Modular Functions and Special Cycles

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Maryna Viazovska

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1. Gutachter: **Prof. Dr. Don B. Zagier**
2. Gutachter: **Prof. Dr. Werner Müller**

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Contents

Summary

In this thesis we study algebraic cycles on Shimura varieties of orthogonal type. Such varieties are a higher dimensional generalization of modular curves and their important feature is that they have natural families of algebraic cycles in all codimesions. We mostly concentrate on low-dimensional examples: Heegner points on modular curves, Hirzebruch-Zagier cycles on Hilbert surfaces, Humbert surfaces on Siegel modular threefolds.

In Chapter 2 we compute the restriction of Siegel Eisenstein series of degree 2 and more generally of Saito-Kurokawa lifts of elliptic modular forms to Humbert varieties. Using these restriction formulas we obtain certain identities for special values of symmetric square L-functions.

In Chapter 3 a more general formula for the restriction of Gritsenko lifts to Humbert varieties is obtained. Using this formula we complete an argument which was given in a conjectural form in [76] (assertion on p. 246) giving a much more elementary proof than the original one of [36] that the generating series of classes of Heegner points in the Jacobian of a modular curve is a modular form.

In Chapter 4 we present computations that relate the heights of Heegner points on modular curves and Heegner cycles on Kugo-Sato varieties to the Fourier coefficients of Siegel Eisenstein series of degree 3. This was the problem originally suggested to me as a thesis topic, and I was able to obtain certain results which are described here. Some of the results of this chapter overlap some of those given in the recent book [53]. succeed in calculating all terms completely, and also, similar results appeared in the recent book [53].

The main result of the thesis is contained in Chapter 5. In this chapter we study CM values of higher Green's functions. Higher Green's functions are real-valued functions of two variables on the upper half-plane which are bi-invariant under the action of a congruence subgroup, have a logarithmic singularity along the diagonal and satisfy $\Delta f = k(1-k)f$, where k is a positive integer. Such functions were introduced in [35]. Also it was conjectured in [35] and [36] that these functions have "algebraic" values at CM points. A precise formulation of the conjecture is given in the introduction. thesis [60]. In Chapter 5 we prove this conjecture for any pair of CM points lying in the same quadratic imaginary field. Our proof has two main parts. First, we show that the regularized Petersson scalar product of a binary theta series with a weight one weakly holomorphic cusp form is the logarithm of the absolute value of an algebraic number. Second, we prove that the special values of weight k Green's function occurring in the conjecture can be written as Petersson product of this type, where the form of weight one is the $(k-1)$-st Rankin-Cohen bracket of an explicit weakly holomorphic modular form of weight $2-2k$ with a binary theta series. The algebraicity of regularized Petersson products was proved independently at about the same time and by different method by W. Duke and Y. Li [23]; however, our result is stronger since we also give a formula for the factorization of the algebraic number in the number field to which it belongs.

Introduction

This thesis is devoted to the study of algebraic cycles and modular forms on Shimura varieties of orthogonal type. The motivating example of a Shimura variety for us will be the modular curve $X(\Gamma)$, constructed as the quotient of the upper half-plane $\mathfrak{H}$ by the by the action of a congruence subgroup Γ of the modular group $\mathrm{SL}_2(\mathbb{Z})$. The extensive study of such curves in nineteenth century lead to the proof of the beautiful "Kronecker's Jugendtraum". Recall, that each point $\tau \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ corresponds to the elliptic curve $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. The endomorphism ring of an elliptic curve is usually $\mathbb{Z}$, but if not, it is an order in an imaginary quadratic number field, and the elliptic curve is then said to have *complex multiplication*. The points of the upper half-plane that lie in an imaginary quadratic field K correspond to elliptic curves with complex multiplication by some order in K, and they are called the *CM points*. The first important result in this subject goes back to Kronecker and Weber, and it states that the Hilbert class field (maximal abelian unramified extension) of an imaginary quadratic field K is generated by the special value $j(\tau)$ of the j-function at any element τ of K lying in the complex upper half-plane and having the fundamental discriminant. Recall that $j(\tau)$ is the unique holomorphic function on the complex upper half-plane invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, having a simple pole with residue 1 at infinity and the unique zero at $\frac{1+\sqrt{-3}}{2}$.

Another important application of the CM-points on modular curves was found by Heegner in his work [38] on the class number problem for imaginary quadratic fields. The significance of these points in the arithmetic of the Jacobians of modular curves was first recognized by Birch. In [8] Birch used these CM-points to construct rational points of infinite order in the Jacobians. In the landmark work [35] Gross and Zagier have found the criterion for a Heegner point on modular elliptic curve to be of infinite order. The criterion is given in terms of L-functions. Combined with the result of Kolyvagin [49] this proves the equality between the rank of an elliptic curve and the order of vanishing of its Hasse-Weil L-function predicted by the Birch and Swinnerton-Dyer conjecture provided the order of vanishing of L-function is less than or equal to 1.

Hilbert emphasized the importance of extending the complex multiplication theory to functions of several variables in the twelfth of his problems at the International Congress in 1900. First steps in this direction were made by Hilbert, Blumenthal and Hecke in their study of Hilbert modular varieties. However, the modern theory of Shimura varieties originated with the development of the theory of abelian varieties with complex multiplication by Shimura, Taniyama and Weil, and with the proof by Shimura of the existence of canonical models for certain families of Shimura varieties. In two fundamental papers [20, 21] Deligne reformulated the theory in the language of abstract reductive groups and extended Shimura's results on canonical models.

A Shimura variety is equipped with a large supply of algebraic cycles provided by sub-Shimura varieties. The simplest example of such cycles would be CM-points on modular curves. For Shimura varieties of orthogonal type a similar pattern of subvarieties arises in all co-dimensions and can be well understood in terms of lattices in corresponding quadratic spaces and their sublattices [51, 76]. This picture gives rise to the following questions: relations between special values of L-functions [40, 44], modularity of generating series of CM-cycles modulo different equivalence relations [76, 81], computation of CM-values of modular functions [13, 33, 62]. In this thesis we address some of these questions.

The thesis is organized as follows. In Chapter 1 we collect necessary facts on the theory of automorphic forms. We recall the definition and main properties of Shimura varieties of orthogonal type. Also in this chapter we give a brief review of the theta correspondence. We consider both the classical theta lift acting between spaces of holomorphic modular forms and the regularized Borcherds lift extended to modular forms with singularities at cusps.

In Chapter 2 we compute the restriction of Siegel Eisenstein series of level 1, degree 2, and arbitrary weight k to Humbert surfaces. More precisely, for each prime discriminant $p > 0$ we consider an embedding ρ of Hilbert modular surface corresponding to p into a Siegel modular threefold. Denote by $\mathcal{N}$ the Naganuma lifting from the space of modular forms of Hecke's Nebentypus $(\frac{\cdot}{p})$ to the space of Hilbert modular forms for $\mathrm{SL}_2(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers in the real quadratic field $\mathbb{Q}(\sqrt{p})$. Then we prove

Theorem 2.1 *The pullback of the Siegel Eisenstein series via the map ϱ defined in* (1.6) *equals*

$$E_k^{\mathrm{Sieg}}(\varrho(\tau_1,\tau_2)) = \sum_{i=1}^{\dim M_k(\Gamma_0(p),\chi)} \lambda_i \mathcal{N}(f_i)(\tau_1,\tau_2),$$

where $f_i(\tau) = \sum_m a_i(m) e^{2\pi i m\tau}$ are the normalized Hecke eigenforms in $M_k(\Gamma_0(p), (\frac{\cdot}{p}))$ and

$$\lambda_i = \frac{2^{8-4k} k!(2k-3)!}{B_k B_{2k-2}} \cdot \left(1 + \frac{a(p)^2}{p^{2k-2}}\right) \cdot \frac{L(\mathrm{Sym}^2 f_i, 2k-2)}{\|f_i\|^2 \pi^{3k-3}}.$$

We illustrate this formula numerically for $p = 5$ and $k = 4, 6, 12$.

In Chapter 3 we generalize this theorem and obtain a formula for the restriction of Gritsenko lifts of arbitrary modular forms of half integral weight to Humbert varieties. Consider an integer N satisfying $(N/p) = 1$. Let $\mathfrak{a}$ be a fractional ideal contained in $\mathfrak{d}^{-1}$, the inverse of the different of K, and suppose that $\mathfrak{d}^{-1}/\mathfrak{a} \cong \mathbb{Z}/N\mathbb{Z}$. In Section 1.2 we describe an embedding of Hilbert surface $\mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a})\backslash\mathfrak{H} \times \mathfrak{H}$ into the Siegel modular threefold $\Gamma_N\backslash\mathfrak{H}^{(2)}$, where Γ_N denotes the level N paramodular group. Denote by $\mathcal{N}_{\mathfrak{a}}$ the Naganuma lifting from the space of modular forms $S_k(\Gamma_0(p), (\frac{\cdot}{p}))$ to the space of Hilbert modular forms $S_k(\mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a}))$.

Theorem 3.1 *Let h be a half-integral modular form in $M^+_{k-1/2}(N)$ and $F \in M_k(\Gamma_N)$ be* *the Gritsenko lift of h. Then the pullback of F via the map ρ defined in* (1.6) *equals*

$$F(\rho(\tau_1,\tau_2)) = \frac{1}{2}\mathcal{N}_{\mathfrak{a}} g(\tau_1,\tau_2),$$

where $g(\tau) = \theta(\tau)h(p\tau)|U_{4N}$.

We give the following application of this formula. In the paper [76] Zagier suggests a method how to deduce the modularity of the generating series of Heegner points on modular curve modulo rational equivalence from the modularity of the generating series of homology classes of modular curves on Hilbert surfaces, which was proved in [41]. However, an important assertion on p. 246 in [76] was left without a proof, and the method was applied only to Heegner points on the modular curve $X_0(37)$. Using Theorem 3.1 we prove this assertion under additional assumptions about the convergence of power series.

Theorem 3.2. *Let h be a holomorphic periodic function on $\mathfrak{H}$ having the Fourier expansion of the form*

$$h(\tau) \;=\; \sum_{\substack{D>0\\ -D\equiv \text{ square mod } 4N}} b(D)\, q^D \qquad\qquad (q=e^{2\pi i\tau})$$

with N prime, and suppose that the power series

$$g_p(\tau) := h(p\tau)\,\theta(\tau)\mid U_{4N} = \sum_{M>0}\Big(\sum_{\substack{x^2<4NM\\ x^2\equiv 4NM (\mathrm{mod}\ p)}} b\Big(\frac{4NM-x^2}{p}\Big)\Big)q^M$$

is a modular form of weight k, level p and Nebentypus $(\frac{p}{\cdot})$ for every prime $p\equiv 1 \pmod 4$ with $(\frac{N}{p})=1$. Then h belongs to $M^+_{k-1/2}(N)$.

Thus, we can apply the method proposed in [76] to all modular curves $X_0(p)$ with prime conductor. In [81] X. Yuan, S.-W. Zhang and W. Zhang extended the idea of [76] to higher dimensional cycles and obtained conditional modularity results for Chow groups of Shimura varieties of orthogonal type.

The main result of the thesis is contained in Chapter 5. In this chapter we employ the theory of Borcherds lift and the idea of a see-saw identity to study CM values of higher Green's functions. For any integer $k>1$ and subgroup $\Gamma\subset \mathrm{PSL}_2(\mathbb{Z})$ of finite index there is a unique function $G_k^{\Gamma\backslash\mathfrak{H}}$ on the product of two upper half planes $\mathfrak{H}\times\mathfrak{H}$ that satisfies the following conditions:

(i) $G_k^{\Gamma\backslash\mathfrak{H}}$ is a smooth function on $\mathfrak{H}\times\mathfrak{H}\setminus\{(\tau,\gamma\tau),\tau\in\mathfrak{H},\gamma\in\Gamma\}$ with values in $\mathbb{R}$.

(ii) $G_k^{\Gamma\backslash\mathfrak{H}}(\tau_1,\tau_2) = G_k^{\Gamma\backslash\mathfrak{H}}(\gamma_1\tau_1,\gamma_2\tau_2)$ for all $\gamma_1,\gamma_2\in\Gamma$.

(iii) $\Delta_i G_k^{\Gamma\backslash\mathfrak{H}} = k(1-k)G_k^{\Gamma\backslash\mathfrak{H}}$, where Δ_i is the hyperbolic Laplacian with respect to the i-th variable, $i=1,2$.

(iv) $G_k^{\Gamma\backslash\mathfrak{H}}(\tau_1,\tau_2) = m\log|\tau_1-\tau_2| + O(1)$ when τ_1 tends to τ_2 (m is the order of the stabilizer of τ_2, which is almost always 1).

(v) $G_k^{\Gamma\backslash\mathfrak{H}}(\tau_1,\tau_2)$ tends to 0 when τ_1 tends to a cusp.

This function is called the *higher Green's function*. Such functions were introduced in [35]. The existence of the Green's function is shown in [35] by an explicit construction and the uniqueness follows from the maximum principle for subharmonic functions. In the case $k=1$ also there exists the unique function $G_1^{\Gamma\backslash\mathfrak{H}}(\tau_1,\tau_2)$ satisfying (i)-(iv) and the condition (v) should be slightly modified. We know from [35] that the values $G_1^{\Gamma\backslash\mathfrak{H}}(\tau_1,\tau_2)$ are essentially the local height pairings at archimedean places between the divisors $(\tau_1)-(\infty)$ and $(\tau_2)-(\infty)$ on $\Gamma\backslash\mathfrak{H}$.

Consider the function

$$G_{k,\boldsymbol{\lambda}}^{\Gamma\backslash\mathfrak{H}} := \sum_{m=1}^{\infty} \lambda_m\, m^{k-1}\, G_k^{\Gamma\backslash\mathfrak{H}}(\tau_1,\tau_2)|T_m$$

where T_m is a Hecke operator and $\boldsymbol{\lambda} = \{\lambda_m\}_{m=1}^{\infty} \in \oplus_{m=1}^{\infty}\mathbb{Z}$ satisfies $\sum_{m=1}^{\infty}\lambda_m a_m = 0$ for any cusp form $f=\sum_{m=1}^{\infty} a_m q^m \in S_{2k}(\Gamma)$. We call such $\boldsymbol{\lambda}$ a relation for $S_{2k}(\Gamma)$.

If $k=1$, then, since the action of the Hecke operators on the Jacobian of $\overline{\Gamma\backslash\mathfrak{H}}$ is the same as that on $S_2(\Gamma)$, the fact that $\boldsymbol{\lambda}$ is a relation for $S_2(\Gamma)$ means that the divisor $\sum_{m=1}^{\infty}\lambda_m T_m((x)-(\infty))$ is principal. Suppose that for $\tau_1,\tau_2\in\Gamma\backslash\mathfrak{H}$ the divisors $(\tau_1)-\infty, (\tau_2)-\infty$ ore defined over $\overline{\mathbb{Q}}$. Then the axioms for the local height pairings imply that the number $G_{1,\boldsymbol{\lambda}}^{\Gamma\backslash\mathfrak{H}}(\tau_1,\tau_2)$ is the logarithm of the absolute value of an algebraic number.

for $G_k^{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}}$.

It was suggested in [35] that for $k>1$ there also should be an interpretation of $G_k^{\Gamma\backslash\mathfrak{H}}(\tau_1,\tau_2)$ as some sort of a height. Such interpretation was given by Zhang in [80], though a complete height theory in this case is still missing. The following conjecture was formulated in [35] and [36].

Conjecture 1. *Suppose $\boldsymbol{\lambda}$ is a relation for $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. Then for any two CM points $\mathfrak{z}_1$, $\mathfrak{z}_2$ of discriminants D_1, D_2 there is an algebraic number α such that*

$$G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1,\mathfrak{z}_2) = (D_1D_2)^{\frac{1-k}{2}}\log|\alpha|.$$

Moreover, D. Zagier has made a more precise conjecture about the field of definition and prime factorization of this number α. This conjecture is stated as Conjecture 2 in Section 5.1.

In many cases (e.g $k=2$, $D_1=-4$ and D_2 arbitrary) Conjecture 1 was proven by A. Mellit in his Ph.D. thesis [60]. In Chapter 5 we prove this conjecture for any pair of CM points lying in the same imaginary quadratic field.

Theorem 5.7 *Let $\mathfrak{z}_1,\mathfrak{z}_2\in\mathfrak{H}$ be two CM points in the same quadratic imaginary field $\mathbb{Q}(\sqrt{-D})$ and let $\boldsymbol{\lambda}$ be a relation on $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ for integer $k>1$. Then there is an algebraic number α such that*

$$G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1,\mathfrak{z}_2) = D^{1-k}\log|\alpha|.$$

Along the way of the proof of Theorem 5.7 we have discovered the following result which is of independent interest. After the results of this paper where first announced, the author learned that a similar result was found independently in a slightly different context by W. Duke and Y. Li [23].

Theorem 5.6 *We let N be an even lattice of signature $(2,0)$ and let f be a weakly holomorphic weight one vector valued modular form transforming with representation ρ_N (this representation is defined in Section 1.4) that has zero constant term and rational Fourier coefficients. Then the regularized Petersson inner product between f and the (vector valued) binary theta series Θ_N satisfies*

$$(f, \Theta_N)_{\mathrm{reg}} = \log|\alpha|$$

for some $\alpha \in \bar{\mathbb{Q}}$.

Moreover, in Theorem 5.8 we find the field of definition and a simple formula for the prime factorization of the number α in the above theorem. This result allows us to prove Conjecture 2.

Our proof of Theorem 5.7 is based on the theory of Borcherds lifts developed in [10] and the notion of see-saw identities introduced in [50]. From [12] we know that the Green's functions can be realized as Borcherds lifts. In Theorem 5.3 we show that higher Green's functions are equal to the Borcherds lift of an eigenfunction of the Laplace operator. This allows us to extend a method given in [62], that is to analyze CM values of Green's function using see-saw identities. Applying see-saw identities in Theorems 5.4 and 5.5 we prove that a CM-value of higher Green's function is equal to the logarithm of a CM-value of a certain meromorphic modular function with algebraic Fourier coefficients. Thus, it follows from the theory of complex multiplication that $G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1, \mathfrak{z}_2)$ is the logarithm of the absolute value of an algebraic number. Finally, we use the theory of local height pairing [34] and the explicit computations of the height pairing between Heegner points made in [35, 36] in order to compute these CM-values and hence prove Conjecture 2.

We finish this section by giving an example for Conjectures 1 and 2.

Example. The space $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ is zero for $k = 1, 2, 3, 4, 5$ and 7. Hence, $\boldsymbol{\lambda} = (1, 0, 0, \ldots)$ is a relation for these spaces. Thus, Conjecture 1 predicts that for $k = 1, \ldots, 5$

$$G_k\Big(\frac{1+\sqrt{-23}}{4}, \frac{-1+\sqrt{-23}}{4}\Big) = 23^{1-k} \log|\alpha_k|,$$

where α_k is an algebraic number.

Consider the following numbers in the Hilbert class field H of $\mathbb{Q}(\sqrt{-23})$. Let ϱ be the real root of the polynomial $X^3 - X - 1$. Define

$$\pi_5 = 2 - \varrho,\ \pi_7 = \varrho + 2,\ \pi_{11} = 2\varrho - 1,\ \pi_{17} = 3\varrho + 2,\ \pi_{19} = 3\varrho + 1, \tag{0.1}$$

$$\pi_{23} = 2\varrho + 3,\ \varpi_{23} = 3 - \varrho\ , \pi_{25} = 2\varrho^2 - \varrho + 1,\ \pi_{49} = \varrho^2 - 2\varrho + 3,$$

where each π_q has norm q.

One can check numerically that

$$\begin{aligned}
\varpi_{23}^{-23}\,\alpha_2 &= \pi_5^{18}\,\pi_{25}^{-42}\,\pi_7^{36}\,\pi_{49}^{-48}\,\pi_{11}^{4}\,\pi_{17}^{-22}\pi_{19}^{-30}\,\varrho^{207},\\
\varpi_{23}^{23^2}\,\alpha_3 &= \pi_5^{-294}\,\pi_{25}^{546}\,\pi_7^{572}\,\pi_{49}^{-100}\,\pi_{11}^{1052}\,\pi_{17}^{166}\,\pi_{19}^{-146}\,\varrho^{187},\\
\varpi_{23}^{-23^3}\,\alpha_4 &= \pi_5^{16878}\,\pi_{25}^{-7182}\,\pi_7^{21276}\,\pi_{49}^{-3168}\,\pi_{11}^{3164}\,\pi_{17}^{-10802}\,\pi_{19}^{-6930}\,\varrho^{120183},\\
\varpi_{23}^{23^4}\,\alpha_5 &= \pi_5^{627354}\,\pi_{25}^{-5446}\,\pi_7^{108156}\,\pi_{49}^{-34084}\,\pi_{11}^{-411844}\,\pi_{17}^{142078}\,\pi_{19}^{239838}\,\varrho^{373939}.
\end{aligned} \tag{0.2}$$

We will prove these identities in Section 5.13 and demonstrate how all the steps of the proof of Theorem 5.7 work.

List of notations

$\mathbb{A}_K$ ring of adeles of a global field K;
$\mathbb{Q}$ field of rational numbers;
$\mathbb{R}$ field of real numbers;
$\mathbb{Z}$ ring of integers;
$\mathbb{C}$ field of complex numbers;
$\Re(z)$ real part of z;
$\Im(z)$ imaginary part of z;
$\mathbf{e}(x) := e^{2\pi i x}$;
V, $(\ ,\ )$ quadratic vector space;
$\mathrm{q}(l) = \frac{1}{2}(l,l)$ norm of a vector $l \in V$;
$\mathrm{O}(V)$ orthogonal group;
$L \subset V$ a lattice;
$L' = \{v \in L \otimes \mathbb{Q} | (v, L) \subseteq \mathbb{Z}\}$ dual lattice;
We say that the lattice $L \subset V$ is even if $\mathrm{q}(l) \in \mathbb{Z}$ for all $l \in L$;
$\mathrm{Aut}(L', L)$ denotes the subgroup of $\mathrm{SO}(V)$ that fixes each element of L'/L;
$\mathfrak{H} = \{z \in \mathbb{C} | \Im z > 0\}$ upper half-plane;
$\mathfrak{H}^{(n)} = \{Z \in \mathrm{Mat}_{n\times n}(\mathbb{C}) | Z = {}^tZ,\ \Im(Z) > 0\}$ Siegel upper half space of degree n;
W, $\langle\ ,\ \rangle$ symplectic vector space;
$\mathrm{Sp}(W)$ symplectic group;
$\mathrm{SL}_2(\mathbb{Z})$ the full modular group;
$\mathrm{Mp}_2(\mathbb{Z})$ the metaplectic cover of $\mathrm{SL}_2(\mathbb{Z})$, defined in Section 1.4;
ρ_L the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ associated to the lattice L, see in Section 1.4;
$\mathfrak{M}_k(\rho)$ the space of real analytic,
$M_k(\rho)$ the space of holomorphic,
$\widehat{M}_k(\rho)$ the space of almost holomorphic,
$M_k^!(\rho)$ the space of weakly holomorphic vector valued modular forms of weight k and representation ρ;
$\mathrm{Gr}^+(V)$ set of b^+-dimensional positive define subspaces of the space $V \otimes \mathbb{R}$ of signature (b^+, b^-);
$\Theta_L(\tau, v^+)$ Siegel theta function, defined in Section 1.8;
$\Phi_L(f, v^+)$ regularized theta lift, defined in Section 1.8;
${}^tZ$ the transpose of the matrix Z.

Chapter 1

Background on modular varieties and modular forms

1.1 Introduction

In this chapter we give necessary background on the theory of automorphic forms.

In Section 1.2 we recall the definition and main properties of *Shimura varieties of orthogonal type*. An essential feature of such varieties is that they have natural families of algebraic cycles in all codimensions. Another important fact about these varieties is that in small dimensions they coincide with classical modular varieties like modular curves and Hilbert modular surfaces. Finally, the construction of automorphic forms on Shimura varieties by means of theta correspondence gives a lot of information about the geometric properties of these varieties [11], [31].

The theta correspondence provides a method to transfer automorphic forms between different reductive groups. Central to the theory is the notion of a *dual reductive pair*. This is a pair of reductive subgroups G and G' contained in an isometry group $\mathrm{Sp}(W)$ of a symplectic vector space W that happen to be the centralizers of each other in $\mathrm{Sp}(W)$. This correspondence was introduced by Roger Howe in [42]. In Section 1.6 we recall the explicit construction of theta correspondence for the reductive pair consisting of the double cover Mp_2 of SL_2 and the orthogonal group $\mathrm{O}(V)$ of a rational quadratic space V of signature $(2, n)$.

The main examples of the theta correspondence for us will be the Shimura, Doi-Naganuma and Gritsenko lifts, considered in Section 1.6, and the Borcherds lift, considered in Section 1.8.

Finally, in Section 1.10 we recall the notion of a *"see-saw dual reductive pair"* introduced by S. Kudla in the paper [50].

1.2 Quotients of Grassmanians and Shimura varieties of orthogonal type

A Shimura variety is a higher-dimensional analogue of a modular curve. It arises as a quotient of a Hermitian symmetric space by a congruence subgroup of a reductive algebraic group defined over $\mathbb{Q}$. Modular curves, Hilbert modular surfaces, and Siegel modular varieties are among the best known classes of Shimura varieties. Special instances of Shimura varieties were originally introduced by Goro Shimura as a part of his generalization of the complex multiplication theory. Shimura showed that while initially defined analytically, they are arithmetic objects, in the sense that they admit models defined over a number field. In two fundamental papers [20, 21], Pierre Deligne created an axiomatic framework for the work of Shimura. Langlands made Shimura varieties a central part of his program, as a source of representations of Galois groups and as tests for the conjecture that all motivic L-functions are automorphic.

We will start with a definition of Shimura varieties. Let $\mathbb{S}$ be $\mathbb{C}^\times$ regarded as a torus over $\mathbb{R}$. A *Shimura datum* is a pair (G, X) consisting of a reductive algebraic group G defined over the field $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \to G_\mathbb{R}$ satisfying, for every $h \in X$:

(SV1) $\mathrm{Ad} \circ h : \mathbb{S} \to \mathrm{GL}(\mathrm{Lie}(G_\mathbb{R}))$ defines a Hodge structure on $\mathrm{Lie}(G_\mathbb{R})$ of type $\{(-1, 1), (0, 0), (1, -1)\}$;

(SV2) ad $h(i)$ is a Cartan involution on G^{ad};

(SV3) G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

These axioms ensure that $X = G(\mathbb{R})/K_\infty$, where K_∞ is the stabilizer of some $h \in X$, is a finite disjoint union of hermitian symmetric domains.

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ be a ring of finite adeles. For a compact open subgroup $K \subset G(\mathbb{A}_f)$ the double coset space

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K)$$

is a finite disjoint union of locally symmetric varieties of the form $\Gamma\backslash X^+$, where the plus superscript indicates a connected component. The varieties $\mathrm{Sh}_K(G, X)$ are complex quasi-projective varieties, which are defined over $\mathbb{Q}$, and they form an inverse system over all sufficiently small compact open subgroups K. The inverse system $(\mathrm{Sh}_K(G, X))_K$ admits a natural right action of $G(\mathbb{A}_f)$. It is called the *Shimura variety* associated with the Shimura datum (G, X) and is denoted $\mathrm{Sh}(G, X)$.

We will give more elementary and explicit description of Shimura varieties in the case when G is the orthogonal group of signature $(2, b^-)$.

Let $(V, (\ ,\ ))$ be a quadratic space over $\mathbb{Q}$ of signature $(2, b^-)$. Denote by $\mathrm{Gr}^+(V)$ the set of positive definite 2-dimensional subspaces v^+ of $V \otimes \mathbb{R}$.

In the case of signature $(2, b^-)$ the Grassmanian $\mathrm{Gr}^+(V)$ carries a structure of a Hermitian symmetric space. If X and Y are an oriented orthogonal base of some element v^+ in $\mathrm{Gr}^+(V)$ then we map v^+ to the point of the complex projective space $\mathbb{P}(V \otimes \mathbb{C})$

represented by $Z = X + iY \in V \otimes \mathbb{C}$. The fact that $Z = X + iY$ has norm 0 is equivalent to saying that X and Y are orthogonal and have the same norm. This identifies $\mathrm{Gr}^+(V)$ with an open subset of the norm 0 vectors of $\mathbb{P}(V \otimes \mathbb{C})$ in a canonical way, and gives $\mathrm{Gr}^+(V)$ a complex structure invariant under the subgroup $\mathrm{O}^+(V \otimes \mathbb{R})$ of index 2 of $\mathrm{O}(V \otimes \mathbb{R})$ of elements preserving the orientation on the 2 dimensional positive definite subspaces. More explicitly, the open subset

$$\mathcal{P} = \{[Z] \in \mathbb{P}(V \otimes \mathbb{C}) \mid (Z, Z) = 0 \text{ and } (Z, \overline{Z}) > 0\}$$

is isomorphic to $\mathrm{Gr}^+(V)$ by mapping $[Z]$ to the subspace $\mathbb{R}\Re(Z) + \mathbb{R}\Im(Z)$.

Consider an even lattice $L \subset V$. Denote by $\mathrm{Aut}(L)$ the group of those isometries of $L \otimes \mathbb{R}$ that fix each element of L'/L. We will study the quotient

$$X_L := \mathrm{Gr}^+(V)/\mathrm{Aut}(L).$$

An important feature of such varieties is that they come with natural families of algebraic cycles in all codimensions, see [51]. These special cycles arise from embeddings of rational quadratic subspaces $U \subset V$ of signature $(2, c^-)$ with $0 \le c^- \le b^-$, since in this case there is a natural embedding of Grassmanians $\mathrm{Gr}^+(U) \hookrightarrow \mathrm{Gr}^+(V)$.

There is a principal $\mathbb{C}^*$ bundle $\mathcal{L}$ over the hermitian symmetric space $\mathcal{P}$, consisting of the norm 0 points $Z = X + iY \in V \otimes \mathbb{C}$. We define an automorphic form of weight k on $\mathrm{Gr}^+(V)$ to be a function Ψ on $\mathcal{L}$ which is homogeneous of degree $-k$ and invariant under some subgroup Γ of finite index of $\mathrm{Aut}(L)$. More generally, if χ is a one dimensional representation of Γ then we say Ψ is an automorphic form of character χ if $\Psi(\sigma(Z)) = \chi(\sigma)\Psi(Z)$ for $\sigma \in \Gamma$.

The following technical construction will give us a convenient "coordinate system" on the space X_L. We choose $m \in L$, $m' \in L'$ such that $m^2 = 0$, $(m, m') = 1$ and denote $V_0 := V \cap m^\perp \cap m'^\perp$. The tube domain

$$\mathcal{H} = \{z \in V_0 \otimes_{\mathbb{R}} \mathbb{C} | (\Im(z), \Im(z)) > 0\} \tag{1.1}$$

is isomorphic to $\mathcal{P}$ by mapping $z \in \mathcal{H}$ to the class in $\mathbb{P}(L \otimes \mathbb{C})$ of

$$Z(z) := z + m' - \frac{1}{2}((z, z) + (m', m'))m.$$

The choice of a vector m is equivalent to choice of a cusp on $X_L = \mathrm{Gr}^+(V)/\mathrm{Aut}(L)$.

Now we consider several low-dimensional examples.

Modular curves

We fix N to be any positive integer (called the level). We let L be the 3-dimensional even lattice of all symmetric matrices

$$l = \begin{pmatrix} C/N & -B/2 \\ -B/2 & A \end{pmatrix}$$

with A, B, C integers, with the norm $\mathrm{q}(l) = \frac{1}{2}(l,l)$ equal to $-N\det(l)$. The dual lattice is the set of matrices

$$l' = \begin{pmatrix} C'/N & -B'/2N \\ -B'/2N & A' \end{pmatrix}$$

with A', B', C' integers, and L'/L can be identified with $\mathbb{Z}/2N\mathbb{Z}$ by mapping a matrix of L' to the value of B' mod $\mathbb{Z}/2N\mathbb{Z}$. The group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) |\ c \ \equiv \ 0 \bmod N \right\}$$

acts on the lattice L by $l \to \gamma l \gamma^t$ for $\gamma \in \Gamma_0(N)$, and under this action it fixes all elements of L'/L. We identify the upper half-plane with points in the Grassmanian $\mathrm{Gr}^+(L \otimes \mathbb{R})$ by mapping $\tau \in \mathfrak{H}$ to the 2-dimensional positive definite space spanned by the real and imaginary parts of the norm 0 vector

$$\begin{pmatrix} \tau^2 & \tau \\ \tau & 1 \end{pmatrix}.$$

For each $d \in \mathbb{Z}_{>0}$ and $\lambda \in L'/L = \mathbb{Z}/2N\mathbb{Z}$ the *Heegner divisor* $P_{d,\lambda}$, is the union of the points orthogonal to norm $\frac{-d}{4N}$ vectors of $L + \lambda$. In terms of points on $\mathfrak{H}$ this Heegner divisor consists of all points $\tau \in \mathfrak{H}$ such that

$$A\tau^2 + B\tau + C = 0$$

for some integers A, B, C (not necessarily coprime) with

$$N|A,\ B \ \equiv \ \lambda \bmod 2N,\ B^2 - 4AC = -d.$$

Hilbert modular surfaces

Fix a squarefree positive integer Δ, and consider the real quadratic field $K = \mathbb{Q}(\sqrt{\Delta})$. Let $\mathfrak{o}$ be the ring of integers of K. We will write x' for the conjugate of an element $x \in K$, $\mathrm{n}(x) := xx'$ for the norm, and $\mathrm{tr}(x) = x + x'$ for the trace. Also we denote by $\mathfrak{d}$ the different of K (i. e. the principal ideal $(\sqrt{\Delta})$).

The group $\mathrm{SL}_2(K)$ acts on $\mathfrak{H} \times \mathfrak{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau_1, \tau_2) = \left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{a'\tau_2 + b'}{c'\tau_2 + d'} \right) \tag{1.2}$$

For the fractional ideal $\mathfrak{a}$ of K we set

$$\mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} |\ a,\ d \in \mathfrak{o},\ b \in \mathfrak{a}^{-1},\ c \in \mathfrak{a} \right\}.$$

The quotient space

$$\mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a}) \backslash \mathfrak{H} \times \mathfrak{H}$$

is called a Hilbert modular surface.

We let L be the even lattice of matrices of the form

$$l = \begin{pmatrix} C & -B \\ -B' & A \end{pmatrix}$$

with $A, C \in Z$, $B \in \mathfrak{o}$, with the norm given by $-2\det(l)$. The group $\mathrm{SL}_2(K)$ acts on the vector space $L \otimes \mathbb{Q}$ of hermitian matrices by $l \to \gamma l \gamma'^t$ for $\gamma \in \mathrm{SL}_2(K)$ and $l \in L \otimes \mathbb{Q}$. The group $\mathrm{SL}_2(\mathfrak{o})$ maps L to itself under this action.

We identify the product of two copies of the upper half-plane with the positive Grassmannian of $L \otimes \mathbb{R}$ by mapping $(\tau_1, \tau_2) \in \mathfrak{H}^2$ to the space spanned by the real and imaginary parts of the norm 0 vector

$$\begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix}.$$

This induces the usual action of $\mathrm{SL}_2(K)$ on $\mathfrak{H}^2$ given by (1.2).

If l is a negative norm vector in L' then we define the curve T_l to be the orthogonal complement of l in the Grassmannian of L. If l is the matrix

$$\begin{pmatrix} C & -B \\ -B' & A \end{pmatrix}$$

then T_l is the set of points $(\tau_1, \tau_2) \in \mathfrak{H}^2$ such that

$$A\tau_1\tau_2 + B'\tau_1 + B\tau_2 + C = 0.$$

The following union of such curves

$$T_N := \bigcup_{\substack{l \in L' \\ \mathrm{q}(l) = -N}} T_l$$

is a Hirzebruch-Zagier divisor considered in [41].

Siegel modular threefolds

If we take L to be a lattice of signature $(2, 3)$ then the positive Grassmanian of L is isomorphic to the Siegel upper half space of genus 2. The divisors on this Siegel upper half space associated to vectors of L (or rather their images in the quotient) are the so-called Humbert surfaces. Recall that the Siegel upper half space of genus 2 is defined as

$$\mathfrak{H}^{(2)} = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \,|\, Z = {}^tZ, \ \Im(Z) > 0\}.$$

Here we write ${}^tZ$ for the transpose of the matrix Z. Let us denote by $\mathcal{A}_N$ the moduli space of abelian surfaces with polarization of the type $(1, N)$

$$\mathcal{A}_N \cong \Gamma_N \setminus \mathfrak{H}^{(2)},$$

where

$$\Gamma_N := \left\{ \begin{pmatrix} * & N* & * & * \\ * & * & * & N^{-1}* \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Q}),\ \text{all } * \in \mathbb{Z} \right\} \tag{1.3}$$

is a paramodular group. These varieties are referred to as Siegel modular varieties.

The Hilbert modular surfaces are the moduli spaces of complex abelian surfaces whose endomorphism ring contains an order from a real quadratic field. In [28] van der Geer describes natural maps of Hilbert modular surfaces to Siegel modular threefolds. The images of these maps are called the *Humbert surfaces*.

Let $\mathfrak{a}$ be a fractional ideal of a real quadratic field $K = \mathbb{Q}(\sqrt{\Delta})$ and suppose that $\mathfrak{a}$ is contained in $\mathfrak{d}^{-1}$. It is explained in [28] that $\mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a})\backslash\mathfrak{H}\times\mathfrak{H}$ is the moduli space of triples (A, j, r), A a polarized n-dimensional complex abelian variety, $j : \mathfrak{o} \to \mathrm{End}(A)$ and r on $\mathfrak{o}$-module isomorphism carrying a Riemann form to the standard form. This Riemann form is equivalent to

$$\begin{pmatrix} 0 & 0 & d_1 & 0 \\ 0 & 0 & 0 & d_2 \\ -d_1 & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 \end{pmatrix},$$

where $d_1|d_2$ are the elementary divisors of the abelian group $\mathfrak{d}^{-1}/\mathfrak{a}$.

For simplicity we assume that $d_1 = 1$ and $d_2 = N$.

Since we can view the varieties $\mathrm{SL}(\mathfrak{o}\oplus\mathfrak{a})\backslash\mathfrak{H}\times\mathfrak{H}$ as moduli spaces of polarized complex abelian varieties with some additional structure there exist “forgetful” maps

$$\mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a})\backslash\mathfrak{H} \times \mathfrak{H} \to \Gamma_N\backslash\mathfrak{H}^{(2)} \tag{1.4}$$

(with $\mathfrak{d}^{-1}/\mathfrak{a} \cong \mathbb{Z}/\mathbb{Z}N$) which are called *modular embeddings*. These maps are described explicitly on p. 209 in [28].

Choose $R \in \mathrm{GL}(2,\mathbb{R})$ such that

$$\begin{pmatrix} R & 0 \\ 0 & {}^tR^{-1} \end{pmatrix} \mathfrak{o} \oplus \mathfrak{a} = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}N, \tag{1.5}$$

where we view $\mathfrak{o} \oplus \mathfrak{a}$ as embedded in $\mathbb{R}^4$ using $K \to \mathbb{R}^2$. Then the following two maps

$$\rho : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}^{(2)}$$

$$(\tau_1, \tau_2) \xrightarrow{\rho} R \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} {}^tR, \tag{1.6}$$

and

$$\phi : \mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a}) \to \Gamma_N$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} R & 0 \\ 0 & {}^tR^{-1} \end{pmatrix} \begin{pmatrix} \tilde\alpha & \tilde\beta \\ \tilde\gamma & \tilde\delta \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ 0 & {}^tR \end{pmatrix},$$

where $\tilde{x} = \begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix}$, describe the modular embedding (1.4) explicitly.

It follows from (1.5) that the matrix R has the form

$$R = \begin{pmatrix} \rho_1 & \rho_1' \\ \rho_2 & \rho_2' \end{pmatrix},$$

where $\rho_1 \in \mathfrak{a},\ \rho_2 \in \mathfrak{d}^{-1}$ and $\det R = \pm 1/\sqrt{\Delta}$.

The image of $\rho(\mathfrak{H} \times \mathfrak{H})$ in the quotient $\Gamma_N \backslash \mathfrak{H}^{(2)}$ does not depend on the choice of R. Moreover the pullback of a Siegel modular form F of weight k on $\mathfrak{H}^{(2)}$ via the map ρ will be a Hilbert modular form of weight k for the group $\mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a})$. A consequence of transformation properties of F is that $F \circ \rho$ does not depend on particular choice of R.

To a non-zero vector $x = (A, B, C, D, E) \in \mathbb{Z}^5$ we associate the subset H_x in a Siegel upper half space

$$H_x := \left\{ \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \in \mathfrak{H}^{(2)} \mid A\tau_1 + Bz + C\tau_2 + D(z^2 - \tau_1\tau_2) + E = 0 \right\}. \tag{1.7}$$

Denote by $V_{N,\Delta}$ the set of all $(A, B, C, D, E) \in \mathbb{Z}^{(5)}$ with

$$C, D \equiv 0 \pmod{N} \text{ and } B^2 - 4AC - 4DE = \Delta. \tag{1.8}$$

The image $\rho(\mathfrak{H} \times \mathfrak{H})$ belongs to the surface given by (1.7) with

$$A = \Delta\, \mathrm{n}(\rho_2),\ B = \Delta\, \mathrm{tr}(\rho_1\rho_2'),\ C = \Delta\, \mathrm{n}(\rho_1),\ D = E = 0.$$

It follows from (1.5) that the relation (1.8) is true for these coefficients.

Denote by $\mathfrak{H}_\Delta$ the image under $\mathfrak{H}^{(2)} \to \Gamma_N \backslash \mathfrak{H}^{(2)}$ of all H_x with $x \in V_{N,\Delta}$ and x primitive. The surface $\mathfrak{H}_\Delta$ is called a *Humbert surface* of invariant Δ in $\Gamma_N \backslash \mathfrak{H}^{(2)}$. The following theorem gives us information about irreducible components of $\mathfrak{H}_\Delta$.

Theorem. ([28] Theorem (2.1)) *Every irreducible component of $\mathfrak{H}_\Delta$ in $\Gamma_N \backslash \mathfrak{H}^{(2)}$ can be represented in $\mathfrak{H}^{(2)}$ by an equation $\tau_1 + bz + cN\tau_2$ with $b^2 - 4Nc = \Delta,\ 0 \le b < 2N$. The number of irreducible components of $\mathfrak{H}_\Delta$ is $\sharp\{\, b \pmod{2N} \mid b^2 \equiv \Delta \pmod{4N} \,\}$.*

If Δ is a fundamental discriminant each irreducible component of $\mathfrak{H}_\Delta$ corresponds to a strict ideal class $[\mathfrak{b}]$ of $\mathfrak{o}$ containing an ideal $\mathfrak{b} \subset \mathfrak{d}^{-1}$ with $\mathfrak{d}^{-1}/\mathfrak{b} \cong \mathbb{Z}/N\mathbb{Z}$.

1.3 Weil representation

The metaplectic group $\mathrm{Mp}(W)$ is a double cover of the symplectic group $\mathrm{Sp}(W)$. It can be defined over either real or p-adic numbers. More generally, the metaplectic group can be constructed over an arbitrary local or finite field, and even the ring of adeles. The metaplectic group has a particularly significant infinite-dimensional linear representation, the *Weil representation* [73]. It was used by André Weil to give a representation-theoretic interpretation of theta functions, and is important in the theory of modular forms of half-integral weight and the theta correspondence.

The Weil representation [73] can be defined for any abelian locally compact group G. We will restrict here to the case of a finite free module W over R equal to a $\mathbb{Q}_p$, $\mathbb{R}$ or

$\mathbb{A}$ respectively. Let V be an R-vector space. Then $W = V \oplus V^*$ becomes a symplectic vector space in a canonical way by

$$\langle (v_1, v_1^*), (v_2, v_2^*) \rangle = v_1^*(v_2) - v_2^*(v_1).$$

Associated with W there is a *Heisenberg group*

$$H := R \times V \times V^*,$$

defined by the group law

$$(r_1, v_1, v_1^*)(r_2, v_2, v_2^*) = (r_1 + r_2 + v_1^*(v_2), v_1 + v_2, v_1^* + v_2^*).$$

Choose any non-trivial additive character χ on R. We get an action of H on $L_2(V^*)$ by

$$(g\phi)(v^*) = \chi(r_1 + v^*(v_1))\phi(v^* + v_1^*)$$

for $g = (r_1, v_1, v_1^*)$ and $\phi \in L_2(V^*)$. This is the unique irreducible representation of H, where R acts through χ. The unicity yields a projective representation of the automorphism group of H. This group is the symplectic group $\mathrm{Sp}(W)$. It acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (r, v, v^*) =$$
$$\Big(\frac{1}{2}\langle cv + dv^*, av + bv^* \rangle - \frac{1}{2}\langle v^*, v \rangle + r, av + bv^*, cv + dv^*\Big).$$

This projective representation can be considered as an honest representation of an extension

$$0 \to \mathbb{C}^* \to \mathrm{Mp}(W) \to \mathrm{Sp}(W) \to 0.$$

It is called the *Weil representation.*

The Weil representation can be described explicitly. Consider the following elements of $\mathrm{Sp}(W)$

$$\begin{aligned} g_a &= \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \\ u_b &= \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \\ j_c &= \begin{pmatrix} 0 & -{}^t c^{-1} \\ c & 0 \end{pmatrix} \end{aligned} \tag{1.9}$$

where $a \in \mathrm{Aut}(V)$, $b \in \mathrm{Hom}(V^*, V)$ is the bilinear form on V^* and $c \in \mathrm{Iso}(V, V^*)$. The elements defined in (1.9) have lifts to $\mathrm{Mp}(W)$ given by the following action on the space $\mathcal{S}(V_R^*)$ of Schwartz-Bruhat functions on V_R^*

$$\begin{aligned} \omega(g_a)(\varphi)(x^*) &= |\det({}^t a)|^{1/2}\varphi(ax^*) \\ \omega(u_b)(\varphi)(x^*) &= \chi((x^*, b(x^*)))\varphi(x^*) \\ \omega(j_c)(\varphi)(x^*) &= |\det(c)|^{-1/2}\int_V \varphi({}^t cx)\chi((x^*, x))dx. \end{aligned} \tag{1.10}$$

Here dx is any measure on V and $|c|$ is the comparison factor between the image under c of the chosen measure on V and the dual of the chosen measure. Note that the last formula does not depend on this choice.

1.4 Vector-valued modular forms

Recall that the group $\mathrm{SL}_2(\mathbb{Z})$ has a double cover $\mathrm{Mp}_2(\mathbb{Z})$ called the *metaplectic group* whose elements can be written in the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau+d}\right)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\sqrt{c\tau+d}$ is considered as a holomorphic function of τ in the upper half-plane whose square is $c\tau+d$. The multiplication is defined so that the usual formulas for the transformation of modular forms of half integral weight work, which means that

$$(A, f(\tau))(B, g(\tau)) = (AB, f(B(\tau))g(\tau))$$

for $A, B \in \mathrm{SL}_2(\mathbb{Z})$ and f, g suitable functions on $\mathfrak{H}$.

Suppose that V is a vector space over $\mathbb{Q}$ and $(\ ,\ )$ is a bilinear form on $V \times V$ with signature (b^+, b^-). For an element $x \in V$ we will write $x^2 := (x,x)$ and $\mathrm{q}(x) = \frac{1}{2}(x,x)$. Let $L \subset V$ be a lattice. The dual lattice of L is defined as $L' = \{x \in V | (x, L) \subset \mathbb{Z}\}$. We say that L is even if $\mathrm{q}(l) \in \mathbb{Z}$ for all $l \in L$. In this case L is contained in L' and L'/L is a finite abelian group.

We let the elements e_ν for $\nu \in L'/L$ be the standard basis of the group ring $\mathbb{C}[L'/L]$, so that $e_\mu e_\nu = e_{\mu+\nu}$. The complex conjugation acts on $\mathbb{C}[L'/L]$ by $\overline{e_\mu} = e_\mu$. Consider the scalar product on $\mathbb{C}[L'/L]$ given by

$$\langle e_\mu, e_\nu \rangle = \delta_{\mu,\nu} \tag{1.11}$$

and extended to $\mathbb{C}[L'/L]$ by linearity. Recall that there is a unitary representation ρ_L of the double cover $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(\widetilde{T})(e_\nu) = \mathbf{e}\big(\mathrm{q}(\nu)\big)\, e_\nu, \tag{1.12}$$

$$\rho_L(\widetilde{S})(e_\nu) = i^{(b^-/2-b^+/2)}\, |L'/L|^{-1/2} \sum_{\mu \in L'/L} \mathbf{e}\big(-(\mu,\nu)\big) e_\mu, \tag{1.13}$$

where

$$\widetilde{T} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \quad \text{and} \quad \widetilde{S} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right) \tag{1.14}$$

are the standard generators of $\mathrm{Mp}_2(\mathbb{Z})$.

