. The corresponding phases involve non-star triangulations of Δ and can be reached by flops. By the rules discussed above we can find the intersection ring and the Mori cone in phases related by flops. We understand also the blowing down of one model. Non reflexivity posses a slight technical difficulty in providing the data for the calculation of the instantons. The fastest way to get the data for all cases is to provide for the models 15 and 16 a simplicial Kähler cone and reach all other cases by flops and blowdowns. We will do this in appendix E.

CHAPTER 4

Wall-crossing holomorphic anomaly and mock modularity of multiple M5-branes

In section 2.8 we reviewed the effective descriptions of M5-branes wrapping a complex surface P as well as previous appearances of the holomorphic anomaly. In the topological $\mathcal{N} = 4$ U(r) theory it was observed [140] that a non-holomorphicity [145] had to be introduced in order to restore S-dualtiy, the resulting holomorphic anomaly that was related in ref. [139] to an anomaly [141] appearing in the context of E-strings. The anomaly was conjectured to take into account contributions coming from reducible connections in $\mathcal{N} = 4$ SYM theory. We will show that the contributions from bound-states as a cause for non-holomorphicity will persist more generally for the class of surfaces we will be studying. The goal of this chapter is to show that wall-crossing in D4-D2-D0 systems leads to an anomaly equation which coincides with the anomalies found before and hence this work complements in some sense this circle of ideas.

4.1 Generating functions from wall-crossing

In section 2.8.3 we have argued that the partition function of $\mathcal{N} = 4$ U(r) SYM theory suffers from a holomorphic anomaly for divisors with $b_2^+(P) = 1$. In fact there exists another way to see the anomaly which is also intimately related to the computation of BPS degeneracies encoded in the elliptic genus and will be the subject of this section. This method relies on wall-crossing formulas and originally goes back to Göttsche and Zagier [136, 312]. In the physics context it has also been employed in [154, 155]. It will be used in section 4.2 to derive the elliptic genus for BPS states and their anomaly rigorously. In the following presentation we will be very sketchy as we merely want to stress the main ideas. We refer to section 4.2 for details.

The starting point is the Kontsevich-Soibelman formula [126] which describes the wall-crossing of bound-states of D-branes. Specifying to the case of two M5-branes and taking the equivalent D4-D2-D0 point of view the Kontsevich-Soibelman formula reduces to the primitive wall-crossing formula

$$\Delta\Omega(\Gamma; J \to J') = \Omega(\Gamma; J') - \Omega(\Gamma; J) = (-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\langle\Gamma_1, \Gamma_2\rangle\, \Omega(\Gamma_1)\, \Omega(\Gamma_2), \tag{4.1}$$

which describes the change of BPS degeneracies of a bound-state with charge vector $\Gamma = \Gamma_1 + \Gamma_2$, once a wall of marginal stability specified by J_W is crossed. The symplectic charge product $\langle\cdot,\cdot\rangle$ is defined by

$$\langle\Gamma_1,\Gamma_2\rangle = -Q_6^{(1)}Q_0^{(2)} + Q_4^{(1)}\cdot Q_2^{(2)} - Q_2^{(1)}\cdot Q_4^{(2)} + Q_0^{(1)}Q_6^{(2)}. \tag{4.2}$$

Hence, for D4-D2-D0 brane configurations $\langle\Gamma_1,\Gamma_2\rangle$ is independent of the D0-brane charge. Further, in eq. (4.1) Γ_1 and Γ_2 are primitive charge vectors such that $\Omega(\Gamma_i)$ do not depend on the moduli. Thus, the Γ_i can be thought of as charge vectors with $r = 1$ whereas Γ corresponds to a charge vector with $r = 2$. Assuming, that the wall of marginal stability does not depend on the D0-brane charge, formula (4.1) can be translated into a generating series $\Delta f^{(2)}_{\mu,J\to J'}$ defined by

$$\Delta f^{(2)}_{\mu,J\to J'} = \sum_{d\geq 0} \Delta\bar{\Omega}(\Gamma;J\to J')\, q^{d-\frac{\chi(P)}{12}}. \tag{4.3}$$

Assuming that there exists a reference chamber J' such that $\bar{\Omega}(\Gamma;J) = 0$, this gives us directly an expression for $f^{(2)}_{\mu,J}$.

As it will turn out in the next section, $\Delta f^{(2)}_{\mu,J\to J'}$ is given in terms of an indefinite theta-function $\Theta^{J,J'}_{\Lambda,\mu}$, which contains the information about the decays due to wall-crossing as one moves from J to J'. Indefinite theta-functions were analysed by Zwegers in his thesis [146]. One of their major properties is that they are not modular as one only sums over a bounded domain of the lattice Λ specified by J and J'. However, Zwegers showed that by adding a non-holomorphic completion the indefinite theta-functions have modular transformation behaviour and fall into the class of mock modular forms.[1]

As described in sections 2.8.1 and 2.8.3 the (MSW) CFT and the $\mathcal{N} = 4$ U(r) SYM partition functions should behave covariantly under modular transformations of the SL$(2,\mathbb{Z})$ acting on τ. Thus, the modular completion outlined above will effect the generating functions $f^{(2)}_{\mu,J}$ through their relation to the indefinite theta-function $\Theta^{J,J'}_{\Lambda,\mu}$, which needs a modular completion to transform covariantly under modular transformations, i.e.

$$\Theta^{J,J'}_{\Lambda,\mu} \mapsto \widehat{\Theta}^{J,J'}_{\Lambda,\mu} \tag{4.4}$$

and consequently $f^{(2)}_{J}$ is replaced by $\hat{f}^{(2)}_{\mu,J}$. Due to (2.186) the counting function of BPS invariants $\hat{f}^{(2)}_{\mu,J}$ and thus the elliptic genus $Z^{(2)}_P$ are going to suffer from a holomorphic anomaly, to which we turn next.

4.2 Wall-crossing and mock modularity

In this section we derive an anomaly equation for two M5-branes wound on a rigid surface/divisor P with $b_2^+(P) = 1$, inside a Calabi-Yau manifold X. We begin by reviewing D4-D2-D0 bound-states in the type IIA picture and their wall-crossing in the context of the Kontsevich-Soibelman formula. Then we proceed by deriving a generating function for rank two sheaves from the Kontsevich-Soibelman formula which is equivalent to Göttsche's formula [312]. This generating function is an indefinite theta-function, which fails to be modular. As a next step we apply ideas of Zwegers to remedy this failure of modularity by introducing a non-holomorphic completion. This leads to a holomorphic anomaly equation of the elliptic genus of two M5-branes that we prove for rigid divisors P.

[1] We review some notions in appendix 2.6.2.

4.2.1 D4-D2-D0 wall-crossing

In the following we take on the equivalent type IIA point of view, adapting the discussion of refs. [124, 154, 155] to describe the relation to the Kontsevich-Soibelman wall-crossing formula [126]. We restrict our attention to the D4-D2-D0 system on the complex surface P and work in the large volume limit with vanishing B-field.

Let us recall that a generic charge vector with D4-brane charge r is given by (see section 2.4.3 for details)

$$\Gamma = (Q_6, Q_4, Q_2, Q_0) = r\left(0,\ [P],\ i_*F(\mathcal{E}),\ \frac{\chi(P)}{24} + \int_P \tfrac{1}{2}F(\mathcal{E})^2 - \Delta(\mathcal{E})\right), \tag{4.5}$$

where $\mathcal{E}$ is a sheaf on the divisor P. Further, we define

$$\Delta(\mathcal{E}) = \frac{1}{r(\mathcal{E})}\left(c_2(\mathcal{E}) - \frac{r(\mathcal{E})-1}{2r(\mathcal{E})}c_1(\mathcal{E})^2\right), \quad \mu(\mathcal{E}) = \frac{c_1(\mathcal{E})}{r(\mathcal{E})}, \quad F(\mathcal{E}) = \mu(\mathcal{E}) + \frac{[P]}{2}. \tag{4.6}$$

We recall that in the large volume regime the notion of D-brane stability is equivalent to μ-stability, see [124] and section 2.4.3. Given a choice of $J \in \mathcal{C}(P)$, a sheaf $\mathcal{E}$ is called μ-semi-stable if for every sub-sheaf $\mathcal{E}'$

$$\mu(\mathcal{E}') \cdot J \leq \mu(\mathcal{E}) \cdot J. \tag{4.7}$$

Moreover, a wall of marginal stability is a co-dimension one subspace of the Kähler cone $\mathcal{C}(P)$ where the following condition is satisfied

$$(\mu(\mathcal{E}_1) - \mu(\mathcal{E}_2)) \cdot J = 0, \tag{4.8}$$

but is non-zero away from the wall. Across such a wall of marginal stability the configuration (4.5) splits into two configurations with charge vectors

$$\begin{aligned}\Gamma_1 &= r_1\left(0,\ [P],\ i_*F_1,\ \frac{\chi(P)}{24} + \int_P \tfrac{1}{2}F_1^2 - \Delta(\mathcal{E}_1)\right),\\ \Gamma_2 &= r_2\left(0,\ [P],\ i_*F_2,\ \frac{\chi(P)}{24} + \int_P \tfrac{1}{2}F_2^2 - \Delta(\mathcal{E}_2)\right),\end{aligned} \tag{4.9}$$

where $r_i = \mathrm{rk}(\mathcal{E}_i)$ and $\mu_i = \mu(\mathcal{E}_i)$. By making use of the identity

$$r\Delta = r_1\Delta_1 + r_2\Delta_2 + \frac{r_1 r_2}{2r}\left(\frac{c_1(\mathcal{E}_1)}{r_1} - \frac{c_1(\mathcal{E}_2)}{r_2}\right)^2, \tag{4.10}$$

one can show that $\Gamma = \Gamma_1 + \Gamma_2$. Therefore, charge-vectors as defined in (4.5) form a vector-space which will be essential for the application of the Kontsevich-Soibelman formula.

Before we proceed, let us note, that the BPS numbers and the Euler numbers of the moduli space of sheaves are related as follows. Denote by $\mathcal{M}_J(\Gamma)$ the moduli space of semi-stable sheaves characterised by Γ. Its dimension reads [210]

$$\dim_{\mathbb{C}}\mathcal{M}_J(\Gamma) = 2r^2 - r^2\chi(\mathcal{O}_P) + 1. \tag{4.11}$$

The relation between BPS invariants and the Euler numbers of the moduli spaces $M_J(\Gamma)$ is then given by [124]

$$\Omega(\Gamma, J) = (-1)^{\dim_{\mathbb{C}} M_J(\Gamma)}\chi(M(\Gamma), J)\,. \tag{4.12}$$

Moreover, for the system of charges we have specified to, the symplectic pairing of charges simplifies to [124]

$$\langle \Gamma_1, \Gamma_2 \rangle = r_1 r_2 (\mu_2 - \mu_1) \cdot [P] \,. \tag{4.13}$$

The holomorphic function $f^{(r)}_{\mu,J}(\tau)$ appearing in eq. (2.331) can now be identified with the generating function of BPS invariants of moduli spaces of semi-stable sheaves. Its wall crossing will be described in the following.

Kontsevich-Soibelman wall-crossing formula

Kontsevich and Soibelman [126] have proposed a formula which determines the jumping behaviour of BPS-invariants $\Omega(\Gamma; J)$ across walls of marginal stability. The Kontsevich-Soibelman wall-crossing formula states that across a wall of marginal stability the following formula holds

$$\prod^{\curvearrowright}_{\Gamma: Z(\Gamma;J)\in V} T_\Gamma^{\Omega(\Gamma;J_+)} = \prod^{\curvearrowright}_{\Gamma: Z(\Gamma;J)\in V} T_\Gamma^{\Omega(\Gamma;J_-)} . \tag{4.14}$$

Restricting to the case $r = 2$ and $r_1 = r_2 = 1$, (4.14) can be truncated to

$$\prod_{Q_{0,1}} T_{\Gamma_1}^{\Omega(\Gamma_1)} \prod_{Q_0} T_\Gamma^{\Omega(\Gamma;J_+)} \prod_{Q_{0,2}} T_{\Gamma_2}^{\Omega(\Gamma_2)} = \prod_{Q_{0,2}} T_{\Gamma_2}^{\Omega(\Gamma_2)} \prod_{Q_0} T_\Gamma^{\Omega(\Gamma;J_-)} \prod_{Q_{0,1}} T_{\Gamma_1}^{\Omega(\Gamma_1)}, \tag{4.15}$$

where Q_0 is the D0-brane charge of Γ and the $Q_{0,i}$ are the D0-brane charges belonging to Γ_i, respectively. The above formula has been derived by setting all Lie algebra elements with D4-brane charge greater than two to zero. Therefore, the element e_Γ is central, using the Baker-Campbell-Hausdorff formula $e^X e^Y = e^Y e^{[X,Y]} e^X$ and the fact that the symplectic product is independent of the D0-brane charge, one finds the following change of BPS numbers across a wall of marginal stability [128, 154]

$$\Delta\Omega(\Gamma) = (-1)^{\langle \Gamma_1, \Gamma_2 \rangle - 1} \langle \Gamma_1, \Gamma_2 \rangle \sum_{Q_{0,1}+Q_{0,2}=Q_0} \Omega(\Gamma_1)\, \Omega(\Gamma_2). \tag{4.16}$$

Moreover, one can deduce that the rank one degeneracies $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$ do not depend on the modulus J.

4.2.2 Relation of Kontsevich-Soibelman to Göttsche's wall-crossing formula

Göttsche has found a wall-crossing formula for the Euler numbers of moduli spaces of rank two sheaves in terms of an indefinite theta-function in ref. [312]. In this section we want to derive a modified version of this formula from the Kontsevich-Soibelman wall-crossing formula associated to D4-D2-D0 bound-states with D4-brane charge equal to two.

We use the short notation $\Gamma = (r, \mu, \Delta)$ to denote a rank r sheaf with the specified Chern classes that is associated to the D4-D2-D0 states. For rank one sheaves the generating function has no chamber dependence and we have already seen that it is given by (2.337). Following the discussion of our last section, higher rank sheaves do exhibit wall-crossing phenomena and therefore do depend on the chamber in moduli space, i.e. on $J \in C(P)$.

Our aim now is to determine the generating function of the D4-D2-D0 system using the primitive wall-crossing formula derived from the KS wall-crossing formula. From now on we restrict our attention to rank two sheaves $\mathcal{E}$. They can split across walls of marginal stability into rank one sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ as outlined in section 4.2.1. Using relation (4.10) we can write

$$d = d_1 + d_2 + \xi \cdot \xi, \tag{4.17}$$

where $\xi = \mu_1 - \mu_2$ and $d = 2\Delta$. Further, a wall is given by (4.8), i.e. the set of walls given a split of charges ξ reads

$$W^\xi = \{J \in C(P) \,|\, \xi \cdot J = 0\}\,. \tag{4.18}$$

Now, consider a single wall $J_W \in W^\xi$ determined by a set of vectors $\xi \in \Lambda + \mu$. Let J_+ approach J_W infinitesimally close from one side and J_- infinitesimally close from the other side. Thus, in our context the primitive wall-crossing formula (4.16) becomes

$$\bar{\Omega}(\Gamma; J_+) - \bar{\Omega}(\Gamma; J_-) = \sum_{Q_{0,1}+Q_{0,2}=Q_0} (-1)^{2\xi\cdot[P]}\, 2\,(\xi \cdot [P])\,\Omega(\Gamma_1)\,\Omega(\Gamma_2), \tag{4.19}$$

where we have used the identity (4.13). Note, that $Q_{0,i}$ and Q_0 are determined in terms of Γ and Γ_i through (4.5) and (4.9). Now, we can sum over the D0-brane charges to obtain a generating series. This yields

$$\begin{aligned}
&\sum_{d\geq 0} (\bar{\Omega}(\Gamma; J_+) - \bar{\Omega}(\Gamma; J_-)) q^{d - \frac{\chi(P)}{12}} \\
= \;& \sum_{d_1, d_2 \geq 0, \xi} (-1)^{2\xi\cdot[P]}\,(\xi \cdot [P])\,\Omega(\Gamma_1)\Omega(\Gamma_2) q^{d_1 + d_2 + \xi^2 - \frac{2\chi(P)}{24}} \\
= \;& (-1)^{2\mu\cdot[P]-1} \frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta(\tau)^{2\chi(P)}} \sum_{\xi} (\xi \cdot [P])\, q^{\xi^2},
\end{aligned} \tag{4.20}$$

where for the first equality use has been made of the identities (4.17, 4.19), and for the second equality the identity (2.337) has been used. The last line can be rewritten as

$$(-1)^{2\mu\cdot[P]-1} \frac{1}{2} \frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta(\tau)^{2\chi(P)}} \mathrm{Coeff}_{2\pi i y}(\Theta^{J_+, J_-}_{\Lambda,\mu}(\tau, [P]y)), \tag{4.21}$$

where we have introduced the indefinite theta-function

$$\Theta^{J,J'}_{\Lambda,\mu}(\tau, x) := \frac{1}{2} \sum_{\xi\in\Lambda+\mu} (\mathrm{sgn}\langle J, \xi\rangle - \mathrm{sgn}\langle J', \xi\rangle)\, e^{2\pi i \langle \xi, x\rangle}\, q^{Q(\xi)}, \tag{4.22}$$

with the inner product[2] defined by $\langle x, y\rangle = 2d_{AB}x^A y^B$ and the quadratic form $Q(\xi) = \frac{1}{2}\langle \xi, \xi\rangle$. As these theta-functions obey the cocycle condition [136]

$$\Theta^{F,G}_{\Lambda,\mu} + \Theta^{G,H}_{\Lambda,\mu} = \Theta^{F,H}_{\Lambda,\mu}, \tag{4.23}$$

we finally arrive at the beautiful relation between the BPS numbers in an arbitrary chamber J and those in a chamber J' first found by Göttsche in the case $\Lambda = H^2(P, \mathbb{Z})$:

$$f^{(2)}_{\mu,J'}(\tau) - f^{(2)}_{\mu,J}(\tau) = \frac{1}{2} \frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta^{2\chi(P)}(\tau)} \mathrm{Coeff}_{2\pi i y}(\Theta^{J,J'}_{\Lambda,\mu}(\tau, [P]y)). \tag{4.24}$$

[2] Note, that this is not the symplectic product of D-brane charges defined before.

4.2.3 Holomorphic anomaly at rank two

In this subsection we discuss the appearance of a holomorphic anomaly at rank two and give a proof of it by combing our previous results with results of Zwegers [146].

Elliptic genus at rank two and modularity

An important datum in (4.24) is the choice of chambers $J, J' \in C(P)$, which are any points in the Kähler cone of P. As a consequence, the indefinite theta-series does not transform well under $SL(2,\mathbb{Z})$ in general. However, from the discussion of sect. 2.8.1 we expect, that the generating series $f^{(r)}_{\mu,J}(\tau)$ transforms with weight $-\frac{r(\Lambda)+2}{2}$ in a vector-representation under the full modular group, where $r(\Lambda)$ denotes the rank of the lattice Λ. Hence, there is a need to restore modularity. The idea is as follows.

Following Zwegers [146], it turns out that the indefinite theta-function can be made modular at the cost of losing its holomorphicity. From the definition (4.22) Zwegers smoothes out the sign-functions and introduces a modified function as

$$\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x) = \frac{1}{2}\sum_{\xi\in\Lambda+\mu}\left(E\left(\frac{\langle c, \xi + \frac{\mathrm{Im}\,(x)}{\tau_2}\rangle\sqrt{\tau_2}}{\sqrt{-Q(c)}}\right) - E\left(\frac{\langle c', \xi + \frac{\mathrm{Im}\,(x)}{\tau_2}\rangle\sqrt{\tau_2}}{\sqrt{-Q(c')}}\right)\right)e^{2\pi i\langle\xi,x\rangle}q^{Q(\xi)}, \tag{4.25}$$

where E denotes the incomplete error function

$$E(x) = 2\int_0^x e^{-\pi u^2}du. \tag{4.26}$$

Note, that if c or c' lie on the boundary of the Kähler cone, one does not have to smooth out the sign-function. Zwegers shows, that the non-holomorphic function $\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x)$ satisfies the correct transformation properties of a Jacobi form of weight $\frac{1}{2}r(\Lambda)$. Due to the non-holomorphic pieces it contains mock modular forms, that we want to identify in the following. In order to separate the holomorphic part of $\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x)$ from its shadow we recall the following property of the incomplete error function

$$E(x) = \mathrm{sgn}(x)(1-\beta_{\frac{1}{2}}(x^2)), \tag{4.27}$$

which enables us to split up $\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x)$ into pieces. Here, β_k is defined by

$$\beta_k(t) = \int_t^\infty u^{-k}e^{-\pi u}du. \tag{4.28}$$

Hence, one can write eq. (4.25) as

$$\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x) = \Theta^{c,c'}_{\Lambda,\mu}(\tau,x) - \Phi^{c}_{\mu}(\tau,x) + \Phi^{c'}_{\mu}(\tau,x), \tag{4.29}$$

with

$$\Phi^{c}_{\mu}(\tau,x) = \frac{1}{2}\sum_{\xi\in\Lambda+\mu}\left[\mathrm{sgn}\langle\xi,c\rangle - E\left(\frac{\langle c, \xi + \frac{\mathrm{Im}\,(x)}{\tau_2}\rangle\sqrt{\tau_2}}{\sqrt{-Q(c)}}\right)\right]e^{2\pi i\langle\xi,x\rangle}q^{Q(\xi)}. \tag{4.30}$$

If c belongs to $C(P)\cap\mathbb{Q}^{r(\Lambda)}$, we may write

$$\Phi^{c}_{\mu}(\tau,x) = R(\tau,x)\theta(\tau,x), \tag{4.31}$$

where we decomposed the lattice sum into contributions along the direction of c and perpendicular to c given by R and θ, respectively. Hence, θ is a usual theta-series associated to the quadratic form $Q|\langle c\rangle^{\perp}$, i.e. of weight $(r(\Lambda)-1)/2$. R is the part which carries the non-holomorphicity. It transforms with a weight $\frac{1}{2}$ factor and therefore $\mathrm{Coeff}_{2\pi iy}(R(\tau,[P]y))$ is of weight $\frac{3}{2}$. Following the general idea of Zagier [147] that we recapitulate in appendix 2.6.2, we should encounter the $\beta_{\frac{3}{2}}$ function in the $2\pi iy$-coefficient of Φ. Indeed one can prove the following identity

$$\mathrm{Coeff}_{2\pi iy}\Phi^{c}_{\mu}(\tau,[P]y) = -\frac{1}{4\pi}\frac{\langle c,[P]\rangle}{\langle c,c\rangle}\sum_{\xi\in\Lambda+\mu}|\langle c,\xi\rangle|\beta_{\frac{3}{2}}\left(\frac{\tau_2\langle c,\xi\rangle^2}{-Q(c)}\right)q^{Q(\xi)}. \tag{4.32}$$

Taking the derivative with respect to $\bar\tau$ in order to obtain the shadow and setting $c=-[P]$ corresponds to the attractor point[3] we arrive at the following final expression

$$\partial_{\bar\tau}\mathrm{Coeff}_{2\pi iy}\Phi^{c}_{\mu}(\tau,[P]y) = -\frac{\tau_2^{-\frac{3}{2}}}{8\pi i}\frac{c\cdot[P]}{\sqrt{-c^2}}(-1)^{4\mu^2}\,\theta^{(2)}_{\mu-\frac{[P]}{2},c}(\tau,0)\Bigg|_{c=-[P]}, \tag{4.33}$$

where we define the Siegel-Narain theta-function $\theta^{(r)}_{\mu,c}(\tau,z)$ as in eq. (2.333). For more details on the transformation properties of the indefinite theta-functions we refer the reader to section 2.6.2. Note, that Now, these results can be used to compute the elliptic genus for two M5-branes wrapping the divisor P. Consider

$$f^{(2)}_{\mu,J}(\tau) = f_{\mu,J'}(\tau) - \frac{1}{2}\frac{\vartheta_{\Lambda^{\perp}}(\tau)^2}{\eta^{2\chi(P)}}\,\mathrm{Coeff}_{2\pi iy}\Theta^{J,J'}_{\Lambda,\mu}(\tau,[P]y), \tag{4.34}$$

where $f_{\mu,J'}(\tau)$ is a holomorphic ambiguity given by the generating series in a reference chamber J', which we choose to lie at the boundary of the Kähler cone $J'\in\partial C(P)$. In explicit computations it may be possible to choose J' such that the BPS numbers vanish. In general, however, such a vanishing chamber might not always exist, but since J' is at the boundary of the Kähler cone, $f_{\mu,J'}(\tau)$ has no influence on the modular transformation properties, nor on the holomorphic anomaly. We write the full M5-brane elliptic genus as

$$Z^{(2)}_P(\tau,z) = \sum_{\mu\in\Lambda^*/\Lambda}\hat f^{(2)}_{\mu,J}(\tau)\theta^{(2)}_{\mu,J}(\tau,z), \tag{4.35}$$

where $\hat f^{(2)}_{\mu,J}$ denotes the modular completion as outlined above. We can show using Zwegers' results [146], that the M5-brane elliptic genus transforms like a Jacobi form of bi-weight $(-\frac{3}{2},\frac{1}{2})$. Again, we refer the reader to section 2.6.2 for further details.

Proof of holomorphic anomaly at rank two

Now, we are in position to prove the holomorphic anomaly at rank two for general surfaces P with $b_2^+(P)=1$. The holomorphic anomaly takes the following form

$$\mathcal{D}_2 Z^{(2)}_P(\tau,z) = \tau_2^{-3/2}\frac{1}{16\pi i}\frac{J\cdot[P]}{\sqrt{-J^2}}\left(Z^{(1)}_P(\tau,z)\right)^2\Bigg|_{J=-[P]}, \tag{4.36}$$

[3] For the case that $c\neq-[P]$ the holomorphic anomaly does not hold in the desired form since the derivative would give rise to extra terms, see [300].

where the derivative $\mathcal{D}_k$ is given as

$$\mathcal{D}_k = \partial_{\bar\tau} + \frac{i}{4\pi k}\partial_{z_+}^2, \tag{4.37}$$

and z_+ refers to the projection of z along a direction $J \in C(P)$. For the proof, $\mathcal{D}_2 Z_P^{(2)}$ can be computed explicitly. Using (4.33) we obtain directly

$$\mathcal{D}_2 Z_P^{(2)}(\tau,z) = \tau_2^{-3/2}\frac{1}{16\pi i}\frac{J\cdot[P]}{\sqrt{-J^2}}\frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta(\tau)^{2\chi}}\sum_{\mu\in\Lambda^*/\Lambda}(-1)^{4\mu^2}\theta^{(2)}_{\mu-\frac{[P]}{2},J}(\tau,0)\theta^{(2)}_{\mu,J}(\tau,z)\Bigg|_{J=-[P]}. \tag{4.38}$$

Since the following identity among the theta-functions $\theta_{\mu,J}$ holds

$$\left(\theta^{(1)}_{0,J}(\tau,z)\right)^2 = \sum_{\mu\in\Lambda^*/\Lambda}(-1)^{4\mu^2}\theta^{(2)}_{\mu-\frac{[P]}{2},J}(\tau,0)\theta^{(2)}_{\mu,J}(\tau,z), \tag{4.39}$$

we have proven the holomorphic anomaly equation at rank two for general surfaces P.

4.3 Applications and extensions

In the following we want to apply the previous results to several selected examples. Before doing so, we explain two mathematical facts which will help to fix the ambiguity $f_{\mu,J'}(\tau)$, which are the blow-up formula and the vanishing lemma. After discussing the examples, we turn our attention to a possible extension to higher rank. This leads us to speculations about mock modularity of higher depth and wall-crossing having its origin in a meromorphic Jacobi form.

4.3.1 Blow-up formulae and vanishing chambers

There is a universal relation between the generating functions of stable sheaves on a surface P and on its blow-up $\tilde P$ [140,143,144,312,313]. Let P be a smooth projective surface and $\pi : \tilde P \to P$ the blow-up at a non-singular point with E the exceptional divisor of π. Let $J \in C(P)$, r and μ such that $\gcd(r, r\mu\cdot J) = 1$. Then, the generating series $f^{(r)}_{\mu,J}(\tau;P)$ and $f^{(r)}_{\mu,J}(\tau;\tilde P)$ are related by the blow-up formula

$$f^{(r)}_{\pi^*(\mu)-\frac{k}{r}E,\pi^*(J)}(\tau;\tilde P) = B_{r,k}(\tau) f^{(r)}_{\mu,J}(\tau;P), \tag{4.40}$$

with $B_{r,k}$ given by

$$B_{r,k}(\tau) = \frac{1}{\eta^r(\tau)}\sum_{a\in\mathbb{Z}^{r-1}+\frac{k}{r}} q^{\sum_{i\le j}a_i a_j}. \tag{4.41}$$

The second fact states that for a class of semi-stable sheaves on certain surfaces the moduli space of the sheaves is empty. We refer to this fact as the vanishing lemma [312]. For this let P be a rational ruled surface $\pi : P \to \mathbb{P}^1$ and J be the pullback of the class of a fibre of π. Picking a Chern class μ with $r\mu\cdot J$ odd, we have

$$M((r,\mu,\Delta),J) = \emptyset \tag{4.42}$$

for all d and $r \ge 2$.

4.3.2 Applications to surfaces with $b_2^+ = 1$

The surfaces we are going to consider are $\mathbb{P}^2$, the Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_1$, the del Pezzo surfaces dP_8 and dP_9 ($\frac{1}{2}K3$).

Projective plane $\mathbb{P}^2$

The projective plane $\mathbb{P}^2$ has been discussed quite exhaustively in the literature. The rank one result was obtained by Göttsche [112]

$$Z^{(1)}_{\mathbb{P}^2} = \frac{\vartheta_1(-\bar\tau,-z)}{\eta^3(\tau)}. \tag{4.43}$$

The generating functions of the moduli space of rank two sheaves or SO(3) instantons of SYM theory on $\mathbb{P}^2$ were written down by [140, 142, 143] and are given by

$$\begin{aligned} f_0(\tau) &= \sum_{n=0}^{\infty}\chi(M((2,0,n),J))q^{n-\frac{1}{4}} = \frac{3h_0(\tau)}{\eta^6(\tau)},\\ f_1(\tau) &= \sum_{n=0}^{\infty}\chi(M((2,1,n),J))q^{n-\frac{1}{2}} = \frac{3h_1(\tau)}{\eta^6(\tau)}. \end{aligned} \tag{4.44}$$

Here, $h_j(\tau)$ are mock modular forms given by summing over Hurwitz class numbers $H(n)$

$$h_j(\tau) = \sum_{n=0}^{\infty} H(4n+3j)q^{n+\frac{3j}{4}}, \qquad (j=0,1). \tag{4.45}$$

Their modular completion is denoted by $\hat h_j(\tau)$, where the shadows are given by $\vartheta_{3-j}(2\tau)$ [145]. Explicitly, we have

$$\partial_{\bar\tau}\hat h_j(\tau) = \frac{\tau_2^{-\frac{3}{2}}}{16\pi i}\vartheta_{3-j}(-2\bar\tau). \tag{4.46}$$

Note, that these results are valid for all Kähler classes $J \in H^2(\mathbb{P}^2,\mathbb{Z})$ as there is no wall crossing in the Kähler moduli space of $\mathbb{P}^2$. This leads directly to the following elliptic genus of two M5-branes wrapping the $\mathbb{P}^2$ divisor

$$Z^{(2)}_{\mathbb{P}^2}(\tau,z) = \hat f_0(\tau)\vartheta_2(-2\bar\tau,-2z) - \hat f_1(\tau)\vartheta_3(-2\bar\tau,-2z). \tag{4.47}$$

Denoting by $\mathcal{D}_2 = \partial_{\bar\tau} + \frac{i}{8\pi}\partial_z^2$ one finds the expected holomorphic anomaly equation at rank two, given by[4]

$$\mathcal{D}_2\, Z^{(2)}_{\mathbb{P}^2}(\tau,z) = -\frac{3}{16\pi i}\tau_2^{-\frac{3}{2}}\left(Z^{(1)}_{\mathbb{P}^2}(\tau,z)\right)^2, \tag{4.48}$$

which can be derived directly from the simple fact that

$$\vartheta_1(\tau,z)^2 = \vartheta_2(2\tau)\vartheta_3(2\tau,2z) - \vartheta_3(2\tau)\vartheta_2(2\tau,2z). \tag{4.49}$$

Further note, that the q-expansion of f_0, eq. (4.44), has non-integer coefficients. It was explained in [154] that this is due to the fact that the generating series involves the fractional BPS invariants $\bar\Omega(\Gamma)$, which we encountered before.

[4] This result has already been derived in [156].

Hirzebruch surface $\mathbb{F}_0$

Our next example is the Hirzebruch surface $P = \mathbb{F}_0$. We denote by F and B the fibre and the base $\mathbb{P}^1$'s respectively. For an embedding into a Calabi-Yau manifold one may consult app. B.2. Let us choose $J = F + B$, $J' = B$ and Chern class $\mu = F/2$. The choice $\mu = B/2$ can be treated analogously and leads to the same results. The other sectors corresponding to $\mu = 0$ and $\mu = (F+B)/2$ require a knowledge of the holomorphic ambiguity at the boundary and will not be treated here. One obtains

$$\begin{aligned} f^{(2)}_{\mu,F+B}(\tau) &= \frac{1}{2\eta^8(\tau)}\mathrm{Coeff}_{2\pi i y}(\Theta^{F+B,B}_{\Lambda,\mu}(\tau,[P]y)) \\ &= q^{-\frac{1}{3}}\left(2q + 22q^2 + 146q^3 + 742q^4 + \dots\right), \end{aligned} \tag{4.50}$$

where we denote by μ either $B/2$ or $F/2$. This exactly reproduces the numbers obtained in [314].

We want to compute the shadow of the completion given by adding Φ^{F+B}_μ and Φ^B_μ to the indefinite theta-series $\Theta^{F+B,B}_{\Lambda,\mu}$. Since B is chosen at the boundary, Φ^B_μ vanishes for $\mu = F/2, B/2$. The only relevant contribution has a shadow proportional to $\vartheta_2(\tau)$. Precisely, we obtain

$$\partial_{\bar\tau} f^{(2)}_{\mu,F+B}(\tau) = -\tau_2^{-3/2}\frac{1}{4\pi i\sqrt{2}}\frac{\overline{\vartheta_2(\tau)}\vartheta_2(\tau)}{\eta^8(\tau)} \qquad (\mu = \frac{F}{2},\frac{B}{2}). \tag{4.51}$$

Hirzebruch surface $\mathbb{F}_1$

The next example is the Hirzebruch surface $\mathbb{F}_1$, which is a blow-up of $\mathbb{P}^2$. Again we denote by F and B the fibre and base $\mathbb{P}^1$'s. The $\mathbb{P}^2$ hyperplane is given by the pullback of $F + B$ and B is the exceptional divisor. This example is particularly nice, since we can check our results against the blow-up formula (4.40) or use the results known from $\mathbb{P}^2$ to write generating functions in sectors which are not accessible through the vanishing lemma. Notice, that the holomorphic expansions have been already discussed in ref. [155]. From the general discussion one sees that there are four different choices for the Chern class $\mu \in \{\frac{B}{2}, \frac{F+B}{2}, \frac{F}{2}, 0\}$.

