On the Classification of Cohomology Bott Manifolds

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1. Introduction

The central objects of this thesis are cohomology Bott manifolds which are a generalisation of Bott manifolds. Bott manifolds were defined in [BS58] by Bott and Samelson. The name Bott manifold is due to a paper of Grossberg and Karshon ([GK94]).
By definition a Bott manifold is the total space of an iterated $\mathbb{C}\mathrm{P}^1$-bundle, where each total space is the fibrewise projectivisation of the Whitney sum of an arbitrary complex line bundle and a trivial one. In [GK94] they were examined from the perspective of symplectic geometry. Later on, they came into the focus of toric topologists as one of the main examples for toric manifolds. An $n-$dimensional toric manifold is defined to be a smooth and compact, normal, complex algebraic variety X which contains an algebraic torus $(\mathbb{C}^*)^n \subset X$ as a dense subset and which admits an action $(\mathbb{C}^*)^n \times X \to X$ of the algebraic torus which extends the action of $(\mathbb{C}^*)^n$ on itself (cf. [Ful93]). In 2008 Choi, Masuda and Suh [CMS10] enhanced the interest in Bott manifolds when they started to work on the following conjectures.

Conjectures:

1. Let M and N be two toric manifolds such that their integral cohomology rings are isomorphic. Then M and N are diffeomorphic.
2. Any isomorphism $\phi : H^*(M) \to H^*(N)$ between the integral cohomology rings of M and N can be realised by some diffeomorphism $f\colon N \to M$, i.e. $f^* = \phi$.

The first part of the conjecture is usually referred to as the weak, the second as the strong cohomological rigidity conjecture or problem for toric manifolds, abbreviated by (WCRP) or (SCRP).
Before Choi, Masuda and Suh started to examine this problem Masuda in [Mas08] showed that equivariant cohomology distinguishes toric manifolds as varieties. Hence, the question arose whether ordinary cohomology can distinguish toric manifolds.
Since Bott manifolds are toric manifolds they form a test case for the (WCRP) and the (SCRP). From now on, if we talk about the weak or strong cohomological rigidity problem, we refer to the respective conjectures for Bott manifolds.

So far there is a number of special cases in which the conjecture is proven. Bott manifolds of dimension four were already known by Hirzebruch. In [Hir51], he considers a class of complex surfaces and shows that two of those surfaces are diffeomorphic if either both are *Spin*-manifolds or both are not. If they are *Spin* they are diffeomorphic to $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$; if they are non-*Spin* they are diffeomorphic to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. At the time these complex surfaces formed the first known examples of manifolds which admit infinitely many complex structures. Honoring his work, Bott manifolds of real dimension four are called Hirzebruch surfaces. By explicit construction of all isomorphisms of their cohomology rings, the (SCRP) is known to hold for Hirzebruch surfaces.

Furthermore, the (WCRP) is known for two classes of Bott manifolds, known as $\mathbb{Q}$-*trivial* and *one twist Bott manifolds*, introduced in [CM12] and [CS11a], respectively.
The idea for the proof of the (WCRP) for these two classes of Bott manifolds is to use bundle isomorphisms of the underlying complex vector bundles, i.e. isomorphisms of those vector bundles whose projectivisations are the considered Bott manifolds.
For $\mathbb{Q}$-trivial Bott manifolds this method even allows to prove the (SCRP).

The first class of Bott manifolds for which a different machinery is necessary is the class of Bott manifolds of real dimension six. The proof of the (WCRP) in dimension six uses surgery theoretical results developed in [Wal66] and [Jup73]. Since the cohomology ring of a Bott manifold is torsion free an isomorphism between the integral cohomology rings of two Bott manifolds induces an isomorphism between the cohomology rings with coefficients in $\mathbb{Z}/2$. Therefore, we denote the isomorphism on cohomology with $\mathbb{Z}/2$-coefficients with the same symbol. By the results in [Wal66] and [Jup73] it suffices to show that any isomorphism $\varphi\colon H^*(B) \to H^*(B')$ between the cohomology rings of six-dimensional Bott manifolds B and B' has the following two properties:

1. It preserves the total Stiefel-Whitney classes $w(B)$ and $w(B')$ of B and B', respectively, i.e. $\varphi(w(B)) = w(B')$ and
2. it also preserves the total Pontrjagin classes $p(B)$ and $p(B')$ of B and B', respectively, i.e. $\varphi(p(B)) = p(B')$.

This was proven in [CMS10].

For Bott manifolds of dimension eight there exists a preprint by Choi (cf. [Cho11a]) which shows that the (WCRP) holds for Bott manifolds of dimension eight. Furthermore, he reduces the (SCRP) to the problem, whether four automorphisms of a certain class of Bott manifolds can be realised.

Motivated by these examples which support the cohomological rigidity conjecture and by the methods of the proof for the (WCRP) of six-dimensional Bott manifolds we pose slightly different questions.
Let M be a smooth, simply connected and closed manifold of dimension greater or equal to six. Furthermore, let B be a fixed Bott manifold and let $\varphi\colon H^*(B) \to H^*(M)$ be a ring isomorphism which has the properties that

1. it preserves the total Stiefel-Whitney classes, i.e. $\varphi(w(B)) = w(M)$ and
2. it preserves the total Pontrjagin classes , i.e. $\varphi(p(B)) = p(M)$.

We refer to the class of manifolds M with these properties as *cohomology Bott manifolds* (with respect to B). Note that this definition differs from the one given in [CS11a].
By [CMS10] the first property of φ is automatically fulfilled for any ring isomorphism

of cohomology Bott manifolds. The second property is believed to hold if M is a Bott manifold, too. This was claimed in [Cho11b], but unfortunately there was a gap in the proof.

We ask ourselves the following natural questions about cohomology Bott manifolds.

Questions:

1. Can we say something about diffeomorphism classes of cohomology Bott manifolds?
2. Is it possible that they also fulfil cohomological rigidity?
3. Can we classify them in some way?

Since dimension six is solved by [Wal66], [Jup73] and [CMS10], we consider the next interesting dimension, i.e. we consider cohomology Bott manifolds of dimension eight.

This thesis answers the first two questions and examines the third.

Our method to examine cohomology Bott manifolds is modified surgery theory as developed in [Kre99]. This method enables us to translate the question whether two manifolds are diffeomorphic to the question whether these manifolds represent the same element in a certain bordism group $\Omega_8^{\mathbb{B}}$. Since bordism groups are stable homotopy groups of Thom spectra, by the Pontrjagin-Thom construction, modified surgery theory allows us to examine the diffeomorphism classification of cohomology Bott manifolds with the tools of stable homotopy theory.

Using this method we can answer the first question with Theorem 4.2:

Theorem. *Let B_4 be a Bott manifold of dimension eight. The number of diffeomorphism classes of cohomology Bott manifolds with respect to B_4 is finite.*

The proof of the theorem is based on the fact that we can control the free part of the bordism groups $\Omega_8^{\mathbb{B}}$ by invariants.
As a matter of fact we can even give an upper bound for the number of diffeomorphism classes of cohomology Bott manifolds with respect to B_4 (cf. Corollary 4.8). The upper bound can be deduced from the size of the torsion subgroup of the bordism group.

To answer the second question we construct explicit examples of cohomology Bott manifolds which are not diffeomorphic to a Bott manifold in Theorem 4.10:

Theorem. *Let S be a Bott manifold which admits a String-structure and which fulfills the (SCRP). Then there exists a cohomology Bott manifold F (with respect to S) such that F is not diffeomorphic to any Bott manifold.*

Since there clearly exist Bott manifolds which fulfil the assumptions of the theorem, for example in the class of $\mathbb{Q}$-trivial Bott manifolds, the answer to the second question is

negative: cohomology Bott manifolds are in general not cohomologically rigid.
In a sense this theorem is also a first step towards the answer of the third question. We can hope to classify cohomology Bott manifolds if we understand the torsion subgroup of $\Omega_8^{\mathbb{B}}$ and F gives rise to a non-trivial element in $\Omega_8^{\mathbb{B}}$. Conjecturally the theorem, and in particular the methods we use to construct F can be used to construct more manifolds which represent elements in $\Omega_8^{\mathbb{B}}$.
To prove the theorem we need a codimension two Arf-invariant, that is, the Arf-invariant of a submanifold of codimension two. Another interesting question is, whether cohomology Bott manifolds are rigid if we additionally require them to have the same codimension two Arf-invariants or additionally also the same Arf-invariants for some further codimensions.

Interestingly enough codimension two Arf-invariants are also important for the final part of this thesis, where we examine the (SCRP) in dimension eight. As already mentioned the (SCRP) in dimension eight can be solved (cf. [Cho11a]) if four specific automorphisms on a certain class of Bott manifolds can be realised. In Theorem 5.2 we show that one of these automorphisms can be realised if certain codimension two Arf-invariants vanish.

Organisation of this thesis:

In Section 2 we define Bott manifolds and recall their basic properties, e.g. we compute the cohomology ring of a Bott manifold and introduce two sets of generators of the cohomology ring, we determine the isomorphism class of the tangent bundle of a Bott manifold and their Stiefel-Whitney and Pontrjagin classes.
In Section 3 we recall the basic notions of modified surgery theory and the main theorem of modified surgery theory for even-dimensional manifolds. We adapt the main theorem, i.e. we deduce two corollaries, namely Corollary 3.11 and 3.12 which are convenient for the application to cohomology Bott manifolds.
Furthermore, we introduce tools which we need for the calculation of bordism groups that appear later on.
In Section 4 we use Corollary 3.11 to prove Theorem 4.2, i.e. that the number of diffeomorphism classes of cohomology Bott manifolds is finite. Then we construct the counter examples to cohomological rigidity of cohomology Bott manifolds in Theorem 4.10.
In Section 5 we examine whether one of the automorphisms mentioned above can be realised.

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2. Bott manifolds - Basic properties

In this section, we introduce Bott manifolds and their basic properties. We determine the cohomology ring and the homotopy groups of a Bott manifold. The cohomology ring of a Bott manifold, and later on a cohomology Bott manifold, plays a central role throughout this thesis.
Furthermore, we examine the tangent bundle and point out how the characteristic classes of a Bott manifold are determined by its cohomology ring.

2.1. Definition and cohomology ring

The manifolds, which are now called Bott manifolds, were first introduced in a paper by Bott and Samelson (cf. [BS58]). The name *Bott manifold* is due to [GK94], a paper by Grossberg and Karshon.
Bott manifolds are defined inductively. Given any Bott manifold, we obtain a new Bott manifold by projectivising some appropriate complex rank two vector bundle over the given one.

In this section, we consider fibre bundles obtained by projectivising complex vector bundles in general and then specialise to Bott manifolds which form one class of examples. For the remainder of this section, we fix $p\colon E \to X$ to be a smooth complex vector bundle of rank $r+1$ over a smooth manifold X. Moreover, we denote the fibrewise projectivisation of E by $P(p)\colon P(E) \to X$. We deduce basic properties of Bott manifolds from the general case of a projectivised bundle $P(E)$.
The trivial complex vector bundle of rank r is denoted by $\underline{\mathbb{C}}^r$, i.e. we suppress the projection and base space from notation. Furthermore, we denote a fibre bundle and its total space with the same symbol if the projection map is obvious.

Definition 2.1. Define B_0 to be a point. Assume inductively that B_{j-1} is defined and let $L_{j-1} \to B_{j-1}$ be some complex line bundle over B_{j-1}. Then B_j is the total space of the bundle $P(L_{j-1} \oplus \underline{\mathbb{C}}) \to B_{j-1}$. We obtain a sequence of fibre bundles

$$\begin{array}{ccccccccccc} & & \mathbb{CP}^1_{j+1} & & \mathbb{CP}^1_j & & \mathbb{CP}^1_{j-1} & & & & \\ & & \downarrow{\scriptstyle i_{j+1}} & & \downarrow{\scriptstyle i_j} & & \downarrow{\scriptstyle i_{j-1}} & & & & \\ \dots & \longrightarrow & B_{j+1} & \xrightarrow{\pi_{j+1}} & B_j & \xrightarrow{\pi_j} & B_{j-1} & \longrightarrow & \dots & \longrightarrow & B_0\,. \end{array}$$

We call the whole sequence a *Bott tower* and each B_j a *Bott manifold*.

Note that the first stage B_1 of a Bott tower is the complex projective space since all bundles over the point are trivial.
A Bott tower is not only equipped with a projection between any two stages but also

with a section

$$\begin{array}{ccc} \mathbb{C}\mathrm{P}^1_{j+1} & & \mathbb{C}\mathrm{P}^1_j \,. \\ \downarrow{\scriptstyle i_{j+1}} & & \downarrow{\scriptstyle i_j} \\ B_{j+1} & \underset{s_{j+1}}{\overset{\pi_{j+1}}{\rightleftarrows}} & B_j \end{array}$$

Restricted to an open subset $U \subset B_j$ such that $B_{j+1}|_U \cong U \times \mathbb{C}\mathrm{P}^1_{j+1}$, i.e. restricted to a local trivialisation the section is given by $b \overset{s_{j+1}}{\mapsto} (b, [1 : 0])$. This determines the section on all of B_j since all transition functions are elements in the projectivisation of $U(1) \oplus U(1) \subset U(2)$.

By the existence of the section, the long exact sequence of homotopy groups of the fibration $\mathbb{C}\mathrm{P}^1_j \to B_j \to B_{j-1}$ decomposes into split short exact sequences. Inductively we see

Lemma 2.2. *The homotopy groups of Bott manifolds are determined by the homotopy groups of S^2, namely $\pi_i(B_j) \cong \pi_i(S^2)^j$.*

Consider the more general situation, i.e. $E \to X$ is again a smooth complex vector bundle of rank $r+1$ over an arbitrary smooth manifold X. The total space $P(E)$ admits a *tautological line bundle* $\gamma \to P(E)$, which is defined analogously to the tautological line bundle over the complex projective spaces. Its total space γ consists of pairs $(p, v) \in P(E) \times E$ such that $v \in p$. The projection is given by $(p, v) \mapsto p$.

By calling this bundle *tautological* line bundle we stick to the conventions of [CMS10]. Standard text books as [MS74] refer to this bundle as the *canonical* bundle.

In the case of a Bott manifold $B_j = P(L_{j-1} \oplus \underline{\mathbb{C}})$ we denote the tautological bundle by γ_j, i.e. γ_j consists of the total space

$$\gamma_j := \{(b, v) \in B_j \times (L_{j-1} \oplus \underline{\mathbb{C}}) \mid v \in b\}$$

together with the obvious projection.

The first Chern class of the tautological line bundle $\gamma \to P(E)$ plays a central role for the description of the cohomology of $P(E)$. We denote its negative by $y := -c_1(\gamma)$. In the case of Bott manifolds we write

$$-c_1(\gamma_j) = y_j \in H^2(B_j; \mathbb{Z}). \tag{1}$$

We introduce the sign to ensure $\langle i_j^* y_j, [\mathbb{C}\mathrm{P}^1_j] \rangle = 1$, where $\langle\ ,\ \rangle$ denotes the Kronecker product. Furthermore, it allows an elegant description of the cohomology ring.

Once more, we turn back to the general situation of a complex rank $(r+1)$ vector bundle $E \to X$. Consider the pullback diagram

$$\begin{array}{ccccc} i^*\gamma & \longrightarrow & \gamma & & \\ \downarrow & & \downarrow & & \\ \mathbb{C}P^r & \xrightarrow{i} & P(E) & \longrightarrow & X . \end{array}$$

By definition, the pullback $i^*\gamma$ is the tautological bundle over the fibre. Hence, its first Chern class is a generator of the second cohomology of the fibre.
By the Leray-Hirsch Theorem (cf. [Hat02] Theorem 4D.1) the integral cohomology ring $H^*(P(E))$ is generated as a $H^*(X)$-module by $1, y, ..., y^r$.
From now on all cohomology will be integral cohomology unless otherwise indicated.

Again specialising to Bott manifolds, we see that $H^*(B_j)$ is a $H^*(B_{j-1})$-module on the generators 1 and y_j. Inductively we see that the cohomology groups $H^{2k}(B_j),\ k \leq j$ are generated by cup products $y_{i_1} \cup ... \cup y_{i_k},\ 1 \leq i_1 < ... < i_k \leq j$, where we suppress the pullbacks from notation from now on.

We still need to describe the ring structure. Consider $P(E) \to X$. From the definition of Chern classes using the splitting principle (cf. [Hus94, p. 248]), we get

$$H^*(P(E)) \cong H^*(X)[y]/(\sum_{i=0}^{r+1} c_i(E) y^{r+1-i}) \tag{2}$$

as rings. Owing to our choice of sign, i.e. defining y to be $-c_1(\gamma)$ there do not appear any signs in the sum.

We need some more notation.
For any manifold X the set of isomorphism classes of complex line bundles over X, denoted by $\mathcal{L}_{\mathbb{C}}(X)$, can be endowed with a group structure by the tensor product. The neutral element is the trivial line bundle, the inverse of some bundle is its dual bundle. We denote the inverse of a line bundle $L \to X$ by L^{-1}. With this group structure the first Chern class $c_1 \colon \mathcal{L}_{\mathbb{C}}(X) \to H^2(X)$ is an isomorphism of groups (cf. [Hus94, Theorem 3.4, p.250]).

Recall that each Bott manifold B_j is defined using a line bundle $L_{j-1} \to B_{j-1}$. Since $y_1, ..., y_{j-1}$ generate $H^2(B_{j-1})$ there exist $A^i_j \in \mathbb{Z}$, $i < j$ such that

$$L_{j-1} = \bigotimes_{i=1}^{j-1} \gamma_i^{A^i_j}.$$

Define $\alpha_j := \sum_{i=1}^{j-1} A^i_j y_i = -c_1(L_{j-1})$. Consequently, the total Chern class of $L_{j-1} \oplus \underline{\mathbb{C}}$ is given by $c(L_{j-1} \oplus \underline{\mathbb{C}}) = 1 - \alpha_j$. Hence, by (2) and induction we obtain the cohomology ring of B_j to be

$$\begin{aligned} H^*(B_j) &\cong H^*(B_{j-1})[y_j]/(y_j^2 = \alpha_j y_j) \\ &\cong \mathbb{Z}[y_1, ..., y_j]/(y_i^2 = \alpha_i y_i)_{i=1,...,j}. \end{aligned}$$

In particular, the elements $y_1, ..., y_j$ form a basis of $H^2(B_j)$. We refer to this basis as the *bundle basis* of $H^2(B_j)$.
Note that the cohomology ring was already determined in the paper by Bott and Samelson [BS58] in which Bott manifolds were first studied.

Using a naturality argument we now show that

$$s_j^*(y_j) = \alpha_j. \tag{3}$$

For this purpose, we show that the pullback of γ_j to B_{j-1} along the section is the defining bundle L_{j-1}.
For now, let $p_{j-1}\colon L_{j-1} \to B_{j-1}$ denote the projection of the defining bundle an let $v \in p_{j-1}^{-1}(b)$ be an element in the fibre over $b \in B_{j-1}$. Furthermore, let (b, w) be an element in the total space of $\underline{\mathbb{C}} \to B_{j-1}$. If $(v, w) \neq (0, 0)$ we denote the induced element in $B_j = P(L_{j-1} \oplus \underline{\mathbb{C}})$ by $(b, [v : w])$. Moreover, we denote an element in the total space of γ_j which projects to $(b, [v : w])$ by (b, v', w'), i.e. $(v', w') \in [v : w]$. By the definition of pullback of a fibre bundle,

$$s_j^*(\gamma_j) = \left\{ \left((b, v, w), b'\right) \in \gamma_j \times B_{j-1} \mid (b, [v : w]) = (b', [1 : 0]) \right\}.$$

This only holds if $b = b'$ and $[v : w] = [1 : 0]$. The second equation only holds, if $w = 0$. Hence, there is an isomorphism $f\colon s_j^*\gamma_j \to L_{j-1}$ of vector bundle defined by $(b, (v, 0), b) \mapsto (b, v)$. By naturality Equation (3) follows.

So far we only considered the basis of $H^*(B_j)$ which is most commonly used in the literature, e.g. [CMS10]. But later on, we need another basis which we introduce now.
In a sense, this new basis is very geometric because it is defined by considering homology classes which are induced by embedded submanifolds in B_j. The submanifolds are the fibres $\mathbb{C}\mathrm{P}_i^1,\ i \leq j$ of the Bott tower and the first Bott stage $\mathbb{C}\mathrm{P}_1^1 := B_1$ which are embedded by the appropriate compositions of inclusion of fibres and sections. We denote the induced elements in $H_2(B_j)$ by

$$\begin{aligned} \sigma_1 &:= [s_j \circ \ldots \circ s_2(\mathbb{C}\mathrm{P}_1^1)] \\ \sigma_2 &:= [s_j \circ \ldots \circ s_3 \circ i_2(\mathbb{C}\mathrm{P}_2^1)] \\ &\vdots \\ \sigma_{j-1} &:= [s_j \circ i_{j-1}(\mathbb{C}\mathrm{P}_{j-1}^1)] \\ \sigma_j &:= [i_j(\mathbb{C}\mathrm{P}_j^1)]. \end{aligned}$$

Furthermore, denote their Kronecker duals by $x_k \in H^2(B_j)$, i.e.

$$\langle x_k, \sigma_l \rangle = \delta_{kl}$$

for $k, l \leq j$. We claim that the elements $x_1, ..., x_j \in H^2(B_j)$ form another basis of $H^2(B_j)$ which we call the *geometric basis*. The base change between the bundle basis and the geometric basis is given in the following Lemma.

Lemma 2.3. *Let y_k, α_k and x_k, $k \leq j$ be as above. Then*

$$x_k = y_k - \alpha_k.$$

In particular, this implies that $\{x_1, \ldots, x_j\}$ is another basis for $H^2(B_j)$ and generates $H^*(B_j)$ as a ring. The proof is elementary.

Proof. For this proof it is necessary to spell out all pullbacks. In particular, note that

$$i_l^* \pi_l^* \circ \ldots \circ \pi_{k+1}^* y_k = 0 \tag{4}$$

for $k < l$. Furthermore, recall that $\alpha_k = -c_1(L_{k-1}) \in H^2(B_{k-1})$.
The defining property of the x_k is how they evaluate on σ_l. Hence we only need to show that $\langle \pi_j^* \circ \ldots \circ \pi_{k+1}^*(y_k - \alpha_k), \sigma_l \rangle = \delta_{kl}$.
Let f_l denote s_2 if $l = 1$ and i_l otherwise. There are three different cases: $k < l, k = l$ and $k > l$.
We start with $k > l$:

$$\begin{aligned}
&\langle \pi_j^* \ldots \pi_{k+1}^*(y_k - \pi_k^* \alpha_k), (s_j)_* \ldots (s_{l+1})_* (f_l)_* [\mathbb{C}\mathrm{P}_l^1] \rangle \\
&\qquad = \langle y_k - \pi_k^* \alpha_k, (s_k)_* \ldots (f_l)_* [\mathbb{C}\mathrm{P}_l^1] \rangle \\
&\qquad = \langle s_k^* y_k - \alpha_k, (s_{k-1})_* \ldots (f_l)_* [\mathbb{C}\mathrm{P}_l^1] \rangle \\
&\qquad = 0 \text{ by Equation (3).}
\end{aligned}$$

If $l < k$ the claim holds by Equation (4).
It remains to check that $\langle \pi_j^* \ldots \pi_{k+1}^*(y_k - \pi_k^* \alpha_k), \sigma_k \rangle = 1$:

$$\begin{aligned}
\langle \pi_j^* ... \pi_{k+1}^*(y_k - \pi_k^* \alpha_k), \sigma_k \rangle &= \langle \pi_j^* ... \pi_{k+1}^*(y_k - \pi_k^* \alpha_k), (s_j)_* ... (f_k)_* [\mathbb{C}\mathrm{P}_k^1] \rangle \\
&= \langle f_k^* y_k - f_k^* \pi_k^* \alpha_k, [\mathbb{C}\mathrm{P}_k^1] \rangle \\
&= \langle f_k^* y_k [\mathbb{C}\mathrm{P}_k^1] \rangle - \langle f_k^* \pi_k^* \alpha_k, [\mathbb{C}\mathrm{P}_k^1] \rangle \\
&= 1.
\end{aligned}$$

Here the last equation holds for the following reasons: By definition the pullback $f_k^* y_k$ is the generator of $H^2(\mathbb{C}\mathrm{P}_k^1)$ which is Kronecker dual to $[\mathbb{C}\mathrm{P}_k^1]$. Furthermore, the map f_k is the inclusion of the fibre if $k \neq 1$, i.e. $f_k^* \pi_k^* = 0$ and if $k = 1$ the $f_1^* \pi_1^* = s_2^* \pi_1^* = 0$ since π_1 is the projection to $B_0 = pt$. □

We introduce another nice geometric interpretation of the x_i in the case of a Bott manifold of dimension eight.

Remark 2.4. Let

$$\begin{array}{ccccccc} \mathbb{C}\mathrm{P}^1_4 & & \mathbb{C}\mathrm{P}^1_3 & & \mathbb{C}\mathrm{P}^1_2 & & \\ \downarrow{\scriptstyle i_4} & & \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle i_2} & & \\ B_4 & \xrightarrow{\pi_4} & B_3 & \xrightarrow{\pi_3} & B_2 & \xrightarrow{\pi_2} & \mathbb{C}\mathrm{P}^1_1 \end{array}$$

denote a Bott tower of height four. In addition to the section

$$s_k \colon B_{k-1} \to B_k,\ b \mapsto (b, [1:0])$$

there is a second section

$$s_k^\infty \colon B_{k-1} \to B_k,\ b \mapsto (b, [0:1]).$$

In B_4 there exist the following submanifolds:

$$P_1 := B_4|_{B_3|_{i_2(\mathbb{C}\mathrm{P}^1_2)}}, \quad P_2 := B_4|_{B_3|_{s_2^\infty(\mathbb{C}\mathrm{P}^1_1)}}, \quad P_3 := B_4|_{s_3^\infty(B_2)} \text{ and } \quad P_4 := s_4^\infty(B_3).$$

By definition they are Bott manifolds of dimension six.
Let f_l be as in the proof above. Abbreviate $s_4 \circ .. \circ f_l(\mathbb{C}\mathrm{P}^1_l)$ by $s_l(\mathbb{C}\mathrm{P}^1_l)$. Observe that $s_l(\mathbb{C}\mathrm{P}^1_l)$ and P_k intersect in one point if $l = k$. If $l \neq k$ they are disjoint.
We can consider the induced homology classes $\rho_i := [P_i]$.
Let $\bullet \colon H_6(B_4) \times H_2(B_4) \to H_0(B_4)$ denote the intersection product (cf.[Bre93] Chapter VI.11). If the homology classes in consideration are given by submanifolds Theorem VI 11.9 of [Bre93] allows us to calculate the intersection product by counting intersection points of the underlying submanifolds. Thus,

$$\rho_l \bullet \sigma_k = \delta_{lk}.$$

Let M be an oriented, connected, closed manifold of dimension n. Sticking to the notation of [Bre93] let $D \colon H_i(M) \to H^{n-i}(M)$ denote the inverse of the Poincaré duality isomorphism. By definition $\rho_l \bullet \sigma_k = D(\rho_l) \cap \sigma_k = \delta_{lk}$. Thus, x_l as in the Lemma is the Poincaré dual of ρ_l

$$x_l \cap [B_4] = \rho_l.$$

2.2. Tangent bundle and characteristic classes

Using that $s_j^* \gamma_j = \alpha_j$ (cf. Equation (3)), we can determine the tangent bundle TB_j. We claim

$$TB_j \cong \bigoplus_{i=1}^{j} \gamma_i^{-2} \otimes L_{i-1}.$$

Since $TB_j \cong \pi_j^* TB_{j-1} \oplus T_{fib}B_j$ determining the tangent bundle amounts to determining the fibrewise tangent bundle of the fibre bundle $B_j \to B_{j-1}$. Here, the fibrewise tangent bundle is a complex line bundle, i.e. it is determined by its first Chern class

$$c_1(T_{fib}B_j) =: \sum_{i=1}^{j} \lambda_i y_i \in H^2(B_j).$$

Let y denote the negative of $c_1(\gamma)$ where γ is the tautological bundle over $\mathbb{C}\mathrm{P}^1_j$.
By definition the fibrewise tangent bundle pulls back to the tangent space of the fibre under the inclusion of the fibre. Consequently, we obtain

$$2y = c_1(T\mathbb{C}\mathrm{P}^1_j) = c_1(i_j^* T_{fib}B_j) = i^*(\sum_{i=1}^{j} \lambda_i y_i) = \lambda_j y,$$

where the last equality holds by Equation (4). Hence $\lambda_j = 2$.
On the other hand, we can consider the pullback of the fibrewise tangent bundle along the section $s_j \colon B_{j-1} \to B_j = P(L_{j-1} \oplus \underline{\mathbb{C}}),\ b \mapsto (b, [1,0])$ and obtain the normal bundle $\nu(B_{j-1} \stackrel{s_j}{\to} B_j) \cong L_{j-1}^{-1}$ (cf. Section 2.3). Therefore,

$$\begin{aligned} \alpha_j &= c_1(L_{j-1}^{-1}) = c_1(s_j^*(T_{fib}B_j)) \\ &= s_j^*(c_1(T_{fib}B_j)) = s_j^*(2y_j) + \sum_{i=1}^{j-1} \lambda_i y_i. \end{aligned}$$

Since $\alpha_j = s_j^* y_j$ we obtain $\sum_{i=1}^{j-1} \lambda_i y_i = -\alpha_j$. Hence, the first Chern class $c_1(T_{fib}B_j)$ equals $2y_j - \alpha_j$, i.e. $T_{fib}B_j \cong \gamma_j^{-2} \otimes L_{j-1}$

Another way to determine the fibrewise tangent bundle is to use Borel and Hirzebruch's general formula (cf. [BH58]) for the total Chern class of the fibrewise tangent bundle of $P(E) \to X$, for E and X as before. They determine

$$c(T_{fib}P(E)) = \sum_{q=0}^{r+1} (1+y)^{r+1-q} c_q(E),$$

which leads to the same result as above.

By the Whitney sum formula the total Chern class of a Bott manifold with tangent bundle TB_j is

$$c(TB_j) = c(\bigoplus_{i=1}^{j} T_{fib}B_i) = \prod_{i=1}^{j} (1 + 2y_i - \alpha_i).$$

The odd Stiefel-Whitney classes are the mod two reduction of the Chern classes. Thus, we see that the Stiefel-Whitney classes are determined by the α_i. Similarly, the Pontrjagin classes are determined by

$$\begin{aligned} c(TB_j \otimes \mathbb{C}) &= c(\bigoplus_{i=1}^{j} T_{fib}B_i) \cup c(\bigoplus_{i=1}^{j} \overline{T_{fib}B_i}) \\ &= \prod_{i=1}^{j}(1 + 2y_i - \alpha_i) \cup \prod_{i=1}^{j}(1 - 2y_i + \alpha_i) = \prod_{i=1}^{j}(1 - \alpha_i^2). \end{aligned}$$

In this sense the cohomology ring determines the Stiefel-Whitney and Pontrjagin classes.

2.3. Global description

One way to understand the normal bundle of the section $s_i \colon B_{i-1} \to B_i$ is to use the global description of Bott manifolds. Global description here means that we introduce a Bott manifold B_i as a quotient of $(\mathbb{C}^2/\{0\})^i$. Apparently, this was first done in [CM12].

The complex projective space is very well-known to be

$$\mathbb{C}^2 - \{0\}/\sim,$$

where two points in $(p_1, q_1), (\widetilde{p}_1, \widetilde{q}_1) \in \mathbb{C}^2 - \{0\}$ are equivalent if $(p_1, q_1) = z_1(\widetilde{p}_1, \widetilde{q}_1)$, for some $z_1 \in \mathbb{C}^*$. As usual, we denote the equivalence classes under this relation by $[p_1 : q_1]$.
The A_2^1-th tensor power of the tautological line bundle is then given by

$$\mathbb{C}^2 - \{0\} \times \mathbb{C}/\sim,$$

where two points (p_1, q_1, p_2) and $(\widetilde{p}_1, \widetilde{q}_1, \widetilde{p}_2)$ are equivalent if

$$(p_1, q_1, p_2) = (z_1\widetilde{p}_1, z_1\widetilde{q}_1, z_1^{-A_2^1} p_2),$$

for some $z_1 \in \mathbb{C}^*$. Adding a trivial line bundle amounts to adding a fourth coordinate q_2. After projectivising we get a stage two Bott manifold

$$B_2 = (\mathbb{C}^2 - \{0\})^2/\sim,$$

where two points (p_1, q_1, p_2, q_2) and $(\widetilde{p}_1, \widetilde{q}_1, \widetilde{p}_2, \widetilde{q}_2)$ are equivalent if

$$(p_1, q_1, p_2, q_2) = (z_1\widetilde{p}_1, z_1\widetilde{q}_1, z_2 z_1^{-A_2^1}\widetilde{p}_2, z_2\widetilde{q}_2)$$

for some $z_1,\ z_2 \in \mathbb{C}^*$.
Generalising this procedure, we obtain a Bott manifold by

$$B_j = (\mathbb{C}^2 - \{0\})^j/\sim,$$

where $((p_1, q_1), ..., (p_j, q_j))$ and $\left(z_1(p_1, q_1), z_2(z_1^{-A_2^1} p_2, q_2), ..., z_j(\prod_{i=1}^{j-1} z_i^{-A_j^i} p_j, q_j)\right)$ are equivalent for all $(z_1, ..., z_j) \in (\mathbb{C}^*)^j$.
We can also say that the Bott manifold B_j is the orbit space of the free, proper and smooth action $(\mathbb{C}^*)^j \times (\mathbb{C}^2 - \{0\})^j \to (\mathbb{C}^2 - \{0\})^j$ defined by

$$((z_1, ..., z_j), ((p_1, q_1), ..., (p_j, q_j))) \mapsto \left(z_1(p_1, q_1), ..., z_j(\prod_{i=1}^{j-1} z_i^{-A_j^i} p_j, q_j)\right).$$

The A_{j+1}^j-st power of the tautological line bundle over B_j is given by

$$\gamma_j^{A_{j+1}^j} = \{((p_1, q_1), ..., (p_j, q_j), p_{j+1})) \in (\mathbb{C}^2/\{0\})^j \times \mathbb{C}\}/\sim,$$

where $((p_1, q_1), ..., (p_j, q_j), p_{j+1})$ and $\left(z_1(p_1, q_1), ... z_j(\prod_{i=1}^{j-1} z_i^{-A_j^i} p_j, q_j), z_j^{-A_{j+1}^j} p_{j+1}\right)$ are equivalent for $z_i \in \mathbb{C}^*$, $i \leq j$. We denote the equivalence classes by brackets again.
In this setting, the section $s_j \colon B_{j-1} \to B_j$ is given by

$$[p_1 : q_1 : ... : p_{j-1} : q_{j-1}] \mapsto [p_1 : q_1 : ... : p_{j-1} : q_{j-1} : 1 : 0].$$

It is obviously well-defined.
Moreover, we see that a tubular neighbourhood of $s_j(B_{j-1})$, i.e. a disk bundle of the normal bundle of $D(\nu(B_{j-1} \to B_j))$ consists of points which admit preferred representatives of the form $(p_1, q_1, ..., p_{j-1}, q_{j-1}, 1, p_j^{-1} q_j)$. Changing the representative for a point in $(p_1, q_1, ..., p_{j-1}, q_{j-1}) \in B_{j-1}$ by the action of some $(z_1, ..., z_{j-1}) \in (\mathbb{C}^*)^{j-1}$ amounts to changing the last coordinate $p_j^{-1} q_j$ of the preferred representative by $\prod_{i=1}^{j-1} z_i^{A_j^i}$. This is one way to see that the normal bundle $\nu(B_{j-1} \to B_j)$ is isomorphic to $L_{j-1}^{-1} \cong \bigotimes \gamma_i^{-A_j^i}$.

3. Modified surgery theory

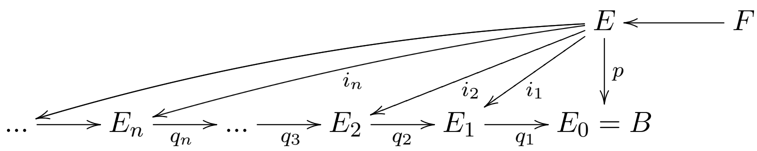
In this section, we introduce the methods we use to examine eight-dimensional cohomology Bott manifolds on the one hand, and the strong cohomological rigidity conjecture in dimension eight on the other hand.
For the reader's convenience, we summarise the most important notions of modified surgery theory as developed in [Kre99]. Then we adapt the main theorem of modified surgery theory for even-dimensional manifolds to our situation.
Afterwards, we develop some tools to calculate bordism groups that appear in this setting.

3.1. Postnikov decompositions

This section recalls the notion of a Postnikov decomposition of a fibration and introduces some necessary notation.
Furthermore, we present a result that connects a differential in the Leray-Serre spectral sequence of a principal fibration with fibre an Eilenberg-MacLane space, to the classifying map of the fibration.

Consider a fibration $F \to E \xrightarrow{p} B$ of path-connected CW-spaces. Then there exists a *Postnikov decomposition* (cf. [Bau77] p. 306 ff.).

Theorem 3.1. *For a fibration $F \to E \xrightarrow{p} B$ as above, there exists a commutative diagram*

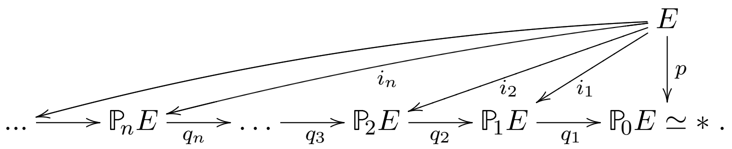


such that for all $j \geq 0$

- *the maps $q_{j+1}\colon E_{j+1} \to E_j$ are fibrations with fibre the Eilenberg-MacLane space $K(\pi_{j+1}(F), j)$,*
- *the maps $i_j\colon E \to E_j$ are $(j+1)$-connected, i.e. for $k \leq j$ they induce isomorphisms $\pi_k(E) \to \pi_k(E_j)$ and an epimorphism $\pi_{j+1}(E) \to \pi_{j+1}(E_j)$,*
- *the maps $p_j := q_1 \circ \ldots \circ q_j\colon E_j \to B$ are $(j+1)$-co-connected, i.e. for $k \geq j+1$ they induce isomorphisms $\pi_k(E_j) \to \pi_k(B)$ and a monomorphism $\pi_j(E_j) \to \pi_j(B)$.*

Each space E_j is unique up to fibre homotopy. It is called the j-th Postnikov stage *of the fibration $E \to B$. The whole tower is called* Postnikov decomposition *of the fibration $E \to B$.*

Instead of considering an arbitrary fibration $E \to B$ as above, we now restrict to the case $B \simeq *$ to get the well-known *Postnikov tower* of a path-connected space E. In this situation we denote each stage by $\mathbb{P}_j E$. The diagram simplifies to



From now on, we assume that E is simply connected. Note that the Postnikov tower can be constructed functorially under this assumption. The functorial construction can be found in Chapter IX of [Whi78].
Furthermore, each fibration $\mathbb{P}_j E \to \mathbb{P}_{j-1} E$ is a principal fibration (cf. [Hat02], Theorem 4.69). Therefore, there exists a map $k_j \colon \mathbb{P}_{j-1} E \to K(\pi_j(E), j+1)$, which is called the *j-th k-invariant* of the Postnikov tower, such that the fibration $q_j \colon \mathbb{P}_j E \to \mathbb{P}_{j-1} E$ is the pullback of the path-loop fibration

$$K(\pi_j(E), j) = \Omega K(\pi_j(E), j+1) \to PK(\pi_j(E), j+1) \to K(\pi_j(E), j+1)$$

along k_j.

Recall that for us cohomology is always integral cohomology unless otherwise indicated. Later on, we need tools to calculate the cohomology of a Postnikov stage $H^k(\mathbb{P}_j E)$. For $k \leq j$ this is rather easy.

Lemma 3.2. *Let $\mathbb{P}_j E$ be the j-th Postnikov stage of a simply connected space E. Then $H^k(\mathbb{P}_j E) \cong H^k(E)$ for $k \leq j$.*

Proof. First we apply the mapping cylinder construction to turn the map $i_j \colon E \to \mathbb{P}_j(E)$ into an inclusion. Hence we can consider the pair $(\mathbb{P}_j E, E)$. Since i_j is $(j+1)$-connected the relative homotopy groups $\pi_k(\mathbb{P}_j E, E)$ vanish for $k \leq j+1$.
By assumption E is simply connected, whence we can apply the relative Hurewicz theorem. We deduce that $H_k(\mathbb{P}_j E, E)$ also vanishes for $k \leq j+1$.
The universal coefficient theorem implies $H^k(\mathbb{P}_j E, E) = 0$ for $k \leq j+1$. Thus, we obtain the lemma by the long exact sequence in cohomology of the pair $(\mathbb{P}_j E, E)$. □

One approach to the calculation of the cohomology groups $H^k(\mathbb{P}_j E)$ for $k > j$ is given by the application of the cohomological Leray-Serre spectral sequence with integral coefficients. But to successfully determine $H^k(\mathbb{P}_j E)$, at least in a range, we need to understand the differentials.

Consider the fibration $K(\pi_j(E), j) \to \mathbb{P}_j E \to \mathbb{P}_{j-1} E$. The fibre $K(\pi_j(E), j)$ is $(j-1)$-connected. Again applying the Hurewicz theorem and the universal coefficient theorem we see

$$\widetilde{H}^k(K(\pi_j(E),j);\pi_j(E)) \cong \begin{cases} 0 & \text{if } k < j \\ \mathrm{Hom}(\pi_j(E),\pi_j(E)) & \text{if } k = j. \end{cases}$$

Let $E_{j+1}^{p,q}$ be the (p,q)-entry on the $(j+1)^{st}$ page of the Leray-Serre spectral sequence of the fibration $K(\pi_j(E),j) \to \mathbb{P}_j E \to \mathbb{P}_{j-1}E$ with coefficients in $\pi_j(E)$.
All differentials $d_k\colon E_k^{0,j} \to E_k^{k,j-k+1}$ for $k \leq j$ vanish, since $\widetilde{H}^l(K(\pi_j(E),j);\pi_j(E)) = 0$ for $l < j$. Thus $E_{j+1}^{0,j} \cong H^j(K(\pi_j(E),j);\pi_j(E))$ and

$$d_{j+1}\colon E_{j+1}^{0,j} \to E_{j+1}^{j+1,0}$$

is the first differential, with this domain, that need not vanish. It is also the first differential that can hit $E_{j+1}^{j+1,0}$. Thus we can identify $E_{j+1}^{j+1,0} \cong H^{j+1}(\mathbb{P}_{j-1}E;\pi_j(E))$.

Lemma 55 in [Mül09a] connects the differential $d_{j+1}\colon E_{j+1}^{0,j} \to E_{j+1}^{j+1,0}$ to the k-invariant of $\mathbb{P}_j E \to \mathbb{P}_{j-1}E$. Let Δ denote the canonical generator of $H^{j+1}(K(\pi_j(E),j);\pi_j(E))$ and let $\kappa_j \in H^{j+1}(\mathbb{P}_{j-1}E;\pi_j(E))$ denote the cohomology class that corresponds to the k-invariant $k_j\colon \mathbb{P}_{j-1}E \to K(\pi_j(E),j+1)$ under the isomorphism

$$[\mathbb{P}_{j-1}E, K(\pi_j(E),j+1)] \to H^{j+1}(\mathbb{P}_{j-1}E,\pi_j(E)).$$

Lemma 3.3. *The differential $d_{j+1}\colon E_{j+1}^{0,j} \to E_{j+1}^{j+1,0}$ maps the canonical generator Δ of $H^j(K(\pi_j(E),j);\pi_j(E))$ to $\kappa_j \in H^{j+1}(\mathbb{P}_{j-1}E;\pi_j(E))$.*

Now, let $E_{j+1}^{p,q}$ be the (p,q)-entry on the $(j+1)^{st}$ page of the Leray-Serre spectral sequence with integral coefficients.
Later on, we need the integral Leray-Serre spectral sequence for fibrations of the form $K(\pi_j(E),j) \to \mathbb{P}_j E \to \mathbb{P}_{j-1}E$. Fortunately, Proposition 4.4 in [Mül09b] determines the differential $d_{j+1}\colon E_{j+1}^{0,j} \to E_{j+1}^{j+1,0}$ for principal fibrations $P \to B$ whose fibre is an Eilenberg-MacLane space, under some conditions.

Proposition 3.4. *[Mül09b, Proposition 4.4] Let π be a finitely generated, free abelian group. Furthermore, let $P \to B$ be a principal fibration with fibre $K(\pi,j)$ such that B is homotopy equivalent to a CW-complex and assume that $H_i(B)$ is finitely generated for $i \leq j+2$.*
Let $k\colon B \to K(\pi,j+1)$ be the classifying map for $P \to B$ and let $\kappa \in H^{j+1}(B;\pi)$ be the induced class in cohomology.
Then there exists a natural isomorphism

$$\Psi\colon H^{j+1}(B;\pi) \to \mathrm{Hom}(H^j(K(\pi,j)), H^{j+1}(B))$$

such that $\Psi(\kappa) = (d_{j+1}\colon E_{j+1}^{0,j} \to E_{j+1}^{j+1,0})$.

Let M be a closed, simply connected manifold with finitely generated, free abelian homotopy group $\pi_j(M)$. As stated above, the fibration $K(\pi_j(M), j) \to \mathbb{P}_j M \to \mathbb{P}_{j-1} M$ is principal with classifying map $k = k_j$, where k_j is the j-th k-invariant. Thus, we can apply the proposition to this setting.