For an integer $n \in \mathbb{Z}$ we denote by $L(n)$ the lattice L equipped with a quadratic form $q^{(n)}(l) := n\mathrm{q}(l)$. In the case $n = -1$ the lattices $L'(-1)$ and $(L(-1))'$ coincide and hence the groups L'/L and $L(-1)'/L(-1)$ are equal. Both representations ρ_L and $\rho_{L(-1)}$ act on $\mathbb{C}[L'/L]$ and for $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$ we have $\rho_{L(-1)}(\gamma) = \overline{\rho_L(\gamma)}$.

A vector valued modular form of half-integral weight k and representation ρ_L is a function $f : \mathfrak{H} \to \mathbb{C}[L'/L]$ that satisfies the following transformation law

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \sqrt{c\tau+d}^{\,2k} \rho_L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau+d}\right) f(\tau)$$

for each $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau+d}\right) \in \mathrm{Mp}_2(\mathbb{Z})$.

We will use the notation $\mathfrak{M}_k(\rho_L)$ for the space of real analytic, $M_k(\rho_L)$ for the space of holomorphic, $\widehat{M}_k(\rho_L)$ for the space of almost holomorphic, and $M^!_k(\rho_L)$ for the space of weakly holomorphic modular forms of weight k and representation ρ_L.

Now we recall some standard maps between the spaces of vector valued modular forms of associated to different lattices [15].

If $M \subset L$ is a sublattice of finite index then a vector valued modular form $f \in \mathfrak{M}_k(\rho_L)$ can be naturally viewed as a vector valued modular form in $f \in \mathfrak{M}_k(\rho_M)$. Indeed, we have the inclusions

$$M \subset L \subset L' \subset M'$$

and therefore

$$L/M \subset L'/M \subset M'/M.$$

We have the natural map $L'/M \to L'/L$, $\mu \to \bar{\mu}$.

Lemma 1.1. *For $\mathcal{M} = \mathfrak{M}, M, \widehat{M}$ or $M^!$ there are two natural maps*

$$\mathrm{res}_{L/M} : \mathcal{M}_k(\rho_L) \to \mathcal{M}_k(\rho_M),$$

and

$$\mathrm{tr}_{L/M} : \mathcal{M}_k(\rho_M) \to \mathcal{M}_k(\rho_L),$$

given by

$$\big(\mathrm{res}_{L/M}(f)\big)_\mu = \begin{cases} f_{\bar{\mu}}, & \text{if } \mu \in L'/M \\ 0 & \text{if } \mu \notin L'/M \end{cases}, \qquad \big(f \in \mathcal{M}_k(\rho_L),\ \mu \in M'/M\big) \tag{1.15}$$

and

$$\big(\mathrm{tr}_{L/M}(g)\big)_\lambda = \sum_{\mu \in L'/M:\, \bar{\mu}=\lambda} g_\mu, \qquad \big(g \in \mathcal{M}_k(\rho_M),\ \ \lambda \in L'/L\big). \tag{1.16}$$

Now suppose that M and N are two even lattices and $L = M \oplus N$. Then we have

$$L'/L \cong (M'/M) \oplus (N'/N).$$

Moreover

$$\mathbb{C}[L'/L] \cong \mathbb{C}[M'/M] \otimes \mathbb{C}[N'/N]$$

as unitary vector spaces and naturally

$$\rho_L = \rho_M \otimes \rho_N.$$

Lemma 1.2. *For two modular forms $f \in \mathcal{M}_k(\rho_L)$ and $g \in \mathcal{M}_l(\rho_{M(-1)})$ the function*

$$h := \langle f, g\rangle_{\mathbb{C}[M'/M]} = \sum_{\nu \in N'/N} e_\nu \sum_{\mu \in M'/M} f_{\mu\oplus\nu}\, g_\mu$$

belongs to $\mathcal{M}_{k+l}(\rho_N)$.

1.5 Jacobi forms and Kohnen's plus space

In this section we explain a relation between vector valued modular forms and more classical objects: Jacobi forms and scalar valued modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

A Jacobi form of weight k and index N is a holomorphic function $\phi : \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the transformation law

$$\phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (c\tau+d)^k \, \mathbf{e}\Big(\frac{Ncz^2}{c\tau+d}\Big)\, \phi(\tau, z) \tag{1.17}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and

$$\phi(\tau, z+\tau m+n) = \mathbf{e}\big(-Nm^2\tau - 2mz\big)\, \phi(\tau, z) \tag{1.18}$$

for $m, n \in \mathbb{Z}$. Such function ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ r^2 \leq 4Nn}} c(n,r)\, \mathbf{e}(n\tau + rz),$$

where $c(n,r)$ depends only on $r^2 - 4Nn$ and on the residue class of $r(\mathrm{mod}\ 2N)$. The systematic theory of such functions is developed in [25]: in this monograph Jacobi cusp forms, Eisenstein series, the Petersson scalar product, Hecke operators, and new forms are defined.

It is shown in Theorem 5.1 of [25] that the space $J_{k,N}$ of Jacobi forms of weight k and level N is isomorphic to the space of vector valued modular forms $M_{k-1/2}(\rho_{\mathbb{Z}(-N)})$, where $\mathbb{Z}(-N)$ is the lattice $\mathbb{Z}$ equipped with the quadratic form $\mathrm{q}(l) := -Nl^2$, $l \in \mathbb{Z}$. The connection between the spaces of vector valued modular forms related to other lattices and and Jacobi forms is explained in [17], [69].

In this section we show that the space of vector valued modular forms is isomorphic to a space of certain real-analytic functions similar to Jacobi forms. Let L be an even lattice of signature (b^+, b^-). Let v^+ be a positive b^+-dimensional subspace of $L \otimes \mathbb{R}$. Denote by v^- the orthogonal complement of v^+. For a vector $l \in L$ denote by l_{v^+} and l_{v^-} its projections on v^+ and v^-.

For $\lambda \in L'/L$ we define

$$\theta^{\mathrm{J}}_{L+\lambda}(\tau, z; v^+) := \sum_{l\in\lambda+L} \mathbf{e}\big(\mathrm{q}(l_{v^+})\tau + \mathrm{q}(l_{v^-})\bar{\tau} + (l, z)\big),$$

where $\tau \in \mathfrak{H}$, $z \in L \otimes \mathbb{C}$, and $v_+ \in \mathrm{Gr}^+(L)$. It follows from Theorem 4.1 of [10] that this function satisfies the following transformation properties

$$\theta^{\mathrm{J}}_{L+\lambda}\Big(\frac{-1}{\tau}, \frac{z_{v^+}}{\tau} + \frac{z_{v^-}}{\bar{\tau}}; v^+\Big) = \tag{1.19}$$

$$i^{(b^-/2-b^+/2)}\,|L'/L|^{-1/2}\,\tau^{b^+/2}\,\bar\tau^{b^-/2}\,\mathbf{e}\Big(\frac{\mathrm{q}(z_{v^+})}{\tau}+\frac{\mathrm{q}(z_{v^-}^2)}{\bar\tau}\Big)\sum_{\mu\in L'/L}\mathbf{e}(-(\lambda,\mu))\,\theta^{\mathrm{J}}_{\mu+L}(\tau,z;v^+).$$

For a vector valued modular form $f=(f_\lambda)_{\lambda\in L'/L}\in M_k(\mathrm{SL}_2(\mathbb{Z}),\rho_L)$ we consider the function

$$F(\tau,z;v^+):=\sum_{\lambda\in L'/L}f_\lambda(\tau)\,\overline{\theta_\lambda(\tau,z;v^+)}.$$

Equation (1.19) implies that F satisfies the following transformation properties similar to (1.17), (1.18), namely

$$F\Big(\frac{-1}{\tau},\frac{z_{v^+}}{\tau}+\frac{z_{v^-}}{\bar\tau},v^+\Big)=i^{(k+b^-/2-b^+/2)}\,\tau^{k+b^+/2}\,\bar\tau^{b^-/2}\,\mathbf{e}\Big(\frac{\mathrm{q}(z_{v^+})}{\tau}+\frac{\mathrm{q}(z_{v^-})}{\bar\tau}\Big)\,F(\tau,z,v^+),$$

and

$$F(\tau,z+m\tau+n,v^+)=\mathbf{e}\big(-2\mathrm{q}(m_{v^+})\tau-2\mathrm{q}(m_{v^-})\bar\tau-(z,m)\big)\,F(\tau,z,v^+)$$

for $m\in L'$, $n\in L$. In particular, when the lattice L is negative definite the function F is a holomorphic Jacobi form. For positive definite lattices L the function F becomes a *skew holomorphic Jacobi form*. These forms were introduced by Skoruppa in [68].

Modular functions of half-integral weight are defined like forms of integral weight, except that the automorphy factor is more complicated.

Let

$$\theta(\tau)=\sum_{n\in\mathbb{Z}}e^{2\pi in^2}\tag{1.20}$$

be the standard theta function. If $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ belongs to $\Gamma_0(4)$, we have

$$\theta(Az)=j(A,z)\,\theta(z),$$

where $j(A,z)$ is the "θ-multiplier" of A. Recall (cf. for instance [65]) that, if $c\neq 0$, we have

$$j(A,z)=\varepsilon_d\Big(\frac{c}{d}\Big)(cz+d)^{1/2},$$

where

$$\varepsilon_d=\begin{cases}1 & \text{if } d\equiv 1 \bmod 4\\ i & \text{if } d\equiv -1 \bmod 4\end{cases}$$

and $(cz+d)^{1/2}$ is the "principal" determination of the square root of $cz+d$, i.e. the one whose real part is >0.

A function h on $\mathfrak{H}$ is called a modular form of weight $k/2$ on $\Gamma_0(4N)$ if :

a) $h(\tau)/\theta^k(\tau)$ is invariant under $\Gamma_0(4N)$;

b) ϕ is holomorphic, both on $\mathfrak{H}$ and at the cusps (see [65]).

We denote the space of such forms by $M_{k/2}(N)$. Shimura developed an extensive theory of such forms in [64, 65]. Kohnen introduced the following subspace of $M_{k/2}$, the so-called "+"-space [47],

$$M^+_{k-1/2}(N) = \Big\{ h \in M_{k-1/2}(N) \,\Big|\, h(\tau) = \sum_{-D \,\equiv\, \text{square mod } 4N} b(D)\, \mathbf{e}(D\tau) \Big\}.$$

For N prime the following map

$$\sum_{-D \,\equiv\, \text{square mod } 4N} b(D)\, \mathbf{e}(D\tau) \to \sum_{\substack{n,r\in\mathbb{Z} \\ 4Nn-r^2\geq 0}} b(4Nn - r^2)\, \mathbf{e}(n\tau + rz), \tag{1.21}$$

gives an isomorphism between $M^+_{k-1/2}(N)$ and $J_{k,N}$. Thus, the following spaces are isomorphic

$$M^+_{k-1/2}(N) \cong J_{k,N} \cong M_{k-1/2}(\rho_{\mathbb{Z}(-N)}).$$

1.6 Theta correspondence

In this section we briefly recall the theta (Howe) correspondence. For a commuting pair of subgroups in the metaplectic group there is a correspondence between representations of the two subgroups, obtained by decomposing Weil representation of the metaplectic group into a sum of tensor products of representations of the two subgroups. As some representations of groups over the adeles tend to correspond to automorphic forms, we can get a correspondence between automorphic forms on these two groups.

Firstly, we would like to describe the logical structure of the theta correspondences on the level of abstract representations [42], [43].

Let W be a vector space over the number field k endowed with a symplectic form $\langle\ ,\ \rangle$. Let

$$\mathrm{Mp}(W_{\mathbb{A}}) \to \mathrm{Sp}(W_{\mathbb{A}})$$

be the nontrivial 2-fold central extension of the adelization of $\mathrm{Sp}(W_{\mathbb{A}})$. In Section 1.3 we have consider the representation ω of $\mathrm{Mp}(W_{\mathbb{A}})$ called the Weil representation. The Hilbert space on which the representation ω is realized is $L_2(V^*_{\mathbb{A}})$. The space of smooth vectors is the space $\mathcal{S}(V^*_{\mathbb{A}})$ of Schwartz-Bruhat functions on $V^*_{\mathbb{A}}$ [73].

In [73] it is shown that there is a certain linear functional Θ on $\mathcal{S}(V^*_{\mathbb{A}})$ such that

$$\Theta(\omega(\gamma)\varphi) = \Theta(\varphi) \quad \gamma \in \mathrm{Sp}(W_k), \quad \varphi \in \mathcal{S}(V^*_{\mathbb{A}}). \tag{1.22}$$

This linear functional is defined as follows. In the symplectic vector space W, choose two maximal isotropic subspaces V and V^* such that $W = V \oplus V^*$. We will assume V and V^* are in fact k-rational subspaces of W. We then call the pair (V, V^*) a k-rational complete polarization. The functional Θ of formula (1.22) is given by

$$\Theta(\varphi) = \sum_{x\in V^*_{\mathbb{Q}}} \varphi(x), \quad \varphi \in \mathcal{S}(V^*_{\mathbb{A}}).$$

Property (1.22) is a generalization of the Poisson summation formula.

Recall that a *reductive pair* in $\mathrm{Sp}(W)$ is a pair (G, G') of reductive subgroups of $\mathrm{Sp}(W)$ each of which is the full centralizer of the other. Let $\widetilde{G}(\mathbb{A})$ and $\widetilde{G'}(\mathbb{A})$ denote the inverse images of $G(\mathbb{A})$ and $G'(\mathbb{A})$ in $\mathrm{Mp}(W_{\mathbb{A}})$. Given $\varphi \in \mathcal{S}(V^*_{\mathbb{A}})$, we can define a function θ_φ on $\widetilde{G}_{\mathbb{A}} \times \widetilde{G'}_{\mathbb{A}}$ by the rule

$$\theta_\varphi(g, g') = \Theta(\omega(g)\omega(g')(\varphi)) \quad g \in \widetilde{G}(\mathbb{A}), \;\; g' \in \widetilde{G'}(\mathbb{A}).$$

The function θ_φ is referred as the *θ-kernel* corresponding to φ.

Let f be a cusp form on $\widetilde{G}(\mathbb{A})$. If φ satisfies certain finiteness conditions (see [43]), then

$$\theta_\varphi(f)(g') = \int\limits_{G(k)\backslash G(\mathbb{A})} f(g)\, \theta_\varphi(g, g')\, dg \tag{1.23}$$

is an automorphic form on $\widetilde{G'}(\mathbb{A})$. This automorphic form is called the φ-lift of f.

In the next two sections we will give several examples of the realization of theta correspondence for concrete subspaces of automorphic forms.

1.7 Shimura, Doi-Naganuma, Saito-Kurokawa and Gritsenko lift

Let V be a finite dimensional vector space over $\mathbb{Q}$, and let be $(\ ,\ )$ an inner product on V. Let $\mathrm{O}(V)$ denote the group of linear isometries of $(\ ,\ )$. Let W denote another finite-dimensional vector space over $\mathbb{Q}$ and let $\langle\ ,\ \rangle$ denote a symplectic form on W. Set $W' := V \otimes W$. The tensor product of the forms $(\ ,\ )$ and $\langle\ ,\ \rangle$ defines a symplectic form $\langle\ ,\ \rangle'$ on W'. The groups $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$ act on W' in the obvious way. Their action clearly preserves the form $\langle\ ,\ \rangle'$, and each group clearly commutes with the other. In fact, each of the groups $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$ is the full centralizer of the other in $\mathrm{Sp}(W')$, so that $(\mathrm{O}(V), \mathrm{Sp}(W))$ forms a dual pair in $\mathrm{Sp}(W')$.

In this section we consider theta lifts for dual reductive pair $(\mathrm{O}(V), \mathrm{Sp}(W))$ in the case when the symplectic space W has dimension 2 and the quadratic space V has signature $(2, n)$. In many particular cases such theta lifts were found before the general theory was developed. Since it is difficult to compute the action of representation ω given by (1.10), finding theta kernel $\theta_\varphi(g, g')$ in (1.23) becomes a nontrivial computation. In this section we give examples of the theta kernel for some concrete subspaces of modular forms (see (1.31), (1.35)). Another important task is to compute the Fourier expansion of the theta lift $\theta_\varphi(f)$ from the Fourier expansion of f (see (1.24), (1.33), (1.34)).

Shimura lifting

Shimura's correspondence introduced in [64, 65] takes modular forms of half integral weight $k + 1/2$ modular forms of integral weight $2k$, which can be thought of as modular forms of weight k for the group $\mathrm{O}_{2,1}(\mathbb{R})$. In the simplest case, when $f(\tau) = \sum c(n)q^n$ is a modular form for $\Gamma_0(4)$ of weight $k+1/2$ such that $c(n)$ vanishes unless $n \equiv 0, 1 \bmod 4$ then

$$F(z) := \frac{-c(0)B_k}{2k} + \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1}\, c(n^2/d^2)\, \mathbf{e}(nz)$$

is a modular form of weight $2k$.

Doi-Naganuma lifting

In [48] K. Doi and H. Naganuma discovered a lifting from the space of ordinary modular forms to the space of Hilbert modular forms for real quadratic field. In [61] Naganuma extended these ideas to the case of modular forms of Hecke's Nebentypus. More precisely, in [48, 61] Doi and Naganuma proved the following. Let p be the prime equal 1 modulo 4, $\mathfrak{o}_p$ denotes the ring of integers in $\mathbb{Q}(\sqrt{p})$ and let $f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}$ be the Hecke eigenform in the space of cusp forms for $\Gamma_0(p)$ and the character $\chi = (p/\ )$. Then, if $\mathfrak{o}_p$ is Euclidean, so that $\mathrm{SL}_2(\mathfrak{o}_p)$ is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \epsilon & \epsilon^{-1}\mu \\ 0 & \epsilon^{-1} \end{pmatrix}$ (μ an integer of $\mathbb{Q}(\sqrt{p})$, ϵ a unit of $\mathbb{Q}(\sqrt{p})$), the product

$$\Big(\sum a_n n^{-s}\Big) \Big(\sum \bar{a}_n n^{-s}\Big)$$

is the Mellin transform of a Hilbert modular form for $\mathrm{SL}_2(\mathfrak{o}_p)$. Employing a later result of Vaserstein (see [28] Chapter IV.6) on generators of Hilbert modular groups, the proof can be generalized to all primes with class number 1.

Let $S_k(\Gamma_0(\Delta), \chi)$ be the space of cusp forms for $\Gamma_0(\Delta)$ and the character $\chi = (\Delta/\ )$ and $S_k^{\mathrm{Hilb}}(\Delta)$ the space of cusp forms for the Hilbert modular group $\mathrm{SL}_2(\mathfrak{o}_\Delta)$. We denote the Naganuma map by

$$\mathcal{N} : S_k(\Gamma_0(\Delta), \chi) \to S_k^{\mathrm{Hilb}}(\Delta).$$

In [74] D. Zagier gave an alternative definition of the map $\mathcal{N}$ and showed that the lifting exists for all positive discriminants $\Delta \equiv 1 \bmod 4$. In the simplest case when $\Delta = p$ is prime the lifting of $f = \sum a(n)q^n \in S_k(\Gamma_0(p), \chi)$ equals

$$\mathcal{N} f(\tau_1, \tau_2) = \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} c\big((\nu\mathfrak{d})\big)\, \mathbf{e}(\nu\tau_1 + \nu'\tau_2), \tag{1.24}$$

where for each ideal $\mathfrak{a}$ the coefficient $c(\mathfrak{a})$ is defined as

$$c(\mathfrak{a}) = \sum_{r|\mathfrak{a}} r^{k-1} \hat{a}\Big(\frac{\mathrm{n}(\mathfrak{a})}{r^2}\Big), \tag{1.25}$$

with

$$\hat{a}(n) := \begin{cases} a(n) & \text{if } p \nmid n \\ a(n) + \bar{a}(n) & \text{if } p | n. \end{cases}$$

To show this Zagier constructed the kernel function $\Omega(\tau,\tau_1,\tau_2)$ for the map $\mathcal{N}$. The kernel function has the property that for each $f\in S_k(\Gamma_0(\Delta),\chi)$ the identity

$$\mathcal{N}(f)(\tau_1,\tau_2)\doteq\int\limits_{\Gamma_0(\Delta)\backslash\mathfrak{H}} f(\tau)\,\overline{\Omega(\tau,\tau_1,\tau_2)}\,\Im(\tau)^{k-2}\,d\tau. \tag{1.26}$$

holds. For each $m\geq 0$ consider the function

$$\omega_m(\tau_1,\tau_2)=\sum_{\substack{a,b\in\mathbb{Z},\ \lambda\in\mathfrak{d}^{-1}\\ \lambda\lambda'-ab=m/\Delta}}\frac{1}{(a\tau_1\tau_2+\lambda\tau_1+\lambda'\tau_2+b)^k}. \tag{1.27}$$

Then the function Ω is defined by

$$\Omega(\tau_1,\tau_2;\tau)=\sum_{m=1}^{\infty}m^{k-1}\omega_m(\tau_1,\tau_2)\,\mathbf{e}(m\tau). \tag{1.28}$$

It is a Hilbert modular form in variables (τ_1,τ_2) and a modular form for $\Gamma_0(\Delta)$ and the character $\chi=(\Delta/\ )$ in variable τ.

In [28] the Naganuma map is defined for all groups $\mathrm{SL}(\mathfrak{o}\oplus\mathfrak{a})$. For each fractional ideal $\mathfrak{a}$ of K the function

$$\omega_{m,\mathfrak{a}}(\tau_1,\tau_2)=\sum_{\substack{a,b\in\mathbb{Z},\ \lambda\in\mathfrak{a}^{-1}\mathfrak{d}^{-1}\\ \lambda\lambda'\mathrm{n}(\mathfrak{a})-ab=m/\Delta}}\frac{1}{(a\tau_1\tau_2+\lambda\tau_1+\lambda'\tau_2+b/\mathrm{n}(\mathfrak{a}))^k} \tag{1.29}$$

is a Hilbert modular form of weight k in $\mathrm{SL}(\mathfrak{o}\oplus\mathfrak{a})$. It follows from (1.29) that

$$\omega_{m,\mathfrak{a}}\left(\frac{-1}{\mathrm{n}(\mathfrak{a})\tau_1},\frac{-1}{\mathrm{n}(\mathfrak{a})\tau_2}\right)=\ \mathrm{n}(\mathfrak{a})^k\,\tau_1^k\,\tau_2^k\,\omega_{m,\mathfrak{a}}(\tau_1,\tau_2). \tag{1.30}$$

In analogy with (1.31) the kernel function is defined by

$$\Omega_{\mathfrak{a}}(\tau_1,\tau_2;\tau)=\sum_{m=1}^{\infty}m^{k-1}\omega_{m,\mathfrak{a}}(\tau_1,\tau_2)\,\mathbf{e}(m\tau). \tag{1.31}$$

It is proved in Theorem 3.1 of [28] that $\Omega_{\mathfrak{a}}(\tau_1,\tau_2;\tau)$ in variable τ is a cusp form of weight k on $\Gamma_0(\Delta)$ and character $(\cdot/\Delta)$.

The lift $\mathcal{N}_{\mathfrak{a}}:S_k(\Gamma_0(\Delta),\chi)\to S_k(\mathrm{SL}(\mathfrak{o}\oplus\mathfrak{a}))$ is defined by

$$\mathcal{N}_{\mathfrak{a}}(f)(\tau_1,\tau_2)\doteq\int\limits_{\Gamma_0(\Delta)\backslash\mathfrak{H}} f(\tau)\,\overline{\Omega_{\mathfrak{a}}(\tau,\tau_1,\tau_2)}\,\Im(\tau)^{k-2}\,d\tau. \tag{1.32}$$

In this case the analog of the formula (1.24) is

$$\mathcal{N}_{\mathfrak{a}}\,f(\tau_1,\tau_2)=\sum_{\substack{\nu\in\mathfrak{a}\mathfrak{d}^{-1}\\ \nu\gg 0}}c((\nu)\mathfrak{d}\mathfrak{a}^{-1})\,\mathbf{e}(\nu\tau_1+\nu'\tau_2). \tag{1.33}$$

This identity follows from Theorem (4.2) in [28] and the proof of Theorem 5 in [74].

Gritsenko lift

The Gritsenko lift of a Jacobi form

$$\phi(\tau, z) = \sum_{n,r} b(n,r)\,\mathbf{e}(n\tau + rz) \in J_{k,N}$$

is defined as

$$F(Z) = \sum_{T} B(T)\,\mathbf{e}\big(\mathrm{tr}(TZ)\big) \in M_k^{\mathrm{Sieg}}(\Gamma_N),$$

where

$$B\begin{pmatrix} m & r/2 \\ r/2 & nN \end{pmatrix} = \sum_{l|(m,n,r)} l^{k-1} c\Big(\frac{mn}{l^2}, \frac{r}{l}\Big), \tag{1.34}$$

and

$$B\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} = 0 \text{ if } N \nmid n.$$

It is proved in [37] that $F(Z)$ is a Siegel modular form for a paramodular group Γ_N introduced in (1.3).

1.8 Borcherds lift

In this section we recall the definition of regularized theta lift given by Borcherds in [10].

We let L be an even lattice of signature $(2, b^-)$ with dual L'. The (positive) Grassmannian $\mathrm{Gr}^+(L)$ is the set of positive definite two dimensional subspaces v^+ of $L \otimes \mathbb{R}$. We write v^- for the orthogonal complement of v^+, so that $L \otimes \mathbb{R}$ is the orthogonal direct sum of the positive definite subspace v^+ and the negative definite subspace v^-. The projection of a vector $l \in L \otimes \mathbb{R}$ into a subspaces v^+ and v^- is denoted by l_{v^+} and l_{v^-}, respectively, so that $l = l_{v^+} + l_{v^-}$.

The vector valued Siegel theta function $\Theta_L : \mathfrak{H} \times \mathrm{Gr}^+(L) \to \mathbb{C}[L'/L]$ of L is defined by

$$\Theta_L(\tau, v^+) = y^{b^-/2} \sum_{\lambda \in L'/L} e_\lambda \sum_{l \in L+\lambda} \mathbf{e}\big(\mathrm{q}(l_{v^+})\tau + \mathrm{q}(l_{v^-})\bar{\tau}\big). \tag{1.35}$$

Remark 1.1. Our definition of Θ_L differs from the one given in [10] by the multiple $y^{b^-/2}$.

Theorem 4.1 in [10] says that $\Theta_L(\tau, v^+)$ is a real-analytic vector valued modular form of weight $1 - b^-/2$ and representation ρ_L with respect to variable τ.

We suppose that f is some $\mathbb{C}[L'/L]$-valued function on the upper half-plane $\mathfrak{H}$ transforming under $\mathrm{SL}_2(\mathbb{Z})$ with weight $1 - b^-/2$ and representation ρ_L. Define a *regularized theta integral* as

$$\Phi_L(v^+, f) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}}^{\mathrm{reg}} \langle f(\tau), \overline{\Theta_L(\tau, v^+)}\rangle\, y^{-1-b^-/2}\, dx\, dy \tag{1.36}$$

(the product of Θ_L and f means we take their inner product using $\langle e_\mu, e_\nu\rangle = 1$ if $\mu = \nu$ and 0 otherwise.)

The integral is often divergent and has to be regularized as follows. We integrate over the region $\mathcal{F}_t$, where

$$\mathcal{F}_\infty = \{\tau \in \mathfrak{H} | -1/2 < \Re(\tau) < 1/2 \text{ and } |\tau| > 1\}$$

is the usual fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathcal{F}_t$ is the subset of $\mathcal{F}_\infty$ of points τ with $\Im(\tau) < t$. Suppose that for $\Re(s) \gg 0$ the limit

$$\lim_{t\to\infty} \int_{F_t} \langle f(\tau), \overline{\Theta_L(\tau, v^+)}\rangle\, y^{-1-b^-/2-s}\, dx\, dy$$

exists and can be continued to a meromorphic function defined for all complex s. Then we define

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}}^{\mathrm{reg}} \langle f(\tau), \overline{\Theta_L(\tau, v^+)}\rangle\, y^{-1-b^-/2}\, dx\, dy$$

to be the constant term of the Laurent expansion of this function at $s = 0$.

As i Section 1.2, we denote by $\mathrm{Aut}(L)$ the group of those isometries of $L\otimes\mathbb{R}$ that fix all elements of L'/L. The regularized integral $\Phi_L(v^+, f)$ is a function on the Grassmannian $\mathrm{Gr}^+(L)$ that is invariant under $\mathrm{Aut}(L)$.

Suppose that $f \in \widehat{M}^!(\mathrm{Mp}_2(\mathbb{Z}), \rho_L)$ has the Fourier expansion

$$f_\mu(\tau) = \sum_{n\in\mathbb{Q}} \sum_{t\in\mathbb{Z}} c_\mu(n,t)\, \mathbf{e}(n,\tau)\, y^{-t}$$

and the coefficients $c_\mu(n,t)$ vanish whenever $n \ll 0$ or $t < 0$ or $t \gg 0$.

We will say that a function f has singularities of type g at a point if $f - g$ can be redefined on a set of codimension at least 1 so that it becomes real analytic near the point.

Then the following theorem which is proved in [10] describes the singularities of regularized theta lift $\Phi_L(v^+, f)$.

Theorem B1.([10] Theorem 6.2) *Near the point $v_0^+ \in \mathrm{Gr}^+(L)$, the function $\Phi_L(v^+, f)$ has a singularity of type*

$$\sum_{t\geq 0} \sum_{\substack{l\in L'\cap v_0^- \\ l\neq 0}} -c_{l+L}\big(\mathrm{q}(l), t\big)\, \big(-4\pi\, \mathrm{q}(l_{v^+})\big)^t \log\big(\mathrm{q}(l_{v^+})\big)/t!\,.$$

In particular Φ_L is nonsingular (real analytic) except along a locally finite set of codimension 2 sub Grassmannians (isomorphic to $\mathrm{Gr}^+(2, b^- - 1)$) of $\mathrm{Gr}^+(L)$ of the form $l^\perp$ for some negative norm vectors $l \in L$.

Recall that in Section 1.2 we have shown that the open subset

$$\mathcal{P} = \{[Z] \in \mathbb{P}(L\otimes\mathbb{C}) | (Z,Z) = 0 \text{ and } (Z,\overline{Z}) > 0\}$$

is isomorphic to $\mathrm{Gr}^+(L)$ by mapping $[Z]$ to the subspace $\mathbb{R}\Re(Z) + \mathbb{R}\Im(Z)$.
We choose $m \in L$, $m' \in L'$ such that $\mathrm{q}(m) = 0$, $(m, m') = 1$. Denote $V_0 := L\otimes\mathbb{Q}\cap m^\perp\cap m'^\perp$. The tube domain

$$\mathcal{H} = \{z \in V_0 \otimes_{\mathbb{R}} \mathbb{C} | (\Im(z), \Im(z)) > 0\} \tag{1.37}$$

is isomorphic to $\mathcal{P}$ by mapping $z \in \mathcal{H}$ to the class in $\mathbb{P}(L \otimes \mathbb{C})$ of

$$Z(z) = z + m' - \frac{1}{2}((z, z) + (m', m'))m.$$

We consider the lattices $M = L \cap m^\perp$ and $K = (L \cap m^\perp)/\mathbb{Z}m$, and we identify $K \otimes \mathbb{R}$ with the subspace $L \otimes \mathbb{R} \cap m^\perp \cap m'^\perp$.

We write N for the smallest positive value of the inner product (m, l) with $l \in L$, so that $|L'/L| = N^2|K'/K|$.

Suppose that $f = \sum_\mu e_\mu f_{L+\mu}$ is a modular form of type ρ_L and half integral weight k. Define a $\mathbb{C}[K'/K]$-valued function

$$f_K(\tau) = \sum_{\kappa \in K'/K} f_{K+\kappa}(\tau)\, e_\kappa$$

by putting

$$f_{K+\kappa}(\tau) = \sum_{\substack{\mu \in L'/L: \\ \mu|M = \kappa}} f_{L+\mu}(\tau)$$

for $\kappa \in K$. The notation $\lambda|M$ means the restriction of $\lambda \in \mathrm{Hom}(L, \mathbb{Z})$ to M, and $\gamma \in \mathrm{Hom}(K, \mathbb{Z})$ is considered as an element of $\mathrm{Hom}(M, \mathbb{Z})$ using the quotient map from M to K. The elements of L' whose restriction to M is 0 are exactly the integer multiples of m/N.

For $z \in \mathcal{H}$ denote by w^+ the following positive definite subspace of V_0

$$w^+(z) = \mathbb{R}\Im(z) \in \mathrm{Gr}^+(K). \tag{1.38}$$

Theorem 7.1 in [10] gives the Fourier expansion of the regularized theta lift and in the case when lattice L has signature $(2, b^-)$ this theorem can be reformulated at the following form.

Theorem B2. *Let L, K, m, m' be defined as above. Suppose*

$$f = \sum_{\mu \in L'/L} e_\mu \sum_{m \in \mathbb{Q}} c_\mu(m, y)\, \mathbf{e}(mx)$$

is a modular form of weight $1 - b^-$ and type ρ_L with at most exponential growth as $y \to \infty$. Assume that each function $c_\mu(m, y) \exp(-2\pi|m|y)$ has an asymptotic expansion as $y \to \infty$ whose terms are constants times products of complex powers of y and nonnegative integral powers of $\log(y)$. Let $z = u + iv$ be an element of a tube domain $\mathcal{H}$. If (v, v) is sufficiently large then the Fourier expansion of $\Phi_L(v^+(z), f)$ is given by the constant term of the Laurent expansion at $s = 0$ of the analytic continuation of

$$\sqrt{\mathrm{q}(v)}\Phi_K(w^+(z), f_K) + \frac{1}{\sqrt{\mathrm{q}(v)}} \sum_{l \in K'} \sum_{\substack{\mu \in L'/L: \\ \mu|M = l}} \sum_{n > 0} \mathbf{e}\big((nl, u - m') + (n\mu, m')\big) \times \tag{1.39}$$

$$\times \int_{y>0} c_\mu\left(\mathrm{q}(l), y\right) \exp\Big(-\frac{\pi n^2 \mathrm{q}(v)}{y} - \pi y\Big(\frac{(l,v)^2}{\mathrm{q}(v)} - 2\mathrm{q}(l)\Big)\Big)\, y^{-s-3/2}\, dy$$

(which converges for $\Re(s) \gg 0$ to a holomorphic functions of s which can be analytically continued to a meromorphic function of all complex s).

The lattice K has signature $(1, b^- - 1)$, so $\mathrm{Gr}^+(K)$ is real hyperbolic space of dimension $b^- - 1$ and the singularities of Φ_K lie on hyperplanes of codimension 1. Then the set of points where Φ_K is real analytic is not connected. The components of the points where Φ_K is real analytic are called the Weyl chambers of Φ_K. If W is a Weyl chamber and $l \in K$ then $(l, W) > 0$ means that l has positive inner product with all elements in the interior of W.

1.9 Infinite products

We see from Theorem B1 that the theta lift of a weakly holomorphic modular form has logarithmic singularities along special divisors. In [10] Borcherds shows that it's possible to exponentiate this function. The following theorem relates regularized theta lifts with infinite products introduced in the earlier paper [9].

THEOREM B3([10], Theorem 13.3) *Suppose that $f \in M^!_{1-b^-/2}(\mathrm{SL}_2(\mathbb{Z}), \rho_L)$ has the Fourier expansion*

$$f(\tau) = \sum_{\lambda \in L'/L} \sum_{n \gg -\infty} c_\lambda(n)\, \mathbf{e}(n\tau)\, e_\lambda.$$

and the Fourier coefficients $c_\lambda(n)$ are integers for $n \leq 0$. Then there is a meromorphic function $\Psi_L(Z, f)$ on $\mathcal{L}$ with the following properties.

1. *Ψ is an automorphic form of weight $c_0(0)/2$ for the group $\mathrm{Aut}(L, f)$ with respect to some unitary character of $\mathrm{Aut}(L, f)$*

2. *The only zeros and poles of Ψ_L lie on the rational quadratic divisors $l^\perp$ for $l \in L$, $\mathrm{q}(l) < 0$ and are zeros of order*

$$\sum_{\substack{x \in \mathbb{R}^+: \\ xl \in L'}} c_{xl}\big(\mathrm{q}(xl)\big)$$

3. $$\Phi_L(Z, f) = -4 \log|\Psi_L(Z, f)| - 2c_0(0)(\log|Y| + \Gamma'(1)/2 + \log\sqrt{2\pi}).$$

4. *For each primitive norm 0 vector m of L and for each Weyl chamber W of K the restriction $\Psi_m(Z(z), f)$ has an infinite product expansion converging when z is in a neighborhood of the cusp of m and $\Im(z) \in W$ which is some constant of the absolute value*

$$\prod_{\substack{\delta \in \mathbb{Z}/N\mathbb{Z} \\ \delta \neq 0}} (1 - \mathbf{e}(\delta/N))^{c_{\delta m/N}(0)/2}$$

times

$$\mathbf{e}((Z,\rho(K,W,f_K)))\prod_{\substack{k\in K':\\(k,W)>0}}\prod_{\substack{\mu\in L'/L:\\ \mu|M=k}}(1-\mathbf{e}((k,Z)+(\mu,m')))^{c_\mu(k^2/2)}.$$

The vector $\rho(K,W,f_K)$ is the Weyl vector, which can be evaluated explicitly using the theorems in Section 10 of [10].

Remark 1.2. In the case then L has no primitive norm 0 vectors Fourier expansions of Ψ do not exist.

1.10 See-saw identities

In the paper [50] S. Kudla introduced the notion of a *see-saw dual reductive pair* and proved a wide family of identities between inner products of automorphic forms on different groups, now called *see-saw identities.* His construction clarified the source of identities of this type which appeared in many places in the literature, often obtained from complicated manipulations.

We will use the same notations as in Section 1.6. Consider a dual reductive pair (G,G') in $\mathrm{Sp}(W)$. For automorphic forms f_1,f_2 on $\widetilde{G}(\mathbb{A})$ with f_1 a cusp form denote

$$\langle f_1,f_2\rangle_G=\int\limits_{\widetilde{Z}(\mathbb{A})G(k)\backslash G(\mathbb{A})} f_1(g)\,\overline{f_2(g)}\,dg$$

where dg is the Tamagawa measure and $\widetilde{Z}(\mathbb{A})$ is a center of $G(\mathbb{A})$. Let f and f' be a pair of cusp forms on $\widetilde{G}(\mathbb{A})$ and $\widetilde{G'}(\mathbb{A})$ respectively. For the theta lifts of f and f' given by (1.23) one obviously has the following adjointness formula:

$$\langle\theta_\varphi(f),f'\rangle_G=\langle f,\overline{\theta_\varphi(\overline{f'})}\rangle_{G'}. \tag{1.40}$$

A *see-saw dual pair* in $\mathrm{Sp}(W)$ is a pair (G,H'), (H,G') of dual pairs in $\mathrm{Sp}(W)$ such that

$$G\supset H \text{ and } G'\subset H.$$

The *"see-saw" identity* associated to such a pair is an immediate generalization of the adjointness formula (1.40). Let f and f' be cusp forms on $H(\mathbb{A})$ and $H'(\mathbb{A})$ respectively, then

$$\langle\theta_\varphi(f),f'\rangle_G=\langle f,\overline{\theta_\varphi(\overline{f'})}\rangle_{G'} \tag{1.41}$$

where functions $\theta_\varphi(f)$ and $\theta_\varphi(f')$ are restricted to H and H' respectively.

In this thesis the see-saw pair $\big(\mathrm{Sp}(W),\mathrm{O}(V)\big)$, $\big(\mathrm{Sp}(W)\times\mathrm{Sp}(W),\mathrm{O}(V')\times\mathrm{O}(V'')\big)$ plays an important role, where the $\dim W=2$ and $V=V'\oplus V''$. In Theorems 3.1 and 5.2 we prove identities associated to this pair for certain concrete subspaces of automorphic form. We should say that we rephrase identity (1.41) in the following way. In Sections 3.2

and 5.3 we prove that there exits a map $T : M(\mathrm{Sp}(W)) \to M'(\mathrm{Sp}(W))$ between certain subspaces of modular forms which will be specified in Sections 3.2 and 5.3 such that

$$\theta_\varphi(f)|_{\mathrm{O}(V')} = \theta_\varphi(T(f)). \tag{1.42}$$

Such reformulation of (1.41) is especially useful in the case of the regularized theta lift, when the Petersson scalar product might not converge.

Chapter 2

The restriction of Siegel Eisenstein series to Humbert surfaces

2.1 Introduction

In this chapter we compute the restriction of Siegel Eisenstein series from the Siegel half-space $\mathfrak{H}^{(2)}$ to Humbert surfaces.

The Siegel half-space $\mathfrak{H}^{(2)}$ is a multidimensional generalization of the Poincare half-plane $\mathfrak{H}$ consisting of complex 2×2 matrices with positive-definite imaginary part

$$\mathfrak{H}^{(2)} := \Big\{ Z = \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \Bigm| \Im(Z) > 0 \Big\}.$$

The Siegel Eisenstein series is defined as

$$E_k^{\mathrm{Sieg}}(Z) := \sum_{\{C,D\}} \det(CZ + D)^{-k},$$

where the sum is taken over the equivalence classes of coprime symmetric pairs.

The quotient space $\mathfrak{H}^{(2)}/\mathrm{Sp}_4(\mathbb{Z})$ is the moduli space of principally polarized abelian varieties of dimension 2. This manifold has a rich geometry and contains a lot of subvarieties with remarkable arithmetic and geometric properties [27].

For instance, for each $\Delta > 0$ there is a Hilbert modular surface of the real quadratic field $\mathbb{Q}(\sqrt{\Delta})$ in $\mathfrak{H}^{(2)}/\mathrm{Sp}_4(\mathbb{Z})$, called the Humbert variety $\mathfrak{H}_\Delta$. In the simplest case $\Delta = 1$ the surface $\mathfrak{H}_1$ is the image of all diagonal matrices $\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \in \mathfrak{H}^{(2)}$ and is isomorphic to $\mathfrak{H}/\mathrm{SL}_2(\mathbb{Z}) \times \mathfrak{H}/\mathrm{SL}_2(\mathbb{Z})$. The restriction formula for Siegel Eisenstein series in this case is well known (e.g. [26]) and reads

$$E_k^{\mathrm{Sieg}} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = \sum_{i=1}^{\dim M_k} \mu_i \, g_i(\tau_1) \, g_i(\tau_2).$$

Here the coefficients μ_i are related to a special value of the symmetric square L-function of the normalized eigenforms $g_i \in M_k(\mathrm{SL}_2(\mathbb{Z}))$

$$\mu_i = \frac{2^{8-4k} \, k! \, (2k-3)!}{B_k \, B_{2k-2}} \cdot \frac{D(g_i, 2k-2)}{\|g_i\|^2 \, \pi^{3k-3}},$$

where $D(g,s)$ is defined in Section 2.3 for cusp forms, and $\mu_1 = (2k/B_k)^2$ for the Eisenstein series $g_1 = G_k = -B_k/2k + \sum_n(\sum_{m|n} m^{k-1})q^n$.

In Theorem 2.1 we generalize this formula for $\mathfrak{H}_\Delta$, when Δ is prime and equals 1 modulo 4.

2.2 Eisenstein series

In this Chapter we consider several different spaces of modular forms. Each of them can be decomposed into two parts: the space of cusp forms and the space of Eisenstein series. These two subspaces are orthogonal to each other with respect to Peterson scalar product.

In the case of ordinary modular forms $M_k(SL_2(\mathbb{Z}))$ the space of Eisenstein series is one-dimensional and it is spanned by the function

$$E_k(\tau) = \frac{1}{2}\sum_{(c,d)}{}' \frac{1}{(c\tau+d)^k},$$

where the summation is taken over all coprime pairs of integers. We will use the following different normalization of the Eisenstein series

$$G_k(\tau) = \frac{-B_k}{2k}E_k(\tau) = \frac{-B_k}{2k} + \sum_n \Big(\sum_{m|n} m^{k-1}\Big) e^{2\pi i n\tau}.$$

The function G_k is a normalized Hecke eigenform, i.e. it is a common eigenvector for all Hecke operators and the Fourier coefficient at $e^{2\pi i\tau}$ equals 1.

Denote by $M_k(\Gamma_0(p),\chi)$ the space of modular forms on the group $\Gamma_0(p)$ of weight k and character $\chi = (\frac{\cdot}{p})$ (see Section 1.7 for the definition). The space of Eisenstein series in $M_k(\Gamma_0(p),\chi)$ has dimension 2 if p is prime. It is spanned by the Hecke eigenforms

$$E_{\chi,k} = \frac{L(1-k,\chi)}{2} + \sum_{n=1}^{\infty}\sum_{m|n} m^{k-1}\,\chi_d(m)\,q^n$$

and

$$G_{\chi,k} = \sum_{n=1}^{\infty}\sum_{m|n} m^{k-1}\,\chi(n/m)\,q^n.$$

For a fundamental discriminant $\Delta > 0$ denote by $M_k(\mathrm{SL}_2(\mathfrak{o}_\Delta))$ the space of Hilbert modular forms for the group $\mathrm{SL}_2(\mathfrak{o}_\Delta)$. The definition of Hilbert modular forms is given in Section 1.7. The dimension of the space of Hilbert Eisenstein series in $M_k(\mathrm{SL}_2(\mathfrak{o}_\Delta))$ is equal to the class number of $\mathbb{Q}(\sqrt{\Delta})$. More explicitly, for each ideal class C of K, set

$$E_k^{\mathrm{Hilb}}(\tau_1,\tau_2;C) = N(\mathfrak{a})^k \sum_{(\mu,\nu)\in(\mathfrak{a}\times\mathfrak{a}-\{(0,0)\})/\mathfrak{o}^*} \frac{1}{(\mu\tau_1+\nu)^k\,(\mu'\tau_2+\nu')^k},$$

where τ_1,τ_2 are in $\mathfrak{H}$ and $\mathfrak{a}$ is any ideal in C. The Eisenstein series $E_k^{\mathrm{Hilb}}(\cdot,C)$, $C\in\mathrm{CL}(K)$, are linearly independent and span the space $\mathcal{E}_k(SL_2(\mathfrak{o}_\Delta))$. Consider the finite sum

$$E_k^{\mathrm{Hilb}}(\tau_1,\tau_2) := \sum_C E_k^{\mathrm{Hilb}}(\tau_1,\tau_2;C).$$

Recall that in Section 1.7 we have defined the map $\mathcal{N} : S_k(\Gamma_0(\Delta), \chi) \to S_k^{\mathrm{Hilb}}(SL_2(\mathfrak{o}_\Delta))$. The map $\mathcal{N}$ can be defined not only for cusp forms but also can be extended to the whole space

$$M_k(\Gamma_0(\Delta), \chi) = \mathcal{E}_k(\Gamma_0(\Delta), \chi) \oplus S_k(\Gamma_0(\Delta), \chi).$$

The space $\mathcal{E}_k(\Gamma_0(\Delta), \chi)$ is spanned by two functions

$$E_k^{\pm} = \frac{L(1-k, \chi_d)}{2} + \sum_{n=1}^{\infty} \sum_{m|n} m^{k-1}(\chi_d(m) \pm \chi_d(n/m))q^n.$$

For them we have $\mathcal{N}(E_k^-) = 0$ and $\mathcal{N}(E_k^+) \doteq E_k^{\mathrm{Hilb}}$.