First, we choose $J = F + B$, $J' = F$ and Chern class $\mu = B/2$. We then obtain

$$\begin{aligned} f^{(2)}_{\mu,F+B}(\tau) &= \frac{1}{2\eta^8(\tau)}\mathrm{Coeff}_{2\pi i y}(\Theta^{F+B,F}_{\Lambda,B}(\tau,[P]y)) \\ &= q^{-\frac{1}{12}}\left(-\frac{1}{2} - q + \frac{15}{2}q^2 + 91q^3 + 558q^4 + \dots\right). \end{aligned} \tag{4.52}$$

A check of this result against the blow-up formula (4.40) applied to $\mathbb{P}^2$ yields

$$\frac{3h_0(\tau)}{\eta^6(\tau)}\frac{\vartheta_2(2\tau)}{\eta^2(\tau)} = q^{-\frac{1}{12}}\left(-\frac{1}{2} - q + \frac{15}{2}q^2 + 91q^3 + 558q^4 + \dots\right) = f^{(2)}_{\mu,F+B}(\tau). \tag{4.53}$$

Further, we calculate the shadow by differentiating $\hat f^{(2)}$ with respect to $\bar\tau$

$$\partial_{\bar\tau}\hat f^{(2)}_{\mu,F+B}(\tau) = \frac{3}{16\pi i}\tau_2^{-3/2}\frac{\overline{\vartheta_3(2\tau)}\vartheta_2(2\tau)}{\eta^8(\tau)}, \tag{4.54}$$

which also is in accord with the blow-up formula. Note, that (4.52) has half-integer expansion coefficients, since $J = B + F$ lies on a wall for the Chern class $\mu = B/2$.

As a second case we choose $J = F + B$, $J' = F$ and Chern class $\mu = (F+B)/2$ and obtain

$$\begin{aligned} f^{(2)}_{\mu,F+B}(\tau) &= \frac{1}{2\eta^8(\tau)}\mathrm{Coeff}_{2\pi i y}(\Theta^{F+B,F}_{\Lambda,F+B}(\tau,[P]y)) \\ &= q^{-\frac{7}{12}}\left(q + 13q^2 + 93q^3 + 496q^4 + \dots\right), \end{aligned} \tag{4.55}$$

which we again can check against the blow-up formula (4.40) for $\mathbb{P}^2$

$$\frac{3h_1(\tau)}{\eta^6(\tau)}\frac{\vartheta_3(2\tau)}{\eta^2(\tau)} = q^{-\frac{7}{12}}\left(q + 13q^2 + 93q^3 + 496q^4 + \dots\right) = f^{(2)}_{\mu,F+B}(\tau). \tag{4.56}$$

Calculating the shadow yields

$$\partial_{\bar\tau}\hat{f}^{(2)}_{\mu,F+B}(\tau) = \frac{3}{16\pi i}\tau_2^{-3/2}\frac{\overline{\vartheta_2(2\tau)}\vartheta_3(2\tau)}{\eta^8(\tau)}, \tag{4.57}$$

which is also in accord with the blow-up formula.

The last two sectors $\mu = F/2, 0$ are not accessible via the vanishing lemma. However, using a blow-down to $\mathbb{P}^2$ we observe, that the above two cases reproduce correctly the two Chern classes in the cases of rank two sheaves on $\mathbb{P}^2$. Using the blow-up formulas once more we finally arrive at

$$\begin{aligned} f^{(2)}_{(0,0),J}(\tau) &= \frac{3h_0(\tau)}{\eta^6(\tau)}\frac{\vartheta_3(2\tau)}{\eta^2(\tau)}, \\ f^{(2)}_{(\frac{1}{2},0),J}(\tau) &= \frac{3h_1(\tau)}{\eta^6(\tau)}\frac{\vartheta_2(2\tau)}{\eta^2(\tau)}, \\ f^{(2)}_{(0,\frac{1}{2}),J}(\tau) &= \frac{3h_0(\tau)}{\eta^6(\tau)}\frac{\vartheta_2(2\tau)}{\eta^2(\tau)}, \\ f^{(2)}_{(\frac{1}{2},\frac{1}{2}),J}(\tau) &= \frac{3h_1(\tau)}{\eta^6(\tau)}\frac{\vartheta_3(2\tau)}{\eta^2(\tau)}, \end{aligned} \tag{4.58}$$

where $J = F + B$ and $\mu = (a,b) = aF + bB$. Note, that in the cases $f^{(2)}_{(0,0),J}$ and $f^{(2)}_{(0,\frac{1}{2}),J}$ the blow-up formula is not valid since we violate the gcd-condition, as $\pi_*\mu = 0$ in these cases. However, for rank two sheaves on $\mathbb{F}_1$ the blow-up formula seems to work anyway, since the generating series using the blow-up procedure and the indefinite theta-function description coincide for the Chern class $\mu = (0, \frac{1}{2})$.

Del Pezzo surface dP_8

As in [137] we embed the surface dP_8 in a certain free $\mathbb{Z}_5$ quotient[5] of the Fermat quintic $\tilde{X} = \{\sum_{i=1}^5 x_i^5 = 0\}$ in $\mathbb{P}^4$. The action of the group $G = \mathbb{Z}_5$ on the projective coordinates of the ambient space is given by $x_i \sim \omega^i x_i$, where $\omega = e^{2\pi i/5}$. For the hyperplane section, denoted P, we observe that $P^3 = 1$, as for the Fermat quintic the five points of intersection of three hyperplanes $\{x_i = x_j = x_k = 0\}$ are identified under the action of the group G. The Euler character of the hyperplane is given by $\chi(P) = 11$. It can be shown that the divisor P is rigid and has $b_2^+ = 1$. We observe that $H^2(P,\mathbb{Z}) = \mathbb{Z} \oplus (-E_8)$ as is explained

[5] The only freely acting group actions for the quintic are a $\mathbb{Z}_5^2$ and the above $\mathbb{Z}_5$.

in [137]. The elliptic genus of a single M5-brane is then fixed by the modular weights

$$Z_P^{(1)}(\tau, z) = \frac{E_4(\tau)}{\eta^{11}(\tau)}\,\vartheta_1(-\bar{\tau}, -z). \tag{4.59}$$

The form of $Z_P^{(2)}$ can now be calculated as for $\mathbb{P}^2$ and is given by

$$Z_P^{(2)}(\tau, z) \sim \frac{E_4(\tau)^2}{\eta(\tau)^{22}}\,(\hat{h}_0(\tau)\,\vartheta_2(-2\bar{\tau}, -2z) - \hat{h}_1(\tau)\,\vartheta_3(-2\bar{\tau}, -2z)). \tag{4.60}$$

The holomorphic anomaly equation fulfilled by $Z_P^{(2)}(\tau, z)$ can be obtained as in the $\mathbb{P}^2$ case

$$\mathcal{D}_2\, Z_P^{(2)}(\tau, z) \sim \frac{\tau_2^{-\frac{3}{2}}}{16\pi i}\left(Z_P^{(1)}(\tau, z)\right)^2. \tag{4.61}$$

Del Pezzo surface dP_9, the $\frac{1}{2}$K3

We end our examples by returning and commenting on $\frac{1}{2}$K3 or dP_9 which was the example of section (2.8.3), as M5-branes wrapping on it give rise to the multiple E-strings. The dP_9 surface can be understood as a $\mathbb{P}^2$ blown up at nine points (see appendix B for details) or a rational elliptic surface. This case is interesting as one can map via T-duality along the elliptic fibration the computation of the modified elliptic genus to the computation of the partition function of topological string theory on the same surface [139], which we explore further in section 5.5. The middle dimensional cohomology lattice of dP_9 is given by $H^2(dP_9, \mathbb{Z}) = \Lambda^{1,1} \oplus E_8$ and the Euler number can be computed to $\chi(dP_9) = 12$. Modularity then fixes the form of the elliptic genus at rank one to

$$Z_{dP_9}^{(1)}(\tau, z) = \frac{E_4(\tau)}{\eta(\tau)^{12}}\,\theta_{0,J}^{(1)}(\tau, z), \tag{4.62}$$

where $\theta_{0,J}^{(1)}(\tau, z)$ is the theta-function associated to the lattice $\Lambda^{1,1}$ with standard intersection form

$$(-d_{AB}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.63}$$

Choosing the Kähler form $J = (R^{-2}, 1)^T$, where $(1, 0)^T$ is the class of the elliptic fibre, one can show that

$$\theta_{0,J}^{(1)}(\tau, 0) \to \frac{R}{\sqrt{\tau_2}} \quad \text{as} \quad R \to \infty. \tag{4.64}$$

In this limit of small elliptic fibre one recovers the results of sect. 2.8.3. The factor $E_4(\tau)$ is precisely the theta-function of the E_8 lattice. The results obtained from the anomaly for higher wrappings of refs. [139, 141] were proven mathematically for double wrapping in ref. [315]. In this analysis the Weyl group of the E_8 lattice was used to perform the theta-function decomposition.

4.3.3 Extensions to higher rank and speculations

In the following sections we want to discuss the extension of our results to higher rank. Partial results for rank three can be found already in the literature [155, 282, 314, 316, 317]. Thereafter, we discuss a possible generalisation of mock modularity and speculate about a contour description which stems from a relation to a meromorphic Jacobi form.

Higher rank anomaly and mock modularity of higher depth

We want to focus on the holomorphic anomaly equation at general rank as conjectured in [139]. We recall that its form is given by

$$\mathcal{D}_r Z_P^{(r)}(\tau, z) \sim \sum_{n=1}^{r-1} n(r-n) Z_P^{(n)}(\tau, z) Z_P^{(r-n)}(\tau, z), \tag{4.65}$$

where $Z_P^{(r)}(\tau, z)$ can be decomposed into Siegel-Narain theta-functions as described in section 2.8.1. One may thus ask the question what it implies for the functions $\hat{f}_{\mu,J}^{(r)}(\tau)$ for general r. In order to extract this information we want to compare the coefficients in the theta-decomposition on both sides of (4.65). For this we need a generalisation of the identity (4.39). A computation shows that

$$\theta_{\nu,J}^{(n)}(\tau, z)\, \theta_{\lambda,J}^{(r-n)}(\tau, z) = \sum_{\mu \in \Lambda^*/\Lambda} c_{\nu\lambda}^{\mu}(\tau)\, \theta_{\mu,J}^{(r)}(\tau, z), \tag{4.66}$$

where $c_{\nu\lambda}^{\mu}$ are Siegel-Narain theta-functions themselves given by

$$c_{\nu\lambda}^{\mu}(\tau) = \delta_g(\mu) \sum_{\xi \in \Lambda + \mu + \frac{g}{r}(\nu - \lambda)} \bar{q}^{-\frac{rn(r-n)}{2g^2}\xi_+^2} q^{\frac{rn(r-n)}{2g^2}\xi_-^2} \tag{4.67}$$

with $g = \gcd(n, r-n)$ and $\delta_g(\mu)$ yields one if $r\mu$ is divisible by g and vanishes otherwise. With this input one finds

$$\partial_{\bar{\tau}} \hat{f}_{\mu,J}^{(r)}(\tau) \sim \sum_{n=1}^{r-1} n(r-n) \sum_{\nu,\lambda \in \Lambda^*/\Lambda} \hat{f}_{\nu,J}^{(n)}(\tau) \hat{f}_{\lambda,J}^{(r-n)}(\tau) c_{\nu\lambda}^{\mu}(\tau), \tag{4.68}$$

which sheds some light into the question about the modular properties of generating functions at higher rank as follows.

The structure of eq. (4.68) indicates, that an appropriate description of the generating function $\hat{f}_{\mu,J}^{(r)}$ needs a generalisation of the usual notion of mock modularity. This results from the fact, that on the right hand side of the anomaly equation (4.68), mock modular forms appear, such that the shadow of $\hat{f}_{\mu,J}^{(r)}$ is a mock modular form itself. Therefore, it is also subject to a holomorphic anomaly equation. This would lead to the notion of mock modularity of higher depth [318], similar to the case of almost holomorphic modular forms of higher depth. These are functions like $\widehat{E}_2(\tau)$ and powers thereof, which can be written as a polynomial in τ_2^{-1} with coefficients being holomorphic functions.

A further motivation for this comes from the observation that the generating functions $\hat{f}_{\mu,J}^{(r)}$ could be obtained from an indefinite theta-function as in the case of two M5-branes. The lattice, however, that is summed over in these higher rank indefinite theta-functions will be of higher signature. In the case of r M5-branes one would expect a signature $(r-1, (r-1)(r(\Lambda)-1))$ due to the $r-1$ relative D2-brane charges of the possible r decay products of D4-D2-D0 bound-states [152, 154].

The contour description

The elliptic genus of r M5-branes wrapping P is denoted by $Z_P^{(r)}(\tau, z)$, where we don't indicate any dependence of $Z_P^{(r)}$ on a Kähler class/ chamber $J \in C(P)$. The basic assumption is that the elliptic genus does not depend on such a choice. We simply think about $Z_P^{(r)}$ as being a *meromorphic* Jacobi form, which has poles as a function of the elliptic variable z. We assume, that it is of bi-weight $(-\frac{3}{2}, \frac{1}{2})$. In the following we want to exploit the implications of this statement.

It is known that a Jacobi form has an expansion into theta-functions with coefficients being modular forms. Since Zwegers [146], we also know that a meromorphic Jacobi form with one elliptic variable has a similar expansion, where the coefficients are mock modular see also section 2.6.2. Using our Siegel-Narain theta-function $\theta_{\mu,J}^{(r)}(\tau, z)$, eq. (2.333), we conjecture the following expansion

$$Z_P^{(r)}(\tau, z) = \sum_{\mu \in \Lambda^*/\Lambda} f_{\mu,J}^{(r)}(\tau)\theta_{\mu,J}^{(r)}(\tau, z) + \text{Res}, \tag{4.69}$$

with J a point in the Kähler cone which is related to a point $z_J \in \Lambda_{\mathbb{C}}$ where the decomposition is carried out. Note, that in eq. (4.69) the term “Res” should be given as a finite sum over the residues of $Z_P^{(r)}(\tau, z)$ in the fundamental domain $z_J + e\tau + e$ with $e = [0, 1]^{r(\Lambda)}$.

Let’s see how the dependence on J comes about. Doing a Fourier transform we can write

$$f_{\mu,J}^{(r)}(\tau) = (-1)^{r\mu\cdot[P]}\bar{q}^{\frac{r}{2}\mu_+^2}q^{-\frac{r}{2}\mu_-^2}\int_{C_J} Z_P^{(r)}(\tau, z)e^{-2\pi i r(\mu+\frac{[P]}{2})\cdot z}dz, \tag{4.70}$$

where C_J is a contour which has to be specified since $Z_P^{(r)}$ is meromorphic. Due to the periodicity in the elliptic variable C_J can be given as $z_J + e$ for some point z_J. Now, suppose we have a parallelogram $\mathcal{P} = z_J + ez_{J'} + e$ and that there is a single pole of $Z_P^{(r)}$ inside $\mathcal{P}$, say at $z = z_0$. Then, we obtain by integrating over the boundary of $\mathcal{P}$

$$f_{\mu,J}^{(r)}(\tau) - f_{\mu,J'}^{(r)}(\tau) = 2\pi i\, \alpha_\mu(\tau) \operatorname*{Res}_{z=z_0}\left(Z_P^{(r)}(\tau, z)e^{-2\pi i r(\mu+\frac{[P]}{2})\cdot z}\right), \tag{4.71}$$

where we abbreviate

$$\alpha_\mu(\tau) = (-1)^{r\mu\cdot[P]}\bar{q}^{\frac{r}{2}\mu_+^2}q^{-\frac{r}{2}\mu_-^2}. \tag{4.72}$$

That is, the coefficients of the Laurent expansion of the elliptic genus encode the jumping of the BPS numbers across walls of marginal stability and the walls are in one-to-one correspondence with the positions of the poles of $Z_P^{(r)}$. An analogous dependence on a contour of integration for wall-crossing of $\mathcal{N} = 4$ dyons was introduced in refs. [149, 230].

Moreover, the shadow of $f_{\mu,J}^{(r)}$ should be determined in terms of the residues of $Z_P^{(r)}$, since a generalizations of the ideas of [146] should show, that it is contained in the factor “Res” of eq. (4.69). Thus, combining this result with the interpretation of eq. (4.71) one expects, that the shadow not only renders $f_{\mu,J}^{(r)}$ modular, but also encodes the decay of bound-states and hence knows about the jumping of BPS invariants across walls of marginal stability.

It is tempting to speculate even further. When comparing our results to the case of dyon state counting in $\mathcal{N} = 4$ theories [150, 232] one might suspect that there is an analog of the Igusa cusp form Φ_{10} in our setup. In the $\mathcal{N} = 4$ dyon case there are meromorphic Jacobi forms, often denoted ψ_m, which are summed up to give Φ_{10} [150]. In analogy, it may be useful to introduce another parameter $\rho \in \mathbb{H}$ and to study the object

$$\phi_P^{-1}(\tau, \rho, z) = \sum_{r\geq 1} Z_P^{(r)}(\tau, z)e^{2\pi i r\rho}. \tag{4.73}$$

CHAPTER 5

Quantum geometry of elliptic fibrations

The aim of this chapter is to describe the quantum geometry of elliptic fibrations. For this we introduce the needed notions from topological string theory and discuss two examples with base $\mathbb{P}^2$ and $\mathbb{F}_1$. One of our key findings is the holomorphic anomaly (5.15). The first section finishes with a discussion of the modular properties due to the monodromy group.

From the data provided in section 3.3, namely the Mori cone and the intersection numbers, follow differential equations as well as particular solutions, which allow to calculate the instanton numbers as established mathematically for genus zero by Givental, Lian, Liu and Yau [93, 94]. These can be calculated very efficiently using the program described in [305]. In the cases at hand one can evaluate the genus one data using the genus zero results, the holomorphic anomaly equation for the Ray Singer Torsion, boundary conditions provided by the evaluation of $\int_M J_a c_2$ and the behaviour of the discriminant at the conifold to evaluate the instantons of the elliptic fibrations.

The higher genus curves are less systematically studied on compact threefolds. However, if the total space of the elliptic fibration over a base class is a contractable rational surface, one can shrink the latter and obtain a local model on which the modular structure of higher genus amplitudes has been intensively studied. The explicit data suggest that that this structure is maintained for all classes in the base.

We summarise in the next subsection the strategy to obtain the instanton data and based on the results we propose a general form of the instanton corrected amplitude in terms of modular forms coming from the elliptic geometry of the fibre and a simple and general holomorphic anomaly formula, which governs the all genus instanton corrected amplitudes for the above discussed class of models.

In the following subsection we use the B-model to prove some aspects of the proposed statements. This can establish the A-model results for genus zero and one, since mirror symmetry is proven and the B-model techniques apply. Higher genus B-model calculations have been first extended to compact multi-moduli Calabi-Yau manifolds in [319] and for the case of elliptic Calabi-Yau manifolds [320].

5.1 Quantum cohomology, modularity and the anomaly equations

The basic object, the instanton corrected triple intersections $C_{abc}(t^a)$ are due to special geometry all derivable from the holomorphic prepotential $F^{(0)}$, which also fixes the Kähler potential $K(t^a, \bar{t}^a)$ for the metric on the vector moduli space. At the point of maximal unipotent monodromy [91, 305]

$$F^{(0)}(t^a) = (X^0)^2 \left[-\frac{t^3}{3!} + A_{ab}t^a t^b + c_a t^a + \frac{\chi(M)\zeta(3)}{2(2\pi i)^3} + \sum_{\gamma \in H_2(M,\mathbb{Z})} n^{(0)}_\gamma \, \mathrm{Li}_3(\exp(2\pi i \gamma_a t^a)) \right], \tag{5.1}$$

where

$$t^3 = d_{abc}t^a t^b t^c \tag{5.2}$$

with d_{abc} the triple intersection numbers,

$$c_a = \frac{1}{24}\int_M c_2 J_a \tag{5.3}$$

and $\chi(M)$ is the Euler number of M. By J_a, $a = 1, \ldots, h^{1,1}(M)$, we denote harmonic $(1,1)$ forms, which form a basis of the Kähler cone and the complexified Kähler parameter

$$t^a = \int_{C_a} (i\omega + b), \tag{5.4}$$

where C_a is a curve class in the Mori cone dual to the Kähler cone and b is the Neveu-Schwarz $(1,1)$-form $b-$field. The real coefficients A_{ab} are not completely fixed. They are unphysical in the sense that the Kähler potential $K(t^a, \bar{t}^a)$ and $C_{abc}(t^a)$ do not depend on them. The upper index (0) on the $F^{(0)}$ indicates the genus of the instanton contributions. The triple couplings receive only contributions from genus 0. At the point of maximal unipotent monodromy, the classical topological data provide us with the B-model period integrals

$$\Pi = \begin{pmatrix} F_I \\ X^I \end{pmatrix} = \begin{pmatrix} \int_{\beta^I} \Omega \\ \int_{A_I} \Omega \end{pmatrix} \tag{5.5}$$

over an integral symplectic basis of 3-cycles of the mirror geometry $\tilde{M}$: (A_I, B^I), $I = 0, \ldots, h^{2,1}(\tilde{M})$. This is achieved by matching the $b_3(\tilde{M})$ solutions to the Picard-Fuchs equation with various powers of $\log(z_a) \sim t^a$, with the expected form of the A-model period vector

$$\Pi = X^0 \begin{pmatrix} 2F^{(0)} - t^a \partial_{t^a} F^{(0)} \\ \partial_{t^a} F^{(0)} \\ 1 \\ t^a \end{pmatrix} = X^0 \begin{pmatrix} \frac{t^3}{3!} + c_a t^a - i\chi(M)\frac{\zeta(3)}{(2\pi)^3} + 2f(p) - t^a \partial_{t^a} f(p) \\ -\frac{d_{abc}t^b t^c}{2} + A_{ab}t^b + c_a + \partial_{t^a} f(p) \\ 1 \\ t^a \end{pmatrix}, \tag{5.6}$$

where the lower case indices run from $a = 1, \ldots, h^{1,1}(M)$. In the following we will set $X^0 = 1$.

One can define a generating function for the free energy in terms of a genus expansion in the coupling g_s

$$F(g_s, t^a) = \sum_{g=0}^{\infty} g_s^{2g-2} F^{(g)}(t^a), \tag{5.7}$$

where the upper index $F^{(g)}(t^a)$ indicates as before the genus.

According to the split of the homology $H_2(M, \mathbb{Z})$ into the base and the fibre homology, we define

$$p^\beta = \prod_{k=1}^{b_2(B)} \exp(2\pi i \int_\beta i\omega + b) = \exp(2\pi i \sum_{i=1}^{b_2(B)} \beta_i t^i), \tag{5.8}$$

where $\beta \in H_2(M,\mathbb{Z})$ lies in the image of the map $\sigma_* : H_2(B,\mathbb{Z}) \hookrightarrow H_2(M,\mathbb{Z})$ induced by the embedding $\sigma : B \hookrightarrow M$; and we define

$$q = \exp(2\pi i \int_e i\omega + b) = \exp(2\pi i \tau), \tag{5.9}$$

where $e \in H_2(M,\mathbb{Z})$ is the curve representing the fibre. Now we define the following objects

$$F^{(g)}_\beta(\tau) = \mathrm{Coeff}(F^{(g)}(t^a), p^\beta) \,. \tag{5.10}$$

We have the following universal sectors

$$F^{(0)}_0(\tau) = \left(\int_B c_1^2\right)\frac{t^3}{3!} + \chi(M)\frac{\zeta(3)}{2(2\pi i)^3} - \chi(M)\sum_{n=1}^{\infty}\mathrm{Li}_3(q^n), \tag{5.11}$$

$$F^{(1)}_0(\tau) = \left(\frac{\int_B c_2}{24}\right)\mathrm{Li}_1(q), \quad F^{(g>1)}_0(\tau) = (-1)^g\frac{\chi(M)}{2}\frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!}. \tag{5.12}$$

We note that it follows from the expression for $F^{(0)}_0(q)$ that

$$C_{\tau\tau\tau} = \int_B c_1^2 + \frac{\chi(M)}{2}\zeta(-3) - \frac{\chi(M)}{2}\zeta(-3)E_4(\tau). \tag{5.13}$$

The $F^{(g)}_\beta(\tau)$ have distinguished modular properties. The $F^{(g)}_\beta(\tau)$ can be written in the following general form [166, 167]:

$$F^{(g)}_\beta(\tau) = \left(\frac{q^{\frac{1}{24}}}{\eta(\tau)}\right)^{12\sum_i a_i\beta^i} P_{2g+6\sum_i a_i\beta^i-2}(E_2(\tau), E_4(\tau), E_6(\tau)), \tag{5.14}$$

where $\eta(\tau)$ is the Dedekind eta function (2.133) and $P_{2g+6\sum_i a_i\beta^i-2}(E_2, E_4, E_6)$ are (quasi)-modular forms [321] of weight $2g + 6\sum_i a_i\beta^i - 2$, where $E_{2k}(\tau)$ are the Eisenstein series of weight $2k$ (2.123). The functions P_m are quasi-modular forms since, besides being a function of the true modular forms E_4 and E_6, they are also a function of E_2 which does not transform as a true modular form. For the sectors $\beta > 0$, which describe non-trivial dependence on the Kähler class of the base, the E_2 dependence satisfies the following recursive condition

$$\frac{\partial F^{(g)}_\beta(\tau)}{\partial E_2} = \frac{1}{24}\sum_{h=0}^{g}\sum_{\beta'+\beta''=\beta}(\beta'\cdot\beta'')\,F^{(h)}_{\beta'}(\tau)\,F^{(g-h)}_{\beta''}(\tau) + \frac{1}{24}\beta\cdot(\beta-K_B)\,F^{(g-1)}_\beta(\tau)\,. \tag{5.15}$$

We derive the above relations in section 5.4. For the other types of elliptic fibrations E_7, E_6, & D_5, the right-hand side is divided by $a = 2, 3, 4$ respectively. Eq. (5.15) generalizes the similar equation (1.2) in [167], to arbitrary classes in the base and types of fibres. In particular, if one restricts to elliptic fibrations over the blow up of $\mathbb{P}^2$ and the Hirzebruch surface $\mathbb{F}_1$ the rational fibre class in the base (5.15) becomes the equation of [167] counting curves of higher genus on the E_8, E_7, E_6, & D_5 del Pezzo surfaces. The form (5.14) and its relation to [167] has been observed in [166] for the Hirzebruch surface $\mathbb{F}_0$ as base. A derivation of the equation (5.15) is given in section 5.4.

5.2 The B-model approach to elliptically fibred Calabi-Yau spaces

We continue the discussion with some B-model aspects of elliptically fibred Calabi-Yau manifolds. We assume some familiarity with the formalism developed in [304,305] and concentrate on features relevant and common to the B-model geometry of elliptic fibrations and how they emerge from the topological data of the A-model discussed in section 3.3.

The vectors $l^{(i)}$ are the generators of the Mori cone, i.e. the cone dual to the Kähler cone. As such they reflect classical properties of the Kähler moduli space and the classical intersection numbers, like the Euler number and the evaluation of $\int_M c_2\omega_a$ on the basis of Kähler forms on the elliptic fibration.

On the other hand the differential operators

$$\left(\prod_{l_i^{(r)}>0} \partial_{a_i}^{l_i^{(r)}} - \prod_{l_i^{(r)}<0} \partial_{a_i}^{-l_i^{(r)}}\right)\tilde{\Pi} = 0, \tag{5.16}$$

annihilate the periods $\tilde{\Pi} = \frac{1}{a_0}\Pi$ of the mirror $\tilde{M}$. Here the a_i are the coefficients of the monomials in the equation defining $\tilde{M}$. They are related to the natural large complex structure variables of $\tilde{M}$ by

$$z_r = (-1)^{l_0^r} \prod_i a_i^{l_i^r} \, . \tag{5.17}$$

Note that Π is well defined on $\tilde{M}$, while $\tilde{\Pi}$ is not an invariant definition of periods on $\tilde{M}$. However by commuting out a_0^{-1} one can rewrite the equations (5.16) so that they annihilate Π. Further they can be expressed in the independent complex variables z_r using the gauge condition

$$\theta_{a_i} = \sum_r l_i^k \theta_{z_r}, \tag{5.18}$$

where $\theta_x = x\frac{d}{dx}$ denotes the logarithmic derivative. Equations (5.16) reflect symmetries of the holomorphic $(3,0)$ form and every positive l in the Mori cone (5.16) leads to a differential operator annihilating Π. The operators obtained in this way are contained in the left differential ideal annihilating Π, but they do not generate this ideal. There is however a factorisation procedure, basically factoring polynomials $P(\theta)$ to the left, that leads in our examples to a finite set of generators which determines linear combinations of periods as their solutions. It is referred to as a complete set of Picard-Fuchs operators. In this way properties of the instanton corrected moduli space of M, often called the quantum Kähler moduli space are intimately related to the $l^{(r)}$ and below we will relate some of it properties to the topology of M.

In particular the Mori generator $l^{(e)}$ determines to a large extend the geometry of the elliptic fibre modulus. As one sees from (3.42) the mixing between the base and the fibre is encoded in the z row of $l^{(i)}$, $i = 1, \ldots, h^{1,1}(B)$ and $l^{(e)}$. Let us call this the z-component of $l^{(i)}$ and the corresponding variable a_z.

Following the procedure described above, one obtains after factorizing from $l^{(e)}$ a second order generator Picard-Fuchs operator. For the fibrations types introduced before it is given by

$$\mathcal{L}_e^k = \theta_e(\theta_e - \sum_i a^i\theta_i) - \mathcal{D}^K \tag{5.19}$$

where $k = E8, E7, E6, D5$ refers to the fibration type and $\mathcal{D}^K$ contains the dependence on the type

$$\begin{array}{llll} \mathcal{D}^{E8} & = 12(6\theta_e - 1)(6\theta_e - 5)z_e, & \mathcal{D}^{E7} & = 4(4\theta_e - 1)(4\theta_e - 3)z_e, \\ \mathcal{D}^{E6} & = 3(3\theta_e - 1)(3\theta_e - 2)z_e, & \mathcal{D}^{D5} & = 4(2\theta_e - 1)^2 z_e\,. \end{array} \tag{5.20}$$

Formally setting $\theta_i = 0$ corresponds to the large base limit. Then the equation (5.19) becomes the Picard-Fuchs operator, which annihilates the periods over the standard holomorphic differential on the corresponding family of elliptic curves.

In the limit of large fibre one gets as local model the total space of the canonical line bundle

$$\mathcal{O}(K_B) \to B \tag{5.21}$$

over the Fano base B. Local mirror symmetry associates to such non-compact Calabi-Yau manifolds a genus one curve with a meromorphic 1-form λ that is the limit of the holomorphic $(3,0)$-form. The local Picard-Fuchs system $\mathcal{L}_i^B$ annihilating the periods Π_{loc} of λ can be obtained as a limit of the compact Picard-Fuchs system for $l^{(i)}$, $i = 1, \ldots, h^{1,1}(B)$ by formally setting $\theta_e = 0$. It follows directly from (5.16), since the Mori generators of the base have vanishing first entry and commuting out a_0^{-1} becomes trivial. Differently then for the elliptic curve of the fibre, these Picard-Fuchs operators do not annihilate the periods over holomorphic differential one form of the elliptic curve, which are $\frac{1}{a_z}\Pi_{loc}$. Given the local Picard-Fuchs system the dependence on θ_e can be restored by replacing θ_{a_z} by $\theta_e - \sum_i a_i\theta_i$ instead of $-\sum_i a_i\theta_i$. Since $l^{(i)}$ is negative θ_e appears in $\mathcal{L}_b^i$ only multiplied by at least one explicit z_i^b factor.

There are important conclusions that follow already from the general form of the Picard-Fuchs system. To see them it is convenient to rescale $x_e = c_k z_e$, where

$$\begin{aligned} c_{E8} &= 432, c_{E7} = 64, \\ c_{E6} &= 27, c_{D5} = 16. \end{aligned} \tag{5.22}$$

It is often useful to also rescale the z_i in a similar fashion and call them x_i. The effect of this is that the symbols of the Picard-Fuchs system become the same for all fibre types. From this we can conclude that for all fibre types the Yukawa-couplings and the discriminants are identical in the rescaled variables.

The second conclusion is that the Picard-Fuchs equation of the compact Calabi-Yau is invariant under the $\mathbb{Z}_2$ variable transformation

$$x_e \to (1 - x_e), \qquad x_i \to \left(-\frac{x_e}{1 - x_e}\right)^{a^i} x_i\,. \tag{5.23}$$

This means that there is always a $\mathbb{Z}_2$ involution acting on the moduli space parametrized by (x_e, x_i), which must be divided out to obtain the truly independent values of the parameters.

Another consequence of this statement is that the discriminants $\Delta_i(x_j)$ of the base Picard-Fuchs system determine the discriminant locus of the global system apart from the fibre related $\Delta(x_e)$ components. The former contains always a conifold component $\Delta_c(x_j)$ and only that one, if there are no points on the edges of the 2d polyhedron. Points on the edges correspond to SU(2) or SU(3) gauge symmetry enhancement discriminants which contain only x_i variables dual to Kähler classes, whose $a^i = 0$. They are therefore invariant under (5.23). Moreover the lowest order term in the conifold discriminant is a constant and highest terms are weighted monomials of degree $\chi(B)$ with weights for the x_i a^i or 1 if $a^i = 0$. It follows by (5.23) that the transformed conifold discriminant

$$\Delta_c'(x_j) \sim (1 - x_e)^{\chi(B)} + \mathcal{O}(x_i). \tag{5.24}$$

Examples: elliptic fibrations over $\mathbb{P}^2$ and $\mathbb{F}_1$

Let us demonstrate the above general statements with a couple of examples. We discuss the $E8$ elliptic fibration with base $\mathbb{P}^2$ and with base $\mathbb{F}_1$.
For the first example the Mori vectors are given as

$$\begin{aligned} l^{(e)} &= (-6, 3, 2, \quad 1, 0, 0, 0), \\ l^{(1)} &= (\ \ 0, 0, 0, -3, 1, 1, 1). \end{aligned} \tag{5.25}$$

Form this we can derive the following set of Picard Fuchs equations, where we denote $\theta_i = z_i \partial_{z_i}$.

$$\begin{aligned} \mathcal{L}_e &= \theta_e(\theta_e - 3\theta_2) - 12 z_e(6\theta_e + 5)(6\theta_e + 1), \\ \mathcal{L}_1 &= \theta_2^3 - z_2(\theta_e - 3\theta_2)(\theta_e - 3\theta_2 - 1)(\theta_e - 3\theta_2 - 2). \end{aligned} \tag{5.26}$$

The Yukawa couplings for this example read as follows, where we use $z_e = \frac{x_e}{432}$, $z_1 = \frac{x_1}{27}$ and the discriminants $\Delta_e = 1 - 3x_e + 3x_e^2 - x_e^3 - x_e^3 x_1$ and $\Delta_1 = 1 + x_1$

$$\begin{aligned} C_{eee} &= \frac{9}{x_e^3 \Delta_e}, \\ C_{ee1} &= -\frac{3(-1 + x_e)}{x_e^2 x_1 \Delta_1}, \\ C_{e11} &= \frac{(-1 + x_e)^2}{x_e x_1^2 \Delta_e}, \\ C_{111} &= \frac{1 - 3x_e + 3x_e^2}{3 x_1^2 \Delta_e \Delta_1}. \end{aligned} \tag{5.27}$$

The second example over $\mathbb{F}_1$ has the following three generators of the Mori cone

$$\begin{aligned} l^{(e)} &= (-6\ ,\ 3,\ \ 2,\ \ 1,\ \ 0,\ \ 0,\ \ 0,\ \ 0), \\ l^{(1)} &= (\ \ 0\ ,\ 0,\ \ 0, -1, -1,\ \ 0,\ \ 1,\ \ 1), \\ l^{(2)} &= (\ \ 0\ ,\ 0,\ \ 0, -2,\ \ 1,\ \ 1,\ \ 0,\ \ 0), \end{aligned} \tag{5.28}$$

and gives rise to the following Picard-Fuchs equations

$$\begin{aligned} \mathcal{L}_e &= \theta_e(\theta_e - 2\theta_2 - \theta_1) - 12 z_e(6\theta_e + 5)(6\theta_e + 1), \\ \mathcal{L}_1 &= \theta_1^2 - z_1(\theta_1 - \theta_2)(2\theta_2 + \theta_1 - \theta_e), \\ \mathcal{L}_2 &= \theta_2(\theta_2 - \theta_1) - z_2(2\theta_2 + \theta_1 - \theta_e)(2\theta_2 + \theta_1 - \theta_e + 1). \end{aligned} \tag{5.29}$$

This example contains the rational elliptic surface, which we discuss in detail in section 5.6. Furthermore we focus on this example to give a proof of the holomorphic anomaly at genus zero by using mirror symmetry in Section 5.4.1.