3.2. Main theorem of modified surgery theory for even-dimensional manifolds

In this section, we recall some definitions and statements of [Kre99] for the convenience of the reader. Some of the definitions are rather algebraic. Another reference for the algebraic part is [CS11b].

We start with the definition of the normal k-type of a manifold.
Let M be a smooth n-dimensional manifold. By Whitney's embedding theorem there exists a smooth embedding of M in $\mathbb{R}^{n+r}$ for $r \geq n$. Such an embedding is unique up to isotopy if $r \geq n+1$ (cf. [Wu58]).
The normal Gauss map $\nu_r \colon M \to BO_r$ of an embedding $\varphi \colon M \to \mathbb{R}^{n+r}$ is a representative for the homotopy class of maps which classify the normal bundle $\nu(M \xrightarrow{\varphi} \mathbb{R}^{n+r})$.
Let BO be the direct limit of all BO_r, $i_r \colon BO_r \to BO$ the inclusion. We call $i_r \circ \nu_r$ the stable normal Gauss map of the embedding φ. Since, for $N \geq 2n+1$, any two embeddings into $\mathbb{R}^N$ are isotopic their stable normal Gauss maps are homotopic. Thus, the stable normal Gauss map is unique up to homotopy.

Definition 3.5. [Kre99, p. 711] Let M be a smooth n-dimensional manifold and let $\nu \colon M \to BO$ be its stable normal Gauss map. Furthermore, let $p \colon \mathbb{B} \to BO$ be a fibration. If there exists a lift of the stable normal Gauss map, i.e. if there exists a map $\widetilde{\nu} \colon M \to \mathbb{B}$ such that the diagram

$$\begin{array}{ccc} & & \mathbb{B} \\ & \overset{\widetilde{\nu}}{\nearrow} & \downarrow p \\ M & \xrightarrow{\nu} & BO \end{array}$$

commutes up to homotopy, then M admits a *normal* $\mathbb{B}$*-structure*.
If $\widetilde{\nu} \colon M \to \mathbb{B}$ is $(k+1)$-connected, i.e. if $\widetilde{\nu}_* \colon \pi_i(M) \to \pi_i(\mathbb{B})$ is an isomorphism for $i \leq k$ and onto for $i = k+1$, we call $\widetilde{\nu}$ a *normal k-smoothing* into $\mathbb{B}$.
If, furthermore, $p \colon \mathbb{B} \to BO$ is $(k+1)$-co-connected, i.e. if $p_* \colon \pi_i(\mathbb{B}) \to \pi_i(BO)$ is injective for $i = k+1$ and an isomorphism for $i \geq k+2$, we call $\mathbb{B}$ the *normal k-type* of M.
Let M_0 and M_1 be two n-dimensional manifolds which admit normal $\mathbb{B}$-structures $\widetilde{\nu}_i \colon M_i \to \mathbb{B}$, then M_0 and M_1 are *normally* $\mathbb{B}$*-bordant* if there exists a compact manifold W of dimension $n+1$ and a normal $\mathbb{B}$-structure $\widetilde{\nu} \colon W \to \mathbb{B}$ such that $\partial W = M_0 \cup M_1$ and $\widetilde{\nu}|_{M_i} = \widetilde{\nu}_i$.

Being $\mathbb{B}$-bordant is an equivalence relation on manifolds which admit a normal $\mathbb{B}$-structure. The set of equivalence classes of all k-dimensional manifolds which admit a $\mathbb{B}$-structure turns out to be a group, the $\mathbb{B}$-*bordism group* which we denote by $\Omega_k^{\mathbb{B}}$. For more details on $\mathbb{B}$-structures and $\mathbb{B}$-bordism we refer the reader to [Sto68].
Employing the pathspace construction, we can consider the stable normal Gauss map as a fibration. Thus, we see that the normal k-type of a manifold is the k-th stage of the Postnikov decomposition of its stable normal Gauss map. Hence, by Theorem 3.1, the normal k-type of M is unique up to fibre homotopy equivalence. Therefore, we denote it by $\mathbb{B}_k(M)$. Normal k-smoothings, however, are not unique in general.

Before we can cite the main theorem of modified surgery theory for even-dimensional manifolds, we still need to define the surgery obstruction in the setting of modified surgery theory. It is an element in the so-called "little l"-monoid, which we define next.

Let π be a group together with a homomorphism $w\colon \pi \to \mathbb{Z}/2$ and let $\Lambda := \mathbb{Z}[\pi]$ be its integral group ring. On Λ, there exists an involution defined by

$$\begin{aligned} \bar{\ }\colon \Lambda &\to \Lambda \\ \sum_{g\in\pi} \lambda_g \cdot g &\mapsto \overline{\sum_{g\in\pi} \lambda_g \cdot g} := \sum_{g\in\pi} \lambda_g w(g) g^{-1}. \end{aligned}$$

Here, $w(g)$ acts by sign.
We work with left Λ-modules. Applying the involution we can turn every right Λ-module into a left one.

Let $\epsilon \in \{\pm 1\}$ and consider $S := \{s - \epsilon\overline{s} \mid s \in \Lambda\}$. The maps

$$\begin{aligned} \Lambda/S \times \Lambda/S &\to \Lambda/S,\ ([x],[y]) \mapsto [x+y], \\ \Lambda \times \Lambda/S &\to \Lambda/S,\ (x,[y]) \mapsto [xy\overline{x}] \text{ and} \\ \Lambda/S &\to \Lambda,\ [x] \mapsto x + \epsilon\overline{x} \end{aligned}$$

are well-defined. From now on we omit the brackets in the notation.

Definition 3.6. [Kre99, p. 725] Let $\epsilon \in \{\pm 1\}$ and let V be a left $\Lambda-$module. An ϵ-*quadratic form* is a triple (V, λ, μ), where $\lambda\colon V \times V \to \Lambda$ and $\mu\colon V \to \Lambda/S$ are maps such that, for all $v, w \in V$ and $x \in \Lambda$:

i) $\lambda_v\colon V \to \Lambda,\ w \mapsto \lambda_v(w) := \lambda(w, v)$ is an element in $Hom_\Lambda(V, \Lambda)$,

ii) $\lambda(v, w) = \epsilon\overline{\lambda(w, v)}$,

iii) $\mu(v + w) = \mu(v) + \mu(w) + \lambda(v, w) \in \Lambda/S$,

iv) $\mu(xv) = x\mu(v)\overline{x}$ and

v) $\lambda(v,v) = \mu(v) + \epsilon\overline{\mu(v)} \in \Lambda$.

We call λ *intersection form* and μ *quadratic refinement.*

Note that there is a natural way to add quadratic forms. Let (V_i, λ_i, μ_i), $i = 1, 2$ be two ϵ-quadratic forms. We define the sum of the two intersection forms, $\lambda_1 \oplus \lambda_2 \colon V_1 \oplus V_2 \to \Lambda$, by $(\lambda_1 \oplus \lambda_2)(u,v) := \lambda_1(u_1, v_1) + \lambda_2(u_2, v_2)$ for $(u_1, u_2) = u$ and $(v_1, v_2) = v$ two elements of $V_1 \oplus V_2$. The same works for the quadratic refinements.
We denote the sum of two quadratic forms by

$$(V_1, \lambda_1, \mu_1) \perp (V_2, \lambda_2, \mu_2) := (V_1 \oplus V_2, \lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2).$$

Let e_1, e_2 denote the standard basis for the $\Lambda-$module $\Lambda \oplus \Lambda$.
For us, the most important example of an ϵ-quadratic form is the triple $(\Lambda \oplus \Lambda, \lambda_\epsilon, \mu_\epsilon)$, where λ_ϵ is defined by $\lambda_\epsilon(e_i, e_i) = 0$ for $i = 1, 2$ and $\lambda_\epsilon(e_1, e_2) = 1$ and $\mu_\epsilon(e_i) = 0$ for $i = 1, 2$. This form is called the *standard ϵ-hyperbolic form.* Let $(\Lambda \oplus \Lambda, \lambda_\epsilon, \mu_\epsilon)^{\perp_r}$ denote its r-fold sum.

Recall that Λ is the group ring $\mathbb{Z}[\pi]$ and that $w \colon \pi \to \mathbb{Z}/2$ is a homomorphism.
We consider two bases of a $\Lambda-$module as equivalent if the matrix of the base change vanishes in the Whitehead group $Wh(\pi)$ which is defined in [Lüc02, Chapter 2.1].
A $\Lambda-$module is called *based* if it is equipped with an equivalence class of bases. Let V and V' be based $\Lambda-$modules. An isomorphism $\phi \colon V \to V'$ is called *simple* if its matrix with respect to the two equivalence classes of bases vanishes in $Wh(\pi)$.

The objects of the "little l"-monoids are represented by tuples $((\Lambda \oplus \Lambda, \lambda_\epsilon, \mu_\epsilon)^{\perp_r}, V)$ of the r-fold sum of the hyperbolic form and a based, half-rank direct summand V of Λ^{2r}.
Next, we define a equivalence relation on such tuples.

First of all we stabilise, i.e. we identify the tuples

$$\left((\Lambda \oplus \Lambda, \lambda_\epsilon, \mu_\epsilon)^{\perp_r}, V\right) \text{ and } \left((\Lambda \oplus \Lambda, \lambda_\epsilon, \mu_\epsilon)^{\perp_{r+1}}, V \oplus (\Lambda \times \{0\})\right).$$

Following Wall [Wal70] we define $TU^\epsilon(\Lambda^{2r})$ to be the group of those isometries

$$\phi \colon (\Lambda \oplus \Lambda, \lambda_\epsilon, \mu_\epsilon)^{\perp_r} \to (\Lambda \oplus \Lambda, \lambda_\epsilon, \mu_\epsilon)^{\perp_r},$$

of the r-fold sum of the standard ϵ-hyperbolic form, whose restriction to $\Lambda^r \times \{0\}$ is a simple isomorphism.
We denote the direct limit with respect to the inclusions $TU^\epsilon(\Lambda^{2r}) \to TU^\epsilon(\Lambda^{2r+2})$ by $TU^\epsilon(\Lambda)$.
Let $\sigma \colon \Lambda \oplus \Lambda \to \Lambda \oplus \Lambda$ be defined by $\sigma(e_1) = \epsilon e_2$ and $\sigma(e_2) = e_1$ for $\{e_1, e_2\}$ the standard basis of $\Lambda \oplus \Lambda$, as before. We call σ the *flip map.*
Finally, let $RU^\epsilon(\Lambda)$ be the group generated by elements in $TU^\epsilon(\Lambda)$ and by the flip map.

Now we have assembled all objects necessary to define the "little l"-monoid.

Definition 3.7. [Kre99, p. 733] Let $\left((\Lambda \oplus \Lambda, \lambda_\epsilon, \mu_\epsilon)^{\perp r}, V\right)$ and $\left((\Lambda \oplus \Lambda, \lambda_\epsilon, \mu_\epsilon)^{\perp r'}, V'\right)$ be two stable tuples of sums of hyperbolic forms and based, half-rank direct summands V and V', respectively. Two such stable tuples are equivalent if there exists an element $A \in RU^\epsilon(\Lambda)$ such that, stably, the image of V under A is V', i.e. there exists $l \in \mathbb{N}$ such that $A(V \oplus (\Lambda^l \times \{0\})) = V' \oplus (\Lambda^{r-r'+l} \times \{0\})$.
For $\epsilon := (-1)^q$ the *little l-monoid* $l_{2q+1}(\pi, w)$ is defined to be the set of equivalence classes $[((\Lambda \oplus \Lambda), \lambda_\epsilon, \mu_\epsilon)^{\perp r}, V]$ of stable tuples.
Together with the sum operation $\perp$, which is well-defined on the equivalence classes, this set becomes a monoid.

If the action of w on π is trivial we omit it in the notation. In particular, if π is the trivial group, we denote its integral group ring by $\mathbb{Z}$. Thus, its little l-monoid is denoted by $l_{2q+1}(\mathbb{Z})$.

To get from the algebraic setting to topology, we need Proposition 4 of [Kre99]. From now on, if we talk about surgery on a compact $\mathbb{B}$- manifold W, we always refer to surgery which is compatible with the $\mathbb{B}$-structure.

Proposition 3.8. *Let W be a smooth, compact manifold of dimension $2q$ or $2q+1$ for $q \geq 2$. Let $\mathbb{B} \to BO$ be a fibration whose total space $\mathbb{B}$ is connected and has a finite q-skeleton.*
If W admits a normal $\mathbb{B}$-structure $\widetilde{\nu}\colon W \to \mathbb{B}$, we can change $(W, \widetilde{\nu})$ to $(W', \widetilde{\nu}')$ such that $\widetilde{\nu}'$ is a normal $(q-1)$-smoothing by a finite sequence of surgeries.

This type of surgery is known as *surgery below the middle dimension.*

We are finally ready to define the surgery obstruction. The manifolds we are interested in are all of dimension $2q$ for q even. Consequently, we restrict to q even from now on. Thereby, we avoid some technicalities in dimensions 6 and 14.

Let $\mathbb{B} \to BO$ be a fibration as in Proposition 3.8, $\pi := \pi_1(\mathbb{B})$ and $w := w_1(\mathbb{B})$, where $w_1(\mathbb{B})$ is the first Stiefel-Whitney class of $\mathbb{B}$. The surgery obstruction for $\mathbb{B}$-bordism is an element in $l_{2q+1}(\pi_1(\mathbb{B}), w_1(\mathbb{B}))$. Therefore, we start by constructing a $\Lambda-$module which admits an $\epsilon-$quadratic form and a based, half-rank direct summand.

Let M_0 and M_1 be two connected manifolds of dimension $2q$, $q \geq 2$ and q even, with the same Euler characteristic. Furthermore, let $f\colon \partial M_0 \to \partial M_1$ be a diffeomorphism.
Assume there exist normal $(q-1)$-smoothings $\widetilde{\nu}_i\colon M_i \to \mathbb{B}$ which are compatible with f, i.e. $\widetilde{\nu}_0|_{\partial M_0} \simeq \widetilde{\nu}_1 \circ f$. Furthermore, assume that there exists a zero-bordism W of $M_0 \cup_f M_1$ which admits a normal $\mathbb{B}$-structure $\widetilde{\nu}\colon W \to \mathbb{B}$ that fulfils $\widetilde{\nu}|_{M_i} = \widetilde{\nu}_i$.
By surgery below the middle dimension as in Proposition 3.8, we can assume that $\widetilde{\nu}$ is a normal $(q-1)-$smoothing, too. Hence, the first non-vanishing homotopy group of the pair $(\mathbb{B}, W)$ is $\pi_{q+1}(\mathbb{B}, W)$.

Choose embeddings $f_j\colon S^q_j \hookrightarrow \mathrm{int}W$ which generate $\mathrm{im}\,(d\colon \pi_{q+1}(\mathbb{B}, W) \to \pi_q(W))$. By the transversality theorem we can always arrange the embeddings to have empty intersection.
Since they are in the image of d the induced elements $S^q_j \hookrightarrow W \xrightarrow{\widetilde{\nu}} \mathbb{B}$ in $\pi_q(\mathbb{B})$ are trivial. In particular, $[S^q_j \xhookrightarrow{f_j} W \xrightarrow{\widetilde{\nu}} \mathbb{B} \to BO]$ vanishes in $\pi_q(BO)$. This implies that $\nu(W)|_{S^q_j}$ is trivial. Therefore, the normal bundle of $S^q_j \to W$ is trivial and we obtain disjoint embeddings $S^q_j \times D^{q+1} \hookrightarrow \mathrm{int}W$ by the tubular neighbourhood theorem.
Let $U := \bigcup_j (S^q_j \times D^{q+1})$. Since M_i and ∂U are disjoint $H_k(\partial U; \Lambda) \cong H_k(\partial U \cup M_i, M_i; \Lambda)$ for all k. Consider the triple

$$M_i \subset M_i \cup \partial U \subset W - int(U)$$

and its long exact sequence

$$\to H_{k+1}(W - int(U), \partial U \cup M_i;\ \Lambda) \to \underbrace{H_k(\partial U \cup M_i, M_i;\ \Lambda)}_{\cong H_k(\partial U;\Lambda)} \to H_k(W - int(U), M_i; \Lambda) \to\ .$$

Since $\partial U \cong S^q \times S^q$ the homology groups $H_k(\partial U;\ \Lambda)$ vanish for all $k \neq 0, q, 2q$.
Recall that $H_q(\partial U; \Lambda)$ is equipped with a quadratic form given by the geometric intersection and self-intersection form (cf. [Wal70]). It actually is $(-1)^q$−hyperbolic. We denote it by $(H_q(\partial U; \Lambda), \lambda_U, \mu_U)$.
Standard arguments in algebraic topology show that

$$\mathrm{rank}\, H_{q+1}(W - int(U), \partial U \cup M_i; \Lambda) = \mathrm{rank}\, H_q(W - int(U), M_i; \Lambda).$$

Further arguments show that the homology groups $H_{k+1}(W - int(U), \partial U \cup M_i; \Lambda)$ and $H_k(W - int(U), M_i; \Lambda)$ vanish for $k \neq 0, q, 2q$. Thus, we can apply Lemma 2.3 of [Wal70] which implies that all modules in

$$0 \to H_{q+1}(W - int(U), \partial U \cup M_0; \Lambda) \to H_q(\partial U; \Lambda) \to H_q(W - int(U), M_0; \Lambda) \to 0.$$

are stably free and can be equipped with a preferred equivalence class of bases. Thus, we obtain a short exact sequence of based Λ-modules. Since the rank of the left and the right modules is equal, the image $\mathrm{im}(H_{q+1}(W - int(U), \partial U \cup M_0; \Lambda) \to H_q(\partial U; \Lambda))$ is a based, half-rank direct summand. We are now ready to define the surgery obstruction. Its well-definedness will follow from the theorem below the definition.

Definition 3.9. (cf. [Kre99] p. 734) Let $q \geq 2$, q even, let W of dimension $2q + 1$ be a $\mathbb{B}$-bordism between M_0 and M_1 as above and let

$$V := \mathrm{im}(H_{q+1}(W - int(U), \partial U \cup M_0; \Lambda) \to H_q(\partial U; \Lambda)).$$

The *surgery obstruction* is defined to be

$$\Theta(W,\widetilde{\nu}) := [(H_q(\partial U;\Lambda),\lambda_U,\mu_U),V] \in l_{2q+1}(\pi_1(\mathbb{B}),w_1(\mathbb{B})).$$

Let $\widetilde{V}$ be a based, half-rank direct summand of $(\Lambda\oplus\Lambda)^{\perp_r}$ such that $\widetilde{V}\oplus(\{0\}\times\Lambda^r)\cong(\Lambda\oplus\Lambda)^r$ and such that the basis of $\widetilde{V}$ together with the standard basis of $\{0\}\times\Lambda^r$ is equivalent to the standard basis of $(\Lambda\oplus\Lambda)^r$.
The surgery obstruction is *elementary* if there exists $\widetilde{V}$ as above such that $\Theta(W,\widetilde{\nu})$ is equivalent to $\Big((\Lambda\oplus\Lambda,\lambda_\epsilon,\mu_\epsilon)^{\perp_r},\widetilde{V}\Big)$.

Next we state Theorem 4 of [Kre99] for q even.

Theorem 3.10. *Let M_0 and M_1 be two connected manifolds of dimension $2q$, $q\geq 2$ even, which have the same Euler characteristic $\chi(M_0)=\chi(M_1)$ and let $f\colon\partial M_0\to\partial M_1$ be a diffeomorphism.*
Furthermore, let $\mathbb{B}\to BO$ be a fibration and $\widetilde{\nu}_i\colon M_i\to\mathbb{B}, i=0,1$ normal $(q-1)$-smoothings such that $\widetilde{\nu}_0\simeq\widetilde{\nu}_1\circ f$.
Assume that there exists a zero-bordism W of $M_0\cup_f M_1$ which admits a normal $\mathbb{B}$-structure $\widetilde{\nu}\colon W\to\mathbb{B}$ that fulfils $\widetilde{\nu}|_{M_i}=\widetilde{\nu}_i$. Then

$$\Theta(W,\widetilde{\nu})\in l_{2q+1}(\pi_1(\mathbb{B}),w_1(\mathbb{B}))$$

is invariant under bordism relative to the boundary.
Moreover, (W,Θ) is bordant, relative to the boundary, to a relative s-cobordism if and only if $\Theta(W,\widetilde{\nu})$ is elementary.

Note that the existence of $(W,\widetilde{\nu})$ as in the theorem is equivalent to the statement that $[M_0\cup_f M_1,\widetilde{\nu}_0\cup_f\widetilde{\nu}_1]$ vanishes in $\Omega^{\mathbb{B}}_{2q}$.

For our purposes we specialise Theorem 3.10 in a number of ways. In our applications all manifolds will be simply connected and we control $\pi_q(M_i)$, in particular we know it is finite. As we will see this controls the surgery obstruction.
We now state two corollaries. Then, we prove both in one go.

Corollary 3.11. *Let M_0 and M_1 be two simply connected, closed $2q-$dimensional manifolds, $q>3$ even, which fulfil $\chi(M_0)=\chi(M_1)$ and which have finite homotopy groups $\pi_q(M_i)$ for $i=0,1$.*
Furthermore, let $\mathbb{B}\to BO$ be a fibration and $\widetilde{\nu}_i\colon M_i\to\mathbb{B}, i=0,1$ normal $(q-1)$-smoothings.
Assume that M_0 and M_1 are $\mathbb{B}$-bordant, i.e. assume that there exists a bordism W between M_0 and M_1 which admits a normal $\mathbb{B}$-structure $\widetilde{\nu}\colon W\to\mathbb{B}$ which fulfils $\widetilde{\nu}|_{M_i}=\widetilde{\nu}_i$. Then there exists a diffeomorphism $m\colon M_0\to M_1$ such that $\widetilde{\nu}_1\circ m\simeq\widetilde{\nu}_0$.

In Section 4 we use this corollary to prove that the number of diffeomorphism classes of cohomology Bott manifolds in dimension eight is finite.
Apart from this classification result, we also examine (cf. Section 5) if a certain automorphism of the cohomology ring of specific eight-dimensional Bott manifolds is realisable. To do this, we use the relative setting of Theorem 3.10, that is, we use that it can be applied to manifolds with boundary.
The underlying idea is to consider closed manifolds X_i, for $i = 0, 1$, which admit a decomposition $X_i = M_i \cup_{h_i} N_i$, where M_i and N_i are smooth manifolds and h_i is a diffeomorphism of the boundaries $\partial M_i \to \partial N_i$. If we have a diffeomorphism n between N_0 and N_1 we can apply Theorem 3.10 to $M_0 \cup_{h_1 \circ n \circ h_0^{-1}} M_1$ to obtain

Corollary 3.12. *Let M_i and N_i, for $i = 0, 1$, be simply connected, compact $2q-$dimensional manifolds with boundary such that $\chi(M_0) = \chi(M_1)$ and such that $\pi_q(M_i)$ is finite. Let $h_i\colon \partial M_i \to \partial N_i$ and $n\colon N_0 \to N_1$ be diffeomorphisms.*
Furthermore, let $\widetilde{\nu}_i$, $i = 0, 1$ be normal $(q-1)$-smoothings of M_0 and M_1 into the same fibration $\mathbb{B} \to BO$ such that

$$\widetilde{\nu}_0|_{\partial M_0} \simeq \widetilde{\nu}_1 \circ h_1^{-1} \circ n \circ h_0.$$

If there exists a zero-bordism W of $M_0 \cup_{h_1^{-1} \circ n \circ h_0} M_1$ which admits a $\mathbb{B}$-structure $\widetilde{\nu}\colon W \to \mathbb{B}$ which restricts to $\widetilde{\nu}_i$ on M_i , then there exists a diffeomorphism $m\colon M_0 \to M_1$ which extends n, i.e. $M_0 \cup_{h_0} N_0$ is diffeomorphic to $M_1 \cup_{h_1} N_1$ under $m \cup n$.

To deduce the corollaries from Theorem 3.10, we essentially need to show that the surgery obstruction is elementary under the assumptions of the corollaries, namely the assumption that the homotopy groups in the middle dimension are finite and the assumption that both M_i are simply connected. In the proof, we use Proposition 8 of [Kre99] which essentially states that the surgery obstruction, in our situation, is an element in Wall's L-group.
For the proof we stick to the notation of Corollary 3.11 but the arguments work exactly the same in the setting of Corollary 3.12.

Proof. By [Kre99, p. 733] there exists a subgroup $L_{2q+1}(\pi, w)$ in the monoid $l_{2q+1}(\pi, w)$ which consists of those elements $\big((\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu)^{\perp_r}, W\big) \in l_{2q+1}(\pi, w)$ whose intersection form and quadratic refinement vanish on W. As the notation indicates this group is connected to Wall's L-groups $L_{2q+1}^{\text{Wall}}(\pi, w)$. Let $Wh(\pi)$ denote the Whitehead group of π. There exists a homomorphism $L_{2q+1}(\pi, w) \to Wh(\pi)$ whose kernel is $L_{2q+1}^{\text{Wall}}(\pi, w)$. Since our fundamental group is trivial the Whitehead group vanishes and we can identify $L_{2q+1}(\mathbb{Z})$ with $L_{2q+1}^{\text{Wall}}(\mathbb{Z})$. But the odd L-groups are well-known to vanish in the simply connected setting (cf. [Wal70, Theorem 13A]), i.e. $L_{2q+1}(\mathbb{Z}) = 0$.
Thus, it suffices to show that the surgery obstruction $\Theta(W, \widetilde{\nu})$ is an element of $L_{2q+1}(\mathbb{Z})$.

Let $\big((\mathbb{Z} \oplus \mathbb{Z}, \lambda_\epsilon, \mu_\epsilon)^{\perp_r}, V\big)$ be a representative for $\Theta(W, \widetilde{\nu})$. We need to show that λ_ϵ and

μ_ϵ vanish on V.

Wall's intersection and self-intersection form induce an intersection form and a quadratic refinement, in the sense of Definition 3.6, on $K := \ker(\pi_q(M_0) \to \pi_q(\mathbb{B}))$ (cf. [Kre99, p. 727]). By Proposition 8 of [Kre99], there exists a surjective isometry of quadratic forms $V \to K$. Consequently, it suffices to show that the intersection form and the quadratic refinement vanish on K.
From the properties of the intersection form and the quadratic refinement we deduce that both of them vanish on elements $x \in K$ whose order is finite.
Indeed, assume that $k \cdot x = 0$ for $k \neq 0$, i.e. $\lambda(kx, y) = 0$ for all $y \in K$. By property i) of a quadratic form this implies that $k\lambda(x, y) = 0 \in \mathbb{Z}$ whence $\lambda(x, y) = 0$. Since q is even, $\Lambda/S \cong \mathbb{Z}$. Therefore, the analogous proof works for the self-intersection form if we use property iv) instead of property i).
By assumption of the corollaries $\pi_q(M_0)$ is finite. Hence, K is finite, as well. Therefore, the intersection form and the quadratic refinement both vanish on K, implying that the surgery obstruction is an element of the trivial group $L^{\mathrm{Wall}}_{2q+1}(\mathbb{Z})$. Thus, the surgery obstruction vanishes.

Consider the setting of Corollary 3.11. Then, by Theorem 3.10, there exists an s-cobordism W between M_0 and M_1. By the s-cobordism theorem W is diffeomorphic, relative to the boundary, to $M_0 \times I$. In particular, there exists a diffeomorphism $f\colon M_0 \to M_1$ such that $\widetilde{\nu}_0 \simeq \widetilde{\nu}_1 \circ f$.

Consider the setting of Corollary 3.12. There, Theorem 3.10 implies that there exists an s-cobordism of $M_0 \cup_{h_1^{-1} \circ n \circ h_0} M_1$. Thus, there exists a diffeomorphism $m\colon M_0 \to M_1$ such that $m|_{\partial M_0} = h_1^{-1} \circ n \circ h_0$. □

3.3. Twisted Bordism

Let M be an n-dimensional manifold. The goal of this section is to construct fibrations $\mathbb{B}$ over BO which admit normal smoothings $M \to \mathbb{B}$. Similar constructions can be found in [KS91], [Tei93] and [Olb07].
Let $p_m\colon BO\langle m\rangle \to BO$ denote the $(m-1)$-connected cover of BO. By definition the homotopy groups of $BO\langle m\rangle$ are either $\pi_i(BO\langle m\rangle) \cong \pi_i(BO)$ for $i \geq m$, where the isomorphism is induced by p_m, or $\pi_i(BO\langle m\rangle) = 0$ for $i < m$.
One naive way to try to construct a fibration which admits k-smoothings for k sufficiently large is to take the product of a Postnikov stage $\mathbb{P}_l M$ of M and a sufficiently high connected cover $BO\langle m\rangle$, for $m > l + 1$, of BO:

$$\mathbb{P}_l M \times BO\langle m\rangle \xrightarrow{p_m \circ pr_2} BO\ .$$

Here pr_2 denotes the projection onto the second factor.
A space M always admits a map into all its Postnikov stages and the map j_l into the

l-th stage induces isomorphisms for all homotopy groups of degree lower or equal l and an epimorphism in $l+1$. Therefore, if $\nu\colon M \to BO$ admits a lift $\widetilde{\nu}$ to $BO\langle m\rangle$, the map $M \xrightarrow{\Delta} M \times M \xrightarrow{j_l \times \widetilde{\nu}} \mathbb{P}_l M \times BO\langle m\rangle$ is an l-smoothing. If $m = l+2$, the fibration even is the normal l-type.
If we do not need the l-type but only a fibration that admits a normal $l-$smoothing we can replace $\mathbb{P}_l M$ by any space X that admits a map $\iota\colon M \to X$ which is $(l+1)-$connected. Of course, a lift $\widetilde{\nu}\colon M \to BO\langle m\rangle$ does not exist in general.

In all cases that are of interest to us we can solve this problem by "twisting" the fibration. Let BO_r denote the classifying space of real vector bundles of rank r and, furthermore, let $i_r\colon BO_r \to BO$ denote the inclusion into the direct limit BO.

Definition 3.13. Let $E \to X$ be an oriented vector bundle of rank r over a CW-complex X. By abuse of notation we denote the classifying map of $E \to X$ by $E\colon X \to BO_r$, too. Let $\gamma^u \to BO$ be the stable universal vector bundle and let $\oplus : BO \times BO \to BO$ be the classifying map of $\gamma^u \times \gamma^u \to BO \times BO$.
Consider the map

$$X \times BO\langle m\rangle \xrightarrow{i_r \circ E \times p_m} BO \times BO \xrightarrow{\oplus} BO$$

and replace its domain by a homotopy equivalent space such that the map becomes a fibration which we denote by $X \widetilde{\times} BO\langle m\rangle$. Normal bordism with respect to this fibration is called *twisted bordism.* We denote its normal bordism groups by $\Omega_n^{O\langle m\rangle}(X, E)$.
If the bundle E is trivial we obtain ordinary $O\langle m\rangle$-bordism.

Knowing that we can replace maps by fibrations we, from now on, will not distinguish between the map $\oplus \circ (i_r \circ E \times p_m)$ and the fibration we can replace it with.

In the next lemma we specify manifolds which admit l-smoothings into $X \widetilde{\times} BO\langle m\rangle$.

Lemma 3.14. *Let M be a manifold that admits a map $\iota\colon M \to X$ which is $(l+1)$-connected. Furthermore, let $E \to X$ be an oriented real vector bundle of rank r and let $-E$ denote its K-theory inverse.*
If the classifying map of $\iota^*(-E) \oplus \nu(M)$ admits a lift $\mu\colon M \to BO\langle m\rangle$ for $m > l+1$, then $(\iota \times \mu) \circ \Delta\colon M \to X \widetilde{\times} BO\langle m\rangle$ is a normal $l-$smoothing.

Proof. It is obvious that $(\iota \times \mu) \circ \Delta$ is $(l+1)$-connected since $\pi_i(BO\langle m\rangle) = 0$ for $i \leq l+1$. It remains to show that it is a lift of the stable Gauss map $\nu\colon M \to BO$.
By definition $\oplus^*\gamma^u \cong \gamma^u \times \gamma^u$. Thus, $(i_r \circ E \times p_m)^*(\gamma^u \times \gamma^u)$ is stably isomorphic to $E \times \gamma^u_m$, where $\gamma^u_m = p^*_m \gamma^u$ is the universal bundle over $BO\langle m\rangle$.
By assumption $(\iota \times \mu)^*(E \times \gamma^u_m) \cong \iota^* E \times (\iota^*(-E) \oplus \nu(M))$. Finally, we pull back along the diagonal map Δ, which corresponds to taking the Whitney sum, and obtain $\iota^*(E \oplus (-E)) \oplus \nu(M)$ which is stably isomorphic to $\nu(M)$. □

For the classification problems we are interested in, this construction will always suffice. The next step is to develop tools to calculate the bordism groups of twisted fibrations over BO.

3.4. Computing twisted bordism groups

We want to be able to use the methods of stable homotopy theory to calculate twisted bordism groups. Consequently, we need to construct spectra whose stable homotopy groups are isomorphic to the twisted bordism groups which we want to determine. The construction follows along the lines of Chapter 12 in [Swi02].

We obtain spectra for twisted bordism by modifying the construction of Thom spectra for $BO\langle m\rangle$-bordism slightly. In order to distinguish Thom spaces from Thom spectra we denote the first by $Th(.)$ and the latter by $M(.)$. For any map f of vector bundles, we denote by $Th(f)$ the induced map between the Thom spaces of the bundles.

To construct spectra as in Chapter 12 of [Swi02] we need (strictly) commutative diagrams

$$\begin{array}{ccc} BO\langle m\rangle_{n-1} & \xrightarrow{o_{n,m}} & BO\langle m\rangle_n \\ \downarrow{\scriptstyle pr_{n-1,m}} & & \downarrow{\scriptstyle pr_{n,m}} \\ BO_{n-1} & \xrightarrow{o_n} & BO_n \end{array} . \tag{5}$$

We obtain those by using a functorial construction for the $(m-1)$-connected cover of a simply connected space. Instead of working over BO we can always work over BSO since there exist commutative diagrams

$$\begin{array}{ccc} BSO_{n-1} & \longrightarrow & BSO_n \\ \downarrow & & \downarrow \\ BO_{n-1} & \longrightarrow & BO_n \end{array} , \tag{6}$$

where all maps are induced by the respective inclusions of the underlying groups.
By [Whi78], Chapter IX the Postnikov tower of a simply connected CW complex can be constructed functorially.
Applying the construction to BSO_n we obtain maps $BSO_n \to \mathbb{P}_{m-1}BSO_n$. By the long exact sequence of homotopy groups of a fibration, the homotopy fibre of this map is the $(m-1)$-connected cover $BO\langle m\rangle_n$ of BSO_n. Since the homotopy fibre can also be constructed functorially we obtain commutative squares

$$\begin{array}{ccc} BO\langle m\rangle_{n-1} & \xrightarrow{o_{n,m}} & BO\langle m\rangle_n \\ \downarrow{\scriptstyle \widetilde{pr}_{n-1,m}} & & \downarrow{\scriptstyle \widetilde{pr}_{n,m}} \\ BSO_{n-1} & \xrightarrow{o_{n,1}} & BSO_n \end{array} . \tag{7}$$

Combining the commutative squares of diagram (6) and (7) we obtain the commutative square in diagram (5). Using the observations above, we can now sketch the construction of the Thom spectrum for BO- and $BO\langle m\rangle$-bordism denoted by MO and $MO\langle m\rangle$, respectively.
Let $\gamma_n^u \to BO_n$ denote the universal vector bundle of rank n and let $\underline{\mathbb{R}} \to BO_n$ denote the trivial line bundle. There exist a bundle map $\overline{o_n}\colon o_n^*\gamma_n^u \to \gamma_n^u$ covering o_n and there exists an isomorphism $f_n\colon \gamma_{n-1}^u \oplus \underline{\mathbb{R}} \to o_n^*\gamma_n^u$. Composing both and passing to the Thom spaces we obtain a map

$$\sigma_n := Th(\overline{o_n} \circ f_n)\colon Th(\gamma_{n-1}^u) \wedge S^1 \to Th(\gamma_n^u)\ .$$

The spectrum MO consists of the spaces $Th(\gamma_n)$ together with the maps σ_n.

Denote the pullback of γ_n^u along $pr_{n,m}$ by $\gamma_{n,m}^u$. Completely analogously to the construction above we obtain the spectra $MO\langle m\rangle$:
There exists a bundle map $\overline{o_{n,m}}\colon o_{n,m}^*\gamma_{n,m}^u \to \gamma_{n,m}^u$ and, by commutativity of the diagram (5), a bundle isomorphism $\widetilde{f}_n\colon (\gamma_{n-1,m}^u \oplus \mathbb{R}) \to o_{n,m}^*\gamma_{n,m}^u$.
By composing both maps and by passing to the Thom spaces we obtain maps

$$\sigma_{n,m} := Th(\overline{o_{n,m}} \circ \widetilde{f}_n)\colon Th(\gamma_{n-1,m}^u) \wedge S^1 \to Th(\gamma_{n,m}^u)\ .$$

The spectrum $MO\langle m\rangle$ consists of the Thom spaces $Th(\gamma_{n,m}^u)$ together with the maps $\sigma_{n,m}$.

Now we come to the construction for twisted bordism.
Recall that the fibration $X\widetilde{\times}BO_{n-r}$ is defined by

$$X \times BO\langle m\rangle \xrightarrow{i_r\circ E\times p_m} BO \times BO \xrightarrow{\oplus} BO\ ,$$

for E the classifying map of a vector bundle of rank r over X. We denote the total space of the bundle by E, too. There are commutative diagrams

$$\begin{array}{ccc}
X \times BO\langle m\rangle_{n-1-r} & \xrightarrow{\mathbb{1}_X\times o_{n-r,m}} & X \times BO\langle m\rangle_{n-r} \\
\Big\downarrow{\scriptstyle E\times pr_{n-1-r,m}} & & \Big\downarrow{\scriptstyle E\times pr_{n-r,m}} \\
BO_r \times BO_{n-r-1} & \xrightarrow{\mathbb{1}_{BO_r}\times o_{n-r}} & BO_r \times BO_{n-r} \\
\Big\downarrow{\scriptstyle \oplus} & & \Big\downarrow{\scriptstyle \oplus} \\
BO_{n-1} & \xrightarrow{o_n} & BO_n\ .
\end{array}$$

Thus, we obtain another sequence of Thom spaces

$$X_n := Th(E \times \gamma_{n-r,m}^u) \cong Th(E) \wedge Th(\gamma_{n-r,m}^u).$$

The maps $\mathbb{1}_E \times \overline{o_{n-r,m}} \circ \widetilde{f}_{n-r}$ induce

$$\mathbb{1}_{Th(E)} \wedge \sigma_{n-r,m} \colon X_{n-1} \wedge S^1 \to X_n.$$

Let $M(E \times \gamma^u)$ be the spectrum consisting of the spaces X_n together with the maps $\mathbb{1}_{Th(E)} \wedge \sigma_{n-r,m}$. By the Pontjagin-Thom construction we know

$$\Omega_n^{O\langle m\rangle}(X, E) \cong \pi_n^{st}(M(E \times \gamma^u)).$$

Thus, we can now apply the Adams spectral sequence to compute $\pi_n^{st}(M(E \times \gamma^u))$, i.e. $\Omega_n^{O\langle m\rangle}(X, E)$

In addition, it is helpful to have a modified version of an Atiyah-Hirzebruch spectral sequence which we introduce now. It is particularly helpful, because we can use it to determine which torsion can appear in $\Omega_n^{O\langle m\rangle}(X, E)$. For this purpose, we use it, e.g. in Sections 4.3 and 5.4.

We denote the t-fold suspension of a spectrum A by $\Sigma^t A$.
By construction $M(E \times \gamma^u) \simeq Th(E) \wedge \Sigma^{-r} MO\langle m\rangle \simeq \Sigma^{-r}(Th(E) \wedge MO\langle m\rangle)$. Hence,

$$\begin{aligned} \Omega_n^{O\langle m\rangle}(X, E) &\cong \pi_n^{st}(\Sigma^{-r}(Th(E) \wedge MO\langle m\rangle)) \\ &\cong \pi_{n+r}^{st}(Th(E) \wedge MO\langle m\rangle) \cong \Omega_{n+r}^{O\langle m\rangle}(Th(E), pt). \end{aligned}$$

Remark 3.15. On a geometric level the isomorphism T which is given by the composition

$$\Omega_n^{O\langle m\rangle}(X, E) \to \Omega_{n+r}^{O\langle m\rangle}(Th(E), pt) \to \Omega_{n+r}^{O\langle m\rangle}(D(E), S(E))$$

maps an element $[M, f \times \alpha]$ to $[(D(f^*E), S(f^*E)), \widetilde{f} \times \widetilde{\alpha}]$. Here $\widetilde{\alpha}$ is the $O\langle m\rangle$-structure on $D(f^*E)$, obtained by composing the projection of the disc bundle with α, and $\widetilde{f}$ is the bundle map covering f.

To compute $\Omega_*^{O\langle m\rangle}(Th(E), pt)$ we can use the usual Atiyah-Hirzebruch spectral sequence converging to the reduced ordinary $O\langle m\rangle$-bordism groups, i.e.

$$E^2_{pq} = \widetilde{H}_p(Th(E); \Omega_q^{O\langle m\rangle}(pt)) \Rightarrow \Omega_{p+q}^{O\langle m\rangle}(Th(E), pt).$$

By the Thom isomorphism for oriented vector bundles

$$H_p(X; \Omega_q^{O\langle m\rangle}(pt)) \cong \widetilde{H}_{p+r}(Th(E); \Omega_q^{O\langle m\rangle}(pt)).$$

Since $\Omega_{p+q}^{O\langle m\rangle}(X, E) \cong \Omega_{p+q+r}^{O\langle m\rangle}(Th(E), pt)$ there is a spectral sequence with E^2-page

$$E^2_{pq} \cong H_p(X; \Omega_q^{O\langle m\rangle}(pt))$$

converging to $\Omega_{p+q}^{O\langle m\rangle}(X,E)$.
We refer to this as the *twisted Atiyah-Hirzebruch spectral sequence.*

Note, that the E^2-page for the twisted Atiyah-Hirzebruch spectral sequence is exactly the same E^2-page as the one of the Atiyah-Hirzebruch spectral sequence converging to the ordinary $O\langle m\rangle$-bordism group $\Omega_{p+q}^{O\langle m\rangle}(X)$. But even d_2-differentials can differ if $w_2(E)\neq 0$, as we will see below.

Recall that, for $m\geq 4$, we have $\Omega_0^{O\langle m\rangle}(pt)\cong\mathbb{Z}$ and $\Omega_i^{O\langle m\rangle}(pt)\cong\mathbb{Z}/2$ for $i=1,2$. Thus, entries in the 0-, 1- and 2-line on the (twisted) Atiyah-Hirzebruch spectral sequence are given by homology with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/2$, respectively.
In general we do not know the differentials in the (twisted) Atiyah-Hirzebruch spectral sequence. But we can say something about some of the d_2-differentials. The following lemma is due to [Tei93] for $m=4$ but the proof follows completely analogously for $m>4$.

Lemma 3.16. *Let X be a CW-complex and $E\to X$ a (possibly trivial) twisting bundle. Consider the (twisted) Atiyah-Hirzebruch spectral sequence converging to $\Omega_n^{O\langle m\rangle}(X,E)$ for $m\geq 4$.*

1. *Let $w_2:=w_2(E)$. The differential $d_2\colon E^2_{p+2,1}\to E^2_{p,2}$ is dual to*

$$\begin{aligned} Sq^2_{w_2}\colon H^p(X;\mathbb{Z}/2) &\to H^{p+2}(X;\mathbb{Z}/2),\\ x &\mapsto Sq^2(x)+x\cup w_2. \end{aligned}$$

2. *Let $Sq^2_{w_2}$ be defined as above. The differential $d_2\colon E^2_{p+2,0}\to E^2_{p,1}$ is given by the composition $d_2=(Sq^2_{w_2})^*\circ red$, where $red\colon H_{p+2}(X;\mathbb{Z})\to H_{p+2}(X;\mathbb{Z}/2)$ is the reduction mod two.*

Another helpful tool for computations is a sequence that allows us to compare twisted $O\langle l\rangle$-bordism of $\mathbb{C}P^m$ and $\mathbb{C}P^{m-1}$. We start with the definition of a map which appears within the sequence but also in a more general setting.
Let $\xi\to X$ be a real bundle over a manifold X which contains a codimension r submanifold $Y\subset X$ and let ξ' denote the normal bundle of $Y\hookrightarrow X$.

Definition 3.17. Let M be a smooth, closed n-dimensional manifold and $f\colon M\to X$ a map such that $\nu M\oplus f^*(-\xi)$ admits a $O\langle l\rangle$-structure $\alpha\colon M\to BO\langle l\rangle$. Consider the induced element $[M,f\times\alpha]\in\Omega_n^{O\langle l\rangle}(X,\xi)$, We can always assume that $f\pitchfork Y$. Let N be the preimage $f^{-1}(Y)$. Since

$$\nu(N)\cong\nu(N\hookrightarrow M)\oplus\nu(M)|_N\cong(f|_N)^*\xi'\oplus\nu(M)|_N,$$

the Whitney sum of $\nu(N)$ and $(f|_N)^*(-(\xi|_Y\oplus\xi'))$ fulfills

$$\nu(N)\oplus(f|_N)^*(-(\xi|_Y\oplus\xi'))\cong\nu(M)|_N\oplus(f|_N)^*(-\xi|_Y).$$

Consequently $\alpha|_N$ defines a $O\langle l\rangle$-structure twisted by $\xi|_Y \oplus \xi'$.
Thus, we can define

$$t\colon \Omega_n^{O\langle l\rangle}(X,\xi) \to \Omega_{n-r}^{O\langle l\rangle}(Y,\xi|_Y \oplus \xi'), \text{ by } [M, f\times\alpha] \mapsto [N, (f\times\alpha)|_N].$$

This map is well-defined by the following observation. Given two representatives of $[M, f\times\alpha] \in \Omega_n^{O\langle l\rangle}(X,\xi)$ they are, by definition, bordant in $\Omega_n^{O\langle l\rangle}(X,\xi)$. We apply the construction in the definition of t to the bordism and obtain that the images of the representatives under t are bordant in $\Omega_{n-r}^{O\langle l\rangle}(Y,\xi|_Y \oplus \xi')$.