The Siegel Eisenstein series is given by

$$E_k^{\mathrm{Sieg}}(Z) = \sum_{\{C,D\}} \det(CZ+D)^{-k},$$

where the sum is taken over the equivalence classes of coprime symmetric pairs. We recall that the pair of matrices is called symmetric if $C^tD = D^tC$. We say that two pairs $\{C_1, D_1\}$ and $\{C_2, D_2\}$ are equivalent if there exists a unimodular matrix U such that $UC_1 = C_2$ and $UD_1 = D_2$. And finally, the pair $\{C, D\}$ is coprime if the matrices XC, XD are integral only for X integral. We will also use a different normalization of Siegel Eisenstein series

$$G_k^{\mathrm{Sieg}} = \frac{\zeta(1-k)\,\zeta(3-2k)}{2} E_k^{\mathrm{Sieg}}.$$

Siegel Eisenstein series posseses the Fourier expansion of the form

$$G_k^{\mathrm{Sieg}}(Z) = \sum_T A(T)\,\mathbf{e}(\mathrm{tr}TZ),$$

where the sum is taken over semi-definite half-integral symmetric matrices. The coefficients $A(T)$ are defined in the following way. If D is a fundamental discriminant denote by $L_D(s)$ the L-series $L(s, (\frac{D}{\cdot}))$. For all $D \in \mathbb{Z}$ we define

$$L_D(s) = \begin{cases} 0 & \text{if } D \equiv 2,3 \bmod 4, \\ \zeta(1-2s) & \text{if } D = 0, \\ L_{D_0}(s) \sum_{d|f} \mu(d)(\frac{D_0}{d}) d^{-s} \sigma_{1-2s}(f/d) & \text{if } D \equiv 0,1 \bmod 4,\ D \neq 0, \end{cases}$$

where in the last line $D = D_0 f^2$ with $f \in \mathbb{N}$ and D_0 equals to a discriminant of $\mathbb{Q}(\sqrt{D})$. The values $L_D(2-k)$ are well-known to be rational and non-zero. They were extensively studied by H. Cohen [18], who used the notation

$$H(k-1, |D|) = L_D(2-k).$$

Then the Fourier coefficients of Siegel Eisenstein series equal

$$A\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} = \sum_{l|(m,n,r)} l^{k-1} H\Big(k-1, \frac{4mn-r^2}{l^2}\Big),$$

$$A\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \frac{\zeta(1-k)}{2}.$$

2.3 Symmetric square L-function

Let $f = \sum a(n)e^{2\pi i n\tau}$ be an eigenform in the space $S_k(\Gamma_0(\Delta), \chi)$. For each prime q let α_q and β_q be the roots of the polynomial

$$x^2 - a_q x + \chi(q)q^{k-1}.$$

Then the symmetric square L-function attached to f is defined by the Euler product

$$D(f,s) := \prod_q \left((1-\alpha_q^2 q^{-s})(1-\beta_q^2 q^{-s})(1-\alpha_q\beta_q q^{-s})\right)^{-1}.$$

The following identity holds

$$\sum_m a(m^2)m^{-s} = \frac{D(f,s)}{\zeta(2s-2k+2)}.$$

2.4 Restriction formula

Futher we assume $\Delta = p$ to be prime and equal 1 modulo 4.

Theorem 2.1. *The pullback of the Siegel Eisenstein series via the map ϱ defined in* (1.6) *equals*

$$E_k^{\mathrm{Sieg}}(\varrho(\tau_1,\tau_2)) = \sum_{i=1}^{\dim M_k(\Gamma_0(p),\chi)} \lambda_i \mathcal{N}(f_i)(\tau_1,\tau_2),$$

where $f_i(\tau) = \sum_m a_i(m)e^{2\pi i m\tau}$ are the normalized Hecke eigenforms in $M_k(\Gamma_0(p),\chi)$ and

$$\lambda_i = \frac{2^{8-4k}k!(2k-3)!}{B_k B_{2k-2}} \cdot \left(1 + \frac{a(p)^2}{p^{2k-2}}\right) \cdot \frac{D(f_i, 2k-2)}{\|f_i\|^2\pi^{3k-3}}. \tag{2.1}$$

If $p \equiv 1 \bmod 4$ we can choose the basis of $\mathfrak{o}$ to be $\rho_1 = \frac{1+\sqrt{p}}{2}$ and $\rho_2 = \frac{1-\sqrt{p}}{2}$, in this case

$$R = \begin{pmatrix} \frac{1+\sqrt{p}}{2} & \frac{1-\sqrt{p}}{2} \\ \frac{1-\sqrt{p}}{2} & \frac{1+\sqrt{p}}{2} \end{pmatrix} \text{ and } \det R = \sqrt{p}.$$

Let $\{C, D\}$ be a symmetric pair. Then the matrix $\tilde{R}\tilde{C}D^t\tilde{R}$ has the form

$$\tilde{R}\tilde{C}D^t\tilde{R} =: \begin{pmatrix} \sigma & s \\ s & \sigma' \end{pmatrix}, \tag{2.2}$$

where $\sigma \in \mathfrak{o}$, and $s \in \mathbb{Z}$ (we denote by $\tilde{X}$ the adjoint of the matrix X).

Let $\mathcal{Q}$ be the set of all triples (a, b, Q), where $a, b \in \mathbb{Z}$, Q is an integral symmetric matrix and $\det Q = ab$. We will denote by $\mathrm{cont}(Q)$ the content of the matrix Q, i.e. the greatest common divisor of elements of Q. Let $\mathcal{Q}'$ be the set of all elements $(a, b, Q) \in \mathcal{Q}$ with $\gcd(a, b, \mathrm{cont}Q) = 1$. In order to prove Theorem 1 we will need the following technical statement.

Lemma 2.1. *The map φ defined by*

$$\{C, D\} \xrightarrow{\varphi} (\det C, \det D, \tilde{C}D)\ , \tag{2.3}$$

gives a one-to-one correspondence between the set of equivalence classes of symmetric coprime pairs and $\mathcal{Q}'/\sim$.

Proof. First we show that the map φ is well-defined. It is clear that equivalent pairs are mapped to equivalent triples. We have to show that the image of a coprime pair belongs to $\mathcal{Q}'$. Assume that for the coprime pair $\{C, D\}$ some prime q divides the greatest common divisor of $(\det C, \det D, \mathrm{cont}(\tilde{C}D))$. Since the pair $\{C, D\}$ is coprime, q can not divide both $\mathrm{cont}(C)$ and $\mathrm{cont}(D)$. Without loss of generality suppose that $q \nmid \mathrm{cont}(C)$ (otherwise we can change $\{C, D\}$ by the pair $\{{}^tD, {}^tC\}$) and consider the matrix $X = \frac{1}{q}\tilde{C}$. The matrix X is not integral, although the matrices XC, XD are. Thus, we have got a contradiction with the assumption that the pair $\{C, D\}$ is coprime.

The next step is to show that the map φ is surjective. Fix some $(a, b, Q) \in \mathcal{Q}'$. We have to show that there exists a coprime symmetric pair $\{C, D\}$ such that

$$\det C = a, \quad \det D = b \quad \text{and} \quad \tilde{C}D = Q. \tag{2.4}$$

For some unimodular matrices U_1, U_2 the matrix U_1QU_2 is diagonal. We can write this matrix as a product over prime numbers $U_1QU_2 = \prod_q Q_q$, where $Q_q = \begin{pmatrix} q^{\alpha_q} & 0 \\ 0 & q^{\beta_q} \end{pmatrix}$ (we will use the notation $q^\infty = 0$). Let $q^{\alpha'}||a$ and $q^{\beta'}||b$. Since $\gcd(a, b, \mathrm{cont}Q) = 1$, one of the numbers $\alpha, \beta, \alpha'_q, \beta'_q$ should be zero. Consider these four cases:

- if $\alpha'_q = 0$, we define $C_q := E$ and $D_q := Q_q$,
- if $\beta'_q = 0$, we define $C_q := Q_q$ and $D_q := E$,
- if $\alpha_q = 0$, we define $C_q := \begin{pmatrix} q^{\alpha'} & 0 \\ 0 & 1 \end{pmatrix}$ and $D_q := \begin{pmatrix} 1 & 0 \\ 0 & q^{\beta'} \end{pmatrix}$,
- if $\beta_q = 0$, we define $C_q := \begin{pmatrix} 1 & 0 \\ 0 & q^{\alpha'} \end{pmatrix}$ and $D_q := \begin{pmatrix} q^{\beta'} & 0 \\ 0 & 1 \end{pmatrix}$,

It is easy to see that in all these cases the pair $\{C_q, D_q\}$ defined above is coprime. Thus we can define the pair $\{C, D\}$ as

$$C := U_2^{-1}(\prod_q C_q)U_1^{-1} \text{ and } D := U_1^{-1}(\prod_q D_q)U_2^{-1}.$$

Obviously, this pair is symmetric, coprime and satisfies conditions (2.4).

Finally, it remains to show that the pair corresponding to $(a, b, Q) \in \mathcal{Q}'$ is unique up to equivalence. Assume that there exist two such pairs $\{C_1, D_1\}$ and $\{C_2, D_2\}$. In this case $C_1^{-1}D_1 = C_2^{-1}D_2$. The matrix $U := C_2C_1^{-1}$ has determinant 1 and

$$UC_1 = C_2 \quad \text{and} \quad UD_1 = D_2. \tag{2.5}$$

Since the pair $\{C_1, D_1\}$ is coprime, the matrix U is integral, and hence, unimodular. Hence, it follows from (2.5) that the pairs $\{C_1, D_1\}$ and $\{C_2, D_2\}$ are equivalent. $\square$

For an integer $m \geq 0$ denote by $\mathcal{T}_m$ the set of triples (a, b, σ) where $a \in \mathbb{Z},\ b \in \mathbb{Z},\ \sigma \in \mathfrak{o}$ and

$$\sigma\sigma' - pab = m^2. \tag{2.6}$$

Let $\mathcal{T} := \bigcup_m \mathcal{T}_m$. Now we prove

Lemma 2.2. *The map* ψ

$$(a, b, Q) \xrightarrow{\psi} (a, b, \sigma)\ , \tag{2.7}$$

where σ *is defined by* $\tilde{R}Q\,{}^t\tilde{R} = \left(\begin{smallmatrix}\sigma & s\\ s & \sigma'\end{smallmatrix}\right)$, *maps* $\mathcal{Q}$ *to* $\mathcal{T}$. *Each element in* $\bigcup_{p\nmid m} \mathcal{T}_m$ *has exactly* 1 *preimage and each element in* $\bigcup_{p|m} \mathcal{T}_m$ *has* 2 *preimages.*

Proof. First we show that ψ maps $\mathcal{Q}$ to $\mathcal{T}$. Let $(a, b, Q) \in \mathcal{Q}$ be mapped to the triple (a, b, σ) by ψ. It is easy to see, that $\sigma \in \mathfrak{o}$ and $s \in \mathbb{Z}$. Since

$$\det \left(\begin{smallmatrix}\sigma & s\\ s & \sigma'\end{smallmatrix}\right) = pab, \tag{2.8}$$

the triple (a, b, σ) is an element of $\mathcal{T}_{|s|}$.

Let $(a, b, \sigma) \in \mathcal{T}_m$. It follows from (2.8) that (a, b, σ) can only have 2 preimages $(a, b, Q_\pm)$, where

$$Q_\pm = \frac{1}{p} R \begin{pmatrix}\sigma & \pm m\\ \pm m & \sigma'\end{pmatrix}\ {}^tR.$$

It follows from (2.6) that

$$m^2 - (\mathrm{tr}(\sigma)/2)^2 \equiv 0 \bmod p.$$

Define s to be either m or $-m$ so that $s + \mathrm{tr}(\sigma)/2 \equiv 0 \bmod p$. For such choice of s the matrix

$$Q := \frac{1}{p} R \left(\begin{smallmatrix}\sigma & s\\ s & \sigma'\end{smallmatrix}\right)\ {}^tR$$

is integral, $(a, b, Q) \in \mathcal{Q}$ and $\psi(a, b, Q) = (a, b, \sigma)$. The lemma is proved. □

Proof of Theorem 2.1. A corollary of Lemma 2.1 is

$$E_k^{\mathrm{Sieg}}(Z) = \frac{1}{2} \sum_{(a,b,Q)\in\mathcal{Q}'} a^k \det(aZ + Q)^{-k}.$$

Since each non-zero element $(a, b, Q) \in \mathcal{Q}$ can be uniquely written as $(\lambda a, \lambda b, \lambda Q)$ for some $(a, b, Q) \in \mathcal{Q}'$ and $\lambda \in \mathbb{N}$, we have

$$E_k^{\mathrm{Sieg}}(Z) = \frac{1}{2\zeta(k)} \sum_{(a,b,Q)\in\mathcal{Q}} a^k \det(aZ + Q)^{-k}. \tag{2.9}$$

Let $(a, b, \sigma) \in \mathcal{T}$ corresponds (a, b, Q) via the map ψ (see (2.7)). From a simple computation we get

$$a^{-1} \det(a\rho(\tau_1, \tau_2) + Q) = ap\tau_1\tau_2 + \sigma\tau_1 + \sigma'\tau_2 + b.$$

Hence it follows from Lemma 2.2 that

$$E_k^{\mathrm{Sieg}}(\rho(\tau_1,\tau_2)) = \frac{1}{2\zeta(k)} \sum_{p\nmid m} \sum_{(a,b,\sigma)\in\mathcal{T}_m} \frac{1}{(ap\tau_1\tau_2+\sigma\tau_1+\sigma'\tau_2+b)^k} + \tag{2.10}$$
$$+ \frac{1}{\zeta(k)} \sum_{p|m} \sum_{(a,b,\sigma)\in\mathcal{T}_m} \frac{1}{(ap\tau_1\tau_2+\sigma\tau_1+\sigma'\tau_2+b)^k}.$$

For the fundamental unit ϵ the number $\eta := \epsilon\sqrt{p}$ is totally positive and has norm p. We can write

$$ap\tau_1\tau_2+\sigma\tau_1+\sigma'\tau_2+b = a\eta\tau_1\eta'\tau_2+\lambda\eta\tau_1+\lambda'\eta'\tau_2+b,$$

where $\lambda = \sigma/\eta \in \mathfrak{D}^{-1}$. Thus from definitions (1.6) and (1.27) we get

$$E_k^{\mathrm{Sieg}}(\varrho(\tau_1,\tau_2)) = \frac{1}{2\zeta(k)}\left(\sum_{m=0}^{\infty}\omega_{m^2}(\tau_1,\tau_2)+\sum_{p|m}\omega_{m^2}(\tau_1,\tau_2)\right).$$

We can reformulate identities the (1.27) and (1.31) to obtain

$$\omega_m \doteq m^{-k+1}\sum_{i=1}^{\dim S_k(\Gamma_0(p),\chi)} \frac{a_i(m)}{\|f_i\|^2}\mathcal{N}(f_i), \quad m \geq 1,$$

where $f_i = \sum_{n=1}^{\infty} a_i(n)q^n$, $i = 1,\dots,\dim S_k(\Gamma_0(p),\chi)$ denote the normalized Hecke eigenforms. From [74] p. 30 we find that the right-hand side should be multiplied by $c_k = \frac{(-1)^{k/2}\pi}{2^{k-3}(k-1)}$ in order to get exact equality. Hence we arrive at

$$E_k^{\mathrm{Sieg}}(\varrho(\tau_1,\tau_2)) = \frac{c_k}{\zeta(k)}\omega_0(\tau_1,\tau_2) + \sum_{i=1}^{\dim S_k(\Gamma_0(p),\chi)} \lambda_i\mathcal{N}(f_i)(\tau_1,\tau_2),$$

where

$$\lambda_i = \frac{c_k}{2\zeta(k)\|f_i\|^2}\left(\sum_{m=1}^{\infty}\frac{a_i(m^2)}{m^{2k-2}}+\sum_{m=1}^{\infty}\frac{a_i(m^2p^2)}{m^{2k-2}p^{2k-2}}\right).$$

It is shown in [74] that ω_0 is a multiple of the Hecke-Eisenstein series

$$\omega_0(\tau_1,\tau_2) = \frac{\zeta(k)}{\zeta_K(k)}\,E_k^{\mathrm{Hilb}}(\tau_1,\tau_2).$$

This finishes the proof. □

2.5 Example $p = 5$

In this section we check the restriction formula numerically in the case $p = 5$.

The dimensions of the spaces of cusp forms of small weight are given in the following table.

k	2	4	6	8	10	12	14	16	18	20
$\dim S_k(SL_2(\mathbb{Z}))$	0	0	0	0	0	1	0	1	1	1
$\dim S_k(\Gamma_0(5),\chi)$	0	0	2	2	4	4	6	6	8	8
$\dim S_k(SL_2(\mathfrak{o}))$	0	0	1	1	2	3	3	4	5	7

For each even $k > 2$ the dimensions of corresponding spaces of Eisenstein series equal

$$\dim \mathcal{E}_k(SL_2(\mathbb{Z})) = 1, \ \ \dim \mathcal{E}_k(\Gamma_0(5),\chi) = 2, \ \ \dim \mathcal{E}_k(SL_2(\mathfrak{o})) = 1.$$

In this section we will check the identity

$$\frac{2}{\zeta(3-2k)} G_k^{\text{Sieg}} \circ \varrho = \frac{1}{L_\chi(-k+1)} G_k^{\text{Hilb}} + \sum_{i=1}^{\dim S_k(\Gamma_0(5),\chi)} \lambda_i \mathcal{N} f_i \tag{2.11}$$

for several values of k. The Fourier expansion of Hilbert Eisenstein series is given by

$$G_k^{\text{Hilb}}(\tau_1,\tau_2) = \zeta_{\mathbb{Q}(\sqrt{5})}(-k+1) + \sum_{\substack{\nu\in\mathfrak{D}^{-1}\\ \nu\gg 0}} \sum_{\mathfrak{b}|(\nu)\mathfrak{D}} N(\mathfrak{b})^{k-1} e^{2\pi i(\nu\tau_1+\nu'\tau_2)}.$$

We will compare Fourier coefficients of both sides of (2.11) for several $\nu \in \mathfrak{D}^{-1}$. The Fourier coefficients of the functions involved in (2.11) are given in the following table:

$\sqrt{5}\nu$	$G_k^{\text{Sieg}} \circ \varrho$	G_k^{Hilb}	$\mathcal{N} f$
0	$\frac{\zeta(1-k)\zeta(3-2k)}{2}$	$\zeta_{\mathbb{Q}(\sqrt{5})}(-k+1)$	0
$\frac{1+\sqrt{5}}{2}$	$H(k-1,0) = \zeta(3-2k)$	1	1
$1+\sqrt{5}$	$H(k-1,-3) + (2^{k-1}+1)H(k-1,0)$	$4^{k-1}+1$	$a(2)^2+2^k$
$\sqrt{5}$	$H(k-1,-4)$	$5^{k-1}+1$	$a(5)+\bar{a}(5)$
$\frac{1+3\sqrt{5}}{2}$	$H(k-1,-7)+H(k-1,-8)$	$11^{k-1}+1$	$a(11)$

Case $\mathbf{k=4}$. In this case the equation (2.11) becomes

$$\frac{2}{\zeta(-5)} G_4^{\text{Sieg}} \circ \varrho = \frac{1}{L_\chi(-3)} G_4^{\text{Hilb}}. \tag{2.12}$$

We compute the values of L-functions

$$\zeta(-5) = \frac{-1}{252}, \qquad L_\chi(-3) = \frac{1}{2},$$

and the values of $H(3,D)$ are given in the table below

$H(3,0)$	$H(3,-3)$	$H(3,-4)$	$H(3,-7)$	$H(3,-8)$
$\frac{-1}{252}$	$\frac{-2}{9}$	$\frac{-1}{2}$	$\frac{-16}{7}$	-3

Substituting these values into the table we verify that the identity (2.12) holds:

$\sqrt{5}\nu$	$\frac{1}{\zeta(-5)} G_4^{\mathrm{Sieg}} \circ \varrho$	G_4^{Hilb}
0	$\frac{1}{240}$	$\frac{1}{240}$
$\frac{1+\sqrt{5}}{2}$	1	1
$1+\sqrt{5}$	65	65
$\sqrt{5}$	126	126
$\frac{1+3\sqrt{5}}{2}$	1332	1332

Case k = 6.

The space $S_6(\Gamma_0(5), \chi)$ is spanned by the two eigenforms $f = \sum a(n) q^n$ and $f^\rho = \sum \overline{a(n)} q^n$, where the first coefficients of f are given in the table

n	$a(n)$
1	1
2	$2\sqrt{-11}$
3	$-6\sqrt{-11}$
4	-12
5	$-45+10\sqrt{-11}$
6	132
7	$-18\sqrt{-11}$
8	$40\sqrt{-11}$
9	153
10	$-220-90\sqrt{-11}$
11	252

The equation (2.11) can be written as

$$\frac{2}{\zeta(-9)} G_6^{\mathrm{Sieg}} \circ \varrho = \frac{1}{L_\chi(-5)} G_6^{\mathrm{Hilb}} + 2\lambda \mathcal{N}(f),$$

where λ is a special value of the symmetric square L-function.

We find that

$$\zeta(-9) = \frac{-1}{132}, \qquad L_\chi(-5) = \frac{-67}{10},$$

and the values of $H(5, D)$ are given in the following table

$H(5,0)$	$H(5,-3)$	$H(5,-4)$	$H(5,-7)$	$H(5,-8)$
$\frac{-1}{132}$	$\frac{2}{3}$	$\frac{5}{2}$	32	57

The Fourier coefficients of the Eisenstein series and the Doi-Naganuma lift are given in the table:

$\sqrt{5}\nu$	$\frac{1}{\zeta(-9)} G_6^{\text{Sieg}} \circ \varrho$	G_6^{Hilb}	$\mathcal{DN} f$
0	$-\frac{1}{504}$	$\frac{67}{2520}$	0
$\frac{1+\sqrt{5}}{2}$	1	1	1
$1+\sqrt{5}$	-55	1025	20
$\sqrt{5}$	-330	3126	-90
$\frac{1+3\sqrt{5}}{2}$	-11748	161052	252

From that we find

$$\lambda = \frac{72}{67}.$$

This agrees with the value found directly from (2.1).

Case k = 12. This is the first interesting case, when the space $S_k(SL_2(\mathfrak{o}))$ is not spanned by Naganuma lifts of elements from $S_k(\Gamma_0(5),\chi)$.

The space $S_{12}(\Gamma_0(5),\chi)$ has dimension 4. It is spanned by the functions $f_i = \sum a_i(n)q^n$ and $f_i^\rho = \sum \overline{a_i(n)}q^n,\ i = 1,2$, where the first coefficients of f_i are given in the table

n	$a_i(n)$
1	1
2	α_i
3	$\frac{-1}{112}\alpha_i^3 - \frac{725}{28}\alpha_i$
4	$\alpha_i^2 + 2048$
5	$\frac{5}{112}\alpha_i^3 + 5\alpha_i^2 + \frac{3345}{28}\alpha_i + 10255$
6	$11\alpha_i^2 + 22288$
7	$-\frac{17}{16}\alpha_i^3 - \frac{12949}{4}\alpha_i$
8	$\alpha_i^3 + 4096\alpha_i$
9	$-78\alpha_i^2 - 154785$
10	$5\alpha_i^3 - 65\alpha_i^2 + 10225\alpha_i - 111440$
11	$-220\alpha_i^2 - 536108$

Here α_1, $\overline{\alpha}_1$ and α_2, $\overline{\alpha}_2$ are solutions of the equation

$$a^4 + 4132a^2 + 2496256 = 0.$$

Thus, we check the identity

$$\frac{2}{\zeta(-21)} G_{12}^{\text{Sieg}} \varrho = \frac{1}{L_\chi(-11)} G_{12}^{\text{Hilb}} + 2\lambda_1 \mathcal{N} f_1 + 2\lambda_2 \mathcal{N} f_2.$$

We compute

$$\zeta(-21) = -\frac{77683}{276},$$
$$L_\chi(-11) = \frac{1150921}{2},$$

and

$$\frac{1}{\zeta(-11)} = \frac{32760}{691} \qquad \frac{2}{\zeta_{\mathbb{Q}(\sqrt{5})}(-11)} = \frac{131040}{795286411}.$$

The values of $H(11, D)$ are given in a table

$H(11,0)$	$H(11,-3)$	$H(11,-4)$	$H(11,-7)$	$H(11,-8)$
$-\frac{77683}{276}$	$-\frac{3694}{3}$	$-\frac{50521}{2}$	-9006448	-36581523

The Fourier expansions of the functions are equal to

$\sqrt{5}\nu$	$G_{12}^{\mathrm{Sieg}} \circ \varrho$	G_{12}^{Hilb}	$\mathcal{DN} f_1$	$\mathcal{DN} f_2$
0	$-\dfrac{53678953}{18083520}$	$-\dfrac{795286411}{65520}$	0	0
$\dfrac{1+\sqrt{5}}{2}$	$-\dfrac{77683}{276}$	1	1	1
$1+\sqrt{5}$	$-\dfrac{159512315}{276}$	4194305	$2030+30\sqrt{1969}$	$2030-30\sqrt{1969}$
$\sqrt{5}$	$-\dfrac{50521}{2}$	48828126	$-150+300\sqrt{1969}$	$-150-300\sqrt{1969}$
$\dfrac{1+3\sqrt{5}}{2}$	-45587971	285311670612	$-81588-6600\sqrt{1969}$	$-81588+6600\sqrt{1969}$

From the table we find

$$\lambda_1 = \frac{140\left(-7102265 - 10797937\sqrt{1969}\,\right)}{89406996043\sqrt{1969}} = \frac{1}{L_\chi(-11)\zeta(-21)}\,\frac{15\left(7102265 + 10797937\sqrt{1969}\,\right)}{2\sqrt{1969}},$$

$$\lambda_2 = \frac{140\left(7102265 - 10797937\sqrt{1969}\,\right)}{89406996043\sqrt{1969}} = \frac{1}{L_\chi(-11)\zeta(-21)}\,\frac{15\left(-7102265 + 10797937\sqrt{1969}\,\right)}{2\sqrt{1969}}.$$

An interesting observation is that the denominator of the product

$$\lambda_1\,\lambda_2 = -2^9\cdot 3\cdot 5^2\cdot 7^4\cdot 11^{-1}\cdot 79\cdot 131^{-2}\cdot 179^{-1}\cdot 593^{-2}\cdot 536651\cdot 1150921^{-1}$$

is divisible by the prime 1150921, which also divides the special value $L_\chi(-11) = \frac{1150921}{2}$. A similar phenomenon holds also in the case $k = 6$. In the level one case congruences of this type are discussed in [24].

Chapter 3

Modular surfaces, modular curves, and modular points

3.1 Introduction

In this chapter we generalize the see-saw identity obtained in the previous section to the case of the paramodular group.

Using this identity we give a much simpler proof of the modularity of the generating series $\sum P_d q^d$ of Heegner points on the modular curve $X_0(N)$. We follow the idea explained in [76], which has been applied to $X_0(37)$ there. In Section 3.3 we prove Theorem 3.2 which was formulated in conjectural form in [76]. This allows to apply the method developed in [76] to any prime level N.

3.2 Pullbacks of Gritsenko lifts

In this section we compute the restriction of the Gritsenko lift of a half intergral weight modular form to the Humbert surfaces, which were defined in Section 1.2.

Let

$$h(\tau) = \sum b(n)\,\mathbf{e}(n\tau) \in M^{+}_{k-1/2}\big(\Gamma_0(4N)\big)$$

be a modular form of half-integral weight and let

$$F(Z) = \sum_{T} B(T)\,\mathbf{e}\big(\mathrm{tr}(TZ)\big) \in M_k^{\mathrm{Sieg}}(\Gamma_N)$$

be the Gritsenko lift of h. It follows from (1.34) and (1.21) that

$$B\begin{pmatrix} m & r/2 \\ r/2 & Nn \end{pmatrix} = \sum_{l\mid(m,n,r)} l^{k-1}\, b\Big(\frac{4Nmn - r^2}{l^2}\Big). \tag{3.1}$$

In what follows we will use the operator U_N defined by

$$\sum_{n=0}^{\infty} a(n)\,\mathbf{e}(n\tau) \mid U_N := \sum_{n=0}^{\infty} a(Nn)\,\mathbf{e}(n\tau).$$

For $M \mid N$ this operator maps $M_k(\Gamma_0(M))$ to $M_k(\Gamma_0(N))$, but in certain situations it can even decrease the level. For example, see [77].

Lemma 3.1. *Let $h \in M^+_{k-1/2}(\Gamma_0(4N))$ be a half-integral weight modular form and let θ be the standard theta-function* (1.36)*. Fix a positive prime discriminant p with $\left(\frac{N}{p}\right) = 1$. Then the function*

$$g := U_{4N}[\theta(\tau)h(p\tau)]$$

belongs to $M_k(\Gamma_0(p), \chi)$.

Let $N, p, \mathfrak{a}$ be as in Section 1.7. Denote by $\mathcal{H}_N$ the set of semi-definite half-integral matrices of the form $\begin{pmatrix} m & r/2 \\ r/2 & Nn \end{pmatrix}$, and let $\mathcal{S}_{\mathfrak{a}}$ be the set of couples (σ, s) such that

$$\sigma \in \mathfrak{a},\ \sigma \gg 0,\ s \in \frac{1}{p}\mathbb{Z},$$

$$4\sigma\sigma' - s^2 \geq 0 \text{ and } (\mathrm{tr}(\sigma) - s) \in \mathbb{Z}.$$

Let

$$T = \begin{pmatrix} m & r/2 \\ r/2 & nN \end{pmatrix} \in \mathcal{H}_N \text{ and } {}^tRTR = \begin{pmatrix} \sigma & s/2 \\ s/2 & \sigma' \end{pmatrix}, \tag{3.2}$$

where the matrix R is defined in (1.6). Further assume that p is prime.

Lemma 3.2. *Denote by ι the map*

$$T \xrightarrow{\ \iota\ } (\sigma, s)\ , \tag{3.3}$$

where σ and s are defined from (3.2)*. Then*

a) the map ι gives a one-to-one correspondence between $\mathcal{H}_N$ and $\mathcal{S}_{\mathfrak{a}}$.

b) for each integer l we have

$$l|(m, n, r) \Leftrightarrow l|(p\sigma, ps, s - \mathrm{tr}(\sigma)). \tag{3.4}$$

Proof. Suppose that $T \in \mathcal{H}_N$ satisfies (3.2). It follows from (3.2) that

$$\begin{aligned} \sigma =& \rho_1^2 m + \rho_1\rho_2 r + \rho_2^2 nN, \\ s =& 2\rho_1\rho_1' m + (\rho_1\rho_2' + \rho_1'\rho_2)r + 2\rho_2\rho_2' Nn. \end{aligned} \tag{3.5}$$

Note that from $\mathfrak{d}^{-1}/\mathfrak{a} \cong \mathbb{Z}/N\mathbb{Z}$ we see that $N/\sqrt{p}$ is an element of $\mathfrak{a}$. Since $\rho_1 \in \mathfrak{a}, \rho_2 \in \mathfrak{d}^{-1}$ it is clear that $\sigma \in \mathfrak{a}$ and $s \in \frac{1}{p}\mathbb{Z}$. It follows from the identity

$$\sigma + \sigma' - s = (\rho_1 - \rho_1')^2 m + (\rho_1 - \rho_1')(\rho_2 - \rho_2')r + (\rho_2 - \rho_2')^2 n$$

that $\mathrm{tr}(\sigma) - s \in \mathbb{Z}$. Finally, since $T \geq 0$ we obtain inequalities the $\sigma \gg 0$ and $4\sigma\sigma' - s^2 \geq 0$. Hence, (σ, s) is an element of $\mathcal{S}_{\mathfrak{a}}$.

Now we assume that (σ, s) belongs to $\mathcal{S}_{\mathfrak{a}}$. In this case the preimage of (σ, s) under the map ι is the matrix

$$\begin{pmatrix} m & r/2 \\ r/2 & nN \end{pmatrix} = p\ {}^tR^* \begin{pmatrix} \sigma & s/2 \\ s/2 & \sigma' \end{pmatrix} R^*,$$

where

$$\begin{aligned} m =& p\left((\rho_2')^2\sigma - \rho_2\rho_2' s + \rho_2^2\sigma'\right), \\ n =& p\left((\rho_1')^2\sigma - \rho_1\rho_1' s + \rho_1^2\sigma'\right), \\ r =& p\left(-2\rho_1\rho_2'\sigma + (\rho_1\rho_2' + \rho_1'\rho_2)s - 2\rho_1'\rho_2\sigma'\right). \end{aligned}$$

It follows from the identities

$$\begin{aligned} m =& p\left((\mathrm{tr}\rho_2)^2(\mathrm{tr}\sigma - s) - \mathrm{tr}\rho_2(\rho_2 - \rho_2')(\sigma - \sigma')/2 + (\rho_2 - \rho_2')^2(\mathrm{tr}\sigma + s)/4\right), \\ n =& p\left((\mathrm{tr}\rho_1)^2(\mathrm{tr}\sigma - s) - \mathrm{tr}\rho_1(\rho_1 - \rho_1')(\sigma - \sigma')/2 + (\rho_1 - \rho_1')^2(\mathrm{tr}\sigma + s)/4\right), \\ r =& p\left(\mathrm{tr}(\rho_1\rho_2')(\mathrm{tr}\sigma + s) + (\rho_1\rho_2' - \rho_1\rho_2')(\sigma - \sigma')\right), \end{aligned} \tag{3.6}$$

that numbers the m, n, r are integers. Since n belongs to $\mathfrak{a} \cap \mathbb{Z}$ it is divisible by N. Thus, part (a) of the Lemma is proved. Part (b) follows from (3.5) and (3.6). $\square$

Assume that N and p are prime and $(\frac{p}{N}) = 1$. Let $\mathfrak{a}$ be a fractional ideal contained in $\mathfrak{d}^{-1}$ with $\mathfrak{d}^{-1}/\mathfrak{a} \cong \mathbb{Z}/N\mathbb{Z}$. Then the following theorem holds.

Theorem 3.1. *Suppose that h is a half-integral modular form in $M^+_{k-1/2}(N)$ and let $F \in M_k(\Gamma_N)$ be the Gritsenko lift of h. Then, the pullback of F via the map ρ defined in* (1.6) *equals*

$$F(\rho(\tau_1, \tau_2)) = \frac{1}{2}\mathcal{N}_{\mathfrak{a}} g(\tau_1, \tau_2),$$

where $g(\tau) = \theta(\tau)h(p\tau) \mid U_{4N}$ and $\mathcal{N}_{\mathfrak{a}}$ is the Naganuma lift defined by (1.33).

Proof. First we will compute the Fourier expansion of $F(\rho(\tau_1, \tau_2))$. Suppose that $h(\tau)$ has the Fourier expansion

$$h(\tau) = \sum b(D)\,\mathbf{e}(D\tau).$$

The Gritsenko lift of h equals

$$F(Z) = \sum_T B(T)\,\mathbf{e}(\mathrm{tr}TZ),$$

where $B(T)$ are given by (1.34). It follows from the definition of the map ρ that

$$F(\rho(\tau_1, \tau_2)) = \sum_{\substack{T \text{ half-integral} \\ T \geq 0}} B(T)\,\mathbf{e}\left(\mathrm{tr}\left[{}^tRTR \cdot \begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}\right]\right).$$

From Lemma 3.2 we obtain

$$F(\rho(\tau_1,\tau_2)) = \sum_{\substack{\sigma\in\mathfrak{o}\\ \sigma\gg 0}} \sum_{{}^tRTR=\begin{pmatrix}\sigma & s/2\\ s/2 & \sigma'\end{pmatrix}} B(T)\,\mathbf{e}(\sigma\tau_1+\sigma'\tau_2),$$

where the last summation is taken over all semi-definite half-integral matrices T that satisfy ${}^tRTR = \begin{pmatrix}\sigma & s/2\\ s/2 & \sigma'\end{pmatrix}$ for some s. It follows from formula (3.1) and Lemma 3.2 that for fixed $\sigma\in\mathfrak{a}\mathfrak{d}^{-1}$

$$\sum_{{}^tRTR=\begin{pmatrix}\sigma & s/2\\ s/2 & \sigma'\end{pmatrix}} B(T) = \sum_{\substack{4\sigma\sigma'-s^2\geq 0\\ l|s-\mathrm{tr}\sigma\\ l|(p\sigma,ps)}} l^{k-1}\, b\Big(\frac{(4\sigma\sigma'-s^2)p}{l^2}\Big),$$

where $s\in\mathbb{Z}$ and $l\in\mathbb{N}$. So, we get

$$F(\rho(\tau_1,\tau_2)) = \sum_{\substack{\sigma\in\mathfrak{a}\mathfrak{d}^{-1}\\ \sigma\gg 0}} \widetilde{a}(\sigma)\,\mathbf{e}(\sigma\tau_1+\sigma'\tau_2),$$

where

$$\widetilde{a}(\sigma) = \sum_{\substack{4\sigma\sigma'-s^2\geq 0\\ p\,l|s-\mathrm{tr}\sigma\\ l|(\sigma,s)}} l^{k-1}\, b\Big(\frac{(4\sigma\sigma'-s^2)p}{l^2}\Big). \tag{3.7}$$

Our second step is to compute the Fourier expansion of $\mathcal{N}(g)(\tau_1,\tau_2)$. It follows from the definition (5.3) that

$$g(\tau) = \sum_{n=0}^{\infty} c(n)\,\mathbf{e}(n\tau),$$

where

$$c(n) = \sum_{\substack{4Nn-t^2\geq 0\\ p|4Nn-t^2}} b\Big(\frac{4Nn-t^2}{p}\Big). \tag{3.8}$$

For an integer n define

$$\delta_p(n) := \begin{cases} 1 & \text{if } p\nmid n,\\ 2 & \text{if } p\mid n.\end{cases}$$

The Naganuma lift of $g(\tau)$ has the Fourier expansion

$$\mathcal{N}_{\mathfrak{a}} g(\tau_1,\tau_2) = \sum_{\substack{\sigma\in\mathfrak{a}\mathfrak{d}^{-1}\\ \sigma\gg 0}} \widetilde{b}(\sigma)\,\mathbf{e}(\sigma\tau_1+\sigma'\tau_2)$$

for some numbers $\widetilde{b}(\sigma) \in \mathbb{C}$. Since the coefficients $c(n)$ are real, from the additive formula (1.24) for the Naganuma map we get

$$\widetilde{b}(\sigma) = \sum_{l|(\sigma)\mathfrak{d}\mathfrak{a}^{-1}} l^{k-1}\, \delta_p\Big(\frac{\mathrm{n}(\sigma)p^2}{Nl^2}\Big)\, c\Big(\frac{\mathrm{n}(\sigma)p^2}{Nl^2}\Big).$$

It follows from (3.8) that

$$\widetilde{b}(\sigma) = \sum_{l|\sigma} \sum_{\substack{4\sigma\sigma'/l^2-t^2\geq 0\\ p|4\sigma\sigma'/l^2-t^2}} l^{k-1}\, \delta\Big(\frac{\sigma\sigma'}{l^2}\Big)\, b\Big(\frac{(4\sigma\sigma'-l^2t^2)p}{l^2}\Big).$$

Note that for any sequence $\{\gamma(n)\}_{n\in\mathbb{Z}}$ we have

$$\sum_{n\equiv a \bmod p} \gamma(n^2) = \frac{1}{2}\,\delta(a) \sum_{n\equiv \pm a \bmod p} \gamma(n^2).$$

Thus, we have

$$\sum_{\substack{4\sigma\sigma'/l^2-t^2\geq 0\\ p|4\sigma\sigma'/l^2-t^2}} l^{k-1}\delta\Big(\frac{\sigma\sigma'}{l^2}\Big)\, b\Big(\frac{4\sigma\sigma'-l^2t^2}{p\cdot l^2}\Big) = \frac{1}{2} \sum_{\substack{4\sigma\sigma'-l^2t^2\geq 0\\ p\cdot l|\mathrm{tr}\sigma - l\cdot t\\ l|\sigma}} l^{k-1}\, b\Big(\frac{4\sigma\sigma'-l^2t^2}{p\cdot l^2}\Big).$$

Hence, we arrive at

$$\mathcal{N}_{\mathfrak{a}} g(\tau_1,\tau_2) = \sum_{\substack{\sigma\in\mathfrak{a}\mathfrak{d}^{-1}\\ \sigma\gg 0}} \widetilde{b}(\sigma)\, \mathbf{e}(\sigma\tau_1+\sigma'\tau_2),$$

where

$$\widetilde{b}(\sigma) = \frac{1}{2} \sum_{\substack{4\sigma\sigma'-l^2t^2\geq 0\\ p\cdot l|\mathrm{tr}\sigma - l\cdot t\\ l|\sigma}} l^{k-1}\, b\Big(\frac{4\sigma\sigma'-l^2t^2}{p\cdot l^2}\Big). \tag{3.9}$$

Comparing the Fourier expansions (3.7) and (5.3) we finish the proof of the Theorem. □

3.3 Modularity of Heegner Points

The following statement is formulated as a conjecture in [76]

Theorem 3.2. *Let $h : \mathfrak{H} \to \mathbb{C}$ be a periodic holomorphic function having a Fourier expansion of the form*

$$h(\tau) = \sum_{\substack{D>0\\ -D\equiv \text{ square mod } 4N}} b(D)\, q^D \qquad \big(q = e^{2\pi i\tau}\big)$$

with N prime, and suppose that the power series

$$g_p(\tau) := h(p\tau)\,\theta(\tau)|_{U_{4N}}$$

is a modular form of weight k, level p and Nebentypus $(\frac{p}{\cdot})$ for every prime $p \equiv 1 \pmod 4$ with $(\frac{N}{p}) = 1$. Then h belongs to $M^+_{k-1/2}(N)$.

In this section we prove Theorem 3.2 under additional assumptions on h. Namely, we assume that coefficients $b(D)$ have moderate growth and power series h converges for $|q| < 1$. Then $h(\tau)$ is a holomorphic function on $\mathfrak{H}$.

Although we don't know that the function h is a modular form, we can define its Gritsenko lift by formula (3.1)

$$F(Z) = \sum_T B(T)\,\mathbf{e}\big(\mathrm{tr}(TZ)\big).$$

For the function F we prove the following

Lemma 3.3. *Let $h(\tau)$ be a holomorphic function on $\mathfrak{H}$. Suppose that the function h satisfies the hypotheses of Theorem 2. Then the function F defined above is a Siegel modular form for the paramodular group Γ_N.*

Proof. We have to show that

$$F|_k M(Z) = F(Z) \tag{3.10}$$

for all $M \in \Gamma_N$ and $Z \in \mathfrak{H}^{(2)}$. Suppose that $Z = \rho(\tau_1, \tau_2) \in \mathfrak{H}^{(2)}$ for some map ρ defined in (1.6) and some $(\tau_1, \tau_2) \in \mathfrak{H} \times \mathfrak{H}$. We will check the identity (3.10) for the generators of Γ_N. It is shown in [32] that the paramodular group is generated by

$$J_N = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & N^{-1} \\ -1 & 0 & 0 & 0 \\ 0 & -N & 0 & 0 \end{pmatrix}$$

and the elements of $\Gamma_N \cap \Gamma_\infty(\mathbb{Q})$, where

$$\Gamma_\infty(\mathbb{Q}) = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q}) \right\}.$$

First we show that for $Z = \rho(\tau_1, \tau_2)$ the identity

$$F|_k J_N(Z) = F(Z) \tag{3.11}$$

holds. In this case

$$F|_k J_N(Z) = N^{-k} \det Z^{-k} F\left(\begin{pmatrix} 1 & 0 \\ 0 & N^{-1} \end{pmatrix} {}^tR^{-1} \begin{pmatrix} \frac{-1}{\tau_1} & 0 \\ 0 & \frac{-1}{\tau_2} \end{pmatrix} R^{-1} \begin{pmatrix} 1 & 0 \\ 0 & N^{-1} \end{pmatrix} \right).$$

It follows from the definition of the Gritsenko lift (1.34) that

$$F\begin{pmatrix}\tau_1 & z\\ z & \tau_2\end{pmatrix} = F\begin{pmatrix}N\tau_2 & -z\\ -z & N^{-1}\tau_1\end{pmatrix} = \tag{3.12}$$

$$= F\left(\begin{pmatrix}0 & -N^{1/2}\\ N^{-1/2} & 0\end{pmatrix}\begin{pmatrix}\tau_1 & z\\ z & \tau_1\end{pmatrix}\begin{pmatrix}0 & N^{-1/2}\\ -N^{1/2} & 0\end{pmatrix}\right).$$

Thus, we can write

$$F(J_N(Z)) = F\left(\begin{pmatrix}1 & 0\\ 0 & N^{-1}\end{pmatrix}\ {}^tR^{-1}\begin{pmatrix}\frac{-1}{\tau_1} & 0\\ 0 & \frac{-1}{\tau_2}\end{pmatrix}R^{-1}\begin{pmatrix}1 & 0\\ 0 & N^{-1}\end{pmatrix}\right) =$$

$$= F\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\ {}^tR^{-1}\begin{pmatrix}\frac{-1}{N\tau_1} & 0\\ 0 & \frac{-1}{N\tau_2}\end{pmatrix}R^{-1}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right).$$

Note that

$${}^tR^{-1} = \det(R^{-1})\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}R\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Using this identity we arrive at

$$F(J_N(Z)) = F\left(R\begin{pmatrix}\frac{-p}{N\tau_2} & 0\\ 0 & \frac{-p}{N\tau_1}\end{pmatrix}\ {}^tR\right) =$$

$$= F\circ\rho\Big(\frac{-p}{N\tau_2}, \frac{-p}{N\tau_1}\Big).$$

It follows from Theorem 3.1 that $F\circ\rho = \frac{1}{2}\mathcal{N}_{\mathfrak{a}}(g)$. It follows from (1.30) that

$$\frac{1}{2}\mathcal{N}_{\mathfrak{a}}(g)\Big(\frac{-p}{N\tau_2}, \frac{-p}{N\tau_1}\Big) = \frac{1}{2}\,N^k\,\tau_1^k\,\tau_2^k\,\mathcal{N}_{\mathfrak{a}}(g)\,(\tau_1,\tau_2) =$$

$$= N^k\,\tau_1^k\,\tau_2^k\,F\circ\rho(\tau_1,\tau_2).$$

Now it remains to prove (3.10) for $M\in\Gamma_N\cap\Gamma_\infty(\mathbb{Q})$. Each element of $M\in\Gamma_N\cap\Gamma_\infty(\mathbb{Q})$ can be written as $M = M_U M_S$, where

$$M_U = \begin{pmatrix}U & 0\\ 0 & U\end{pmatrix},\ \text{with } U\,{}^tU = E,\ U = \begin{pmatrix}* & N*\\ N* & *\end{pmatrix},$$

and

$$M_S = \begin{pmatrix}U & 0\\ 0 & U\end{pmatrix},\ \text{with } S = {}^tS,\ S = \begin{pmatrix}* & *\\ * & N^{-1}*\end{pmatrix}.$$

Here all $*$ are in $\mathbb{Z}$. The matrix $P = UR$ satisfies (1.5) and defines a map $\pi(\tau_1, \tau_2) = P \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} {}^tP$. Obviously,

$$F(M_U(Z)) = F \circ \pi(\tau_1, \tau_2).$$

It follows from Theorem 3.1 that

$$F \circ \pi(\tau_1, \tau_2) = \frac{1}{2}\mathcal{N}_{\mathfrak{a}}(g)(\tau_1, \tau_2) = F \circ \rho(\tau_1, \tau_2).$$

Thus, we obtain

$$F|_k M_U(Z) = F(Z). \tag{3.13}$$

Finally, it follows easily from (3.1) that

$$F|_k M_S(Z) = F(Z + S) = F(Z). \tag{3.14}$$

It follows from (3.11), (3.13), and (3.14) that (3.10) holds for all $M \in \Gamma_N$ and $Z = \rho(\tau_1, \tau_2)$. The set of all matrices $Z = \rho(\tau_1, \tau_2)$ for all primes p, maps ρ defined by (1.6) and all points $(\tau_1, \tau_2) \in \mathfrak{H} \times \mathfrak{H}$ is dense in $\mathfrak{H}^{(2)}$. Since the function F is continuous, the identity (3.10) holds for all $Z \in \mathfrak{H}^{(2)}$. Lemma 3.3 is proved.

□

Using Theorem 3.2 we can deduce the modularity of the generating series of the classes of the Heegner divisors in the Jacobian of $X_0(N)$ from the modularity of the generating series of the classes of Hirzebruch-Zagier curves in the homology group of a certain Hilbert surface. This idea is explained in [76] and is applied there in the case $N = 37$. The Hirzebruch-Zagier curve T_N on the Hilbert surface $Y_\delta = \mathfrak{H} \times \mathfrak{H} \backslash \mathrm{SL}_2(\mathfrak{o})$ is given by the equations

$$A\tau_1\tau_2 + \frac{\lambda}{\sqrt{p}}\tau_1 - \frac{\lambda'}{\sqrt{p}}\tau_2 + B = 0$$

with $AB - \lambda\lambda'/p = N$. The curve T_N is isomorphic to $X_0(N)$. Denote by $[T_N^c]$ the (compact) homology class of T_N. It is proved in [41] that $\sum_{N=1}^{\infty} [T_N] q^N$ is a modular form of weight 2, level p and nebentypus $(\frac{\cdot}{p})$. Since the surface Y_p is simply connected, the generating series $\sum_{N=1}^{\infty} T_N q^N$ of classes of T_N in its first Chow group is also a modular form. Since

$$T_N \cap T_M = \bigcup_{\substack{r^2 < 4MN \\ p \mid 4MN - r^2}} P_{\frac{4MN - r^2}{p}},$$

the series

$$\sum_M \left(\sum_{\substack{4MN - r^2 > 0 \\ p \mid 4MN - r^2}} P_{\frac{4MN - r^2}{p}} \right) q^M$$

is also a modular form of weight 2, level p and nebentypus $(\frac{\cdot}{p})$.