5.3 Modular subgroup of monodromy group

The deeper origin of the appearance of modular forms is the monodromy group of the Calabi-Yau threefold. Ref. [322] explains that in the large volume limit of $X_{18}[1, 1, 1, 6, 9]$, which corresponds to the elliptic fibration over $\mathbb{P}^2$, the monodromy group reduces to an $SL(2,\mathbb{Z})$ monodromy group. This section recalls the appearance of this modular group and how it generalises to other elliptic fibrations. The moduli space of $X_{18}[1,1,1,6,9]$ with the degeneration loci is portrayed in Fig. 5.1.

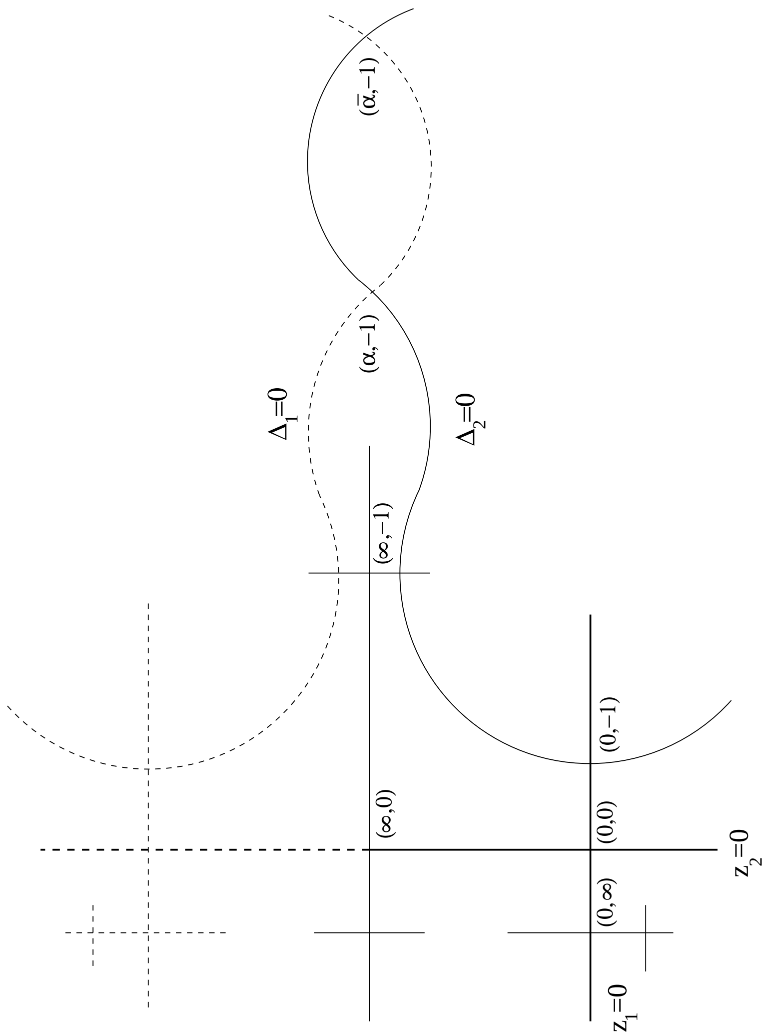

Figure 5.1: The moduli space for the elliptic Calabi-Yau fibration over $\mathbb{P}^2$ $X_{18}[1,1,1,6,9]$.

We continue by recalling the monodromy for the model in [322] adapted to our discussion. The fundamental solution is given by

$$\begin{aligned} w_0(z_e,z_1) &= \sum_{m,n=0}^{\infty} \frac{(18n+6m)!}{(9n+3m)!\,(6n+2m)!\,(n!)^3\,m!}\, z_e^{3n+m}\, z_1^m \\ &= \sum_{k=0}^{\infty} \frac{(6k)!}{k!\,(2k)!\,(3k)!}\, z_e^k\, U_k(z_1). \end{aligned} \tag{5.30}$$

with

$$U_\nu(z_1) = z_1^\nu \sum_{n=0}^{\infty} \frac{\nu!}{(n!)^3\,\Gamma(\nu-3n+1)}\, z_1^{-3n} \tag{5.31}$$
$$= z_1^\nu \sum_{n=0}^{\infty} \frac{\Gamma(3n-\nu)}{\Gamma(-\nu)\,(n!)^3}\, z_1^{-3n},$$

which is a finite polynomial for positive integers ν, since $\Gamma(\nu - 3n + 1) = \infty$ for sufficiently large n. The translation to the parameters in [322] is $(z_e, z_1) = (\,(18\psi)^{-6}, -3\phi)$. The natural coordinates obtained from toric methods are $\tilde{z}_e = z_e z_1$ and $\tilde{z}_1 = z_1^{-3}$. Note that the second line (5.30) makes manifest the presence of the elliptic curve in the geometry. For this regime of the parameters one can easily find logarithmic solutions by taking derivates to k and n [304]

$$2\pi i\, w_e^{(1)}(z_e, z_1) = \log(z_e z_1)\, w_0 + \ldots, \tag{5.32}$$
$$2\pi i\, w_1^{(1)}(z_e, z_1) = -3\log(z_1)\, w_0 + \ldots.$$

The periods are defined by $\tau = w_e^{(1)}/w_0$ and $t_1 = w_1^{(1)}/w_0$ and $q = e^{2\pi i\tau}$, $q_1 = e^{2\pi i t_1}$. The two monodromies which generate the modular group are

$$M_0: \quad (z_e, z_1) \to (e^{2\pi i} z_e, z_1), \;\; z_e \text{ small}, \; z_1 \text{ large},$$
$$M_\infty: \quad (z_e, z_1) \to (e^{2\pi i} z_e, z_1), \;\; z_e \text{ large}, \; z_1 \text{ large}.$$

The monodromy around $z_e = 0$ follows directly from (5.32), it acts as

$$\mathbf{M}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{5.33}$$

on $(w_e^{(1)}, w_0)^{\mathrm{T}}$. To determine the action on the periods of M_∞, we need to analytically continue w_0 and $w_1^{(1)}$ to large z_e. To this end, we write w_0 as a Barnes integral

$$w_0(z_e, z_1) = \frac{1}{2\pi i} \int_C ds\, \frac{\Gamma(-s)\,\Gamma(6s+1)}{\Gamma(2s+1)\,\Gamma(3s+1)}\, e^{\pi i s}\, z_e^s\, U_s(z_1), \tag{5.34}$$

where C is the vertical line from $-i\infty - \varepsilon$ to $i\infty - \varepsilon$. For small $|z_e|$ the contour can be deformed to the right giving back the expression in (5.30). For large $|z_e|$ one instead obtains the expansion

$$w_0(z_e, z_1) = \frac{1}{6\pi^2} \sum_{r=1,5} \sin(\pi r/3) \sum_{k=0}^{\infty} a_r(k)\, (-z_e)^{-k-\frac{r}{6}}\, U_{-k-r/6}(z_1), \tag{5.35}$$

with

$$a_r(k) = (-1)^k \frac{\Gamma(k + r/6)\Gamma(2k + r/3)\Gamma(3k + r/2)}{\Gamma(6k + r)}.$$

The logarithmic solution $w_e^{(1)}$ is given similarly by

$$w_e^{(1)}(z_e, z_1) = \frac{1}{2\pi i}\int_C ds\, \frac{\Gamma(-s)^2\,\Gamma(6s+1)\,\Gamma(s+1)}{\Gamma(2s+1)\,\Gamma(3s+1)}\, e^{2\pi i s}\, z_e^s\, U_s(z_1), \tag{5.36}$$
$$= \frac{1}{6\pi^2 i}\sum_{r=1,5} e^{-\pi i r/6}\cos(\pi r/6)\sum_{k=0}^{\infty} a_r(k)\,(-z_e)^{-k-\frac{r}{6}}\, U_{-k-r/6}(z_1).$$

To determine the action of M_∞, we define the basis

$$f_r(z_e, z_1) = \sum_{n=0}^{\infty} a_r(k)\,(-z_e)^{-k-\frac{r}{6}}\, U_{-k-r/6}(z_1) \tag{5.37}$$

for $r = 1, 5$, and the matrix $\mathbf{A}$ which relates the two bases

$$(w_e^{(1)}, w_1^{(1)})^{\mathrm{T}} = \mathbf{A}\,(f_1, f_5)^{\mathrm{T}}. \tag{5.38}$$

Clearly, M_∞ acts diagonally on the f_r: $\mathbf{T} = \mathrm{diag}(\alpha^{-1}, \alpha^{-5})$ with $\alpha = e^{2\pi i/6}$, which gives for $\mathbf{M}_\infty$

$$\mathbf{M}_\infty = \mathbf{A}\mathbf{T}\mathbf{A}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{5.39}$$

This gives for the monodromy around the conifold locus

$$\mathbf{M}_1 = \mathbf{M}_0\mathbf{M}_\infty^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{5.40}$$

The generator $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{Z})$ corresponds to $\mathbf{M}_0\mathbf{M}_\infty^{-1}$. The large volume limit is such that $u = q^{3/2}p \to 0$. We see that M_0 and M_∞ map small u to small u. The monodromies act on u by [322]

$$M_0 u = -u, \qquad M_\infty u = u \tag{5.41}$$

Thus we have established an action of $\mathrm{SL}_2(\mathbb{Z})$ on the boundary of the moduli space. The above analysis can be extended straightforwardly to the other types of fibrations using the expansions (E.9). The matrix $\mathbf{M}_0$ is for all fibre types the same. We find that

$$\mathbf{M}_\infty = \begin{pmatrix} 1-a & -1 \\ a & 1 \end{pmatrix} \in \Gamma_0(a) \tag{5.42}$$

for $a = 2, 3$ and 4 corresponding to the fibre types E_7, E_6 and D_5. Note that $\mathbf{M}_\infty$ has order 4 and 3 for $a = 2$ and 3 respectively, while the order is infinite for $a = 4$. Generalisation to other base surfaces B is also straightforward. In case of multiple 2-cycles in the base, it is natural to define parameters for each base class: $u_i = q^{a_i/2}p_i$, with $p_i = \exp(2\pi i\, t^i)$, $i = 1, \ldots, b_2(B)$. This is precisely the change of parameters given by (3.34). These transform under the action of M_0 and M_∞ as:

$$M_0 u_i = (-)^{a_i} u_i, \qquad M_\infty u_i = u_i. \tag{5.43}$$

5.4 Derivation of the holomorphic anomaly equation

In this section we discuss the holomorphic anomaly equation that arises for elliptically fibred Calabi-Yau threefolds. By using mirror symmetry the anomaly for genus 0 is proven for the base being $\mathbb{F}_1$. We further establish a derivation from the BCOV holomorphic anomaly equations.

5.4.1 The elliptic fibration over $\mathbb{F}_1$

We start by deriving the holomorphic anomaly equation at genus zero by adapting the proof which appeared in Ref. [167] for a similar geometry. For this we determine the solution to the Picard-Fuchs equation and express the solution in terms of modular forms. After determining the mirror map we find a recursive relation in the functions $c_m(x_e)$, which are related to the genus zero topological amplitude. This can be used to prove the holomorphic anomaly equation for genus zero. Furthermore the genus zero topological amplitude can be expressed in terms of standard Eisenstein series of the elliptic parameter. We start by studying the Picard-Fuchs operator associated to the elliptic fibre $X_6[1,2,3]$ only. Denoting by $\theta_e = z_e \partial_{z_e}$ the Picard-Fuchs operator can be written as

$$\mathcal{L} = \theta_e^2 - 12 z_e (6\theta_e + 5)(6\theta_e + 1). \tag{5.44}$$

One can immediately write down two solutions as power series expansions around $z_e = 0$. They are given by

$$\phi(z_e) = \sum_{n \geq 0} a_n z_e^n, \quad \tilde{\phi}(z_e) = \log(z_e)\phi(z_e) + \sum_{n \geq 0} b_n z_e^n, \tag{5.45}$$

with

$$a_n = \frac{(6n)!}{(3n)!(2n)!n!}, \quad b_n = a_n(6\psi(1+6n) - 3\psi(1+3n) - 2\psi(1+2n) - \psi(1+n)), \tag{5.46}$$

where $\psi(z)$ denotes the digamma function. The mirror map is thus given by

$$2\pi i \tau = \frac{\tilde{\phi}(z_e)}{\phi(z_e)}. \tag{5.47}$$

Using standard techniques from the Gauss-Schwarz theory for the Picard-Fuchs equation (cf. [94]) one observes

$$j(\tau) = \frac{1}{z_e(1 - 432 z_e)}, \tag{5.48}$$

which can be inverted to yield

$$z_e(\tau) = \frac{1}{864}(1 - \sqrt{1 - 1728/j(\tau)}) = q - 312 q^2 + O(q^3). \tag{5.49}$$

Further, the polynomial solution $\phi(x_e)$ can be expressed in terms of modular forms as

$$\phi(z_e) = {}_2F_1\left(\frac{5}{6}, \frac{1}{6}, 1; 432 z_e\right) = \sqrt[4]{E_4(\tau)}, \tag{5.50}$$

from which one can conclude that

$$
\begin{aligned}
E_4(\tau) &= \phi^4(z_e), \\
E_6(\tau) &= \phi^6(z_e)(1-864z_e), \\
\Delta(\tau) &= \phi^{12}(z_e)z_e(1-432z_e), \\
\frac{1}{2\pi i}\frac{dz_e}{d\tau} &= \phi^2(z_e)z_e(1-432z_e).
\end{aligned}
\tag{5.51}
$$

Let us now examine the periods of the mirror geometry $\tilde{M}$ in the limit that the fibre f over the Hirzebruch surface $\mathbb{F}_1$ becomes small. At $z_e = z_1 = z_2$ we have a point of maximal unipotent monodromy. In particular this implies that one has among the eight periods one holomorphic one, $w_0(z_e, z_1, z_2)$, one that starts with $\log(z_e)$, $w_e^{(1)}(z_e, z_1, z_2)$, and one that starts with $\log(z_1)$, $w_1^{(1)}(z_e, z_1, z_2)$. In the limit of small fibre f, i.e. $z_2 = 0$, one obtains for these periods [167]

$$
\begin{aligned}
w_0(z_e, z_1, 0) &= \phi(z_e), \\
w_e^{(1)}(z_e, z_1, 0) &= \tilde{\phi}(z_e), \\
w_1^{(1)}(z_e, z_1, 0) &= \log(z_1)\phi(z_e) + \xi(z_e) + \sum_{m\geq 1}(\mathcal{L}_m\phi(z_e))z_1^m,
\end{aligned}
\tag{5.52}
$$

with

$$
\xi(z_e) = \sum_{n\geq 0} a_n(\psi(1+n) - \psi(1))z_e^n, \tag{5.53}
$$

and

$$
\mathcal{L}_m = \frac{(-)^m}{m(m!)}\prod_{k=1}^{m}(\theta_{z_e} - k + 1). \tag{5.54}
$$

This can be obtained by applying the Frobenius method to derive the period integrals, see e.g. [305]. The mirror map reads

$$
2\pi i t_j = \frac{w_j^{(1)}(z_e, z_1, 0)}{w_0(z_e, z_1, 0)}, \quad j = e, 1. \tag{5.55}
$$

Comparing this with our previous discussion about the Picard-Fuchs operator of the elliptic fibre we see that for $t_e = \tau$ there is nothing left to discuss. Hence, let's study the mirror map associated to $t_1 = t$. We observe that by formally inverting, the inverse mirror map can be determined iteratively through the relation [167]

$$
z_1(q, p) = p\,\zeta e^{-\sum_{m\geq 1} c_m(z_e)z_1^m}, \tag{5.56}
$$

where $\zeta = e^{-\frac{\xi(z_e)}{\phi(z_e)}}$, $p = \exp(2\pi i\, t)$ and

$$
c_m(z_e) = \frac{\mathcal{L}_m\phi(z_e)}{\phi(z_e)}. \tag{5.57}
$$

Using eq. (5.51) $c_1(z_e)$ is given by

$$
\begin{aligned}
c_1(z_e) &= -\frac{1}{12}(f_1 - 2) - \frac{f_1}{12}\frac{E_2(\tau)}{\phi^2(z_e)} \\
&= -\frac{1}{\phi^6}\frac{f_1}{12}(E_2E_4 - E_6),
\end{aligned}
\tag{5.58}
$$

where we introduced $f_1 = (1-432z_e)^{-1}$. In order to obtain the other $c_m(z_e)$ one uses

$$\begin{aligned} \theta_e f_1 &= f_1(f_1-1), \\ \theta_e\left(\frac{E_2}{\phi^2}\right) &= -\frac{1}{\phi^8}\frac{f_1}{12}\left(E_2^2E_4 - 2E_2E_6 + E_4^2\right), \\ \theta_e\left(\frac{E_6}{\phi^6}\right) &= -\frac{1}{\phi^{12}}\frac{f_1}{12}\left(6E_4^3 - 6E_6^2\right), \end{aligned} \tag{5.59}$$

and finds the following kind of structure. One can show inductively that

$$c_m(z_e) = \frac{1}{\phi^{6m}}\left(\frac{f_1}{12}\right)^m Q_{6m}(E_2, E_4, E_6), \tag{5.60}$$

where Q_{6m} is a quasi-homogeneous polynomial of degree $6m$ and type $(2,4,6)$, i.e.

$$Q_{6m}(\lambda^2 z_e, \lambda^4 z_1, \lambda^6 z_2) = \lambda^{6m} Q_{6m}(z_e, z_1, z_2).$$

Also by induction, it follows from (5.58) and (5.59) that Q_{6m} is linear in E_2. This allows to write a second structure which is analogous to the one appearing in ref. [167] and given by

$$c_m(z_e) = B_m\frac{E_2}{\phi^2} + D_m, \tag{5.61}$$

where the coefficients B_m, D_m obey the following recursion relation

$$\begin{aligned} B_{m+1} &= -\frac{m}{(m+1)^2}\left[(\theta_{z_e} - m)B_m + D_1B_m - B_1D_m\right], \\ D_{m+1} &= -\frac{m}{(m+1)^2}\left[(\theta_{z_e} - m)D_m - D_1D_m + B_1B_m\right], \end{aligned} \tag{5.62}$$

with $B_1 = -\frac{f_1}{12}$ and $D_1 = -\frac{1}{12}(f_1-2)$. A formal solution to the recursion relation (5.62) can be given by

$$\begin{aligned} B_m &= -\frac{f_m}{12}, \\ D_m &= \frac{1}{f_1}\left[\frac{(m+1)^2}{m}f_{m+1} + (\theta_e - m - \frac{1}{12}(f_1-2))f_m\right], \end{aligned} \tag{5.63}$$

where we define f_m to be

$$f_m(z_e) = \tilde{\phi}(z_e)\mathcal{L}_m\phi(z_e) - \phi(z_e)\mathcal{L}_m\tilde{\phi}(z_e). \tag{5.64}$$

Due to the relations (5.59) we conclude, that the f_m as well as B_m and D_m are polynomials in f_1. Since f_1 is a rational function of z_e, it transforms well under modular transformations. Therefore modular invariance is broken only by the E_2 term in c_m. We express this via the partial derivative of c_m

$$\frac{\partial c_m(z_e)}{\partial E_2} = -\frac{1}{12}\frac{f_m(z_e)}{\phi^2(z_e)}. \tag{5.65}$$

In order to prove the holomorphic anomaly equation (5.15) one first shows using the general results about the period integrals in [305] that the instanton part of the prepotential can be expressed by the functions $f_m(z_e)$. A tedious calculation reveals

$$\frac{1}{2\pi i}\frac{\partial}{\partial t}F^{(0)}(\tau,t)=\sum_{m\geq 1}\frac{f_m(z_e)}{\phi^2(z_e)}z_1^m. \tag{5.66}$$

Using the inverse function theorem and eqs. (5.65), (5.56) yields

$$\frac{\partial z_1}{\partial E_2}=\frac{1}{12}\left(\frac{1}{2\pi i}\frac{\partial z_1}{\partial t}\right)\left(\frac{1}{2\pi i}\frac{\partial F^{(0)}}{\partial t}\right). \tag{5.67}$$

Now, we have

$$\frac{\partial}{\partial E_2}\left(\frac{1}{2\pi i}\frac{\partial F^{(0)}}{\partial t}\right)=\frac{1}{12}\left(\frac{\partial^2 F^{(0)}}{\partial(2\pi i t)^2}\right)\left(\frac{1}{2\pi i}\frac{\partial F^{(0)}}{\partial t}\right), \tag{5.68}$$

which implies that up to a constant term in p one arrives at

$$\frac{\partial F^{(0)}}{\partial E_2}=\frac{1}{24}\left(\frac{1}{2\pi i}\frac{\partial F^{(0)}}{\partial t}\right)^2. \tag{5.69}$$

By definition of $F_n^{(0)}$, Eq. (5.10), we have $\frac{1}{2\pi i}\frac{\partial}{\partial t}F^{(0)}(\tau,t)=\sum_{m\geq 1} m\,F_m^{(0)}\,p^m$ and hence obtain by resummation

$$\frac{\partial F_n^{(0)}}{\partial E_2}=\frac{1}{24}\sum_{s=1}^{n-1}s(n-s)F_s^{(0)}F_{n-s}^{(0)}. \tag{5.70}$$

This almost completes the derivation of (5.15). We still need to determine the explicit form of $F_n^{(0)}$. To achieve this we proceed inductively. Using (5.51), (5.66) and (5.56) one obtains

$$F_1^{(0)}=\frac{\zeta f_1}{\phi^2}=q^{\frac{1}{2}}\frac{E_4}{\eta^{12}}. \tag{5.71}$$

Employing the structure (5.60) one can evaluate (5.66) and calculate that

$$\begin{aligned} F_n^{(0)} &= \frac{\zeta^n f_1^n}{\phi^{6n}}P_{6n-2}(E_2,E_4,E_6),\\ &= \left(\frac{\zeta f_1}{\phi^2}\right)^n\frac{1}{\phi^{4n}}P_{6n-2}(E_2,E_4,E_6),\\ &= \frac{q^{\frac{n}{2}}}{\eta^{12n}}P_{6n-2}(E_2,E_4,E_6), \end{aligned} \tag{5.72}$$

where P_{6n-2} is of weight $6n-2$ and is decomposed out of (parts of) Q_m's. This establishes a derivation of the holomorphic anomaly equation (5.15) at genus zero for the elliptic fibration over Hirzebruch surface $\mathbb{F}_1$ with large fibre class. We collect some results for the other fibre types in Appendix E.1.

5.4.2 Derivation from BCOV

The last section provided a derivation of the anomaly equation (5.15) for genus zero from the mirror geometry. More fundamental is a derivation purely within the context of moduli spaces of maps from Riemann surfaces to a Calabi-Yau manifold. This is the approach taken BCOV [101] to derive holomorphic anomaly equations for genus g n-point correlation function with $2g-2+n>0$. The correlation functions are given by covariant derivatives to the free energies $F^{(g)}$:

$$C^{(g)}_{a_1 a_2 \ldots a_n} = D_{a_1} \ldots D_{a_n} F^{(g)}, \tag{5.73}$$

with D_a covariant derivatives of for sections of the bunde $\mathcal{L}^{2-2g} \otimes \mathrm{Sym}^n\, T$, with T the tangent bundle of the coupling constant moduli space, and $\mathcal{L}$ a line bundle over this space whose Chern class corresponds to $G_{a\bar{b}}$. The holomorphic anomaly equation reads for the n-point functions

$$\begin{aligned}\bar{\partial}_a C^{(g)}_{a_1\ldots a_n} &= \frac{1}{2}\bar{C}_{\bar{a}\bar{b}\bar{c}} e^{2K} G^{b\bar{b}} G^{c\bar{c}} C^{(g-1)}_{bca_1\ldots a_n} + \\ &+ \frac{1}{2}\bar{C}_{\bar{a}\bar{b}\bar{c}} e^{2K} G^{b\bar{b}} G^{c\bar{c}} \sum_{r=0}^{g}\sum_{s=0}^{n} \frac{1}{s!(n-s)!} \sum_{\sigma\in S_n} F^{(r)}_{ba_{\sigma(1)}\ldots a_{\sigma(s)}} C^{(g-r)}_{ca_{\sigma(s+1)}\ldots a_{\sigma(n)}} \\ &- (2g-2+n-1) \sum_{s=1}^{n} G_{a\bar{a}_s} C^{(g)}_{a_1\ldots a_{s-1}a_{s+1}\ldots a_n}.\end{aligned} \tag{5.74}$$

This equation can be summarised in terms of the generating function

$$F(g_s, t^a; x^a) = \sum_{g=0}^{\infty}\sum_{n=0}^{\infty} g_s^{2g-2} \frac{1}{n!} C^{(g)}_{a_1\ldots a_n} x^{a_1} \ldots x^{a_n} + \left(\frac{\chi(M)}{24} - 1\right)\log g_s. \tag{5.75}$$

Contrary to [101], we take the terms with $2g - 2 + n \leq 0$ as given by $D_1 \ldots D_n F^{(g)}$ instead of setting them to 0. Eq. (5.74) implies that F satisfies

$$\bar{\partial}_a \exp(F) = \left[\frac{g_s^2}{2}\bar{F}_{\bar{a}\bar{b}\bar{c}} e^{2K} G^{b\bar{b}} G^{c\bar{c}} \frac{\partial^2}{\partial x^b \partial x^c} - G_{\bar{a}b} x^b \left(g_s \frac{\partial}{\partial g_s} + x^c \frac{\partial}{\partial x^c}\right)\right]\exp(F). \tag{5.76}$$

To relate (5.76) to the holomorphic anomaly Eq. (5.15) for this geometry, we split again the Kähler parameters t^a into the fibre parameter τ and base parameters t^i. Then we write $F(g_s, \tau, t^i; x^i)$ as a Fourier expansion instead of a Taylor expansion in x^i

$$F(g_s, \tau, t^i; x^i) = \sum_{g=0}^{\infty}\sum_{\beta\in H_2(B,\mathbb{Z})} g_s^{2g-2} F^{(g)}_\beta(\tau)\, f^{(g)}_\beta(x^i, t^i)\, e^{2\pi i \beta\cdot x} p^\beta + \left(\frac{\chi(M)}{24} - 1\right)\log g_s, \tag{5.77}$$

with as before $p^\beta = \exp(2\pi i\, t^i \beta_i)$, $q = \exp(2\pi i\, \tau)$. Moreover, the functions $f^{(g)}_\beta(x^i, t^i)$ satisfy

$$D_i F|_{x=0} = \partial_{x_i} F|_{x=0} \tag{5.78}$$

and $f^{(g)}_\beta(0, t^i) = 1$. In the large volume limit, the covariant derivatives D_i become flat derivatives $\frac{\partial}{\partial t^i}$ and thus $f^g_\beta(x^i, t^i) \to 1$. Therefore, to deduce (5.15) from (5.76) we can set $x^i = 0$ and replace the $\frac{\partial}{\partial x^i}$ by $\frac{\partial}{\partial t^i}$.

Eq. (5.15) follows now by considering $\frac{1}{2\pi i}\partial_{\bar{\tau}} \exp(F)$ on the right hand side of (5.76). As discussed earlier, all $\bar{\tau}$ dependence arises from completing the weight 2 Eisenstein series $\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}$, which gives

$$\frac{\partial}{\partial E_2} = \frac{4\pi^2\tau_2^2}{3} \frac{\partial}{2\pi i \partial\bar{\tau}}. \tag{5.79}$$

We first discuss how the right-hand side of (5.15) can be derived from Eq. (5.76) for the geometry $X_{18}[1, 1, 1, 6, 9]$. We use the basis (3.29) and choose as parameters the "base" parameter $t = B + iJ$ and the fibre parameter $\tau = \tau_1 + i\tau_2$. We are interested in the large volume limit $\tau \to i\infty$, $t \to i\infty$ in such a way that $J \gg \tau_2$. In this limit, the Kähler potential is well approximated by the polynomial form

$$K \approx -\log(\tfrac{4}{3}\,\tilde{d}_{abc}J^aJ^bJ^c) = -\log(\tfrac{4}{3}(\alpha\tau_2^3 + 3\tau_2 J^2)), \tag{5.80}$$

with $\alpha = \tilde{\mathcal{K}}_e^3$ (3.29). This gives for the metric

$$\begin{pmatrix} G_{\tau\bar{\tau}} & G_{t\bar{\tau}} \\ G_{\tau\bar{t}} & G_{t\bar{t}} \end{pmatrix} \approx \begin{pmatrix} \frac{1}{4\tau_2^2} & \frac{\alpha\tau_2}{3J^3} \\ \frac{\alpha\tau_2}{3J^3} & \frac{1}{2J^2} \end{pmatrix},$$

which gives for the matrix $e^K G^{i\bar{j}}$

$$e^K G^{-1} \approx \begin{pmatrix} \frac{1}{J^2} & -\frac{2\alpha\tau_2^2}{3J^3} \\ -\frac{\alpha\tau_2^2}{3J^3} & \frac{1}{2\tau_2} \end{pmatrix}. \tag{5.81}$$

Thus in the limit $J \to \infty$, one finds that only $e^K G^{t\bar{t}} \approx \frac{1}{2\tau_2}$ does not vanish.[1] Therefore,

$$\bar{C}_{\bar{\tau}\bar{b}\bar{c}}e^{2K}G^{b\bar{b}}G^{c\bar{c}}\frac{\partial^2}{\partial x^b \partial x^c} \approx \frac{1}{4\pi^2}\frac{1}{4\tau_2^2}\frac{\partial^2}{\partial x^t \partial x^t}. \tag{5.82}$$

Using (5.79), this shows that (5.76) reduces to:

$$\frac{\partial}{\partial E_2}\exp(F) = \frac{g_s^2}{24}\left(p\frac{\partial}{\partial p}\right)^2 \exp(F). \tag{5.83}$$

Expansion of both sides in p and taking the p^n coefficient gives a holomorphic anomaly equation as (5.15) for $g = 0$. It also gives the correct (5.15) for $g > 0$ except for the appearance of K_B. We belief that a more thorough analysis of the covariant derivatives will explain this term. Assuming the form

$$f_\beta^{(g)}(x,t) \to 1 + x^2\beta \cdot K_B + \dots \tag{5.84}$$

would give the shift in (5.15). The derivation is very similar for the other types of fibres discussed in Section 3.3. The right hand side of Eq. (5.83) is simply divided by a, in agreement with [167].

5.5 T-duality on the fibre

In this section we discuss properties of T-duality on the elliptic fibre in order to relate the result from our period calculations to $D4$-brane counting. One can perform two T-dualities around the circles of the elliptic fibre. Due to the freedom in choosing the circles, this leads to an $\mathrm{SL}(2,\mathbb{Z})$ (or a congruence subgroup) group of dualities mapping IIA branes to IIA branes. This duality group is equal to the modular subgroup of the monodromy group which leave invariant the $F_\beta^{(g)}$'s discussed in section 5.3.

Let $D2_{e/\beta}$ be a $D2$-brane wrapped either on the elliptic fibre e or on a class β in the base. Moreover, we denote by $D4_e$ a D4-brane wrapped around the base and by $D4_\beta$ a D4-brane wrapped around the cycle β in the base and the elliptic fibre e. The double T-duality on both circles of the elliptic fibre

[1] The factor $\frac{1}{4\pi^2}$ appears due to a factor $-2\pi i$ between the moduli in [101] and ours.

transforms pairs of D-brane charges heuristically in the following way:

$$\begin{pmatrix} \mathrm{D6} \\ \mathrm{D4}_e \end{pmatrix} = \gamma \begin{pmatrix} \tilde{\mathrm{D6}} \\ \tilde{\mathrm{D4}}_e \end{pmatrix}, \quad \begin{pmatrix} \mathrm{D4}_\beta \\ \mathrm{D2}_\beta \end{pmatrix} = \gamma \begin{pmatrix} \tilde{\mathrm{D4}}_\beta \\ \tilde{\mathrm{D2}}_\beta \end{pmatrix}, \quad \begin{pmatrix} \mathrm{D2}_e \\ \mathrm{D0} \end{pmatrix} = \gamma \begin{pmatrix} \tilde{\mathrm{D2}}_e \\ \tilde{\mathrm{D0}} \end{pmatrix}, \tag{5.85}$$

with γ in $\mathrm{SL}(2,\mathbb{Z})$ or a congruence subgroup. See for more a more formal treatment of T-duality on Calabi-Yau manifolds [171, 172]. In the following sections, we will always consider the element

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.86}$$

T-duality is not valid for every choice of the Kähler parameter. One way to see this is that the BPS invariants of $D2$ branes do not depend on the choice of the Kähler moduli but those of $D4$ and $D6$ branes do through wall-crossing. The choice where the two are related by T-duality is sufficiently close to the class of the elliptic fibre, this is called a suitable polarisation in the literature [323]. Sufficiently close means that no wall is crossed between the fibre class and the suitable polarisation.

The equality of invariants of D0 branes and D2 branes wrapping the fibre can be easily verified. The BPS invariant of an arbitrary number $n > 0$ of D0 branes is known to be equal to the Euler number of the Calabi-Yau manifold X [126]:

$$\Omega(\,(0,0,0,n), X) = -\chi(X). \tag{5.87}$$

One can verify that these equal the BPS invariants of n D2 branes wrapping the E_8 elliptic fibre of M. See for example [304]. If the modular group is a congruence subgroup of level k then only the BPS invariant corresponding to $n = 0 \mod k$ D2 branes wrapping the fibre equals (5.87).