Now we come to the special case of $X = \mathbb{C}P^m$ and $Y = \mathbb{C}P^{m-1}$. We obtain a long exact sequence relating twisted bordism of $\mathbb{C}P^m$ and $\mathbb{C}P^{m-1}$ which is well-known to the experts. Applications of a similar sequence can be found in [Kre09]. But there does not seem to be a published proof. Therefore, we also give a proof.

Lemma 3.18. *Let $\xi \to \mathbb{C}P^m$ be an oriented real vector bundle of finite rank. Let $i\colon \Omega_k^{O\langle l\rangle}(pt) \to \Omega_k^{O\langle l\rangle}(\mathbb{C}P^m,\xi)$ be the map induced by the inclusion of a point into $\mathbb{C}P^m$ and let $H := \nu(\mathbb{C}P^{m-1} \to \mathbb{C}P^m)$ denote the Hopf bundle. Then the following sequence is exact:*

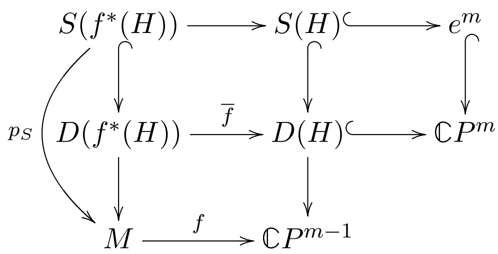
$$\ldots \to \Omega_n^{O\langle l\rangle}(pt) \xrightarrow{i} \Omega_n^{O\langle l\rangle}(\mathbb{C}P^m,\xi) \xrightarrow{t} \Omega_{n-2}^{O\langle l\rangle}(\mathbb{C}P^{m-1},\xi|_{\mathbb{C}P^{m-1}} \oplus H) \xrightarrow{s} \Omega_{n-1}^{O\langle l\rangle}(pt) \to \ldots \quad .$$

The map s will be constructed in the proof.

To prove this lemma we need the following easy observation.

Lemma 3.19. *Let $\xi \to \mathbb{C}P^m$ be a real vector bundle, M a compact, smooth manifold and $f\colon M \to \mathbb{C}P^{m-1}$ a map. Furthermore, let $p_S\colon S(f^*H) \to M$ denote the sphere bundle of f^*H.*
Then the bundle $p_S^* f^*(\xi|_{\mathbb{C}P^{m-1}})$ is trivial.

Proof. Let e^m denote the top cell of $\mathbb{C}P^m$. We obtain the following commutative diagram of total spaces



Recall that $\mathbb{C}P^m = D(H) \cup_{S(H)} e^m$, i.e. $S(H)$ bounds the top disc e^m. Every bundle ξ over $\mathbb{C}P^m$ becomes trivial under restriction to e^m and thus, under restriction to $S(H)$. By the commutativity of the diagram the pullback $p_S^* f^* \xi|_{\mathbb{C}P^{m-1}}$ is also trivial. □

Now we are ready to prove Lemma 3.18. Within the proof we suppress the decoration $O\langle l\rangle$ in the notation of the twisted bordism groups.

Proof. We start with the definition of $s\colon \Omega_{n-2}(\mathbb{C}P^{m-1}, \xi|_{\mathbb{C}P^{m-1}} \oplus H) \to \Omega_{n-1}(pt)$. We claim that

$$\begin{aligned} s\colon \Omega_{n-2}(\mathbb{C}P^{m-1}, \xi|_{\mathbb{C}P^{m-1}} \oplus H) &\rightarrow \Omega_{n-1}(pt) \text{ defined by} \\ [M, f \times \alpha] &\mapsto [S(f^*H), pt \times (\alpha \circ p_S)], \end{aligned}$$

is a well-defined map. Here pt denotes the constant map to a point. We need to check that $[S(f^*H), pt \times \alpha \circ p_S]$ is an element in $\Omega_{n-1}(pt)$, i.e. we need to show that $\alpha \circ p_S$ is an $O\langle l\rangle$-structure on the total space of the sphere bundle $p_S\colon S(f^*H) \to M$.
The stable tangent bundle of $S(f^*H)$ is isomorphic to $p_S^*TM \oplus p_S^*(f^*H)$, where $p_S^*(f^*H)$ is trivial by Lemma 3.19 since we can consider the Hopf bundle over $\mathbb{C}P^{m-1}$ as restriction of the Hopf bundle over $\mathbb{C}P^m$. Thus, the stable normal bundle $\nu(S(f^*H))$ is isomorphic to $p_S^*(\nu(M))$. By Lemma 3.19 the bundle $p_S^* f^*(-(H \oplus \xi)|_{\mathbb{C}P^{m-1}})$ is also trivial. Consequently, we obtain

$$\nu(S(f^*H)) \cong p_S^*\big(\nu(M) \oplus f^*(-(H \oplus \xi)|_{\mathbb{C}P^{m-1}})\big).$$

By assumption, α is an $O\langle l\rangle$-structure on $\nu(M) \oplus f^*(-(H \oplus \xi)|_{\mathbb{C}P^{m-1}})$. Hence, the composition $\alpha \circ p_S$ is a $O\langle l\rangle$-structure on the normal bundle $\nu(S(f^*H))$.

Well-definedness of s follows, again, by applying the construction of s to a twisted $\mathbb{C}P^{m-1}\widetilde{\times}BO\langle l\rangle$-bordism between two representatives of $[M, f \times \alpha]$.

Now, we prove the exactness. We start by showing that $\operatorname{im}(i) \subset \ker(t)$.
Let $[M, pt \times \alpha] \in \Omega_n(pt)$. The map $pt\colon M \to \mathbb{C}P^m$ is transversal to $\mathbb{C}P^{m-1} \subset \mathbb{C}P^m$ if $pt \notin \mathbb{C}P^{m-1}$. Thus, by definition of t, the composition $t \circ i$ vanishes.

Next, we show $\ker(t) \subset \operatorname{im}(i)$.
Let $[M, f \times \alpha] \in \ker(t) \subset \Omega_n(\mathbb{C}P^m, \xi)$ and let $[N, (f \times \alpha)|_N] := t([M, f \times \alpha])$. Since, by assumption, $[N, (f \times \alpha)|_N] = 0$ there exists a $\mathbb{C}P^{m-1}\widetilde{\times}BO\langle l\rangle$ zero-bordism W, i.e. $\partial W = N$ and there exists a map $F \times \beta\colon W \to \mathbb{C}P^{m-1} \times BO\langle l\rangle$ such that $(F \times \beta)|_N = (f \times \alpha)|_N$. By definition, β is an $O\langle l\rangle$-structure on the Whitney sum $\nu(W) \oplus F^*(-(\xi \oplus H))$.
Let D denote the total space of the disc bundle $p_D\colon D(F^*H) \to W$. Since the normal bundle of D is isomorphic to $p_D^*(\nu(W) \oplus F^*(-H))$, $\beta \circ p_D$ is an $O\langle l\rangle$-structure on the sum $\nu(D) \oplus p_D^* F^*(-\xi)$. Furthermore, there are isomorphisms $D|_N \cong D((f|_N)^*H) \cong D\nu(N \hookrightarrow M) =: D'$. In particular, there is an embedding $D' \hookrightarrow M \times \{1\}$. Thus, we can construct a bordism

$$W' := M \times I \cup_{D'} D.$$

It admits a twisted $O\langle l\rangle$-structure and a map to $\mathbb{C}P^m$ whose restriction to $M \times \{0\} \subset W'$ is $f \times \alpha$.

The other boundary component of W' is $M' = (M - D') \cup S(F^*H)$. By construction there is a map $\overline{F}\colon S(F^*H) \to S(H) \subset e^m$, covering F, which is homotopic to the constant map. By Lemma 3.19 $\beta \circ p_S$ is an $O\langle l\rangle$-structure on $\nu(S(F^*H))$.
The map $f|_{M-D'}$ is also homotopic to the constant map since $\mathrm{im}(f|_{M-D'}) \subset e^m \subset \mathbb{C}P^m$. Thus, the restriction $f^*(-\xi)|_{M-D'}$ is trivial. Consequently, $\alpha|_{M-D'}$ is an $O\langle l\rangle$-structure on $\nu(M - D') = \nu(M)|_{M-D'}$. Hence, we obtain an element $[M', pt \times (\alpha|_{M-D'} \cup \beta \circ p_S)]$ in $\Omega_n(pt)$ whose image under i is, by W', bordant to $[M, f \times \alpha] \in \Omega_n(\mathbb{C}P^m, \xi)$.

We proceed by showing that $\mathrm{im}(t) \subset \ker(s)$.
Let $[M, f \times \alpha] \in \Omega_n(\mathbb{C}P^m, \xi)$ and let $[N, (f \times \alpha)|_N] := t([M, f \times \alpha])$. We need to show that $S := S((f|_N)^*H)$ is zero-bordant in $\Omega_{n-1}(pt)$.
Note that $S \cong S(\nu(N \hookrightarrow M))$. Thus, it is the boundary of $W := M - D(\nu(N \hookrightarrow M))$. By construction we again obtain that $\mathrm{im}(f|_W) \subset e^m \subset \mathbb{C}P^m$ and thus, $f|_W$ is homotopic to the constant map implying that $f^*(-\xi)|_W$ is trivial. Consequently, the restriction $\alpha|_W$ is an $O\langle l\rangle$-structure on $\nu(M)|_W \cong \nu(W)$. Hence, W is a zero-bordism of S in $\Omega_{n-1}(pt)$.

Of course, the next step is to show that $\ker(s) \subset \mathrm{im}(t)$.
Now assume that $[N, g \times \beta]$ is in the kernel of s, i.e. $[S(g^*(H)), pt \times (p_S \circ \beta)]$ is zero-bordant by some bordism W with $O\langle l\rangle$-structure α which restricts to $p_S \circ \beta$.
Let D denote the total space of the disc bundle $p_D\colon D(g^*H) \to N$. Its normal bundle is $\nu(D) \cong p_D^*(\nu(N) \oplus g^*(-H))$. Thus, $\beta \circ p_D$ is an $O\langle l\rangle$-structure on $\nu(D) \oplus p_D^* g^*(-\xi)$. Furthermore, there is a bundle map $\overline{g}\colon D(g^*H) \to D(H) \hookrightarrow \mathbb{C}P^m$ covering g. As before, the image of $\overline{g}|_{S(g^*H)}$ is contained in the the top cell $e^m \subset \mathbb{C}P^m$, i.e. $\overline{g}|_{S(g^*H)} \simeq pt$. Consider $M := W \cup_{S(g^*H)} D$. It admits a map $f := pt \cup \overline{g}\colon M \to \mathbb{C}P^m$. Furthermore, $\alpha \cup \beta \circ p_D$ is an $O\langle l\rangle$-structure on $\nu(M) \oplus f^*(-\xi)$. By construction $t([M, f \times (\alpha \cup p \circ \beta)]) = [N, g \times \beta]$.

It remains to show that $\mathrm{im}(s) = \ker(i)$.
Let $[M, f \times \alpha] \in \Omega_{n-2}(\mathbb{C}P^{m-1}, (\xi \oplus H)|_{\mathbb{C}P^{m-1}})$. We need to construct a zero-bordism of $i \circ s([M, f \times \alpha]) = i([S(f^*H), pt \times \alpha])$ in $\Omega_{n-1}(\mathbb{C}P^m, \xi)$. Consider the disc bundle with total space $D := D(f^*H)$ and projection p_D, together with the covering map $\overline{f}\colon D(f^*H) \to D(H) \subset \mathbb{C}P^m$. Since $\nu(D) \cong \nu(M) \oplus p_D^* f^*(-H)$, $\alpha \circ p_D$ is an $O\langle l\rangle$-structure on $\nu(D) \oplus p_D^* f^*(-\xi)$. Thus, D together with $\overline{f}$ and $\alpha \circ p_D$ is our zero-bordism.

Finally, we show that $\ker(i) \subset \mathrm{im}(s)$.
For this purpose, let $[M, pt \times \alpha] \in \ker(i) \subset \Omega_{n-1}(pt)$. Let W be a zero-bordism of $i([M, pt \times \alpha]) \in \Omega_{n-1}(\mathbb{C}P^m, \xi)$, i.e. there exists $F\colon W \to \mathbb{C}P^m$ such that $\nu(W) \oplus F^*(-\xi)$ admits an $O\langle l\rangle$-structure β which restricts to α. Assume that F is transversal to $\mathbb{C}P^{m-1} \subset \mathbb{C}P^m$ and let $N := F^{-1}(\mathbb{C}P^{m-1})$ and $F|_N =: f$. Then $\nu(N) \cong \nu(W) \oplus f^*H$ and $\beta|_N$ is an $O\langle l\rangle$-structure for $\nu(N) \oplus f^*(-(\xi \oplus H))$. Consequently, $[N, f \times \beta|_N]$ is an element in $\Omega_{n-2}(\mathbb{C}P^{m-1}, \xi \oplus H)$.
Consider $W' := W - D(f^*H)$. Then β restricts to an $O\langle l\rangle$-structure on $\nu(W')$ since

$\mathrm{im}(F|_{W'}) \subset e^m \subset \mathbb{C}P^m$, i.e. $(F|_{W'})^*(-\xi)$ is trivial. Thus, W' is a bordism between $s([N, f \times \beta|_N])$ and $[M, pt \times \alpha]$. □

4. Eight-dimensional cohomology Bott manifolds

In this section, we consider cohomology Bott manifolds of dimension eight, i.e. simply connected, closed, smooth, eight-dimensional manifolds M which admit an ring isomorphism between the integral cohomology ring of M and the integral cohomology ring of a Bott manifold B which preserves the Stiefel-Whitney and Pontrjagin classes. We will make this notion precise subsequently.

Recall that the weak cohomological rigidity problem asks if two Bott manifolds are diffeomorphic if their integral cohomology rings are isomorphic. Our motivation to consider the class of cohomology Bott manifolds is based on the observation that the proof of the weak cohomological rigidity problem in dimension six only uses that any cohomology ring isomorphism of Bott manifolds preserves the Stiefel-Whitney and Pontrjagin classes. It does not use the toric structure of Bott manifolds in any way. The underlying results which are used in the proof are classification results of Wall and Jupp (cf. [Wal66] and [Jup73]) on simply connected six-dimensional manifolds. One consequence of their results is that cohomology Bott manifolds of dimension six are rigid. Thus, we consider the next interesting dimension, which is dimension eight.

After we make the notion of a cohomology Bott manifold precise, we show that the number of diffeomorphism classes of cohomology Bott manifolds, with respect to some Bott manifold, is finite. This is the content of Theorem 4.2.
In Theorem 4.10 we then show that there exist cohomology Bott manifolds which are not diffeomorphic to any Bott manifold, i.e. cohomology Bott manifolds are not rigid. The proof of this theorem gives first clues towards a classification of cohomology Bott manifolds.

We start with the definition of a cohomology Bott manifold.

Let M be a smooth, closed manifold with torsion-free integral cohomology. Since $H^*(M)$ is torsion free, an isomorphism of integral cohomology induces an isomorphism of cohomology with coefficients in $\mathbb{Z}/2$. We denote both isomorphisms by the same symbol.
The total Stiefel-Whitney and Pontrjagin classes of a smooth manifold M are denoted by $w(M)$ and $p(M)$, respectively.

Definition 4.1. Let M and N be two smooth, closed manifolds with torsion-free integral cohomology. A *polarisation map* $g_{M,N}$ of M and N is an isomorphism of rings $g_{M,N}\colon H^*(M) \to H^*(N)$ such that $g(p(M)) = p(N)$ and $g(w(M)) = w(N)$.
The *polarised structure set* $S^p(M)$ of M is defined to be

$$S^p(M) := \{(N, g_{M,N}) \mid N \text{ smooth manifold, } g_{M,N} \text{ polarisation map}\}/\sim,$$

where $(N, g) \sim (N', g')$ if and only if there exists a diffeomorphism $h\colon N \to N'$ such that

$$\begin{array}{ccc} H^*(N') & \xrightarrow{\quad h^* \quad} & H^*(N) \\ & {}^{g'}\nwarrow \qquad \nearrow^{g} & \\ & H^*(M) & \end{array}$$

commutes. Obviously, $\sim$ is an equivalence relation.
If M is simply connected we, in addition, demand that all elements in the polarised structure set $S^p(M)$ are simply connected.
We call a smooth, simply connected manifold N a *cohomology Bott manifold* (with respect to a Bott manifold B_j) if there exists a polarisation map $g_{B_j,N}$, i.e. if $[N, g_{B_j,N}] \in S^p(B_j)$.

If $H^*(M)$ is generated as a ring by elements in $H^r(M)$ the condition on the total Stiefel-Whitney class is automatically fulfilled by Lemma 8.1 in [CMS10]. Since elements in $H^2(B_j)$ generate $H^*(B_j)$ this, in particular, holds for cohomology Bott manifolds.
Some of the toric topologists conjecture that a ring isomorphism between the integral cohomology rings of two Bott manifolds automatically is a polarisation map, i.e. they conjecture that a ring isomorphism between the cohomology rings of two Bott manifolds also preserves the Pontrjagin classes. There was an attempt to prove this conjecture in [Cho11b] but it turned out that there was a gap in the proof.

From now on, we concentrate on eight-dimensional cohomology Bott manifolds.

Theorem 4.2. *Let B_4 be a Bott manifold of dimension eight. The cardinality of the polarised structure set $|S^p(B_4)|$ is finite, i.e. the number of diffeomorphism classes of cohomology Bott manifolds with respect to B_4 is finite.*

Note that the statement that $|S^p(B_4)|$ is finite, is stronger that the statement that the number of diffeomorphism classes of the underlying manifolds is finite. If $|S^p(B_4)|$ is finite the number of diffeomorphism that are not realisable is also finite.

The proof of this theorem will take the next sections. The proof strategy, of course, is to use modified surgery theory, in particular, Corollary 3.11. To use Corollary 3.11 we need to compare manifolds up to $\mathbb{B}$-bordism, for a convenient fibration $\mathbb{B} \to BO$. Thus, we construct $\mathbb{B}$ in Section 4.1. Then we determine the (co)homological properties of $\mathbb{B}$ in Section 4.2. Using the (co)homological properties of $\mathbb{B}$ we can approximate the $\mathbb{B}$-bordism groups in Section 4.3. Finally, we assemble everything into the proof of Theorem 4.2 in Section 4.4.

4.1. The normal three-type for elements of $S^p(B_4)$

For the classification of eight-dimensional cohomology Bott manifolds, one convenient fibtration $\mathbb{B}$ is the normal three-type of B_4.
Recall that we denote the universal stable real vector bundle over BO by γ^u. Furthermore, let $\oplus\colon BO \times BO \to BO$ denote the classifying map of $\gamma^u \times \gamma^u \to BO \times BO$. We change our notation slightly. Instead of denoting the seven-connected cover of BO by $BO\langle 8\rangle$, we denote it by its more common name $BString$.

Proposition 4.3. *Let B_4 be an eight-dimensional Bott manifold.*

1. *If $w_2(B_4) = 0$ and $p_1(B_4) = 0$ the normal three-type of B_4 is given by*

$$\mathbb{B}_3(B_4) \simeq \mathbb{P}_3 B_4 \times BString \xrightarrow{p_8 \circ pr_2} BO\ .$$

   Here pr_2 is the projection to the second factor and p_8 is the usual projection map $BString \to BO$.

2. *Otherwise, the normal three-type is given by*

$$\mathbb{B}_3(B_4) \simeq \mathbb{P}_3 B_4 \widetilde{\times} BString \xrightarrow{-\bigoplus l_i \times p_8} BO \times BO \xrightarrow{\oplus} BO\ .$$

   The map $-\bigoplus l_i$ is the classifying map for the bundle $-\bigoplus l_i \to \mathbb{P}_3 B_4$ which is constructed in the proof.

Recall that the first Pontrjagin class of a *Spin*-manifold M is always divisible by two. The obstruction to the existence of a normal *String*-structure on a M is $\frac{1}{2}p_1(M)$. In our setting M has torsion-free cohomology. Thus, if it suffices to show that $p_1(M)$ vanishes.

Proof. Consider the first part of the lemma. Since $w_2(B_4) = 0 = p_1(B_4)$ the stable normal Gauss map admits a lift $\widetilde{\nu}$ to $BString$. By definition of $\mathbb{P}_3 B_4$ there exists a four-connected map $B_4 \to \mathbb{P}_3 B_4$. Since $BString$ is seven-connected the product of this map and $\widetilde{\nu}$ is also four-connected, whereas the projection is four-co-connected. Consequently, the first part of the Proposition follows.

Thus, let B_4 be as in its second part.
By the definitions of $BString$ and Postnikov stage $\oplus \circ (-\bigoplus l_i \times p_8)\colon \mathbb{B}_3(B_4) \to BO$ is four-co-connected in this setting, too.
Let $\iota_3\colon B_4 \to \mathbb{P}_3 B_4$ denote an arbitrary four-connected map. By Lemma 3.14 it only remains to show that there exists a vector bundle E of finite rank such that the classifying map of $\iota_3^*(-E) \oplus \nu(B_4)$ admits a lift to $BString$.

Recall the following result on complex line bundles:
Let X be a space which has the homotopy type of a CW complex and let $\mathcal{L}_{\mathbb{C}}(X)$ denote

the group of isomorphism classes of complex line bundles over X. The first Chern class constitutes an isomorphism $c_1 \colon \mathcal{L}_{\mathbb{C}}(X) \to H^2(X)$ (cf. [Hus94, Theorem 3.4, p.250]).
By Section 2.2 the tangent bundle of B_4 is a Whitney sum of complex line bundles $\widetilde{l_i}$ for $i = 1, .., 4$, i.e. $TB_4 = \bigoplus \widetilde{l_i}$. In particular, $\widetilde{l_i}$ is determined by $c_1(\widetilde{l_i})$. Since ι_3 is four-connected ι_3^* is an isomorphism in second cohomology. Therefore, we can always find classes $\omega_i \in H^2(\mathbb{P}_3 B_4)$ such that $\iota_3^*(\omega_i) = c_1(\widetilde{l_i})$ for $i = 1, .., 4$. We define line bundles l_i by $c_1(l_i) = \omega_i$. By construction $\iota_3^*(\bigoplus l_i) \oplus \nu(B_4)$ is isomorphic to $TB_4 \oplus \nu(B_4)$. This bundle is trivial and, therefore, clearly admits a *String* structure.
Thus, the twisting bundle E as in Lemma 3.14 should be related to $-\bigoplus l_i$, the K-theoretic inverse of $\bigoplus l_i$. Unfortunately this, a priori, need not be a vector bundle of finite rank, which is assumed in Lemma 3.14.
Each complex line bundle l over a CW-complex X is the pullback of the tautological line bundle $\gamma \to \mathbb{C}P^\infty$ along a map $c \colon X \to \mathbb{C}P^\infty$, where c corresponds to the first Chern class $c_1(l)$. For each finite dimensional skeleton $X^{(k)}$ the map factors through $\mathbb{C}P^k$, i.e. we obtain $c^{(k)} \colon X^{(k)} \to \mathbb{C}P^k$.
Let $\gamma^\perp \to \mathbb{C}P^k$ denote the bundle which is perpendicular to γ, i.e. the bundle with total space $\gamma^\perp = \big\{(z, v) \in \mathbb{C}P^k \times \mathbb{C}^{k+1} \mid v^\perp \in z\big\}$ and projection $(z, v) \mapsto z$. Since $\gamma \oplus \gamma^\perp$ is trivial we can define $-l|_{X^{(k)}} := (c^{(k)})^* \gamma^\perp$.
We are interested in maps from eight-dimensional manifolds to $\mathbb{P}_3 B_4$. Therefore, we can always assume that the image of those maps is contained in the k-skeleton of $\mathbb{P}_3 B_4$ for some sufficiently large k. Thus, we can define the twisting bundle $-\bigoplus l_i$ to be the Whitney sum of the pullbacks of $\gamma^\perp$ along the maps induced by $c_1(l_i)$. □

Let $(N, g_{B_4,N})$ be a cohomology Bott manifold together with a polarisation map.
In order to apply modified surgery theory with respect to the control space $\mathbb{B}_3(B_4)$ to all elements in the polarised structure set, we need to show that there exists a normal three-smoothing $N \to \mathbb{B}_3(B_4)$ for each pair $(N, g_{B_4,N})$ as above.

Proposition 4.4. *Let B_4 be a Bott manifold and let $(N, g_{B_4,N})$ be a representative of an element in $S^p(B_4)$. Furthermore, let $\mathbb{B}_3(B_4) \to BO$ be the fibration induced by the maps introduced in Proposition 4.3. Then, there exists a normal three-smoothing $N \to \mathbb{B}_3(B_4)$.*

Proof. The proof is structured in the following way. We start by constructing the Postnikov tower $\mathbb{P}_3 B_4 \to \mathbb{P}_2 B_4$ more explicitly. Then, we show that there exists a map $j_3 \colon N \to \mathbb{P}_3 B_4$ which is four-connected. The last step is to show that the classifying map of $j_3^*(\bigoplus l_i) \oplus \nu(N)$ admits a lift α to *BString*. Then $j_3 \times \alpha \colon N \to \mathbb{B}_3(B_4)$ is a normal three-smoothing, again by Lemma 3.14.

The homotopy groups of B_4 are $\pi_i(B_4) \cong \pi_i(S^2)^4$, for all $i \in \mathbb{N}$, by Lemma 2.2.
Therefore, B_4 is simply connected and $\pi_2(B_4) \cong \mathbb{Z}^4$. Hence, the second Postnikov stage is

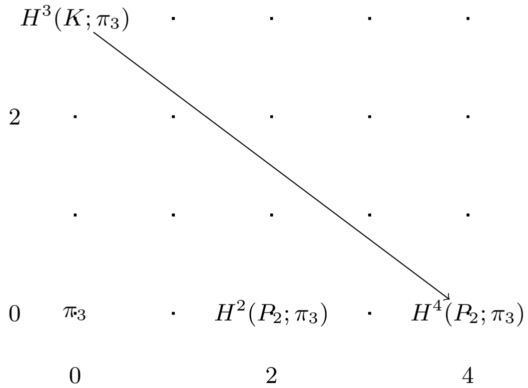
$$\mathbb{P}_2(B_4) \simeq (\mathbb{C}P^\infty)^4.$$

Let a_i, $i \leq 4$, denote a basis of $H^2((\mathbb{C}P^\infty)^4)$, such that each a_i is the pullback of a generator of the cohomology of one factor. In Section 2.1 Equation (1) we introduced generators y_i, $1 \leq i \leq 4$ for $H^*(B_4)$.
Since $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$ there exists a map $\iota_2 \colon B_4 \to (\mathbb{C}P^\infty)^4$ such that $i_2^*(a_i) = y_i$. Because $\pi_3(B_4) \cong \mathbb{Z}^4$, we see that $\mathbb{P}_3 B_4$ is the total space of a fibration over $\mathbb{P}_2 B_4$ with fiber $K(\mathbb{Z}^4, 3)$. The fibration $\mathbb{P}_3 B_4 \to \mathbb{P}_2 B_4$ is the pullback of the pathspace fibration over $K(\mathbb{Z}^4, 4)$ by the third k-invariant $k_3 \colon \mathbb{P}_2 B_4 \to K(\mathbb{Z}^4, 4)$ (cf. Section 3.1). Our next goal is to understand k_3.

To determine the k-invariant we use Lemma 3.3 which connects the k-invariant to a differential in the cohomological Leray-Serre spectral sequence with coefficients in π_3.
By Lemma 3.2 $H^3(\mathbb{P}_3 B_4) \cong H^3(B_4) = 0$. Thus, the universal coefficient theorem implies $H^3(\mathbb{P}_3 B_4; \pi_3(B_4)) = 0$.
Consider the fourth page E_4^{pq} of the cohomological Leray-Serre spectral sequence of the fibration

$$K := K(\pi_3(B_4), 3) \to \mathbb{P}_3 B_4 \xrightarrow{p} \mathbb{P}_2 B_4 =: P_2$$

with coefficients in $\mathbb{Z}^4 \cong \pi_3(B_4) =: \pi_3$ for $p + q \leq 4$ and $q \leq 3$.



The indicated differential must be injective since we already know that $H^3(\mathbb{P}_3 B_4; \pi_3) = 0$. Consequently, Lemma 3.3 implies that

$$k_3^* \colon H^4(K(\pi_3, 4); \pi_3) \to H^4(\mathbb{P}_2 B_4; \pi_3)$$

is also injective. Since π_3 is free it follows that $k_3^* \colon H^4(K(\pi_3, 4)) \to H^4(\mathbb{P}_2 B_4)$ is injective, too.
Consider a lift $\iota_3 \colon B_4 \to \mathbb{P}_3 B_4$ of $\iota_2 \colon B_4 \to \mathbb{P}_2 B_4$ which exists by the definition of a Postnikov tower. In particular, the k-invariant k_3 has the property that the composition $k_3 \circ \iota_2 \colon B_4 \to K(\pi_3, 4)$ is homotopic to the constant map, i.e. $i_2^* \circ k_3^*$ is the zero-map in cohomology with coefficients in π_3.
Again by the universal coefficient theorem and the fact that π_3 is free, this implies that

k_3 fulfills $i_2^* \circ k_3^* = 0$ in integral cohomology.
Recall (cf. Section 2.1) that the generators y_j, $1 \leq j \leq 4$ of the cohomology ring of B_4 fulfil $y_j^2 = \alpha_j y_j = \sum_{i<j} A_j^i y_i y_j$. Consequently, a basis for $\ker(i_2^* \colon H^4(\mathbb{P}_2 B_4) \to H^4(B_4))$ is given by $a_j^2 - \sum_{i<j} A_j^i a_i a_j$.
Let z_i for $1 \leq i \leq 4$ denote a basis of $H^4(K(\pi_3, 4)) \cong \mathbb{Z}^4$. Since $K(\pi_3, 4)$ is an Eilenberg-MacLane space, maps into $K(\pi_3, 4)$ are determined, up to homotopy, by the cohomology class they induce. We choose k_3 to be the map defined by

$$k_3^*(z_i) = a_i^2 - \sum_{i<j} A_j^i a_i a_j.$$

Thus, the fibration $\mathbb{P}_3 B_4 \to \mathbb{P}_2 B_4$ is the pullback of the pathspace fibration by k_3.

Next, we show that N also admits a map to $\mathbb{P}_3 B_4$ which is a four-equivalence.
Since $\mathbb{P}_2 B_4$ is an Eilenberg-MacLane space we obtain a map $N \to \mathbb{P}_2 B_4$ by fixing classes in $H^2(N)$. Let $j_2 \colon N \to \mathbb{P}_2 B_4$ be the map defined by $a_i \mapsto g_{B_4,N}(y_i)$. Thus, j_2^* is an isomorphism on second cohomology. Since $g_{B_4,N}$ is an isomorphism of rings we have

$$j_2^*(a_j^2 - \sum_{i<j} A_j^i a_i a_j) = g_{B_4,N}(y_j^2 - \sum_{i<j} A_j^i y_i y_j) = 0.$$

Consequently, there exists a lift of j_2 which we denote by $j_3 \colon N \to \mathbb{P}_3 B_4$.

Now we show that this map is four-connected.
We turn $N \to \mathbb{P}_2 B_4$ into an inclusion by the mapping cylinder construction and consider the long exact sequence of the pair $(\mathbb{P}_2 B_4, N)$ in cohomology

$$H^0(N) \overset{\cong}{\to} H^0(\mathbb{P}_2 B_4) \to H^1(\mathbb{P}_2 B_4, N) \to \overset{=0}{H^1(\mathbb{P}_2 B_4)} \to \overset{=0}{H^1(N)} \to H^2(\mathbb{P}_2 B_4, N) \to H^2(\mathbb{P}_2 B_4) \overset{\cong}{\to}$$
$$H^2(N) \to H^3(\mathbb{P}_2 B_4, N) \to \underset{=0}{H^3(\mathbb{P}_2 B_4)} \to \underset{=0}{H^3(N)} \to H^4(\mathbb{P}_2 B_4, N) \to \underset{\cong \mathbb{Z}^{10}}{H^4(\mathbb{P}_2 B_4)} \overset{j_2^*}{\to} \underset{\cong \mathbb{Z}^6}{H^4(N)}\ .$$

We deduce that $H^i(\mathbb{P}_2 B_4, N) = 0$ for $i = 1, 2, 3$ and $H^4(\mathbb{P}_2 B_4, N) \cong \ker(j_2^*)$, which is torsion free. By the universal coefficient theorem $H_i(\mathbb{P}_2 B_4, N)$ vanishes for $i = 1, 2, 3$ and by the relative Hurewicz theorem $\pi_i(\mathbb{P}_2 B_4, N)$ vanishes, too. Furthermore, the relative Hurewicz theorem implies $\pi_4(\mathbb{P}_2 B_4, N) \cong H_4(\mathbb{P}_2 B_4, N)$. Now we consider the long exact sequence of homotopy groups

$$\underbrace{\pi_4(\mathbb{P}_2 B_4)}_{=0} \to \pi_4(\mathbb{P}_2 B_4, N) \to \pi_3(N) \to \underbrace{\pi_3(\mathbb{P}_2 B_4)}_{=0}\ ,$$

to obtain $\pi_4(\mathbb{P}_2 B_4, N) \cong \pi_3(N)$. We assemble all isomorphisms, use the universal coefficient theorem once more, and obtain

$$\pi_3(N) \cong \pi_4(\mathbb{P}_2 B_4, N) \cong H_4(\mathbb{P}_2 B_4, N) \cong H^4(\mathbb{P}_2 B_4, N) \cong \ker(j_2^*) \cong \mathbb{Z}^4.$$

Note that $p_*\colon \pi_2(\mathbb{P}_3B_4) \to \pi_2(\mathbb{P}_2B_4)$ and $(j_2)_*\colon \pi_2(N) \to \pi_2(\mathbb{P}_2B_4)$ are isomorphisms. The map $(j_3)_*\colon \pi_2(N) \to \pi_2(\mathbb{P}_3B_4)$ is an isomorphism, too, since $p \circ j_3 = j_2$. The same holds for the induced map $(j_3)_*$ on homology. By turning j_3 into an inclusion we obtain

$$\begin{array}{ccccccccc} \pi_3(N) & \xrightarrow{(j_3)_*} & \pi_3(\mathbb{P}_3B_4) & \longrightarrow & \pi_3(\mathbb{P}_3B_4, N) & \longrightarrow & \pi_2(N) & \xrightarrow{\cong} & \pi_2(\mathbb{P}_3B_4) \\ & & & & \downarrow\cong & & & & \\ 0 = H_3(N) & \longrightarrow & H_3(\mathbb{P}_3B_4) & \longrightarrow & H_3(\mathbb{P}_3B_4, N) & \longrightarrow & H_2(N) & \xrightarrow{\cong} & H_2(\mathbb{P}_3B_4)\ . \end{array}$$

By Lemma 3.2 $H_3(\mathbb{P}_3B_4)$ vanishes, hence $H_3(\mathbb{P}_3B_4, N) = 0$. Thus, the map $(j_3)_*$ is onto. Its domain and target fulfil $\pi_3(N) \cong \mathbb{Z}^4 \cong \pi_3(\mathbb{P}_3B_4)$. Consequently, $(j_3)^*$ is an isomorphism.

It remains to show that $j_3^*(\bigoplus l_i) \oplus \nu(N))$ admits a *String*-structure, i.e. it remains to show that $w_2(j_3^*(\bigoplus l_i) \oplus \nu(N))$ and $p_1(j_3^*(\bigoplus l_i) \oplus \nu(N))$ both vanish.
By definition of j_3 and the fact that $g_{B_4,N}$ is a polarisation map

$$j_3^*(w_2(\bigoplus l_i)) = g_{B_4,N}(w_2(TB_4)) = w_2(TN) = w_2(\nu(N)).$$

Since $H^1(N;\mathbb{Z}/2) = 0$ the last equality follows by applying the Whitney sum formula to $w_2(TN \oplus \nu(N)) = 0$. Analogously we obtain $j_3^*(p_1(\bigoplus l_i)) = p_1(TN)$.
There also exists a Whitney sum formula for Pontrjagin classes (cf. [MS74] p. 175). For two vector bundles ξ and η over the same base space the total Pontrjagin classes fulfil

$$2(p(\xi \oplus \eta) - p(\xi) \cup p(\eta)) = 0.$$

In our situation the base space is N. Since $H^*(N)$ is torsion free $p(\xi \oplus \eta) = p(\xi) \cup p(\eta)$, in particular $p_1(\xi) + p_1(\eta) = p_1(\xi \oplus \eta)$. Therefore, we have

$$p_1(j_3^*(\bigoplus l_i) \oplus \nu(N)) = p_1(TN) \oplus p_1(\nu(n)) = p_1(TN \oplus \nu(N)) = 0.$$

Thus, there exists a lift $\widetilde{\nu}_N$ of the classifying map of $j_3^*(\bigoplus l_i \oplus \nu_N)$, i.e. by Lemma 3.14 $j_3 \times \widetilde{\nu}_N$ is a normal three-smoothing of N. □

Subsequently we denote the elements in $\Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i)$ induced by a cohomology Bott manifold N and a polarisation map $g_{B_4,N}$, as constructed in the proof, by $[N, j_3 \times \widetilde{\nu}_N]$.

Note that the lift $j_3\colon N \to \mathbb{P}_3B_4$ is unique since $H^3(N;\pi_3) = 0$.
Let $\bigoplus l_i \oplus \nu_N$ be oriented. Then, the lift $\widetilde{\nu}_N$ is also unique.

Observe that the induced element $[N, j_3 \times \widetilde{\nu}_N]$ is independent of the choice of representative of an equivalence class $[\widetilde{N}, \widetilde{g}] \in S^p(B_4)$:
Let $(N, g_{B_4,N})$ and $(N', g_{B_4,N'})$ be two representatives of $[\widetilde{N}, \widetilde{g}]$. By definition there exists a diffeomorphism $f\colon N' \to N$ inducing a commutative triangle on cohomology as

in Definition 4.1, i.e. $g_{B_4,N'} \circ f^* = g_{B_4,N}$. The map j_2 is defined by the polarisation map and thus unique up to homotopy, as is the lift j_3 since the number of choices of lifts is determined by $H^3(N) = 0$. Thus, by construction $(j_3 \times \widetilde{\nu}_N) \circ f \simeq j_3' \times \nu_{N'}$, i.e. $N \times I \cup_f N'$ is a $\mathbb{B}_3(B_4)$-bordism between N and N'.
The construction in Proposition 4.4 results in a map $S^p(B_4) \to \Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i)$. It is well-defined by the observation above. We prove Theorem 4.2 by showing that the map is injective and has finite image.

4.2. Homology of $\mathbb{P}_3B_4$

In Section 3.4 we introduced the twisted Atiyah-Hirzebruch spectral sequence. In the next section, we use it to calculate $\Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i)$. Recall that this includes the untwisted case if $\bigoplus l_i$ is the trivial bundle.
In order to calculate the E^2-page of the twisted Atiyah-Hirzebruch spectral sequence we need the homology groups of $\mathbb{P}_3B_4$, at least up to dimension eight.
Since $\mathbb{P}_3B_4$ is the total space of a fibration we can apply the Leray-Serre spectral sequence with integral coefficients. Even though we are interested in the first eight homology groups of $\mathbb{P}_3B_4$ we use the cohomological Leray-Serre spectral sequence, because there we can employ the multiplicative structure on cohomology. For this purpose, we need the integral cohomology of the fiber. In [Hat04] on page 30 we find

i	0	1	2	3	4	5	6	7	8	9
$H^i(K(\mathbb{Z},3))$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	$\mathbb{Z}/2$

.

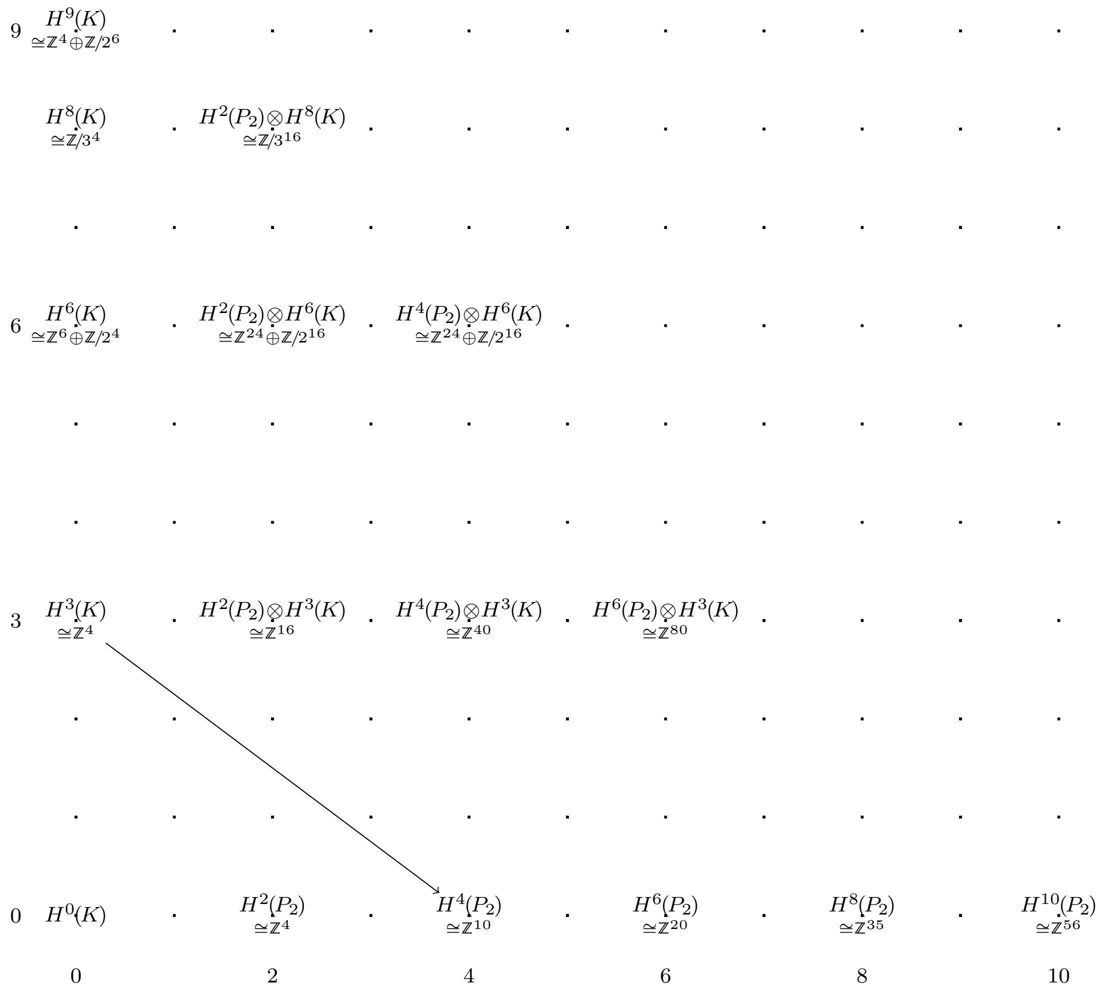
With the more general form of the Künneth theorem for cohomology as presented in [HW60] we calculate

i	0	1	2	3	4	5	6	7	8	9
$H^i(K(\mathbb{Z}^4,3))$	$\mathbb{Z}$	0	0	$\mathbb{Z}^4$	0	0	$\mathbb{Z}^6 \oplus \mathbb{Z}/2^4$	0	$\mathbb{Z}/3^4$	$\mathbb{Z}^4 \oplus \mathbb{Z}/2^6$

Within the calculation of $H^*(\mathbb{P}_3B_4)$ we abbreviate the fibration $K(\mathbb{Z}^4, 3) \to \mathbb{P}_3B_4 \to \mathbb{P}_2B_4$ with $K \to P_3 \to P_2$.

For $p+q \leq 9$, there does not exist a page E_j^{pq} in the Leray-Serre spectral sequence with integral coefficients that admits non-vanishing differentials with target or domain E_i^{08} or E_i^{28} since these entries contain the only appearing odd torsion.
Thus, the indicated d_4-differential in the E_4-page below is the first possible to appear in this range. It is determined by the third k-invariant (cf. Proposition 3.4).
By the Künneth theorem for cohomology presented in [HW60] products of the generators in $H^3(K)$ generate $E_4^{06} = H^6(K)$ and $E_4^{09} = H^9(K)$, whence the Leibniz rule determines the differentials with domain E_4^{i6} and E_4^{i9}.

Find below the E_4-page of the cohomological Leray-Serre spectral sequence with integral coefficients of the fibration $K \to P_3 \to P_2$ for $q \leq 9$ and $p+q \leq 10$.



