

The Impact of Bounded Rationality on Equity-Linked Life  
Insurance and Technical Trading

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# Introduction

This dissertation is concerned with the behavior of investors on financial and insurance markets. Following Shefrin (2008), one departs from the neoclassical view on decision making in financial markets, behavioral financial economists argue in favor of two major systematic errors investors make: How investors form *beliefs* and how investor *preferences* depart from rationality.

Investor's formation of beliefs departs from rational behavior because empirically, humans do not follow Bayes law when updating their beliefs (cf. Barberis and Thaler (2003)). Instead of forming beliefs according to Bayes law, investors use a representativeness heuristic (Kahneman and Tversky (1974)): The investor takes the available information as representative, anchors his belief around some (arbitrary) initial value, and takes even small samples of a distribution as representative sample which captures all relevant information. Rabin (2002) labels this small sample representativeness as *The law of small numbers*. This irrationality in forming beliefs as summarized by Barberis and Thaler (2003) has consequences for financial markets. For example, Barberis, Shleifer, and Vishny (1998) develop a theory of investor sentiment. This model of investor sentiment explains both short-term under-reaction to a particular piece of news and long-term over-reaction of stock prices to a series of new information.

Investor preferences depart from rationality because they violate the theory of expected utility. Expected utility theory describes how investors should make decisions under uncertainty. Empirically, investors are not rational and depart from the predictions of expected utility theory as laid out in Gilboa (2009). For example, the prospect theory of Kahneman and Tversky (1979) and the improved version of cumulative prospect theory in Tversky and Kahneman (1992) describes people's decision making as follows: People value risky choices against a reference value labeled reference point instead of drawing utility from the absolute value of outcome. The reference point splits outcomes into gains and losses. Investors are loss averse: The impact of losses on utility is larger than the impact of gains. Furthermore, people cannot judge probabilities objectively, but

distort probabilities. On the one hand, low probabilities are overweight. For example, an objective probability of 1% feels like a probability of 15%. On the other hand, investors cannot distinguish between intermediate probabilities: An investor can hardly tell the difference between a probability of 40% and 45%.

Closely related to the issue of investor preferences on risky choices is the question to which choices the preferences apply to. Empirically, people *frame* risky choices *narrowly*. Following Tversky and Kahneman (1981), investors depart from expected utility theory because they do not evaluate the risk they face combined by considering total wealth or consumption. Instead, investors isolate risks and ignore interdependencies with other risky choices they face.

This dissertation is concerned with the impact of investor behavior for life insurance and technical trading. In this dissertation, I focus on the second deviation of the behavioral finance approach to investor behavior and study the impact of investor preferences. The first part of this dissertation is concerned with equity-linked life insurance contracts while the second part of this dissertation is concerned with technical trading of stocks.

**Part I** of this dissertation is split into two chapters. In **Chapter One**, I study the effect of secondary markets on equity-linked life insurance contracts with surrender guarantees for policyholders with bounded rationality. Many equity-linked life insurance products offer the possibility to surrender policies prematurely. Secondary markets for policies with surrender guarantees influence both policyholders and insurers. I show that secondary markets lead to a gap in policy value between insurer and policyholder. Insurers increase premiums to adjust for higher surrender rates of customers and optimized surrender behavior by investors acquiring the policies on secondary markets. Hence, the existence of secondary markets is not necessarily profitable for the primary policyholders. The result depends on the demand for and the supply of the contracts brought to the secondary markets.

In **Chapter Two**, I study the effect of policyholder's risk preferences on equity-linked life insurance contracts with surrender guarantees. While the first chapter takes the reasons for bounded rationality as exogenous, in the second chapter I model the risk preferences of the policyholder explicitly. I value equity-linked life insurance contracts with surrender guarantee for boundedly rational policyholders with loss averse preferences as in Tversky and Kahneman (1991) Our policyholders' surrender behavior deviates from both the standard optimal stopping approach and expected utility implied surrender behavior in two ways: Firstly, the equity level for exercise changes as our policyholders surrender earlier if the underlying equity underperforms. Secondly, policyholders sur-

render to avoid the insurance becoming a loss, which reshapes the surrender area and creates surrender peaks if the surrender benefit reaches the policyholders' reference point.

**Part II** of this dissertation is concerned with technical trading of stocks. **Chapter Three** studies technical analysis from the perspective of Cumulative Prospect Theory. Technical analysts, or chartists, aim at predicting future prices from past prices. Sometimes they draw resistance levels and Moving Average (MA) lines into stock price charts. I show that the widely employed MA cross-over rule is consistent with prospect theory preferences even when prices do not move in trends and when stock trading is unattractive to all rational expected utility maximizers. While chartists often argue that market participants being less than fully rational explains why technical analysis is profitable, this chapter shows that technical analysis may be attractive - even when not profitable - to investors who are less than fully rational.

This thesis has benefited from numerous comments and suggestions of coauthors, readers, seminar and conference participants, journal referees, and editors. Furthermore, it has been proof-read by various people. Chapter One has been developed jointly with Jing Li and Alexander Szimayer and is based on Hilpert, Li, and Szimayer (2013). An earlier version of this model has been my masters thesis. Chapter Two is single-authored and based on Hilpert (2014). Chapter Three is joint work with Sebastian Ebert and based on Ebert and Hilpert (2013).

Each of the remaining three chapters is self-contained.



**Part I**

**Life Insurance**





# Chapter 1

## The Effect of Secondary Markets on Equity-Linked Life Insurance with Surrender Guarantees

### 1.1 Introduction

The world market for life insurance contracts is huge with a premium volume of 2,332 billion US dollars in 2009 (SwissRe (2010)). Roughly 50% of these contracts in most developed countries are terminated before the maturity date (Gatzert (2010), Bundesverband Vermögensanlagen im Zweitmarkt Lebensversicherung e.V. (BVZL (2010))). At the premature termination of contract, policyholders are offered predefined lump-sum payments by the primal insurers called surrender guarantees. Usually, these surrender guarantees are financially unattractive relative to the fair value of the policy. In recent years, secondary markets for insurance contracts have developed. They provide the policyholders with an alternative way to give up their contracts prematurely by selling the contracts to third parties on these markets for higher prices. In the US and the UK, which are the world's largest and third largest life insurance markets, respectively, these secondary markets have a long history and have been growing in the last few decades. In other countries, like Japan and Germany, secondary markets for life insurance contracts have been established recently and a substantial increase in the trading volume on these markets can be observed.<sup>1</sup>

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<sup>1</sup>For the UK secondary markets for life insurance products can be traced back to 1844, for the US to 1911. The US market has a volume of 2 million US dollar in 1990, 12 billion US dollar in 2008, and BVZL estimates a traded volume of 30 billion dollars for 2017. On the UK secondary market, 20,000 contracts with a price volume of 200 million GBP have been traded in 1996, which increased to 200,000 contracts with a price volume of 500 million GBP in 2003. The price volume of traded policies in Germany raised from €50 million in 2000 to €1.4 billion in 2007, while the total volume of terminated contracts increased

In this paper, we study the pricing of equity-linked life insurance contracts when a secondary market exists. In our model, the secondary market increases the contract value under the following two assumptions: Firstly, primary policyholders surrender contracts either for reasons exogenous to the conditions of the policy or because other investment opportunities seem more advantageous. Secondly, contract buyers on the secondary market are assumed to be financial experts being able to optimally exercise the surrender option entitled by the insurers. The first assumption is in line with the empirical study of Kuo, Tsai, and Chen (2003) who show in a cointegration analysis that both endogenous (interest rate) and exogenous (unemployment) reasons affect the lapse rate.<sup>2</sup> To model this behavior, we use the setup which can be found, e.g., in Albizzati and Geman (1994), Stanton (1995), De Giovanni (2010), and Li and Szimayer (2010). We call this approach the bounded monetary rationality setup. In this setting, the policyholders are not stopping the contingent claims optimally in monetary terms, but make monetarily suboptimal surrender decisions. These surrenders can still be rational, but due to reasons beyond the model. The second assumption implies that the contracts become the pure American-style type once sold on the secondary market. Their prices can be derived within the American-style contingent claim framework which is discussed, e.g., in Grosen and Jorgensen (1997, 2000), Bacinello (2003, 2005), and Bacinello, Biffis, and Millosovich (2010). We show that the added value created by the secondary market as a whole increases the surrender risk borne by the insurers, and hence, increases the contract value from the insurers' perspective. However, depending on the equilibrium out of the demand for and the supply of the contracts brought to the secondary market, policyholders may only receive part of the added value. The contract value from the policyholders' perspective is thus lower than the value for the insurers. Overall, we find that the policyholders may only profit partly from the secondary market. Although the introduction of the secondary market may increase the payout to the policyholders, it is not necessarily beneficial for them if the premium increases at the same time. We demonstrate that the secondary market is not profitable for any policyholder if the contract demand on the secondary market is very sensitive to the traded price of the contracts. Even when the contract demand is less sensitive to the traded price, only policyholders who are informed of its existence may profit from it, while uninformed policyholders bear the costs incurred by it.

Secondary markets for life insurance have rarely been examined in the literature. Gatzert (2010) compares the major secondary markets, evaluates the respective market

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from €8.2 billion (2000) to €13.8 billion (2009). See BVZL (2010), Gatzert (2010).

<sup>2</sup>There are two ways of lapsing policies, either by surrendering the policy or by not paying premiums. See Kuo et al. (2003). In our paper we only consider single premium contracts, and hence, the lapse rate is equivalent to the surrender rate.

potential, and points out possible effects on these markets, but does not address the quantification of these effects. Gatzert, Hoermann, and Schmeiser (2009) simulate surrender profits in a model of heterogeneous mortality of insurance holders, and analyze the effect of asymmetric surrender behavior on the secondary market. Giacolone (2001) reviews the secondary market for life insurance contracts with viatical transactions.<sup>3</sup> Dothery and Singer (2003) focus on welfare aspects of secondary markets in this case.

The remainder of this article is organized as follows. In the “Model” section, we introduce the framework of both the finance and insurance market, and describe the surrender behavior of a representative policyholder from an insurer’s perspective. In the “Contract Valuation” section, we provide the necessary pricing formulas. In the section “Numerical Analysis”, we analyze the effect of a secondary market through various numerical examples. Finally, we conclude. Proofs of the main results are given in the Appendix.

## 1.2 Model

In order to price equity-linked life insurance contracts, a model for both the financial and the insurance market is necessary. Our model extends Li and Szimayer (2014) by a secondary market on which policyholders may sell their life insurance contracts. The model for the decision process is motivated and formalized for a representative policyholder.

### 1.2.1 Financial and Insurance Market

We formulate the financial and insurance market directly under the risk neutral measure  $\mathbb{Q}$ . For a detailed but rather technical discussion supporting our approach, we refer the reader to Bielecki and Rutkowski (2004).

The financial market is defined on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$  and consists of a risk-less money market account with price process  $B$  and a non-dividend paying risky asset with price process  $S$ . The risky asset plays the role of the reference fund for the equity-linked life insurance contract. We fix a time horizon  $T > 0$  and define the dynamics of the price processes by

$$dB_t = r(t) B_t dt, \quad \text{for } 0 \leq t \leq T, \text{ and } B_0 = 1, \quad (1.1)$$

$$dS_t = r(t) S_t dt + \sigma(t, S_t) S_t dW_t, \quad \text{for } 0 \leq t \leq T, \text{ and } S_0 = s > 0. \quad (1.2)$$

---

<sup>3</sup>Viaticors are policyholders with sharply reduced life expectancy due to severe illness. The first secondary markets for life insurance products have been established in the US for people with drastically reduced life expectancy, in particular persons inflicted by HIV (See Giacolone (2001) ).

The deterministic function  $r$  denotes the short rate, the function  $\sigma > 0$  is the volatility of the risky asset, and  $W$  is a Wiener process under  $\mathbb{Q}$ . Both price processes are assumed to be Markovian, i.e. they do not depend on past events, only on the present state. The Wiener process  $W$  generates the filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  which reflects all the information available on the financial market. By definition, the financial market is arbitrage-free and complete, i.e. the equivalent martingale measure  $\mathbb{Q}$  exists and is unique.

The insurance market is modeled by two random times  $\tau$  and  $\lambda$  potentially ending the financial contract. The time  $\tau$  refers to the death time of an individual aged  $y$  at time  $t = 0$  when the contract is signed. The time  $\lambda$  refers to the time when the policyholder decides to surrender the contract. Surrender is here understood as both ways the policyholder can walk away from the contract, i.e. by exercising the surrender option and by selling it on the secondary market. In case the contract is sold on the secondary market the contract is still alive. However, the primary policyholder is no longer holding the contract. The jump process associated with  $\tau$  is  $H$  with  $H_t = \mathbb{1}_{\{\tau \leq t\}}$ , for  $0 \leq t \leq T$ , and generates the filtration  $\mathbb{H} = (\mathcal{H}_t)_{0 \leq t \leq T}$ . The hazard rate of the random time  $\tau$  (or the mortality intensity) is denoted by  $\mu$  and is assumed to be a deterministic function. Under this assumption, the mortality risk can be diversified over a large pool of policyholders. The jump process associated with  $\lambda$  is  $J$  with  $J_t = \mathbb{1}_{\{\lambda \leq t\}}$ , for  $0 \leq t \leq T$ . It generates the filtration  $\mathbb{J} = (\mathcal{J}_t)_{0 \leq t \leq T}$ . The hazard rate of the random time  $\lambda$  is denoted by  $\gamma$ , and is also called the surrender intensity.

By introducing the random time  $\lambda$ , and correspondingly, the surrender intensity  $\gamma$ , we can actually represent a large family of insurance contracts. For the degenerate case where  $\gamma = 0$ , the insurance contracts are European style. When  $\gamma$  is allowed to take strictly positive and finite values, the policyholder can walk away from the contract. In contrast to the mortality intensity  $\mu$ , the surrender intensity  $\gamma$  is not deterministic but depends on the monetary rationality of the policyholder in making surrender decisions by comparing the contract value and the surrender value. Since the contract value and eventually also the surrender value are linked to the risky asset  $S$ ,  $\gamma$  is assumed to be  $\mathbb{F}$ -measurable. The exact form of  $\gamma$  will be specified in the next section.

The nature of equity-linked life insurance policies is that they are linking the financial market and the insurance market. To model information on the linked market, the filtrations  $\mathbb{F}$ ,  $\mathbb{H}$  and  $\mathbb{J}$  are combined by  $\mathbb{G} = \mathbb{F} \vee \mathbb{H} \vee \mathbb{J}$ . The necessary technicalities are given by Bielecki and Rutkowski (2004).

## 1.2.2 Surrender Behavior of the Representative Policyholder

We assume that there is a large pool of policyholders. The policyholders of the equity-linked life insurance contracts can choose to continue the contract or to end the contracts

either be exercising the surrender option or by selling the contracts on the secondary market. The policyholders potentially face financial constraints, e.g., liquidity needs caused by personal financial distress. Besides, they are not all finance experts and may not always realize whether they are better off by giving up the contracts or by keeping the contracts. Among these policyholders, some of them know about the existence of the secondary market, while others do not.

In this section, we summarize the surrender behaviors of the whole pool of policyholders by simply looking at a representative policyholder. We assume that at the portfolio level there are proportion of  $p$  of policyholders who are informed of the existence of the secondary market. Reducing to the single representative policyholder, the probability that the representative agent is informed of the existence of the secondary market is  $p \in [0, 1]$  under  $\mathbb{Q}$ .<sup>4</sup> This is captured by the random variable  $X$  that is independent of the enlarged filtration  $\mathcal{G}_T$ , reflecting the information of both the financial and the insurance market, and is taking the value 1 with probability  $p$  (informed of the existence of the secondary market) and the value 0 with probability  $1 - p$  (no information about the existence of the secondary market).<sup>5</sup>

If at time  $t$  the representative policyholder gives up the contract, i.e.  $\lambda = t$ , by surrendering the contract to the insurer, i.e.  $X = 0$ , he receives the predefined surrender benefit  $L(t)$  from the insurer. If instead, the policyholder is able to access the secondary market, i.e.,  $X = 1$ , then price and quantity actually are determined by an equilibrium rationale.

We assume the contract buyer on the secondary market to be an agent in a perfect financial market with no frictions and access to all relevant information. Consequently, the new contract buyer exercises the surrender option financially optimal and the contract value at time  $t$  is then the price of the corresponding American-style contingent claim in the presence of diversifiable mortality risk, which we denote as  $V_t^{Am}$  per contract. The price for the primary policyholder to sell the contract on the secondary market at time  $t$  thus cannot exceed  $V_t^{Am}$ . As well, the price cannot drop below the surrender value  $L(t)$ , otherwise the policyholder would rather exercise the surrender option than selling the contract on the secondary market. Accordingly, any admissible price can be written as  $(1 - \kappa)L(t) + \kappa V_t^{Am}$ , with  $\kappa \in [0, 1]$ . And  $\kappa$  can be interpreted as standardized traded price on the secondary market per contract. For the new contract buyer on the secondary market, a higher  $\kappa$  indicates a lower profit from the transaction. The incentive to buy the

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<sup>4</sup>We assume that  $p$  remains identical under both the risk neutral and the real world measure. This invariance of the probability is based on the underlying assumption that the associated risk is unsystematic and hence diversifiable.

<sup>5</sup>Formally, we have to enlarge the filtration  $\mathbb{G}$  to also include the information generated by the process  $(X 1_{\{\lambda \leq t\}})_{0 \leq t \leq T}$  revealing  $X$  at  $\lambda$ .

contract, which is inferred from the new contract buyer's probability to buy the contract, is hence decreasing with  $\kappa$ . At the portfolio level, this probability corresponds to the total contract demand at the portfolio level. In other words, the inverse demand function is decreasing in the contract demand. We further assume that the policyholder who is informed of the existence of the secondary market would first make the decision about whether to enter the secondary market. The probability to enter the secondary market is equivalent to the total contract supply at the portfolio level. For the policyholder, entering the secondary market leads to the personal costs of searching a contract buyer. Even though he knows that the price offered by the secondary market is no less than the surrender guarantee offered by the insurer, he does not know whether the benefit obtained from the secondary market can compensate his search costs. On the other hand, the policyholder would have a certain expectation about the secondary market situation which he gains from his own information sources. These two effects, the search costs and the expectation about the secondary market, will lead to the fact that the probability of entering the secondary market increases with  $\kappa$ , or equivalently, the inverse supply function is increasing in the contract supply.

Now we formalize the above assumptions. We denote the (inverse) demand function by  $\kappa^D : [0, 1] \rightarrow [0, 1], q \mapsto \kappa^D(q)$  and the (inverse) supply function by  $\kappa^S : [0, 1] \rightarrow [0, 1], q \mapsto \kappa^S(q)$ , respectively. Assume that  $\kappa^D$  is continuous and strictly decreasing,  $\kappa^S$  is continuous and strictly increasing with  $\kappa^D(0) \geq \kappa^S(0)$  and  $\kappa^D(1) \leq \kappa^S(1)$ , then a unique equilibrium  $(\kappa, q)$  exists. Here  $q$  can be interpreted as standardized quantity. The transacted price  $\kappa$  and quantity  $q$  can be determined from demand and supply.

For a more (less) competitive secondary market, i.e. a market with more (less) potential buyers, the demand increases (decreases) at a given price and we would expect an upward movement of the (inverse) demand curve. With regard to the policyholder, if the search costs decrease (increase), we would expect that the informed policyholder is more (less) likely to be ready to sell the contract at a given price. This would lead to the downward (upward) movement of the (inverse) supply curve. In the extreme case that the search costs are reduced to 0 or the policyholder's expectation about the secondary market is extremely high, the policyholder will enter the secondary market with certainty, i.e.,  $q = 1$ . In this case, the transacted price is purely determined by the demand on the secondary market.

Based on the above assumptions, given that the primary policyholder is informed of the existence of the secondary market and he decides to give up the contract, he will surrender the contract to the insurer at the price  $L(t)$  with probability  $1 - q$  and sell the contract on the secondary market at the price  $L(t) + \kappa (V_t^{Am} - L(t))$  with probability  $q$ . The expected benefit that the primary policyholder obtains when giving up the contract

at  $t$  is then  $L(t) + \kappa q (V_t^{Am} - L(t))$ . Taking into account that the policyholder knows about the secondary market with probability  $p$ , the expected contract value when giving up the contract at  $t$  is then

$$\tilde{L}_t = (1 - p) L(t) + p (L(t) + \kappa q [V_t^{Am} - L(t)]) = L(t) + \kappa p q (V_t^{Am} - L(t)). \quad (1.3)$$

On the other hand, when the contract is not surrendered at time  $t$ , its value for the policyholder is  $V_t^C$ . The surrender behavior of the policyholder is represented by the surrender intensity  $\gamma$ , whose value is determined by the relationship between the active contract value  $V_t^C$  and the expected surrender value  $\tilde{L}_t$ . If  $V_t^C > \tilde{L}_t$ , it is on average not advantageous for the policyholder to surrender the contract. However, there are always exogenous reasons, e.g., caused by the policyholder's personal financial constraints which force the policyholder to surrender the contract with a certain probability. In this case, the surrender intensity is expected to take a lower bound value  $\underline{\rho}$ . If, on the contrary,  $V_t^C \leq \tilde{L}_t$ , surrendering the contract is a more attractive alternative in this market situation. The policyholder is more likely to surrender contract with the upper bound surrender intensity  $\bar{\rho}$ . However, since the policyholder could not capture this best surrender time for sure due to his limited ability to judge the contract value exactly, the insurer will not observe the absolute surrender action in this case, i.e.,  $\bar{\rho} < \infty$ . We summarize the surrender intensity  $\gamma$  as follows.

$$\gamma_t = \begin{cases} \underline{\rho}, & \text{for } \tilde{L}_t < V_t^C, \\ \bar{\rho}, & \text{for } \tilde{L}_t \geq V_t^C. \end{cases} \quad (1.4)$$

The intensity difference  $\bar{\rho} - \underline{\rho}$  can be interpreted as the level of monetary rationality of the representative policyholder. This way of modeling the surrender behavior follows Li and Szimayer (2014) that is dating back to Stanton (1995).<sup>6</sup>

### 1.3 Contract Valuation

Given the market model and the policyholder's surrender behavior, the pricing of the insurance contract is now possible. This pricing is carried out first from the perspective of the representative policyholder to obtain  $V^C$ . Then the value from the perspective of the insurance company  $V^I$  is derived using as input the behavior of the representative policyholder captured by  $\gamma$ . The difference in the contract values is the premium (or

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<sup>6</sup>The surrender intensity here is modeled under the risk neutral measure. The bounds for the surrender intensities do not change if we work under the real world measure what is motivated by diversification reasons, see Li and Szimayer (2011) for details.

cost) for introducing the secondary market.

We consider the case of single premium contracts. The results can be extended to the case of continuous premiums without much difficulty. The payoff structure of the insurance contract is divided into three parts: the benefit at maturity, denoted  $\Phi(S_T)$ , the benefit at death  $\Psi(\tau, S_\tau)$ , and the benefit if the contract is terminated early, given by  $L(\lambda)$ .<sup>7</sup> The payoff functions considered here are

$$\Phi(T, S_T) = P \max \left( \alpha (1 + g)^T, \left( \frac{S_T}{S_0} \right)^k \right), \quad (1.5)$$

$$\Psi(\tau, S_\tau) = P \max \left( \alpha (1 + g_d)^\tau, \left( \frac{S_\tau}{S_0} \right)^{k_d} \right), \quad (1.6)$$

$$L(\lambda) = (1 - \beta_\lambda) P (1 + h)^\lambda. \quad (1.7)$$

Here,  $\alpha$  is the fraction of the premium guaranteed to yield the minimum rate  $g$ , usually smaller than the risk free rate,  $P$  the single premium and  $k$  the participation coefficient specifying the degree to which the policyholder participates in gains of the risky asset underlying the insurance contract. Mostly,  $g = g_d$  and  $k = k_d$ , i.e. death is not penalized by the insurer (Bernard and Lemieux (2008)). Surrendering the contract early is penalized however, captured by the time dependent function  $\beta$ . In most cases,  $\beta$  is a function decreasing in time, aiming to punish early surrender over continuation of the contract.

**Remark 1.** *The value of an American option in the presence of mortality risk then describes the maximal possible value of the contract and assumes that the contract holder is facing no financial constraints in an efficient market. On  $\{t < \tau \wedge \lambda \wedge T\}$  the value of the American option is  $V_t^{Am} = v^{Am}(t, S_t)$  where  $v^{Am} : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfies the free boundary value problem given by the PDE*

$$0 = \frac{\partial v^{Am}}{\partial t}(t, s) + r(t)s \frac{\partial v^{Am}}{\partial s}(t, s) + \frac{1}{2} \sigma^2(t, s) s^2 \frac{\partial^2 v^{Am}}{\partial s^2}(t, s) + \mu(t) \Psi(t, s) - (r(t) + \mu(t)) v^{Am}(t, s),$$

with constraint  $v^{Am}(t, s) \geq L(t)$  on  $[0, T] \times \mathbb{R}_+$  and terminal condition  $v^{Am}(T, s) = \Phi(s)$ , for  $s \in \mathbb{R}_+$ .

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<sup>7</sup>The payoff structure is taken from Bernard and Lemieux (2008) and is also employed by Li and Szimayer (2014).



### 1.3.1 Representative Policyholder's Contract Value

In the following, we derive the pricing PDE for the contract value from the perspective of the policyholder  $V^C$ . Our derivation extends Li and Szimayer (2014) to allow for a secondary market. The basis of the derivation remains the balance law as stated by Dai, Kwok, and You (2007). The expected return of the contingent claim specified by the insurance contract has to equal the risk free rate under the risk-neutral measure as this is a no-arbitrage condition. Contracts that have not terminated for any reason, i.e. on  $\{t < \lambda \wedge \tau \wedge T\}$  with  $x \wedge y := \min(x, y)$ , satisfy

$$r(t)V_t^C dt = \mathbb{E}^{\mathbb{Q}} [dV_t^C | \mathcal{G}_t] . \quad (1.8)$$

On the above set, the following possible cases may happen:

1. The conditional probability that death occurs over  $(t, t + dt)$  while surrender does not is  $\mu(t) dt (1 - \gamma_t p dt - \gamma_t (1 - p) dt) = \mu(t) dt$ . The decision to access a secondary market is irrelevant here.
2. The conditional probability that death does not occur over  $(t, t + dt)$  while surrender does and the secondary market is used is  $\gamma_t p dt (1 - \mu(t) dt) = \gamma_t p dt$ .
3. The conditional probability that death does not occur over  $(t, t + dt)$  while surrender does and the secondary market is not used is  $\gamma_t (1 - p) dt (1 - \mu(t) dt) = \gamma_t (1 - p) dt$ .
4. The conditional probability that both surrender and death happen over  $(t, t + dt)$  is 0 as in Li and Szimayer (2010). Again, the decision to enter a secondary market is irrelevant.