Our interest is in the D4 branes which can be obtained from $\mathrm{D2}_\gamma$ with $\gamma = \beta + ne$ by T-duality. These D4 branes wrap classes in the base times the fibre, and have $D0$ brane charge n. D4 branes on Calabi-Yau manifolds correspond to black holes in 4-dimensional space-time and are well studied [138], in particular M-theory relates the degrees of freedom of D4 brane black holes to those of a $\mathcal{N} = (0,4)$ CFT, which we discussed in 2.8.1. To make a comparison we denote by P the four-cycle wrapped by the D4 brane in this setup and the generating function is denotes accordingly with respect to the D4 brane charge.

We continue by specialising to the elliptic Calabi-Yau fibrations. We denote the four-cycle obtained by T-duality from a curve β_i in the base by β^i or simply β. Whether β denotes a two- or four-cycle should be clear from the context. One derives from the triple intersection numbers (3.35) that these four-cycles β^i, have a vanishing triple intersection number $(\beta)^3 = 0$. These D4 branes correspond therefore not to large black holes, but to "small" black holes [193]. This means that only after addition of higher derivative corrections to the supergravity effective action one finds a non-vanishing area of the horizon. It is very intriguing that we can obtain detailed knowledge about the spectrum of these black holes using mirror symmetry.

Let $\iota : \beta \to M$ be the embedding of the divisor β into M, which provides a pull back map on the second cohomology $\iota^* : H^2(M,\mathbb{Z}) \to H^2(\beta,\mathbb{Z})$. Since the divisors β are not positive, this map is not injective. One deduces from (3.35) that the rank of the quadratic form $D_{ab} = d_{abj}\beta^j$ is 2 independent of

M. Therefore, the image of the map ι^* is two-dimensional, and we find consequently that the modular weight of $f_\mu^{(\beta)}(\tau)$ is equal to -2. More details of the modular properties of $f_\mu^{(\beta)}(\tau)$ can be derived. If β is primitive, i.e. $\beta/n \notin H_4(M,\mathbb{Z})$ for $n \geq 2$, one can find another divisor $\tilde{\beta}$ such that $\mathcal{K}\cdot\beta\cdot\tilde{\beta} = 1$. The quadratic form D_{ab} then takes the form:

$$\begin{pmatrix} 0 & 1 \\ 1 & \mathcal{K}\cdot\tilde{\beta}^2 \end{pmatrix}. \tag{5.88}$$

With this information one can precisely determine the modular transformation properties of the vector-valued modular form $f_\mu^{(\beta)}(\tau)$. See Eq. (4.17) of [324]. The elements of the modular vector are modular forms of a congruence subgroup $\Gamma(m)$.

The genus 0 free energies $F_\beta^{(0)}$ (5.14) give a prediction $f_{\mathbf{0}}^{(\beta),\mathrm{pr}}(\tau)$ for $f_{\mathbf{0}}^{(\beta)}(\tau)$. Correcting the power by which non-primitive charges are weighted, we find for $f_{\mathbf{0}}^{(\beta),\mathrm{pr}}(\tau)$ in terms of $F_\beta^{(0)}(\tau)$

$$f_{\mathbf{0}}^{(\beta),\mathrm{pr}}(\tau) = \sum_{m|\beta}\frac{1}{m^2}\left(\sum_{n|m}\frac{\mu(n)}{n}\right)F_{\beta/m}^{(0)}(m\tau), \tag{5.89}$$

where $\mu(m)$ is the Möbius function, which is defined for $m \geq 1$ by

$$\mu(m) = \begin{cases} (-1)^\ell & m \text{ is the product of } \ell \text{ distinct primes } \geq 2, \\ \mu(m) = 0 & \text{otherwise.} \end{cases} \tag{5.90}$$

The modular properties of $f_{\mathbf{0}}^{(P)}(\tau)$ defined this way are precisely consistent with the structure found for the genus 0 amplitudes obtained from the mirror periods, see Eq. (5.14) combined with Eq. (3.30). As explained in section 5, the free energy $F_\beta^{(0)}$ is a modular form of weight -2, in agreement with the weight of $f_\mu^{(\beta)}(\tau)$. Due to contributions to $f_{\mathbf{0}}^{(\beta),\mathrm{pr}}(\tau)$ (5.89) of $F_\beta^{(0)}(m\tau)$ with $m > 1$, $f_{\mathbf{0}}^{(\beta),\mathrm{pr}}(\tau)$ is in general an element of the congruence subgroup $\Gamma(m)$ in agreement with the analysis of the modular properties of the supergravity partition function. Generically, one cannot determine uniquely from $f_{\mathbf{0}}^{(\beta),\mathrm{pr}}(\tau)$ the other elements of the modular vector, but in simple examples this can be done.

Besides verifying that the modular properties of $f_{\mathbf{0}}^{(\beta),\mathrm{pr}}(\tau)$ agree with $f_{\mathbf{0}}^{(\beta)}(\tau)$, it is also possible to verify the agreement for the first few coefficients, for small $D0$ and $D4$-brane charge, the BPS invariants can be computed either from the microscopic D-brane perspective or the supergravity context [123, 137, 287, 289, 303, 325, 326]. For example from the microscopic point of view, the moduli space of a single $D4$-brane is given by projective space $\mathbb{P}^n$. Using index theorems one can compute that $n = \frac{1}{6}P^3 + \frac{1}{12}c_2\cdot P - 1$ [138]. Therefore, the first coefficient of $f_{\mathbf{0}}^{(P)}(\tau)$ is expected to be

$$\Omega_P(-\tfrac{1}{24}c_L) = \tfrac{1}{6}P^3 + \tfrac{1}{12}c_2\cdot P. \tag{5.91}$$

The second coefficient corresponds to adding a unit of (anti) $D0$-brane charge. Now the linear system for the divisor of the $D4$-brane is constrained to pass through the $D0$-brane. This gives with Eq. (5.87) [137]

$$\Omega_P(1 - \tfrac{1}{24}c_L) \cong \chi(M)(\tfrac{1}{12}c_2\cdot P - 1). \tag{5.92}$$

Here we have written a "≅" instead of "=" since if $1 - \frac{1}{24}c_L \geq 0$ the formula for the horizon area gives a positive value, such that the BPS states might correspond to black holes with intrinsically gravitational degrees of freedom which are less well understood.

Continuing with two units of $D0$ charge, one finds

$$\Omega_P(2 - \tfrac{1}{24}c_L) \cong \tfrac{1}{2}\chi(M)(\chi(M) + 5)(\tfrac{1}{12}c_2 \cdot P - 2). \tag{5.93}$$

One can in principle continue along these lines, which becomes increasingly elaborate for three reasons. First effects of $D2$-branes become important, second single center black holes contribute for $\hat{q}_0 > 0$ and third the index might depend on the background moduli t.

We now briefly explain which bound states appear in the supergravity picture for small $D0/4$-brane charge. The first terms in the q-expansion cannot correspond to single center black holes since $\hat{q}_0 < 0$. The first terms correspond to bound states of $D6$ and $\overline{D6}$-branes [123]. If P is an irreducible cycle (i.e. it cannot be written as $P = P_1 + P_2$ with P_1 and P_2 effective classes) then the charges Γ_1 and Γ_2 of the constituents are

$$\Gamma_1 = (1, P, \tfrac{1}{2}P^2 - \tfrac{c_2}{24}, \tfrac{1}{6}P^3 + \tfrac{c_2 \cdot P}{24}), \qquad \Gamma_2 = (-1, 0, \tfrac{c_2}{24}, 0), \tag{5.94}$$

The index of a 2-center bound state is given by:

$$\langle \Gamma_1, \Gamma_2 \rangle \; \Omega(\Gamma_1)\, \Omega(\Gamma_2),$$

with $\langle \Gamma_1, \Gamma_2 \rangle = -P_1^0 Q_{0,2} + P_1 \cdot Q_2 - P_2 \cdot Q^1 + P_2^0 Q_{0,1}$ the symplectic inner product. Since the constituents are single D6-branes with a non-zero flux, their index is $\Omega(\Gamma_i; t) = 1$. Therefore, $\Omega_P(-\frac{1}{24}c_L) = \langle \Gamma_1, \Gamma_2 \rangle = \frac{1}{6}P^3 + \frac{1}{12}c_2 \cdot P$, which reproduces Eq. (5.91).

One can continue in a similar fashion with adding other constituents to compute indices with higher charge. For example, BPS states with charge $\Gamma = (0, 2P, 0, \frac{1}{3}P^3 + \frac{c_2 \cdot P}{12})$ corresponds to Γ_1 as in (5.94) and

$$\Gamma_2 = (-1, P, -\tfrac{1}{2}P^2 + \tfrac{c_2}{24}, \tfrac{1}{6}P^3 + \tfrac{c_2 \cdot P}{24}). \tag{5.95}$$

One obtains then $\Omega_{2P}(-\frac{1}{24}c_L) = \frac{8}{6}P^3 + \frac{2}{12}c_2 \cdot P$. Similarly, one could also add $\overline{D0}$ charges, and find the right hand sides of Eqs. (5.91) to (5.93) with P replaced by $2P$.

Example: $X_{18}[1, 1, 1, 6, 9]$

We now consider the periods for $X_{18}[1, 1, 1, 6, 9]$, i.e. a elliptic fibration over $\mathbb{P}^2$ and compare with the above discussion. This Calabi-Yau has a 2-dimensional Kähler cone, and lends it self well to studies of D4 branes. We consider D4 branes wrapping the divisor whose Poincaré dual is the hyperplane class H of the base surface $\mathbb{P}^2$. The number of wrappings is denoted by r.

As explained in section 5, the genus zero GW invariants are well-studied [304, 322]. Using (5.89), the $F_\beta^{(0)}(\tau)$ provide the following predictions for $f_{\mathbf{0}}^{(r)}(\tau)$:

$$\begin{aligned}
f_{\mathbf{0}}^{(1),\mathrm{pr}}(\tau) &= \frac{31E_4^4 + 113E_4E_6^2}{48\eta(\tau)^{36}} \\
&= q^{-3/2}(3 - 1080\,q + 143770\,q^2 + 204071184\,q^3 + \dots), \\
f_{\mathbf{0}}^{(2),\mathrm{pr}}(\tau) &= \frac{-196319E_4E_6^5 - 755906E_4^4E_6^3 - 208991E_4^7E_6}{221184\,\eta(\tau)^{72}} - \frac{1}{24}E_2 f_0^{(1),\mathrm{pr}}(\tau)^2 + \frac{1}{8}f_0^{(1),\mathrm{pr}}(2\tau) \\
&= q^{-3}(-6 + 2700\,q - 574560\,q^2 + \cdots) + \frac{1}{4}f_0^{(1),\mathrm{pr}}(2\tau), \\
f_{\mathbf{0}}^{(3),\mathrm{pr}}(\tau) &= q^{-9/2}(27 - 17280\,q + 5051970\,q^2 + \cdots) + \frac{1}{9}f_0^{(1),\mathrm{pr}}(3\tau).
\end{aligned}$$

We want to compare this to the expressions derived above from the point of view of D4 branes. For

$r = 1$, we have

$$\Omega(\Gamma; J) = \frac{1}{12} c_2 \cdot H = 3, \tag{5.96}$$

in agreement with the first coeficient of $f^{(1)}(\tau)$. The second term in the q-expansion corresponds to

$$\Omega(1, \tfrac{1}{2}H, -1) = \chi(M)(\tfrac{1}{12}c_2 \cdot P - 1) = 1080, \tag{5.97}$$

which is also in agreement with the periods. For two D0 branes we find a small discrepancy, one finds:

$$\tfrac{1}{2}(\tfrac{1}{12}c_2 \cdot P - 2)\chi(M)\,(\chi(M) + 5) = 144450. \tag{5.98}$$

This is an excess of $1080 = -2\chi(M)$ states compared to the 3rd coefficient in $f^{(1)}(\tau)$. This number is very suggestive of a bound state picture, possibly involving D2 branes. Since $\hat{q}_0 > 0$ one could argue that these states are due to intrinsic gravitational degrees of freedom, but it seems actually a rather generic feature if we consider other elliptic fibrations (e.g. over $\mathbb{F}_1$).

For $r = 2$, also the first two coefficients of the spectrum match with the D4 brane indices, and the 3rd differs by $-6\chi(M)$. Something non-trivial happens for $r = 3$. We leave an interpretation of these indices from multi-center solutions for a future work, and continue with the example of the local elliptic surface [139].

5.6 BPS invariants of the rational elliptic surface

This section continues with the comparison of the D4 and D2 brane spectra for the E_8 elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$ which was first addressed by refs. [139, 315]. Let $\sigma : \mathbb{F}_1 \to M$ be the embedding of $\mathbb{F}_1$ into the Calabi-Yau threefold. The surface $\mathbb{F}_1$ is itself a fibration $\pi : \mathbb{F}_1 \to C \cong \mathbb{P}^1$ with fibre $f \cong \mathbb{P}^1$, with intersections $C^2 = -1$, $C \cdot f = 1$ and $f^2 = 0$. The Kähler cone of M is spanned by the elliptic fibre class J_1, and the classes $J_2 = \sigma_*(C + f)$ and $J_3 = \sigma_*(f)$. The Calabi-Yau intersections and Chern classes are given by (3.39).

A few predictions from the periods for the D4-brane partition functions are

$$\begin{aligned}
f^{(1),\mathrm{pr}}_{C,\mathbf{0}}(\tau) &= \frac{E_4(\tau)}{\eta(\tau)^{12}} = q^{-1/2}(1 + 252q + \dots), \\
f^{(1),\mathrm{pr}}_{f,\mathbf{0}}(\tau) &= \frac{2E_4(\tau)E_6(\tau)}{\eta(\tau)^{24}} \\
&= -2q^{-1} + 480 + 282888q + \cdots, \\
f^{2,\mathrm{pr}}_{C,\mathbf{0}}(\tau) &= \frac{E_2(\tau)E_4(\tau)^2 + 2E_4(\tau)E_6(\tau)}{24\eta(\tau)^{24}} + \frac{1}{8} f^{(1),\mathrm{pr}}_{C,0}(2\tau) \\
&= -9252\,q - 673760\,q^2 + \cdots + \frac{1}{4} f_C(2\tau), \\
f^{(3),\mathrm{pr}}_{C,\mathbf{0}}(\tau) &= \frac{54E_2^2E_4^3 + 216E_2E_4^2E_6 + 109E_4^4 + 197E_4E_6^2}{15552\eta^{36}} + \frac{2}{27} f^{(1),\mathrm{pr}}_{C,0}(3\tau) \\
&= 848628\,q^{3/2} + 115243155\,q^{5/2} + \cdots + \frac{1}{9} f^{(1),\mathrm{pr}}_{C,0}(3\tau).
\end{aligned} \tag{5.99}$$

Since $c_2(M) \cdot f = 24$, explicit expressions in terms of modular forms for the divisors $f_{C+nf}(\tau)$ become rather lengthy. Interestingly, one finds that for this class the first coefficients (checked up to $n = 12$), are given by $1 + 2n$ in agreement with Eq. (5.91). Moreover, the second and third coefficients are respectively given by

$$\chi(M)\left(\frac{1}{12}c_2 \cdot P - 1\right) \tag{5.100}$$

and

$$\tfrac{1}{2}\chi\,(\chi + 9)\,\left(\tfrac{1}{12}c_2 \cdot P - 2\right) \tag{5.101}$$

as long as the corresponding $\hat{q}_0 < 0$.

Another interesting class are r D4 branes wrapped on the divisor C, which is however not an ample divisor since $C = J_2 - J_3$. The Euler number of this divisor is $c_2 \cdot C = 12$, it is in fact the rational elliptic surface dP_9, which is the 9-point blow-up of the projective plane $\mathbb{P}^2$, or equivalently, the 8-point blow-up of $\mathbb{F}_1$. For r D4 branes we have $P = rC$. Eq. (3.39) shows that the quadratic form $D_{abc}P^c$ restricted to J_1 and J_3 is

$$r\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.102}$$

The other 8 independent classes of $H^2(P,\mathbb{Z})$ are not "visible" to the computation based on periods, since these 2-cycles of P do not pull back to 2-cycles of M. We continue by confirming the expressions found from the periods with a computation of the Euler numbers of the moduli spaces of semi-stable sheaves as in Refs. [139, 315]. The algebraic computations are more naturally performed in terms of Poincaré polynomials, and thus give more refined information about the moduli space [301]. Moreover, the eight independent classes which are not visible from the Calabi-Yau point of view, can be distinguished from this perspective.

One might wonder whether the extra parameter appearing with the Poincaré polynomial is related to the higher genus expansion of topological strings. However, the refined information of the genus expansion is different. Roughly speaking, the D2-brane moduli space is a torus fibration over a base manifold [114]. The genus expansion captures the cohomology of the torus, whereas the D4 brane moduli space gives naturally the cohomology of the total moduli space. For $r = 1$, Ref. [167] argues that the torus fibration is also present for moduli spaces of rank 1 sheaves on dP_9, but it is non-trivial to continue this to higher rank. Another approach to verify the Fourier-Mukai transform at a refined level is consider the refined topological string partition function with parameters ϵ_1 and ϵ_2, and then take the Nekrasov-Shatashvili limit $\epsilon_1 = 0$, $\epsilon_2 \ll 1$ instead of the topological string limit $\epsilon_1 = -\epsilon_2 = g_s$.

The structure described in section 5.5 for D4 brane partition functions simplifies when one specializes to a (local) surface. The charge vector Γ becomes $(r, \mathrm{ch}_1, \mathrm{ch}_2)$ with r the rank and ch_i the Chern characters of the sheaf and recall $\Delta = \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2)$, and $\mu = c_1/r \in H^2(S,\mathbb{Q})$.

This section verifies the agreement of the BPS invariants obtained from the periods and vector bundles of dP_9 for $f^{(r)}_{c_1,J_{m,n}}(\tau, z)$ for $r \leq 3$. The results for $r \leq 2$ are due to Göttsche [112] and Yoshioka [315]. The computations apply notions and techniques from algebraic geometry as Gieseker stability, HN filtrations and the blow-up formula. We refer to [155, 301] for further references and details. The most crucial difference between the computations for dP_9 and those for Hirzebruch surfaces in [155, 301] is that the lattice arising from $H^2(dP_9,\mathbb{Z})$ is now 10 dimensional. We continue therefore with giving a detailed description of different bases of $H^2(dP_9,\mathbb{Z})$, gluing vectors and theta functions.

5.6.1 The lattice $H^2(dP_9,\mathbb{Z})$

The second cohomology $H^2(dP_9,\mathbb{Z})$ gives naturally rise to a unimodular basis, it is in fact isomorphic to the unique unimodular lattice with signature $(1, 9)$, which we denote in the following by $\Lambda_{1,9}$. Three different bases (**C**, **D** and **E**) of $\Lambda_{1,9}$ are useful. The first basis is the geometric basis **C**, which keeps manifest that dP_9 is the 9-point blow-up of the projective plane $\mathbb{P}^2$. The basis vectors of **C** are H

(the hyperplane class of $\mathbb{P}^2$) and $\mathbf{c}_i$ (the exceptional divisors of the blow-up)[2]. The quadratic form is $\mathrm{diag}(1, -1, \ldots, -1)$. The canonical class K_9 of dP_9 is given in terms of this basis by

$$K_9 = -3H + \sum_{i=1}^{9} \mathbf{c}_i. \tag{5.103}$$

One can easily verify that $K_9^2 = 0$. Note that $-K_9$ is numerically effective but not ample.

The second basis $\mathbf{D}$ parametrizes $\Lambda_{1,9}$ as a gluing of the two non-unimodular lattices A and D. The basis $\mathbf{D}$ is given in terms of $\mathbf{C}$ by

$$\begin{aligned} &\mathbf{a}_1 = -K_9, \qquad \mathbf{a}_2 = H - \mathbf{c}_9, \\ &\mathbf{d}_i = \mathbf{c}_i - \mathbf{c}_{i+1}, \qquad 1 \le i \le 7, \\ &\mathbf{d}_8 = -H + \mathbf{c}_7 + \mathbf{c}_8 + \mathbf{c}_9. \end{aligned} \tag{5.104}$$

The $\mathbf{a}_i$ are basis elements of A and $\mathbf{d}_i$ of D. Since A and D are not unimodular, integral lattice elements of $\mathbf{C}$ do not correspond to integral elements of D. For example, $\mathbf{c}_9$ is given by

$$\mathbf{c}_9 = \frac{1}{2}\left(\mathbf{a}_1 + \mathbf{a}_2 + \sum_{i=1}^{6} i\mathbf{d}_i + 3\mathbf{d}_7 + 4\mathbf{d}_8\right). \tag{5.105}$$

The other $\mathbf{c}_i$ are easily determined using $\mathbf{c}_9$. The quadratic form Q_A of the lattice A is

$$Q_A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \tag{5.106}$$

and Q_D of the lattice D is minus the D_8 Cartan matrix

$$Q_D = -Q_{D_8} = -\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}. \tag{5.107}$$

Gluing of A and D to obtain $\Lambda_{1,9}$ corresponds to an isomorphism between A^*/A and D^*/D. This isomorphism is given by 4 gluing vectors $\boldsymbol{g}_i$, since the discriminants of A and D are equal to 4. We choose them to be

$$\begin{aligned} \boldsymbol{g}_0 &= \mathbf{0}, \\ \boldsymbol{g}_1 &= \tfrac{1}{2}(1,0,1,0,1,0,1,0,0,1), \\ \boldsymbol{g}_2 &= \tfrac{1}{2}(0,1,0,0,0,0,0,0,1,1), \\ \boldsymbol{g}_3 &= \tfrac{1}{2}(1,1,1,0,1,0,1,0,1,0). \end{aligned}$$

Theta functions which sum over D will play an essential role later in this section. The theta functions

[2] We will use in general boldface to parametrize vectors.

$\Theta_{rD_8,\boldsymbol{\mu}}(\tau)$ are defined by

$$\Theta_{rD_8,\mu}(\tau) = \sum_{\mathbf{k}=\boldsymbol{\mu} \mod r\mathbb{Z}} q^{\frac{\mathbf{k}^2}{2r}}. \qquad (5.108)$$

Such sums converge rather slowly. Therefore, we also give their expression in terms of unary theta functions $\theta_i(\tau) = \theta_i(0,\tau)$. For $r = 1$ and the glue vectors $\boldsymbol{g}_i$ one has

$$\begin{aligned}
\Theta_{D_8,\boldsymbol{g}_0}(\tau) &= \tfrac{1}{2}\left(\theta_3(\tau)^8 + \theta_4(\tau)^8\right),\\
\Theta_{D_8,\boldsymbol{g}_1}(\tau) &= \tfrac{1}{2}\theta_2(\tau)^8,\\
\Theta_{D_8,\boldsymbol{g}_2}(\tau) &= \tfrac{1}{2}\left(\theta_3(\tau)^8 - \theta_4(\tau)^8\right),\\
\Theta_{D_8,\boldsymbol{g}_3}(\tau) &= \tfrac{1}{2}\theta_2(\tau)^8.
\end{aligned}$$

For $r = 2$, the μ in the $\Theta_{2D_8,\mu}(\tau)$ take values in $D/2D$. The 2^8 elements are naturally grouped in 6 classes with multiplicities 1, 56, 140, 1, 56 and 2 depending on the corresponding theta function $\Theta_{2D_8,\mu}(\tau)$. We choose as representative for each class

$$\begin{aligned}
\boldsymbol{d}_0 &= \mathbf{0},\\
\boldsymbol{d}_1 &= (1,0,0,0,0,0,0,0),\\
\boldsymbol{d}_2 &= (1,0,1,0,0,0,0,0),\\
\boldsymbol{d}_3 &= (0,0,0,0,0,0,1,1),\\
\boldsymbol{d}_4 &= (1,0,1,0,1,0,0,0),\\
\boldsymbol{d}_5 &= (1,0,1,0,1,0,1,0).
\end{aligned}$$

Elements $\boldsymbol{\mu} \in \boldsymbol{g}_i + D/2D$ fall similarly in conjugacy classes corresponding to their theta functions. We let $m_{i,j}$ denote the number of elements in the class represented by $\boldsymbol{g}_i + \boldsymbol{d}_j$. The non-vanishing $m_{i,j}$ are given in table 5.1. The corresponding theta functions are given by

$m_{i,j}$	0	1	2	3	4	5
0	1	56	140	1	56	2
1	128			128		
2	16	112	112		16	
3	128			128		

Table 5.1: The number of elements $m_{i,j}$ in $\boldsymbol{g}_i + D/2D$ with equal theta functions $\Theta_{2D_8,\boldsymbol{g}_i+\boldsymbol{d}_j}(\tau)$.

$$\begin{aligned}
\Theta_{2D_8,\boldsymbol{d}_0}(\tau) &= \tfrac{1}{2}\left(\theta_3(2\tau)^8 + \theta_4(2\tau)^8\right),\\
\Theta_{2D_8,\boldsymbol{d}_1}(\tau) &= \tfrac{1}{16}\left(\theta_3(\tau)^8 - \theta_4(\tau)^8\right) - \tfrac{1}{2}\theta_2(2\tau)^6\theta_3(2\tau)^2,\\
\Theta_{2D_8,\boldsymbol{d}_2}(\tau) &= \tfrac{1}{32}\theta_2(\tau)^8,\\
\Theta_{2D_8,\boldsymbol{d}_3}(\tau) &= \tfrac{1}{2}\left(\theta_3(2\tau)^8 - \theta_4(2\tau)^8\right),\\
\Theta_{2D_8,\boldsymbol{d}_4}(\tau) &= \tfrac{1}{2}\theta_2(2\tau)^6\theta_3(2\tau)^2,\\
\Theta_{2D_8,\boldsymbol{d}_5}(\tau) &= \tfrac{1}{2}\theta_2(2\tau)^8,
\end{aligned} \qquad (5.109)$$

For $\boldsymbol{g}_1$

$$\begin{aligned}\Theta_{2D_8,\boldsymbol{g}_1}(\tau) &= \tfrac{1}{8}\theta_2(\tau)^4\left(\theta_3(2\tau)^4-\tfrac{1}{2}\theta_4(2\tau)^4\right),\\ \Theta_{2D_8,\boldsymbol{g}_1+\boldsymbol{d}_3}(\tau) &= \Theta_{2D_8,\boldsymbol{d}_2}(\tau),\end{aligned}$$

for $\boldsymbol{g}_2$

$$\begin{aligned}\Theta_{2D_8,\boldsymbol{g}_2}(\tau) &= \tfrac{1}{4}\theta_2(\tau)^2\,\theta_3(2\tau)^6,\\ \Theta_{2D_8,\boldsymbol{g}_2+\boldsymbol{d}_1}(\tau) &= \tfrac{1}{16}\theta_2(\tau)^6\,\theta_3(2\tau)^2,\\ \Theta_{2D_8,\boldsymbol{g}_2+\boldsymbol{d}_2}(\tau) &= \tfrac{1}{16}\theta_2(\tau)^6\left(\theta_3(2\tau)^2-\theta_4(\tau)^2\right),\\ \Theta_{2D_8,\boldsymbol{g}_2+\boldsymbol{d}_4}(\tau) &= \tfrac{1}{4}\theta_2(2\tau)^6\theta_2(\tau)^2,\end{aligned} \tag{5.110}$$

and for $\boldsymbol{g}_3$:

$$\begin{aligned}\Theta_{2D_8,\boldsymbol{g}_3}(\tau) &= \Theta_{2D_8,\boldsymbol{g}_1}(\tau),\\ \Theta_{2D_8,\boldsymbol{g}_3+\boldsymbol{d}_3}(\tau) &= \Theta_{2D_8,\boldsymbol{g}_1+\boldsymbol{d}_1}(\tau).\end{aligned}$$

The third basis is basis **E** corresponding to the representation of $\Lambda_{1,9}$ as the direct sum of the two lattices B and E, whose basis vectors $\mathbf{b}_i$ and $\mathbf{e}_i$ are

$$\begin{aligned}&\mathbf{b}_1=-K_9, \qquad \mathbf{b}_2=\mathbf{c}_9,\\ &\mathbf{e}_i=\mathbf{c}_i-\mathbf{c}_{i+1}, \qquad 1\le i\le 7,\\ &\mathbf{e}_8=-H+\mathbf{c}_6+\mathbf{c}_7+\mathbf{c}_8.\end{aligned} \tag{5.111}$$

The element H of basis **C** is in terms of this basis: $H=(3,3,3,6,9,12,15,10,5,2)$. The intersection numbers for $\mathbf{b}_i$ are $\mathbf{b}_1^2=0$, $\mathbf{b}_2^2=-1$ and $\mathbf{b}_1\cdot\mathbf{b}_2=1$. The quadratic form Q_E for E is minus the E_8 Cartan matrix, which is given by

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1\\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \tag{5.112}$$

The 256 elements in $E/2E$ fall in 3 inequivalent Weil orbits with vectors of length 0, 2 and 4, which have multiplicities $m_0=1$, $m_1=120$ and $m_2=135$ respectively. We choose as representatives

$$\begin{aligned}\boldsymbol{e}_0 &= \mathbf{0},\\ \boldsymbol{e}_1 &= (1,0,0,0,0,0,0,0),\\ \boldsymbol{e}_2 &= (1,0,1,0,0,0,0,0).\end{aligned}$$

The corresponding theta functions $\Theta_{rE_8,\boldsymbol{e}_0}$ are for $r = 1, 2$:

$$\begin{aligned}
\Theta_{E_8,\boldsymbol{e}_0}(\tau) &= E_4(\tau),\\
\Theta_{2E_8,\boldsymbol{e}_0}(\tau) &= E_4(2\tau),\\
\Theta_{2E_8,\boldsymbol{e}_1}(\tau) &= \frac{1}{240}\left(E_4(\tau/2) - E_4(\tau/2+1/2)\right),\\
\Theta_{2E_8,\boldsymbol{e}_2}(\tau) &= \frac{1}{15}\left(E_4(\tau) - E_4(2\tau)\right).
\end{aligned}$$

5.6.2 BPS invariants for $r \leq 3$

Rank 1
The results from the periods for $f^{(1),\mathrm{pr}}_{C,\mathbf{0}}(\tau)$ is (5.99)

$$f^{(1),\mathrm{pr}}_{C,\mathbf{0}}(\tau) = \frac{E_4(\tau)}{\eta(\tau)^{12}}. \tag{5.113}$$

This can easily be verified with the results for sheaves on surfaces. The result for $r = 1$ and a complex, simply connected surface S is [112]

$$f^{(1)}_{c_1}(z,\tau;S) = \frac{i}{\vartheta_1(2z,\tau)\,\eta(\tau)^{b_2(S)-1}}. \tag{5.114}$$

The dependence on J can be omitted for $r = 1$ since all rank 1 sheaves are stable. If we specialize to $S = dP_9$, take the limit $w \to -1$, and sum over all $c_1 \in E = H^2(dP_9,\mathbb{Z})/\iota^* H^2(M,\mathbb{Z})$ one obtains

$$f^{(1)}_{C,\mathbf{0}}(\tau) = \frac{E_4(\tau)}{\eta(\tau)^{12}} \tag{5.115}$$

in agreement with Eq. (5.113).