Next, an estimate of the naive height of points P_d in the Jacobian shows that the power series

$$\sum_d P_d\, q^d$$

converges for $|q| < 1$.

Hence, it follows from Theorem 3.2 that $\sum_d P_d\, q^d$ is a modular form of half integral weight. This is one of the main results of [36], where it is proved by a much more difficult computation of the height pairings of Heegner points.

Chapter 4

Heegner points and Siegel Eisenstein series

4.1 Intoduction

In this chapter we study a relation between Fourier coefficients of the degree 3 Siegel Eisenstein series of weight 2 and heights of Heegner points on modular curves. Such a relation was conjectured by B. Gross and S. Kudla many years ago and it became one of the motivating examples of Kudla's program connecting special cycles on Shimura varieties with Eisenstein series. In this chapter we explain how this idea can give a new approach to Gross-Kohen-Zagier formula that is both easier and more conceptual than the original one. Some of our results overlap with results given in [53].

In the paper [1] Gross and Zagier showed that the height of Heegner point on an elliptic curve E is an explicit(and in general non-zero) multiple of the derivative $L'(E/\mathbb{Q}, 1)$. This implies that for any given elliptic curve E with $\mathrm{ord}_{s=1} L(E/\mathbb{Q}, s) = 1$ there are Heegner points of non-zero height, which therefore are non-torsion, in particular the rank of $E(\mathbb{Q})$ is then at least one. In a subsequent paper [36] the same authors and W. Kohnen proved a more general formula involving $L'(E(\mathbb{Q}), 1)$ and height pairings between two different Heegner points. We now explain this in more detail.

Let $X_0(N)$ be the modular curve with complex points $\overline{\Gamma_0(N)\backslash\mathfrak{H}}$ and J^* be the Jacobian of $X_0^*(N)$, the quotient of $X_0(N)$ by the Fricke involution w_N. For each imaginary quadratic field K whose discriminant D is a square modulo N and to each $r \in \mathbb{Z}/2N\mathbb{Z}$ with $r^2 \equiv D(\mathrm{mod}\, 2N)$, we associate a Heegner divisor $y^*_{D,r} \in J^*$ as follows. If $\tau \in \mathfrak{H} = \{z \in \mathbb{C} |\ \Im(z) > 0\}$ is the root of a quadratic equation

$$a\tau^2 + b\tau + c = 0, \quad a, b, c \in \mathbb{Z}, \quad a > 0,$$

$$a \equiv 0(\mathrm{mod}\ N), \quad b \equiv r(\mathrm{mod}\ 2N), \quad b^2 - 4ac = D$$

then the image of τ in $\mathfrak{H}/\Gamma_0(N) \subset X_0(N)(\mathbb{C})$ is defined over H, the Hilbert class field of K. There are exactly $h = [H : K]$ such images and their sum is a divisor $P_{D,r}$ of degree h defined over K. We write $y_{D,r}$ for the divisor $P_{D,r} - h \cdot (\infty)$ of degree 0 on $X_0(N)$ and for its class in the Jacobian, and $y^*_{D,r}$ for the image of $y_{D,r}$ in J^*. The action of the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$ on $\mathrm{Tr}_{H/K}((y))$ is the same as that of w_N, therefore the image $y^*_{D,r}$ of $y_{D,r}$ in J^* is defined over $\mathbb{Q}$. Its f-component is non-trivial only if f is a modular form on $\Gamma^*(N)$, and this is the case precisely when $L(f,s)$ has a minus sign in its functional equation and hence a zero (of odd order) at $s=1$.

A striking coincidence is that the Heegner divisors $y^*_{D,r}$ and Fourier coefficients of Jacobi cusp forms of weight 2 and level N are indexed by the same set of pairs (D,r), where D is a square modulo N and $r \in \mathbb{Z}/2N\mathbb{Z}$ satisfies $r^2 \cong D(\mathrm{mod}\ 2N)$. Moreover, it is shown in [67] that the new part of the space of Jacobi cusp forms $J^{\mathrm{cusp}}_{k,N}$ is isomorphic as a Hecke module to the new part of the space $S_{2k-2}(N)^-$ space of cusp forms of weight $2k-2$ on $\Gamma_0(N)$ with eigenvalue -1 under the involution $f(z) \to (-Nz^2)^{-k+1}f(-1/Nz)$. This led the authors of [36] to guess that the height pairing of the f-components of $y^*_{D_0,r_0}$ and $y^*_{D_1,r_1}$ for different discriminants D_0 and D_1 should be related to the product $L'(f,1)\,c(n_0,r_0)\,c(n_1,r_1)$, where $D_i = r_i^2 - 4Nn_i$ and $c(n_i,r_i)$ are the Fourier coefficients of a unique up to scalar Jacobi form $\phi \in J^{\mathrm{cusp}}_{2,N}$ having the same eigenvalues as f under all Hecke operators.

More precisely, let D_0, $D_1 < 0$ be coprime fundamental discriminants, $D_i = r_i^2 - 4Nn_i$, and $f \in S_2(\Gamma_0^*(N))$ a normalized newform, in [36] the authors prove the formula for the height pairings of the f-eigencomponents of $y^*_{D_0,r_0}$ and $y^*_{D_1,r_1}$

$$\langle (y^*_{D_0,r_0})_f, (y^*_{D_1,r_1})_f \rangle = \frac{L'(f,1)}{4\pi\|\phi\|^2}\, c(n_0,r_0)\, c(n_1,r_1). \tag{4.1}$$

where $c(n_i,r_i)$ denote the coefficient of $\mathbf{e}(n_i\tau + r_i z)$ in $\phi \in J^{\mathrm{cusp}}_{2,N}$, the Jacobi form corresponding to f $(i=0,1)$.

In the case when f is a modular form of weight $2k > 2$ P. Deligne has found a definition of Heegner vectors S_x in the stalks above Heegner points x of the local coefficient system $\mathrm{Sym}^{2k-2}(H^1)(H^1 =$ first cohomology group of the universal elliptic curve over $X_0(N))$ and suggested an interpretation of the right-hand side of (4.1) as some sort of height pairing between these Heegner vectors. In [16] Brylinsky worked out some definitions of local heights suggested by Deligne. In [80] Zhang extended the result of [36] to higher weights by using the arithmetic intersection theory of Gillet and Soulé [30]. More precisely, for a CM-divisor on $X_0(N)_{\mathbb{Z}}$ Zhang defined a CM-cycle $S_k(x)$ on a certain Kuga-Sato variety. He defined the (global) height pairing between CM-cycles in these Kuga-Sato varieties, and showed an identity between the height pairings of Heegner cycles and coefficients of certain cusp forms of higher weights. We consider the following two generating functions

$$\mathcal{H}_{k,N}(\tau_0,z_0;\tau_1,z_1) := \sum_{n_0,r_0,n_1,r_1} \langle S_{k-1}(y_{D_0,r_0}), S_{k-1}(y_{D_1,r_1})\rangle\, \mathbf{e}(n_0\tau_0 + r_0z_0 + n_1\tau_1 + r_1z_1)$$

and

$$\mathcal{L}_{k,N}(\tau_0,z_0;\tau_1,z_1) := \sum_{i=1}^{\dim S_{2k-2}(N)^-} \frac{L'(f_i,k-1)}{4\pi\|\phi_i\|^2}\, \phi_i(\tau_0,z_0)\, \phi_i(\tau_1,z_1), \tag{4.2}$$

where the sum is taken over a set of normalized Hecke eigenforms $f_i \in S_{2k-2}(N)^-$ and $\phi_i \in J_{k,N}$ are the corresponding Jacobi forms.

The Gross-Kohnen-Zagier formula (4.1) says that the Fourier coefficients of $\mathcal{H}_{2,N}$ and $\mathcal{L}_{2,N}$ at $\mathbf{e}(n_0\tau_0 + r_0 z_0 + n_1\tau_1 + r_1 z_1)$ coincide if the discriminants D_0, D_1 are fundamental and coprime. This formula is proved in [36] by showing that both $\mathcal{H}$ (the value related to height pairing of two Heegner points) and $\mathcal{L}$ (the special value of L-function) are equal to a complicated "seventeen term" expression X. The idea of B. Gross and S. Kudla was show that $\mathcal{H}$ and $\mathcal{L}$ are both equal to $\mathcal{E}$, the Fourier coefficients of a certain Siegel Eisenstein series. This idea is shown symbolically in the following picture:

$$\begin{tikzcd} \mathcal{H} \arrow[rr, equal, "\text{Gross}-\text{Kohnen}-\text{Zagier}", "\text{formula}"'] \arrow[dr, equal] \arrow[ddr, equal, dashed] & & \mathcal{L} \arrow[dl, equal] \arrow[ddl, equal, dashed] \\ & X & \\ & \mathcal{E} & \end{tikzcd} \tag{4.3}$$

Denote by $E_2^{(3)}(\,\cdot\,; s)$ the non-holomorphic Siegel Eisenstein series of degree 3 and weight 2. The definition of Siegel Eisenstein series is given in Section 4.6. The upper half-space $\mathfrak{H}^{(3)}$ consists of symmetric 3×3 complex matrices

$$Z = \begin{pmatrix} \tau_0 & z & z_0 \\ z & \tau_1 & z_1 \\ z_0 & z_1 & \tau \end{pmatrix}$$

with positive definite imaginary part.

The starting point of our research is the observation made many years ago by B. Gross and S. Kudla that

$$E_2^{(3)}(Z; 0) \equiv 0, \quad Z \in \mathfrak{H}^{(3)}. \tag{4.4}$$

We will show that the single function of 6 variables

$$E^*(Z) := \frac{\partial}{\partial s} E_2^{(3)}(Z; s)\bigg|_{s=0}$$

encodes information about the height of Heegner points and Heegner cycles for all weights k and levels N.

The function $E^*(Z)$ is naturally related to both sides of the Gross-Kohnen-Zagier formula. Firstly, using the Rankin-Selberg method one can find a connection between $E^*(Z)$ and a derivatives of L-functions. This has been shown by T. Arakawa and B. Heim [5]. Secondly, the height pairing of two divisors is defined as a sum of local heights for all primes including infinity (see Section 4.3). At the same time the Fourier coefficients of $E_2^{(3)}(\,\cdot\,; s)$ can be written as a product of local densities (see Section 4.6). The miraculous identity (4.4) leads to a natural decomposition of the the Fourier coefficients of E^* into a sum of local contributions (see Theorems 4.1 and 4.2). Thus, we hope to restore from the function E^* not only the global height of Heegner points but also the local contribution for each place of $\mathbb{Q}$. In this chapter we show that this is the case when the discrminants of these two Heegner points are fundamental and coprime.

4.2 Statement of results

The function $E^*(Z)$ has the Fourier expansion

$$E^*(Z) = \sum_{H \in H_3(\mathbb{Z})} A^*(H, Y; s)\, \mathbf{e}(\operatorname{tr} HZ),$$

where $H_3(\mathbb{Z})$ denotes the set of half integral 3×3 matrices.

In Section 4.8 we recall the Maass operator $\mathcal{M}_k$ that maps Siegel modular forms of weight 2 to the space of Siegel modular forms of weight k. We define

$$E_k^*(Z) := \mathcal{M}_k E^*(Z).$$

It has the Fourier expansion

$$E_k^*(Z) = \sum_{H \in H_3(\mathbb{Z})} A_k^*(H, Y)\, \mathbf{e}(\operatorname{tr} HZ).$$

In Section 4.9 we compute the Fourier expansion of $E_k^*(Z)$ with respect to the variable τ. The following limit exists

$$E_{k,N}^{*,(2)}\left(\begin{pmatrix} \tau_0 & z \\ z & \tau_1 \end{pmatrix}, (z_0, z_1)\right) := \lim_{v \to \infty} v^{k/2-1} \int_{iv}^{iv+1} E_k^*(Z)\, \mathbf{e}(-N\tau)\, d\tau$$

and transforms like a holomorphic Jacobi form of degree 2 weight k and level N. Set

$$E_{k,N}^*(\tau_0, z_0; \tau_1, z_1) := E_{k,N}^{*,(2)}\left(\begin{pmatrix} \tau_0 & 0 \\ 0 & \tau_1 \end{pmatrix}, (z_0, z_1)\right).$$

In Section 4.10 we calculate the holomorphic projection of $E_{k,N}^*$ to the space $J_{k,N}^{\text{cusp}} \otimes J_{k,N}^{\text{cusp}}$. The resulting function

$$\mathcal{E}_{k,N} := \pi_{\text{hol}}(E_{k,N}^*)$$

can be used to prove the Gross-Kohnen-Zagier formula.

For a half-integral matrix

$$H = \frac{1}{2}\begin{pmatrix} 2n_0 & r & r_0 \\ r & 2n_1 & r_1 \\ r_0 & r_1 & 2N \end{pmatrix}$$

we define

$$\mathbf{A}_k^*(H) := (D_0 D_1)^{k-3/2} \int_0^\infty \int_0^\infty \lim_{v \to \infty} \left(v^{(k-2)/2} A_k^*(H, Y)\right) v_0^{k-5/2}\, v_1^{k-5/2}\, e^{2\pi(D_0 v_0 + D_1 v_1)}\, dv_0\, dv_1, \tag{4.5}$$

where

$$Y = \begin{pmatrix} v_0 & 0 & y_0 \\ 0 & v_1 & y_1 \\ y_0 & y_1 & v \end{pmatrix} \quad \text{and} \quad D_i = r_i^2 - 4Nn_i, (i = 0, 1).$$

The following proposition is proved in Section 4.10.

Proposition 4.1. *For $N \in \mathbb{N}$ and even $k > 2$ the function $\mathcal{E}_{k,N}$ has the Fourier expansion*

$$\mathcal{E}_{k,N} = \sum c(n_0, r_0; n_1, r_1)\, \mathbf{e}(n_0\tau_0 + r_0 z_0 + n_1\tau_1 + r_1 z_1),$$

where

$$c(n_0, r_0; n_1, r_1) = \sum_{r \in \mathbb{Z}} \mathbf{A}_k^*(H(r))$$

and

$$H(r) = H_{n_0,r_0,n_1,r_1}(r) = \frac{1}{2}\begin{pmatrix} 2n_0 & r & r_0 \\ r & 2n_1 & r_1 \\ r_0 & r_1 & 2N \end{pmatrix}.$$

In Section 4.11 we prove the following formula for finite primes

Theorem 4.1. *Fix $N \in \mathbb{N}$ and even $k \geq 2$. Let D_0, $D_1 < 0$ be coprime fundamental discriminants and $n_0, n_1, r_0, r_1 \in \mathbb{Z}$ satisfy $D_i = r_i^2 - 4n_iN$, then for a finite prime p*

$$\langle S_{k-1}(y_{D_0,r_0}), S_{k-1}(y_{D_1,r_1})\rangle_p = \mathrm{const} \sum_r^p \mathbf{A}_k^*(H(r)),$$

where the sum is taken over all integers r such that the matrix $H(r)$ is positive definite and anisotropic over the field $\mathbb{Q}_p$ of p-adic numbers.

Theorem 4.2. *Fix $N \in \mathbb{N}$ and even $k > 2$. Let D_0, $D_1 < 0$ be fundamental discriminants coprime to N and $n_0, n_1, r_0, r_1 \in \mathbb{Z}$ satisfy $D_i = r_i^2 - 4n_iN$. Then*

$$\langle S_{k-1}(y_{D_0,r_0}), S_{k-1}(y_{D_1,r_1})\rangle_\infty = \mathrm{const} \sum_r^\infty \mathbf{A}_k^*(H(r)), \tag{4.6}$$

where the sum is taken over all integers r such that the matrix $H(r)$ is indefinite.

Note that for $k = 2$ the infinite sum (4.6) does not converge. However the next theorem holds for all even $k \geq 2$.

Theorem 4.3. *The Fourier coefficients of $\mathcal{E}_{k,N}$ and $\mathcal{H}_{k,N}$ at $\mathbf{e}(n_0\tau_0 + r_0 z_0 + n_1\tau_1 + r_1 z_1)$ coincide if $D_i = r_i^2 - 4Nn_i$, $i = 0, 1$, are fundamental and coprime.*

The main result of [4] combined with Theorem 2.6 in [39] gives us a connection between the values of L-functions and the Siegel Eisenstein series.
Theorem. (Arakawa, Heim) *Let $\phi \in J_{k,N}^{\mathrm{cusp}}$ be a Hecke-Jacobi newform and let $E_{k,N}^J$ be the degree 2 Jacobi-Eisenstein series defined in Section 4.9, then*

$$\left\langle \phi(\tau_0, z_0); E_{k,N}^J\left(\begin{pmatrix} \tau_0 & 0 \\ 0 & \tau_1 \end{pmatrix}, (z_0, z_1), s\right)\right\rangle_J =$$
$$c(k, N, s)L(2k - 3 + 2s, \phi)\phi(\tau_1, z_1),$$

where

$$c(k, N, s) = \zeta(4s + 2k - 2)^{-1} \prod_{p \mid N} (1 + p^{-s+1})^{-1}.$$

The following identity is an immediate corollary of the above restriction formula.

Theorem 4.4. *For $N \in \mathbb{N}$ and even $k \geq 2$*

$$\mathcal{E}_{k,N}^{\mathrm{new}} = \mathcal{L}_{k,N}^{\mathrm{new}},$$

where $\mathcal{E}_{k,N}^{\mathrm{new}}$ and $\mathcal{L}_{k,N}^{\mathrm{new}}$ denotes the projection of functions $\mathcal{E}_{k,N}$ and $\mathcal{L}_{k,N}$ into the space of new forms.

Now the Gross-Zagier formula and its analog for $k > 2$ follow from Theorems 4.3 and 4.4.

Remark 4.1. In a recent book [53] Theorems 4.2 and 4.1 are proved (in a different way) in the case $k = 2$ without the additional assumptions on D_0 and D_1.

4.3 Local and global heights on curves

In this section we recall the basic ideas of Néron's theory. A more detailed overview of this topic is given in [34]. Let X be a non-singular, complete, geometrically connected curve over the locally compact field F_v. We normalize the valuation map $|\ |_v : F_v \to \mathbb{R}_+^\times$ so that for any Haar measure dx on F_v we have the formula $\alpha^*(dx) = |\alpha|_v \cdot dx$.

Let a and b denote divisors of degree zero on X over F_v with disjoint support. Then Néron defines a *local symbol* $\langle a, b\rangle_v$ with values in $\mathbb{R}$ which is

(i) bi-additive,

(ii) symmetric,

(iii) continuous,

(vi) satisfies the property $\langle \sum m_x(x), (f)\rangle_v = \log |\prod f(x)^{m_x}|_v$, when $b = (f)$ is principal.

These properties characterize the local symbol completely.

When v is archimedean, one can compute the Néron symbol as follows. Associated to b is a Green's function G_b on the Riemann surface $X(\overline{F_v}) - |b|$ which satisfies $\partial\overline{\partial} G_b = 0$ and has logarithmic singularities at the points in $|b|$. More precisely, the function $G_b - \mathrm{ord}_z(b) \log |\pi|_v$, is regular at every point z, where π is a uniformizing parameter at z. These conditions characterize G_b up to the addition of a constant, as the difference of any two such functions would be globally harmonic. The local formula for $a = \sum m_x(x)$ is then

$$(a, b)_v = \sum m_x\, G_b(x).$$

This is well-defined since $\sum m_x = 0$ and satisfies the required properties since if $b = (f)$ we could take $G_b = \log |f|$.

If v is a non-archimedean place, let $\mathfrak{o}_v$ denote the valuation ring of F_v and q_v the cardinality of the residue field. Let $\mathcal{X}$ be a regular model for X over $\mathfrak{o}_v$ and extend the divisors a and b to divisors A and B of degree zero on $\mathcal{X}$. These extensions are not unique, but if we require that A have zero intersection with each fibral component of $\mathcal{X}$ over the residue field, then the intersection product $(A \cdot B)$ is well defined. We have the formula

$$\langle a, b\rangle_v = -(A \cdot B) \log q_v.$$

Finally, if X, a, and b are defined over the global field F we have $(a,b)_v = 0$ for almost all completions F_v and the sum

$$\langle a, b\rangle = \sum_v \langle a, b\rangle_v \tag{4.7}$$

depends only on the classes of a and b in the Jacobian. This is equal to the global height pairing of Néron and Tate.

It is desirable to have an extension of the local pairing to divisors a and b of degree 0 on X which are not relatively prime. At the loss of some functoriality, this is done in [34] as follows.

At each point x in the common support, choose a basis $\frac{\partial}{\partial t}$ for the tangent space and let π be a uniformizing parameter with $\frac{\partial \pi}{\partial t} = 1$. Any function $f \in F_v(X)^*$ then has a well-defined "value" at x:

$$f[x] = \frac{f}{z^m}(x) \text{ in } F_v^*,$$

where $m = \mathrm{ord}_x f$. This depends only on $\frac{\partial}{\partial t}$, not on π. Clearly we have

$$fg[x] = f[x]\, g[x].$$

To pair a with b we may find a function f on X such that $b = \mathrm{div}(f) + b'$, where b' is relatively prime to a. We then define

$$\langle a, b\rangle_v = \log |f[a]|_v + \langle a, b'\rangle. \tag{4.8}$$

This definition is independent of the choice of f used to move b away from a. The same decomposition formula (4.7) into local symbols can be used even when the divisors a and b have a common support provided that the uniformizing parameter π at each point of their common support is chosen over F.

4.4 Arithmetic intersection theory

Let us review the arithmetic intersection theory of Gillet and Soulé [30]. Let F be a number field with with the ring of integers $\mathfrak{o}_F$. Let Y be a regular arithmetic scheme of dimension d over Spec $\mathfrak{o}_F$. This means that the morphism $Y \to \mathrm{Spec}\mathfrak{o}_F$ is projective and and that Y is regular. For any integer $p \geq 0$, let $A^{p,p}(Y)$ (respectively $D^{p,p}(Y)$) denote the real vector space of real differential forms α which are of type (p,p) on $Y(\mathbb{C})$ and such that $F_\infty^* \alpha = (-1)^p \alpha$, where $F_\infty : Y(\mathbb{C}) \to Y(\mathbb{C})$ denotes the complex conjugation.

A cycle of codimension p on Y with real coefficients is a finite formal sum

$$Z = \sum_i r_i Z_i,$$

where $r_i \in \mathbb{R}$, and Z_i are closed irreducible subvarieties of codimension p in Y. Such a cycle defines a current of integration $\delta_Z \in D^{p,p}(Y_\mathbb{R})$, whose value on a form η of complementary degree is

$$\delta_Z(\eta) = \sum_i r_i \int_{Z_i(\mathbb{C})} \eta.$$

A *Green's current* for Z is any current $g \in D^{p-1,p-1}(Y_{\mathbb{R}})$ such that the curvature

$$h_Z = \delta_Z - \frac{\partial\overline{\partial}}{\pi i} g$$

is a smooth form in $A^{p,p}(Y) \subset D^{p,p}(Y)$.

The *(real) arithmetic Chow group* of codimension p is the real vector space $\widehat{\mathrm{Ch}^p}(Y)_{\mathbb{R}}$ generated by pairs (Z, g), where Z is a real cycle of codimension p on Y and g is a Green's current for Z, the addition being defined componentwise, with the following relation over $\mathbb{R}$. Firstly, any pair $(0; \partial u + \overline{\partial} v)$ is trivial in $\widehat{\mathrm{Ch}^p}(Y)_{\mathbb{R}}$. Secondly, if $Y \subset Y'$ is an irreducible subscheme of codimension $p-1$ on Y, $f \in F^*(Y')$ is a nonzero rational function on Y, then the pair $(\mathrm{div}(f), -\log|f|\delta_{Y(\mathbb{C})})$ is zero in $\widehat{\mathrm{Ch}^p}(Y)_{\mathbb{R}}$.

It is shown in [13] that there is an associative and commutative intersection product

$$\widehat{\mathrm{Ch}^p}(Y)_{\mathbb{R}} \otimes \widehat{\mathrm{Ch}^q}(Y)_{\mathbb{R}} \to \widehat{\mathrm{Ch}^{p+q}}(Y)_{\mathbb{R}}$$

such that, if (Z_1, g_1) and (Z_2, g_2) are two cycles of codimension p and q, then

$$(Z_1, g_1) \cdot (Z_2, g_2) := (Z_1 \cdot Z_2, g_2 \delta_{Z_1(\mathbb{C})} + h_{Z_2} g_1).$$

We can identify $\widehat{\mathrm{Ch}^d}(Y)_{\mathbb{R}}$ with $\mathbb{R}$ by taking intersection with Y, then the intersection product of cycles with complementary degrees gives the intersection pairing of these cycles.

Let $\widehat{Z}_1 = (Z_1, g_1)$ and $\widehat{Z}_2 = (Z_2, g_2)$ be two arithmetic cycles of Y of co-dimensions p and $d-p$. We would like to decompose $\widehat{Z}_1 \cdot \widehat{Z}_2$ into the local intersections $(\widehat{Z}_1 \cdot \widehat{Z}_2)_v$ for places v of F

$$\widehat{Z}_1 \cdot \widehat{Z}_2 = \sum_v (\widehat{Z}_1 \cdot \widehat{Z}_2)_v \epsilon_v$$

If Z_1 and Z_2 are disjoint at the generic fiber then the intersection $Z_1 \cdot Z_2$ with support defines an element in $\mathrm{Ch}^d_{|Z_1| \cap |Z_2|}(Y)$ (see Section 4.1.1 in [30]). Since $|Z_1| \cap |Z_2|$ is supported in special fibers, one has well defined $x_v \in \mathrm{Ch}^d_{|Y \otimes F_v|}(Y)$ for each finite place v such that

$$Z_1 \cdot Z_2 = \sum_v x_v.$$

We define

$$(\widehat{Z}_1 \cdot \widehat{Z}_2)_v = \deg x_v$$

if v is finite, and

$$(\widehat{Z}_1 \cdot \widehat{Z}_2)_v = \int_{Z_{2v}(\mathbb{C})} g_1 + \int_{Y_v(\mathbb{C})} g_2 h_{Z_1}$$

if v is infinite, where Y_v denotes $Y \otimes_{\mathfrak{o}_F, \sigma} \mathbb{C}$ for an embedding $\sigma : F \to \mathbb{C}$ inducing v and Z_{2v} is the pullback of Z_2 on Y_v.

4.5 Heegner cycles on Kuga-Sato varieties

In this section we recall the definition of CM-cycles on Kuga-Sato varieties given by Zhang in [80].

For an elliptic curve E with a CM by $\sqrt{D}$, let $Z(E)$ denote the divisor class on $E \times E$ of $\Gamma - E \times \{0\} - D\{0\} \times E$, where Γ is the graph of $\sqrt{D}$. Then for a positive integer k $Z(E)^{k-1}$ is a cycle of codimension $k-1$ in E^{2k-2}. Denote by $S_k(E)$ the cycle

$$c \sum_{g \in G_{2k-2}} \mathrm{sign} g^*(Z(E)^{k-1}),$$

where G_{2k-2} denotes the symmetric group of $2k-2$ letters which acts on E^{2k-2} by permuting the factors, and c is a real number such that the self-intersection of $S_k(E)$ on each fiber is $(-1)^{k-1}$.

For N a product of two relatively prime integers ≥ 3, one can show that the universal elliptic curve over the non-cuspidal locus of $X(N)_{\mathbb{Z}}$ can be extended uniquely to a regular semistable elliptic curve $\mathcal{E}(N)$ over whole $X(N)$. The *Kuga-Sato variety* $Y = Y_k(N)$ is defined to be a canonical resolution of the $(2k-2)$-tuple fiber product of $\mathcal{E}(N)$ over $X(N)$. If y is a CM- point on $X(N)$, the CM-cycle $S_k(y)$ over y is defined to be $S_k(\mathcal{E}_y)$ in Y. If x a CM-divisor on $X_0(N)_{\mathbb{Z}}$ the CM-cycle $S_k(x)$ over x is defined to be $\sum S_k(x_i)/\sqrt{\deg p}$, where p denotes the canonical morphism from $X(N)$ to $X_0(N)$, and $\sum x_i = p^*x$. One can show that $S_k(x)$ has zero intersection with any cycle of Y supported in the special fiber of $Y_{\mathbb{Z}}$, and that the class of $S_k(x)$ in $H^{2k}(Y(\mathbb{C}); \mathbb{C})$ is zero. Therefore, there is a Green's current $g_k(x)$ on $Y(\mathbb{C})$ such that

$$\frac{\partial\bar{\partial}}{\pi i} g_k(x) = \delta_{S_k(x)}.$$

The arithmetic CM-cycle $\hat{S}_k(x)$ over x, in the sense of Gillet and Soulé [30], is defined to be

$$\hat{S}_k(x) = (S_k(x), g_k(x)).$$

If x and y are two CM-points on $X_0(N)$, then the height pairing of the CM-cycles $S_k(x)$ and $S_k(y)$ is defined as the intersection product

$$\langle S_k(x), S_k(y) \rangle := (-1)^k \hat{S}_k(x) \cdot \hat{S}_k(y),$$

which was considered in the previous section.

4.6 Siegel Eisenstein series

For matrix Z in Siegel upper half-space $\mathfrak{H}^n = \{Z = {}^tZ |\ \Im(Z) \text{ is positive definite}\}$ and $s \in \mathbb{C}$, $\Re(s) \gg 0$, the non-analytic Siegel Eisenstein series of degree n and weight k are defined as

$$E_k^{(n)}(Z; s) = \det(\Im Z)^s \sum_{\{C,D\}} \det(CZ+D)^{-k} |\det(CZ+D)|^{-2s}.$$

Here the summation is taken over the set of equivalence classes of coprime symmetric pairs. We know from [55] that the series $E_k^{(n)}(Z,s)$ have a meromophic continuation to the whole s-plane and satisfy a functional equation. Set

$$\widehat{E}_k^{(n)}(Z,s) = \Gamma_k^{(n)}(s)\, E_k^{(n)}(Z,s),$$

where

$$\Gamma_k^{(n)}(s) = \frac{\Gamma_n(s+k)}{\Gamma_n(s+k/2)}\,\widehat{\zeta}(2s+k)\prod_{j=1}^{[n/2]}\widehat{\zeta}(4s+2k-2j),$$

$$\Gamma_n(s) = \pi^{n(n-1)/4}\prod_{j=0}^{n-1}\Gamma(s-j/2),$$

$$\widehat{\zeta}(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s).$$

The completed Eisenstein series satisfy the following functional equation

$$\widehat{E}_k^{(n)}(Z,s) = \widehat{E}_k^{(n)}(Z,-k+\frac{n+1}{2}-s). \tag{4.9}$$

We are most interested in the case $n=3, k=2$. In this case

$$\Gamma_2^{(3)}(s) = s\,(s+1/2)\,(s+1)\,\widehat{\zeta}(2s+2)\,\widehat{\zeta}(4s+2).$$

It is known that $E_2^{(3)}(Z,s)$ is holomorphic at $s=0$. Hence it follows from (4.9) that

$$E_2^{(3)}(Z,0)\equiv 0.$$

For a commutative ring R denote by $S_n(R)$ the set of symmetric $n\times n$ matrices with entries in R and by $H^n(R)$ the set of symmetric half-integral matrices over R. The Siegel Eisenstein series posses the Fourier expansion

$$E_k^{(n)}(Z;s) = \sum_{H\in H^n(\mathbb{Z})} A_k(H,Y;s)\,\mathbf{e}(\operatorname{tr} HZ),\quad Z = X+iY.$$

For a non-degenerate $H\in H^n(\mathbb{Z})$ we have a decomposition

$$A_k(H,Y;s) = W_k(H,Y;s)\,B(H;k+2s).$$

Here

$$W_k(H,Y;s) = \int\limits_{S_n(\mathbb{R})} \det Y^s \det(X+iY)^{-k}\,|\det(X+iY)|^{-2s}\,\mathbf{e}(-\operatorname{tr} HZ)\,dX \tag{4.10}$$

is a generalized Whittaker function (or a confluent hypergeometric function). The analytic properties of such functions where studied in detail by Shimura [66]. The Siegel series $B(H;s)$ are defined as

$$B(H;s) = \sum_R \nu(R)^{-s}\,\mathbf{e}(\operatorname{tr}\, RH),$$

where R runs over a complete set of representatives of $S_n(\mathbb{Q})/S_n(\mathbb{Z})$ and $\nu(R)$ is the product of denominators of elementary divisors of R. To investigate the Siegel series, for a prime number p and a half-integral matrix H of degree n define the local Siegel series $B_p(H,s)$ by

$$B_p(H,s) = \sum_{R \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} p^{-\mathrm{ord}(\nu(R))s}\, \mathbf{e}(\mathrm{tr}\, HR).$$

It is easy to see that

$$B(H,s) = \prod_p B_p(H,s).$$

An explicit form of $B_p(H,s)$ for any non-degenerate half-integral matrix H over $\mathbb{Z}_p$ is given in [45]. More precisely, the local densities are equal to

$$B_p(H,s) = (1-p^{-s})(1-p^{2-2s}) f_p(p^{2-s}), \tag{4.11}$$

where f_p is a polynomial depending on the p-adic nature of H. To define it we need the local invariants of H at p. Define nonnegative integers α, β, γ and δ by

$$\alpha = \mathrm{ord}_p(m(2H)), \qquad \beta = \mathrm{ord}_p(m(\widetilde{2H})) - 2\alpha, \qquad \delta = 3\alpha + 2\beta + \gamma = \mathrm{ord}_p(4 \det H),$$

where $m(2H)$ and $m(\widetilde{2H})$ denote the content(greatest integer dividing) $2H$ and its adjoint $\widetilde{2H}$ in the lattice of even integral matrices and the set of ternary quadratic forms representing only integers congruent to 0 or 3 modulo 4, respectively. We also have $c_p(=\pm 1)$, the Hasse invariant of $m(H)^{-1}H$, as well as a further invariant $\epsilon = \pm 1$ to be defined below in the case that β is even and $\gamma \neq 0$. If p is odd, then we can diagonalize H over $\mathbb{Z}_p$ as $\mathrm{diag}(p^\alpha A, p^{\alpha+\beta} B, p^{\alpha+\beta+\gamma} C)$ with $p \nmid ABC$, and $c_p = (\frac{-1}{p})^{\alpha\beta} (\frac{-AC}{p})^{\alpha} (\frac{-BC}{p})^{\beta}$.

In case $\alpha = \beta = \gamma = 0$ the p-rank of H is 3 and $f_{0,0,0}(X) = 1$.

In case $\alpha = \beta = 0$, $\gamma > 0$ the p-rank of H is 2. Then $\tilde{H}$ has p-rank 1, so $-\tilde{H}$ represents numbers D prime to p but these all have the same value ± 1 of $(\frac{D}{p})$. This common value is denoted by ε. Then

$$f_{0,0,\gamma}(X) = 1 + \varepsilon X + \cdots + \varepsilon^\gamma X^\gamma. \tag{4.12}$$

In other cases we get more complicated recursive formula given in [45].

The H-th coefficient of the Eisenstein series of even weight $k \geq 2$ on $\mathrm{Sp}(3,\mathbb{Z})$ is given by

$$A_k(H) = \frac{4}{\zeta(1-k)\zeta(3-2k)} \prod_p f_p(p^{2-k}). \tag{4.13}$$

4.7 Quaternion algebras and local densities

We will start this section with a brief overview of quaternion algebra theory. An introduction to this subject can be found in [2, 63, 72].

Let K be a field of characteristic different from 2. A *quaternion K-algebra* $\mathbb{B}$ is a central simple K-algebra of dimension 4 over K.

Over a field K of characteristic different from 2, every quaternion algebra $\mathbb{B}$ has K-basis $\{1, i, j, ij\}$ satisfying the relations $i^2 = a,\ j^2 = b$, and $ij = -ji$, for some $a, b \in K^*$.

A quaternion $\omega = x + yi + zj + tij$ in $\mathbb{B}$ is called pure if $x = 0$. We denote by $\mathbb{B}_0$ the K-vector space of pure quaternions.

Every quaternion K-algebra $\mathbb{B}$ is provided with a K-endomorphism which is an involutive antiautomorphism called conjugation; for $\omega = x + yi + zj + tij$ it is defined by $\bar{\omega} := x - yi - zj - tij$. The reduced norm and reduced trace are defined by $\mathrm{n}(\omega) := \omega\bar{\omega}$ and $\mathrm{tr}(\omega) := \omega + \bar{\omega}$, respectively. There is a symmetric bilinear K-form on $\mathbb{B}$ given by $(a, b) := \mathrm{tr}(ab)$. For a place v of K we define

$$\mathbb{B}_v := K_v \otimes \mathbb{B}.$$

If $\mathbb{B}_v$ is a division algebra, we say that $\mathbb{B}$ is ramified at v; otherwise we say that $\mathbb{B}$ is non-ramified at v. The following theorem is well known.

Theorem.

(i) $\mathbb{B}$ *is ramified at a finite even number of places*

(ii) Two quaternion K-algebras are isomorphic if and only if they are ramified at the same places.

A subset $S \subset \mathbb{B}$ is called a $\mathbb{Z}$-order if it is a $\mathbb{Z}$-ideal and a ring. By definition an Eichler $\mathbb{Z}$-order is an intersection of two maximal $\mathbb{Z}$-orders. We associate to each half integral 3×3 matrix an order in a quaternion algebra. Let

$$H = \frac{1}{2}\begin{pmatrix} 2n_0 & r & r_0 \\ r & 2n_1 & r_1 \\ r_0 & r_1 & 2N \end{pmatrix}.$$

Put

$$n = r_0 r_1 - 2Nr,\ D_i = r_i^2 - 4n_i N\ (i = 0, 1). \tag{4.14}$$

We define a quaternion algebra $\mathbb{B}(H)$ over $\mathbb{Q}$ with basis $\langle 1, e_0, e_1, e_0 e_1\rangle$ satisfying multiplicative relations

$$e_0^2 = D_0,\ e_1^2 = D_1,\ e_0 e_1 + e_1 e_0 = 2n.$$

We now introduce the order

$$\mathbb{S}(H) := \mathbb{Z} + \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_0\alpha_1 \tag{4.15}$$

in the quaternion algebra $\mathbb{B}(H)$, where $\alpha_i := \frac{r_i + e_i}{2}$.

The following Lemma gives us a connection between the quaternion algebra $\mathbb{B}_p(H)$ and the local density $B_p(H, s)$ defined in Section 4.6.

Lemma 4.1. *If $\mathbb{B}(H)$ is ramified at p then the local density $B_p(H, 2) = 0$.*

Proof. Denote by $S_{p^n}(H)$ the number of solutions modulo p^n of the equation

$$2R\,{}^tR \equiv 2H(\bmod p^n). \tag{4.16}$$

It is proved in [70] that

$$B_p(H,2) = \lim_{n\to\infty} p^{-6n} S_{p^n}(H).$$

Assume that $B_p(H,2) \neq 0$. Then there is a solution of (4.16) in $\mathbb{Q}_p$. Hence H is isotropic in $\mathbb{Q}_p$ for $p \neq 2$ (the identity matrix $\mathbb{I}_3$ is isotropic over $\mathbb{Q}_p$).

Now assume that $\mathbb{B}(H)$ is ramified at p. Then $q(a) = -a^2$ is an anisotropic quadratic form on the space of pure quaternions $\mathbb{B}^0(H)_p$. In the basis $\langle e_0, e_1, e_0e_1 - n\rangle$ this quadratic form is given by the matrix

$$G = \begin{pmatrix} -D_1 & n & 0 \\ n & -D_0 & 0 \\ 0 & 0 & 4MN \end{pmatrix}, \quad M = 4\det H.$$

For the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_0 & r_1 & 2N \end{pmatrix}$$

we have

$$G = A\,\widetilde{2H}\,{}^tA.$$

We have got a contradiction, since G is anisotropic and $\widetilde{H}$ is isotropic over $\mathbb{Q}_p$. $\square$

Recall that an order R of $\mathbb{B}$ is an Eichler order of index N if for all primes $p \nmid N$ the localization $R_p = R \otimes \mathbb{Z}_p \subseteqq \mathbb{H}_p = \mathbb{H} \otimes \mathbb{Q}_p$ is a maximal order and for all primes $p \mid N$ there is an isomorphism from $\mathbb{H}_p$ to $M_2(\mathbb{Q}_p)$ which maps R_p to the order

$$\left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}_p \right\}.$$

Denote by $\rho_p(H)$ the number of Eichler orders of index N in $\mathbb{B}_p(H)$ that contain $\mathbb{S}_p(H)$. The next two lemmas follow from identity (4.11) and the proof of Proposition 2 in Section I.3 of [36]. Here we assume that D_0 and D_1 defined by (4.14) are coprime.

Lemma 4.2. *If $\mathbb{B}(H)$ is non-ramified at prime q then the local density $B_q(H,2)$ is equal to $(1-q^{-2})^2\,\rho_q(H)$.*

Lemma 4.3. *If $\mathbb{B}(H)$ is ramified at p then*

$$\frac{\partial}{\partial s} B_p(H, 2+2s)\bigg|_{s=0} = (1-p^{-2})^2 \cdot \log p \cdot (\mathrm{ord}_p(M) + 1),$$

where $M = 4\det H$. In this case $\rho_p(H) = 1$.

4.8 Maass differential operator

In this section we recall a differential operator introduced in [56] that raises the weight of Siegel modular forms. We slightly modify the operator, so that the function obtained from a holomorphic modular form transforms like a holomorphic modular form of higher weight, however it is not holomorphic.

Suppose that $Z = X + iY$ is an element of $\mathfrak{H}^3$. This matrix can be written as $Z = \{z_{ij}\}_{i,j=1}^3$, where $z_{ij} = z_{ji}$. Consider the matrix $\frac{\partial}{\partial Z} := \{(1+\delta_{ij})\frac{\partial}{\partial z_{ij}}\}_{i,j=1}^3$. Then $\det(\frac{\partial}{\partial Z})$ is a differential operator of order 3. Define the operator

$$\mathcal{D}_k := \det Y^{1-k} \det(\frac{\partial}{\partial Z}) \det Y^{k-1}.$$

It is proved in [56] on p. 309 that this differential operator has the following property

$$\mathcal{D}_k(F|_k M) = \mathcal{D}_k(F)|_{k+2} M$$

for an arbitrary smooth enough function F on $\mathfrak{H}^{(3)}$ and an arbitrary symplectic matrix M. Therefore, if the function F transforms like a holomorphic Siegel modular form of weight k then $\mathcal{D}_k F$ transforms like a a holomorphic Siegel modular form of weight $k+2$. For $k \in 2\mathbb{Z}$ the following operator

$$\mathcal{M}_k := \mathcal{D}_{k-2} \circ \cdots \circ \mathcal{D}_2$$

maps Siegel modular forms of weight 2 to Siegel modular forms of weight k. It is shown in Section 19 of [56] that

$$\mathcal{M}_k(E_2(Z,s)) = \epsilon(k,s)\, E_k(Z, s-(k-2)/2),$$

where

$$\epsilon(k,s) = \prod_{h=2}^{(k-2)/2} (h+s)(h+s-\frac{1}{2})(h+s-1).$$

Set

$$E_k^*(Z) := \mathcal{M}_k(E^*(Z)).$$

Since $E_2(Z,0) \equiv 0$, then the following is true

$$\mathcal{M}_k(E^*(Z)) = \epsilon(k,0)\, \frac{\partial}{\partial s} E_k(Z, s+1-k/2)\bigg|_{s=0}. \tag{4.17}$$

4.9 Jacobi Eisenstein series of degree two

The notion of Jacobi forms can be generalized to higher dimensions in the following way. The Heisenberg group $\mathrm{H}_n(\mathbb{R})$ is defined to be the set of triples $(\lambda,\mu,\kappa) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with group law

$$(\lambda,\mu,\kappa)(\lambda',\mu',\kappa') = (\lambda+\lambda', \mu+\mu', \kappa+\kappa'+\lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

The group $\mathrm{Sp}(2n,\mathbb{R})$ operates on $\mathrm{H}_n(\mathbb{R})$ from the right by

$$(\lambda,\mu,\kappa)\circ M = ((\lambda,\mu)M,\kappa).$$

The semidirect product $\mathrm{G}_n(\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R}) \ltimes \mathrm{H}_n(\mathbb{R})$ is called a generalized Jacobi group. An element

$$\gamma = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda,\mu,\kappa)\right) \in \mathrm{G}_n(\mathbb{R})$$

acts on $\mathfrak{H}^{(n)} \times \mathbb{C}^n$ by

$$\gamma(T,z) = ((AT+B)(CT+D)^{-1}, (z+\lambda T+\mu)(CT+D)^{-1}).$$

We define

$$j_{k,N}(\gamma;T,z) := \det(CT+D)^{-k} \times$$
$$\times \mathbf{e}^N\left(-z(CT+D)^{-1}C\,{}^tz + 2z(CT+D)^{-1}\,{}^t\lambda + \lambda(AT+B)(CT+D)^{-1}\,{}^t\lambda\right).$$

For $k,N\in\mathbb{Z}$ the group $G_n(\mathbb{R})$ acts on functions $f:\mathfrak{H}^{(n)}\times\mathbb{C}^n\to\mathbb{C}$ by

$$f(T,z)|_{k,N}\,\gamma = j_{k,N}(\gamma;T,z)f(\gamma(T,z)).$$

Set $\Gamma^{\mathrm{J}} := \mathrm{G}_n(\mathbb{Z})$ and denote by $\Gamma^{\mathrm{J}}_\infty$ be the stabilizer group of the constant function 1. We define the *Eisenstein-Jacobi series* $E^{\mathrm{J}}_{k,N}(T,z;s)$ on $\mathfrak{H}^n\times\mathbb{C}^n$ as

$$E^{\mathrm{J}}_{k,N}(T,z;s) = \sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J} \det(\Im(T))^s \,|_{k,N}\, \gamma.$$

Note that we consider z as a row-vector. We have the Fourier expansion

$$E^{\mathrm{J}}_{k,N}(T,z,s) = \sum_{H',\mu} A^{\mathrm{J}}_{k,N}(H',\mu,V;s)\,\mathbf{e}(\mathrm{tr}TH' + z\,{}^t\mu),\quad T = U+iV,$$

where H' runs over symmetric half-integral 2×2 matrices and μ runs over $\mathbb{Z}^2$.

As in the case of Siegel Eisenstein series considered in Section 4.6 the Fourier coefficients $A^{\mathrm{J}}_{k,N}(H,\mu;V;s)$ also can be decomposed into "analytic" and "arithmetic" parts. Consider the following singular series

$$B^{\mathrm{J}}_N(H,\mu;s) := \sum_{M\in P_{n,0}\backslash\Gamma^*_n/P^J_{n,0}} \sum_{\lambda\in\mathbb{Z}^n/\mathbb{Z}^nC} \det C^{-s}\,\mathbf{e}^N(\lambda AC^{-1}\,{}^t\lambda)\,\mathbf{e}(C^{-1}DH + \mu C^{-1}\,{}^t\lambda),$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ with $A,B,C,D\in\mathrm{M}_{n\times n}(\mathbb{Z})$ and

$$P_{0,n} = \left\{\begin{pmatrix} A' & B' \\ 0_{n,n} & D'\end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z})\right\},$$

$$\Gamma^*_n = \left\{\left(\begin{smallmatrix} A' & B' \\ C' & D\end{smallmatrix}\right)\Big| \, C'\neq 0\in\mathrm{Sp}_{2n}(\mathbb{Z})\right\},$$

$$P_{0,n}^J = \left\{ \begin{pmatrix} 1_n & B' \\ 0_{n,n} & D' \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}) \right\}.$$

We will use the following notations

$$Z = \begin{pmatrix} T & {}^t z \\ z & \tau \end{pmatrix}, \qquad Z = X + iY \in \mathfrak{H}^{n+1}, \tag{4.18}$$

$$T = U + iV \in \mathfrak{H}^n,\ z = x + iy \in \mathbb{C}^n,\ \tau = u + iv \in \mathbb{C},$$

$$H = \begin{pmatrix} H' & \frac{1}{2}\,{}^t\mu \\ \frac{1}{2}\mu & N \end{pmatrix} \in H_{n+1}(\mathbb{Z}), \qquad H' \in H_n(\mathbb{Z}), \qquad H'' = H' - \frac{1}{4N}\,{}^t\mu\mu.$$

The following statement is proved by T. Arakawa and B. Heim.