The contract value at time  $t \leq \lambda \wedge \tau \wedge T$  is assumed to take the form

$$\begin{aligned} V_t^C = & \mathbb{1}_{\{t < \tau \wedge \lambda\}} v(t, S_t) + \mathbb{1}_{\{t = \tau \leq \lambda\}} \Psi(\tau, S_\tau) + \mathbb{1}_{\{t = \lambda < \tau, X=0\}} L(\lambda) \\ & + \mathbb{1}_{\{t = \lambda < \tau, X=1\}} \left[ L(\lambda) + \kappa q \left( V_\lambda^{Am} - L(\lambda) \right) \right], \end{aligned} \quad (1.9)$$

where  $v$  is a suitably differentiable function  $v : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ . By (1.4) we see that  $\gamma$  depends on the current continuation value  $V^C$  and the expected termination payoff driven by  $V^{Am}$ .  $V^{Am}$  can be expressed as a function of time  $t$  and price of the risky asset  $s$ , see Remark 1, and therefore  $\gamma$  can be written as a function of  $(t, s, v)$ , i.e.  $\gamma : [0, T] \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ , and set  $\gamma_t = \gamma(t, S_t, v(t, S_t))$ . As given by Li and Szimayer (2014), the occurrence of the event of death changes the contracts payoff  $\Psi(t, s)$  leaving a change in payment that is  $\Psi(t, s) - v(t, s)$ . Basically, the claim to receive  $v$  is lost and replaced by the payment of  $\Psi$ . Similarly, the payment liability is effected by the

surrender action. The secondary market changes the payoff to be dependent on the state of the decision variable  $X$ . The change in payment remains to be  $L(t) - v(t, S_t)$  if  $X = 0$ , but if the secondary market is accessed then  $X = 1$  and the payoff change is  $L(t) + \kappa q [v^{Am}(t, S_t) - L(t)] - v(t, S_t)$ .

Using the above changes in payment liabilities, the balance law (1.8) can be written as

$$\begin{aligned} r(t)v(t, S_t)dt = & \mathbb{E}^{\mathbb{Q}} [dv(t, S_t) | \mathcal{F}_t] + [L(t) + \kappa q (V_t^{Am} - L(t)) - v(t, s)] p\gamma_t dt \\ & + [L(t) - v(t, S_t)] (1 - p)\gamma_t dt + [\Psi(t, S_t) - v(t, S_t)] \mu(t) dt. \end{aligned} \quad (1.10)$$

Equation (1.10) carries economic interpretation: The change in the contract's value can be split up in the changes in value due to the different surrender and death events and the change in value originating in the continuation value. All these components have to equal the risk free rate in total, since the pricing takes place under the risk neutral measure. The expected change in the continuation value is based on filtration  $\mathbb{F}$  as, in economic terms, the death risk process does not influence the stock prices. Expanding the increment of the continuation value by Itô's Lemma gives

$$\mathbb{E}^{\mathbb{Q}} [dv(t, S_t) | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}} \left[ \mathcal{L}v(t, s) dt + \sigma(t, S_t) \frac{\partial v}{\partial s}(t, S_t) dW_t \middle| \mathcal{F}_t \right] = \mathcal{L}v(t, S_t) dt, \quad (1.11)$$

where the differential operator  $\mathcal{L}$  is given by

$$\mathcal{L}f(t, s) = \frac{\partial f}{\partial t}(t, s) + r(t)s \frac{\partial f}{\partial s}(t, s) + \frac{1}{2} \sigma^2(t, s) s^2 \frac{\partial^2 f}{\partial s^2}(t, s).$$

Thus, the balance law produces

$$\begin{aligned} 0 = & \mathcal{L}v(t, s) + \mu(t)\Psi(t, s) + \gamma(t, s, v(t, s))(1 - p)L(t) \\ & + \gamma(t, s, v(t, s))p [L(t) + \kappa q (v^{Am}(t, s) - L(t))] - (r(t) + \mu(t) + \gamma(t, s, v(t, s))) v(t, s). \end{aligned}$$

A further no-arbitrage condition is  $v(T, s) = \Phi(s)$ , for all  $s > 0$ , i.e. the value of the contract that has survived up to maturity will be the same as the value of the payout specified for this case. This completes the derivation of the following proposition:

**Proposition 1** (Pricing PDE, Representative Policyholder). *For the contract value  $V^C$  given by (1.9) the price function  $v$  is the solution of the partial differential equation*

$$\begin{aligned} 0 = & \mathcal{L}v(t, s) + \mu(t)\Psi(t, s) + \gamma(t, s, v(t, s)) [L(t) + \kappa pq (v^{Am}(t, s) - L(t))] \\ & - [r(t) + \mu(t) + \gamma(t, s, v(t, s))]v(t, s), \end{aligned} \quad (1.12)$$

for  $(t, s) \in [0, T) \times \mathbb{R}_+$  with terminal condition  $v(T, s) = \Phi(s)$ , for  $s \in \mathbb{R}_+$ . The solution of (1.12) together with Remark 1 and equation (1.4) then characterize the intensity  $\gamma$ .

The following stochastic representation formula is obtained via the Feynman-Kac:

**Corollary 1** (Stochastic Representation Formula, Representative Policyholder). *Suppose the surrender intensity  $\gamma$  is given. Then the value of the contract  $V^C$  can be represented on  $\{t < \tau \wedge \lambda \wedge T\}$  by*

$$\begin{aligned} V_t^C = & \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r(y) + \mu(y) + \gamma_y dy} \Phi(S_T) \middle| \mathcal{F}_t \right] \\ & + \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-\int_t^u r(y) + \mu(y) + \gamma_y dy} \mu(u) \Psi(u, S_u) du \middle| \mathcal{F}_t \right] \\ & + \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-\int_t^u r(y) + \mu(y) + \gamma_y dy} \gamma_u \left[ L(u) + \kappa p q \left( V_u^{Am} - L(u) \right) \right] du \middle| \mathcal{F}_t \right]. \end{aligned} \quad (1.13)$$

### 1.3.2 Insurance Company's Contract Value

The derivation of the contract value for the insurance company is broadly similar to that for the representative policyholder. However, there are some distinct differences. The contract value from the perspective of the insurance company  $V^I$  depends on the behavior of the representative policyholder as described by  $\gamma$ . Thus  $\gamma$  and indirectly also  $v$  serve here as an input parameter. Further, in case the contract is sold on the secondary market the insurance company has to account for the full costs.

In the spirit of (1.9) we express the contract value  $V^I$  by

$$\begin{aligned} V_t^I = & \mathbb{1}_{\{t < \tau \wedge \lambda\}} u(t, S_t) + \mathbb{1}_{\{t = \tau \leq \lambda\}} \Psi(\tau, S_\tau) + \mathbb{1}_{\{t = \lambda < \tau, X = 0\}} L(\lambda) \\ & + \mathbb{1}_{\{t = \lambda < \tau, X = 1\}} \left[ q V_\lambda^{Am} + (1 - q) L(\lambda) \right], \end{aligned} \quad (1.14)$$

where  $u : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is the related value function.

**Proposition 2** (Pricing PDE, Insurance Company). *Suppose that the contract value for the representative policyholder is given by  $v$  and the intensity is given by  $\gamma$ , respectively, both according to Proposition 1. For the contract value  $V^I$  given by (1.14) the price function  $u$  is the solution of the partial differential equation*

$$\begin{aligned} 0 = & \mathcal{L}u(t, s) + \mu(t) \Psi(t, s) + \gamma(t, s, v(t, s)) \left[ L(t) + p q (v^{Am}(t, s) - L(t)) \right] \\ & - [r(t) + \mu(t) + \gamma(t, s, v(t, s))] u(t, s) \end{aligned} \quad (1.15)$$

for  $(t, s) \in [0, T) \times \mathbb{R}_+$  with terminal condition  $u(T, s) = \Phi(s)$  for  $s \in \mathbb{R}_+$ .

Again, we have an immediate corollary giving the stochastic representation of the price function.

**Corollary 2** (Stochastic Representation Formula, Insurance Company). *Suppose the surrender intensity  $\gamma$  is given. Then the value of the contract  $V^I$  can be represented on  $\{t < \tau \wedge \lambda \wedge T\}$  by*

$$\begin{aligned} V_t^I = & \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r(y) + \mu(y) + \gamma_y dy} \Phi(S_T) \middle| \mathcal{F}_t \right] \\ & + \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-\int_t^u r(y) + \mu(y) + \gamma_y dy} \mu(u) \Psi(u, S_u) du \middle| \mathcal{F}_t \right] \\ & + \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-\int_t^u r(y) + \mu(y) + \gamma_y dy} \gamma_u [L(u) + pq(V_u^{Am} - L(u))] du \middle| \mathcal{F}_t \right]. \end{aligned} \quad (1.16)$$

The relationship between the insurance company's value and the representative policyholder's value is that, as expected, the insurance company's value is greater than that of the representative policyholder. This is made precise below and follows directly from Corollary 1 and Corollary 2.

**Corollary 3.** *The value difference of the contract from the perspective of the insurance company and from the perspective of the representative policyholder, respectively, can be represented on  $\{t < \tau \wedge \lambda \wedge T\}$  by*

$$V_t^I - V_t^C = (1 - \kappa) pq \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-\int_t^u r(y) + \mu(y) + \gamma_y dy} \gamma_u [V_u^{Am} - L(u)] du \middle| \mathcal{F}_t \right], \quad (1.17)$$

and is non-negative.

**Remark 2.** *The pricing PDE and the stochastic representation formula can be extended to incorporate continuous premiums. Further, the constant surrender parameters  $\underline{\rho}$  and  $\bar{\rho}$  can be allowed to be functions of the time and the price of the risky asset. Then the results are still valid under the extended setup.*

In the traditional sense, an equity-linked life insurance is fair if and only if the expected payment to the policyholder equals the premium paid by the policyholder at the initial date. Such a fair contract does not necessarily exist if the insurer charges  $V^I$  but the policyholder is paid only  $V^C$  in expectation.

**Proposition 3.** *If a fair equity-linked life insurance contract exists on the insurance market with a secondary market, then one of the following conditions must be satisfied<sup>8</sup>*

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<sup>8</sup>For  $q = 0$ , i.e. no equilibrium exists on the secondary market, also a fair contract exists. However, we assumed the existence of a unique equilibrium at the beginning of the paper.

- 1)  $p = 0$ , i.e., there is no possibility to access the secondary market;
- 2)  $\kappa = 1$ , i.e., the secondary market is completely competitive;
- 3)  $(\underline{\rho}, \bar{\rho}) = (0, \infty)$ , i.e., the policyholder faces no financial constraints and acts monetarily rational.

### 1.3.3 Comparative Statics

In the following, the effects of changes in the model parameters are analyzed.

For the representative policyholder we can derive a comprehensive set of comparative statics. The more rational the representative policyholder is, or, the less financial constraints he is facing, the higher is the contract value. Thus the contract value is increasing for increasing monetary rationality ( $\bar{\rho}$ ) and for decreasing likelihood of exogenous surrender ( $\underline{\rho}$ ). The impact of the secondary market parameters is that the increasing probability of access to the secondary market ( $p$ ), an increasing equilibrium quantity ( $q$ ) and an increasing equilibrium price ( $\kappa$ ) result in an increasing contract value.

**Proposition 4** (Comparative Statics for Representative Policyholder). *For  $0 \leq \underline{\rho} \leq \bar{\rho}$  and  $0 \leq \kappa, p, q \leq 1$  denote by  $v$  the representative policyholder's value function, and for the set of parameters  $0 \leq \underline{\rho}' \leq \bar{\rho}'$  and  $0 \leq \kappa', p', q' \leq 1$ , denote the respective value function by  $v'$ , both as given in Proposition 1. Suppose that  $\underline{\rho}' \leq \underline{\rho}$ ,  $\bar{\rho}' \geq \bar{\rho}$ , and  $\kappa' p' q' \geq \kappa p q$ , then  $v'(t, s) \geq v(t, s)$ , for all  $(t, s) \in [0, T] \times \mathbb{R}_+$ .*

The above proposition states that a contract is more valuable, if the option to surrender the contract is used “more rationally”. Such an insurance contract has higher value compared to a second one if the number of surrenders during periods in which it is rational to lapse the contract, is higher (i.e. surrender takes place after a shorter period of waiting). We have a further natural interpretation of the above results relating to the secondary market. Given all else remains constant, an increased willingness to access a secondary market for the contract raises the value of this life insurance. Further, increased quantity sold on the secondary market and a raise in the policyholder's share of the profits made through optimal exercise increases the contracts value also.

For the insurance company's contract value the dependence on the parameters is complex since the policyholder's contract value also has an impact via the policyholder's behavior ( $\gamma$ ). We can provide the following result.

**Proposition 5** (Comparative Statics for Insurance Company). *For  $0 \leq \underline{\rho} \leq \bar{\rho}$  and  $0 \leq p, \kappa \leq 1$  denote by  $u$  the insurance company's value function, and for the set of parameters  $0 \leq \underline{\rho}' \leq \bar{\rho}'$  and  $0 \leq p', \kappa' \leq 1$ , denote the respective value function by  $u'$ , both*

as given in Proposition 2. Suppose that  $\underline{\rho}' = \underline{\rho}$ ,  $\bar{\rho}' = \bar{\rho}$ ,  $\kappa'p'q' = \kappa pq$ , and  $\kappa' \leq \kappa$ , then  $u'(t, s) \geq u(t, s)$ , for all  $(t, s) \in [0, T] \times \mathbb{R}_+$ .

## 1.4 Numerical Analysis

This section studies numerical examples of the contracts analyzed in the previous section. Firstly, we study the effect of the secondary market on policyholders' surrender behavior, which at the same time depends on the policyholders' monetary rationality degree. Secondly, we compare the contract values with and without a secondary market for both the insurance company and the policyholders and investigate the impact of the secondary market for both parties.

The parametrization is specified as follows: The risky asset has a volatility of  $\sigma = 0.2$  and  $S_0 = 1000$  as a starting value. The interest rate is taken to be constant and equal to  $r = 4\%$ . The single premium is  $P = 100$ , and the time to maturity is  $T = 10$  years. The percentage of the premium covered by the guarantee is  $\alpha = 0.85$ , while the guaranteed rates for both the final payment as well as for premature termination, whether due to death or surrender, are  $g = g_d = h = 2\%$ . The participation coefficient for gains of the underlying asset is  $k = k_d = 0.9$ . The penalty function for early surrender,  $\beta$ , is assumed to have the penalty rates  $\beta_1 = 0.05$ ,  $\beta_2 = 0.04$ ,  $\beta_3 = 0.02$ ,  $\beta_4 = 0.01$  and  $\beta_t = 0$  for  $t \geq 5$ . The policyholder is assumed to be forty years old when he enters the contract. The deterministic mortality intensity takes the form  $\mu(t) = A + Bc^{y+t}$ , where  $A = 5.0758 \times 10^{-4}$ ,  $B = 3.9342 \times 10^{-5}$ , and  $c = 1.1029$ .

For the secondary market, we specify the inverse supply and demand functions to be

$$\kappa^S(q) = q^2, \quad \kappa^D(q) = \max(1 - a \cdot q^2, 0),$$

which produces the equilibrium price-quantity tuple  $\left(\frac{1}{1+a}, \sqrt{\frac{1}{1+a}}\right)$ . Here,  $a$  can be interpreted as the sensitivity of the contract demand to the traded price  $\kappa$  on the secondary market.

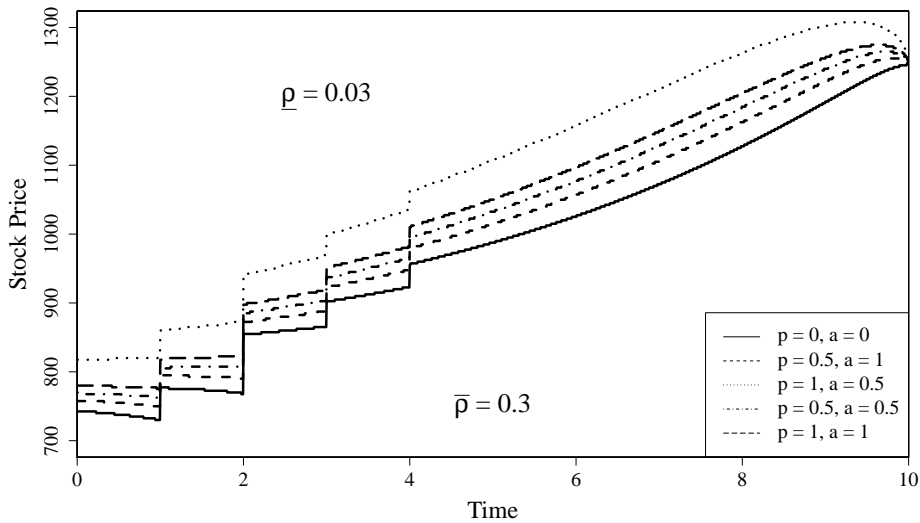
To solve the PDE specified in Propositions 1 and 2 via finite differences, we use the Crank-Nicolson Scheme originally proposed in Crank and Nicolson (1947). The Crank-Nicolson method is unconditionally stable and under some regularity assumptions, the order of the method is  $O(\Delta s^2) + O(\Delta t^2)$ , with  $\Delta s$  and  $\Delta t$  being the step sizes of the mesh in the  $s$  and  $t$  direction, respectively. In our study, we use  $\Delta s = 10.2$  and  $\Delta t = 0.02$ . A detailed discussion of the Crank-Nicolson method is beyond the scope of our paper. For an extensive discussion c.f. for example Seydel (2012).

### 1.4.1 Effect of Secondary Market on Surrender Behavior

In Figure 1.1 we show for  $(\underline{\rho}, \bar{\rho}) = (0.03, 0.3)$  that policyholders are more likely to surrender their contracts when the underlying asset prices are relatively low and less likely to do so when the underlying asset prices are relatively high. The boundary which separates these two regions is called the separating boundary.

As we have addressed above, the secondary market is featured by the policyholder's access probability into the market  $p$  and the demand and supply of the contract on the secondary market. The surrender behavior is affected by the product of  $p$ ,  $\kappa$  and  $q$  in our model. Based on the supply and demand functions we have specified for our numerical example, we can see that  $\kappa$  and  $q$  depend solely on the parameter  $a$ . By examining the separating boundaries with different values of  $p$  and  $a$ , we shed some light on the effect of secondary markets on policyholders' surrender behavior.

**Figure 1.1:** Separating Boundary for  $p = a = 0$ ;  $p = a = 0.5$ ,  $p = 0.5$  and  $a = 1$ ;  $p = 1$  and  $a = 0.5$ ;  $p = 1$  and  $a = 1$ .



When  $p = 0$ , we are back to the model of Li and Szimayer (2014) where a secondary market is not accessible to the policyholders. For  $p > 0$ , the secondary market comes into play. The surrender intensity now captures both the policyholder exercising the surrender option and hence giving back the contract to the insurer and the policyholder selling the contract on the secondary market. We see from Figure 1.1 that compared to the case with  $p = 0$ , the  $\bar{\rho}$ -region is enlarged while the  $\underline{\rho}$ -region shrinks for all the cases with  $p > 0$ . In the  $\bar{\rho}$ -region, more policyholders give up their contracts as it is monetarily better to do so. Further, since a fixed proportion (indicated by  $p \times q$ ) among these policyholders

sell the contracts on the secondary market successfully, more contracts are surrendered monetarily optimal in the future. At the same time, the shrink of the  $\underline{\rho}$ -region indicates that fewer policyholders will give up the contracts when it is monetarily disadvantageous to do so. Moreover, a proportional amount among them go to the secondary market which triggers the optimal surrender later on. Both of these aspects indicate that the insurer bears more risk when a secondary market emerges. In addition, we observe from Figure 1.1 that the separating boundary moves upwards either when more policyholders are informed of the existence of the secondary market, i.e., when  $p$  increases, or when the demand is more sensitive to the price change of the contracts, i.e., when  $a$  decreases. In both cases, we expect to see the increase in the number of contracts traded on the secondary market. This indicates that the risk borne by the insurer increases in both cases. As mentioned in Li and Szimayer (2014), the kinks displayed in all the graphs are due to the discontinuous levels of penalties for early surrender  $\beta$ . If early surrender is not penalized, e.g. by  $\beta_t = 0$  for all  $t \geq 0$ , the graphs turn out to be smooth. This case does not alter the economic interpretation offered above in any way, however.

## 1.4.2 Risk Analysis for the Insurer

In this section, we quantify the magnitude of the increased risk borne by the insurer due to the introduction of the secondary market. Besides, we study the interacting effect of policyholder's monetary rationality with the secondary market on the contract values.

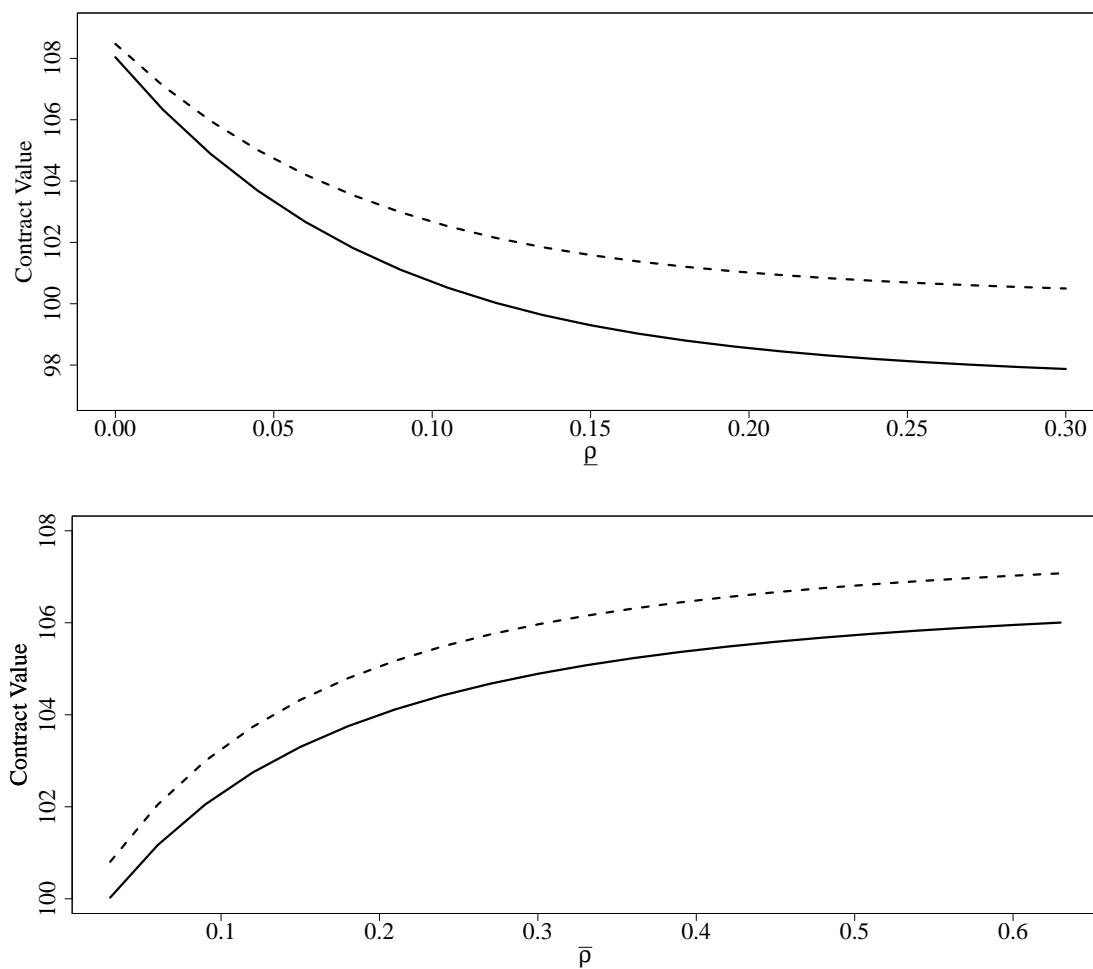
In Table 1.1 we present the contract values for  $\underline{\rho} \in \{0, 0.03, 0.3\}$ ,  $\bar{\rho} \in \{0, 0.03, 0.3, \infty\}$ ,  $p \in \{0, 0.25, 0.5, 0.75, 1\}$  and  $a \in \{0, 0.5, 1, 2, \infty\}$ . The first row with  $(p, a) = (0, 0)$  displays the contract values when there is no secondary market. We use it as benchmark to investigate the risk for the insurer caused by the secondary market. For  $(\underline{\rho}, \bar{\rho}) = (0, 0)$ , the contract is actually of European type and is not influenced by the secondary market. For  $(\underline{\rho}, \bar{\rho}) = (0, \infty)$ , both the policyholder and the secondary market are supposed to be able to exercise the surrender option monetarily optimally. Hence, it does not matter who is actually to exercise it, and the secondary market is irrelevant in this case. For the other cases we always observe the increase of the contract values whenever the secondary market is introduced.

For given  $(p, a)$  combinations, we always observe that the contract value increases monotonically with the decrease of  $\underline{\rho}$  and the increase of  $\bar{\rho}$ . This can be observed more clearly in the top graph of Figure 1.2, where we show the contract values for different  $\underline{\rho}$  values and the bottom graph of Figure 1.2, where the contract values for different  $\bar{\rho}$  are displayed.

For the insurance company, the lower  $\underline{\rho}$  is, the lower would be the probability that the policyholder surrenders the contract suboptimally which increases the contract value.



**Figure 1.2:** Contract Values from the policyholder's (solid line) and the insurers' (dashed line) perspective for varying  $\rho$  (upper graph) and  $\bar{\rho}$  (lower graph) with parameters  $p = a = 0.5$ . For the upper graph,  $\bar{\rho} = 0.3$ . For the lower graph,  $\underline{\rho} = 0.03$ .



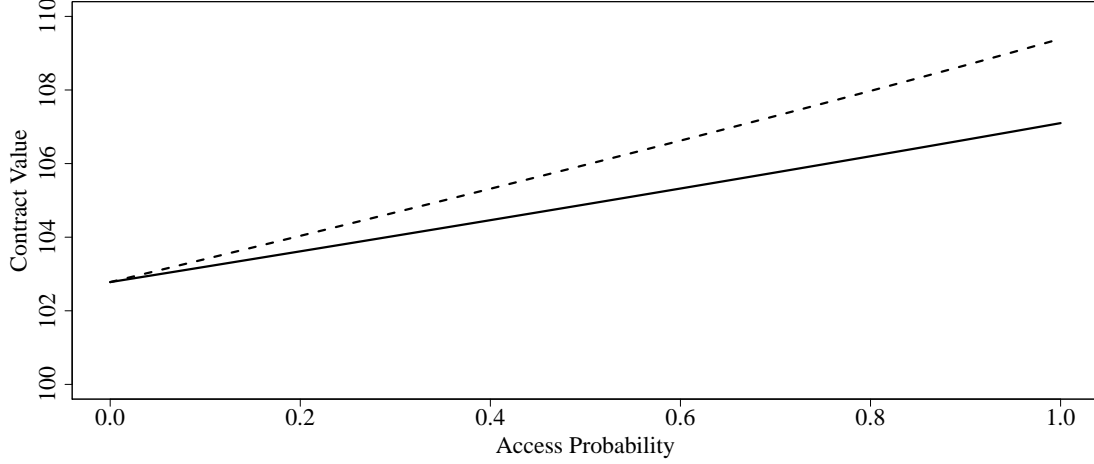
On the other hand, the lower would be the probability that the contract is sold to the secondary market and hence the lower is the chance that the optimal surrender is triggered. This aspect tends to decrease the contract value for given  $(p, a)$ . From the table we infer that the first effect dominates. With regard to  $\bar{\rho}$ , the higher  $\bar{\rho}$  is, the higher is the probability that the contract is surrendered optimally. Hence, a higher  $\bar{\rho}$  indicates a higher contract value.