Up to extension problems we obtain $H^i(\mathbb{P}_3 B_4)$ for $i \leq 9$ from the Leray-Serre spectral sequence above. The extension problems are considered in Appendix A. They can be solved by comparison with the cohomology with $\mathbb{Z}/2$-coefficients $H^*(\mathbb{P}_3 B_4; \mathbb{Z}/2)$. In Appendix A we determine $H^*(\mathbb{P}_3 B_4; \mathbb{Z}/2)$ using the Leray-Serre spectral sequence with coefficients in $\mathbb{Z}/2$.
After solving the extension problems in cohomology, the Künneth Theorem determines $H_i(\mathbb{P}_3 B_4)$ for $i \leq 8$.

Lemma 4.5. *The integral (co)homology groups of $\mathbb{P}_3 B_4$, abbreviated by P_3, are*

i	0	1	2	3	4	5	6	7	8	9
$H^i(P_3)$	$\mathbb{Z}$	0	$\mathbb{Z}^4$	0	$\mathbb{Z}^6$	0	$\mathbb{Z}^4\oplus\mathbb{Z}/2^4$	0	$\mathbb{Z}\oplus\mathbb{Z}/2^{16}\oplus\mathbb{Z}/3^4$	0
$H_i(P_3)$	$\mathbb{Z}$	0	$\mathbb{Z}^4$	0	$\mathbb{Z}^6$	$\mathbb{Z}/2^4$	$\mathbb{Z}^4$	$\mathbb{Z}/2^{16}\oplus\mathbb{Z}/3^4$	$\mathbb{Z}$	

In Appendix A we also compute the product structure of the integral cohomology and of cohomology with coefficients in $\mathbb{Z}/2$, in a range. For the latter we also determine the Steenrod-module structure.

4.3. The twisted bordism group $\Omega_8^{String}(\mathbb{P}_3 B_4, -\bigoplus l_i)$

The E^2-page of the (twisted) Atiyah-Hirzebruch spectral sequence converging to $\mathbb{B}$-bordism, with $\mathbb{B}$ either a twisted fibration $X \widetilde{\times} BO\langle m \rangle$ or the product fibration $X \times BO\langle m \rangle$ over BO is in both cases is given by $E^2_{p,q} \cong H_p(X; \Omega_q^{O\langle m \rangle}(pt))$, by the construction in Section 3.4. Of course, the differentials depend on the bundle we twist with.
We apply the (twisted) Atiyah-Hirzebruch spectral sequence to our situation. Therefore, we need the coefficients $\Omega_i^{String}(pt)$. For dimensions less or equal 16 they were calculated in [Gia71]. For $i \leq 9$ they are:

i	0	1	2	3	4	5	6	7	8	9
$\Omega_i^{String}(pt)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2^2$

Since we are interested in $\Omega_8^{String}(\mathbb{P}_3 B_4; -\bigoplus l_i)$ we only depict the seventh, eighth and ninth diagonal and the coefficients.

9	$\mathbb{Z}/2^2$	·	·	·	·	·	·	·	·	·
	$\mathbb{Z} \oplus \mathbb{Z}/2$	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·
6	$\mathbb{Z}/2$	·	$\mathbb{Z}/2^4$	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·
3	$\mathbb{Z}/24$	·	·	·	$\mathbb{Z}/24^6$	$\mathbb{Z}/2^4$	$\mathbb{Z}/2^8$	·	·	·
	$\mathbb{Z}/2$	·	·	·	·	$\mathbb{Z}/2^4$	$\mathbb{Z}/2^8$	$\mathbb{Z}/2^{16}$	·	·
	$\mathbb{Z}/2$	·	·	·	·	·	$\mathbb{Z}/2^8$	$\mathbb{Z}/2^{16}$	$\mathbb{Z}/2^{17}$	·
0	$\mathbb{Z}$	·	·	·	·	·	·	$\mathbb{Z}/2^{16} \oplus \mathbb{Z}/3^4$	$\mathbb{Z}$	T
	0		2		4		6		8	

Here T is a finite group we cannot and need not determine exactly.
By Lemma 3.16 there is a tool to compute the differential $d_2\colon E^2_{p+2,q} \to E^2_{p,q+1}$ for $q = 0, 1$. In order to use Lemma 3.16 we need to know the second Steenrod square $Sq^2\colon H^k(\mathbb{P}_3 B_4; \mathbb{Z}/2) \to H^{k+2}(\mathbb{P}_3 B_4; \mathbb{Z}/2)$ for $k = 5, 6, 7$, furthermore, the reduction modulo two $red\colon H^k(\mathbb{P}_3 B_4) \to H^k(\mathbb{P}_3 B_4; \mathbb{Z}/2)$ for $k = 8, 9$ and, in the twisted case, the cup product with $w_2(-\bigoplus l_i)$.
We determine the complete Steenrod module structure of $H^k(\mathbb{P}_3 B_4; \mathbb{Z}/2)$ for $k \leq 10$ (Appendix A). There we also obtain a sufficient part of the product structure in the same range to determine the cup product with $w_2(-\bigoplus l_i)$.
We consider the case $w_2(B_4) \neq 0$ first.

We start with $d_2\colon E^2_{71} \to E^2_{52}$ which, by Lemma 3.16 is dual to the map

$$\begin{aligned} Sq^2_w\colon H^5(\mathbb{P}_3 B_4; \mathbb{Z}/2) &\to H^7(\mathbb{P}_3 B_4; \mathbb{Z}/2) \\ x &\mapsto Sq^2(x) + x \cup w_2. \end{aligned}$$

Knowing the Steenrod module structure we, in particular, know that the Steenrod square $Sq^2\colon H^5(\mathbb{P}_3 B_4; \mathbb{Z}/2) \to H^7(\mathbb{P}_3 B_4; \mathbb{Z}/2)$ vanishes (compare Equation (15) in Appendix A), i.e. $Sq^2_w(x) = Sq^2(x) + x \cup w_2 = x \cup w_2$. Thus, the cup product with w_2 determines Sq^2_w. By the product structure of $H^7(\mathbb{P}_3 B_4; \mathbb{Z}/2)$ we know Sq^2_w is injective. Hence,

$$\ker(d_2\colon E^2_{71} \to E^2_{52}) = \mathbb{Z}/2^{12}.$$

By a slightly more tedious argument we obtain

$$\operatorname{im}(d_2\colon E^2_{81} \to E^2_{62}) \cong \mathbb{Z}/2^4.$$

The calculation here is a bit more involved because $Sq^2\colon H^6(\mathbb{P}_3 B_4; \mathbb{Z}/2) \to H^8(\mathbb{P}_3 B_4; \mathbb{Z}/2)$ does not vanish (compare Table A and Equations (14) and (16) in Appendix A). But in the end the cup product with w_2 determines Sq^2_w and thus d_2.

The last differential whose image we determine is $d_2\colon E^2_{90} \to E^2_{71}$. Here the result depends on the coefficients A^i_j, for $1 \leq j \leq 4$ and $i < j$, which determine the defining line bundles for a Bott tower of height four (cf. Section 2.1).
In Section 2.2 we show $c(TB_4) = \prod(1 - 2y_i + \alpha_i)$, where $\alpha_i = \sum A^i_j y_j$ for $i < j$. The second Stiefel-Whitney class $w_2(TB_4)$ is the reduction modulo two of $c_1(TB_4)$. Thus,

$$w_2(B_4) = (A^1_2 + A^1_3 + A^1_4)y_1 + (A^2_3 + A^2_4)y_2 + A^3_4 y_3 \mod 2.$$

Let $p_3\colon \mathbb{P}_3 B_4 \to \mathbb{P}_2 B_4$ denote the projection and let a_i for $i = 1, .., 4$ denote the basis of $H^2(\mathbb{P}_2 B_4)$ as before. Recall that the isomorphism $\iota_3^*\colon H^2(\mathbb{P}_3 B_4) \to H^2(B_4)$ has the properties $p_3^* a_i \mapsto y_i$ and $\iota_3^*(\bigoplus l_i) \cong TB_4$. Hence, we obtain

$$w_2(-\bigoplus l_i) = w_2(\bigoplus l_i) = (A^1_2 + A^1_3 + A^1_4)p_3^* a_1 + (A^2_3 + A^2_4)p_3^* a_2 + A^3_4 p_3^* a_3 \mod 2.$$

Here, $\mathrm{im}(d_2\colon E^2_{90} \to E^2_{71})$ depends on the coefficients A^i_j reduced mod two, for $1 \le j \le 4$ and $i < j$.
Let $a_ia_j,\ 1 \le i, j \le 4,\ i \ne j$ be the basis of $H^4(\mathbb{P}_3B_4; \mathbb{Z}/2)$ which consists of pullbacks of $a_ia_j \in H^4(\mathbb{P}_2B_4; \mathbb{Z}/2)$ and let $b_k,\ 1 \le k \le 4$ denote a basis of $H^5(\mathbb{P}_3B_4; \mathbb{Z}/2)$ (cf. Appendix A, Table A).
By Equations (14), (15) and the product structure of $H^*(\mathbb{P}_3B_4; \mathbb{Z}/2)$ given in Table A the image of $Sq^2_w\colon H^7(\mathbb{P}_3B_4; \mathbb{Z}/2) \to H^9(\mathbb{P}_3B_4; \mathbb{Z}/2)$ is generated by

$$\begin{array}{ll} \big((A^2_3 + A^2_4)a_1a_2 + A^3_4a_1a_3\big)\, b_k, & \big((A^1_2(A^2_3 + A^2_4) + A^1_3 + A^1_4)a_1a_2 + A^3_4a_2a_3\big)\, b_k, \\ \big((A^1_2 + A^1_3)a_1a_4 + A^2_3a_2a_4\big)\, b_k \quad \text{and} & \big((A^1_2 + A^1_3A^3_4 + A^1_4)a_1a_3 + (A^2_3A^3_4 + A^2_4)a_2a_3\big)\, b_k. \end{array}$$

By Lemma 3.16 $(Sq^2_w)^* \circ red = d_2$ for $q = 0$. The map $red\colon H^9(\mathbb{P}_3B_4) \to H^9(\mathbb{P}_3B_4; \mathbb{Z}/2)$ is onto which can be seen by using the Bockstein long exact sequence of $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$. Thus, $\ker(Sq^2_w)$ is either $\mathbb{Z}/2^{16}, \mathbb{Z}/2^8$ or $\mathbb{Z}/2^4$, i.e.

$$\mathrm{im}(d_2\colon E^2_{90} \to E^2_{71}) = 0, \mathbb{Z}/2^8 \text{ or } \mathbb{Z}/2^{12}.$$

If $w_2(B_4) = 0$ the differentials $d_2\colon E^2_{71} \to E^2_{52}$ and $d_2\colon E^2_{81} \to E^2_{62}$ vanish. But again, the differential $d_2\colon E^2_{90} \to E^2_{71}$ depends on the A^i_j. In this case the image $\mathrm{im}(Sq^2\colon H^7(\mathbb{P}_3B_4; \mathbb{Z}/2) \to H^9(\mathbb{P}_3B_4; \mathbb{Z}/2))$ is generated by

$$A^1_2a_1a_2b_k, \quad (A^1_3a_1a_3 + A^2_3a_2a_3)b_k \quad \text{and} \quad (A^1_4a_1a_3 + A^2_4a_2a_4)b_k.$$

Thus, $\ker(Sq^2)$ is either $\mathbb{Z}/2^{16}, \mathbb{Z}/2^8$ or $\mathbb{Z}/2^4$, i.e.

$$\mathrm{im}(d_2\colon E^2_{90} \to E^2_{71}) = 0, \mathbb{Z}/2^8 \text{ or } \mathbb{Z}/2^{12}.$$

Note that, for the twisted and the untwisted case as well, there cannot appear any differentials that kill the integral part of the eighth diagonal. Thus, we have

Lemma 4.6. *Let R denote the torsion subgroup of $\Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i)$. Then*

$$\Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i) \cong \mathbb{Z}^2 \oplus R,$$

and

1. *for $w_2 \ne 0$ we obtain $|R| \le |\mathbb{Z}/2^k|$, where $k = 25, 17$ or 13 for $\mathrm{im}(d_2\colon E^2_{90} \to E^2_{71}) = 0, \mathbb{Z}/2^8$ or $\mathbb{Z}/2^{12}$, respectively, and*

2. *for $w_2 = 0$ we obtain $|R| \le |\mathbb{Z}/2^k|$, where $k = 33, 25$ or 21 if $\mathrm{im}(d_2\colon E^2_{90} \to E^2_{71}) = 0, \mathbb{Z}/2^8$ or $\mathbb{Z}/2^{12}$, respectively.*

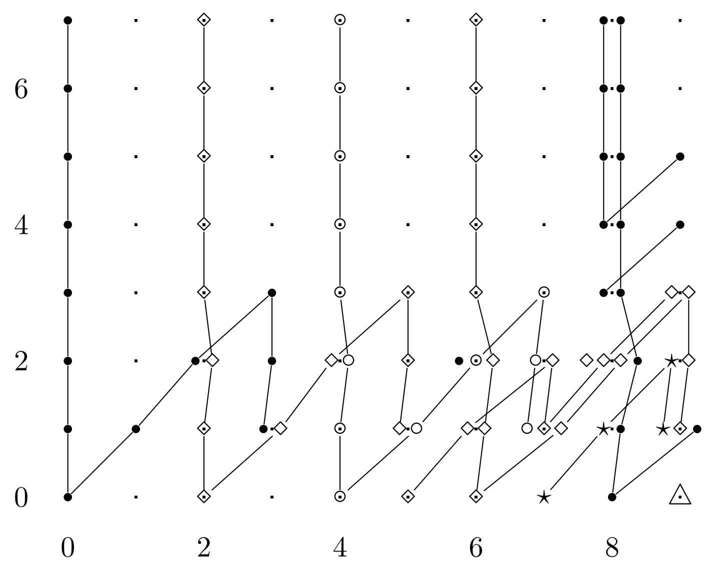
We did not find any way to determine further differentials in the (twisted) Atiyah-Hirzebruch spectral sequence. Furthermore, we tried to apply the Adams spectral sequence but this, in general, did not help to determine the twisted bordism group

$\Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i)$ on the nose. Even the calculation of the E_2-page of the Adams spectral sequence is quite tedious, since it depends on the Steenrod module structure which in turn depends on the A^i_j.
None the less we, at least, want to give one example of a calculation with the Adams spectral sequence because it allows a statement about the form of the torsion subgroup.

Assume we are in the first case of Lemma 4.3, i.e. consider a Bott manifold which is *String*. Additionally, assume that $A^i_j = 0 \mod 2$ for $i < j \leq 4$. By the Pontrjagin-Thom construction $\Omega_k^{String}(\mathbb{P}_3B_4) \cong \pi_k^{st}(\mathbb{P}_3B_{4+} \wedge MString)$ and we can apply the Adams spectral sequence.
From the Atiyah-Hirzebruch spectral sequence we know that the only torsion which can appear for $\Omega_8^{String}(\mathbb{P}_3B_4)$ is torsion at the prime two. Thus, it suffices to consider the Adams spectral sequence converging to $\pi_{t-s}^{st}(\mathbb{P}_3B_{4+} \wedge MString)$/non-2-torsion for $t - s \leq 9$. Its E_2-page is given by

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(\mathbb{P}_3B_{4+} \wedge MString;\ \mathbb{Z}/2), \mathbb{Z}/2).$$

To calculate the E_2-page for $t - s \leq 9$ we use the method of minimal resolutions as introduced in [Sto85]. Later on we calculate a resolution for one example explicitly (cf. Appendix B). Here, we just depict the E_2-page for $t - s \leq 9$. We calculated it with the computer algorithm developed by Bruner (cf. [Bru93] and [Bru]). We use the following notation: • denotes a $\mathbb{Z}/2$, ⋄ a $\mathbb{Z}/2^4$, ∘ a $\mathbb{Z}/2^6$, ⋆ a $\mathbb{Z}/2^{16}$ and △ a $\mathbb{Z}/2^{24}$. The multiplicative structure on the E_2 is indicated in the same way as in Example 6.19 of [Sto85].



Since we did not find a way to determine the d_2-differential $d_2\colon E_{90} \to E_{82}$, the Adams spectral sequence also does not determine $\Omega_8^{String}(\mathbb{P}_3B_4)$. It does, however, imply, that the torsion subgroup T of $\Omega_8^{String}(\mathbb{P}_3 B_4)$ is a direct sum of $\mathbb{Z}/2$-summands for this class of Bott manifolds. If we drop the condition $A^i_j = 0 \mod 2$ the E_2-page still indicates that the torsion subgroup is a sum of $\mathbb{Z}/2$-summands. In general, this need not be true.

4.4. Proof of Theorem 4.2 by modified surgery theory

Now we are ready to apply Corollary 3.11 to prove that two cohomology Bott manifolds with respective polarisation maps represent the same element in the polarised structure set $S^p(B_4)$ if they induce the same element in $\Omega_8^{String}(\mathbb{P}_3 B_4, -\bigoplus l_i)$.

Lemma 4.7. *Let N and N' be two eight-dimensional cohomology Bott manifolds with polarisation maps $g\colon H^*(B_4) \to H^*(N)$ and $g'\colon H^*(B_4) \to H^*(N')$, respectively. Their induced elements in $\Omega_8^{String}(\mathbb{P}_3 B_4, -\bigoplus l_i)$, as constructed in the proof of Proposition 4.4, are denoted by $[N, j_3 \times \widetilde{\nu}_N]$ and $[N', j_3' \times \widetilde{\nu}_{N'}]$. If*

$$[N, j_3 \times \widetilde{\nu}_N] = [N', j_3' \times \widetilde{\nu}_{N'}] \in \Omega_8^{String}(\mathbb{P}_3 B_4, -\bigoplus l_i),$$

then

$$[N, g] = [N', g'] \in S^p(B_4).$$

Proof. By the definition of the polarised structure set, we need to find a diffeomorphism $f\colon N' \to N$ such that $f^* \circ g' = g$.
To apply Corollary 3.11 it remains to show that the fourth homotopy groups of N and N' are finite. We show that this holds for all cohomology Bott manifolds of dimension eight.
Recall that $p_3\colon \mathbb{P}_3 B_4 \to \mathbb{P}_2 B_4$ is the projection of the Postnikov tower and that $p_3^* a_i$ generate $H^2(\mathbb{P}_3 B_4)$, where a_i, $1 \le i \le 4$, are a basis for $H^2(\mathbb{P}_2 B_4)$.
In Appendix A we deduce that $H^4(\mathbb{P}_3 B_4)$ is generated by products $a_i a_j$ for $i \neq j$. Thus, the map $j_3^*\colon H^4(\mathbb{P}_3 B_4) \to H^4(N)$ is an isomorphism by construction. The universal coefficient theorem implies that $(j_3)_*\colon H_4(N) \to H_4(\mathbb{P}_3 B_4)$ is an isomorphism, too. Here, we use that $H_k(\mathbb{P}_3 B_4)$ is torsion free for $k \le 4$.
By the long exact sequence

$$\underbrace{H_5(N)}_{=0} \to H_5(\mathbb{P}_3 B_4) \xrightarrow{q_*} H_5(\mathbb{P}_3 B_4, N) \to H_4(N) \xrightarrow{\cong} H_4(\mathbb{P}_3 B_4) \to H_4(\mathbb{P}_3 B_4, N) \to \underbrace{H_3(N)}_{=0}$$

q_* is an isomorphism on fifth homology and $H_4(\mathbb{P}_3 B_4, N) = 0$. Since j_3 is 3-connected the lower homology groups of the pair vanish as well. Thus, by Hurewicz's theorem and the long exact sequences of the pair $(\mathbb{P}_3 B_4, N)$ in homotopy there are isomorphisms

$$H_5(\mathbb{P}_3 B_4) \cong H_5(\mathbb{P}_3 B_4, N) \cong \pi_5(\mathbb{P}_3 B_4, N) \cong \pi_4(N).$$

In Lemma 4.5 we showed that $H_5(\mathbb{P}_3 B_4)$ is finite. Thus, since $[N, g]$ and $[N', g']$ are bordant by assumption, there exists a diffeomorphism $f\colon N \to N'$ such that $j_3' \circ f = j_3$

by Corollary 3.11.
By construction j_3 and j_3' are lifts of j_2 and j_2', respectively. Thus, $j_2' \circ f = j_2$.
By definition (cf. proof of Proposition 4.4), $j_2^*(a_i) = g(y_i)$ and $(j_2')^*(a_i) = g'(y_i)$. Since the cohomology of N is generated by elements in degree two f^* fulfils $f^* \circ g' = g$. □

Now we are ready to prove Theorem 4.2.

Proof. We know $\Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i) \cong \mathbb{Z}^2 \oplus R$, where R denotes the torsion subgroup. By the Lemma above it suffices to show that all elements in the bordism group that are induced by cohomology Bott manifolds have the same image under projection to the integral part of the controlled bordism group.
We need two invariants which detect $\mathbb{Z}^2 \subset \Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i)$, the first is the Thom homomorphism and the second is related to the Pontrjagin numbers.

Consider the Thom homomorphism

$$\begin{aligned} \mathcal{T} \colon \Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i) &\to H_8(\mathbb{P}_3B_4) \\ [M, h \times \beta] &\mapsto h_*[M]. \end{aligned}$$

It is also possible to describe the Thom homomorphism as edge homomorphism

$$\Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i) \twoheadrightarrow E_{80}^{\infty} \hookrightarrow E_{80}^2 \cong H_8(\mathbb{P}_3B_4)$$

of the twisted Atiyah-Hirzebruch spectral sequence. This follows from [Arl96]. Note that there, the Thom homomorphism is called generalised Hurewicz homomorphism.
Thus, the Thom homomorphism detects the integral summand $\mathbb{Z} \cong E_{p0}^{\infty}$.

Next, we show that $\mathcal{T}([N, j_3 \times \widetilde{\nu}]) = \mathcal{T}([N', j_3' \times \widetilde{\nu}'])$ for any two cohomology Bott manifold N and N' with maps j_3 and j_3' induced by the polarisation maps g and g', respectively.
Recall that $a_i, 1 \leq i \leq 4$ denote the generators of $H^*(\mathbb{P}_2B_4)$ and that $p_3 \colon \mathbb{P}_3B_4 \to \mathbb{P}_2B_4$ denotes the projection. The class $p_3^*(a_1 \cup a_2 \cup a_3 \cup a_4) =: a$ is a generator of $\mathbb{Z} \subset H^8(\mathbb{P}_3B_4)$ by a spectral sequence argument (cf. Leray-Serre spectral sequence in Section 4.5 and Appendix A).
Let (N, g) and $[N, j_3 \times \widetilde{\nu}]$ be as above. By construction of j_3

$$j_3^*(a) = j_2^*(a_1 \cup a_2 \cup a_3 \cup a_4) = g(y_1 \cup y_2 \cup y_3 \cup y_4) = [N]^*,$$

where $[N]^*$ denotes the generator of $H^8(N)$ dual to the fundamental class $[N] \in H_8(N)$.
Since $H^9(\mathbb{P}_3B_4) = 0$ the universal coefficient theorem implies $(j_3)_*[N] = a^*$, where a^* is the dual of a, for all elements in $\Omega_8^{String}(\mathbb{P}_3B_4, -\bigoplus l_i)$ induced by elements in $S^p(B_4)$.
This holds for all cohomology Bott manifolds with respect to B_4. Hence, any two cohomology Bott manifolds have the same image under the Thom homomorphism.

Recall that $E^{\infty}_{08} \cong \mathbb{Z} \oplus \mathbb{Z}/2$ or $\mathbb{Z}$, depending on the differentials, and that the bordism group is $\Omega_8^{String}(\mathbb{P}_3 B_4; -\bigoplus l_i) \cong R \oplus E^{\infty}_{08}/tor \oplus \mathbb{Z}$. The second (integral) summand is given by $E^{\infty}_{08}/tor \hookrightarrow \Omega_8(\mathbb{P}_3 B_4; -\bigoplus l_i)$. We claim that $E^{\infty}_{08}/tor \cong \Omega_8^{String}(pt)/tor$. To see this, let $E^t_{p,q}(X,E)$ denote the (p,q)-entry on the t-th page of the (twisted) Atiyah-Hirzebruch spectral sequence converging to $\Omega^{String}_{p+q}(X,E)$, where we omit E from the notation if it is trivial. Furthermore, let r be the rank of $-\bigoplus l_i$.
By definition of the twisted Atiyah-Hirzebruch spectral sequence

$$E^2_{08}(\mathbb{P}_3 B_4, -\bigoplus l_i) = E^2_{r,8}(Th(-\bigoplus l_i)).$$

The r-skeleton of $Th(-\bigoplus l_i)$ is S^r. The inclusion of the r-skeleton induces a map $E^2_{r,8}(S^r) \to E^2_{r,8}(Th(-\bigoplus l_i))$. But $E^2_{r,8}(S^r) \cong \widetilde{\Omega}^{String}_{r+8}(S^r) \cong \Omega^{String}_8(pt)$. Thus, the claim follows.
Now we consider the situation on a geometric level. Define

$$\begin{aligned} incl\colon \Omega_8^{String}(pt) &\to \Omega_8^{String}(\mathbb{P}_3 B_4, -\bigoplus l_i) \text{ by} \\ [M, \widetilde{\nu}] &\mapsto [M, pt \times \widetilde{\nu}]. \end{aligned}$$

Furthermore, let pr_8 be the projection $BString \to BSO$ and

$$\begin{aligned} pr_*\colon \Omega_8^{String}(\mathbb{P}_3 B_4, -\bigoplus l_i) &\to \Omega_8^{SO}(pt) \cong \mathbb{Z} \oplus \mathbb{Z}, \\ [M, f \times \widetilde{\nu}] &\mapsto [M, (-\bigoplus l_i \oplus pr_8) \circ (f \times \widetilde{\nu})]. \end{aligned}$$

The composition $pr_* \circ incl$ equals $(pr_8)_*$ and is well-known to have kernel $\mathbb{Z}/2$. Thus, $\mathbb{Z}$ is contained in $\mathrm{im}(incl)$ and $pr_*|_{\mathbb{Z} \subset \mathrm{im}(incl)}\colon \mathbb{Z} \to \Omega_8^{BSO}$ is injective.
The Pontrjagin numbers with respect to p_2 and p_1^2 are a complete set of invariants for $\Omega_8^{SO}(pt)$. Since these are fixed in the polarised structure set, this finishes the proof. □

Note that the proof of the theorem and Lemma 4.6 lead to the following corollary:

Corollary 4.8. *Let B_4 be a Bott manifold and let R denote the torsion subgroup of $\Omega_8^{String}(\mathbb{P}_3 B_4, -\bigoplus l_i)$. Then the number of diffeomorphism classes of cohomology Bott manifolds is bounded by $|R|$ and we obtain*

1. *for $w_2 \neq 0$ $|R| \leq |\mathbb{Z}/2^k|$, where $k = 25, 17$ or 13 for $\mathrm{Im}(d_2\colon E^2_{90} \to E^2_{71}) = 0, \mathbb{Z}/2^8$ or $\mathbb{Z}/2^{12}$, respectively, and*
2. *for $w_2 = 0$ $|R| \leq |\mathbb{Z}/2^k|$, where $k = 33, 25$ or 21 if $\mathrm{Im}(d_2\colon E^2_{90} \to E^2_{71}) = 0, \mathbb{Z}/2^8$ or $\mathbb{Z}/2^{12}$, respectively.*

Here, the differentials are differentials of the (twisted) Atiyah-Hirzebruch sequence on page 49.

Remark 4.9. To conclude, we want to outline the classification of cohomology Bott manifolds of real dimension ten.

For a classification of ten-dimensional manifolds we need normal four-smoothings. We can construct the normal four-type of a Bott manifold B_5 of dimension ten.
The tangent bundle of a ten-dimensional Bott manifolds B_5 is a sum of complex line bundles $\bigoplus \widetilde{l}_i$, where $1 \leq i \leq 5$. The normal three-type of B_5 is given by $\mathbb{P}_3 B_5 \widetilde{\times} BString$. Here the twist bundle is $-\bigoplus l_i$, for $1 \leq i \leq 5$, where l_i is defined by the property that the pullback of l_i under the map $B_5 \to \mathbb{P}_3 B_5$ is $\widetilde{l}_i$.
By Lemma 2.2 $\pi_5(B_5)$ is isomorphic to $\mathbb{Z}/2^5$. By Section 3.1 the fourth Postnikov stage of B_5 is a fibration $K(\mathbb{Z}/2^5, 4) \to \mathbb{P}_4 B_5 \to \mathbb{P}_3 B_5$ which is classified by a map to $K(\mathbb{Z}/2^5, 5)$. The normal four-type of B_5 is given by $\mathbb{B}_4(B_5) := \mathbb{P}_4 B_5 \widetilde{\times} BString$, where the twist bundle is the same as the one for the normal three-type.

Analogously to Proposition 4.4 we see that a representative $(N, g_{B_5,N})$ of an element in $S^p(B_5)$ admits a normal three-smoothing $j_3 \times \widetilde{\nu}_N$ into $\mathbb{B}_3(B_5)$. Since $H^5(N; \mathbb{Z}/2^5) = 0$ the obstruction to the existence of a lift of j_3 to $\mathbb{P}_4 B_5$ vanishes. Thus, N admits a normal four-smoothing into $\mathbb{P}_4 B_5 \widetilde{\times} BString = \mathbb{B}_4(B_5)$. To get a result similar to Theorem 4.2 we need to compute $\Omega_{10}^{\mathbb{B}_4(B_5)}$, at least we need to determine the integral subgroups.

By Lemma 3.2 the cohomology groups $H^k(\mathbb{P}_4 B_5)$ are isomorphic to $H^k(B_5)$ for $k \leq 4$. Therefore, the non-vanishing cohomology groups in this range are $H^k(\mathbb{P}_4 B_5) \cong \mathbb{Z}, \mathbb{Z}^5$ and $\mathbb{Z}^{10}$ for $k = 0, 2, 4$, respectively. From the cohomological Leray-Serre spectral sequence of $\mathbb{P}_4 B_5 \to \mathbb{P}_3 B_5$ we deduce that the only further free subgroups in $H_k(\mathbb{P}_4 B_5)$ are $\mathbb{Z}^{10}$ for $k = 6$, $\mathbb{Z}^5$ for $k = 8$ and $\mathbb{Z}$ for $k = 10$.
Consider the E^2-page of the (twisted) Atiyah-Hirzebruch spectral sequence

$$E^2_{pq} = H_p(\mathbb{P}_4 B_5; \Omega_q^{String}(pt))$$

converging to $\Omega_{p+q}^{String}(\mathbb{P}_4 B_5, -\bigoplus l_i)$ for $p + q = 10$. The only free subgroups of the coefficients are contained in $\Omega_0^{String}(pt) \cong \mathbb{Z}$ and $\Omega_8^{String}(pt) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. This implies that only the entries E_2^{28} and $E_2^{10,0}$ contain integral summands, namely $\mathbb{Z}^5$ and $\mathbb{Z}$, respectively. Furthermore, there are no differentials which can kill these groups since all integral entries appear in E^r_{pq} with p and q even and $p + q \leq 11$.

The Thom homomorphism is an invariant for $\mathbb{Z} \subset E_2^{10,0}$, as before. Furthermore, the image of a representative $(N, g_{B_5,N})$ of an element in $S^p(B_5)$ is fixed as in the eight-dimensional case. Therefore, it remains to find invariants for $\mathbb{Z}^5 \subset E_\infty^{28}$.

There is a map

$$p \times p_8 \colon \mathbb{P}_4 B_5 \times BString \to \mathbb{P}_2 B_5 \times BSO$$

which induces a map $(p \times p_8)_* \colon \Omega_{10}^{String}(\mathbb{P}_4 B_5, -\bigoplus l_i) \to \Omega_{10}^{SO}(\mathbb{P}_2 B_5) \cong \Omega_{10}^{SO}((\mathbb{C}P^\infty)^5)$. This map is injective on $\mathbb{Z}^5 \subset E_\infty^{28} \subset \Omega_{10}^{String}(\mathbb{P}_4 B_5, -\bigoplus l_i)$ since the underlying map on

homology is injective in second homology. Let $a_j \in H^2((\mathbb{C}P^\infty)^5)$ for $1 \leq j \leq 5$ denote a basis of $H^2((\mathbb{C}P^\infty)^5)$, furthermore, let $p_1 \in H^4(BSO)$ and $p_2 \in H^8(BSO)$ denote the universal Pontrjagin classes and let $[M, f \times \widetilde{\nu}]$ be an element in $\Omega_{10}^{SO}((\mathbb{C}P^\infty)^5)$. Then we can define invariants $a_j p_i$, for $1 \leq j \leq 5, i = 1,2$, on $\Omega_{10}^{SO}((\mathbb{C}P^\infty)^5)$ by $[M, f \times \widetilde{\nu}] \mapsto (f^*(a_j) \cup \widetilde{\nu}^*(p_1)^2) \cap [M]$ and by $[M, f \times \widetilde{\nu}] \mapsto (f^*(a_j) \cup \widetilde{\nu}^*(p_2)) \cap [M]$.
Since the Pontrjagin classes and the cohomology ring of a cohomology Bott manifold are fixed, all of these invariants have the same image for cohomology Bott manifolds of dimension ten.

Therefore, the polarised structure set $S^p(B_5)$ is finite, i.e. the number of cohomology Bott manifolds in dimension ten is also finite. A straight forward calculation will determine an upper bound k for the rank of the torsion subgroup R of $\Omega_{10}^{String}(\mathbb{P}_4 B_5, -\bigoplus l_i)$. Thereby the rank of $S^p(B_5)$ is also bounded by k.

We conclude with the following observation. An upper bound for the number of diffeomorphism classes of Bott manifolds (or toric manifolds) whose cohomology ring is isomorphic to $H^*(B_5)$ now only depends on k and the number of ring isomorphisms which do not preserve the Pontrjagin classes.

4.5. A cohomology Bott manifold which is not diffeomorphic to a Bott manifold

For the remainder of this section, T denotes a Bott manifold for which $p_1(T) = 0$, $w_2(T) = 0$ and for which the strong cohomological rigidity problem holds, i.e. for any other Bott manifold B_4 such that there exists a ring isomorphism $\Psi\colon H^*(T) \to H^*(B_4)$ there exists a diffeomorphism $f\colon B_4 \to T$ such that $f^* = \Psi$. Such Bott manifolds exist by [CM12] and [Cho11a].
For each T as above we construct an explicit counterexample to the cohomological rigidity of eight-dimensional cohomology Bott manifolds, i.e. we construct a manifold in the polarised structure set of $S^p(T)$ which is not diffeomorphic to any Bott manifold, in particular, not to T.

Theorem 4.10. *For each Bott manifold T as above there exists a cohomology Bott manifold $F \in S^p(T)$ such that F is not diffeomorphic to a Bott manifold.*

The strategy of the proof is to find a manifold, which we denote by K_p, which induces a non-trivial element in $\Omega_8^{String}(\mathbb{C}P^\infty)$. Furthermore, K_p has the property that the parametric connected sum $T\#_{\mathbb{C}\mathrm{P}^1} K_p$, which we explain subsequently, is a cohomology Bott manifold. The assumption that $T\#_{\mathbb{C}\mathrm{P}^1} K_p$ and T are diffeomorphic leads to a contradiction.

We start by constructing K_p.
It is well-known that the Kervaire manifold $S^3 \times S^3 =: K$ together with the String

structure $L\colon K \to BString$ obtained by the Lie-group framing, is the non-trivial element $\Omega_6^{String}(pt) \cong \Omega_6^{fr}(pt) \cong \mathbb{Z}/2$, since its Arf-invariant (or Kervaire-Arf-invariant) is non-trivial. This was already shown in [KM63]. For a definition of the Arf-invariant we refer the reader to Chapter 6 in [Lüc02]. We denote the non-trivial element in $\Omega_6^{String}(pt)$ by $[K, L]$.
Now we consider $K \times \mathbb{C}\mathrm{P}^1$ together with the map $L \times pt\colon K \times \mathbb{C}\mathrm{P}^1 \to BString$. By abuse of notation we denote this map by L, too. Since $T(L \times \mathbb{C}\mathrm{P}^1) \cong TL \oplus T\mathbb{C}\mathrm{P}^1$ the normal bundle is given by $\nu(L \times \mathbb{C}\mathrm{P}^1) \cong \nu(L) \oplus \nu(\mathbb{C}\mathrm{P}^1)$. Since the latter summand is stably trivial L induces a normal String structure on $K \times \mathbb{C}\mathrm{P}^1$.

Lemma 4.11. *Let $pr_2\colon K \times \mathbb{C}P^1 \to \mathbb{C}P^1$ be the projection upon the second factor. The element $\omega := [K \times \mathbb{C}P^1, pr_2 \times L] \in \Omega_8^{String}(\mathbb{C}P^1)$ is non-trivial. It is of finite order.*

The construction which we use to prove the first part of the lemma is also known as codimension two Arf-invariant.

Proof. In Definition 3.17 we introduce the homomorphism

$$\begin{array}{rcl} t\colon \Omega_8^{String}(\mathbb{C}\mathrm{P}^1) & \to & \Omega_6^{String}(pt) \\ [M, f \times \alpha] & \mapsto & [f^{-1}(f \pitchfork pt)(f \times \alpha)|_{f^{-1}(f \pitchfork pt)}]. \end{array}$$

By construction $t([K \times \mathbb{C}\mathrm{P}^1, pr_2 \times L]) = [K, L] \neq 0$. Thus, the preimage $[K \times \mathbb{C}\mathrm{P}^1, L \times pr_2]$ must be non-trivial in $\Omega_8^{String}(\mathbb{C}\mathrm{P}^1)$.
The map t vanishes on $\mathrm{im}(\Omega_8^{String}(pt) \hookrightarrow \Omega_8^{String}(\mathbb{C}\mathrm{P}^1))$ by the exact sequence of Lemma 3.18. Thus, ω must be non-trivial under the projection to the reduced bordism group $\widetilde{\Omega}_8^{String}(\mathbb{C}\mathrm{P}^1) \cong \mathbb{Z}/2$.

Next we show that ω is of finite order.
As in the proof of Theorem 4.2 we use the map $pr_8\colon BString \to BSO$ which induces a map $\mathbb{Z} \oplus \mathbb{Z}/2 \cong \Omega_8^{String}(pt) \to \Omega_8^{SO}(pt)$ whose kernel is $\mathbb{Z}/2$.
The Pontrjagin numbers are a complete set of invariants of $\Omega_8^{SO}(pt)$. The first Pontrjagin class $p_1(K \times \mathbb{C}\mathrm{P}^1)$ is an element in $H^4(K \times \mathbb{C}\mathrm{P}^1)$ which vanishes. Hence, the second Pontrjagin number $p_{(2)}(K \times \mathbb{C}\mathrm{P}^1) := \langle p_2(K \times \mathbb{C}\mathrm{P}^1), [K \times \mathbb{C}\mathrm{P}^1]\rangle$ is, by the signature theorem, determined by the signature of $K \times \mathbb{C}\mathrm{P}^1$. But $H^4(K \times \mathbb{C}\mathrm{P}^1) = 0$ implies that the signature vanishes. Thus, $p_{(2)}(K \times \mathbb{C}\mathrm{P}^1)$ vanishes, as well. This shows that the element $\omega \in \Omega_8^{String}(\mathbb{C}\mathrm{P}^1)$ is contained in $\Omega_8^{String}(pt)/\mathbb{Z} \oplus \widetilde{\Omega}_8^{String}(\mathbb{C}\mathrm{P}^1)$ which is the finite group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. □

Now we change $[K \times \mathbb{C}\mathrm{P}^1, pr_2 \times L]$ by surgery. By Proposition 4 of [Kre99] (which we cite in Proposition 3.8) we can turn $pr_2 \times L$ into a four-equivalence by surgery below the middle dimension. Since $\widetilde{H}_k(\mathbb{C}\mathrm{P}^1 \times BString) = 0$ for $2 \neq k \leq 7$ we obtain a manifold

K_p with

$$H_k(K_p) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, 2, 6, 8 \\ 0 & \text{else.} \end{cases}$$

We denote this representative of $[K \times \mathbb{C}\mathrm{P}^1, pr_2 \times L] \in \Omega_8^{String}(\mathbb{C}\mathrm{P}^1)$ by $(K_p, \kappa \times \widetilde{\nu}_{K_p})$.

Later on we want to be able to compare elements induced by T and the parametric connected sum - which we still need to explain - of T and K_p in $\Omega_8^{String}(\mathbb{C}P^\infty)$. To be able to do this we need to understand K_p as an element in $\Omega_8^{String}(\mathbb{C}P^\infty)$.

Lemma 4.12. *The inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty$ induces a monomorphism*

$$\Omega_8^{String}(\mathbb{C}P^1) \to \Omega_8^{String}(\mathbb{C}P^\infty).$$

In particular, $[K_p, \kappa \times \widetilde{\nu}_{K_p}]$ gives rise to a non-trivial element in $\widetilde{\Omega}_8^{String}(\mathbb{C}P^\infty)$.

The strategy of the proof is the following. The Pontrjagin-Thom construction results in an isomorphism

$$\Omega_8^{String}(\mathbb{C}P^\infty) \cong \pi_8^{st}(\mathbb{C}P_+^\infty \wedge MString).$$

Thus, we can apply the Adams spectral sequence to calculate $\Omega_8^{String}(\mathbb{C}P^\infty)$. Then we can compare the Atiyah-Hirzebruch spectral sequences of $\Omega_8^{String}(\mathbb{C}\mathrm{P}^1)$ and $\Omega_8^{String}(\mathbb{C}P^\infty)$ since the inclusion $\mathbb{C}\mathrm{P}^1 \to \mathbb{C}P^\infty$ induces a map on the respective E^2-pages. By the calculations of $\Omega_8^{String}(\mathbb{C}\mathrm{P}^1)$ and $\Omega_8^{String}(\mathbb{C}P^\infty)$ we also know the infinity pages. This enables us to deduce that the map

$$\Omega_8^{String}(\mathbb{C}\mathrm{P}^1) \to \Omega_8^{String}(\mathbb{C}P^\infty)$$

induced by the inclusion $\mathbb{C}\mathrm{P}^1 \to \mathbb{C}P^\infty$ is injective.

Later on we calculate the E^2-page of the Atiyah-Hirzebruch spectral sequence. There we see, that the only torsion that appears is two-primary. Consequently it is justified to restrict to the Adams spectral sequence at the prime two. Recall that the E_2-page of the Adams spectral sequence, converging to $\pi_{t-s}^{st}(\mathbb{C}P_+^\infty \wedge MString)$, at the prime two, has entries

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(\mathbb{C}P_+^\infty \wedge MString;\ \mathbb{Z}/2), \mathbb{Z}/2).$$

To calculate the E_2-page for $t-s \leq 9$ we use the method of minimal resolutions as introduced in [Sto85]. We calculate the necessary data for the method of minimal resolutions, i.e. the Steenrod module structure of $H^*(\mathbb{C}P^\infty \wedge MString;\ \mathbb{Z}/2)$ in the proof. For this one example we also calculate the resolution explicitly in Appendix B. We checked the result by a computer algorithm developed by Bruner (cf. [Bru93] and [Bru]).

Proof. To calculate the minimal resolution we need $H^k(\mathbb{C}P^\infty_+ \wedge MString;\ \mathbb{Z}/2)$ for $k \leq 10$. The ring $H^*(\mathbb{C}P^\infty,\ \mathbb{Z}/2)$ is generated by $a \in H^2(\mathbb{C}P^\infty;\ \mathbb{Z}/2)$.
The cohomology $H^*(BString;\ \mathbb{Z}/2)$ is determined in [Sto63]. By the Thom isomorphism the only non-vanishing cohomology groups of $MString$ in degree less or equal ten are $H^k(MString;\ \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $k = 0, 8$. The generator in degree zero is the Thom class u, the one in degree eight is uw_8. Here w_8 denotes the pullback of the eighth universal Stiefel-Whitney class in $H^8(BO; \mathbb{Z}/2)$ to $H^8(BString; \mathbb{Z}/2)$. By Chapter 8 in [MS74] we know $Sq^8(u) = uw_8$.
We consider the pullback of the classes $u, uw_8, a, a^2, ...$ to $H^*(\mathbb{C}P^\infty_+ \wedge MString)$, apply the Künneth theorem and obtain

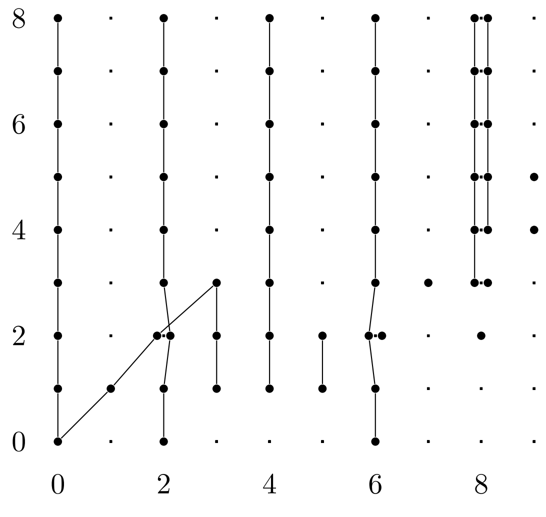
i	0	2	4	6	8	10
$H^i(\mathbb{C}P^\infty_+ \wedge MString;\ \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2^2$	$\mathbb{Z}/2^2$
generators	u	ua	ua^2	ua^3	uw_8, ua^4	ua^5, w_8a

.