Rank 2
The prediction by the periods for $r = 2$ is given by $f^{(2),\mathrm{pr}}_{C,\mathbf{0}}(\tau)$ in (5.99). This is a sum over all BPS invariants for $c_1 \cdot \mathbf{a}_i = 0$, $i = 1, 2$. In order to verify this result, it is useful to decompose $f^{(2),\mathrm{pr}}_{C,\mathbf{0}}(\tau)$ according to the three conjugacy classes of $E/2E \in H_2(dP_9)/(\iota^* H_2(M,\mathbb{Z}))$. One obtains

$$f^{\mathrm{pr}}_{C,\mathbf{0}}(\tau) = \sum_{i=0}^{2} m_i\, f^{(2),\mathrm{pr}}_{\boldsymbol{e}_i,\mathbf{a}_1}(\tau)\,\Theta_{2E_8,\boldsymbol{e}_i}(\tau) \tag{5.116}$$

with [139]:

$$\begin{aligned}
f^{(2),\mathrm{pr}}_{\boldsymbol{e}_0,\mathbf{a}_1}(\tau) &= \frac{1}{24\,\eta(\tau)^{24}}\left[E_2(\tau)\,\Theta_{2E_8,\boldsymbol{e}_0}(\tau) + \left(\vartheta_3(\tau)^4\vartheta_4(\tau)^4 - \tfrac{1}{8}\vartheta_2(\tau)^8\right)\left(\vartheta_3(\tau)^4 + \vartheta_4(\tau)^4\right)\right],\\
&\quad + \frac{1}{8} h^{\mathrm{pr}}_{1,\boldsymbol{e}_0}(2\tau),\\
f^{(2),\mathrm{pr}}_{\boldsymbol{e}_1,\mathbf{a}_1}(\tau) &= \frac{1}{24\,\eta(\tau)^{24}}\left[E_2(\tau)\,\Theta_{2E_8,\boldsymbol{e}_1}(\tau) - \tfrac{1}{8}E_4(\tau)\,\vartheta_2(\tau)^4\right],\\
f^{(2),\mathrm{pr}}_{\boldsymbol{e}_2,\mathbf{a}_1}(\tau) &= \frac{1}{24\,\eta(\tau)^{24}}\left[E_2(\tau)\,\Theta_{2E_8,\boldsymbol{e}_2}(\tau) - \tfrac{1}{8}\vartheta_2(\tau)^8\left(\vartheta_3(\tau)^4 + \vartheta_4(\tau)^4\right)\right].
\end{aligned} \tag{5.117}$$

Verification of the expressions (5.117) is much more elaborate then for $r = 1$. We will use the approach of [142, 143, 315]. The main issues are the determination of the BPS invariants for a polarisation close to the class $\mathbf{a}_2$ (a suitable polarisation) and wall-crossing from the suitable polarisation to

$$J = -K_9 = \mathbf{a}_1. \tag{5.118}$$

These issues are dealt with for the Hirzebruch surfaces [142, 143], and for dP_9 in [315]. The main difficulty for dP_9 compared to the Hirzebruch surfaces is that that the class f and K_9 span the lattice A, which is related to $\Lambda_{1,9}$ by a non-trivial gluing with the lattice D. Before turning to the explicit expressions, we briefly outline the computation; we refer for more details about the used techniques to [301]. The polarisation J is parametrized by

$$J_{m,n} = m\,\mathbf{a}_1 + n\,\mathbf{a}_2. \tag{5.119}$$

In order to determine the BPS invariants for the suitable polarisation $J_{\varepsilon,1}$, view dP_9 as the 8-point blow-up of the Hirzebruch surface $\mathbb{F}_1$: $\phi : dP_9 \to \mathbb{F}_1$. We choose to perform this blow-up for the polarisation $J_{\mathbb{F}_1} = f$, with f the fibre class of the Hirzebruch surface. The pull back of this class to dP_9 is $\phi^* f = J_{0,1}$. The generating function of the BPS invariants for this choice takes a relatively simple form: it either vanishes or equals a product of eta and theta functions [143, 301] depending on the Chern classes. This function represents the sheaves whose restriction to the rational curve $\mathbf{a}_2$ is semi-stable. The generating function $f^{(r)}_{c_1,J_{0,1}}(\tau, z)$ is therefore this product formula multiplied by the factors due to blowing-up the 8 points. To obtain the BPS invariants from this function, one has to change $J_{0,1}$ to $J_{\varepsilon,1}$ and subtract the contribution due to sheaves which became (Gieseker) unstable due to this change [301]. Consequently, we can determine the BPS invariants for any other choice of J by the wall-crossing formula [126, 144, 285]. In particular, we determine the invariants for $J_{1,0} = -K_9$ and change to the basis $\mathbf{E}$ in order to compare with the expression from the periods. We continue with determining the BPS invariants for $J = J_{0,1}$. The BPS invariants vanish for $c_1 \cdot \mathbf{a}_2 = 1 \mod 2$

$$f^{(2)}_{c_1,J_{\varepsilon,1}}(z,\tau) = 0, \qquad c_1 \cdot \mathbf{a}_2 = 1 \mod 2. \tag{5.120}$$

Since BPS invariants depend on $c_1 \mod 2\Lambda_{1,9}$, we distinguish further $c_1 \cdot \mathbf{a}_2 = 0 \mod 4$ and $c_1 \cdot \mathbf{a}_2 = 2 \mod 4$. For these cases, we continue as in [301] using the (extended) HN filtration. A sheaf F which is unstable for $J_{\varepsilon,1}$ but semi-stable for $J_{0,1}$, can be described as a HN-filtration of length 2 whose quotients we denote by E_i, $i = 1, 2$. If we parametrize the first Chern class of E_2 by $\mathbf{k} = (\mathbf{k}_A, \mathbf{k}_D)$, then the discriminant $\Delta(F)$ is given by

$$2\Delta(F) = \Delta(E_1) + \Delta(E_2) - \frac{1}{4}(2\mathbf{k}_A - c_1|_A)^2 - \frac{1}{4}(2\mathbf{k}_D - c_1|_D)^2. \tag{5.121}$$

The choice of $\mathbf{k}_D$ does not have any effect on the stability of F as long as J is spanned by $J_{0,1}$ and $J_{1,0}$. Therefore (5.121) shows that the sum over $\mathbf{k}_D$ gives rise to the theta functions $\Theta_{2D_8,\mu}(\tau)$. The condition for semi-stability for $J_{0,1}$ but unstable for $J_{\varepsilon,1}$ implies $(c_1(E_1) - c_1(E_2)) \cdot \mathbf{a}_2 = 0$. This combined with $c_1 \cdot \mathbf{a}_2 = 0 \mod 4$ gives for $c_1(E_i) = 0 \mod 2$, which shows that $c_1(E_i) = \boldsymbol{g}_j \mod 2\Lambda_{1,9}$ only for

$j = 0, 2$. One obtains after a detailed analysis for $c_1 \cdot \mathbf{a}_2 = 0 \mod 4$

$$\begin{aligned} f^{(2)}_{c_1,J_{\varepsilon,1}}(z,\tau) &= \frac{-i\,\eta(\tau)}{\vartheta_1(2z,\tau)^2\,\vartheta_1(4z,\tau)} \prod_{i=1}^{8} B_{2,\ell_i}(z,\tau) \\ &+ \left(\frac{w^{4\{(\frac{1}{2}\boldsymbol{g}_0 - \frac{1}{4}c_1)\cdot \mathbf{a}_1\}}}{1-w^4} - \frac{1}{2}\delta_{0,\{(\frac{1}{2}\boldsymbol{g}_0-\frac{1}{4}c_1)\cdot\mathbf{a}_1\}} \right) \Theta_{2D_8,c_1-2\boldsymbol{g}_0}(\tau)\, f^{(1)}_{\mathbf{0}}(z,\tau)^2 \\ &+ \left(\frac{w^{4\{(\frac{1}{2}\boldsymbol{g}_2 - \frac{1}{4}c_1)\cdot \mathbf{a}_1\}}}{1-w^4} - \frac{1}{2}\delta_{0,\{(\frac{1}{2}\boldsymbol{g}_2-\frac{1}{4}c_1)\cdot\mathbf{a}_1\}} \right) \Theta_{2D_8,c_1-2\boldsymbol{g}_2}(\tau)\, f^{(1)}_{\mathbf{0}}(z,\tau)^2, \end{aligned} \tag{5.122}$$

where $\{\lambda\} = \lambda - \lfloor\lambda\rfloor$ and $\ell_i = c_1 \cdot \mathbf{c}_i$. The right hand side on the first line correspond to the sheaves whose restriction to $\mathbf{a}_2$ are semi-stable. The functions

$$B_{2,\ell}(z,\tau) = \sum_{n\in\mathbb{Z}+\ell/2} q^{n^2} w^n / \eta(\tau)^2 \tag{5.123}$$

are due to the blow-up formula [136, 144, 301, 313]. The second and third line are the subtractions due to sheaves which are unstable for $J_{\varepsilon,1}$.

Similarly one obtains for $c_1 \cdot \mathbf{a}_2 = 2 \mod 4$

$$\begin{aligned} f^{(2)}_{c_1,J_{\varepsilon,1}}(z,\tau) &= \frac{-i\,\eta(\tau)}{\vartheta_1(2z,\tau)^2\,\vartheta_1(4z,\tau)} \prod_{i=1}^{8} B_{2,\ell_i}(z,\tau) \\ &+ \left(\frac{w^{4\{(\frac{1}{2}\boldsymbol{g}_1 - \frac{1}{4}c_1)\cdot \mathbf{a}_1\}}}{1-w^4} - \frac{1}{2}\delta_{0,\{(\frac{1}{2}\boldsymbol{g}_1-\frac{1}{4}c_1)\cdot\mathbf{a}_1\}} \right) \Theta_{2D_8,c_1-2\boldsymbol{g}_1}(\tau)\, f^{(1)}_{\mathbf{0}}(z,\tau)^2 \\ &+ \left(\frac{w^{4\{(\frac{1}{2}\boldsymbol{g}_3 - \frac{1}{4}c_1)\cdot \mathbf{a}_1\}}}{1-w^4} - \frac{1}{2}\delta_{0,\{(\frac{1}{2}\boldsymbol{g}_3-\frac{1}{4}c_1)\cdot\mathbf{a}_1\}} \right) \Theta_{2D_8,c_1-2\boldsymbol{g}_3}(\tau)\, f^{(1)}_{\mathbf{0}}(z,\tau)^2. \end{aligned} \tag{5.124}$$

What remains is to change the polarisation J from $J_{\varepsilon,1}$ to $J_{1,0}$ and determine the change of the invariants using wall-crossing formulas. For $J = (m, n, \mathbf{0}) \in A \oplus D$, we obtain the following expression

$$\begin{aligned} f^{(2)}_{c_1,J_{m,n}}(z,\tau) &= \frac{-i\,\eta(\tau)}{\vartheta_1(2z,\tau)^2\,\vartheta_1(4z,\tau)} \prod_{i=1}^{8} B_{2,\ell_i}(z,\tau) \\ &+ \sum_{j=0,\dots,3} f^{(2),A}_{c_1-2\boldsymbol{g}_j,;J_{m,n}}(z,\tau)\,\Theta_{2D,c_1-2\boldsymbol{g}_j}(\tau), \end{aligned} \tag{5.125}$$

with

$$\begin{aligned} f^{(2),A}_{c_1,J_{m,n}}(z,\tau) &= f^{(2),A}_{c_1,J_{\varepsilon,1}}(z,\tau) + \tfrac{1}{2} \sum_{(a_1,a_2)\in A+c_1} \tfrac{1}{2}\left(\operatorname{sgn}(a_1 n + a_2 m) - \operatorname{sgn}(a_1 + a_2\varepsilon)\right) \\ &\times \left(w^{4a_2} - w^{-4a_2}\right) q^{-4a_1a_2} f^{(1)}_{\mathbf{0}}(z,\tau)^2. \end{aligned}$$

The functions $f^{(2),A}_{c_1,J_{\varepsilon,1}}(z,\tau)$ are rational functions in w multiplied by $f^{(1)}_{\mathbf{0}}(z,\tau)^2$ which can easily be read off from Eq. (5.122). For $J = J_{1,0}$ the functions can be expressed in terms of modular functions.

Table 5.2 presents the BPS invariants for $J = J_{1,0}$. As expected, the Euler numbers are indeed in agreement with the predictions (5.117). One can also verify that for increasing c_2, the Betti numbers asymptote to those of $r = 1$ or equivalently the Hilbert scheme of points of dP_9.

c_1	c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	χ
$\boldsymbol{e}_0$	2	1	10	55							132
	3	1	11	76	396	1356					3680
	4	1	11	78	428	1969	7449	20124			60120
	5	1	11	78	430	2012	8316	30506	95498	221132	715968
$\boldsymbol{e}_1$	1	1	9								20
	2	1	11	75	309						792
	3	1	11	78	426	1843	5525				15768
	4	1	11	78	430	2010	8150	27777	68967		214848
$\boldsymbol{e}_2$	1	1									2
	2	1	11	60							144
	3	1	11	78	404	1386					3760
	4	1	11	78	430	1981	7495	20244			60480

Table 5.2: The Betti numbers b_n (with $n \leq \dim_{\mathbb{C}} \mathcal{M}$) and Euler numbers χ of the moduli spaces of semi-stable sheaves on dP_9 with $r = 2$, $c_1 = \boldsymbol{e}_i$, and $1 \leq c_2 \leq 4$ for $J = J_{1,0}$.

We define the functions $f_{c_1}^{(2),A}(z,\tau) := f_{c_1,J_{1,0}}^{(2),A}(z,\tau)$, which only depend on $c_1|_A = \alpha_1\,\mathbf{a}_1 + \alpha_2\,\mathbf{a}_2$ with $\alpha_1, \alpha_2 \in 0, \frac{1}{2}, 1, \frac{3}{2}$. One finds for $\alpha_2 = 0 \mod 4$

$$\frac{f_{c_1}^{(2),A}(z,\tau)}{f_0^1(z,\tau)^2} = -\frac{1}{8}\frac{1}{2\pi i}\frac{\partial}{\partial z}\ln\left(\vartheta_1(4\tau, 4z+2\alpha_1)\,\vartheta_1(4\tau, 4z-2\alpha_1)\right), \tag{5.126}$$

and for $\alpha_2 \neq 0 \mod 4$

$$\frac{f_{c_1}^{(2),A}(z,\tau)}{f_0^{(1)}(z,\tau)^2} = \frac{i}{2}\frac{q^{-\alpha_1\alpha_2}\,\eta(4\tau)^3}{\vartheta_1(4\tau,2\alpha_2\tau)}\left(\frac{w^{-2\alpha_2}\,\vartheta_1(4\tau,4z+2(\alpha_1-\alpha_2)\tau)}{\vartheta_1(4\tau,4z+2\alpha_1\tau)} - \frac{w^{2\alpha_2}\,\vartheta_1(4\tau,-4z+2(\alpha_1-\alpha_2)\tau)}{\vartheta_1(4\tau,4z+2\alpha_1\tau)}\right). \tag{5.127}$$

To prove the agreement of the Euler numbers with the periods, we specialise to $w = -1$. Let

$$D_k = \frac{1}{2\pi i}\frac{\partial}{\partial\tau} - \frac{k}{12}E_2(\tau) \tag{5.128}$$

be the differential operator which maps weight k modular forms to modular forms of weight $k+2$. Then one can write $f_{c_1,J_{1,0}}^{(2)}(\tau)$ as

$$\begin{aligned} f_{c_1,J_{1,0}}^{(2)}(\tau) \;=\; & \frac{1}{\eta(\tau)^{24}}\left(\frac{1}{2}\delta_{c_1\cdot\mathbf{a}_2,0}D_4(\vartheta_3(2\tau)^m\vartheta_2(2\tau)^{8-m})\right. \\ & \left. + \sum_{i=0,\ldots,3} f_{c_1-2\boldsymbol{g}_j}^{A}(\tau)\,\Theta_{2D,c_1-2\boldsymbol{g}_j}(\tau)\right), \end{aligned} \tag{5.129}$$

with

$$\begin{aligned}
f^A_{0,0}(\tau) &= \tfrac{1}{8}\vartheta_3(2\tau)^4 + \tfrac{1}{24}E_2(\tau),\\
f^A_{-\frac12,\frac12}(\tau) &= \tfrac{1}{2}\vartheta_2(2\tau)\,\vartheta_3(2\tau)^3,\\
f^A_{\frac12,\frac12}(\tau) &= \tfrac{1}{2}\vartheta_2(2\tau)^3\,\vartheta_3(2\tau),\\
f^A_{1,0}(\tau) &= \tfrac{1}{12}\vartheta_2(2\tau)^4 - \tfrac{1}{24}\vartheta_3(2\tau)^4 + \tfrac{1}{24}E_2(\tau),\\
f^A_{0,1}(\tau) &= \tfrac{1}{24}\vartheta_2(2\tau)^4 - \tfrac{1}{12}\vartheta_3(2\tau)^4,\\
f^A_{1,1}(\tau) &= -\tfrac{1}{8}\vartheta_2(2\tau)^4.
\end{aligned}$$

If $c_1|_B = 0$, this reproduces the functions in [139, 315] depending on whether the classes in lattice E are even or odd.

Modularity

Electric-magnetic duality of $\mathcal{N} = 4$ $U(r)$ SYM theory implies modular properties for its partition function [140]. Determination of the modular properties gives therefore insight about the realisation of electric-magnetic duality at the quantum level.

The expression in equation (5.125) does not transform as a modular form for generic choices of J. However, using the theory of indefinite theta functions [146], the functions can be completed to a function $\widehat{f}^{(2)}_{c_1,J}(z,\tau)$ by addition of a non-holomorphic term, such that $\widehat{f}^{(2)}_{c_1,J}(z,\tau)$ does transform as a modular form [300]. Interestingly, equation (5.129) shows that $f^{(2)}_{c_1,J}(z,\tau)$ becomes a quasi-modular form for

$$\lim_{J\to J_{1,0}} \widehat{f}^{(2)}_{c_1,J}(z,\tau), \tag{5.130}$$

i.e. it can be expressed in terms of modular forms and Eisenstein series of weight 2. In some cases it becomes even a true modular form. This is due to the special form of Q_A.

The transition from mock modular to quasi-modular can be made precise. Due to the gluing vectors, the function

$$f_{2,c_1}(z,\tau;J) = f^{(2)}_{c_1,J}(z,\tau)/f^{(1)}_{c_1}(z,\tau)^2 \tag{5.131}$$

takes the form

$$\begin{aligned}
f_{2,c_1}(z,\tau;J_{m,n}) \;=\; &\sum_{\mu} f^{(2),A}_{(c_1-2\mu)_A,J_{m,n}}(z,\tau)\,\Theta_{2D,(2\mu-c_1)_D}(\tau)\\
&+\delta_{c_1\cdot\mathbf{a}_2,0}\,\frac{i\,\eta(\tau)^3}{\vartheta_1(\tau,4z)}\vartheta_3(2\tau,2z)^k\vartheta_2(2\tau,2z)^{8-k},
\end{aligned} \tag{5.132}$$

where k is the number of $c_1\cdot\mathbf{c}_i = 1 \mod 2$ for $1\le i\le 8$. The completed generating function $\widehat{f_{2,c_1}}(z,\tau;J)$ is a slight generalisation of the equations in Section 3.2 in Ref [300]:

$$\begin{aligned}
\widehat{f_{2,c_1}}(\tau;J) = f_{2,c_1}(\tau;J) + \\
\sum_{\substack{\mathbf{c}\in -c_1\\ +H^2(\Sigma_9,2\mathbb{Z})}} \left(\frac{K_9\cdot J}{4\pi\sqrt{J^2\,\tau_2}}\,e^{-\pi\tau_2\mathbf{c}_+^2} - \frac{1}{4}K_9\cdot\mathbf{c}\,\mathrm{sgn}(\mathbf{c}\cdot J)\,\beta_{\frac12}(\mathbf{c}_+^2\,\tau_2)\right)(-1)^{K_9\cdot\mathbf{c}}q^{-\mathbf{c}^2/4},
\end{aligned} \tag{5.133}$$

where $\tau_2 = \mathrm{Im}(\tau)$ and $\beta_\nu(x) = \int_x^\infty u^{-\nu}e^{-\pi u}du$. We parametrize J by $b_1 + v\,b_2$ and carefully study the limit $v\to 0$ (this corresponds to the limit $R\to\infty$ in [139]). In this limit, J approaches $-K_9$. Moreover, $J\cdot K_9 = -v$ and $J^2 = v(2-v)$. If one parametrizes $\mathbf{c}$ by $(n_0, n_1, \mathbf{c}_\perp)$, only terms with $n_1 = 0$ contribute to the sum in the limit $v\to 0$. Therefore the term with $\beta_{\frac12}(\mathbf{c}_+^2\,\tau_2)$ does not contribute to the anomaly. After a Poisson resummation on n_0, one finds that the limit is finite and given by

$$\widehat{f_{2,c_1}}(\tau; J_{1,0}) = f_{2,c_1}(\tau; J_{1,0}) - \frac{\delta_{c_1\cdot \mathbf{a}_1,0}}{8\pi\,\tau_2} \sum_{\substack{\mathbf{c}\in -c_1|_E\\ +2E}} q^{-\mathbf{c}_\perp^2/4}. \tag{5.134}$$

This is in good agreement with equation (5.117) if $c_1 \cdot \mathbf{a}_1 = 0$. The lattice sum over $\mathbf{c}$ gives precisely the theta functions $\Theta_{2E_8,\mathbf{e}_i}(\tau)$. Recalling the modular completion of the weight 2 Eisenstein series: $\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}$, we see that the non-holomorphic term implies that in the holomorphic part of $\widehat{f_{2,c_1}}(\tau; J_{1,0})$, $E_2(\tau)$ is multiplied by the $\Theta_{2E_8,\mathbf{e}_i}(\tau)/24$ as in equation (5.117). We have thus verified that the non-holomorphic dependence of D4 brane partition functions is indeed consistent with (5.117) and therefore with (5.15) for topological strings as implied by T-duality. Note that for $c_1\cdot\mathbf{a}_1 = 1 \mod 2$, the non-holomorphic dependence of $f_{2,c_1}(\tau; J)$ vanishes in the limit $J \to J_{1,0}$, in agreement with (5.129).

Rank 3

Similarly as for $r = 2$, Ref. [139] also decomposes $f^{(3),\mathrm{pr}}_{C,\mathbf{0}}(\tau)$ into different Weyl orbits. We will restrict in the following to the $\boldsymbol{e}_0 = \mathbf{0}$ orbit in $E/3E$ since the expressions become rather lengthy. In order to present $f^{(3),\mathrm{pr}}_{\boldsymbol{e}_0,\mathbf{a}_1}(\tau)$, define

$$b_{3,\ell}(\tau) = \sum_{m,n\in\mathbb{Z}+\ell/3} q^{m^2+n^2+mn}. \tag{5.135}$$

Then $f^{(3),\mathrm{pr}}_{\boldsymbol{e}_0,\mathbf{a}_1}(\tau)$ is given by [139]

$$\begin{aligned} f^{(3),\mathrm{pr}}_{\boldsymbol{e}_0,\mathbf{a}_1}(\tau) \;=\; & \frac{1}{2592\,\eta^{36}}\Big[\big(51\,b_{3,0}^{12} - 184\,b_{3,0}^9 b_{3,1}^3 + 336\,b_{3,0}^6 b_{3,1}^6 + 288\,b_{3,0}^3 b_{3,1}^9 + 32\,b_{3,1}^{12}\big) \\ & + E_2 b_{3,0}\big(36\,b_{3,0}^9 - 112\,b_{3,0}^6 b_{3,1}^3 + 32\,b_{3,0}^6 b_{3,1}^3 - 64\,b_{3,1}^9\big) \\ & + E_2^2 b_{3,0}^2\big(9\,b_{3,0}^6 - 16\,b_{3,0}^3 b_{3,1}^3 + 16\,b_{3,0}^6\big)\Big]. \end{aligned} \tag{5.136}$$

In order to verify this expression, we extend the analysis for $r = 2$ to $r = 3$. For $c_1 \cdot \mathbf{a}_2 = \pm 1 \mod 3$ the BPS invariants vanish for a suitable polarization

$$f^{(3)}_{c_1,J_{\varepsilon,1}}(z,\tau) = 0. \tag{5.137}$$

The HN-filtrations for the sheaves which are unstable for $J_{\varepsilon,1}$ but semi-stable for $J_{0,1}$ have length 2 or 3. From those of length 2, one obtains rational functions in w multiplied by $f^{(1)}_0(z,\tau)\, f^{(2)}_{\boldsymbol{\mu}}(z,\tau)\, \Theta_{2D_8,\boldsymbol{\mu}}(\tau)$, with $\boldsymbol{\mu} = \mathbf{0}$, $\mathbf{a}_2$, $\boldsymbol{d}_i$ and $\boldsymbol{d}_i + \mathbf{a}_2$. The theta function arising from the sum over the D_8 lattice is more involved for filtrations of length 3. Instead of a direct sum, a "twisted" sum of 2 D_8-lattices appears; we will denote this lattice by D_8^{t}

$$\Theta_{2D_8^{\mathrm{t}};\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(\tau) = \sum_{\mathbf{k}_i\in D_8+\boldsymbol{\mu}_i, i=1,2} q^{\mathbf{k}_1^2+\mathbf{k}_1\cdot\mathbf{k}_2+\mathbf{k}_2^2} \tag{5.138}$$

$$= \sum_i m_i\, \Theta_{2D_8,\boldsymbol{\mu}_1+\boldsymbol{\mu}_2+\boldsymbol{d}_i}(\tau)\, \Theta_{2D_8,\boldsymbol{\mu}_1-\boldsymbol{\mu}_2+\boldsymbol{d}_i}(3\tau) \tag{5.139}$$

where m_i are the multiplicities of the theta characteristics $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{d}_i$, thus for $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 \in D$, $i = 1, \ldots, 6$, and for $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 \in D/2$, $i = 1, \ldots, 4$. For numerical computations the second line is considerably faster than the first line. We obtain after a careful analysis

$$\begin{aligned} f^{(3)}_{0,J_{\varepsilon,1}}(z,\tau) \;=\;& \frac{i\eta(\tau)^3}{\vartheta_1(2z,\tau)^2\,\vartheta_1(4z,\tau)^2\,\vartheta_1(6z,\tau)}\,B_{3,0}(z,\tau)^8 \\ &+2\left(\frac{1}{1-w^{12}}-\frac{1}{2}\right) f^{(1)}_0(z,\tau)\sum_{i=0,3} f^{(2)}_{(0,0,\boldsymbol{d}_i),J_{\varepsilon,1}}(z,\tau)\Theta_{2D_8,\boldsymbol{d}_i}(3\tau) \\ &+2\left(\frac{w^6}{1-w^{12}}\right) f^{(1)}_0(z,\tau)\sum_{i=0,3} f^{(2)}_{(0,1,\boldsymbol{d}_i),J_{\varepsilon,1}}(z,\tau)\Theta_{2D_8,\boldsymbol{d}_i}(3\tau) \\ &+2\left(\frac{1}{1-w^{6}}-\frac{1}{2}\right) f^{(1)}_0(z,\tau)\sum_{i=1,2,4,5} m_{0,i}\, f^{(2)}_{(0,0,\boldsymbol{d}_i),J_{\varepsilon,1}}(z,\tau)\Theta_{2D_8,\boldsymbol{d}_i}(3\tau) \\ &+2\left(\frac{w^3}{1-w^{6}}\right) f^{(1)}_0(z,\tau)\sum_{i=0,1,2,4} m_{2,i}\, f^{(2)}_{\boldsymbol{g}_2+\boldsymbol{d}_i,J_{\varepsilon,1}}(z,\tau)\Theta_{2D_8,\boldsymbol{g}_2+\boldsymbol{d}_i}(3\tau) \\ &-\left(\frac{1+w^{12}}{(1-w^8)(1-w^{12})}-\frac{1}{1-w^{12}}+\frac{1}{6}\right) f^{(1)}_0(z,\tau)^3\,\Theta_{2D_8^{\mathrm{t}};0,0}(\tau) \\ &-2\left(\frac{w^6}{(1-w^4)(1-w^{12})}-\frac{w^6}{1-w^{12}}\right) f^{(1)}_0(z,\tau)^3\,\Theta_{2D_8^{\mathrm{t}};\boldsymbol{g}_2,0}(\tau) \\ &-\left(\frac{w^4+w^{16}}{(1-w^8)(1-w^{12})}\right) f^{(1)}_0(z,\tau)^3\,\Theta_{2D_8^{\mathrm{t}};0,0}(\tau). \end{aligned} \tag{5.140}$$

The functions due to the blowing-up of 8 points are now given by

$$B_{3,k}(z,\tau)=\sum_{m,n\in\mathbb{Z}+k/3}\frac{q^{m^2+n^2+mn}w^{4m+2n}}{\eta(\tau)^3}. \tag{5.141}$$

We have used in (5.140) that $f^{(2)}_{c_1,J_{m,n}}(z,\tau)$ only depends on the conjugacy class of c_1 in $D/2D$, and moreover that

$$f^{(2)}_{c_1,J_{m,n}}(z,\tau)=f^{(2)}_{c_1',J_{m,n}}(z,\tau) \tag{5.142}$$

if $c_1=(0,0,\boldsymbol{d}_i)$ and $c_1'=(0,1,\boldsymbol{d}_i)$ for $i=1,2,4,5$ (but not for $i=0,3$) and $c_1=(0,0,\boldsymbol{d}_i)+\boldsymbol{g}_2$ and $c_1'=(0,1,\boldsymbol{d}_i)+\boldsymbol{g}_2$.

Having determined $f^{(3)}_{0,J_{\varepsilon,1}}(z,\tau)$, what rests is to perform the wall-crossing from $J_{\varepsilon,1}$ to $J_{1,0}$. To this end we define

$$\begin{aligned} f^{(3);A}_{c_1,J}(z,\tau) \;=\;& \sum_{\mathbf{a}=c_1|_A \mod 2A} \tfrac{1}{2}\left(\mathrm{sgn}(a_1n+a_2m)-\mathrm{sgn}(a_1+a_2\varepsilon)\right) \\ & \left(w^{6a_2}-w^{-6a_2}\right) q^{-3a_1a_2}\, f^{(2)}_{(\mathbf{a},c_1|_D),J_{|a_1|,|a_2|}}(z,\tau)\, f^{(1)}_0(z,\tau), \end{aligned} \tag{5.143}$$

with $\mathbf{a}=(a_1,a_2)$. Then $f^{(3)}_{0,J}(z,\tau)$ is given by [154, 155]

$$f^{(3)}_{0,J}(z,\tau;J)=h^{(3)}_{0,J_{\varepsilon,1}}(z,\tau)+\sum_{\mathbf{a}\in 2A/A} m_{i,j}\, f^{A}_{3,\mathbf{a}+\boldsymbol{g}_i+\boldsymbol{d}_j}(z,\tau;J)\,\Theta_{2D_8,\boldsymbol{g}_i+\boldsymbol{d}_j}(3\tau).$$

The Betti numbers for $J=J_{1,0}$ and small c_2 are presented in Table 5.3, and indeed agree with the Euler numbers computed from the periods.

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	χ
3	1	10	65	320	1025	1226							4068
4	1	11	77	417	1902	7372	23962	57452	68847				251235
5	1	11	78	429	2002	8260	30710	103867	316586	836221	1706023	2029416	8037792

Table 5.3: The Betti numbers b_n (with $n \leq \dim_{\mathbb{C}} \mathcal{M}$) and the Euler number χ of the moduli spaces of semi-stable sheaves on dP_9 with $r = 3$, $c_1 = 0$, and $3 \leq c_2 \leq 5$ for $J = J_{1,\varepsilon}$.

One might wonder how to derive the modular properties $f^{(3)}_{0,J}(z,\tau)$. The completion takes in general a very complicated form due to the quadratic condition on the lattice points [155]. One can show however that for $J = J_{1,0}$ the quadratic condition disappears from the generating function due to a special symmetry of the lattice A, and therefore one again obtains quasi-modular forms at this point.[3]

[3] We thank S. Zwegers for providing this argument.

CHAPTER 6

Conclusions and Outlook

In this thesis we investigated the wall-crossing holomorphic and modular anomaly of generating functions of BPS invariants in string theory. These results are obtained by studying multiple M5 branes that wrap a divisor P inside a Calabi-Yau threefold and using methods from topological string theory, algebraic geometry and the theory of modular forms. In particular it is possible to perform explicit calculations of the generating function for higher rank using stability conditions. For the case of elliptically fibred Calabi-Yau threefolds we established a new holomorphic anomaly equation, which is recursive in the genus and the base within the framework of topological string theory generalising earlier observations by Hosono. By using T-duality a relation to the holomorphic anomaly of D4-D2-D0 BPS invariants is established. This recursive structure allows to determine the generating function for higher rank and genus.

We first investigated r M5 branes on divisors P with $b_2^+ = 1$ and focused on the case $r = 2$. In the type IIA language this corresponds to a system with D4-D2-D0 BPS states. Upon further compactification on $P \times T^2$ the setup can be described in two limits depending on the size of the compactification spaces. For the case that P is small compared to T^2 we obtain a $(0,4)$ CFT description which for the case of $r = 1$ is known to be the MSW CFT. In the case that T^2 is small compared to P the resulting theory corresponds to a topological $\mathcal{N} = 4\,\mathrm{U}(r)$ SYM theory. The main object of our studies is the modified elliptic genus $Z_P^{(r)}$ and its properties. This modified elliptic genus can be decomposed into a vector-valued modular form $f_{\mu,J}^{(r)}(\tau)$, which contains the information about the BPS states, and a Siegel-Narain theta function. We used the Kontsevich-Soibelmann wall-crossing formula to calculate the change in $f_{\mu,J}^{(r)}(\tau)$ when moving from J to J' inside the Kähler cone. Using earlier results of Göttsche, we express this change in terms of an indefinite theta function $\Theta_{\Lambda,\mu}^{J,J'}(\tau, z)$. However, though holomorphy is still guaranteed at this point, modularity is spoiled due to the indefinite theta function and would hence spoil S-duality invariance. It is restored at the cost of holomorphy by regularising the indefinite theta function as prescribed by Zwegers in his work on mock modular forms. The basic idea is to replace the discontinuous sign function by the continuous error function. With this insight we use the language of mock modular forms to prove the holomorphic anomaly for two M5 branes if one sets one of the Kähler parameters to $-[P]$ which corresponds to the attractor point. The holomorphic anomaly can be interpreted in terms of bound states of the M5 branes.

We checked the holomorphic anomaly and calculated the modified elliptic genus explicitly for the divisor being $\mathbb{P}^2, \mathbb{F}_0, \mathbb{F}_1, dP_8$ and $dP_9 = \frac{1}{2}K3$. Other surfaces are obtained from these by the blow up procedure and the blow up formulae for the generating functions. For $r > 2$ one would need the theory of mock modular forms of higher depth to discover a similar connection between wall-crossing and modularity. However, such a theory is not developed and using insights from physics by using wall-crossing techniques helps to construct generating functions. First attempts of this were made in [155]. The lattice of such an indefinite theta function would be of signature $(r-1)(b_2^+, b_2^-)$.

For the case of $dP_9 = \frac{1}{2}\mathrm{K3}$ we obtained the higher rank results by calculating coefficients of the prepotential $F^{(0)}$. In particular in this case the holomorphic anomaly is with respect to quasi-modular forms and the anomaly is captured by the second Eisenstein series E_2. We investigated this anomaly further and generalised it to the case of elliptically fibred Calabi-Yau threefolds.

We provide the possible geometric constructions of elliptically fibred Calabi-Yau manifolds by means of toric geometry and discuss various properties of these geometries. The quantum geometry is explored by means of the A-model of topological string theory. In particular we find a holomorphic anomaly in the topological amplitudes with respect to the base $F_\beta^{(g)}$ which is recursive both in the genus g and in the base β. Furthermore the $F_\beta^{(g)}$ can be expressed in terms of quasi-modular forms. Using B-model techniques we discuss the elliptic fibrations over $\mathbb{P}^2$ and $\mathbb{F}_1$. For the explicit case of the elliptic fibration over $\mathbb{P}^2$ we trace the appearance of modular forms back to the modular subgroup of the monodromy group. For the other fibrations similar results hold. We proof the holomorphic anomaly by using mirror symmetry for the explicit example of the base $\mathbb{F}_1$. Using insights from BCOV on the holomorphic anomaly equations for n-point functions we derive the discovered holomorphic anomaly equation.

The results for elliptically fibred Calabi-Yau threefolds obtained via topological string calculations are related to those of multiple M5 branes via double T-duality/ the Fourier-Mukai transform on the elliptic fibre. We check this explicitly for the case of $dP_9 = \frac{1}{2}\mathrm{K3}$ by calculating the generating functions for $r \leq 3$ by using techniques from the study of stability of sheaves.

The new discovered holomorphic anomaly has been checked in the base degree and for genus zero and one. However, due to a symmetry in the moduli space of the elliptically fibred Calabi-Yau threefolds it should be possible to determine the higher genus contributions as this symmetry allows to fix the holomorphic ambiguity in the determination of the $F^{(g)}$. This would provide an interesting check of our anomaly and furthermore it might be possible to have an integrable model. First higher genus checks were performed in [320].

Though we have been mainly concerned with the obtained results in the field theory limit of our setup, it is also an interesting question if our results can be used to gain new ideas of the MSW CFT for multiple M5 branes. These results could be used to gain a better understanding of multiple M5 branes and on the microscopic description of the black hole entropy. In particular the physics of the anomaly and the factor of $\tau_2^{-\frac{3}{2}}$ could represent a boson separating the two M5 branes.

New insights might also be within reach for the OSV conjecture, which states that the black partition function is proportional to the square of the topological string partition function. Our results for the topological string theory on elliptically fibred Calabi-Yau threefolds can be related via T-duality to our setups with D4-D2-D0 branes and hence provide a new check for the OSV conjecture.

One possible realisation of mock modular forms is that of meromorphic Jacobi forms. Such a meromorphic Jacobi form is known to count the black hole microstates in the case of $\mathcal{N} = 4$ CHL compactifications. Of course it would be interesting to have such an object as well for D4-D2-D0 bound states and the mock modular form would appear in the Fourier-Jacobi development of that meromorphic form. The generating function could then be expressed as the Fourier coefficient of such a meromorphic Jacobi form and information about the wall structure would be encoded in the path of integration. However, at the present point this is purely speculative.

Another direction of future research that is more mathematically oriented concerns the study of mock modular forms of higher depth. Using the techniques presented in this thesis it is possible to consider examples of mock modular forms of higher depth. However, the general structure is still unclear and it would be very interesting to give a general definition as well as applications.

In a nutshell, the interplay between wall-crossing, holomorphic anomalies and modularity provides a fruitful testing ground of string theory and its interaction with mathematics with many discoveries still to be made.

Appendix

APPENDIX A

Complex Geometry

In this section we provide some basic notions from complex geometry which are important for studying string theory. We start with the definition of a complex manifold and focus on the case of Calabi-Yau manifolds which appear in the context of compactifications of string theory. $K3$ surfaces and its properties are also introduced as they provide a Calabi-Yau twofold which is used for compactifications and orbifolds thereof. We follow [327], other standard texts are [89, 328–330].