$p, a$	$\rho, \bar{\rho}$								
	0, 0	0, 0.03	0, 0.3	0, $\infty$	0.03, 0.03	0.03, 0.3	0.03, $\infty$	0.3, 0.3	0.3, $\infty$
0, 0	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.25, 0	101.48	102.93	107.97	112.67	99.90	104.72	108.19	97.45	99.43
0.5, 0	101.48	103.22	108.83	112.67	101.33	106.73	109.62	102.27	103.83
0.75, 0	101.48	103.66	109.89	112.67	102.75	108.92	111.05	107.09	108.26
1, 0	101.48	104.38	112.12	112.67	104.18	112.08	112.67	111.91	112.67
0, 0.5	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.25, 0.5	101.48	102.88	107.82	112.67	99.64	104.35	107.93	96.56	98.62
0.5, 0.5	101.48	103.09	108.47	112.67	100.80	105.97	109.09	100.50	102.23
0.75, 0.5	101.48	103.35	109.20	112.67	101.97	107.63	110.26	104.43	105.83
1, 0.5	101.48	103.70	110.04	112.67	103.13	109.39	111.42	108.37	109.43
0, 1	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.25, 1	101.48	102.85	107.74	112.67	99.48	104.14	107.77	96.03	98.14
0.5, 1	101.48	103.02	108.28	112.67	100.49	105.52	108.78	99.44	101.26
0.75, 1	101.48	103.21	108.86	112.67	101.50	106.93	109.78	102.85	104.38
1, 1	101.48	103.44	109.48	112.67	102.51	108.37	110.80	106.26	107.50
0, 2	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.25, 2	101.48	102.82	107.64	112.67	99.30	103.89	107.58	95.41	97.53
0.5, 2	101.48	102.95	108.07	112.67	100.12	105.00	108.41	98.19	100.12
0.75, 2	101.48	103.09	108.51	112.67	100.94	106.13	109.24	100.97	102.66
1, 2	101.48	103.24	108.97	112.67	101.77	107.27	110.06	103.76	105.21
0, $\infty$	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.25, $\infty$	101.48	102.70	107.22	112.67	98.48	102.79	106.75	92.64	95.03
0.5, $\infty$	101.48	102.70	107.23	112.67	98.48	102.79	106.76	92.65	95.05
0.75, $\infty$	101.48	102.70	107.23	112.67	98.49	102.80	106.76	92.70	95.06
1, $\infty$	101.48	102.70	107.23	112.67	98.49	102.80	106.77	92.69	95.08

**Table 1.1:** Contract values from the insurer's perspective.

Now, we further study the impact of  $p$  and  $a$  separately. The insurer does not care about how the profits are shared between the secondary market and the original policyholder. The insurer is concerned about the total amount of extra money which is to flow out of the company due to the existence of the secondary market. The parameter  $p$  has both direct and indirect effects on the profits generated by the secondary market. The indirect effect comes from the change in the surrender behavior. As we have observed

**Figure 1.3:** Contract values from the policyholder's (solid line) and the insurers' (dashed line) perspective for varying probability to access the secondary market. We have  $\underline{\rho} = 0.03$ ,  $\bar{\rho} = 0.3$ , and  $a = 0.5$ .

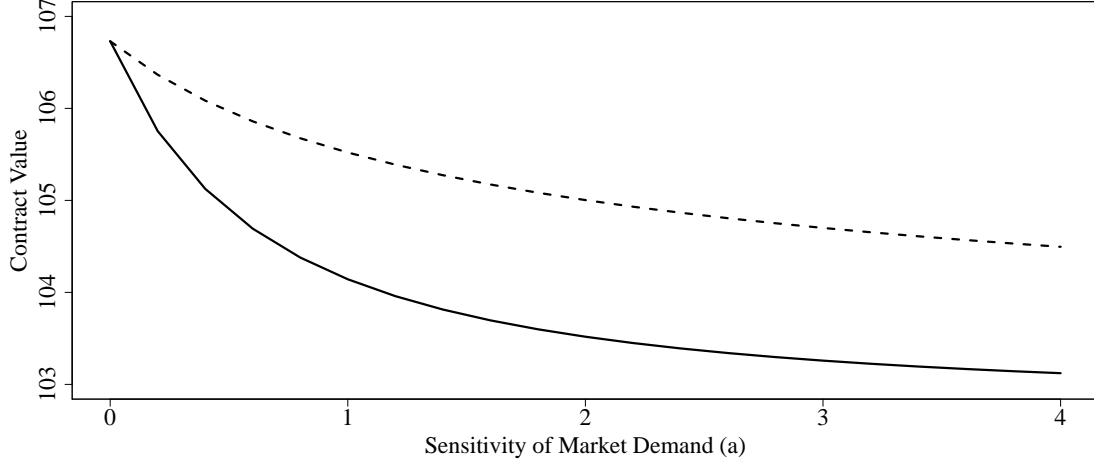


in Figure 1.1, a higher  $p$  leads to the upward movement of the surrender boundary and hence the expansion of the  $\bar{\rho}$ -region. Both a higher surrender intensity  $\gamma$  and a higher access probability  $p$  itself increase the number of policyholders going to the secondary market for given  $a$ . The two effects of  $p$  can be inferred from equations (1.15) and (1.16). As an example, we display in Figure 1.3 the contract values for different  $p$  values while the other parameters are kept constant.

The parameter  $a$  reflects the contract demand on the secondary market. Based on the demand function we have applied in the example, a lower value of  $a$  indicates less sensitive buyers on the secondary market to the traded price of the contract. This would lead to both a higher traded price of the contracts indicated by  $\kappa$  and a higher trade volume of the contracts indicated by  $q$ . The higher  $\kappa$  value increases the contract value indirectly through the expansion of the  $\bar{\rho}$ -region. While the higher  $q$  value has both the indirect effect (through the change in the surrender intensity) and the direct effect on the contract value. Consequently, we observe that increase of the contract value with the decrease of the parameter  $a$ . Figure 1.4 displays this relationship. Here we also point out that when  $\underline{\rho} = \bar{\rho}$ , the policyholder's surrender decision is, independent of the values of  $p$  and  $a$ , exogenously determined. Hence, the indirect effects of  $p$  and  $a$  vanishes, while the direct effects solely determine the contract values.

To study the interaction of the policyholder's monetary rationality with the secondary market, we present in Table 1.2, for policyholders with different degrees of monetary rationality  $(\underline{\rho}, \bar{\rho})$ , the relative deviation of the contract values when there is a secondary

**Figure 1.4:** Contract values from the policyholder's (solid line) and the insurers' (dashed line) perspective for varying demand sensitivity of the secondary market. We have  $\underline{\rho} = 0.03$ ,  $\bar{\rho} = 0.3$ , and  $p = 0.5$ .



market from the contract values when a secondary market does not exist. Comparing the columns for  $(\underline{\rho}, \bar{\rho}) = (0.03, 0.03), (0.03, 0.3), (0.03, \infty)$ , we see that the impact of the secondary market first increases with the rise of the endogenous surrender intensity and then decreases with it for given  $(p, a)$  combinations. A similar pattern can be observed for  $(\underline{\rho}, \bar{\rho}) = (0, 0.03), (0, 0.3)$ . This pattern is the joint work of the endogenous surrender intensity  $\bar{\rho}$  and the margin from the secondary market  $(V^{Am} - V)$  where  $V$  refers to the contract value without the secondary market. Although the increase of  $\bar{\rho}$  indicates that the policyholder is more likely to surrender the contract to the secondary market when it is monetarily rational to do so, the margin decreases when policyholders are more capable of surrendering the contract optimally by themselves. In which degree the contract value increases due to the introduction of the secondary market depends on the change of  $\bar{\rho}(V^{Am} - V)$  with  $\bar{\rho}$  at any time when the contract is likely to be surrendered endogenously by the policyholder to either the insurer or the secondary market. With the increase of  $\bar{\rho}$ , at the beginning, its raise dominates the decrease of the margin  $(V^{Am} - V)$  so that the relative deviation increases. When  $\bar{\rho}$  increases further, its raise is dominated by the decrease of the margin. This causes the decrease of the relative deviation. Note that  $(\underline{\rho}, \bar{\rho}) = (0, \infty)$  results in a relative deviation of zero, which we have not displayed in the table. Differently, we observe the monotonic increase of the relative deviation with the exogenous surrender intensity  $\underline{\rho}$  when we compare the columns with  $(\underline{\rho}, \bar{\rho}) = (0, 0.3), (0.03, 0.3), (0.3, 0.3)$ . A higher  $\underline{\rho}$  indicates the lower monetary rationality degree of the policyholder. Hence, the margin from the secondary market increases. The

**Table 1.2:** Deviation of contract values from the insurer's perspective with secondary market relative to the contract value without secondary market (in percent).

$p, a$	$\rho, \bar{\rho}$						
	0, 0.03	0, 0.3	0.03, 0.03	0.03, 0.3	0.03, $\infty$	0.3, 0.3	0.3, $\infty$
0.25, 0	0.22%	0.70%	1.45%	1.88%	1.35%	5.20%	4.64%
0.5, 0	0.51%	1.49%	2.90%	3.84%	2.69%	10.41%	9.27%
0.75, 0	0.93%	2.49%	4.35%	5.97%	4.03%	15.61%	13.93%
1, 0	1.63%	4.56%	5.80%	9.05%	5.55%	20.82%	18.58%
0, 0.5	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.25, 0.5	0.17%	0.56%	1.18%	1.53%	1.11%	4.25%	3.79%
0.5, 0.5	0.38%	1.17%	2.37%	3.10%	2.20%	8.50%	7.58%
0.75, 0.5	0.63%	1.84%	3.55%	4.72%	3.29%	12.75%	11.38%
1, 0.5	0.97%	2.63%	4.73%	6.43%	4.38%	17.00%	15.17%
0, 1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.25, 1	0.15%	0.48%	1.03%	1.32%	0.96%	3.68%	3.28%
0.5, 1	0.31%	0.99%	2.05%	2.67%	1.91%	7.36%	6.57%
0.75, 1	0.50%	1.53%	3.08%	4.04%	2.84%	11.04%	9.85%
1, 1	0.72%	2.11%	4.10%	5.44%	3.79%	14.72%	13.14%
0, 2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.25, 2	0.12%	0.39%	0.84%	1.08%	0.77%	3.00%	2.65%
0.5, 2	0.24%	0.79%	1.67%	2.16%	1.56%	6.01%	5.36%
0.75, 2	0.38%	1.20%	2.51%	3.26%	2.33%	9.01%	8.04%
1, 2	0.52%	1.63%	3.35%	4.37%	3.10%	12.02%	10.73%
0, $\infty$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.25, $\infty$	0.00%	0.00%	0.00%	0.01%	0.00%	0.02%	0.01%
0.5, $\infty$	0.00%	0.00%	0.01%	0.01%	0.01%	0.03%	0.03%
0.75, $\infty$	0.00%	0.01%	0.01%	0.02%	0.01%	0.05%	0.04%
1, $\infty$	0.00%	0.01%	0.02%	0.02%	0.02%	0.07%	0.06%

double effect, i.e.,  $\rho(V^{Am} - V)$ , leads to the increase of the relative deviation of the contract value with the secondary market from the contract value without the secondary market.

### 1.4.3 Benefit Analysis for the Policyholder

In this section we study the effect of secondary markets for the representative policyholder. In Table 1.3 we present the contract values from the policyholder's perspective. We compare the values when the secondary market exists and when it does not. Similar to Table 1.1, we do not observe its effect when  $(\rho, \bar{\rho}) = (0, 0), (0, \infty)$ . In the other cases, the policyholder's benefit increases with the increase of  $p$  and the decrease of  $a$ , as displayed

in Figures 1.3 and 1.4. Besides the effects analyzed in section 1.4.2 which also have influence on the policyholder's benefit, we note that  $\kappa$  also increases the policyholder's share of the profits generated by the secondary market through the decrease of  $a$ . This also plays a role in the increase of the contract value.

**Table 1.3:** Contract values from the policyholder's perspective.

$p, a$	$\underline{\rho}, \bar{\rho}$								
	0, 0	0, 0.03	0, 0.3	0, $\infty$	0.03, 0.03	0.03, 0.3	0.03, $\infty$	0.3, 0.3	0.3, $\infty$
0, 0	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.25, 0	101.48	102.93	107.97	112.67	99.90	104.72	108.19	97.45	99.43
0.5, 0	101.48	103.22	108.82	112.67	101.33	106.73	109.62	102.27	103.83
0.75, 0	101.48	103.66	109.89	112.67	102.75	108.92	111.05	107.09	108.26
1, 0	101.48	104.38	112.12	112.67	104.18	112.08	112.67	111.91	112.67
0, 0.5	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.25, 0.5	101.48	102.82	107.62	112.67	99.25	103.83	107.54	95.25	97.42
0.5, 0.5	101.48	102.95	108.04	112.67	100.03	104.89	108.31	97.87	99.82
0.75, 0.5	101.48	103.10	108.49	112.67	100.80	105.98	109.09	100.50	102.23
1, 0.5	101.48	103.29	108.98	112.67	101.58	107.10	109.87	103.12	104.63
0, 1	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.25, 1	101.48	102.78	107.48	112.67	98.98	103.46	107.27	94.33	96.58
0.5, 1	101.48	102.86	107.74	112.67	99.48	104.14	107.77	96.03	98.14
0.75, 1	101.48	102.94	108.02	112.67	99.99	104.84	108.27	97.74	99.70
1, 1	101.48	103.04	108.30	112.67	100.49	105.54	108.78	99.44	101.26
0, 2	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.25, 2	101.48	102.74	107.36	112.67	98.75	103.15	107.02	93.55	95.83
0.5, 2	101.48	102.78	107.50	112.67	99.02	103.52	107.31	94.48	96.72
0.75, 2	101.48	102.83	107.64	112.67	99.30	103.89	107.59	95.41	97.57
1, 2	101.48	102.87	107.79	112.67	99.57	104.26	107.86	96.34	98.42
0, $\infty$	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.25, $\infty$	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.5, $\infty$	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
0.75, $\infty$	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02
1, $\infty$	101.48	102.70	107.22	112.67	98.47	102.78	106.75	92.62	95.02

Furthermore, to study the interaction of policyholder's monetary rationality with the  $(p, a)$ -combinations and its effect on the benefit of the representative policyholder, we present in Table 1.4 the relative deviation of the contract values when there is secondary market from the values when there is no secondary market. We observe the same pattern as is demonstrated in Table 1.2, i.e., the relative deviation increases first with  $\bar{\rho}$  and then decreases with it for  $\bar{\rho} \in \{0.03, 0.3, \infty\}$ , and the relative deviation increases monotonically with  $\underline{\rho}$ . The reason for this phenomenon is the same as is analyzed in the previous section, see also Proposition 4.

Now, we compare Tables 1.1 and 1.3. When  $a = 0$ , we have  $\kappa = 1$ . The policyholder obtains all the profits generated by the secondary market. There is no difference between the contract values from the insurer's and the policyholder's perspectives. Moreover, since the secondary market has no effect for  $(\underline{\rho}, \bar{\rho}) = (0, 0), (0, \infty)$ , we do not see the difference in these cases either. In the other cases, we observe that the true contract values for the policyholder are always lower than the values for the insurer, because the profits generated by the secondary market are shared with contract buyer. Besides, the difference between them are higher for higher  $p$ . This is because *certeris paribus* a higher  $p$  indicates that the policyholder is more likely to go to the secondary market and more profits are to be generated by the secondary market due to its competence to exercise the surrender option optimally. On the contrary, the difference between the two values first increases and then decreases with the increase of the parameter  $a$ . This is the result of the following two effects. On the one hand,  $\kappa$  decreases with  $a$ , which indicates that the benefits obtained by the policyholder moves farther away from the premium charged by the insurer. On the other hand, the traded amount  $q$  (given that the policyholder is aware of the existence of the secondary market) decreases with  $a$ . A lower traded volume indicates fewer profits that can be created by the secondary market. This will lead to the shrink of the difference between the two contract values. We see from the result that the effect of  $\kappa$  dominates the effect of  $q$  at the beginning but is dominated by  $q$  when  $a$  further increases. The same result can be drawn when we look closely into equation (1.17). *Certeris paribus*, the difference between the two values is determined by  $(1 - \kappa)q$ . Since  $(\kappa, q) = (\frac{1}{1+a}, \sqrt{\frac{1}{1+a}})$ , we can prove that  $(1 - \kappa)q$  is a concave function in  $a$  which takes its maximum value at  $a = 2$ . This corresponds to our observation in the numerical part. Through the above comparison, we see that the introduction of the secondary market is not necessarily profitable for the policyholder if the increase of the benefit is associated with the increase of the premium. If the insurer takes the secondary market into account when calculating the premium, then the secondary market is only desirable for the representative policyholder when the demand on the secondary market is hardly sensitive to the traded price of the contract, i.e., when  $a$  converges to 0.

We know that our policyholder actually represents a large pool of policyholders. The surrender behavior of the representative policyholder summarizes the average behavior of the pool of policyholders. We may argue that within this pool some policyholders are more informed of the existence of the secondary market than the others. When the insurer charges a higher premium, those policyholders who are better informed may benefit from the secondary market and those who are less informed have to bear the costs caused by the secondary market. Now we study which policyholders are really profiting from the secondary market.

**Table 1.4:** Deviation of contract values from the policyholder's perspective with secondary market relative to the contract value without secondary market (in percent).

$p, a$	$\underline{\rho}, \bar{\rho}$						
	0, 0.03	0, 0.3	0.03, 0.03	0.03, 0.3	0.03, $\infty$	0.3, 0.3	0.3, $\infty$
0.25, 0	0.22%	0.70%	1.45%	1.88%	1.35%	5.20%	4.64%
0.5, 0	0.51%	1.49%	2.90%	3.84%	2.69%	10.41%	9.27%
0.75, 0	0.93%	2.49%	4.35%	5.97%	4.03%	15.61%	13.93%
1, 0	1.63%	4.56%	5.80%	9.05%	5.55%	20.82%	18.58%
0, 0.5	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.25, 0.5	0.11%	0.37%	0.79%	1.02%	0.74%	2.83%	2.53%
0.5, 0.5	0.24%	0.76%	1.58%	2.05%	1.47%	5.67%	5.06%
0.75, 0.5	0.39%	1.18%	2.37%	3.11%	2.19%	8.50%	7.58%
1, 0.5	0.57%	1.64%	3.16%	4.20%	2.92%	11.33%	10.11%
0, 1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.25, 1	0.07%	0.24%	0.51%	0.66%	0.49%	1.84%	1.64%
0.5, 1	0.15%	0.48%	1.03%	1.33%	0.96%	3.68%	3.28%
0.75, 1	0.24%	0.74%	1.54%	2.00%	1.42%	5.52%	4.93%
1, 1	0.33%	1.01%	2.05%	2.68%	1.90%	7.36%	6.57%
0, 2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.25, 2	0.04%	0.13%	0.28%	0.36%	0.26%	1.00%	0.86%
0.5, 2	0.08%	0.26%	0.56%	0.72%	0.53%	2.00%	1.79%
0.75, 2	0.12%	0.39%	0.84%	1.08%	0.79%	3.00%	2.68%
1, 2	0.17%	0.53%	1.12%	1.45%	1.04%	4.01%	3.58%
0, $\infty$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.25, $\infty$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.5, $\infty$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.75, $\infty$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1, $\infty$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

We assume there are two types of policyholders. Say, 50% of the policyholders are of type 1 and they have no access to the secondary market. The other 50% are of type 2, who have full access to the secondary market. This indicates that  $p = 0.5$ . Moreover, we assume that the policyholders of the two types have on average the same degree of monetary rationality, namely,  $(\underline{\rho}, \bar{\rho}) = (0.03, 0.3)$ . We look at the secondary markets with different sensitivities of demand to the traded price,  $a = 0, 1, 2$ . Since the secondary market is irrelevant for the type 1 policyholders, the contract value for this policyholder type is the same for different  $a$  values, namely, 102.78. When  $a = 0$ , the contract value for the type 2 policyholders is 112.08, see the  $\{(p, a), (\underline{\rho}, \bar{\rho})\} = \{(1, 0), (0.03, 0.3)\}$  entry in Table 1.3. This is higher than the premium at the amount of 106.73, which is to be charged by the insurer when taking the secondary market into account, see the



$\{(p, a), (\underline{\rho}, \bar{\rho})\} = \{(0.5, 0), (0.03, 0.3)\}$  entry in Table 1.1. Thus, the type 2 policyholders benefit from the secondary market, while the type 1 policyholders bear the costs incurred by the secondary market. This is also the case for  $a = 1$ , where the contract value for the type 2 policyholder (105.54) is higher than the premium charged by the insurer (105.52). However, as the demand on the secondary market becomes more sensitive to the traded price, the existence of the secondary market could harm both types of the policyholders. For instance, when  $a = 2$ , we find that the contract value for the type 2 policyholder is 104.26, which is lower than the premium (105.00) charged by the insurer. The benefits from the secondary market are completely absorbed by the secondary market itself. This indicates that when the secondary market is more sensitive to the traded price of the contract, its existence is less attractive to the policyholders.

## 1.5 Conclusion

With regard to equity-linked life insurance contracts, the secondary market increases the value of existing contracts due to its ability to surrender the contracts at a better moment than the primary policyholders. The extra value is shared between the secondary market and the original policyholders, so that the surrender behavior of the policyholders changes. Overall, the risk borne by the insurer increases. If the insurer takes this factor into account, the premiums must be increased. Since the entire extra value is relevant for the insurer while only a part of it is relevant for the policyholders, the premiums could even increase to a level higher than the benefits the policyholders have gained. In this case, the policyholders do not necessarily benefit from the secondary market. Overall, we find that the policyholders may only profit partly from the secondary market. Although the introduction of the secondary market may increase the payout to the policyholders, it is not necessarily beneficial for them if the premium increases at the same time. On a secondary market where the contract demand is insensitive to the traded contract price, those policyholders who are informed of its existence may profit from it, while uninformed policyholders bear the costs incurred by it.

The secondary market brings a challenge to the insurance companies in regard to managing these contracts. Although the primary policyholders may not rush to surrender their contracts simultaneously, the secondary market may be able to do so due to its expertise in managing the contracts. Hence, once the secondary market comes into play, the insurance companies have to deal with the potential liquidity problem caused by the simultaneous surrender of the contracts. We leave the analysis of this liquidity issue to future research.

## 1.6 Appendix to Chapter 1

### 1.6.1 Proof of Proposition 3

*Proof.* From Corollary 3 we see that  $V^I = V^C$  if one of the given conditions is met. The contract parameters can then be specified so that the expected payment to the policyholder equals the expected premium paid by the policyholder at the initial date.  $\square$

### 1.6.2 Proof of Proposition 4

*Proof.* The pairs  $(v, \gamma)$  and  $(v', \gamma')$  are solutions to the PDE (1.12) with respective parameters, see Proposition 1. Consider the difference  $z = v' - v$ . First, the boundary condition of  $z$  is computed, i.e.  $z(T, s) = v'(T, s) - v(T, s) = \Phi(s) - \Phi(s) = 0$ , for all  $s \in \mathbb{R}_+$ . By taking the difference of (1.12) for  $v'$  and  $v$  we obtain the PDE describing  $z$  on  $[0, T) \times \mathbb{R}_+$ , i.e.

$$0 = \mathcal{L}z(t, s) + [\gamma'(t, s, v'(t, s)) - \gamma(t, s, v(t, s))] \left[ (L(t) + \kappa' p' q' (v^{Am}(t, s) - L(t))) - v'(t, s) \right] \\ + \gamma(t, s, v(t, s)) (\kappa' p' q' - \kappa p q) [v^{Am}(t, s) - L(t)] - [r(t) + \mu(t) + \gamma(t, s, v(t, s))] z(t, s).$$

We see that the sign of  $z$  depends on the sign of  $A$  given by

$$A(t, s) = (\gamma'(t, s, v'(t, s)) - \gamma(t, s, v(t, s))) \left( [L(t) + \kappa' p' q' (v^{Am}(t, s) - L(t))] - v'(t, s) \right) \\ + \gamma(t, s, v(t, s)) (\kappa' p' q' - \kappa p q) [v^{Am}(t, s) - L(t)],$$

what can be inferred, e.g., from the Feynman-Kac stochastic representation formula, i.e.

$$z(t, s) = \mathbb{E}_{t,s}^{\mathbb{Q}} \left[ \int_t^T e^{-\int_t^u r(y) + \mu(y) + \gamma(y, S_y, v(y, S_y)) dy} A(u, S_u) du \right].$$

Now,  $A \geq 0$  implies exactly what we want to show, i.e.  $z \geq 0$ , or, equivalently  $v' \geq v$ . To establish this we analyze the first component of  $A$ . For the case  $v' > L + \kappa' p' q' (v^{Am} - L)$  we have  $\gamma' = \underline{\rho}'$  by (1.4). Thus  $(\gamma' - \gamma) \leq \underline{\rho}' - \underline{\rho} \leq 0$  by assumption. Both factors constituting the first component of  $A$  are non-positive, hence their product is non-negative. For the case  $v' \leq L + \kappa' p' q' (v^{Am} - L)$  we have  $\gamma' = \bar{\rho}'$  by (1.4). And then  $(\gamma' - \gamma) \geq \bar{\rho}' - \bar{\rho} \geq 0$  by assumption. Now, both factors are non-negative and so is their product. It remains to investigate the second component of  $A$ . Note that  $\gamma \geq 0$ ,  $\kappa' p' q' \geq \kappa p q$  by assumption, and  $v^{Am} \geq L$ , to see that also the second component of  $A$  is non-negative. This finishes the proof.  $\square$

### 1.6.3 Proof of Proposition 5

*Proof.* First note that the corresponding contract values from the perspective of the representative policyholder we have  $v' = v$  by Proposition 1 and assumption  $\underline{\rho}' = \underline{\rho}$ ,  $\bar{\rho}' = \bar{\rho}$ ,  $\kappa' p' q' = \kappa p q$ . Consequently, we have that  $\gamma' = \gamma$ , where we have also used (1.4). Now, we can apply Corollary 2 for  $u'$  and  $u$ . Taking the difference we see that the first two summands cancel out and we obtain

$$u'(t, s) - u(t, s) = (p' q' - p q) \mathbb{E}_{t,s}^{\mathbb{Q}} \left[ \int_t^T e^{-\int_t^u r(y) + \mu(y) + \gamma y dy} (v^{Am}(u, S_u) - L(u)) \gamma_u du \right].$$

Observe that  $\kappa' \leq \kappa$  implies  $p' q' \geq p q$  by the assumed constraint  $\kappa' p' q' = \kappa p q$ . Finally, it follows that  $u' \geq u$ .  $\square$



## Chapter 2

# The Effect of Risk Preferences on Equity-Linked Life Insurance with Surrender Guarantees

### 2.1 Introduction

Equity-linked life insurance products combine a classical term life insurance and a savings component, which invests the policyholder's premiums in equity. Equity-linked life insurance contracts form a large market in many developed countries, for example, the US market has a volume of over \$100 billion annual sales, c.f. Hardy (2003). Commonly, equity-linked life insurance products include a capital guarantee plus a minimum interest rate guarantee, and the option to surrender the contract. This investment guarantee is risky for insurers because, unlike mortality risk, the equity risk cannot be diversified across cohorts as it is identical for a large number of insurance contracts (Hardy (2003)). In particular, in combination with the equity risk, the surrender of insurance policies is an important risk for insurers as identified by European Union regulators. (Eling and Kochanski (2012)).