The other groups $H^i(\mathbb{C}P^\infty_+ \wedge MString;\ \mathbb{Z}/2)$ vanish for $i \leq 10$. Now, a straight forward calculation shows that the only non-vanishing operations of Steenrod squares Sq^i in this range are:

$$Sq^8u = uw_8, \qquad Sq^2ua = ua^2,$$
$$Sq^8ua = uw_8a, \qquad Sq^4ua^2 = ua^4,$$
$$Sq^2ua^3 = ua^4, \qquad Sq^4ua^3 = ua^5.$$

From this data we calculate the minimal resolution in Appendix B and obtain the following E_2-page. Again, we indicate the multiplicative structure on the E_2-page as in Example 6.19 of [Sto85].



The entries for $t-s$=9 correspond to the coefficients $\Omega_9^{String}(pt)$. Consequently, they must survive to the E_∞-page. Hence, there cannot be any differential that hits the column $(t-s)=8$ and we obtain

$$\Omega_8^{String}(\mathbb{C}P^\infty) \cong \Omega_8^{String}(pt) \oplus \mathbb{Z} \oplus \mathbb{Z}/2.$$

Since $\mathbb{C}\mathrm{P}^1 \cong S^2$ we see

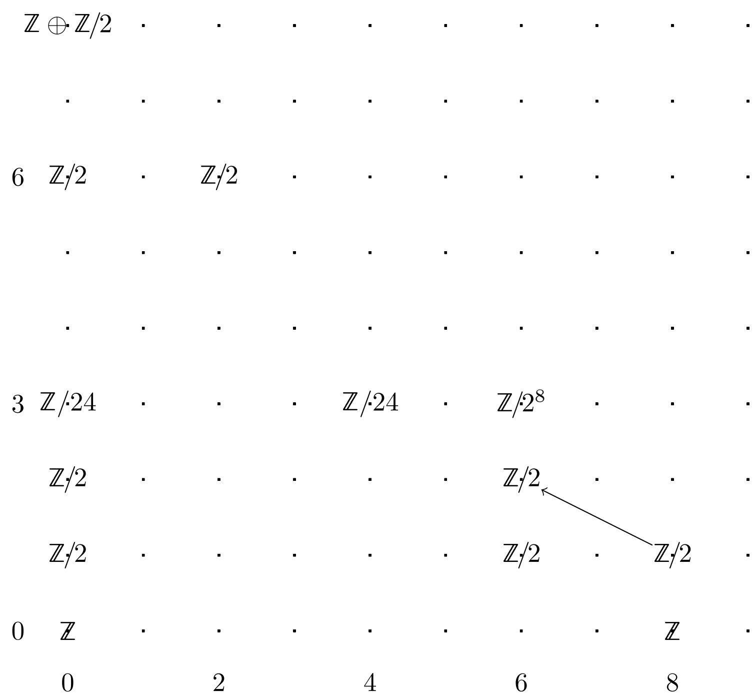
$$\Omega_8^{String}(\mathbb{C}\mathrm{P}^1) \cong \Omega_8^{String}(pt) \oplus \Omega_8^{String}(S^2, pt) \cong \Omega_8^{String}(pt) \oplus \Omega_6^{String}(pt)$$

is isomorphic to $\Omega_8^{String}(pt) \oplus \mathbb{Z}/2$.

Now we start the comparison of the E^2-pages of the Atiyah-Hirzebruch spectral sequences converging to $\Omega_8^{String}(\mathbb{C}\mathrm{P}^1)$ and $\Omega_8^{String}(\mathbb{C}P^\infty)$. For this we use that $\mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{C}P^\infty$ induces an injective map on homology groups.
Consider the Atiyah-Hirzebruch spectral sequence with E^2-page

$$E^2_{pq}(\mathbb{C}P^\infty) \cong H_p(\mathbb{C}P^\infty; \Omega_8^{String}(pt))$$

converging to $\Omega_{p+q}^{String}(\mathbb{C}P^\infty)$. Since we are only interested in $p+q=8$ we only depict the seventh, eighth and ninth diagonal and the coefficients.



Since $Sq^2\colon H^6(\mathbb{C}P^\infty;\mathbb{Z}/2) \to H^8(\mathbb{C}P^\infty;\mathbb{Z}/2)$ is an isomorphism, the indicated differential is an isomorphism by Lemma 3.16 as well. Thus, on the E^3-page, there only remain two entries containing a $\mathbb{Z}/2$. Since $\Omega_8^{String}(\mathbb{C}P^\infty) \cong \Omega_8^{String}(pt) \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ both entries must survive to the E^∞-page.
We now compare the E^2-pages. Denote the entry of the Atiyah-Hirzebruch spectral sequence converging to $\Omega_8^{String}(\mathbb{C}\mathrm{P}^1)$ by $E^2_{pq}(\mathbb{C}\mathrm{P}^1) = H_p(\mathbb{C}\mathrm{P}^1;\Omega_q^{String}(pt))$. The inclusion $\mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{C}P^\infty$ induces injective maps

$$\begin{aligned} H_0(\mathbb{C}\mathrm{P}^1;\Omega_8^{String}(pt)) \cong E^2_{08}(\mathbb{C}\mathrm{P}^1) &\to E^2_{08}(\mathbb{C}P^\infty) \cong H_0(\mathbb{C}P^\infty;\Omega_8^{String}(pt)) \text{ and} \\ H_2(\mathbb{C}\mathrm{P}^1;\Omega_6^{String}(pt)) \cong E^2_{26}(\mathbb{C}\mathrm{P}^1) &\to E^2_{26}(\mathbb{C}P^\infty) \cong H_2(\mathbb{C}P^\infty;\Omega_6^{String}(pt)). \end{aligned}$$

Since all these entries survive to E^∞, this proves the Lemma. □

Now we construct the parametric connected sum $T\#_{\mathbb{C}\mathrm{P}^1}K_p$ and show that it actually is a cohomology Bott manifold.

By Hurewicz's Theorem all classes in $H_2(K_p) \cong \mathbb{Z}$ are spherical. Thus, we can fix an embedding $i\colon S^2 \hookrightarrow K_p$ such that $i_*[S^2]$ generates $H_2(K_p)$. Since K_p is a String manifold $\nu(S^2 \hookrightarrow K_p)$ is stably trivial. The rank of $\nu(S^2 \hookrightarrow K_p)$ is bigger than the dimension of the sphere, whence the normal bundle is actually trivial. Consequently, we obtain an embedding $S^2 \times D^6 \hookrightarrow K_p$.
The same holds for T, where we take the embedding to be $s_4 \circ s_3 \circ s_2\colon \mathbb{C}\mathrm{P}^1_1 \to T$ as defined in Section 2.1. Thus, we can cut $S^2 \times D^6$ out of T and K_p and identify the boundaries along the identity. We call this the *parametric connected sum* and denote it by $F := T\#_{\mathbb{C}\mathrm{P}^1}K_p$, where F stands for **f**ake Bott manifold.

Note that we could also construct F using the embeddings given by the appropriate compositions of sections and inclusions of the fiber, e.g. $s_4 \circ i_3$.

Lemma 4.13. *The parametric connected sum F is a cohomology Bott manifold.*

Recall that a cohomology Bott manifold is defined to be a manifold which admits a polarisation map $g\colon H^*(T) \to H^*(F)$. We construct such a map g in the proof.

Proof. First we prove that there exists a ring isomorphism $g\colon H^*(T) \to H^*(F)$.
Let C_K be the complement $K_p - S^2 \times D^6$ and let C_T be the complement $T - (S^2 \times D^6)$.
By the Mayer-Vietoris sequence of $K_p = S^2 \times D^6 \cup C_K$ we get $H^j(C_K) \cong H^j(S^2 \times D^6)$ for all j.
Similarly, by the Mayer-Vietoris sequence of $T = C_T \cup S^2 \times D^6$ and $F = C_T \cup C_K$, we obtain isomorphisms $H^k(T) \to H^k(C_T)$ and $H^k(F) \to H^k(C_T)$ for $k = 0, 2, 4$. Since these are induced by the inclusions $i\colon C_T \to T$ and $j\colon C_T \to F$ they are natural with respect to the cup product.
Combining both isomorphisms we obtain isomorphisms

$$\phi_k := i^* \circ (j^*)^{-1}\colon H^k(T) \to H^k(F)$$

for $k \leq 4$ such that, for all $x, y \in H^2(T)$, $\phi_2(x) \cup \phi_2(y) = \phi_4(x \cup y)$.

Let $D^6_\epsilon \subset D^6$ be a disk such that the closure of D^6_ϵ is contained in the interior of D^6. Thus, we can apply excision to $S^2 \times D^6_\epsilon \subset S^2 \times D^6 \subset T$ and obtain an isomorphism $H^k(T, S^2 \times D^6) \cong H^k(C_T, \partial C_T)$.
Similarly we can construct $\widetilde{C}_K \subset C_K$ such that we can apply excision to $\widetilde{C}_K \subset C_K \subset F$, whence $H^k(F, C_K) \cong H^k(C_T, \partial C_T)$.
Furthermore, by the long exact sequence of the pairs $(T, S^2 \times D^6)$, (F, C_K) and $(C_T, \partial C_T)$, we obtain a commutative diagram, where, for $k = 4$ all maps are isomorphisms:

$$\begin{array}{ccccc} H^k(T) & \dashrightarrow & H^k(C_T) & \dashleftarrow & H^k(F) \\ \uparrow & & \uparrow & & \uparrow \\ H^k(T, S^2 \times D^6) & \dashrightarrow & H^k(C_T, \partial C_T) & \dashleftarrow & H^k(F, C_K)\ . \end{array}$$

For $k = 4$ the dotted arrows form ϕ_4. Composing the dashed arrows we also obtain ϕ_4 by commutativity. For $k = 8$ the dashed arrows are also natural isomorphisms. We denote their composition by $\phi_8 \colon H^8(T) \to H^8(F)$. By naturality $\phi_4(x) \cup \phi_4(y) = \phi_8(x \cup y)$ for all $x, y \in H^4(T)$. In particular, this determines the intersection form on F.

Since all odd cohomology groups of F vanish, it remains to construct an isomorphism $\phi_6 \colon H^6(T) \to H^6(F)$ such that together all ϕ_k, for $k = 0, 2, 4, 6, 8$, constitute an isomorphism of rings.
Recall that $x_m := y_m - \alpha_m$ is a basis for $H^2(T)$ by Lemma 2.3.
For $1 \leq i, j, l, m \leq 4$, $i < j$ and $l < m$ we obtain

$$\phi_2(y_i) \cup \phi_2(y_j) \cup \phi_2(y_l) \cup \phi_2(x_m) = \phi_4(y_i \cup y_j) \cup \phi_4(y_l \cup x_m) = \phi_8(y_i \cup y_j \cup y_l \cup x_m).$$

An explicit calculation shows that $y_i \cup y_j \cup y_l \cup x_m$ is a generator, and thereby that $\phi_8(y_i \cup y_j \cup y_l \cup x_m)$ is a generator, if and only if $\{i, j, l, m\} = \{1, 2, 3, 4\}$ and zero else.
Let $l \notin \{i, j, k\}$. By the above observation the products $\phi_2(y_i) \cup \phi_2(y_j) \cup \phi_2(y_k)$ for $i < j < k$, form the basis of $H^6(F)$ consisting of the Kronecker duals of $\phi_2(x_l) \cap [F]$.
Define $\phi_6(y_i \cup y_j \cup y_k) := \phi_2(y_i) \cup \phi_2(y_j) \cup \phi_2(y_k)$ for $i < j < k$. Together the ϕ_k constitute a ring isomorphism $g \colon H^*(T) \to H^*(F)$.

It remains to show that g is a polarisation map, i.e. that it preserves the Stiefel-Whitney and Pontrjagin classes.

To prove that g preserves the characteristic classes in degree less or equal four, we use the inclusions $i \colon C_T \to T$ and $j \colon C_T \to F$. By construction ϕ_2 and ϕ_4 are $(j^*)^{-1} \circ i^*$. The tangent bundles of T and F both pull back to the tangent bundle of C_T under i and j, respectively. Thus, by naturality, ϕ_2 and ϕ_4 respect the second and fourth Stiefel-Whitney class and the first Pontrjagin class.

The Euler class of F and T is even. Thus, the top Stiefel-Whitney class vanishes since it is just the mod two reduction of the Euler class. The signatures of F and T agree and $p_1(T) = 0 = p_1(F)$. Hence, their second Pontrjagin classes are preserved under g by the signature theorem.
Consequently, it remains to show that $g(w_6(T)) = w_6(F)$. We use the Wu classes to check this.
Recall that the Wu classes $v_i(X)$ of an $n-$dimensional, compact, smooth manifold X are defined by $v_k(X) \cup x = Sq^k(x)$ for all $x \in H^*(X;\mathbb{Z}/2)$. In particular, $v_k = 0$ for $2k > n$. The Wu classes are connected to the Stiefel-Whitney-classes by the Wu formula (cf. [MS74] p.132)

$$w_i(X) = \sum_{i+j=k} Sq^i(v_j(X)).$$

Since $w_2(T) = 0 = w_2(F)$ and $H^i(T;\ \mathbb{Z}/2) = H^i(F;\ \mathbb{Z}/2) = 0$ for $i = 1, 3$ the first three Wu-classes vanish. Consequently $w_4(T) = v_4(T)$ and $w_6(T) = Sq^2(w_4(T))$.
The even Stiefel-Whitney classes are the mod two reductions of the corresponding Chern classes. We use Section 2.2 to calculate $w_4(T) = \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 \mod 2$. A straight forward calculation shows that this term simplifies to a multiple of y_1y_2. Thus, $w_6(T) = 0$ since $Sq^2(y_1y_2) = 0$.
We already established that $g(w_4(T)) = w_4(F)$, i.e. it is a multiple of $g(y_1)g(y_2)$. In particular, it is a product of classes in degree two. Since $Sq^2(x) = x^2$ if the degree of x is two, the multiplicative structure of $H^*(F)$ determines the square $Sq^2(g(y_1)g(y_2))$. It vanishes, whence $w_6(F) = 0$, too.
This finishes the proof that g is a polarisation map. □

Lemma 8.1 in [CMS10] implies that any ring isomorphism of cohomology Bott manifolds fixes the Stiefel-Whitney classes. Its proof works along the lines of our explicit calculations to show that g preserves the Stiefel-Whitney classes.

Remark 4.14. We already mentioned that we can also build the parametric connected sum $F_i := T\#_{S^2}K_p$, for $i = 2, 3$ and 4, using the embeddings given by $s_2 := s_4 \circ s_3 \circ i_2$, $s_3 := s_4 \circ i_3$ and $s_4 := i_4$, respectively. The proof of Lemma 4.13 does not use the explicit form of the embedding. Consequently, it also works for F_i.

We are now ready to prove Theorem 4.10.

Proof. First of all we want to understand F, together with appropriate maps, as an element in $\Omega_8^{String}(\mathbb{C}P^\infty)$. Let $\widetilde{\nu}_T$ be a lift of the stable normal Gauss of T.
Then $[T, y_1 \times \widetilde{\nu}_T]$ is an element in $\Omega_8^{String}(\mathbb{C}P^\infty)$, where $y_1 \colon T \to \mathbb{C}P^\infty$ is a representative of the homotopy class of maps that corresponds to the generator y_1.
Recall, that the embedding $S^2 \to B_4$ is the composition $s := s_4 \circ s_3 \circ s_2 \colon \mathbb{C}\mathrm{P}^1_1 \to T$. In particular, $s^*(y_1)$ is a generator of S^2. Thus, $S^2 \times D^6 \to T \to \mathbb{C}P^\infty$ corresponds to a generator of $H^2(S^2)$. Note, that we obtain the other generator by choosing $-y_1$ instead

of y_1.
We turn to K_p. The surgery by which we obtain K_p from $K \times \mathbb{C}\mathrm{P}^1$ leaves a neighbourhood of the 2-skeleton invariant. Thus, in a tubular neighbourhood $S^2 \times D^6 \hookrightarrow K_p$ of $i\colon S^2 \hookrightarrow K_p$ the map $\kappa\colon K_p \to \mathbb{C}\mathrm{P}^1$ is the projection to $\mathbb{C}\mathrm{P}^1$. The map to $\mathbb{C}P^\infty$ is given by composition with the inclusion

$$\widetilde{\kappa}\colon K_p \to \mathbb{C}\mathrm{P}^1 \to \mathbb{C}P^\infty.$$

Again precomposing with the embedding $S^2 \times D^6 \to K_p$, we obtain a map that also corresponds to a generator of $H^2(S^2)$.
Thus, the maps $T \to \mathbb{C}P^\infty$ and $K_p \to \mathbb{C}P^\infty$ can be chosen compatibly on $S^2 \times D^6$.
Since $\pi_2(BString)$ is trivial $\widetilde{\nu}_{K_p}|_{S^2\times D^6} \simeq * \simeq \widetilde{\nu}_T|_{S^2\times D^6}$, i.e. the maps $T \to BString$ and $K_p \to BString$ are also compatible on $S^2 \times D^6$.
Hence, the parametric connected sum is a well-defined element

$$[F, \underbrace{(\widetilde{\kappa} \times \widetilde{\nu}_{K_p})\#_{\mathbb{C}\mathrm{P}^1}(y_1 \times \widetilde{\nu}_T)}_{:=h\times\widetilde{\nu}_F}] \in \Omega_8^{String}(\mathbb{C}P^\infty).$$

Consider $S^2 \times D^6 \times D^1$ together with the String structure given by the constant map $pt\colon S^2 \times D^6 \times D^1 \to BString$ and the map $S^2 \times D^6 \times D^1 \to \mathbb{C}P^\infty$ corresponding to a generator of $H^2(S^2 \times D^6 \times D^1)$. We obtain a controlled bordism

$$W := T \times I \cup K_p \times I \cup S^2 \times D^6 \times D^1$$

between $F = T\#_{\mathbb{C}\mathrm{P}^1}K_p$ and the disjoint union $T \cup K_p$. Thus,

$$[F, h \times \widetilde{\nu}_F] = [K_p, \widetilde{\kappa} \times \widetilde{\nu}_{K_p}] + [T, y_1 \times \widetilde{\nu}_T] \in \Omega_8^{String}(\mathbb{C}P^\infty).$$

We prove the Theorem by contradiction.
Assume that there exists a diffeomorphism $f\colon T \to F$. Then

$$[T, (h \times \widetilde{\nu}_F) \circ f] = [F, h \times \widetilde{\nu}_F] = [K_p, \widetilde{\kappa} \times \widetilde{\nu}_{K_p}] + [T, y_1 \times \widetilde{\nu}_T] \in \Omega_8^{String}(\mathbb{C}P^\infty).$$

By Lemma 4.12 $[K_p, \widetilde{\kappa} \times \widetilde{\nu}_{K_p}]$ is a non-trivial element in $\Omega_8^{String}(\mathbb{C}P^\infty)$. Consequently, it only remains to show that $[T, (h \times \widetilde{\nu}_F) \circ f] = [T, y_1 \times \widetilde{\nu}_T]$.

By construction the maps $h \circ f$ and y_1 to $\mathbb{C}P^\infty$ correspond to primitive elements a and y_1 of $H^2(T)$ that square to zero. Thus, there exists an automorphism $\Psi\colon H^*(T) \to H^*(T)$ such that $\Psi(a) = y_1$. By assumption all automorphisms on $H^*(T)$ are realisable by a self-diffeomorphism f'. Consequently, there exists a controlled bordism $T \times I \cup_{f'} T \times I$ between $[T, (h \times \widetilde{\nu}_F) \circ f]$ and $[T, y_1 \times \widetilde{\nu}_T]$. Thus, the bordism classes are equal.

Assume F is diffeomorphic to another Bott manifold B_4 by some diffeomorphism φ. Then $H^*(B_4)$ and $H^*(T)$ are isomorphic by the composition of φ^* and the polarisation map between $H^*(F)$ and $H^*(T)$. By assumption B_4 and T are diffeomorphic, implying that F and T are diffeomorphic which is a contradiction. □

Remark 4.15. We can modify this proof such that we see that each cohomology Bott manifold F_k, for $k = 2, 3, 4$, as in Remark 4.14 is not diffeomorphic to any Bott manifold, either. For this purpose we replace $[T, y_1 \times \widetilde{\nu}_T]$ in the proof by $[T, y_k \times \widetilde{\nu}_T]$ such that $s_k^*(y_k)$ is a generator of $H^2(\mathbb{C}\mathrm{P}^1_k)$. With this modifications the proof works the same.

Of course each parametric connected sum, $F =: F_1$ as well as each F_k, induces an element in $\Omega_8^{String}(\mathbb{P}_3T)$. Recall that we have a map $\mathbb{P}_3T \to \mathbb{P}_2T \simeq (\mathbb{C}P^\infty)^4$. We can compose the map $\mathbb{P}_3T \to (\mathbb{C}P^\infty)^4$ with the projection to one of the factors. This composition induces four maps

$$q_k\colon \Omega_8^{String}(\mathbb{P}_3T) \to \Omega_8^{String}(\mathbb{C}P^\infty).$$

Let N be a cohomology Bott manifold in the polarised structure set of T.
Recall the following construction and notation: In Proposition 4.4 we construct a map $j_3\colon N \to \mathbb{P}_3T$ such that the pullback of the generators a_i of $H^2(\mathbb{P}_3T)$ are generators of $H^2(N)$. The induced element in $\Omega_8^{String}(\mathbb{P}_3T)$ is denoted by $[N, j_3 \times \widetilde{\nu}_N]$.
The element $[T, y_k \times \widetilde{\nu}_T] - [F_k, (j_k)_3 \times \widetilde{\nu}_{F_k}]$ is non-trivial under q_k. We can use this to show that the $[F_k, (j_k)_3 \times \widetilde{\nu}_{F_k}]$, for $k = 1, ..., 4$ are non-trivial, distinct elements in $\Omega_8^{String}(\mathbb{P}_3T)$. Thus, we obtain representatives for a subgroup of order $|\mathbb{Z}/2^4|$ in $\Omega_8^{String}(\mathbb{P}_3T)$.

We conjecture that we can generalise this construction to cohomology Bott manifolds with respect to arbitrary Bott manifolds B_4 which fulfil the (SCRP), i.e. we drop the assumption that the Bott manifold must be *String*.

5. On the realisation of some automorphism on $H^*(B_4)$

In the previous section we examine cohomology Bott manifolds. Now we return to the original cohomological rigidity problem, to be more precise, the strong cohomological rigidity problem, i.e. the question whether an isomorphism between the cohomology rings of two Bott manifolds can be realised by a diffeomorphism of the underlying spaces. For Bott manifolds of dimension smaller than or equal to six the strong cohomological rigidity conjecture holds by [Cho11a] and [CM12]. In the latter paper, it is also proven for the so-called $\mathbb{Q}$-trivial Bott manifolds.

In [Cho11a] this question is studied for eight-dimensional Bott manifolds and reduced to the question, whether four automorphisms of the cohomology ring of a special class of Bott manifolds can be realised by a diffeomorphism.

We start by introducing the class of Bott manifolds. Let B_4 be the fourth stage of a Bott tower of the form

$$
\begin{array}{ccl}
\mathbb{C}\mathrm{P}^1_4 \xrightarrow{i_4} & B_4 & = P(\gamma_3 \otimes \gamma_2^{-\frac{1}{2}A_3^2} \otimes \gamma_1^{-\frac{1}{2}A_3^1} \oplus \underline{\mathbb{C}}) =: P(L_3 \oplus \underline{\mathbb{C}}) \\
& \pi_4 \downarrow \uparrow s_4 & \\
\mathbb{C}\mathrm{P}^1_3 \xrightarrow{i_3} & B_3 & = P(\gamma_2^{A_3^2} \otimes \gamma_1^{A_3^1} \oplus \underline{\mathbb{C}}) =: P(L_2 \oplus \underline{\mathbb{C}}) \\
& \pi_3 \downarrow \uparrow s_3 & \\
\mathbb{C}\mathrm{P}^1_2 \xrightarrow{i_2} & B_2 & = P(\gamma_1^{A_2^1} \oplus \underline{\mathbb{C}}) =: P(L_1 \oplus \underline{\mathbb{C}}) \\
& \pi_2 \downarrow \uparrow s_2 & \\
& \mathbb{C}\mathrm{P}^1_1 &
\end{array}
$$

where $c_1(L_1)$ is arbitrary while $c_1(L_2) = -A_3^2 y_2 - A_2^1 y_1 = -\alpha_3$ must be divisible by two since $c_1(L_3) = -\alpha_4 = \frac{1}{2}\alpha_3 - y_3$. Here, we use the notation introduced in Section 2.1, i.e. $y_i = -c_1(\gamma_i)$, where γ_i is the tautological bundle over B_i.
For the remainder of the section, B_4 will denote Bott manifolds of this form.

We consider the realisation question for one of the four automorphisms introduced in [Cho11a]. Next, we recall those four automorphisms.
In [Cho11a] these automorphism are defined using the bundle basis, i.e. using the basis consisting of $y_i = -c_1(\gamma_i)$, for $1 \leq i \leq 4$. But to attack the realisation question we must understand the automorphism on the basis elements x_i, for $1 \leq i \leq 4$, of the geometric basis introduced in Section 2.1. Recall that the geometric basis is given by the Kronecker duals of the homology classes defined by the $\mathbb{C}\mathrm{P}^1_i$, embedded along the appropriate compositions of inclusions of the fibres and sections.

By Proposition 2.3 we get the following base changes between the geometric and the bundle basis for B_4:

$$\begin{aligned} x_1 =& y_1, & y_1 =& x_1 \\ x_2 =& y_2 - \alpha_2, & y_2 =& x_2 + A_2^1 x_1 \\ x_3 =& y_3 - \alpha_3, & y_3 =& x_3 + A_3^2 x_2 + (A_2^1 A_3^2 + A_3^1) x_1 \\ x_4 =& y_4 - \alpha_4, & y_4 =& x_4 + x_3 + \frac{1}{2} A_3^2 x_2 + \frac{1}{2}(A_2^1 A_3^2 + A_3^1) x_1. \end{aligned} \tag{8}$$

We abbreviate $A_3^2 x_2 + (A_2^1 A_3^2 + A_3^1) x_1$ by $\widetilde{\alpha}_3$. This notation is justified since $x_i^2 = -\widetilde{\alpha}_i x_i$.

For the sake of completeness we now recall all four automorphisms ϕ_i of $H^*(B_4)$, defined in [Cho11a] for $i = 1, 2, 3, 4$, in the bundle basis and in the geometric basis even though we only examine ϕ_1 later on.
For the two bases the automorphisms are defined by $\phi_i(y_j) = y_j$ and $\phi_i(x_j) = x_j$ for $j = 1, 2$ and by:

$$\begin{aligned} \phi_1(y_3) =& 2y_4 - y_3 + \alpha_3, & \phi_1(x_3) =& 2x_4 + x_3 \\ \phi_1(y_4) =& y_4, & \phi_1(x_4) =& - x_4 \\ \\ \phi_2(y_3) =& 2y_4 - y_3 + \alpha_3, & \phi_2(x_3) =& 2x_4 + x_3 \\ \phi_2(y_4) =& y_4 - y_3 + \frac{\alpha_3}{2}, & \phi_2(x_4) =& - x_4 - x_3 - \frac{\widetilde{\alpha}_3}{2} \\ \\ \phi_3(y_3) =& - 2y_4 + y_3, & \phi_3(x_3) =& - 2x_4 - x_3 - \widetilde{\alpha}_3 \\ \phi_3(y_4) =& - y_4, & \phi_3(x_4) =& x_4 \\ \\ \phi_4(y_3) =& - 2y_4 + y_3, & \phi_4(x_3) =& - 2x_4 - x_3 - \widetilde{\alpha}_3 \\ \phi_4(y_4) =& - y_4 + y_3 - \frac{\alpha_3}{2}, & \phi_4(x_4) =& x_4 + x_3 + \frac{\widetilde{\alpha}_3}{2}. \end{aligned} \tag{9}$$

We consider the first automorphism ϕ_1. Its easy form in the geometric basis allows us to apply Corollary 3.12.

Recall that, to apply Corollary 3.12, we need to decompose the manifold on which we want to realise a diffeomorphism. Let $B_4 =: M \cup_{h_\partial} N$, where $h_\partial \colon \partial M \to \partial N$ is a diffeomorphism. We determine the explicit forms of M and N later in this section. In particular, we see that $H^2(N) \cong H^2(B_4)$. Thus, we can attempt to realise ϕ_1 on N. This is actually possible by a diffeomorphism $n \colon N \to N$ which we also construct. Finally, we use Corollary 3.12 to examine whether the diffeomorphism can be extended over M.

Recall that $\Omega_8^{String}(pt) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ (cf. [Gia71]), where the two-torsion is generated by an element $\widetilde{\theta}_8$.

Lemma 5.1. *The generator $\widetilde{\theta}_8$ of $\mathbb{Z}/2 \subset \Omega_8^{String}(pt)$ is the exotic eight-sphere Θ_8 considered as an element in String-bordism.*

Proof. Since $H^k(\Theta_8)$ vanishes for $k \neq 0, 8$ there is no obstruction to the existence of a String structure ϑ_8 on Θ_8. Thus, $[\Theta_8, \vartheta_8] =: \widetilde{\theta}_8$ clearly is an element in $\Omega_8^{String}(pt)$.

It remains to show, that $\widetilde{\theta}_8$ is non-trivial and of order two.
Assume that $\widetilde{\theta}_8$ vanishes in $\Omega_8^{String}(pt)$. Then, there exists a bordism W, together with a String-structure $\nu_8\colon W \to BString$ such that $\partial W = \Theta_8$ and $\nu_8|_{\Theta_8} = \vartheta_8$. This will result in a contradiction to the non-existence of a parallelisable manifold whose boundary is Θ_8.

By surgery below the middle dimension in the interior of W we can turn ν_8 into a four-equivalence. Thus, we can assume W to be three-connected which implies $H^8(W) = 0$. The obstruction for the existence of a lift of ν_8 to $BO\langle 9\rangle \to BO\langle 8\rangle = BString$ is an element in $H^8(W)$. Hence, we know that ν_8 admits a lift $\nu_9\colon W \to BO\langle 9\rangle$.
The obstruction to the existence of a lift $\nu_{10}\colon BO\langle 9\rangle \to BO\langle 10\rangle$ is $w_9(W)$.
Recall that the Wu classes $v_i(X)$ of an $n-$dimensional, compact, smooth manifold X are defined by $v_k(X) \cup x = Sq^k(x)$ for all $x \in H^*(X;\mathbb{Z}/2)$. In particular, $v_k = 0$ for $2k > n$. The Wu classes are connected to the Stiefel-Whitney-classes by the Wu formula (cf. [MS74] p.132)

$$w_i(X) = \sum_{i+j=k} Sq^i(v_j(X)).$$

Since W is three-connected the formula for $w_9(W)$ simplifies to

$$\begin{aligned} w_9(W) &= Sq^0(v_9(W)) + Sq^4(v_5(W)) + Sq^5(v_4(W)) + Sq^9(v_0(W)) \\ &= Sq^0(v_9(W)) + Sq^4(v_5(W)), \end{aligned}$$

where the second equality holds for dimension reasons. But $v_5(W)$ and $v_9(W)$ correspond to $Sq^i\colon H^k(W;\mathbb{Z}/2) \to H^{k+i}(W;\mathbb{Z}/2)$ for $i = 5, 9$, respectively. Consequently, they also vanish.
This implies that ν_{10} exists. Since W is of dimension nine all further obstruction to W being parallelisable vanish. But since bP_9 is trivial Θ_8 cannot bound a parallelisable manifold and we have a contradiction. Hence, $\widetilde{\theta}_8$ is non-trivial in $\Omega_8^{String}(pt)$.
Since $\Theta_8 \# \Theta_8 = S^8$ it is of order two. □

Let $X \times BString \xrightarrow{E \times p_8} BO \times BO \xrightarrow{\oplus} BO$ denote a twisted fibration over BO. The inclusion $pt \hookrightarrow X$ induces a map $\Omega_8^{String}(pt) \to \Omega_8^{String}(X, E)$. Let θ_8 denote the image of $\widetilde{\theta}_8$ under this map.

Our goal is to prove the following theorem.

Theorem 5.2. *Let $\phi_1\colon H^*(B_4) \to H^*(B_4)$ be the automorphism of Equation (9) and e^8 an eight-cell. Then there exist*

- *a twisted fibration*
$$\mathbb{B} := (\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \cup e^8) \times BString \xrightarrow{E \times p_8} BO \times BO \xrightarrow{\oplus} BO\ ,$$
- *a decomposition $B_4 = M \cup_{h_\partial} N$ into manifolds with boundary and a diffeomorphism $n\colon N \to N$,*
- *two normal three-smoothings $\widetilde{\nu}_1, \widetilde{\nu}_2\colon M \to \mathbb{B}$ fulfilling $\widetilde{\nu}_1 \circ h_\partial^{-1} \circ n \circ h_\partial \simeq \widetilde{\nu}_2|_{\partial M}$ which give rise to an element $[M \cup_{h_\partial^{-1} \circ n \circ h_\partial} M, \widetilde{\nu}_1 \cup \widetilde{\nu}_2] =: \omega \in \Omega_8^{String}(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \cup e^8, E)$ and*
- *invariants $a_1, a_2 : \Omega_8^{String}(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \cup e^8, E) \to \mathbb{Z}/2$*

such that ϕ_1 is realisable if $a_1(\omega) = 0 = a_2(\omega)$ and $\omega \neq \theta_8$.

All objects will be constructed in a very explicit way subsequently.
The invariants a_1 and a_2 are so-called codimension two Arf-invariants. We will see that they allow a very nice geometric description. Roughly, they associate to elements in the torsion subgroup of $\Omega_8^{String}(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \cup e^8, E)$ the Arf-invariant of some codimension two submanifold.

The second cohomology of N turns out to be isomorphic to $H^2(B_4)$. The conjugation of n^* with this isomorphism realises ϕ_1 on a subspace of B_4.
To attack the realisation problem for the automorphisms ϕ_i for $i = 2, 3, 4$ in Equation (9) using Corollary 3.12 is more difficult, if at all possible, since there is no obvious decomposition of B_4 into manifolds M' and N' such that ϕ_i can be realised on N' in some way.

The proof takes the remainder of the section. It consists of two parts. First, we construct the objects whose existence is claimed in the theorem. Then we use modified surgery theory, in particular Corollary 3.12, to examine if we can extend the self-diffeomorphism, which we can construct on a subspace of B_4, all over B_4. The subsequent sections can be summarised as follows:
In Section 5.1 we construct M, N and the diffeomorphism $h_\partial\colon \partial M \to \partial N$. In Section 5.2 we construct the diffeomorphism $n\colon N \to N$. In Section 5.3 we construct the fibration $\mathbb{B}$ and the normal smoothings $\widetilde{\nu}_1, \widetilde{\nu}_2\colon N \to \mathbb{B}$. This finishes the constructive part of the proof.
Next, we compute the twisted bordism group $\Omega_8^{String}(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \cup e^8, E)$ in Section 5.4. In Section 5.5 we assemble all objects into a proof of Theorem 5.2. The key of the proof is to develop the invariants a_1 and a_2.

5.1. A suitable description for B_4

First of all, we change our perspective on the Bott manifold B_4 slightly. So far we considered B_4 as $\mathbb{C}\mathrm{P}^1$-fibre bundle over B_3. We change that now.
For this purpose, we use Ehresmann's theorem (cf. [Voi07] Chapter 9.9.1). Let B, E_i and F_i for $i = 1, 2$ be smooth manifolds, $p_1\colon E_1 \to B$ a smooth fibre bundle with fibre F_1 and $p_2\colon E_2 \to E_1$ a smooth fibre bundle with fibre F_2. Then Ehresmann's theorem states that the composition $p := p_1 \circ p_2 : E_2 \to B$ is again a smooth fibre bundle if $p^{-1}(pt)$ is compact. In particular, the fibre of $p\colon E_2 \to B$ is $E_2|_{F_1}$, i.e. it is the total space of a fibre bundle $F_2 \to F \to F_1$.

In our situation we consider the bundles $\pi_4\colon B_4 \to B_3$ and $\pi_3\colon B_3 \to B_2$. By Ehresmann's Theorem $\pi := \pi_3 \circ \pi_4$ is again a fibre bundle if $\pi^{-1}(pt)$ is compact.
Let $i_3\colon \mathbb{C}\mathrm{P}^1_3 \to B_3$ denote the inclusion of the fibre. To determine what the restriction of $B_4 = P(L_3 \oplus \underline{\mathbb{C}})$ to $i_3(\mathbb{C}\mathrm{P}^1_3)$ is, we determine the restriction of the defining bundle L_3, i.e. we consider the pullback $i_3^*L_3$. By definition of the tautological line bundle over some stage B_j, this is just the tautological line bundle $\gamma = i_3^*(\gamma_3)$ over $\mathbb{C}\mathrm{P}^1_3$. Thus,

$$\pi^{-1}(pt) = \pi_4^{-1}(i_3(\mathbb{C}\mathrm{P}^1_3)) = P(\gamma \oplus \underline{\mathbb{C}}).$$

In particular, $\pi^{-1}(pt)$ is compact. Hence, $\pi\colon B_4 \to B_2$ is a fibre bundle with fibre $P(\gamma \oplus \underline{\mathbb{C}})$.

We have another description for $P(\gamma \oplus \underline{\mathbb{C}})$.
In [Hir51] Hirzebruch already showed that two Bott manifolds $H := P(\gamma_1^a \oplus \underline{\mathbb{C}})$ and $H' := P(\gamma_1^{a'} \oplus \underline{\mathbb{C}})$ of dimension four are diffeomorphic if and only if $a = a' \mod 2$. If $a = 0 \mod 2$, then H is diffeomorphic to $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ and we denote it by $\mathcal{H}_0$. Otherwise, i.e. if $a = 1 \mod 2$, then H is diffeomorphic to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ which we denote by $\mathcal{H}_1$. Honoring his work we still call Bott manifolds of dimension four *Hirzebruch surfaces.*

In Section 2.2 we determine the Stiefel-Whitney classes of a Bott manifold. These results show that $w_2(P(\gamma \oplus \underline{\mathbb{C}})) \neq 0$. Consequently, the fibre of the bundle $\pi\colon B_4 \to B_2$ is diffeomorphic to the non-trivial Hirzebruch surface $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. Note that the base space B_2 is a Hirzebruch surface, too. But since A^1_2 is arbitrary, we do not know which one, so we stick to the notation B_2.

The next step is to use the description of B_4 as total space of $\mathcal{H}_1 \to B_4 \to B_2$ to obtain the manifolds M and N of Theorem 5.2.

The idea to obtain M and N is the following:
We decompose the base space B_2 into two parts. One part is the a tubular neighbourhood of the two-skeleton $S^2 \vee S^2$ of B_2 which we denote by Pl, the other part is the complement $B_2 - Pl$, which is the top disc of B_2. We can restrict B_4 to both parts and obtain a decomposition of B_4. Finally, we show that one of this parts admits a diffeomorphism

to $Pl \times \mathcal{H}_1$.

Now let us construct the decomposition in more detail. Since $\mathcal{H}_1 \to B_4 \to B_2$ is a locally trivial fibre bundle, the normal bundle of the inclusion of the fibre is trivial. Thus, there exists an embedding $\Psi\colon D^4 \times \mathcal{H}_1 \hookrightarrow B_4$ and we can choose D^4 such that $\pi \circ \Psi(D^4 \times \mathcal{H}_1) = D^4 \subset B_2$.
We can consider the complement $B_4 - \Psi(D^4 \times \mathcal{H}_1)$. This complement is the restriction of the bundle $B_4 \to B_2$ to $B_2 - D^4$.
The Hirzebruch surface B_2 is homotopy equivalent to a CW-complex $(S^2 \vee S^2) \cup e^4$. Thus, if we cut out the top disc, we obtain a space which is homotopy equivalent to $S^2 \vee S^2 \cong \mathbb{C}\mathrm{P}^1 \vee \mathbb{C}\mathrm{P}^1$.
More precisely, we can choose the two-skeleton to consist of $B_1 = \mathbb{C}\mathrm{P}^1_1$, embedded by the section $s_2\colon \mathbb{C}\mathrm{P}^1_1 \to B_2$, and $\mathbb{C}\mathrm{P}^1_2$, embedded by the inclusion of the fibre $i_2\colon \mathbb{C}\mathrm{P}^1_2 \to B_2$. Their normal bundles are $\nu(\mathbb{C}\mathrm{P}^1_1 \hookrightarrow B_2) \cong \gamma_1^{-A_2^1}$ and $\nu(\mathbb{C}\mathrm{P}^1_2 \hookrightarrow B_2) \cong \underline{\mathbb{C}}$, i.e. the trivial bundle. By the tubular neighbourhood theorem there exist embeddings

$$s\colon D\nu(\mathbb{C}\mathrm{P}^1_1) \to B_2 \text{ and } i\colon D\nu(\mathbb{C}\mathrm{P}^1_2) \to B_2.$$

The images of the embeddings intersect in an embedded $D^2 \times D^2$. We identify $x_1 \in D\nu(\mathbb{C}\mathrm{P}^1_1)$ and $x_2 \in D\nu(\mathbb{C}\mathrm{P}^1_2)$ if $s(x_1) = i(x_2)$ and obtain the *pl*umbing

$$Pl = D\nu(\mathbb{C}\mathrm{P}^1_1) \natural D\nu(\mathbb{C}\mathrm{P}^1_2)$$

of $D\nu(\mathbb{C}\mathrm{P}^1_1)$ and $D\nu(\mathbb{C}\mathrm{P}^1_2)$ together with an embedding $s\natural i\colon Pl \to B_2$.
A priori a plumbing as Pl is not a smooth manifold but a manifold with corners. Fortunately we can smoothen the corners by standard methods as described in Appendix A of [Kre10].
We obtain a decomposition $D^4 \cup (s\natural i)(Pl) = B_2$.

If we restrict the composition of $s_4 \circ s_3$ to $B_4|_{(s\natural i)(Pl)}$ we still obtain embeddings of $\mathbb{C}\mathrm{P}^1_1$ and $\mathbb{C}\mathrm{P}^1_2$ into $B_4|_{(s\natural i)(Pl)}$. If we choose the inclusions of the fibres $\mathbb{C}\mathrm{P}^1_3$ and $\mathbb{C}\mathrm{P}^1_4$ to be inclusions over a point in $(s\natural i)(Pl)$ and $B_3|_{(s\natural i)(Pl)}$, respectively, they are also still embedded in $B_4|_{(s\natural i)(Pl)}$. All four embeddings together induce a basis of $H_2(B_4|_{(s\natural i)(Pl)})$. Consequently, their Kronecker duals form a basis for $H^2(B_4|_{(s\natural i)(Pl)})$.
Under the inclusion $B_4|_{(s\natural i)(Pl)} \hookrightarrow B_4$ the basis of embedded $\mathbb{C}\mathrm{P}^1$ maps, by definition, to the basis elements $\sigma_i, 1 \leq i \leq 4$ of $H_2(B_4)$ (cf. Section 2.1). The Kronecker duals of the σ_i are the basis elements $x_i, 1 \leq i \leq 4$ of the geometric basis. We denote the pullbacks of the x_i to $H^2(B_4|_{(s\natural i)(Pl)})$ by x_i, too. They are Kronecker duals of embedded $\mathbb{C}\mathrm{P}^1_i$, too.

In a sense, we now have a decomposition of B_4 into pieces, namely $D^4 \times \mathcal{H}_1 = B_4|_{D^4}$ and $B_4|_{(s\natural i)(Pl)}$ which are identified along the identity on the boundary. Now we want to understand $B_4|_{(s\natural i)(Pl)}$ better, to be able to realise ϕ_1 there. Here, realisation means that we find a self-diffeomorphism of $B_4|_{(s\natural i)(Pl)}$ which realises ϕ_1 on $H^2(B_4|_{Pl}) \cong H^2(B_4)$.

Our next goal is to construct a diffeomorphism $B_4|_{(s\natural i)(Pl)} \cong Pl \times \mathcal{H}_1$.
The construction works along the following lines:
Since $D\nu(\mathbb{C}\mathrm{P}^1_i)$ is homotopy equivalent to $\mathbb{C}\mathrm{P}^1_i$ the restriction of B_4 to the embedded $D\nu(\mathbb{C}\mathrm{P}^1_i)$ is determined by the restriction of B_4 to the embedded $\mathbb{C}\mathrm{P}^1_i$, for $i = 1, 2$. We will show below that the restrictions of B_4 to the embedded $\mathbb{C}\mathrm{P}^1_i$ are Bott manifolds of dimension six which are diffeomorphic to $\mathbb{C}\mathrm{P}^1_i \times \mathcal{H}_1$. We can extend these diffeomorphisms to ones between $B_4|_{s(D\nu(\mathbb{C}\mathrm{P}^1_1))}$ and $D\nu\mathbb{C}\mathrm{P}^1_1 \times \mathcal{H}_1$, and $B_4|_{i(D\nu(\mathbb{C}\mathrm{P}^1_2))}$ and $D\nu\mathbb{C}\mathrm{P}^1_2 \times \mathcal{H}_1$, respectively. Finally, we "glue" the diffeomorphisms to obtain one diffeomorphism $B_4|_{Pl} \cong Pl \times \mathcal{H}_1$.