A.1 Complex manifolds

In the following we assume that M is a real manifold of dimension $2m$. From this we obtain a complex manifold by the following definition:

Definition: Let $\{U_i\}$ be an open covering of the manifold M and let $\phi_i : U_i \to \mathbb{C}^m$ be a homeomorphism to an open subset of $\mathbb{C}^m$. Now $(M, \{U_i, \phi_i\})$ is a complex manifold if for all $U_i \cap U_j \neq 0$ the transition function

$$\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \tag{A.1}$$

is holomorphic. ◇

So the transition functions have to fullfill

$$\bar{\partial}_{\bar{k}} \phi^l_{ij} = \frac{\partial}{\partial \bar{z}^k} \phi^l_{ij} = 0 \quad \forall k, l, \tag{A.2}$$

where we denote by $z^k = x^k + iy^k$ complex coordinates and (A.2) is just the Cauchy-Riemann equation. The complex dimension of the complex manifold M is m.

Examples:

1. The simplest complex manifold is $\mathbb{C}^m$.
2. The torus T^2 is a complex manifold.
3. The complex projective space $\mathbb{C}P^m = \mathbb{P}^m$. This is constructed in the following way. We take points $(z^0, z^1, z^2, \ldots, z^m) \in \mathbb{C}^{m+1}$ and perform the following identification

$$(z^0, z^1, z^2, \ldots, z^m) \sim \lambda(z^0, z^1, z^2, \ldots, z^m),\ \lambda \in \mathbb{C}^*. \tag{A.3}$$

For all coordinates $(z^0, z^1, \dots, z^m)$ obeying (A.3) we introduce homogeneous coordinates by writing

$$[z^0 : z^1 : \dots : z^m]. \tag{A.4}$$

From the definition it follows that every complex manifold is also a real manifold. The converse is in general not true as it is not clear how one should assign complex coordinates. Therefore one needs the notion of a complex structure.

Definition: An almost complex structure J on a $2m$ real dimensional manifold M is a smooth tensor field $J \in \Gamma(TM \otimes TM^*)$ such that

$$J^a_b J^b_c = -\delta^a_c. \tag{A.5}$$

◇

An almost complex structure is the generalisation of multiplication with i known from complex analysis in $\mathbb{C}$. This can be seen from (A.5) leading to $J^2 = -\mathbb{1}_{TM}$.
A $2m$ dimensional real manifold M with an almost complex structure J is also called an almost complex manifold.

Definition: The Nijenhuis tensor $N_J(v, w)$ of two vector fields v, w with respect to the almost complex structure J is defined by

$$N_J(v, w) = [Jv, Jw] - J[v, Jw] - J[Jv, w] - [v, w], \tag{A.6}$$

where $[\cdot, \cdot]$ denotes the Lie bracket between two vector fields. ◇

In local coordinates the Nijenhuis tensor is given by

$$N^a_{bc} = J^d_b(\partial_d J^a_c - \partial_c J^a_d) - J^d_c(\partial_d J^a_b - \partial_b J^a_d). \tag{A.7}$$

Definition: An almost complex manifold M with almost complex structure J is a complex manifold, if and only if the Nijenhuis tensor vanishes $N_J \equiv 0$. Then J is called complex structure. ◇

The two definitions of a complex manifold are equivalent as can be shown by using a theorem by Newlander and Nirenberg. The theorem states that our first definition of a complex manifold holds, if and only if the complex structure J satisfies an integrability condition. This integrability condition is stating that the Lie bracket of two holomorphic vector fields is always holomorphic.

In order to understand this, we consider the complexified tangent space

$$T_pM \otimes \mathbb{C} =: T^{\mathbb{C}}_p M, \tag{A.8}$$

at each point p of the $2m$-dimensional real manifold M. The eigenvalues of the complex structure J are $\pm i$ as $J^2_p = -\mathbb{1}_{T_pM}$. The corresponding eigenspaces are $T^{1,0}_p M$ for i and $T^{0,1}_p M$ for $-i$. This gives a decomposition into holomorphic and antiholomorphic tangent spaces of the complexified tangent space at every point of the manifold

$$TM \otimes \mathbb{C} =: T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M. \tag{A.9}$$

$T^{1,0}M$ is called the holomorphic tangent bundle and $T^{0,1}M$ the anti-holomorphic tangent bundle.
The vanishing of the Nijenhuis tensor is equivalent to an integrability condition. Denoting by $P = \frac{1-iJ}{2}$

the projection on $T^{1,0}M$ and by $\bar{P} = \frac{1+iJ}{2}$ the projection on $T^{0,1}M$, the integrability condition reads

$$\bar{P}[Pv, Pw] = 0. \tag{A.10}$$

The complexified cotangent bundle also splits into holomorphic and anti-holomorphic bundles.

$$T^*M \otimes \mathbb{C} =: T^*_{\mathbb{C}}M = T^{*1,0}M \oplus T^{*0,1}M. \tag{A.11}$$

Form this it is possible to construct tensor fields as sections of tensor products of the tangent bundle with the the cotangent bundle.

A.2 Homology and cohomology

We consider k-forms on a m-dimensional real manifold M given by smooth sections of the k-th exterior power of the cotangent bundle $\Lambda^k T^*M$, i.e. they are totally antisymmetric tensors of type $(0, k)$. If m is the dimension of the underlying manifold M, then the dimension of $\Lambda^k T^*M$ is $\binom{m}{k}$.
The space of k forms is denoted as $\Omega^k(M)$. The wedge product between a k- and a l-form leads to a $k+l$ form. In coordinates this can be written as follows

$$\begin{aligned} \alpha = \alpha_{i_1\dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \beta = \beta_{j_1\dots j_l} dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}, \\ \alpha \wedge \beta = \alpha_{i_1\dots i_k} \beta_{i_{k+1}\dots i_{k+l}} dx^{i_1} \wedge dx^{i_k} \wedge dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_{k+l}}. \end{aligned} \tag{A.12}$$

The exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ of a k-form α is in local coordinates given by

$$d\alpha = \frac{\partial}{\partial x^{i_1}} \alpha_{i_2\dots i_{k+1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{k+1}}. \tag{A.13}$$

An important property is that the total derivative squares to zero,

$$d(d\alpha) = 0, \quad \forall \alpha \in \Omega^k(M). \tag{A.14}$$

A k-form α is closed if

$$d\alpha = 0, \tag{A.15}$$

and it is called exact if there exists a $(k-1)$ form β such that

$$\alpha = d\beta, \quad \beta \in \Omega^{k-1}(M). \tag{A.16}$$

It follows that every exact form is closed.
The fact that $d^2 = 0$ leads to a very important notion - the notion of cohomology. Let us first give the general definition.

Definition: Let $A_0, A_1, \dots$ be abelian groups connected by homomorphisms $d_n : A_n \to A_{n+1}$ such that $d_{n+1} \circ d_n = 0$, $\forall n$. The cochain complex is given by

$$0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} \dots . \tag{A.17}$$

The cohomology groups H^k are defined as

$$H^k = \frac{\mathrm{Ker}(d_k : A_k \to A_{k+1})}{\mathrm{Im}(d_{k-1} : A_{k-1} \to A_k)}. \tag{A.18}$$

◇

The k-forms on a real manifold M together with the properties of the exterior derivative d can be used to construct the so called de Rahm complex.

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0, \tag{A.19}$$

with the de Rahm cohomology groups $H^k_{dR}(M,\mathbb{R})$

$$H^k_{\mathrm{dR}}(M,\mathbb{R}) = \frac{\mathrm{Ker}(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}. \tag{A.20}$$

So within the the cohomology $H^k_{\mathrm{dR}}(M,\mathbb{R})$ two k-forms α and β are equal if they only differ by an exact form $d\gamma$, with $\gamma \in \Omega^{k-1}(M)$.

$$\alpha = \beta + d\gamma \tag{A.21}$$

Therefore we have cohomology classes $[\alpha]$. In the following we will simply write α for the corresponding cohomology class.

Definition: The dimension of the de Rahm cohomology group $H^k_{dR}(M,\mathbb{R})$ is called the k-th Betti number b^k

$$b^k = \dim H^k_{dR}(M,\mathbb{R}). \tag{A.22}$$

◇

Definition: The Euler characteristic χ is given by

$$\chi = \sum_{k=0}^{n} (-1)^k b^k. \tag{A.23}$$

◇

Now we want to discuss the case of complex manifolds. For complex manifolds we have to take care of holomorphic and anti-holomorphic pieces and we will have (p,q) forms given as sections $\Gamma(\Lambda^p T^{*(1,0)}M \otimes \Lambda^q T^{*0,1}M)$. The space of k-forms $\Omega^k(M)$ can be decomposed as

$$\Omega^k(M) = \bigoplus_{j=0}^{k} \Omega^{j,k-j}(M). \tag{A.24}$$

We have introduced the space of so called (p,q)-forms $\Omega^{p,q}$ defined as

$$\Omega^{p,q} = \Lambda^p T^{*1,0}M \otimes \Lambda^q T^{*0,1}M. \tag{A.25}$$

So a (p,q)-form consists of p holomorphic forms and q anti-holomorphic forms. The exterior derivative

d also has to be decomposed with respect to holomorphic and anti-holomorphic parts.

$$\begin{aligned} d &= \partial + \bar{\partial}, \\ \partial &: \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \\ \bar{\partial} &: \Omega^{p,q}(M) \to \Omega^{p,q+1}(M), \end{aligned} \tag{A.26}$$

By using $d^2 = 0$ the holomorphic ∂ and anti-holomorphic $\bar{\partial}$ exterior derivatives fulfill

$$\partial^2 = 0,\ \bar{\partial}^2 = 0,\ \partial\bar{\partial} + \bar{\partial}\partial = 0. \tag{A.27}$$

The Dolbeault cohomology is given by the following complex

$$0 \xrightarrow{\bar{\partial}} \Omega^{p,0}(M) \xrightarrow{\bar{\partial}} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{p,m}(M) \xrightarrow{\bar{\partial}} 0, \tag{A.28}$$

and the Dolbeault cohomology groups are given by

$$H^{p,q}_{\bar{\partial}}(M) = \frac{\mathrm{Ker}(\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\mathrm{Im}(\bar{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}. \tag{A.29}$$

Instead of using $\bar{\partial}$ we also could have used ∂. Furthermore the Dolbeault cohomology depends on the choice of complex structure.

Definition: The Hodge numbers $h^{p,q}$ are the complex dimensions of the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}$.

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}. \tag{A.30}$$

◇

The Hodge numbers can be organised in the so called Hodge diamond.

$$\begin{array}{ccccccc} & & & h^{m,m} & & & \\ & & h^{m,m-1} & \vdots & h^{m-1,m} & & \\ & \iddots & & & & \ddots & \\ h^{m,0} & \cdots & & & & \cdots & h^{0,m} \\ & \ddots & & & & \iddots & \\ & & h^{1,0} & \vdots & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} \tag{A.31}$$

In order to study the dependence of the Hodge numbers we first need to define a metric on a complex manifold leading to the notion of a Kähler manifold.

A.3 Kähler manifolds

Definition: Let M be a complex manifold with $\dim_{\mathbb{C}} M = m$ equipped with complex structure J. A Hermitian metric is a positive-definite inner product $g : T^{1,0}M \otimes T^{0,1}M \to \mathbb{C}$ at every point of the manifold M. ◇

This is equivalent to saying that g is viewed as Riemannian metric on M fulfilling

$$g(Jv, Jw) = g(v, w), \quad \forall v, w \in \Gamma(TM). \tag{A.32}$$

The Hermitian (1,1) form ω is defined via

$$\omega(v, w) = g(v, Jw), \tag{A.33}$$

and reads in local coordinates

$$\omega = i g_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu}. \tag{A.34}$$

Definition: Let M be a complex manifold with complex structure J and g a Hermitian metric with the Hermitian form ω. The hermitian metric g is called a Kähler metric if

$$d\omega = 0. \tag{A.35}$$

Then ω is called Kähler form and the manifold (M, J, G) is called Kähler manifold. $\diamond$

The Kähler condition can be rewritten in coordinates as

$$\begin{aligned} \partial_{\mu} g_{\nu\bar{\alpha}} &= \partial_{\nu} g_{\mu\bar{\alpha}} \\ \bar{\partial}_{\bar{\rho}} g_{\mu\bar{\alpha}} &= \bar{\partial}_{\bar{\alpha}} g_{\mu\bar{\rho}} \end{aligned} \tag{A.36}$$

From the Kähler condition one concludes that locally it is always possible to give a function K, called the Kähler potential, such that

$$g_{\alpha\bar{\beta}} = \partial_{\alpha} \bar{\partial}_{\bar{\beta}} K. \tag{A.37}$$

As $d\omega = 0$, ω is an element of $H^{1,1}_{\bar{\partial}}(M)$ and in the de Rahm cohomology we have $\omega \in H^2_{dR}(M, \mathbb{R})$, also called Kähler class. Taking the m-th power of ω this is proportional to the volume form.

$$\int_M \omega^m = \frac{1}{m!} \mathrm{vol}(M) \tag{A.38}$$

As for compact manifolds the volume is positive, the cohomology class ω is non-zero and therefore $h^{m,m} \geq 1$. Furthermore one can ask the question which classes in $H^{1,1}_{\bar{\partial}}$ lead to valid Kähler forms. The metric should be positive definite

$$\int_{c_k} \omega^k > 0, \quad c_k \in H_{2k}(M, \mathbb{R}), k = 0, \ldots, m, \tag{A.39}$$

and if ω fulfills this condition, so does $\lambda\omega$ with λ a positive number. Hence these classes form a cone, the so called Kähler cone. Furthermore one concludes that there are $h^{1,1}$ deformations of the Kähler form.

Next we want to study some properties of forms on Kähler manifolds. First we give the definition of the Hodge star $\star$ on complex manifolds.

Definition: Let α and β be complex k-forms on a complex Kähler manifold M of real dimension $2m$

with Kähler metric g. Define pointwise an inner product[1] by

$$(\alpha,\beta) = \alpha_{\mu_1\dots\mu_k}\overline{\beta_{\nu_1\dots\nu_k}}g^{\mu_1\bar{\nu}_1}\dots g^{\mu_k\bar{\nu}_k},\ \alpha,\beta \in \Omega^{k,0}(M). \tag{A.40}$$

The Hodge star $\star$ on Kähler manifolds is an isomorphism

$$\star : \Lambda^k T^*_{\mathbb{C}}M \to \Lambda^{2m-k}T^*_{\mathbb{C}}M, \tag{A.41}$$

such that for a complex k-form β we have that $\star\beta$ is the unique $(2m-k)$-form such that for every k-form α

$$\alpha \wedge \star\beta = (\alpha,\beta)dV. \tag{A.42}$$

◇

Therefore the action of the Hodge star is

$$\star : \Omega^{p,q}(M) \to \Omega^{m-p,m-q}(M). \tag{A.43}$$

By using the Hodge star and complex conjugation we get the following relations for the Hodge numbers on Kähler manifolds

$$\begin{aligned} h^{p,q} &= h^{m-p,m-q}, \\ h^{p,q} &= h^{q,p}. \end{aligned} \tag{A.44}$$

Of course with the help of a metric one can calculate now the curvature tensor, the Ricci-tensor and Ricci scalar. The Ricci tensor $R_{\mu\bar{\nu}}$ is a tensor of type $(1,1)$ and to this one associates the Ricci form $\mathcal{R}$ given in local coordinates by

$$\begin{aligned} \mathcal{R} &= iR_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^{\bar{\nu}} \\ &= \frac{i}{2}d(\partial - \bar{\partial}) \log\det g. \end{aligned} \tag{A.45}$$

Now we introduce the notion of homology. Denote by ∂ the boundary operator mapping a compact k-dimensional submanifold β of a manifold M triangulated in simplices to its boundary $\partial\beta$. Then the statement $\partial\beta = 0$ means that β has no boundary and if $\beta = \partial\alpha$, then it is the boundary of the submanifold α. This of course also implies that $\partial^2 = 0$. This can be generalized to the following notion of homology. Denote the space of k- dimensional submanifolds by $C_k(M)$ called a k-chain. Then consider the sequence of k-chains with the boundary operators $\partial_k : C_k(M) \to C_{k-1}(M)$ and $\partial_k \circ \partial_{k+1} = 0$.

$$\dots \xrightarrow{\partial_{k+1}} C_k(M) \xrightarrow{\partial_k} C_{k-1}(M) \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} C_0(M) \equiv 0 \tag{A.46}$$

Definition: The k-th homology group $H_k(M)$ is given by

$$H_k(M) = \frac{\mathrm{Ker}(\partial_k : C_k(M) \to C_{k-1}(M))}{\mathrm{Im}(\partial_{k+1} : C_{k+1}(M) \to C_k(M))} \tag{A.47}$$

◇

Note that k-chains β satisfying $\partial\beta = 0$ are called k-cycles.
The connection to cohomology is given by de Rahms theorem stating that $H_k(M)$ and $H^k(M)$ are iso-

[1] here we consider the forms as forms on the $2m$ dimensional manifold M

morphic to each other. This can be seen by using the following pairing

$$\begin{aligned} H_k(M) \times H^k(M) &\to \mathbb{C} \\ (\beta, \alpha) &\mapsto \int_\beta \alpha \end{aligned} \tag{A.48}$$

By using Stokes' theorem it can be shown that this pairing does not depend on the choice of representatives for α and β.
Taking a k-form $\alpha \in H^k(M)$ and a $n-k$-form $\beta \in H^{n-k}(M)$ we define an inner product by using the n-form $\alpha \wedge \beta$.

$$\begin{aligned} H^k(M) \times H^{n-k}(M) &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto \int_M \alpha \wedge \beta \end{aligned} \tag{A.49}$$

As the pairing is non degenerate we conclude that $H^k(M)$ and $H^{n-k}(M)$ are isomorphic to each other, a result known as Poincaré duality

$$H^k(M) \simeq H^{n-k}(M). \tag{A.50}$$

This can also be used to state that a cycle $\beta \in H_k(M)$ is dual to a $n-k$-form $\alpha \in H^{n-k}(M)$.

A.4 Chern classes

In the following we consider a fibre bundle $E \xrightarrow{\pi} M$ with structure group G. Then the total Chern character is given by the following definition:

Definition: Let $E \xrightarrow{\pi} M$ be a complex vector bundle, and let $F = dA + A \wedge A$ be the curvature two form of a connection A on E. The total Chern class $c(E)$ is given by

$$c(E) = \det\left(1 + \frac{i}{2\pi} F\right). \tag{A.51}$$

The Chern classes $c_k(E) \in H^{2k}(M, \mathbb{R})$ are given by the expansion of $c(E)$ in forms of even degrees

$$c(E) = 1 + c_1(E) + c_2(E) + \ldots . \tag{A.52}$$

◇

Note that different two forms F and F' only differ by an exact form, so in the cohomology class they are equal. For a m-dimensional manifold the Chern class $c_k(E)$ with $2k > m$ vanishes and $c_k(E) = 0$ for $k > r$ with r being the rank of the bundle E. Explicit formula for the Chern classes are given by

$$\begin{aligned} c_0(E) &= 1, \\ c_1(E) &= \frac{i}{2\pi} \mathrm{Tr} F, \\ c_2(E) &= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 (\mathrm{Tr} F \wedge \mathrm{Tr} F - \mathrm{Tr} F \wedge F). \end{aligned} \tag{A.53}$$

Consider a short exact sequence of complex vector bundles

$$0 \longrightarrow E \longrightarrow V \longrightarrow F \longrightarrow 0. \tag{A.54}$$

From this we get $V = E \oplus F$ and the total Chern class is given by Whitney's product formula

$$c(V) = c(E \oplus F) = c(E) \wedge c(F). \tag{A.55}$$

Write the total Chern class of a complex rank r vector bundle E as

$$c(E) = \prod_{i=1}^{r}(1 + x_i), \tag{A.56}$$

with x_i being the eigenvalues of $\frac{i\pi F}{2\pi}$. Then the Chern character $ch(E)$ is given by

$$ch(E) = \sum_{i=1}^{r} e^{x_i}, \tag{A.57}$$

which can be expanded as

$$ch(E) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)) + \ldots. \tag{A.58}$$

The Chern character has the following properties

$$\begin{aligned} ch(E \oplus F) &= ch(E) + ch(F), \\ ch(E \otimes F) &= ch(E)ch(F). \end{aligned} \tag{A.59}$$

A.5 Line bundles

We give the definition of a holomorphic vector bundle and then we discuss line bundles.

Definition: A holomorphic vector bundle consists of complex vector spaces E_p at every point p of a complex manifold M. These form the complex manifold E which is equipped with a natural projection

$$E \xrightarrow{\pi} M. \tag{A.60}$$

E is a holomorphic vector bundle with fiber $\mathbb{C}^k$ if $\pi : E \to M$ is a holomorphic map and for all $p \in M$ there exists an open neighborhood $U \subset M$ and a biholomorphic map

$$\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k, \tag{A.61}$$

such that for $u \in U$ we have an isomorphism between E_u and $\mathbb{C}^k$

$$\phi_U : E_u \mapsto \{u\} \times \mathbb{C}^k. \tag{A.62}$$

We call k the rank of the bundle. ◇

Definition: A holomorphic line bundle is a holomorphic vector bundle where the fibre is $\mathbb{C}$ and hence the rank is 1. ◇

Consider the holomorphic line bundle $\Lambda^{m,0}M$. Sections of it are holomorphic $(m, 0)$ - forms. This line bundle is also referred to as the canonical bundle K_M. This concept can be generalised to any vector bundle E.
For any holomorphic vector bundle E of rank k one can construct the determinant line bundle $\Lambda^k M$ with transition functions given by the determinant of the transition functions of E.

The tensor product $L \otimes L'$ of two line bundles L and L' is again a line bundle. This can be proven by dimensional analysis of the fibres which leads to the result, that the dimension of the corresponding fibre of $U \otimes V$ is 1. This makes it possible to construct a lot of line bundles. The set of complex line bundles of a complex manifold M has the structure of a group with multiplication given by the tensor product. The inverse element is given by the dual line bundle L^{-1} and the neutral element is $L \otimes L{-}1$. This group is called the Picard group.

In the case of $\mathbb{P}^n$ we have the tautological line bundle L^{-1} where a point $l \in \mathbb{P}^n$ is represented as a line in $\mathbb{C}^{n+1}$

$$L^{-1} = O_{\mathbb{P}^n}(-1) = \{(l, z) \in \mathbb{P} \times \mathbb{C}^{n+1} | z \in l\} . \tag{A.63}$$

The dual of this line bundle L^{-1} is the hyperplane line bundle $L = O(1)$. By using these two line bundles, we can construct line bundles $L^k, k \in \mathbb{Z}$ by building tensor products. Instead of writing L^k one writes $O(k)$. The notion $O(n)$ is used for the bundle as well and $O_X(E)$ denotes the sheaf of sections. The canonical bundle $K_{\mathbb{P}^m}$ is isomorphic to $O(-m - 1)$.

We want to calculate the Chern class for the complex projective space $\mathbb{P}^n$. This is done by using the short exact Euler sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow O_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T^{1,0}\mathbb{P}^n \longrightarrow 0 . \tag{A.64}$$

We then have

$$\begin{aligned} c(\mathbb{P}^n) &= c(O_{\mathbb{P}^n}(1)^{\oplus(n+1)} \oplus \mathbb{C}) \\ &= c(O_{\mathbb{P}^n}(1))^{n+1} \\ &= (1 + x)^{n+1}, \end{aligned} \tag{A.65}$$

where we have defined $x = c_1(O_{\mathbb{P}^n}(1))$.

A.6 Holonomy

If we parallel transport a vector in the tangent space T_pM around a loop then this gives rise to the so called holonomy group $\mathrm{Hol}_p(M)$. Let us formulate this in more detail.

Definition: Let M be a m-dimensional Riemannian manifold with metric g and affine connection ∇. Let $p \in M$ and consider the set of loops around p, $\mathrm{loop}_p = \{c(t) | 0 \leq t \leq 1, c(0) = c(1) = p\}$. Take a vector $v \in T_pM$ and parallel transport it around $c(t)$ by using the metric connection ∇. This induces a linear, invertible transformation $P_c : T_pM \to T_pM$. The holonomy group $\mathrm{Hol}_p(M)$ is the set of all these transformations

$$\mathrm{Hol}_p(M) = \{P_c : T_pM \to T_pM | c \in \mathrm{loop}_p\} \tag{A.66}$$

◇

The maximal possible holonomy group is of course $GL(n, \mathbb{R})$. Furthermore the holonomy group is independent of the choice of the base point p for a connected manifold, as $\mathrm{Hol}_p(M) \simeq \mathrm{Hol}_q(M)$. This can be seen by using a curve connecting p and q inducing a map $A : T_pM \to T_qM$, under which the holonomy groups are related by

$$\mathrm{Hol}_p(M) = A^{-1}\mathrm{Hol}(M)_q A. \tag{A.67}$$

Therefore one simply writes $\mathrm{Hol}(M)$.
For a $2m$-real dimensional Kähler manifold with metric g the holonomy group is a subgroup of $U(m)$. This follows form the fact, that a holomorphic vector gets mapped into another holomorphic vector and furthermore the length is preserved under parallel transport with a metric connection, i.e. $\nabla_{\dot{c}}\, g = 0$. .

A.7 Calabi-Yau manifolds

Now we can give the definition of a Calabi-Yau manifold. From a mathematical point of view the starting point is given by Yau's theorem stating that for a Kähler manifold with vanishing first Chern Class $c_1(M) = 0$ there exists a Kähler metric with zero Ricci-form. From a physics point of view Calabi-Yau manifolds are important in the context of compactifications of superstring theories in order to ensure a supersymmetric vacuum. Standard references on Calabi-Yau manifolds are [253, 331].

Definition: A Calabi-Yau manifold X is a $2m$-real dimensional compact Kähler manifold (X, J, g) such that the first Chern class c_1 vanishes $c_1 = 0$. $\diamond$

Of course there are also different but equivalent definitions. Instead of demanding, that the first Chern class vanishes, on can also require that

- X is Ricci flat, i.e. $\mathcal{R} = 0$,
- the holonomy group is $\mathrm{Hol}(X) = SU(m)$,
- the canonical bundle is trivial,
- there exists a nowhere vanishing holomorphic m-form $\Omega^{m,0}$.

We are not going to show that these are equivalent and refer to the detailed discussion in [89, 327].

Example: The two torus T^2 is a compact Calabi-Yau onefold. The Hodge diamond is given by

$$\begin{matrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{matrix}\ . \tag{A.68}$$

The only one form is given by dz and its conjugate. $\diamond$
We focus on Calabi-Yau threefolds, which are Calabi-Yau manifolds of complex dimension 3. These have six real dimensions and will allow for 4-dimensional Minkowski space-time in the compactification

of 10-dimensional superstring theories. The Hodge diamond is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{A.69}$$

In here we have used the properties of the Hodge numbers (A.44). In order for ensure the existence of a unique holomorphic three form $\Omega^{3,0}$ we have $h^{3,0} = 1$. From this we can conclude that $h^{0,0} = h^{3,3} = 1$ because given a $(0, p)$-form α there is a $(0, 3 - p)$-form β and by using Poincaré duality we have $h^{0,p} = h^{p,3-p}$. Furthermore one can show that $h^{1,0} = 0$. Therefore the only two independent Hodge numbers are $h^{1,1}$ and $h^{2,1}$ and the Euler characteristic reads

$$\chi = 2(h^{1,1} - h^{2,1}). \tag{A.70}$$

A.8 Moduli space of Calabi-Yau manifolds

A Calabi-Yau manifold is specified by naming the topology, the complex structure and the Kähler class. The space of all possible Calabi-Yau manifolds with a given topology, called the moduli space, depends only on the complex structure and the Kähler class. More details can be found in [89, 332]. The Calabi-Yau condition states, that $\mathcal{R}(g) = 0$ which is of course dependent on the metric g. If we vary the metric

$$g_{mn} \to g_{mn} + \delta g_{mn}, \tag{A.71}$$

the Calabi-Yau condition should still be fulfilled, in order for δg to be a modulus we have

$$\mathcal{R}(g + \delta g) = 0. \tag{A.72}$$

This gives rise to the so called Lichnerowitz equation. As we do not want to consider ordinary coordinate transformations we choose the gauge

$$\nabla^m \delta g_{mn} = \frac{1}{2} \nabla_n g^{mp} \delta g_{mp}, \tag{A.73}$$

and the Lichnerowicz equation reads.

$$\nabla^k \nabla_k \delta g_{mn} + 2R_m{}^p{}_n{}^q \delta g_{pq} = 0. \tag{A.74}$$

This can now be studied independently for variations with mixed index structure $\delta g_{\mu\bar{\nu}}$ and for variations with equal index structure $\delta g_{\mu\nu}$. For the mixed structure one associates a variation of the Kähler form, i.e. a harmonic $(1, 1)$ form given by

$$\delta J = i \delta g_{\mu\bar{\nu}} dz^\mu d\bar{z}^\nu. \tag{A.75}$$

Therefore $h^{1,1}$ counts the possible complex structure deformations.
For the equal index structure one associates variations of the complex structure by associating the har-

monic (2, 1) form χ^a, where Ω denotes the (3, 0) form.

$$\delta z^a \chi_a = \Omega_{\mu\nu\kappa} g^{\kappa\bar{\lambda}} \delta g_{\bar{\lambda}\bar{\delta}} dz^\mu \wedge dz^\nu \wedge d\bar{z}^{\bar{\delta}}. \tag{A.76}$$

Therefore the complex structure deformations are associated to $H^{2,1}(X)$ and their dimension is $h^{2,1}$. In order to describe the moduli space of a Calabi-Yau one needs so called special geometry, which we are not going to introduce here. For more details see e.g. [24, 332].

A.9 $K3$ surfaces

The subject of this section are $K3$ surfaces. These are Calabi-Yau twofolds, which play an important role in the context of compactifications to six dimensions and heterotic-type II duality. A good reference on the various aspects of $K3$ and its relation to string theory is [333].

Definition: A $K3$ surface X is a compact, complex Kähler manifold of complex dimension two with

$$\begin{aligned} h^{1,0}(X) &= 0, \\ c_1(TX) &= 0. \end{aligned} \tag{A.77}$$

◇

The second statement comes from the fact, that for a $K3$ surface the canonical bundle K is trivial. The Hodge diamond of K3 is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} . \tag{A.78}$$

Therefore the Euler characteristic can be computed to be

$$\chi(K3) = 24. \tag{A.79}$$

In order to understand the Hodge diamond we will look at a realisation of $K3$ as a hypersurface of degree n in $\mathbb{P}^3$

$$x_0^n + x_1^n + x_2^n + x_3^n = 0. \tag{A.80}$$

If we embed $K3$ into $\mathbb{P}^3$ the tangent bundle of $T\mathbb{P}^3|_{K3}$ splits into the tangent bundle and the normal bundle

$$\begin{aligned} T\mathbb{P}^3|_{K3} &= TK3 \oplus NK3 \\ c(T\mathbb{P}^3)|_{K3} &= c(TK3)c(NK3) \end{aligned} \tag{A.81}$$

The total Chern class for $\mathbb{P}^3$ is given by

$$c(T\mathbb{P}^3) = (1 + x)^4, \tag{A.82}$$

as follows from (A.65) and $c_1(N_{K3}) = 1 + c_1(NK3) = 1 + nx$. Therefore we conclude that

$$\begin{aligned} c(T_{K3}) &= \frac{c(T\mathbb{P}^3)}{c(NK3)} \\ &= \frac{(1+x)^4}{1+nx} \\ &= 1 + (4-n)x + (6-4n+n^2)x^2\,. \end{aligned} \tag{A.83}$$

For $c_1(TK3)$ to vanish we see from (A.83) one has $n = 4$. Therefore the Euler number is

$$\begin{aligned} \chi(K3) &= \int_{K3} c_2(TK3) \\ &= \int_{\mathbb{P}^3} c_2(TK3)c_1(NK3) \\ &= 6x^2 4x \\ &= 24. \end{aligned} \tag{A.84}$$

Giving the result from (A.79) and using $h^{1,0} = 0$ the Hodge diamond of $K3$ is given by (A.78).

Now we want to study the moduli space of complex structures. First we notice that $b_2(K3) = 22$ and therefore the Homology group is isomorphic to $\mathbb{Z}^{22}$. We understand $H_2(K3,\mathbb{Z})$ as a lattice $\Lambda^{m,n}$ with signature (m,n) by defining an inner product of elements $\alpha_i \in H_2(K3,\mathbb{Z})$ by the intersection number of the cycles

$$\alpha_i.\alpha_j = \#(\alpha_i \cap \alpha_j). \tag{A.85}$$

Note that m denotes the negative eigenvalues and n the positive ones. In order to determine the signature of the lattice $\Lambda^{m,n}$ we use the Hirzebruch signature complex. Given a lattice $\Lambda^{m,n}$ the Hirzebruch signature τ is defined by

$$\tau = m - n. \tag{A.86}$$

Furthermore it can be calculated via an index theorem as

$$\begin{aligned} \tau &= \int_{K3} -\frac{2}{3}c_2 \\ &= -16. \end{aligned} \tag{A.87}$$

Therefore $H_2(K3,\mathbb{Z})$ is a lattice of signature $(19,3)$ and we write $H_2(K3,\mathbb{Z}) \simeq \Lambda^{19,3}$. By Poincaré duality, we can find a dual basis e_j^* to a basis $e_i \in H_2(K3,\mathbb{Z})$ such that

$$e_i^* \cdot e_j = \delta_{ij}. \tag{A.88}$$

As the e_i^* are also a basis of $H_2(K3,\mathbb{Z})$ the lattice is self-dual $\Lambda^{19,3} = \Lambda^{19,3*}$ and by introducing the metric g_{ij} by

$$g_{ij} = e_i.e_j \tag{A.89}$$

we see that the lattice is unimodular, i.e. $\sqrt{|\det g| = 1}$.
Furthermore the lattice $\Lambda^{19,3}$ is even, that is for all $v \in \Lambda^{19,3}$

$$v.v \in 2\mathbb{Z}, \tag{A.90}$$

which follows from the fact that $c_1(TK3) = 0$. For an even self-dual lattice one also has the requirement

$$m - n = 0 \bmod 8, \tag{A.91}$$

and than for $m, n > 0$ the lattice is unique up to isometries.
For $H_2(K3, \mathbb{Z})$ we can write

$$H_2(K3, \mathbb{Z}) = \Lambda^{E_8} \oplus \Lambda^{E_8'} \oplus \Lambda^{1,1} \oplus \Lambda^{1,1} \oplus \Lambda^{1,1} \tag{A.92}$$

where $\Lambda^{E_8} = -(\mathbf{E_8})$ with $\mathbf{E_8}$ the Cartan matrix of the E_8 group and

$$\Lambda^{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.93}$$

Now we can take $\Omega \in H^2(K3, \mathbb{C})$ and write $\Omega = x + iy, x, y \in H^2(K3, \mathbb{R})$. The two vectors[2] x and y are linearly independent and span a space-like two plane Ω as can be seen by looking at

$$\begin{aligned} \int_{K3} \Omega \wedge \Omega &= 0, \\ \int_{K3} \Omega \wedge \bar{\Omega} &> 0. \end{aligned} \tag{A.94}$$

Therefore the choice of a complex structure corresponds to choosing a lattice of signature $\Lambda^{19,3} \subset \mathbb{R}^{19,3}$ and a two-plane Ω. Changing the complex structure will rotate the plane Ω in the lattice $\Lambda^{19,3}$.
Therefore the moduli space of complex structures is given by

$$\mathcal{M}_c = O^+(\Lambda^{19,3}) \backslash O^+(19, 3\mathbb{R}) / (O(2, \mathbb{R}) \times O(19, 1, \mathbb{R}))^+. \tag{A.95}$$

The space moduli space of Einstein metrics is given by

$$\mathcal{M} = O(\Lambda^{19,3}) \backslash O(19, 3) / (O(19, \mathbb{R}) \times O(3, \mathbb{R})). \tag{A.96}$$

This can be seen from Yau's theorem, implying that the choice of a two plane Ω and of the Kähler form ω spanning a 3-plane Σ in $H_2(K3, \mathbb{Z})$ specifies a unique Ricci-flat or Einstein metric.