A large insurance literature considers how to price equity-linked life insurance contracts based on the Black and Scholes (1973) option pricing theory. This literature considers the equity-linked life insurance contract in isolation of the policyholder's general economic situation. The embedded surrender guarantee of the insurance contract takes the form of an American option, which Bacinello (2005) prices in a Binomial model. Many other works price life insurance contracts with and without surrender guarantees in continuous time: Grosen and Jorgensen (1997) price early exercise interest guarantees and apply it to life insurance contracts in Grosen and Jorgensen (2000). Shen and Xu (2005) use a partial differential equation approach to value equity-linked life insurance

contracts with surrender guarantees. All these previous works assume a policyholder who has unlimited access to the perfect financial market. In particular, the policyholder can follow any financial strategy and replicate any derivative.

Empirically, the policyholder's surrender behavior depends on the economic situation of the policyholder in general, and the financial performance of the insurance contract, which depends on the financial markets. A cointegration approach to the surrender rate of US policyholders by Kuo et al. (2003) links the surrender rate to both the unemployment rate and the interest rate. A high unemployment increases surrenders, because policyholders cannot afford premiums and need access to their savings. In situations with high interest rates, the policyholder's surrender rate increases, because alternative investments perform better compared to the equity-linked life insurance contract.

Recently, Li and Szimayer (2014) and De Giovanni (2010) include bounded rationality into the pricing of the equity-linked life insurance, but do not endogenize this bounded rationality. Instead, they describe the policyholders surrender behavior by a doubly stochastic Poisson process. This Cox process produces financially suboptimal surrender behavior. These studies extend Albizzati and Geman (1994), who analyze the surrender guarantee from the perspective of a representative policyholder and derive the dependence of the surrender decision on stochastic interest rates.

In this paper, I analyze the influence of policyholder's risk preferences on the valuation and surrender behavior of equity-linked life insurance contracts with surrender guarantees. The literature on behavioral finance and economics documents the impact of bounded rationality on human decision in a financial context. For example, cumulative prospect theory of Tversky and Kahneman (1992), which refines the original prospect theory of Kahneman and Tversky (1979) explains decision making under uncertainty. Barberis (2012) explains how gamblers behave in casinos if they follow cumulative prospect theory and shows how gamblers stop gambling differently from expected utility maximizers.

To study the impact of behavioral preferences on surrender behavior, I extend the bounded rationality approach for a representative policyholder in spirit of Albizzati and Geman (1994). The policyholder considers his life insurance contract in isolation of his other investments as he uses mental accounting for his life insurance contract. This mental account consists of the value of the life insurance from the policyholder's perspective as it moves over time. Mental accounting is introduced by Thaler (1980, 1985) and applied to finance by Shefrin and Statman (1985), who explain how investors use isolated mental accounts for each stock investment they own. This mental account also is in line with the narrow framing of Tversky and Kahneman (1981).

In my paper, the policyholder considers his insurance value in his mental account over time, from the signing of the policy up to the maturity date. If the policyholder

surrenders his insurance policy or the policy reaches maturity, he receives the respective benefit from the insurance company. Then, the policyholder closes his mental account “life insurance” and evaluates the payoff he receives. The policyholder only draws utility from realizing payments as gains and losses, but not from observing the insurance contract’s performance during the lifetime of the contract (c.f. Barberis and Xiong (2012)). The policyholder makes his surrender choice based on how his realization utility of the realized payment develops: If the utility of immediate surrender exceeds the expected utility for later times, he surrenders.

In my model, the policyholder follows loss averse preferences given by Tversky and Kahneman (1991) to analyze the payoff he receives when he closes his mental account “life insurance”. He considers his life insurance policy in comparison to a reference point. This reference point splits the outcome of the equity-linked life insurance contract into gains and losses. The policyholder takes the insurance as a loss, if its payoff falls short of his reference point. The reference point of the policyholder is for example his expectation of the insurance payoff, or his initial investment in the insurance contract. Empirically, losses loom larger than gains. Furthermore, the preferences exhibit diminishing sensitivity of returns, which produce in combination with loss aversion the S-shape of the policyholder’s utility function. As a special case, I also consider standard expected utility theory in form of the power utility function.

The main contribution of this paper is twofold: Technically, I derive the expected utility of the policyholder under loss averse preferences by applying dynamic programming. The resulting optimal control problem requires a viscosity solution for the resulting Hamilton-Jacobi-Bellman equation. Ebert, Koos, and Schneider (2013) and Doskeland and Nordahl (2008) consider equity-linked life insurance and pension insurance contracts, respectively, in combination with loss averse preferences. These papers explain the demand for products with investment guarantees by cumulative prospect theory. Furthermore, expected utility with standard preferences alone cannot explain the demand for guarantees in life and pension insurance. However, they do not analyze surrender guarantees. Chen, Hentschel, and Klein (2013), however, show expected utility policyholders to prefer guarantees in presence of mortality risk.

In economic terms, I endogenize the surrender behavior of the policyholder and explain the exogenously assumed increased surrender behavior of policyholders in the literature, e.g. by Li and Szimayer (2014) by risk preferences. Compared to the financially optimal surrender behavior implied by the option pricing literature, I show that the policyholder deviates from this financially optimal surrender behavior in two ways: Firstly, the policyholder surrenders contracts earlier if he is sufficiently risk-averse. The life insurance’s embedded guarantee usually offers a lower return compared to the risk-free return. If

the underlying fund of the equity-linked contract performs poorly, the policyholder surrenders earlier because he wants to secure the risk-free return for his premium, but the potential chance to receive gains from a recovery of the fund plays a small role. This effect is also present for the optimal stopping in the Black and Scholes (1973) sense, but it is more pronounced for risk-averse policyholders.

Furthermore, I show the importance of the reference point in combination with loss aversion to the policyholder's surrender behavior. The reference point controls an important part of the surrender behavior: Over the lifetime of the contract, the surrender becomes relatively less attractive because the policyholder receives higher risk-free rate for a shorter period of time compared to the guaranteed rate of the contract. If the reference point allows potentially both gains and losses with high probability, then the surrender value will switch from being a gain to a loss at some time. At this time, the policyholder's loss aversion triggers surrender for higher fund levels, because the policyholder wants to secure the contract as a gain. If considered for a pool of policyholders, this triggers a surrender peak of the representative policyholders as the policyholder wants to avoid the loss.

Finally, I quantify the loss of the policyholder due to the surrender following his risk preferences. I show that the policyholder loses up to a quarter of his insurance value because he surrenders sub-optimally.

The remainder of this paper is organized as follows: Firstly, I introduce the model of the financial market underlying the asset linked to the insurance by the equity-linked contract and the policyholders preferences in Section 2.2. Secondly, in Section 2.3 I evaluate the contract from the perspective of the policyholder both fully rational and under risk preferences. Section 4 studies the differences in the contract valuation due to the different exercise strategies. Section 5 concludes.

## **2.2 A Model of Equity-Linked Life Insurance and Policyholder's Risk Preferences**

Our model consists of two parts: The first part models the insurance contract which is linked to equity in form of a risky asset in the Black and Scholes (1973) financial market model. The second part models the insurance policyholder who is described by his risk preferences through a utility function.



## 2.2.1 Equity-Linked Life Insurance Contract

The insurance contract is traded on a Black and Scholes (1973)-type financial market which is defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, \mathbb{P})$ . The financial market consists of a risky fund  $S$ , and a risk-free bond  $B$  with the dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}, \quad S_0 > 0, \quad (2.1)$$

$$dB_t = r B_t dt, \quad B_0 = 1. \quad (2.2)$$

The risky asset  $S$  depends on the drift  $\mu$ , volatility  $\sigma$ , and is driven by a standard Brownian motion  $W^{\mathbb{P}}$  under the real world probability measure  $\mathbb{P}$ . The financial market is complete and arbitrage free.

I consider a typical equity-linked life insurance contract. Hardy (2003) provides an overview for the different popular variations of this type of insurance. Our equity-linked contract's performance is linked to the performance of the underlying fund  $S$  according to the participation coefficient  $k$ , which in practice satisfies  $0 < k < 1$ . The participation coefficient describes how heavily contract's return depends on the fund  $S$ : The higher  $k$ , the more the contract's return depends on  $S$ . The contract is embedded with a guaranteed rate of  $g$  for a fraction of  $\alpha$  on the initial premium  $P$  paid upfront. Hence, this fraction of the capital is protected from losses of the fund; the policyholder always receives this guaranteed rate  $g$ . If the participation part of the contract exceeds the guaranteed rate, the policyholder gets his participation share of the fund. Thus, the contract has the terminal payment of

$$\Phi(S_T) = \alpha P \cdot \max \left\{ (1 + g)^T, \left( \frac{S_T}{S_0} \right)^k \right\}. \quad (2.3)$$

Furthermore, the policyholder has the right to surrender his policy early at any time for a surrender benefit of  $L$ , which satisfies

$$L(t) = \alpha P \cdot (1 + h)^t \cdot (1 - \beta_t). \quad (2.4)$$

Here, the upfront premium is compounded with the termination rate  $h$ , but the equity-linked part is removed. If the policyholder chooses to surrender early, he loses his terminal payment of  $\Phi$ . On the guaranteed rate  $h$ , there usually are legal restrictions. For example in Canada it may not fall below the guaranteed rate of the contract  $g$  (Bernard and Lemieux (2008)). Furthermore, early termination of the contract is punished by the penalty function  $\beta$ , which is decreasing in time.

For simplicity, there is no mortality in the model. This paper focuses on the impact

of surrender on life insurance contracts and more generally the policyholder’s perception of insurance contracts as investment products. Chen et al. (2013) model mortality from a utility perspective, where the policyholder’s death benefit earns the risk-free rate until maturity. The effect of mortality is delicate because the policyholder has to anticipate his own mortality probability. Andersson and Lundborg (2007) show empirically that people underestimate their own death risk, for example road traffic risk. For our equity-linked life insurance, this approach delivers the policyholder a death benefit, which is identical to his terminal benefit, early with a low probability. The probability of death is low for usual customers of this type of life insurance, as they usually are employees who use the insurance to save for retirement and do not use it primarily as life insurance. Hence, the effect of mortality in this framework can safely be ignored, cf. Doskeland and Nordahl (2008).

## 2.2.2 Policyholder’s Risk Preferences

Our investor considers his life insurance in isolation of his other investment activities. This way of evaluation is consistent with the concept of *narrow framing* and *mental accounting*. People do not combine risky choices, but consider each decision in isolation, which Tversky and Kahneman (1981) label as narrow framing. The similar concept of mental accounting (Thaler (1980, 1985)) explains how people use mental accounts for separate activities to keep order in their finance. Furthermore, people use mental accounting as a tool to avoid self-control problems. Thaler (1985) gives the example of a household financing decision: A couple does not repay a credit with their savings for a holiday home, because they are afraid they will never replace the savings, although the credit has a higher interest rate as the couples’ saving account. Shefrin and Statman (1985) explain how investors use isolated mental accounts for each stock investment they own. The investor does not mentally connect his stock investments, in particular, the investor does not use interdependencies. In our context, the policyholder has a mental account “life insurance”, which consists of the running value of his insurance. If the policyholder receives the terminal value  $\Phi$  or surrenders the contract for  $L$ , he closes his mental account and evaluates the outcome. Hence, he derives utility from realizing the value of his mental account. This concept is termed realization utility by Barberis and Xiong (2012).

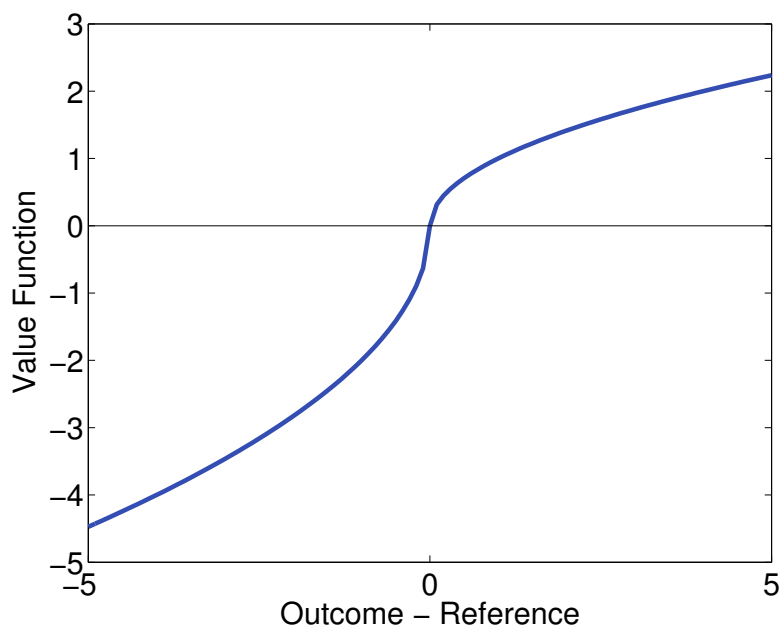
The next part of our model for the policyholder’s reception of the life insurance is *how* he judges his mental account for his insurance. We describe this part of the policyholder’s decision making by his risk preferences. While mainstream economics uses the concept of risk aversion for a long time, the descriptive power of a purely risk-averse decision maker in experimental and real world data is limited. A popular alternative to model

risk preferences is to include loss aversion. Hence, according to Tversky and Kahneman (1991)'s theory of loss aversion, the policyholder has a utility function  $U : \mathbb{R}^+ \rightarrow \mathbb{R}$ , which has the following form: <sup>1</sup>

$$U(x) = \begin{cases} (x - R)^\gamma, & x \geq R \\ -\lambda(-(x - R))^\gamma, & x < R \end{cases} \quad (2.5)$$

Hereby, the policyholder's perception of a random outcome  $x$  of a lottery depends on a reference point  $R$ . The reference point models how people do not evaluate outcomes of random processes in absolute terms, but relative to some reference value, say their expected outcome or the status quo.

**Figure 2.1:** Example of a utility function as a function of the outcome of a gamble relative to the policyholder's reference point. The utility function has a  $S$ -shape parameter of  $\gamma = 0.5$  and a loss aversion of  $\lambda = 2$ .



Furthermore, losses loom larger than gains: the policyholder feels the pain of losses stronger than the joy over gains. The loss aversion parameter  $\lambda$  measures how much more pain from losses the policyholder feels compared to joy over gains. Finally, the utility function displays diminishing marginal sensitivity of both gains and losses. The parameter  $\gamma$  captures this diminishing sensitivity of returns and creates the  $S$ -shape of prospect theory. Figure 2.1 illustrates the features of the loss averse utility function:

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<sup>1</sup>I use the notion of utility function instead of Tversky and Kahneman (1991)'s value function.

It plots the utility function for a sample parametrization which delivers a pronounced  $S$ -shape with a  $S$ -shape parameter of  $\gamma = 0.5$  and a strong loss aversion by  $\lambda = 2$ .

The nature of the equity-linked life insurance contract with surrender guarantees implies payments potentially spread out over the whole time period between the signing of the insurance policy up to the maturity. As the policyholder can surrender the contract at any point in time, the time of the payment is uncertain. This aspect requires the policyholder to compare payments at different time points and evaluate them according to his preferences. To achieve comparability, the policyholder takes into account only payments at maturity. The policyholder transfers any payment he receives before maturity to the terminal time  $T$  by investing in the risk-free bank account. This approach alters Ahn and Wilmott (1998) because the policyholder considers the end value of his mental account instead of the present value. The end value is a natural choice for life insurance, because policyholders benchmark the contract to their final outcome. This time transfer of money eliminates the need to use time dependent references.

## 2.3 Contract Valuation

The first part of the problem of evaluating the contract is the rational valuation with full access to the financial market in the Black and Scholes (1973) model to establish a benchmark. This rational valuation refers to the optimal exercise strategy of the American option embedded in the insurance contract.

The second subsection derives the policyholder's valuation under loss averse preferences and the behavioral exercise strategy. Here, the policyholder cannot access the financial market directly except through the equity-linked life insurance contract. Consequently, the policyholder cannot replicate the insurance contract and use risk-neutral pricing. As shown by Ahn and Wilmott (1998), the policyholder then does not want to stop optimally according to the Black-Scholes strategy, and also cannot lock in potential profits due to his limited access to the underlying.<sup>2</sup>

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<sup>2</sup>In this paper, the policyholder cannot trade the risky asset to replicate his own contract. Alternatively, the policyholder is not smart enough to replicate. The lock in strategy works as follows: At surrender, the optimally stopped contract has a value  $V^{opt} \geq V_t$ , the current value. If  $V_{opt} = V_t$ , the policyholder surrenders and thus stops optimally. If  $V_{opt} > V_t$ , the policyholder replicates the payoff by adequate positions in the underlying and the risk free bond according to Delta hedging. He sells a fraction of this portfolio to give him the current value  $V_t$  now, and keeps the rest until maturity. With this strategy, the policyholder does not lose anything compared to the American stopping strategy.

### 2.3.1 Benchmark Solution: Valuation as American Option

The policyholder has unlimited access to the financial market, in particular, the policyholder can trade both the risk-free and the risky asset without limitations in continuous time, which allows the pricing of the insurance contract. Because the financial market is complete and arbitrage free by assumption, there exists a unique risk-neutral probability measure  $\mathbb{Q}$ , under which the policyholder prices the contract as an American style contingent claim in the Black-Scholes framework.

The standard approach to price this equity-linked life insurance with surrender guarantee is to derive the pricing partial differential equation (PDE) for the contract using risk-neutral pricing approach, for example as in Shen and Xu (2005). The optimal stopping time is  $\tau^{Am}$ , which is the first time where the surrender benefit exceeds the value of continuing the contract. In the remainder of the paper, I label the optimal stopping time  $\tau^{Am}$  the *American stopping time*. Formally, we have for a continuation value  $V(t, S_t)$  of the insurance value the following representation of the American stopping time  $\tau^{Am}$ :

$$\tau^{Am} := \inf \{t \in (0, T] : L(t) \geq V(t, S_t)\} \quad (2.6)$$

This formulation allows us to write the pricing value of the insurance contract as the solution to the free boundary problem on  $t < \tau^{Am} \wedge T$  with  $x \wedge y$  denoting the minimum of  $x$  and  $y$ :

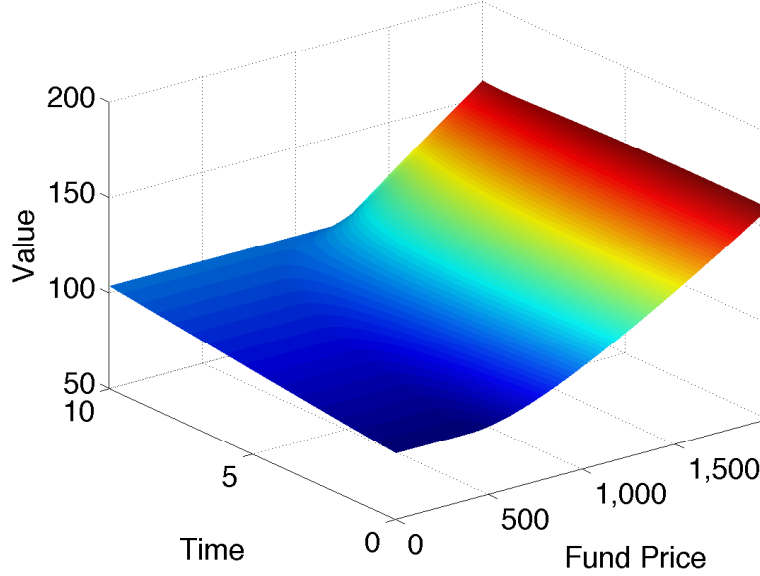
**Proposition 6** (Pricing PDE). *The value of an American-style option describes the maximal possible value of the contract and assumes no financial constraints in and unlimited access to an efficient financial market. On  $\{t < \tau^{Am} \wedge T\}$  the value of the American option is  $V_t = v(t, S_t)$  where  $v : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is continuously differentiable in time, twice continuously differentiable in price, and satisfies the free boundary value problem given by the PDE*

$$\frac{\partial v}{\partial t}(t, s) + rs \cdot \frac{\partial v}{\partial s}(t, s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2}(t, s) - rv(t, s) = 0 \quad (2.7)$$

with constraint  $v(t, s) \geq L(t)$  on  $[0, T] \times \mathbb{R}_+$  and terminal condition  $v(T, s) = \Phi(s)$ , for  $s \in \mathbb{R}_+$ .

Since the value of this contract cannot be established in closed form, we use the finite difference methodology to price this contract. The Crank-Nicolson Scheme, derived in Crank and Nicolson (1947), allows us to evaluate this free boundary problem effectively. We use the Crank-Nicolson Scheme to solve the grid of 2,000 price steps and 2,000 time steps. The Crank-Nicolson method is unconditionally stable and under some regularity assumptions, the order of the method is  $O(\Delta s^2) + O(\Delta t^2)$ , with  $\Delta s$  and  $\Delta t$  being the step

**Figure 2.2: Rational pricing partial differential equation.** The grid displays the contract value for all time-fund price combinations if priced as an American option. The grid value at time zero gives the current insurance price for an initial fund level of 1000. The contracts value is 103.5921.



sizes of the mesh in the  $s$  and  $t$  direction, respectively. In our study, we use  $\Delta s = 0.0025$  and  $\Delta t = 0.0050$ . A detailed discussion of the Crank-Nicolson method is beyond the scope of our paper. For an extensive discussion c.f. for example Seydel (2012).

Figure 2.2 displays the pricing PDE of Proposition 6 for the established grid for a sample parametrisation of a maturity of 10 years, an upfront premium  $P = 100$ , a participation coefficient of 90%, a guaranteed rate  $g = 2\%$ , and a guaranteed amount of  $\alpha = 85\%$  for the contract. The guaranteed rate for surrenders is also  $h = 2\%$ . The financial market offers a risk-free interest rate of  $r = 3.5\%$ , and a fund with volatility of  $\sigma = 30\%$ , which gives a contract value of 103.5921.

The exercise boundary for the contract is the set of  $(s, t)$ -tupels for which the exercise of the surrender guarantee is beneficial to the investor. Formally, we define the early exercise region to be

$$H := \{(s, t) \in \mathbb{R}^+ \times [0, T] : L(t) \geq V(s, t)\}. \quad (2.8)$$

Because of the risk-neutral valuation, this surrender behavior is a worst case scenario in terms of hedging. The price of the hedging strategy is the value of the contract derived in Proposition 6. Because this strategy gives the maximal contract value, any deviation from this exercise for example due to bounded rationality or risk aversion lowers the price of the contract.

**Figure 2.3: Exercise boundary for the rational pricing approach.** The graph displays the optimal exercise boundary  $H$ : Above the graph are the time-fund value combinations for which the insurance contract's surrender is not optimal. Below the graph are the time-fund value combinations for which surrender gives higher value than continuation. The initial fund value is 1000.

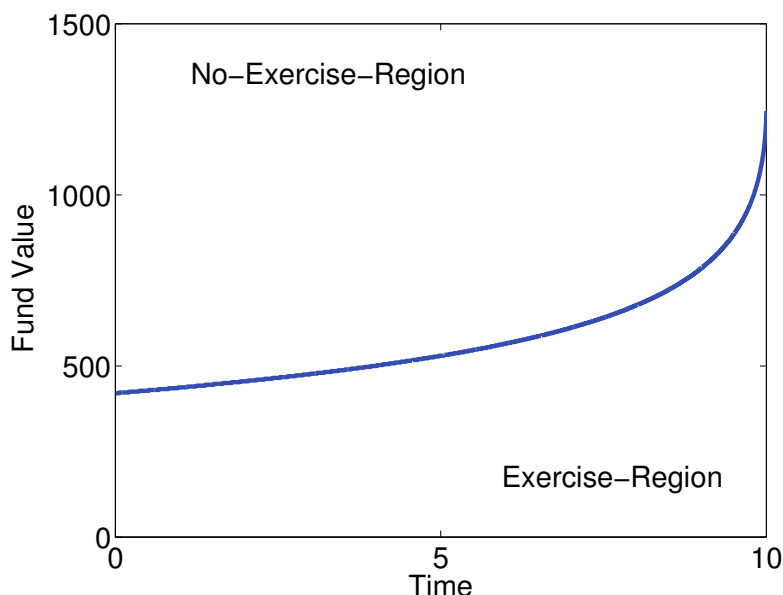


Figure 2.3 shows the exercise boundary of the insurance contract for the above parametrization. Firstly, the guaranteed rate of the contract implies an embedded put option for the equity-linked part of the insurance contract. Figure 2.3 shows that it is rational to exercise the contract if the underlying fund drops far enough because the contract offers a lower guaranteed rate than the financial market offers on the risk-free account. Hence, if it is unlikely for the equity-component to end up in the money, the insurance value is lower compared to the surrender benefit.

Secondly, the exercise boundary is rising in time. If the contract is closer to maturity, the policyholder surrenders the insurance contract for higher prices of the underlying fund. This effect occurs because of the guaranteed rate of the insurance contract is lower than the risk-free rate. Hence, if the underlying fund of the insurance is substantially below the initial investment value, it's probability to end up above the guaranteed rate during the contract's remaining lifetime decreases in time. If the probability of this event is low enough, it is preferable for the investor to surrender the contract and invest his money for the risk-free rate.