We start by considering the restrictions of B_4 to the embedded $\mathbb{C}\mathrm{P}^1_1$ and $\mathbb{C}\mathrm{P}^1_2$. Recall, that

$$B_4 = P(L_3 \oplus \underline{\mathbb{C}}) \to P(L_2 \oplus \underline{\mathbb{C}}) \to P(L_1 \oplus \underline{\mathbb{C}}) = B_2.$$

with $L_1 = \gamma_1^{A^1_2}$, $L_2 = \gamma_1^{A^1_3} \otimes \gamma_2^{A^2_3}$ and $L_3 = \gamma_1^{-\frac{1}{2}A^1_3} \otimes \gamma_2^{-\frac{1}{2}A^2_3} \otimes \gamma_3$. Recall that A^1_3 and A^2_3 are divisible by two, by assumption. We determine the restrictions $B_4|_{\mathbb{C}\mathrm{P}^1_i}$ by first considering the underlying defining bundles $L_i \oplus \underline{\mathbb{C}}$, for $i = 2, 3$, and their restrictions, i.e. their pullbacks.
We start with the pullback of $L_2 \oplus \underline{\mathbb{C}}$ to $\mathbb{C}\mathrm{P}^1_i$ along s_2 and i_2 for $i = 1, 2$, respectively. By Section 2.1 we know

$$s_2^*(L_2 \oplus \underline{\mathbb{C}}) = \gamma_1^{(A^1_2 A^2_3 + A^1_3)} \oplus \underline{\mathbb{C}} \text{ and } i_2^*(L_2 \oplus \underline{\mathbb{C}}) = (\gamma_2|^{A^2_3}_{\mathbb{C}\mathrm{P}^1_2} \oplus \underline{\mathbb{C}}).$$

After projectivisation we obtain submanifolds $\widehat{B}_2$ and $\overline{B}_2$ that are Bott manifolds themselves. Consequently, there are commutative squares

$$\begin{array}{ccccc} B_3 & \xleftarrow{\widehat{s_2}} & \widehat{B}_2 & \longleftarrow & \mathbb{C}\mathrm{P}^1_3 \\ \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi_3|_{\widehat{B}_2}} & & \\ B_2 & \xleftarrow{s_2} & \mathbb{C}P^1_1 & & \end{array} \qquad \begin{array}{ccccc} B_3 & \xleftarrow{\overline{i_2}} & \overline{B}_2 & \longleftarrow & \mathbb{C}\mathrm{P}^1_3 \\ \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi_3|_{\overline{B}_2}} & & \\ B_2 & \xleftarrow{i_2} & \mathbb{C}P^1_2 & & \end{array}$$

Then we repeat the procedure with $L_3 \oplus \underline{\mathbb{C}}$, i.e. we pull back $L_3 \oplus \underline{\mathbb{C}}$ to $\widehat{B}_2$ and $\overline{B}_2$ along $\widehat{s_2}$ and $\overline{i_2}$ respectively. Again after projectivisation we obtain the following two Bott towers:

$$\begin{array}{ccc} B_4 & \longleftarrow & P((\gamma_1^{-\frac{1}{2}(A^1_2 A^2_3 + A^1_3)} \otimes \gamma_3)|_{\widehat{B}_2} \oplus \underline{\mathbb{C}}) =: \widehat{B}_3 \\ \downarrow & & \downarrow \\ B_3 & \longleftarrow & \widehat{B}_2 \\ \downarrow & & \downarrow \\ B_2 & \longleftarrow & \mathbb{C}P^1_1 \end{array} \quad (\widehat{\pi}: \widehat{B}_3 \to \mathbb{C}P^1_1) \qquad \begin{array}{ccc} B_4 & \longleftarrow & P((\gamma_2^{-\frac{1}{2}A^2_3} \otimes \gamma_3)|_{\overline{B}_2} \oplus \underline{\mathbb{C}}) =: \overline{B}_3 \\ \downarrow & & \downarrow \\ B_3 & \longleftarrow & \overline{B}_2 \\ \downarrow & & \downarrow \\ B_2 & \longleftarrow & \mathbb{C}P^1_2 \end{array} \quad (\overline{\pi}: \overline{B}_3 \to \mathbb{C}P^1_2).$$

Thus, we identified $B_4|_{\mathbb{C}\mathrm{P}^1_i}$ to be $\widehat{B}_3$ and $\overline{B}_3$ for $i = 1, 2$, respectively. Therefore, we are now ready to construct diffeomorphisms $\widehat{h}\colon \widehat{B}_3 \to \mathbb{C}\mathrm{P}^1_1 \times \mathcal{H}_1$ and $\overline{h}\colon \overline{B}_3 \to \mathbb{C}\mathrm{P}^1_2 \times \mathcal{H}_1$.

Observe that the Bott towers of $\widehat{B}_3$ and $\overline{B}_3$ are both of the form

$$P(\gamma_1^{-k} \otimes \gamma_2 \oplus \underline{\mathbb{C}}) \to P(\gamma_1^{2k} \oplus \underline{\mathbb{C}}) \to \mathbb{C}\mathrm{P}^1$$

for $k = \frac{1}{2}(A^1_3 + A^1_2 A^2_3)$ and $k = \frac{1}{2}A^2_3$, respectively. Denote $P(\gamma_1^{-k} \otimes \gamma_2 \oplus \underline{\mathbb{C}})$ by $\widetilde{B}_3$. Consequently, it suffices to construct $\widetilde{h}\colon \widetilde{B}_3 \to \mathbb{C}\mathrm{P}^1 \times \mathcal{H}_1$ to obtain $\widehat{h}$ and $\overline{h}$.
Let $v_1, v_2, v_3 \in H^2(\widetilde{B}_3)$ denote the bundle basis for cohomology, i.e. the basis given the negative of the first Chern classes of the respective tautological line bundles. Using the results of Section 2.1 we get,

$$v_1^2 = 0, \qquad v_2^2 = 2kv_1v_2 =: \vartheta_2 v_2 \qquad \text{and} \qquad v_3^2 = -kv_1v_3 + v_2v_3 =: \vartheta_3 v_3.$$

In particular $\vartheta_i^2 = 0$ for $i = 1, 2$. This is important because we are now in the setting of so-called $\mathbb{Q}-$trivial Bott manifolds, where, abstractly, the existence of diffeomorphisms realising isomorphisms of the cohomology ring, is known.

By definition a Bott manifold B_j is $\mathbb{Q}$-trivial (cf. [CM12]) if and only if the rational cohomology rings $H^*(B_j;\mathbb{Q})$ and $H^*((\mathbb{C}\mathrm{P}^1)^j;\mathbb{Q})$ are isomorphic as graded rings.
Let $y_i \in H^*(B_j)$ be the bundle basis, which, in particular, fulfils $y_i^2 = \alpha_i y_i$ for $1 \le i \le j$. Proposition 3.1 in [CM12] shows that B_j is $\mathbb{Q}-$trivial if and only if $\alpha_i^2 = 0$ for all $1 \le i \le j$.
Furthermore, Corollary 5.2 in [CM12] states that each cohomology ring isomorphism between $\mathbb{Q}$-trivial manifolds can be realised by a diffeomorphism.

Let $p\colon \mathcal{H}_1 \to \mathbb{C}\mathrm{P}^1$ denote the fibre bundle projection. The product $\mathbb{C}\mathrm{P}^1 \times \mathcal{H}_1$ is a Bott manifold and the third stage in the tower

$$\mathbb{C}\mathrm{P}^1 \times \mathcal{H}_1 \xrightarrow{\mathbb{1}\times p} \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1.$$

Let b_1, b_2, b_3 denote the bundle basis of $\mathbb{C}\mathrm{P}^1 \times \mathcal{H}_1$. Then,

$$b_1^2 = 0, \quad b_2^2 = 0 \text{ and } b_3^2 = b_1 b_3.$$

Consequently, the bundle $\mathbb{C}\mathrm{P}^1 \times \mathcal{H}_1$ is a $\mathbb{Q}-$trivial Bott manifold, too.

The cohomology rings of $\widetilde{B}_3$ and $\mathbb{C}\mathrm{P}^1 \times \mathcal{H}_1$ are isomorphic as rings. It is easy to check, that

$$\begin{aligned} \varphi_k\colon H^2(\mathbb{C}\mathrm{P}^1 \times \mathcal{H}_1) &\to H^2(\widetilde{B}_3) \\ b_1 &\mapsto v_1 \\ b_2 &\mapsto v_2 - kv_1 \\ b_3 &\mapsto v_3 \end{aligned} \tag{10}$$

induces an isomorphism of rings with inverse

$$\begin{aligned} \varphi_k^{-1}\colon H^2(\widetilde{B}_3) &\to H^2(\mathbb{C}\mathrm{P}^1 \times \mathcal{H}_1) \\ v_1 &\mapsto b_1 \\ v_2 &\mapsto b_2 + kb_1 \\ v_3 &\mapsto b_3. \end{aligned}$$

Thus, by [CM12, Corollary 5.2], there exist diffeomorphisms

$$\widehat{h}\colon \widehat{B}_3 \to \mathbb{C}\mathrm{P}^1_1 \times \mathcal{H}_1 \text{ and } \overline{h}\colon \overline{B}_3 \to \mathbb{C}\mathrm{P}^1_2 \times \mathcal{H}_1$$

that induce φ_k for $k = \frac{1}{2}(A^1_3 + A^1_2 A^2_3)$ and $k = \frac{1}{2}A^2_3$, respectively.

Instead of just stating the existence we explicitly construct these diffeomorphisms. We claim that they are induced by an isomorphism of vector bundles. The general result for $\mathbb{Q}$-trivial Bott manifolds is based on similar constructions.

Recall that there is a bundle isomorphism G between $\gamma_1^{2k} \oplus \underline{\mathbb{C}} \to \mathbb{C}\mathrm{P}^1$ and $\gamma_1^k \oplus \gamma_1^k \to \mathbb{C}\mathrm{P}^1$ (follows from Corollary 3.5 in [Hus94]) which covers the identity $\mathbb{1}_{\mathbb{C}\mathrm{P}^1}$. The projectivisation $P(\gamma_1^k \oplus \gamma_1^k)$ is diffeomorphic to $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. The projectivisation of G is a diffeomorphism $\widetilde{g}$ such that the diagram

$$\begin{array}{ccc} P(\gamma_1^{2k} \oplus \underline{\mathbb{C}}) & \xrightarrow{\widetilde{g}} & \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \\ \downarrow & & \downarrow \\ \mathbb{C}\mathrm{P}^1 & \xrightarrow{\mathbb{1}_{\mathbb{C}\mathrm{P}^1}} & \mathbb{C}\mathrm{P}^1 \end{array},$$

commutes. By the commutativity $\widetilde{g}^*(b_1) = v_1$. Since $\widetilde{g}$ is a diffeomorphism it induces a ring isomorphism in cohomology. The image $g^*(b_2)$ must be $\pm v_2 \mp kv_1$, otherwise g^* would not be a ring isomorphism.
To determine the sign we consider the tautological line bundle η_2 over $P(\gamma^k \oplus \gamma^k)$, i.e. the bundle whose negative first Chern class is b_2. Let $\iota\colon \mathbb{C}\mathrm{P}^1 \to P(\gamma_1^{2k} \oplus \underline{\mathbb{C}})$ denote the inclusion of the fiber. The pullback of η_2 along the composition $\iota \circ \widetilde{g}$ must be the tautological bundle over the fiber. Thus, the sign in front of v_2 must be positive. Here, we use the isomorphism between cohomology in degree two and isomorphism classes of complex line bundles (cf. [Hus94, Theorem 3.4]).
Consequently, $\widetilde{g}$ realises $\varphi_k|_{\mathbb{Z}\langle b_1,b_2\rangle}$ and the pullback $\widetilde{g}^*(\eta_2)$ is isomorphic to $\gamma_1^{-k} \otimes \gamma_2$. This implies that there is a bundle isomorphism $g'\colon \gamma_1^{-k} \otimes \gamma_2 \to \eta_2$ covering $\widetilde{g}$.
Furthermore, there is a bundle isomorphism between the trivial bundles over $P(\gamma^{2k} \oplus \underline{\mathbb{C}})$ and $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ given by

$$\begin{aligned} k\colon P(\gamma^{2k} \oplus \mathbb{C}) \times \mathbb{C} &\to (\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1) \times \mathbb{C} \\ (x, z) &\mapsto (\widetilde{g}(x), z)\ . \end{aligned}$$

The projectivisation of the Whitney sum $g' \oplus k$, which is an isomorphism of vector bundles, is the bundle diffeomorphism $\widetilde{h}\colon \widetilde{B}_3 \to \mathbb{C}\mathrm{P}^1 \times \mathcal{H}_1$ which we want to construct.
It remains to check that $\widetilde{h}^*(b_3) = v_3$. Up to sign this follows, again, from the fact that $\widetilde{h}$ induces an isomorphism of rings. The sign is again determined by the pullback along the inclusion of the fibre.

We take $\widehat{h}$ and $\overline{h}$ to be the diffeomorphism $\widetilde{h}$ constructed above, for $k = \frac{1}{2}(A_3^1 + A_2^1 A_3^2)$and $k = \frac{1}{2}A_3^2$, respectively. In particular, $\widehat{h}$ and $\overline{h}$ are both diffeomorphisms of fibre bundles and cover $\widehat{g}$ and $\overline{g}$. Here, $\widehat{g}$ and $\overline{g}$ are the diffeomorphisms corresponding to $\widetilde{g}$ as above for the respective k.

Next, we extend the diffeomorphism to $D\nu\mathbb{C}\mathrm{P}_1^1$ and $D\nu\mathbb{C}\mathrm{P}_2^1$, respectively.
Let $p_i\colon D\nu\mathbb{C}\mathrm{P}_i^1 \to \mathbb{C}\mathrm{P}_i^1$ be the projections to the base space, t_i the zero-sections of both bundles and $pr_2\colon \mathbb{C}\mathrm{P}_1^1 \times \mathcal{H}_1 \to \mathcal{H}_1$ the projection to the second coordinate. Recall that $s\colon D\nu\mathbb{C}\mathrm{P}_1^1 \to B_2$ and $i\colon D\nu\mathbb{C}\mathrm{P}_2^1 \to B_2$ denote the embeddings of the tubular neighbourhoods, i.e. $s \circ t_1 = s_2$ and $i \circ t_2 = i_2$.
By definition $p_1^*(\widehat{B}_3) = \Big\{(d,b) \in D\nu\mathbb{C}\mathrm{P}_1^1 \times \widehat{B}_3 \mid p_1(d) = \widehat{\pi}(b)\Big\}$, whence we define a diffeomorphism

$$\begin{array}{rcl} \widehat{h}_1\colon p_1^*(\widehat{B}_3) & \to & D\nu\mathbb{C}\mathrm{P}_1^1 \times \mathcal{H}_1 \text{ by} \\ (d,b) & \mapsto & (d, pr_2 \circ \widehat{h}(b)) \end{array}$$

with inverse

$$\begin{array}{rcl} \widehat{h}_1^{-1}\colon D\nu\mathbb{C}\mathrm{P}_1^1 \times \mathcal{H}_1 & \to & p_1^*(\widehat{B}_3) \text{ by} \\ (d,h) & \mapsto & (d, \widehat{h}^{-1}(p_1(d), h)). \end{array}$$

Note that $p_1^*(\widehat{B}_3)|_{\mathbb{C}\mathrm{P}_1^1} = \widehat{B}_3$ and $\widehat{h}_1|_{\widehat{B}_3} = \widehat{h}$.
Since $t_1 \circ p_1$ is homotopic to the identity $\mathbb{1}_{D\nu(\mathbb{C}\mathrm{P}_1^1)}$ the restriction $B_4|_{s(D\nu(\mathbb{C}\mathrm{P}_1^1))} = s^*B_4 = \mathbb{1}^*_{D\nu(\mathbb{C}\mathrm{P}_1^1)} s^*B_4$ is isomorphic to $p_1^* t_1^* s^* B_4 = p_1^* B_4|_{s \circ t_1(\mathbb{C}\mathrm{P}_1^1)} = p_1^* \widehat{B}_3$ as fibre bundles. Composing this diffeomorphism with $\widehat{h}_1$ we obtain a diffeomorphism

$$h_1\colon B_4|_{s(D\nu\mathbb{C}\mathrm{P}_1^1)} \to p_1^*(\widehat{B}_3).$$

By construction the restriction of h_1 to $B_4|_{s(\mathbb{C}\mathrm{P}_1^1)} = \widehat{B}_3 \simeq B_4|_{s(D\nu\mathbb{C}\mathrm{P}_1^1)}$ is $\widehat{h}$. Consequently, h_1 realises $\varphi_{\frac{1}{2}(A_3^1 + A_2^1 A_3^2)}$ as in Equation (10).

Analogously we define $h_2\colon B_4|_{i(D\nu\mathbb{C}\mathrm{P}_2^1)} \to p_2^*(\overline{B}_3)$ using $\overline{h}$ which realises $\varphi_{\frac{1}{2}A_3^2}$ as in Equation (10).

Now we combine both diffeomorphisms to obtain $h\colon B_4|_{Pl} \to Pl \times \mathcal{H}_1$. To do that we recall the construction of $Pl = D\nu(\mathbb{C}\mathrm{P}^1_1)\natural D\nu(\mathbb{C}\mathrm{P}^1_2) = D\nu(\mathbb{C}\mathrm{P}^1_1) \amalg D\nu(\mathbb{C}\mathrm{P}^1_2)/\sim$, where $x \sim y$ if $s(x) = i(y)$. Define

$$Pl \times \mathcal{H}_1 := D\nu\mathbb{C}\mathrm{P}^1_1 \times \mathcal{H}_1 \amalg D\nu\mathbb{C}\mathrm{P}^1_2 \times \mathcal{H}_1/\sim,$$

where we identify $x \sim y$, for $x \in D\nu(\mathbb{C}\mathrm{P}^1_1) \times \mathcal{H}_1$ and $y \in D\nu(\mathbb{C}\mathrm{P}^1_2) \times \mathcal{H}_1$, if they fulfil $h_2^{-1}(y) = h_1^{-1}(x)$. The notation as a product, i.e. the trivial $\mathcal{H}_1$-bundle over Pl is justified since h_1 and h_2 are diffeomorphisms of fibre bundles. Therefore, $Pl \times \mathcal{H}_1$ really is the trivial $\mathcal{H}_1$-bundle over Pl. We smoothen the corners of Pl in the same way as above and obtain

$$h\colon B_4|_{(s\natural i)(Pl)} \to Pl \times \mathcal{H}_1,$$

defined by $h(x) = h_1(x)$ if $x \in B_4|_{s(D\nu\mathbb{C}\mathrm{P}^1_1)}$ and $h(y) = h_2(y)$ if $y \in B_4|_{i(D\nu\mathbb{C}\mathrm{P}^1_2)}$ which is a well-defined diffeomorphism of smooth fibre bundles.

Recall that, by construction, the diffeomorphisms $\widehat{h}$ and $\overline{h}$ cover the diffeomorphisms $\widehat{g}\colon B_3|_{s(\mathbb{C}\mathrm{P}^1_1)} \to \mathbb{C}\mathrm{P}^1_1 \times \mathbb{C}\mathrm{P}^1_3$ and $\overline{g}\colon B_3|_{i(\mathbb{C}\mathrm{P}^1_2)} \to \mathbb{C}\mathrm{P}^1_2 \times \mathbb{C}\mathrm{P}^1_3$. Analogously to the construction of h_1, h_2 and h we obtain g_1, g_2 and g. In particular, there is a commutative diagram

$$\begin{array}{ccc} B_4|_{(s\natural i)(Pl)} & \xrightarrow{h} & Pl \times \mathcal{H}_1 \\ \downarrow & & \downarrow \\ B_3|_{(s\natural i)(Pl)} & \xrightarrow{g} & Pl \times \mathbb{C}\mathrm{P}^1_1 \end{array} \qquad (11)$$

To construct a map on $Pl \times \mathcal{H}_1$ that gives rise to a realisation of ϕ_1 on second cohomology $H^2(B_4|_{(s\natural i)(Pl)}) \cong H^2(B_4)$, we need to understand h^* on second cohomology. Since we do understand $\widehat{h}^*$ and $\overline{h}^*$ we can deduce h^* from an easy diagram chase in the Mayer-Vietoris sequences of $B_4|_{(s\natural i)(Pl)} \simeq \widehat{B}_3 \cup \overline{B}_3$ and $Pl \times \mathcal{H}_1 \simeq \mathbb{C}\mathrm{P}^1_1 \times \mathcal{H}_1 \cup \mathbb{C}\mathrm{P}^1_2 \times \mathcal{H}_1$:

$$\begin{array}{ccccccccc} 0 & \to & H^2(Pl \times \mathcal{H}_1) & \to & H^2(\mathbb{C}\mathrm{P}^1_1 \times \mathcal{H}_1) \oplus H^2(\mathbb{C}\mathrm{P}^1_2 \times \mathcal{H}_1) & \to & H^2(\mathcal{H}_1) & \to & 0 \\ & & \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle \widehat{h}^* \oplus \overline{h}^*} & & \downarrow & & \\ 0 & \to & H^2(B_4|_{(s\natural i)(Pl)}) & \to & H^2(\widehat{B}_3) \oplus H^2(\overline{B}_3) & \to & H^2(\mathcal{H}_1) & \to & 0\,. \end{array}$$

Let $w_1, .., w_4$ denote the bundle generators of $H^*(Pl \times \mathcal{H}_1)$. The diffeomorphism h induces the isomorphism given by

$$\begin{array}{rcl} h^* : H^2(Pl \times \mathcal{H}_1) & \to & H^2(B_4|_{(s\natural i)(Pl)}) \\ w_i & \mapsto & y_i \text{ for } i = 1, 2, 4 \text{ and} \\ w_3 & \mapsto & y_3 - \frac{1}{2}(A^1_3 y_1 + A^2_3 y_2). \end{array}$$

An analogous calculation shows $g^* = h^*|_{\mathbb{Z}\langle w_1, w_2, w_3\rangle}$.

5.2. Realisation of ϕ_1 on $B_4|_{(s\natural i)(Pl)}$

Now, B_4 is decomposed into $N = B_4|_{(s\natural i)(Pl)}$ and $M = D^4 \times \mathcal{H}_1$, identified along the identity on the boundary, and we can start with the realisation of ϕ_1. For abbreviation we omit the embedding in the notation of $B_4|_{(s\natural i)(Pl)}$ from now on.
In Equation (9) we saw that ϕ_1 has a very nice form if we change to the geometric basis. Thus, we need the isomorphism h^* in terms of the geometric basis of $B_4|_{Pl}$, given by the Kronecker duals of the appropriately embedded $\mathbb{C}\mathrm{P}^1_i, 1 \leq i \leq 4$.

Let $v_1, .., v_4$ denote the geometric basis of $Pl \times \mathcal{H}_1$. By Lemma 2.3 $v_i = w_i$, for $i \leq 3$ and $v_4 = w_4 - w_3$.
In Equation (8) we recalled the base change for $H^2(B_4)$. The base change for $B_4|_{Pl}$ is slightly different. Since the base space Pl is homotopy equivalent to $\mathbb{C}\mathrm{P}^1_1 \vee \mathbb{C}\mathrm{P}^1_2$ the generator y_2 is Kronecker dual to $[\mathbb{C}\mathrm{P}^1_2]$, i.e. $x_2 = y_2$. The base change between y_i and x_i for $i = 1, 3, 4$ is exactly the same as the base change of the corresponding elements in $H^2(B_4)$.
In the geometric basis h^* is given by

$$\begin{aligned} v_i &\mapsto x_i \text{ for } i = 1, 2, 4 \text{ and} \\ v_3 &\mapsto x_3 + A_3^1 x_1 + A_3^2 x_2 \end{aligned}$$

with inverse

$$\begin{aligned} x_i &\mapsto v_i \text{ for } i = 1, 2, 4 \text{ and} \\ x_3 &\mapsto v_3 - A_3^1 v_1 - A_3^2 v_2 \end{aligned}$$

To realise ϕ_1 on $B_4|_{Pl}$ we need to realise $\widetilde{\phi_1} := (h^*)^{-1} \circ \phi_1 \circ h^*$ on $Pl \times \mathcal{H}_1$. A straight forward calculation shows:

Lemma 5.3. *On $H^2(Pl \times \mathcal{H}_1)$ we need to realise $\widetilde{\phi_1}$ defined by*

$$\begin{aligned} v_i &\mapsto v_i \text{ for } i = 1, 2 \\ v_3 &\mapsto v_3 + 2v_4 \text{ and} \\ v_4 &\mapsto -v_4. \end{aligned}$$

We claim that there exists a diffeomorphism of the form $\mathbb{1}_{Pl} \times f \colon Pl \times \mathcal{H}_1 \to Pl \times \mathcal{H}_1$ that induces $\widetilde{\phi_1}$. It remains to determine f which is a diffeomorphism on a Hirzebruch surface. Thus, we briefly discuss some diffeomorphisms on the non-trivial Hirzebruch surface $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. We use Lemma 2 of [Wal64].

Lemma 5.4. *Let M_1 and M_2 be two closed manifolds of dimension n and $k_i \colon M_i \to M_i$ two orientation preserving self-diffeomorphisms. Then there exists a diffeomorphism $k \colon M_1 \# M_2 \to M_1 \# M_2$ whose induced map on cohomology is $k^* = k_1^* \oplus k_2^*$.*

Wall constructs k by changing k_1 and k_2 up to isotopy such that there exist embedded discs which are fixed by the modified k_i. Thus he can build the connected sum along these discs and obtain k as "honest" connected sum of both maps. Therefore, we denote k as in the lemma by $k_1 \# k_2$.

Let $c\colon \mathbb{C}P^2 \to \mathbb{C}P^2$ be given by complex conjugation on the homogeneous coordinates of $\mathbb{C}P^2$, i.e. $[z_0 : z_1 : z_2] \mapsto [\overline{z_0} : \overline{z_1} : \overline{z_3}]$. Consider the embedding $e\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}P^2$, which is defined by $[z_0 : z_1] \mapsto [z_0 : z_1 : 0]$. Let $a := e_*[\mathbb{C}\mathrm{P}^1]$. Since $c|_{e(\mathbb{C}\mathrm{P}^1)}$ is the reflection on $\mathbb{R}P^1 \hookrightarrow \mathbb{C}P^1$, i.e. the map of degree -1, we obtain $c_*(a) = -a$.
Analogously we obtain $\overline{c}\colon \overline{\mathbb{C}P^2} \to \overline{\mathbb{C}P^2}$.
Together with Lemma 5.4 this enables us to realise some automorphisms of $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$, in particular we claim that we can realise the automorphism which we need.

Embed two copies of $\mathbb{C}\mathrm{P}^1$ into $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ on the one hand as two-skeleton of $\mathbb{C}P^2$, on the other as two-skeleton of $\overline{\mathbb{C}P^2}$. This gives rise to the standard basis of $H_2(\mathcal{H}_1)$, we denote it by s_1 and s_2. Let t_1, t_2 denote the Poincaré duals of s_1 and s_2. They form a basis of $H^2(\mathcal{H}_1)$. With respect to this basis the intersection form is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With respect to the basis v_3, v_4 the intersection form is

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let ϵ_i be ± 1 for $i = 1, 2$. Using the intersection forms we obtain the following base change between v_3, v_4 and t_1, t_2

$$v_3 = \epsilon_1 t_1 - \epsilon_2 t_2 \text{ and } v_4 = \epsilon_2 t_2.$$

An easy computation shows $\widetilde{\phi}_1(t_1) = t_1$ and $\widetilde{\phi}_1(t_2) = -t_2$ in the basis of the t_i. In particular, $\widetilde{\Phi}$ is realised by $\mathbb{1}_{Pl} \times \mathbb{1} \# \overline{c}\colon Pl \times \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \to Pl \times \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. Therefore, we proved the following Lemma.

Lemma 5.5. *On $B_4|_{Pl}$ the automorphism ϕ_1 is realised by the self-diffeomorphism*

$$h^{-1} \circ (\mathbb{1}_{Pl} \times f) \circ h, \text{ where } f = \mathbb{1} \# \overline{c}.$$

In other words we have now have $h^{-1} \circ (\mathbb{1}_{Pl} \times f) \circ h =: n$, where $n\colon N \to N$ is the diffeomorphism of Theorem 5.2.

5.3. Preparing the setting for modified surgery

Now we finally come to the application of modified surgery theory, in particular, of Corollary 3.12. We have a decomposition $B_4 = B_4|_{Pl} \cup_{Id} B_4|_{D^4} = N \cup M$ and a diffeomorphism $n = h^{-1} \circ (\mathbb{1} \times f) \circ h$ on $B_4|_{Pl}$, which we want to extend over $B_4|_{D_4} = D^4 \times \mathcal{H}_1$.

By Lemma 2.2, we know $\pi_4(D^4 \times \mathcal{H}_1) \cong \mathbb{Z}/2^2$. Thus, the first assumption of Corollary 3.12, i.e. the finiteness assumption on the homotopy group in middle dimension, is fulfilled.

In order to apply modified surgery theory, it remains to construct a fibration $\mathbb{B} \to BO$ such that there exist two normal three-smoothings $D^4 \times \mathcal{H}_1 \to \mathbb{B}$ which, on the boundary $\partial D^4 \times \mathcal{H}_1 = S^3 \times \mathcal{H}_1$, are compatible with the diffeomorphism $h^{-1} \circ (\mathbb{1} \times f) \circ h|_{S^3 \times \mathcal{H}_1}$. Let e^8 be an 8-cell, $\rho \in \pi_7(\mathcal{H}_1)$ and $\iota\colon \mathcal{H}_1 \to \mathcal{H}_1 \cup_\rho e^8$ the inclusion. We will see that for some choice of ρ the total space of $\mathbb{B}$ is $(\mathcal{H}_1 \cup_\rho e^8) \times BString$. The fibration is a twisted fibration over BO. The twist is made explicit subsequently.

Before we start the construction of the twisted fibration over BO or even its total space, we need one further observation.
Let $\gamma \to \mathbb{C}\mathrm{P}^1$ denote the tautological line bundle. So far we only need the abstract knowledge that $P(\gamma \oplus \underline{\mathbb{C}})$ is the non-trivial Hirzebruch surface $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. Now we need the identification a bit more explicit. Recall that the embedding $e\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}P^2$ has normal bundle $\nu(e) = \gamma^{-1}$ and that we can identify $D\nu(e)$ with $\mathbb{C}P^2 - D^4$, where D^4 is the top disc.
Since $\mathcal{H}_1$ is a Bott manifold it, in particular, admits a section $\sigma_2\colon \mathbb{C}\mathrm{P}^1 \to \mathcal{H}_1$ as defined above Lemma 2.2 in Section 2.1. Let $\sigma\colon Pl \times \mathbb{C}\mathrm{P}^1_3 \to Pl \times \mathcal{H}_1$ be $\mathbb{1} \times \sigma_2$.
We know (cf. Section 2.3) that the normal bundle of the section $\sigma_2\colon \mathbb{C}\mathrm{P}^1 \to P(\gamma \oplus \underline{\mathbb{C}})$ is γ^{-1}. Thus, we can identify a tubular neighbourhood of $\sigma_2(\mathbb{C}\mathrm{P}^1)$, which is an embedded $D\nu(s)$, with $\mathbb{C}P^2 - D^4$. Under this identification the maps σ_2 and e are equal.
Recall that there exist sections $s_4\colon B_3|_{Pl} \to B_4|_{Pl}$ and $s_3\colon Pl \to B_3|_{Pl}$.

Lemma 5.6. *Let $h\colon B_4|_{Pl} \to Pl \times \mathcal{H}_1$ and $g\colon B_3|_{Pl} \to Pl \times \mathbb{C}P^1_3$ be the diffeomorphisms constructed in the last section. This diffeomorphisms fulfil*

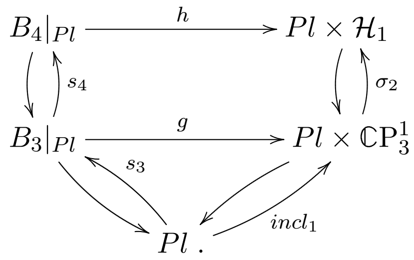
$$h \circ s_4 = (\mathbb{1} \times \sigma_2) \circ g = (\mathbb{1} \times e) \circ g.$$

Furthermore, we obtain $g \circ s_3 = incl_1$, for $incl_1\colon Pl \to Pl \times \mathbb{C}P^1_3$ the inclusion into the first factor.

Recall that there is a commutative diagram

$$\begin{array}{ccccc} B_4|_{Pl} & \xrightarrow{\pi_4} & B_3|_{Pl} & & \\ \downarrow h & & g \downarrow & \searrow & \\ Pl \times \mathcal{H}_1 & \xrightarrow[\mathbb{1} \times p]{} & Pl \times \mathbb{C}\mathrm{P}^1_3 & \xrightarrow[pr_1]{} & Pl\,. \end{array}$$

The upshot of the lemma is that the following diagram is also commutative:



To show this we use that we constructed h as the projectivisation of an isomorphism of the underlying vector bundles. Unfortunately, the sections s_4 and σ_2 are not induced by sections of the underlying vector bundles in an obvious way. Thus, we are forced to change the perspective on B_4 and $\mathcal{H}_1$ slightly.

Proof. The equation $(\mathbb{1} \times \sigma_2) \circ g = (\mathbb{1} \times e) \circ g$ follows from the observations previous to the lemma.
For the first equation we use Lemma 2.1 of [CMS10]. Let B be any smooth manifold, $E \to B$ a complex vector bundle and $L \to B$ a line bundle. By Lemma 2.1 of [CMS10] the fibre bundles $P(E)$ and $P(E \otimes L)$ are isomorphic. Thus, their total spaces are diffeomorphic.
Recall that B_4 is the projectivisation $P(L_4 \oplus \underline{\mathbb{C}})$. By the lemma we have a diffeomorphism between $P(L_4 \oplus \underline{\mathbb{C}})$ and $P(\underline{\mathbb{C}} \oplus L_4^{-1})$. Analogously $\mathcal{H}_1$ is diffeomorphic to $P(\underline{\mathbb{C}} \oplus \gamma^{-1})$.
Thus, we can consider the following section

$$\widetilde{s}_4 \colon B_3 \to \underline{\mathbb{C}} \oplus L_4^{-1}$$

given by the direct sum of the constant section into $\underline{\mathbb{C}}$ and the zero-section into L_4^{-1}. The constant section is given by $b \mapsto (b, z) \in \underline{\mathbb{C}} = B_3 \times \mathbb{C}$ for $z \in \mathbb{C}$ a fixed, non-vanishing complex number.
After projectivisation and identification of $P(\underline{\mathbb{C}} \oplus L_4^{-1})$ with $P(L_4 \oplus \underline{\mathbb{C}})$ this is exactly our section s_4.
In the same way, we can construct σ_2 as the projectivisation of the sum of the constant section into $\underline{\mathbb{C}} = \mathbb{C}\mathrm{P}^1_3 \times \mathbb{C}$ - with respect to the same constant $z \in \mathbb{C}$ - and the zero-section into γ^{-1}.

Let $pr_2 \colon Pl \times \mathbb{C}\mathrm{P}^1_3 \to \mathbb{C}\mathrm{P}^1_3$ denote the projection to the second factor. Furthermore, let $\eta_3 = pr_2^* \gamma$ denote the tautological bundle over $Pl \times \mathbb{C}\mathrm{P}^1_3$, i.e. $-c_1(\eta_3) = w_3$, where w_3 is the third generator of the bundle basis of $Pl \times \mathbb{C}\mathrm{P}^1_3$.
Recall that we constructed g such that $g^*(w_3) = y_3 - \frac{1}{2}(A_3^1 y_1 + A_3^2 y_2)$. Thus, the pullback $g^* \eta_3^{-1}$ is isomorphic to L_4^{-1} as bundle.
The bundle map $g' \colon g^* \eta_3^{-1} \to \eta_3^{-1}$ over g maps the zero-section in the pullback bundle to

the zero-section in η_3^{-1}. On the other hand the isomorphism $g^*\eta_3^{-1} \cong L_4^{-1}$ also preserves the zero-section.
To obtain h we also consider the constant map between the trivial bundles

$$\begin{aligned} k\colon (Pl \times \mathbb{C}\mathrm{P}^1_3) \times \mathbb{C} &\to B_3|_{Pl} \times \mathbb{C}, \\ (x, y) &\mapsto (g^{-1}(x), y) \end{aligned}$$

which preserves the constant section. By construction the projectivisation of $k \oplus g'$ is h and preserves the projectivisation of the sum of the constant and the zero-section. Since h covers g we, for $b \in B_3|_{Pl}$, obtain $h(s_4(b)) = (\mathbb{1} \times \sigma_2)(g(b))$.
The proof for the equality $g \circ s_3 = incl_1$ works the same way. □

Now we begin the construction of the normal smoothings, by constructing highly connected maps into $\mathcal{H}_1 \cup_\rho e^8$.

Lemma 5.7. *Let $\iota\colon \mathcal{H}_1 \to \mathcal{H}_1 \cup_\rho e^8$ denote the inclusion and let $pr\colon D^4 \times \mathcal{H}_1 \to \mathcal{H}_1$ denote the projection onto $\mathcal{H}_1$. The following maps are six-connected*

$$\begin{aligned} &i_1\colon D^4 \times \mathcal{H}_1 \to \mathcal{H}_1 \cup_\rho e^8,\ i_1 := \iota \circ pr \text{ and} \\ &i_2\colon D^4 \times \mathcal{H}_1 \to \mathcal{H}_1 \cup_\rho e^8,\ i_2 := \iota \circ pr \circ (\mathbb{1}_{D^4} \times f) \end{aligned}$$

for any element $\rho \in \pi_7(\mathcal{H}_1)$. Furthermore, there exists $\rho \in \pi_7(\mathcal{H}_1)$ such that the diagram

$$\begin{array}{ccc} & \mathcal{H}_1 \cup_\rho e^8 & \\ {\scriptstyle i_2|_{S^3\times\mathcal{H}_1}}\nearrow & & \nwarrow{\scriptstyle i_1|_{S^3\times\mathcal{H}_1}} \\ S^3 \times \mathcal{H}_1 & \xrightarrow[h^{-1}\circ(\mathbb{1}_{S^3}\times f)\circ h|_{S^3\times\mathcal{H}_1}]{} & S^3 \times \mathcal{H}_1 \end{array}$$

commutes up to homotopy.

Proof. It is obvious that i_1 and i_2 are six-connected.

For the remainder of the proof, we always consider all maps restricted to $S^3 \times \mathcal{H}_1$ unless otherwise indicated.

To show the commutativity of the diagram we first show

$$pr \circ h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h|_{S^3 \vee \mathcal{H}_1} = pr \circ (\mathbb{1}_{S^3} \times f)|_{S^3 \vee \mathcal{H}_1}. \qquad (12)$$

Let $\operatorname{Diff} \mathcal{H}_1$ denote the group of self-diffeomorphisms of the non-trivial Hirzebruch surface $\mathcal{H}_1$. By construction $h\colon S^3 \times \mathcal{H}_1 \to S^3 \times \mathcal{H}_1$ is a diffeomorphism of fibre bundles. Thus, it is of the form $h(t, b) = (t, \widetilde{h}(t)(b))$ for some $\widetilde{h}$ which represents an element in $\pi_3(\operatorname{Diff} \mathcal{H}_1)$. The elements of $\pi_3(\operatorname{Diff} \mathcal{H}_1)$ are induced by base point preserving maps $S^3 \to \operatorname{Diff} \mathcal{H}_1$, where the base point of $\operatorname{Diff}\mathcal{H}_1$ is the identity. Thus, there is a base point s_0 of the sphere which always maps to the identity $\mathbb{1}_{\mathcal{H}_1}$. In particular, this also holds for $\widetilde{h}$. Hence, Equation (12) holds for all points in $(s_0, x) \in \{s_0\} \times \mathcal{H}_1$.

The next step is to show that there exists a point $y_0 \in \mathcal{H}_1$ such that Equation (12) holds for all $(t, y_0) \in S^3 \times \mathcal{H}_1$.
First we consider $pr \circ h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h$. Let (t,x) be a point in the image of $s_4(S^3 \times \mathbb{C}\mathrm{P}^1) = S^3 \times e(\mathbb{C}\mathrm{P}^1)$. By Lemma 5.6 we know $h(t,x) \in \sigma(\{t\} \times \mathbb{C}\mathrm{P}^1_3)$. By definition of $f = \mathbb{1}\#\bar{c}$ the map $\mathbb{1}_{S^3} \times f$ is the identity on $\sigma(\{t\} \times \mathbb{C}\mathrm{P}^1_3) = \{t\} \times e(\mathbb{C}\mathrm{P}^1_3)$. Thus, we obtain $pr \circ h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h(t,x) = pr(t,x) = x$ for all $(t,x) \in S^3 \times e(\mathbb{C}\mathrm{P}^1)$.
Now we consider $pr \circ (\mathbb{1}_{S^3} \times f)$.
Let $(t,x) \in s_4(S^3 \times \mathbb{C}\mathrm{P}^1_3) = S^3 \times e(\mathbb{C}\mathrm{P}^1)$. Again, by definition of $f = \mathbb{1}\#\bar{c}$ we know $pr \circ (\mathbb{1}_{S^3} \times f)(t,x) = x$. Therefore, Equation (12) holds on $S^3 \times \{y_0\}$ for all $y_0 \in e(\mathbb{C}\mathrm{P}^1_3)$ and thereby on $S^3 \vee \mathcal{H}_1$.
In other words, the diagram in the lemma commutes on $S^3 \vee \mathcal{H}_1$ without composing with ι. This implies commutativity on the four-skeleton at least up to homotopy, i.e. there exists a homotopy $h_t \colon (S^3 \vee \mathcal{H}_1) \times I \to \mathcal{H}_1 \cup e_8$ such that $h_0 = pr \circ h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h|_{S^3 \vee \mathcal{H}_1}$ and $h_1 = pr \circ (\mathbb{1}_{S^3} \times f)|_{S^3 \vee \mathcal{H}_1}$.

The next step is to use obstruction theory to extend the homotopy h_t over the six-skeleton, i.e. to a homotopy $H_t \colon (S^3 \times \mathcal{H}_1)^{(6)} \times I \to \mathcal{H}_1$ which fulfils

$$\begin{aligned} H_0 &= pr \circ h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h|_{(S^3 \times \mathcal{H}_1)^{(6)}}, \\ H_1 &= pr \circ (\mathbb{1}_{S^3} \times f)|_{(S^3 \times \mathcal{H}_1)^{(6)}} \text{ and} \\ h_t &= H_t|_{S^3 \vee \mathcal{H}_1}. \end{aligned} \tag{13}$$

Obstruction theory implies that we can extend the homotopy h_t all over $S^3 \times \mathcal{H}_1 \times I$ if the obstruction classes ω_k in $H^{k+1}((S^3 \times \mathcal{H}_1) \times I, (S^3 \vee \mathcal{H}_1) \times I;\ \pi_k(\mathcal{H}_1))$ vanish. For $k \leq 3$ there is nothing to show since the cohomology groups themselves vanish by the long exact sequence of the pair.

Recall that $\pi_j(\mathcal{H}_1) \cong \pi_j(\mathbb{C}\mathrm{P}^1)^2$ by Lemma 2.2, furthermore, that $\mathbb{P}_j\mathcal{H}_1$ denotes the $j-$th Postnikov stage of $\mathcal{H}_1$ and that $k_{j+1} \colon \mathbb{P}_j\mathcal{H}_1 \to K(\pi_{j+1}(\mathcal{H}_1), j+2)$ denotes the $(j+1)$-st k-invariant.

We know that h_t exists. Thus, for all j, there exists a map $h_t^j \colon (S^3 \vee \mathcal{H}_1) \times I \to \mathbb{P}_j\mathcal{H}_1$ such that $k_{j+1} \circ h_t^j$ is null-homotopic. Consequently, we obtain a map from the cone $C := C\big((S^3 \vee \mathcal{H}_1) \times I\big) \to \mathbb{P}_j\mathcal{H}_1$ which we denote by $C(h_t^j)$. If we can extend this map to $(S^3 \times \mathcal{H}_1 \times I) \cup C$, the obstruction class vanishes by definition.

Since $\omega_k = 0$ for $k \leq 3$ we have a map $j \colon ((S^3 \times \mathcal{H}_1) \times I) \cup C \to \mathbb{P}_3\mathcal{H}_1$ extending $C(h_t^3)$. We show that $k_4 \circ j \simeq pt$, where $k_4 \colon \mathbb{P}_3\mathcal{H}_1 \to K(\mathbb{Z}/2^2, 5)$ is the fourth k-invariant.
Let $pr_i \colon K(\mathbb{Z}/2^2, 5) = K(\mathbb{Z}/2, 5) \times K(\mathbb{Z}/2, 5) \to K(\mathbb{Z}/2, 5)$, for $i = 1, 2$, denote the pro-

jection upon the first, respectively second factor. The map $k_4 \circ j$ is null-homotopic if and only if $pr_i \circ k_4 \circ j$ are null-homotopic for $i = 1, 2$.
Thus, we can consider j^* on cohomology with $\mathbb{Z}/2-$coefficients, where we have Steenrod operations.
We know (cf. Appendix B) that $Sq^2 \colon H^5(\mathbb{P}_3\mathcal{H}_1; \mathbb{Z}/2) \to H^7(\mathbb{P}_3\mathcal{H}_1;\ \mathbb{Z}/2)$ vanishes identically. By the long exact sequence of the pair $H^5(S^3 \times \mathcal{H}_1; \mathbb{Z}/2) \cong H^5(S^3 \times \mathcal{H}_1 \cup C; \mathbb{Z}/2)$. But in $H^5(S^3 \times \mathcal{H}_1; \mathbb{Z}/2)$ there exist elements whose second Steenrod square do not vanish, namely $s(w_4 + kw_3)$ for $k = 0, 1$, where s is the pullback of a generator of $H^3(S^3; \mathbb{Z}/2)$. By naturality we thereby know that these elements cannot be hit, i.e. the image of j^* is contained in $\mathbb{Z}/2\langle sw_3\rangle$.
Now we consider $s_4 \colon S^3 \times \mathbb{C}\mathrm{P}^1_3 \hookrightarrow B_4|_{\partial Pl} = S^3 \times \mathcal{H}_1$. By definition sw_3 pulls back to a generator of $H^5(S^3 \times \mathbb{C}\mathrm{P}^1_3; \mathbb{Z}/2)$. On the other hand we already know that our original diagram commutes on $S^3 \times e(\mathbb{C}\mathrm{P}^1_3)$ by the considerations above, so there all obstruction classes vanish. Since sw_3 injects into $H^5(S^3 \times \mathbb{C}\mathrm{P}^1_3)$, $\mathrm{im} j^*$ cannot contain sw_3.
Thus, $\mathrm{im} j^* = 0$, i.e. $(k_4 \circ j)^* = 0$, and there exists a lift $S^3 \times \mathcal{H}_1 \cup C \to \mathbb{P}_4\mathcal{H}_1$. The next obstruction class is an element in $H^6(S^3 \times \mathcal{H}_1 \cup C; \pi_5(\mathcal{H}_1))$ which vanishes by the long exact sequence of the pair $(S^3 \times \mathcal{H}_1 \times I, (S^3 \vee \mathcal{H}_1) \times I)$. Consequently, we even find a lift $\widetilde{j} : S^3 \times \mathcal{H}_1 \times I \to \mathbb{P}_5\mathcal{H}_1$ which extends the map $(S^3 \vee \mathcal{H}_1) \times I \to \mathbb{P}_5\mathcal{H}_1$.