An algebraic $K3$ surface is a $K3$ surface described by an embedding[3] into $\mathbb{P}^n$. The Picard group is defined by

$$\mathrm{Pic}(K3) = H^2(K3, \mathbb{Z}) \cap H^{1,1}(K3). \tag{A.97}$$

The rank of this group is called the Picard number ρ. The moduli space is given by

$$\mathcal{M}_{\text{algebraic } K3} = \frac{O(20 - \rho, 2, \mathbb{R})}{O(20 - \rho, \mathbb{R}) \times O(2, \mathbb{R})} \tag{A.98}$$

Now we want to study orbifolds. In the moduli space it can happen that we are dealing with $K3$ surfaces that are given at their orbifold point. In this context the following definition of an orbifold will hold:

[2] viewed as elements of $\mathbb{R}^{19,3}$

[3] We could also embed into a weighted projective space or into a toric space

Definition: Let M be a manifold and $\mathcal{G}$ a discrete group. Then an orbifold $\mathcal{O}$ of M by $\mathcal{G}$ is given by

$$\mathcal{O} = M/\mathcal{G}. \tag{A.99}$$

◇

In the moduli space of $K3$ there exists the so called orbifold point, where

$$K3 \simeq T^4/\mathbb{Z}_2. \tag{A.100}$$

Consider the T^4 as a two dimensional complex manifold with coordinates (z^1, z^2) then the action of $\mathbb{Z}_2$ is given by

$$z^i \mapsto -z^i, i = 1, 2. \tag{A.101}$$

This gives rise to 16 fix points. Note that the holomorphic two-form $\Omega = dz^1 \wedge dz^2$ is left invariant and also the Kähler form this we conlcude that this is a $K3$ surface. We want to understand the fact that $T^4/\mathbb{Z}_2 \simeq K3$ by giving its Hodge diamond. On T^4 we have the following $(1,0)$ and $(0,1)$ forms

$$dz^i, d\bar{z}^i \quad i = 1, 2\,. \tag{A.102}$$

These transform under the action of the orbifold group as

$$dz^i \mapsto -dz^i,\; d\bar{z}^i \mapsto -\bar{d}z^i, \tag{A.103}$$

Therefore the Hodge diamond of the $T^4/\mathbb{Z}_2$ in the untwisted sector is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 4 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}\,. \tag{A.104}$$

In the twisted sector we note that we have 16 fixed points due to the orbifold action. This are singular points which can be removed by blowing them up. For singularities on $K3$ there exists an ADE classification. For more details see [333, 334]. Each of these singularities contributes one $(1,1)$ form and therefore we have

$$h^{1,1}_{\text{twist}} = 16. \tag{A.105}$$

So we find that the Hodge diamond in total is the one of $K3$

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}\,. \tag{A.106}$$

APPENDIX B

Divisors in Calabi-Yau spaces

Let us recapitulate here some facts of the geometry of smooth divisors P in a Calabi-Yau three-fold X.

B.1 General facts about rigid divisors

We start with some facts about complex surfaces. The Riemann Roch formula relates the signature σ and arithmetic genus χ_0 to Chern class integrals

$$\sigma = \sum_i (b_{2i}^+ - b_{2i}^-) = \frac{1}{3}\int_P (c_1^2 - 2c_2), \qquad \chi_0 = \sum_i (-1)^i h_{i,0} = \frac{1}{12}\int_P (c_1^2 + c_2). \tag{B.1}$$

Regarding the embedding one has the distinction whether P is very ample or not, i.e. if the line bundle $\mathcal{L}_P$ is generated by its global sections or not. In the former case P has $h^0(X, \mathcal{L}_P) - 1$ deformations and there exists an embedding $j : X \to \mathbb{P}^n_P$ so that $\mathcal{L}_P = j^*(\mathcal{O}(1))$, i.e. P can be described by some polynomial. This situation has been considered in [138], where the deformations and b^+, b^- have been given. Generically one has $h^{2,0}(P) = \frac{1}{2}(b_2^+ - 1)$, which is positive in the very ample case.

In this work we consider mainly rigid smooth divisors. In this case one has no deformations and locally the Calabi-Yau manifold can be written as the total space of the canonical line bundle $\mathcal{O}(K_P) \to P$ and the latter can be globalised to a elliptic fibration over P, see section B.2, for $P = \mathbb{F}_n$. In this case $\Lambda_P = \Lambda$, compare section 2.8.1.

As X is a Calabi-Yau manifold and to allow no section, P has to have a positive $D^2 > 0$ anti-canonical divisor class $D = -K_P$, which is also required to be nef, i.e. $D.C \geq 0$ for any irreducible curve C. This defines a weak del Pezzo surface. If $D.C > 0$, then D is ample and P is a del Pezzo surface [335]. Del Pezzo surfaces are either dP_n, which are blow-ups of $\mathbb{P}^2$ in $n \leq 8$ points or $\mathbb{P}^1 \times \mathbb{P}^1$. We can also allow the Hirzebruch surface $\mathbb{F}_2$ which is weak del Pezzo.

As $h_{1,0} = h_{2,0} = 0$ one has $\chi_0(dP_n) = 1$ for all surfaces under consideration. As the Euler number $\chi(dP_n) = 3 + n$ one has by (B.1) that $\int_P c_1^2 = 9 - n$, which implies that $n = 9$ is the critical case for positive anti-canonical class, and $(b_2^+, b_2^-) = (1, n)$. The case $n = 9$ is called $\frac{1}{2}$K3 and we also denote it as dP_9. We include this semi-rigid situation.

In more detail the homology of dP_n is generated by the hyperplane class H of $\mathbb{P}^2$ and the exceptional divisors of the blow-ups e_i, with the non-vanishing intersections $H^2 = 1 = -e_i^2$. The anti-canonical class is given by $-K_{dP_n} = 3H - \sum_{i=1}^n e_i$. Defining the lattice generated by this element in $H_2(P, \mathbb{Z})$ as $\mathbb{Z}_{K_{dP_n}}$ and $E_n^* = (\mathbb{Z}_{K_{dP_n}})^\perp$ one sees that E_1^* is trivial and E_n^* are the lattices of the Lie algebras $(A_1, A_1 \times$

$A_2, A_4, D_5, E_6, E_7, E_8$) for $n = 2, \ldots, 8$. The corresponding basis in terms of (H, e_i) is worked out in [335] and used in section 5.6. The homology lattice for dP_9 is $\Lambda^{1,1} \oplus E_8$, where $\Lambda^{1,1}$ is the hyperbolic lattice with standard metric.

In order to study topological string theory in Calabi-Yau backgrounds realised in simple toric ambient spaces, one has to consider situations in which $\Lambda \subset \Lambda_P$, which is the case for the $\frac{1}{2}$K3 realised in the toric ambient space discussed in the next section.

B.2 Toric data of Calabi-Yau manifolds containing Hirzebruch surfaces $\mathbb{F}_n$

Let X be an elliptic fibration over $\mathbb{F}_n$ for $n = 0, 1, 2$ given by a generic section of the anti-canonical bundle of the ambient spaces specified by the following vertices

$$D_0 = (0,0,0,0), \quad D_1 = (0,0,0,1), \quad D_2 = (0,0,1,0), \quad D_3 = (0,0,-2,-3)$$
$$D_4 = (0,-1,-2,-3), \quad D_5 = (0,1,-2,-3), \quad D_6 = (1,0,-2,-3), \quad D_7 = (-1,-n,-2,-3).$$

One finds large volume phases with the following Mori-vectors

	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	
$l^1 =$	-6	3	2	1	0	0	0	0	C^1
$l^2 =$	0	0	0	-2	1	1	0	0	C^2
$l^3 =$	0	0	0	$n-2$	$-n$	0	1	1	C^3.

We choose a basis $\{C^A, A = 1,2,3\}$ of $H_2(X,\mathbb{Z})$. Let K_A be a Poincaré dual basis of the Chow group of linearly independent divisors of X, i.e. $\int_{C^A} K_B = \delta^A_B$. The divisors $D_i = l^A_i K_A$ have intersections with the cycles C^A given by $D_i.C^A = l^A_i$. We have the following non-vanishing intersections of the divisors given by

$$K_1 \cdot K_2 \cdot K_3 = 1, \quad K_1 \cdot K_2^2 = n, \quad K_1^2 \cdot K_2 = n+2, \quad K_1^2 \cdot K_3 = 2, \quad K_1^3 = 8. \tag{B.2}$$

The divisor giving the Hirzebruch surface inside the Calabi-Yau manifold corresponds to

$$[\mathbb{F}_n] = D_3 = K_1 - 2K_2 - (2-n)K_3. \tag{B.3}$$

Thus, the metric on $H^2(\mathbb{F}_n, \mathbb{Z})$ coming from the intersections in the Calabi-Yau manifold is

$$(K_A \cdot K_B \cdot [\mathbb{F}_n]) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & n & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{B.4}$$

Projecting out the direction corresponding to the elliptic fibre we reduce the problem to the Hirzebruch surface itself. We denote by $F = K_3$ and $B = K_2 - nK_3$ the class of the fibre and base, respectively. Thus, the canonical class reduces to $[\mathbb{F}_n] = -(2+n)F - 2B$. The intersection numbers are given as follows

$$\begin{pmatrix} F \cdot F & F \cdot B \\ B \cdot F & B \cdot B \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}. \tag{B.5}$$

Hence, the Kähler cone is spanned by the two vectors F and $2B + nF$, i.e.

$$C(\mathbb{F}_n) = \{J \in H^2(\mathbb{F}_n, \mathbb{R}) \,|\, J = t_1 F + t_2(2B + nF),\ t_1, t_2 > 0\}. \tag{B.6}$$

For $n = 1$ the geometry admits also an embedding of a K3 and a dP_9 surface.

APPENDIX C

Modular forms

C.1 Notation and conventions

Let us collect the definitions of various modular forms appearing in the main body text. We denote the following standard theta-functions by

$$\begin{aligned}
\vartheta_1(\tau,\nu) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n\nu}, \\
\vartheta_2(\tau,\nu) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}n^2} e^{2\pi i n\nu}, \\
\vartheta_3(\tau,\nu) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i n\nu}, \\
\vartheta_4(\tau,\nu) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n\nu}.
\end{aligned} \tag{C.1}$$

In the case that $\nu = 0$ we simply denote $\vartheta_i(\tau) = \vartheta_i(\tau, 0)$ (notice that $\vartheta_1(\tau) = 0$). Under modular transformations the theta functions $\vartheta_i(\tau)$ behave as vector-valued modular forms of weight $\frac{1}{2}$. They transform as

$$\vartheta_2(-1/\tau) = \sqrt{\frac{\tau}{i}}\vartheta_4(\tau), \qquad \vartheta_2(\tau+1) = e^{\frac{i\pi}{4}}\vartheta_2(\tau), \tag{C.2}$$

$$\vartheta_3(-1/\tau) = \sqrt{\frac{\tau}{i}}\vartheta_3(\tau), \qquad \vartheta_3(\tau+1) = \vartheta_4(\tau), \tag{C.3}$$

$$\vartheta_4(-1/\tau) = \sqrt{\frac{\tau}{i}}\vartheta_2(\tau), \qquad \vartheta_4(\tau+1) = \vartheta_3(\tau). \tag{C.4}$$

Further, the eta-function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \tag{C.5}$$

and transforms according to

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau), \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau). \tag{C.6}$$

The Eisenstein series are defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\frac{n^{k-1}q^n}{1-q^n}, \tag{C.7}$$

where B_k denotes the k-th Bernoulli number. E_k is a modular form of weight k for $k > 2$ and even.

Poisson Resummation

The technique of Poisson resummation is very useful and it is applied at several places in this thesis, for example when we check modularity. We want to present it in detail here. The Poisson resummation formula which is used is the following

$$\sum_{n=-\infty}^{\infty} e^{\pi a n^2 + 2\pi i b n} = \frac{1}{\sqrt{a}}\sum_{m=-\infty}^{\infty} e^{-\pi\frac{m-b^2}{a}}\,. \tag{C.8}$$

We give a proof of the more general Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} f(nT) = \frac{1}{T}\sum_{p=-\infty}^{\infty}\tilde{f}\left(\frac{p}{T}\right), \tag{C.9}$$

where $\tilde{f}$ denotes the Fourier transform of f. We introduce an auxiliary, periodic function $F(x)$ with period T:

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+nT). \tag{C.10}$$

The Fourier expansion of $F(x)$ is given in equation (C.11)

$$\begin{aligned} F(x) &= \sum_{p=-\infty}^{\infty} e^{-\frac{2\pi i x p}{T}}\tilde{F}\left(\frac{p}{T}\right), \\ \tilde{F}(p) &= \frac{1}{T}\int_0^T dz\, e^{2\pi i z p}F(z). \end{aligned} \tag{C.11}$$

We plug the second equation of (C.11) into the first one and comparing with (C.10) leads to the Poisson resummation formula for the case of $x = 0$

$$\sum_{n=-\infty}^{\infty} f(nT) = \frac{1}{T}\sum_{p=-\infty}^{\infty}\tilde{f}\left(\frac{p}{T}\right). \tag{C.12}$$

This can also be generalised to functions of the form

$$f(A) = \sum_{m\in\mathbb{Z}^m} e^{-\pi m\cdot Am}, \tag{C.13}$$

with A a positive-definite, symmetric matrix. Then the Poisson resummation formula generalises to

$$f(A) = \frac{1}{\sqrt{\det A}} \tilde{f}(A^{-1}). \tag{C.14}$$

This is the expression one uses for a lattice Λ with associated quadratic form A in order to perform checks of modularity.

C.2 Modular properties of the elliptic genus

We denote by $Z_P^{(r)}(\tau, z)$ the elliptic genus of r M5-branes wrapping P as defined previously in sect. 2.8.1. The elliptic genus should transform like a Jacobi form of bi-weight $(-\frac{3}{2}, \frac{1}{2})$ and bi-index $(\frac{r}{2}(d_{AB} - \frac{J_A J_B}{J^2}), \frac{r}{2}\frac{J_A J_B}{J^2})$ under the full modular group. In particular, we impose

$$\begin{aligned} Z_P^{(r)}(\tau + 1, z) &= \varepsilon(T)\, Z_P^{(r)}(\tau, z), \\ Z_P^{(r)}(-\frac{1}{\tau}, \frac{z_-}{\tau} + \frac{z_+}{\bar\tau}) &= \varepsilon(S)\, \tau^{-\frac{3}{2}} \bar\tau^{\frac{1}{2}} e^{\pi i r(\frac{z_-^2}{\tau} + \frac{z_+^2}{\bar\tau})}\, Z_P^{(r)}(\tau, z), \end{aligned} \tag{C.15}$$

where ε are certain phases [324].

Siegel-Narain theta-function and its properties

Let us start by recalling the definition of the Siegel-Narain theta-function of equation (2.333)

$$\theta_{\mu,J}^{(r)}(\tau, z) = \sum_{\xi \in \Lambda + \frac{[P]}{2}} (-)^{r(\xi+\mu)\cdot[P]} \bar{q}^{-\frac{r}{2}(\xi+\mu)_+^2} q^{\frac{r}{2}(\xi+\mu)_-^2} e^{2\pi i r(\xi+\mu)\cdot z}, \tag{C.16}$$

where we define

$$\xi_+^2 = \frac{(\xi \cdot J)^2}{J \cdot J}, \quad \xi_-^2 = \xi^2 - \xi_+^2. \tag{C.17}$$

Note, that $\xi_+^2 < 0$ if J lies in the Kähler cone.

If we denote by $\mathcal{D}_k = \partial_{\bar\tau} + \frac{i}{4\pi k}\partial_{z_+}^2$, the theta-function fulfils the heat equation

$$\mathcal{D}_r\, \theta_{\mu,J}^{(r)}(\tau, z) = 0. \tag{C.18}$$

Further, we denote by Λ^* the dual lattice to Λ w.r.t. the metric $r d_{AB}$. For $\mu \in \Lambda^*/\Lambda$, we can deduce the following set of transformation rules

$$\begin{aligned} \theta_{\mu,J}^{(r)}(\tau + 1, z) &= (-1)^{r(\mu + \frac{[P]}{2})^2} \theta_{\mu,J}^{(r)}(\tau, z), \\ \theta_{\mu,J}^{(r)}(-\frac{1}{\tau}, \frac{z_+}{\bar\tau} + \frac{z_-}{\tau}) &= \frac{(-1)^{r\frac{[P]^2}{2}}}{\sqrt{|\Lambda^*/\Lambda|}} (-i\tau)^{\frac{r(\Lambda)-1}{2}} (i\bar\tau)^{\frac{1}{2}} e^{\pi i r(\frac{z_-^2}{\tau} + \frac{z_+^2}{\bar\tau})} \sum_{\delta \in \Lambda^*/\Lambda} e^{-2\pi i r \mu\cdot\delta} \theta_{\delta,J}^{(r)}(\tau, z). \end{aligned} \tag{C.19}$$

Rank one

At rank one we have the universal answer

$$f_{\mu,J}^{(1)}(\tau) = \frac{\vartheta_{\Lambda^\perp}(\tau)}{\eta(\tau)^{\chi}}. \tag{C.20}$$

The transformation rules are simply given by for the eta-function and for $\vartheta_{\Lambda^\perp}$ we obtain (assuming $\Lambda^\perp$ even and self-dual)

$$\begin{aligned}\vartheta_{\Lambda^\perp}(\tau+1) &= \vartheta_{\Lambda^\perp}(\tau),\\ \vartheta_{\Lambda^\perp}(-\frac{1}{\tau}) &= \left(\frac{\tau}{i}\right)^{\frac{r(\Lambda^\perp)}{2}} \vartheta_{\Lambda^\perp}(\tau).\end{aligned} \tag{C.21}$$

Rank two

Using Zwegers' theta-function with characteristics $\vartheta^{c,c'}_{a,b}(\tau)$ given in def. 2.1 of his thesis [146], we can write

$$\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x) = q^{-\frac{1}{2}\langle a,a\rangle} e^{-2\pi i\langle a,b\rangle} \vartheta^{c,c'}_{a+\mu,b}(\tau), \tag{C.22}$$

where $x = a\tau + b$, i.e.

$$a = \frac{\mathrm{Im}(x)}{\mathrm{Im}(\tau)}, \quad b = \frac{\mathrm{Im}(\bar{x}\tau)}{\mathrm{Im}(\tau)}. \tag{C.23}$$

Following Corollary 2.9 of Zwegers [146], we can deduce the following set of transformations

$$\begin{aligned}\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau+1,x) &= (-1)^{\langle\mu,\mu\rangle}\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x),\\ \widehat{\Theta}^{c,c'}_{\Lambda,\mu}(-\frac{1}{\tau},\frac{x}{\tau}) &= \frac{i(-i\tau)^{r(\Lambda)/2}}{\sqrt{|\Lambda^*/\Lambda|}} e^{\pi i\frac{\langle x,x\rangle}{\tau}} \sum_{\delta\in\Lambda^*/\Lambda} e^{-2\pi i\langle\delta,\mu\rangle}\,\widehat{\Theta}^{c,c'}_{\Lambda,\delta}(\tau,x).\end{aligned} \tag{C.24}$$

This input enables us to write down the transformation rules for $\hat{f}^{(2)}_{\mu,J}$. They read

$$\begin{aligned}\hat{f}^{(2)}_{\mu,J}(\tau+1) &= (-1)^{\frac{\chi}{6}+2\mu^2}\hat{f}^{(2)}_{\mu,J}(\tau),\\ \hat{f}^{(2)}_{\mu,J}(-\frac{1}{\tau}) &= -\frac{(-i\tau)^{-\frac{r(\Lambda)+2}{2}}}{\sqrt{|\Lambda^*/\Lambda|}} \sum_{\delta\in\Lambda^*/\Lambda} e^{4\pi i\delta\cdot\mu}\hat{f}^{(2)}_{\delta,J}(\tau).\end{aligned} \tag{C.25}$$

This gives the conjectured transformation properties (C.15).

The blow-up factor

For completeness we elaborate on the transformation properties of the blow-up factor. We define

$$B_{r,k}(\tau) = \eta(\tau)^{-r} \sum_{a_i\in\mathbb{Z}+\frac{k}{r}} q^{\sum_{i\le j\le r-1} a_i a_j}. \tag{C.26}$$

We can deduce the following set of transformation rules

$$\begin{aligned}B_{r,k}(\tau+1) &= (-1)^{\frac{r}{12}+\frac{k^2(r-1)}{r}} B_{r,k}(\tau),\\ B_{r,k}(-\frac{1}{\tau}) &= \frac{1}{\sqrt{r}}\left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \sum_{0\le l\le r-1} (-1)^{\frac{2kl(r-1)}{r}} B_{r,l}(\tau).\end{aligned} \tag{C.27}$$

APPENDIX D

Results on elliptic genera

D.1 Elliptic genera of K3 and $\frac{1}{2}$K3

In the following we give some further examples of elliptic genera of multiple M5-branes wrapping the K3 and $\frac{1}{2}$K3 surfaces within the geometry of ref. [165]. The expressions for the elliptic genera can be read off from the instanton part of the prepotential of the geometry (see section 2.8.3) and were given in ref. [336], the $\frac{1}{2}$K3 expressions were known previously in refs. [139, 141].

Elliptic genera of multiply wrapping the K3

These are obtained by setting $q_2 \to 0$ and can all be obtained from $Z^{(1)}$ by the Hecke transformation.

$$
\begin{aligned}
Z^{(1)} &= -\frac{2E_4E_6}{\eta^{24}} \\
Z^{(2)} &= -\frac{E_4E_6\left(17E_4^3+7E_6^2\right)}{96\eta^{48}} \\
Z^{(3)} &= -\frac{\left(9349E_4^7E_6+16630E_4^4E_6^3+1669E_4E_6^5\right)}{373248\eta^{72}} \\
Z^{(4)} &= -\frac{E_4E_6\left(11422873E_4^9+46339341E_4^6E_6^2+21978651E_4^3E_6^4+880703E_6^6\right)}{2579890176\eta^{96}} \\
Z^{(5)} &= -\frac{E_4E_6\left(27411222535E_4^{12}+198761115620E_4^9E_6^2+222886195242E_4^6E_6^4\right)}{30958682112000\eta^{120}} \\
&\quad -\frac{E_4E_6\left(45368414180E_4^3E_6^6+911966215E_6^8\right)}{30958682112000\eta^{120}}
\end{aligned}
$$

Elliptic genera of $\frac{1}{2}$K3, E-string bound-states

These are obtained by setting $q_3 \to 0$, the polynomials containing E_2 represent the part coming from bound-states. The polynomial appearance of E_2 at higher wrapping is an example of the appearance of mock modular forms of higher depth at higher wrapping.

$$
\begin{aligned}
Z^{(1)} &= \frac{E_4\sqrt{q}}{\eta^{12}}\\
Z^{(2)} &= \frac{E_4(E_2E_4+2E_6)q}{24\eta^{24}}\\
Z^{(3)} &= \frac{E_4\left(54E_2^2E_4^2+109E_4^3+216E_2E_4E_6+197E_6^2\right)q^{3/2}}{15552\eta^{36}}\\
Z^{(4)} &= \frac{E_4\left(24E_2^3E_4^3+109E_2E_4^4+144E_2^2E_4^2E_6+272E_4^3E_6+269E_2E_4E_6^2+154E_6^3\right)q^2}{62208\eta^{48}}\\
Z^{(5)} &= \frac{E_4\left(18750E_2^4E_4^4+150000E_2^3E_4^3E_6+1250E_2^2\left(109E_4^5+341E_4^2E_6^2\right)\right)q^{5/2}}{373248000\eta^{60}}\\
&+\frac{E_4\left(1000E_2\left(653E_4^4E_6+505E_4E_6^3\right)+116769E_4^6+772460E_4^3E_6^2+207505E_6^4\right)q^{5/2}}{373248000\eta^{60}}
\end{aligned}
$$

D.2 One-parameter models

Here we collect details on the computation of the modified elliptic genus for the hyperplane sections in a number of 1-parameter examples along the lines of [137, 287]. In most cases we were able to write down a basis for the modular representations. However, we could not reliable check our results for the non-polar terms of the elliptic genus with the for these geometries available counting arguments, i.e. geometric, AdS_3/CFT_2 and attractor flow tree (see [123, 137, 287–289] for a discussion of the subtleties involved). The modified elliptic genus can be decomposed as

$$Z(\tau,\overline{\tau},y)=\sum_{\mu\in\Lambda^*/\Lambda} f_\mu(\tau)\,\theta_\mu(\tau,\overline{\tau},y) \tag{D.1}$$

where $f_\mu(\tau)$ are the components of a vector valued modular form. For the 1-parameter examples considered here the components enjoy a q-expansion of the form

$$f_\mu(\tau)=\sum_{n\geq 0} a_{\mu,n}\,q^{n-\Delta_\mu} \tag{D.2}$$

where $\Delta_\mu=\frac{c_L}{24}-\frac{\mu}{2}(\frac{\mu}{6D}+1)$, $\mu\in\{0,\dots\left[\frac{6D}{2}\right]\}$ denotes the M2-brane charge and $6D=P^3$ the triple intersection of the hyperplane section. To fix the vector valued modular form one has to specify the polar terms with $n-\Delta_\mu<0$. In the following table we collect the topological data of the models, which are needed for the computation of the modified elliptic genus and the polar terms one has to fix. In the table $\mathbb{P}_{d_1,\dots d_n}[w_i^{h_i}]$ denotes the degree $(d_1,\dots d_n)$ complete intersection in a weighted projective space, where the weight w_i appears h_i-times.

We will restrict to one-parameter models in the following and determine the modified elliptic genus

Model	$h^{1,1}$	$h^{2,1}$	$\chi(X)$	$6D$	$c_2\cdot P$	b_2^+	b_2^-	c_L	c_R	polar terms
$\mathbb{P}_{6,6}[1^2,2^2,3^2]$	1	61	−120	1	22	3	18	23	12	$a_{0,0}$
$\mathbb{P}_{4,4}[1^4,2^2]$	1	73	−144	4	40	7	35	44	24	$a_{0,0},a_{0,1},a_{0,2}$ $a_{1,0},a_{1,1}$
$\mathbb{P}_{2,2,2,2}[1^8]$	1	65	−128	16	64	15	63	80	48	$a_{0,0},a_{0,1},a_{0,2},a_{0,3}$ $a_{0,4},a_{0,5}$ $a_{1,0},a_{1,1},a_{1,2},a_{1,3}$ $a_{2,0},a_{2,1},a_{2,2},a_{2,3}$ $a_{3,0}$
$\mathbb{P}_{4,2}[1^6]$	1	89	−176	8	56	11	51	64	36	$a_{0,0},a_{0,1},a_{0,2},a_{0,3}$ $a_{1,0},a_{1,1},a_{1,2}$ $a_{2,0},a_{2,1}$ $a_{3,0}$
$\mathbb{P}_{3,2,2}[1^7]$	1	73	−144	12	60	13	57	72	42	$a_{0,0},a_{0,1},a_{0,2},a_{0,3}$ $a_{0,4}$ $a_{1,0},a_{1,1},a_{1,2},a_{1,3}$ $a_{2,0},a_{2,1}$ $a_{3,0}$
$\mathbb{P}_{4,3}[1^5,2]$	1	79	−156	6	48	9	43	54	30	$a_{0,0},a_{0,1},a_{0,2}$ $a_{1,0}$ $a_{2,0}$
$\mathbb{P}_{6,2}[1^5,3]$	1	129	−256	4	52	9	45	56	30	$a_{0,0},a_{0,1},a_{0,2}$ $a_{1,0},a_{1,1}$ $a_{2,0}$
$\mathbb{P}_{6,4}[1^3,2^2,3]$	1	79	−156	2	32	5	27	34	18	$a_{0,0},a_{0,1}$ $a_{1,0}$

Table D.1: Geometric data of the considered one-parameter models and the polar terms, that have to be determined.

by determining the polar terms in the expansion (2.331). The relevant Θ-functions that appear are [287]

$$\begin{aligned}
\Theta^{(m)}_{1,k}(\tau,y) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}+\frac{k}{m}} (-)^{mn} q^{\frac{m}{2}n^2} z^{mn}\\
\Theta^{(m)}_{2,k}(\tau,y) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}+\frac{k}{m}} q^{\frac{m}{2}n^2} z^{mn}\\
\Theta^{(m)}_{3,k}(\tau,y) &= \sum_{n\in\mathbb{Z}+\frac{k}{m}} q^{\frac{m}{2}n^2} z^{mn}\\
\Theta^{(m)}_{4,k}(\tau,y) &= \sum_{n\in\mathbb{Z}+\frac{k}{m}} (-)^{mn} q^{\frac{m}{2}n^2} z^{mn}.
\end{aligned} \tag{D.3}$$

For a modular form h of weight w we define the modular derivative $\mathcal{D}_2$ by

$$\mathcal{D}_2 h(\tau) = \frac{1}{2\pi i}\eta(\tau)^{2w}\partial_\tau(\eta(\tau)^{-2w}h(\tau)) \tag{D.4}$$

As seeding functions we use

$$\begin{aligned}\chi_i^{m,4l-\frac{m-1}{2}} &= \vartheta_3(\tau)^{8l-m}\Theta^{(m)}_{3,i}(\tau) + \vartheta_4(\tau)^{8l-m}\Theta^{(m)}_{4,i}(\tau) + \vartheta_2(\tau)^{8l-m}\Theta^{(m)}_{2,i}(\tau), \qquad m \quad \text{odd}\\ \chi_i^{m,4l-\frac{m-1}{2}} &= \vartheta_3(\tau)^{8l-m}\Theta^{(m)}_{3,i}(\tau) + (-)^i\vartheta_4(\tau)^{8l-m}\Theta^{(m)}_{3,i}(\tau) + \vartheta_2(\tau)^{8l-m}\Theta^{(m)}_{2,i}(\tau), \quad m \quad \text{even.}\end{aligned} \tag{D.5}$$

Note the following properties under modular transformations

$$\begin{aligned}&\Theta^{(m)}_2 \overset{S}{\longleftrightarrow} \Theta^{(m)}_3 \overset{T}{\longleftrightarrow} \Theta^{(m)}_4, \qquad m \quad \text{odd}\\ &\Theta^{(m)}_2 \overset{S}{\longleftrightarrow} \Theta^{(m)}_3 \overset{T}{\longleftrightarrow} (-)^k\Theta^{(m)}_3, \quad m \quad \text{even.}\end{aligned} \tag{D.6}$$

An example: the bisextic

As an example, which is restricted to the single wrapping $M5$-brane in a Calabi-Yau manifold, we consider the one-parameter manifold given by the bi-sextic $X_{6,6}$ in $\mathbb{P}(1^2, 2^2, 3^2)$. We fix the first coefficient via the Euler number of the moduli space of the divisor, which is $\chi(\mathbb{P}^1) = 2$ and obtain the modified elliptic genus via modular invariance as

$$Z(\tau,\bar\tau,y) = f_0(\tau)\Theta^{(1)}_{1,0}(\bar\tau,y) = -\frac{E_6\vec{\chi}_0}{\eta^{23}}\Theta^{(1)}_{1,0}(\bar\tau,y) = -2\eta^{-23}E_4E_6\Theta^{(1)}_{1,0}(\bar\tau,y), \tag{D.7}$$

with

$$f_0(\tau) = q^{-\frac{23}{24}}(-2 + 482q + 282410q^2 + 16775192q^3 + \dots). \tag{D.8}$$

Counting:[1]

Δq_0	Δq_1	geometry	configuration	split flow
Polar state:				
0	0	$-\chi(\mathbb{P}^1) = -2$	–	$(-1)\cdot 2\cdot N_{0,0}\cdot N_{0,0} = -2$
Non-polar states:				
1	0	$-\chi(X) = 120$	D0	$(-1)^0\cdot 1\cdot N_{0,1}\cdot N_{0,0} = 120$
		$n_{1,1} = 360$	$F = J - C_{1,1}$	$\langle\Gamma_1,\Gamma_2\rangle = 0$
2	0	??	2D0	$\langle\Gamma_1,\Gamma_2\rangle = 0$

We note, that the total contribution of 480 does not exactly match the factor of 482. The difference is just 2 which is remarkably two times the number of $(d = 1, g = 2)$ curves in the manifold.

The Quintic $X_5[1^5]$

We discuss the example of the quintic Calabi-Yau. It has the following geometric data

$h^{1,1}(X)$	$h^{2,1}(X)$	$\chi(X)$	P^3	$c_2\cdot P$	b_2^+	b_2^-	c_L	c_R
1	101	-200	5	50	9	44	55	30

[1] We denote by $n_{d,g}$ and $N_{\beta,n}$ the GV and DT invariants, respectively.