### 2.3.2 Behavioral Solution: Valuation under Risk Preferences

We now consider the policyholder with loss averse risk preferences. Hence, the insurance policyholder has a different way to value his contract. Firstly, he maximizes his utility function introduced in Section 2.2.2 instead of the financial value. Secondly, the policyholder is not a financial professional. In particular, he cannot price the contract optimally in the sense of an American option like in Section 2.3.1. The policyholder does not have access to the financial market outside of a bank account and the equity-linked contract. The policyholder cannot trade the underlying fund of the insurance contract. Hence, the American stopping time computed in the previous section is not necessarily optimal for the policyholder. Following the argument of Ahn and Wilmott (1998), a financial professional able to trade the underlying fund could use the American exercise strategy to hedge the payoff which results from the optimal stopping according to his preferences and receive an additional profit from hedging his contract. However, the policyholder's lack of access to the financial market prevents this strategy.

In this section, I apply optimal stopping to value the insurance contract from the perspective of the policyholder under loss averse preferences. The policyholder chooses the behavioral stopping time  $\tau^B$  to maximize his expected utility from the contract. The stopping time  $\tau^B$  carries the label *behavioral stopping time* for the remainder of the paper. We allow  $\tau^B$  to take values in  $\tau^B \in [t, T]$  for an alive contract at  $t$ . The case of  $\tau^B = t$  represents immediate surrender, while the policyholder keeps the contract until maturity if  $\tau^B = T$ . The policyholder never prefers to have the surrender value  $L(T)$  at  $T$  by assumption, because the payoff at maturity  $T$  is  $\Phi(S_T)$ , which is always at least as high as the surrender value. Formally, we have

$$\max \{ \Phi(S_T), L(T) \} = \Phi(S_T). \quad (2.9)$$

The policyholder choice variable is the behavioral stopping time  $\tau^B$  at time  $t \in (0, \tau^B \wedge T)$  for not stopped contracts. We write the policyholder's gain criterion as

$$J(t, S_t, \tau^B) = \mathbb{E}_{t, S_t} \left[ U \left( e^{r(T-\tau^B)} \Psi(\tau^B, S_{\tau^B}) \right) \right]. \quad (2.10)$$

Here, policyholder's payments are transferred to maturity using the bank account. The function  $\Psi$  is the contract's payoff at a given time and takes the surrender payoff for premature surrender and the terminal payoff at maturity as its value. We can write it



formally as

$$\Psi(\tau^B, S_{\tau^B}) = \begin{cases} L(\tau^B) & \tau^B < T \\ \max\{\Phi(S_T), L(T)\} = \Phi(S_T), & \tau^B = T. \end{cases} \quad (2.11)$$

The policyholder's behavioral stopping time differs from the American stopping time  $\tau^{Am}$ .

The policyholder faces the optimal stopping problem

$$u(t, S_t) = \sup_{\tau^B \in \mathcal{T}} J(t, S_t, \tau^B). \quad (2.12)$$

This optimal stopping problem allows us to derive the dynamic programming equation in the sense of a viscosity solution. Because the utility function is not continuously differentiable in contrast to the contract value in Proposition 6, the approach to derive our pricing PDE in Proposition 6 does not extend to the policyholder's optimal stopping problem in Equation (2.12). Hence, we use the concept of viscosity solutions, which relaxes the assumptions on differentiability of both the value function of the control problem, and the argument of the expectation in the gain criterion.<sup>3</sup>

The function in the expectation of Equation (2.10) is continuous, and the target function of the optimal stopping problem (2.12) is locally bounded. According to Touzi (2013), we thus have the dynamic programming equation in the following proposition:

**Proposition 7** (Expected Utility PDE). *On  $\{t < \tau^B \wedge T\}$  the expected utility of the equity-linked life insurance with surrender benefit is  $u_t = u(t, S_t)$  where  $u : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfies the obstacle problem*

$$\min \left\{ - \left( \frac{\partial u}{\partial t}(t, s) + \mu s \cdot \frac{\partial u}{\partial s}(t, s) + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 u}{\partial s^2}(t, s) \right), u_t - U(e^{r(T-t)} L(t)) \right\} = 0 \quad (2.13)$$

with terminal condition  $u(T, s) = U(\Phi(s))$ , for  $s \in \mathbb{R}_+$ .

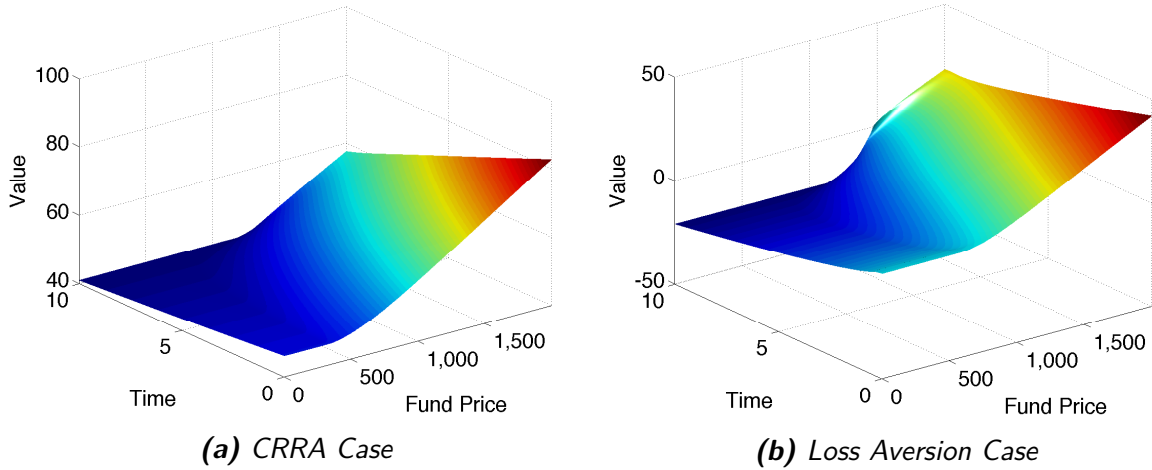
Figure 2.4 displays the policyholder's expected utility PDE derived in Proposition 7 for the contract of the previous section.<sup>4</sup> In Figure 2.4a, the utility function is power-utility

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<sup>3</sup>A viscosity solution is a solution concept for stochastic control problems. The viscosity solution approach solves Hamilton-Jacobi-Bellman equations for locally bounded functions. Pham (2009) gives a detailed account of viscosity solutions.

<sup>4</sup>The Crank-Nicolson scheme derived in the previous section provides the viscosity solution of Proposition 7. In general, the numerical schemes for PDEs do not necessarily extend to sensible numerical schemes for viscosity solutions. In our example, the Crank-Nicolson method works under the regularity condition  $\mu = \sigma^2$ . If this assumption is violated, the Crank-Nicolson scheme no longer converges to the viscosity solution and the implicit scheme is used. For a detailed discussion of the numerical solution of optimal control problem via PDEs for viscosity solutions in finance, see Forsyth and Labahn (2007).

**Figure 2.4: Policyholders utility partial differential equation.** Graph a) displays the utility PDE for a policyholder with constant relative risk aversion, graph b) displays the utility PDE for a policyholder with an S-shaped utility function. The graphs display the expected utility for the time-fund value combinations over the lifetime of the insurance contract.



with a risk aversion parameter of  $\gamma = 0.8$ . In Figure 2.4b, the policyholder follows a utility function with loss aversion of  $\lambda = 2.25$ , a S-shape parameter of  $\gamma = 0.8$ , and a reference point of the initial premium invested at the risk-free rate. Formally,  $R = \alpha P(1 + r)^T$ .

Figure 2.4a displays the diminishing marginal sensitivity of returns to the policyholder. Figure 2.4b shows additionally the pronounced kink at the reference point which separates gains and losses. Here, you can see the large impact of losses on expected utility. The policyholders surrender behavior under risk aversion has a similar structure as the surrender behavior induced by the optimal surrender strategy. However, the level and shape of the policyholder’s exercise boundary depends on his risk preferences and his loss aversion.

## 2.4 The Impact of Risk Preferences

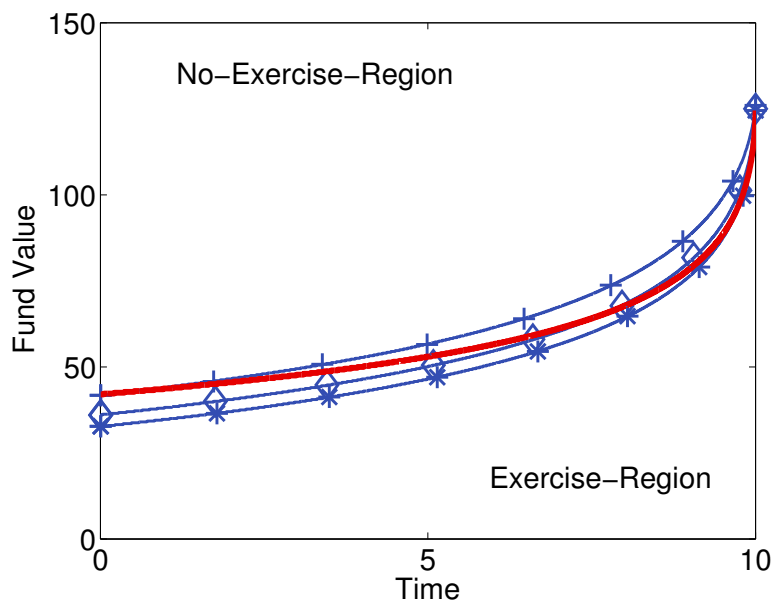
In this section, we take the above behavioral solution and show how the behavioral risk preferences change the policyholder’s surrender behavior. This bounded rationality approach implies a surrender behavior different from both the optimal stopping and the expected utility based surrender behavior. Firstly, risk aversion changes the surrender level. Dependent on the policyholder’s risk aversion, the exercise boundary increases while the shape of the exercise boundary is unchanged. Secondly, loss averse preferences change the level of the exercise boundary and shape the exercise boundary and produce a surrender peak at the reference point. Finally, we show how costly bounded rationality

is to the policyholder.

## 2.4.1 Risk Aversion

In this section, we analyze the impact of risk aversion on the surrender behavior. For standard expected utility preferences the surrender behavior changes dependent on the level of risk aversion. However, the shape of the exercise region remains similar.

**Figure 2.5: Exercise boundary for varying risk aversion.** The graph displays the exercise and no-exercise regions for a policyholder with constant relative risk aversion for the risk aversion levels of  $\gamma = 0.6$  (+),  $\gamma = 0.8$  ( $\diamond$ ),  $\gamma = 1$  (\*), and as a benchmark, the exercise region for the optimal surrender (bold). For each line, the time-fund value combinations above the line have a higher continuation value than surrender value, while below the lines, it is optimal to surrender the insurance for the policyholder with the respective preferences.



Let us consider the case of standard expected utility, for example power utility. Figure 2.5 displays the exercise boundary from the policyholder's perspective for the previous parametrization.

The policyholder's risk aversion implies surrender at higher fund values for higher risk aversion. This increased exercise region occurs because the policyholder has diminishing sensitivity of wealth. If he has high constant relative risk aversion parameter  $\gamma$ , he profits less from high returns of the fund but gains a lot of utility from certain gains. He is willing to switch to the risk-free bank account if the policy will end up with the guaranteed rate with high probability. However, for a low level of risk aversion the policyholder surrenders for lower fund values compared to the optimal strategy, as the policyholder works under

the real world probability measure  $\mathbb{P}$ , in particular the policyholder calculates with the real world drift  $\mu$  of the fund, which is usually higher than the risk-free rate.

**Result 1 (Risk Aversion).** *If the policyholder is risk averse, the fund level for which policyholder's surrenders, increases in the risk aversion. The shape of the exercise boundary is close to the shape of the optimal exercise boundary.*

If the policyholder cannot use the Black-Scholes surrender strategy, the maximal expected utility of a risk-neutral policyholder provides the optimal surrender strategy for the policyholder. The exercise boundary of this risk-neutral policyholder is different from the optimal stopping exercise boundary with  $\tau^{Am}$ .

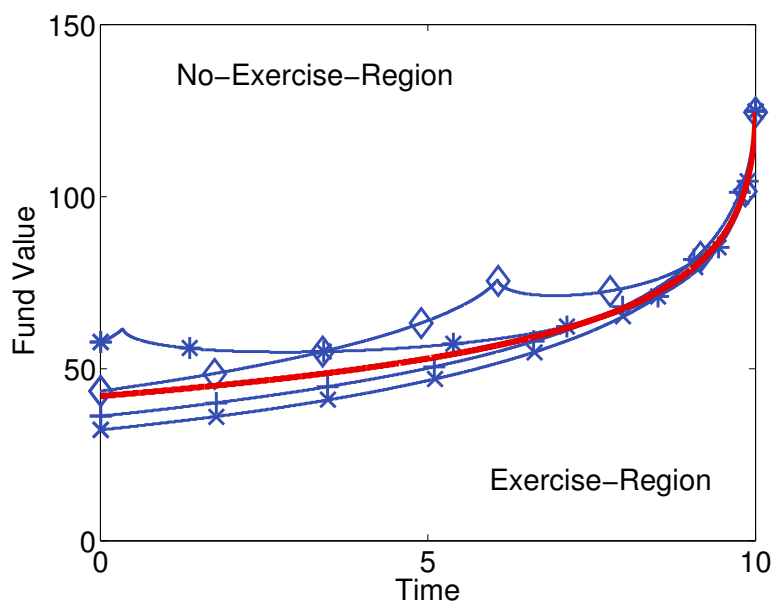
## 2.4.2 Loss aversion and reference point

Given our analysis of the power utility case, we can now turn to the impact of loss aversion and the reference point. The introduction of risk aversion alone only changes the fund level at which the policyholder surrenders his contract, but it does not systematically change the shape of the exercise boundary. If we take a look at the case of loss averse policyholders, we see a changed shape of the exercise boundary around the reference point. The reference point captures the expectations of the policyholder for his insurance policy, in particular the reference point determines whether the insurance investment is a gain or a loss. For the policyholder, we distinguish three possible scenarios: Two reference points in the area of the initial investment or the risk-free rate, a reference point below the lowest payment of the contract, and a reference point which is much higher than anything the market offers.

Figure 2.6 displays four cases of reference points with the respective exercise boundaries in comparison to the optimal surrender boundary for the previous parametrisation. The four reference points are  $R = 20$  (+),  $R = 110$  ( $\diamond$ ),  $R = 120$  (\*), and  $R = 300$  (x).

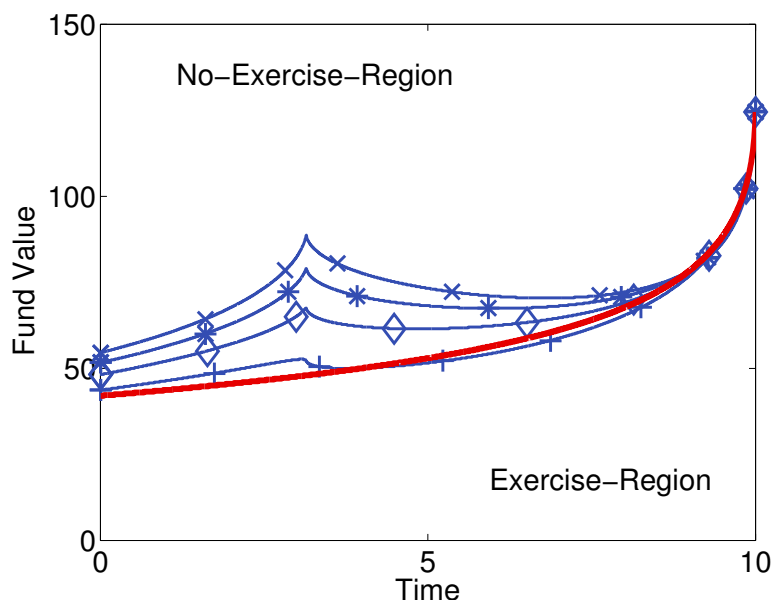
Consider the cases of  $R = 110$  ( $\diamond$ ) and  $R = 120$  (\*) in Figure 2.6. The policyholder has reasonable expectations about the insurance contract, for example his benchmark is the risk-free rate or his initial investment in the contract. These reference points produce a sharp difference to the classical expected utility case with power utility. The surrender value influences the exercise boundary mostly when the surrender value is close to the reference point. Close to the contract's beginning, the surrender value reaches the gain area, as the surrender benefit earns the risk-free rate up to maturity, which is higher than the guaranteed rate. As time progresses, the surrender benefit earns the lower guaranteed rate for a longer period, which pushes the surrender value down to the loss area. If the contract's terminal payment will fall short of the reference point because of the fund's underperformance, the policyholder will surrender the contract at higher fund

**Figure 2.6: Exercise boundary for varying reference points.** The graph displays the exercise and no-exercise regions for a policyholder with S-shape preferences for the reference points of  $R = 20$  (+),  $R = 110$  ( $\diamond$ ),  $R = 120$  (\*),  $R = 300$  (x), and as a benchmark, the exercise region for the optimal surrender (bold). For each line, the time-fund value combinations above the line have a higher continuation value than surrender value, while below the lines, it is optimal to surrender the insurance for the policyholder with the respective preferences.



levels to avoid this loss. After the surrender value has passed the gain area and gives the policyholder a loss under certainty, the policyholder tries to reduce his loss. The policyholder will then surrender the contract only if the loss from the contract will be higher than the loss from the surrender. Furthermore, the policyholder sticks with the insurance contracts risk of a lower fund longer, because he gambles on the funds recovery. The diminishing sensitivity for losses implies this behavior in the policyholder.

**Figure 2.7: Exercise boundary for varying loss aversion.** The graph displays the exercise and no-exercise regions for a policyholder with S-shape preferences for the loss aversion parameters  $\lambda = 1$  (+),  $\lambda = 2$  ( $\diamond$ ),  $\lambda = 3$  (\*),  $\lambda = 4$  (x), and as a benchmark, the exercise region for the optimal surrender (bold). The reference point is  $R = 115$ . For each line, the time-fund value combinations above the line have a higher continuation value than surrender value, while below the lines, it is optimal to surrender the insurance for the policyholder with the respective preferences.



If the policyholder has a low reference point compared to the contract payments, all payments of the contract are gains to the policyholder. The reference point  $R = 20$  (+) displays the low reference point case in Figure 2.6. Since all payments are gains, the policyholder's preference for this lottery is almost identical to the power utility case. However, this case is economically unreasonable.

If the reference point of the policyholder is high, the fund can only end up in the policyholder's gain region with low probability. In this case,  $R = 300$  (x) in Figure 2.6, the policyholder's exercise is close to the optimal surrender because the policyholder tries to maximize the contract's payoff in order to avoid high losses. However, this case is not realistic in economic terms, because this scenario requires the policyholder to expect annual returns well above the stock return.

**Result 2** (Impact of Reference Point). *The policyholder's reference point controls the shape of the exercise boundary. The reference point induces a surrender peak when the surrender benefit flips from gain to loss. If the reference point is so low or high, the reference point implies only gains or losses. In this case, the reference point does not imply a surrender peak anymore.*

The loss aversion  $\lambda$  of the policyholder controls the reference point's impact magnitude. A higher loss aversion increases the difference in utility the policyholder feels between gains and losses. Hence, the switch from the surrender value from gain to loss has a higher opportunity cost to the policyholder. Figure 2.7 displays for the given parametrization the impact of the overweighting of losses  $\lambda$  relative to the gains compared to optimal surrender. Clearly, the higher loss aversion induces a substantially increased fund level for early surrender. Furthermore the loss aversion and the reference point induce a sharp peak in surrenders at the time when the surrender value flips from gain to loss.

**Result 3** (Impact of Loss Aversion). *The policyholder's loss aversion controls the size of the surrender peak and the fund level for which the policyholder surrender his contract. A high loss aversion produces a sharp surrender peak.*

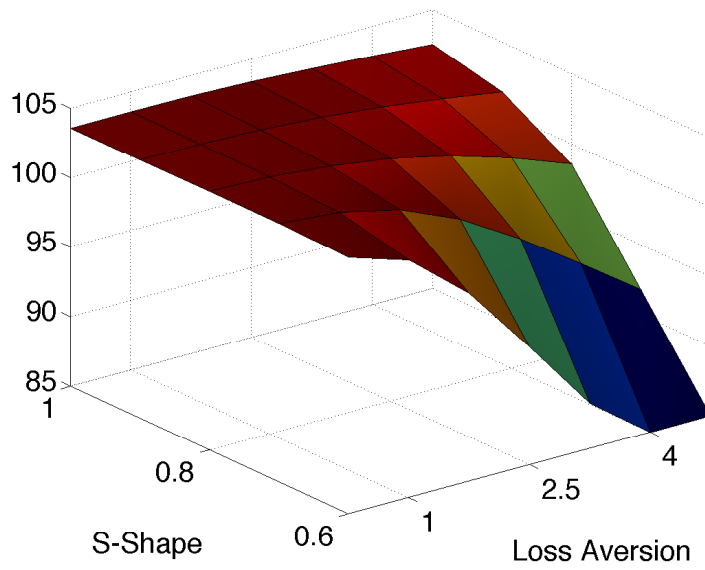
### 2.4.3 Costs of Risk Preferences

In this section, I derive the financial losses a policyholder incurs because of his risk preferences. I measure the price difference of an optimally stopped contract under the conditions of the Black Scholes model with optimal stopping and a contract surrendered optimally given the risk preferences of the policyholder.

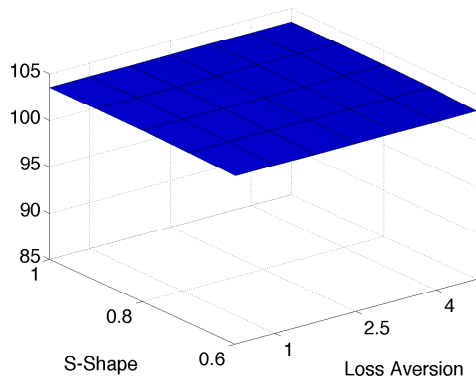
The policyholder does not evaluate the contract purely on the financial value, but also takes the riskiness of the insurance policy into account. Furthermore, he considers the stock and the insurance contract under the statistical measure, not under the risk-neutral probability measure. Hence, the policyholder works with a higher drift which leads to a different surrender behavior.

Figure 2.8 plots the monetary contract value with different reference points as a function of the loss aversion parameter  $\lambda$  and the S-shape parameter  $\gamma$ . The monetary contract value is the price of the contract for a given surrender strategy. The reference point has a pronounced impact: If the reference point is far from reasonable expectations for the insurance contract's performance, the risk attitude does not play a large role to the surrender behavior. Figure 2.8b) and 2.8c) show an unrealistically low and

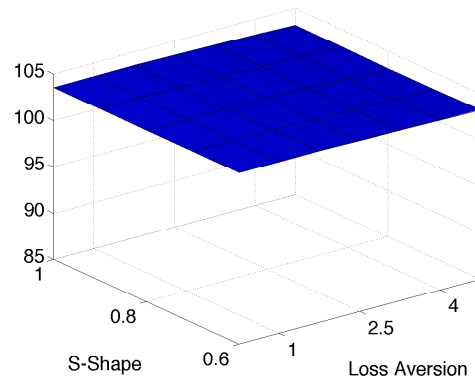
**Figure 2.8: Contract Value for utility optimal boundary.** The graphs display the monetary value of the strategy given by the S-shaped preferences induced surrender behavior for the S-shape parameter  $\gamma$  and the loss aversion parameter  $\lambda$  for a) an intermediate reference point of  $\alpha \cdot P \cdot (1+r)^T$ , b) a low reference point of 100, and c) a high reference point of 130.



**(a)** Reference Point  $\alpha \cdot P \cdot (1+r)^T$ .



**(b)** Reference Point 100.



**(c)** Reference Point 130.



high reference point, respectively. However, in the case of a reference point which divides the contract's performance in gains and losses, the policyholder deviates from the optimal stopping strategy given by the Black Scholes approach. The policyholder loses a substantial amount of money from his insurance contract in this case.

The limited access to the financial market alone does not result in a substantial loss in welfare. A policyholder which has risk preferences close to risk-neutrality uses a surrender strategy which is close to the optimal strategy as given by the Black-Scholes hedging, even though the policyholder works under the statistical measure  $\mathbb{P}$  instead of the risk-neutral measure  $\mathbb{Q}$ .

## 2.5 Conclusion

In this paper, I have derived the impact of risk and loss aversion on the surrender behavior of policyholders of equity-linked life insurance contracts. While the surrender occurs earlier for risk-averse policyholders on average, the surrender depends on the fund level, but is stable and smooth over time. The policyholder's evaluates the insurance contract according to the principles of loss aversion and relative to a reference point. This preferences destroy the smoothness of his exercise boundary: Once the surrender benefit of the insurance contract reaches the reference point and flips from gain to loss, there is a sharp peak in surrenders depending on the level of loss aversion. This peak is important for insurers, as it implies both surrender runs and, as a consequence, a severe drop in liquidity due to bundled surrenders. Furthermore, the policyholder's risk preferences imply an exercise behavior which is costly to policyholders. As the policyholder's risk aversion and loss aversion deviate from risk-neutrality, the surrender behavior induces loss of a substantial fraction of the contract's value under the Black-Scholes approach.

A further interesting aspect to study for the surrender behavior is the aspect of probability weighting to analyze the impact of the Tversky and Kahneman (1992)'s full cumulative prospect theory. However, the incorporation of probability weighting is technically challenging. While the theory of Kahneman and Tversky is the most prominent descriptive theory of decision making under risk, further behavioral biases, for example policyholder's regret and emotional aspects of the policyholder, are left out in this paper, but are potentially interesting to study.



**Part II**

**Technical Trading**



# Chapter 3

## The Trend is Your (Imaginary) Friend: A Behavioral Perspective on Technical Analysis

### 3.1 Introduction

*Technical analysis* has the objective to predict future prices from historical price data, trading volume, and statistics thereof. Fundamental analysis, on the other hand, is based on economic fundamentals as given by macroeconomic or balance sheet data. According to a recent survey of 678 fund managers by Menkhoff (2010), 86% of fund managers rely on technical analysis. For 26% of surveyed fund managers technical analysis is the most important criterion when making investment decisions. Many daily newspapers regularly present technical analysis of selected assets. Online brokers present price charts of the securities that customers select, and provide tools to visualize technical indicators. The *Journal of Technical Analysis* is a practitioners' journal on technical analysis since 1978.

At the same time, the drawing of resistance and support levels as well as moving average (MA) lines into stock price charts to predict future prices appears odd to many people. Academics are generally skeptical towards technical analysis. Predicting future prices from past prices constitutes a futile attempt in light of the efficient market hypothesis even in its weak form (Fama 1970, Jensen 1978).

Against this criticism, technical analysts bring forward that research in behavioral finance has questioned market efficiency. Markets are not fully rational, assets may be mispriced, and such mispricings may exist for a long time. Indeed, prices may differ from their fundamental value because of limits to arbitrage. Possible reasons are noise trader risk (e.g. De Long, Shleifer, Summers, and Vishny 1990) or agency problems that cause arbitrageurs to act myopically (Shleifer and Vishny 1997). Various models predict

momentum in asset returns that could potentially be exploited through trend-chasing technical trading strategies.<sup>1</sup>

The actual profitability of technical analysis is heavily debated. Jegadeesh and Titman (1993, 2001) document the profitability of a variety of momentum strategies and provide some cautious support for the behavioral models that predict predictability of stock returns. Menkhoff and Taylor (2007) survey technical analysis in foreign exchange markets. They provide evidence that technical analysis can give additional information over fundamental factors, and thus be profitable.<sup>2</sup> The survey of Park and Irwin (2007) provides an overview of studies on the profitability of technical trading. There are a number of older studies that report positively on technical analysis, but often suffer from econometric deficiencies like data-snooping bias. Overall, empirical results are mixed, and people tend to have strong opinions for—or against—technical analysis.