Lemma([5], Proposition 2.1) *Assume that H', H'', μ, V are defined as in* (4.18) *and H is a non-degenerate matrix. Then*

$$A_{k,N}^{\mathrm{J}}(H', \mu; V; s) = W_{k-1/2}(H''; V; s)\, B_N^{\mathrm{J}}(H', \mu; k+2s),$$

where $W_k(H, Y)$ is a generalized Whittaker function defined in Section 4.6.

The following result due to Kohnen relates Jacobi-Eisenstein series $E_{k,N}^{\mathrm{J},n}$ with Siegel Eisenstein series $E_k^{(n+1)}$.

Theorem.([47], eq.(20)) *Let Z be as above (4.18) and N be squarefree then*

$$\lim_{v\to\infty} v^{-s}\, e^{2\pi N v} \int_0^1 E_k^{(n+1)}(Z; s)\, \mathbf{e}^{-N}(u)\, du = \mu_{k,N,s}\, E_{k,N}^{(n)}(T, z; s),$$

where $\mu_{k,N,s} = (-1)^{k/2}\, N^{k+2s-1}\, (2\pi)^{k+s-1}/\Gamma(k+s)$.

Lemma 4.4. *Let H, H'' and Y, V, v be as in* (4.18)*. Then*

$$\lim_{v\to\infty} v^s\, W_k(H, Y; s) = \mu_{k,N,s}\, W_{k-1/2}(H'', V; s).$$

Proof. Using the identity

$$\det \begin{pmatrix} T & {}^t z \\ z & \tau \end{pmatrix} = (\tau - zT^{-1}\,{}^t z)\, \det T$$

and integral representation (4.10) we can write

$$W_k(H, Y; s) = e^{2\pi \mathrm{tr} HY} \int_{S_{n+1}(\mathbb{R})} \det Z^{-k}\, |\det Z|^{-2s}\, \mathbf{e}(-\mathrm{tr} HX)\, dX =$$

$$= e^{2\pi \mathrm{tr}(H'V)} \int_{S_n \times \mathbb{R}^n} \det T^{-k}\ |\det T|^{-2s}\ \mathbf{e}(-\mathrm{tr} H'U - z\,{}^t\mu) \times$$

$$\times e^{2\pi N v} \int_{\mathbb{R}} (\tau - zT^{-1}\,{}^t z)^{-k}\, |\tau - zT^{-1}\,{}^t z|^{-2s}\, \mathbf{e}^{-N}(u)\, du\, dU\, dz, \tag{4.19}$$

where $S_n = S_n(\mathbb{R})$. From formulas 13.1.33 and 13.5.2 in [1] we have an asymptotic representation

$$\int_{\mathbb{R}} (u+iv)^{-k}|u+iv|^{-2s}\,\mathbf{e}(-Nu)\,du = v^{-s}\,e^{-2\pi Nv}\left(\mu_{k,N,s} + \underline{o}(\frac{1}{v})\right),\ v \to +\infty.$$

Since $\Im(zT^{-1}z) \le C$ for some C depending only on $y = \Im z$ and $V = \Im T$ the expression

$$v^s e^{2\pi Nv}\int_{\mathbb{R}} (u - zT^{-1}\,{}^tz + iv)^{-k}|u - zT^{-1}\,{}^tz + iv|^{-2s}\mathbf{e}(-Nu)\,du$$

is bounded uniformly in v, $U = \Re(T)$, and $x = \Re(z)$ and tends to $\mathbf{e}(-NzT^{-1}\,{}^tz)$ as v tends to infinity. For $s \gg 0$ the integral (4.19) converges absolutely and uniformly in v on intervals $[v_0, \infty)$. So we can interchange limit with integration

$$\lim_{v\to\infty} v^s\,W_k(H,Y;s)$$

$$= \mu_{k,N,s}\,e^{2\pi(\mathrm{tr}H'V)}\int\limits_{S_n\times\mathbb{R}^n} \det T^{-k}\ |\det T|^{-2s}\ \mathbf{e}(-\mathrm{tr}H'U - z\,{}^t\mu - NzT^{-1}\,{}^tz)\,dU\,dz$$

$$= (2i)^{-n/2}\,\mu_{k,N,s}\,e^{2\pi\mathrm{tr}(H''V)}\int\limits_{S_n} \det T^{-k+1/2}\ |\det T|^{-2s}\ \mathbf{e}(-\mathrm{tr}H''U)\,dU.$$

The lemma is proved. □

Using the theorem of Kohnen and the above lemma we can compute the Fourier coefficients of $E^{(n)}_{k,N}$.

Proposition 4.2. *For squarefree N and non-degenerate H*

$$A^{\mathrm{J}}_{k,N}(H,\mu;V;s) = W_{k-\frac{1}{2}}\big(H'';V;s\big)\,B^{(n+1)}_{k+2s}(H).$$

4.10 Holomorphic projection

In this section we compute the holomorphic projection of the function

$$E^{\mathrm{J}}_{k,N}\left(\begin{pmatrix}\tau_0 & 0\\ 0 & \tau_1\end{pmatrix}, (z_0,z_1), s\right)$$

to the space $J^{\mathrm{cusp}}_{k,N} \otimes J^{\mathrm{cusp}}_{k,N}$. We start with a definition of holomorphic projection.

Let ϕ and ψ be two Jacobi cusp forms of weight k and level N, then the Petersson scalar product of ϕ and ψ is defined as

$$\langle\phi,\psi\rangle = \int\limits_{\Gamma^{\mathrm{J}}\backslash\mathfrak{H}\times\mathbb{C}} \phi(\tau,z)\,\overline{\psi(\tau,z)}\,v^k\,e^{-4\pi Ny^2/v}\,dV, \tag{4.20}$$

where

$$\tau = u + iv, \quad z = x + iy, \quad dV = v^{-3}\, du\, dv\, dx\, dy.$$

Let f be a function on $\mathfrak{H} \times \mathbb{C}$ that transforms like a holomorphic Jacobi form of weight k and level N and rapidly decays at infinity, so that the integral (4.20) is well defined for any cusp form $\psi \in J_{k,N}^{\mathrm{cusp}}$. Then the *holomorphic projection* of f is a unique holomorphic function $\pi_{\mathrm{hol}} f \in J_{k,N}^{\mathrm{cusp}}$ that satisfies

$$\langle f, \psi \rangle = \langle \pi_{\mathrm{hol}} f, \psi \rangle$$

for all $\psi \in J_{k,N}^{\mathrm{cusp}}$.

Lemma 4.5. *Assume that the function $f(\tau, z)$ transforms like a holomorphic Jacobi form of weight k and level N and it has a Fourier expansion*

$$f(\tau, z) = \sum_{n,r} c(n, r, v)\, \mathbf{e}(n\tau + rz),$$

where $\tau = u + iv$. Let for μ modulo $2N$ the function h_μ be defined as in (5.66). *Assume that for all μ modulo $2N$*

$$h_\mu(\tau) = c_\mu + O(v^{-\epsilon}) \text{ as } v \to \infty$$

for some numbers $c_\mu \in \mathbb{C}$ and $\epsilon > 0$. Then

$$\pi_{\mathrm{hol}}(f) = \sum_{n,r} c(n, r)\, \mathbf{e}(n\tau + rz),$$

where

$$c(n, r) = \alpha_{k,N}\, (4Nn - r^2)^{k-\frac{3}{2}} \int_0^\infty v^{k-\frac{5}{2}}\, c(n, r, v)\, e^{-\pi(4Nn - r^2)v/N}\, dv, \tag{4.21}$$

and

$$\alpha_{k,N} = \frac{N^{k-2}\, \Gamma(k - 3/2)}{2\pi^{k-3/2}}.$$

Proof. For integers n, r with $r^2 < 4Nn$ there is a unique function $P_{n,r} \in J_{k,N}^{\mathrm{cusp}}$ depending only on $r^2 - 4Nn$ and on $r(\bmod 2N)$, such that

$$\langle \phi, P_{n,r} \rangle = \alpha_{k,N}\, (4Nn - r^2)^{k+3/2}\, b(n, r) \tag{4.22}$$

for all $\phi = \sum b(n, r)\, \mathbf{e}(n\tau + rz) \in J_{k,N}^{\mathrm{cusp}}$.

It is shown in [36] on p. 519 that

$$P_{n,r}(\tau, z) = \sum_{\gamma \in \Gamma_\infty^{\mathrm{J}} \backslash \Gamma^{\mathrm{J}}} \mathbf{e}(n\tau + rz)|_{k,N}\, \gamma,$$

where

$$\Gamma_\infty^{\mathrm{J}} = \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (\mu, 0) \right) \;\middle|\; n, \mu \in \mathbb{Z} \right\}$$

is the stabilizer of the function $\mathbf{e}(n\tau + rz)$ in the full Jacobi group $\Gamma^J = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. By the usual unfolding argument, we see that the Petersson product of f and $P_{n,r}$ equals

$$\int\limits_{\Gamma^{\mathrm{J}}_\infty \backslash \mathfrak{H} \times \mathbb{C}} f(\tau, z)\, \overline{\mathbf{e}(n\tau + rz)}\, v^{k-3}\, e^{-4\pi N y^2/v} dx\, dy\, du\, dv.$$

Putting in the Fourier expansion of f and observing that a fundamental domain for the action of $\Gamma^{\mathrm{J}}_\infty$ on $\mathfrak{H} \times \mathbb{C}$ is $([0,\infty) \times [0,1]) \times (\mathbb{R} \times [0,1])$, we find that the integral equals

$$\sum_{n',r'} \int\limits_{[0,\infty)\times[0,1]\times\mathbb{R}\times[0,1]} c(n', r', v)\, \mathbf{e}\big((n'-n)u + (r'-r)x\big)\, e^{-2\pi((n'+n)v+(r'+r)y)} \times$$

$$\times v^{k-3} e^{-4Ny^2/v}\, du\, dv\, dx\, dy$$

$$= \int\limits_0^\infty c(n, r, v)\, e^{-4\pi n v}\, v^{k-3} \Bigg(\int\limits_{-\infty}^{\infty} e^{-4\pi(ry + Ny^2/v)}\, dy \Bigg) dv.$$

The inner integral equals $(\frac{v}{4N})^{1/2} e^{\pi r^2 v/N}$, so the scalar product equals

$$1/\sqrt{4N} \int\limits_0^\infty c(n, r, v)\, v^{k-5/2}\, e^{-\pi(4n - r^2/N)v}\, dv.$$

This proves our claim. $\square$

For a matrix $H'' = \begin{pmatrix} a_0 & a \\ a & a_1 \end{pmatrix}$ and $\kappa \in \mathbb{Z}/2$ define

$$\mathbf{W}_{\kappa,s}(H'') := a_0^{\kappa-1}\, a_1^{\kappa-1} \int\limits_0^\infty \int\limits_0^\infty W_\kappa\Bigg(H'', \begin{pmatrix} v_0 & 0 \\ 0 & v_1 \end{pmatrix}; s\Bigg)\, v_0^{\kappa-2}\, v_1^{\kappa-2}\, e^{-2\pi(a_0 v_0 + a_1 v_1)}\, dv_0\, dv_1, \tag{4.23}$$

where $W_\kappa(H'', V; s)$ is a generalized Whittaker function defined in Section 4.6.

Proposition 4.3. *The holomorphic projection of Jacobi-Eisenstein series has a Fourier expansion*

$$\pi_{\mathrm{hol}} E^J_{k,N}\Bigg(\begin{pmatrix} \tau_0 & 0 \\ 0 & \tau_1 \end{pmatrix}, (z_0, z_1); s\Bigg) = \alpha_{k,N} \sum c(n_0, r_0, n_1, r_1)\, \mathbf{e}(n_0\tau_0 + r_0 z_0 + n_1\tau_1 + r_1 z_1),$$

where the coefficients $c(n_0, r_0, n_1, r_1)$ are given by the formula

$$c(n_0, r_0, n_1, r_1) = \sum_{r\in\mathbb{Z}} \mathbf{W}_{k-\frac{1}{2},s}\left(\frac{1}{4N}\begin{pmatrix} D_0 & n \\ n & D_1 \end{pmatrix}\right) B_{k+2s}\left(\frac{1}{2}\begin{pmatrix} 2n_0 & r & r_0 \\ r & 2n_1 & r_1 \\ r_0 & r_1 & 2N \end{pmatrix}\right),$$

with $D_i = 4N_i n_i - r_i^2$ $(i = 0, 1)$ and $n = 2Nr - r_0 r_1$. Here the generalized Whittaker function $W_{k,s}$ and singular series B_s are defined in Section 4.6 and $\alpha_{N,k}$ is as in Lemma 4.5.

Proof. The statement follows from Lemmas 4.2 and 4.5. □

Set

$$w_{\kappa,s}(t) := \mathbf{W}_{\kappa,s}\begin{pmatrix} 1 & t/2 \\ t/2 & 1 \end{pmatrix}. \tag{4.24}$$

The following identity is a consequence of the elementary properties of Whittaker functions

$$\mathbf{W}_{\kappa,s}\left(\frac{1}{4N}\begin{pmatrix} D_0 & n \\ n & D_1 \end{pmatrix}\right) = \left(\frac{D_0\, D_1}{4N^2}\right)^{\kappa+s-\frac{3}{2}} w_{\kappa,s}\left(\frac{2n}{\sqrt{D_0 D_1}}\right). \tag{4.25}$$

The following lemma shows that $w_{\kappa,s}(t)$ is a Mellin transform (with respect to s) and a Fourier transform (with respect to t) of a J-Bessel function.

Lemma 4.6. *For $\kappa+2s>3/2$ and $2s<1$ the following identity holds*

$$w_{\kappa,s}(t) = \lambda_{\kappa,s}\int\limits_{-\infty}^{\infty} |x|^{-\kappa-2s+1}\, J_{\kappa-1}(4\pi x)\, \mathbf{e}(-tx)\, dx, \tag{4.26}$$

where $J_{\kappa-1}$ denotes the Bessel function of the first kind and $\lambda_{\kappa,s} = \pi^{2\kappa}2^{-2s+2}\Gamma(\kappa-1)(\pi^{1/2}\Gamma(\kappa+s-1)/\Gamma(\kappa+s)-2\pi i)/(\kappa+s-1)$.

Proof. Identities (4.10), (4.23), and (4.24) imply

$$w_{\kappa,s}(t) = \int\limits_{\Pi} y_0^{\kappa+s-2}\, y_1^{\kappa+s-2}\, (z_0 z_1 - x^2)^{-\kappa-s}\, (\bar{z}_0\bar{z}_1 - x^2)^{-s}\, \mathbf{e}(-\bar{z}_0-\bar{z}_1-tx)\, d\varpi, \tag{4.27}$$

where $z_j = x_j + iy_j$ $(j=0,1)$,

$$\Pi = [0,\infty)^2 \times (-\infty,\infty)^3$$

and

$$d\varpi = dy_0\, dy_1\, dx_0\, dx_1\, dx.$$

This integral converges absolutely for $s<\frac{1}{2}$ and $\kappa+2s>2$. Indeed, we have

$$\int\limits_{\Pi} y_0^{\kappa+s-2}\, y_1^{\kappa+s-2}\, |z_0 z_1 - x^2|^{-\kappa-2s}\, e^{-2\pi y_0 - 2\pi y_1}\, d\varpi =$$

$$= \int\limits_0^{\infty}\int\limits_0^{\infty} y_0^{\kappa+s-2}\, y_1^{\kappa+s-2}\, e^{-2\pi y_0-2\pi y_1}\, dy_0\, dy_1 \int\limits_{S_2(\mathbb{R})} |\det(X+iE)|^{-\kappa-2s}\, dX,$$

where E denotes the identity matrix.

After the change of variables $z_1 = -\frac{x^2}{z_2}$ in (4.27) we get

$$w_{\kappa,s}(t) = \int\limits_{\Pi'} y_0^{\kappa+s-2}\, y_2^{\kappa+s-2}\, |x|^{-2s}\, \bar{z}_2^{-\kappa}(z_0+z_2)^{-\kappa-s}\, (\bar{z}_0+\bar{z}_2)^{-s}\, \mathbf{e}(-\bar{z}_0 + x^2/\bar{z}_2 - tx)\, d\varpi',$$

where $z_2 = x_2 + iy_2$,

$$\Pi' = [0,\infty)^2 \times (-\infty,\infty)^3$$

and

$$d\varpi' = dy_0\, dy_2\, dx_0\, dx_2\, dx.$$

It follows from a standard presentation of the Bessel function of the first kind [1] that

$$\int_{-\infty}^{\infty} \bar{z}_2^{-\kappa}\, \mathbf{e}(\bar{z}_2 + x^2/\bar{z}_2)\, dx_2 = 2\pi i^{-\kappa} x^{-\kappa+1} J_{\kappa-1}(4\pi x).$$

And finally, after the change of variables $x_3 = x_0 + x_2$, the identity

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} y_0^{\kappa+s-2}\, y_2^{\kappa+s-2}\, (x_3+iy_0+iy_2)^{-\kappa-s}\, (x_3-iy_0-iy_2)^{-s} \mathbf{e}(-x_3+iy_0+iy_2)\, dx_3\, dy_0\, dy_2 =$$

$$\int_0^1 (1-u)^{\kappa+s-2}\, u^{\kappa+s-2}\, du \int_{-\infty}^{\infty} (v+i)^{-\kappa-s}(v-i)^{-s} \left(\int_0^{\infty} y^{\kappa-2}\, \mathbf{e}(-yv+iy)\, dy\right) dv =$$

$$= (2\pi)^{2\kappa}\left(\frac{1}{2(\kappa+s-1)} - i\,\pi^{1/2}\,\frac{\Gamma(\kappa+s-1/2)}{\Gamma(\kappa+s)}\right)\frac{\Gamma(\kappa-1)\Gamma(\kappa+s-1)^2}{\Gamma(2\kappa+2s-2)} = \lambda_{\kappa,s}$$

finishes the proof. □

Using presentation (4.26) and integration by parts we obtain the following result.

Lemma 4.7. *The function $w = w_{\kappa,s}(t)$ satisfies the following second order differential equation*

$$(1-t^2)\, w'' - (2-2\lambda)\, t\, w' + (\kappa - 1/2 - \lambda)\, (\kappa - 3/2 + \lambda)\, w,$$

where $\lambda = \kappa + 2s - 3/2$.

The integral (4.26) is computed in the closed form in [7] (vol I, p. 45, eq. (13)); hence, as a corollary of Lemma 5.5 we obtain the identity

$$\frac{w_{\kappa,s}(t)}{\lambda_{\kappa,s}} = \begin{cases} (2\pi)^{\kappa+2s-2}\dfrac{\Gamma(\frac{1}{2}-s)}{\Gamma(\kappa+s-\frac{1}{2})}\, {}_2F_1\Big(\dfrac{1}{2}-s, \dfrac{3}{2}-\kappa-s; \dfrac{1}{2}; \dfrac{t^2}{4}\Big) & 4-t^2>0 \\ (2\pi)^{\kappa+2s-2}\dfrac{\Gamma(2\kappa+2s-1)}{\Gamma(\kappa)}\, \sin(s\pi)\, t^{2s-1}\, {}_2F_1\Big(\dfrac{1}{2}-s, 1-s; \kappa; \dfrac{4}{t^2}\Big) & 4-t^2<0, \end{cases}$$

where ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. Using identities for hypergeometric series [6](vol I, eq. (15) on p. 150 and eq. (8) on p. 122) we obtain the following.

Lemma 4.8. *For $k \in 2\mathbb{Z}$ the following identities hold*

$$w_{k-\frac{1}{2},1-\frac{k}{2}}(t) = -\lambda_{k-\frac{1}{2},1-\frac{k}{2}}\, P_{k-2}(t/2), \quad 4-t^2>0$$

and

$$\frac{\partial}{\partial s} w_{k-\frac{1}{2},s}(t)\Big|_{s=1-\frac{k}{2}} = \lambda_{k-\frac{1}{2},1-\frac{k}{2}}\, Q_{k-2}(t/2), \quad 4-t^2<0,$$

where $P_k(t) = (2^k k!)^{-1} \frac{d^k}{dt^k}(t^2-1)^k$ is the Legendre function of the first kind and $Q_{k-1}(t) = \int_0^\infty (t+\sqrt{t^2-1}\cosh u)^{-k} du$ is the Legendre function of the second kind.

The final result of this section is the following.

Theorem 4.5. *For $N \in \mathbb{N}$ and even $k > 2$ the function $\mathcal{E}_{k,N}$ has the Fourier expansion*

$$\mathcal{E}_{k,N} = \sum c(n_0, r_0; n_1, r_1)\, \mathbf{e}(n_0\tau_0 + r_0 z_0 + n_1\tau_1 + r_1 z_1),$$

where

$$c(n_0, r_0; n_1, r_1) = \sum_{r\in\mathbb{Z}} \mathbf{A}_k^*(H(r)).$$

Here for the matrix

$$H = H(r) = \frac{1}{2}\begin{pmatrix} 2n_0 & r & r_0 \\ r & 2n_1 & r_1 \\ r_0 & r_1 & 2N \end{pmatrix},$$

we have

$$\mathbf{A}_k^*(H) = \begin{cases} (D_0D_1)^{k/2-1}\, P_{k-2}\Big(\dfrac{n}{\sqrt{D_0D_1}}\Big)\, \dfrac{\partial}{\partial s} B(H, 2+s)\Big|_{s=0} & H>0, \\[2ex] (D_0D_1)^{k/2-1}\, Q_{k-2}\Big(\dfrac{n}{\sqrt{D_0D_1}}\Big) B(H,2) & H \text{ is indefinite.} \end{cases}$$

4.11 Computation of non-archimedean local height

Proof of Theorem 4.1:
Case $\mathbf{k = 2}$.
We should say, that a proof of this theorem in the case $k = 2$ is already contained in [36] but is not emphasized there.

Let $\underline{X}$ be a modular model for $X_0(N)$ over $\mathbb{Z}$. Denote by $\mathcal{X}^*$ the minimal desingularization of the quotient of $\underline{X}$ by the Fricke involution. We let $\underline{P}^*_{D,r}$ denote the multi-section of $\mathcal{X}^*$ over $\mathbb{Z}_p$ which extends $P^*_{D,r}$, the image of the divisor $P_{D,r}$ on $X^*(N)$. Let W be the completion of the maximal unramified extension of $\mathbb{Z}_p$.

It is shown in [36] that

$$(\underline{P}^*_{D_0,r_0} \cdot \underline{P}^*_{D_1,r_1})_p = (\underline{P}_{D_0,r_0} \cdot (\underline{P}_{D_1,r_1} + \underline{P}_{D_1,-r_1}))_W.$$

A point $x = (\phi : E \to E')$ lies in the divisor $\underline{P}_{D_0,r_0}$ if $\mathfrak{o}_0 = \mathbb{Z} + \mathbb{Z}\frac{D_0+\sqrt{D_0}}{2}$ embeds into $\mathrm{End}(x)$ and $\alpha_0 = \frac{r_0+\sqrt{D_0}}{2} \in \mathrm{End}(x)$ annihilates $\ker\phi$.

Suppose that $(\underline{x}\cdot\underline{y})_W > 0$ for $\underline{x} \in \underline{P}_{D_0,r_0}$ and $\underline{y} \in \underline{P}_{D_1,r_1}$. Then our diagramms

$$\underline{x} = (\phi : E \to E') \ \text{ and } \ \underline{y} = (\psi : F \to F')$$

reduce to the same isogeny z on $X \otimes W/pW$. Write R for the endomorphism ring $\mathrm{End}_{W/pW}(z)$. The reduction of endomorphisms gives injections

$$\mathrm{End}_W(\underline{x}) \hookrightarrow R, \ \ \mathrm{End}_W(\underline{y}) \hookrightarrow R.$$

It follows from Deuring's theory that $R \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$ ramified only at p and ∞, and that R is an Eichler order of index N in this quaternion algebra. It is shown in [36] on p. 549 that the embeddings of $\mathrm{End}_W(\underline{x})$ and $\mathrm{End}_W(\underline{y})$ give elements $\sqrt{D_i}$ and $\alpha_i = (r_i + \sqrt{D_i})/2$ $(i = 0, 1)$ in R satisfying

$$\sqrt{D_0}\sqrt{D_1} + \sqrt{D_1}\sqrt{D_0} = 2n \ \text{ for some } \ n \in \mathbb{Z}, \tag{4.28}$$

$$n \equiv r_0 r_1 \pmod{2N}, n^2 < r_0 r_1.$$

Thus, we get an embedding of the Clifford order

$$S = \mathbb{Z} + \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_0\alpha_1$$

into R. The order S corresponds to a half-integral matrix

$$H = \frac{1}{2}\begin{pmatrix} 2n_0 & r & r_0 \\ r & 2n_1 & r_1 \\ r_0 & r_1 & 2N \end{pmatrix}, \text{ where } \qquad r = \frac{r_0 r_1 - n}{2N}, \tag{4.29}$$

in a sense of equation (4.15). By Proposition 6.1 in [35] the intersection number $(\underline{x}\cdot\underline{y})_W$ is equal to

$$(\underline{x}\cdot\underline{y})_W = \frac{1}{2}\sum_{i\geq 1} \mathrm{Card}\,\mathrm{Hom}_{W/p^iW}(\underline{x},\underline{y})_{\deg 1},$$

and it is shown in [36] that

$$(\underline{x}\cdot\underline{y})_W = \frac{1}{2}(\mathrm{ord}_p(M) + 1), \quad \text{where } \ M = \frac{D_0 D_1 - 4n^2}{4N}.$$

Thus, it follows from Lemma 4.3 that

$$\log p\,(\underline{x}\cdot\underline{y})_W = \frac{\partial}{\partial s} B_p(H, 2+s)\bigg|_{s=0}. \tag{4.30}$$

On the other hand for a fixed matrix H, it is shown in [36] that the given embeddings of $\mathfrak{o}_0$ and $\mathfrak{o}_1$ into R correspond to points $\underline{x} \in \underline{P}_{D_0,r_0}$ and $\underline{y} \in \underline{P}_{D_1,r_1}$ which reduce to $z \pmod p$ and are congruent modulo p^k where $k = (\mathrm{ord}_p(M) + 1)/2$. It is also proved in [36] that for a given matrix H the number of such points is equal to the number of embeddings of $S(H)$ into Eichler orders of index N in $\mathbb{B}(H)$. From Lemma 4.2 we know that the number of such embeddings is equal to $\prod_{q\neq p} B_q(H)$. Thus, we have

$$(\underline{P}^*_{D_0,r_0}\cdot\underline{P}^*_{D_1,r_1})_p \log p = \sum_{\substack{\underline{x}\in\underline{P}^*_{D_0,r_0}\\ \underline{y}\in\underline{P}^*_{D_1,r_1}}} (\underline{x}\cdot\underline{y})_W \log p = \sum_{r\in\mathbb{Z}}^{p} \frac{\partial}{\partial s} B_p(H(r),2+s)\Big|_{s=0} \prod_{q\neq p} B_q(H(r)).$$

Here the sum is taken over the matrices $H(r)$ of the form (4.29) such that the quaternion algebra $\mathbb{B}(H(r))$ is ramified at p and ∞. Since $W_2(H,Y;0)=1$ for $H>0$ this sum is equal to

$$\sum_{r\in\mathbb{Z}}^{p} \frac{\partial}{\partial s} A_2(H(r),s)\Big|_{s=0}.$$

Theorem 4.1 in the case $k=2$ is proved.

Case $\mathbf{k>2}$. Suppose that $\underline{x}\in\underline{P}_{D_0,r_0}$ and $\underline{y}\in\underline{P}_{D_1,r_1}$ satisfy condition (4.28). It follows from equation (3.3.1) and Proposition 3.3.3 in [80] that

$$(S^{k-1}(\underline{x})\cdot S^{k-1}(\underline{y}))_W = (\underline{x}\cdot\underline{y})_W\,(D_0D_1)^{k/2-1}\,P_{k-2}\Big(\frac{n}{\sqrt{D_0D_1}}\Big),$$

where $P_{k-2}(t)$ denote a constant multiple of $\frac{d^{k-2}}{dt^{k-2}}(t^2-1)^{k-2}$ such that $P_{k-2}(1)=1$. Set

$$H''(r) = \frac{1}{4N}\begin{pmatrix} D_0 & n \\ n & D_1\end{pmatrix}.$$

From equation (4.25) and Lemma 4.8 we see that

$$(S^{k-1}(\underline{x})\cdot S^{k-1}(\underline{y}))_W = (\underline{x}\cdot\underline{y})_W\,\mathbf{W}_{k-1/2,1-k/2}(H''(r)).$$

Similarly to the case $k=2$ considered above we arrive at the identity

$$(S_{k-1}(\underline{P}_{D_0,r_0})\cdot S_{k-1}(\underline{P}_{D_1,r_1}))_p \log p = \sum_{\substack{\underline{x}\in\underline{P}^*_{D_0,r_0}\\ \underline{y}\in\underline{P}^*_{D_1,r_1}}} (S^{k-1}(\underline{x})\cdot S^{k-1}(\underline{y}))_W \log p =$$

$$\sum_{r\in\mathbb{Z}}^{p} \mathbf{W}_{k-1/2,1-k/2}(H''(r))\,\frac{\partial}{\partial s} B_p(H(r),2+s)\Big|_{s=0} \prod_{q\neq p} B_q(H(r)).$$

Here the sum is taken over the matrices $H(r)$ of the form (4.29) such that the quaternion algebra $\mathbb{B}(H)$ is ramified at p and ∞. This is equal to

$$\sum_{r\in\mathbb{Z}}^{p} A_k^*(H(r)).$$

This finishes the proof of Theorem 4.1.

4.12 Computation of archimedean local height

Proof of Theorem 4.2: For $k > 2$ and $\tau_0, \tau_1 \in \mathfrak{H}$ define

$$g_k(\tau_0, \tau_1) := -2Q_{k-2}\left(1 + \frac{|\tau_0 - \tau_1|^2}{2\Im\tau_0\Im\tau_1}\right).$$

Consider the function on $\mathfrak{H} \times \mathfrak{H} \setminus \{(\tau_0, \tau_1) | \tau_0 = \tau_1\}$ given by

$$G_k(\tau_0, \tau_1) = \sum_{\gamma \in \Gamma_0(N)} g_k(\tau_0, \gamma(\tau_1)).$$

It is proved in Proposition 3.4.1 of [80] that for any two CM-points x and y on $X_0(N)$ one has

$$\langle S_{k-1}(x), S_{k-1}(y)\rangle = \frac{1}{2} G_k(x, y).$$

Thus, it follows from Proposition 2 on p. 545 in [36] that

$$\langle S_{k-1}(y^*_{D_0,r_0}), S_{k-1}(y^*_{D_1,r_1})\rangle_\infty = -2 \sum_{\substack{n > \sqrt{D_0 D_1} \\ n \equiv -r_0 r_1 (\mathrm{mod} 2N)}} \rho(n) Q_{k-2}(\frac{n}{\sqrt{D_0 D_1}}). \tag{4.31}$$

For $n \in \mathbb{Z}$, $n^2 \equiv D_0 D_1 \pmod{4N}$, where

$$\rho(n) = \sum_{d | \frac{n^2 - D_0 D_1}{4N}} \varepsilon(d),$$

and ε is associated to the quadratic form $[D_0, -2n, D_1]$ as in Sect. 3 of Chap. I in [36]. Take $r := (n + r_0 r_1)/2N$ and $n_i := (D_i + r_i^2)/4N,\ i = 0, 1$. It the case when the matrix $H(r)$ defined by (4.29) is indefinite we have

$$B(H(r), 2) = \rho(n). \tag{4.32}$$

By Theorem 4.5 the number $\mathbf{A}^*_k(H(r))$ in this case is equal to

$$\mathbf{A}^*_k(H(r)) = (D_0 D_1)^{k/2-1} Q_{k-2}\Big(\frac{n}{\sqrt{D_0 D_1}}\Big) B(H, 2). \tag{4.33}$$

Thus, equations (4.31)-(4.33) imply the statement of Theorem 4.2. □

4.13 Computation of global height

Proof of Theorem 4.3: In the case $k = 2$ Theorem 4.3 follows from the results of [53]. Hence, here we concentrate on the case $k > 2$.

Let D_0, D_1 be fundamental coprime discriminants and $y^*_{D_0,r_0},\ y^*_{D_1,r_1}$ be two Heegner divisors on $X_0(N)$. From the arithmetic intersection theory reviewed in Section 4.4 we know that

$$\langle S_{k-1}(y^*_{D_0,r_0}), S_{k-1}(y^*_{D_1,r_1})\rangle = \sum_p \langle S_{k-1}(y^*_{D_0,r_0}), S_{k-1}(y^*_{D_1,r_1})\rangle_p$$

where the sum is taken over all the places of $\mathbb{Q}$. For an integer r we consider a half integral matrix

$$H(r) = \frac{1}{2}\begin{pmatrix} 2n_0 & r & r_0 \\ r & 2n_1 & r_1 \\ r_0 & r_1 & 2N \end{pmatrix}. \tag{4.34}$$

By Theorems 4.1 and 4.2 we have

$$\langle S_{k-1}(y^*_{D_0,r_0}), S_{k-1}(y^*_{D_1,r_1})\rangle_p = \text{const} \sum_r{}^p \mathbf{A}^*_k(H(r)),$$

where and the sum is taken over all integers r such that the quaternion algebra $\mathbb{B}(H(r))$ (see Section 4.7 for the definition) is ramified at p and ∞ if p is finite and unramified at all places if $p = \infty$.

Each quaternion algebra $\mathbb{B}(H(r))$ is ramified at an even number of places. Moreover, Lemma 4.1 implies that $\mathbf{A}^*(H(r)) = 0$ in the case when $\mathbb{B}(H(r))$ is ramified at least at two finite primes. Thus, we conclude

$$\sum_p \sum_{r\in\mathbb{Z}}{}^p \mathbf{A}^*_k(H(r)) = \sum_{r\in\mathbb{Z}} \mathbf{A}^*_k(H(r)).$$

By Proposition 4.1 this number is equal to the Fourier coefficient $c(n_0, r_0, n_1, r_1)$ of the function $\mathcal{E}_{k,N}$. $\square$

Chapter 5

CM values of higher Green's functions

5.1 Introduction

For any integer $k > 1$ there is a unique function G_k on the product of two upper half planes $\mathfrak{H} \times \mathfrak{H}$ which satisfies the following conditions:

(i) G_k is a smooth function on $\mathfrak{H} \times \mathfrak{H} \setminus \{(\tau, \gamma\tau), \tau \in \mathfrak{H}, \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$ with values in $\mathbb{R}$.

(ii) $G_k(\tau_1, \tau_2) = G_k(\gamma_1\tau_1, \gamma_2\tau_2)$ for all $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$.

(iii) $\Delta_i G_k = k(1-k)G_k$, where Δ_i is the hyperbolic Laplacian with respect to the i-th variable, $i = 1, 2$.

(iv) $G_k(\tau_1, \tau_2) = m \log|\tau_1 - \tau_2| + O(1)$ when τ_1 tends to τ_2 (m is the order of the stabilizer of τ_2, which is almost always 1).

(v) $G_k(\tau_1, \tau_2)$ tends to 0 when τ_1 tends to a cusp.

This function is called the Green's function.

Let f be a modular function. Then the action of the *Hecke operator* T_m on f is given by

$$\big(f \mid T_m\big)(\tau) \;=\; m^{-1} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{M}_m} f\left(\frac{a\tau + b}{c\tau + d}\right),$$

where $\mathcal{M}_m$ denotes the set of 2×2 integral matrices of determinant m.

The Green's functions G_k have the property

$$G_k(\tau_1, \tau_2) \mid T_m^{\tau_1} = G_k(\tau_1, \tau_2) \mid T_m^{\tau_2},$$

where $T_m^{\tau_i}$ denotes the Hecke operator with respect to variable τ_i, $i = 1, 2$. Therefore, we will simply write $G_k(\tau_1, \tau_2)| \, T_m$.

Denote by $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ the space of cusp forms of weight $2k$ on the full modular group.

Proposition 5.1. *Let $k > 1$ and $\boldsymbol{\lambda} = \{\lambda_m\}_{m=1}^{\infty} \in \oplus_{m=1}^{\infty}\mathbb{Z}$. Then the following are equivalent*

(i) $\sum_{m=1}^{\infty} \lambda_m a_m = 0$ for any cusp form

$$f = \sum_{m=1}^{\infty} a_m q^m \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$$

(ii) There exists a weakly holomorphic modular form

$$g_{\boldsymbol{\lambda}}(\tau) = \sum_{m=1}^{\infty} \lambda_m q^{-m} + O(1) \in M^!_{2-2k}(\mathrm{SL}_2(\mathbb{Z})).$$

The proof of this proposition can be found, for example, in Section 3 of [11]. The space of obstructions to finding modular forms of weight $2-2k$ with given singularity at the cusp and the space of holomorphic modular forms of weight $2k$ can be both identified with cohomology groups of line bundles over a modular curve. The statement follows from Serre duality between these spaces. An elementary proof that (ii) implies (i) is given by noticing that if $f = \sum_m a_m q^m \in S_{2k}$ then $f(\tau)\, g_\lambda(\tau)\, d\tau$ is a meromorphic form on $\mathfrak{H}\backslash\mathrm{SL}_2(\mathbb{Z})$ with no poles except at ∞, thus its residue $\sum_m \lambda_m a_m$ at ∞ vanishes by the residue theorem.

We call a $\boldsymbol{\lambda}$ with the properties, given in the above proposition, a *relation* for $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. Note that the function $g_{\boldsymbol{\lambda}}$ in (ii) is unique and has integral Fourier coefficients.

For a relation $\boldsymbol{\lambda}$ denote

$$G_{k,\boldsymbol{\lambda}} := \sum_{m=1}^{\infty} \lambda_m\, m^{k-1}\, G_k(\tau_1, \tau_2) \mid T_m.$$

The following conjecture was formulated in [35] and [36].

Conjecture 1. *Suppose $\boldsymbol{\lambda}$ is a relation for $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. Then for any two CM points $\mathfrak{z}_1$, $\mathfrak{z}_2$ of discriminants D_1, D_2 there is an algebraic number α such that*

$$G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1, \mathfrak{z}_2) = (D_1 D_2)^{\frac{1-k}{2}} \log|\alpha|.$$

Moreover, D. Zagier has made a more precise conjecture about the field of definition and prime factorization of this number α. We will have to introduce some notations before we can state the conjecture.

Assume that the discriminant $-D < 0$ is prime and consider the imaginary quadratic field $K := \mathbb{Q}(\sqrt{-D})$. For an integral ideal $\mathfrak{a} \subset \mathfrak{o}_K$ and $m \in \mathbb{Z}$ denote by $r_{\mathfrak{a}}(m)$ the number of integral ideals of the norm m in the ideal class of $\mathfrak{a}$. We will write $r(m)$ for the total number of integral ideals of the norm m in K. Let H be a Hilbert class field of K. Denote by h the class number of K. For an ideal class $\mathfrak{a} \in \mathrm{CL}(K)$ we denote by $\sigma_{\mathfrak{a}}$ the element of $\mathrm{Gal}(H/K)$ corresponding to $\mathfrak{a}$ under the Artin isomorphism. Fix an embedding $\imath : H \to \mathbb{C}$. Let p be a rational prime with $\left(\frac{p}{D}\right) = -1$. Let $\mathcal{P}_p = \{\mathfrak{P}_i\}_{i=0}^{h}$ be the set of prime ideals of H lying above p. Complex conjugation acts on this set. Since the class number h is odd, there exists a unique prime ideal in $\mathcal{P}_p$, say $\mathfrak{P}_1$, with $\mathfrak{P}_1 = \overline{\mathfrak{P}}_1$. For a prime ideal $\mathfrak{P} \in \mathcal{P}_p$ there exists a unique element $\sigma \in \mathrm{Gal}(H/K)$ such that

$$\mathfrak{P}^\sigma = \mathfrak{P}_1. \tag{5.1}$$

Denote by $\mathfrak{a} = \mathfrak{a}(\mathfrak{P})$ a fractional ideal of K whose class corresponds to σ under the Artin isomorphism.

The following precise version of Conjecture 1 was made by D. Zagier.

Conjecture 2. *Let $\mathfrak{z}_1,\ \mathfrak{z}_2 \in \mathfrak{H}$ be two CM-points of discriminant $-D$ and let $\mathfrak{b} := \mathfrak{z}_1\mathbb{Z}+\mathbb{Z}$, $\mathfrak{c} := \mathfrak{z}_2\mathbb{Z} + \mathbb{Z}$ be the corresponding fractional ideals of $\mathfrak{o}_K$. Then*

$$G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1, \mathfrak{z}_2) = D^{1-k} \log|\alpha|,$$

where α lies in the Hilbert class field H. Moreover, for a rational prime p with $\left(\frac{p}{D}\right) \neq 0$ and a prime ideal $\mathfrak{P}$ lying above p in H we have

$$\mathrm{ord}_{\mathfrak{P}}(\alpha) = 0$$

in case $\left(\frac{p}{D}\right) = 1$ and

$$\mathrm{ord}_{\mathfrak{P}}(\alpha) = \sum_{m=1}^{\infty}\sum_{n=0}^{Dm} \lambda_m\, m^{k-1}\, P_{k-1}\Big(1 - \frac{2n}{Dm}\Big)\, r_{\mathfrak{b}\bar{\mathfrak{c}}}(Dm-n)\, r_{\mathfrak{b}\mathfrak{c}\mathfrak{a}^2}\Big(\frac{n}{p}\Big)\,(1+\mathrm{ord}_p(n)) \tag{5.2}$$

in case $\left(\frac{p}{D}\right) = -1$. Here $P_k(x) = (2^k k!)^{-1}\frac{d^k}{dx^k}(x^2-1)^k$ is the k-th Legendre polynomial and the ideal class $\mathfrak{a}$ is defined as above.

In this chapter we present a proof Conjecture 1 in the case when $\mathfrak{z}_1, \mathfrak{z}_2$ lie in the same imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ and a proof of Conjecture 2.

Two main ingredients of our proof are the theory of Borcherds lift developed in [10] and a notion of see-saw identities introduced in [50]. Firstly, following ideas given in [12] we prove in Theorem 5.3 that the Green's function can be realized as a Borcherds lift of an eigenfunction of Laplace operator. This allows as to extend a method given in [62], that is to analyze CM values of Green's function using see-saw identities. In Theorem 5.4 we prove that a CM-value of higher Green's function is equal to the regularized Petersson product of a weakly holomorphic modular form of weight 1 and a binary theta series. In Theorem 5.5 we use an embedding trick and show that the regularized Petersson product of any weakly holomorphic modular form of weight 1 and a binary theta series is equal to a CM-value of a certain meromorphic modular function. Thus, from Theorems 5.4 and 5.5 we see that that a CM-value of higher Green's function is equal to the logarithm of a CM-value of a meromorphic modular function with algebraic Fourier coefficients. Finally, we use the theory of local height pairing [34] and the explicit computations of the height pairing between Heegner points made in [35, 36] in order to compute these CM-value and hence prove Conjecture 2.

5.2 Differential operators

For $k \in \mathbb{Z}$ denote by R_k and L_k the Maass raising and lowering differential operators

$$R_k = \frac{1}{2\pi i}\left(\frac{\partial}{\partial \tau} + \frac{k}{\tau - \bar{\tau}}\right), \quad L_k = \frac{1}{2\pi i}(\tau - \bar{\tau})^2 \frac{\partial}{\partial \bar{\tau}},$$

which send real-analytic modular forms of weight k to real-analytic modular forms of weight $k+2$ and $k-2$, respectively. Then the weight k Laplace operator is given by

$$\Delta_k = -4\pi^2 R_{k-2} L_k = -4\pi^2 (L_{k+2} R_k - k) = (\tau - \bar{\tau})^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} + k(\tau - \bar{\tau}) \frac{\partial}{\partial \bar{\tau}}.$$

For integers l, k we denote by $F_{l,k}$ the space of functions of weight k satisfying

$$\Delta f = (l - k/2)(1 - l - k/2) f.$$

Proposition 5.2. *The spaces $F_{l,k}$ satisfy the following properties:*

(i) The space $F_{l,k}$ is invariant under the action of the group $\mathrm{SL}_2(\mathbb{R})$*,*

(ii) The operator R_k maps $F_{l,k}$ to $F_{l,k+2}$,

(iii) The operator L_k maps $F_{l,k}$ to $F_{l,k-2}$.

For a modular form f of weight k we will use the notation

$$R^r f = R_{k+2r-2} \circ \cdots \circ R_k f.$$

Denote $f^{(s)} := \frac{1}{(2\pi i)^s} \frac{\partial^s}{\partial \tau^s} f$. We have (see equation (56) in [14])

$$R^r(f) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \frac{(k+s)_{r-s}}{(4\pi y)^{r-s}} f^{(s)}, \tag{5.3}$$

where $(a)_m = a(a+1)\cdots(a+m-1)$ is the Pochhammer symbol. For modular forms f and g of weight k and l the Rankin-Cohen bracket is defined by

$$[f, g] = l f' g - k f g',$$

and more generally

$$[f, g]_r = [f, g]_r^{k,l} = \sum_{s=0}^{r} (-1)^s \binom{k+r-1}{s} \binom{l+r-1}{r-s} f^{(r-s)} g^{(s)}. \tag{5.4}$$

The function $[f, g]_r$ is a modular form of the weight $k + l + 2r$. Note that

$$\binom{k}{s} := \frac{(k-s+1)_s}{s!}$$

is defined for $s \in \mathbb{N}$ and arbitrary k.

We will need the following proposition.

Proposition 5.3. *Suppose that f and g are modular forms of weight k and l respectively. Then, for an integer $r \geq 0$ we have*

$$R^r(f)\, g = a[f,g]_r + R\bigg(\sum_{s=0}^{r-1} b_s\, R^s(f)\, R^{r-s-1}(g)\bigg)$$

where

$$a = \binom{k+l+2r-2}{r}^{-1}$$

and b_s are some rational numbers.

Proof. The operator R satisfies the following property

$$R(fg) = R(f)g + fR(g).$$

Thus, the sum

$$\sum_{i+j=r} a_i R^i(f) R^j(g)$$

can be written as

$$R\bigg(\sum_{i+j=r-1} b_i\, R^i(f)\, R^j(g)\bigg)$$

for some numbers b_i if and only if $\sum_{i=0}^{r}(-1)^i a_i = 0$. For the Rankin-Cohen brackets the following identity holds

$$[f,g]_r = \sum_{s=0}^{r}(-1)^s \binom{k+r-1}{s}\binom{l+r-1}{r-s} R^{(r-s)}(f) R^s(g). \tag{5.5}$$

We will use the following standard identity

$$\sum_{s=0}^{r}\binom{k+r-1}{s}\binom{l+r-1}{r-s} = \binom{k+l+2r-2}{r}.$$

It follows from the above formula and (5.5) that the sum

$$\binom{k+l+2r-2}{r} R^r(f)g - [f,g]_r$$

can be written in the form

$$R\bigg(\sum_{i+j=r-1} b_i\, R^i(f)\, R^j(g)\bigg).$$

This finishes the proof. $\square$

Proposition 5.4. *Suppose that f is a real analytic modular form of weight $k-2$ and g is a holomorphic modular form of weight k. Then, for a compact region $F \subset \mathfrak{H}$ we have*

$$\int_F R_{k-2}(f)\, \bar{g}\, y^{k-2}\, dx\, dy = \int_{\partial F} f\, \bar{g}\, y^{k-2}\, (dx - idy).$$

Proof. This result is well known [12] and it follows easily from Stokes's theorem. □

Denote by K_ν the K-Bessel function

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}, \quad K_\nu(x) = \frac{\pi}{2}\, \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}.$$

The function K_ν becomes elementary for $\nu \in \mathbb{Z} + \frac{1}{2}$. It can be written as

$$K_{k+\frac{1}{2}}(x) = \frac{(\pi/2)^{\frac{1}{2}}}{x^{k+\frac{1}{2}}}\, e^{-x}\, h_k(x),$$

for $k \in \mathbb{Z}_{\geq 0}$, where h_k is the polynomial

$$h_k(x) = \sum_{r=0}^{k} \frac{(k+r)!}{2^r r!(k-r)!} x^{k-r}. \tag{5.6}$$

The equation (5.3) implies immediately the following statement.