Now we consider the inclusion of the six-skeleton $\iota_6 \colon (S^3 \times \mathcal{H}_1)^{(6)} \times I \to (S^3 \times \mathcal{H}_1) \times I$. We obtain a map into $\mathbb{P}_5\mathcal{H}_1$ by composing ι_6 with $\widetilde{j}$. All higher cohomology groups of $H^*((S^3 \times \mathcal{H}_1)^{(6)} \times I, (S^3 \vee \mathcal{H}_1) \times I)$ vanish. Therefore, we can lift $\widetilde{j} \circ \iota_6$ through the whole Postnikov tower, i.e. we know there exists H_t as in Equation (13).

The final step is to extend the homotopy over the seven-skeleton. So far we neither needed the 8-cell attached to $\mathcal{H}_1$ nor did we specify the map by which it is attached. We now collapse the six-skeleton of $S^3 \times \mathcal{H}_1$ whence we get two induced maps $pr \circ h^{-1} \circ (\mathbb{1}_{S^3} \times \widetilde{f}) \circ h : S^7 \to \mathcal{H}_1$ and $pr \circ (\mathbb{1}_{S^3} \times f) : S^7 \to \mathcal{H}_1$ which induce elements α and β in $\pi_7(\mathcal{H}_1)$. By attaching the 8-cell along $\rho := \alpha - \beta$ we ensure commutativity of the diagram in the lemma. □

As in Section 4 we will consider a convenient bordism group, subsequently. For this purpose, we need a map to *BString*. Since our manifold $D^4 \times \mathcal{H}_1$ has non-vanishing second Stiefel-Whitney class we cannot lift the normal Gauss map to *BString* even though $\frac{p_1}{2}$ vanishes. As before, we resolve the problem by twisting with a vector bundle over $\mathcal{H}_1 \cup_\rho e^8$.

By Section 2.2 there exist line bundles $\widetilde{l_1}$ and $\widetilde{l_2}$ over $\mathcal{H}_1$ such that $\widetilde{l_1} \oplus \widetilde{l_2} \cong T\mathcal{H}_1$. Since the inclusion $\iota \colon \mathcal{H}_1 \to \mathcal{H}_1 \cup e^8$ induces an isomorphism $\iota^* \colon H^2(\mathcal{H}_1 \cup e^8) \to H^2(\mathcal{H}_1)$ there exist l_1 and l_2 such that $\iota^* l_i = \widetilde{l_i}$, i.e. $\iota^*(l_1 \oplus l_2) \cong T\mathcal{H}_1$.

Recall that, by the definition of twisted bordism in Section 3.3, we need that the twisting

bundle is of finite rank. Here, the twisting bundle will be the sum of $-l_1$ and $-l_2$. Hence, we want that $-l_1$ and $-l_2$ are of finite rank. We claim that they are:
The Chern classes $c_1(l_1)$ and $c_1(l_2)$ determine maps to $\mathbb{C}P^\infty$ which are unique up to homotopy. After making them cellular, we obtain maps to some finite $\mathbb{C}P^n$ since $\mathcal{H}_1 \cup e^8$ is a finite dimensional complex.
Over $\mathbb{C}P^n$ there exists an additive inverse to the canonical line bundle $\gamma \to \mathbb{C}P^n$ given by the perpendicular bundle with total space $\gamma^\perp = \left\{(z,v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | v^\perp \in z\right\}$ and projection $(z,v) \mapsto z$. Thus, we can pull back $\gamma^\perp$ along the maps determined by $c_1(l_1)$ and $c_1(l_2)$ and obtain inverse bundles $-l_1$ and $-l_2$ of rank n.

Lemma 5.8. *Let $pt\colon D^4 \times \mathcal{H}_1 \to BString$ denote the constant map and, furthermore, let $\mathbb{B} := \mathcal{H}_1 \cup e^8 \widetilde{\times} BString$. Consider the following fibration over BO:*

$$\mathbb{B} \xrightarrow{-(l_1 \oplus l_2) \times p_{Str}} BO \times BO \xrightarrow{\oplus} BO .$$

Then $i_1 \times pt$ and $i_2 \times pt$ are normal six-smoothings $D^4 \times \mathcal{H}_1 \to \mathbb{B}$.
Under restriction of all maps to $S^3 \times \mathcal{H}_1$ the diagram

$$\begin{array}{ccc} & \mathcal{H}_1 \cup e^8 \times BString & \\ {\scriptstyle i_1 \times pt} \nearrow & & \nwarrow {\scriptstyle i_2 \times pt} \\ S^3 \times \mathcal{H}_1 & \xrightarrow{h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h} & S^3 \times \mathcal{H}_1 \end{array}$$

commutes up to homotopy.

Proof. By Lemma 5.7 and the fact that $\pi_i(BString) = 0$ for all $i \leq 7$ we know that the maps are seven-connected. Therefore, it only remains to show that they really are lifts of the stable normal Gauss map.
The pullback $i_2^*(l_1 \oplus l_2) \cong (\mathbb{1}_{D^4} \times f)^* pr^* \iota^*(l_1 \oplus l_2) \cong (\mathbb{1}_{D^4} \times f)^* pr^*(T\mathcal{H}_1)$ is isomorphic to $T\mathcal{H}_1$ since f is a diffeomorphism on $\mathcal{H}_1$. Since $h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h$ is a diffeomorphism, a bundle isomorphic to the tangent bundle pulls back to a bundle that is isomorphic to the tangent bundle again.
Since $l_1 \oplus l_2$ pulls back to the tangent bundle, its inverse $-(l_1 \oplus l_2)$ pulls back to the stable normal bundle. □

Therefore the, maps $i_1 \times pt$ and $i_2 \times pt$ are the normal three-smoothings $\widetilde{\nu}_1$ and $\widetilde{\nu}_2$ of Theorem 5.2.

By Corollary 3.12 the diffeomorphism on $B_4|_{Pl}$ admits an extension over $D_4 \times \mathcal{H}_1$ if the element induced by $Y := D^4 \times \mathcal{H}_1 \cup_{h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h} D^4 \times \mathcal{H}_1$ together with the map

$$\widetilde{\nu}_1 \cup \widetilde{\nu}_2 \colon Y \to \mathcal{H}_1 \cup e^8 \times BString$$

is trivial in the twisted bordism group $\Omega_8^{String}(\mathcal{H}_1 \cup_\rho e^8, -(l_1 \oplus l_2))$.
From now on we will denote the bundle $-(l_1 \oplus l_2)$ by E and $\widetilde{\nu}_1 \cup \widetilde{\nu}_2$ by $\widetilde{\nu}$.

5.4. The twisted bordism group $\Omega_8^{String}(\mathcal{H}_1 \cup_\rho e^8, E)$

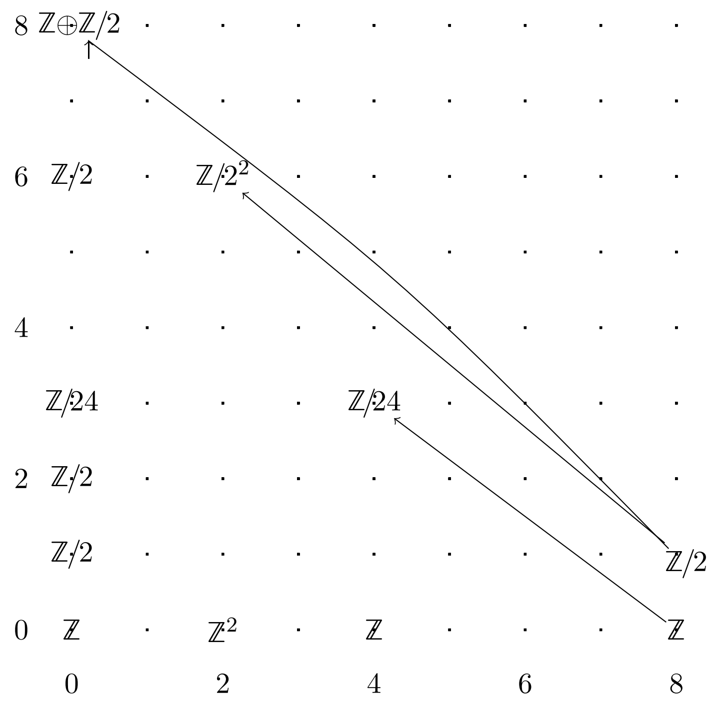
For the proof of Theorem 5.2, we calculate the twisted bordism group $\Omega_8^{String}(\mathcal{H}_1 \cup_\rho e^8, E)$ which is the goal of this section. In the next section, we develop invariants which detect whether $[Y, \widetilde{\nu}]$ vanishes in $\Omega_8^{String}(\mathcal{H}_1 \cup_\rho e^8, E)$.

We first use the twisted Atiyah-Hirzebruch sequence to check that the only appearing torsion is two-torsion. Then we apply the Adams spectral sequence to calculate the twisted bordism group.

Recall that the E^2-page of the twisted Atiyah-Hirzebruch spectral sequence is given by $E^2_{p,q} \cong H_p(\mathcal{H}_1 \cup_\rho e^8; \Omega_q^{String}(pt))$. The non-vanishing homology of $\mathcal{H}_1 \cup e^8$ is

i	0	2	4	8
$H_i(\mathcal{H}_1 \cup e^8; \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$

For the coefficients we refer the reader to page 48. Thus, we can write down the E^2-page of the twisted Atiyah-Hirzebruch spectral sequence converging to $\Omega_8^{String}(\mathcal{H}_1 \cup_\rho e^8, E)$. We only depict the coefficients and the seventh, eighth and ninth diagonal. Furthermore, we depict all possibly non-trivial differentials which have domain in, or target on the eights diagonal. For the sake of brevity we place all differentials in one diagram even though they are differentials of different pages in the twisted Atiyah-Hirzebruch spectral sequence.



As claimed the only possible torsion in $\Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$ is torsion at the prime two. But we cannot determine the group since we cannot determine the differentials.

Let $Th(E)$ denote the Thom space of E. Since $\operatorname{rank} E = 2n$ the twisted bordism group $\Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$ is isomorphic to $\pi_{8+2n}^{st}(Th(E) \wedge MString)$. By the consideration above it suffices to consider the Adams spectral sequence with E_2-page

$$E_2^{s,t+s} = Ext_{\mathcal{A}}^{s,t+s}(H^*(Th(E) \wedge MString;\ \mathbb{Z}/2), \mathbb{Z}/2)$$

converging to $\pi_t^{st}(Th(E) \wedge MString)$.

We use the method of minimal resolutions as developed in Section 6 of [Sto85]. More precisely we use Bruner's computer algorithm (cf. [Bru93] and [Bru]) which implements the method of minimal resolutions. The input for the algorithm is the Steenrod module structure of $H^*(Th(E) \wedge MString;\ \mathbb{Z}/2)$ for $* \leq 10 + 2n$. This range for $*$ suffices by the method of minimal resolutions since we are only interested in $\pi_{8+2n}^{st}(Th(E) \wedge MString)$.

Next, we determine the Steenrod module structure on $H^*(\mathcal{H}_1 \cup e^8; \mathbb{Z}/2)$.
We denote by $\widetilde{y}_i,\ i = 1, 2$ those generators in $H^2(\mathcal{H}_1 \cup e^8; \mathbb{Z}/2)$ which fulfil $\iota^*(\widetilde{y}_i) = y_i$, for $y_i,\ i = 1, 2$, the generators of the bundle base of $H^2(\mathcal{H}_1; \mathbb{Z}/2)$. By naturality and since $\iota\colon \mathcal{H}_1 \to \mathcal{H}_1 \cup e^8$ induces an isomorphism on the four lowest cohomology groups we know that $Sq^2\widetilde{y}_i = \widetilde{y}_i^2 = (i-1)\widetilde{y}_1\widetilde{y}_2$ for $i = 1, 2$. For dimension reasons $Sq^6(\widetilde{y}_i)$ vanishes. The generators in fourth cohomology are products, namely $\widetilde{y}_1\widetilde{y}_2$. Therefore, we know that $Sq^4(\widetilde{y}_1\widetilde{y}_2) = 0$.
The reduced cohomology of the Thom space $Th(E)$ is determined by the Thom isomorphism, i.e. each class $x \in H^*(\mathcal{H}_1 \cup e^8; \mathbb{Z}/2)$ corresponds to a class xu, where u denotes the Thom class.
Let

$$Sq = \sum_i Sq^i \text{ and } w = \sum_i w_i$$

denote the total Steenrod square and the total Stiefel-Whitney class, respectively.
By Wu's formula (cf. [MS74] p.132) the Steenrod-operations on u are determined by the total Stiefel-Whitney classes of E, namely

$$Sq(u) = w \cup u.$$

The twisting bundle E is defined such that the pullback $\iota^*(-E) \cong T\mathcal{H}_1$. By Section 2.2 the total Stiefel-Whitney class of $T\mathcal{H}_1$ is $1 + y_1$. Thus, naturality determines the total Stiefel-Whitney class of $-E$ to be $w(-E) = 1 + \widetilde{y}_1$. Since $-E \oplus E$ is trivial $w(-E \oplus E) = 1$. Consequently, the total Stiefel-Whitney class of E is $w(E) = 1 + \widetilde{y}_1$. Hence, the total Steenrod square of $u \in H^*(Th(E); \mathbb{Z}/2)$ is

$$Sq(u) = u + \widetilde{y}_1 u.$$

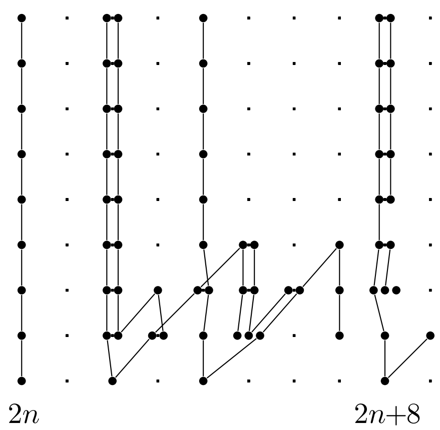
Let $\widetilde{y}_8$ be a generator of $H^8(\mathcal{H}_1 \cup e^8; \mathbb{Z}/2)$. We can calculate the total Steenrod squares on the other cohomology classes in $H^*(Th(E); \mathbb{Z}/2)$. They are:

$$\begin{aligned} Sq(u\widetilde{y}_1) &= Sq(u)Sq(y_1) = (u + u\widetilde{y}_1)\widetilde{y}_1 = u\widetilde{y}_1, \\ Sq(u\widetilde{y}_2) &= Sq(u)Sq(y_2) = (u + u\widetilde{y}_1)(\widetilde{y}_2 + \widetilde{y}_1\widetilde{y}_2) = u\widetilde{y}_2, \\ Sq(u\widetilde{y}_1\widetilde{y}_2) &= Sq(u)Sq(\widetilde{y}_1\widetilde{y}_2) = (u + u\widetilde{y}_1)\widetilde{y}_1\widetilde{y}_2 = u\widetilde{y}_1\widetilde{y}_2 \text{ and} \\ Sq(u\widetilde{y}_8) &= Sq(u)Sq(\widetilde{y}_8) = (u + u\widetilde{y}_1)\widetilde{y}_8 = u\widetilde{y}_8 \ . \end{aligned}$$

Therefore, we now know the Steenrod module structure of $H^*(Th(E); \mathbb{Z}/2)$.
The cohomology $H^*(MString; \mathbb{Z}/2)$, for ≤ 10, is generated by u_{Str}, the Thom class of $MString$ and $w_8 u_{str}$, where w_8 is the pullback of the universal Stiefel-Whitney class in BO to $BString$. Thus, $Squ_{Str} = u_{Str} + u_{Str}w_8$ in $H^*(MString, \mathbb{Z}/2)$ for $* \leq 10$.
We apply the Künneth theorem to calculate $H^*(Th(E) \wedge MString, \mathbb{Z}/2)$. There, we obtain the following non-trivial Steenrod-operations in $H^*(Th(E) \wedge MString; \mathbb{Z}/2)$ for $* \leq 10 + 2n$:

$$\begin{aligned} Sq(uu_{Str}) &= uu_{Str} + u\widetilde{y}_1 u_{Str} + uu_{Str}w_8 + u\widetilde{y}_1 u_{Str}w_8, \\ Sq(u\widetilde{y}_1 u_{Str}) &= u\widetilde{y}_1 u_{Str} + u\widetilde{y}_1 u_{Str}w_8, \\ Sq(u\widetilde{y}_2 u_{Str}) &= u\widetilde{y}_2 u_{Str} + u\widetilde{y}_2 u_{Str}w_8 \text{ and} \\ Sq(uu_{Str}w_8) &= uw_8 + u\widetilde{y}_1 w_8. \end{aligned}$$

We use Bruner's program and obtain the following E_2-page for the Adams spectral sequence. Again, we indicate the multiplicative structure on the E_2-page as in Example 6.19 of [Sto85].



By the multiplicative structure, there cannot appear any non-vanishing differentials, that kill the torsion, whence

$$\Omega_8^{String}(\mathcal{H}_1 \cup e^8) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2.$$

In particular, all differentials in the twisted Atiyah-Hirzebruch spectral sequence converging to $\Omega_8^{String}(\mathcal{H}_1 \cup e^8)$, as depicted on page 5.4, vanish.

5.5. Proof of Theorem 5.2

Now we are ready to prove Theorem 5.2. As already mentioned we want to apply Corollary 3.12. By Lemma 2.2 $\pi_4(D^4 \times \mathcal{H}_1)$ is finite. Furthermore, we have two normal three-smoothings $\widetilde{\nu}_0$ and $\widetilde{\nu}_1$ into the same fibration $\mathcal{H}_1 \cup e^8 \widetilde{\times} BString \to BO$ which are compatible with the diffeomorphism $n = h^{-1} \circ (\mathbb{1}_{Pl} \times f) \circ h$. Recall that Y denotes the manifold

$$D^4 \times \mathcal{H}_1 \cup_{h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h} D^4 \times \mathcal{H}_1$$

and $\widetilde{\nu}$ denotes the map into $\mathcal{H}_1 \cup e^8 \widetilde{\times} BString$ obtained from $\widetilde{\nu}_1$ and $\widetilde{\nu}_2$ constructed in the proof of Lemma 5.7. Corollary 3.12 implies that we can extend n over $D^4 \times \mathcal{H}_1$, i.e. that we can realise ϕ_1, if $\omega := [Y, \widetilde{\nu}]$ vanishes in $\Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$.
Hence, our goal is to find invariants that detect whether ω vanishes in $\Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$.

Proof. Let E^∞_{pq} denote the (p, q)-entry on the E^∞-page of the twisted Atiyah-Hirzebruch spectral sequence converging to $\Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$. By the calculation of Section 5.4 we know

$$\Omega_8^{String}(\mathcal{H}_1 \cup e^8, E) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus E^\infty_{80} \oplus E^\infty_{08}/tor,$$

where $E^\infty_{80} \cong \mathbb{Z}$ and $E^\infty_{08} \cong \mathbb{Z} \oplus \mathbb{Z}/2$. The $\mathbb{Z}/2$ summand in E^∞_{08} extends one of the $\mathbb{Z}/2$ summands of $E^\infty_{26} \cong \mathbb{Z}/2^2$ non-trivially and gives rise to the $\mathbb{Z}/4$ summand. We start by considering the two integral summands of $\Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$, i.e. we consider $\mathbb{Z} \subset E^\infty_{08}$ and E^∞_{80}.

To show that ω vanishes on the $\mathbb{Z} \subset E^\infty_{08}$ we compare *String*-bordism and oriented bordism as in the proof of Theorem 4.2.
Let $pr_{Str} : BString \to BSO$ denote the projection and let p denote the composition

$$\mathcal{H}_1 \cup e^8 \times BString \xrightarrow{E \times p_{Str}} BSO \times BSO \xrightarrow{\oplus} BSO.$$

Consider the map $\Omega_8^{String}(pt) \xrightarrow{j} \Omega_8^{String}(\mathcal{H}_1 \cup e^8; E)$ induced by the inclusion of a point $pt \to \mathcal{H}_1 \cup e^8$. Recall that composing j with the map p induces the map

$$(p_8)_* : \underbrace{\Omega_8^{String}(pt)}_{\mathbb{Z} \oplus \mathbb{Z}/2} \to \underbrace{\Omega_8^{SO}(pt)}_{\mathbb{Z} \oplus \mathbb{Z}}.$$

In dimension eight this map is well-known to have kernel $\mathbb{Z}/2$. In particular, the composition

$$\mathbb{Z} \hookrightarrow \Omega_8^{String}(pt) \cong E^\infty_{08} \hookrightarrow \Omega_8^{String}(\mathcal{H}_1 \cup e^8; E) \to \Omega_8^{SO}(pt)$$

is injective. A manifold induces the trivial element in $\Omega_8^{SO}(pt)$ if its Pontrjagin numbers vanish. Consequently, the element $\omega \in \Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$ vanishes on $\mathbb{Z} \subset E_{08}^{\infty}$ if the Pontrjagin numbers of the underlying manifold Y vanish.

By construction $h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h|_{S^3 \times \mathcal{H}_1} : S^3 \times \mathcal{H}_1 \to S^3 \times \mathcal{H}_1$ is a fibre bundle map. Consequently, the manifold Y is the total space of a fibre bundle over S^4 with fibre $\mathcal{H}_1$. We denote the bundle projection by $\pi : Y \to S^4$ and the inclusion of the fibre by $incl : \mathcal{H}_1 \to Y$.
To determine the Pontrjagin numbers we calculate $H^*(Y)$. We use the cohomological Leray-Serre spectral sequence to determine the integral cohomology of the total space of the fiber bundle $\mathcal{H}_1 \to Y \to S^4$. Its E_2-page is given by $E_2^{pq} = H^p(\mathcal{H}_1; H^q(S^4))$, i.e. we obtain

$$\begin{array}{cccccc}
4 & \mathbb{Z} & \cdot & \cdot & \cdot & \mathbb{Z} \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \mathbb{Z}^2 & \cdot & \cdot & \cdot & \mathbb{Z}^2 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \mathbb{Z} & \cdot & \cdot & \cdot & \mathbb{Z} \\
 & 0 & & 2 & & 4
\end{array} .$$

Obviously, there are no non-trivial differentials. In particular, we obtain $H^4(Y) \cong \mathbb{Z}^2$. Using the edge homomorphisms we see that $H^4(Y)$ is generated by two classes a, b, where $a = \pi^*(s)$ and $incl^*(b)$ is a generator of $H^4(\mathcal{H}_1)$.
We turn to the Pontrjagin numbers. The tangent bundle of Y decomposes into the direct sum $TY \cong \pi^* TS^4 \oplus T_{fib}Y$. Since $p_1(S^4) = 0$ the first Pontrjagin class of the total space is$p_1(TY) = p_1(T_{fib}Y)$.
The pullback $incl^* T_{fib}Y$ is isomorphic to $T\mathcal{H}_1$ whose first Pontrjagin class $p_1(T\mathcal{H}_1)$ vanishes (cf. Section 2.2). By naturality $incl^*(p_1(T_{fib}Y))$ vanishes, too. Thus, $p_1(T_{fib}Y)$ is a multiple of $\pi^*(s)$. But the square $\pi^*(s) \cup \pi^*(s) = \pi^*(s \cup s)$ vanishes. This implies that the Pontrjagin number

$$p_{(1,1)}(Y) := \langle p_1(Y) \cup p_1(Y), [Y] \rangle = 0.$$

The only other Pontrjagin number in dimension eight is $\langle p_2(Y), [Y] \rangle =: p_{(2)}(Y)$. To show that it also vanishes we use the following statement.
Let $F \to E \to B$ be a fibre bundle with F, E and B connected and compact and $\pi_1(B) = 0$. By [CHS57] the signature of the total space of such a fibre bundle is multiplicative, i.e. $\sigma(E) = \sigma(B) \cdot \sigma(F)$. Applied to our bundle this implies $\sigma(Y) = \sigma(S^4) \cdot \sigma(\mathcal{H}_1) = 0$. Let $p_{(2)}(Y)$ denote $\langle p_2(Y), [Y] \rangle$. By Hirzebruch's signature theorem

$$\frac{1}{45} \left(-p_{(1,1)}(Y) + 7 p_{(2)}(Y) \right) = \sigma(Y) = 0.$$

Hence, $p_{(2)}(Y)$ is trivial, too. Consequently, the element ω vanishes on the first integral summand E^∞_{08}/tor.

Now we use the Thom homomorphism $\mathcal{T}\colon \Omega_8^{String}(\mathcal{H}_1 \cup e^8, E) \to H_8(\mathcal{H}_1 \cup e^8)$ to show that ω vanishes in the second integral summand E^∞_{80}. The Thom homomorphism admits two descriptions; it is the edge homomorphism in the Atiyah-Hirzebruch spectral sequence and it has a geometric descrition.
By its description as the edge homomorphism of the Atiyah-Hirzebruch spectral sequence E^∞_{80} injects into $H_8(\mathcal{H}_1 \cup e^8)$, i.e. ω vanishes on E^∞_{80} if $T(\omega)$ vanishes in $H_8(\mathcal{H}_1 \cup e^8)$.
To determine $\mathcal{T}(\omega)$ we use the geometric description. Consider an element $[M, f_1 \times f_2]$ in $\Omega_8^{String}(\mathcal{H}_1 \cup e^8; E)$, where $f_1\colon M \to \mathcal{H}_1 \cup e^8$. The image of $[M, f_1 \times f_2]$ under the Thom homomorphism is $(f_1)_*[M]$.

By definition $\widetilde{\nu} = (i_1 \times pt) \cup (i_2 \times pt) =: i \times pt$ for i_1 and i_2 as defined in Lemma 5.7, i.e. $i_1 = \iota \circ pr$ and $i_2 = \iota \circ pr \circ (\mathbb{1}_{D^4} \times f)$. To show that $i_*([Y]) = 0 \in H_8(\mathcal{H}_1 \cup e^8)$ we apply the Mayer-Vietoris Sequence.
By the Leray-Serre spectral sequence above the group $H^7(Y)$ vanishes. Since Y has vanishing $7-$th homology we can find a CW-structure on Y such that $Y \simeq Y^{(6)} \cup \widetilde{e}^8$, where we denote the eight-cell with a tilde to distinguish it from the eight-cell we attach to $\mathcal{H}_1$. We now compare the Mayer-Vietoris sequences of Y and $\mathcal{H}_1 \cup e^8$:

$$\begin{array}{ccccccc}
H_8(Y^{(6)}) \oplus H_8(\widetilde{e}^8) & \longrightarrow & H_8(Y) & \longrightarrow & H_7(Y^{(6)} \cap \widetilde{e}^8) & \longrightarrow & H_7(Y^{(6)}) \oplus H^7(\widetilde{e}^8) \\
\downarrow & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle (i|_{Y^{(6)}\cap\widetilde{e}^8})_*} & & \downarrow \\
H_8(\mathcal{H}_1) \oplus H_8(e^8) & \longrightarrow & H_8(\mathcal{H}_1 \cup e^8) & \longrightarrow & H_7(\mathcal{H}_1 \cap e^8) & \longrightarrow & H_7(\mathcal{H}_1) \oplus H^7(e^8)
\end{array}$$

The left- and right-most entries vanish for dimension reasons. Therefore, we are left with the middle square in which the horizontal arrows are isomorphisms. But the domain of $i|_{Y^{(6)}\cap\widetilde{e}^8}$ is a subset of the six-skeleton. By Equation (13) in the proof of Lemma 5.7 we know

$$pr \circ h^{-1} \circ (\mathbb{1}_{S^3} \times f) \circ h|_{(S^3 \times \mathcal{H}_1)^{(6)}} \simeq pr \circ (\mathbb{1}_{S^3} \times f)|_{(S^3 \times \mathcal{H}_1)^{(6)}}.$$

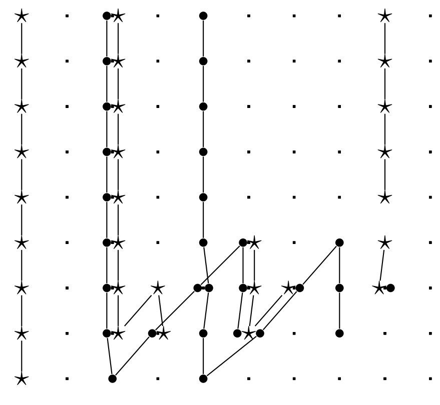
Thus, the map $(i|_{Y^{(6)}\cap\widetilde{e}^8})_*$ factors through $H_7(\mathcal{H}_1)$ which vanishes. Hence, the map $i_*\colon H_8(Y) \to H_8(\mathcal{H}_1 \cup e^8)$ vanishes identically.

To finish the proof we now need to determine the invariants a_1 and a_2 on the torsion subgroup of $\Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$.

First of all note that the inclusion $\iota\colon \mathcal{H}_1 \to \mathcal{H}_1 \cup e^8$ induces a homomorphism

$$\Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1) \xrightarrow{\iota_*} \Omega_8^{String}(\mathcal{H}_1 \cup e^8, E).$$

We claim that ι_* is injective. This follows from comparing the E^2-pages of the Atiyah-Hirzebruch spectral sequences converging to $\Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1)$ and $\Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$, respectively, by the induced map ι_* on homology with coefficients. The E^2-page of the spectral sequence converging to $\Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1)$ differs from the one depicted on page 85 only by the E^2_{80}-entry which vanishes in this case. To solve the extension problem we also repeat the Adams spectral sequence computation and obtain the E_2-page below, where $\bullet$ and $\star$ depict a $\mathbb{Z}/2$. We need the distinction later on. Furthermore the labels for the columns indicate the degree of the twisted bordism group which differs from the degree of the stable homotopy group by the rank n of the twisting bundle since $\Omega_k(\mathcal{H}_1, -T\mathcal{H}_1) \cong \pi^{st}_{k+n}(Th(E) \wedge MString)$.



Therefore, we can deduce

$$\Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4.$$

On homology the map $\iota_*\colon \mathcal{H}_1 \to \mathcal{H}_1 \cup e^8$ is injective. All entries E^2_{pq} with $p+q=8$ on the E^2-page of the twisted Atiyah-Hirzebruch spectral sequence survive to E^∞, by the Adams spectral sequence calculation. Thus, the map on bordism groups is also injective. In particular, the image of ι_* is $\Omega_8^{String}(\mathcal{H}_1 \cup e^8)/E^\infty_{80}$. By the Thom homomorphism we already know that ω vanishes on $E^\infty_{80} \cong H_8(\mathcal{H}_1 \cup e^8)$. Hence, it is contained in $\mathrm{im}\iota_*$. Consequently, it suffices to find invariants for the torsion subgroup of $\Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1)$.

We consider two maps into $\mathcal{H}_1$. The inclusion of the fiber $i_2 : \mathbb{C}\mathrm{P}^1 \to \mathcal{H}_1$ and the section from the base space $\sigma_2 : \mathbb{C}\mathrm{P}^1 \to \mathcal{H}_1$. By the properties of the tautological bundles over a Bott manifold (cf. Section 2.1 Equation (3)) we obtain

$$\sigma_2^*(T\mathcal{H}_1) \cong \gamma_1^{-2} \oplus \gamma_1^{-1}.$$

Over $\mathbb{C}\mathrm{P}^1_1 \cong S^2$ there exist only two stable real vector bundles since $\pi_2(BO) \cong \mathbb{Z}/2$. The non-trivial stable vector bundle is induced by the tautological bundle γ_1. Its square γ_1^2 is stably trivial. Therefore, $\gamma_1^{-2} \oplus \gamma_1^{-1}$ is isomorphic to γ_1 as a stable real vector bundle.

Consequently, the pullback of $-T\mathcal{H}_1$ is isomorphic to the inverse of γ_1. But that is again stably isomorphic to γ_1.
Since $i_2^*(T\mathcal{H}_1) \cong T\mathbb{C}\mathrm{P}_2^1$ which, as a real vector bundle, is stably trivial, $i_2^*(-T\mathcal{H}_1)$ is trivial, too. Consequently, the maps σ_2 and i_2 induce homomorphisms

$$\begin{aligned}(\sigma_2)_*\colon \Omega_8^{String}(\mathbb{C}\mathrm{P}_1^1, \gamma_1) &\to \Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1) \text{ and}\\ (i_2)_*\colon \Omega_8^{String}(\mathbb{C}\mathrm{P}_2^1) &\to \Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1).\end{aligned}$$

We denote reduced bordism groups by a tilde. We know

$$\Omega_8^{String}(\mathbb{C}\mathrm{P}_2^1) \cong \Omega_8^{String}(pt) \oplus \widetilde{\Omega}_8^{String}(S^2) \cong \Omega_8^{String}(pt) \oplus \Omega_6^{String}(pt) \cong \Omega_8^{String}(pt) \oplus \mathbb{Z}/2.$$

We obtain $\Omega_8^{String}(\mathbb{C}\mathrm{P}_1^1, \gamma_1) \cong \widetilde{\Omega}_{10}^{String}(\mathbb{C}P^2) \cong \mathbb{Z} \oplus \mathbb{Z}/4 \cong \pi_{10}^{st}(\mathbb{C}P^2 \wedge MString)$ by another Adams spectral sequence calculation (The E_2-page is given by the $\star$-entries in the Adams spectral sequence above). Then we compare the Atiyah-Hirzebruch spectral sequences, using that σ_2 and i_2 induce injective maps on homology. Furthermore,

$$\begin{aligned}\mathrm{im}\left((i_2)_*\colon H_2(\mathbb{C}\mathrm{P}_2^1;\mathbb{Z}/2) \to H_2(\mathcal{H}_1;\mathbb{Z}/2)\right) \oplus \mathrm{im}\left((\sigma_2)_*\colon H_2(\mathbb{C}\mathrm{P}_1^1;\mathbb{Z}/2)\right) \to H_2(\mathcal{H}_1;\mathbb{Z}/2))\\ = H_2(\mathcal{H}_1;\mathbb{Z}/2).\end{aligned}$$

All entries on the (twisted) Atiyah spectral sequences converging to $\Omega_8^{String}(\mathbb{C}\mathrm{P}_1^1, \gamma_1)$, $\Omega_8^{String}(\mathbb{C}\mathrm{P}_2^1)$ and $\Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1)$ survive to E^∞. We claim that

$$\Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1) \cong \widetilde{\Omega}_8^{String}(\mathbb{C}\mathrm{P}_2^1) \oplus \Omega_8^{String}(\mathbb{C}\mathrm{P}_1^1, \gamma_1).$$

Let $\pi\colon \mathcal{H}_1 \to \mathbb{C}\mathrm{P}_1^1$ denote the projection. That $\widetilde{\Omega}_8^{String}(\mathbb{C}\mathrm{P}_2^1)$ splits off follows, on the one hand, by considering the filtration groups of the spectral sequences converging to $\widetilde{\Omega}_8^{String}(\mathbb{C}\mathrm{P}_2^1)$ and $\Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1)$ and, on the other hand, by considering the induced map π_* on $\mathrm{im}(i_2)*$ and on $\Omega_8^{String}(pt) \subset \Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1)$.

First, we consider elements in $\mathrm{im}((i_2)_*)$. By Lemma 3.18 we have an exact sequence of the form

$$\ldots \to \Omega_n^{String}(pt) \xrightarrow{i} \Omega_n^{String}(\mathbb{C}P^m, \xi) \xrightarrow{t} \Omega_{n-2}^{String}(\mathbb{C}P^{m-1}, \xi \oplus H) \xrightarrow{s} \Omega_{n-1}^{String}(pt) \to \ldots \quad .$$

Here, we are interested in the case where $m = 1$ and where ξ is the trivial bundle. It follows that

$$t : \widetilde{\Omega}_8^{String}(\mathbb{C}\mathrm{P}_2^1) \to \Omega_6^{String}(pt)$$

is an isomorphism. Furthermore, the non-trivial element in $\Omega_6^{String}(pt)$ is detected by the Arf-invariant (cf. Chapter 6 of [Lüc02] for a definition). We can define an invariant on $\Omega_8^{String}(\mathcal{H}_1, -T\mathcal{H}_1) \subset \Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$ by first projecting to $\Omega_8^{String}(\mathbb{C}\mathrm{P}_2^1)$, then applying t and finally using the Arf-invariant in $\Omega_6^{String}(pt)$. We denote this invariant by

$$a_1 : \Omega_8^{String}(\mathcal{H}_1 \cup e^8, E) \to \mathbb{Z}/2,$$

and call it a codimension two Arf-invariant.

It remains to find an invariant for $\Omega_8^{String}(\mathbb{C}\mathrm{P}_1^1, \sigma_2^*\iota^* E) \cong \Omega_8^{String}(\mathbb{C}\mathrm{P}_1^1, \gamma_1)$. This time we apply Lemma 3.18 to $m = 1$ and $\xi = \gamma_1$. We obtain an epimorphism

$$t\colon \Omega_8^{String}(\mathbb{C}\mathrm{P}_1^1, \gamma_1) \to \Omega_6^{String}(pt)$$

with kernel $\ker(t) = \mathrm{im}\left(\Omega_8^{String}(pt) \xrightarrow{i} \Omega_8^{String}(\mathbb{C}\mathrm{P}_1^1, \gamma_1)\right)$ and can again use the Arf-invariant on $\Omega_6^{String}(pt)$ to obtain a second codimension two Arf-invariant

$$a_2\colon \Omega_8^{String}(\mathcal{H}_1 \cup e^8, E) \to \mathbb{Z}/2.$$

Here, we project to $\Omega_8^{String}(\mathbb{C}\mathrm{P}_1^1, \sigma_2^*\iota^* E)$, then apply t to map to $\Omega_6^{String}(pt)$ and take the Arf-invariant there.

We already showed that ω vanishes on $\mathbb{Z} \subset \ker(t)$ using the Pontrjagin numbers. By Lemma 5.1 the generator of the finite subgroup of $\ker(t)$ it the exotic eight sphere. To my knowledge there is no invariant that detects the exotic eight sphere in $\Omega_8^{String}(pt)$. We denote its image in $\Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$ by θ_8.
Thus, we come to the last part of the proof. If the codimension two Arf-invariants vanish on $\omega = [Y, i]$, then $\omega = 0$ if $\omega \neq \theta_8$. Thus, we obtain the final condition of the theorem $\omega \neq \theta_8 \in \Omega_8^{String}(\mathcal{H}_1 \cup e^8, E)$. □

A. The cohomology of $\mathbb{P}_3 B_4$

In this appendix we calculate $H^k(\mathbb{P}_3 B_4; \mathbb{Z}/2)$ together with its product and Steenrod module structure for $k \leq 10$. The extension problem for integral cohomology will follow from the calculation of $\mathbb{Z}/2$-cohomology.

Recall that $\mathcal{H}_1$ denotes the non-trivial Hirzebruch surface which is a four-dimensional Bott manifold B_2. In Section 5.3 we need the Steenrod module structure of $\mathbb{P}_3\mathcal{H}_1$ which is the total space of a fibration $K_2 := K(\mathbb{Z}^2, 3) \to \mathbb{P}_3\mathcal{H}_1 \to \mathbb{P}_2\mathcal{H}_1 \simeq (\mathbb{C}P^\infty)^2$. The calculation is completely analogous to the one for $\mathbb{P}_3 B_4$, only less tedious since the number of generators is smaller. We will not write down the calculations, only the intermediate steps and results.

We start with the calculation of $H^k(\mathbb{P}_3 B_4; \mathbb{Z}/2)$ for $k \leq 10$. Since $\mathbb{P}_3 B_4$ is the total space of the fibration $K_4 := K(\mathbb{Z}^4, 3) \to \mathbb{P}_3 B_4 \to \mathbb{P}_2 B_4 \simeq (\mathbb{C}P^\infty)^4$, we can use the cohomological Leray-Serre spectral sequence with coefficients in $\mathbb{Z}/2$, i.e.

$$E_2^{pq} = H^p(\mathbb{P}_2 B_4; H^q(K_4; \mathbb{Z}/2)) \Rightarrow H^{p+q}(\mathbb{P}_3 B_4; \mathbb{Z}/2).$$

Since the integral cohomology $H^q(\mathbb{P}_2 B_4)$ is torsion-free and finitely generated, the E_2-page simplifies to $E_2^{pq} = H^p(\mathbb{P}_2 B_4) \otimes H^q(K_4; \mathbb{Z}/2)$ by the universal coefficient theorem. The $\mathbb{Z}/2$-cohomology of $K(\mathbb{Z}, 3)$ is very well-understood (cf. [McC01], Theorem 6.19). We abbreviate Steenrod operations $Sq^i Sq^j$ by Sq^{ij}. For $i \leq 10$ the non-vanishing groups $H^i(K(\mathbb{Z}, 3); \mathbb{Z}/2)$ and their generators are

i	0	3	5	6	8	9	10
$H^i(K(\mathbb{Z},3);\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
generators	1	ι	$Sq^2\iota$	ι^2	$\iota \cup Sq^2\iota$	$Sq^{42}\iota$	$Sq^2\iota \cup Sq^2\iota$

Since $K(\mathbb{Z}, 3)^4 \simeq K(\mathbb{Z}^4, 3) = K_4$ and $K(\mathbb{Z}, 3)^2 \simeq K(\mathbb{Z}^2, 3) = K_2$, respectively, we can apply the Künneth theorem to obtain the $\mathbb{Z}/2$-cohomology of the product. In the following table $1 \leq l \leq m \leq n \leq 4$, $1 \leq i, j \leq 4$ and $1 \leq r \leq s \leq t \leq 2$, $1 \leq a, b \leq 2$ and we suppress the cup products from notation.

i	0	3	5	6	8	9	10
$H^i(K_2;\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2^2$	$\mathbb{Z}/2^2$	$\mathbb{Z}/2^3$	$\mathbb{Z}/2^4$	$\mathbb{Z}/2^6$	$\mathbb{Z}/2^4$
generators	1	ι_r	$Sq^2\iota_r$	$\iota_r\iota_s$	$\iota_a Sq^2\iota_b$	$Sq^{42}\iota_r, \iota_r\iota_s\iota_t$	$Sq^2\iota_r Sq^2\iota_s$
$H^i(K_4;\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2^4$	$\mathbb{Z}/2^4$	$\mathbb{Z}/2^{10}$	$\mathbb{Z}/2^{16}$	$\mathbb{Z}/2^{24}$	$\mathbb{Z}/2^{10}$
generators	1	ι_l	$Sq^2\iota_l$	$\iota_l\iota_m$	$\iota_i Sq^2\iota_j$	$Sq^{42}\iota_m, \iota_l\iota_m\iota_n$	$Sq^2\iota_l Sq^2\iota_m$

We denote the generators of $H^i(\mathbb{P}_2 B_4; \mathbb{Z}/2)$ by a_j for $1 \leq j \leq 4$. Since K_4 is 2-connected the first non-trivial differential in the integral Leray-Serre spectral sequence of the fibration $K_4 \to \mathbb{P}_3 B_4 \to \mathbb{P}_2 B_4$ is $d_4^{\mathbb{Z}} : E_4^{03} \to E_4^{40}$. Recall that we already determined this differential $d_4^{\mathbb{Z}} : E_4^{03} \to E_4^{40}$ via the k-invariant and Proposition 3.4. For $1 \leq i < j$ and

$2 \leq j \leq 4$ let A^i_j denote the integers that determine a Bott tower of height four (cf. Section 2.1) as before. If z_j, $1 \leq j \leq 4$ denotes a basis of $H^4(K_4)$, the differential is

$$d_4^{\mathbb{Z}}(z_j) = a_j^2 - \sum_{i<j} A^i_j a_i a_j.$$

Furthermore, $d_4^{\mathbb{Z}} : E_4^{03} \to E_4^{40}$ can be identified with the transgression (cf. [McC01] Theorem 6.8). The integral transgression determines the $\mathbb{Z}/2$-transgression which then determines the $\mathbb{Z}/2$-differential $d_4^{\mathbb{Z}/2} : E_4^{03} \to E_4^{40}$. Thus,

$$\begin{aligned} d_4^{\mathbb{Z}/2}(\iota_j) &= a_j^2 - \sum_{i<j} A^i_j a_i a_j \mod 2 \\ &=: a_j^2 + \widetilde{\alpha}_j a_j. \end{aligned}$$

By the Leibniz-formula this determines differentials on all products of i_l or i_l and a_j.