In this paper, we are agnostic about whether technical analysis is profitable or not. We will also not take a stand on whether the field of behavioral finance indeed lends support to the application of technical analysis. In this paper, we argue that the popularity of technical analysis may also be explained by its advocates being less than fully rational. Unlike previous behavioral work that investigates the predictability of stock returns that may result in technical trading being profitable, our argument is not concerned with—and independent of—profitability. In fact, even when—by assumption—technical trading is not profitable and cannot be reconciled with rational, risk averse expected utility theory preferences, it may be attractive to investors with less than fully rational preferences. This result matches the intuition of many academics who are skeptical towards technical analysis: trading based on technical indicators is consistent with its advocates being less than fully rational.

The investors we consider are less than fully rational because they have non-standard preferences as given by cumulative prospect theory (CPT, Tversky and Kahneman 1992, Kahneman and Tversky 1979). Unlike expected utility theory (EUT), CPT posits that investors do not evaluate absolute wealth levels, but gains and losses relative to a reference point. In evaluating the risky returns of an investment, the reference point is often zero return, the risk-free return, or expected return. The CPT utility function is concave for gains and convex for losses, which implies diminishing sensitivity to both gains and losses. In addition, the utility function is kinked at the reference point and steeper over losses than over gains. This implies that people are more sensitive to losses than to gains,

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<sup>1</sup>Momentum may arise, for example, because investors under- and overreact to news (Barberis et al. 1998) because of cognitive biases that affect their belief formation. Grinblatt and Han (2005) argue that investors' tendency to hold on to assets that have lost in value and holding on to those that have gained in value may generate momentum in asset returns.

<sup>2</sup>Menkhoff, Sarno, Schmeling, and Schrimpf (2012) find strong momentum in a cross-section of currency returns, and also give an extensive literature review on momentum trading.

which is referred to as loss aversion. Moreover, prospect theory investors behave as if their decisions are based on transformed probabilities. These transformed probabilities are obtained from objective probabilities by applying a so-called weighting function. The main effect of this probability weighting is that small probabilities of extreme returns are overweighted which results in a preference for lottery-like, or right-skewed, payoffs.

Why may trading based on technical indicators be attractive to prospect theory investors even when prices do not move in trends? We show that trend-chasing trading strategies shape the distribution of trading proceeds in a way which is desirable for CPT investors. This effect is so strong that even technical trading of a stock with zero excess return—which is not attractive according to any trading strategy to a risk-averse EUT investor—*is* attractive to CPT investors.

We will illustrate this result along the lines of the arguably most popular technical trading strategy, the *moving average cross-over strategy* (MA strategy, for short). However, from our analysis it will become evident that similar results also apply to other strategies that try to chase momentum. A MA line indicates how the average of a fixed number of most recent closing prices of a stock develops over time. The MA cross-over strategy is based on the crossings of two such MA lines which indicate a short-term and a long-term average, respectively. The cross-over MA strategy gives a buy signal when the short-term MA line crosses the long-term MA line *from below*, which is the case when recent prices have increased sharply. When subsequently the short-term MA crosses the long-term MA line *from above*, this corresponds to a sell signal. The MA strategy thus chases momentum as it buys recent winners and sells recent losers.<sup>3</sup> When an investor trades a stock as suggested by the MA strategy, we will also say that he “trades the stock MA.”

The result that technical trading is consistent with CPT even when it cannot be reconciled under rational preferences is made in two steps. The first step consists of obtaining the distribution from trading according to the MA for various financial market models as well as for actual data. The second step consists of verifying that this distribution is indeed attractive to prospect theory investors.

In the first step, for comparative reasons, we benchmark the MA strategy to a totally random trading strategy. This benchmark strategy is a random buy-and-hold strategy whose buy and sell times are identical in distribution to that generated by the MA

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<sup>3</sup>We want to emphasize that this paper is quite different in focus from recent prospect theory papers on the disposition effect such as Barberis and Xiong (2009, 2012) or Henderson (2012). The disposition effect refers to investors being more inclined to sell stocks that have gained in value and hold on to stocks that have lost in value. These models largely focus on the *optimal selling decision*, which is determined endogenously from preferences. Moreover, these models are without probability weighting. In the current paper, we show that an exogenously specified technical trading strategy is attractive to CPT investors, and this effect is largely driven by probability weighting.

strategy. Therefore, also the holding time is identical in distribution to that of MA. However, unlike is the case for the MA strategy, for the benchmark strategy buy and sell time are independent of the stock price evolution. This allows us to study in isolation the effect of timing as suggested by MA on the distribution of trading proceeds.

The main result we obtain in step 1 is that trading MA skews the distribution of trading proceeds strongly to the right. Skewness in asset returns implies a large probability of small negative returns and a small probability of very high returns. In the famous Black-Scholes model, for example, the trading proceeds from buying and selling a stock are log-normally distributed and thus right-skewed. Not surprisingly, also our benchmark buy-and-hold strategy yields a right-skewed distribution in this case. Trading the stock MA, however, skews the distribution much further to the right.

This result holds accordingly for short-selling. For example, short-selling a log-normal stock buy-and-hold results in a left-skewed distribution of trading proceeds. Shorting the stock based on MA buy and sell signals, however, skews the distribution to the right such that this distribution in fact becomes right-skewed. The result that trading MA strongly skews the distribution of trading proceeds to the right is robust towards different underlying stock price dynamics with and without trend or autocorrelation, different time horizons, and is also found in real data on the STOXX Europe 600 stocks.

In the paper, we also explain *why* trading MA skews the return distribution to the right, and illustrate why the effect is not specific to the dynamics of the stock price process. Trading MA features a subtle asymmetry in the way stocks are sold subsequent to a buy signal. A stock whose price rises is not sold until until it experiences a significant loss. Thus the right tail of the return distribution is pronounced. After a sudden price decline, the stock is sold immediately. This limits the downside of the investment by shortening and thickening the left tail.

The second step of our analysis consists of verifying that the MA return distribution is attractive to CPT investors, even when it is not attractive to rational investors. To this end, we employ the numerical methodology developed in Barberis (2012), which also allows us to determine the drivers of this result. Recent theoretical papers have argued that prospect theory implies a strong preference for skewed payoffs; e.g. Barberis and Huang (2008), Ebert and Strack (2012), or Barberis (2013). There is a growing amount of evidence on skewness preference in financial decision situations, ranging from CEO's project choices (Schneider and Spalt 2012) to initial public offerings (Green and Hwang 2012). In this spirit, our results show that the popularity of technical trading strategies like MA may be explained by strong skewness preference. Schneider and Spalt (2012) also use the term "longshot bias" to refer to CEO's excessive skewness preference.

On a more general level, this paper offers a new explanation for why technical analysis



and trend-chasing are popular; being that its advocates and applicants have less than fully rational preferences. This argument is very different from a well-known behavioral argument on technical analysis, one which is often made by its advocates; that technical analysis may be valuable because markets are not efficient, potentially due to *other* market participants being less than fully rational. Having this in mind, we want to emphasize that both arguments are by no means mutually exclusive. As we illustrate through various robustness checks, the effect that trading MA shapes the return distribution in a way desirable to prospect theory investors holds irrespective of whether the assumptions for why MA is applied are met. If MA is highly profitable because prices do move in trends, then the rational reason to trade MA (MA being highly profitable) is reinforced by the less than fully rational we point out (prospect theory preferences). For the main part of the paper we wish to disentangle these effects and study the less than fully rational reason we point out in isolation, and show that it is sufficient to make MA attractive to CPT investors when rational EUT traders do not trade the stock at all. Also other, non-behavioral arguments in favor of technical analysis may be valid. Our conclusion is that boundedly rational investor preferences are one potential—however sufficient—reason for the popularity of technical analysis. This reason may or may not be accompanied other, rational reasons.

The paper proceeds as follows. Section 3.2 defines the MA strategy within a simple financial market model as well as investor preferences. In Section 3.3, we show, and explain why, trading MA skews the distribution of trading proceeds to the right. In Section 3.4 we show that trading MA is attractive to prospect theory investors even when prices do not move in trends and when MA is not profitable. In Section 3.5, we show that an MA short-sale strategy likewise skews to the right and is attractive to prospect theory investors. Section 3.6 then shows that these results are robust towards different specifications of the time horizon, variations of the MA strategy, and alternative stock price dynamics. In particular, our results stay qualitatively the same when prices do move in trends, and they are also found in real data on STOXX Europe 600 stocks. Section 3.7 concludes.

## **3.2 Technical Trading Strategies and Investor Preferences**

We first define the double-crossover MA strategy in Subsection 3.2.1 and illustrate it by means of an example. Subsection 3.2.2 defines the investor’s preferences. In Subsection 3.2.3, we discuss MA in a simple, geometric Brownian Motion model that will be

analyzed in the main part of this paper.

### 3.2.1 The double-crossover MA trading strategy

Let  $S_t$  denote the time  $t$  price of a stock  $S$ . The value of the MA line of stock  $S$  at a given time  $t$  with length  $n$  is the average closing price of the past  $n$  closing prices:

$$MA(t, n) = \frac{1}{n} \sum_{i=1}^n S_{t-i}. \quad (3.1)$$

As time moves on, the observations used to calculate the average are replaced with newer observations of the stock price. The MA (line) of a fixed length  $n$  refers to the function  $t \mapsto MA(t, n)$  and indicates how the recent average stock price evolves over time. Short-term MAs ( $n$  small) are more responsive to recent price changes than long-term MAs ( $n$  large), because more recent observations have higher weight in the computation of the average in equation (3.1).

MAs smooth out short-term fluctuations in the stock price and thus indicate, depending on their length, short-term or long-term trends in the stock price. The perhaps best-known trading strategy based on MAs is the *double-crossover MA strategy*. In this paper, we refer to it as “MA strategy” for short. The MA double-cross-over strategy is based on two MAs with different time spans  $n_{long} > n_{short}$ . A *buy signal* occurs when the short-term MA crosses the long-term MA *from below*, which indicates a short-term upward trend (relative to the long-term trend). Similarly, a *sell signal* occurs when the short-term MA crosses the long-term MA from above, indicating that the short-term upward trend is over. Formally:

**Definition 1** (MA Strategy). *The MA strategy indicates a buy signal at time  $t$  if the short-term MA line crosses the long-term MA line from below:*

$$MA(t-1, n_{short}) < MA(t-1, n_{long}) \text{ and } MA(t, n_{short}) \geq MA(t, n_{long}) \quad (3.2)$$

*The MA strategy indicates a sell signal at time  $t$  if the short-term MA line crosses the long-term MA line from above, that is,*

$$MA(t-1, n_{short}) > MA(t-1, n_{long}) \text{ and } MA(t, n_{short}) \leq MA(t, n_{long}). \quad (3.3)$$

*If we set the time a buy-signal occurs to  $t = 0$ , the holding time  $\tau_{MA}$  denotes the first time after  $t = 0$  when a sell-signal occurs.*

A popular parametrization is  $n_{short} = 50$  for the length of the short-term MA and

$n_{long} = 200$  for length of the long-term MA.<sup>4</sup> As MA lines are exclusively based on time series data, they cannot indicate trends that result from cross-sectional phenomena like a negative return correlation for winners and losers. They may indicate upward trends that are due to autocorrelation, or trends that are simply the result of the stock having a high unconditional mean.

Figure 3.1 illustrates the MA strategy along the lines of the Walmart common stock (ISIN: US9311421039) trajectory of daily closing prices on NYSE for 2008–2013. The figure also shows the prices for which the MA strategy indicates buy and sell signals. A profitable trade according to the MA strategy would have been buying the stock at the buy signal after July 2011 for \$58 and selling it for \$69 in March 2013. A losing trade would have resulted from buying for \$54 in late 2010 and selling for \$53.40 in early summer of 2011. Minor gains would have resulted from buying in October 2009 for \$51 and selling for \$53 in late 2010.

**Figure 3.1: MA buy and sell signal for Walmart from January 2008 to April 2013.** This graph shows the 50-day MA (dashed line) and 200-day MA (solid line) lines computed from the Walmart stock’s daily closing prices (solid line) on NYSE from January 2008 to April 2013. The graph also shows the buy and sell prices of the MA strategy.



### 3.2.2 Investor Preferences

We consider an investor who thinks of his investment history in terms of different investing episodes. Each episode is defined by the name of the investment, the purchase price, and

<sup>4</sup>While 200 is clearly the standard for the long-term MA, technicians sometimes use smaller values like 22 or 37 for  $n_{short}$ . Using shorter values than 50 for  $n_{short}$  does not change the principal argument we make in this paper, but rather amplifies our results.

the sale price; cf. Barberis and Xiong (2012). Rather than taking a portfolio view, the investor evaluates the investment in isolation. This view is consistent with the ideas of narrow framing and mental accounting. Shefrin and Statman (1985), for example, argue that when an investor sells a stock, he closes a mental account that was opened when he first bought the stock. The moment of sale is therefore a natural time at which to evaluate the investment episode. Episodes that result in gains bring positive utility, while losses bring negative utility. In this paper, we study how an investor evaluates an episode when buying and selling according to the MA strategy. In particular, the investor has CPT preferences over returns from trading MA.

The CPT investor evaluates the absolute returns on his investment relative to a reference return  $r$ . Reasonable choices for  $r$  are zero return, risk-free return, or expected return. We will generally assume that the investor views positive absolute returns as gains and negative absolute returns as losses. This implies that the reference return is zero, i.e., any trade at which the stock is sold for more than the purchase price brings positive utility. In the robustness section, we also discuss other reference points.

Consider a prospect (i.e., a risky return)  $X$  with ordered outcomes  $x_1 > x_2 > \dots > x_k \geq r > x_{k+1} > \dots > x_n$  and respective probabilities  $p_1, \dots, p_n$ . Returns  $x_1, \dots, x_k$  are larger than the reference return  $r$  and thus referred to as gains. Outcomes  $x_{k+1}, \dots, x_n$  are referred to as losses. The CPT-utility is given by

$$CPT(X) = \sum_{i=1}^n \pi_i u(x_i - r) \quad (3.4)$$

with *decision weights*  $\pi_i$  and a *utility function*  $u$  defined as follows. The value function  $u$  is given by

$$u(x - r) = \begin{cases} (x - r)^\alpha & \text{if } x \geq r \\ -\lambda(-(x - r))^\alpha & \text{if } x < r \end{cases}$$

with “loss aversion parameter”  $\lambda \geq 1$  and utility curvature parameter  $\alpha \in (0, 1]$ .  $\alpha \in (0, 1]$  implies that the utility function is concave over the region of gains and convex over the region of losses. This implies diminishing sensitivity towards both gains and losses.  $\lambda \geq 1$  implies that the utility function is steeper in the region of losses than in the region of gains, which implies that losses loom larger than gains. The left panel of Figure 3.2 shows the value function for pronounced loss aversion ( $\lambda = 2.5$ ) and pronounced curvature  $\alpha = 0.5$ . Estimates in the literature vary significantly both within studies and within subjects, but benchmark values often reported are the median values of the subjects in the Tversky and Kahneman (1992) study:  $\lambda = 2.25$  and  $\alpha = 0.88$ .

Finally, CPT does not process objective probabilities linearly as does EUT, but replaces the objective probabilities  $p_i$  with decision weights  $\pi_i$ . The  $\pi_i$  in equation (3.4) are given as<sup>5</sup>

$$\begin{aligned}\pi_i &= w(p_1 + \dots + p_i) - w(p_1 + \dots + p_{i-1}), & i = 1, \dots, k \\ \pi_i &= w(p_i + \dots + p_n) - w(p_{i+1} + \dots + p_n), & i = k + 1, \dots, n\end{aligned}$$

where  $w$  is called (*probability*) *weighting function*. Tversky and Kahneman (1992) propose the following weighting function:

$$w(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}}$$

with a probability distortion parameter  $\delta \in (0, 1]$ . The right panel of Figure 3.2 shows plots of the weighting function for different values of  $\delta$ . For  $\delta = 1$  (solid line) the weighting function is linear which implies  $\pi_i = p_i$  for all  $i = 1, \dots, n$ , i.e., no probability distortion. For  $\delta < 1$  the function is inverse-S-shaped. This results in decision weights that overweight the probabilities of small and extreme outcomes, i.e., the tails of the evaluated distribution are overweighted. Tversky and Kahneman (1992) estimated a value of  $\delta = 0.65$  for their median subject which implies a pronounced inverse-S-shape for the weighting function  $w$ .

### 3.2.3 MAs in a simple financial market model

Suppose the stock price follows a geometric Brownian motion, i.e.,

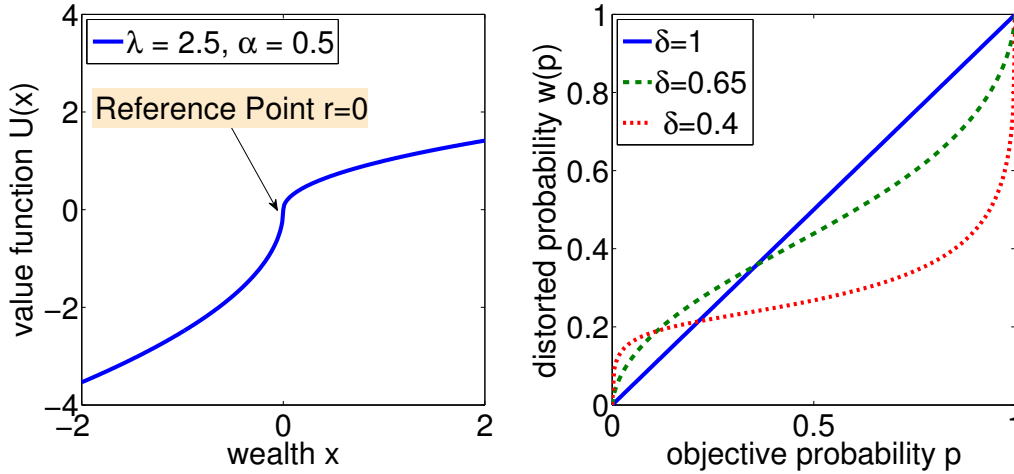
$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \text{with } S_0 > 0 \quad (3.5)$$

where  $W$  is a standard Brownian motion,  $S_0$  is the initial stock value, and  $\mu$  and  $\sigma$  represent the drift and volatility of the stock, respectively. In the simulations we discuss in the main part of the paper, we consider an unattractive stock without drift, that is,  $\mu = 0$ . If the investor does not invest in the stock, he earns a risk-free return of zero. In this setup, which is clearly artificial, the stock earns zero excess return. While our results also hold for other, more realistic specifications, they are best illustrated in this simple setup. We will illustrate the robustness of our results towards more realistic setups in Section 3.6.<sup>6</sup>

<sup>5</sup>In the computations of  $\pi_1$  and  $\pi_n$ , let  $w(p_1 + \dots + p_0) = w(p_{n+1} + \dots + p_n) \equiv 0$ .

<sup>6</sup>Similarly, Barberis (2012) considers a casino which offers fair gambles. Also this stylized assumption does not affect any of his results, but rather allows for clear-cut interpretations and for identifying the drivers of the model's results.

**Figure 3.2: CPT value and weighting functions.** The left hand graph shows the CPT-value function for a loss aversion parameter  $\lambda = 2.5$  and a S-shape parameter of  $\alpha = 0.5$ . The right hand graph shows the CPT-probability distortion function for the case without probability distortion (solid line), an intermediate probability weighting with  $\delta = 0.65$  (dashed line), and a strong probability weighting of  $\delta = 0.4$  (dotted line).



Why do we choose to illustrate our results in this abstract setup? The geometric Brownian motion is Markovian and therefore, by assumption, the stock price does not move in trends. This renders the original purpose of trend-chasing trading strategies like MA moot. When, in addition, a stock offers zero excess return, the holding time of the stock does not affect expected returns. Therefore, by Doob's optional sampling theorem it follows that every trading strategy (and MA in particular) results in an expected return of zero. Therefore, no rational risk-averse expected utility maximizer would invest in the stock. However, and this is what we show in this paper, trading MA will still be beneficial for investors with less than fully rational (CPT) preferences. In other words, we show that CPT can explain the popularity of technical trading *even when, by assumption, the fundamental reasons for it are not met.*

When in Section 3.6 we relax or vary some of these assumptions, results change in the expected directions. For example, when the stock offers excess return, then trading MA will be even more attractive to CPT investors, and will also be attractive to rational investors who are not too risk-averse. If we assume that prices are autocorrelated, then trading MA becomes profitable, and more attractive to CPT and EUT investors alike. Disentangling the drivers that make stock investment attractive in these settings is, however, difficult, which is why for our main illustrations we prefer the simple and abstract setup outlined above.

For comparative reasons, we also analyze the investor's utility from trading stocks randomly according to a *buy-and-hold strategy*, also referred to as *benchmark strategy*. The

buy and sell times of this benchmark strategy are uncertain and identical in distribution to those generated by the MA strategy.<sup>7</sup> Therefore, also the holding time of the buy-and-hold strategy is identical in distribution to that of MA. The only difference is that the benchmark strategy buy and sell times are *independent* of the stock price evolution. Consideration of the benchmark strategy allows us to study in isolation the effect of timing as suggested by MA on the return distribution.<sup>8</sup>

### 3.3 Trading MA when Prices do not Move in Trends

In this section and the next, we show that trading a stock MA is consistent with CPT preferences even when prices do not move in trends and when rational, risk-averse EUT investors do not invest in the stock. To this end, we first compute the distribution of trading proceeds that results from trading MA. Then, in the next section, we analyze the utility of this distribution to EUT and CPT investors.

Note that, even for simple asset dynamics like the geometric Brownian motion, the return distribution from trading MA is not available in closed form. The MA is constructed as the average of lognormally distributed random variables, whose distribution is not known. This makes recovering the distribution implied by the MA strategy, which is based on the crossings of two such MAs, difficult.<sup>9</sup> As regards the CPT analysis of trading MA we will pursue on the next section, the nonlinear probability weighting embedded in CPT poses a further obstacle to analytical treatment. In fact, it is fairly obvious that under probability distortion an analysis of the MA trading strategy or even one of its simpler variants will not be analytically tractable.

We thus analyze our model (and also the robustness checks discussed in Section 3.6) numerically for a Monte Carlo sample of size  $M = 100,000$ . In the main part of this, paper we assume that the stock follows a zero-drift geometric Brownian motion with volatility  $\sigma = 30\%$ . The short-term MA has a length of 50 days and the long-term MA has a length of 200 days. To obtain the MA return distribution we thus simulate prices for at least 200 days so that the short-term and long-term MAs are defined. We then continue simulating until the MA strategy gives a buy signal and, subsequently, a sell signal. This way we also obtain the distribution of the waiting time until the MA strategy

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<sup>7</sup>In the Markovian setting specified in this section, the buy time is immaterial. For some of the robustness checks we conduct in Section 3.6, however, the buy time matters.

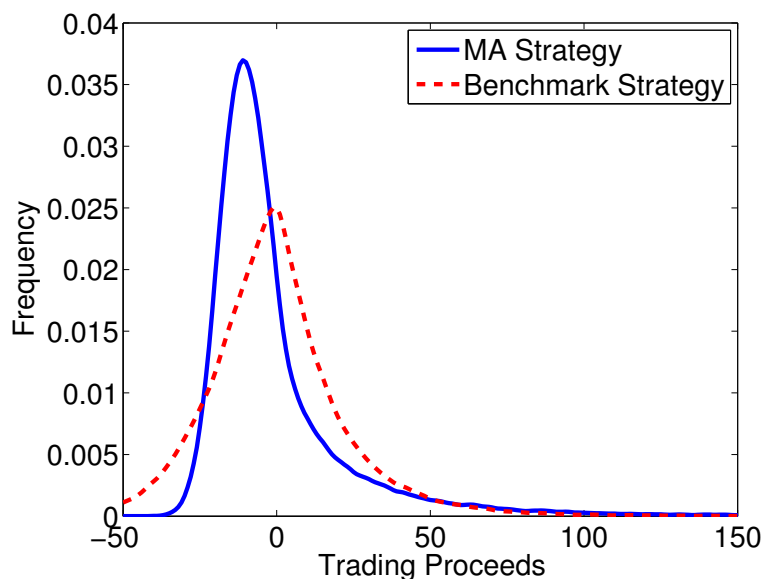
<sup>8</sup>In Section 3.6, we study MA in a finite time horizon framework, and also compare to a buy-and-hold strategy subject to a random, exponentially distributed liquidity shock. This does not affect our results quantitatively.

<sup>9</sup>In specific settings, however, it is possible to obtain analytical results. Zhu and Zhou (2009) compute the expected log-utility for a moving average strategy based on just one, geometric MA. For general power utility they obtain approximately analytical results.

gives a buy signal (referred to as the buy time distribution) as well as the distribution of the holding time (sell time minus buy time).

Figure 3.3 shows the distribution of trading proceeds for the MA strategy (solid line) and the benchmark strategy (dashed line). Since the stock has zero drift and does not

**Figure 3.3: Return Distribution of MA strategy.** This graph shows the return distribution for the MA(50,200) strategy (solid line) and the benchmark strategy (dashed line). The stock follows a geometric Brownian motion with drift  $\mu = 0$  and a volatility of  $\sigma = 0.3$ . The expected return of both strategies is zero, and the average holding time is 148 days. The skewness of the MA(50,200) strategy is 5.52 and the skewness of the benchmark strategy is 1.33.