Proposition 5.5. *For $k \in \mathbb{Z}_{>0}$ the following identity holds*

$$R^k_{-2k}(\mathbf{e}(n\tau)) \;=\; 2\, y^{\frac{1}{2}}\, n^{k+\frac{1}{2}}\, K_{k+1/2}(2\pi n y)\, \mathbf{e}(nx) \;=\; y^{-k}\, h_k(2\pi n y)\, \mathbf{e}(n\tau).$$

5.3 A see-saw identity

In the paper [50] S. Kudla introduced the notion of a "see-saw dual reductive pair." It gives rise in a systematic way to a family of identities between inner products of automorphic forms on different groups, thus clarifying the source of identities of this type which appear in many places in the literature. In this section we prove a see-saw identity for the regularized theta integrals described in Section 1.8.

Suppose that (V, q) is a rational quadratic space of signature $(2, b)$ and $L \subset V$ is an even lattice. Let $V = V_1 \oplus V_2$ be a rational orthogonal splitting of (V, q) such that the space V_1 has the signature $(2, b-d)$ and the space V_2 has the signature $(0, d)$. Consider the two lattices $N := L \cap V_1$ and $M := L \cap V_2$. We have two orthogonal projections

$$\mathrm{pr}_M : L \otimes \mathbb{R} \to M \otimes \mathbb{R} \text{ and } \mathrm{pr}_N : L \otimes \mathbb{R} \to N \otimes \mathbb{R}.$$

Let M' and N' be the dual lattices of M and N. We have the following inclusions

$$M \subset L, \quad N \subset L, \quad M \oplus N \subseteq L \subseteq L' \subseteq M' \oplus N',$$

and equalities of the sets

$$\mathrm{pr}_M(L') = M', \quad \mathrm{pr}_N(L') = N'.$$

Consider a rectangular $|L'/L| \times |N'/N|$ dimensional matrix $T_{L,N} = T_{L,N}(\tau)$ with entries

$$\vartheta_{\lambda,\nu}(\tau) = \sum_{\substack{m\in M':\\ m+\nu\in\lambda+L}} \mathbf{e}\big(-\mathrm{q}(m)\tau\big) \qquad (\lambda \in L'/L, \nu \in N'/N, \tau \in \mathfrak{H}).$$

This sum is well defined since $N \subset L$. Note that the lattice M is negative definite and hence the series converges. For a function $f = (f_\lambda)_{\lambda\in L'/L} \in M_{k+d/2}(\rho_N)$ we define $g = (g_\nu)_{\nu\in N'/N}$ by

$$g_\nu(\tau) = \sum_{\lambda\in L'/L} \vartheta_{\lambda,\nu}(\tau)\, f_\lambda(\tau). \tag{5.7}$$

In other words

$$g = T_{L,N} f \tag{5.8}$$

where f and g are considered as column vectors.

Theorem 5.1. *Suppose that the lattices L, M and N and functions f, g are defined as above. Then the function g belongs to $M_{k+d/2}(\rho_N)$. Thus, there is a map $T_{L,N} : M_k(\rho_L) \to M_{k+d/2}(\rho_N)$ defined by* (5.8).

Proof. Consider the function

$$\Theta_{M(-1)}(\tau) = \overline{\Theta_M(\tau)} = \sum_{\mu\in M'/M} e_\mu \sum_{m\in M+\mu} \mathbf{e}(-\mathrm{q}(m)\tau)$$

that belongs to $M_{d/2}(\rho_{M(-1)})$. It follows from (5.7) and (1.15) that

$$T_{L,N}(f) = \big\langle \mathrm{res}_{L/M\oplus N}(f), \Theta_{M(-1)} \big\rangle_{\mathbb{C}[M'/M]}.$$

Thus, from Lemma 1.2 we deduce that $T_{L,N}(f)$ is in $M_{k+d/2}(\rho_N)$. □

Theorem 5.2. *Let L, M, N be as above. Denote by $i : \mathrm{Gr}^+(N) \to \mathrm{Gr}^+(L)$ a natural embedding induced by inclusion $N \subset L$. Then, for $v^+ \in \mathrm{Gr}^+(N)$ the theta lift of a function $f \in \widehat{M}^!_{1-b/2}(\mathrm{SL}_2(\mathbb{Z}), \rho_L)$ the following holds*

$$\Phi_L(i(v^+), f) = \Phi_N(v^+, T_{L,N}(f)). \tag{5.9}$$

Proof. For a vector $l \in L'$ denote $m = \mathrm{pr}_M(l)$ and $n = \mathrm{pr}_N(l)$. Recall that $m \in M'$ and $n \in N'$. Since v^+ is an element of $\mathrm{Gr}^+(N)$ it is orthogonal to M. We have

$$\mathrm{q}(l_{v^+}) = \mathrm{q}(n_{v^+}),\ \mathrm{q}(l_{v^-}) = \mathrm{q}(m) + \mathrm{q}(n_{v^-}).$$

Thus for $\lambda \in L'/L$ we obtain

$$\Theta_{\lambda+L}(\tau, v^+) = \sum_{l\in\lambda+L} \mathbf{e}\big(\mathrm{q}(l_{v^+})\tau + \mathrm{q}(l_{v^-})\bar\tau\big) =$$

$$\sum_{\substack{m\in M',\, n\in N':\\ m+n\in\lambda+L}} \mathbf{e}\big(\mathrm{q}(n_{v^+})\tau + \mathrm{q}(n_{v^-})\bar\tau + \mathrm{q}(m)\bar\tau\big).$$

Since $N \subset L$ we can rewrite this sum as

$$\Theta_{\lambda+L}(\tau, v^+) = \sum_{\nu\in N'/N} \Theta_{\nu+N}(\tau, v^+)\,\overline{\vartheta_{\nu,\lambda}(\tau)}.$$

Thus, we see that for $f = (f_\lambda)_{\lambda\in L'/L}$ the following scalar products are equal

$$\langle f, \overline{\Theta_L(\tau, v^+)}\rangle = \langle T_{L,N}(f), \overline{\Theta_N(\tau, v^+)}\rangle.$$

Therefore, the regularized integrals (1.36) of both sides of the equality are also equal. $\square$

Remark 5.1. Theorem 5.2 works even in the case when v^+ is a singular point of $\Phi_L(v^+, f)$. If the constant terms of f and $T_{L,N}(f)$ are different, then subvariety $\mathrm{Gr}^+(N)$ lies in singular locus of $\Phi_L(v^+, f)$. On the other hand, if constant terms of f and $T_{L,N}(f)$ are equal then, singularities cancel at the points of $\mathrm{Gr}^+(N)$.

Remark 5.2. The map $T_{M,N}$ is essentially the *contraction map* defined in §3.2 of [62].

5.4 Lattice $M_2(\mathbb{Z})$

Consider the lattice of integral 2×2 matrices, denoted by $M_2(\mathbb{Z})$. Equipped with the quadratic form $\mathrm{q}(x) := -\det x$ it becomes an even unimodular lattice (recall that in our notations the corresponding bilinear form is defined by $(x, x) = 2\mathrm{q}(x)$).

The Grassmannian $\mathrm{Gr}^+(M_2(\mathbb{Z}))$ turns out to be isomorphic to $\mathfrak{H}\times\mathfrak{H}$. This isomorphism can be constructed in the following way. For the pair of points $(\tau_1, \tau_2) \in \mathfrak{H}\times\mathfrak{H}$ consider the element of the norm zero

$$Z = \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} \in M_2(\mathbb{Z}) \otimes \mathbb{C}.$$

Define $v^+(\tau_1, \tau_2)$ be the vector subspace of $M_2(\mathbb{Z})\otimes\mathbb{R}$ spanned by two vectors $X = \Re(Z)$ and $Y = \Im(Z)$. The map

$$(\tau_1, \tau_2) \;\to\; v^+(\tau_1, \tau_2) := \mathbb{R}X + \mathbb{R}Y \tag{5.10}$$

gives an isomorphism between hermitian domains $\mathfrak{H}\times\mathfrak{H}$ and $\mathrm{Gr}^+(M_2(\mathbb{Z}))$.

The group $\mathrm{SL}_2(\mathbb{Z})\times\mathrm{SL}_2(\mathbb{Z})$ acts on $M_2(\mathbb{Z})$ by $(\gamma_1, \gamma_2)(x) = \gamma_1 x\, {}^t\gamma_2$ and preserves the norm.The action of $\mathrm{SL}_2(\mathbb{Z})\times\mathrm{SL}_2(\mathbb{Z})$ on the Grassmannian agrees with the action on $\mathfrak{H}\times\mathfrak{H}$ by fractional linear transformations

$$(\gamma_1, \gamma_2)(v^+(\tau_1, \tau_2)) = v^+(\gamma_1(\tau_1), \gamma_2(\tau_2)).$$

We have

$$(X, X) = (Y, Y) = \frac{1}{2}(Z, \overline{Z}) = -\frac{1}{2}(\tau_1 - \bar\tau_1)(\tau_2 - \bar\tau_2),$$

$$(X,Y) = \frac{1}{2i}(Z,Z) = 0.$$

For $l = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ and $v^+ = v^+(\tau_1,\tau_2)$ we have

$$\mathrm{q}(l_{v^+}) = \frac{(l,Z)(l,\overline{Z})}{(Z,\overline{Z})} = \frac{|\, d\tau_1\tau_2 - c\tau_1 - b\tau_2 + a\,|^2}{-(\tau_1-\bar\tau_1)(\tau_2-\bar\tau_2)}.$$

Denote

$$\Theta(\tau;\tau_1,\tau_2) := \Theta_{M_2(\mathbb{Z})}\big(\tau, v^+(\tau_1,\tau_2)\big)$$

where $\tau = x+iy$. Considered as a function of τ Θ belongs to $\mathfrak{M}_0(\mathrm{SL}_2(\mathbb{Z}))$ and we can explicitly write this function as

$$\Theta(\tau;\tau_1,\tau_2) = y \sum_{a,b,c,d\in\mathbb{Z}} \mathbf{e}\left(\frac{|a\tau_1\tau_2 + b\tau_1 + c\tau_2 + d|^2}{-(\tau_1-\bar\tau_1)(\tau_2-\bar\tau_2)}(\tau-\bar\tau) - (ad-bc)\bar\tau\right)$$

$$= y \sum_{a,b,c,d\in\mathbb{Z}} \mathbf{e}\left(\frac{|a\tau_1\tau_2 + b\tau_1 + c\tau_2 + d|^2}{-(\tau_1-\bar\tau_1)(\tau_2-\bar\tau_2)}\tau - \frac{|a\tau_1\bar\tau_2 + b\tau_1 + c\bar\tau_2 + d|^2}{-(\tau_1-\bar\tau_1)(\tau_2-\bar\tau_2)}\bar\tau\right).$$

5.5 Higher Green's functions as theta lifts

The key point of our proof is the following observation:

Proposition 5.6. *Denote by Δ^z the hyperbolic Laplacian with respect to variable z. For the function Θ defined in the previous section the following identities hold*

$$\Delta^{\tau}\Theta(\tau;\tau_1,\tau_2) = \Delta^{\tau_1}\Theta(\tau;\tau_1,\tau_2) = \Delta^{\tau_2}\Theta(\tau;\tau_1,\tau_2).$$

This identity can be proved by a straightforward computation. Identities of this kind are the general feature of theta kernels [42], which was used in [12] in order to show that Green's functions can be realized as theta lifts. In this section we show how this general principle can be applied to higher Green's functions introduced in Section 5.1.

Suppose that $\boldsymbol{\lambda} = \{\lambda_m\}_{m=1}^{\infty}$ is a relation on $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ (the definition is given in the introduction). Then there exists a unique weakly holomorphic modular form $g_{\boldsymbol{\lambda}}$ of weight $2-2k$ with Fourier expansion of the form

$$\sum_m \lambda_m\, q^{-m} + O(1).$$

Consider the function $h_{\boldsymbol{\lambda}} := R^{k-1}(g_{\boldsymbol{\lambda}})$ which belongs to $\widehat{M}_0^!(\mathrm{SL}_2(\mathbb{Z}))$.

Theorem 5.3. *The following identity holds*

$$G_{k,\boldsymbol{\lambda}}(\tau_1,\tau_2) = \Phi_{M_2(\mathbb{Z})}(v^+(\tau_1,\tau_2), h_{\boldsymbol{\lambda}}).$$

Here

$$\Phi_{M_2(\mathbb{Z})}(v^+(\tau_1,\tau_2), h_{\boldsymbol{\lambda}}) = \lim_{t\to\infty}\int_{F_t} h_{\boldsymbol{\lambda}}(\tau)\,\overline{\Theta(\tau;\tau_1,\tau_2)}\, y^{-2} dx\, dy. \tag{5.11}$$

Proof. We verify that the function $\Phi_{M_2(\mathbb{Z})}(v^+(\tau_1,\tau_2),h_{\boldsymbol{\lambda}})$ satisfies conditions (i)-(iv) listed in the introduction.

Firstly, we verify Property (i). Let $T_m \subset \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \times \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ be the m-th Hecke correspondence. For a relation $\boldsymbol{\lambda}$ consider a divisor

$$D_{\boldsymbol{\lambda}} := \sum_m \lambda_m T_m.$$

Denote by $S_{\boldsymbol{\lambda}}$ the support of $D_{\boldsymbol{\lambda}}$. It follows from the properties (i), (iv) of Green's function given at the introduction that the singular locus of $G_{k,\boldsymbol{\lambda}}$ is equal to $S_{\boldsymbol{\lambda}}$. It follows from Theorem B1 of Section 1.8 that the limit (5.11) exists for all $(\tau_1,\tau_2)\in\mathfrak{H}\times\mathfrak{H}\backslash S_{\boldsymbol{\lambda}}$, moreover, it defines a real-analytic function on this set. For the convenience of the reader we repeat the argument given in [10]. The function $h_{\boldsymbol{\lambda}}$ has the Fourier expansion

$$h_{\boldsymbol{\lambda}}(\tau) = \sum_{\substack{n\in\mathbb{Z}\\ n\gg-\infty}} c(n,y)\,\mathbf{e}(n\tau).$$

Fix $v^+ = v^+(\tau_1,\tau_2)$ for some $\tau_1,\tau_2\in\mathfrak{H}\times\mathfrak{H}$. For $t>1$ the set F_t can be decomposed into two parts $F_t = F_1\cup\Pi_t$ where Π_t is a rectangle $\Pi_t = [-1/2,1/2]\times[1,t]$. It suffices to show that the limit

$$\lim_{t\to\infty}\int_{\Pi_t} h_{\boldsymbol{\lambda}}(\tau)\,\overline{\Theta_{M_2(\mathbb{Z})}(\tau;v^+)}\,y^{-2}dx\,dy$$

exists for all $(\tau_1,\tau_2)\notin S_{\boldsymbol{\lambda}}$. We split the integral over Π_t into two parts

$$\begin{aligned}\int_{\Pi_t} h_{\boldsymbol{\lambda}}(\tau)\,\overline{\Theta_{M_2(\mathbb{Z})}(\tau;v^+)}\,y^{-2}\,dx\,dy &= \int_{\Pi_t}\sum_{m=1}^{\infty} c(-m,y)\mathbf{e}(-m\tau)\,\overline{\Theta_{M_2(\mathbb{Z})}(\tau;v^+)}\,y^{-2}\,dx\,dy\\ &+\int_{\Pi_t}\Big(h_{\boldsymbol{\lambda}}(\tau)-\sum_{m=1}^{\infty}c(-m,y)\,\mathbf{e}(-m\tau)\Big)\,\overline{\Theta_{M_2(\mathbb{Z})}(\tau;v^+)}\,y^{-2}\,dx\,dy.\end{aligned} \tag{5.12}$$

The first integral can be estimated as

$$\begin{aligned}&\int_{\Pi_t}\sum_{m=1}^{\infty}c(-m,y)\,\mathbf{e}(-m\tau)\,\overline{\Theta_{M_2(\mathbb{Z})}(\tau;v^+)}\,y^{-2}\,dx\,dy\\ &=\sum_{m=1}^{\infty}\int_1^t\sum_{l\in M_2(\mathbb{Z}),\ \mathrm{q}(l)=-m} c(-m,y)\,\exp\big(-4\pi\,\mathrm{q}(l_{v^+})\,y\big)\,y^{-1}\,dy.\end{aligned} \tag{5.13}$$

Note that $l_{v^+}>0$ if $c(\mathrm{q}(l),y)\neq 0$ and $v^+\notin S_{\boldsymbol{\lambda}}$. Hence, the limit of (5.13) as $t\to\infty$ is finite for $v^+\notin S_{\boldsymbol{\lambda}}$. Using the asymptotic estimates

$$h_{\boldsymbol{\lambda}}(\tau)-\sum_{m=1}^{\infty}c(-m,y)\,\mathbf{e}(-m\tau) = O(y^{1-k}),\ \ y\to\infty,$$

and

$$\Theta_{M_2(\mathbb{Z})}(\tau; v^+) = O(y),\ y \to \infty,$$

we see that the second summand at the right hand side of (5.12) tends to a finite limit as t goes to infinity.

Properties (i) and (iv) follow from Theorem B1 stated in Section 1.8.

Property (ii) is obvious since the function $\Theta(\tau; \tau_1, \tau_2)$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant in the variables τ_1 and τ_2.

Property (iii) formally follows from the property of the theta kernel given in Proposition 5.6 and the fact that the Laplace operator is self adjoint with respect to Petersson scalar product. More precisely, we have

$$\Delta^{\tau_1}\Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol{\lambda}}, v^+(\tau_1,\tau_2)) = \lim_{t\to\infty} \int\limits_{F_t} h_{\boldsymbol{\lambda}}(\tau)\, \overline{\Delta^{\tau_1}\Theta(\tau;\tau_1,\tau_2)}\, y^{-2} dx\, dy.$$

Using Proposition 5.6 we arrive at

$$\Delta^{\tau_1}\Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol{\lambda}}, v^+(\tau_1,\tau_2)) = \lim_{t\to\infty} \int\limits_{F_t} h_{\boldsymbol{\lambda}}(\tau)\, \overline{\Delta^{\tau}\Theta(\tau;\tau_1,\tau_2)}\, y^{-2}\, dx\, dy.$$

It follows from the Stokes theorem that

$$\int\limits_{F_t} h_{\boldsymbol{\lambda}}(\tau)\, \overline{\Delta^{\tau}\Theta(\tau;\tau_1,\tau_2)}\, y^{-2}\, dx\, dy - \int\limits_{F_t} \Delta h_{\boldsymbol{\lambda}}(\tau)\, \overline{\Theta(\tau;\tau_1,\tau_2)}\, y^{-2}\, dx\, dy =$$

$$\int\limits_{-1/2}^{1/2} (h_{\boldsymbol{\lambda}}\, \overline{L_0(\Theta)} - L_0(h_{\boldsymbol{\lambda}})\, \overline{\Theta}) y^{-2} dx \Bigg|_{y=t}.$$

This expression tends to zero as t tends to infinity. Since $g_{\boldsymbol{\lambda}} \in F_{k,2-2k}$ it follows from Proposition 5.2 that $\Delta h_{\boldsymbol{\lambda}} = k(1-k)h_{\boldsymbol{\lambda}}$. Thus, we see that the theta lift $\Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol{\lambda}}, v^+)$ satisfies the desired differential equation

$$\Delta^{\tau_i}\Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol{\lambda}}, v^+(\tau_1,\tau_2)) = k(1-k)\Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol{\lambda}}, v^+(\tau_1,\tau_2)), \quad i = 1, 2.$$

It remains to prove (v). To this end we compute the Fourier expansion of $\Phi_{M_2(\mathbb{Z})}(h_{\boldsymbol{\lambda}}, v^+(\tau_1,\tau_2))$. This can be done using Theorem B2 of Section 1.8. We select a primitive norm zero vector $m := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z})$ and choose $m' := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ so that $(m, m') = 1$. For this choice of vectors m, m' the tube domain $\mathcal{H}$ defined by equation (1.37) is isomorphic to $\mathfrak{H} \times \mathfrak{H}$ and the map between $\mathfrak{H} \times \mathfrak{H}$ and Grassmanian $\mathrm{Gr}^+(M_2(\mathbb{Z}))$ is given by (5.10). The lattice $K = (M_2(\mathbb{Z}) \cap m^\perp)/m$ can be identified with

$$M_2(\mathbb{Z}) \cap m^\perp \cap m'^\perp = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \middle| \, b, c \in \mathbb{Z} \right\}.$$

Denote $x_i = \Re(\tau_i)$ and $y_i = \Im(\tau_i)$ for $i = 1, 2$. The subspace $w^+(\tau_1, \tau_2) \in \mathrm{Gr}^+(K)$ defined by equation (1.38) in Section 1.8 is equal to

$$\mathbb{R}\begin{pmatrix} 0 & y_1 \\ y_2 & 0 \end{pmatrix}.$$

Suppose that the function $g_{\boldsymbol{\lambda}}$ has the Fourier expansion

$$g_{\boldsymbol{\lambda}} = \sum_{n\in\mathbb{Z}} a(n)\,\mathbf{e}(n\tau).$$

It follows from Proposition 5.5 that

$$h_{\boldsymbol{\lambda}}(\tau) = \sum_{n\in\mathbb{Z}} c(n,y)\,\mathbf{e}(n\tau)$$

where

$$c(n,y) = a(n)\,y^{1/2}\,n^{k-1/2}\,K_{k-1/2}(2\pi n y)\exp(2\pi n y).$$

Using (5.6) we can write

$$c(n,y) = \sum_{t=0}^{k-1} b(n,t) y^{-t},$$

where

$$b(n,t) = a(n)\,n^{k-1-t}\,\frac{(k+t-1)!}{2\,\pi^t\,t!\,(k-t-1)!}.$$

We can rewrite (4.9) as

$$\begin{aligned}
\Phi_{M_2(\mathbb{Z})}(v^+, h_{\boldsymbol{\lambda}}) =& \frac{1}{\sqrt{2}\,|m_{v^+}|}\Phi_K(w^+, h) + \sqrt{2}\,|m_{v^+}| \sum_{l\in K}\sum_{n>0} \mathbf{e}\big((nl,u)\big)\times \\
&\times \int_0^\infty c\big(\mathrm{q}(l), y\big)\exp\big(-\pi n^2/4\mathrm{q}(m_{v^+})y - 4\pi\mathrm{q}(l_{w^+})y\big) y^{-3/2}dy \\
=& \sqrt{y_1y_2}\Phi_K(w^+(\tau_1,\tau_2), f_K) + \frac{1}{\sqrt{y_1y_2}}\sum_{l\in K}\sum_{n>0}\mathbf{e}((nl,u))\times \\
&\times \int_0^\infty c\big(\mathrm{q}(l), y\big)\exp\Big(-\frac{\pi n^2 y_1 y_2}{y} - \pi y\frac{(l,v)^2}{y_1y_2}\Big)\,y^{-3/2}dy,
\end{aligned} \tag{5.14}$$

where $v^+ = v^+(\tau_1,\tau_2)$, $w^+ = w^+(\tau_1,\tau_2)$, $u = \Re\begin{pmatrix} 0 & \tau_1 \\ \tau_2 & 0\end{pmatrix}$, and $v = \Im\begin{pmatrix} 0 & \tau_1 \\ \tau_2 & 0\end{pmatrix}$. We choose a primitive norm 0 vector $r = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} \in K$. It follows from Theorem 10.2 [10] that for $y_1 > y_2$

$$\begin{aligned}\Phi_K(w^+, h_{\boldsymbol{\lambda}}) =& \sum_{t=0}^{k-1} (2r_{w^+}^2)^{t+1/2}\, b(0,t)\, (-4\pi)^{t+1}\, B_{2t+2}\, t!/(2t+2)! \\ =& \sum_{t=0}^{k-1} (y_2/y_1)^{t+1/2}\, b(0,t)\, (-4\pi)^{t+1}\, B_{2t+2}\, t!/(2t+2)! \\ =& \Big(\frac{y_2}{y_1}\Big)^{k-1/2} (-1)^k\, 2\pi\, \frac{a(0)\, B_{2k}}{2k(2k-1)}.\end{aligned} \tag{5.15}$$

In the case $l_{w^+} \neq 0$ it follows from Lemma 7.2 of [11]

$$\begin{aligned}&\int_{y>0} c\big(\mathrm{q}(l), y\big)\, \exp\big(-\pi n^2/4y\mathrm{q}(m_{v^+}) - 2\pi y \mathrm{q}(l_{w^+})\big)\, y^{-3/2}\, dy \\ &= \sum_{t=0}^{k-1} 2b\big(\mathrm{q}(l), t\big)(2|m_{v^+}|\, |l_{w^+}|/n)^{t+1/2} K_{-t-1/2}(2\pi n |l_{w^+}|/|m_{v^+}|).\end{aligned} \tag{5.16}$$

In case $l_{w^+} = 0$ it follows from Lemma 7.3 of [11]

$$\begin{aligned}&\int_{y>0} c\big(\mathrm{q}(l), y\big)\, \exp\big(-\pi n^2/4y\mathrm{q}(m_{v^+}) - 4\pi y \mathrm{q}(l_{w^+})\big)\, y^{-3/2}\, dy \\ &\qquad = \sum_{t=0}^{k-1} b\big(\mathrm{q}(l), t\big)\, \big(4\mathrm{q}(m_{v^+})/\pi n^2\big)^{t+1/2} \Gamma(t+1/2).\end{aligned} \tag{5.17}$$

Substituting formulas (5.15)-(5.17) into (5.14) we obtain

$$\begin{aligned}\Phi_{M_2(\mathbb{Z})}(v^+(\tau_1,\tau_2), h_{\boldsymbol{\lambda}}) =& (-1)^{k+1}\, \frac{(2\pi y_2)^k}{(2\pi y_1)^{k-1}}\, \frac{a(0)\, B_{2k}}{2k(2k-1)} \\ &+ (4\pi^2 y_1 y_2)^{1-k}\, a(0)\, \zeta(2k-1)\, \Big(\frac{2k!}{k!}\Big)^2 \\ &+ 4 \sum_t \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \sum_{n>0} (y_1y_2)^{-t} b(cd,t) n^{-2t-1} \times \\ &\times \mathbf{e}(ncx_1 + ndx_2) |ncy_1 + ndy_2|^{t+1/2} K_{-t-1/2}(2\pi |ncy_1 + ndy_2|).\end{aligned} \tag{5.18}$$

We see from (5.18) that $\Phi_M(v^+(\tau_1,\tau_2), h_{\boldsymbol{\lambda}}) \to 0$ as $y_1 \to \infty$. This finishes the proof. □

Remark 5.3. The Fourier expansion of higher Green's functions is computed using a different method by Zagier in an unpublished paper [79].

5.6 CM values as regularized Petersson products

Now we can analyze the CM values of $G_{k,\boldsymbol{\lambda}}$ using the see-saw identity (5.9) .

Let $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{H}$ be two CM points in the same quadratic imaginary field $\mathbb{Q}(\sqrt{-D})$. Let $v^+(\mathfrak{z}_1, \mathfrak{z}_2)$ be the two-dimensional positive definite subspace of $M_2(\mathbb{R})$ defined as

$$v^+(\mathfrak{z}_1, \mathfrak{z}_2) = \mathbb{R}\Re\begin{pmatrix} \mathfrak{z}_1\mathfrak{z}_2 & \mathfrak{z}_1 \\ \mathfrak{z}_2 & 1 \end{pmatrix} + \mathbb{R}\Im\begin{pmatrix} \mathfrak{z}_1\mathfrak{z}_2 & \mathfrak{z}_1 \\ \mathfrak{z}_2 & 1 \end{pmatrix}. \tag{5.19}$$

In the case when $\mathfrak{z}_1$ and $\mathfrak{z}_2$ lie in the same quadratic imaginary field the subspace $v^+(\mathfrak{z}_1, \mathfrak{z}_2)$ defines a rational splitting of $M_2(\mathbb{Z}) \otimes \mathbb{Q}$. Therefore, we can consider the following two lattices

$$N := v^+(\mathfrak{z}_1, \mathfrak{z}_2) \cap M_2(\mathbb{Z}) \quad \text{and} \quad M := v^-(\mathfrak{z}_1, \mathfrak{z}_2) \cap M_2(\mathbb{Z}).$$

The Grassmannian $\mathrm{Gr}^+(N)$ consists of a single point $N \otimes \mathbb{R}$ and its image in $\mathrm{Gr}^+(M_2(\mathbb{Z}))$ is $v^+(\mathfrak{z}_1, \mathfrak{z}_2)$.

Since the lattice N has signature $(2, 0)$ the theta lift of a function $f \in \widehat{M}_1^!(\mathrm{SL}_2(\mathbb{Z}), \rho_N)$ is just a number and it is equal to the regularized integral

$$\Phi_N(f) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}}^{\mathrm{reg}} \langle f(\tau), \overline{\Theta_N(\tau)} \rangle \, y^{-1} \, dx \, dy =: (f, \Theta_N)_{\mathrm{reg}}. \tag{5.20}$$

Here Θ_N is a usual (vector valued) theta function of the lattice N. The matrix $T_{M_2(\mathbb{Z}),N} = (\vartheta_{0,\nu})_{\nu \in N'/N}$ becomes a vector in this case and it is given by

$$\vartheta_{0,\nu}(\tau) = \sum_{m \in M' \cap (-\nu + M_2(\mathbb{Z}))} \mathbf{e}(-\tau m^2/2).$$

Till the end of this section we will simply write $\vartheta_\nu(\tau)$ for $\vartheta_{0,\nu}(\tau)$.

Theorem 5.4. *Suppose that two CM-points $\mathfrak{z}_1, \mathfrak{z}_2$ and a lattice $N \subset M_2(\mathbb{Z})$ are defined as above. Let $\boldsymbol{\lambda}$ be a relation for $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ and let $g_{\boldsymbol{\lambda}} \in M_{2-2k}^!(\mathrm{SL}_2(\mathbb{Z}))$ be the corresponding weakly holomorphic form defined in Proposition 5.1. Then, if $(\mathfrak{z}_1, \mathfrak{z}_2) \notin S_{\boldsymbol{\lambda}}$ we have*

$$G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1, \mathfrak{z}_2) = (f, \Theta_N)_{\mathrm{reg}},$$

where $f = (f_\nu)_{\nu \in N'/N} \in M_1^!(\mathrm{SL}_2(\mathbb{Z}), \rho_N)$ is given by

$$f_\nu = [\, g_{\boldsymbol{\lambda}}, \vartheta_\nu]_{k-1}.$$

Proof. Recall that by Theorem 5.3

$$G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1, \mathfrak{z}_2) = \Phi_{M_2(\mathbb{Z})}(v^+(\mathfrak{z}_1, \mathfrak{z}_2), R^{k-1}(g_{\boldsymbol{\lambda}})).$$

For $(\mathfrak{z}_1, \mathfrak{z}_2) \notin S_{\boldsymbol{\lambda}}$ the constant term (with respect to $\mathbf{e}(x)$) of the product

$$\langle R^{k-1}(g_{\boldsymbol{\lambda}})(\tau), \overline{\Theta(\tau; \mathfrak{z}_1, \mathfrak{z}_2)} \rangle$$

is equal to

$$\sum_{l \in M_2(\mathbb{Z})} a_{l^2/2}(y) \exp(-2\pi y l_{v^+}^2) y$$

and decays as $O(y^{2-k})$ as $y \to \infty$. Thus,

$$\Phi_{M_2(\mathbb{Z})}\big(v^+(\mathfrak{z}_1,\mathfrak{z}_2), R^{k-1}(g_{\boldsymbol{\lambda}})\big) = \lim_{t\to\infty} \int_{F_t} R^{k-1}(g_{\boldsymbol{\lambda}})(\tau)\, \overline{\Theta(\tau;\mathfrak{z}_1,\mathfrak{z}_2)}\, y^{-2} dx\, dy.$$

It follows from the see-saw identity (5.9)

$$\Phi_{M_2(\mathbb{Z})}\big(v^+(\mathfrak{z}_1,\mathfrak{z}_2), R^{k-1}(g_{\boldsymbol{\lambda}})\big) = \lim_{t\to\infty} \int_{F_t} \langle R^{k-1}(g_{\boldsymbol{\lambda}})\, \vartheta, \overline{\Theta}_N\rangle\, y^{-1}\, dx\, dy.$$

By Proposition 5.3

$$R^{k-1}(g_{\boldsymbol{\lambda}})\, \vartheta_\nu = (-1)^{k-1}[g_{\boldsymbol{\lambda}}, \vartheta_\nu]_{k-1} + R\Big(\sum_{s=0}^{k-2} b_s R^s(g_{\boldsymbol{\lambda}})\, R^{k-2-s}(\vartheta_\nu)\Big), \tag{5.21}$$

where b_s are some rational numbers. For $\nu \in N'/N$ denote

$$\psi_\nu(\tau) := \sum_{s=0}^{k-2} b_s R^s(g_{\boldsymbol{\lambda}})\, R^{k-2-s}(\vartheta_\nu).$$

Using identity (5.21) we write

$$\lim_{t\to\infty} \int_{F_t} \langle R^{k-1}(g_{\boldsymbol{\lambda}})\, \vartheta, \overline{\Theta}_N\rangle\, y^{-1}\, dx\, dy =$$

$$(-1)^{k-1} \lim_{t\to\infty} \int_{F_t} \langle [g_{\boldsymbol{\lambda}}, \vartheta]_{k-1}, \overline{\Theta}_N\rangle\, y^{-1}\, dx\, dy + \lim_{t\to\infty} \int_{F_t} \langle R(\psi), \overline{\Theta}_N\rangle\, y^{-1}\, dx\, dy.$$

It follows from Proposition 5.4 that

$$\lim_{t\to\infty} \int_{F_t} \langle R(\psi), \overline{\Theta_N}\rangle\, y^{-1}\, dx\, dy =$$
$$\lim_{t\to\infty} \int_{-1/2}^{1/2} \langle \psi(x+it), \overline{\Theta_N(x+it)}\rangle\, t^{-1}\, dx = 0.$$

This finishes the proof. □

5.7 Embedding trick

Theorem 5.5. *We let N be an even lattice of signature $(2,0)$ and let $f \in M_1^!(\rho_N)$ be a modular form with zero constant term and rational Fourier coefficients. Then there exists an even lattice P of signature $(2,1)$ and a function $h \in M_{1/2}^!(\rho_P)$ such that*

(i) there is an inclusion $N \subset P$;

(ii) the lattice P contains a primitive norm zero vector;

(iii) the function h has rational Fourier coefficients;

(iv) the constant term of h is zero;

(v) we have $T_{P,N}(h) = f$ for the map $T_{P,N}$ defined in Theorem 5.2.

Proof. We adopt the method explained in [10], Lemma 8.1.

Consider two even unimodular definite lattices of dimension 24, for example three copies of E_8 root lattice $E_8 \oplus E_8 \oplus E_8$ and the Leech lattice Λ_{24}. We can embed both lattices into $\frac{1}{16}\mathbb{Z}^{24}$. To this end we use the standard representation of E_8 in which all vectors have half integral coordinates and the standard representation of the Leech lattice togather with the norm doubling map defined on p.242 of Chapter 8 in [19].

Denote by M_1 and M_2 the negative definite lattices obtained from $E_8 \oplus E_8 \oplus E_8$ and Λ_{24} by multiplying the norm with -1 and assume that they are embedded into $\frac{1}{16}\mathbb{Z}^{24}$. Denote by M the negative definite lattice $16\mathbb{Z}^{24}$. The theta functions of lattices M_1 and M_2 are modular forms of level 1 and weight 12 and their difference is 720Δ, where $\Delta = q - 24q^2 + 252q^3 + O(q^4)$ is the unique cusp form of level 1 and weight 12.

Consider the function g in $M^!_{-11}(\mathrm{SL}_2(\mathbb{Z}), \rho_{N\oplus M})$ defined as

$$g := \mathrm{res}_{(N\oplus M_1)/N\oplus M}(f/\Delta) - \mathrm{res}_{(N\oplus M_2)/N\oplus M}(f/\Delta).$$

The maps

$$\mathrm{res}_{(N\oplus M_i)/N\oplus M} : M^!_{-11}(\mathrm{SL}_2(\mathbb{Z}), \rho_{N\oplus M_i}) \to M^!_{-11}(\mathrm{SL}_2(\mathbb{Z}), \rho_{N\oplus M}),\ i = 1, 2,$$

are defined as in Lemma 1.1. It is easy to see from the definitions (1.15) and (5.7) that

$$T_{N\oplus M,N}(g) = T_{N\oplus M,N}\big(\mathrm{res}_{(N\oplus M_1)/N\oplus M}(f/\Delta) - \mathrm{res}_{(N\oplus M_2)/N\oplus M}(f/\Delta)\big) =$$
$$T_{N\oplus M_1,N}(f/\Delta) - T_{N\oplus M_2,N}(f/\Delta) =$$
$$\frac{f}{\Delta}(\bar{\Theta}_{M_1} - \bar{\Theta}_{M_2}) = 720f.$$

Suppose that g has the Fourier expansion

$$g_\mu(\tau) = \sum_{m\in\mathbb{Q}} c_\mu(m)\,\mathbf{e}(m\tau),\ \mu \in (N' \oplus M')/(N \oplus M).$$

By the construction of g its constant term is zero. Consider the following set of vectors in M'

$$S := \{l \in M' | c_{(0,l+M)}\big(\mathrm{q}(l)\big) \neq 0\},$$

where $(0, l + M)$ denotes an element in $(N' \oplus M')/(N \oplus M)$. Note that this set is finite and does not contain the zero vector. Choose a vector $p \in M$ such that
1. the lattice $N \oplus \mathbb{Z}p$ contains a primitive norm 0 vector;
2. $(p, l) \neq 0$ for all $l \in S$.
Consider the lattice $P := N \oplus \mathbb{Z}p$. It follows from Theorem B1 that the subvariety $\mathrm{Gr}^+(P)$ of $\mathrm{Gr}^+(N \oplus M)$ is not contained in the singular locus of $\Phi_{N\oplus M}(v^+, g)$. Moreover, the restriction of $\Phi_{N\oplus M}(v^+, g)$ to $\mathrm{Gr}^+(P)$ is nonsingular at the point $\mathrm{Gr}^+(N)$.

Define $h := \frac{1}{720} T_{N\oplus M,P}(g)$. The constant term of h is nonzero and h has rational (with denominator bounded by 720) Fourier coefficients. We have

$$T_{P,N}(h) = \frac{1}{720} T_{P,N}\big(T_{N\oplus M,P}(g)\big) = \frac{1}{720} T_{N\oplus M,N}(g) = f.$$

This finishes the proof. □

Theorem 5.6. *We let N be an even lattice of signature $(2,0)$ and let $f \in M^!_1(\rho_N)$ be a modular form with zero constant term and rational Fourier coefficients. Then the regularized Petersson product*

$$(f, \Theta_N)_{\mathrm{reg}} := \int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} f(\tau)\, \overline{\Theta_N(\tau)}\, y^{-1} dx\, dy$$

satisfies

$$(f, \Theta_N)_{\mathrm{reg}} = \log|\alpha|$$

for some $\alpha \in \bar{\mathbb{Q}}$.

Proof. By the definition of the regularized theta lift

$$\Phi_N(f) = (f, \Theta_N)_{\mathrm{reg}}.$$

Choose a lattice P of signature $(2,1)$ and a function $h \in M^!_{1/2}(\rho_P)$ that satisfy Theorem 5.5. By Theorem 5.2 the conditions (iv) and (v) of Theorem 5.5 imply

$$\Phi_N(f) = \Phi_P(\mathrm{Gr}^+(N), h).$$

There exists an integer n such that all negative Fourier coefficients of nh are integers. The function nh satisfies the assumptions of Theorem B3 in Section 1.9. Hence we can write

$$\Phi_P(\mathrm{Gr}^+(N), h) = -4\log|\Psi_P(\mathrm{Gr}^+(N), h)|,$$

where $\Psi_P(\,\cdot\,, nh)$ is the meromorphic infinite product defined in Theorem B3. Since the constant term of h is zero, from Theorem B3 we know that

$$\Phi_P(\mathrm{Gr}^+(N), nh) = -4\log|\Psi_P(\tau_N, nh)|,$$

where $\Psi_P(\tau, nh)$ is a meromorphic modular function on $\mathfrak{H}$ for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with respect to some unitary character and $\tau_N \in \mathfrak{H}$ is a CM point. Theorem 14.1 of [11] says that this unitary character is finite. Theorem B3 Part 3 implies that $\Psi_P(\tau, nh)$ has rational Fourier coefficients. Thus, it follows from the theory of complex multiplication that $\alpha := \Psi_P(\tau_N, h)^{1/n}$ is an algebraic number. □

5.8 Main Theorem

Theorem 5.7. *Let $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{H}$ be two CM points in the same quadratic imaginary field $\mathbb{Q}(\sqrt{-D})$ and let $\boldsymbol{\lambda}$ be a relation on $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ for an integer $k > 1$. Then there is an algebraic number α such that*

$$G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1, \mathfrak{z}_2) = \log|\alpha|.$$

Proof. Let $g_{\boldsymbol{\lambda}}$ be the weakly holomorphic modular form of weight $2-2k$ defined by Proposition 5.1. Consider the function $h_{\boldsymbol{\lambda}} = R^{k-1}(g_{\boldsymbol{\lambda}})$. In Theorem 5.3 we showed that

$$G_{k,\boldsymbol{\lambda}}(\tau_1, \tau_2) = \Phi_{M_2(\mathbb{Z})}(v^+(\tau_1, \tau_2), h_{\boldsymbol{\lambda}}) \tag{5.22}$$

for $(\tau_1, \tau_2) \in \mathfrak{H} \times \mathfrak{H} \setminus S_{\boldsymbol{\lambda}}$.

Let $v^+(\mathfrak{z}_1, \mathfrak{z}_2)$ be the two-dimensional positive definite subspace of $M_2(\mathbb{R})$ defined in (5.19). In the case when $\mathfrak{z}_1$ and $\mathfrak{z}_2$ lie in the same quadratic imaginary field the subspace $v^+(\mathfrak{z}_1, \mathfrak{z}_2)$ defines a rational splitting of $M_2(\mathbb{Z}) \otimes \mathbb{R}$. Hence, the lattice $N := v^+(\mathfrak{z}_1, \mathfrak{z}_2) \cap M_2(\mathbb{Z})$ has signature $(2, 0)$.

It follows from Theorem 5.4 that

$$\Phi_{M_2(\mathbb{Z})}(v^+(\mathfrak{z}_1, \mathfrak{z}_2), R^{k-1}(g_{\boldsymbol{\lambda}})) = \Phi_N(f), \tag{5.23}$$

where $f = (f_\nu)_{\nu \in N'/N} \in M_1^!(\mathrm{SL}_2(\mathbb{Z}), \rho_N)$ is given by

$$f_\nu = [g_{\boldsymbol{\lambda}}, \vartheta_\nu]_{k-1}.$$

Let P and h be as in Theorem 5.5. Theorem 5.6 implies

$$\Phi_N(f) = \Phi_P(\mathrm{Gr}^+(N), h) = -4\log|\Psi_P(\mathrm{Gr}^+(N), h)|.$$

Thus, from the theory of complex multiplication we know that

$$\Phi_N(f) = \log|\alpha| \tag{5.24}$$

for some $\alpha \in \bar{\mathbb{Q}}$. The statement of the theorem follows from equations (5.22) – (5.24). □

5.9 Prime factorization of regularized Petersson products

In this section we find a field of definition and prime factorization of the algebraic number α defined in Theorem 5.6. For simplicity, we will restrict ourself to the case when the lattice N has prime discriminant.

Theorem 5.8. *Let N be an even lattice of signature $(2, 0)$ and prime discriminant D. Suppose that N is isomorphic to a fractional ideal $\mathfrak{b} \subset K$ equipped with the quadratic form $\frac{1}{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b})}\mathrm{N}_{K/\mathbb{Q}}(\cdot)$. Let $f \in M_1^!(\rho_N)$ be a weakly holomorphic modular form with the Fourier expansion*

$$f = \sum_{\nu \in N'/N} \sum_{t \gg -\infty} c_\nu(t)\, q^t\, e_\nu,$$

where $c_\nu(t) \in \mathbb{Z}$ *and* $c_0(0) = 0$. *Then*

$$(f, \Theta_N)_{\mathrm{reg}} = \log|\alpha|,$$

where $\alpha \in H$. *Moreover, for a rational prime* p *and an ideal* $\mathfrak{P}$ *of* H *lying above* p *we have*

$$\mathrm{ord}_{\mathfrak{P}}(\alpha) = \sum_{t<0} \sum_{\nu \in N'/N} c_\nu(t)\, r_{\mathfrak{b}\mathfrak{a}^2}\Big(\frac{-Dt}{p}\Big)\big(1 + \mathrm{ord}_p(t)\big) \quad \text{in the case } \Big(\frac{p}{D}\Big) = -1, \qquad (5.25)$$

$$\mathrm{ord}_{\mathfrak{P}}(\alpha) = 0 \quad \text{in the case } \Big(\frac{p}{D}\Big) = 1.$$

We will prove Theorem 5.8 in Section 5.11.

Theorem 5.8 is compatible with, but stronger than the result of J. Schofer [62]. More precisely, Theorem 4.1 of [62] states that the *sum* over all isomorphism classes of even lattices of discriminant $-D$ of the identity (5.8) is true.

In Section 5.12 we will show that Theorem 5.8 implies Conjecture 2.

Theorem 5.9. *The factorization formula for the CM-values of higher Green's functions given in Conjecture 2 in Section 5.1 is true.*

5.10 Lattices and fractional ideals

In this section we collect some facts about lattices and fractional ideals of quadratic imaginary fields. They will play an important role in our proof of Theorems 5.8 and 5.9.

In this section $-D$ is a negative prime discriminant. Recall the following well-known resuts about fractional ideals of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$.

Lemma 5.1. *Suppose that* $-D < 0$ *is a square-free discriminant and* $[a, b, c]$ *is a primitive quadratic form of disctiminant* $-D$. *Let* $\mathfrak{z}$ *be a solution of the equation* $a\mathfrak{z}^2 + b\mathfrak{z} + c = 0$. *Then the lattice*

$$\mathfrak{c} = \mathbb{Z} + \mathfrak{z}\mathbb{Z}$$

is a fractional ideal of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$. *Moreover, this ideal satisfies*

$$\mathfrak{c}\bar{\mathfrak{c}} = (a)^{-1}. \qquad (5.26)$$

Lemma 5.2. *Let* $\mathfrak{c} \subset K$ *be a fractional ideal. Consider the quadratic form* $\mathrm{q}(\cdot)$ *on* K *given by* $\mathrm{q}(\beta) = N_{K/\mathbb{Q}}(\beta)$. *Then the dual lattice of* $\mathfrak{c}$ *with respect to this quadratic form is equal to* $(N_{K/\mathbb{Q}}(\mathfrak{c}))^{-1}\, \mathfrak{c}\, \mathfrak{d}^{-1}$. *Here* $\mathfrak{d}$ *denotes the different of* $\mathfrak{o}_K$, *i. e. the principal ideal* $(\sqrt{-D})$.

Our next goal is to find a convenient lattice of signature $(2,1)$ that contains the positive definite lattice associated to the ideal $\mathfrak{b}$ as a sublattice. Consider the lattice

$$L = \left\{ \begin{pmatrix} A/D & B \\ B & C \end{pmatrix} \Big| A, B, C \in \mathbb{Z} \right\} \qquad (5.27)$$

equipped with the quadratic form $\mathrm{q}(x) := -D\det(x)$. Its dual lattice L' is given by

$$L' = \left\{ \begin{pmatrix} A'/D & B'/2D \\ B'/2D & C' \end{pmatrix} \Big| A', B', C' \in \mathbb{Z} \right\}. \tag{5.28}$$

For $\ell \in L'$ with $\mathrm{q}(\ell) < 0$ denote by $\mathfrak{z}_\ell$ the point in $\mathfrak{H}$ corresponding to the positive definite subspace $\ell^\perp$ via (5.45). More explicitly, for the vector

$$\ell = \begin{pmatrix} \gamma & -\beta/2 \\ -\beta/2 & \alpha \end{pmatrix}$$

the point $\mathfrak{z}_\ell$ is a root of the quadratic equation

$$\alpha \mathfrak{z}_\ell^2 + \beta \mathfrak{z}_\ell + \gamma = 0. \tag{5.29}$$

The following two lemmas are crucial to show that for each fractional ideal $\mathfrak{b}$ the positive definite lattice associated to it is contained in L as a lower rank sublattice.