The modified elliptic genus reads

$$
\begin{aligned}
Z(\tau,\bar\tau,y) &= \sum_{k=0}^{4} f_k \Theta_k(\bar q, z)\\
&= \frac{1}{2985984\eta^{55}} 5\Big(66895E_4^6\vec{\chi}_0 + 541110E_4^3E_6^2\vec{\chi}_0 + 20987E_6^4\vec{\chi}_0\\
&\quad -1816200E_4^4E_6\vec{\chi}_1 - 1294200E_4E_6^3\vec{\chi}_1 + 3798000E_4^5\vec{\chi}_2 + 2422800E_4^2E_6^2\vec{\chi}_2\Big)\\
f_0(\tau) &= q^{-\frac{55}{24}}\Big(5 - 800q + 58500q^2 + 5817125q^3 + 75474060100q^4 + 28096675153255q^5 + \dots\Big)\\
f_1(\tau) &= q^{-\frac{203}{120}}\Big(8625q - 1138500q^2 + 3777474000q^3 + 3102750380125q^4 + 577727215123000q^5 + \dots\Big)\\
f_2(\tau) &= q^{-\frac{107}{120}}\Big(-1218500q + 441969250q^2 + 953712511250q^3 + 217571250023750q^4 + \dots\Big)
\end{aligned}
\tag{D.9}
$$

with

$$
\Theta_k(\bar q, z) \sum_{n\in\mathbb{Z}} (-1)^{n+k} \bar q^{\frac{1}{2}5(n+\frac{k}{5}+\frac{1}{2})^2} z^{5n+k+\frac{5}{2}} \tag{D.10}
$$

Counting:

Δq_0	Δq_1	geometry	configuration	split flow
Polar state:				
0	0	$-\chi(\mathbb{P}^4) = 5$	–	$(-1)^4 \cdot 5 \cdot N_{0,0} \cdot N_{0,0} = 5$
1	0	$-\chi(\mathbb{P}^3)\chi(X) = -800$	D0	$(-1)^3 4 \cdot 200 \cdot 1 = -800$
2	0	$\chi(\mathbb{P}^2)\chi(\mathrm{Sym}^2) + \chi(\mathbb{P}^3)\chi X$	2D0	$(-1)^2 \cdot 3$
0	1	$n_{1,1} = 0$		
1	1	$-\chi(\mathbb{P}^2) n_{1,0} = -8625$	$F = C_1$	$(-1)^2 3 N_{\mathrm{DT}}(1,1) N_{\mathrm{DT}}(0,0) = 8625$
1	2	$n_{1,0} = 0$		0
Non-polar states:				
3	0	$\chi(\mathbb{P}^1)\chi(\mathrm{Sym}^3(X)) + \chi(\mathbb{P}^2)\chi(X)^2 + \chi(\mathbb{P}^3)\chi(X)$ $n_{1,0} = 2875, 2875 \cdot 2874$	3 D0 $C_1 - C_1'$	

$X_{4,4}[1^4, 2^2]$

Consider the bi-quartic in $\mathbb{P}(1^4, 2^2)$.

$$
\begin{aligned}
Z(\tau,\bar\tau,y) &= \sum_{k=0}^{3} f_k(\tau)\Theta^{(4)}_{1,k}(\bar\tau, y)\\
&= -\frac{\eta^{-44}}{3240}\sum_{k=0}^{3}\Big(1895E_4^3E_6\chi_k^{4,\frac52} + 625E_6^3\chi_k^{4,\frac52} - 3504E_4^4\mathcal{D}_2\chi_k^{4,\frac52}\\
&\quad - 7728E_4E_6^2\mathcal{D}_2\chi_k^{4,\frac52} + 20736E_4^2E_6\mathcal{D}_2^2\chi_k^{4,\frac52}\Big)\Theta^{(4)}_{1,k}(\bar\tau, y)
\end{aligned}
\tag{D.11}
$$

$$
\begin{aligned}
f_0(\tau) &= q^{-\frac{11}{6}}(-4 + 432q - 10032q^2 + 148611456q^3 + \dots) \\
f_1(\tau) &= q^{-\frac{11}{6}+\frac{5}{8}}(-7424q + 7488256q^2 + 7149513728q^3 + \dots) \\
f_2(\tau) &= q^{-\frac{11}{6}+\frac{3}{2}}(-2816 + 2167680q + 3503031296q^2 + \dots)
\end{aligned}
\tag{D.12}
$$

Counting:

Δq_0	Δq_1	geometry	configuration	split flow
Polar states:				
0	0	$\chi(\mathbb{P}^3) = 4$	–	$(-1)^3 \cdot 4 \cdot N_{0,0} \cdot N_{0,0} = -4$
0	1	$n_{1,1} = 0$	$C_{1,1}$	$(-1)^2 \cdot 3 \cdot N_{1,0} \cdot N_{0,0} = 0$
1	0	$\chi(\mathbb{P}^2)\chi(X) = -432$	1 D0	$(-1)^2 \cdot 3 \cdot N_{0,1} \cdot N_{0,0} = 432$
1	1	$\chi(\mathbb{P}^1)n_{1,0} = 7424$	$C_{1,0}$	$(-1)^1 \cdot 2 \cdot N_{1,1} \cdot N_{0,0} = -7424$
0	2	$\chi(\mathbb{P}^1)n_{2,1} = 2816$	$C_{2,1}$	$(-1)^1 \cdot 2 \cdot N_{2,0} \cdot N_{0,0} = -2816$
Non-polar states:				
2	0	$\chi(\mathbb{P}^1)\frac{\chi(X)(\chi(X)-1)}{2} + \chi(\mathbb{P}^1)\chi(\mathbb{P}^2)\chi(X) = 20016$	2 D0	$(-1)^1 \cdot 2 \cdot N_{0,2} \cdot N_{0,0} = -20016$
		$n_{1,1} = 0$	$C_{1,0} - C_{1,1}$	$(-1)^0 \cdot 1 \cdot N_{1,1} \cdot N_{1,0} = 0$
		?	$C_{2,1} - C_{2,1}$	$\langle \Gamma_1, \Gamma_2 \rangle = 0$
2	1	?	$C_{1,0}$ + D0	$(-1)^0 \cdot 1 \cdot N_{1,2} \cdot N_{0,0} = 527104$
1	2	?	$C_{2,0}$	$(-1)^0 \cdot 1 \cdot N_{2,1} \cdot N_{0,0} = 1185216$
3	0	?	?	?

$X_{2,2,2,2}[1^8]$

Consider the quadri-conic in $\mathbb{P}^7$.

Δq_0	Δq_1	geometry	configuration	split flow
Polar states:				
0	0	$\chi(\mathbb{P}^7) = 8$	–	$(-1)^7 \cdot 8 \cdot N_{0,0} \cdot N_{0,0} = -8$
1	0	$\chi(\mathbb{P}^6)\chi(X) = -896$	1 D0	$(-1)^6 \cdot 7 \cdot N_{0,1} \cdot N_{0,0} = 896$
2	0	$\chi(\mathbb{P}^5)\frac{\chi(X)(\chi(X)-1)}{2} + \chi(\mathbb{P}^5)\chi(\mathbb{P}^2)\chi(X) = 47232$	2 D0	$(-1)^5 \cdot 6 \cdot N_{0,2} \cdot N_{0,0} = -47232$
3	0		3 D0	$(-1)^4 \cdot 5 \cdot N_{0,3} \cdot N_{0,0} = 1544960$
			$C_{1,0} - C'_{1,0}$	$(-1)^3 \cdot 4 \cdot N_{1,1} \cdot N_{1,1} = -1048576$
0	1	$n_{1,1} = 0$	$C_{1,1}$	$(-1)^6 \cdot 7 \cdot N_{1,0} \cdot N_{0,0} = 0$
1	1	$\chi(\mathbb{P}^5)n_{1,0} = -3072$	$C_{1,0}$	$(-1)^5 \cdot 6 \cdot N_{1,1} \cdot N_{0,0} = -3072$
1	2	$\chi(\mathbb{P}^4)n_{2,0} = 48640$	$C_{2,0}$	$(-1)^4 \cdot 5 \cdot N_{2,1} \cdot N_{0,0} = 48640$
0	2	$n_{2,1} = 0$	$C_{2,1}$	$(-1)^5 \cdot 6 \cdot N_{2,0} \cdot N_{0,0} = 0$
2	1	$n_{1,0}\chi(X)\chi(\mathbb{P}^4) = -327680$	$C_{1,0}$ + D0	$(-1)^4 \cdot 5 \cdot N_{1,2} \cdot N_{0,0} = 322560$
2	2			
0	3	$n_{3,1} = 0$	$C_{3,1}$	$(-1)^4 \cdot 5 \cdot N_{3,0} \cdot N_{0,0} = 0$
1	3			
2	3			
0	4	$\chi(\mathbb{P}^3)n_{4,1} = 59008$	$C_{4,1}$	$(-1)^3 \cdot 4 \cdot N_{4,0} \cdot N_{0,0} = -59008$
0	5	$\chi\mathbb{P}^2 n_{5,1} = 26348544$	$C_{5,1}$	$(-1)^2 \cdot 3 \cdot N_{5,0} \cdot N_{0,0} = 26348544$
Non-polar states:				
0	6	$\chi\mathbb{P}^1 n_{6,1} = 53440009216$	$C_{6,1}$	$(-1)^1 \cdot 2 \cdot N_{6,0} \cdot N_{0,0} = -5715280896$
0	7		$C_{7,1}$	$(-1)^0 \cdot 1 \cdot N_{6,0} \cdot N_{0,0} = 934638858240$

$X_{4,2}[1^6]$

Consider the degree (4, 2) complete intersection in $\mathbb{P}^5$.

Counting:

Δq_0	Δq_1	geometry	configuration	split flow
Polar states:				
0	0	$\chi(\mathbb{P}^5) = 6$	–	$(-1)^5 \cdot 6 \cdot N_{0,0} \cdot N_{0,0} = -6$
1	0	$\chi(\mathbb{P}^4)\chi(X) = -880$	1 D0	$(-1)^4 \cdot 5 \cdot N_{0,1} \cdot N_{0,0} = 880$
2	0	$\chi(\mathbb{P}^3)\frac{\chi(X)(\chi(X)-1)}{2} + \chi(\mathbb{P}^3)\chi(\mathbb{P}^2)\chi(X) = 60192$	2 D0	$(-1)^3 \cdot 4 \cdot N_{0,2} \cdot N_{0,0} = -60192$
3	0		3 D0	$(-1)^2 \cdot 3 \cdot N_{0,3} \cdot N_{0,0} = 276864$
			$C_{1,0} - C'_{1,0}$	
0	1	$n_{1,1} = 0$	$C_{1,1}$	$(-1)^4 \cdot 5 \cdot N_{1,0} \cdot N_{0,0} = 0$
0	2	$n_{2,1} = 0$	$C_{2,1}$	$(1)^{\cdot} 4 \cdot N_{2,0} \cdot N_{0,0} = 0$
1	1	$\chi(\mathbb{P}^3)n_{1,0} = 5120$	$C_{1,0}$	$(-1)^3 \cdot 4 \cdot N_{1,1} \cdot N_{0,0} = -5120$
2	1	$n_{1,0} \cdot \chi(X)\chi(\mathbb{P}^2) = -675840$	$C_{1,0} + D0$	$(-1)^2 \cdot 3 \cdot N_{1,2}N_{0,0} = 668160$
1	2	$\chi(\mathbb{P}^2)n_{2,0} = 276864$	$C_{2,0}$	$(-1)^2 \cdot 3 \cdot N_{2,1} \cdot N_{0,0} = 276864$
0	3	$\chi(\mathbb{P}^2) \cdot n_{3,1} = 7680$	$C_{3,1}$	$(-1)^2 \cdot 3 \cdot N_{3,0} \cdot N_{0,0} = 7680$

$X_{3,2,2}[1^7]$

Consider the degree (3, 2, 2) complete intersection in $\mathbb{P}^6$. Counting:

Δq_0	Δq_1	geometry	configuration	split flow
Polar states:				
0	0	$\chi(\mathbb{P}^6) = 7$	–	$(-1)^5 \cdot 6 \cdot N_{0,0} \cdot N_{0,0} = 7$
1	0	$\chi(\mathbb{P}^5)\chi(X) = -864$	D0	$(-1)^5 \cdot 6 \cdot N_{0,1} \cdot N_{0,0} = -864$
2	0	$\chi(\mathbb{P}^4)\frac{\chi(X)(\chi(X)-1)}{2} + \chi(\mathbb{P}^4)\chi(\mathbb{P}^2)\chi(X) = 50040$	2 D0	$(-1)^4 \cdot 5 \cdot N_{0,2} \cdot N_{0,0} = 50040$
3	0		3 D0	$(-1)^3 \cdot 4 \cdot N_{0,3} \cdot N_{0,0} = 1785216$
0	1	$n_{1,1} = 0$	$C_{1,1}$	$(-1)^5 \cdot 6 \cdot N_{1,0}N_{0,0} = 0$
0	2	$n_{2,1} = 0$	$C_{2,1}$	$(-1)^4 \cdot 5 \cdot N_{2,0} \cdot N_{0,0} = 0$
1	1	$\chi(\mathbb{P}^4) \cdot n_{1,0} = 3600$	$C_{1,0}$	$(-1)^4 \cdot 5 \cdot N_{1,1}N_{0,0} = 3600$
2	1	$n_{1,0}\chi(\mathbb{P}^3)\chi(X) = -414720$	$C_{1,0} + D0$	$(-1)^3 \cdot 4 \cdot N_{1,2} \cdot N_{0,0} = -408960$
1	2	?	$C_{2,0}$	$(-1)^3 \cdot 4 \cdot N_{2,1}N_{0,0} = -89712$
0	3	?	$C_{3,1}$	$(-1)^3 \cdot 4 \cdot N_{3,0}N_{0,0} = -256$
1	3	?	$C_{3,0}$	$(-1)^2 \cdot 3 \cdot N_{3,1}N_{0,0} = 4862160$
0	4	?	$C_{4,1}$	$(-1)^2 \cdot 3 \cdot N_{4,0}N_{0,0} = 795339$

$X_{4,3}[1^5, 2]$

Consider the degree (4, 3) complete intersection in $\mathbb{P}(1^5, 2)$.

$$Z(\tau,\bar\tau,y) = \sum_{k=0}^{4} f_k(\tau)\Theta^{(5)}_{1,k}(\bar\tau,y)$$
$$= \frac{1}{1548288\eta^{54}}\Big(114695E_4^6\vec\chi_0 + 1069286E_4^3E_6^2\vec\chi_0$$
$$-233581E_6^4\vec\chi_0 - 3960480E_4^4E_6\vec\chi_1 - 1983840E_4E_6^3\vec\chi_1$$
$$+9176832E_4^5\vec\chi_2 + 14047488E_4^2E_6^2\vec\chi_2 - 40089600E_4^3E_6\vec\chi_3 - 39536640E_6^3\vec\chi_3\Big) \quad \text{(D.13)}$$

$$f_0(\tau) = q^{-\frac{9}{4}}\left(5 - 624q + 35334q^2 + 30774450q^3 + 75188479200q^4 + 23750896418568q^5 +\right)$$
$$f_1(\tau) = q^{-\frac{2}{3}}\left(9720q + 287226q^2 + 3972088854q^3 + 2507626066824q^4 + 426841073597520q^5 +\right)$$
$$f_2(\tau) = q^{-11/12}\left(81 - 673515q + 397393128q^2 + 588860623845q^3 + 127692745959339q^4 +\right)$$
$$f_3(\tau) = q\left(-532610 + 123274056q + 355818034800q^2 + 84538116868308q^3 + 8789673155673504q^4 +\right) \quad \text{(D.14)}$$

Counting:

Δq_0	Δq_1	geometry	configuration	split flow
0	0	$\chi(\mathbb{P}^4) = 5$	-	$(-1)^4 \cdot 5N_{0,0} \cdot N_{0,0} = 5$
0	1	$n_{1,1} = 0$	$C_{1,1}$	$(-1)^3 \cdot 4 \cdot N_{1,0} \cdot N_{0,0} = 0$
1	0	$\chi(\mathbb{P}^3)\chi(X) = -624$	1 D0	$(-1)^3 \cdot 4 \cdot N_{0,1} \cdot N_{0,0} = -624$
0	2	$\chi(\mathbb{P}^2)n_{2,0} = 81$	$C_{1,0}$	$(-1)^2 \cdot 3 \cdot N_{1,1} \cdot N_{0,0} = 81$
1	1	?		$(-1)^4 \cdot 5 \cdot N_{1,1}N_{0,0} = 9720$
2	0	$\chi(\mathbb{P}^2)\frac{(\chi(X)-1)\chi(X)}{2} + \chi(\mathbb{P}^2)\chi(\mathbb{P}^2)\chi(X) = 35334$	2 D0	$(-1)^2 \cdot 3 \cdot N_{0,2} \cdot N_{0,0} = 35334$

$X_{6,2}[1^5, 3]$

Consider the degree $(6,2)$ complete intersection in $\mathbb{P}(1^5,3)$.

$$Z(\tau,\bar\tau,y) = \sum_{k=0}^{3} f_k(\tau)\Theta^{(4)}_{1,k}(\bar\tau,y) \quad \text{(D.15)}$$

$$f_0(\tau) = q^{-\frac{7}{3}}(5 + 1024q + 96384q^2 + 172524512q^3 + \dots)$$
$$f_1(\tau) = q^{-\frac{7}{3}+\frac{5}{8}}(14976q + 9676064q^2 + 11594977568q^3 + \dots) \quad \text{(D.16)}$$
$$f_2(\tau) = q^{-\frac{7}{3}+\frac{3}{2}}(-4608 - 2228284q + 4265161280q^2 + \dots).$$

Counting:

Δq_0	Δq_1	geometry	configuration	split flow
0	0	$\chi(\mathbb{P}^4) = 5$	-	$(-1)^4 \cdot 5 \cdot N_{0,0} \cdot N_{0,0} = 5$
0	1	$n_{1,1} = 0$	$C_{1,1}$	$(-1)^3 \cdot 4 \cdot N_{1,0} \cdot N_{0,0} = 0$
1	0	$\chi(\mathbb{P}^3)\chi(X) = -1024$	1 D0	$(-1)^3 \cdot 4 \cdot N_{0,1} \cdot N_{0,0} = 1024$
0	2	$n_{2,1} = -504$	$C_{2,1}$	$(-1)^2 \cdot 3 \cdot N_{2,0} \cdot N_{0,0} = -4608$
1	1	$\chi(\mathbb{P}^2)n_{1,0} = 14976$	$C_{1,0}$	$(-1)^2 \cdot 3 \cdot N_{1,1}N_{0,0} = 14976$
2	0	$\chi(\mathbb{P}^2)\frac{\chi(X)(\chi(X)-1)}{2}$	2 D0	
		$+\chi(\mathbb{P}^2)\chi(\mathbb{P}^2)\chi(X) = 96384$		$(-1)^2 \cdot 3 \cdot N_{0,2} \cdot N_{0,0} = 96384$
1	2	$\chi(\mathbb{P}^1)n_{2,0} = 4777536$	$C_{2,0}$	$(-1) \cdot 2 \cdot N_{2,1}N_{0,0} = -4258360$
2	1	$n_{1,0}(\chi(X) - \chi(C))\chi(\mathbb{P}^1)$	$C_{1,0}$ + D0	
		$+n_{1,0}\chi(C)\chi(\mathbb{P}^2)\chi(\mathbb{P}^2) = -2535936$		$(-1) \cdot 2 \cdot N_{1,2}N_{0,0} = 2535936$

$X_{6,4}[1^3, 2^2, 3]$

Consider the degree $(6, 4)$ complete intersection in $\mathbb{P}(1^3, 2^2, 3)$.

$$\begin{aligned} Z(\tau, \bar{\tau}, y) &= \sum_{k=0}^{1} f_k(\tau)\Theta^{(2)}_{1,k}(\bar{\tau}, y) \\ &= \frac{62335\mathrm{E}4^3\chi_0 + 295673\mathrm{E}6^2\chi_0 + 1196352\mathrm{E}4\mathrm{E}6\chi_1}{6048\eta^{34}} \end{aligned} \tag{D.17}$$

$$\begin{aligned} f_0(\tau) &= q^{-\frac{17}{12}}(3 - 156q + 21256959q^2 + 1028459492q^3 + 28640911694q^4) \\ f_1(\tau) &= q^{-\frac{17}{12}+\frac{3}{4}}\left(16784 - 6242560q - 353820224q^2 - 3896941952q^3 + 119356094048q^4\right). \end{aligned} \tag{D.18}$$

Counting:

Δq_0	Δq_1	geometry	configuration	split flow
0	0	$\chi(\mathbb{P}^2) = 3$	-	$(-1)^2 \cdot 3 \cdot N_{0,0} \cdot N_{0,0} = 3$
0	1	$n_{1,1}\chi(\mathbb{P}^1) + n_{3,4} + n_{1,0} = 15520$?	$C_{1,0}, C_{1,0}$	$(-1)^1 \cdot 2 \cdot N_{1,0}N_{0,0} + (-1)^0 \cdot 1 \cdot N_{1,1} \cdot N_{0,0} = 16784$
1	0	$\chi(X)\chi(\mathbb{P}^1) + n_{2,2} = -184$?	1 D0	$(-1)^1 \cdot 2 \cdot N_{0,1} \cdot N_{0,0} = -156$

APPENDIX E

Toric data for the elliptic hypersurfaces

Here we collect the toric data necessary to treat all models discussed. We list the Mori cones in the star triangulation for the bases of model 8-15 of figure 1

Δ_B	8(4)				9(4)					10(4)				11(5)					
ν_i^B	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(1)}$	$l^{(1)}$	$l^{(2)}$	$l^{(1)}$	$l^{(2)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$
z	0	−1	0	−1	−1	−1	0	−1	−1	0	−1	0	0	0	−1	0	0	−1	−1
1	0	0	0	1	−1	1	0	0	0	1	0	0	0	1	0	0	0	0	1
2	1	0	0	0	1	−1	1	0	0	−2	1	0	0	−2	1	0	0	0	0
3	−2	1	0	0	0	1	−2	1	0	1	−1	1	0	1	−1	1	0	0	0
4	1	−1	1	0	0	0	1	−1	1	0	1	−2	1	0	1	−2	1	0	0
5	0	1	−2	1	0	0	0	1	−1	0	0	1	−2	0	0	1	−2	1	0
6	0	0	1	−1	1	0	0	0	1	0	0	0	1	0	0	0	1	−1	1
														0	0	0	0	1	−1
ex			7					12				4						16	

Δ_B	12(5)							13(6)							14(6)							
ν_i^B	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(8)}$
z	−1	−1	−1	0	−1	0	−1	0	−1	0	0	0	−1	0	−1	0	−1	0	0	−1	0	−1
1	1	1	0	0	0	0	0	−2	1	0	0	0	0	0	−1	1	0	0	0	0	0	1
2	0	−1	1	0	0	0	0	1	−1	1	0	0	0	0	1	−2	1	0	0	0	0	0
3	0	1	−1	1	0	0	0	0	1	−2	1	0	0	0	0	1	−1	1	0	0	0	0
4	0	0	1	−2	1	0	0	0	0	1	−2	1	0	0	0	0	1	−2	1	0	0	0
5	0	0	0	1	−1	1	0	0	0	0	1	−2	1	0	0	0	0	1	−1	1	0	0
6	1	0	0	0	1	−2	−2	0	0	0	0	1	−1	1	0	0	0	0	1	−1	1	0
7	−1	0	0	0	0	1	1	0	0	0	0	0	1	−2	0	0	0	0	0	1	−2	1
8								1	0	0	0	0	0	1	1	0	0	0	0	0	1	−1
ex						29							20								43	

Δ_B	15(5)								16(7)								
ν_i^B	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(8)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(8)}$	$l^{(9)}$
z	0	−1	0	−1	0	−1	0	−1	0	0	−1	0	0	−1	0	0	−1
1	−2	1	0	0	0	0	0	1	−2	1	0	0	0	0	0	0	1
2	1	−1	1	0	0	0	0	0	1	−2	1	0	0	0	0	0	0
3	0	1	−2	1	0	0	0	0	0	1	−1	1	0	0	0	0	0
4	0	0	1	−1	1	0	0	0	0	0	1	−2	1	0	0	0	0
5	0	0	0	1	−2	1	0	0	0	0	0	1	−2	1	0	0	0
6	0	0	0	0	1	−1	1	0	0	0	0	0	1	−1	1	0	0
7	0	0	0	0	0	1	−2	1	0	0	0	0	0	1	−2	1	0
8	1	0	0	0	0	0	1	−1	0	0	0	0	0	0	1	−2	1
9									1	0	0	0	0	0	0	1	−1
ex								53								59	

The simplicial mori cone for the model 15 and 16 occur e.g. for the triangulation depicted here

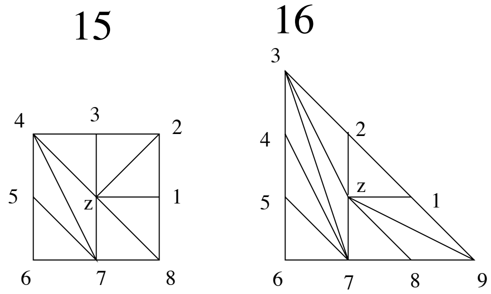

Figure E.1: Nonstar triangulations of the basis of model 15 and 16, which lead to simplicial Kähler cone for the Calabi-Yau space

For the model 15 the moricone reads

$$\begin{aligned} l^{(e)} &= (-6,0,0,0,0,1,-1,1,0,0,2,3), & l^{(1)} &= (0,-2,1,0,0,0,0,0,1,0,0,0) \\ l^{(2)} &= (0,1,-1,1,0,0,0,0,0,-1,0,0), & l^{(3)} &= (0,0,1,-2,1,0,0,0,0,0,0,0), \\ l^{(4)} &= (0,0,0,1,-1,1,0,0,0,-1,0,0), & l^{(5)} &= (0,0,0,0,0,-1,0,1,-1,1,0,0), \\ l^{(6)} &=,0,0,0,0,0,0,1,-2,1,0,0,0) \end{aligned} \tag{E.1}$$

This yields the intersection numbers

$$\begin{aligned} \mathcal{R} =& 4J_e^3 + 2J_e^2J_2 + 4J_e^2J_3 + J_eJ_2J_3 + 2J_eJ_3^2 + 3J_e^2J_4 + J_eJ_2J_4 + 2J_eJ_3J_4 + J_eJ_4^2 + \\ & 2J_e^2J_5 + J_eJ_2J_5 + 2J_eJ_3J_5 + J_eJ_4J_5 + 6J_e^2J_6 + 2J_eJ_2J_6 + 4J_eJ_3J_6 + J_2J_3J_6 + \\ & 2J_3^2J_6 + 3J_eJ_4J_6 + J_2J_4J_6 + 2J_3J_4J_6 + J_4^2J_6 + 2J_eJ_5J_6 + J_2J_5J_6 + 2J_3J_5J_6 + \\ & J_4J_5J_6 + 6J_eJ_6^2 + 2J_2J_6^2 + 4J_3J_6^2 + 3J_4J_6^2 + 2J_5J_6^2 + 6J_6^3 + 5J_e^2J_7 + 2J_eJ_2J_7 + \\ & 4J_eJ_3J_7 + J_2J_3J_7 + 2J_3^2J_7 + 3J_eJ_4J_7 + J_2J_4J_7 + 2J_3J_4J_7 + J_4^2J_7 + 2J_eJ_5J_7 + \\ & J_2J_5J_7 + 2J_3J_5J_7 + J_4J_5J_7 + 6J_eJ_6J_7 + 2J_2J_6J_7 + 4J_3J_6J_7 + 3J_4J_6J_7 + 2J_5J_6J_7 + \\ & 6J_6^2J_7 + 5J_eJ_7^2 + 2J_2J_7^2 + 4J_3J_7^2 + 3J_4J_7^2 + 2J_5J_7^2 + 6J_6J_7^2 + 5J_7^3 \end{aligned} \tag{E.2}$$

and the evaluation of c_2 on the basis J_i

$$\begin{aligned} & c_2J_e = 52, \quad c_2J_1 = 24, \quad c_2J_2 = 48, \quad c_2J_3 = 36, \\ & c_2J_4 = 24, \quad c_2J_5 = 72, \quad c_2J_6 = 62. \end{aligned} \tag{E.3}$$

The same data for the model 16

$$\begin{aligned} l^{(e)} &= (-6,0,0,0,0,1,-1,1,0,0,0,2,3), \quad l^{(1)} = (0,-2,1,0,0,0,0,0,0,1,0,0,0),\\ l^{(2)} &= (0,1,-2,1,0,0,0,0,0,0,0,0,0), \quad l^{(3)} = (0,0,0,-1,1,0,0,-1,0,0,1,0,0),\\ l^{(4)} &= (0,0,0,1,-2,1,0,0,0,0,0,0,0), \quad l^{(5)} = (0,0,0,0,1,-2,1,0,0,0,0,0,0),\\ l^{(6)} &= (0,0,0,0,0,0,0,1,-2,1,0,0,0), \quad l^{(7)} = (0,1,0,0,0,0,0,0,1,-1,-1,0,0), \end{aligned} \tag{E.4}$$

and the intersection by

$$\begin{aligned} \mathcal{R} =& 3J_e^3 + 4J_e^2J_2 + 2J_eJ_2^2 + 2J_e^2J_3 + J_eJ_2J_3 + 6J_e^2J_4 + 4J_eJ_2J_4 + 2J_2^2J_4 + 2J_eJ_3J_4 +\\ & J_2J_3J_4 + 6J_eJ_4^2 + 4J_2J_4^2 + 2J_3J_4^2 + 6J_4^3 + 5J_e^2J_5 + 4J_eJ_2J_5 + 2J_2^2J_5 + 2J_eJ_3J_5 +\\ & J_2J_3J_5 + 6J_eJ_4J_5 + 4J_2J_4J_5 + 2J_3J_4J_5 + 6J_4^2J_5 + 5J_eJ_5^2 + 4J_2J_5^2 + 2J_3J_5^2 + 6J_4J_5^2 +\\ & 5J_5^3 + 4J_e^2J_6 + 4J_eJ_2J_6 + 2J_2^2J_6 + 2J_eJ_3J_6 + J_2J_3J_6 + 6J_eJ_4J_6 + 4J_2J_4J_6 + 2J_3J_4J_6 +\\ & 6J_4^2J_6 + 5J_eJ_5J_6 + 4J_2J_5J_6 + 2J_3J_5J_6 + 6J_4J_5J_6 + 5J_5^2J_6 + 4J_eJ_6^2 + 4J_2J_6^2 + 2J_3J_6^2 + \end{aligned} \tag{E.5}$$

$$\begin{aligned} & 6J_4J_6^2 + 5J_5J_6^2 + 4J_6^3 + 3J_e^2J_7 + 2J_eJ_2J_7 + J_eJ_3J_7 + 3J_eJ_4J_7 + 2J_2J_4J_7 + J_3J_4J_7 +\\ & 3J_4^2J_7 + 3J_eJ_5J_7 + 2J_2J_5J_7 + J_3J_5J_7 + 3J_4J_5J_7 + 3J_5^2J_7 + 3J_eJ_6J_7 + 2J_2J_6J_7 + J_3J_6J_7 +\\ & 3J_4J_6J_7 + 3J_5J_6J_7 + 3J_6^2J_7 + J_eJ_7^2 + J_4J_7^2 + J_5J_7^2 + J_6J_7^2 + 6J_e^2J_8 + 4J_eJ_2J_8 +\\ & 2J_eJ_3J_8 + 6J_eJ_4J_8 + 4J_2J_4J_8 + 2J_3J_4J_8 + 6J_4^2J_8 + 6J_eJ_5J_8 + 4J_2J_5J_8 + 2J_3J_5J_8 +\\ & 6J_4J_5J_8 + 6J_5^2J_8 + 6J_eJ_6J_8 + 4J_2J_6J_8 + 2J_3J_6J_8 + 6J_4J_6J_8 + 6J_5J_6J_8 + 6J_6^2J_8 + 3\\ & J_eJ_7J_8 + 3J_4J_7J_8 + 3J_5J_7J_8 + 3J_6J_7J_8 + 6J_eJ_8^2 + 6J_4J_8^2 + 6J_5J_8^2 + 6J_6J_8^2 \end{aligned} \tag{E.6}$$

and the evaluation on c_2 is

$$\begin{aligned} c_2J_e &= 42, \quad c_2J_1 = 48, \quad c_2J_2 = 24, \quad c_2J_3 = 72,\\ c_2J_4 &= 62, \quad c_2J_5 = 52, \quad c_2J_6 = 36, \quad c_2J_7 = 72. \end{aligned} \tag{E.7}$$

E.1 Results for the other fibre types with $\mathbb{F}_1$ base

We give some results of the periods for the different fibre types with base $\mathbb{F}_1$. The corresponding Picard-Fuchs operators read [295]

$$\begin{aligned} \mathcal{L}_{E7} &= \theta^2 - 4z(4\theta+3)(4\theta+1),\\ \mathcal{L}_{E6} &= \theta^2 - 3z(3\theta+2)(3\theta+1)\\ \mathcal{L}_{D5} &= \theta^2 - 4z(2\theta+1)^2 \end{aligned} \tag{E.8}$$

The solutions read as follows

$$\begin{aligned}
\phi_{E7} &= \sum_{n\geq 0}^{\infty} \frac{(4n)!}{(n!)^2(2n)!} z^n = {}_2F_1(\frac{3}{4}, \frac{1}{4}, 1, 64z), \\
\phi_{E6} &= \sum_{n\geq 0}^{\infty} \frac{(3n)!}{(n!)^3} z^n = {}_2F_1(\frac{2}{3}, \frac{1}{3}, 1, 27z), \\
\phi_{D5} &= \sum_{n\geq 0}^{\infty} \frac{(2n)!^2}{(n!)^4} z^n = {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, 16z),
\end{aligned} \tag{E.9}$$

with:

$${}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \tag{E.10}$$

where $(a)_n = a(a+1)\ldots(a+n-1)$ denotes the Pochhammer symbol.
The j-functions read for these read

$$\begin{aligned}
1728 j_{E7} &= \frac{(1+192z)^3}{z(1-64z)^2} \\
1728 j_{E6} &= \frac{(1+216z)^3}{z(1-27z)^3} \\
1728 j_{D5} &= \frac{(1+244z+256z^2)}{z(-1+16z)^4}
\end{aligned} \tag{E.11}$$

We collect the expressions for the solutions in terms of modular forms

$$\begin{aligned}
\phi_{E7}(z(q))^2 &= 1 + 24q + 24q^2 + 96q^3 + \cdots = -E_2(\tau) + 2E_2(2\tau) \\
\phi_{E6}(z(q)) &= 1 + 6q + 6q^3 + \cdots = \sum_{m,n\in\mathbb{Z}} q^{m^2+n^2+mn} = \theta_2(\tau)\theta_2(3\tau) + \theta_3(\tau)\theta_3(3\tau) \\
\phi_{D5}(z(q)) &= 1 + 4q + 4q^2 + \cdots = \theta_3(2\tau)^2
\end{aligned} \tag{E.12}$$

Following analogous steps presented in section 5.2, one can again proof the holomorphic anomaly equation for genus 0.

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Acronyms

BCOV Bershadsky-Cecotti-Ooguri-Vafa. 10, 66, 69, 122, 125

BPS Bogomol'nyi–Prasad–Sommerfield. 1, 7–10, 13, 15–17, 19–24, 26, 30–35, 53, 55, 70, 73–77, 79–83, 99–103, 105, 107, 112, 128–130, 132, 136–138, 141, 145

CFT conformal field theory. 7, 8, 13, 30, 58–60, 63, 66, 70–74, 100, 128, 145, 146, 209

CHL Cadhuri-Hockney-Lykken. 53, 57, 146

FI Fayet-Iliopoulos. 86

GSY Gaiotto-Strominger-Yin. 77

GUT grand unified theory. 4, 5

GW Gromov-Witten. 5, 10, 62, 63, 70, 76, 130

HN Harder-Narasimhan. 26, 29, 132, 137, 141

KSWCF Kontsevich-Soibelman wall-crossing formula. 22, 24–26, 30

MSW Maldacena-Strominger-Witten. 13, 70–74, 100, 145, 146

OSV Ooguri-Strominger-Vafa. 8, 146

SUSY supersymmetry. 14, 15, 209

SYM super Yang-Mills. 9, 13, 31, 70, 79, 99, 100, 107, 140, 145

SYZ Strominger-Yau-Zaslow. 6