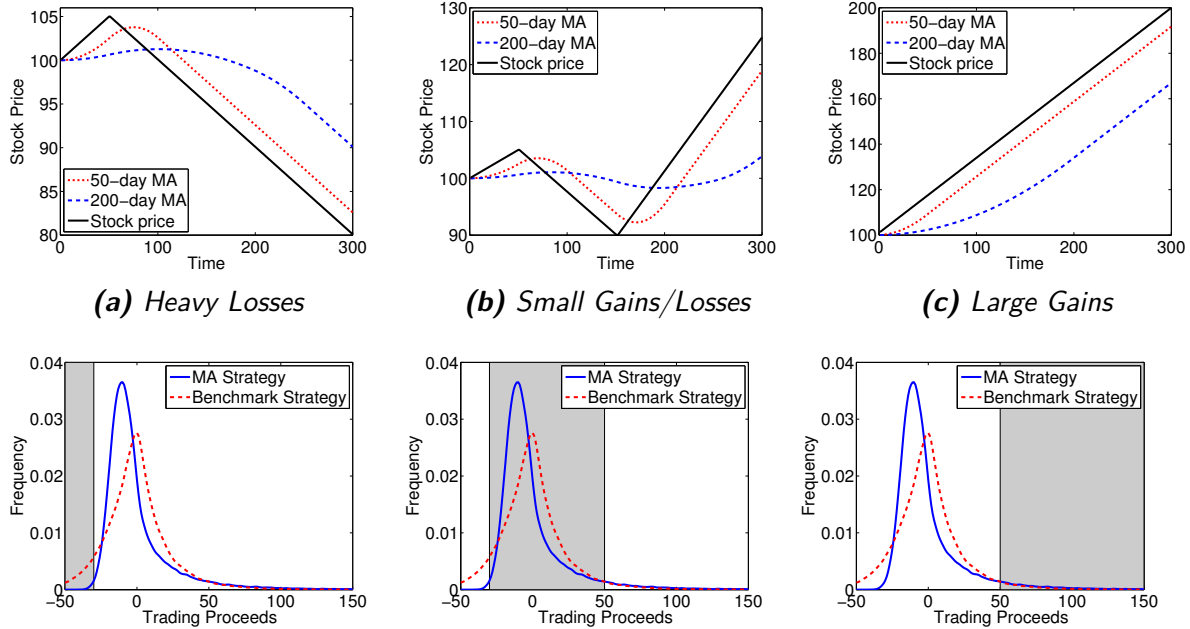
move in trends, by Doob's optional sampling theorem the expected return of both the MA and the benchmark strategy are zero.

While trading MA cannot generate excess return in this setting, it does have an effect on the shape of the return distribution. The overall effect of trading MA is that, as compared to random trading as is in the case of the benchmark strategy, the distribution of trading proceeds is strongly skewed to the right. Trading MA reduces the probability of returns that are less than  $-35\%$  to almost zero (0.029%). At the same time, the probability of moderately negative returns is substantial: While the left tail of the distribution is short, it is also very thick. The right tail of the distribution, on the other hand, is long but very lean. The far right tail lies slightly above that of the benchmark strategy, while the probability of moderate gains is smaller. The third standardized central moments compute to 5.52 and 1.33 for the benchmark and the MA strategy, respectively.

Why does trading MA skew the distribution of trading proceeds to the right? What is the intuition behind the asymmetric effect that trading MA has on the tails of the return distribution? Figure 3.4 provides the intuition for this result by considering three



**Figure 3.4: Sales timing implied by trading MA.** The top row shows stylized stock price trajectories (solid line) and corresponding evolutions of 50-day (dotted line) and 200-day (dashed line) MA lines. The bottom row highlights in gray different parts of the MA(50,200) return distribution shown in Figure 3.3. The shape of the return distribution in the highlighted parts can be related to the time of sale that the MA strategy dictates for stock price evolutions as in the above panel.



stylized stock price trajectories subsequent to a buy signal (shown in the upper panels of Figures 3.4a, 3.4b, and 3.4c).<sup>10</sup> For each of the three scenarios, the lower panel shows again the MA return distribution from Figure 3.3. Each scenario explains why the MA return distribution differs in the highlighted, gray area from that of random trading as given by the benchmark strategy.

In the scenario depicted in Figure 3.4a, the stock price increases at first and then declines substantially. The 50-day MA line picks up the initial upward trend and rises quickly, while the increase of the 200-day MA is less pronounced. Likewise, once the stock price starts to decline, the 50-day MA adjusts quickly and turns downward. The 200-day MA is yet increasing when the price decline starts; only slowly is the 200-day MA turning downward. The sell signal occurs when the 50-day MA crosses the 200-day MA at a value above 100. At that time, the stock price has decreased to about 98. Therefore, trading MA yields a mild negative return of  $-2\%$  in this scenario. The benchmark strategy would have resulted in a substantially negative return of  $-20\%$ . This example illustrates that the MA strategy sells a stock after an episode of sufficiently rapid price decline. This explains why there is almost no probability mass on substantial losses (the highlighted

<sup>10</sup>On the 199 trading days prior to the buy signal, for each of the three scenarios, the stock traded constantly at 100 while on day 200 before the buy signal it traded at 50. Note that this indeed results in a buy signal at  $t = 0$ . The holding time of the buy-and-hold strategy is 300 days.

area in the lower panel of Figure 3.4a). Only when prices decline steadily for a long time, at a rate small enough to avoid a sell-signal, MA may result in large losses.

The scenario in Figure 3.4b starts out just like that in Figure 3.4a, but after the decline in prices from day 51 to 150 the stock recovers. As the stock has been sold by then, the investor misses upon a quite large return of 25%. Instead, he realizes a moderate loss just as in the previous scenario. This explains why trading MA shifts probability mass from moderate gains to moderate losses. Again, whenever the stock price decreases sufficiently within a short period of time the stock is sold. Thus all scenarios with such an early price decline result in a mildly negative return. This effect takes probability mass from both severe losses (scenarios as depicted in the upper panel of Figure 3.4a) and moderate gains (scenarios as depicted in the upper panel of Figure 3.4b) to mild losses (the lower panel of Figure 3.4b). This explains the thick but short left tail of the MA return distribution. In fact, the probability of earning a return between -30% and 0% in Figure 3.3 equals 69.25 %.

Finally, Figure 3.4c shows that for continuously increasing stock prices without large setbacks, the MA lines do not cross. The sluggish 200-day MA just never catches up with the 50-day MA line. Therefore, the fact that the MA strategy buys into the stock when the short-term MA lies above the long-term MA *and* that the short-term MA is more flexible in adjusting to recent stock price movements implies a systematic asymmetry in the way stocks are sold. Stocks that keep rising and rising are never sold, which implies a pronounced extreme upside. Losing stocks are sold quickly, which implies a restricted downside. Effectively, trading MA thus results in skewing the distribution of the overall trading experience to the right. This asymmetry in when stocks are sold has nothing to do with chasing a trend that may potentially be predicted from past prices or the MA buy signal in particular. The sell signal is based on future prices—trend-following or not. Therefore, while some investors might argue that they trade MA because they believe in prices moving in trends, it may also be the case that they appreciate the right-skewed trading experience MA subtly imposes.

## 3.4 Cumulative Prospect Theory Utility of MA Trading

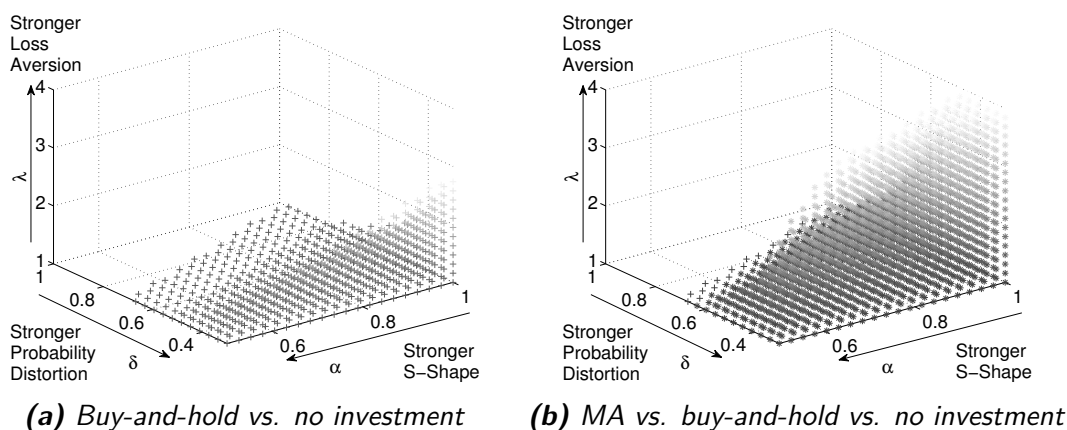
In this section, we verify that trading MA can be reconciled with CPT preferences. To this end, we evaluate the return distribution that results from trading MA with CPT as defined in Section 3.2. As explained before, no risk-averse expected utility maximizer will trade stocks in this setting. Because of the non-linear probability weighting embedded

in prospect theory, for CPT we have to evaluate the MA return distribution numerically. Our methodology for this part of the analysis is similar to that in Barberis (2012).

We restrict our attention to parameter values  $(\alpha, \delta, \lambda)$  for which  $\alpha \in (0, 1]$ ,  $\delta \in [0.3, 1]$ , and  $\lambda \in [1, 4]$ .<sup>11</sup> This parameter range covers well the parameter values that are typically estimated in empirical studies or experiments. The median estimate of Tversky and Kahneman (1992) is given by  $(\alpha, \delta, \lambda) = (0.88, 0.65, 2.25)$ . For each parameter, we discretize the parameter range specified above into 20 different, equally distant values. We then compute the CPT utility of the return distributions implied by MA and buy-and-hold for  $20^3 = 8000$  different values. The CPT utility of not investing is zero in our setting.

The results of these computations are visualized in Figure 3.5. Figure 3.5a only

**Figure 3.5: Utility of Trading MA.** This graph illustrates the parameter values for which a CPT investor finds trading MA attractive. For comparative reasons, part (a) indicates with a “+” the 915 out of the 8000 considered parameter values for which trading the stock randomly according to the benchmark strategy is preferred over no investment. Part (b) compares preference between a stock investment according to the benchmark strategy, the MA strategy, and no stock investment. In that case, the benchmark strategy is optimal for only 15 parameter combinations (indicated by a “+”) while the MA(50,200) strategy is preferred for 2541 parameter combinations (indicated by a “\*”).



compares the benchmark buy-and-hold strategy with the possibility of no investment. A “+” sign marks the parameter triples  $(\alpha, \delta, \lambda)$  for which the benchmark strategy yields a higher CPT utility than no investment, i.e., positive CPT utility.

Figure 3.5a shows that trading the zero return stock buy-and-hold is attractive to only few CPT investors (for 915 out of the 8000 parameter triples). These investors’ preferences are characterized by a mildly S-shaped utility function (large  $\alpha$ ), pronounced probability distortion (small  $\delta$ ), and mild loss aversion (small  $\lambda$ ). The intuition is straightforward. Stock investment according to the benchmark strategy is risky and may result

<sup>11</sup> $\delta \geq 0.3$  is imposed to ensure that the weighting function is increasing as first noted by Rieger and Wang (2006).

in losses bounded by the initial investment, but also in unbounded gains. A mild S-shape of the utility function implies that the marginal utility of gains decreases less rapidly. Probability weighting implies that the unbounded right tail of the distribution is significantly overweighted. And low loss aversion ensures that the investor is not too afraid of the potential losses of a risky investment.

Figure 3.5b illustrates the investors' decision when his investment opportunity set is enriched by the MA strategy. In Figure 3.5b, a "+" marks the parameter triples for which buy-and-hold yields the highest CPT utility. A "\*" symbol marks the parameter triples for which the CPT utility of trading MA is highest. If the space for a parameter triple is left blank, the investor decides against investment because the CPT utility of buying the stock is negative for both trading it MA or buy-and-hold.

From Figure 3.5b we see that, firstly, investing in the stock is attractive for much more CPT preference parameter triples (2556 of 8000). Secondly, almost all (2541) of these triples are "\*" triples. This means that those investors who prefer to invest in the stock prefer trading it MA to buy-and-hold. Only 15 out of the 2556 stock investors prefer to stay with trading it buy-and-hold when also MA is available. The key to understanding this result is to recall what trading MA does to the distribution of trading proceeds (Figure 3.3).

Both MA and buy-and-hold result in a right-skewed distribution of trading proceeds. This is why both strategies are generally favored by investors with a mildly S-shaped utility function, mild loss aversion, and pronounced probability weighting. Trading MA, however, results in a much more right-skewed distribution than trading buy-and-hold. This increased skewness makes MA attractive to investors with pronounced probability weighting even when they are quite loss averse or when they have a pronounced diminishing sensitivity towards large gains. Probability weighting is in fact necessary for skewness preference. Indeed, loss-averse-investors with very little probability weighting may even prefer left-skewed return distributions; this will be explained in the next section. The fact that CPT investors with mild probability weighting prefer less skewness is also the reason why for 15 investors we find that the mildly skewed buy-and-hold is more attractive than MA. Note that these 15 parameter triples indeed feature moderate levels of probability weighting.

### 3.5 Short Selling

In this section, we repeat the analysis of the previous two sections for the short-sale MA strategy. This is interesting because shorting a stock results in a left-skewed return

distribution.<sup>12</sup> This is in part due to the stock price dynamics we assume, but also lies in the nature of the short-sale. The upside return of a short position is limited to 100% (in case the stock becomes worthless), while the downside is unbounded (because the potential stock price increase is unbounded). This is just the opposite to the long position where the downside return is limited to  $-100\%$  and the upside is unbounded. The analysis of the short-sale MA strategy thus also sheds light on whether our observations depend upon the fact that we started out with a right-skewed return distribution as generated by geometric Brownian motion. A possible hypothesis could be that trading MA skews left-skewed distributions more to the left and right-skewed distributions more to the right; or that it has no systematic effect; or that trading MA skews returns to the right no matter what.

The MA short-sale strategy tries to exploit negative momentum in the stock by shorting it when the 50-day MA crosses the 200-day MA from above, thereby capturing a downward trend.

**Definition 2** (Short-Selling-MA Strategy). *The short sale-MA strategy opens a short position in the stock when the MA lines indicate a sell signal, and closes the short position when the MA lines indicates a buy signal.*

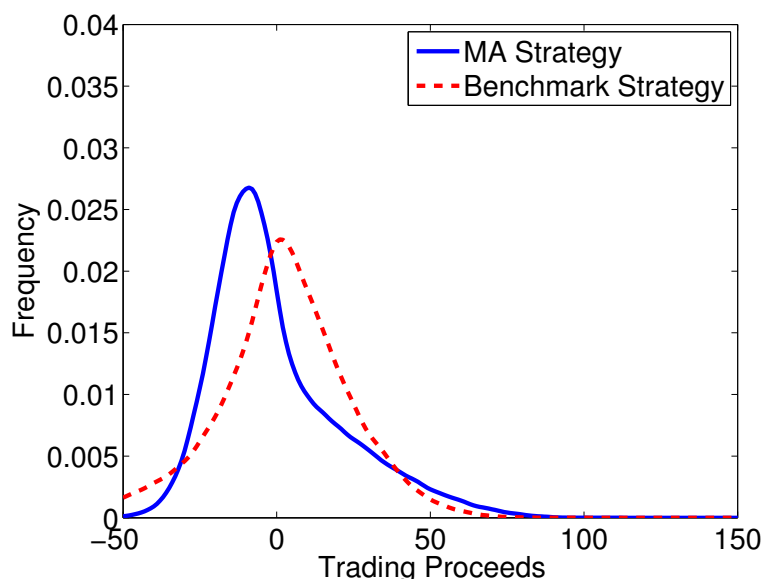
Figure 3.6 shows the short MA return distribution (solid line) and that of the benchmark short strategy (dashed line). The graph shows that also the short-sale MA skews the distribution of trading proceeds to the right, even though the benchmark buy-and-hold strategy is indeed left-skewed. The third standardized central moments are given by  $-1.63$  for the buy-and-hold and by  $+1.02$  for the MA. This result is in fact not surprising. The same logic explained along the lines of Figure 3.4 also applies in the case of short-sale. MA implies that the position in the stock—now a short position—is closed quickly once its value deteriorates sufficiently strongly and quickly. When the value of the position continuously rises (in case of the short sale, prices decrease steadily), the MA strategy does not yield a sell signal. Thus the upside implied by the MA strategy is more pronounced than that of the benchmark strategy. Therefore, also short-sale MA results in a right-skewed return distribution. As, due to the nature of the short-sale, the upside is less pronounced than in the case of a long position, the short-sale MA strategy results in a comparatively less right-skewed distribution.

Figure 3.7 illustrates the CPT utility of MA and buy-and-hold short-selling. Overall, stock investment is attractive for a lot less preference parameter triples. Short-sale according to the benchmark strategy (Figure 3.7a) is preferred over no investment in the

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<sup>12</sup>The short-sale benchmark strategy is analogous to the benchmark strategy for the long position case. It buys the stock at a random time which is identical in distribution to the buy time of short-sale MA, and holds it for a time period which is identical in distribution to that of short-sale MA.

**Figure 3.6: Return distribution of short-sale MA.** This graph shows the distribution of trading proceeds that results from short-sale MA (solid line) and the benchmark short-sale strategy (dashed line). The skewness of the short-selling-MA strategy is 1.02 and the skewness of the benchmark strategy is -1.63. The average holding time is 180 days.

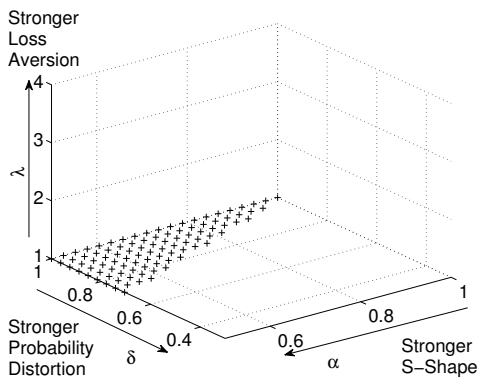


case of no loss aversion, mild probability weighting, and pronounced diminishing sensitivity. The left-skewed benchmark distribution implies a small probability of a large loss. But when utility is convex over losses and when the associated probabilities are not over-weighted, large losses are not too frightening. Likewise, for these preference parameters, a high probability of small gains is not underweighted. Moreover, small gains with high probability are preferred over large gains with small probability in the sense that large gains are less exciting under diminishing sensitivity. For these reason, CPT investors with low levels of loss aversion and probability weighting as well as with pronounced diminishing sensitivity may prefer left-skewed distributions.

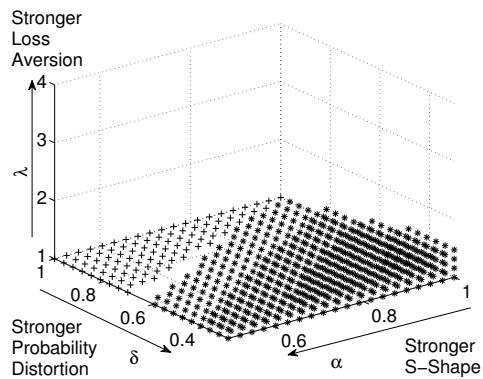
Figure 3.7b illustrates the investment decision when short MA is available. The short MA strategy results in a right-skewed distribution. As discussed, this is preferred for mild loss aversion, mild diminishing sensitivity, and pronounced probability weighting; this is indeed consistent with the position of the “\*” marks in Figure 3.7b.

Thus our results for the short-sale MA strategy are in line with those we obtained for the case of a long position. No matter whether one employs MA to chase a downward or an upward trend, trading MA skews the distribution of trading proceeds to the right, which is preferred by most CPT investors. The only difference is that, due to the nature of the short-sale, the short-sale MA return distribution is comparatively less right-skewed. This is why, compared to the MA strategy that opens a long position, fewer CPT investors find the short-sale MA strategy attractive.

**Figure 3.7: Utility of Trading MA.** This graph illustrates the parameter values for which a CPT investor finds short-selling a stock according to MA attractive. For comparative reasons, part (a) shows the 88 parameter values (indicated by “+”) for which trading the stock randomly according to the short-sale benchmark strategy is preferred over no investment. Part (b) compares a short stock investment according to the benchmark strategy, a short stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 88 parameter combinations (indicated by a “+”) while the MA(50,200) strategy is preferred for 548 parameter combinations (indicated by a “\*”).



(a) Buy-and-hold vs. no investment



(b) MA vs. buy-and-hold vs. no investment

## 3.6 Robustness

In this section, we conduct some additional analysis to evaluate the robustness of our results. In Subsection 3.6.1, we analyze our model for different time horizons. In Subsection 3.6.2, we employ different stock price models with stochastic volatility and jumps. Subsection 3.6.3 repeats the analysis for a stock with strictly positive expected return and for different reference points. Subsection 3.6.4 studies the case of autocorrelated price dynamics where MA is clearly profitable. Subsection 3.6.5 studies MA trading on STOXX Europe 600 stocks since 1990. Subsection 3.6.6 employs the MA strategy for short-term MAs of 22 and 37 days. Overall, our observation that trading MA results in a strongly right-skewed return distribution which is appealing to prospect theory investors is qualitatively unaffected by these modifications.

### 3.6.1 Indefinite and finite time horizon

In the main part of the paper, we assumed that each MA trade is completed. That is, liquidation of the stock exclusively happens at the time of a sell signal, even if the signal is not observed for a long time. In this subsection, we consider the case where the investor may be forced to sell early due to an exogenous liquidity shock. We refer to this time horizon as case 2 while the infinite time horizon considered so far is case 1. Case 3 then analyzes the situation where the investor has a fixed trading horizon, that is, a final time

$T$  at which the stock must be sold at the latest. Finally, in case 4 we compare trading MA with a benchmark buy-and-hold strategy whose holding time is constant and equal to the expected holding time of MA.

Table 3.1 summarizes these four cases.  $\tau_{MA}$  and  $\tilde{\tau}_{MA}$  denote the stopping times that refer to the MA and the benchmark strategy, respectively.  $\tau_{MA}$  is defined as in Definition 1.  $\tilde{\tau}_{MA}$  is identical in distribution to  $\tau_{MA}$ , but independent of the the stock price dynamics  $S_t$ . The liquidity shock  $\tau_{exp}$  is given by the first arrival time of a Poisson process with parameter  $1/(148 \cdot 1.5) = 1/222$ . Therefore, the liquidity shock is exponentially distributed and occurs on average after 222 days, which is 1.5 times the expected holding time of the MA strategy. We also chose the limited trading horizon in cases 3 to be  $T = 222$  days. For case 4 we consider a benchmark strategy with a constant holding time which is equal to the average holding time of MA, i.e.,  $\bar{T} = E[\tau_{MA}] = 148$ .

**Table 3.1:** MA strategy for different Time horizons

Time Horizon/Strategy	Double-Crossover MA	Benchmark Buy-and-Hold
(1) Infinite (Main Model)	$\tau_{MA}$	$\tilde{\tau}_{MA}$
(2) Indefinite (Liquidity Shock)	$\tau_{MA} \wedge \tau_{exp}$	$\tilde{\tau}_{MA} \wedge \tau_{exp}$
(3) Finite (fixed trading horizon)	$\tau_{MA} \wedge T$	$\tilde{\tau}_{MA} \wedge T$
(4) Finite (Constant Benchmark)	$\tau_{MA}$	$\bar{T} = E[\tau_{MA}]$

Figures 3.8, 3.9, and 3.10 replicate the analysis for the main model (case 1), as shown in Figures 3.3 and 3.5, for cases (2), (3), and (4), respectively. We observe that none of these variations changes our observations from the main model qualitatively. As long as the (expected) time horizon for trading is not too short so that a relevant number of MA trades is completed, MA retains its effect of strongly skewing to the right. Further simulations show that, when the (expected) time horizon becomes shorter and shorter, the MA skewness effect diminishes more and more.

### 3.6.2 Stochastic volatility and jumps

In this subsection, we relax the assumption of geometric Brownian motion and study MA when prices evolve according to the popular stochastic volatility model of Heston (1993) and the jump-diffusion model of Merton (1976). According to the Heston model, the stock price dynamics are given by

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^S, S_0 > 0 \quad (3.6)$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_{vol} \sqrt{V_t} dW_t^V, V_0 > 0. \quad (3.7)$$



Here,  $\mu$  denotes the drift of the stock price  $S_t$  which is driven by a Brownian motion  $W^S$  whose volatility is stochastic. The volatility process  $V$  is driven by a Brownian motion  $W^V$  with a volatility of volatility  $\sigma_{vol}$ , mean reversion level  $\theta$ , and speed of reversion  $\kappa$ . The two Brownian motions are correlated through a parameter  $\rho$ . Figure 3.11 illustrates our results for the Heston model with  $\mu = 0$ ,  $\sigma_{vol} = 0.5$ ,  $\rho = -0.7$ ,  $\theta = 0.3$ ,  $\kappa = 3$ , and  $V_0 = 0.5$ .

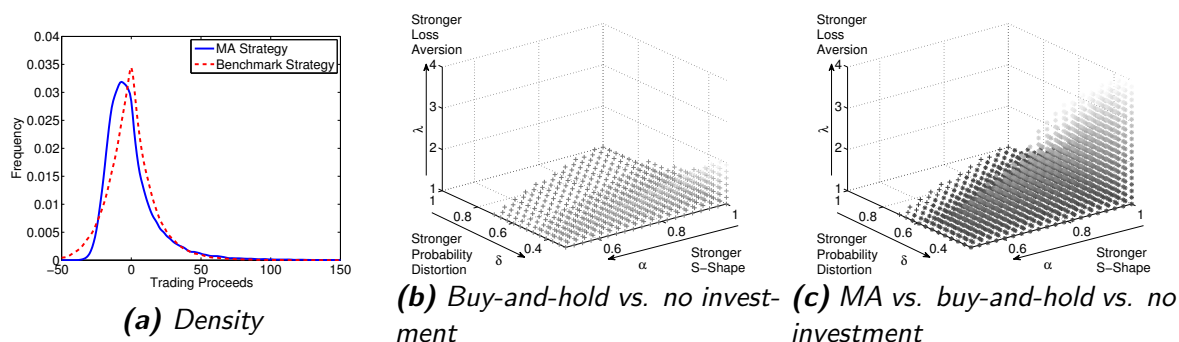
In the jump-diffusion model of Merton (1976), the stock price dynamics and the solution to the stochastic differential equation are as follows:

$$dS(t) = (\mu - \gamma\kappa)S(t-)dt + \sigma_S S(t-)dW_t^S + (Y(t) - 1)dN(t), \quad S_0 > 0 \quad (3.8)$$

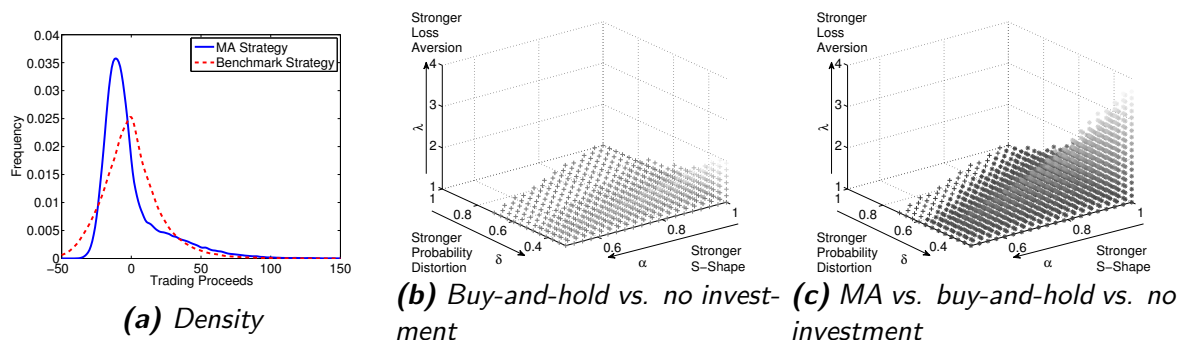
$$S(t) = S_0 \exp\left(\left(\mu - \gamma\kappa - \frac{\sigma_S^2}{2}\right)t + \sigma_S W(t)\right) \prod_{i=1}^{N(t)} Y(t_i) \quad (3.9)$$

Here,  $\mu$  denotes the drift of the stock price  $S_t$  which is driven by the Brownian motion  $W^S$  with volatility  $\sigma_S$ .  $N(t)$  is a Poisson process, where  $N_t$  denotes the number of jumps until time  $t$  which are computed with jump intensity of  $\gamma$ , that is the mean number of arrivals per year. The jump size  $Y_i$  of jump  $i$  follows a lognormal distribution with mean zero and a volatility of  $\sigma_{jump}$ . Figure 3.12 presents the results for a Merton model with  $\mu = 0$ ,  $\sigma = 0.3$ ,  $\sigma_{jump} = 0.15$  and  $\gamma = 1$ .  $\kappa = \mathbb{E}[Y - 1]$  is the expected relative change in the stock upon arrival of a price jump.