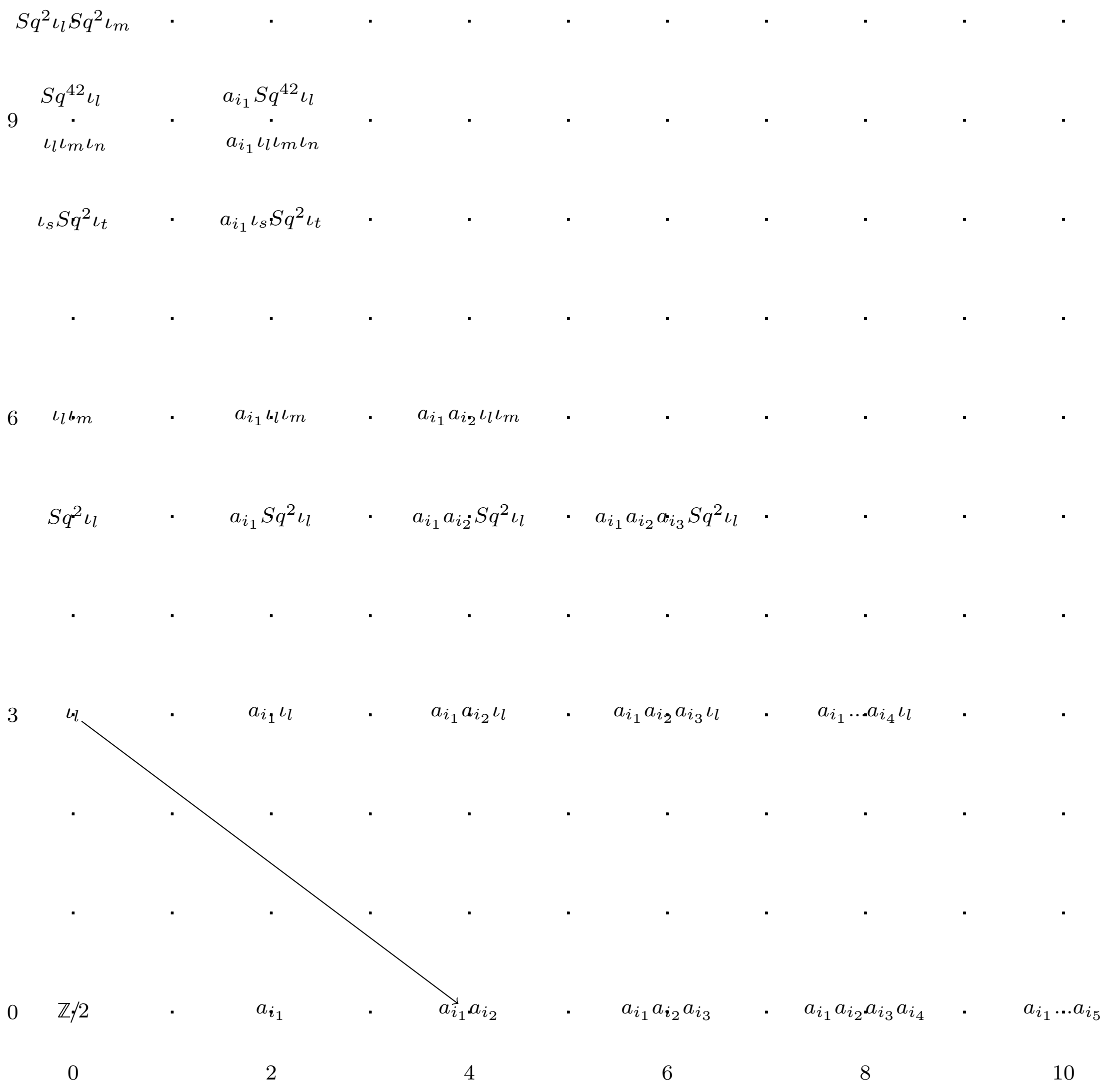
It remains to determine the differentials on $Sq^2\iota_l$ and $Sq^{42}\iota_l$. To understand this calculations more easily, take a look at the E_4- page of the cohomological Leray-Serre spectral sequence converging to $H^*(\mathbb{P}_3 B_4; \mathbb{Z}/2)$ on the next page.
The only differential $d_j \colon E_j^{05} \to E_j^{j,5-j}$ that can be non-trivial is $d_6 \colon E_6^{05} \to E_6^{06}$. In particular, we can identify E_6^{05} with $H^5(K_4; \mathbb{Z}/2)$. By Corollary 6.9 in [McC01] we know $d_6(Sq^2\iota_l) = Sq^2 \circ d_4(\iota_l) \in E_6^{60} = H^6(P_2; \mathbb{Z}/2)/\mathrm{im}(d_4)$. Since

$$Sq^2 \circ d_4(\iota_l) = Sq^2(a_l^2 + \widetilde{\alpha}_l a_l) = Sq^2(\widetilde{\alpha}_2) a_l + \alpha_l Sq^2 a_l = \widetilde{\alpha}_l(\widetilde{\alpha}_l a_l + a_l^2) \in \mathrm{im}(d_4 \colon E_4^{23} \to E_4^{60})$$

the differential $d_6(Sq^2\iota_l)$ vanishes for all $1 \leq l \leq 4$.

Thus, it only remains to determine the differential on $Sq^{42}\iota_l \in E_j^{09}$. We use [Ara57], where the idea of Steenrod operations on cohomology is extended to Steenrod operations on the Leray-Serre spectral sequence.
Let $F \to E \to B$ be a fibration with connected fibre and simply connected base space. Then, on the E_2-page, the Steenrod operations on the spectral sequence, as defined in [Ara57], coincide with the ordinary Steenrod operations of the fibre and base on E_2^{0q} and E_2^{p0}, respectively. Furthermore, the Steenrod operations on the spectral sequence commute, in some sense, with the differentials (see [Ara57] p.89/90).
This enables us to determine $d_j(Sq^{42}\iota_l) = 0$ for all $2 \leq j \leq 10$, i.e. $Sq^{42}\iota_l$ survives to E_∞.

Next we write down the E_2-page for $K_4 \to \mathbb{P}_3 B_4 \to \mathbb{P}_2 B_4$ with $\mathbb{Z}/2$-coefficients and entries E_2^{pq} with $q \leq 10$ and $p+q \leq 11$. Instead of writing down the groups E_2^{pq}, we write down a basis for each entry. Let $1 \leq l \leq m \leq n \leq 4$, $1 \leq i_1 \leq ... \leq i_5 \leq 4$ and $1 \leq s, t \leq 4$. Since the first non-trivial differentials appear on the E_4-page, the E_2-page and the E_4-page agree. Therefore, we indicate the d_4-differential which determines all other differentials.



Note that the E_4-page of the cohomological Leray-Serre spectral sequence with coefficients in $\mathbb{Z}/2$, which converges to $H^*(\mathbb{P}_3\mathcal{H}_1;\mathbb{Z}/2)$ looks almost the same. The only difference is, that all indices run between 1 and 2 and not between 1 and 4.

Combining all our knowledge on the differentials, we obtain E_∞^{pq} for all p, q with $q \leq 11$ and $p + q \leq 10$. Again we write down a basis for each entry E_∞^{pq}. We only depict rows which are non-empty in our range.

On the E_∞-page, let the indices be given by $1 \leq l \leq m \leq 4$ and $1 \leq i_1 < i_2 < i_3 \leq 4$.

10	$Sq^2\iota_l Sq^2\iota_m$	·	·	·	·	·	·	·	·
9	$Sq^{42}\iota_l$	·	·	·	·	·	·	·	·
6	ι_l^2	·	$a_{i_1}\iota_l^2$	·	$a_{i_1}a_{i_2}\iota_l^2$	·	·	·	·
5	$Sq^2\iota_l$	·	$a_{i_1}Sq^2\iota_l$	·	$a_{i_1}a_{i_2}Sq^2\iota_l$	·	·	·	·
0	$\mathbb{Z}/2$	·	a_{i_1}	·	$a_{i_1}a_{i_2}$	·	$a_{i_1}a_{i_2}a_{i_3}$	·	$a_1a_2a_3a_4$
	0		2		4		6		8

Again, the E_∞-page for $\mathbb{P}_3\mathcal{H}_1$ looks similar. The indices still run between 1 and 2. Since there cannot be indices fulfilling $1 \leq i_1 < i_2 < i_3 \leq 2$ the entries E_∞^{60} and E_∞^{80} vanish for $H^*(\mathbb{P}_3\mathcal{H}_1; \mathbb{Z}/2)$. From now on we abbreviate $\mathbb{P}_3\mathcal{H}_1$ by Q_3.

To determine generators of $H^*(P_3; \mathbb{Z}/2)$ and $H^*(Q_3; \mathbb{Z}/2)$, respectively, we take into account the cup product structure. For the rest of this calculation let $1 \leq i < j < h \leq 4$ and $1 \leq l, m \leq 4$ for P_3. For the statement of results the indices for Q_3 are $1 \leq i < j \leq 2$ and $1 \leq l, m \leq 2$.
We start by using that $p^*\colon H^*(P_2; \mathbb{Z}/2) \to H^*(P_3; \mathbb{Z}/2)$ is the edge homomorphism $H^p(P_2; \mathbb{Z}/2) \cong E_2^{p0} \twoheadrightarrow E_\infty^{p0} \hookrightarrow H^p(P_3; \mathbb{Z}/2)$. Thus, the elements $p^*(a_i)$ and $p^*(a_i)p^*(a_j)$ form a basis for $H^2(P_3, \mathbb{Z}/2)$ and $H^4(P_3; \mathbb{Z}/2)$, respectively. By the properties of d_4 we obtain $p^*(a_i)^2 = p^*(\widetilde{\alpha}_i a_i)$.
Furthermore, $p^*(a_i a_j a_h)$ and $p^*(a_1a_2a_3a_4)$ are generators for $H^6(P_3; \mathbb{Z}/2)$ and $H^8(P_3; \mathbb{Z}/2)$, respectively. The relation $p^*(a_i)^2 = p^*(\widetilde{\alpha}_i)p^*(a_i)$ determines all products of generators in the image of p^*. From now on we suppress the pullback p^* in the notation of the generators.
Let $k\colon K_4 \to P_3$ denote the inclusion of the fibre. There is a second edge homomorphism $H^q(K_4; \mathbb{Z}/2) \twoheadrightarrow E_\infty^{q0} \hookrightarrow E_2^{q0} \cong H^q(K_4; \mathbb{Z}/2)$ which equals k^*. Thus, there is a basis b_l, $1 \leq l \leq 4$ of $H^5(P_3; \mathbb{Z}/2)$ with the property that $k^*(b_l) = Sq^2\iota_l$.
The cup product structure on the cohomology of the total space $H^*(P_3; \mathbb{Z}/2)$ induces a cup product structure on E_∞^{**}. Thus, if there is a non-trivial cup product on the E_∞-page there must be a corresponding non-trivial cup product on $H^*(P_3; \mathbb{Z}/2)$.

The elements $a_i b_l$ form a basis for $H^7(P_3;\mathbb{Z}/2) \cong E_\infty^{25}$, since $E_\infty^{05} \times E_\infty^{20} \to E_\infty^{25}$ is an isomorphism.
By the edge homomorphism argument there exists generators $c_l \in H^6(P_3;\mathbb{Z}/2)$ which fulfil $k^*(c_l) = \iota_l^2$. We specify this generators subsequently. Together with $a_i a_j a_h$ they form a basis for $H^6(P_3;\mathbb{Z}/2)$. Furthermore, there again is an isomorphism $E_\infty^{06} \times E_\infty^{20} \to E_\infty^{26}$. Thus, $c_l a_i$, together with $a_1 a_2 a_3 a_4$ form a basis for $H^8(P_3;\mathbb{Z}/2)$.
In the end, we obtain the following table of non-trivial reduced cohomology groups and generators

i	2	4	5	6	7	8	9	10
$\widetilde{H}^i(Q_3;\mathbb{Z}/2)$	$\mathbb{Z}/2^2$	$\mathbb{Z}/2$	$\mathbb{Z}/2^2$	$\mathbb{Z}/2^2$	$\mathbb{Z}/2^4$	$\mathbb{Z}/2^4$	$\mathbb{Z}/2^6$	$\mathbb{Z}/2^5$
generators	a_i	$a_1 a_2$	b_l	c_l	$a_i b_l$	$a_i c_l,$	$a_1 a_2 b_l, e_l$	$a_1 a_2 c_l, b_l b_m$
$\widetilde{H}^i(P_3;\mathbb{Z}/2)$	$\mathbb{Z}/2^4$	$\mathbb{Z}/2^6$	$\mathbb{Z}/2^4$	$\mathbb{Z}/2^8$	$\mathbb{Z}/2^{16}$	$\mathbb{Z}/2^{17}$	$\mathbb{Z}/2^{28}$	$\mathbb{Z}/2^{34}$
generators	a_i	$a_i a_j$	b_l	$a_i a_j a_h, c_l$	$a_i b_l$	$a_i c_l, a_1 a_2 a_3 a_4$	$a_i a_j b_l, e_l$	$a_i a_j c_l, b_l b_m$

Here, e_l has the property that $k^* e_l = Sq^{42}\iota_l$. Note that all products are determined by the products indicated in the table and by $a_i^2 = \widetilde{\alpha}_i a_i$.

The calculation implies that the extension problems for the integral cohomology groups of degree less or equal ten are trivial. Recall that integrally $H^5(P_3) = 0$ and $H^6(P_3)$ is either $\mathbb{Z}^4$ or $\mathbb{Z}^4 \oplus \mathbb{Z}/2^4$ (cf. spectral sequence on page 47). For $H^5(P_3;\mathbb{Z}/2)$ to be $\mathbb{Z}/2^4$ the group $H^6(P_3)$ must split off $\mathbb{Z}/2^4$ by the universal coefficient theorem. Analogously $H^8(P_3)$ must split off $\mathbb{Z}/2^{16}$. Calculating $H^*(P_3;\mathbb{Z}/3)$ one can see, that $H^8(P_3)$ must also have a direct summand $\mathbb{Z}/3^4$.

Next we turn to the Steenrod module structure, in particular to the calculation of Sq^2. Let Sq denote the total Steenrod square. For any two elements x and y in cohomology with $\mathbb{Z}/2$-coefficients $Sq(xy) = Sq(x)Sq(y)$. Thus, it suffices to determine the Steenrod operations on each factor. Since $Sq^2(a_i) = a_i^2 = \widetilde{\alpha}_i a_i$ we have

$$Sq(a_i) = a_i + \widetilde{\alpha}_i a_i. \tag{14}$$

We turn to $b_l \in H^5(P_3;\mathbb{Z}/2)$ ($b_l \in H^5(Q_3;\mathbb{Z}/2)$, respectively). By naturality $k^*(Sq^1(b_l)) = Sq^1(k^* b_l) = Sq^1 Sq^2 \iota_l = \iota_l^2$. Hence, we can define $Sq^1 b_l =: c_l$. Analogously we obtain $Sq^4(b_l) =: e_l$ and $Sq^5(b_l) = b_l^2$.
Determining $Sq^2 b_l$ is slightly harder. Consider $\vee_4 S^2 := S^2 \vee S^2 \vee S^2 \vee S^2$ and $S^2 \vee S^2$. There are obvious maps $f_4\colon \vee_4 S^2 \to (\mathbb{C}P^\infty)^4 = P_2$ such that $f_4^*(a_i) =: \widetilde{a}_i$ is a basis of $H^2(\vee_4 S^2;\mathbb{Z}/2)$ and $f_2\colon S^2 \vee S^2 \to (\mathbb{C}P^\infty)^2 = \mathbb{P}_2\mathcal{H}_1 =: Q_2$ such that $f_2^*(a_i) =: \widetilde{a}_i$ is a basis of $H^2(S^2 \vee S^2;\mathbb{Z}/2)$. The pullback $f_4^*(P_3)$ is the product fibration $\vee_4 S^2 \times K_4 \to \vee_4 S^2$, whereas the pullback $f_2^*(Q_3)$ is the product fibration $S^2 \vee S^2 \times K_2 \to S^2 \vee S^2$. We obtain

commutative diagrams of fibrations

$$\begin{array}{ccc} K_4 & \xrightarrow{\mathbb{1}} & K_4 \\ \downarrow & & \downarrow \\ \vee_4 S^2 \times K_4 & \xrightarrow{\widetilde{f_4}} & P_3 \\ \downarrow & & \downarrow \\ \vee_4 S^2 & \xrightarrow{f} & P_2 \end{array} \qquad \begin{array}{ccc} K_2 & \xrightarrow{\mathbb{1}} & K_2 \\ \downarrow & & \downarrow \\ S^2 \vee S^2 \times K_2 & \xrightarrow{\widetilde{f_2}} & Q_3 \\ \downarrow & & \downarrow \\ S^2 \vee S^2 & \xrightarrow{f} & Q_2 \end{array} .$$

In particular, we obtain a map $(f_k^*)_r$ for $k = 2, 4$ between the E_r-pages of the cohomological Leray-Serre spectral sequences of both pairs of fibrations.
Note that the first three collumns of the E_2-page of $\vee_4 S^2 \times K_4 \to \vee_4 S^2$ agree with the first three columns of the E_2-page of $P_3 \to P_2$ and the first three collumns of the E_2-page of $S^2 \vee S^2 \times K_2 \to S^2 \vee S^2$ agree with the first three columns of the E_2-page of $Q_3 \to Q_2$. By the Künneth theorem all differentials in the spectral sequence of $\vee_4 S^2 \times K_4 \to \vee_4 S^2$ and $S^2 \vee S^2 \times K_2 \to S^2 \vee S^2$ vanish. Since all extensions are trivial we obtain

$$\begin{array}{rcl} \widetilde{f_4}^*(b_l) & = & Sq^2 \iota_l \in H^5(\vee_4 S^2 \times K_4; \mathbb{Z}/2) \text{ and} \\ \widetilde{f_2}^*(b_l) & = & Sq^2 \iota_l \in H^5(S^2 \vee S^2 \times K_2; \mathbb{Z}/2). \end{array}$$

Furthermore, the maps

$$\begin{array}{rcl} \widetilde{f_4}^* : H^7(P_3; \mathbb{Z}/2) & \to & H^7(\vee_4 S^2 \times K_4; \mathbb{Z}/2) \text{ and} \\ \widetilde{f_2}^* : H^7(Q_3; \mathbb{Z}/2) & \to & H^7(S^2 \vee S^2 \times K_2; \mathbb{Z}/2) \end{array}$$

are isomorphisms. Since $Sq^2(Sq^2 \iota_l \otimes 1) = 0$, we obtain $Sq^2 b_l = 0$. By the Adem relations $Sq^3(b_l) = Sq^1 Sq^2(b_l) = 0$ and, hence, the total Steenrod square is

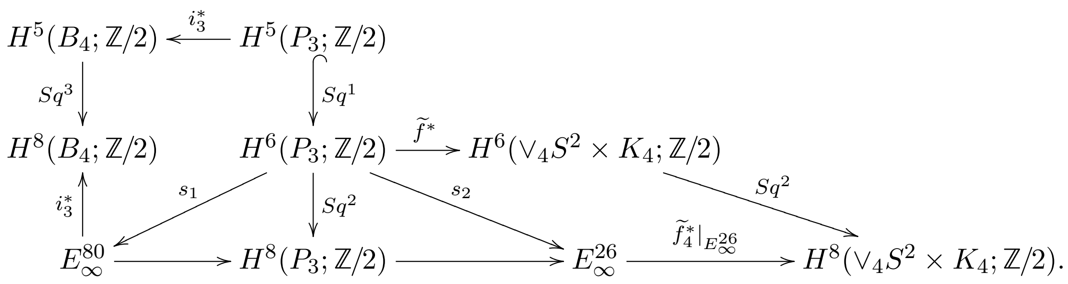
$$Sq(b_l) = b_l + c_l + e_l + b_l^2, \tag{15}$$

for $b_l \in H^5(P_3; \mathbb{Z}/2)$ and $b_l \in H^5(Q_3; \mathbb{Z}/2)$, respectively.
This is all we need to know of the Steenrod module structure of Q_3. Thus, we do not consider it any further.

Since $c_l = Sq^1 b_l$ we instantly obtain $Sq^1 c_l = 0$. Again, Sq^2 is harder. To calculate it, we, amongst other things, need to compare $H^*(P_3; \mathbb{Z}/2)$ and $H^*(B_4; \mathbb{Z}/2)$. We already saw that $p^*(a_1 a_2 a_3 a_4) \in E_\infty^{80} \subset H^8(P_3; \mathbb{Z}/2)$ is a generator. Recall that we have a commutative triangle

$$\begin{array}{ccc} & & P_3 \\ & \overset{i_3}{\nearrow} & \downarrow p \\ B_4 & \xrightarrow{i_2} & P_2 \end{array} .$$

By definition $i_2^*(a_1a_2a_3a_4)$ is a generator of $H^8(B_4;\mathbb{Z}/2)$. Thus, $i_3^*|_{E_\infty^{80}}$ is an isomorphism. To determine $Sq^2(c_l)$ we consider the following commutative diagram



Since $H^5(B_4;\mathbb{Z}/2) = 0$ and since $i_3^*|_{E_\infty^{08}}$ is an isomorphism, $s_1(Sq^1 b_l) = 0$. Additionally, s_2 also vanishes since $Sq^2\colon H^6(\vee_4 S^2 \times K_4;\mathbb{Z}/2) \to H^8(\vee_4 S^2 \times K_4;\mathbb{Z}/2)$ vanishes. Consequently, $Sq^2(Sq^1 b_l) = Sq^2 c_l = 0$. Hence, $Sq^3 c_l$ vanishes, too. Finally, $k^*(Sq^4 c_l) = Sq^4(Sq^3 c_l) = Sq^5 Sq^2 \iota_l = (Sq^2 \iota_l)^2 = k^*(b_l^2)$ and we obtain the total Steenrod square to be

$$Sq(c_l) = c_l + b_l^2 + x, \tag{16}$$

where x is an element of $H^i(P_3;\mathbb{Z}/2)$ for $i \geq 11$.
Observe that we now have assembled the complete Steenrod module structure of $H^i(P_3;\mathbb{Z}/2)$ for $i \leq 10$.

B. Calculation of a minimal resolution

In this appendix we explicitly construct a minimal resolution to calculate the E_2-page of an Adams spectral sequence at the prime two. The group we want to calculate is $\pi_8^{st}(\mathbb{C}P_+^\infty \wedge MString)$. Thus, we need the E_2-page

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(\mathbb{C}P_+^\infty \wedge MString;\ \mathbb{Z}/2), \mathbb{Z}/2).$$

for $t - s \leq 9$.

We construct the minimal resolution as described in Section 6.18 of [Sto85]. For this purpose we need $H^k(\mathbb{C}P_+^\infty \wedge MString;\ \mathbb{Z}/2)$ for $k \leq 10$. Recall that we denote the generator of the ring $H^*(\mathbb{C}P^\infty,\ \mathbb{Z}/2)$ by $a \in H^2(\mathbb{C}P^\infty;\ \mathbb{Z}/2)$, the generator of $H^0(MString; \mathbb{Z}/2)$ by u and the one of $H^8(MString; \mathbb{Z}/2)$ by uw_8. Here, u is the Thom class of $MString$ and w_8 is the pullback of the eighth universal Stiefel-Whitney class in $H^8(BO; \mathbb{Z}/2)$ to $H^8(BString; \mathbb{Z}/2)$.
We consider the pullback of the classes $u, uw_8, a, a^2, \ldots$ to $H^*(\mathbb{C}P_+^\infty \wedge MString)$, apply the Künneth theorem and obtain

i	0	2	4	6	8	10
$H^i(\mathbb{C}P^\infty \wedge MString;\ \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2^2$	$\mathbb{Z}/2^2$
generators	u	ua	ua^2	ua^3	uw_8, ua^4	ua^5, w_8a

.

The other groups $H^i(\mathbb{C}P_+^\infty \wedge MString;\ \mathbb{Z}/2)$ vanish for $i \leq 10$. An easy calculation shows that the only non-vanishing operations of Steenrod squares Sq^i in this range are:

$$Sq^8u = uw_8, \qquad Sq^8ua = w_8a, \qquad Sq^2ua = ua^2,$$
$$Sq^4ua^2 = ua^4, \qquad Sq^2ua^3 = ua^4, \qquad Sq^4ua^3 = ua^5.$$

Now we can calculate the minimal almost free resolution (cf. Definition 6.2 and 6.12 in [Sto85])

$$\ldots \to M_i \to M_{i-1} \to \ldots \to M_1 \to M_0 \to H^i := H^i(\mathbb{C}P^\infty \wedge MString; \mathbb{Z}/2).$$

We stick to the notation of [Sto85], in particular

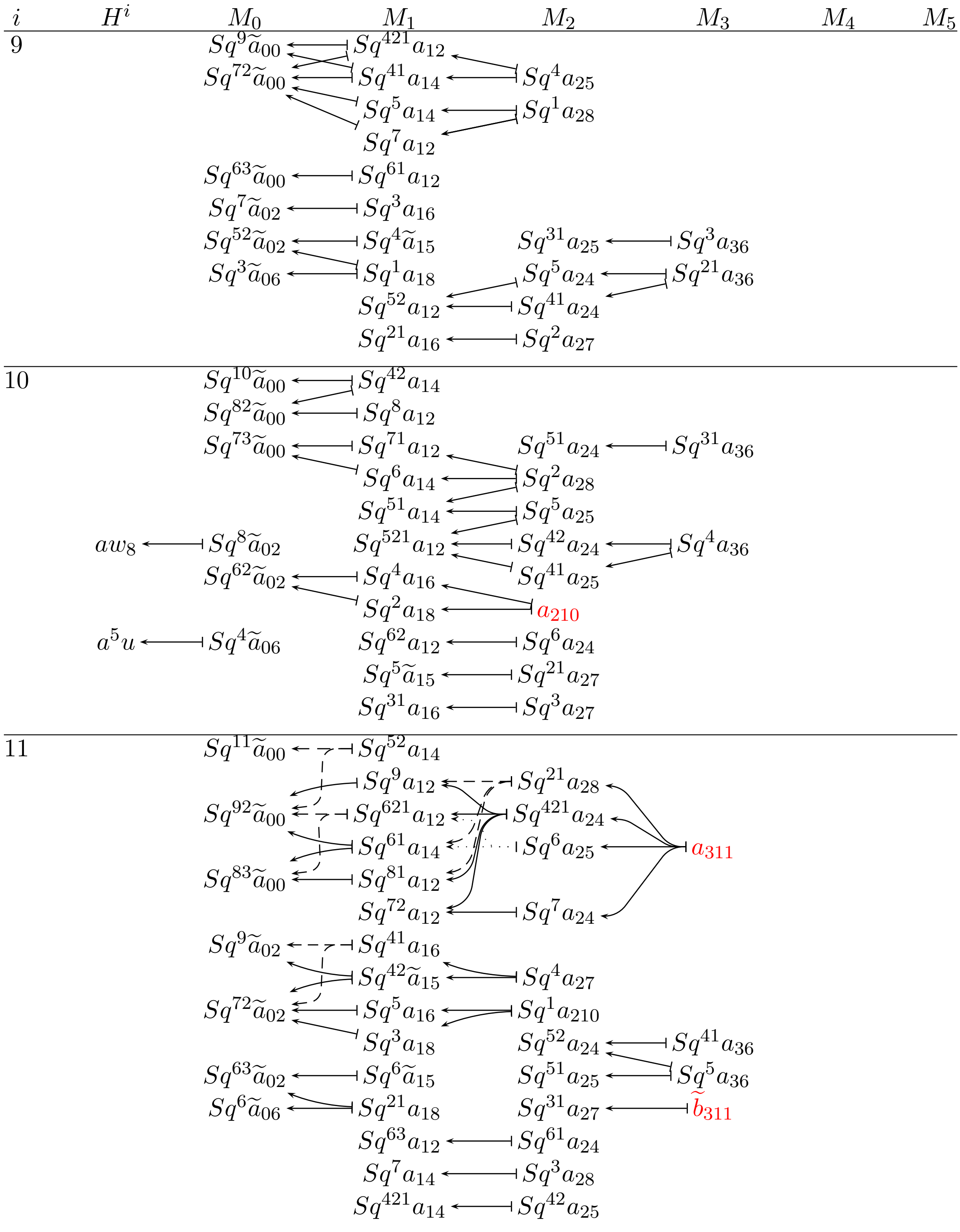
$$a_{st} \text{ and } \widetilde{a}_{st} \in \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(\mathbb{C}P_+^\infty \wedge MString;\ \mathbb{Z}/2), \mathbb{Z}/2),$$

generate M_s, i.e.

$$M_s \cong \bigoplus_t \mathcal{A}a_{st} \oplus \bigoplus_t \mathcal{A}/(\mathcal{A}Sq^1)\widetilde{a}_{st}.$$

Then, Proposition 6.14 in [Sto85] states how these generators of the modules in the resolution of H^i form a $\mathbb{Z}/2$-basis of $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(\mathbb{C}P_+^\infty \wedge MString;\ \mathbb{Z}/2), \mathbb{Z}/2)$.
Find the resolution on the next pages.

i	H^i	M_0	M_1	M_2	M_3	M_4	M_5
0	u	$\widetilde{a}_{00}$					
1	0						
2	au	$\widetilde{a}_{02}$					
		$Sq^2\widetilde{a}_{00}$	a_{12}				
3		$Sq^3\widetilde{a}_{00}$	Sq^1a_{12}				
4	a^2u	$Sq^2\widetilde{a}_{02}$					
		$Sq^4\widetilde{a}_{00}$	a_{14}				
			Sq^2a_{12}	a_{24}			
5		$Sq^5\widetilde{a}_{00}$	$Sq^{21}a_{12}$	a_{25}			
			Sq^1a_{14}				
		$Sq^3\widetilde{a}_{02}$	$\widetilde{a}_{15}$				
			Sq^3a_{12}	Sq^1a_{24}			
6	a^3u	$\widetilde{a}_{06}$	$Sq^{31}a_{12}$	Sq^2a_{24}	a_{36}		
		$Sq^6\widetilde{a}_{00}$	Sq^2a_{14}	Sq^1a_{25}			
		$Sq^{42}\widetilde{a}_{00}$	Sq^4a_{12}				
		$Sq^4\widetilde{a}_{02}$	a_{16}				
7		$Sq^7\widetilde{a}_{00}$	Sq^3a_{14}	Sq^3a_{24}	Sq^1a_{36}		
		$Sq^{52}\widetilde{a}_{00}$	Sq^5a_{12}	$Sq^{21}a_{24}$			
			$Sq^{41}a_{12}$				
		$Sq^5\widetilde{a}_{02}$	$Sq^2\widetilde{a}_{15}$	a_{27}			
			Sq^1a_{16}				
			$Sq^{21}a_{14}$	Sq^2a_{25}			
8	$u \cup w_8$	$Sq^8\widetilde{a}_{00}$	Sq^6a_{12}	a_{28}			
		$Sq^{62}\widetilde{a}_{00}$	Sq^4a_{14}				
		$Sq^6\widetilde{a}_{02}$	Sq^2a_{16}	$Sq^{21}a_{25}$	Sq^2a_{36}		
			$Sq^{51}a_{12}$	$Sq^{31}a_{24}$			
	a^3u	$Sq^{42}\widetilde{a}_{02}$	a_{18}				
		$Sq^2\widetilde{a}_{06}$					
			$Sq^{42}a_{12}$	Sq^4a_{24}			
			$Sq^{21}a_{14}$	Sq^3a_{25}			
			$Sq^3\widetilde{a}_{15}$	Sq^1a_{27}			



i | H^i | M_0 | M_1 | M_2 | M_3 | M_4 | M_5
9 Sq^9 ã00 | Sq^421 a12
Sq^72 ã00 | Sq^41 a14 | Sq^4 a25
Sq^5 a14 | Sq^1 a28
Sq^7 a12
Sq^63 ã00 | Sq^61 a12
Sq^7 ã02 | Sq^3 a16
Sq^52 ã02 | Sq^4 ã15 | Sq^31 a25 | Sq^3 a36
Sq^3 ã06 | Sq^1 a18 | Sq^5 a24 | Sq^21 a36
Sq^52 a12 | Sq^41 a24
Sq^21 a16 | Sq^2 a27
10 Sq^10 ã00 | Sq^42 a14
Sq^82 ã00 | Sq^8 a12
Sq^73 ã00 | Sq^71 a12 | Sq^51 a24 | Sq^31 a36
Sq^6 a14 | Sq^2 a28
Sq^51 a14 | Sq^5 a25
aw8 | Sq^8 ã02 | Sq^521 a12 | Sq^42 a24 | Sq^4 a36
Sq^62 ã02 | Sq^4 a16 | Sq^41 a25
Sq^2 a18 | a210
a^5 u | Sq^4 ã06 | Sq^62 a12 | Sq^6 a24
Sq^5 ã15 | Sq^21 a27
Sq^31 a16 | Sq^3 a27
11 Sq^11 ã00 | Sq^52 a14
Sq^9 a12 | Sq^21 a28
Sq^92 ã00 | Sq^621 a12 | Sq^421 a24
Sq^61 a14 | Sq^6 a25 | a311
Sq^83 ã00 | Sq^81 a12
Sq^72 a12 | Sq^7 a24
Sq^9 ã02 | Sq^41 a16
Sq^42 ã15 | Sq^4 a27
Sq^72 ã02 | Sq^5 a16 | Sq^1 a210
Sq^3 a18 | Sq^52 a24 | Sq^41 a36
Sq^63 ã02 | Sq^6 ã15 | Sq^51 a25 | Sq^5 a36
Sq^6 ã06 | Sq^21 a18 | Sq^31 a27 | b̃311
Sq^63 a12 | Sq^61 a24
Sq^7 a14 | Sq^3 a28
Sq^421 a14 | Sq^42 a25

i	H^i	M_0	M_1	M_2	M_3	M_4	M_5
12			$Sq^{82}a_{12}$	$Sq^{8}a_{24}$			
			$Sq^{10}a_{12}$				
			$Sq^{62}a_{14}$	$Sq^{4}a_{28}$			
			$Sq^{71}a_{14}$	$Sq^{31}a_{28}$			
			$Sq^{91}a_{12}$	$Sq^{7}a_{25}$	$Sq^{1}a_{311}$		
				$Sq^{521}a_{24}$			
			$Sq^{721}a_{12}$	$Sq^{421}a_{25}$	$Sq^{42}a_{36}$		
			$Sq^{73}a_{12}$	$Sq^{71}a_{24}$			
			$Sq^{631}a_{12}$	$Sq^{62}a_{24}$	$Sq^{6}a_{36}$		
				$Sq^{61}a_{25}$			
			$Sq^{521}a_{14}$	$Sq^{52}a_{25}$			
			$Sq^{6}a_{16}$		$Sq^{51}a_{36}$	$\widetilde{a}_{412}$	
			$Sq^{31}a_{18}$	$Sq^{2}a_{210}$			
			$Sq^{51}a_{16}$	$Sq^{5}a_{27}$			
			$Sq^{52}\widetilde{a}_{15}$	$Sq^{41}a_{27}$			
13				$Sq^{72}a_{24}$	$Sq^{7}a_{36}$		
				$Sq^{71}a_{25}$			
				$Sq^{9}a_{24}$	$Sq^{421}a_{36}$	a_{413}	
				$Sq^{81}a_{24}$	$Sq^{2}a_{311}$		
				$Sq^{621}a_{24}$			
				$Sq^{521}a_{25}$	$Sq^{52}a_{36}$		
				$Sq^{51}a_{27}$	$Sq^{2}\widetilde{b}_{311}$		
				$Sq^{63}a_{24}$	$Sq^{61}a_{36}$		
14					$Sq^{521}a_{36}$	$Sq^{1}a_{413}$	
					$Sq^{3}a_{311}$	$Sq^{2}\widetilde{a}_{412}$	a_{514}

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Zusammenfassung

Diese Dissertation befasst sich mit der Klassifikation von Kohomologie Bott Mannigfaltigkeiten, einer Verallgemeinerung von Bott Mannigfaltigkeiten, mit den Methoden der modifizierten Chirurgietheorie. Bott Mannigfaltigkeiten wurden in der Arbeit [BS58] von Bott und Samelson eingeführt. Eine Bott Mannigfaltigkeit ist der Totalraum eines iterierten $\mathbb{C}P^1$-Bündels, bei dem jedes Bündel die Projektivierung eines komplexen Vektorbündels vom Rang zwei ist. Ihren Namen haben Sie von Grossberg und Karshon in [GK94] erhalten.
Bott Mannigfaltigkeiten der komplexen Dimension n stellen ein wichtiges Beispiel für torische Mannigfaltigkeiten dar. Eine torische Mannigfaltigkeit X ist eine glatte, kompakte, normale, komplexe Varietät, die einen algebraischen Torus $(\mathbb{C}^*)^n$ als dichte Teilmenge enthält, so dass es eine Wirkung des algebraischen Torus $(\mathbb{C}^*)^n$ auf X gibt, welche die natürliche Wirkung des algebraischen Torus auf $(\mathbb{C}^*)^n \subset X$ fortsetzt. Für sie wurden 2008 von Choi, Masuda und Suh in [CMS10] die folgenden Vermutungen formuliert:

1. Zwei torische Mannigfaltigkeiten M und N sind genau dann diffeomorph, wenn ihre Kohomologieringe isomorph sind.

2. Für jeden Isomorphismus $\phi : H^*(M) \to H^*(N)$ der ganzzahligen Kohomologieringe zweier torischer Mannigfaltigkeiten M und N gibt es einen Diffeomorphismus $f\colon N \to M$, so dass $f^* = \phi$, d.h. so dass ϕ von f realisiert wird.

Man spricht im ersten Fall von der schwachen, im zweiten Fall von der starken kohomologischen Starrheitsvermutung. Diese beiden überraschenden Vermutungen wurden am Beispiel der Bott Mannigfaltigkeiten untersucht. Von nun an beziehen wir uns, wenn wir von den Vermutungen sprechen, immer auf die entsprechende Vermutung für Bott Mannigfaltigkeiten.

Bott Mannigfaltigkeiten der reellen Dimension vier sind sogenannte *Hirzebruchflächen.* Wie der Name impliziert wurden diese bereits von Hirzebruch in [Hir51] untersucht. Sie erfüllen die starke Starrheitsvermutung.

Weiterhin gibt es die Klassen der sogenannten $\mathbb{Q}$-*trivialen* und der *einfach verdrillten* Bott Mannigfaltigkeiten. Sie wurden in [CM12] beziehungsweise [CS11a] eingeführt und untersucht. Für $\mathbb{Q}$-triviale Bott Mannigfaltigkeiten gilt die starke, für einfach verdrillte Bott Mannigfaltigkeiten die schwache Vermutung.

Für die bisher genannten Bott Mannigfaltigkeiten kann man die schwache Vermutung beweisen, indem man im Wesentlichen Isomorphismen der unterliegenden Vektorbündel betrachtet.

Eine weitere Klasse von Bott Mannigfaltigkeiten, für die die schwache Vermutung gilt, ist

die der sechs-dimensionalen Bott Mannigfaltigkeiten. Diese wurden in [CMS10] untersucht. Um die schwache Vermutung hier zu beweisen ist allerdings eine andere Technik nötig, nämlich die der Chirurgietheorie. Der Beweis nutzt Klassifikationsresultate für einfach zusammenhängende Mannigfaltigkeiten, die in [Wal66] und [Jup73] bewiesen wurden.
Um diese Klassifikationsresultate nutzen zu können, mussten Choi, Masuda und Suh im Wesentlichen prüfen, ob ein Isomorphismus $\varphi\colon H^*(B) \to H^*(B')$ zwischen den Kohomologieringen zweier sechs-dimensionaler Bott Mannigfaltigkeiten B und B' die folgenden zwei Bedingungen erfüllt:

1. Der Isomorphismus, der durch φ auf Kohomologie mit $\mathbb{Z}/2$-Koeffizienten induziert wird bildet die totalen Stiefel-Whitney Klassen aufeinander ab, d.h. es gilt $\varphi(w(B)) = w(B')$.
2. Auch die totalen Pontrjagin Klassen werden aufeinander abgebildet, d.h. es gilt $\varphi(p(B)) = p(B')$.

Diese Beobachtung hat uns dazu veranlasst eine allgemeinere Klasse von Mannigfaltigkeiten zu untersuchen, die *Kohomologie Bott Mannigfaltigkeiten*. Es sei B eine Bott Mannigfaltigkeit. Eine Kohomologie Bott Mannigfaltigkeit M bezüglich einer Bott Mannigfaltigkeit B ist eine glatte, geschlossene und einfach zusammenhängende Mannigfaltigkeit, für die es einen Ringisomorphismus $\varphi\colon H^*(B) \to H^*(M)$ gibt,

1. so dass die totalen Stiefel-Whitney Klassen aufeinander abgebildet werden, d.h. sie erfüllen $\varphi(w(B)) = w(M)$ und
2. so dass die totalen Pontrjagin Klassen aufeinander abgebildet werden, d.h. sie erfüllen $\varphi(p(B)) = p(M)$.

Für Kohomologie Bott Mannigfaltigkeiten fragen wir uns,

1. ob wir etwas über die Diffeomorphismusklassen von Kohomologie Bott Mannigfaltigkeiten sagen können,
2. ob die Möglichkeit besteht, dass sie, wie im Fall der Dimension sechs, kohomologisch starr sind und
3. ob wir sie irgendwie klassifizieren können.

Mit diesen Fragen beschäftigt sich diese Arbeit. Da die Dimension sechs bereits gelöst ist, haben wir uns diese Fragen für acht-dimensionale Bott Mannigfaltigkeiten gestellt. In dieser Dimension ist die schwache Starrheitsvermutung für Bott Mannigfaltigkeiten nach einem Preprint von Choi (siehe [Cho11a]) gelöst. Über Kohomologie Bott Mannigfaltigkeiten ist allerdings nichts bekannt.
In Theorem 4.2 beantworten wir die erste Frage. Wir stellen fest, dass die Anzahl an Kohomologie Bott Mannigfaltigkeiten immer endlich ist.

Theorem. *Es sei B_4 eine Bott Mannigfaltigkeit der Dimension acht. Die Anzahl von Diffeomorphismusklassen von Kohomologie Bott Mannigfaltigkeiten, deren Kohomologiering isomorph zu B_4 ist, ist endlich.*

Die Antwort auf die zweite Frage ist, dass Kohomologie Bott Mannigfaltigkeiten nicht kohomologisch starr sind. In Theorem 4.10 zeigen wir:

Theorem. *Es sei S eine Bott Mannigfaltigkeit, für die die starke kohomologische Starrheitsvermutung gilt und die außerdem eine String-Struktur zulässt. Dann existiert eine kohomologie Bott Mannigfaltigkeit F (bzgl. S), so dass F nicht diffeomorph zu einer Bott Mannigfaltigkeit ist, insbesondere nicht zu S.*

Die Theorie, die wir für den Beweis beider Theoreme nutzen ist die der modifizierten Chirurgietheorie. Dabei führen wir die Frage, ob zwei Mannigfaltigkeiten deren Normalenbündel eine gewisse Zusatzstruktur tragen, die wir normale $\mathbb{B}$-Struktur nennen, diffeomorph sind, auf die Frage zurück, ob sie $\mathbb{B}$-bordant sind, d.h. bordant durch einen Bordisums, dessen Normalenbündel ebenfalls eine $\mathbb{B}$-Struktur trägt, welche sich auf die $\mathbb{B}$-Strukturen der beiden Mannigfaltigkeiten einschränkt. Die Menge aller $\mathbb{B}$-Bordismusklassen von n-dimensionalen Mannigfaltigkeiten mit normaler $\mathbb{B}$-Struktur bildet eine Gruppe, die sogenannte $\mathbb{B}$-Bordismusgruppe $\Omega_n^{\mathbb{B}}$. Können wir also die $\mathbb{B}$-Bordismusgruppen kontrollieren, so erlaubt uns dies eine Aussage über die Diffeomorphismusklassen.

Um das erste Theorem zu beweisen, zeigen wir, dass alle acht-dimensionalen Kohomologie Bott Mannigfaltigkeiten eine gewisse $\mathbb{B}$-Struktur tragen. Wir berechnen die $\mathbb{B}$-Bordismusgruppen und finden Invarianten, die zeigen, dass durch Kohomologie Bott Mannigfaltigkeiten nur endlich viele Elemente in $\Omega_8^{\mathbb{B}}$ erzeugt werden.

Für den Beweis des zweiten Theorems nutzen wir eine Konstruktion, die man als Kodimension zwei Arf-Invariante bezeichnen kann, d.h. wir nutzen die Arf-Invariante einer Untermannigfaltigkeit der Kodimension zwei. Darauf aufbauend knnen wir S und F in einer geeigneten $\mathbb{B}$-Bordismusgruppe unterscheiden. Wir zeigen, dass Sie damit insbesondere nicht diffeomorph sein können.
Wir vermuten, dass die Kodimension zwei Arf-Invarianten ein erster Ansatzpunkt für die Klassifikation von Kohomologie Bott Mannigfaltigkeiten ist.

Interessanter Weise begegnet uns diese Art Invariante auch im letzten Kapitel dieser Arbeit. Dort befassen wir uns mit dem starken Starrheitsproblem. Wie bereits erwähnt ist die schwache Vermutung für acht-dimensionale Bott Mannigfaltigkeiten gelöst. Die starke Vermutung wird, ebenfalls in [Cho11a], auf die Frage reduziert, ob vier Automorphismen auf dem Kohomologiering einer speziellen Klasse von Bott Mannigfaltigkeiten realisiert werden können. Einen dieser vier Automorphismen untersuchen wir im letzten Teil der Arbeit. Wir stellen in Theorem 5.2 fest, dass dieser Automorphismus realisiert werden kann, falls bestimmte Kodimension zwei Arf-Invarianten verschwinden.