Lemma 5.3. *For each ideal class $\mathfrak{c} \in \mathrm{CL}_K$ there exists a vector $m \in L'$ such that $\mathrm{q}(m) = -1/4$ and $\mathfrak{z}_m\mathbb{Z} + \mathbb{Z} \subset K$ is a fractional ideal in $\mathfrak{c}$.*

Proof. The classical correspondence between fractional ideals of $\mathfrak{o}_K$ and binary quadratic forms of discriminant $-D$ implies that for each ideal class $\mathfrak{c} \in \mathrm{CL}_K$ there exist $A, B, C \in \mathbb{Z}$ such that

$$B^2 - 4AC = -D$$

and for $\mathfrak{z} \in \mathfrak{H}$, satisfying

$$A\mathfrak{z}^2 + B\mathfrak{z} + C = 0,$$

the subset $\mathfrak{z}\mathbb{Z} + \mathbb{Z}$ of K is a fractional ideal in the ideal class $\mathfrak{c}$. Or equivalently, there exists a half-integral matrix

$$l = \begin{pmatrix} C & -B/2 \\ -B/2 & A \end{pmatrix}$$

with

$$\mathfrak{z}_l\mathbb{Z} + \mathbb{Z} \in \mathfrak{c}.$$

For each $x \in \mathrm{SL}_2(\mathbb{Z})$ the fractional ideal $\mathfrak{z}_{xlx^t}\mathbb{Z} + \mathbb{Z}$ is equivalent to $\mathfrak{z}_l\mathbb{Z} + \mathbb{Z}$. It is easy to see, that the matrix l is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to some matrix of the form

$$\tilde{l} = \begin{pmatrix} \tilde{C} & -\tilde{B}/2 \\ -\tilde{B}/2 & \tilde{A} \end{pmatrix}, \quad \tilde{A} \in D\mathbb{Z}, \tilde{B} \in D\mathbb{Z}, \tilde{C} \in \mathbb{Z}.$$

Then the matrix $m =: \tilde{l}/D$ belongs to L', has norm $-1/4$, and since $\mathfrak{z}_m = \mathfrak{z}_{\tilde{l}}$ the fractional ideal $\mathfrak{z}_m\mathbb{Z} + \mathbb{Z}$ belongs to the ideal class $\mathfrak{c}$. Lemma is proved. $\square$

Lemma 5.4. *Let $m \in L'$ be a vector of norm $-1/4$. Set $N := L \cap m^{\perp}$. Denote by $\mathfrak{c}$ the fractional ideal $\mathfrak{z}_m\mathbb{Z} + \mathbb{Z}$. Then the following holds*
(i) the lattice N is isomorphic to the fractional ideal $\mathfrak{c}^2$ equipped with the quadratic form $\mathrm{q}(\gamma) = \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{c}^2)} N_{K/\mathbb{Q}}(\gamma)$;
(ii) $L = N \oplus 2m\mathbb{Z}$.

Proof. First we prove part (i). Each element of L' can be written as

$$m = \frac{1}{D}\begin{pmatrix} c & -b/2 \\ -b/2 & a \end{pmatrix}$$

for some $a \in D\mathbb{Z}, b \in \mathbb{Z},\ c \in \mathbb{Z}$. The condition $4D\mathrm{q}(m) = b^2 - 4ac = -D$ implies that $b \in D\mathbb{Z}$. Set

$$Z := \frac{a}{D}\begin{pmatrix} \mathfrak{z}_m^2 & \mathfrak{z}_m \\ \mathfrak{z}_m & 1 \end{pmatrix}.$$

This element of $L \otimes \mathbb{C}$ satisfies

$$\mathrm{q}(Z) = \mathrm{q}(\overline{Z}) = 0 \quad \text{and} \quad (Z, \overline{Z}) = 1.$$

Moreover, the elements Z and $\overline{Z}$ are both orthogonal to m. Consider the map

$$\imath : K \to N \otimes \mathbb{Q}$$

defined by

$$s \to \overline{s}Z + s\overline{Z}.$$

This map is an isometry, assuming that the quadratic form on K is given by $\mathrm{q}(\beta) = N_{K/\mathbb{Q}}(\beta)$ and the quadratic form on $N \otimes \mathbb{Q}$ is given by $\mathrm{q}(\ell) = -D\det(\ell)$. We have

$$\begin{aligned}
\imath(a) &= \frac{a}{D}\begin{pmatrix} \mathfrak{z}_m^2 + \overline{\mathfrak{z}_m}^2 & \mathfrak{z}_m + \overline{\mathfrak{z}_m} \\ \mathfrak{z}_m + \overline{\mathfrak{z}_m} & 2 \end{pmatrix} = \frac{1}{D}\begin{pmatrix} (b^2 - D)/2 & -ab \\ -ab & 2a^2 \end{pmatrix}, \\
\imath(a\mathfrak{z}_m) &= \frac{a}{D}\begin{pmatrix} \mathfrak{z}_m\overline{\mathfrak{z}_m}(\mathfrak{z}_m + \overline{\mathfrak{z}_m}) & \mathfrak{z}_m^2 + \overline{\mathfrak{z}_m}^2 \\ \mathfrak{z}_m^2 + \overline{\mathfrak{z}_m}^2 & \mathfrak{z}_m + \overline{\mathfrak{z}_m} \end{pmatrix} = \frac{1}{D}\begin{pmatrix} -bc & (b^2 - D)/2 \\ (b^2 - D)/2 & ab \end{pmatrix}, \\
\imath(a\mathfrak{z}_m^2) &= \frac{a}{D}\begin{pmatrix} \mathfrak{z}_m^2\overline{\mathfrak{z}_m}^2 & \mathfrak{z}_m\overline{\mathfrak{z}_m}(\mathfrak{z}_m + \overline{\mathfrak{z}_m}) \\ \mathfrak{z}_m\overline{\mathfrak{z}_m}(\mathfrak{z}_m + \overline{\mathfrak{z}_m}) & \mathfrak{z}_m^2 + \overline{\mathfrak{z}_m}^2 \end{pmatrix} = \frac{1}{D}\begin{pmatrix} 2c^2 & -bc \\ -bc & (b^2 - D)/2 \end{pmatrix}.
\end{aligned}$$

Using that $a, b \in D\mathbb{Z}$ and $b \equiv D (\mathrm{mod}\ 2)$, we see that

$$\imath(a\,\mathbb{Z} + a\,\mathfrak{z}_m\,\mathbb{Z} + a\,\mathfrak{z}_m^2\mathbb{Z}) \subseteq N. \tag{5.30}$$

On the other hand

$$(a)\mathfrak{c}^2 = a\mathbb{Z} + a\mathfrak{z}_m\mathbb{Z} + a\mathfrak{z}_m^2\mathbb{Z}. \tag{5.31}$$

Lemma 5.1 implies that the ideal $(a)\mathfrak{c}^2$ has norm 1. Hence, by Lemma 5.2 the dual lattice of $(a)\mathfrak{c}^2$ in K is equal to $\mathfrak{d}^{-1}(a)\mathfrak{c}^2$. Since $\imath$ is an isometry, the dual of $\imath((a)\mathfrak{c}^2)$ is $\imath(\mathfrak{d}^{-1}(a)\mathfrak{c}^2)$. We have the inclusions

$$\imath((a)\mathfrak{c}^2) \subseteq N \subset N' \subseteq \imath(\mathfrak{d}^{-1}(a)\mathfrak{c}^2). \tag{5.32}$$

Since $(\mathfrak{d}^{-1}(a)\mathfrak{c}^2)/((a)\mathfrak{c}^2) \cong \mathbb{Z}/D\mathbb{Z}$ we find that $|N'/N|$ is a divisor of D. Since a positive definite 2-dimensional even lattice can not be unimodular, we deduce that $|N'/N| = D$. Thus the symbols "$\subseteq$" in (5.32) should be replaced by "$=$". Part (i) of Lemma 5.4 is proved.

Now we prove (ii). The condition $\mathrm{q}(m) = -1/4$ implies that $b \in D\mathbb{Z}$. Hence, the element $2m$ belongs to L. Set $M := 2m\mathbb{Z}$. We have the following inclusions

$$M' \oplus N' \subseteq L' \subset L \subseteq M \oplus N.$$

Observe that

$$|L'/L| = 2D, \quad |M'/M| = 2, \quad |N'/N| = D.$$

Thus, $L = M \oplus N$ and $L' = M' \oplus N'$. □

We combine the previous two lemmas in the following theorem.

Theorem 5.10. *For each ideal class $\mathcal{B}$ of K there exists a vector $m \in L'$ such that*
(i) $\mathrm{q}(m) = -1/4$*;*
(ii) the lattice $N := L \cap m^\perp$ is isomorphic to the lattice $N_{\mathcal{B}}$ defined as $(\mathfrak{b}, \mathrm{N}_{K/\mathbb{Q}}(\cdot)/\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}))$ *for some $\mathfrak{b} \in \mathcal{B}$;*
(iii) $L = N \oplus 2m\mathbb{Z}$.

Proof. Since D is prime, the class number of K is odd. Thus, each ideal class $\mathfrak{b}$ is equal to $\mathfrak{c}^2$ for some $\mathfrak{c} \in \mathrm{CL}_K$. Let $m \in L'$ be the vector constructed in Lemma 5.3, which satisfies $\mathfrak{z}_m\mathbb{Z} + \mathbb{Z} \in \mathfrak{c}$. Then Lemma 5.4 readily implies that m satisfies the conditions of the theorem. □

The following lemma will play an important role in the proof of Conjecture 2.

Lemma 5.5. *Let $\mathfrak{z}_1,\ \mathfrak{z}_2 \in \mathfrak{H}$ be two CM-points of discriminant D and let $\mathfrak{b} := \mathfrak{z}_1\mathbb{Z} + \mathbb{Z}$, $\mathfrak{c} := \mathfrak{z}_2\mathbb{Z} + \mathbb{Z}$ be fractional ideals of $\mathfrak{o}_K$. Consider the lattices $N = \mathrm{M}_2(\mathbb{Z}) \cap v^+(\mathfrak{z}_1, \mathfrak{z}_2)$ and $M = \mathrm{M}_2(\mathbb{Z}) \cap N^\perp$. Then the lattice N is isomorphic to the fractional ideal $\mathfrak{b}\mathfrak{c}$ equipped with the quadratic form $\frac{\mathrm{N}_{K/\mathbb{Q}}(\cdot)}{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}\mathfrak{c})}$ and the lattice M is isomorphic to the fractional ideal $\mathfrak{b}\bar{\mathfrak{c}}$ equipped with the quadratic form $\frac{-\mathrm{N}_{K/\mathbb{Q}}(\cdot)}{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}\bar{\mathfrak{c}})}$.*

Proof. Firstly, we compute the lattice N. Suppose that

$$a_1\mathfrak{z}_1^2 + b_1\mathfrak{z}_1 + c_1 = 0$$

and

$$a_2\mathfrak{z}_2^2 + b_2\mathfrak{z}_2 + c_2 = 0,$$

where $a_1, b_1, c_1 \in \mathbb{Z}$, $\gcd(a_1, b_1, c_1) = 1$, $a_1 > 0$ and $a_2, b_2, c_2 \in \mathbb{Z}$, $\gcd(a_2, b_2, c_2) = 1$, $a_2 > 0$. Consider the map

$$\jmath : K \to M_2(\mathbb{Q})$$

defined by

$$x + y\sqrt{-D} \to \begin{pmatrix} \frac{-b_1 - b_2}{4a_1a_2}x - \frac{b_1b_2 - D}{4a_1a_2}y & \frac{1}{2a_1}x + \frac{b_1}{2a_1}y \\ \frac{1}{2a_2}x + \frac{b_2}{2a_2}y & -2y \end{pmatrix}.$$

It maps K to $v^+(\mathfrak{z}_1, \mathfrak{z}_2)$ and is an isometry, provided that the quadratic form on K is given by $\mathrm{q}(\beta) = N_{K/\mathbb{Q}}(\beta)$ and the quadratic form on $M_2(\mathbb{Q})$ is given by $\mathrm{q}(l) = -\det(l)$. We have

$$\begin{aligned} \jmath(1) =& \frac{-1}{4a_1a_2} \begin{pmatrix} b_1 + b_2 & -2a_2 \\ -2a_1 & 0 \end{pmatrix}, \\ \jmath(\mathfrak{z}_1) =& \frac{-1}{4a_1a_2} \begin{pmatrix} -2c_1 & 0 \\ b_1 - b_2 & 2a_2 \end{pmatrix}, \\ \jmath(\mathfrak{z}_2) =& \frac{-1}{4a_1a_2} \begin{pmatrix} -2c_2 & -b_1 + b_2 \\ 0 & 2a_1 \end{pmatrix}, \\ \jmath(\mathfrak{z}_1\,\mathfrak{z}_2) =& \frac{-1}{4a_1a_2} \begin{pmatrix} 0 & 2c_1 \\ 2c_2 & b_1 + b_2 \end{pmatrix}. \end{aligned}$$

Thus, we have

$$M_2(\mathbb{Z}) \cap v^+(\mathfrak{z}_1, \mathfrak{z}_2) = 2a_1a_2\,\jmath(1)\mathbb{Z} + 2a_1a_2\,\jmath(\mathfrak{z}_1)\mathbb{Z} + 2a_1a_2\,\jmath(\mathfrak{z}_2)\mathbb{Z} + 2a_1a_2\,\jmath(\mathfrak{z}_1\mathfrak{z}_2)\mathbb{Z}. \tag{5.33}$$

On the other hand

$$\mathfrak{b}\mathfrak{c} = \mathbb{Z} + \mathfrak{z}_1\mathbb{Z} + \mathfrak{z}_2\mathbb{Z} + \mathfrak{z}_1\,\mathfrak{z}_2\mathbb{Z}. \tag{5.34}$$

Note that $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}) = \frac{1}{a_1}$ and $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{c}) = \frac{1}{a_2}$. The quadratic form on $\mathfrak{b}\mathfrak{c}$ is given by $\mathrm{q}(\gamma) = a_1\,a_2\,N_{K/\mathbb{Q}}(\gamma)$. Hence, we check that

$$\begin{aligned} \mathrm{q}(1) &= \mathrm{q}\big(2a_1a_2\,\jmath(1)\big), \\ \mathrm{q}(\mathfrak{z}_1) &= \mathrm{q}\big(2a_1a_2\,\jmath(\mathfrak{z}_1)\big), \\ \mathrm{q}(\mathfrak{z}_2) &= \mathrm{q}\big(2a_1a_2\,\jmath(\mathfrak{z}_2)\big), \\ \mathrm{q}(\mathfrak{z}_1\mathfrak{z}_2) &= \mathrm{q}\big(2a_1a_2\,\jmath(\mathfrak{z}_1\mathfrak{z}_2)\big). \end{aligned}$$

Now the statement of the lemma follows from the equations (5.33) and (5.34). The lattice M can be computed along the same lines. $\square$

5.11 Proof of Theorem 5.8

Our next goal is to find a preimage of a function $f \in M_1^!(\rho_N)$ under the map $T_{L,N}$ defined in Theorem 5.2.

Recall that $N'/N \cong \mathbb{Z}/D\mathbb{Z}$ and $L'/L \cong \mathbb{Z}/2D\mathbb{Z}$. Moreover, we can choose isomorphisms $\imath_N : \mathbb{Z}/D\mathbb{Z} \to N'/N$ and $\imath_L : \mathbb{Z}/2D\mathbb{Z} \to L'/L$ such that $\mathrm{q}(\imath_N(\nu)) \equiv \nu^2/D (\mathrm{mod}\ \mathbb{Z})$ for each $\nu \in \mathbb{Z}/D\mathbb{Z}$ and $\mathrm{q}(\imath_L(\lambda)) \equiv \lambda^2/4D (\mathrm{mod}\ \mathbb{Z})$ for each $\lambda \in \mathbb{Z}/2D\mathbb{Z}$. Suppose that $f = (f_\nu)_{\nu \in N'/N}$ belongs to $M_k^!(\rho_N)$ for some odd k. Then, the transformation property (1.12) for $\widetilde{R} = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right)$ and $\widetilde{T} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \in \mathrm{Mp}_2(\mathbb{Z})$ implies that

$$f_\nu = f_{-\nu}, \quad \nu \in \mathbb{Z}/D\mathbb{Z}$$

and f_ν has the Fourier expansion of the form

$$f_\nu = \sum_{t \equiv \nu^2 (\mathrm{mod}\ D)} c(t)\, \mathbf{e}\Big(\frac{t}{D}\tau\Big).$$

Since D is prime, the Fourier expansion of f can be written as

$$f(\tau) = \sum_{\nu \in \mathbb{Z}/D\mathbb{Z}} e_\nu \sum_{t \equiv \nu^2 (\mathrm{mod}\ D)} c(t)\, \mathbf{e}\Big(\frac{t}{D}\tau\Big).$$

Similarly, we see that for $l \in 1/2 + 2\mathbb{Z}$ each modular form $h \in M_l^!(\rho_L)$ has the Fourier expansion of the form

$$h(\tau) = \sum_{\lambda \in \mathbb{Z}/2D\mathbb{Z}} e_\lambda \sum_{d \equiv \lambda^2 (\mathrm{mod}\ D)} b(d)\, \mathbf{e}\Big(\frac{d}{4D}\tau\Big).$$

Theorem 5.11. *Let the lattices L, N, and the vector m be as in Theorem 5.10. Suppose that $f \in M_1^!(\rho_N)$ is a modular form with zero constant term and rational Fourier coefficients. Then there exists a function $h \in S_{1/2}^!(\rho_L)$ such that:*

(i) the function $h(\tau) = \sum_{\lambda \in \mathbb{Z}/2D\mathbb{Z}} e_\lambda \sum_{d \equiv \lambda^2 (\mathrm{mod}\ 4D)} b(d)\, \mathbf{e}(\frac{d}{4D}\tau)$ has rational Fourier coefficients;

(ii) the Fourier coefficients of h satisfy $b(-Ds^2) = 0$ for all $s \in \mathbb{Z}$;

(iii) $T_{L,N}(h) = f$.

Proof. Denote by S the lattice $\mathbb{Z}$ equipped with the quadratic form $\mathrm{q}(x) := -x^2$. For this lattice we have $S'/S \cong \mathbb{Z}/2\mathbb{Z}$. Lemma 5.4 implies that $L \cong N \oplus S$. Note that $L'/L \cong S'/S \times N'/N$ and $\rho_L = \rho_S \otimes \rho_N$. Set

$$\theta_0(\tau, z) = \sum_{n \in \mathbb{Z}} \mathbf{e}(n^2\tau + 2nz), \quad \theta_1(\tau, z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} \mathbf{e}(n^2\tau + 2nz)$$

and

$$\theta_\kappa(\tau) = \theta_\kappa(\tau, 0), \quad \kappa = 0, 1.$$

It follows from the definition of $T_{L,N}$ that

$$(T_{L,N}(h))_\nu = \sum_{\kappa \in S'/S} h_{(\kappa,\nu)} \theta_\kappa.$$

Let $\tilde{\phi}_{-2,1}$, $\tilde{\phi}_{0,1}$ be the weak Jacobi forms defined on p. 108 of [25]. These functions can be written as

$$\begin{aligned}
\tilde{\phi}_{-2,1}(\tau, z) &= \psi_0(\tau)\, \theta_0(\tau, z) + \psi_1(\tau)\, \theta_1(\tau, z), \\
\tilde{\phi}_{0,1}(\tau, z) &= \varphi_0(\tau)\, \theta_0(\tau, z) + \varphi_1(\tau)\, \theta_1(\tau, z)
\end{aligned}$$

where

$$\begin{aligned}
\psi_0 &= -2 - 12q - 56q^2 - 208q^3 + \cdots, \qquad (5.35) \\
\psi_1 &= q^{-1/4} + 8q^{3/4} + 39q^{7/4} + 152q^{11/4} + \cdots
\end{aligned}$$

$$\begin{aligned}
\varphi_0 &= 10 + 108q + 808q^2 + 4016q^3 + \cdots, \\
\varphi_1 &= q^{-1/4} - 64q^{3/4} - 513q^{7/4} - 2752q^{11/4} + \cdots.
\end{aligned}$$

The vector-valued functions (ψ_0, ψ_1) and (φ_0, φ_1) belong to the spaces $M^!_{-5/2}(\rho_S)$ and $M^!_{-1/2}(\rho_S)$ respectively, and they satisfy

$$\begin{aligned}
\tilde{\phi}_{-2,1}(\tau, 0) &= \psi_0(\tau)\, \theta_0(\tau) + \psi_1(\tau)\, \theta_1(\tau) = 0, \qquad (5.36) \\
\tilde{\phi}_{0,1}(\tau, 0) &= \varphi_0(\tau)\, \theta_0(\tau) + \varphi_1(\tau)\, \theta_1(\tau) = 12.
\end{aligned}$$

First, we construct a function $g \in M^!_{1/2}(\rho_L)$ that satisfies conditions (i) and (iii). Define

$$g_{(\kappa,\nu)} := \frac{1}{12} \varphi_\kappa f_\nu, \quad (\kappa, \nu) \in S'/S \times N'/N. \qquad (5.37)$$

This function satisfies

$$\begin{aligned}
T_{L,N}(g) =& \frac{1}{12} \sum_{\nu \in N'/N} e_\nu \left(g_{(0,\nu)} \theta_0 + g_{(1,\nu)} \theta_1 \right) \\
=& \frac{1}{12} \sum_{\nu \in N'/N} e_\nu\, f_\nu (\varphi_0 \theta_0 + \varphi_1 \theta_1) \\
=& f.
\end{aligned}$$

Next, we will add a correction term to g and construct a function that satisfies also (ii). Fix an integer $s > 0$. Our next goal is to construct a supplementary function $\tilde{g}(\tau) = \sum_{\lambda \in \mathbb{Z}/2D\mathbb{Z}} \sum_{d \equiv \lambda^2 \pmod{4D}} \tilde{a}(d)\, \mathbf{e}(d\tau) \in M^!_{1/2}(\rho_L)$ with the following properties:

$$\begin{aligned}
&\tilde{a}(-Ds^2) \neq 0 \text{ and } \tilde{a}(-Dr^2) = 0 \text{ for all } r > s, \qquad (5.38) \\
&T_{L,N}(\tilde{g}) = 0 \qquad (5.39) \\
&\tilde{g} \text{ has rational Fourier coefficients.} \qquad (5.40)
\end{aligned}$$

To this end we consider the following theta function

$$\widetilde{\Theta} := \sum_{\nu \in \mathbb{Z}/D\mathbb{Z}} e_\nu \sum_{a \in \mathfrak{o}+\nu/\sqrt{-D}} (a^2 + \overline{a}^2)\, \mathbf{e}(a\overline{a}\tau).$$

By Theorem 4.1 in [10] his theta function belongs to $S_3(\rho)$. We define

$$\tilde{g}_{(\kappa,\nu)} := \psi_\kappa\, \widetilde{\Theta}_\nu\, j^{\frac{s^2-t}{4}+1}, \quad (\kappa,\nu) \in S'/S \times N'/N, \tag{5.41}$$

where

$$t = \begin{cases} 0 & \text{if } s \equiv 0 \bmod 2, \\ 1 & \text{otherwise,} \end{cases}$$

and j is the j-invariant. First we check that the function $\tilde{g}$ satisfies condition (5.38). For $D \neq 3$ we have

$$\Theta_0 = 4q + O(q^2), \quad q = \mathbf{e}(\tau).$$

Hence, from (5.41) we find that for s even

$$\begin{aligned} \tilde{g}_{(0,0)} &= -8q^{-s^2/4} + O(q^{-s^2/4+1}), \\ \tilde{g}_{(1,0)} &= 4q^{-s^2/4-1/4} + O(q^{-s^2/4+3/4}), \end{aligned}$$

and for s odd

$$\begin{aligned} \tilde{g}_{(0,0)} &= -8q^{-s^2/4+1/4} + O(q^{-s^2/4+5/4}), \\ \tilde{g}_{(1,0)} &= 4q^{-s^2/4} + O(q^{-s^2/4+1}). \end{aligned}$$

This proves (5.38). The function $\tilde{g}$ satisfies

$$\begin{aligned} T_{L,N}(\tilde{g}) &= \sum_{\nu \in N'/N} e_\nu \left(\tilde{g}_{(0,\nu)}\theta_0 + \tilde{g}_{(1,\nu)}\theta_1\right) \\ &= \sum_{\nu \in N'/N} e_\nu\, \widetilde{\Theta}_{\mathfrak{o}+\nu}\, j^{\frac{s^2-t}{4}+1} \left(\psi_0\theta_0 + \psi_1\theta_1\right) \\ &= 0. \end{aligned}$$

This proves (5.39). The property (5.40) is obvious.

By subtracting from g a suitable linear combination of functions $\tilde{g}$ for different s we find a function

$$h(\tau) = \sum_{\lambda \in \mathbb{Z}/D\mathbb{Z}} e_\lambda \sum_{d \equiv \lambda^2 (\mathrm{mod}\ 4D)} b(d)\, \mathbf{e}\Big(\frac{d}{4D}\tau\Big) \in M^!_{1/2}(\rho_L)$$

such that

$$b(-Dr^2) = 0 \text{ for all } r \in \mathbb{Z}\backslash 0, \tag{5.42}$$

$$T_{L,N}(h) = f, \tag{5.43}$$

$$h(\tau) \text{ has rational Fourier coefficients.} \tag{5.44}$$

The final step is to show that $b(0) = 0$. Identity (5.43) implies that

$$h_{(0,0)}\theta_0 + h_{(1,0)}\theta_1 = f_0.$$

Hence, the constant terms of these functions are equal. By the assumptions of the theorem

$$\mathrm{CT}(f_0) = 0.$$

On the other hand

$$\mathrm{CT}(h_{(0,0)}\theta_0 + h_{(1,0)}\theta_1) = \sum_{s\in\mathbb{Z}} b(-Ds^2) = b(0).$$

Thus, the function h satisfies the conditions (i)-(iii) of the theorem. This finishes the proof. $\square$

We observe that the Grassmanian $\mathrm{Gr}^+(L)$ is isomorphic to the upper half-plane $\mathfrak{H}$. There is a map $\mathfrak{H} \to \mathrm{Gr}^+(L)$ given by

$$z \quad \to \quad v^+(z) := \Re\begin{pmatrix} z^2 & z \\ z & 1 \end{pmatrix}\mathbb{R} + \Im\begin{pmatrix} z^2 & z \\ z & 1 \end{pmatrix}\mathbb{R} \subset L \otimes \mathbb{R}. \tag{5.45}$$

The group $\Gamma_0(D)$ acts on L' and fixes all the elements of L'/L. Denote by $X_0(D)$ the modular curve $\overline{\Gamma_0(D)\backslash\mathfrak{H}}$.

Suppose that the vector $m \in L'$, the lattice N and the point $\mathfrak{z}_m \in \mathfrak{H}$ are defined as in Theorem 5.10. Let h be the modular form $h \in S^!_{1/2}(\rho_L)$ satisfying

$$T_{L,N}(h) = f, \tag{5.46}$$

that was constructed in the previous theorem. It follows from (5.46) and Theorem 5.2 that

$$\Phi_L(h, \mathfrak{z}_m) = \Phi_N(f).$$

Recall that by definition

$$\Phi_N(f) = (f, \Theta_{\mathfrak{b}})^{\mathrm{reg}}.$$

Without loss of generality we assume that h has integral negative Fourier coefficients. The infinite product $\Psi(z) := \Psi_L(h, z)$ introduced in Section 1.9 defines a meromorphic function on $X_0(D)$. Theorem B3 in Section 1.9 implies

$$(f, \Theta_{\mathfrak{b}})^{\mathrm{reg}} = \log|\Psi_L(h, \mathfrak{z}_m)|. \tag{5.47}$$

It also follows from Theorem B3 that the divisor of Ψ_L is supported at Heegner points.

Next we compute the local height pairing between Heegner divisors. These calculations are carried out in the celebrated series of papers [35], [36]. For the convenience of the reader we recall the main steps of the computation in what follows.

First, let as recall the definition of Heegner points and the way they can be indexed by the vectors of the lattice L'.

For $\ell \in L'$ with $\mathrm{q}(\ell) < 0$ denote by x_ℓ the divisor $(\mathfrak{z}_\ell) - (\infty)$ on the modular curve $X_0(D)$. The divisor x_ℓ is defined over the Hilbert class field of $\mathbb{Q}(\sqrt{D\mathrm{q}(\ell)})$.

For any integer $d > 0$ such that $-d$ is congruent to a square modulo $4D$, choose a residue $\beta(\bmod 2D)$ with $-d \equiv \beta^2(\bmod 4D)$ and consider the set

$$L_{d,\beta} = \left\{ \ell = \begin{pmatrix} a/D & b/2D \\ b/2D & c \end{pmatrix} \in L' \,\middle|\, \mathrm{q}(\ell) = -\frac{d}{4D},\ b \equiv \beta(\bmod 2D) \right\}$$

on which $\Gamma_0(D)$ acts. Define the Heegner divisor

$$\mathrm{y}_{d,\beta} = \sum_{\ell \in \Gamma_0(D) \backslash L_{d,\beta}} \mathrm{x}_\ell.$$

The Fricke involution acts on L' by

$$\ell \to \frac{1}{D} \begin{pmatrix} 0 & 1 \\ -D & 0 \end{pmatrix} \ell \begin{pmatrix} 0 & -D \\ 1 & 0 \end{pmatrix}$$

and maps $L_{d,\beta}$ to $L_{d,-\beta}$. Set

$$\mathrm{y}_d^* = \mathrm{y}_{d,\beta} + \mathrm{y}_{d,-\beta}. \tag{5.48}$$

The divisor y_d^* is defined over $\mathbb{Q}$ ([36] p. 499.)

Now we would like to compute the local height pairings between the divisor x_ℓ and a Heegner divisor. The definition of the local height pairing is given in Section 4.3. The divisors x_ℓ and y_d^* have the point ∞ at their common support. In order to define the height pairing between these divisors we must fix a uniformizing parameter π at this cusp. We let π denote the Tate parameter q on the family of degenerating elliptic curves near ∞. This is defined over $\mathbb{Q}$. Over $\mathbb{C}$ we have $q = \mathbf{e}(z)$ on $X_0^*(D) = \Gamma_0^*(D)\backslash\overline{\mathfrak{H}}$, where $z \in \mathfrak{H}$ with $\Im(z)$ sufficiently large. The following theorem can be deduced from the computations in Section IV.4 in [36].

Theorem 5.12. *Let $d_1,\ d_2 > 0$ be two integers and $\beta_1,\ \beta_2$ be two elements of $\mathbb{Z}/2D\mathbb{Z}$ with $-d_1 \equiv \beta_1^2(\bmod 4D)$ and $-d_2 \equiv \beta_2^2(\bmod 4D)$. Suppose that d_1 is fundamental and d_2/d_1 is not a full square. Fix a vector $\ell \in L_{d_1,\beta_1}$. Let p be a prime with $\gcd(p, D) = 1$. Choose a prime ideal $\mathfrak{P}$ lying above p in the Hilbert class field of $\mathbb{Q}(\sqrt{-d_1})$. Then the following formula for the local height holds:*
in the case $\left(\frac{p}{d_1}\right) = 1$ we have

$$\langle \mathrm{x}_\ell, \mathrm{y}_{d_2}^* \rangle_{\mathfrak{P}} = 0, \tag{5.49}$$

in the case $\left(\frac{p}{d_1}\right) = -1$ we have

$$\langle \mathrm{x}_\ell, \mathrm{y}_{d_2}^* \rangle_{\mathfrak{P}} = \log(p) \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \beta_1\beta_2 \bmod 2}} \delta_{d_1}(r)\, r_{\mathfrak{n}\bar{\mathfrak{c}}^2\mathfrak{a}^2}\left(\frac{d_1 d_2 - r^2}{4Dp}\right) \mathrm{ord}_p\left(\frac{d_1 d_2 - r^2}{4D}\right). \tag{5.50}$$

Here $\mathfrak{c} = \mathbb{Z}\mathfrak{z}_\ell + \mathbb{Z}$, $\mathfrak{n} = \mathbb{Z}D + \mathbb{Z}\frac{\beta_1 + \sqrt{-d_1}}{2}$, $\mathfrak{a}$ is any ideal in the ideal class $\mathbb{A}$ defined by (5.1), and

$$\delta_d(r) = \begin{cases} 2 & \text{for } r \equiv 0 \bmod d; \\ 1 & \text{otherwise.} \end{cases}$$

Proof. The curve $X_0(D)$ may be described over $\mathbb{Q}$ as the compactification of the space of moduli of elliptic curves with a cyclic subgroup of order D [35]. Over a field k of characteristic zero, the points y of $X_0(D)$ correspond to diagrams

$$\psi : F \to F',$$

where F and F' are (generalized) elliptic curves over k and ψ is an isogeny over k whose kernel is isomorphic to $\mathbb{Z}/D\mathbb{Z}$ over an algebraic closure $\overline{k}$.

The point $\mathfrak{z}_\ell \in \mathfrak{H}$ defines the point $\mathrm{x} \in X_0(D)$. Then $\mathrm{x} = (\phi : E \to E')$ and over $\mathbb{C}$ this diagram is isomorphic to

$$\mathbb{C}/\mathfrak{c} \xrightarrow{\mathrm{id}_{\mathbb{C}}} \mathbb{C}/\mathfrak{cn}\,.$$

Following the calculations in [35] we reduce the computation of local heights to a problem in arithmetic intersection theory. Let us set up some notations. Denote by v the place of H_{d_1}, the Hilbert class field of $\mathbb{Q}(\sqrt{-d_1})$, corresponding to the prime ideal $\mathfrak{P}$. Denote by Λ_v the ring of integers in the completion $H_{d_1,v}$ and let π be an uniformizing parameter in Λ_v. Let W be the completion of the maximal unramified extension of Λ_v. Let $\underline{X}$ be a regular model for X over Λ_v and $\underline{\mathrm{x}}$, $\underline{\mathrm{y}}$ be the sections of $\underline{X} \otimes \Lambda_v$ corresponding to the points x and y. A model that has a modular interpretation is described in Section III.3 of [35]. The general theory of local height pairing [34] implies

$$\langle \mathrm{x}, \mathrm{y} \rangle_v = -(\underline{\mathrm{x}} \cdot \underline{\mathrm{y}}) \log p.$$

The intersection product is unchanged if we extend scalars to W. By Proposition 6.1 in [35]

$$(\underline{\mathrm{x}} \cdot \underline{\mathrm{y}})_W = \frac{1}{2} \sum_{n \geq 1} \mathrm{CardHom}_{W/\pi^n}(\underline{\mathrm{x}}, \underline{\mathrm{y}})_{\mathrm{deg}1}.$$

Denote by R the ring $\mathrm{Hom}_{W/\pi}(\underline{\mathrm{x}_\ell})$. On p. 550 of [36] the following formula for the intersection number is obtained

$$(\underline{\mathrm{x}_\ell} \cdot \underline{\mathrm{y}^*_{d_2}})_W = \frac{1}{4} \sum_{\substack{r^2 < d_1 d_2 \\ r \equiv \beta_1 \beta_2 (\mathrm{mod}\ 2D)}} \mathrm{Card}\big\{ S_{[d_1,2r,d_2]} \to R \bmod R^\times \big\} \,\mathrm{ord}_p \left(\frac{d_1 d_2 - r^2}{4D} \right), \tag{5.51}$$

where $S_{[d_1,2r,d_2]}$ is the Clifford order

$$S_{[d_1,2r,d_2]} = \mathbb{Z} + \mathbb{Z}\frac{1+e_1}{2} + \mathbb{Z}\frac{1+e_2}{2} + \mathbb{Z}\frac{(1+e_1)(1+e_2)}{4},$$

$$e_1^2 = -d_1, \quad e_2^2 = -d_2, \quad e_1 e_2 + e_2 e_1 = 2r.$$

In the case $\left(\frac{p}{d_1}\right) = 1$ the ring R is isomorphic to an order in $\mathfrak{o}_{d_1}$. Since d_1/d_2 is not a full square the ring R can not contain the Clifford order $S_{[d_1,2r,d_2]}$. Hence, $(\underline{\mathrm{x}_\ell} \cdot \underline{\mathrm{y}^*_{d_2}})_W = 0$. This proves (5.49).

Now we consider the case $\left(\frac{p}{d_1}\right) = -1$. Formula (9.3) in [35] gives us a convenient description of the ring R. Namely, for $a, b \in \mathbb{Q}(\sqrt{-d_1})$ denote

$$[a, b] = \begin{pmatrix} a & b \\ p\bar{b} & \overline{a} \end{pmatrix}$$

and consider the quaternion algebra over $\mathbb{Q}$

$$B = \left\{ [a, b] \,\middle|\, a, b \in \mathbb{Q}(\sqrt{-d_1}) \right\}.$$

Then R is an Eichler order of index D in this quaternion algebra given by

$$R = \left\{ [a, b] \,\middle|\, a \in \mathfrak{d}^{-1},\ b \in \mathfrak{d}^{-1}\mathfrak{n}\overline{\mathfrak{a}\mathfrak{c}\mathfrak{a}}^{-1}\mathfrak{c}^{-1}, a \equiv b \bmod \mathfrak{o}_{d_1} \right\},$$

where $\mathfrak{d}$ is the different of $\mathbb{Q}(\sqrt{-d_1})$.

By the same computations as in Lemma 3.5 of [33] we find that the number of embeddings of $S_{[d_1, 2r, d_2]}$ into R, normalized so that the image of e_1 is $[\sqrt{-d_1}, 0]$, is equal to

$$\delta_{d_1}(r)\, r_{\mathfrak{n}\bar{\mathfrak{c}}^2\mathfrak{a}^2}\left(\frac{d_1 d_2 - r^2}{4Dp}\right) \operatorname{ord}_p\left(\frac{d_1 d_2 - r^2}{4D}\right).$$

This finishes the proof of the theorem. □

Proof of Theorem 5.8. Since the discriminant $-D$ is prime, the class number of K is odd and there exists an ideal class $\mathfrak{c}$ such that $\mathfrak{b} = \bar{\mathfrak{c}}^2$ in the ideal class group. The class $\mathfrak{c}$ contains an ideal of the form

$$\mathfrak{c} = \mathfrak{z}\mathbb{Z} + \mathbb{Z}, \tag{5.52}$$

where $\mathfrak{z}$ is a CM point of discriminant $-D$. Property (5.52) is preserved when we act on $\mathfrak{z}$ by elements of $\mathrm{SL}_2(\mathbb{Z})$. As we have explained in the proof of Theorem 5.10, we may assume that $\mathfrak{z}$ satisfies the quadratic equation

$$a\mathfrak{z}^2 + b\mathfrak{z} + c = 0$$

for $a \in D\mathbb{Z}, b \in D\mathbb{Z}, c \in \mathbb{Z}$ and $b^2 - 4ac = -D$. The matrix

$$m = \frac{1}{D}\begin{pmatrix} c & -b/2 \\ -b/2 & a \end{pmatrix}$$

belongs to the lattice L' and has the norm $-1/4$. Lemma 5.4 implies that the lattice $N := L \cap m^{\perp}$ is isomorphic to the fractional ideal $\mathfrak{c}^2$ equipped with the quadratic form $\mathrm{q}(\gamma) = \mathrm{N}_{K/\mathbb{Q}}(\gamma)/\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{c}^2)$ and moreover, the lattice L splits as $L = N \oplus 2m\mathbb{Z}$.

Next, by Theorem 5.11 we find a weak cusp form $h \in S^!_{1/2}(\rho_L)$ satisfying

$$T_{L,N}(h) = f, \tag{5.53}$$

where $T_{L,N}$ is defined as in Theorem 5.2. Function h has the Fourier expansion

$$h(\tau) = \sum_{\beta \in \mathbb{Z}/2D\mathbb{Z}} e_\beta \sum_{d \equiv \beta^2 (\bmod\ 4D)} b(d)\, \mathbf{e}\Big(\frac{d}{4D}\tau\Big).$$

It follows from (5.53) and Theorem 5.2 that

$$(f, \Theta_N)_{\mathrm{reg}} = \Phi_L(h, \mathfrak{z}).$$

From Theorem B3 in Section 1.9 we know that

$$\Phi_L(h, \mathfrak{z}) = \log|\Psi_L(h, \mathfrak{z})|, \tag{5.54}$$

where $\Psi(z) = \Psi_L(h, z)$ is a meromorphic function. Theorem B3 also implies that

$$\operatorname{div}(\Psi) = \sum_{d=0}^{\infty} b(-d)\, \mathrm{y}_d^*, \tag{5.55}$$

where y_d^* is the Heegner divisor defined in (5.48).

Set $\mathrm{x} = (\mathfrak{z}) - (\infty)$. The condition (ii) of Theorem 5.11 implies that the function $\Phi_L(h, \cdot)$ is real analytic at the point $\mathfrak{z}$. Thus, the only point in the common support of x and $\operatorname{div}(\Psi)$ is ∞. Recall, that we have fixed the uniformizing parameter π at this cusp to be the Tate parameter q on the family of degenerating elliptic curves near ∞.

Recall that the divisors x and $\operatorname{div}(\Psi)$ are defined over H. The axioms of local height (listed in Section 4.3) together with the refined definition (4.8) imply that for each prime $\mathfrak{P}$ of H

$$\operatorname{ord}_{\mathfrak{P}}\big(\Psi(\mathfrak{z})\big)\log p - \operatorname{ord}_{\mathfrak{P}}\big(\Psi[\infty]\big)\log p = \big\langle \mathrm{x}, \sum_{d=1}^{\infty} b(-d)\, \mathrm{y}_d^* \big\rangle_{\mathfrak{P}}. \tag{5.56}$$

From the infinite product of Theorem 13.3 in [10] we find that $\Psi[\infty] = 1$ for the choice of the uniformizing parameter at ∞ as above. Theorem 5.11 part (ii) implies that d/D is not a full square provided $b(-d) \neq 0$. Thus, by Theorem 5.12 for each prime $\mathfrak{P}$ of H lying above a rational prime p with $\big(\frac{p}{D}\big) \neq 0$ we obtain

$$\langle \mathrm{x}, \mathrm{y}_d^* \rangle_{\mathfrak{P}} = 0$$

in the case $\big(\frac{p}{D}\big) = 1$, and

$$\langle \mathrm{x}, \mathrm{y}_d^* \rangle_{\mathfrak{P}} = \log(p) \sum_{\substack{n \in \mathbb{Z} \\ n \equiv d (\mathrm{mod}\ 2)}} r_{\bar{\mathfrak{c}}^2 \mathfrak{a}^2}\Big(\frac{d - Dn^2}{4p}\Big) \operatorname{ord}_p\Big(\frac{d - Dn^2}{4}\Big) \tag{5.57}$$

in the case $\big(\frac{p}{D}\big) = -1$. We observe that the sum

$$\sum_{d=0}^{\infty} b(-d) \sum_{\substack{n \in \mathbb{Z} \\ n \equiv d (\mathrm{mod}\ 2)}} r_{\bar{\mathfrak{c}}^2 \mathfrak{a}^2}\Big(\frac{d - Dn^2}{4p}\Big) \operatorname{ord}_p\Big(\frac{d - Dn^2}{4}\Big)$$

is equal to the constant term with respect to $\mathbf{e}(\tau)$ of the following series

$$\sum_{\nu\in\mathbb{Z}/D\mathbb{Z}} \left(\left(h_{(0,\nu)}\theta_0 + h_{(1,\nu)}\theta_1 \right) \sum_{t\equiv\nu \bmod D} r_{\mathfrak{b}\mathfrak{a}^2}\left(\frac{t}{p}\right) \operatorname{ord}_p(t)\, \mathbf{e}\left(\frac{t}{D}\tau\right) \right).$$

The equation (5.53) implies

$$f_\nu = h_{(0,\nu)}\theta_0 + h_{(1,\nu)}\theta_1, \quad \nu \in \mathbb{Z}/D\mathbb{Z}. \tag{5.58}$$

Hence, combining the equations (5.57) and (5.58) we arrive at

$$\left\langle \mathrm{x}, \sum_{d=0}^{\infty} b(-d)\, \mathrm{y}_d^* \right\rangle_{\mathfrak{P}} = \log p \sum_{\nu\in\mathbb{Z}/D\mathbb{Z}} \sum_{t=0}^{\infty} c_\nu(-t)\, r_{\mathfrak{b}\mathfrak{a}^2}\left(\frac{t}{p}\right) \operatorname{ord}_p(t).$$

Finally, the equations (5.54) and (5.56) imply

$$\operatorname{ord}_{\mathfrak{P}}(\alpha) = \operatorname{ord}_{\mathfrak{P}}(\Psi_L(h,\mathfrak{z})) = \frac{1}{\log p} \left\langle \mathrm{x}, \sum_{d=0}^{\infty} b(-d)\, \mathrm{y}_d^* \right\rangle_{\mathfrak{P}} =$$

$$= \sum_{\nu\in\mathbb{Z}/D\mathbb{Z}} \sum_{t=0}^{\infty} c_\nu(-t)\, r_{\mathfrak{b}\mathfrak{a}^2}\left(\frac{t}{p}\right) \operatorname{ord}_p(t).$$

This finishes the proof of Theorem 5.8. □

5.12 Theorem 5.8 implies Conjecture 2

Proof of Theorem 5.9: Recall that for a relation $\boldsymbol{\lambda}$ and two CM-points $\mathfrak{z}_1,\ \mathfrak{z}_2 \in \mathfrak{H}$ lying in the same quadratic field $K = \mathbb{Q}(\sqrt{-D})$ Theorems 5.3 and 5.4 imply

$$G_{k,\boldsymbol{\lambda}}(\mathfrak{z}_1,\mathfrak{z}_2) = (f, \Theta_N)_{\mathrm{reg}}. \tag{5.59}$$

Here the lattices N and M are defined as

$$N := v^+(\mathfrak{z}_1,\mathfrak{z}_2) \cap M_2(\mathbb{Z}), \quad M := N^\perp \cap M_2(\mathbb{Z})$$

and the function $f =\in M_1^!(\mathrm{SL}_2(\mathbb{Z}), \rho_N)$ is given by

$$f = [\,g_{\boldsymbol{\lambda}}, \Theta_{M(-1)}]_{k-1}.$$

Firstly, we compute the lattices $N = \mathrm{M}_2(\mathbb{Z}) \cap v^+(\mathfrak{z}_1,\mathfrak{z}_2)$ and $M = \mathrm{M}_2(\mathbb{Z}) \cap N^\perp$. By Lemma 5.5 the lattice N is isomorphic to the fractional ideal $\mathfrak{b}\mathfrak{c}$ equipped with the quadratic form $\frac{1}{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}\mathfrak{c})}\mathrm{N}_{K/\mathbb{Q}}(\cdot)$ and the lattice M is isomorphic to the fractional ideal $\mathfrak{b}\bar{\mathfrak{c}}$ equipped with the quadratic form $\frac{-1}{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}\bar{\mathfrak{c}})}\mathrm{N}_{K/\mathbb{Q}}(\cdot)$.

Next, we compute the negative Fourier coefficients of the function

$$f = [g_{\boldsymbol{\lambda}}, \Theta_{M(-1)}]_{k-1}.$$

The function f has the Fourier expansion of the form

$$f = \sum_{\nu \in N'/N} e_\nu \sum_{\substack{t \in \frac{1}{D}\mathbb{Z} \\ t \gg -\infty}} c_\nu(t)\, \mathbf{e}(t\tau).$$

For a definite even lattice L, an element $\lambda \in L'/L$ and a rational number t we denote by $R_{L,\lambda}(t)$ the number of elements of norm t in $\lambda + L$. The formula (5.4) implies

$$\begin{aligned}
[\mathbf{e}(-m\tau), \mathbf{e}(n\tau)]_{k-1}^{2-2k,1} &= \sum_{s=0}^{k-1} (-1)^s \binom{k-1}{k-1-s} \binom{-k}{s} (-m)^{(k-1-s)}\, n^s\, \mathbf{e}\big((-m+n)\tau\big) \\
&= (-m)^{k-1} P_{k-1}\Big(1 - \frac{2n}{m}\Big)\, \mathbf{e}\big((-m+n)\tau\big),
\end{aligned}$$

where $P_k(x) = (2^k k!)^{-1} \frac{d^k}{dx^k}(x^2-1)^k$ is the k-th Legendre polynomial. Hence, for $t < 0$ and $\nu \in M'/M$ the Fourier coefficients of f are equal to

$$c_\nu(t) = (-1)^{k-1} \sum_m \lambda_m\, m^{k-1}\, P_{k-1}\Big(-1 - \frac{2t}{m}\Big)\, R_{M(-1),\nu}(m+t). \tag{5.60}$$

Finally, we recall that by the standard argument

$$r_{\mathfrak{f}}(n) = \sum_{\lambda \in L'/L} R_{L,\lambda}\big(\frac{n}{D}\big),$$

where D is square free, $\mathfrak{f}$ is a fractional ideal in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ and L is the lattice $\mathfrak{a}$ with quadratic form $\frac{1}{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{f})}\mathrm{N}_{K/\mathbb{Q}}(\cdot)$. Note that $\mathrm{q}(\lambda) \in \frac{1}{D}\mathbb{Z}$ for all $\lambda \in L$. Thus, we can rewrite (5.60) for $n > 0$ as

$$\sum_{\nu \in N'/N} c_\nu\big(\frac{-n}{D}\big) = \sum_m \lambda_m\, m^{k-1}\, r_{\mathfrak{b}\bar{\mathfrak{c}}}(Dm - n)\, P_{k-1}\Big(1 - \frac{2n}{Dm}\Big). \tag{5.61}$$

Now, after we have computed the lattice N and the function f in (5.59), applying Theorem 5.8 to the right hand side of (5.59) we obtain the statement of Conjecture 2.□

5.13 Numerical examples

In this section we give examples and numerical computations to Theorems 5.7, 5.8, and 5.9.