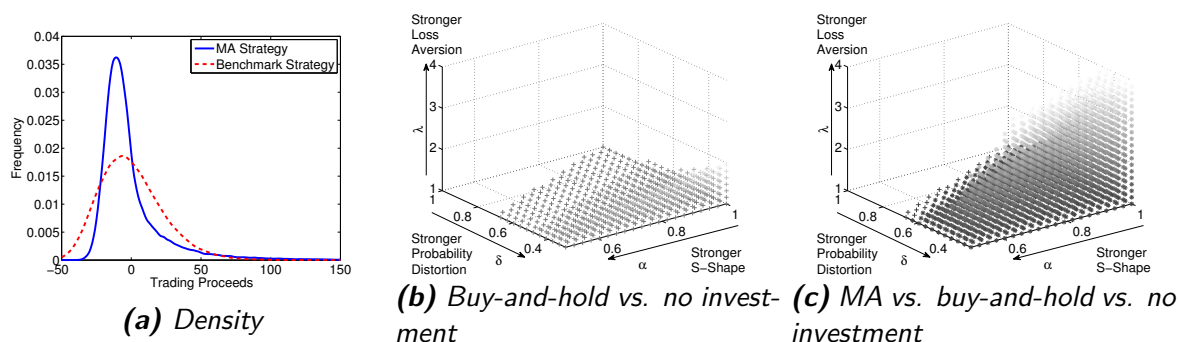
**Figure 3.8: MA trading in the presence of liquidity shocks.** Part (a) of this figure illustrates the return distributions of MA (solid line) and the benchmark strategy (dashed line) when investing is subject to a liquidity shock that, on average, occurs after 222 days (case 2 in Table 3.1). The skewness of the MA (benchmark) return distribution is 2.92 (1.02) and the expected holding time of both strategies is 94 days. Part (b) shows the 730 CPT parameter values (indicated by “+”) for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 24 parameter combinations (indicated by a “+”) while the MA(50,200) strategy is preferred for 1869 parameter combinations (indicated by a “\*”).



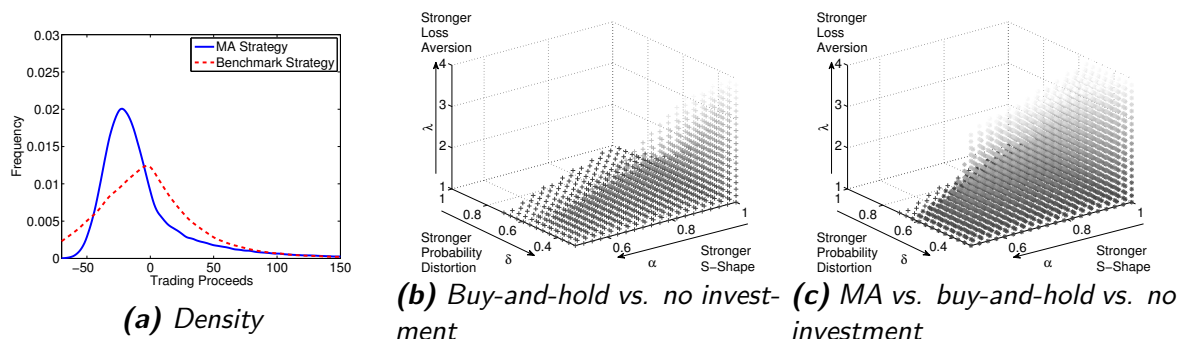
**Figure 3.9: MA trading with a fixed trading horizon.** Part (a) of this figure illustrates the return distributions of MA (solid line) and the benchmark strategy (dashed line) when the investor has a limited trading horizon 222 days (case 3 in Table 3.1). The skewness of the MA (benchmark) return distribution is 2.01 (0.88) and the expected holding time of both strategies is 123 days. Part (b) shows the 733 CPT parameter values (indicated by “+”) for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 44 parameter combinations (indicated by a “+”) while the MA(50,200) strategy is preferred for 1425 parameter combinations (indicated by a “\*”).



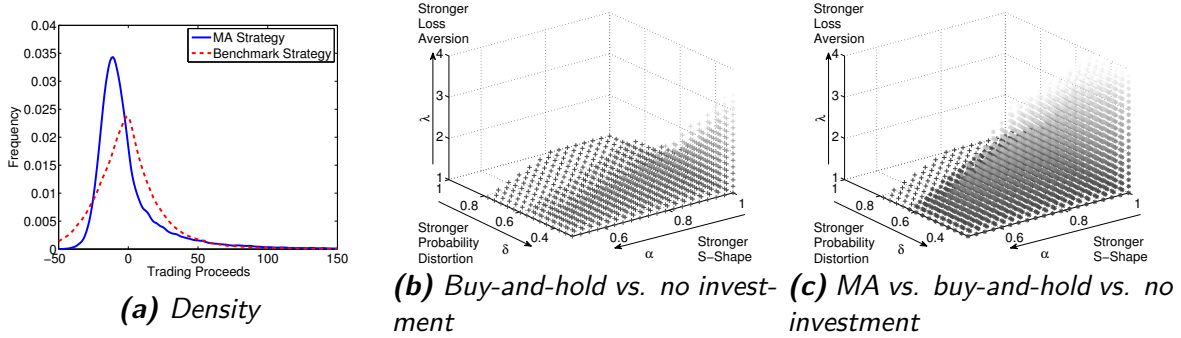
**Figure 3.10: MA trading for a constant Benchmark.** Part (a) of this figure illustrates the return distributions of MA (solid line) and a buy-and-hold benchmark strategy (dashed line) with a constant holding time equal to 148 days (case 4 in Table 3.1). The skewness of the MA (benchmark) return distribution is 4.61 (0.7) and the expected holding time of both strategies is 148 days. Part (b) shows the 683 CPT parameter values (indicated by “+”) for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 25 parameter combinations (indicated by a “+”) while the MA(50,200) strategy is preferred for 2486 parameter combinations (indicated by a “\*”).



**Figure 3.11: MA trading in the Heston Model.** Part (a) of this figure shows the return distributions of MA (solid line) and the benchmark strategy (dashed line) when the stock price follows the dynamics of a Heston model. The skewness of the MA (benchmark) return distribution is 10.65 (2.17) and the expected holding time of both strategies is 135 days. Part (b) shows the 1519 CPT parameter values (indicated by “+”) for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 15 parameter combination while MA(50,200) is preferred for 3008 parameter combinations (indicated by a “\*”).



**Figure 3.12: MA trading in the Merton jump-diffusion Model.** Part (a) of this figure shows the return distributions of MA (solid line) and the benchmark strategy (dashed line) when the stock price follows the dynamics of a Heston model. The skewness of the MA (benchmark) return distribution is 5.69 (1.64) and the expected holding time of both strategies is 149 days. Part (b) shows the 1194 CPT parameter values (indicated by “+”) for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 60 parameter combinations while MA is preferred for 2680 parameter combinations (indicated by a “\*”).

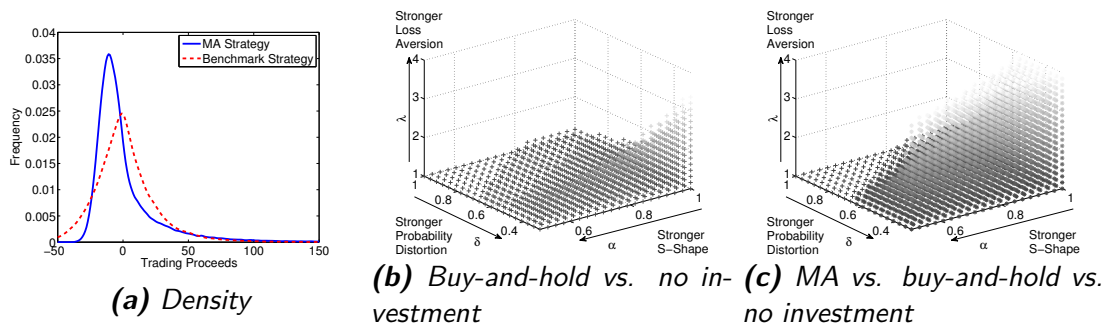


### 3.6.3 Excess return

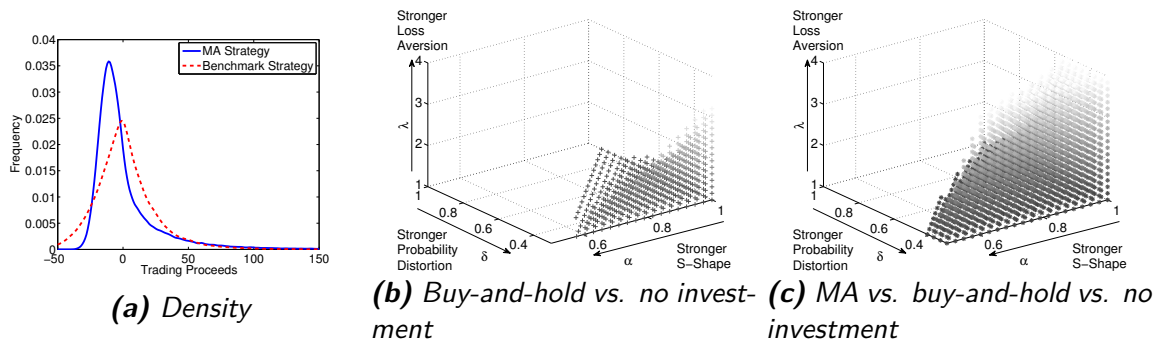
In this section, we study the more realistic situation when the stock offers excess return. In this case, Doob’s optional sampling theorem does not apply and various trading strategies offer different expected returns. We thus present the CPT analysis for a reference point of zero (Figure 3.13) and also for the case where the reference point equals the expected return of the MA strategy (Figure 3.14).

As the stock now offers an excess return, both the MA and the benchmark strategy are (unconditionally) profitable. In the studied scenario, the MA strategy (the benchmark strategy) yields an expected return of  $E[\tau_{MA,}] = 2.09\%$  ( $E[\tau_{MA}] = 1.92\%$ ) over an expected holding period of 155 days. The observation that trading MA results in a strongly skewed return distribution remains. A comparison of Figures 3.13 and 3.14 evidences that differences in reference points (in magnitudes that are economically meaningful) are not crucial to our result that the CPT investor has a pronounced preference for the right-skewed MA distribution.

**Figure 3.13: MA trading in the GBM model with trend and a reference point of zero.** Part (a) of this figure shows the return distributions of MA (solid line) and the benchmark strategy (dashed line) when the stock price follows the dynamics of a GBM model with a positive trend of  $\mu = 0.03$ . The skewness of the MA (benchmark) return distribution is 7.88 (1,62) and the expected holding time of both strategies is 155 days. Part (b) shows the 1592 CPT parameter values (indicated by "+") for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 120 parameter combinations while MA is preferred for 3174 parameter combinations (indicated by a "\*").



**Figure 3.14: MA trading in the GBM model with trend when the reference point equals expected return.** Part (a) of this figure shows the return distributions of MA (solid line) and the benchmark strategy (dashed line) when the stock price follows the dynamics of a GBM model with a positive trend of  $\mu = 0.03$ ; cf. Figure 3.13a. Part (b) shows the 719 CPT parameter values (indicated by "+") for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for zero parameter combinations while MA is preferred for 2355 parameter combinations (indicated by a "\*").



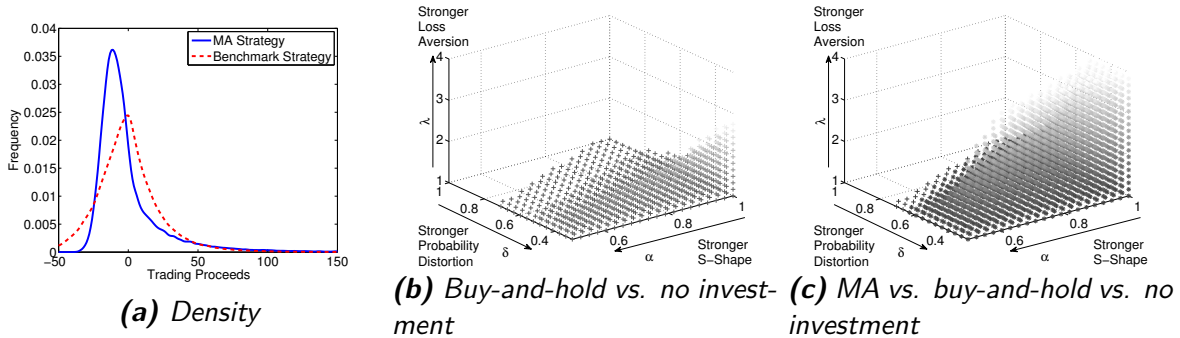
### 3.6.4 Autocorrelated returns

In this section, we assume stock price dynamics that imply that prices do move in trends over a time period of several months such that the MA strategy indeed is indeed profitable. Specifically, we employ a Markov-switching model as an extension of the geometric Brownian motion model: The stock  $S$  has the following dynamics:

$$dS_t = S(t) [\mu \cdot \eta(t)dt + \sigma dW(t)]$$

Here, the stock drift  $\mu$  is a function of the regime  $\eta$ .  $\eta$  can take the values  $+1$  and  $-1$ , specifying a positive or negative regime. The switching probabilities are equal, that is, a switch from positive to negative regime and from negative to positive regime are equal. The switching probabilities are 0.5% per day, resulting in a mean of 1.3 switches per year. For the study in Figure 3.15 we set  $\mu = 0.1$  and  $\sigma = 0.3$ .

**Figure 3.15: MA trading in a model of autocorrelated returns.** Part (a) of this figure shows the return distributions of MA (solid line) and the benchmark strategy (dashed line) when the stock price evolves according to an autocorrelated Markov switching model. The return of the MA (benchmark) strategy is TBD (zero). The skewness of the MA (benchmark) return distribution is 5.42 (1.45) and the expected holding time of both strategies is 148 days. Part (b) shows the 977 CPT parameter values (indicated by “+”) for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 19 parameter combinations while MA is preferred for 2726 parameter combinations (indicated by a “\*”).



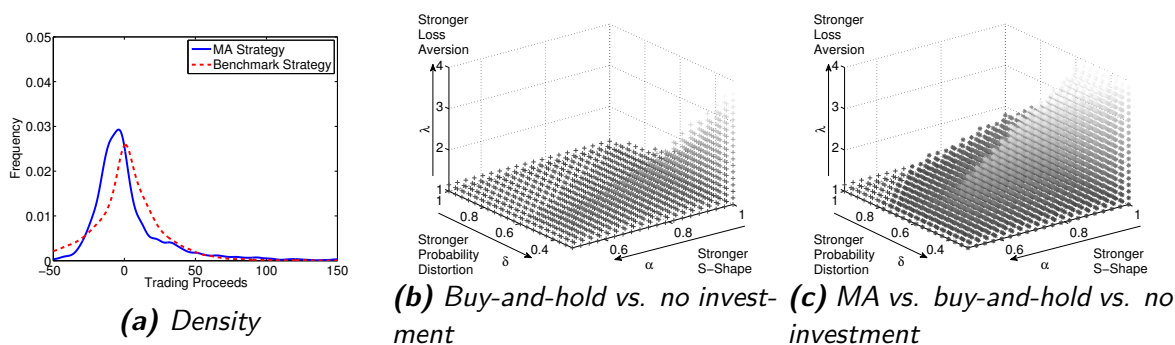
### 3.6.5 Trading MA on real data

In this section, we compute and analyze the MA return distribution based on real stock price data. We consider all stocks that were part of the STOXX Europe 600 at January 1, 2000 and monitor these stocks until June 30, 2013 or until de-listed. The MA return distribution is obtained as the empirical distribution of all possible MA trades of all 600 stocks within this sample period. For each stock we observe on average 5.1 buy signals



that are followed by a sell signal in the sample period, which implies a total number of 3070 trades. The average holding time of a trade equals 180 days. The return distribution for the random buy-and-hold benchmark strategy is constructed by sampling the same number of 4283 trades as follows: pick a stock at random, draw a buy time uniformly over the sample period and draw a holding time from the same distribution as the MA strategies' holding time.

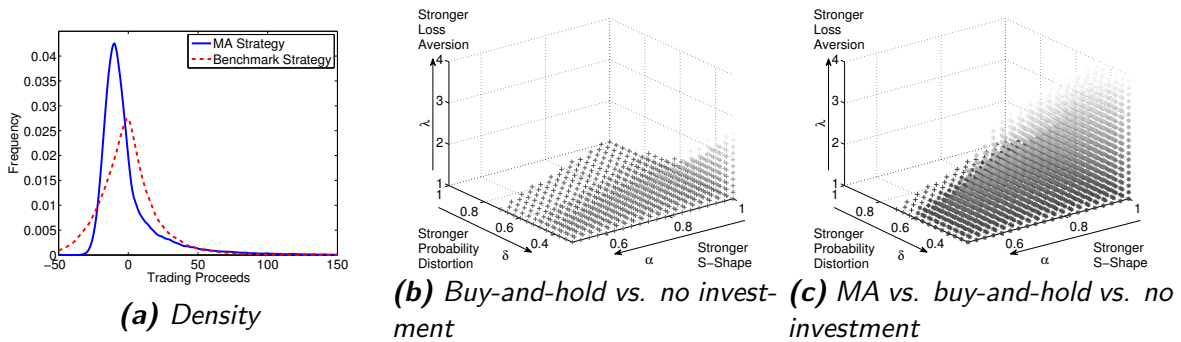
**Figure 3.16: MA trading on STOXX Europe 600 stocks.** Part (a) of this figure shows the return distributions of MA (solid line) and the benchmark strategy (dashed line) obtained from trading MA all stocks that were part of the STOXX Europe 600 at 1 January, 2000 until 30 June, 2013 or until de-listed. The mean return for the MA (benchmark) strategy is 5.9% (2.44%) for the benchmark strategy for an average holding time of 180 days. The skewness of the MA (benchmark) return distribution is 6.73 (3.47). Part (b) shows the 1728 CPT parameter values (indicated by “+”) for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 58 parameter combinations while MA is preferred for 3632 parameter combinations (indicated by a “\*”).



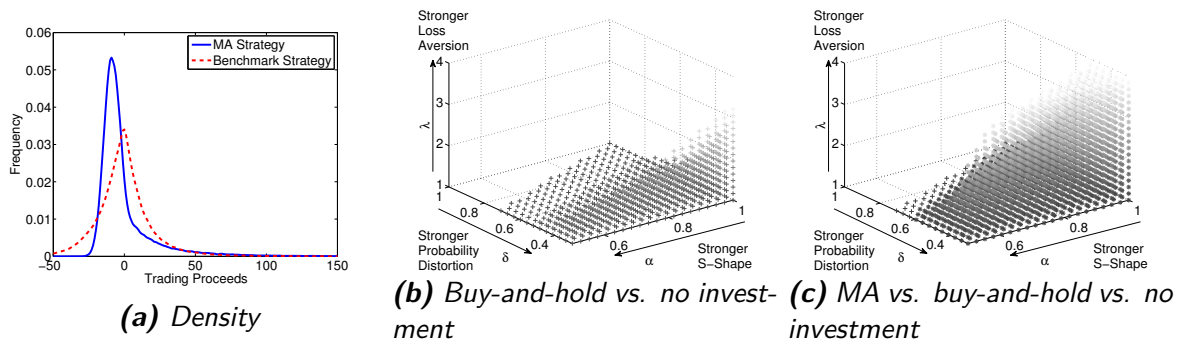
### 3.6.6 Timing of the MA strategy

In this section, we repeat our analysis for a MA strategy that is based on shorter short-term MA lines of length 22 and 37, respectively, while the the long-term MA line remains at 200 days. Furthermore, we repeat our analysis for a MA strategy that has a short long-term MA line of length 100, while the short-term MA remains at 50. The results are shown in Figures 3.17, 3.18, and 3.20. Shorter short-term MAs actually amplify the skewness effect of MA we observe, as stocks that start losing in value are sold even more quickly. The general observation is that the larger the difference in length between the long-term and short-term MA, i.e., the more flexible the short-term MA and the more sluggish the long-term MA, the more does MA skew to the right. This we illustrate also in Figure 3.19.

**Figure 3.17: MA trading with a 37-day short-term MA.** Part (a) of this figure shows the return distributions of MA(37,200) (solid line) and the benchmark strategy (dashed line). The skewness of the MA (benchmark) return distribution is 5.13 (1.27) and the expected holding time of both strategies is 132 days. Part (b) shows the 895 CPT parameter values (indicated by “+”) for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 26 parameter combinations while MA is preferred for 2586 parameter combinations (indicated by a “\*”).

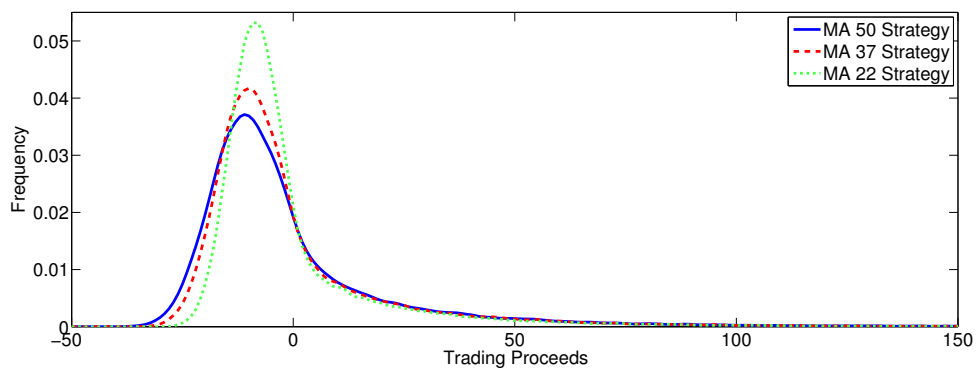


**Figure 3.18: MA trading with a 22-day short-term MA.** Part (a) of this figure shows the return distributions of MA(22,200) (solid line) and the benchmark strategy (dashed line). The skewness of the MA (benchmark) return distribution is 5.68 (1.56) and the expected holding time of both strategies is 109 days. Part (b) shows the 863 CPT parameter values (indicated by “+”) for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 29 parameter combinations while MA is preferred for 2769 parameter combinations (indicated by a “\*”).

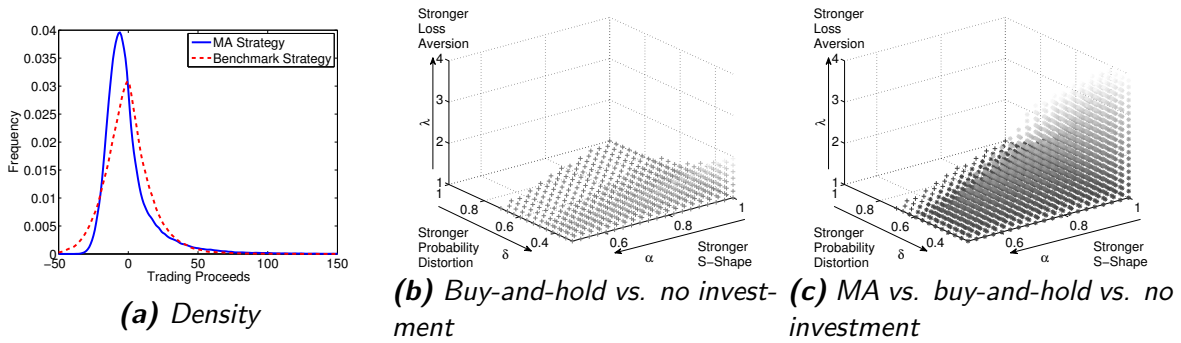




**Figure 3.19: Comparison of distributions from different MA strategies.** This graph shows the densities implied by the MA(50,200) (solid), the MA(37,200) (dashed), and the MA(22,200) (dotted) strategies.



**Figure 3.20: MA trading with a 100-day long-term MA.** Part (a) of this figure shows the return distributions of MA(50,100) (solid line) and the benchmark strategy (dashed line). The skewness of the MA (benchmark) return distribution is 4.08 (0.90) and the expected holding time of both strategies is 88 days. Part (b) shows the 679 CPT parameter values (indicated by “+”) for which the benchmark strategy is preferred over no investment. Part (c) compares stock investment according to the benchmark strategy, stock investment according to MA, and no stock investment. In that case, the benchmark strategy is preferred for 27 parameter combinations while MA is preferred for 2138 parameter combinations (indicated by a “\*”).



### 3.7 Conclusion

We have analyzed one of the most popular, trend-chasing technical trading strategies—the double-crossover moving average strategy—in a behavioral setting. Even when prices do not move in trends, when trading MA is not profitable, and when rational expected utility maximizers thus not trade MA, the popularity of MA can be reconciled with cumulative prospect theory preferences. While academics are generally skeptical towards technical analysis, we believe that this paper is the first to show that technical analysis can be exclusively reconciled from less than fully rational preferences.

Our result is a consequence of the observation that trading MA skews the distribution of trading proceeds to the right. This is attractive to many prospect theory investors who have strong preferences for right-skewed payoff distributions.

While we obtain this result explicitly for the MA strategy, we believe that the same rationale also applies to other trading strategies that chase momentum. That is, it lies in the nature of trend-chasing that one sells quickly upon a downward trend and holds on to the position in case of an upward trend. Chasing momentum thus makes small losses likely but also generates a small possibility of large gains. In other words, chasing trends results in right-skewed payoff distributions. The prevalence of momentum chasing may thus not only be explained with prices actually moving in trends—or a belief therein—but also by investors having strong preferences for positive skewness.

We conducted our analysis for the arguably most prominent model of less than fully

rational investor preferences: prospect theory. For most parametrizations that are observed in empirical settings, prospect theory implies skewness preference. However, other behavioral theories that imply skewness preference may just as well explain the attractiveness of trend-chasing. For example, the salience model of choice under risk by Bordalo, Gennaioli, and Shleifer (2012, 2013) also implies a taste for positive skewness and is thus a promising candidate to imply a preference for trend-chasing.



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