Computation of $G_2(\mathfrak{z}_1, \mathfrak{z}_2)$

In this subsection we explain how to compute $G_2(\mathfrak{z}_1, \mathfrak{z}_2)$ for two CM points $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{H}$ lying in the same quadratic imaginary field $\mathbb{Q}(\sqrt{-D})$, and then give an example that is worked out in detail.

Step 1. Find the function $g \in M^!_{-2}(\mathrm{SL}_2(\mathbb{Z}))$ with Fourier expansion $g = q^{-1} + O(1)$. This function is

$$g = \frac{E_4 E_6}{\Delta}.$$

Step 2. Consider the vector space $v^+ = v^+(\mathfrak{z}_1, \mathfrak{z}_2)$. Compute the lattices $N = v^+ \cap M_2(\mathbb{Z})$, $M = N^\perp \cap M_2(\mathbb{Z})$ and the corresponding finite abelian groups N'/N, M'/M.

Step 3. Compute $T_{M_2(\mathbb{Z}),N} = (\vartheta_\nu)_{\nu \in N'/N}$, where ϑ_ν is the binary theta series

$$\vartheta_\nu(\tau) = \vartheta_{0,\nu}(\tau) = \sum_{m \in M' \cap (-\nu + M_2(\mathbb{Z}))} \mathbf{e}\big(-\tau \mathrm{q}(m)\big).$$

Step 4. Compute $f \in M^!_1(\mathrm{SL}_2(\mathbb{Z}), \rho_N)$ given by

$$f_\nu = [g, \vartheta_\nu],\ \nu \in N'/N.$$

Step 5. The vectors

$$l_1 = \Re \begin{pmatrix} \mathfrak{z}_1\mathfrak{z}_2 & \mathfrak{z}_1 \\ \mathfrak{z}_2 & 1 \end{pmatrix} \qquad l_2 = \frac{1}{\sqrt{D}} \Im \begin{pmatrix} \mathfrak{z}_1\mathfrak{z}_2 & \mathfrak{z}_1 \\ \mathfrak{z}_2 & 1 \end{pmatrix}$$

satisfy

$$N \otimes \mathbb{Q} = l_1\mathbb{Q} + l_2\mathbb{Q},$$

$$\mathrm{q}(l_1) = D\,\Im(\mathfrak{z}_1)\,\Im(\mathfrak{z}_2), \quad \mathrm{q}(l_2) = \Im(\mathfrak{z}_1)\,\Im(\mathfrak{z}_2), \quad (l_1, l_2) = 0.$$

For simplicity assume that there exist a $\mathbb{Q}$-basis m_1, m_2 of $N \otimes \mathbb{Q}$ such that

$$\mathrm{q}(m_1) = D, \quad \mathrm{q}(m_2) = 1, \quad (m_1, m_2) = 0.$$

This assumption holds in the examples we consider below. Define $K = m\mathbb{Z}$ where $\mathrm{q}(m) = -1$ and set

$$P := N \oplus K.$$

It follows from the assumption that the lattice P contains a norm 0 primitive vector. Moreover, there is an isomorphism between rational quadratic spaces $(P \otimes \mathbb{Q}, q)$ and $(S_2(\mathbb{Q}), -D\det(\cdot))$. For example, the isomorphism given by

$$x m_1 + y m_2 + z m \quad \to \quad \begin{pmatrix} \dfrac{z+y}{D} & x \\ x & z-y \end{pmatrix}.$$

The group $\mathrm{SL}_2(\mathbb{Q})$ acts on $P \otimes \mathbb{Q}$ by $x \to \gamma x \gamma^t$ and preserves the norm $\mathrm{q}(\cdot)$. Finally, we find a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ that fixes all elements of P'/P.

Step 6. Next we must find $h \in M^!_{1/2}(\mathrm{SL}_2(\mathbb{Z}), \rho_P)$ with constant term 0 satisfying

$$T_{P,N}(h) = f, \tag{5.62}$$

where $T_{P,N}$ is defined as in Theorem 5.2. We do this as follows.

Note that $P'/P \cong K'/K \times N'/N$ and $\rho_P = \rho_K \otimes \rho_N$. For K as above we have $K'/K \cong \mathbb{Z}/2\mathbb{Z}$.

Denote

$$\theta_0(\tau, z) = \sum_{n \in \mathbb{Z}} \mathbf{e}(n^2\tau + 2nz), \quad \theta_1(\tau, z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} \mathbf{e}(n^2\tau + 2nz)$$

and

$$\theta_\kappa(\tau) = \theta_\kappa(\tau, 0), \quad \kappa = 0, 1.$$

Recall that by the definition of $T_{P,N}$

$$(T_{P,N}(h))_\nu = \sum_{\kappa \in K'/K} h_{(\kappa,\nu)}\theta_\kappa.$$

Let $\tilde{\phi}_{-2,1},\ \tilde{\phi}_{0,1}$ be the weak Jacobi forms defined in the book [25] p.108.

We can write

$$\begin{aligned}
\tilde{\phi}_{-2,1}(\tau, z) &= \psi_0(\tau)\,\theta_0(\tau, z) + \psi_1(\tau)\,\theta_1(\tau, z),\\
\tilde{\phi}_{0,1}(\tau, z) &= \varphi_0(\tau)\,\theta_0(\tau, z) + \varphi_1(\tau)\,\theta_1(\tau, z)
\end{aligned}$$

where

$$\begin{aligned}
\psi_0 &= -2 - 12q - 56q^2 - 208q^3 + \cdots,\\
\psi_1 &= q^{-1/4} + 8q^{3/4} + 39q^{7/4} + 152q^{11/4} + \cdots\\
\\
\varphi_0 &= 10 + 108q + 808q^2 + 4016q^3 + \cdots,\\
\varphi_1 &= q^{-1/4} - 64q^{3/4} - 513q^{7/4} - 2752q^{11/4} + \cdots.
\end{aligned} \tag{5.63}$$

The vector valued functions (ψ_0, ψ_1) and (φ_0, φ_1) belong to $M^!_{-5/2}(\mathrm{SL}_2(\mathbb{Z}), \rho_K)$ and $M^!_{-1/2}(\mathrm{SL}_2(\mathbb{Z}), \rho_K)$ respectively, and one has

$$\begin{aligned}
\tilde{\phi}_{-2,1}(\tau, 0) &= \psi_0(\tau)\vartheta_0(\tau) + \psi_1(\tau)\vartheta_1(\tau) = 0,\\
\tilde{\phi}_{0,1}(\tau, 0) &= \varphi_0(\tau)\vartheta_0(\tau) + \varphi_1(\tau)\vartheta_1(\tau) = 12.
\end{aligned} \tag{5.64}$$

Define the supplementary function $\tilde{f} \in M^!_3(\mathrm{SL}_2(\mathbb{Z}), \rho_N)$ as

$$\tilde{f}_\nu = [g, \vartheta_\nu]_2, \ \nu \in N'/N.$$

We have

$$f_0 = q^{-1} + 0 + O(q), \text{ and } \tilde{f}_0 = q^{-1} + 0 + O(q). \tag{5.65}$$

Consider the function $h \in M^!_{1/2}(\mathrm{SL}_2(\mathbb{Z}), \rho_P)$ defined by

$$h_{(\kappa,\nu)} := \frac{3}{4}\psi_\kappa \tilde{f}_\nu + \frac{1}{9}\varphi_\kappa f_\nu,\ (\kappa,\nu) \in K'/K \times N'/N. \tag{5.66}$$

It follows from (5.63) and (5.65) that h has the constant term 0. The equations (5.64) imply that h satisfies (5.62).

Step 7. Compute the infinite product corresponding to h defined in Theorem B3(see Section 1.9). Note that the negative Fourier coefficients of h might be not integral. Denote by n the common denominator of all negative Fourier coefficients of h. The function $nh(\tau)$ satisfies the conditions of Theorem B3 and only the function $\Psi_P(nh)$ is well defined.

Using results from [10], which are repeated in Section 1.9, we can find the level of $\Psi_P(nh)$, its zeros and poles, and its Fourier expansion at cusps. Knowing this information we can compute the value of $\Psi_P(nh)$ at the CM-point $\mathrm{Gr}^+(N)$.

Step 8. In the final step we compute $G_2(\mathfrak{z}_1, \mathfrak{z}_2)$. Theorems 5.3 and 5.4 tell us that

$$G_2(\mathfrak{z}_1, \mathfrak{z}_2) = (f, \Theta_N)_{\mathrm{reg}}.$$

Theorem 5.5 implies

$$(f, \Theta_N)_{\mathrm{reg}} = \Phi_P(\mathrm{Gr}^+(N), h).$$

Since the constant term of h is zero, from Theorem B3 we know that

$$\Phi_P(\mathrm{Gr}^+(N), h) = -\frac{4}{n}\log(\Psi_P(\mathrm{Gr}^+(N), nh)).$$

Computation of $G_2\left(\frac{1+\sqrt{-23}}{4}, \frac{-1+\sqrt{-23}}{4}\right)$

Now we apply the algorithm described above to the pair of CM points

$$\mathfrak{z}_1 = \frac{1+\sqrt{-23}}{4},\ \mathfrak{z}_2 = \frac{-1+\sqrt{-23}}{4}.$$

Step 1. Recall that

$$g = \frac{E_4E_6}{\Delta} = q^{-1} - 240 - 141444q - 8529280q^2 - 238758390q^3 + \cdots$$

is the unique function in $M^!_{-2}$ with the Fourier expansion $q^{-1} + O(1)$.
Step 2. The lattice $N = v^+(\mathfrak{z}_1, \mathfrak{z}_2) \cap M_2(\mathbb{Z})$ is equal to

$$N = n_1\mathbb{Z} + n_2\mathbb{Z}$$

where

$$n_1 = \begin{pmatrix} 3 & -11 \\ -12 & -2 \end{pmatrix},\ n_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The dual lattice is given by

$$N' = \frac{1}{23}n_1\mathbb{Z} + n_2\mathbb{Z}$$

and the Gram matrix of N is

$$\begin{pmatrix} 276 & -23 \\ -23 & 2 \end{pmatrix}.$$

Step 3. We compute $T_{M_2(\mathbb{Z}),N} = (\vartheta_\nu)_{\nu \in N'/N}$ defined by

$$\vartheta_\nu(\tau) = \sum_{m \in M' \cap (-\nu + M_2(\mathbb{Z}))} \mathbf{e}\big(-\tau \mathrm{q}(m)\big).$$

The lattice $M = N^\perp \cap M_2(\mathbb{Z})$ is equal to

$$M = m_1\mathbb{Z} + m_2\mathbb{Z}$$

where

$$m_1 = \begin{pmatrix} -11 & 2 \\ -2 & -8 \end{pmatrix}, \quad m_2 = \begin{pmatrix} -7 & 1 \\ -1 & -5 \end{pmatrix}.$$

The dual lattice is given by

$$M' = \frac{1}{23}m_1\mathbb{Z} + m_2\mathbb{Z}$$

and the Gram matrix of M is

$$\begin{pmatrix} -184 & -115 \\ -115 & -72 \end{pmatrix}.$$

We identify $\mathbb{Z}/23\mathbb{Z}$ with N'/N by sending r to $\frac{r}{23}n_1$. Then

$$\vartheta_\nu(\tau) = \sum_{x \in 6r + 23\mathbb{Z},\, y \in \mathbb{Z}} \mathbf{e}\big(\tau(\frac{4}{23}x^2 + 5xy + 36y^2)\big).$$

Step 4. We compute $f \in M_1^!(\mathrm{SL}_2(\mathbb{Z}), \rho_N)$ given by

$$f_\nu = [g, \vartheta_\nu], \ \nu \in N'/N.$$

The Fourier expansion of $23f$ is given below.

$\nu \in N'/N$	$23\, f_\nu$	
0	$-23x^{-23}$	$-3253074x^{23} + \cdots$
± 1	$-11x^{-17}$	$-2845x^{6} - 4964298x^{29} + \cdots$
± 2		$75x - 34489x^{24} + \cdots$
± 3	$-7x^{-15}$	$-3801x^{8} - 5530941x^{31} + \cdots$
± 4	$-15x^{-19}$	$-1889x^{4} - 4397647x^{27} + \cdots$
± 5	$2x^{-11}$	$-11520x^{12} - 13295643x^{35} + \cdots$
± 6	$-5x^{-14}$	$-4238x^{9} - 5829924x^{32} + \cdots$
± 7	$26x^{-5}$	$-17221x^{18} - 16709862x^{41} + \cdots$
± 8	$18x^{-7}$	$-15305x^{16} - 15577459x^{39} + \cdots$
± 9	$-17x^{-20}$	$-1411x^{3} - 4114356x^{26} + \cdots$
± 10	$-19x^{-21}$	$-960x^{2} - 3818769x^{25} + \cdots$
± 11	$3x^{-10}$	$-6093x^{13} - 6982596x^{36} + \cdots$

Here $x = \mathbf{e}(\tau/23)$.

Step 5. Consider the lattice

$$L = \left\{ \begin{pmatrix} a/23 & b \\ b & c \end{pmatrix} \Big| a, b, c \in \mathbb{Z} \right\}$$

equipped with the quadratic form $\mathrm{q}(l) := -23\det(l)$. Choose the vector

$$l_1 = \begin{pmatrix} 12/23 & 1 \\ 1 & 2 \end{pmatrix}.$$

The vector l_1 has the norm $\mathrm{q}(l_1) = -1$ and its orthogonal complement $L \cap l_1^{\perp}$ is isomorphic to N. Moreover, L splits into a direct sum $L \cong p_1\mathbb{Z} \oplus N$. Denote

$$l_2 = \begin{pmatrix} 6 & 12 \\ 12 & 23 \end{pmatrix}, \quad l_3 = \begin{pmatrix} 11/23 & -1 \\ -1 & -1 \end{pmatrix}.$$

The lattice L is equal to $l_1\mathbb{Z} + l_2\mathbb{Z} + l_3\mathbb{Z}$ and the dual lattice L' is equal to $\frac{1}{2}l_1\mathbb{Z} + \frac{1}{23}l_2\mathbb{Z} + l_3\mathbb{Z}$. The group $\Gamma_0^*(23)$ acts on L' by $x \to \gamma x \gamma^t$ and fixes all elements of L'/L.

Step 6. We compute the function $h \in M^!_{1/2}(\mathrm{SL}_2(\mathbb{Z}), \rho_L)$ defined by (5.66). The Fourier expansion of $529h(\tau)$ is given in the following table.

$\mu \in L'/L$	$529h_\mu(\tau)$
$(0,0)$	$4232x^{-92} + O(x)$
$(0,1)$	$-5290x^{-115} - 4232x^{-23} + O(x)$
$(\pm 1,1)$	$-2854x^{-91} + O(x)$
$(\pm 1,0)$	$2672x^{-68} + O(x)$
$(\pm 2,1)$	$1698x^{-19} + O(x)$
$(\pm 2,0)$	$O(x)$
$(\pm 3,1)$	$-2186x^{-83} + O(x)$
$(\pm 3,0)$	$2440x^{-60} + O(x)$
$(\pm 4,1)$	$-3594x^{-99} - 12943x^{-7} + O(x)$
$(\pm 4,0)$	$3048x^{-76} + O(x)$
$(\pm 5,1)$	$-2132x^{-67} + O(x)$
$(\pm 5,0)$	$4816x^{-44} + O(x)$
$(\pm 6,1)$	$-1879x^{-79} + O(x)$
$(\pm 6,0)$	$2378x^{-56} + O(x)$
$(\pm 7,1)$	$148x^{-43} + O(x)$
$(\pm 7,0)$	$6880x^{-20} + O(x)$
$(\pm 8,1)$	$-468x^{-51} + O(x)$
$(\pm 8,0)$	$5904x^{-28} + O(x)$
$(\pm 9,1)$	$-3991x^{-103} - 16870x^{-11} + O(x)$
$(\pm 9,0)$	$3290x^{-80} + O(x)$
$(\pm 10,1)$	$-4406x^{-107} - 17224x^{-15} + O(x)$
$(\pm 10,0)$	$3568x^{-84} + O(x)$
$(\pm 11,1)$	$-831x^{-63} + O(x)$
$(\pm 11,0)$	$2490x^{-40} + O(x)$

with x as before .
Step 7. There is a map $\mathfrak{H} \to \mathrm{Gr}^+(L)$ given by

$$z \quad \to \quad v^+(z) := \Re \begin{pmatrix} z^2 & z \\ z & 1 \end{pmatrix} \mathbb{R} + \Im \begin{pmatrix} z^2 & z \\ z & 1 \end{pmatrix} \mathbb{R} \subset L \otimes \mathbb{R}. \tag{5.67}$$

For the theta integral of the vector valued function $h \in M^!_{1/2}(\mathrm{SL}_2(\mathbb{Z}), \rho_L)$ we write

$$\Phi_L(z,h) := \Phi_L(v^+(z),h),$$

and for the corresponding meromorphic function(infinite product)

$$\Psi_L(z,h) := \Psi_L(v^+(z),h). \tag{5.68}$$

Note that

$$\mathrm{Gr}^{+}(N) = v^{+}\Big(\frac{23+\sqrt{-23}}{46}\Big). \tag{5.69}$$

All the elements of L'/L are fixed by the group $\Gamma_0(23)$. Hence, the theta integral $\Phi_L(z,h)$ is invariant under $\Gamma_0(23)$. The infinite product $\Psi_L(z,h)$ is an automorphic modular function for $\Gamma_0(23)$ with some unitary character χ. This character has finite order (see [11], Theorem 4.1).

The curve $\overline{\Gamma_0^*(23)\backslash\mathfrak{H}}$ has genus 0 and only one cusp. Let $j_{23}^*(z)$ be the Hauptmodul for $\Gamma_0^*(23)$ having the Fourier expansion $j_{23}^*(z) = q^{-1} + O(q)$, where $q = \mathbf{e}(z)$. This function is given explicitly by

$$\begin{aligned} j_{23}^*(z) &= \frac{1}{\eta(z)\eta(23z)} \sum_{m,n\in\mathbb{Z}} \mathbf{e}((m^2+mn+6n^2)z) - 3 \\ &= q^{-1} + 4q + 7q^2 + 13q^3 + 19q^4 + 33q^5 + 47q^6 + 74q^7 + \cdots . \end{aligned}$$

For any integer $d > 0$ such that $-d$ is congruent to a square modulo 92, choose an integer $\beta(\bmod 46)$ with $-d \equiv \beta^2(\bmod 92)$ and consider the set

$$L_{d,\beta} = \left\{ l = \begin{pmatrix} a/23 & b/46 \\ b/46 & c \end{pmatrix} \in L' \,\middle|\, \mathrm{q}(l) = -d/92, b \equiv \beta(\bmod 46) \right\}$$

on which $\Gamma_0(23)$ acts. The Fricke involution acts on L' by

$$l \to \frac{1}{23}\begin{pmatrix} 0 & 1 \\ -23 & 0 \end{pmatrix} l \begin{pmatrix} 0 & -23 \\ 1 & 0 \end{pmatrix}$$

and maps $L_{d,\beta}$ to $L_{d,-\beta}$

For $l \in L'$ with $\mathrm{q}(l) < 0$ denote by $\mathfrak{z}_l$ the point in $\mathfrak{H}$ corresponding the positive definite subspace $l^\perp$ via (5.67). The following equation holds

$$23a\mathfrak{z}_l^2 + b\mathfrak{z}_l + c = 0 \text{ for } l = \begin{pmatrix} c/23 & -b/46 \\ -b/46 & a \end{pmatrix}.$$

We define a polynomial $\mathcal{H}_{d,23}(X)$ by

$$\mathcal{H}_{d,23}(X) = \prod_{l\in L_{d,\beta}} (X - j_{23}^*(\mathfrak{z}_l))^{1/|\mathrm{Stab}(l)|}.$$

It follows from Theorem B3 part 2 that

$$\Psi(z,h) = \lambda(h) \prod_{d\ll\infty} \mathcal{H}_{d,23}(j_{23}^*(z))^{B(d)},$$

where

$$\lambda(h) = 2^{16}$$

and the numbers $B(d)$ can be found from the Fourier expansion of h given in the table on page 122. For example we find

$$B(7) = -12943, \quad B(11) = -16870, \quad B(15) = -17224, \quad B(19) = 1698.$$

The full list of the numbers $B(d)$ is given in the table on page 126.

Step 8. The last step is to compute the value $\Psi_L\Big(\mathrm{Gr}^+(N), h\Big)$. The equation (5.69) implies

$$\Psi_L\Big(\mathrm{Gr}^+(N), h\Big) = \Psi_L\Big(\frac{23+\sqrt{-23}}{46}, h\Big)$$

Consider the following algebraic numbers. Let ϱ be the real root of the polynomial $X^3 - X - 1$, and let π_q, $(q = 5, 7, 11, 17, 19, 25, 49)$ be the numbers of norm q in H given in (0.2). The value of the Hauptmodul j^*_{23} at the point $\frac{23+\sqrt{-23}}{46}$ is equal to $-\varrho - 2$. The values of $\mathcal{H}_{d,23}\Big(j^*\big(\frac{23+\sqrt{-23}}{46}\big)\Big)$ are given in the following table.

d	$529\,B(d)$	$\mathcal{H}_{d,23}(X)$	$\mathcal{H}_{d,23}(-2-\varrho)$
7	-12943	$(X+2)^2$	ϱ^2
11	-16870	$(X+1)^2$	ϱ^6
15	-17224	$(X^2+3X+3)^2$	ϱ^{10}
19	1698	$(X+3)^2$	ϱ^{-8}
20	6880	$(X^2+4X+5)^2$	$\pi_5^2\,\varrho^{10}$
23	-4232	$X^3+6X^2+11X+7$	0
28	5904	$X^2(X+2)^2$	$\pi_7^2\,\varrho^2$
40	2490	$(X^2+2X+3)^2$	$\pi_{25}^2\,\varrho^6$
43	148	$(X-1)^2$	$\pi_5^4\,\varrho^{16}$
44	4816	$(X+1)^2(X^3+7X^2+17X+13)^2$	$\pi_{11}^2\,\varrho^{10}$
51	-468	$(X^2+4X+7)^2$	$\pi_7^4\,\varrho^{-6}$
56	2378	$(X^4+4X^3-16X-17)^2$	$\pi_{49}^2\,\varrho^{12}$
60	2440	$(X^2+3X+3)^2(X^2+7X+13)^2$	π_{25}^2
63	-831	$(X+2)^2(X^4+5X^3+12X^2+20X+19)^2$	$\pi_{25}^4\,\varrho^8$
67	-2132	$(X-3)^2$	$\pi_{11}^4\,\varrho^6$
68	2672	$(X^4+10X^3+34X^2+46X+25)^2$	$\pi_{17}^2\,\varrho^{-6}$
76	3048	$(X+3)^2(X^3-X^2-9X-9)^2$	$\pi_{19}^2\,\varrho^4$

d	$529\,B(d)$	$\mathcal{H}_{d,23}(X)$	$\mathcal{H}_{d,23}(-2-\varrho)$
79	-1879	$(X^5+10X^4+43X^3+90X^2+90X+27)^2$	$\pi_{49}^4\,\varrho^{16}$
80	3290	$(X^2+4X+5)^2(X^4+6X^3+20X^2+30X+17)^2$	$\pi_5^2\,\pi_{25}^2\,\varrho^{28}$
83	-2186	$(X^3-X^2-13X-19)^2$	$\pi_{25}^4\,\varrho^6$
84	3568	$(X^4+2X^3+6X^2+14X+13)^2$	$\pi_{49}^2\,\varrho^{26}$
91	-2854	$(X^2-4X-9)^2$	$\pi_{17}^4\,\varrho^{-6}$
92	4232	$(X^3-2X^2-17X-25)\times\mathcal{H}_{23,23}(X)$	$-2^2\,\varpi_{23}\,\varrho^5\ \times 0$
99	-3594	$(X+1)^2(X^2+8X+19)^2$	$\pi_{19}^4\,\varrho^{-8}$
103	-3991	$(X^5+4X^4+7X^3+33X^2+99X+81)^2$	$\pi_5^4\,\pi_{25}^4\,\varrho^{18}$
107	-4406	$(X^3+5X^2+19X+31)^2$	$\pi_{49}^4\,\varrho^8$
115	-5290	$(X+5)^2$	ϖ_{23}^2

Finally we arrive at

$$\frac{23}{12}\log\Big|\Psi_L\Big(\frac{23+\sqrt{-23}}{46},h\Big)\Big| = \log\big|\pi_5^{18}\,\pi_{25}^{-42}\,\pi_7^{36}\,\pi_{49}^{-48}\,\pi_{11}^4\,\pi_{17}^{-22}\,\pi_{19}^{-30}\,\varpi_{23}^{-23}\,\varrho^{-9\cdot 23}\big|.$$

This proves the result (0.2) for $k=2$ obtained numerically from the Fourier expansion (5.18). The same argument works for $k=3,4,5,7$.

Numerical verification of Theorem 5.9

In this subsection we check the factorization formula (5.2).

The ideal class group of the field $K=\mathbb{Q}(\sqrt{-23})$ consists of three elements

$$\mathrm{CL}_K=\{\mathfrak{o},\mathfrak{b},\mathfrak{b}^{-1}\}.$$

Each rational prime p that is inert in K splits in the Hilbert class field H as

$$(p)=\mathfrak{P}_1\,\mathfrak{P}_2\,\overline{\mathfrak{P}}_2$$

for some prime ideals $\mathfrak{P}_1$, $\mathfrak{P}_2$ with $\mathfrak{P}_1=\overline{\mathfrak{P}}_1$. Theorem 5.7 implies that for $k=1,\dots,5$ and 7

$$G_k\Big(\frac{1+\sqrt{-23}}{4},\frac{-1+\sqrt{-23}}{4}\Big)=23^{1-k}\,\log|\alpha_k|,$$

where α_k is an algebraic number. Conjecture 2 proved in Section 5.12 predicts that $\alpha_k\in H$ and gives the factorization of α_k. Specifically it says that no prime factor of l in H occurs in α_k if l is split in K, while if l is inert in K we have

$$\mathrm{ord}_{\mathfrak{P}_1}(\alpha_k)=23^{k-1}\sum_{n=0}^{23}P_{k-1}\Big(1-\frac{2n}{23}\Big)\,r_{\mathfrak{b}}(23-n)\,r_{\mathfrak{o}}\Big(\frac{n}{p}\Big)\,(1+\mathrm{ord}_p(n)),\tag{5.70}$$

$$\mathrm{ord}_{\mathfrak{P}_2}(\alpha_k)=\mathrm{ord}_{\overline{\mathfrak{P}}_2}(\alpha_k)=23^{k-1}\sum_{n=0}^{23}P_{k-1}\Big(1-\frac{2n}{23}\Big)\,r_{\mathfrak{b}}(23-n)\,r_{\mathfrak{b}}\Big(\frac{n}{p}\Big)\,(1+\mathrm{ord}_p(n)).\tag{5.71}$$

We verify these identities in the following table.

p	5				7		
$23-n$	18	13	8	3	16	9	2
n	5	10	15	20	7	14	21
$r_{\mathfrak{b}}(23-n)$	2	1	1	1	2	1	1
$r_{\mathfrak{b}}\left(\frac{n}{p}\right)$	0	1	1	1	0	1	1
$r_{\mathfrak{o}}\left(\frac{n}{p}\right)$	1	0	0	1	1	0	0
$23P_1\left(1-\frac{2n}{23}\right)$	13	3	-7	-17	9	-5	-19
$23^2P_2\left(1-\frac{2n}{23}\right)$	-11	-251	-191	169	-143	-227	277
$23^3P_3\left(1-\frac{2n}{23}\right)$	-4823	-2313	4697	1207	-5319	3655	-2071
$23^4P_4\left(1-\frac{2n}{23}\right)$	-105359	87441	18241	-102959	-27039	58081	-41039

p	11		17	19	23	
$23-n$	12	1	6	4	23	0
n	11	22	17	19	0	23
$r_{\mathfrak{b}}(23-n)$	2	0	1	1	0	1/2
$r_{\mathfrak{b}}\left(\frac{n}{p}\right)$	0	1	0	0	1/2	0
$r_{\mathfrak{o}}\left(\frac{n}{p}\right)$	1	0	1	1	1/2	1
$23P_1\left(1-\frac{2n}{23}\right)$	1	-21	-11	-15	23	-23
$23^2P_2\left(1-\frac{2n}{23}\right)$	-263	397	-83	73	23^2	23^2
$23^3P_3\left(1-\frac{2n}{23}\right)$	-791	-6489	5401	3465	23^3	-23^3
$23^4P_4\left(1-\frac{2n}{23}\right)$	102961	80961	-71039	-119919	23^4	23^4

For example, for $p = 7$ and $k = 2$ we have

$$\mathfrak{P}_1 = (\pi_7), \quad \mathfrak{P}_2\overline{\mathfrak{P}}_2 = (\pi_{49}),$$

where p_7 and p_{49} are defined in (0.1). We find

$$\begin{aligned}\mathrm{ord}_{\mathfrak{P}_1}(\alpha_2) =&23\sum_{n=0}^{23} P_1\Big(1-\frac{2n}{23}\Big)\,\delta(n)\,r_{\mathfrak{b}}(23-n)\,r_{\mathfrak{o}}\Big(\frac{n}{p}\Big)\,\big(1+\mathrm{ord}_p(n)\big)\\ =&23\bigg(P_1\Big(\frac{19}{23}\Big)\,r_{\mathfrak{b}}(2)\,r_{\mathfrak{o}}(3)+P_1\Big(\frac{5}{23}\Big)\,r_{\mathfrak{b}}(9)\,r_{\mathfrak{o}}(2)+P_1\Big(\frac{-9}{23}\Big)\,r_{\mathfrak{b}}(16)\,r_{\mathfrak{o}}(1)\bigg)\\ =&36,\end{aligned}$$

$$\begin{aligned}\mathrm{ord}_{\mathfrak{P}_2}(\alpha_2)=\mathrm{ord}_{\overline{\mathfrak{P}}_2}(\alpha_2) =&23\sum_{n=0}^{23} P_1\Big(1-\frac{2n}{23}\Big)\,\delta(n)\,r_{\mathfrak{b}}(n)\,r_{\mathfrak{b}}\Big(\frac{23-n}{p}\Big)\,\big(1+\mathrm{ord}_p(n)\big)\\ =&23\bigg(P_1\Big(\frac{19}{23}\Big)\,r_{\mathfrak{b}}(2)\,r_{\mathfrak{b}}(3)+P_1\Big(\frac{5}{23}\Big)\,r_{\mathfrak{b}}(9)\,r_{\mathfrak{b}}(2)+P_1\Big(\frac{-9}{23}\Big)\,r_{\mathfrak{b}}(16)\,r_{\mathfrak{b}}(1)\bigg)\\ =&-48.\end{aligned}$$

This agrees with formula (0.2) found by numerical computations.

Numerical verification of Theorem 5.8

In this subsection we illustrate Theorem 5.8 with several examples coming from the computation of CM values of higher Green's functions. As before, let N and M be the lattices

$$N=v^+(\mathfrak{z}_1,\mathfrak{z}_2)\cap M_2(\mathbb{Z})\quad\text{and}\quad M=v^+(\mathfrak{z}_1,\mathfrak{z}_2)\cap M_2(\mathbb{Z}),$$

where $\mathfrak{z}_1=\frac{1+\sqrt{-23}}{4}$, $\mathfrak{z}_2=\frac{-1+\sqrt{-23}}{4}$. For $k=2,3,4,5,7$ let g_k be the unique element of $M^!_{2-2k}$ with the Fourier expansion $g_k=q^{-1}+O(1)$. Denote

$$f_k=23^{k-1}[g_k,\Theta_{M(-1)}]_{k-1}.$$

Theorems 5.3 and 5.4 imply that

$$\log(\alpha_k)=23^{k-1}G_k\left(\frac{1+\sqrt{-23}}{4},\frac{-1+\sqrt{-23}}{4}\right)=(f_k,\Theta_N)_{\mathrm{reg}}.$$

Functions f_k have Fourier expansions of the form

$$f_k=\sum_{\nu\in N'/N}e_\nu\sum_{t\in\mathbb{Z}}c_\nu^{(k)}(t)\,\mathbf{e}\Big(\frac{t}{23}\tau\Big).$$

From Theorem 5.8 we deduce

$$\mathrm{ord}_{\mathfrak{P}_1}(\alpha_k)=\sum_{n=0}^{\infty}\sum_{\nu\in N'/N}c_\nu^{(k)}(-n)\,r_{\mathfrak{o}}\Big(\frac{n}{p}\Big)\,\big(1+\mathrm{ord}_p(n)\big),$$

$$\mathrm{ord}_{\mathfrak{P}_2}(\alpha_k)=\mathrm{ord}_{\overline{\mathfrak{P}}_2}(\alpha_k)=\sum_{n=0}^{\infty}\sum_{\nu\in N'/N}c_\nu^{(k)}(-n)\,r_{\mathfrak{b}}\Big(\frac{n}{p}\Big)\,\big(1+\mathrm{ord}_p(n)\big).$$

For the primes $p=5,7,11,17,19$ we get the following table.

p	n	ν	$r_{\mathfrak{b}}\big(\frac{n}{p}\big)$	$r_{\mathfrak{o}}\big(\frac{n}{p}\big)$	$2c_\nu^{(2)}\big(-n\big)$	$2c_\nu^{(3)}\big(-n\big)$	$2c_\nu^{(4)}\big(-n\big)$
5	5	± 7	0	1	-52	44	19292
	10	± 11	1	0	-6	-502	4626
	15	± 3	1	0	14	-382	-382
	20	± 9	1	1	34	338	-2414
7	7	± 8	0	1	-36	-572	21276
	14	± 6	1	0	10	-454	-7310
	21	± 10	1	0	38	554	4142
11	11	± 5	0	1	-4	-1052	3164
17	17	± 1	0	1	22	-166	-10802
19	19	± 4	0	1	30	146	-6930

The prime ideals of H lying above $p = 5$ satisfy

$$\mathfrak{P}_1 = (p_5), \quad \mathfrak{P}_2 \overline{\mathfrak{P}}_2 = (p_{25}).$$

We compute

$$\begin{aligned} \mathrm{ord}_{\mathfrak{P}_1}(\alpha_2) =& 2\big(c^{(2)}(-5)\, r_{\mathfrak{o}}(1) + c^{(2)}(-10)\, r_{\mathfrak{o}}(2) + c^{(2)}(-15)\, r_{\mathfrak{o}}(3) + c^{(2)}(-20)\, r_{\mathfrak{o}}(4)\big) \\ =& -42, \end{aligned}$$

$$\begin{aligned} \mathrm{ord}_{\mathfrak{P}_2}(\alpha_2) &= \mathrm{ord}_{\overline{\mathfrak{P}}_2}(\alpha_2) \\ &= 2\big(c^{(2)}(-5)\, r_{\mathfrak{b}}(1) + c^{(2)}(-10)\, r_{\mathfrak{b}}(2) + c^{(2)}(-15)\, r_{\mathfrak{b}}(3) + c^{(2)}(-20)\, r_{\mathfrak{b}}(4)\big) \\ &= 18. \end{aligned}$$

For $p = 7$ we have

$$\mathfrak{P}_1 = (p_7), \quad \mathfrak{P}_2 \overline{\mathfrak{P}}_2 = (p_{49}).$$

Thus, we arrive at

$$\begin{aligned} \mathrm{ord}_{\mathfrak{P}_1}(\alpha_2) =& 2\big(c^{(2)}(-7)\, r_{\mathfrak{o}}(1) + c^{(2)}(-14)\, r_{\mathfrak{o}}(2) + c^{(2)}(-21)\, r_{\mathfrak{o}}(3)\big) \\ =& 36, \\ \mathrm{ord}_{\mathfrak{P}_2}(\alpha_2) =& \mathrm{ord}_{\overline{\mathfrak{P}}_2}(\alpha_2) \\ =& 2\big(c^{(2)}(-7)\, r_{\mathfrak{b}}(1) + c^{(2)}(-14)\, r_{\mathfrak{b}}(2) + c^{(2)}(-21)\, r_{\mathfrak{b}}(3)\big) \\ =& -48. \end{aligned}$$

This agrees with the numerical computations (0.2).

Bibliography

[1] M. Abramowitz, I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* National Bureau of Standards Applied Mathematics Series, 55 For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1964 xiv+1046 pp.

[2] M. Alsina, Quaternion orders, quadratic forms, and Shimura curves. American Mathematical Soc., 2004.

[3] A. N. Andrianov, *Modular descent and the Saito-Kurokawa conjecture.* Inv. Math. 53 (1979) 267-280.

[4] T. Arakawa, *Jacobi Eisenstein series and a basis problem for Jacobi forms*, Comm. Math. Universitatis Sancti Pauli 43 (1994), No. 2, 181-216.

[5] T. Arakawa, B. Heim,*Real analytic Jacobi Eisenstein series and Dirichlet series attached to three Jacobi forms,* Max-Planck-Institut für Mathematik, Preprint series 1998 (66).

[6] H. Bateman, *Higher Transcendental Functions.* Mc Grawl-Hill Book Company, 1954.

[7] H. Bateman, *Tables of integral transforms.* Mc Grawl-Hill Book Company, 1955.

[8] B. J. Birch, Elliptic curves and modular functions, Symp. Math. Inst. Alta Math. 4, 27-32 (1970).

[9] R. E. Borcherds, *Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products*, Invent. Math. 120, 161-213 (1995).

[10] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. 132, 491-562 (1998).

[11] R. E. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J. 97, 219233 (1999).

[12] J. H. Bruinier, *Borcherds products on $O(2, l)$ and Chern classes of Heegner divisors* - 2000 - Springer LNM 1780.

[13] J. Bruinier, S. Kudla, and T. Yang, *Special values of Green functions at big CM points*, International Mathematics Research Notices, accepted for publication

[14] J. H. Bruinier, G. van der Geer, G. Harder and D. Zagier, *The 1-2-3 of modular forms*, Universitext, Springer-Verlag, Berlin, Heidelberg, 2008.

[15] J. H. Bruinier, T. Yang, *Faltings heights of CM cycles and derivatives of L-functions*, Invent. math. 177, 631-681 (2009).

[16] J.-L. Brylinski, *Heights for local system on curves*, Duke Math. J. 59, 1-26 (1989).

[17] F. Cléry, V. Gritsenko, *Modular forms of orthogonal type and Jacobi theta-series.* arXiv:1106.4733v1 [math.AG]

[18] H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters.* Math. Ann. 217, pp. 271-285, (1975).

[19] J. H. Conway, N. J. A. Sloan, *Sphere Packings, Lattices and Groups*, second edition, Grundlehren der mathematischen Wissenschaften 290, Springer-Verlag 1992.

[20] P. Deligne, *Travaux de Shimura.* Séminaire Bourbaki, 23éme année (1970/71), Exp. No. 389, pp. 123165. Lecture Notes in Math., Vol. 244, Springer, Berlin, 1971.

[21] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modéles canoniques*, in Automorphic forms, representations and L-functions, Proc. Sympos. Pure Math., XXXIII (Corvallis, OR, 1977), Part 2, pp. 247289, Amer. Math. Soc., Providence, R.I., 1979.

[22] K. Doi, H. Naganuma, *On the functional equation of certain Dirichlet series.* Invent. math. 9, pp. 1-14, (1969).

[23] W. Duke, Y. Li, *Mock-modular forms of weight one*, preprint, http://www.math.ucla.edu/ wdduke/preprints/mock weight one.pdf, (2012).

[24] N. Dummigan, *Symmetric square L-Functions and Shafarevich -Tate Groups.* Experimental Mathematics 10:3, pp. 383-400, (2001).

[25] M. Eichler, D. Zagier, *The theory of Jacobi forms*, Birkhäuser, 1985.

[26] P. Garrett, *Pullbacks of Eisenstein series; Applications.* Automorphic forms of several variables. Taniguchi symposium 1983, Birkhäuser, 1984.

[27] G. van der Geer, *On the geometry of Siegel modular threefold.* Math. Ann. 260, pp. 317-350, (1982).

[28] G. van der Geer, *Hilbert modular surfaces*, Springer Verlag 1987.

[29] S. Gelbart, *Examples of dual reductive pairs*, Proc. Symp. Pure Math. XXXIII, A.M.S.,

[30] H. Gillet, C. Soulé, *Arithmetic intersection theory,* Publ. Math. I. H. E. S. 72, pp. 94-174, (1990).

[31] V. A. Gritsenko, *Arithmetical lifting and its applications*, Séminaire de Théorie des Nombres, Paris 1992-93, Birkhauser, pp. 103-126.

[32] V. A. Gritsenko, *Modulformen zur Pammodulgruppe und Modulräume der Abdschen Varietäten*, Mathematica Gottingensis Schrift. des SFB Geometrie und Analysis, Heft 12 (1995), pp. 1-89.

[33] B. Gross and D. Zagier, *On singular moduli*, J. Reine Angew. Math. 355, pp. 191-220, (1985).

[34] B. Gross, *Local Heights on Curves*, Arithmetic Geometry, SpringerVerlag, New York, 1986

[35] B. Gross, D. Zagier, *Heegner points and derivatives of L-series.* Invent. math. 84, pp. 225-320, (1986).

[36] B. Gross, W. Kohnen, D.Zagier, *Heegner points and derivatives of L-series. II* Math. Ann. 278, pp. 497-562, (1987).

[37] V. Gritsenko, *Modulformen zur Paramodulgruppe und Modulräume der abelschen Varietäten*, Mathematica Goettingensis Schrittenreihe des Sonderforschungsbereichs Geometrie und Analysis, Heft 12 (1995).

[38] K. Heegner, *Diophantische Analysis und Modulfunktionen*, Math. Z., 56, pp. 227-253, (1952).

[39] B. Heim, *L-functions for Jacobi forms and the basis problem*, Manuscripta math. 106, pp. 489-503, (2001).

[40] B. Heim, *A trace formula of special values of automorphic L-functions*, Multiple Dirichlet series, L-functions and automorphic forms, 173191, Progr. Math., 300, Birkhäuser/Springer, New York, 2012.

[41] F. Hirzebruch, D. Zagier, *Intersection numbers of curves on Hibert modular surfaces and modular forms of Nebentypus*, - Invent. math., (1976).

[42] R. Howe, *θ-series and invariant theory*, in Borel, Armand; Casselman, W., Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Providence, R.I.: American Mathematical Society, pp. 275285, (1979).

[43] R. Howe and I. I. Piatetski-Shapiro,*Some examples of automorphic forms on* Sp_4, Duke Math. J. Volume 50, Number 1, pp. 55-106, (1983).

[44] A. Ichino, *Pullbacks of Saito-Kurokawa lifts*, Inv. Math. 162, 551-647 (2005)

[45] H. Katsurada, *An explicit formula for siegel series.* American Jornal of Mathematics 121, 415-452, (1999).

[46] W. Kohnen, *Modular forms of half-integral weight on* $\Gamma_0(4)$. Math. Ann. 248, pp. 249-266, (1980).

[47] W. Kohnen, *Nonholomorphic Poincare-type series on Jacobi groups.* J. Number Theory 46, no. 1, pp. 70-99, (1994).

[48] K. Doi, H. Naganuma, *On the functional equation of certain Dirichlet series.*

[49] V. A. Kolyvagin, *Euler systems*, The Grothendieck Festschrift. Prog. in Math., Boston, Birkhauser (1990).

[50] S. Kudla, *Seesaw dual reductive pairs.* Automorphic forms of several variables (Katata, 1983), pp. 244268, Progr. Math., 46, Birkha"user Boston, Boston, MA, 1984.

[51] S. Kudla, Algebraic cycles on Shimura varieties of orthogonal type. Duke Math. J. 86, no. 1, pp. 3978, (1997).

[52] S. Kudla, *Central derivatives of Eisenstein series and height pairing.* Annals of Mathematics 146, pp. 545-646, (1997).

[53] S. Kudla, M. Rapoport and T. Yang, Modular forms and special cycles on Shimura curves, Annals of Math. Studies series, vol. 161, Princeton Univ. Publ., 2006.

[54] N. Kurokawa, *Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two.* Invent. math. 49, pp. 149-165, (1978).

[55] R. P. Langlands, *On the functional equations satisfied by Eisenstein series.* Berlin: Springer-Verlag, 1976.

[56] H. Maass, *Siegel's modular forms and Dirichlet series.* Lecture Notes in Mathematics, Vol. 216. Springer-Verlag, Berlin-New York, 1971.

[57] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades.* Inv. Math. 52, pp. 95-104, (1979).

[58] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades, II.* Inv. Math. 53, pp. 249-253, (1979).

[59] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades, III.* Inv. Math. 53, pp. 255-265, (1979).

[60] A. Mellit, *Higher Greens functions for modular forms*, PhD thesis, 2008.

[61] H. Naganuma, *On the coincedence of two Dirichlet series associated with cusp forms of Hecke's "Neben"-type and Hildert modular forms over a real quadratic field*, J. Math. Soc. Japan 25, pp. 547-555 (1973).

[62] J. Schofer, *Borcherds forms and generalizations of singular moduli*, J. Reine Angew. Math. 629 (2009), 136.

[63] J.-P. Serre, A course in arithmetic. Translated from the French. Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg, 1973.

[64] G. Shimura, *Modular forms of half integral weight*, in Modular Functions of one Variable, I. Lecture Notes N 320, Springer-Verlag, Berlin/Heidelberg/New York (1973), pp. 57-74.

[65] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. 97 (1973), pp. 440-481.

[66] G. Shimura, *Confluent hypergeometric functions on tube domains,* Math. Ann. 260, pp. 269-302, (1982).

[67] N. Skoruppa, D. Zagier, *Jacobi forms and a certain space of modular forms,* Invent. Math. 94 (1988), pp. 113-146.

[68] N. P. Skoruppa, *Explicit formulas for the Fourier coefficients of Jacobi and elliptic modular forms.* Invent. Math. 102, no. 3, pp. 501-520, (1990).

[69] N. Skoruppa, *Jacobi forms of critical weight and Weil representations.* Modular forms on Schiermonnikoog, 239-266, Cambridge Univ. Press, Cambridge, 2008.

[70] C. L. Siegel, *Einfürung in die Theorie der Modulfunktionen n-en Grades.* Mathematische Annalen 116, pp. 617-656, (1939).

[71] M. F. Vignéras, Séries theta des formes quadratiques indéfinies. (French) Séminaire Delange-Pisot-Poitou, 17e année (1975/76), The'orie des nombres: Fasc. 1, Exp. No. 20, 3 pp. Secrétariat Math., Paris, 1977.

[72] M. F. Vignéras, *Arithmétique des algébres de quaternions.* (French) [Arithmetic of quaternion algebras] Lecture Notes in Mathematics, 800. Springer, Berlin, 1980.

[73] A. Weil, *Sur certains groupes d'operateurs unitares*, Acta Math. 111, pp. 143-211, (1964).

[74] D. Zagier, *Modular Forms Associated to Real Quadratic Fields.* Inventiones math. 30, pp. 1-46 (1975).

[75] D. Zagier, *Sur la conjecture de Saito-Kurokawa (d'apres H. Maass),* Seminare Delagne-Pisot-Poitou 1979-1980, Math. 12., Birkhäuser-Verlag, pp. 371-394, (1980).

[76] D. Zagier, *Modular points, modular curves, modular surfaces and modular forms.* LNM 1111, (1984), pp. 225-248.

[77] D. Zagier, *Introduction to modular forms*, in From Number Theory to Physics, eds. M. Waldschmidt et al, Springer-Verlag, Heidelberg (1992) 238-291

[78] D. Zagier, *Traces of singular moduli*, Max-Planck-Institut für Mathematik, Preprint series 2000 (8).

[79] D. Zagier, *Green's functions for quotients of the upper half-plane*, unpublished.

[80] S. -W. Zhang, *Heights of Heegner cycles and derivatives of L-series*. Invent. math. 130, (1997).

[81] X. Yuan, S.-W. Zhang, W. Zhang, *The Gross-Kohnen-Zagier theorem over totally real fields*. Compositio Math. 145, no. 5, pp. 1147-1162, (2009).